Implied Volatility: Theory and Empirical Method

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Abstract

The estimation of the implied volatility is the one of most important topics in option pricing research. The main purpose of this chapter is to review the different theoretical methods used to estimate implied standard deviation and to show how the implied volatility can be estimated in empirical work. The OLS method for estimating implied standard deviation is first introduced, and the formulas derived by applying a Taylor series expansion method to Black–Scholes option pricing model are also described. Three approaches of estimating

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implied volatility are derived from one, two, and three options, respectively. Regarding to these formulas with the remainder terms, the accuracy of these formulas depends on how an underlying asset is close to the present value of exercise price in an option. The formula utilizing three options for estimating implied volatility is more accurate rather than other two approaches.

In empirical work, we use call options on S&P 500 index futures in 2010 and 2011 to illustrate how MATLAB can be used to deal with the issue of convergence in estimating implied volatility of future options. The results show that the time series of implied volatility significantly violates the assumption of constant volatility in Black–Scholes option pricing model. The volatility parameter in the option pricing model fluctuates over time and therefore should be estimated by the time series and cross-sectional model.

Keywords
Implied volatility • Implied standard deviation (ISD) • Option pricing model • MATLAB • Taylor series expansion • Ordinary least squares (OLS) • Black–Scholes model • Options on S&P 500 index futures

90.1 Introduction

The volatility of the return of the underlying asset is the important factor in option pricing model (see Merton 1973; Black and Scholes 1973). However, the standard deviation of the underlying asset return cannot be observed directly (See Merton 1976; Macbeth et al. 1979; Chance 1986). The estimation of the implied volatility of the underlying asset in option framework becomes the one of most important topics in option pricing research. There are two main methods developed in the finance literature to estimate the standard deviation of the underlying asset in option framework: (1) the historical standard deviation and (2) the implied standard deviation (called ISD hereafter) derived from the Black–Scholes' option pricing model framework (See Hull 2011).

Garman and Klass (1980) study the historical standard deviation by using open, high, low prices, and closed prices' data to estimate the standard deviation. To support the use of historical standard deviation for implied standard deviation in option pricing model requires that the underlying asset's rate of return is stationary over the option's life, which contradicts the time-varying standard deviation documented by Schwert (1989).

Since the Black–Scholes' option pricing model is a nonlinear equation, an explicit analytic solution for the ISD is not available in the literature (except for at-the-money call), and numerical methods are used to approximate the ISD (see Latane and Rendleman 1976; Beckers 1981; Manaster and Koehler 1982; Brenner and Subrahmanyam 1988; Lai et al. 1992; Chance 1996; Hallerback 2004; Corrado and Miller 1996, 2004; Li 2005). The derivation and use of the ISD for an option as originated by Latane and Rendleman (1976) have become a widely used methodology for variance estimation. By applying the Newton-Raphson method, Manaster and Koehler (1982) provide an iterative algorithm for the ISD. Brenner and Subrahmanyam (1988) applied Taylor series expansion at zero base to the cumulative normal function in pricing Black–Scholes option pricing model up to the first-order term and set the underlying asset price to equal the present value of exercise price to solve the ISD. Lai et al. (1992) derive a closed-form solution for the ISD in terms of the delta $\partial C/\partial S$ and $\partial C/\partial E$. Following the same approach as Brenner and Subrahmanyam, Corrado and Miller (1996, 2004) utilize the cumulative normal function at zero to the first-order term to derive a quadratic equation of the ISD. Then the ISD can be obtained by solving the quadratic equation. Hallerback (2004) also derives an improved formula, which is similar to Corrado and Miller's formula (1996), to compute the ISD. Later, Li (2005) bases on Brenner and Subrahmanyam's approach to expand the expression to third-order term and solve for the ISD with a cubic equation. Since Li includes third-order term in the Taylor expansion on the cumulative normal distribution in his derivation, Li claims that his formula of ISD provides a consistently more accurate estimate of the true ISD than that of Brenner and Subrahmanyam's formula.

However, the fact that there are as many estimated ISD of an underlying asset as the number of different exercise price in options violates the constant variance assumption used in deriving the Black–Scholes' option pricing model. Chance (1996) assumes different exercise prices result in different ISDs, which violate the constant variance assumption used in deriving the Black–Scholes' option pricing model. Under the existence of a call at-the-money assumption, Chance uses Brenner and Subrahmanyam's formula to calculate the at-the-money's ISD. Chance then applies Taylor series expansion to the difference of the call options in terms of the first and the second-order terms. The drawback of Chance's method is the constraint of the use only for at-the-money option price. In other words, if the underlying asset price deviates from the present value of the exercise price and the call option price is not available (or unobservable) in the market, then Chance's formula for the ISD may not apply. Later, Ang et al. (2009, 2012) relax the constraint in Chance's method and develop three formulas which depend on a Taylor series expansion and utilize single, two, and three options, respectively, to estimate implied volatility.

The purpose of this chapter is to review the different theoretical methods used to estimate ISD and to show how the implied volatility can be estimated in empirical work. We use the data from options on S&P 500 index futures in 2010 and 2011 to illustrate how MATLAB can be used to deal with the issue of convergence in estimating implied volatility of options on index futures. This chapter is organized as follows. In Sect. 90.2, we review the OLS method used in estimation of the ISD in Black–Scholes' option pricing model and expand this method to estimate the implied volatility of the underlying asset for options on the index futures. Then, in Sect. 90.3, we introduce the formulas of implied volatility developed by Ang et al. (2012) which apply a Taylor series expansion to the Black–Scholes option pricing model. The process and results of empirical work on estimating ISD for options on S&P 500 index futures are shown in Sect. 90.4. Finally, Sect. 90.5 represents the conclusions of this study.

90.2 Estimating the Implied Standard Deviation with OLS Method

Black and Scholes (1973) and Merton (1973) derive the European call option pricing model on a stock as follows:

$$\begin{aligned} &C = SN(d_1) - Ke^{-r\tau}N(d_2) \\ &d_1 = [1\mathrm{n}(S/K) + (r+\sigma^2/2)\tau]/\sigma\sqrt{\tau} \\ &d_2 = d_1 - \sigma\sqrt{\tau}, \end{aligned} \tag{90.1}$$

where C is the call premium, S is the underlying stock price, K is the exercise price, r is the instantaneous risk-free rate, τ is the time to the maturity, σ is the standard deviation of the underlying asset rate of return on annual basis, and N(x) is the standard cumulative normal distribution function up to x.

The sensitivities, or first partial derivatives, of the call option formula in Eq. 90.1 with respect to the change of the volatility of the underlying stock can be derived as

$$\frac{\partial C}{\partial \sigma} = S\sqrt{\tau}N'(d_1) = \frac{S\sqrt{\tau}}{S\sqrt{2\pi}}e^{-d_1^2/2} \tag{90.2}$$

where N′(x) is the standard normal probability density function at value x. Equation 90.2 shows the positive relationship between the call option price and the volatility of the underlying stock. Since a call option has no downside risk (except for its cost), increasing risk of the underlying stock simply enlarges the probability that the option will end up in the money by expiration (hence, with a larger intrinsic value).

The OLS method for estimating implied standard deviation is first proposed by Whaley (1982). Although Whaley's original intent for this method was to improve upon the existing weighting techniques, his ordinary least squares (OLS) approach can also be used to derive the implied standard deviations (ISD) for call options. To begin the development of his method, Whaley applies a Taylor series expansion around some initial value of the standard deviation and omits higher-order terms. Mathematically, this is expressed as

$$C^M_{j,t} = C^T_{j,t}(\sigma_0) + \left(\frac{\partial C^T_{j,t}}{\partial \sigma}\bigg| \sigma = \sigma_0\right)(\sigma_s - \sigma_0) + e_{j,t}, \tag{90.3}$$

$$(j = 1, 2, \ldots, \mathrm{J})$$

where $C^M_{j,t}$ denotes the market price for the option j at time t, $C^T_{j,t}$ is the theoretical model price estimated by Eq. 90.1 for the option j at time t based on an estimated value for the ISD (σ_0), σ_0 is the estimated ISD evaluated from some initialization value up to some minimum level of tolerance of error, σ_s denote the true or actual

ISD which we are looking for, and $e_{j,t}$ is the random disturbance term for option j at time t. By rearranging Eq. 90.3, we can obtain

$$\left[C^M_{j,t} - C^T_{j,t}(\sigma_0)\right] + \sigma_0\left(\frac{\partial C^T_{j,t}}{\partial \sigma}\bigg|\sigma = \sigma_0\right) = \sigma_s\left(\frac{\partial C^T_{j,t}}{\partial \sigma}\bigg|\sigma = \sigma_0\right) + e_{j,t} \qquad (90.4)$$
$$(j = 1, 2, \ldots, \mathrm{J})$$

since $C^M_{j,\,t}$ is observable in market $\left(\frac{\partial C^T_{j,t}}{\partial \sigma}\Big|\sigma = \sigma_0\right)$ and $C^T_{j,\,t}(\sigma_0)$ can be evaluated at any given value σ_0 by using Eqs. 90.1 and 90.2.

Whaley (1982) then applies OLS, which minimizes the sum of squared residuals, to achieve a single, weighted σ from the options on a particular stock. The actual estimation procedures begin from a linearization of the option pricing model around 0, and then OLS is applied to Eq. 90.4. The process thus proceeds in an iteration manner until the estimated ISD $\hat{\sigma}_s$ satisfies an acceptable tolerance of

$$\left|\frac{\hat{\sigma}_s - \sigma_0}{\sigma_0}\right| < Q, \qquad (90.5)$$

where Q is a small positive number where Whaley(1982) uses Q equal to 0.0001 as the acceptable tolerance of estimated error and $\hat{\sigma}_s$ is the estimate for the true ISD σ_s for the market option price. If the tolerance criterion is not satisfied, $\hat{\sigma}_s$ becomes the new initialization value and the OLS procedure is repeated.

The OLS method also can be applied to estimate the ISD for options on index future with the similar procedure of a Taylor series expansion (See Wolf 1982; Park et al. 1985; Ramaswamy et al. 1985; Brenner et al. 1985). The call options on index future derived by Black (1975, 1976) are given by

$$\begin{aligned} C^F_t &= e^{-r\tau}[F_t N(d_3) - KN(d_4)] \\ d_3 &= \left[1\mathrm{n}(F_t/K) + \left(\sigma_f^2/2\right)\tau\right]/\sigma_f\sqrt{\tau} \\ d_4 &= d_3 - \sigma_f\sqrt{\tau}, \end{aligned} \qquad (90.6)$$

where C^F_t is the model price for a call option on index future at time t, F_t is the underlying index future price at time t, K is the exercise price of the call option on index future, τ is the option's remaining time to maturity in terms of a year, r is the continuous annualized risk-free rate, σ_f^2 is the instantaneous variance of returns of the underlying index future contract over the remaining life of the option, and N(x) is the standard cumulative normal distribution function up to x.

There is similar procedure with Whaley's method to calculate the ISD for options on S&P 500 index futures. The ISD is obtained by first choosing an initial estimate, σ_0, and then using Eq. 90.7 to iterate towards the correct value as follows:

$$C^F{}_{t,j} - C^F{}_{t,j}(\sigma_0) = (\sigma_1 - \sigma_0)\left(\frac{\partial C^F_{j,t}}{\partial \sigma}\bigg|\sigma = \sigma_0\right), \qquad (90.7)$$

where $C^F_{t,j}$ denotes the market price of call option j at time t, $C^F_{t,j}(\sigma_0)$ is the theoretical price of call option j at time t given σ equal to σ_0, σ_0 is the initialized estimate of the ISD, σ_1 = estimate of the true ISD from iteration, $\left(\frac{\partial C^F_{t,j}}{\partial \sigma}\middle| \sigma = \sigma_0\right)$ is the partial derivative of the call option on index future with respect to the standard deviation evaluated at a σ_0. In the context of the Black (1976) option pricing model, the partial with respect to the standard deviation of the underlying index future can be expressed explicitly as

$$\frac{\partial C^F_{t,j}}{\partial \sigma} = F_t e^{-r\tau}\sqrt{\tau}N'(d_3) = F_t e^{-r\tau}\frac{\sqrt{\tau}}{\sqrt{2\pi}}e^{-d_3^2/2}, \tag{90.8}$$

where d_3 is defined as in Eq. 90.6 and N′(x) is the standard normal probability density function at value x. The partial derivative formula in Eq. 90.8 is also called Vega of a call option on index futures which is represented the rate of change of the value of a call option on index futures with respect to the volatility of the underlying index futures. The iteration proceeds by reinitializing σ_0 to equal σ_1 at each successive stage until an acceptable tolerance level in Eq. 90.5 is attained.

90.3 Estimating the Implied Standard Deviation with Taylor Series Expansion Method

In this section, we first introduce the exact closed-form solution in for the ISD under the condition that the underlying asset price equals the present value of the exercise price. Then we discuss Ang et al.'s (2012) alternative formulas to estimate the ISD by applying a Taylor series expansion to the Black–Scholes option pricing model under the relaxation of the previous restrictive condition.

When the underlying stock price equals the present value of the exercise price (i.e., $S = Ke^{-r\tau}$), the Eq. 90.1 can be reduced as follows:

$$\begin{aligned} C &= S[N(\sigma\sqrt{\tau}/2) - N(-\sigma\sqrt{\tau}/2)] \\ &= S[1 - 2N(-\sigma\sqrt{\tau}/2)] \\ &= S[2N(\sigma\sqrt{\tau}/2) - 1]. \end{aligned} \tag{90.9}$$

Based on the characteristics of existence and uniqueness of the inverse cumulative normal distribution, an exact closed-form solution for the ISD in Eq. 90.9 can be derived as

$$\sigma\sqrt{\tau} = 2N^{-1}[(S + C)/(2S)]. \tag{90.10}$$

Ang et al. (2012) apply Taylor's formula to the cumulative normal functions in Eq. 90.1 at base $\ln(S/Ke^{-r\tau})/(\sigma\sqrt{\tau})$ up to the second-order terms, then the

European call option in Eq. 90.1 can be rearranged as a quadratic equation of $\sigma\sqrt{\tau}$ plus the remainder term as follows[1]:

$$\sigma^2\tau[4(S+Ke^{-r\tau})-(S-Ke^{-r\tau})\ln(S/Ke^{-r\tau})]-4\sigma\sqrt{\tau}\sqrt{2\pi}(2C-S+Ke^{-r\tau}) \\ +8\ln(S/Ke^{-r\tau})\left[(S-Ke^{-r\tau})\left(1+(\ln(S/Ke^{-r\tau})/4)^2\right)-(S+Ke^{-r\tau})\ln(S/Ke^{-r\tau})/4\right]+\varepsilon=0. \tag{90.11}$$

Dropping the remainder term ε, the ISD can be obtained by solving the root of quadratic equation function in Eq. 90.11. Since Ang et al. (2012) utilize four times of a Taylor series expansion method to derive the quadratic function of a European call option and the remainder terms are omitted, the ISD calculated by Eq. 90.11 is not an exact formula. Therefore, the effectiveness of using Eq. 90.11 to estimate the ISD depends on the deviation of the underlying stock price (S) from the present value of exercise price ($Ke^{-r\tau}$).

Moreover, Ang et al. (2012) derive the second alternative formula for estimating ISD by using two call options,C_1 and C_2, on the same underlying stock but at different exercise, K_1 and K_2, respectively (here we assume $K_1 < K_2$). By applying Taylor series expansion to Eq. 90.1 for two call options at K_2 for C_1 and at K_1 for C_2, respectively, we can obtain

$$C_1 = C_2 - e^{-r\tau}N\left(\ln(S/K_2e^{-r\tau})/\left(\sigma\sqrt{\tau}\right)-\sigma\sqrt{\tau}/2\right)(K_1-K_2)+\varepsilon_1, \tag{90.12}$$

$$C_2 = C_1 - e^{-r\tau}N\left(\ln(S/K_1e^{-r\tau})/\left(\sigma\sqrt{\tau}\right)-\sigma\sqrt{\tau}/2\right)(K_2-K_1)+\varepsilon_2. \tag{90.13}$$

Here ε_1 and ε_2 are the remainder terms of C_1 at K_2 and C_2 at K_1 from Eq. 90.1. Dividing both sides of Eqs. 90.12 and 90.13 by $e^{-r\tau}$ (K_2-K_1) and simple manipulations produce the same left-hand side of $(C_1–C_2)/e^{-r\tau}(K_2-K_1)$.

Then applying the inverse function of cumulative normal function on both sides and after using the Taylor's formula yields the following equations:

$$N^{-1}[(C_1-C_2)/e^{-r\tau}(K_2-K_1)] = \ln(S/K_1e^{-r\tau})/\left(\sigma\sqrt{\tau}\right)-\sigma\sqrt{\tau}/2+\eta_1, \tag{90.14}$$

$$N^{-1}[(C_1-C_2)/e^{-r\tau}(K_2-K_1)] = 1\text{n}(S/K_2e^{-r\tau})/\left(\sigma\sqrt{\tau}\right)-\sigma\sqrt{\tau}/2+\eta_2, \tag{90.15}$$

where η_1 and η_2 are the remainder terms of Taylor's formulas derived from Eqs. 90.12 and 90.13, respectively. After combining Eqs. 90.14 and 90.15 and dropping the remainder terms (η_1 + η_2), the quadratic function of $\sigma\sqrt{\tau}$ can be shown as

[1]The details of the derivation of Eq. 90.10 can be found in Ang et al. (2012) paper.

$$\left(\sigma\sqrt{\tau}\right)^2 + 2N^{-1}[(C_1 - C_2)/e^{-r\tau}(K_2 - K_1)]\left(\sigma\sqrt{\tau}\right) - \ln\left[S^2/\left(e^{-2r\tau}K_1K_2\right)\right] = 0. \tag{90.16}$$

Thus, the ISD can be solved as

$$\sigma\sqrt{T} = \begin{cases} -N^{-1}[(C_1 - C_2)/e^{-r\tau}(K_2 - K_1)] + \sqrt{\zeta} & \text{when S} > \text{K}_1 \\ -N^{-1}[(C_1 - C_2)/e^{-r\tau}(K_2 - K_1)] - \sqrt{\zeta} & \text{when S} \le \text{K}_1 \le \text{K}_2 \end{cases}$$
$$\zeta = \left[N^{-1}((C_1 - C_2)/e^{-r\tau}(K_2 - K_1))\right]^2 + \ln\left(S^2/e^{-2r\tau}K_1K_2\right). \tag{90.17}$$

It is clear that if stock price is less than the lower exercise K_1 (i.e., then both call options are out of the money), and if we had chosen the value with the plus sign of $\sqrt{\zeta}$ in Eq. 90.17, ISD calculated by Eq. 90.17 will be overstated. The advantage of this formula is that a sufficient condition to calculate ISD by Eq. 90.17 only requires that there existed any two consecutive call option values with different exercise prices. But, the accuracy of this formula will depend on the magnitude of the deviation between these two exercise prices.

Ang et al. (2012) further extend this approach to include a third option to derive the third formula. Similar to Eq. 90.16, if there is a third call option C_3 with the exercise price K_3, then the following Eq. 90.18 must hold for K_2, K_3 and C_2, C_3.

$$\left(\sigma\sqrt{\tau}\right)^2 + 2N^{-1}[(C_2 - C_3)/e^{-r\tau}(K_3 - K_2)]\left(\sigma\sqrt{\tau}\right) - \ln\left[S^2/\left(e^{-2r\tau}K_2K_3\right)\right] = 0. \tag{90.18}$$

Given the constant variance assumption in Black and Scholes option model, the following Eq. 90.19 is thus derived by subtracting Eq. 90.18 from Eq. 90.16 as follows:

$$\sigma\sqrt{\tau} = \ln(K_3/K_1)/\left[2\left(N^{-1}((C_1 - C_2)/e^{-r\tau}(K_2 - K_1)) - N^{-1}((C_2 - C_3)/e^{-r\tau}(K_3 - K_2))\right)\right]. \tag{90.19}$$

An advantage of using Eq. 90.19 rather than Eq. 90.17 to estimate the ISD is to circumvent the sign issue that appears in Eq. 90.17. However, a drawback of using Eq. 90.19 is that there must exist at least three instead of two call options for Eq. 90.17. Equation 90.19 provides a simple formula to calculate ISD because all option values and exercise price are given and the inverse function of the standard cumulative normal function also available in the Excel spreadsheet. Ang et al. (2012) state that this third formula in Eq. 90.19 is more accurate method for estimating ISD based on their simulation results.

90.4 Illustration of Estimating Implied Standard Deviation by MATLAB

The data for this study for estimating ISD include the call options on the S&P 500 index futures which are traded at the Chicago Mercantile Exchange (CME).[2] According to Eq. 90.6, we need the information of market call option price on S&P 500 index, the annualized risk-free rate, S&P 500 index futures price, exercise price, and maturity date on the contracts as input variables to calculate the ISD of call option on S&P 500 index futures. Daily closed-price data of S&P 500 index futures and options on S&P 500 index futures was gathered from Datastream for two periods of time: the options expired on March, June, and September, 2010; options expired on March, June, and September, 2011; and the S&P 500 index future from October 1, 2008, to November 4, 2011. The S&P 500 spot price is based on the closed price of S&P 500 index on Yahoo! Finance[3] during the same period of S&P 500 index future data. The risk-free rate used in Black model is based on 3-month Treasury bill from Federal Reserve Bank of St. Louis.[4] The selection of these futures option contracts is based on the length of trading days. The futures options expired on March, June, September, and December have over 1 year trading date (above 252 observations), and other options only have more or less 100 observations. Therefore, we only choose the futures options with longer trading period to investigate the distributional statistics of these ISD series. Studying two different time periods (2010 and 2011) of call options on S&P 500 index futures will allow the examination of ISD characteristics and movements over time as well as the effects of different market climates.

The tolerance level used is the same formula as shown in Eq. 90.5, and let the tolerance level Q equal to 0.000001 as follows:

$$\left|\frac{\sigma_1 - \sigma_0}{\sigma_0}\right| < .000001$$

This chapter utilized financial toolbox in MATLAB to calculate the implied volatility for futures option that the code of function is as follows[5]:

$$\text{Volatility} = \text{blsimpv}(\text{Price},\ \text{Strike},\ \text{Rate},\ \text{Time}, \text{Value}, \text{Limit},\ \text{Tolerance},\ \text{Class})$$

[2]Nowadays Chicago Mercantile Exchange (CME), Chicago Board of Trade (CBOT), New York Mercantile Exchange (NYMEX), and Commodity Exchange (COMEX) are merged and operate as designated contract markets (DCM) of the CME Group which is the world's leading and most diverse derivatives marketplace. Website of CME group: http://www.cmegroup.com/

[3]Website of Yahoo! Finance is as follows: http://finance.yahoo.com

[4]Website of Federal Reserve Bank of St. Louis: http://research.stlouisfed.org/

[5]The syntax and the code from m-file source of MATLAB for Implied Volatility Function of Futures Options are represented in Appendix 1. The detailed information of the function and example of calculating the implied volatility for futures option also can be referred on MathWorks website: http://www.mathworks.com/help/toolbox/finance/blkimpv.html

where the blsimpv is the function name in MATLAB; Price, Strike, Rate, Time, Value, Limit, Tolerance, and Class are input variables; Volatility is the annualized ISD (also called implied volatility). The advantages of this function are the allowance of the upper bound of implied volatility (Limit variable) and the adjustment of the implied volatility termination tolerance (Tolerance variable), in general, equal to 0.000001.

A summary of the ISD distributional statistics for S&P 500 index futures call options in 2010 and 2011 appears in Table 90.1. The most noteworthy feature from this table is the significantly different mean values of the ISD that occur for different exercise prices. The means and variability of the ISD in 2010 and 2011 appear to be inversely related to the exercise price. Comparing the mean ISDs across time periods, it is quite evident that the ISDs in 2011 are significantly smaller. Also, the time-to-maturity effect observed by Park and Sears (1985) can be identified. The September options in 2011 possess higher mean value of the ISD than those maturing in June and March with the same strike price.

The other statistical measures listed in Table 90.1 are the relative skewness and relative kurtosis of the ISD series, along with the studentized range. Skewness measures lopsidedness in the distribution and might be considered indicative of a series of large outliers at some point in the time series of the ISDs. Kurtosis measures the peakedness of the distribution relative to the normal and has been found to affect the stability of variance (see Lee and Wu 1985). The studentized range gives an overall indication as to whether the measured degrees of skewness and kurtosis have significantly deviated from the levels implied by a normality assumption for the ISD series.

Although an interpretation of the effects of skewness and kurtosis on the ISD series needs more accurate analysis, a few general observations are warranted at this point. Both 2010 and 2011 ISD's statistics present a very different view of normal distribution, certainly challenging any assumptions concerning normality in Black–Scholes option pricing model framework. Using significance tests on the results of Table 90.1 in accordance with Jarque–Bera test, the 2010 and 2011 skewness and kurtosis measures indicate a higher proportion of statistical significance. We also utilize simple back-of-the-envelope test based on the studentized range to identify whether the individual ISD series approximate a normal distribution. The studentized range larger than 4 in both 2010 and 2011 indicates that a normal distribution significantly understates the maximum magnitude of deviation in individual ISD series.

As a final point to this brief examination of the ISD skewness and kurtosis, note the statistics for MAR10 1075, MAR11 1200, and MAR11 1250 contracts. The relative size of these contract's skewness and kurtosis measures reflect the high degree of instability that its ISD exhibited during the last 10 days of the contract's life. Such instability is consistent across contracts. However, these distortions remain in the computed skewness and kurtosis measures only for these particular contracts to emphasize how a few large outliers can magnify the size of these statistics. For example, the evidence that S&P 500 future price jumped on

Table 90.1 Distributional statistics for the ISD series of call options on S&P 500 index futures

Option series[a]	Mean	Std. dev.	CV[b]	Skewness	Kurtosis	Studentized range[c]	Observations
Call futures options in 2010							
MAR10 1075 (C070WC)	0.230	0.032	0.141	2.908	14.898	10.336	251
JUN10 1050 (B243UE)	0.263	0.050	0.191	0.987	0.943	6.729	434
JUN10 1100 (B243UF)	0.247	0.047	0.189	0.718	−0.569	4.299	434
SEP10 1100 (C9210T)	0.216	0.024	0.111	0.928	1.539	6.092	259
SEP10 1200 (C9210U)	0.191	0.022	0.117	0.982	2.194	6.178	257
Call futures options in 2011							
MAR11 1200 (D039NR)	0.206	0.040	0.195	5.108	36.483	10.190	384
MAR11 1250 (D1843V)	0.188	0.027	0.145	3.739	25.527	10.636	324
MAR11 1300 (D039NT)	0.176	0.021	0.118	1.104	4.787	8.588	384
JUN11 1325 (B513XF)	0.165	0.016	0.095	−1.831	12.656	10.103	200
JUN11 1350 (A850CJ)	0.161	0.018	0.113	−0.228	1.856	8.653	234
SEP11 1250 (B9370T)	0.200	0.031	0.152	2.274	6.875	7.562	248
SEP11 1300 (B778PK)	0.185	0.024	0.131	2.279	6.861	7.399	253
SEP11 1350 (B9370V)	0.170	0.025	0.147	2.212	5.848	6.040	470

[a]Option series contain the name and code of futures options with information of the strike price and the expired month, for example, SEP11 1350 (B9370V) represents that the futures call option is expired on September 2011 with the strike price $1,350, and the parentheses is the code of this futures option in Datastream

[b]CV represents the coefficient of variation that is standard deviation of option series divided by their mean value

[c]Studentized range is the difference of the maximum and minimum of the observations divided by the standard deviation of the sample

January 18, 2010, and plunged on February 2, 2011, causes the ISD of these particular contracts sharply increasing on that dates. Thus, while still of interest, any skewness and kurtosis measures must be calculated and interpreted with caution.

One difficulty in discerning the correct value for the volatility parameter in the option pricing model is due to its fluctuation over time. Therefore, since an accurate estimate of this variable is essential for correctly pricing an option, it would seem that time series and cross-sectional analysis of this variable would be as important as the conventional study of security price movements. Moreover, by examining the ISD series of each call options on S&P 500 index futures over time as well as within different time sets, the unique relationships between the underlying stochastic process and the pricing influences of differing exercise prices, maturity dates, and market sentiment (and, indirectly, volume), might be revealed in a way that could be modeled more efficiently. Therefore, we should consider autoregressive–moving-average (ARMA) models or cross-sectional time series regression models to analyze the ISD series and forecast the price of call options on S&P 500 index futures by predicting the future ISD of these options.

90.5 Summary and Concluding Remarks

The research in estimation of the implied volatility becomes the one of most important topics in option pricing research because the standard deviation of the underlying asset return, which is the important factor in Black–Scholes' option pricing model, cannot be observed directly. The purpose of this chapter is to review the different theoretical methods used to estimate implied standard deviation and to show how the implied volatility can be estimated in empirical work. We review the OLS method and a Taylor series expansion method for estimating the ISD in previous literature. Three formulas for the estimation of the ISD by applying a Taylor series expansion method to Black–Scholes option pricing model can be derived from one, two, and three options, respectively. Regarding to these formulas with the remainder terms in a Taylor series expansion method, the accuracy of these formulas depends on how an underlying asset is close to the present value of exercise price in an option.

In empirical work, we illustrate how MATLAB can be used to deal with the issue of estimating implied volatility for call options on S&P 500 index futures in 2010 and 2011. The results show that the time series of implied volatility significantly violate the assumption of constant volatility in Black–Scholes option pricing model. The skewness and kurtosis measures reflect the instability and fluctuation of the ISD series over time. Therefore, in the future research in the ISD, we should consider autoregressive–moving-average (ARMA) models or cross-sectional time series regression models to analyze and predict the ISD series to forecast the future price of call options on S&P 500 index futures.

Appendix 1: The Syntax and Code for Implied Volatility Function of Futures Options in MATLAB

The function name of estimating implied volatility for European call options on index futures in this chapter are as below:

Syntax

$$\text{Volatility} = \text{blsimpv}(\text{Price}, \text{Strike}, \text{Rate}, \text{Time}, \text{Value}, \text{Limit}, \ \ldots \text{Tolerance}, \text{Class})$$

The input variables that can be a scalar, vector, or matrix in the function of estimating implied volatility are described in Table 90.2

The code from m-file source of MATLAB for implied volatility function of futures options is shown as below:

```
function volatility = blkimpv(F, X, r, T, value, varargin)
% BLKIMPV Implied volatility from Black's model for futures options.
% Compute the implied volatility of a futures price from the market
% value of European futures options using Black's model.
%
% Volatility = blkimpv(Price, Strike, Rate, Time, Value)
% Volatility = blkimpv(Price, Strike, Rate, Time, Value, Limit, ...
% Tolerance, Class)
%
% Optional Inputs: Limit, Tolerance, Class.
%
% Inputs:
% Price - Current price of the underlying asset (i.e., a futures contract).
%
```

Table 90.2 The description of input variables used in blsimpv function in MATLAB

Price	Current price of the underlying asset (a futures contract)
Strike	Exercise price of the futures option
Rate	Annualized, continuously compounded risk-free rate of return over the life of the option, expressed as a positive decimal number
Time	Time to expiration of the option, expressed in years
Value	Price of a European futures option from which the implied volatility of the underlying asset is derived
Limit (optional)	Positive scalar representing the upper bound of the implied volatility search interval. If Limit is empty or unspecified, the default = 10, or 1,000 % per annum
Tolerance (optional)	Implied volatility termination tolerance. A positive scalar. Default = 1e-6
Class (optional)	Option class (call or put) indicating the option type from which the implied volatility is derived. May be either a logical indicator or a cell array of characters. To specify call options, set Class = true or Class = {'call'}; to specify put options, set Class = false or Class = {'put'}. If Class is empty or unspecified, the default is a call option

```
% Strike - Strike (i.e., exercise) price of the futures option.
%
% Rate - Annualized continuously compounded risk-free rate of return
% over the life of the option, expressed as a positive decimal number.
%
% Time - Time to expiration of the option, expressed in years.
%
% Value - Price (i.e., value) of a European futures option from which
% the implied volatility is derived.
%
% Optional Inputs:
% Limit - Positive scalar representing the upper bound of the implied
% volatility search interval. If empty or missing, the default is 10,
% or 1000% per annum.
%
% Tolerance - Positive scalar implied volatility termination tolerance.
% If empty or missing, the default is 1e-6.
%
% Class - Option class (i.e., whether a call or put) indicating the
% option type from which the implied volatility is derived. This may
% be either a logical indicator or a cell array of characters. To
% specify call options, set Class = true or Class = {'call'}; to specify
% put options, set Class = false or Class = {'put'}. If empty or missing,
% the default is a call option.
%
% Output:
% Volatility - Implied volatility derived from European futures option
% prices, expressed as a decimal number. If no solution is found, a
% NaN (i.e., Not-a-Number) is returned.
```

```
%
% Example:
% Consider a European call futures option trading at
$1.1166, with an
% exercise prices of $20 that expires in 4 months. Assume
the current
% underlying futures price is also $20 and that the risk-
free rate is 9%
% per annum. Furthermore, assume we are interested in
implied volatilities
% no greater than 0.5 (i.e., 50% per annum). Under these
conditions, any
% of the following commands
%
% Volatility = blkimpv(20, 20, 0.09, 4/12, 1.1166, 0.5)
% Volatility = blkimpv(20, 20, 0.09, 4/12, 1.1166, 0.5,
[], {'Call'})
% Volatility = blkimpv(20, 20, 0.09, 4/12, 1.1166, 0.5,
[], true)
%
% return an implied volatility of 0.25, or 25%, per annum.
%
% Notes:
% (1) The input arguments Price, Strike, Rate, Time, Value,
and Class may be
% scalars, vectors, or matrices. If scalars, then that
value is used to
% compute the implied volatility from all options. If more
than one of
% these inputs is a vector or matrix, then the dimensions of
all
% non-scalar inputs must be the same.
% (2) Ensure that Rate and Time are expressed in consistent
units of time.
%
% See also BLKPRICE, BLSPRICE, BLSIMPV.
% Copyright 1995-2003 The MathWorks, Inc.
% $Revision: 1.4.2.2 $ $Date: 2004/01/08 03:06:15 $
% References:
% Hull, J.C., "Options, Futures, and Other Derivatives",
Prentice Hall,
% 5th edition, 2003, pp. 287-288.
% Black, F., "The Pricing of Commodity Contracts," Journal
of Financial
```

```
% Economics, March 3, 1976, pp. 167-79.
%
%
% Implement Black's model for European futures options as
a wrapper
% around a general Black-Scholes option model.
%
% In this context, Black's model is simply a special case of
a
% Black-Scholes model in which the futures/forward con-
tract is
% the underlying asset and the dividend yield = the risk-
free rate.
%
ifnargin< 5
error('Finance:blkimpv:TooFewInputs', ...
'Specify Price, Strike, Rate, Time, and Value.')
end
switchnargin
case 5
[limit, tol, optionClass] = deal([]);
case 6
[limit, tol, optionClass] = deal(varargin{1}, [], []);
case 7
[limit, tol, optionClass] = deal(varargin{1}, varargin
{2}, []);
case 8
[limit, tol, optionClass] = deal(varargin{1:3});
otherwise
error('Finance:blkimpv:TooManyInputs', 'Too many
inputs.')
end
try
volatility = blsimpv(F, X, r, T, value, limit, r, tol,
optionClass);
catch
errorStruct = lasterror;
errorStruct.identifier = strrep(errorStruct.identifier,
'blsimpv', 'blkimpv');
errorStruct.message = strrep(errorStruct.message,
'blsimpv', 'blkimpv');
rethrow(errorStruct);
end
```

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