# Chapter 20
# Generic Existence of Solutions and Generic Well-Posedness of Optimization Problems

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*Dedicated to Jonathan Borwein on the occasion of his 60th birthday*

**Abstract** We exhibit a large class of topological spaces in which the generic attainability of the infimum by the bounded continuous perturbations of a lower semicontinuous function implies generic well-posedness of the perturbed optimization problems. The class consists of spaces which admit a winning strategy for one of the players in a certain topological game and contains, in particular, all metrizable spaces and all spaces that are homeomorphic to a Borel subset of a compact space.

**Key words:** Baire category • Generic well-posedness of perturbation problems • Perturbed optimization problem • Variational principle • Well-posed problem

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## 20.1 Introduction

Let $X$ be a completely regular topological space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a fixed bounded from below lower semicontinuous function which is proper (the latter means that $f$ has at least one finite value). We say that $f$ *attains its infimum in* $X$, if there exists some $x \in X$ for which $f(x) = \inf_X f$. Denote by $C(X)$ the space

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of all bounded real-valued and continuous functions in $X$ which we equip with the usual sup-norm $\|g\|_\infty = \sup\{|g(x)| : x \in X\}$, $g \in C(X)$. It has been shown in [17] that the set $E(f) = \{g \in C(X) : f+g$ attains its infimum in $X\}$ is dense in $C(X)$. We call the statement "$E(f)$ is dense in $C(X)$" a *variational principle for $f$ with $C(X)$ as a set of perturbations*. The variational principle is called *generic* if the set $E(f)$ is residual in $C(X)$. Recall that $E(f)$ is *residual in* $C(X)$ if its complement is of the first Baire category in $C(X)$. Such a (or similar) setting, with different sets of perturbations, is present in several well-known variational principles–see, e.g., Ekeland [9], Stegall [22], Borwein and Preiss [3] and Deville, Godefroy, and Zizler [7,8] for the case of metric spaces $X$ and [4,5] outside the case of metrizable spaces.

Our aim in this paper is to show that, for a very large class of spaces $X$, the residuality of $E(f)$ in $C(X)$ implies the residuality in the same space of the set $W(f) := \{g \in C(X) : f+g$ is well posed$\}$. Let us recall that a bounded from below function $h : X \to \mathbb{R} \cup \{+\infty\}$ (or more precisely, the problem to minimize $h$ on $X$) is called *well posed* if every minimizing net $(x_\lambda)_\lambda \subset X$ for $h$ has a cluster point. If $h$ is lower semicontinuous and is well posed, then the set $M(h)$ of minimizers of $h$ in $X$ is a nonempty compact set in $X$ and for every open $U \supset M(h)$ there exists $\varepsilon > 0$ for which $\{x \in X : h(x) < \inf_X h + \varepsilon\} \subset U$.

The spaces $X$ for which we prove here that residuality of $E(f)$ (in $C(X)$) implies residuality of $W(f)$ are described by a topological game called *a determination game* and denoted by $DG(X)$. The reasons for this terminology will become clear later. Two players, $\Sigma$ (who starts the game) and $\Omega$, play by choosing at each step $n \geq 1$ nonempty sets $A_n$ (the choices of $\Sigma$) and $B_n$ (the choices of $\Omega$) so that $B_n$ is relatively open in $A_n$ and $A_{n+1} \subset B_n \subset A_n$ for any $n$. Playing this way the players generate a sequence $p = \{A_n, B_n\}_{n\geq 1}$ which is called *a play*. The player $\Omega$ wins the play $p$ if the intersection $\cap_n \overline{A}_n = \cap_n \overline{B}_n$ is either empty or a nonempty compact set such that, for each open set $U$ containing $\cap_n \overline{B}_n$, there is some $n$ with $B_n \subset U$. Otherwise, by definition, player $\Sigma$ is declared to have won the play $p$. A *partial play* in the game $DG(X)$ is any finite sequence of the type $(A_1, B_1, \dots, A_n)$ or $(A_1, B_1, \dots, A_n, B_n)$, $n \geq 1$, where for $i = 1, \dots, n$, the sets $A_i$ and $B_i$ are moves in $DG(X)$ of $\Sigma$ and $\Omega$ correspondingly. A *strategy* $\omega$ for the player $\Omega$ is defined recursively and is a rule which to any possible partial play of the type $(A_1, \dots, A_n)$, $n \geq 1$, puts into correspondence a nonempty set $B_n := \omega(A_1, \dots, A_n) \subset A_n$ which is relatively open in $A_n$. If in a given play $\{A_n, B_n\}_{n\geq 1}$ of the game $DG(X)$ each choice $B_n$ of $\Omega$ is obtained via the strategy $\omega$, that is, $B_n = \omega(A_1, \dots, A_n)$ for every $n \geq 1$, then this play $p$ is called *an $\omega$-play*. The strategy $\omega$ for the player $\Omega$ is called *winning* if the player $\Omega$ wins every $\omega$-play in this game. The notions of strategy and winning strategy for the player $\Sigma$ are introduced in a similar way. The term *the game is favorable (resp. unfavorable)* for some player means that the corresponding player has (resp. does not have) a winning strategy in the game.

In Theorem 20.3 we prove that if the player $\Omega$ has a winning strategy in the game $DG(X)$ and if for some proper bounded from below lower semicontinuous function $f$ the set $E(f)$ is residual in $C(X)$, then the set $W(f)$ is also residual in $C(X)$. In other words, generic attainability of the infimum by the perturbations implies generic well-posedness of the perturbations. Let us mention that the class of spaces $X$ for which $\Omega$ has a winning strategy for the game $DG(X)$ is quite large: it contains all metrizable spaces, all Borel subsets of compact spaces, a large class of fragmentable spaces, etc. (see the Concluding Remarks for more information about this class). There are spaces $X$ however for which the phenomenon does not hold. In Example 20.4 we give a space $X$ and a function $f$ such that $E(f) = C(X)$ and $W(f) = \emptyset$.

The game $DG(X)$ has been used in [11] in order to give sufficient conditions when a semitopological group is, in fact, a topological group and in [12] to study the points of continuity of the so-called quasi-continuous mappings. Variants of $DG(X)$ have been used by Michael [19] (for the study of completeness properties of metric spaces), by Kenderov and Moors [13–15] (for characterization of the fragmentability of topological spaces), and by the authors in [6, 16, 17] (for proving the validity of generic variational principles).

## 20.2 Preliminary Results and Notions

Let $X$ be a completely regular topological space and consider, as above, the Banach space $C(X)$ of all continuous and bounded functions in $X$ equipped with its sup-norm. For a given function $f : X \to \mathbb{R} \cup \{+\infty\}$, the symbol $\mathrm{dom}(f)$ denotes the *effective domain* of $f$, which is the set of points $x \in X$ for which $f(x) \in \mathbb{R}$. For our further considerations we need the following statement:

**Proposition 20.1 ([17], Lemma 2.1).** *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function which is bounded from below and proper. Let $x_0 \in \mathrm{dom}(f)$ and $\varepsilon > 0$ be such that $f(x_0) < \inf_X f + \varepsilon$. Then, there exists a continuous function $g : X \to \mathbb{R}^+$ for which $\|g\|_\infty \leq \varepsilon$ and the function $f + g$ attains its infimum at $x_0$.*

In particular, this proposition shows that the set $E(f) = \{g \in C(X) : f + g$ attains its infimum in $X\}$ is dense in $C(X)$.

Further, any proper function $f : X \to \mathbb{R} \cup \{+\infty\}$ which is bounded from below defines a set-valued mapping $M_f : C(X) \rightrightarrows X$ as follows:

$$M_f(g) := \{x \in X : (f + g)(x) = \inf_X (f + g)\}, \quad g \in C(X),$$

which to each $g \in C(X)$ puts into correspondence the (possibly empty) set of minimizers in $X$ of the perturbation $f + g$. It is known as the *solution mapping* determined by $f$.

We denote by $\mathrm{Gr}(M_f)$ the graph of $M_f$ and by $\mathrm{Dom}(M_f)$ the set $\{g \in C(X) : M_f(g) \neq \emptyset\}$ which is called *effective domain of $M_f$*. The following properties are well known in the case when $f \equiv 0$. For an arbitrary proper bounded from below and lower semicontinuous $f$ the proof of these properties is given in [6]. Recall that, for a set $A \subset X$, the symbol $\overline{A}$ denotes the closure of $A$ in $X$.

**Proposition 20.2 ([6], Proposition 2.4).** *Let $X$ be a completely regular topological space and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper bounded from below lower semicontinuous function. Then the solution mapping $M_f : C(X) \rightrightarrows X$ satisfies the following properties:*

*(a)* $\mathrm{Gr}(M_f)$ *is closed in the product topology in* $C(X) \times X$.
*(b)* $\mathrm{Dom}\,(M_f)$ *is dense in* $C(X)$.
*(c)* $M_f$ *maps* $C(X)$ *onto* $\mathrm{dom}\,(f)$.
*(d) For any two open sets $U$ of $C(X)$ and $W$ of $X$ such that $M_f(U) \cap W \neq \emptyset$ there is a nonempty open set $V \subset U$ such that $M_f(V) \subset W$.*
*(e) If $(V_n)_{n\geq 1}$ is a base of neighborhoods of $g_0 \in C(X)$ then $M_f(g_0) = \cap_n \overline{M(V_n)}$.*

The tool we use to show that a certain set is residual in a topological space is the well-known Banach-Mazur game. Given a topological space $X$ and a set $S \subset X$, two players, denoted by $\alpha$ and $\beta$, play a game by choosing alternatively nonempty open sets $U_n$ (the choices of $\beta$ who starts the game) and $V_n$ (the choices of $\alpha$), $n \geq 1$, with the rule $U_{n+1} \subset V_n \subset U_n$. The player $\alpha$ wins the play $\{U_n, V_n\}_{n\geq 1}$ if $\cap_n U_n = \cap_n V_n \subset S$. Otherwise, $\beta$ wins. The game is known as the Banach-Mazur game and is denoted by $BM(X,S)$. The notions of (winning) strategies for the players are defined as in the game $DG(X)$. It was proved by Oxtoby [20] that the player $\alpha$ has a winning strategy in $BM(X,S)$ if and only if the set $S$ is residual in $X$.

## 20.3 Generic Well-Posedness of Perturbed Optimization Problems

In this section we formulate and prove our main result. Namely, we have the following

**Theorem 20.3.** *Let $X$ be a completely regular topological space which admits a winning strategy for the player $\Omega$ in the determination game $DG(X)$. Suppose that for some proper bounded from below lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$ the set $E(f) = \{g \in C(X) : f+g$ attains its minimum in $X\}$ is residual in $C(X)$. Then the set $W(f) = \{g \in C(X) : f+g$ is well posed$\}$ is also residual in $C(X)$.*

*Proof.* Let $X$ and $f$ be as in the theorem. We will prove that the player $\alpha$ has a winning strategy in the Banach-Mazur game $B(C(X), W(f))$ played in $C(X)$ equipped with the sup-norm. According to the result of Oxtoby cited above this will imply that $W(f)$ is residual in $C(X)$.

First, knowing that $E(f)$ is residual in $C(X)$, let $(O_n)_n$ be a countable family of open and dense subsets of $C(X)$ such that $\cap_n O_n \subset E(f)$. Let us denote by $\omega$ the winning strategy in the game $DG(X)$ for the player $\Omega$. We will construct now a winning strategy $s$ for the player $\alpha$ in the game $BM(C(X), W(f))$.

To this end, let $U_1$ be an arbitrary nonempty open set of $C(X)$ which can be a legal move of the player $\beta$ in this game. Take $A_1 := M_f(U_1)$ which is a nonempty set of

$X$, according to Proposition 20.2 (b). Consider this set as a first move of the player $\Sigma$ in the determination game $DG(X)$. Then put $B_1 := \omega(A_1)$ to be the answer of the player $\Omega$ in the game $DG(X)$ according to his/her strategy $\omega$. Since $B_1$ is relatively open subset of $A_1$ there is some open set $W_1 \subset X$ such that $B_1 = W_1 \cap A_1$. Now, by Proposition 20.2 (d), there is a nonempty open set $V_1$ of $C(X)$ for which $V_1 \subset U_1$ and $M_f(V_1) \subset W_1$. Thus $M_f(V_1) \subset W_1 \cap M_f(U_1) = W_1 \cap A_1 = B_1$. We may think, without loss of generality, that $V_1 \subset O_1$, $\overline{V}_1 \subset U_1$ and that in addition $\mathrm{diam}(V_1) < 1$. Define the value of the strategy $s$ for the player $\alpha$ in the game $BM(C(X), W(f))$ at the set $U_1$ to be $s(U_1) := V_1$. Let further the nonempty open set $U_2 \subset V_1$ be an arbitrary legitimate choice of the player $\beta$ in the game $BM(C(X), W(f))$ at the second step. Put $A_2 := M_f(U_2)$ which is a nonempty set of $X$ according again to Proposition 20.2 (b). Since $A_2 = M_f(U_2) \subset M_f(V_1) \subset B_1$ the set $A_2$ can be a legal move of the player $\Sigma$ in the game $DG(X)$ at the second step. Put $B_2 := \omega(A_1, B_1, A_2)$ to be the answer of the player $\Omega$ according to his/her strategy $\omega$. The set $B_2$ is a nonempty relatively open subset of $A_2$; thus, there is some nonempty open set $W_2 \subset X$ such that $B_2 = W_2 \cap A_2$. Now, using once again Proposition 20.2 (d), there is some nonempty open subset $V_2$ of $U_2$ for which $M_f(V_2) \subset W_2$. Therefore, $M_f(V_2) \subset W_2 \cap M_f(U_2) = B_2$. Moreover, without loss of generality, we may think that $V_2 \subset O_2$, $\overline{V}_2 \subset U_2$ and that $\mathrm{diam}(V_2) < 1/2$. Define the value of the strategy $s$ by $s(U_1, V_1, U_2) := V_2$.

Proceeding by induction we define a strategy $s$ for the player $\alpha$ in the Banach-Mazur game $BM(C(X), W(f))$ such that for any $s$-play $\{U_n, V_n\}_{n\geq 1}$ in this game (i.e., $V_n = s(U_1, V_1, \ldots, U_n)$ for each $n \geq 1$) there exists an associated $\omega$-play $\{A_n, B_n\}_{n\geq 1}$ in the game $DG(X)$ such that the following properties are satisfied for any $n \geq 1$:

(i) $A_n = M_f(U_n)$.
(ii) $M_f(V_n) \subset B_n$.
(iii) $V_n \subset O_n$.
(iv) $\overline{V}_{n+1} \subset U_{n+1} \subset V_n$.
(v) $\mathrm{diam}(V_n) < 1/n$.

Conditions (iv) and (v) ensure that the intersection $\cap_n V_n$ is a one point set, say $g \in C(X)$ and condition (iii) entails that $g \in \cap_n O_n \subset E(f)$. According to Proposition 20.2 (e) and taking into account (i) and (iv) we have

$$M_f(g) = \cap_n \overline{M_f(V_n)} = \cap_n \overline{M_f(U_n)} = \cap_n \overline{A}_n.$$

Since $g \in E(f)$, the set $M_f(g) = \cap_n \overline{A}_n$ is nonempty and therefore, because $\omega$ is a winning strategy for $\Omega$ in the determination game $DG(X)$, this set is compact and the family $(B_n)_n$ behaves like a base for $\cap_n \overline{A}_n = M_f(g)$, that is, for any open set $U$ containing $M_f(g)$ there is some $n$ such that $B_n \subset U$. We will show that $g \in W(f)$ and this will complete the proof. To show that the function $f+g$ is well posed let $(x_\lambda)_\lambda$ be a minimizing net for $f+g$, that is, $f(x_\lambda)+g(x_\lambda) \to \inf_X(f+g)$. We have to show that this net has a cluster point (necessarily lying in $M_f(g)$). For this, having in mind that the set of minima $M_f(g)$ for $f+g$ is nonempty and compact, it is enough to show that if $U$ is an open subset of $X$ so that $M_f(g) \subset U$, then $x_\lambda \in U$ eventually.

Fix $n \geq 1$ so large that $B_n \subset U$. Put $\varepsilon_\lambda := f(x_\lambda) + g(x_\lambda) - \inf_X(f+g) \geq 0$. We may think, without loss of generality, that $\varepsilon_\lambda > 0$ for every $\lambda$. By Proposition 20.1, for each $\lambda$, there is $g_\lambda \in C(X)$ with $\|g_\lambda\|_\infty < \varepsilon_\lambda$ and such that $x_\lambda \in M_f(g+g_\lambda)$. Since $(g_\lambda)_\lambda$ converges uniformly to zero, we have $g+g_\lambda \in V_n$ eventually. Thus, we have (using also (ii) above) $x_\lambda \in M_f(V_n) \subset B_n$ eventually (for $\lambda$). Therefore, $x_\lambda \in U$ eventually, and this completes the proof. ■

The next example shows that there are spaces in which we have generic attainment of the infimum by the perturbations, without having generic well-posedness of the perturbed optimization problems.

*Example 20.4.* Take $Y$ to be the product of uncountably many copies of the unit interval $[0,1]$ with the usual product topology under which it is a compact topological space. Let $X$ be the so-called sigma-product in $Y$, i.e., the subset of those $x \in Y$ for which only countable number of coordinates are different from zero. With the inherited topology $X$ is a sequentially compact space which is not compact. Thus, for any proper bounded from below lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$ we will have $E(f) = C(X)$. In particular, this is so for any function $f \in C(X)$. Fix such a function $f$. In this case all the perturbations $f+g$, $g \in C(X)$ are continuous in $X$. On the other hand, it is easy to see that, for each value $r$ of a continuous function $h$ in $X$, the level set $h^{-1}(r) = \{x \in X : h(x) = r\}$ contains as a closed subset a copy of the sigma-product of uncountably many copies of the interval $[0,1]$. Hence, the set $h^{-1}(r)$ is not compact for $r = \inf_X h$ and thus $W(f) = \emptyset$.

## 20.4 Concluding Remarks

Some versions of the determination game $DG(X)$ have already been used for different purposes. We have in mind games in which the rules for selection of sets are as in $DG(X)$, but the rules for winning a play are different. We consider three of these versions here. In the first one, which is denoted by $G(X)$, $\Omega$ wins a play $\{A_n, B_n\}_{n \geq 1}$ if $\cap_n \overline{A}_n = \cap_n \overline{B}_n \neq \emptyset$. Otherwise $\Sigma$ wins this play. The game $G(X)$ was used by Michael [19] for the study of completeness properties of metric spaces. It was also used by the authors in [6, 17] to show that the existence of a winning strategy for the player $\Omega$ in $G(X)$ ensures the validity of the following generic variational principle.

**Theorem 20.5 ([17], Theorem 3.1).** *If the player $\Omega$ has a winning strategy in the game $G(X)$, then, for any proper bounded from below lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$, the set $E(f) = \{g \in C(X) : f+g$ attains its minimum in $X\}$ is residual in $C(X)$.*

Note however that, for some particular functions $f$, the set $E(f)$ may be residual in $C(X)$ even if the space $X$ does not admit a winning strategy for $G(X)$ (see, e.g., Example 5.2 from [6]).

In the second variant, denoted by $FG(X)$ and called *fragmenting game*, the player $\Omega$ wins a play $\{A_n, B_n\}_{n\geq 1}$ if $\cap_n \overline{A}_n = \cap_n \overline{B}_n$ is either empty or a one point set. Otherwise $\Sigma$ wins this play. The game $FG(X)$ was used in [13–15] for the study of fragmentable spaces. Recall that a topological space $X$ is called *fragmentable* (see Jayne and Rogers [10]) if there is a metric $d$ in $X$ such that for any nonempty set $A$ of $X$ and any $\varepsilon > 0$ there is a relatively open set $B$ of $A$ with the property $d$-diam$(B) < \varepsilon$, with $d$-diam$(B)$ having the usual meaning of the diameter of the set $B$ with respect to the metric $d$. Every metric space is fragmentable by its own metric. There are however interesting examples of nonmetrizable spaces which are fragmentable. For example every weakly compact subset of a Banach space is fragmented by the metric generated by the norm. Every bounded subset of the dual of an Asplund space is fragmented by the metric of the dual norm. The class of fragmentable spaces has proved its usefulness in the study of different problems in topology (e.g., single-valuedness of set-valued maps) and in the geometry of Banach spaces (e.g., differentiability of convex functions)–see [10, 13–15, 21] and the reference therein. It was proved in [13, 14] that

**Theorem 20.6.** *The space $X$ is fragmentable if and only if the player $\Omega$ has a winning strategy in the fragmenting game $FG(X)$.*

Fragmentability is closely related to generic *Tykhonov well-posedness* of minimization problems. Tykhonov well posed are the problems which are well posed and have unique minimizer.

**Theorem 20.7.** *Let $X$ be a topological space which is fragmented by a metric whose topology contains the original topology in $X$. Suppose that for some bounded from below and lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$ which is proper, the set $E(f)$ is residual in $C(X)$. Then the set $T(f) := \{g \in C(X) : f + g$ is Tykhonov well posed$\}$ is residual in $C(X)$ as well.*

*Proof.* Fragmentability by a metric whose topology contains the original topology in $X$ is characterized by the fact that the player $\Omega$ possesses a special winning strategy $\omega$ in the determination game $DG(X)$ such that, for any $\omega$-play $\{A_n, B_n\}_{n\geq 1}$, the set $\cap_n A_n$ is either empty or consists of just one point, say $\{x\}$, for which the family $(B_n)_n$ behaves like a base: for every open $U \ni x$ there is some $n \geq 1$ such that $B_n \subset U$. Further we proceed exactly as in the proof of Theorem 20.3 and construct a strategy $s$ for the player $\alpha$ in the game $BM(C(X), T(f))$ such that, for any $s$-play $\{U_n, V_n\}_{n\geq 1}$, there exists an associate $\omega$-play $\{A_n, B_n\}_{n\geq 1}$ in the game $DG(X)$ with the properties (i)-(v). As above $\cap_n U_n = \cap_n V_n$ is a one point set, say $g \in C(X)$, for which $g \in W(f)$. Since, in addition, we have that the target sets of the corresponding $\omega$-plays are singletons, then we have, in fact, that $g \in T(f)$. And this completes the proof. ■

Let us mention that if in the above theorem the metric which fragments $X$ is also complete, then the set $T(f)$ is residual in $C(X)$–see [18], Theorem 2.3.

The third version of the game $DG(X)$ explains where the name "determination game" comes from. This version is played in a compactification $bX$ of the completely regular topological space $X$. The moves of the players $\Sigma$ and $\Omega$ are as in $DG(bX)$. The player $\Omega$ wins a play $p = \{A_n, B_n\}_{n\geq 1}$ in this new game if the *target set* $T(p) = \cap_n \overline{A}_n^{bX}$ (which is always nonempty in this setting) is either entirely in $X$ or entirely in $bX \setminus X$. In the next statement the term *equivalent games*, for games with the same players, is used in the sense that the games are simultaneously favorable (unfavorable) for any of the players.

**Proposition 20.8 ([12], Proposition 4).** *Let $X$ be a completely regular topological space and $bX$ be any compactification of $X$. Then the game described above in $bX$ and the game $DG(X)$ are equivalent. In particular, if some of the players $\Sigma$ or $\Omega$ has a winning strategy in one compactification of $X$, then he/she has such a winning strategy in any other compactification of $X$.*

In other words, in the game $DG(X)$ the existence of a winning strategy for the payer $\Omega$ determines that, when using this strategy, the target sets of the corresponding plays in the compactification $bX$ will be either entirely in $X$ or entirely in the complement $bX \setminus X$. In a certain sense the game "determines" or "identifies" the space $X$.

Let us turn back to the game $DG(X)$ and denote by $GD$ the class of *game determined* spaces $X$ (for which the player $\Omega$ has a winning strategy in the game $DG(X)$). It has turned out that the class $GD$ is rather large (for the following facts we refer to [12]): it includes all fragmentable spaces which are fragmented by a metric $d$ whose topology contains the original topology in $X$; in particular, the class contains all metrizable spaces; the class $GD$ contains also all $p$-spaces introduced by Arhangel'skii [1] and also all Moore spaces.

The class $GD$ includes also the class of topological spaces introduced in [15] and called spaces with *countable separation*: the completely regular topological space $X$ is said to have *countable separation* if for some compactification $bX$ of $X$ there is a countable family $(U_n)_n$ of open (in $bX$) sets such that for any two points $x, y$ with $x \in X$ and $y \in bX \setminus X$ there is an element $U_n$ of the family which contains exactly one of the points $x$ and $y$. If $X$ has countable separation then the latter property is satisfied in any compactification of $X$. Let us mention that each Borel set of a space with countable separation has again countable separation. Moreover, each set obtained by applying Souslin operations on subsets with countable separation has countable separation as well. The class $GD$ also includes all spaces with *star separation* introduced in [2].

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