Chapter 3
Champernowne's Number, Strong Normality, and the X Chromosome

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This paper is dedicated to Jon Borwein in celebration of his 60th birthday

Abstract Champernowne's number is the best-known example of a normal number, but its digits are far from random. The sequence of nucleotides in the human X chromosome appears nonrandom in a similar way. We give a new asymptotic test of pseudorandomness, based on the law of the iterated logarithm; we call this new criterion "strong normality." We show that almost all numbers are strongly normal and that strong normality implies normality. However, Champernowne's number is not strongly normal. We adapt a method of Sierpiński to construct an example of a strongly normal number.

Key words: Champernowne's number • Law of the iterated logarithm • Normality of numbers • Random walks on digits of numbers • Random walks on nucleotide sequences

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3.1 Normality

We can write a real number α in any integer base $r \geq 2$ as a sum of powers of the base:

$$\alpha = \sum_{j=-d}^{\infty} a_j r^{-j}.$$

The standard "decimal" notation is

$$\alpha = a_{-d}\, a_{-(d-1)} \cdots a_0 \,.\, a_1\, a_2 \, \cdots \, .$$

The sequence of digits $\{a_j\}$ gives the representation of α in the base r, and this representation is unique unless α is rational, in which case α may have two representations. (For example, in the base 10, $0.1 = 0.0999\cdots$.)

We call a subsequence of consecutive digits a *string*. The string may be finite or infinite; we call a finite string of t digits a *t-string*. An infinite string beginning in a specified position we call a *tail*, and we call a finite string beginning in a specified position a *block*.

A number α is *simply normal* in the base r if every 1-string in its base-r expansion occurs with an asymptotic frequency approaching $1/r$. That is, given the expansion $\{a_j\}$ of α in the base r, and letting $m_k(n)$ be the number of times that $a_j = k$ for $j \leq n$, we have

$$\lim_{n\to\infty} \frac{m_k(n)}{n} = \frac{1}{r}$$

for each $k \in \{0, 1, \ldots, r-1\}$. This is Borel's original definition [6].

A number is *normal* in the base r if every t-string in its base-r expansion occurs with a frequency approaching r^{-t}. Equivalently, a number is normal in the base r if it is simply normal in the base r^t for every positive integer t (see [6, 14, 17]).

A number is *absolutely normal* if it is normal in every base. Borel [6] showed that almost every real number is absolutely normal.

In 1933, Champernowne [8] produced the first concrete construction of a normal number. Champernowne's number is

$$\gamma_{10} = .1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ \cdots .$$

The number is written in the base 10, and its digits are obtained by concatenating the natural numbers written in the base 10. This number is likely the best-known example of a normal number.

Generally, the base-r Champernowne number is formed by concatenating the integers 1, 2, 3, ... in the base r. For example, the base-2 Champernowne number is written in the base 2 as

$$\gamma_2 = .1\ 10\ 11\ 100\ 101\ \cdots .$$

For any r, the base-r Champernowne number is normal in the base r. However, the question of its normality in any other base (not a power of r) is open. For example, it is not known whether the base-10 Champernowne number is normal in the base 2.

In 1917, Sierpiński [15] gave a construction of an absolutely normal number (in fact, one such number for each ε with $0 < \varepsilon \leq 1$). A computable version of this construction was given by Becher and Figueira [2].

Most fundamental irrational constants, such as $\sqrt{2}$, $\log 2$, π, and e, appear to be normal, and statistical tests done to date are consistent with the hypothesis that they are normal. (See, for example, Kanada on π [10] and Beyer, Metropolis and Neergard on irrational square roots [5].) However, there is no proof of the normality of any of these constants.

There is an extensive literature on normality in the sense of Borel. Introductions to the literature may be found in [4, 7].

3.2 Walks on the Digits of Numbers and on Chromosomes

In this section we graphically compare two walks on the digits of numbers with a walk on the values of the Liouville λ function and a walk on the nucleotides of the human X chromosome.

The walks are generated on a binary sequence of digits (Figs. 3.1 and 3.2) by converting each 0 in the sequence to -1 and then using digit pairs $(\pm 1, \pm 1)$ to walk $(\pm 1, \pm 1)$ in the plane. The colour or shading in the figures gives a rough indication of the number of steps taken in the walk. The values of the Liouville λ function (Fig. 3.3) are already ± 1.

There are four nucleotides in the X chromosome sequence, and each of the four is assigned one of the values $(\pm 1, \pm 1)$ to create a walk on the nucleotide sequence (Fig. 3.4). The nucleotide sequence is available on the UCSC Genome Browser [16].

A random walk on a million digits is expected to stay within roughly a thousand units of the origin, and this will be seen to hold for the walks on the digits of π and on the Liouville λ function values. On the other hand, the walks on the digits of Champernowne's number and on the X chromosome travel much farther than would be expected of a random walk.

The walk on the Liouville λ function moves away from the origin like $\sqrt{n}$, but it does not seem to move randomly near the origin. In fact, the positive values of λ first outweigh the negative values when $n = 906\,180\,359$ [12], which is not at all typical of a random walk.

3.3 Strong Normality

Mauduit and Sárközy [13] have shown that the digits of the base-2 Champernowne number γ_2 fail two tests of randomness. Dodge and Melfi [9] compared values of an

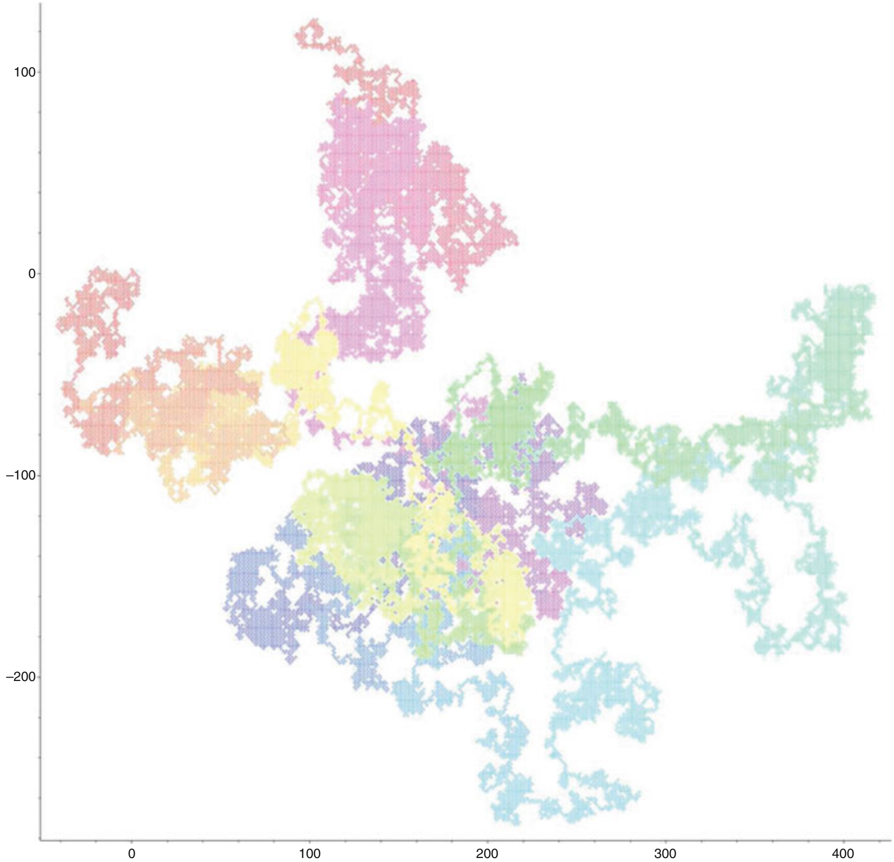

Fig. 3.1 A walk on 10^6 binary digits of π

autocorrelation function for Champernowne's number and π and found that π had the expected pseudorandom properties but that Champernowne's number did not.

Here we provide another test of pseudorandomness and show that it must be passed by almost all numbers. Our test is a simple one, in the spirit of Borel's test of normality, and Champernowne's number will be seen to fail the test.

If the digits of a real number α are chosen at random in the base r, the asymptotic frequency $m_k(n)/n$ of each 1-string approaches $1/r$ with probability 1. However, the *discrepancy* $m_k(n) - n/r$ does not approach any limit, but fluctuates with an expected value equal to the standard deviation $\sqrt{(r-1)n}/r$.

Kolmogorov's law of the iterated logarithm allows us to make a precise statement about the discrepancy of a random number. We use this to define our criterion.

Definition 3.1. For real α, and $m_k(n)$ as above, α is *simply strongly normal* in the base r if for each $k \in \{0, \ldots, r-1\}$

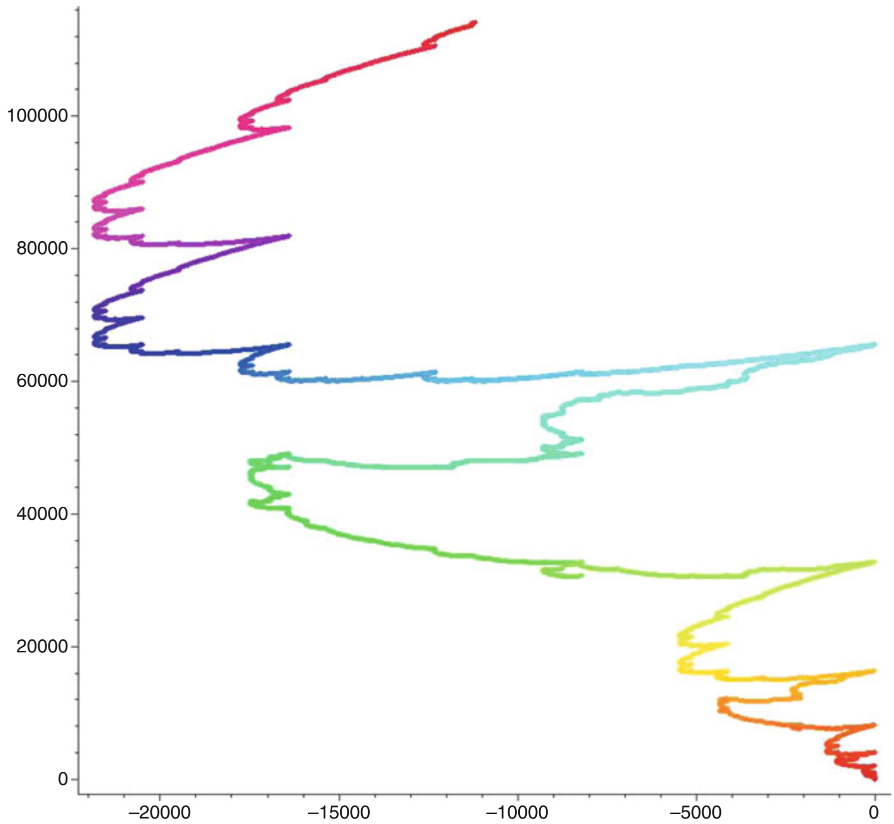

Fig. 3.2 A walk on 10^6 binary digits of the base-2 Champernowne number

$$\limsup_{n\to\infty} \frac{m_k(n) - \frac{n}{r}}{\frac{\sqrt{r-1}}{r}\sqrt{2n\log\log n}} = 1$$

and

$$\liminf_{n\to\infty} \frac{m_k(n) - \frac{n}{r}}{\frac{\sqrt{r-1}}{r}\sqrt{2n\log\log n}} = -1 .$$

We make two further definitions analogous to the definitions of normality and absolute normality.

Definition 3.2. A number is *strongly normal* in the base r if it is simply strongly normal in each of the bases r^j, $j = 1, 2, 3, \ldots$.

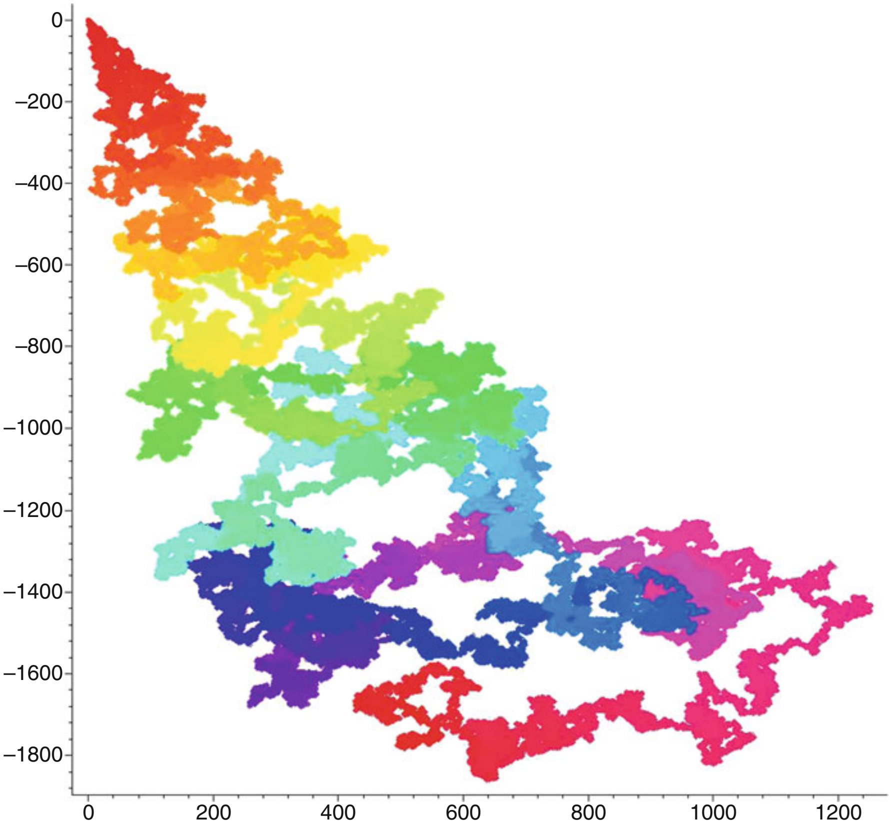

Fig. 3.3 A walk on 10^6 values of the Liouville λ function

Definition 3.3. A number is *absolutely strongly normal* if it is strongly normal in every base.

These definitions of strong normality are sharper than those given by one of the authors in [3].

3.4 Almost All Numbers Are Strongly Normal

Theorem 3.4. *Almost all numbers are simply strongly normal in any base r.*

Proof. Without loss of generality, we consider numbers in the interval $[0,1]$ and fix the integer base $r \geq 2$. We take Lebesgue measure to be our probability measure. For any k, $0 \leq k \leq r-1$, the ith digit of a randomly chosen number is k with probability r^{-1}. For $i \neq j$, the ith and jth digits are both k with probability r^{-2}, so the digits are pairwise independent.

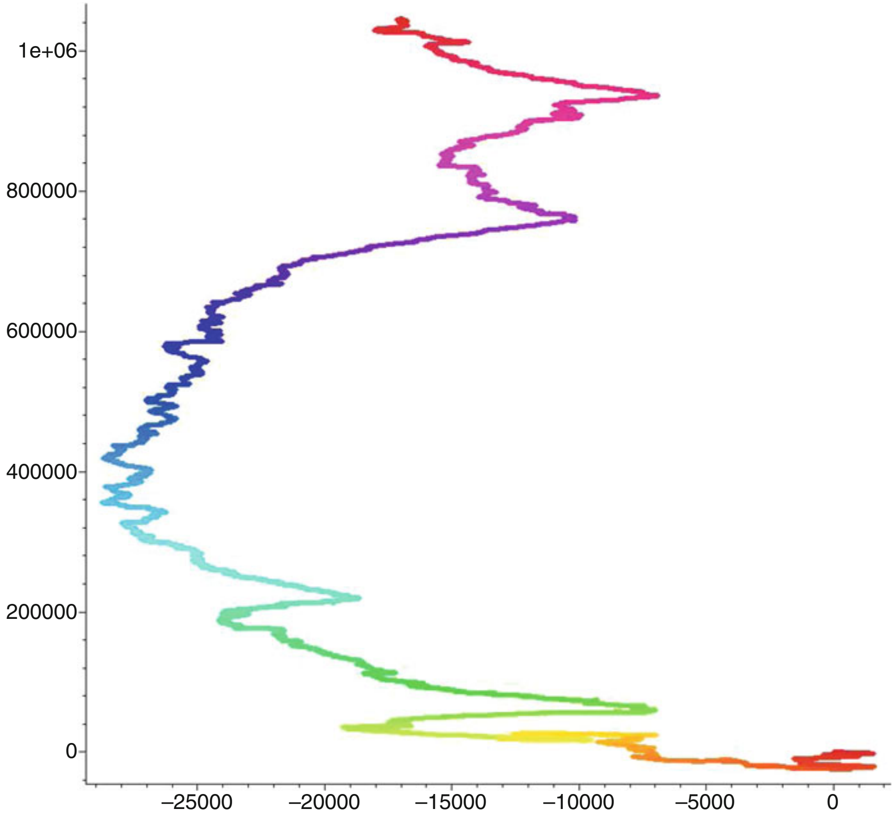

Fig. 3.4 A walk on the nucleotides of the human X chromosome

We define the sequence of random variables X_j by

$$X_j = \sqrt{r-1}$$

if the jth digit is k, with probability $\frac{1}{r}$, and

$$X_j = -\frac{1}{\sqrt{r-1}}$$

otherwise, with probability $\frac{r-1}{r}$.

Then the X_j form a sequence of independent identically distributed random variables with mean 0 and variance 1. Put

$$S_n = \sum_{j=1}^{n} X_j \ .$$

By the law of the iterated logarithm (see, for example, [11]), with probability 1,

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = 1 ,$$

and

$$\liminf_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = -1 .$$

Now we note that if $m_k(n)$ is the number of occurrences of the digit k in the first n digits of our random number, then

$$S_n = m_k(n)\sqrt{r-1} - \frac{n - m_k(n)}{\sqrt{r-1}} .$$

Substituting this expression for S_n in the limits immediately above shows that the random number satisfies Definition 3.1 with probability 1. ■

This is easily extended.

Corollary 3.5. *Almost all numbers are strongly normal in any base r.*

Proof. By the theorem, the set of numbers in $[0,1]$ which fails to be simply strongly normal in the base r^j is of measure zero, for each j. The countable union of these sets of measure zero is also of measure zero. Therefore the set of numbers simply strongly normal in every base r^j is of measure 1. ■

The following corollary is proved in the same way as the last.

Corollary 3.6. *Almost all numbers are absolutely strongly normal.*

The results for $[0,1]$ are extended to $\mathbb{R}$ in the same way.

3.5 Champernowne's Number Is Not Strongly Normal

We begin by examining the digits of Champernowne's number in the base 2,

$$\gamma_2 = 0.1\ 10\ 11\ 100\ 101\ \cdots .$$

Each integer q, $2^{n-1} \le q \le 2^n - 1$, has an n-digit base-2 representation and so contributes an n-block to the expansion of γ_2. In each of these n-blocks, the first digit is 1. If we consider the remaining $n-1$ digits in each of these n-blocks, we see that every possible $(n-1)$-string occurs exactly once. The n-digit integers, concatenated, together contribute a block of length $n2^{n-1}$, and in this block, if we set aside the ones corresponding to the initial digit of each integer, the zeros and ones are equal in number. In the whole block there are $(n-1)2^{n-2}$ zeros and $(n-1)2^{n-2}+2^{n-1}$ ones. The excess of ones over zeros in the entire $\left(n2^{n-1}\right)$-block is just equal to the number of integers, 2^{n-1}, contributing to the block.

As we concatenate the integers from 1 to 2^k-1, we write the first

$$N-1=\sum_{n=1}^{k} n2^{n-1}=(k-1)2^k+1$$

digits of γ_2. The excess of ones in the digits is

$$2^k-1.$$

The locally greatest excess of ones occurs at the first digit contributed by the integer 2^k, since each power of 2 is written as 1 followed by zeros. At this point the number of digits is $N=(k-1)2^k+2$ and the excess of ones is 2^k. That is, the actual number of ones in the first N digits is

$$m_1(N)=(k-2)2^{k-1}+1+2^k.$$

This gives

$$m_1(N)-\frac{N}{2}=2^{k-1}\ .$$

Thus, we have

$$\frac{m_1(N)-\frac{N}{2}}{N^{1/2+\varepsilon}}\geq\frac{2^{k-1}}{\left((k-1)2^k\right)^{1/2+\varepsilon}}.$$

For any sufficiently small positive ε, the right-hand expression is unbounded as $k\to\infty$. We have

$$\limsup_{N\to\infty}\frac{m_1(N)-\frac{N}{2}}{\frac{1}{2}\sqrt{2N\log\log N}}\geq\limsup_{N\to\infty}\frac{m_1(N)-\frac{N}{2}}{N^{1/2+\varepsilon}}=\infty\ .$$

We thus have:

Theorem 3.7. *The base-2 Champernowne number is not strongly normal in the base 2.*

One can show that Champernowne's number also fails the lower limit criterion. In fact, $m_1(N)-\dfrac{N}{2}>0$ for every N.

The theorem can be generalized to every Champernowne number, since there is a shortage of zeros in the base-r representation of the base-r Champernowne number. Each base-r Champernowne number fails to be strongly normal in the base r.

3.6 Strongly Normal Numbers Are Normal

Our definition of strong normality is strictly more stringent than Borel's definition of normality:

Theorem 3.8. *If a number α is simply strongly normal in the base r, then α is simply normal in the base r.*

Proof. It will suffice to show that if a number is not simply normal, then it cannot be simply strongly normal.

Let $m_k(n)$ be the number of occurrences of the 1-string k in the first n digits of the expansion of α in the base r, and suppose that α is not simply normal in the base r. This implies that for some k

$$\lim_{n\to\infty} \frac{rm_k(n)}{n} \neq 1.$$

Then there is some $Q > 1$ and infinitely many n_i such that either

$$rm_k(n_i) > Qn_i$$

or

$$rm_k(n_i) < \frac{n_i}{Q}.$$

If infinitely many n_i satisfy the former condition, then for these n_i,

$$m_k(n_i) - \frac{n_i}{r} > Q\frac{n_i}{r} - \frac{n_i}{r} = n_i P$$

where P is a positive constant.

Then for any $R > 0$,

$$\limsup_{n\to\infty} R\frac{m_k(n) - \frac{n}{r}}{\sqrt{2n\log\log n}} \geq \limsup_{n\to\infty} R\frac{nP}{\sqrt{2n\log\log n}} = \infty,$$

so α is not simply strongly normal.

On the other hand, if infinitely many n_i satisfy the latter condition, then for these n_i,

$$\frac{n_i}{r} - m_k(n_i) > \frac{n_i}{r} - \frac{n_i}{Qr} = n_i P,$$

and once again the constant P is positive. Now

$$\liminf_{n\to\infty} \frac{m_k(n) - \frac{n}{r}}{\sqrt{2n\log\log n}} = -\limsup_{n\to\infty} \frac{\frac{n}{r} - m_k(n)}{\sqrt{2n\log\log n}}$$

and so, in this case also, α fails to be simply strongly normal. ■

The general result is an immediate corollary.

Corollary 3.9. *If α is strongly normal in the base r, then α is normal in the base r.*

3.7 No Rational Number Is Simply Strongly Normal

In light of Theorem 3.8, it will suffice to show that no simply normal rational number can be simply strongly normal.

If α is rational and simply normal in the base r, then if we restrict ourselves to the first n digits in the repeating tail of the expansion, the frequency of any 1-string k is exactly n/r whenever n is a multiple of the length of the repeating string. The excess of occurrences of k can never exceed the constant number of times k occurs in the repeating string. Therefore, with $m_k(n)$ defined as in Sect. 3.3,

$$\limsup_{n\to\infty} \left(m_k(n) - \frac{n}{r}\right) = Q,$$

with Q a constant due in part to the initial non-repeating block and in part to the maximum excess in the tail.

But

$$\limsup_{n\to\infty} \frac{Q}{\sqrt{2n\log\log n}} = 0\,,$$

so α does not satisfy Definition 3.1.

3.8 Construction of an Absolutely Strongly Normal Number

To determine an absolutely strongly normal number, we modify Sierpiński's method of constructing an absolutely normal number [15]. We begin with an easy lemma.

Lemma 3.10. *Let $f(n)$ be a real-valued function of the first n base r digits of a number $\alpha \in [0,1]$, and suppose*

$$\mathbf{P}\left[\limsup_{n\to\infty} f(n) = 1\right] = 1$$

and

$$\mathbf{P}\left[\liminf_{n\to\infty} f(n) = -1\right] = 1 .$$

Given positive $\delta_1 > \delta_2 > \delta_3 > \cdots$, *and* $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \cdots$, *we can find* $M_1 < M_2 < M_3 < \cdots$ *so that*

$$\mathbf{P}\left[\left|\sup_{M_i \le n < M_{i+1}} f(n) - 1\right| > \delta_i \quad \text{or} \quad \left|\inf_{M_i \le n < M_{i+1}} f(n) + 1\right| > \delta_i\right] < \varepsilon_i .$$

Notes. The function $f(n)$ depends on both n and α. The probability is the Lebesgue measure of the set of $\alpha \in [0,1]$ for which f satisfies the condition(s).

The lemma can easily be proved under more general assumptions.

Proof. For sufficiently large M,

$$\mathbf{P}\left[\sup_{n\ge M} f(n) > 1 + \delta_1\right] < \frac{\varepsilon_1}{4} \quad \text{and}$$

$$\mathbf{P}\left[\inf_{n\ge M} f(n) < -1 - \delta_1\right] < \frac{\varepsilon_1}{4} .$$

Set M_1 to be the least such M.

Now, as $M \to \infty$,

$$\mathbf{P}\left[\sup_{M_1 \le n < M} f(n) < 1 - \delta_1\right] \to 0 ,$$

and also

$$\mathbf{P}\left[\inf_{M_1 \le n < M} f(n) > -1 + \delta_1\right] \to 0 .$$

Thus, for sufficiently large M, these four conditions are satisfied:

$$\mathbf{P}\left[\sup_{M_1 \le n < M} f(n) < 1 - \delta_1\right] < \frac{\varepsilon_1}{4} ,$$

$$\mathbf{P}\left[\inf_{M_1 \le n < M} f(n) > -1 + \delta_1\right] < \frac{\varepsilon_1}{4} ,$$

$$\mathbf{P}\left[\sup_{n\ge M} f(n) > 1 + \delta_2\right] < \frac{\varepsilon_2}{4} ,$$

and

$$\mathbf{P}\left[\inf_{n\geq M} f(n) < -1-\delta_2\right] < \frac{\varepsilon_2}{4}\,.$$

We set M_2 to be the least $M > M_1$ satisfying all four conditions. Since

$$\mathbf{P}\left[\sup_{M_1\leq n<M_2} f(n) > 1+\delta_1\right] \leq \mathbf{P}\left[\sup_{n\geq M_1} f(n) > 1+\delta_1\right]$$

and

$$\mathbf{P}\left[\inf_{M_1\leq n<M_2} f(n) < -1-\delta_1\right] \leq \mathbf{P}\left[\inf_{n\geq M_1} f(n) < -1-\delta_1\right]\,,$$

we have

$$\mathbf{P}\left[\left|\sup_{M_1\leq n<M_2} f(n)-1\right| > \delta_1 \quad \text{or} \quad \left|\inf_{M_1\leq n<M_2} f(n)+1\right| > \delta_1\right] < \varepsilon_1\,.$$

We can continue in this way, recursively choosing $M_3, M_4, M_5, \ldots$ so that each M_i is the least satisfying the required conditions. ∎

Now we fix an integer base $r \geq 2$ and a 1-string $k \in \{0,1,\ldots,r-1\}$. For each $\alpha \in [0,1]$, put

$$f(n) = f(\alpha,k,n) = \frac{m_k(n)-\dfrac{n}{r}}{\dfrac{\sqrt{r-1}}{r}\sqrt{2n\log\log n}}\,.$$

Here, as in Definition 3.1 of Sect. 3.3, $m_k(n)$ is the number of occurrences of k in the first n base r digits of α, and α is simply strongly normal in the base r if

$$\limsup_{n\to\infty} f(n) = 1$$

and

$$\liminf_{n\to\infty} f(n) = -1\,.$$

By Theorem 3.4, Sect. 3.4, these conditions hold with probability 1, so f satisfies the conditions of Lemma 3.10.

Now fix $0 < \varepsilon \leq 1$; set $\delta_i = \dfrac{1}{i}$ and $\varepsilon_i = \varepsilon_{r,i} = \dfrac{\varepsilon}{3\cdot 2^i r^3}$. These δ_i and ε_i also satisfy the conditions of Lemma 3.10.

We will construct a set $A_\varepsilon \subset [0,1]$, of measure less than 1, in such a way that every element of A_ε^C is absolutely strongly normal.

Let $M_1 < M_2 < M_3 < \cdots$ be determined as in the proof of Lemma 3.10, so that the conclusion of the lemma holds. We build a set $A_{r,i}$ containing those α for which the first M_{i+1} digits are, in a loose sense, far from simply strongly normal in the base r.

Around each $\alpha = .a_1a_2\cdots a_{M_{i+1}}\cdots$ such that

$$\left| \sup_{M_i \le n < M_{i+1}} f(n) - 1 \right| > \delta_i \tag{3.1}$$

or

$$\left| \inf_{M_i \le n < M_{i+1}} f(n) + 1 \right| > \delta_i \tag{3.2}$$

we construct an open interval containing α:

$$\left(\frac{a_1}{r} + \frac{a_2}{r^2} + \cdots + \frac{a_{M_{i+1}}}{r^{M_{i+1}}} - \frac{1}{r^{M_{i+1}}}, \frac{a_1}{r} + \frac{a_2}{r^2} + \cdots + \frac{a_{M_{i+1}}}{r^{M_{i+1}}} + \frac{2}{r^{M_{i+1}}} \right) .$$

Let $A_{r,k,i}$ be the union of all the intervals constructed in this way. By our construction, the union of the closed intervals consisting of the numbers with initial digits $.a_1a_2\ldots a_{M_{i+1}}$ satisfying one of our two conditions (3.1) or (3.2) has measure less than ε_i, so, denoting Lebesgue measure by μ,

$$\mu\left(A_{r,k,i}\right) < 3\varepsilon_i \; = \frac{\varepsilon}{2^i r^3}.$$

In this way we construct $A_{r,k,i}$ for every base r and 1-string $k \in \{0,1,\ldots,r-1\}$. We let

$$A_\varepsilon = \bigcup_{r=2}^{\infty} \bigcup_{k=0}^{r-1} \bigcup_{i=1}^{\infty} A_{r,k,i} \; ,$$

so

$$\begin{aligned} \mu(A_\varepsilon) &\le \sum_{r=2}^{\infty} \sum_{k=0}^{r-1} \sum_{i=1}^{\infty} \mu\left(A_{r,k,i}\right) \\ &< \sum_{r=2}^{\infty} \sum_{k=0}^{r-1} \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i r^3} \\ &= \left(\frac{\pi^2}{6} - 1 \right) \varepsilon \; . \end{aligned}$$

Let E_ε be the complement of A_ε in $[0,1]$. Since $\mu(A_\varepsilon) < 1$, E_ε is of positive measure. We claim that every element of E_ε is absolutely strongly normal.

For each base r and 1-string $k \in \{0,1,\ldots,r-1\}$, we have specified a set of integers $M_1 < M_2 < M_3 < \cdots$, depending on r and k. By our construction, if $\alpha \in E_\varepsilon$, then, recalling that f depends on α, we have

$$\left| \sup_{M_i \le n < M_{i+1}} f(n) - 1 \right| < \delta_i$$

and

$$\left| \inf_{M_i \le n < M_{i+1}} f(n) + 1 \right| < \delta_i$$

for every i. Clearly for this α, since $\delta_i \to 0$,

$$\limsup_{n\to\infty} f(n) = 1$$

and

$$\liminf_{n\to\infty} f(n) = -1 .$$

This is true for every k, so α is simply strongly normal to the base r, by Definition 3.1 (Sect. 3.3). Thus α is simply strongly normal to every base, and is therefore absolutely strongly normal by Definitions 3.2 and 3.3.

To specify an absolutely strongly normal number, we note that E_ε contains no interval, since, by Sect. 3.7, no rational number is simply strongly normal in any base. Since E_ε is bounded, $\inf E_\varepsilon$ is well defined; and $\inf E_\varepsilon \in E_\varepsilon$ since otherwise $\inf E_\varepsilon$ would be interior to some open interval of A_ε.

For example, $\inf E_1$ is a well-defined absolutely strongly normal number.

3.9 Further Questions

It should be possible to construct a computable absolutely strongly normal number by the method of Becher and Figueira [2].

We conjecture that such naturally occurring constants as the irrational numbers π, e, $\sqrt{2}$, and $\log 2$ are absolutely strongly normal.

On the other hand, we speculate that the binary Liouville λ number, created in the obvious way from the λ function values, may be normal but not strongly normal.

Bailey and Crandall [1] proved normality base 2 for an uncountable class of "generalized Stoneham constants," namely constants of the form

$$\alpha_{2,3}(r) = \sum_{k=0}^{\infty} \frac{1}{3^k 2^{3^k + r_k}},$$

where r_k is the kth binary digit of a real number r in the unit interval. This class of numbers may be a good place to look for examples of strong normality. However, new techniques may be required for this.

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