Modeling Multiple Asset Returns by a Time-Varying *t* Copula Model

52

Long Kang

Contents

Abstract

We illustrate a framework to model joint distributions of multiple asset returns using a time-varying Student's *t* copula model. We model marginal distributions of individual asset returns by a variant of GARCH models and then use a Student's *t* copula to connect all the margins. To build a time-varying structure for the correlation matrix of *t* copula, we employ a dynamic conditional correlation (DCC) specification. We illustrate the two-stage estimation procedures for the model and apply the model to 45 major US stocks returns selected from nine

L. Kang
Department of Finance, Antai College of Economics and Management, Shanghai Jiao Tong University, Shanghai, China

The Options Clearing Corporation and Center for Applied Economics and Policy Research, Indiana University, Bloomington, IN, USA
e-mail: lkang@indiana.edu; kanglong@gmail.com

C.-F. Lee, J. Lee (eds.), *Handbook of Financial Econometrics and Statistics*,
DOI 10.1007/978-1-4614-7750-1_52,

sectors. As it is quite challenging to find a copula function with very flexible parameter structure to account for difference dependence features among all pairs of random variables, our time-varying *t* copula model tends to be a good working tool to model multiple asset returns for risk management and asset allocation purposes. Our model can capture time-varying conditional correlation and some degree of tail dependence, while it also has limitations of featuring symmetric dependence and inability of generating high tail dependence when being used to model a large number of asset returns.

Keywords

Student's *t* copula • GARCH models • Asset returns • US stocks • Maximum likelihood • Two-stage estimation • Tail dependence • Exceedance correlation • Dynamic conditional correlation • Asymmetric dependence

52.1 Introduction

There have been a large number of applications of copula theory in financial modeling. The popularity of copula mainly results from its capability of decomposing joint distributions of random variables into marginal distributions of individual variables and the copula which links the margins. Then the task of finding a proper joint distribution becomes to find a copula form which features a proper dependence structure given that marginal distributions of individual variables are properly specified. Among many copula functions, Student's *t* copula is a good choice, though not perfect, for modeling multivariate financial data as an alternative to a normal copula, especially for a very large number of assets. The *t* copula models are very useful tools to describe joint distributions of multiple assets for risk management and asset allocation purposes. In this chapter, we illustrate how to model the joint distribution of multiple asset returns under a Copula-GARCH framework. In particular, we show how we can build and estimate a time-varying *t* copula model for a large number of asset returns and how well the time-varying *t* copula accounts for some dependence features of real data.

There are still two challenging issues when applying copula theory to multiple time series. The first is how to choose a copula that best describes the data. Different copulas feature different dependence structure between random variables. Some copulas may fit one particular aspect of the data very well but do not have a very good overall fit, while others may have the opposite performance. What criteria to use when we choose from copula candidates is a major question remaining to be fully addressed. Secondly, how to build a multivariate copula which is sufficiently flexible to simultaneously account for the dependence structure for each pair of random variables in joint distributions is still quite challenging. We hope to shed some light on those issues by working through our time-varying *t* copula model.

Under a Copula-GARCH framework, we first model each asset return with a variant of GARCH specification. Based on different properties of asset returns, we choose a proper GARCH specification to formulate conditional distributions of each return. Then, we choose a proper copula function to link marginal distributions of each return to form the joint distribution. As in marginal distributions of each return, the copula parameters can also be specified as being dependent on previous observations to make the copula structure time varying for a better fit of data. In this chapter, we have an AR(1) process for the conditional mean and a GJR-GARCH (1,1) specification for the conditional volatility for each return. We employ a Student's *t* copula with a time-varying correlation matrix (by a DCC specification) to link marginal distributions. Usually the specified multivariate model contains a huge number of parameters, and the estimation by maximum likelihood estimator (MLE) can be quite challenging. Therefore, we pursue a two-stage procedure, where all the GARCH models for each return are estimated individually first and copula parameters are estimated in the second stage with estimated cumulative distribution functions from the first stage.

We apply our model to modeling log returns of 45 major US stocks selected from nine sectors with a time span ranging from January 3, 2000 to November 29, 2011. Our estimation results show that AR(1) and GJR-GARCH(1,1) can reasonably well capture empirical properties of individual returns. The stock returns possess fat tails and leverage effects. We plot the estimated conditional volatility on selected stocks and volatility spikes which happened during the "Internet Bubbles" in the early 2000s and the financial crisis in 2008. We estimate a DCC specification for the time-varying *t* copula and also a normal copula for comparison purposes. The parameter estimates for time-varying *t* copula are statistically significant, which indicates a significant time-varying property of the dependence structure. The time-varying *t* copula yields significantly higher log-likelihood than normal copula. This improvement of data fitness results from flexibility of *t* copula (relative to normal copula) and its time-varying correlation structure.

We plot the time-varying correlation parameter for selected pairs of stocks under the time-varying *t* copula model. The correlation parameters fluctuate around certain averages, and they spike during the 2008 crisis for some pairs. For 45 asset returns, the estimated degree-of-freedom (DoF) parameter of the *t* copula is around 25. Together with the estimated correlation matrix of the *t* copula, this DoF leads to quite low values of tail dependence coefficients (TDCs). This may indicate the limitation of *t* copulas in capturing possibly large tail dependence behavior for some asset pairs when being used to model a large number of asset returns. Nevertheless, the time-varying Student's *t* copula model has a relatively flexible parameter structure to account for the dependence among multiple asset returns and is a very effective tool to model the dynamics of a large number of asset returns in practice.

This chapter is organized as follows. Section 52.2 gives a short literature review on recent applications of copulas to modeling financial time series. Section 52.3 introduces our copula model where we introduce copula theory, Copula-GARCH framework, and estimation procedures. In particular, we elaborate on how to construct and estimate a time-varying *t* copula model. Section 52.4 documents the data source and descriptive statistics for the data set we use. Section 52.5 reports estimation results and Sect. 52.6 concludes.

52.2 Literature Review

Copula-GARCH models were previously proposed by Jondeau and Rockinger (2002) and Patton (2004, 2006a).[1] To measure time-varying conditional dependence between time series, the former authors use copula functions with time-varying parameters as functions of predetermined variables and model marginal distributions with an autoregressive version of Hansen's (1994) GARCH-type model with time-varying skewness and kurtosis. They show for many market indices, dependency increases after large movements and for some cases it increases after extreme downturns. Patton (2006a) applies the Copula-GARCH model to modeling the conditional dependence between exchange rates. He finds that mark-dollar and yen-dollar exchange rates are more correlated during depreciation against dollar than during appreciation periods. By a similar approach, Patton (2004) models the asymmetric dependence between "large cap" and "small cap" indices and examines the economic and statistical significance of the asymmetries for asset allocations in an out-of-sample setting. As in above literature, copulas are mostly used in capturing asymmetric dependence and tail dependence between times series. Among copula candidates, Gumbel's copula features higher dependence (correlation) at upper side with positive upper tail dependence, and rotated Gumbel's copula features higher dependence (correlation) at lower side with positive lower tail dependence. Hu (2006) studies the dependence structure between a number of pairs of major market indices by a mixed copula approach. Her copula is constructed by a weighted sum of three copulas–normal, Gumbel's, and rotated Gumbel's copulas. Jondeau and Rockinger (2006) model the bivariate dependence between major stock indices by a Student's *t* copula where the parameters are assumed to be modeled by a two-state Markov process.

The task of flexibly modeling dependence structure becomes more challenging for *n*-dimensional distributions. Tsafack and Garcia (2011) build up a complex multivariate copula to model four international assets (two international equities and two bonds). In his model, he assumes that the copula form has a regime-switching setup where in one regime he uses an *n*-dimensional normal copula and in the other he uses a mixed copula of which each copula component features the dependence structure of two pairs of variables. Savu and Trede (2010) develop a hierarchical Archimedean copula which renders more flexible parameters to characterize dependency between each pair of variables. In their model, each pair of closely related random variables is modeled by a copula of a particular Archimedean class, and then these pairs are nested by copulas as well. The nice property of Archimedean family easily leads to the validity of the

[1]Alternative approaches are also developed, such as in Ang and Bekaert (2002), Goeij and Marquering (2004), and Lee and Long (2009), to address non-normal joint distributions of asset returns.

joint distribution constructed by this hierarchical structure. (Trivedi and Zimmer 2006) apply trivariate hierarchical Archimedean copulas to model sample selection and treatment effects with applications to the family health-care demand.

Statistical goodness-of-fit tests can provide some guidance for selecting copula models. Chen et al. (2004) propose two simple goodness-of-fit tests for multivariate copula models, both of which are based on multivariate probability integral transform and kernel density estimation. One test is consistent but requires the estimation of the multivariate density function and hence is suitable for a small number of random variables, while the other may not be consistent but requires only kernel estimation of a univariate density function and hence is suitable for a large number of assets. Berg and Bakken (2006) propose a consistent goodness-of-fit test for copulas based on the probability integral transform, and they incorporate in their test a weighting functionality which can increase influence of some specific areas of copulas.

Due to their parameter structure, the estimation of Copula-GARCH models also suffers from "the curse of dimensionality".[2] The exact maximum likelihood estimator (MLE) works in theory.[3] In practice, however, as the number of time series being modeled increases, the numerical optimization problem in MLE will become formidable. Joe and Xu (1996) propose a two-stage procedure, where in the first stage only parameters in marginal distributions are estimated by MLE and then the copula parameters are estimated by MLE in the second stage. This two-stage method is called inference for the margins (IFM) method. Joe (1997) shows that under regular conditions the IFM estimator is consistent and has the property of asymptotic normality and Patton (2006b) also shows similar estimator properties for the two-stage method. Instead of estimating parametric marginal distributions in the IFM method, we can estimate the margins by using empirical distributions, which can avoid the problem of mis-specifying marginal distributions. This method is called canonical maximum likelihood (CML) method by Cherubini et al. (2004). Hu (2006) uses this method and she names it as a semi-parametric method. Based on Genest et al. (1995), she shows that CML estimator is consistent and has asymptotical normality. Moreover, copula models can also be estimated under a nonparametric framework. Deheuvels (1981) introduces the notion of empirical copula and shows that the empirical copula converges uniformly to the underlying true copula. Finally, Xu (2004) shows how the copula models can be estimated with a Bayesian approach. The author shows how a Bayesian approach can be used to account for estimation uncertainty in portfolio optimization based on a Copula-GARCH model, and she proposes to use a Bayesian MCMC algorithm to jointly estimate the copula models.

[2]For a detailed survey on the estimation of Copula-GARCH model, see Chap. 5 of Cherubini et al. (2004).

[3]See Hamilton (1994) and Greene (2003) for more details on maximum likelihood estimation.

52.3 The Model

52.3.1 Copula

We introduce our Copula-GARCH model framework by first introducing the concept of copula. A copula is a multivariate distribution function with uniform marginal distributions as its arguments, and its functional form links all the margins to form a joint distribution of multiple random variables.[4] Copula theory is mainly based on the work of Sklar (1959), and we state the Sklar's theorem for continuous marginal distributions as follows.

Theorem 52.1 *Let $F_1(x_1), \ldots, F_n(x_n)$ be given marginal distribution functions and continuous in $x_1, \ldots, x_n$, respectively. Let H be the joint distribution of $(x_1, \ldots, x_n)$. Then there exists a unique copula C such that*

$$H(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)), \quad \forall (x_1, \ldots, x_n) \in \overline{\mathbb{R}}^n. \tag{52.1}$$

Conversely, if we let $F_1(x_1), \ldots, F_n(x_n)$ be continuous marginal distribution functions and C be a copula, then the function H defined by Eq. 52.1 *is a joint distribution function with marginal distributions $F_1(x_1), \ldots, F_n(x_n)$.*

The above theory allows us to decompose a multivariate distribution function into marginal distributions of each random variable and the copula form linking the margins. Conversely, it also implies that to construct a multivariate distribution, we can first find a proper marginal distribution for each random variable and then obtain a proper copula form to link the margins. Depending on which dependence measure used, the copula function mainly, not exclusively, governs the dependence structure between individual variables. Hence, after specifying marginal distributions of each variable, the task of building a multivariate distribution solely becomes to choose a proper copula form which best describes the dependence structure between variables.

Differentiating Eq. 52.1 with respect to $(x_1, \ldots, x_n)$ leads to the joint density function of random variables in terms of copula density. It is given as

$$h(x_1, \ldots, x_n) = c((F_1(x_1), \ldots, F_n(x_n))\prod_{i=1}^{n} f_i(x_i), \quad \forall (x_1, \ldots, x_n) \in \overline{\mathbb{R}}^n, \tag{52.2}$$

where $c(F_1(x_1), \ldots, F_n(x_n))$ is the copula density and $f_i(x_i)$ is the density function for variable i. Equation 52.2 implies that the log-likelihood of the joint density can be decomposed into components which only involve each marginal density and a component which involves copula parameters. It provides a convenient structure for a two-stage estimation, which will be illustrated in details in the following sections.

[4] See Nelsen (1998) and Joe (1997) for a formal treatment of copula theory, and Bouye et al. (2000), Cherubini et al. (2004), and Embrechts et al. (2002) for applications of copula theory in finance.

To better fit the data, we usually assume the moments of distributions of random variables are time varying and depend on past variables. Therefore, the distribution of random variables at time t becomes a conditional one, and then the above copula theory needs to be extended to a conditional case. It is given as follows.[5]

Theorem 52.2 *Let Ω_{t-1} be the information set up to time t, and let $F_1(x_{1,t}|\Omega_{t-1})$, …, $F_n(x_{n,t}|\Omega_{t-1})$ be continuous marginal distribution functions conditional on Ω_{t-1}. Let H be the joint distribution of $(x_1, \ldots, x_n)$ conditional on Ω_{t-1}. Then there exists a unique copula C such that*

$$H\left(x_1, \ldots, x_n|\Omega_{t-1}\right) = C\left(F_1\left(x_1|\Omega_{t-1}\right), \ldots, F_n\left(x_n|\Omega_{t-1}\right)|\Omega_{t-1}\right), \ \forall(x_1, \ldots, x_n) \in \overline{\mathbb{R}}^n. \tag{52.3}$$

Conversely, if we let $F_1(x_{1,t}|\Omega_{t-1})$, …, $F_n(x_{n,t}|\Omega_{t-1})$ be continuous conditional marginal distribution functions and C be a copula, then the function H defined by Eq. 52.3 is a conditional joint distribution function with conditional marginal distributions $F_1(x_{1,t}|\Omega_{t-1})$, …, $F_n(x_{n,t}|\Omega_{t-1})$.

It is worth noting that for the above theorem to hold, the information set Ω_{t-1} has to be the same for the copulas and all the marginal distributions. If different information sets are used, the conditional copula form on the right side of Eq. 52.3 may not be a valid distribution. Generally, the same information set used may not be relevant for each marginal distributions and the copula. For example, the marginal distributions or the copula may be only conditional on a subset of the universally used information set. At the very beginning of estimation of the conditional distributions, however, we should use the same information set based on which we can test for insignificant explanatory variables so as to stick to a relevant subset for each marginal distribution or the copula.

52.3.2 Modeling Marginal Distributions

Before building a copula model, we need to find a proper specification for marginal distributions of individual asset returns, as mis-specified marginal distributions automatically lead to a mis-specified joint distribution. Let $x_{i,t}$ be asset i return at time t, and its conditional mean and variance are modeled as follows:

$$x_{i,t} = \alpha_{0,i} + \alpha_{1,i}x_{i,t-1} + \varepsilon_{i,t}, \tag{52.4}$$

$$\varepsilon_{i,t} = \sqrt{h_{i,t}}\eta_{i,t}, \tag{52.5}$$

$$h_{i,t} = \beta_{0,i} + \beta_{1,i}h_{i,t-1} + \beta_{2,i}\varepsilon^2_{i,t-1} + \beta_{3,i}\varepsilon^2_{i,t-1}1\left(\varepsilon_{i,t-1} < 0\right). \tag{52.6}$$

[5] See Patton (2004).

As shown in Eqs. 52.4, 52.5 and 52.6, we model the conditional mean as an AR(1) process and the conditional variance as a GJR(1,1) specification.[6] We have parameter restrictions as $\beta_{0,i} > 0$, $\beta_{1,i} \geq 0$, $\beta_{2,i} \geq 0$, $\beta_{2,i} + \beta_{3,i} \geq 0$, and $\beta_{1,i} + \beta_{2,i} + \frac{1}{2}\beta_{3,i} < 1$. $1(\varepsilon_{i,t-1} < 0)$ is an indicator function, which equals one when $\varepsilon_{i,t-1} < 0$ and zero otherwise. We believe that our model specifications can capture the features of the individual stock returns reasonably well. It is worth noting that Eqs. 52.4, 52.5 and 52.6 can include more exogenous variables to better describe the data. Alternative GARCH specifications can be used to describe the time-varying conditional volatility. We assume $\eta_{i,t}$ is *i.i.d.* across time and follows a Student's t distribution with DoF v_i.

Alternatively, to model the conditional higher moments of the series, we can follow Hansen (1994) and Jondeau and Rockinger (2003) who assume a skewed t distribution for the innovation terms of GARCH specifications and find that the skewed t distribution fits financial time series better than normal distribution. Accordingly, we can assume $\eta_{i,t} \sim Skewed\, T(\eta_{i,t}|v_{i,t},\lambda_{i,t})$ with zero mean and unitary variance where $v_{i,t}$ is DoF parameter and $\lambda_{i,t}$ is skewness parameter. The two parameters are time varying and depend on lagged values of explanatory variables in a nonlinear form. For illustration purposes, however, we will only use Student's t distribution for $\eta_{i,t}$ in this chapter.

52.3.3 Modeling Dependence Structure

Normal copula and Student's t copula are two copula functions from elliptical families, which are frequently used in modeling joint distributions of random variables. In this chapter, we also estimate a normal copula model for comparison purposes. Let Φ^{-1} denote the inverse of the standard normal distribution Φ and $\Phi_{\sum,n}$ be n-dimensional normal distribution with correlation matrix $\sum$. Hence, the n-dimensional normal copula is

$$C(\mathbf{u};\Sigma) = \Phi_{\Sigma,N}\left(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n)\right), \tag{52.7}$$

and its density form is

$$c(\mathbf{u};\Sigma) = \frac{\phi_{\Sigma,n}\left(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n)\right)}{\prod_{i=1}^{n} \phi\left(\Phi^{-1}(u_i)\right)}, \tag{52.8}$$

where ϕ and $\phi_{\sum,n}$ are the probability density functions (*pdf*s) of Φ and $\Phi_{\sum,n}$, respectively. It can be shown via Sklar's theorem that normal copula generates standard joint normal distribution if and only if the margins are standard normal.

[6]See Glosten et al. (1993).

On the other hand, let T_v^{-1} be the inverse of standard Student's *t* distribution T_v with DoF parameter[7] $v > 2$ and $T_{R,v}$ be *n*-dimensional Student's *t* distribution with correlation matrix *R* and DoF parameter *v*. Then *n*-dimensional Student's *t* copula is

$$C(\mathbf{u}; R, v) = T_{R,v}\left(T_v^{-1}(u_1), \ldots, T_v^{-1}(u_n)\right), \tag{52.9}$$

and its density function is

$$c(u_1, \ldots, u_n) = \frac{t_{R,v}\left(T_v^{-1}(u_1), \ldots, T_v^{-1}(u_n)\right)}{\prod_{i=1}^{n} t_v\left(T_v^{-1}(u_i)\right)},$$

where t_v and $t_{R,v}$ are the *pdf*s of T_v and $T_{R,v}$, respectively.

Borrowing from the dynamic conditional correlation (DCC) structure of multivariate GARCH models, we can specify a time-varying parameter structure in the *t* copula as follows.[8] For a *t* copula, the time-varying correlation matrix is governed by

$$Q_t = (1 - \alpha - \beta)S + \alpha\left(\varsigma_{t-1}\varsigma_{t-1}'\right) + \beta Q_{t-1}, \tag{52.10}$$

where *S* is the unconditional covariance matrix of $\varsigma_t = (T_v^{-1}(u_{1,t}), \ldots, T_v^{-1}(u_{n,t}))'$ and α and β are nonnegative and satisfy the condition $\alpha + \beta < 1$. We assign $Q_0 = S$ and the dynamics of Q_t is given by Eq. 52.10. Let $q_{i,j,t}$ be the *i*,*j* element of the matrix Q_t, and the *i*,*j* element of the conditional correlation matrix R_t can be calculated as

$$\rho_{i,j,t} = \frac{q_{i,j,t}}{\sqrt{q_{i,i,t}q_{j,j,t}}}. \tag{52.11}$$

Moreover, the specification of Eq. 52.10 guarantees that the conditional correlation matrix R_t is positive definite.

Proposition 52.1 *In Eqs.* 52.10 and 52.11, *if*

(a) $\alpha \geq 0$ and $\beta \geq 0$,
(b) $\alpha + \beta < 1$,
(c) *All eigenvalues of S are strictly positive, then the correlation matrix R_t is positive definite.*

[7] In contrast to the previous standardized Student's t distribution, the standard Student's t distribution here has variance as $v/(v-2)$.

[8] Please see Engle and Sheppard (2001) and Engle (2002) for details on the multivariate DCC-GARCH models.

Proof *First, (a) and (b) guarantee the system for $\varsigma_t\varsigma_t'$ is stationary and S exists. With $Q_0 = S$, (c) guarantees Q_0 is positive definite. With (a) to (c), Q_t is the sum of a positive definite matrix, a positive semi-definite matrix, and a positive definite matrix both with nonnegative coefficients and then is positive definite for all t. Based on the proposition Eq.* 52.1 *in Engle and Sheppard* (2001), *we prove that R_t is positive definite.*

52.3.4 Estimation

We illustrate the estimation procedure by writing out the log-likelihoods for observations. Let $\Theta = \{\theta, \gamma_1, \ldots, \gamma_n\}$ be the set of parameters in the joint distribution where θ is the set of parameters in the copula and γ_t is the set of parameters in marginal distributions for asset *i*. Then the conditional cumulative distribution function (*cdf*) of *n* asset returns at time *t* is given as

$$F\left(x_{1,t}, \ldots, x_{n,t} \middle| \underline{X}_{t-1}, \Theta\right) = C\left(u_{1,t}, \ldots, u_{n,t} \middle| \underline{X}_{t-1}, \theta\right) \tag{52.12}$$

where $\underline{X}_{t-1}$ is a vector of previous observations, $C(\cdot|\underline{X}_{t-1}, \theta)$ is the conditional copula, and $u_{i,t} = F_i(x_{i,t}|X_{t-1}, \gamma_i)$ is the conditional *cdf* of the margins. Differentiating both sides with respect to $x_{1t}, \ldots, x_{n,t}$ leads to the density function as

$$f\left(x_{1,t}, \ldots, x_{n,t} \middle| \underline{X}_{t-1}, \Theta\right) = c\left(u_{1,t}, \ldots, u_{n,t} \middle| \underline{X}_{t-1}, \theta\right) \prod_{i=1}^{n} f_i\left(x_{i,t} \middle| \underline{X}_{t-1}, \gamma_i\right), \tag{52.13}$$

where $c(\cdot|\underline{X}_{t-1}, \theta)$ is the density of the conditional copula and $f_i(x_{i,t}|\underline{X}_{t-1}, \gamma_i)$ is the conditional density of the margins. Accordingly, the log-likelihood of the sample is given by

$$L(\Theta) = \sum_{t=1}^{T} \log f\left(x_{1,t}, \ldots, x_{n,t} \middle| \underline{X}_{t-1}, \Theta\right). \tag{52.14}$$

With Eq. 52.13, the log-likelihood can be written as

$$L(\theta, \gamma_1, \ldots, \gamma_n) = \sum_{t=1}^{T} \log c\left(u_{1,t}, \ldots, u_{n,t} \middle| \underline{X}_{t-1}, \theta\right) + \sum_{t=1}^{T} \sum_{i=1}^{n} f_i\left(x_{i,t} \middle| \underline{X}_{t-1}, \gamma_i\right). \tag{52.15}$$

From Eq. 52.15, we observe that the copula and marginal distributions are additively separate. Therefore, we can estimate the model by a two-stage MLE procedure. In the first stage, the marginal distribution parameters for each asset are estimated by MLE, and then with estimated *cdf* of each asset, we estimate the copula parameters by MLE. Based on Joe (1997) and Patton (2006b), this two-stage estimator is consistent and asymptotically normal.

With our model specifications, we first estimate the univariate GJR-GARCH (1,1) with an AR(1) conditional mean and Student's t distribution by MLE. In the second stage, we need to estimate the parameters for the constant normal copula and the time-varying Student's t copula. Let $\mathbf{x}_t = (\Phi^{-1}(u_{1,t}), \ldots, \Phi^{-1}(u_{n,t}))'$, and we can analytically derive the correlation matrix estimator $\hat{\Sigma}$ which maximizes the log-likelihood of the normal copula density as

$$\hat{\Sigma} = \frac{1}{T}\sum_{t=1}^{T} \mathbf{x}_t \mathbf{x}_t'. \tag{52.16}$$

As there is no analytical solution for MLE of Student's t copula, the numerical maximization problem is quite challenging. Following Chen et al. (2004), however, with $\varsigma_t = (T_v^{-1}(u_{1,t}), \ldots, T_v^{-1}(u_{n,t}))'$, we can calculate the sample covariance matrix of ς_t as $\hat{S}$, which is a function of DoF parameter v. By setting $Q_0 = \hat{S}$, we can express Q_t and R_t for all t in terms of α, β, and v using Eq. 52.10. Then we can estimate α, β, and v by maximizing the log-likelihood of t copula density. In the following sections, we apply our estimation procedure to the joint distribution of 45 selected major US stock returns.

52.4 Data

We apply our model to modeling log returns of 45 major US stocks from nine sectors: Consumer Discretionary, Consumer Staples, Energy, Financials, Health Care, Industrials, Technology, Materials, and Utilities. Table 52.1 shows stock symbols and company names of the selected 45 companies. We select five major companies from each sector to form the stock group. The time span ranges from January 3, 2000 to November 29, 2011 with 2990 observations. We download data from yahoo finance (http://finance.yahoo.com/). The log returns are calculated from daily close stock prices adjusted for dividends and splits.

To save space, we only plot and calculate descriptive statistics of nine stocks with each from one sector. Figure 52.1 plots the log returns of those nine selected stocks, and there are two periods of volatility clusterings due to "Internet Bubbles" in the early 2000s and the financial crisis in 2008, respectively. We observe that during the financial crisis in 2008, major banks, such as Citigroup, incurred huge negative and positive daily returns. Table 52.2 shows the calculated mean, standard deviation, skewness, and kurtosis for the nine stocks. The average returns for the nine stocks are close to zero. Major banks, represented by Citigroup, have significantly higher volatility. Most of the stocks are slightly positively skewed, and only two have slight negative skewness. All the stocks have kurtosis greater than three indicating fat tails, and again major banks have significantly fatter tails. All the descriptive statistics indicate that the data property of individual returns needs to be captured by a variant of GARCH specification.

Table 52.1 Symbols and names of 45 selected stocks from nine sectors

Sector	Consumer discretionary	Consumer Staples	Energy
Stock symbol	MCD: McDonald's	WMT: Wal-Mart Stores Inc.	XOM: Exxon Mobil Corp.
	HD: Home Depot	PG: Procter & Gamble Co.	CVX: Chevron Corp.
	DIS: Walt Disney Co.	KO: Coca-Cola Co.	COP: CONOCOPHILLIPS.
	TGT: Target	WAG: Walgreen Co.	DVN: Devon Energy Corp.
	LOW: Lowe's	MO: Altria Group Inc.	SLB: Schlumberger Limited
Sector	Financials	Health Care	Industrials
Stock symbol	C: Citigroup Inc.	JNJ: Johnson & Johnson	GE: General Electric Co.
	BAC: Bank of America Corp.	PFE: Pfizer Inc.	UNP: Union Pacific Corp.
	JPM: JPMorgan Chase & Co.	ABT: Abbott Laboratories	UTX: United Technologies Corp.
	USB: U.S. Bancorp	MRK: Merck & Co. Inc.	MMM: 3 M Co.
	WFC: Wells Fargo & Co.	AMGN: Amgen Inc.	BA: Boeing Co.
Sector	Technology	Materials	Utilities
Stock symbol	T: AT&T Inc.	NEM: Newmont Mining Corp.	EXC: Exelon Corp.
	MSFT: Microsoft Corp.	DD: E.I. DuPont de Nemours & Co.	FE: FirstEnergy Corp.
	IBM: International Business Machines Corp.	DOW: Dow Chemical Co.	PPL: PPL Corporation
	CSCO: Cisco Systems Inc.	FCX: Freeport-McMoRan Copper & Gold Inc.	D: Dominion Resources, Inc.
	HPQ: Hewlett-Packard Co.	PX: Praxair Inc.	DUK: Duke Energy Corp.

52.5 Empirical Results

52.5.1 Marginal Distributions

We briefly report estimation results for marginal distributions of 45 stock returns. For convenience, we only show the estimates and standard errors (in brackets) for nine selected stocks with each from one sector in Table 52.3. The star indicates statistical significance at a 5 % level. Consistent with our observations in Table 52.2, all the nine stocks have low values of DoF indicating fat tails. The parameter $\beta_{3,i}$ is

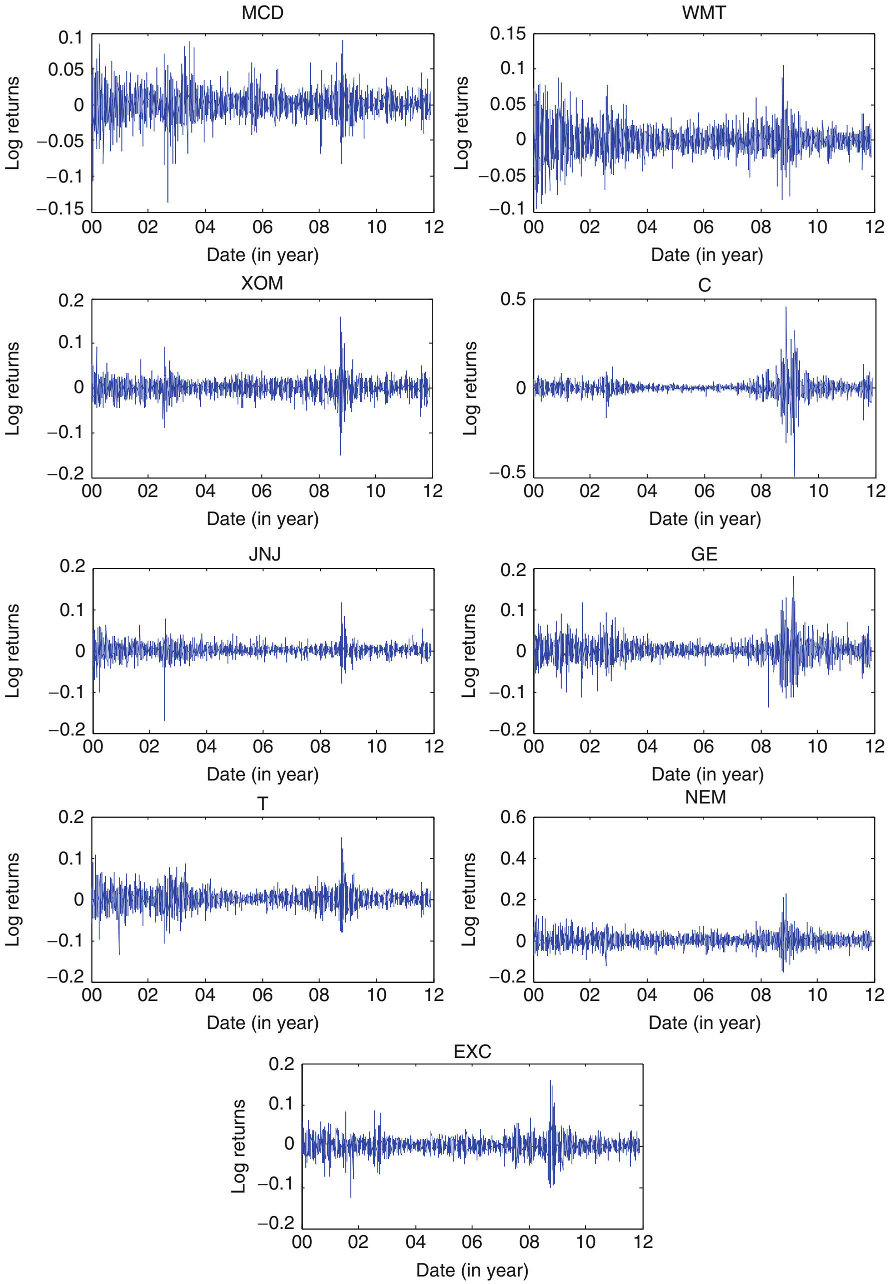

Fig. 52.1 The log returns of the nine of our 45 selected stocks with each from one sector have been plotted

Table 52.2 Descriptive statistics (mean, standard deviation, skewness, and kurtosis) for the nine of our 45 selected stocks with each from each sector

Stock symbol	MCD	WMT	XOM	C	JNJ	GE	T	NEM	EXC
Mean	3.73E-04	7.21E-06	3.15E-04	−8.05E-04	1.96E-04	−2.89E-04	1.23E-05	3.65E-04	4.48E-04
Std. dev.	0.017	0.017	0.017	0.037	0.013	0.022	0.019	0.027	0.018
Skewness	−0.21	0.13	0.02	−0.48	−0.53	0.04	0.12	0.34	0.05
Kurtosis	8.25	7.72	12.52	35.55	17.83	9.99	8.68	8.22	10.58
# obs.	2,990	2,990	2,990	2,990	2,990	2,990	2,990	2,990	2,990

statistically significant at a 5 % level for eight of the nine stocks indicating significant leverage effects for stock returns. The parameters in conditional mean are statistically significant for some stocks and not for others. In Fig. 52.2, we plot estimated conditional volatility for the stocks MCD, WMT, XOM, and C. Consistent with Fig. 52.1, we observe MCD and WMT have significant high volatility in the early 2000s and 2008, while XOM and C have their volatility hikes mainly in 2008 with C, representing Citigroup, having the highest conditional volatility during the 2008 crisis.

52.5.2 Copulas

We report estimation results for the time-varying t copula parameters in Table 52.4. All the three parameters α, β, and v are statistically significant. The estimate α is close to zero and the estimate for β is close to one. The estimate for v is about 25. As our estimation is carried out on the joint distribution of 45 stock returns, the estimate for v shed some light on how much Student's t copula can capture tail dependence when used to fit a relatively large number of variables. We also report the log-likelihood for time-varying Student's t copula and normal copula in Table 52.4. As the correlation matrix in normal copula is estimated by its sample correlation, we did not report it here. We find that time-varying t copula has significantly higher log-likelihood than normal copula, which results from the more flexible parameter structure of t copula and the time-varying parameter structure.

52.5.3 Time-Varying Dependence

Our time-varying t copula features a time-varying dependence structure among all the variables. The DoF parameter, together with the correlation parameters, governs the tail dependence behavior of multiple variables. We plot the estimated

Table 52.3 The GARCH estimation results of individual stock returns for the nine of our selected 45 stocks with each from one sector. Values in brackets are standard errors. The star indicates the statistical significance at a 5 % level

Stock symbol	MCD	WMT	XOM	C	JNJ	GE	T	NEM	EXC
Conditional mean									
$\alpha_{0,i}$	6.69E-04*	−1.77E-05	6.50E-04*	−1.03E-06	1.17E-04	−1.93E-06	2.73E-04	3.31E-04	7.10E-04*
	(2.25E-04)	(2.11E-04)	(2.41E-04)	(2.42E-04)	(1.59E-04)	(2.32E-04)	(2.29E-04)	(3.87E-04)	(2.36E-04)
$\alpha_{1,i}$	−0.030	−0.037*	−0.076*	−0.003	0.004	−0.015	0.009	−0.056*	−0.019
	(0.018)	(0.018)	(0.019)	(0.018)	(0.018)	(0.018)	(0.018)	(0.018)	(0.019)
Conditional variance									
$\beta_{0,i}$	1.69E-06*	8.22E-07*	5.97E-06*	1.96E-06*	1.87E-06*	1.26E-06*	1.37E-06*	3.11E-06*	3.91E-06*
	(5.21E-07)	(3.50E-07)	(1.28E-06)	(5.53E-07)	(4.40E-07)	(3.93E-07)	(4.47E-07)	(1.31E-06)	(9.82E-07)
$\beta_{1,i}$	0.947*	0.952*	0.901*	0.908*	0.905*	0.948*	0.943*	0.959*	0.897*
	(0.007)	(0.007)	(0.013)	(0.009)	(0.011)	(0.007)	(0.007)	(0.007)	(0.012)
$\beta_{2,i}$	0.024*	0.026*	0.025*	0.045*	0.025*	0.017*	0.027*	0.037*	0.066*
	(0.008)	(0.008)	(0.012)	(0.011)	(0.011)	(0.007)	(0.008)	(0.009)	(0.014)
$\beta_{3,i}$	0.046*	0.038*	0.092*	0.094*	0.127*	0.067*	0.051*	−0.001	0.046*
	(0.012)	(0.013)	(0.018)	(0.017)	(0.021)	(0.012)	(0.012)	(0.011)	(0.018)
Degree of freedom									
v_i	6.187* (0.67)	7.002* (0.92)	9.496* (1.59)	6.505* (0.75)	5.796* (0.62)	6.622* (0.66)	8.403* (1.20)	7.831* (1.09)	9.247* (1.28)

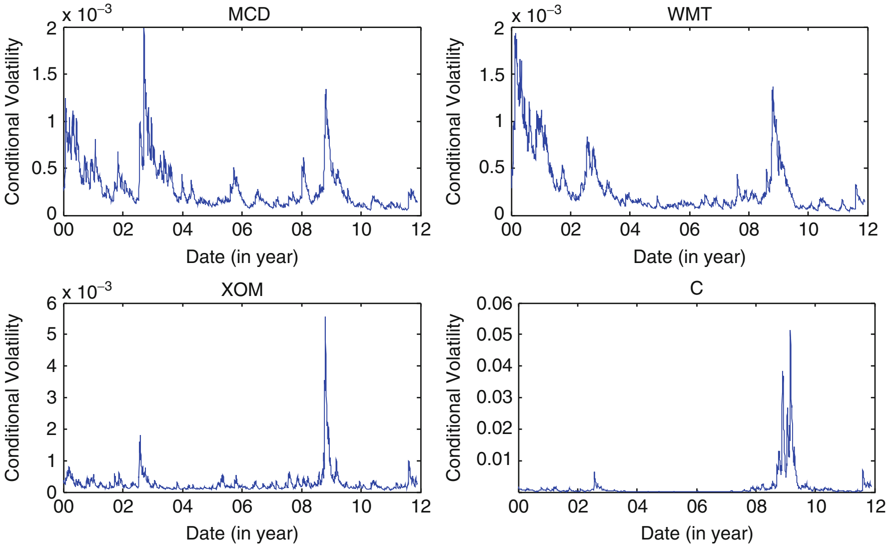

Fig. 52.2 The estimated time-varying conditional volatility for four selected stocks has been plotted

Table 52.4 The estimates and standard errors for time-varying Student's t copula. Values in brackets are standard errors. The star indicates the statistical significance at a 5 % level. We also report the log-likelihood for time-varying t copula and normal copula

	Time-varying t copula	Normal copula
Parameter estimates		
α	0.0031* (0.0002)	
β	0.984* (0.0016)	
v	25.81* (1.03)	
Log-likelihood of copula component		
Log-likelihood	40,445.44	38,096.36

conditional correlation parameters of *t* copula for four selected pairs of stock returns in Fig. 52.3. For those four pairs, the conditional correlation parameter fluctuates around certain positive averages. The two pairs, MCD-WMT and NEM-EXC, experienced apparent correlation spikes during the 2008 financial crisis. Moreover, Fig. 52.4 shows the estimated TDCs for the four pairs. We find that with the DoF around 25, the TDCs for those pairs of stock returns are very low, though some pairs do exhibit TDC spikes during the 2008 crisis. The low values of TDCs indicate possible limitations of *t* copula to account for tail dependence when being used to model a large number of variables.

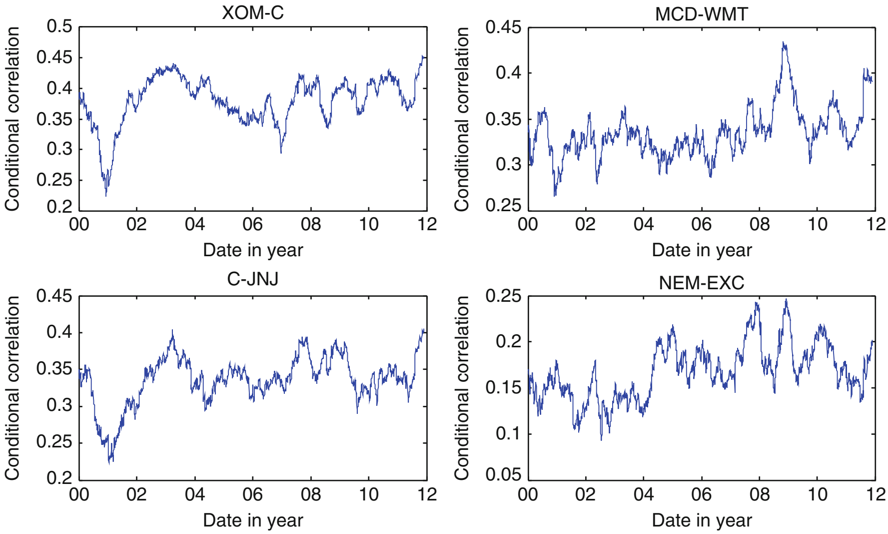

Fig. 52.3 The estimated time-varying correlation parameters in t copula for four selected pairs of stock returns have been plotted

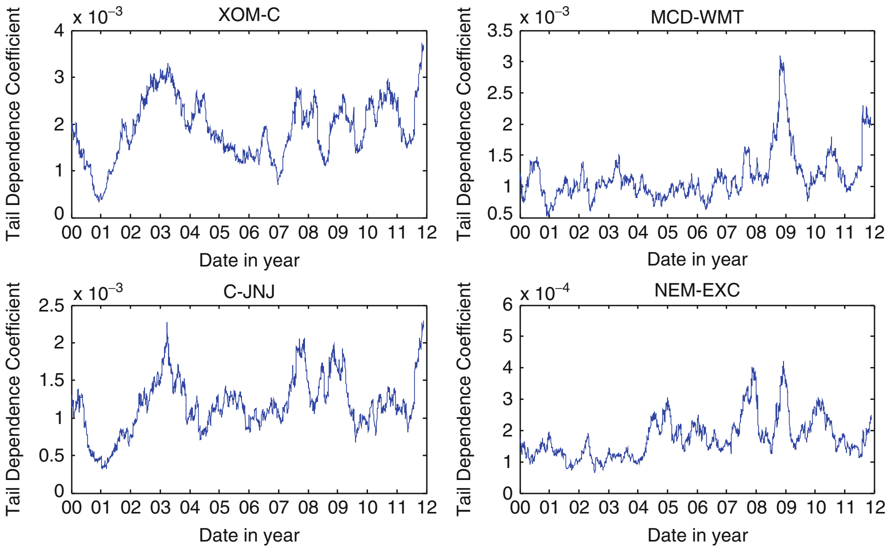

Fig. 52.4 The time-varying tail dependence coefficient (*TDC*) for the four selected pairs of stock returns has been plotted

52.6 Conclusion

We illustrate an effective approach (Copula-GARCH models) to model the dynamics of a large number of multiple asset returns by constructing a time-varying Student's *t* copula model. Under a general Copula-GARCH framework, we specify a proper GARCH model for individual asset returns and use a copula to link the margins to build the joint distribution of returns. We apply our time-varying Student's *t* copula model to 45 major US stock returns, where each stock return is modeled by an AR(1) and GJR-GARCH(1,1) specification and a Student's *t* copula with a DCC dependence structure is used to link all the returns. We illustrate how the model can be effectively estimated by a two-stage MLE procedure, and our estimation results show time-varying *t* copula model has significant better fitness of data than normal copula models.

As it is quite challenging to find a copula function with very flexible parameter structure to account for difference dependence features among all pairs of random variables, our time-varying *t* copula model tends to be a good working tool to model multiple asset returns for risk management and asset allocation purposes. Our model can capture time-varying conditional correlation and some degree of tail dependence, while it also has limitations of featuring symmetric dependence and inability of generating high tail dependence when being used to model a large number of asset returns. Nevertheless, we hope that this chapter provides researchers and financial practitioners with a good introduction on the Copula-GARCH models and a detailed illustration on constructing joint distributions of multiple asset returns using a time-varying Student's *t* copula model.

References

Ang, A., & Bekaert, G. (2002). International asset allocation with regime shifts. *Review of Financial Studies, 15*(4), 1137–1187.

Berg, D., & Bakken, H. (2006). *A copula goodness-of-fit test based on the probability integral transform*. Working Paper at http://www.danielberg.no/publications/Btest.pdf

Bouyé, E., Valdo D., Ashkan N., Gaël R., & Thierry, R. (2000). *Copulas for finance – A reading guide and some applications*. Available at SSRN http://ssrn.com/abstract=1032533. March.

Chen, X., Yanqin F., & Andrew, J. P. (2004). *Simple tests for models of dependence between multiple financial time series, with applications to U.S. Equity returns and exchange rates* (Discussion Paper 483). London School of Economics, Revised July 2004.

Cherubini, U., Luciano, E., & Vecchiato, W. (2004). *Copula methods in finance* (Wiley series in financial engineering). Chichester: Wiley.

De Goeij, P., & Marquering, W. (2004). Modeling the conditional covariance between stock and bond returns: A multivariate GARCH approach. *Journal of Financial Econometrics, 2*(4), 531–564.

Deheuvels, P. (1981). *A Nonparametric test for independence*. Paris: Universite de Paris, Institut de Statistique.

Engle, R. (2002). Dynamic conditional correlation: A simple class of multivariate generalized autoregressive conditional heteroskedasticity models. *Journal of Business & Economic Statistics, 20*(3), 339–350.

Engle, R. F., & Sheppard, K. (2001). *Theoretical and empirical properties of dynamic conditional correlation multivariate GARCH* (NBER Working Paper No. 8554). Cambridge, MA: National Bureau of Economic Research.

Genest, C., Ghoudi, K., & Rivest, L. (1995). A semiparametric estimation procedure of dependence parameters in multivariate families of distributions. *Biometrika, 82*, 543–552.

Glosten, L. R., Ravi, J., & Runkle, D. E. (1993). On the relation between the expected value and the volatility of the nominal excess return on stocks. *Journal of Finance, 48*(5), 1779–1801.

Greene, W. H. (2003). *Econometric analysis*. Upper Saddle River: Prentice Hall.

Hamilton, J. D. (1994). *Time series analysis*. Princeton: Princeton University Press.

Hansen, B. E. (1994). Autoregressive conditional density estimation. *International Economic Review, 35*(3), 705–730.

Hu, L. (2006). Dependence patterns across financial markets: A mixed copula approach. *Applied Financial Economics, 16*, 717–729.

Joe, H. (1997). *Multivariate models and dependence concepts*. London: Chapman and Hall.

Joe, H., & Xu, J. J. (1996). *The estimation method of inference functions for margins for multivariate models* (Technical Report 166). Vancouver: Department of Statistics, University of British Columbia.

Jondeau, E., & Rockinger, M. (2002). *Conditional dependency of financial series: the copula-GARCH model* (FAME Research Paper 69). Geneva: International Center for Financial Asset Management and Engineering.

Jondeau, E., & Rockinger, M. (2003). Conditional volatility, skewness, and kurtosis: Existence, persistence, and comovements. *Journal of Economic Dynamics and Control, 27*(10), 1699–1737.

Jondeau, E., & Rockinger, M. (2006). The copula-GARCH model of conditional dependencies: An international stock market application. *Journal of International Money and Finance, 25*, 827–853.

Lee, T.-H., & Long, X. (2009). Copula-based multivariate GARCH model with uncorrelated dependent errors. *Journal of Econometrics, 150*, 207–218.

Nelson, R. B. (1998). *An introduction to copula*. New York: Springer.

Patton, A. J. (2004). On the out-of-sample importance of skewness and asymmetric dependence for asset allocation. *Journal of Financial Econometrics, 2*(1), 130–168.

Patton, A. J. (2006a). Estimation of multivariate models for time series of possibly different lengths. *Journal of Applied Econometrics, 21*(2), 147–173.

Patton, A. J. (2006b). Modelling asymmetric exchange rate dependence. *International Economic Review, 47*(2), 527–556.

Savu, C., & Trede, M. (2010). Hierarchies of Archimedean copulas. *Quantitative Finance, 10*, 295–304.

Sklar, A. (1959). *Fonctions de repartition an dimensions et leurs marges*. Paris: Publications de l'Institut de statistique de l'Universite de Paris.

Trivedi, P. K., & Zimmer, D. M. (2006). Using trivariate copulas to model sample selection and treatment effects: Application to family health care demand. *Journal of Business and Economic Statistics, 24*(1), 63.

Tsafack, G., & Garcia, R. (2011). Dependence structure and extreme comovements in international equity and bond markets. *Journal of Banking and Finance, 35*, 1954–1970.

Xu, Y. (2004). *Incorporating estimation risk in copula-based portfolio optimization*. Miami: Department of Economics, University of Miami.