Chapter 11
Trading a Mean-Reverting Asset with Regime Switching: An Asymptotic Approach

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11.1 Introduction

This chapter is concerned with mean-reversion trading with regime switching. It is a continuation of the study developed in Zhang and Zhang [15]. In [15], a mean-reversion trading rule was considered. The objective was to buy and sell the asset so as to maximize an overall return. They followed the dynamic programming approach and used the associated HJB equations (quasi-variational inequalities) to characterize the value functions. They showed that the solution to the original optimal stopping problem can be obtained by solving two quasi-algebraic equations. In addition, they obtained sufficient conditions in the form of a verification theorem. Nevertheless, only the basic mean-reversion model with constant equilibrium was considered in [15]. It is important to extend the results to account for more realistic settings. It is the purpose of this chapter to consider the mean-reversion model in which the equilibrium is subject to random jumps governed by a two-state Markov chain and to study the corresponding trading rules.

A mean-reversion model is often used in financial and energy markets to capture price movements that have the tendency to move towards an "equilibrium" level. Studies that support the mean-reversion stock returns can be traced back to the 1930s (see Cowles and Jones [3]) in empirical literature. The research was furthered by many researchers including Fama and French [6], and Gallagher and Taylor [7] among others. In addition to stock markets, mean-reversion models are also used to characterize stochastic volatility (see Hafner and Herwartz [8]) and asset prices in energy markets (see Blanco and Soronow [1] and de Jong and Huisman [4]). See

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also related results in option pricing with a mean-reversion asset by Bos, Ware and Pavlov [2].

Trading rules in financial markets have been studied for many years. For example, an investment capacity expansion/reduction problem was considered in Merhi and Zervos [11]. Under a geometric Brownian motion market model, the authors used the dynamic programming approach and obtained an explicit solution to the singular control problem. A more general diffusion market model was treated by Løkka and Zervos [10] in connection with an optimal investment capacity adjustment problem. More recently, Johnson and Zervos [9] studied an optimal timing of investment problem under a general diffusion market model. The objective was to maximize the expected cash flow by choosing when to enter an investment and when to exit the investment. An explicit analytic solution was obtained in [9]. Recently, Dai et al. [5] provided a theoretical justification of trend following trading. In particular, the underlying stock price was formulated as a geometric Brownian motion with regime switching. Two regimes were considered: the up trend (bull market) and the down trend (bear market). The switching process was modeled as a two-state Markov chain which is not directly observable. The trading decisions were based on current information represented by both the stock price and historical information with the probability in the bull phase conditioning to all available historical price levels as a proxy. Assuming trading one share with a fixed percentage transaction cost, they showed that the strategy that optimizes the discounted expected return is a simple implementable trend following system. This strategy was characterized by two threshold curves for the conditional probability in a bull regime signaling buy and sell, respectively. The main advantage of this approach is that the conditional probability in a bull market can be obtained directly using actual historical stock price data through a differential equation.

In this chapter, we focus on a mean-reversion model in which its equilibrium is subject to random jumps. Such model can be applied to assets with a "staircase" price behavior. We consider trading involving both buying and selling actions. The objective is to buy and sell the underlying asset sequentially in order to maximize a discounted reward function. Slippage cost associated with each transaction is imposed. We assume that a fixed percentage slippage cost is incurred with each transaction. In general, this is a class of challenging problems because a closed-form solution is difficult to obtain. In this chapter, we consider the case in which the underlying Markov chain jumps frequently between its two states. This leads to a class of singular perturbation problems. The idea is to approximate the value functions of the original problem by the value functions of a limiting problem. The limiting problem is easier to solve. The solution of the limiting problem leads to admissible trading rules that are typically as good as the optimal ones for the original problem. There are substantial studies along the line of singular perturbations. We refer the readers to Sethi and Zhang [13] and Yin and Zhang [14] for related literature. In this chapter, we study the problem using the dynamic programming approach and establish the associated HJB equations (quasi-variational inequalities) for the value functions. Following a viscosity solution approach, we establish asymptotic properties of the value functions. Then using a numerical example, we show how the solution for the limiting problem can be used to construct a set of trading rules for the original problem.

This chapter is organized as follows. In Sect. 11.2, we formulate the problem under consideration. In Sect. 11.3, we study properties of the value functions and the associated HJB equations. In Sect. 11.4, we provide asymptotic properties of the value functions and describe the corresponding limiting problem. In Sect. 11.5, we demonstrate further related approximation schemes. A numerical example is given in Sect. 11.6 in which the closed-form solution obtained in [15] is used to construct a trading rule for the original problem. The performance of the trading rule is provided in this example. Finally, some concluding remarks are provided in Sect. 11.7. Some technical definitions and assumption verification details are given in Appendix.

11.2 Problem Formulation

Let $X_t \in \mathbb{R}$ denote a mean-reverting diffusion with regime-switching governed by

$$dX_t = a(b(\alpha_t) - X_t)dt + \sigma(\alpha_t)dW_t,\ X_0 = x, \tag{11.1}$$

where $a > 0$ is the rate of reversion, $b(j)$, $j = 1,2$, is the equilibrium level for each state, $\sigma(j) > 0$, $j = 1,2$, is the volatility, $\alpha_t \in \{1,2\}$ is a two-state Markov chain, and W_t is a standard Brownian motion. In this chapter, we assume that α_t and W_t are independent.

Let $h(x)$ be a smooth function. We consider the model in which the asset price is given by $S_t = h(X_t)$. For example, the function $h(x) = e^x$ is used in Zhang and Zhang [15]. In this chapter, we consider $h(x)$ that equals e^x except when x is large. The main reason for specifying $h(x)$ is to facilitate subsequent analysis without affecting much of the applicability.

Let

$$0 \le \phi_1 \le \psi_1 \le \phi_2 \le \psi_2 \le \cdots \tag{11.2}$$

denote a sequence of stopping times. A buying decision is made at ϕ_k and a selling decision at ψ_k, $k = 1,2,\ldots$.

We consider the case that the net position at any time can be either flat (no stock holding) or long (with one share of stock holding). Let $i = 0,1$ denote the initial net position. If initially the net position is long ($i = 1$), then one should sell the stock before acquiring a share. The corresponding sequence of stopping times is denoted by $\Lambda_1 = (\psi_1, \phi_2, \psi_2, \phi_3, \ldots)$. Likewise, if initially the net position is flat ($i = 0$), then one should first buy a stock before selling a share. The corresponding sequence of stopping times is denoted by $\Lambda_0 = (\phi_1, \psi_1, \phi_2, \psi_2, \ldots)$.

In addition, we consider the problem with at most N round trips of trading. We use the notation $\Lambda_1^n = (\psi_1, \phi_2, \psi_2, \phi_3, \ldots, \phi_n, \psi_n)$ and $\Lambda_0^n = (\phi_1, \psi_1, \phi_2, \psi_2, \ldots, \phi_n, \psi_n)$ to label the corresponding stopping times limited to n round trips for $n = 0,1,\ldots,N$.

Let $0 < K < 1$ denote the percentage of slippage (or commission) per transaction. Given the initial states $X_0 = x$, $\alpha_0 = \alpha$, and initial net position $i = 0, 1$, the reward functions of the decision sequences $\{\Lambda_i^n, n = 0, 1, \ldots, N\}$ are given as follows:

$$J_i^n(x,\alpha,\Lambda_i^n) = \begin{cases} E\left\{ \sum_{k=1}^{n} \left[e^{-\rho\psi_k} S_{\psi_k}(1-K) - e^{-\rho\phi_k} S_{\phi_k}(1+K) \right] \right\}, & \text{if } i = 0, \\ E\left\{ e^{-\rho\psi_1} S_{\psi_1}(1-K) \right. \\ \quad \left. + \sum_{k=2}^{n} \left[e^{-\rho\psi_k} S_{\psi_k}(1-K) - e^{-\rho\phi_k} S_{\phi_k}(1+K) \right] \right\}, & \text{if } i = 1, \end{cases} \tag{11.3}$$

where $\rho > 0$ is the discount factor.

For $i = 0, 1$ and $n = 0, 1, \ldots, N$, let $V_i^n(x,\alpha)$ denote the value functions with the initial state $(X_0, \alpha_0) = (x, \alpha)$ and initial net positions $i = 0, 1$. That is,

$$V_i^n(x,\alpha) = \sup_{\Lambda_i^n} J_i^n(x,\alpha,\Lambda_i^n). \tag{11.4}$$

Remark 11.1. In (29), we allow the equalities, i.e., one is allowed to buy and sell at the same time. Nevertheless, owing to the existence of the positive slippage cost K, simultaneous buying and selling only cause negative returns and therefore are automatically ruled out by optimality conditions.

Let $Q = (q_{ij})$ denote the generator of α_t and let $\mathscr{A}$ denote the generator of (X_t, α_t), i.e.,

$$\mathscr{A} f(x,\alpha) = a(b(\alpha) - x)\frac{\partial f(x,\alpha)}{\partial x} + \frac{\sigma^2(\alpha)}{2}\frac{\partial^2 f(x,\alpha)}{\partial x^2} + Qf(x,\cdot)(\alpha),$$

where $Qf(x,\cdot)(\alpha) = q_{\alpha 1} f(x,1) + q_{\alpha 2} f(x,2)$, $\alpha = 1, 2$.

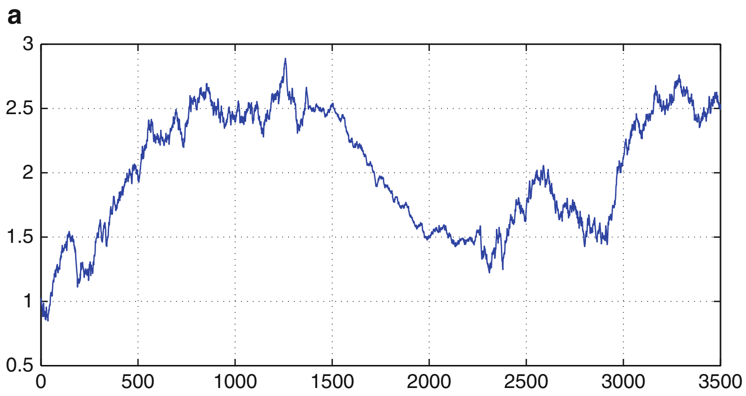
In Fig. 11.1, a sample path of (X_t, α_t) is provided. The picture was generated using the Monte Carlo method with

$$a = 0.8,\ b(1) = 3,\ b(2) = 1,\ \sigma(1) = 0.7,\ \sigma(2) = 0.3,\ Q = \begin{pmatrix} -0.91 & 0.91 \\ 0.62 & -0.62 \end{pmatrix},\ X_0 = 1.$$

It is clear from Fig. 11.1, when $\alpha_t = 1$, the equilibrium $b(1) = 3$ serves as an attractor for X_t pulling it upwards; when α_t switched to 2, the new equilibrium $b(2) = 1$ pulls X_t downwards and so on.

As mentioned in the introduction, a closed-form solution to the problem is difficult to obtain. In this chapter, we consider the case in which the Markov chain jumps frequently between its two states. We aim at the corresponding asymptotic properties. In particular, we consider case where the generator has the following form:

$$Q^\varepsilon = \frac{1}{\varepsilon}\widetilde{Q} + \widehat{Q} = \frac{1}{\varepsilon}\begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix} + \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \mu_1 & -\mu_1 \end{pmatrix},$$



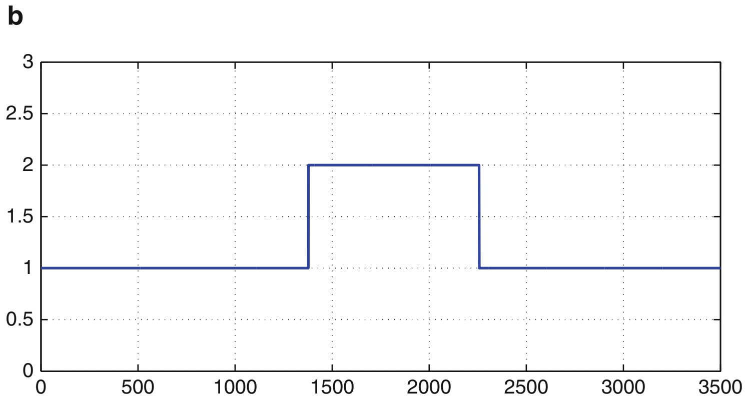


Fig. 11.1 Mean-reversion with regime switching **(a)** A sample path of X_t, **(b)** a sample path of α_t

where $\varepsilon > 0$ is a small parameter, and λ, μ, λ_1, and μ_1 are positive constants. We study the convergence of the problem as $\varepsilon \to 0$. For related Markov models in connection with manufacturing systems, see Sethi and Zhang [13].

Remark 11.2. The Markov chain α_t generated by Q^ε represents the regime of the underlying market. We focus on the market with frequent regime changes in α_t. Such a scenario often arises in a prolonged sideways market such as Dow Jones Industrial Average during the 1960s and 1980s. Its behavior can be captured by our regime-switching model with a relatively small ε. In this chapter, we aim at models with a not-so-small ε and construct near optimal trading rules from the optimal solution of the corresponding limiting problem as $\varepsilon \to 0$. A major advantage of our approach is that one does not have to identify the state of α_t, which is difficult during the period when it is changing rapidly.

The corresponding Markov chain will be labeled as α_t^ε. Similarly, we use X_t^ε for X_t, S_t^ε for S_t, $J_i^{n,\varepsilon}$ for J_i^n, and $V_i^{n,\varepsilon}$ for V_i^n from now on to emphasize the dependence on ε. Using this notation, the optimal trading problem $\mathscr{P}^{N,\varepsilon}$ can be written as follows:

$$\mathscr{P}^{N,\varepsilon}: \begin{cases} \max & J_i^{n,\varepsilon}(x,\alpha,\Lambda_i^n) \\ & = \begin{cases} E\left\{\sum_{k=1}^{n}\left[e^{-\rho\psi_k}S^{\varepsilon}_{\psi_k}(1-K)-e^{-\rho\phi_k}S^{\varepsilon}_{\phi_k}(1+K)\right]\right\}, & \text{if } i=0, \\ E\left\{e^{-\rho\psi_1}S^{\varepsilon}_{\psi_1}(1-K)\right. \\ \quad \left.+\sum_{k=2}^{n}\left[e^{-\rho\psi_k}S^{\varepsilon}_{\psi_k}(1-K)-e^{-\rho\phi_k}S^{\varepsilon}_{\phi_k}(1+K)\right]\right\}, & \text{if } i=1, \end{cases} \\ \text{s.t.} & dX_t^{\varepsilon}=a(b(\alpha_t^{\varepsilon})-X_t^{\varepsilon})dt+\sigma(\alpha_t^{\varepsilon})dW_t,\ X_0^{\varepsilon}=x, \\ \text{value fn } & V_i^{n,\varepsilon}(x,\alpha)=\sup_{\Lambda_i^n}J_i^{n,\varepsilon}(x,\alpha,\Lambda_i^n), \qquad n=0,1,\dots,N, \end{cases}$$

Note that the sequence $\Lambda_0^n=(\phi_1,\psi_1,\dots,\phi_n,\psi_n)$ can be regarded as a combination of a buy at ϕ_1 and then followed by the sequence of stopping times $\Lambda_1^n=(\psi_1,\phi_2,\psi_2,\dots,\phi_n,\psi_n)$. In view of this, we have

$$\begin{aligned} V_0^{n,\varepsilon}(x,\alpha) &\geq J_0^{n,\varepsilon}(x,\alpha,\Lambda_0^n) \\ &= E\left\{e^{-\rho\psi_1}S^{\varepsilon}_{\psi_1}(1-K)+\sum_{k=2}^{n}\left[e^{-\rho\psi_k}S^{\varepsilon}_{\psi_k}(1-K)-e^{-\rho\phi_k}S^{\varepsilon}_{\phi_k}(1+K)\right]\right\} \\ &\quad -Ee^{-\rho\phi_1}S^{\varepsilon}_{\phi_1}(1+K) \\ &= J_1^{n,\varepsilon}(X^{\varepsilon}_{\phi_1},\alpha,\Lambda_1^n)-Ee^{-\rho\phi_1}S^{\varepsilon}_{\phi_1}(1+K). \end{aligned}$$

In particular, setting $\phi_1=0$ (recall that $S_t^{\varepsilon}=h(X_t^{\varepsilon})$), we obtain the inequality

$$V_0^{n,\varepsilon}(x,\alpha)\geq V_1^{n,\varepsilon}(x,\alpha)-h(x)(1+K). \tag{11.5}$$

Similarly, we can show that

$$V_1^{n,\varepsilon}(x,\alpha)\geq V_0^{n-1,\varepsilon}(x,\alpha)+h(x)(1-K). \tag{11.6}$$

Formally, the associated HJB equations should have the form:

$$\begin{aligned} &\min\left\{\rho V_0^{n,\varepsilon}(x,\alpha)-\mathscr{A}V_0^{n,\varepsilon}(x,\alpha),\ V_0^{n,\varepsilon}(x,\alpha)-V_1^{n,\varepsilon}(x,\alpha)+h(x)(1+K)\right\}=0, \\ &\min\left\{\rho V_1^{n,\varepsilon}(x,\alpha)-\mathscr{A}V_1^{n,\varepsilon}(x,\alpha),\ V_1^{n,\varepsilon}(x,\alpha)-V_0^{n-1,\varepsilon}(x,\alpha)-h(x)(1-K)\right\}=0, \end{aligned} \tag{11.7}$$

for $n=1,2,\dots,N$ and $\alpha=1,2$. Here, we follow the convention that $V_0^{0,\varepsilon}(x,\alpha)=0$.

Next, we impose conditions on $h(x)$.

Assumption. $h(x)$, $h'(x)$, $xh'(x)$, and $h''(x)$ are bounded and Lipschitz.

Example 11.1. An immediate example satisfying the above conditions can be given as follows. Let $h_0(x) = \begin{cases} e^x & \text{for } x \le M, \\ e^M & \text{for } x > M, \end{cases}$ for a fixed M. Take $h(x)$ to be the convolution of h_0 with the kernel $\Psi(x) = (1/\sqrt{2\pi})e^{-x^2/2}$. Validation of these conditions is provided in Appendix.

Under these assumptions, we can show, following a similar approach as in Sethi and Zhang [13, Chap. 8], that $V_i^{n,\varepsilon}(x,\alpha)$ are the viscosity solutions (see the definition given in Appendix) of the HJB equations (34).

In this chapter, C (and C_i) are generic positive constants with convention $C + C = C$ and $CC = C$, etc.

11.3 Properties of the Value Functions

In this section, we consider the basic properties of the value functions. In particular, we establish the boundedness and Lipschitz continuity of these functions.

Lemma 11.1. *There exists a constant* C_0 *such that*

$$0 \le V_i^{n,\varepsilon}(x,\alpha) \le C_0,$$

for $\varepsilon > 0$*,* $x \in \mathbb{R}$*,* $\alpha = 1,2$*,* $i = 0,1$*, and* $n = 0,1,\dots,N$*.*

Proof. In view of the definition of $V_i^{n,\varepsilon}(x,\alpha)$, it is clear that they are nonnegative. It remains to establish their upper bounds. Let

$$F(x,\alpha) = a(b(\alpha) - x)h'(x) + \frac{\sigma^2(\alpha)}{2}h''(x) - \rho h(x).$$

Then, using Dynkin's formula, we have

$$Ee^{-\rho\psi_k}S^{\varepsilon}_{\psi_k} - Ee^{-\rho\phi_k}S^{\varepsilon}_{\phi_k} = E\int_{\phi_k}^{\psi_k} e^{-\rho s}F(X_s^{\varepsilon},\alpha_s)ds. \tag{11.8}$$

It is easy to see that the function $F(x,\alpha)$ is bounded above on $\mathbb{R}$ by the boundedness assumptions on $h(x)$. Let C be an upper bound of F. It follows that

$$Ee^{-\rho\psi_k}S^{\varepsilon}_{\psi_k} - Ee^{-\rho\phi_k}S^{\varepsilon}_{\phi_k} \le CE\int_{\phi_k}^{\psi_k} e^{-\rho t}dt. \tag{11.9}$$

Using the definition of $J_0^{n,\varepsilon}(x,\alpha,\Lambda_0^n)$, we have

$$\begin{aligned} J_0^{n,\varepsilon}(x,\alpha,\Lambda_0^n) &\leq \sum_{k=1}^{n}\left(Ee^{-\rho\psi_k}S_{\psi_k}^{\varepsilon} - Ee^{-\rho\phi_k}S_{\phi_k}^{\varepsilon}\right) \\ &\leq \sum_{k=1}^{n} CE\int_{\phi_k}^{\psi_k} e^{-\rho t}dt \\ &\leq C\int_0^{\infty} e^{-\rho t}dt := C_0. \end{aligned}$$

This implies that $0 \leq V_0^{n,\varepsilon}(x,\alpha) \leq C_0$.

Similarly, letting $C_h = \sup|h(x)|$, we have the inequalities

$$J_1^{n,\varepsilon}(x,\alpha,\Lambda_1^n) \leq C_0 + Ee^{-\rho\psi_1}h(X_{\psi_1}^{\varepsilon})(1-K) \leq C_0 + C_h(1-K) := C_0.$$

Therefore, $0 \leq V_1^{n,\varepsilon}(x,\alpha) \leq C_0$. This completes the proof. □

Lemma 11.2. $V_i^{n,\varepsilon}(x,\alpha)$ *are Lipschitz, i.e., there exists* C_0 *such that*

$$|V_i^{n,\varepsilon}(x_1,\alpha) - V_i^{n,\varepsilon}(x_2,\alpha)| \leq C_0|x_1 - x_2|.$$

for $\varepsilon > 0$, $x_1, x_2 \in \mathbb{R}$, $\alpha = 1,2$, $i = 0,1$, *and* $n = 0,1,\ldots,N$.

Proof. Given x_1 and x_2, let X_t^1 and X_t^2 be solutions of (28) with $X_0^1 = x_1$ and $X_0^2 = x_2$, respectively. We claim that: There exists an constant C_0 such that for any stopping time τ,

$$\left|E\left[e^{-\rho\tau}(h(X_\tau^1) - h(X_\tau^2)\right]\right| \leq C_0|x_1 - x_2|. \tag{11.10}$$

Let

$$G(x,y,\alpha) = ab(\alpha)[h'(x) - h'(y)] - a[xh'(x) - yh'(y)] + \frac{\sigma^2(\alpha)}{2}[h''(x) - h''(y)] - \rho[h(x) - h(y)].$$

Then, using the Lipschitz assumptions on $h(x)$, we can see that

$$|G(x,y,\alpha)| \leq C_0|x-y|,$$

for some constant C_0. Then, applying Dynkin's formula, we have

$$E\left[e^{-\rho\tau}(h(X_t^1) - h(X_t^2))\right] = h(x_1) - h(x_2) + E\int_0^{\tau} e^{-\rho t}G(X_t^1, X_t^2, \alpha_t)dt.$$

It follows that

$$\begin{aligned} \left|E\left[e^{-\rho\tau}(h(X_t^1) - h(X_t^2))\right]\right| &\leq |h(x_1) - h(x_2)| + E\int_0^{\infty} e^{-\rho t}|G(X_t^1, X_t^2, \alpha_t)|dt \\ &\leq C_0|x_1 - x_2| + C_0 E\int_0^{\infty} e^{-\rho t}|X_t^1 - X_t^2|dt. \end{aligned}$$

Note that

$$X_t^1 - X_t^2 = x_1 - x_2 - a\int_0^t (X_s^1 - X_s^2)ds.$$

Therefore, $X_t^1 - X_t^2 = (x_1 - x_2)e^{-at}$. In view of this, we have

$$\left|E\left[e^{-\rho\tau}(h(X_t^1) - h(X_t^2))\right]\right| \leq C_0|x_1 - x_2| + C_0 E\int_0^\infty e^{-\rho t}|x_1 - x_2|e^{-at}dt = C_0|x_1 - x_2|,$$

which proves the claim. Using this inequality, for any given Λ_i^n, it is easy to see that

$$|J_i^{n,\varepsilon}(x_1, \alpha, \Lambda_i^n) - J_i^{n,\varepsilon}(x_2, \alpha, \Lambda_i^n)| \leq C_0|x_1 - x_2|. \qquad \square$$

11.4 Asymptotic Properties

In this section, we study the asymptotic properties of the value functions as $\varepsilon \to 0$. We first characterize the limiting problem and then establish the desired convergence.

Lemma 11.3. *For each* (x, α)*, if for some subsequence of* ε*,* $V_i^{n,\varepsilon}(x, \alpha) \to V_i^{n,0}(x, \alpha)$*, then* $V_i^{n,0}(x, \alpha) = V_i^{n,0}(x)$*.*

Proof. Let τ^ε denote the first jump time of α_t^ε. Then $\tau^\varepsilon \to 0$ a.s. as $\varepsilon \to 0$. Following the dynamic programming principle, we have

$$V_i^{n,\varepsilon}(x, \alpha) \geq Ee^{-\rho\tau^\varepsilon} V_i^{n,\varepsilon}(X_{\tau^\varepsilon}, \alpha_{\tau^\varepsilon}).$$

If $\alpha = 1$, then sending $\varepsilon \to 0$, we have

$$V^{n,0}(x, 1) \geq V^{n,0}(x, 2).$$

Similarly,

$$V^{n,0}(x, 2) \geq V^{n,0}(x, 1).$$

Therefore, $V^{n,0}(x, 1) = V^{n,0}(x, 2)$. $\square$

Let (ν_1, ν_2) denote the equilibrium distribution corresponding to $\widetilde{Q}$, i.e.,

$$(\nu_1, \nu_2) = \left(\frac{\mu}{\lambda + \mu}, \frac{\lambda}{\lambda + \mu}\right).$$

so that $(\nu_1, \nu_2)\widetilde{Q} = (0, 0)$. Let $\overline{X}_t$ denote the corresponding mean-reversion process with mean $\overline{b} = \nu_1 b(1) + \nu_2 b(2)$ and volatility $\overline{\sigma} = \sqrt{\nu_1\sigma^2(1) + \nu_2\sigma^2(2)}$. The stock price driven by $\overline{X}_t$ is denoted by $\overline{S}_t = h(\overline{X}_t)$.

Given a sequence of $\sigma\{W_r : r \le t\}$ measurable stopping times

$$0 \le \phi_1 \le \psi_1 \le \phi_2 \le \psi_2 \le \cdots,$$

one can define the set of stopping times Λ_i^n as before for $n = 0, 1, \ldots, N$ and $i = 0, 1$. The limiting problem $\mathscr{P}^{N,0}$ can be defined as follows:

$$\mathscr{P}^{N,0}: \begin{cases} \max & \overline{J}_i^n(x, \Lambda_i^n) \\ & = \begin{cases} E\left\{\sum_{k=1}^{n}\left[e^{-\rho\psi_k}\overline{S}_{\psi_k}(1-K) - e^{-\rho\phi_k}\overline{S}_{\phi_k}(1+K)\right]\right\}, & \text{if } i = 0, \\ E\left\{e^{-\rho\psi_1}\overline{S}_{\psi_1}(1-K) \right. \\ \quad \left. + \sum_{k=2}^{n}\left[e^{-\rho\psi_k}\overline{S}_{\psi_k}(1-K) - e^{-\rho\phi_k}\overline{S}_{\phi_k}(1+K)\right]\right\}, & \text{if } i = 1, \end{cases} \\ \text{s.t.} & d\overline{X}_t = a(\overline{b} - \overline{X}_t)dt + \overline{\sigma}dW_t,\ \overline{X}_0 = x, \\ \text{value fn} & \overline{V}_i^n(x) = \sup_{\Lambda_i^n} \overline{J}_i(x, \Lambda_i^n). \end{cases}$$

Let $\overline{\mathscr{A}}$ denote the generator of $\overline{X}_t$, i.e.,

$$\overline{\mathscr{A}} f(x) = a(\overline{b} - x)\frac{df(x)}{dx} + \frac{\overline{\sigma}^2}{2}\frac{d^2 f(x)}{dx^2}.$$

The associated HJB equations for the limiting problem should have the form:

$$\begin{aligned} &\min\left\{\rho\overline{V}_0^n(x) - \overline{\mathscr{A}}\overline{V}_0^n(x),\ \overline{V}_0^n(x) - \overline{V}_1^n(x) + h(x)(1+K)\right\} = 0, \\ &\min\left\{\rho\overline{V}_1^n(x) - \overline{\mathscr{A}}\overline{V}_1^n(x),\ \overline{V}_1^n(x) - \overline{V}_0^{n-1}(x) - h(x)(1-K)\right\} = 0, \end{aligned} \tag{11.11}$$

for $n = 1, 2, \ldots, N$.

The definition of viscosity solution of the above HJB equations is also given in Appendix. We can show the following lemma, where the uniqueness can be obtained along the line of Pham [12].

Lemma 11.4. $\overline{V}_i^n(x)$ *are the unique viscosity solutions of the HJB equations (38).*

Next, we give the main result of this chapter. We show that the value functions of the original problem converge to those of the limiting problem. This suggests that the optimal solution of the limiting problem can be used to construct a trading rule for the original problem. We refer the readers to Sethi and Zhang [13] for similar approach in connection with manufacturing systems.

Theorem 11.1. *As* $\varepsilon \to 0$, *we have*

$$V_i^{n,\varepsilon}(x, \alpha) \to \overline{V}_i^n(x),$$

for $n = 0, 1, \ldots, N$, $i = 0, 1$, $x \in \mathbb{R}$, *and* $\alpha = 1, 2$.

Proof. Recall the Lipschitz properties of $V_i^{n,\varepsilon}$ in Lemma 11.2. In view of the Arzela–Ascoli Theorem, for each sequence of $\{\varepsilon \to 0\}$, there exists a further subsequence (still indexed by ε) such that $V_i^{n,\varepsilon}(x,\alpha)$ converges. Denote the limit by $V_i^{n,0}(x,\alpha)$. Then by Lemma 11.3, $V_i^{n,0}(x,\alpha) = V_i^{n,0}(x)$. It suffices to show that $V_i^{n,0}(x)$ is a viscosity solution of (38) because Lemma 11.4 implies that $V_i^{n,0}(x) = \overline{V}_i^n(x)$. Following Lemma A.25 in Yin and Zhang [14], for each $i = 0,1$, take a function $\phi_i(x) \in C^2$ such that $V_i^{n,0}(x) - \phi_i(x)$ has a strictly local maximum at any given x_0 in a neighborhood $N(x_0)$. Choose $x_{i,\alpha}^{n,\varepsilon} \in N(x_0)$ such that

$$V_i^{n,\varepsilon}(x_{i,\alpha}^{n,\varepsilon},\alpha) - \phi_i(x_{i,\alpha}^{n,\varepsilon}) = \max_{x\in N(x_0)} \{V_i^{n,\varepsilon}(x,\alpha) - \phi_i(x)\}.$$

Then, $x_{i,\alpha}^{n,\varepsilon} \to x_0$, as $\varepsilon \to 0$. First, fix $i = 0$. We are to show the following inequality:

$$\min\left\{\rho V_0^{n,0}(x_0) - \overline{\mathscr{A}}\phi_0(x_0), V_0^{n,0}(x_0) - V_1^{n,0}(x_0) + h(x_0)(1+K)\right\} \le 0. \quad (11.12)$$

If

$$V_0^{n,0}(x_0) - V_1^{n,0}(x_0) + h(x_0)(1+K) \le 0,$$

then (39) holds. Otherwise,

$$V_0^{n,0}(x_0) - V_1^{n,0}(x_0) + h(x_0)(1+K) > 0.$$

Then there exists $N_0(x_0) \subset N(x_0)$ such that

$$V_0^{n,\varepsilon}(x) - V_1^{n,\varepsilon}(x) + h(x)(1+K) > 0$$

on $N_0(x_0)$ for ε small enough. Recall that $V_i^{n,\varepsilon}$ is a viscosity solution to (34). $V_0^{n,\varepsilon}(x,\alpha)$ must satisfy (45). Necessarily,

$$\rho V_0^{n,\varepsilon}(x_{0,\alpha}^{n,\varepsilon},\alpha) - \mathscr{A}^{\phi_0} V_0^{n,\varepsilon}(x_{0,\alpha}^{n,\varepsilon},\alpha) \le 0,$$

for $\alpha = 1,2$.

It follows that

$$\nu_1(\rho V_0^{n,\varepsilon}(x_{0,1}^{n,\varepsilon},1) - \mathscr{A}^{\phi_0} V_0^{n,\varepsilon}(x_{0,1}^{n,\varepsilon},1)) + \nu_2(\rho V_0^{n,\varepsilon}(x_{0,2}^{n,\varepsilon},2) - \mathscr{A}^{\phi_0} V_0^{n,\varepsilon}(x_{0,2}^{n,\varepsilon},2)) \le 0. \quad (11.13)$$

Note that

$$\begin{aligned}
&\nu_1\left(\frac{\lambda}{\varepsilon}\right)(V_0^{n,\varepsilon}(x_1^\varepsilon,2) - V_0^{n,\varepsilon}(x_1^\varepsilon,1)) + \nu_2\left(\frac{\mu}{\varepsilon}\right)(V_0^{n,\varepsilon}(x_2^\varepsilon,1) - V_0^{n,\varepsilon}(x_2^\varepsilon,2)) \\
&\quad \le \nu_1\left(\frac{\lambda}{\varepsilon}\right)[V_0^{n,\varepsilon}(x_2^\varepsilon,2) - \phi(x_2^\varepsilon) - (V_0^{n,\varepsilon}(x_1^\varepsilon,1) - \phi(x_1^\varepsilon))] \\
&\qquad + \nu_2\left(\frac{\mu}{\varepsilon}\right)[V_0^{n,\varepsilon}(x_1^\varepsilon,1) - \phi(x_1^\varepsilon) - (V_0^{n,\varepsilon}(x_2^\varepsilon,2) - \phi(x_2^\varepsilon))] = 0.
\end{aligned} \quad (11.14)$$

Using this inequality and sending $\varepsilon \to 0$ in (40) to obtain $\rho V_0^{n,0}(x_0) - \overline{\mathscr{A}}\phi_0(x_0) \le 0$, which yields (39). Similarly, we can show

$$\min\left\{\rho V_1^{n,0}(x_0) - \overline{\mathscr{A}}\phi_1(x_0), V_0^{n,0}(x_0) - V_0^{n-1,0}(x_0) - h(x_0)(1-K)\right\} \le 0.$$

Thus, $V_i^{n,0}(x)$ is a viscosity subsolution to (38).

To show that $V_i^{n,0}(x)$ is a viscosity supersolution to (38), note that

$$\min\left\{\rho V_0^{n,\varepsilon}(x_0,\alpha_0) - \mathscr{A}^{\psi} V_0^{n,\varepsilon}(x_0,\alpha_0),\ V_0^{n,\varepsilon}(x_0,\alpha_0) - V_1^{n,\varepsilon}(x_0,\alpha_0) + h(x)(1+K)\right\} \ge 0$$

implies

$$V_0^{n,0}(x_0) - V_1^{n,0}(x_0) + h(x)(1+K) \ge 0.$$

Moreover, following similar argument as in (41), we can show that

$$\rho V_0^{n,0}(x_0) - \overline{\mathscr{A}}\psi_0(x_0) \ge 0.$$

Hence,

$$\min\left\{\rho V_0^{n,0}(x_0) - \overline{\mathscr{A}}\psi_0(x_0),\ V_0^{n,0}(x_0) - V_1^{n,0}(x_0) + h(x)(1+K)\right\} \ge 0.$$

Similarly, we can show the inequality with $i = 1$. Therefore $V_i^{n,0}(x)$ is a viscosity supersolution. This completes the proof. □

11.5 Further Approximations

In this section, we show that the value function $\overline{V}_i^n(x)$ can be further approximated by taking N to be very large and $h(x)$ to be very close to e^x. In this case, we can use the closed-form solution obtained in Zhang and Zhang [15] to come up with an approximate solution for the original problem.

Recall the definition of Λ_i and its N-th round trip truncation Λ_i^N. Let

$$\overline{J}_i(x,\Lambda_i) = \limsup_{N\to\infty} \overline{J}_i^N(x,\Lambda_i^N)$$

and $\overline{V}_i(x) = \sup_{\Lambda_i} \overline{J}_i(x,\Lambda_i)$. It is easy to see that

$$\lim_{N\to\infty} \overline{V}_i^N(x) = \overline{V}_i(x).$$

In fact, for each $\delta > 0$, let $\Lambda_{i,\delta}$ be a sequence of stopping times such that $\overline{J}_i(x,\Lambda_{i,\delta}) \ge \overline{V}_i(x) - \delta$. Then, noticing that $\overline{V}_i^N(x)$ is monotonically increasing in N, we have

$$\overline{V}_i(x) - \delta \le \overline{J}_i(x, \Lambda_{i,\delta}) = \limsup_{N\to\infty} \overline{J}_i^N(x, \Lambda_{i,\delta}^N) \le \limsup_{N\to\infty} \overline{V}_i^N(x) \le \overline{V}_i(x).$$

Next, we consider approximating e^x by particular choices of $h(x)$. Recall Example 11.1 and the definition of $h_0(x)$. For each $\gamma > 0$, let $\Psi_\gamma(x) = (1/\gamma)\Psi(x/\gamma)$ and $h_\gamma(x)$ be the convolution of h_0 and Ψ_γ. Then, $h_\gamma(x) \to h_0(x)$ as $\gamma \to 0$ for all x. Therefore, we can approximate e^x by $h_\gamma(x)$ by choosing a small enough γ on $[-M, M]$.

In view of these, the original problem with a large N can be approximated by the limiting problem with a large N and a large M. In the next section, we study a numerical example demonstrating how these approximations work.

11.6 A Numerical Example

The optimal trading rule in the limiting problem with $N = \infty$ and $h(x) = e^x$ was treated in Zhang and Zhang [15]. The main result can be summarized as follows.

Lemma 11.5. *Let (x_1^*, x_2^*) be a pair satisfying the following conditions:*

$$x_1^* \le \frac{1}{a}\left(\frac{\overline{\sigma}^2}{2} + a\overline{b} - \rho\right) \le x_2^*,\ x_2^* - x_1^* > \log\left(\frac{1+K}{1-K}\right),$$

and

$$\begin{aligned}
&\begin{pmatrix} \int_0^\infty \eta(t)e^{-\kappa(\overline{b}-x_1^*)t}dt & -\int_0^\infty \eta(t)e^{\kappa(\overline{b}-x_1^*)t}dt \\ \int_0^\infty t\eta(t)e^{-\kappa(\overline{b}-x_1^*)t}dt & \int_0^\infty t\eta(t)e^{\kappa(\overline{b}-x_1^*)t}dt \end{pmatrix}^{-1} \begin{pmatrix} e^{x_1^*}(1+K) \\ e^{x_1^*}(1+K)/\kappa \end{pmatrix} \\
&= \begin{pmatrix} \int_0^\infty \eta(t)e^{-\kappa(\overline{b}-x_2^*)t}dt & -\int_0^\infty \eta(t)e^{\kappa(\overline{b}-x_2^*)t}dt \\ \int_0^\infty t\eta(t)e^{-\kappa(\overline{b}-x_2^*)t}dt & \int_0^\infty t\eta(t)e^{\kappa(\overline{b}-x_2^*)t}dt \end{pmatrix}^{-1} \begin{pmatrix} e^{x_2^*}(1-K) \\ e^{x_2^*}(1-K)/\kappa \end{pmatrix}
\end{aligned} \tag{11.15}$$

where $\kappa = \sqrt{2a}/\overline{\sigma}$ and $\eta(t) = t^{(\rho/a)-1}\exp\left(-t^2/2\right)$.

Let

$$\begin{cases} \overline{V}_0(x) = \begin{cases} C_2^* \int_0^\infty \eta(t)e^{\kappa(\overline{b}-x)t}dt & \text{if } x \ge x_1^*, \\ C_1^* \int_0^\infty \eta(t)e^{-\kappa(\overline{b}-x)t}dt - e^x(1+K) & \text{if } x < x_1^*, \end{cases} \\ \overline{V}_1(x) = \begin{cases} C_1^* \int_0^\infty \eta(t)e^{-\kappa(\overline{b}-x)t}dt & \text{if } x < x_2^*, \\ C_2^* \int_0^\infty \eta(t)e^{\kappa(\overline{b}-x)t}dt + e^x(1-K) & \text{if } x \ge x_2^* \end{cases} \end{cases} \tag{11.16}$$

with

$$\begin{pmatrix} C_1^* \\ C_2^* \end{pmatrix} = \begin{pmatrix} \int_0^\infty \eta(t) e^{-\kappa(\overline{b}-x_1^*)t} dt & -\int_0^\infty \eta(t) e^{\kappa(\overline{b}-x_1^*)t} dt \\ \int_0^\infty t\eta(t) e^{-\kappa(\overline{b}-x_1^*)t} dt & \int_0^\infty t\eta(t) e^{\kappa(\overline{b}-x_1^*)t} dt \end{pmatrix}^{-1} \begin{pmatrix} e^{x_1^*}(1+K) \\ e^{x_1^*}(1+K)/\kappa \end{pmatrix}.$$

If, on the interval (x_1^*, x_2^*)*, the following inequalities hold*

$$e^x(1-K) \le \overline{V}_1(x) - \overline{V}_0(x) \le e^x(1+K),$$

then buying when $x \le x_1^*$ *and selling when* $x \ge x_2^*$ *is optimal.*

Example 11.2. In this example, we take

$$a = 0.8, b(1) = 3, b(2) = 1, \sigma(1) = 0.7, \sigma(2) = 0.3,$$

$$\lambda = 0.09, \mu = 0.06, \lambda_1 = 0.01, \mu_1 = 0.02, \rho = 0.5, \text{ and } K = 0.01.$$

Then, $(\nu_1, \nu_2) = (2/5, 3/5)$, $\overline{b} = 9/5$, and $\overline{\sigma} = 0.5$. We solve (42) to obtain $(x_1^*, x_2^*) = (1.115, 1.455)$ and the value functions $\overline{V}_i(x)$. These functions are plotted in Fig. 11.2. In addition, we vary $\varepsilon = 0.1, 0.01$, and 0.001 and solve the corresponding HJB equations in (34) (using the explicit finite difference method) with $N = \infty$. The value functions V_i^ε are also presented in Fig. 11.2. It is clear in this example that V_i^ε can be approximated by $\overline{V}_i$ when ε is small enough.

Next, we use $(x_1^*, x_2^*) = (1.115, 1.455)$ to construct the following trading rules for the original problem:

$$\begin{cases} \text{Buy:} & \text{if } X_t^\varepsilon \le x_1^*, \\ \text{Sell:} & \text{if } X_t^\varepsilon \ge x_2^*. \end{cases} \tag{11.17}$$

Using these trading rules, we generate the corresponding reward functions with a varying ε and $N = \infty$. In particular, we use Monte Carlo simulations based on (28) and generate 10 K sample paths. The corresponding reward functions with $\varepsilon = 1$, 0.01, and 0.0001 are plotted in Fig. 11.3. Combining these two figures, one can see how the constructed trading rules in (44) work for the original problem.

In general, the control policy obtained via a singular perturbation approach not only work when ε is small but also work for the problem with not-so-small ε. The performance with $\varepsilon = 1$ can be seen in Fig. 11.3 in which the corresponding reward functions are fairly close to the value functions of the limiting problem, and therefore, in view of Fig. 11.2, close to those of the original problem.

11.7 Concluding Remarks

In this chapter, we studied the asymptotic properties of the mean-reverting trading problem. We established the convergence of the value functions and demonstrated

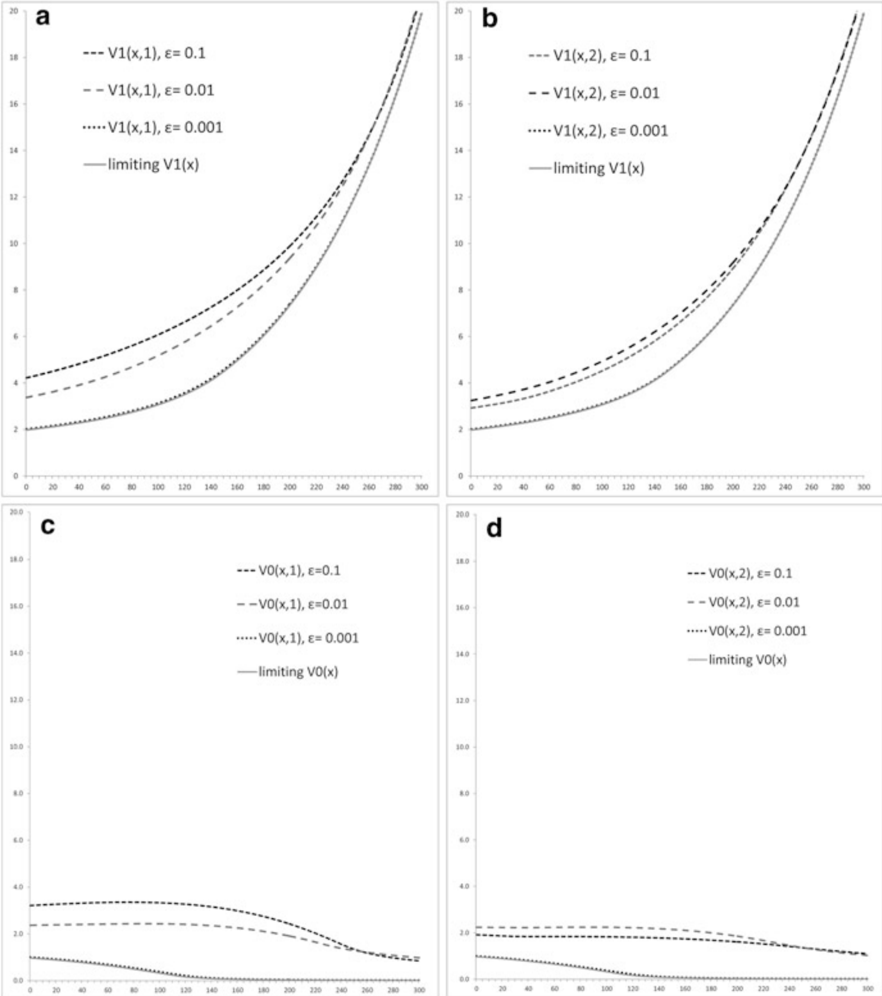


Fig. 11.2 Value function approximation. **(a)** $V_1^{\varepsilon}(x,1)$ and $\overline{V}_1(x)$, **(b)** $V_1^{\varepsilon}(x,2)$ and $\overline{V}_1(x)$, **(c)** $V_0^{\varepsilon}(x,1)$ and $\overline{V}_0(x)$, **(d)** $V_0^{\varepsilon}(x,2)$ and $\overline{V}_0(x)$

how the optimal trading rule for the limiting problem can be used to construct a trading rule for the original problem.

In general, to use an optimal trading rule for the original problem, one needs to determine the mode (or the state of α_t^{ε}). This typically involves nonlinear filtering as in Dai et al. [5]. Nevertheless, in this chapter, we showed that this is not necessary when the jump rates of α_t^{ε} is large because the constructed trading rule does not require the state information of α_t^{ε}.

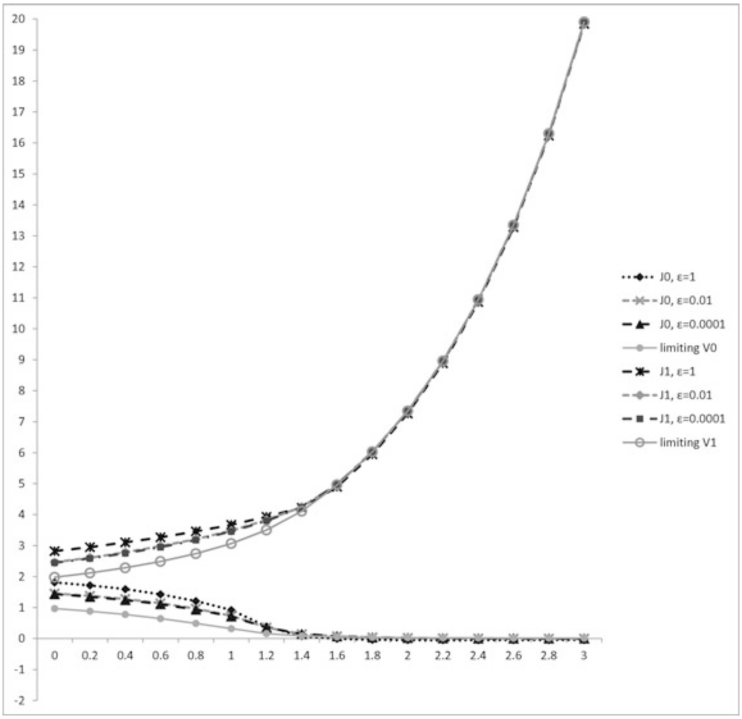


Fig. 11.3 Reward functions under trading rules constructed from that of the limiting problem.

Appendix

In this appendix, we provide the definitions of viscosity solutions of the HJB equations (34) and (38). First, we consider (34). For each $f(x,\alpha)$ and $\phi(x) \in C^2$, let

$$\mathscr{A}^{\phi} f(x,\alpha) = a(b(\alpha)-x)\frac{d\phi(x)}{dx} + \frac{\sigma^2(\alpha)}{2}\frac{d^2\phi(x)}{dx^2} + Qf(x,\cdot)(\alpha).$$

Definition 11.1. $v_i^{n,\varepsilon}(x,\alpha)$ is a *viscosity solution* of (34) if the following hold:

(a) $v_i^{n,\varepsilon}(x,\alpha)$ is uniformly continuous in x;
(b) for any $\alpha_0 \in \{1,2\}$ and x_0,

$$\begin{aligned} &\min\Big\{\rho v_0^{n,\varepsilon}(x_0,\alpha_0) - \mathscr{A}^{\phi_0} v_0^{n,\varepsilon}(x_0,\alpha_0),\\ &\qquad\qquad v_0^{n,\varepsilon}(x_0,\alpha_0) - v_1^{n,\varepsilon}(x_0,\alpha_0) + h(x_0)(1+K)\Big\} \le 0,\\ &\min\Big\{\rho v_1^{n,\varepsilon}(x_0,\alpha_0) - \mathscr{A}^{\phi_1} v_1^{n,\varepsilon}(x_0,\alpha_0),\\ &\qquad\qquad v_1^{n,\varepsilon}(x_0,\alpha_0) - v_0^{n-1,\varepsilon}(x_0,\alpha_0) - h(x_0)(1-K)\Big\} \le 0, \end{aligned} \tag{11.18}$$

for $n = 0, 1, \dots, N$, whenever $\phi_i(x) \in C^2$ and $v_i^{n,\varepsilon}(x,\alpha_0) - \phi_i(x)$ has a local maximum at $x = x_0$; and

(c) for any $\alpha_0 \in \{1,2\}$ and x_0,

$$\begin{aligned}
&\min\Big\{\rho v_0^{n,\varepsilon}(x_0,\alpha_0) - \mathscr{A}^{\psi_0} v_0^{n,\varepsilon}(x_0,\alpha_0),\\
&\qquad\qquad v_0^{n,\varepsilon}(x_0,\alpha_0) - v_1^{n,\varepsilon}(x_0,\alpha_0) + h(x_0)(1+K)\Big\} \geq 0,\\
&\min\Big\{\rho v_1^{n,\varepsilon}(x_0,\alpha_0) - \mathscr{A}^{\psi_1} v_1^{n,\varepsilon}(x_0,\alpha_0),\\
&\qquad\qquad v_1^{n,\varepsilon}(x_0,\alpha_0) - v_0^{n-1,\varepsilon}(x_0,\alpha_0) - h(x_0)(1-K)\Big\} \geq 0,
\end{aligned} \tag{11.19}$$

for $n = 0, 1, \dots, N$, whenever $\psi_i(x) \in C^2$ and $v_i^{n,\varepsilon}(x,\alpha_0) - \psi_i(x)$ has a local minimum at $x = x_0$.

If (a) and (b) (resp. (a) and (c)) hold, we say that v is a *viscosity subsolution* (resp. *viscosity supersolution*).

Finally, we give the definition of viscosity solution of (38). Recall that

$$\overline{\mathscr{A}} f(x) = a(\overline{b} - x)\frac{df(x)}{dx} + \frac{\overline{\sigma}^2}{2}\frac{d^2 f(x)}{dx^2}.$$

Definition 11.2. $v_i^n(x)$ is a *viscosity solution* of (38) if the following hold:

(a) $v_i^n(x)$ is uniformly continuous in x;
(b) for any x_0,

$$\begin{aligned}
&\min\left\{\rho v_0^n(x_0) - \overline{\mathscr{A}}\phi_0(x_0),\ v_0^n(x_0) - v_1^n(x_0) + h(x_0)(1+K)\right\} \leq 0,\\
&\min\left\{\rho v_1^n(x_0) - \overline{\mathscr{A}}\phi_1(x_0),\ v_1^n(x_0) - v_0^{n-1}(x_0) - h(x_0)(1-K)\right\} \leq 0,
\end{aligned} \tag{11.20}$$

for $n = 0, 1, \dots, N$, whenever $\phi_i(x) \in C^2$ and $v_i^n(x) - \phi_i(x)$ has a local maximum at $x = x_0$; and

(c) for any x_0,

$$\begin{aligned}
&\min\left\{\rho v_0^n(x_0) - \overline{\mathscr{A}}\psi_0(x_0),\ v_0^n(x_0) - v_1^n(x_0) + h(x_0)(1+K)\right\} \geq 0,\\
&\min\left\{\rho v_1^n(x_0) - \overline{\mathscr{A}}\psi_1(x_0),\ v_1^n(x_0) - v_0^{n-1}(x_0) - h(x_0)(1-K)\right\} \geq 0,
\end{aligned} \tag{11.21}$$

for $n = 0, 1, \dots, N$, whenever $\psi_i(x) \in C^2$ and $v_i^n(x) - \psi_i(x)$ has a local minimum at $x = x_0$.

If (a) and (b) (resp. (a) and (c)) hold, we say that v is a *viscosity subsolution* (resp. *viscosity supersolution*).

Next, we give a sketch verifying the conditions in Example 11.1, i.e., we show that $h(x)$, $h'(x)$, $xh'(x)$, and $h''(x)$ are bounded and Lipschitz.

First note that $h_0(x)$ is bounded and Lipschitz. The boundedness and Lipschitz properties of h, h', and h'' follow from the equalities:

$$h(x) = \int_{-\infty}^{\infty} h_0(x-u)\Psi(u)du,$$

$$h'(x) = \int_{-\infty}^{\infty} h_0(x-u)\Psi'(u)du,$$

$$h''(x) = \int_{-\infty}^{\infty} h_0(x-u)\Psi''(u)du.$$

Next, we show that $xh'(x)$ is bounded. Note that

$$\begin{aligned} xh'(x) &= x\int_{-\infty}^{\infty} h_0'(u)\Psi(x-u)du, \\ &= x\int_{-\infty}^{M} e^u\Psi(x-u)du, \\ &= x\int_{x-M}^{\infty} e^{x-y}\Psi(y)dy, \text{ (with } y=x-u) \\ &= \frac{xe^x}{\sqrt{2\pi}}\int_{x-M}^{\infty} e^{-y}e^{-y^2/2}dy \\ &\le \frac{xe^x}{\sqrt{2\pi}}\int_{x-M}^{\infty} e^{-y^2/2}dy. \end{aligned} \tag{11.22}$$

Clearly, it is bounded on $(-\infty, M]$. To see it is also bounded on (M,∞), note also that

$$\frac{xe^x}{\sqrt{2\pi}}\int_{x-M}^{\infty} e^{-y^2/2}dy \le \frac{xe^x}{\sqrt{2\pi}}\left(\frac{\exp\left(-(x-M)^2/2\right)}{x-M}\right). \tag{11.23}$$

The boundedness follows.

Finally, to see the Lipschitz property of $xh'(x)$, in view of the Mean Value Theorem, it suffices to show that $xh''(x)$ is bounded. This can be done similarly as in (49) and (50) by noticing

$$xh''(x) = x\int_{-\infty}^{\infty} h_0'(u)\Psi'(x-u)du.$$

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