

# Chapter 17

## Five Classes of Monotone Linear Relations and Operators

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*Dedicated to Jonathan Borwein on the occasion of his 60th birthday*

**Abstract** The relationships among five classes of monotonicity, namely 3\*- , 3-cyclic, strictly, para-, and maximal monotonicity, are explored for linear operators and linear relations in Hilbert space. Where classes overlap, examples are given; otherwise their relationships are noted for linear operators in  $\mathbb{R}^2$ ,  $\mathbb{R}^3$ , and general Hilbert spaces. Along the way, some results for linear relations are obtained.

**Key words:** 3\*-monotone • Cyclic monotone • Cyclically monotone • Linear relations • Maximal monotone • Maximally monotone • Monotone operators • Paramonotone • Rectangular • Strictly monotone

**Mathematics Subject Classifications (2010):** Primary 47H05, 47A06; Secondary 47J20, 47N10.

### 17.1 Introduction

Monotone operators are multi-valued operators  $T : X \rightarrow 2^X$  such that for all  $x^* \in Tx$  and all  $y^* \in Ty$ ,

$$\langle x - y, x^* - y^* \rangle \geq 0. \tag{17.1}$$

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They arise as a generalization of subdifferentials of convex functions and are used extensively in variational inequality (and by reformulation, equilibrium) theory.

Variational inequalities were first outlined in 1966 [23] and have since been used to model a large number of problems.

**Definition 17.1 (Variational Inequality Problem).** Given a nonempty closed convex set  $C$  and a monotone operator  $T$  acting on  $C$ , the *variational inequality problem*,  $VIP(T, C)$ , is to find an  $\bar{x} \in C$  such that for some  $\bar{x}^* \in T(\bar{x})$

$$\langle c - \bar{x}, \bar{x}^* \rangle \geq 0 \text{ for all } c \in C. \quad (17.2)$$

They provide a unified framework for, among others, constrained optimization, saddle point, Nash equilibrium, traffic equilibrium, frictional contact, and complementarity problems. For a good overview of sample problems and current methods used to solve them, see [19] and [20].

Monotone operators are also important for the theory of partial differential equations, where monotonicity both characterizes the vector fields of self-dual Lagrangians [21] and is crucial for the determination of equilibrium solutions (using a variational inequality) for elliptical and evolution differential equations and inclusions (see for instance [1]).

Over the years, various classes of monotone operators have been introduced in the exploration of their theory; however there have been few attempts to comprehensively compare those in use across disciplines.

Five special classes of monotone operators are studied here: strictly monotone, 3-cyclic monotone,  $3^*$ -monotone, paramonotone, and maximal monotone. All possible relationships among these five properties are explored for linear operators in  $\mathbb{R}^2$ ,  $\mathbb{R}^n$ , and in general Hilbert space, and the results are summarized in Tables 17.1 and 17.2 and in Figs. 17.1, 17.2, and 17.3.

**Definition 17.2 (paramonotone).** An operator  $T : X \rightarrow 2^X$  is said to be *paramonotone* if  $T$  is monotone and for  $x^* \in Tx, y^* \in Ty$ ,  $\langle x - y, x^* - y^* \rangle = 0$  implies that  $x^* \in Ty$  and  $y^* \in Tx$ .

A number of iterative methods for solving (17.2) have required paramonotonicity to converge. Examples include an interior point method using Bregman functions [15], an outer approximation method [14], and proximal point algorithms [2, 13]. Often, as in [8], with more work it is possible to show convergence with paramonotonicity where previously stronger conditions, such as strong monotonicity, were required. Indeed, the condition first emerged in this context [12] as a sufficient condition for the convergence of a projected-gradient-like method. For more on the theory of paramonotone operators and why this condition is important for variational inequality problems, see [24] and [31].

**Definition 17.3 (strictly monotone).** An operator  $T : X \rightarrow 2^X$  is said to be *strictly monotone* if  $T$  is monotone and for all  $(x, x^*), (y, y^*) \in \text{gra } T$ ,  $\langle x - y, x^* - y^* \rangle = 0$  implies that  $x = y$ .

**Table 17.1** Monotone linear operators on  $\mathbb{R}^2$ : monotone class relationships

PM	SM	3CM	3*		
0	0	0	0	$\exists$	Example 17.41 ( $R_{\pi/2}$ )
0	*	*	1	$\emptyset$	Proposition 17.33
*	*	1	0	$\emptyset$	Fact 17.9
0	*	1	*	$\emptyset$	Proposition 17.10
0	1	*	*	$\emptyset$	Fact 17.8
1	*	*	0	$\emptyset$	Proposition 17.47
1	0	0	*	$\emptyset$	Remark 17.45
1	0	1	1	$\exists$	Example 17.43 ( $A(x_1, x_2) := (x_1, 0)$ )
1	1	0	1	$\exists$	Example 17.41 ( $R_\theta, \pi/2 >  \theta  > \pi/3$ )
1	1	1	1	$\exists$	$Id$

Where:

“PM” represents paramonotone

“SM” represents strictly monotone

“3CM” represents 3-cyclic monotone

“3\*” represents 3\*-monotone

1 represents that the property is present

0 represents an absence of that property

\* represents that both 0/1 are covered by the result

$\exists$  represents that an example with these properties exists

$\emptyset$  represents that this combination of properties is impossible

**Table 17.2** Monotone linear operators: monotone class relationships

PM	SM	3CM	3*	X		
0	0	0	0	$\mathbb{R}^2$	$\exists$	$R_{\pi/2}$
0	*	*	1	–	$\emptyset$	Proposition 17.33
*	*	1	0	–	$\emptyset$	Fact 17.9
0	*	1	*	–	$\emptyset$	Proposition 17.10
0	1	*	*	–	$\emptyset$	Fact 17.8
1	0	0	0	$\ell_2$	$\exists$	Remark 17.51
1	0	0	1	$\mathbb{R}^2$	$\exists$	Example 17.49
1	0	1	1	$\mathbb{R}$	$\exists$	$\emptyset$
1	1	0	0	$\ell_2$	$\exists$	Example 17.50
1	1	0	1	$\mathbb{R}^2$	$\exists$	Example 17.41 ( $R_\theta, \pi/2 >  \theta  > \pi/3$ )
1	1	1	1	$\mathbb{R}$	$\exists$	$Id$

Where:

“PM” represents paramonotone

“SM” represents strictly monotone

“3CM” represents 3-cyclic monotone

“3\*” represents 3\* monotone

“X” represents the space the operator acts upon

1 represents that the property is present

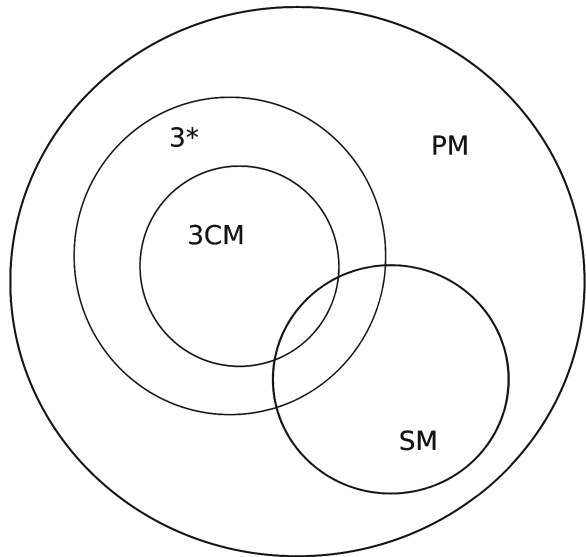
0 represents an absence of that property

\* represents that both 0/1 are covered by the result

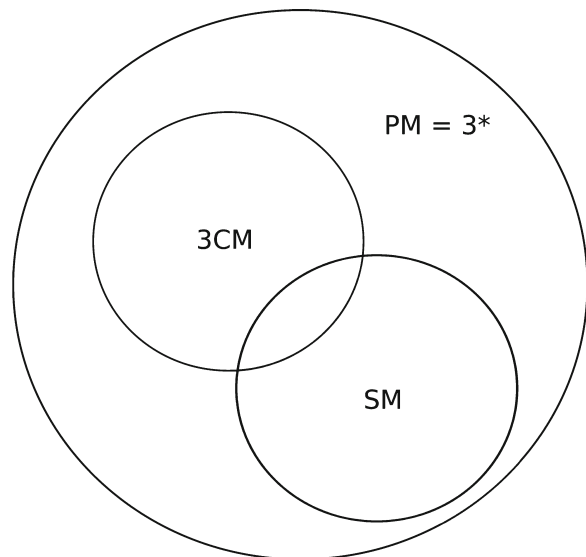
$\exists$  represents that an example with these properties exists

$\emptyset$  represents that this combination of properties is impossible

**Fig. 17.1** Monotone linear operators: monotone class relationships. PM = paramonotone, SM = strictly monotone, 3CM = 3 cyclic monotone,  $3^*$  =  $3^*$ -monotone



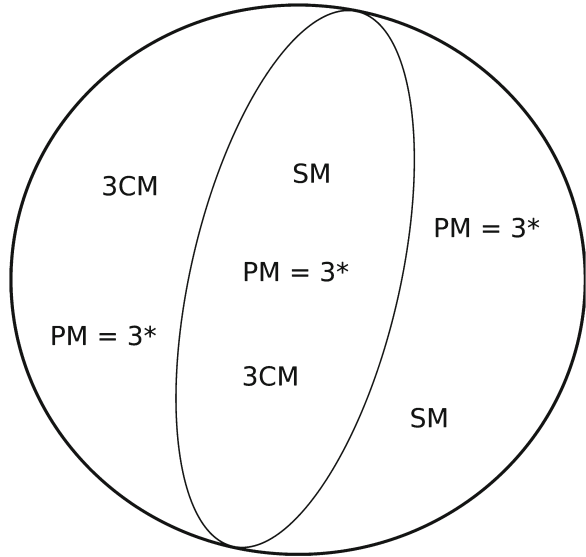
**Fig. 17.2** Monotone linear operators on  $\mathbb{R}^2$ : monotone class relationships. PM = paramonotone, SM = strictly monotone, 3CM = 3 cyclic monotone,  $3^*$  =  $3^*$ -monotone



Strict monotonicity is a stronger condition than paramonotonicity (Fact 17.8), and the strict monotonicity of an operator  $T$  guarantees the uniqueness of a solution to the variational inequality problem (see for instance [19]). These operators are somewhat analogous to the subdifferentials of strictly convex functions.

We adopt the notation of [32] and use the term  $3^*$ -monotone, although this property was first introduced with no name. The property was first referenced simply by “\*” [11] by Brézis and Haraux, and such operators were sometimes called

**Fig. 17.3** Monotone linear operators on  $\mathbb{R}^n$ : monotone class relationships. PM = paramonotone, SM = strictly monotone, 3CM = 3 cyclic monotone,  $3^*$  =  $3^*$ -monotone



(BH)-operators [16] in honour of these original authors. More recently the property has also taken on the name “rectangular” since the domain of the Fitzpatrick function of a monotone operator is rectangular precisely when the operator is  $3^*$ -monotone [29].

**Definition 17.4 ( $3^*$ -monotone).** An operator  $T : X \rightarrow 2^X$  is said to be  $3^*$ -monotone if  $T$  is monotone and for all  $z$  in the domain of  $T$  and for all  $x^*$  in the range of  $T$

$$\sup_{(y,y^*) \in \text{gra} T} \langle z - y, y^* - x^* \rangle < +\infty. \tag{17.3}$$

$3^*$ -monotonicity has the important property in that if  $T_1$  and  $T_2$  are  $3^*$ -monotone, then as long as their sum is maximal monotone, the closure of the sum of their ranges is identical to the closure of the range of their sum. For instance, if two operators are  $3^*$ -monotone, and one is surjective, then if the sum is maximal monotone, it is also surjective. Furthermore, if both are continuous monotone linear operators, and at least one is  $3^*$ -monotone, then the kernel of the sum is the intersection of the kernels [3]. This property can be used, as shown in [11], to determine when solutions to  $T^{-1}(0)$  exist by demonstrating that 0 is in the interior (or is not in the closure) of the sum of the ranges of an intelligent decomposition of a difficult to evaluate maximal monotone operator. It has also been shown for linear relations on Banach spaces that  $3^*$ -monotonicity guarantees the existence of solutions to the primal-dual problem pairs in [27]. It should also be noted that operators with bounded range [32] and strongly coercive operators [11] are  $3^*$ -monotone.

**Definition 17.5 (*n*-cyclic monotone).** Let  $n \geq 2$ . An operator  $T : X \rightarrow 2^X$  is said to be *n*-cyclic monotone if

$$\left. \begin{array}{l} (x_1, x_1^*) \in \text{gra } T \\ (x_2, x_2^*) \in \text{gra } T \\ \dots \in \text{gra } T \\ (x_n, x_n^*) \in \text{gra } T \\ x_{n+1} = x_1 \end{array} \right\} \Rightarrow \sum_{i=1}^n \langle x_i - x_{i+1}, x_i^* \rangle \geq 0. \tag{17.4}$$

A *cyclical monotone* operator is one that is *n*-cyclic monotone for all  $n \in \mathbb{N}$ .

Note that 2-cyclic monotonicity is equivalent to monotonicity. By substituting  $(a_n, a_n^*) := (a_1, a_1^*)$ , it easily follows from the definition that any *n*-cyclic monotone operator is  $(n - 1)$ -cyclic monotone. 1-cyclic monotonicity is not defined, since the  $n = 1$  case for (17.4) is trivial. 3-cyclic monotone operators serve to represent a special case of *n*-cyclic monotone operators that is also a stronger condition than 3\*-monotonicity. Of note, all subdifferentials of convex functions are cyclical monotone [28].

**Definition 17.6 (maximality).** An operator is *maximal n-cyclic monotone* if its graph cannot be extended while preserving *n*-cyclic monotonicity. A *maximal monotone* operator is a maximal 2-cyclic monotone operator. A *maximal cyclical monotone* operator is a cyclical monotone operator such that all proper graph extensions are not cyclical monotone.

There is a rich literature on the theory (see [9] for a good overview) and application (for instance [18]) of maximal monotone operators. Furthermore, it is well known that a maximal monotone operator  $T$  has the property that  $T^{-1}(0)$  is convex, a property shared by paramonotone operators with convex domain (Proposition 17.11), and analogous to the fact that the minimizers of a convex function form a convex set. Maximal monotonicity is also an important property for general differential inclusions [10, 26].

**Definition 17.7 (Five classes of monotone operator).** An operator  $T : X \rightarrow 2^X$  is said to be [Class] (with abbreviation [Code]) if and only if  $T$  is monotone and for every  $(x, x^*), (y, y^*), (z, z^*)$  in  $\text{gra } T$  one has [Condition].

Code	Class	Condition (A)
	Monotone	$\langle x - y, x^* - y^* \rangle \geq 0$
PM	Paramonotone	$\langle x - y, x^* - y^* \rangle = 0 \Rightarrow (x, y^*), (y, x^*) \in \text{gra } T$
SM	Strictly monotone	$\langle x - y, x^* - y^* \rangle = 0 \Rightarrow x = y$
3CM	3-cyclic monotone	$\langle x - y, x^* \rangle + \langle y - z, y^* \rangle + \langle z - x, z^* \rangle \geq 0$
MM	Maximal monotone	$(\forall a \in X)(\forall a^* \in X)$ $\langle x - a, x^* - a^* \rangle \geq 0 \Rightarrow (x, x^*) \in \text{gra } T$
3*	3*-monotone	$\sup_{(a, a^*) \in \text{gra } T} \langle z - a, a^* - x^* \rangle < +\infty$

The order above, PM-SM-3CM-MM-3\*, is fixed to allow a binary label of the classes to which an operator belongs. For instance, an operator with the label 10111 is paramonotone, not strictly monotone, 3-cyclic monotone, maximal monotone, and 3\*-monotone.

After noting some general relationships among these classes in Sect. 17.2, we note in Sect. 17.3 that monotone operators belonging to particular combinations of these classes can be constructed in a product space.

Linear relations are a multi-valued extension of linear operators and are defined by those operators whose graph forms a vector space. This is a natural extension to consider as monotone operators are often multi-valued. We consider linear relations in Sect. 17.4 and explore their characteristics and structure. Of particular note, we fully explore the manner in which linear relations can be multi-valued and remark on a curious property of linear relations whose domains are not closed. Finally, we obtain a generalization to the fact that bounded linear operators that are 3\*-monotone are also paramonotone (a corollary to a result in [11]), with conditions different from those in [7], and demonstrate by example that there is 3\*-monotone linear relation that is not paramonotone.

In Sect. 17.5, we list various examples of linear operators satisfying or failing to satisfy the 5 properties defined above. The examples are chosen to have full domain, low dimension, and be continuous where possible. This is shown to yield a complete characterization of the dependence or independence of these five classes of monotone operator in  $\mathbb{R}^2$ ,  $\mathbb{R}^n$ , and in a general Hilbert space  $X$ . One result of this section is that paramonotone and linear operators in  $\mathbb{R}^2$  are exactly the symmetric or strictly monotone operators in  $\mathbb{R}^2$ .

We assume throughout that  $X$  is a real Hilbert space, with inner product  $\langle \cdot, \cdot \rangle$ . When an operator  $T : X \rightarrow 2^X$  is such that for all  $x \in X$ ,  $Tx$  contains at most one element, such operators are called *single-valued*. When  $T$  is single-valued, for brevity  $Tx$  is at times considered as a point rather than as a set (i.e.,  $x^* \in Tx$ ). The *orthogonal complement* of a set  $C \subset X$  is denoted by  $C^\perp$  and defined by

$$C^\perp := \{x \in X : \langle x, c \rangle = 0 \ \forall c \in C\}. \quad (17.5)$$

Note that for any set  $C \subset X$ , the set  $C^\perp$  is closed in  $X$ . The operator  $P_V$  is the metric projection where  $V$  is a closed subspace of  $X$ . We use the convention that for set addition  $A + \emptyset = \emptyset$ , where  $\emptyset$  is the empty set. A monotone extension  $\tilde{T} : X \rightarrow 2^X$  of a monotone operator  $T : X \rightarrow 2^X$  is a monotone operator such that  $\text{gra}T \subsetneq \text{gra}\tilde{T}$ , where  $\text{gra}T := \{(x, x^*) : x \in \text{dom}T, x^* \in Tx\}$ . An operator  $T : X \rightarrow 2^X$  is said to be *locally bounded* if for every  $x \in \text{dom}T$ , there is a neighbourhood  $V$  of  $x$  and an  $M > 0$  such that for every  $v \in V$ ,  $\sup_{v^* \in Tv} \|v^*\| < M$ . A selection of an operator  $T : X \rightarrow 2^X$  is an operator  $\tilde{T}$  such that  $\text{gra}\tilde{T} \subset \text{gra}T$ , and a single-valued selection of  $T$  is such an operator  $\tilde{T}$  where  $\tilde{T} : X \rightarrow X$ .

## 17.2 Preliminaries

The following arises from the definitions of strict monotonicity and paramonotonicity.

**Fact 17.8.** Any strictly monotone operator  $T : X \rightarrow 2^X$  is also paramonotone.

Two synonymous definitions of 3-cyclic monotonicity are worth explicitly stating. For an operator  $T : X \rightarrow 2^X$  to be 3-cyclic monotone, every  $(x, x^*), (y, y^*), (z, z^*) \in \text{gra} T$  must satisfy

$$\langle x - y, x^* \rangle + \langle y - z, y^* \rangle + \langle z - x, z^* \rangle \geq 0, \tag{17.6}$$

or equivalently

$$\langle z - y, y^* - x^* \rangle \leq \langle x - z, x^* - z^* \rangle. \tag{17.7}$$

From (17.7), the following fact is obvious.

**Fact 17.9.** Any 3-cyclic monotone operator  $T : X \rightarrow 2^X$  is also  $3^*$ -monotone.

Another relationship among these classes of monotone operator was discovered in 2006 (Proposition 3.1 in [22]).

**Proposition 17.10 ([22]).** *If  $T$  is 3-cyclic monotone and maximal (2-cyclic) monotone, then  $T$  is paramonotone.*

*Proof.* Suppose that for some choice of  $(x, x^*), (y, y^*) \in \text{gra}(T)$ ,  $\langle x - y, x^* - y^* \rangle = 0$ , so  $\langle y - x, x^* \rangle = \langle y - x, y^* \rangle$ . Since  $T$  is 3-cyclic monotone, every  $(z, z^*) \in \text{gra}(T)$  satisfies

$$\begin{aligned} 0 &\geq \langle y - x, x^* \rangle + \langle z - y, y^* \rangle + \langle x - z, z^* \rangle \\ &= \langle -x, y^* \rangle + \langle z, y^* \rangle + \langle x - z, z^* \rangle \\ &= \langle z - x, y^* \rangle + \langle x - z, z^* \rangle \\ &= \langle x - z, z^* - y^* \rangle \end{aligned}$$

and so

$$\langle x - z, y^* - z^* \rangle \geq 0 \quad \forall (z, z^*) \in \text{gra}(T).$$

Since  $T$  is maximal monotone,  $y^* \in Tx$ . By exchanging the roles of  $x$  and  $y$  above, it also holds that  $x^* \in T(y)$ , and so  $T$  is paramonotone. ■

When finding the zeros of a monotone operator, it can be useful to know if the solution set is convex or not. It is well known that for a maximal monotone operator  $T$ ,  $T^{-1}(0)$  is a closed convex set (see for instance [4]). A similar result also holds for paramonotone operators.



**Proposition 17.11.** *Let  $T : X \rightarrow 2^X$  be a paramonotone operator with convex domain. Then  $T^{-1}(0)$  is a convex set.*

*Proof.* Suppose  $T^{-1}(0)$  is nonempty. Let  $x, y, z \in X$  such that  $0 \in Tx, 0 \in Tz$ , and  $y = \alpha x + (1 - \alpha)z$  for some  $\alpha \in ]0, 1[$ . Then,  $x - y = (1 - \alpha)(x - z)$  and  $y - z = \alpha(x - z)$ , so  $x - y = \frac{1-\alpha}{\alpha}(y - z)$ . Since  $T$  has convex domain,  $Ty \neq \emptyset$ . By the monotonicity of  $T$ , for all  $y^* \in Ty$

$$0 \leq \langle x - y, -y^* \rangle = \frac{1 - \alpha}{\alpha} \langle y - z, -y^* \rangle \quad \text{and} \quad 0 \leq \langle y - z, y^* \rangle,$$

and so  $\langle y - z, y^* \rangle = 0$ . Therefore, by the paramonotonicity of  $T$ ,  $0 \in T(y)$ , and so the set  $T^{-1}(0)$  is convex. ■

However, if an operator is not maximal monotone, there is no guarantee that  $T^{-1}(0)$  is closed, even if paramonotone, as the operator  $T : \mathbb{R} \rightarrow \mathbb{R}$  below demonstrates:

$$Tx := \begin{cases} -1, & x \leq -1, \\ 0, & x \in ]-1, 1[, \\ 1, & x \geq 1. \end{cases} \tag{17.8}$$

### 17.3 Monotone Operators on Product Spaces

Let  $X_1$  and  $X_2$  be Hilbert spaces, and consider set valued operators  $T_1 : X_1 \rightarrow 2^{X_1}$  and  $T_2 : X_2 \rightarrow 2^{X_2}$ . The product operator  $T_1 \times T_2 : X_1 \times X_2 \rightarrow 2^{X_1 \times X_2}$  is defined as  $(T_1 \times T_2)(x_1, x_2) := \{(x_1^*, x_2^*) : x_1^* \in T_1 x_1 \text{ and } x_2^* \in T_2 x_2\}$ .

**Proposition 17.12.** *If both  $T_1$  and  $T_2$  are monotone, then the product operator  $T_1 \times T_2$  is also monotone.*

*Proof.* For any points  $((x_1, x_2), (x_1^*, x_2^*)), ((y_1, y_2), (y_1^*, y_2^*)) \in \text{gra}(T_1 \times T_2)$ ,

$$\begin{aligned} & \langle (x_1, x_2) - (y_1, y_2), (x_1^*, x_2^*) - (y_1^*, y_2^*) \rangle \\ &= \langle x_1 - y_1, x_1^* - y_1^* \rangle + \langle x_2 - y_2, x_2^* - y_2^* \rangle \geq 0. \end{aligned}$$

Hence,  $T_1 \times T_2$  is monotone. ■

**Proposition 17.13.** *If both  $T_1$  and  $T_2$  are paramonotone, then the product operator  $T_1 \times T_2$  is also paramonotone.*

*Proof.* If  $x_i^* \in T_i x_i, y_i^* \in T_i y_i$  for  $i \in \{1, 2\}$  and

$$\langle (x_1, x_2) - (y_1, y_2), (x_1^*, x_2^*) - (y_1^*, y_2^*) \rangle = 0,$$

then  $\langle x_i - y_i, x_i^* - y_i^* \rangle = 0$  for  $i \in \{1, 2\}$  since both  $T_1$  and  $T_2$  are monotone. By the paramonotonicity of  $T_1$  and  $T_2$ ,  $y_i^* \in T_i x_i$  and  $x_i^* \in T_i y_i$  for  $i \in \{1, 2\}$ , and so  $(x_1^*, x_2^*) \in (T_1 \times T_2)(y_1, y_2)$  and  $(y_1^*, y_2^*) \in (T_1 \times T_2)(x_1, x_2)$ . ■

By following the same proof structure as Proposition 17.13, a similar result immediately follows for some other monotone classes.

**Proposition 17.14.** *If both  $T_1$  and  $T_2$  belong to the same monotone class, where that class is one of strict,  $n$ -cyclic, or  $3^*$ -monotonicity, then so does their product operator  $T_1 \times T_2$ .*

**Proposition 17.15.** *If both  $T_1$  and  $T_2$  are maximal monotone, then the product operator  $T_1 \times T_2$  is also maximal monotone.*

*Proof.* Suppose  $T_1 \times T_2$  is not maximal monotone. Then there exists a point  $((x_1, x_2), (x_1^*, x_2^*)) \notin \text{gra}(T_1 \times T_2)$  such that for all  $((y_1, y_2), (y_1^*, y_2^*)) \in \text{gra}(T_1 \times T_2)$

$$\langle x_1 - y_1, x_1^* - y_1^* \rangle + \langle x_2 - y_2, x_2^* - y_2^* \rangle \geq 0, \tag{17.9}$$

and at least one of  $(x_1, x_1^*) \notin \text{gra}T_1$  or  $(x_2, x_2^*) \notin \text{gra}T_2$ . Suppose without loss of generality that  $(x_1, x_1^*) \notin \text{gra}T_1$ .

By the maximality of  $T_1$ ,  $\langle x_1 - z_1, x_1^* - z_1^* \rangle < 0$  for some  $(z_1, z_1^*) \in \text{gra}T_1$ , and so by setting  $(y_1, y_1^*) := (z_1, z_1^*)$  in (17.9),  $\langle x_2 - y_2, x_2^* - y_2^* \rangle \geq 0$  for all  $(y_2, y_2^*) \in \text{gra}T_2$ . Since  $T_2$  is maximal monotone, it must be that  $(x_2, x_2^*) \in \text{gra}T_2$ . Clearly,  $((z_1, x_2), (z_1^*, x_2^*)) \in \text{gra}(T_1 \times T_2)$ , yet

$$\langle (x_1, x_2) - (z_1, x_2), (x_1^*, x_2^*) - (z_1^*, x_2^*) \rangle < 0.$$

This is a contradiction of (17.9), and so  $T_1 \times T_2$  is maximal monotone. ■

Of course, if an operator  $T_1 : X \rightarrow 2^X$  fails to satisfy the conditions for any of the classes of monotone operator here considered, then the product of that operator with any other operator  $T_2 : Y \rightarrow 2^Y$ , namely  $T_1 \times T_2 : X \times Y \rightarrow 2^{X \times Y}$ , will also fail the same condition. Simply consider the set of points  $P$  in the graph of  $T_1$  which violate a particular condition in  $X$ , and instead consider the set of points  $\tilde{P} := \{(p, a) \times (p^*, a^*) : p \in P\}$  for a fixed arbitrary point  $(a, a^*) \in \text{gra}T_2$ . Clearly  $\tilde{P} \subset \text{gra}T_1 \times T_2$ , and this set will violate the same conditions in  $X \times Y$  that  $P$  violates for  $T_1$  in  $X$ . For instance,

$$\langle (w, a) - (x, a), (y^*, a^*) - (z^*, a^*) \rangle = \langle w - x, y^* - z^* \rangle.$$

In this manner, the lack of a monotone class property (be it  $n$ -cyclic, para-, maximal,  $3^*$ -, nor strict monotonicity) is dominant in the product space.

Taken together, the results of this section are that the product operator  $T_1 \times T_2$  of monotone operators  $T_1$  and  $T_2$  operates with respect to monotone class inclusion as a logical AND operator applied to the monotone classes of  $T_1$  and  $T_2$ . For instance, suppose that  $T_1$  is paramonotone, not strictly monotone, 3-cyclic monotone, maximal monotone, and  $3^*$ -monotone (with binary label 10111), and suppose that

$T_2$  is paramonotone, strictly monotone, not 3-cyclic monotone, maximal monotone, and not  $3^*$ -monotone (with binary label 11010). Then,  $T_1 \times T_2$  is paramonotone, not strictly monotone, not 3-cyclic monotone, maximal monotone, and not  $3^*$ -monotone (with binary label 10010).

## 17.4 Linear Relations

Linear relations are the set-valued generalizations of linear operators, which we define using the nomenclature of R. Cross [17].

**Definition 17.16 (linear relation).** An operator  $A : X \rightarrow 2^X$  is a *linear relation* if  $\text{dom}A$  is a linear subspace of  $X$  and for all  $x, y \in \text{dom}A, \lambda \in \mathbb{R}$

1.  $\lambda Ax \subset A(\lambda x)$ ,
2.  $Ax + Ay \subset A(x + y)$ .

Equivalently, linear relations are exactly those operators  $T : X \rightarrow 2^X$  whose graphs are linear subspaces of  $X \times X$ . The following results on linear relations are well known. Of note, Fact 17.17(1) and (2) are considered basic results and will not be cited in the work below.

**Fact 17.17 ([30]).** For any linear relation  $A : X \rightarrow 2^X$ ,

- (1)  $\lambda Ax = A(\lambda x)$  for all  $x \in \text{dom}A, 0 \neq \lambda \in \mathbb{R}$ ,
- (2)  $Ax + Ay = A(x + y)$  for all  $x, y \in \text{dom}A$ ,
- (3)  $A0$  is a linear subspace of  $X$ ,
- (4)  $Ax = x^* + A0$  for all  $(x, x^*) \in \text{gra}A$ ,
- (5) If  $A$  is single-valued at any point, it is single-valued at every point in its domain.

**Proposition 17.18.** Suppose  $A : X \rightarrow 2^X$  is a linear relation, and let  $x \in \text{dom}A$ . Then,  $P_{A0^\perp}Ax$  is a singleton and

$$Ax \subset P_{A0^\perp}Ax + \overline{A0}. \tag{17.10}$$

If  $A0$  is closed, then there is a unique  $x_0^* \in Ax$  such that  $x_0^* \in A0^\perp$ , where  $x_0^* = P_{A0^\perp}x^*$  for all  $x^* \in Ax$ .

*Proof.* Let  $x \in \text{dom}A$ . Since  $\overline{A0}$  and  $A0^\perp$  are closed subspaces such that  $\overline{A0} + A0^\perp = X$ , then for all  $x^* \in X, x^* = P_{\overline{A0}}x^* + P_{A0^\perp}x^*$ . By Fact 17.17 (4), (17.10) holds and  $P_{A0^\perp}Ax$  is a singleton. If  $A0$  is closed, then for all  $x^* \in Ax$ ,

$$Ax = x^* + A0 = P_{A0^\perp}x^* + A0.$$

Therefore,  $P_{A0^\perp}y^* = P_{A0^\perp}x^*$  for all  $y^* \in Ax$ . Furthermore, since  $0 \in A0$  always,  $P_{A0^\perp}x^* \in Ax$ . ■

**Proposition 17.19.** *Any monotone linear relation  $A : X \rightarrow 2^X$  with full domain is maximal monotone and single-valued.*

*Proof.* Suppose that  $A : X \rightarrow 2^X$  is a linear relation where  $\text{dom}A = X$ . Let  $(z, z^*)$  be a point such that  $\langle z - y, z^* - y^* \rangle \geq 0$  for all  $(y, y^*) \in \text{gra}A$ . Choose an arbitrary  $z_0^* \in Az$ . Let  $y = z - \varepsilon x$  for arbitrary  $(x, x^*) \in \text{gra}A$  and  $\varepsilon > 0$ , so that by linearity  $-\varepsilon x^* \in A(-\varepsilon x)$ . Therefore  $z_0^* - \varepsilon x^* \in Ay$  and so  $\langle \varepsilon x, z^* - z_0^* + \varepsilon x^* \rangle \geq 0$ . Divide out the  $\varepsilon$ , and send  $\varepsilon \rightarrow 0^+$  so that  $\langle x, z^* - z_0^* \rangle \geq 0$  for all  $x \in X$ . Hence  $z^* = z_0^*$  and  $T$  is single-valued and maximal monotone. ■

The following results appear respectively as Proposition 2.2(i) and Proposition 2.4 in [5].

**Proposition 17.20 ([5]).** *If  $A : X \rightarrow 2^X$  is a monotone linear relation, then  $\text{dom}A \subset (A0)^\perp$  and  $A0 \subset (\text{dom}A)^\perp$ .*

**Corollary 17.21 ([5]).** *If a linear relation  $A : X \rightarrow 2^X$  is maximal monotone, then  $(\text{dom}A)^\perp = A0$ , and so  $\overline{\text{dom}A} = (A0)^\perp$  and  $A0$  is a closed subspace.*

This leads to a partial converse result to Proposition 17.19.

**Corollary 17.22.** *If a maximal monotone single-valued linear relation  $A : X \rightarrow X$  is locally bounded, then it has full domain.*

*Proof.* Since  $A$  is single-valued,  $A0 = 0$ , and so by Corollary 17.21,  $\overline{\text{dom}A} = (A0)^\perp = X$ . Choose any point  $x \in X$ . Since  $\text{dom}A$  is dense in  $X$ , there exist a sequence  $(y_n, y_n^*)_{n \in \mathbb{N}} \subset \text{gra}A$  such that  $y_n \rightarrow x$ . Since  $A$  is locally bounded, a subsequence  $(y_{\phi(n)}^*)_{n \in \mathbb{N}}$  of  $(y_n^*)_{n \in \mathbb{N}}$  weakly converges to some point  $x^* \in X$ . Therefore, for all  $(z, z^*) \in \text{gra}A$ ,

$$0 \leq \lim_{n \rightarrow +\infty} \langle y_{\phi(n)} - z, y_{\phi(n)}^* - z^* \rangle = \langle x - z, x^* - z^* \rangle.$$

Since  $A$  is maximal monotone,  $(x, x^*) \in \text{gra}A$ , and so  $A$  has full domain. ■

The following fact appears in Proposition 2.2 in [5].

**Fact 17.23 ([5]).** Let  $A : X \rightarrow 2^X$  be a monotone linear relation. For any  $x, y \in \text{dom}A$ , the set

$$\{ \langle y, x^* \rangle : x^* \in Ax \}$$

is a singleton, the value of which can be denoted simply by  $\langle y, Ax \rangle$ .

*Proof.* Let  $x, y \in \text{dom}A$  and suppose that  $x_1^*, x_2^* \in Ax$ . By Fact 17.17 (4),  $x_2^* - x_1^* \in A0$ . Now, by Proposition 17.20,  $A0 \subset (\text{dom}A)^\perp$ , and so  $x_2^* - x_1^* \in (\text{dom}A)^\perp$ . Since  $y \in \text{dom}A$ ,  $\langle y, x_1^* \rangle = \langle y, x_2^* \rangle$ . ■

Proposition 17.24 below demonstrates that multivalued linear relations are closely related to a number of single-valued linear relations. Note especially that  $V = A0^\perp$  and  $V = \overline{\text{dom}A}$  both satisfy the conditions below.

**Proposition 17.24 (dimension reduction).** *Suppose that  $A : X \rightarrow 2^X$  is a monotone linear relation. Let  $V \subset X$  satisfy*

- (1)  $V$  is a closed subspace of  $X$ ,
- (2)  $\text{dom}A \subset V$ , and
- (3)  $A0 \subset V^\perp$ .

*Define the operator  $\tilde{A} : V \rightarrow 2^V$ , by  $\tilde{A}x := P_V Ax$  on  $\text{dom}A$ , and let  $\tilde{A} = \emptyset$  when  $x \notin \text{dom}A$ . Then,  $\tilde{A}$  is a single-valued monotone linear relation and  $\text{dom}A = \text{dom}\tilde{A}$ . In the case where  $V = A0^\perp$  and  $A0$  is closed, the operator  $\tilde{A}$  is a single-valued selection of  $A$ . If  $A$  is maximal monotone, then  $V = A0^\perp = \overline{\text{dom}A}$  is the only subspace satisfying conditions (17.24)–(17.24) above, and  $\tilde{A}$  is a maximal monotone single-valued selection of  $A$ .*

*Proof.* For any  $x \in X$ ,  $P_V(x) = P_V(P_{A0^\perp}x + P_{A0}x) = P_V(P_{A0^\perp}x)$  as  $\overline{A0} \subset V^\perp$ . By Proposition 17.18,  $\tilde{A}$  is always single-valued, and if  $A0$  is closed,  $P_{A0^\perp}x^* \in Ax$  for each  $(x, x^*) \in \text{gra}A$ , and so if  $V = A0^\perp$ , then  $\tilde{A}$  is a selection of  $A$ . Consider now arbitrary  $(y, \tilde{y}^*), (z, \tilde{z}^*) \in \text{gra}\tilde{A}$ , and  $\lambda \in \mathbb{R}$ . Then, for  $y^* \in Ay$  and  $z^* \in Az$ , we have that  $P_V y^* = \tilde{y}^*$  and  $P_V z^* = \tilde{z}^*$ . Since  $A$  is a linear relation,  $(y + \lambda z, y^* + \lambda z^*) \in \text{gra}A$ . Therefore,  $(y + \lambda z, P_V(y^* + \lambda z^*)) \in \text{gra}\tilde{A}$ , and since  $P_V$  is itself a linear operator,  $P_V(y^* + \lambda z^*) = \tilde{y}^* + \lambda \tilde{z}^*$ , it follows that  $\tilde{y}^* + \lambda \tilde{z}^* \in \tilde{A}(y + \lambda z)$ . Since  $\text{dom}A = \text{dom}\tilde{A}$ , the operator  $\tilde{A}$  is a linear relation. Finally, suppose that  $A$  is maximal monotone, and so from Corollary 17.21 we have that  $A0^\perp = \overline{\text{dom}A}$  and  $A0$  is closed. The only subspace  $V$  satisfying the conditions in this case is  $V = A0^\perp$ . Suppose there exists a point  $(x, x^*)$  where  $x \in V = A0^\perp$ , that is monotonically related to  $\text{gra}\tilde{A}$ . For all  $(z, z^*) \in \text{gra}A$ , there is a  $y \in A0$  such that  $y + P_V z^* = z^*$ . Then, by Fact 17.17 (4),

$$\langle x - z, x^* - z^* \rangle = \langle x - z, x^* - y - P_V z^* \rangle = \langle x - z, x^* - P_V z^* \rangle \geq 0.$$

Therefore,  $(x, x^*)$  is also monotonically related to  $A$ , and since  $A$  is maximal monotone,  $(x, x^*) \in \text{gra}A$ . Since  $x^* \in V$ ,  $P_V x^* = x^*$ , and so  $(x, x^*) \in \text{gra}\tilde{A}$ . Therefore,  $\tilde{A}$  is maximal monotone. ■

From the results in this section so far, we know that monotone linear relations  $A : X \rightarrow 2^X$  can only be multi-valued such that  $A0$  is a subspace of  $X$ ,  $Ax = x^* + A0$  for any  $x^* \in Ax$ , and  $A0 \subset (\text{dom}A)^\perp$ . For the purposes of calculation by the inner product, for any  $x, z \in \text{dom}A$ ,

$$\langle x, Az \rangle = \langle x, \tilde{A}z \rangle, \tag{17.11}$$

where  $\tilde{A}$  is the single-valued operator (a selection of  $A$  if  $A0$  is closed) as calculated in Proposition 17.24 for  $V = A0^\perp$ . In the other direction, any single-valued monotone linear relation  $\tilde{A} : X \rightarrow 2^X$  can be extended to a multivalued monotone linear relation  $A : X \rightarrow 2^X$  by choosing any subspace  $V \subset (\text{dom}A)^\perp$  and setting  $Ax := \tilde{A}x + V$ .

Now, in the unbounded linear case, maximal monotone operators may not have a closed domain. The concept of a *halo* well captures this aspect.

**Definition 17.25 (halo).** The *halo* of a monotone linear relation  $A : X \rightarrow 2^X$  is the set

$$\text{halo } A := \{x \in X : (\exists M)(\forall (y, y^*) \in \text{gra } A)\langle x - y, y^* \rangle \leq M\|x - y\|\}. \tag{17.12}$$

The following is an amalgamation of Proposition 6.2 and Theorem 6.5 in [5].

**Fact 17.26 ([5]).** If  $A : X \rightarrow 2^X$  is a monotone linear relation, then  $\text{dom } A \subseteq \overline{\text{halo } A} \subseteq (A0)^\perp$ . Furthermore,  $A$  is maximal monotone if and only if  $A0^\perp = \overline{\text{dom } A}$  and  $\text{halo } A = \text{dom } A$ .

Now, if the domain of a linear relation is not closed, we have the following curious result. Below,  $A^m$  denotes the iterated operator composition, where for instance  $A^3x = A(A(Ax))$ . Note that if  $\text{dom } A$  is dense in  $X$ , the operator  $P_V A$  is the same as  $A$ .

**Proposition 17.27.** *Suppose a maximal monotone linear relation  $A : X \rightarrow 2^X$  is such that  $\text{dom } A$  is not closed, and let  $V := \overline{\text{dom } A}$ . Then, there is a sequence  $(z_n)_{n \in \mathbb{N}} \subset \text{dom } A$  such that*

$$(P_V A)^m(z_n) \in \text{dom } A, \forall 1 \leq m < n, \tag{17.13}$$

$$(P_V A)^n(z_n) \notin \text{dom } A, \tag{17.14}$$

where for all  $z \in \text{dom } A$ ,  $P_V A z$  is a singleton set.

*Proof.* Since  $A$  is maximal monotone,  $\text{dom } A = \text{halo } A \subseteq \overline{\text{dom } A}$ , and by Corollary 17.21,  $V = A0^\perp$ . Therefore, by Proposition 17.18,  $P_V A z \subset A z$  and is a singleton for every  $z \in \text{dom } A$ . Choose any point  $z_0 \in V$  such that  $z_0 \notin \text{dom } A$ . We shall generate the sequence  $(z_n)_{n \in \mathbb{N}} \subset \text{dom } A$  iteratively as follows. For some  $n \geq 0$ , suppose that  $z_n \in V$ . By Minty’s theorem [25], since  $A$  is maximal monotone,  $\text{ran}(\text{Id} + A) = X$ . Therefore, there exists a  $z_{n+1} \in \text{dom } A$  such that  $z_n \in z_{n+1} + A z_{n+1}$ . Since  $z_n, z_{n+1} \in V$ ,  $z_n \in z_{n+1} + P_V A z_{n+1}$ , and so as  $P_V A z_{n+1}$  is a singleton,

$$P_V A z_{n+1} = \{z_n - z_{n+1}\}.$$

Now, since both  $P_V$  and  $A$  are linear operators, if  $n \geq 2$

$$\begin{aligned} (P_V A)^2 z_{n+1} &= P_V A(z_n - z_{n+1}) \\ &= P_V A z_n - P_V A z_{n+1} \\ &= \{z_{n-1} - 2z_n + z_{n+1}\}, \end{aligned} \tag{17.15}$$

a linear combination of the terms  $z_{n-1}, z_n$ , and  $z_{n+1}$ , with  $z_{n-1}$  appearing with coefficient 1. Similarly, if  $n \geq 3$ ,

$$\begin{aligned}
 (PVA)^3 z_{n+1} &= PVA(z_{n-1} - 2z_n + z_{n+1}) \\
 &= \{z_{n-2} - z_{n-1} - 2z_{n-1} + 2z_n + z_n - z_{n+1}\} \\
 &= \{z_{n-2} - 3z_{n-1} + 3z_n - z_{n+1}\}.
 \end{aligned}
 \tag{17.16}$$

By iterative composition,  $(PVA)^m z_{n+1}$  is linear combination of the terms  $z_p$  for  $n - m + 1 \leq p \leq n + 1$ , with  $z_{n-m+1}$  appearing with coefficient 1, as long as  $n - m + 1 \geq 0$ . Since  $\text{dom}A$  is a linear subspace of  $X$ ,  $(PVA)^m z_{n+1} \in \text{dom}A$  if  $n \geq m$ . However, if  $n + 1 = m$ , the single point in  $(PVA)^m z_{n+1}$  is not in  $\text{dom}A$  since  $z_0 = x \notin \text{dom}A$ . ■

For any linear relation  $A : X \rightarrow 2^X$  where  $\text{dom}A$  is not closed, sequences like those in Proposition 17.27 are plentiful. Every point  $x \in \overline{\text{dom}A}$  such that  $x \notin \text{dom}A$ , including for instance the points  $\lambda x$  for  $\lambda > 0$ , generates a different sequence  $(z_n)_{n \in \mathbb{N}}$  using the method from the proof of Proposition 17.27.

To explore these concepts, consider the following example.

*Example 17.28.* Consider the infinite dimensional Hilbert space  $\ell^2$ , the space of infinite sequences  $\mathbf{x} = (x_k)_{k \in \mathbb{N}}$  such that  $\sum_{k=1}^{+\infty} x_k^2 < +\infty$ . Let  $\mathbf{e}_k$  denote the  $k$ th standard unit vector (the  $k$ th element in the sequence is 1, and all other elements in the sequence are 0). Define the single-valued monotone relation  $A : \ell_2 \rightarrow \ell_2$  for  $\mathbf{x} \in \text{dom}A$  by

$$A\mathbf{x} = A\left(\sum_{k=1}^{+\infty} x_k \mathbf{e}_k\right) := \sum_{k=1}^{+\infty} kx_k \mathbf{e}_k,$$

where

$$\text{dom}A := \{x \in \ell_2 : \exists N \in \mathbb{N} \text{ s.t. } x_k = 0 \ \forall k \geq N\}.$$

Considering the linear relation  $A$  in the example above, the point  $\mathbf{x} := \sum_{k=1}^{+\infty} \frac{1}{k} \mathbf{e}_k$  is not in  $\text{halo}A$ . This is because the sequence  $(\mathbf{y}_n)_{n \in \mathbb{N}} \subset \text{dom}A$  where  $\mathbf{y}_n := \sum_{i=1}^n \frac{1}{2i} \mathbf{e}_i$  eventually violates (17.12) for any choice of  $M > 0$  for a large enough  $n$ . (Therefore we know that  $A$  is not maximal monotone.) However, the point  $\mathbf{z} := \sum_{i=1}^{+\infty} \frac{1}{i^2} \mathbf{e}_i$  is in  $\text{halo}A$ , and  $\text{gra}A$  could be extended by the point  $(\mathbf{z}, \mathbf{x})$  and remain monotone. Since  $\mathbf{x} \in \overline{\text{dom}A}$  but  $\mathbf{x} \notin \text{halo}A$ , yet  $\mathbf{x} = A\mathbf{z}$  and  $\mathbf{z} \in \text{halo}A$ , we have the beginning of a sequence like those in Proposition 17.27 for any monotone extension of  $A$  containing  $(\mathbf{z}, \mathbf{x})$  that is also a linear relation.

Finally, the following result is used later and appears in Proposition 4.6 in [6].

**Proposition 17.29 ([6]).** *Suppose that  $A : X \rightarrow 2^X$  is a linear relation. Then  $A$  is maximal monotone and symmetric if and only if there exists a proper lower semicontinuous convex function  $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$  such that  $A = \partial f$ .*

## 17.5 Monotone Classes of Linear Relations

The recent result for paramonotonicity and  $3^*$ -monotonicity is a portion of the main result in [7].

**Proposition 17.30 ([7]).** *Suppose  $A : X \rightarrow 2^X$  is a maximal monotone linear relation such that  $\text{dom}A$  and  $\text{ran}A_+$  are closed ( $A_+$  is the symmetric part of  $A$ ). Then,  $A$  is  $3^*$ -monotone if and only if  $A$  is paramonotone.*

In this section we use a different approach to that used for Proposition 17.30, where we (while avoiding the use of the Fitzpatrick function) obtain results that apply to all monotone operators regardless maximal monotonicity. This is done by examining the density of  $\text{dom}A$  rather than its closure, further extending these results. First, we characterize paramonotonicity for linear relations with the following two facts.

**Fact 17.31.** *Suppose  $A : X \rightarrow 2^X$  is a monotone linear relation. Then,  $A$  is paramonotone if and only if for all  $x \in X$*

$$\langle x, Ax \rangle = 0 \Rightarrow Ax = A0. \quad (17.17)$$

*Proof.* Suppose that  $A$  is paramonotone and that for some  $x \in \text{dom}A$ ,  $\langle x, Ax \rangle = 0$ . Then,  $\langle x - 0, Ax - A0 \rangle = 0$ , since  $A0 \subset (\text{dom}A)^\perp$  (Proposition 17.20). Therefore, by paramonotonicity, every  $x^* \in Ax$  is also in  $A0$ . By Fact 17.17 (3) and (4),  $Ax = A0$ .

Now, suppose that (17.17) holds for  $A$  and that for some  $(y, y^*), (z, z^*) \in \text{gra}A$ ,

$$\langle y - z, y^* - z^* \rangle = 0.$$

Let  $x = y - z$ . Since  $A$  is a linear relation,  $y^* - z^* \in Ax$ , and so  $\langle x, Ax \rangle = 0$ . Therefore,  $Ax = A0$ , and so  $y^* - z^* \in A0$  and

$$y^* \in z^* + A0; \quad -z^* \in -y^* + A0.$$

By Fact 17.17 (1) and (4),  $-y^* + A0 = -Ay$ . Hence  $y^* \in Az$  and  $z^* \in Ay$ , so  $A$  is paramonotone. ■

**Fact 17.32.** *Suppose  $A : X \rightarrow 2^X$  is a monotone linear relation, and let  $x \in X$ . Then,  $Ax = A0$  if and only if  $0 \in Ax$  and if  $0 \in Ax$ , then  $P_{A0^\perp}Ax = \{0\}$ . If  $A0$  is closed and  $P_{A0^\perp}Ax = \{0\}$ , then  $0 \in Ax$ .*

*Proof.* Let  $Ax = A0$ . Since  $A0$  is a linear subspace of  $X$  (Fact 17.17 (3)),  $0 \in Ax$ . Now, let  $0 \in Ax$ . Then, by Fact 17.17 (4),  $Ax = A0$ .

By Proposition 17.18,  $P_{A0^\perp}Ax$  is a singleton, and since  $0 \in A0^\perp$  by the definition of the orthogonal complement,  $P_{A0^\perp}Ax = \{0\}$ . Now, let  $P_{A0^\perp}Ax = \{0\}$  and suppose that  $A0$  is closed. Then, by Proposition 17.18,  $0 \in Ax$ . ■



**Proposition 17.33.** *Suppose  $A : X \rightarrow 2^X$  is a monotone linear relation such that  $\text{dom}A$  is dense in  $A0^\perp$  and  $A0$  is closed. If  $A$  is  $3^*$ -monotone, then  $A$  is also paramonotone.*

*Proof.* Suppose that  $A$  is not paramonotone. Then, there exists an  $x \in \text{dom}A$  such that  $\langle x, Ax \rangle = 0$  yet  $Ax \neq A0$ . Choose any  $x^* \in Ax$ , and let  $x_0^* = P_{A0^\perp}x^*$ . By Fact 17.32,  $x_0^* \neq 0$  since  $A0$  is closed. If  $x_0^* \in \text{dom}A$ , let  $w = \frac{1}{2}x_0^*$ . If  $x_0^* \notin \text{dom}A$ , there is a sequence  $(y_n)_{n \in \mathbb{N}} \subset \text{dom}A$  converging to  $x_0^*$  since  $\text{dom}A$  is dense in  $A0^\perp$ . In this case, let  $w = y_n$  for some  $n$  such that

$$\langle w, Ax \rangle = \langle y_n, x_0^* \rangle \geq \frac{1}{2} \|x_0^*\|^2.$$

Let  $v = \lambda x$  for some  $\lambda > 0$  and let  $u = 0$  so that

$$\langle w - v, Av - Au \rangle = \langle w - \lambda x, \lambda Ax \rangle \geq \frac{\lambda}{2} \|x_0^*\|^2$$

which is unbounded with respect to  $\lambda$ . Hence,  $A$  is not  $3^*$ -monotone, yielding the contrapositive. ■

We therefore obtain by a different method Proposition 4.5 from [7].

**Corollary 17.34 ([7]).** *If the linear relation  $A : X \rightarrow 2^X$  is maximal monotone and  $3^*$ -monotone, then  $A$  is paramonotone.*

*Proof.* Follows directly from Proposition 17.33 and Corollary 17.21. ■

**Corollary 17.35.** *If the linear relation  $A : X \rightarrow 2^X$  is  $3^*$ -monotone, then the operator  $\tilde{A} : X \rightarrow 2^X$  defined by*

$$\tilde{A}x := Ax + (\text{dom}A)^\perp \tag{17.18}$$

*is a linear relation and is a  $3^*$ -monotone extension of  $A$  that is paramonotone.*

*Proof.* The operator  $\tilde{A}$  is a linear relation since  $A$  is a linear relation, since  $\text{dom}\tilde{A} = \text{dom}A$ , and since  $(\text{dom}A)^\perp$  is a linear subspace. (Recall that we are using the convention that  $\emptyset + S = \emptyset$  for any set  $S$ .) More specifically, for all  $x, y \in \text{dom}\tilde{A} = \text{dom}A$  and for all  $\lambda \in \mathbb{R}$ ,

$$\lambda \tilde{A}x = \lambda Ax + \lambda (\text{dom}A)^\perp \subset A(\lambda x) + (\text{dom}A)^\perp = \tilde{A}(\lambda x),$$

and

$$\tilde{A}x + \tilde{A}y = Ax + (\text{dom}A)^\perp + Ay \subset A(x + y) + (\text{dom}A)^\perp = \tilde{A}(x + y).$$

By the definition of  $(\text{dom}A)^\perp$ , for all  $x, y, z \in \text{dom}\tilde{A}$

$$\langle z - y, \tilde{A}y - \tilde{A}z \rangle = \langle z - y, Ay - Az \rangle.$$

Therefore,  $\tilde{A}$  is monotone and  $3^*$ -monotone because  $A$  is monotone and  $3^*$ -monotone. Since by Proposition 17.20,  $A0 \subset (\text{dom}A)^\perp$ , it follows from Fact 17.17 (4) that  $\tilde{A}$  is a monotone extension of  $A$  and that  $\tilde{A}0 = (\text{dom}A)^\perp$ . Therefore,  $\tilde{A}0^\perp = \overline{\text{dom}A}$ , and so by Proposition 17.33 and since  $\text{dom}A = \text{dom}\tilde{A}$ ,  $\tilde{A}$  is paramonotone. ■

If the linear relation  $A$  from Proposition 17.33 is also a single-valued bounded linear operator, then Proposition 17.33 is a corollary to the stronger result of Proposition 2 in [11].

**Proposition 17.36 ([11]).** *Let  $A : X \rightarrow X$  be a bounded monotone linear operator. Then,  $A$  is  $3^*$ -monotone if and only if there exists an  $\alpha > 0$  such that*

$$\langle x, Ax \rangle \geq \alpha \langle Ax, Ax \rangle = \alpha \|Ax\|^2.$$

**Corollary 17.37.** *If  $A : X \rightarrow X$  is a bounded linear  $3^*$ -monotone operator, then it is paramonotone.*

However, there are  $3^*$ -monotone linear relations that are not paramonotone.

*Example 17.38.* Let  $X = \ell^2$  and define the operators  $\tilde{A}, A : X \rightarrow 2^X$  for  $\mathbf{x} = (x_1, x_2, \dots) \in \ell_2$  by

$$\tilde{A}\mathbf{x} := \sum_{k=1}^{+\infty} x_{2k} \mathbf{e}_{2k} \tag{17.19}$$

and

$$A\mathbf{x} := x_1 \mathbf{u} + \tilde{A}\mathbf{x} + A0, \tag{17.20}$$

where

$$\mathbf{u} := \left( \sum_{k=1}^{\infty} \frac{1}{k} \mathbf{e}_{2k+1} \right), \tag{17.21}$$

$$A0 := \{ \mathbf{x} \in \ell_2 : \exists N \in \mathbb{N} \text{ s.t. } x_k = 0 \ \forall k \geq N \text{ and } x_{2k+1} = 0 \ \forall k \in \mathbb{N} \}, \tag{17.22}$$

and

$$\text{dom}A = \text{dom}\tilde{A} = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_6, \dots\}. \tag{17.23}$$

Then,  $A$  is a  $3^*$ -monotone linear relation, but it is not paramonotone.

*Proof.* Both  $A$  and  $\tilde{A}$  are by definition linear relations. Note that  $\tilde{A} = \mathbf{0} \times J$  where  $J$  is a subgraph of  $\text{Id}$ . Therefore,  $\tilde{A}$  is  $3^*$ -monotone as both  $\text{Id}$  and  $\mathbf{0}$  are  $3^*$ -monotone. Also,  $A0$  is a dense subspace of  $\text{span}\{\mathbf{e}_{2k+1} : k \in \mathbb{N}\}$ , and so  $A0^\perp = \text{span}\{\mathbf{e}_{2k} : k \in \mathbb{N}\}$ . Therefore,  $P_{A0^\perp}A\mathbf{x} = \tilde{A}\mathbf{x}$  as  $\mathbf{u} \in (\text{dom}A)^\perp$ . Since  $\overline{A0} \subset (\text{dom}A)^\perp$  (Proposition 17.20), for all  $(\mathbf{x}, \mathbf{x}^*), (\mathbf{y}, \mathbf{y}^*), (\mathbf{z}, \mathbf{z}^*) \in \text{gra}A$ ,

$$\langle \mathbf{z} - \mathbf{y}, \mathbf{y}^* - \mathbf{x}^* \rangle = \langle \mathbf{z} - \mathbf{y}, P_{A0^\perp} \mathbf{y}^* - P_{A0^\perp} \mathbf{x}^* \rangle = \langle \mathbf{z} - \mathbf{y}, A\mathbf{y} - A\mathbf{x} \rangle,$$

and so  $A$  is also  $3^*$ -monotone. Now,

$$A\mathbf{e}_1 = \mathbf{u} + A0 \not\subseteq A0,$$

and so  $A\mathbf{e}_1 \neq A0$ . However,  $\langle \mathbf{e}_1, A\mathbf{e}_1 \rangle = \langle \mathbf{e}_1, \tilde{A}\mathbf{e}_1 \rangle = 0$ . Therefore,  $A$  is not paramonotone. ■

## 17.6 Monotone Classes of Linear Operators

A *linear operator* is a single-valued linear relation with full domain, which is maximal monotone by Proposition 17.19. Although being single-valued and having full domain are restrictive conditions, when it comes to monotone classes, linear operators are highly characteristic of linear relations with closed domain.

If a monotone linear relation  $A : X \rightarrow 2^X$  has closed domain, which is always the case if  $X = \mathbb{R}^n$ , then  $\text{dom}A$  is itself a Hilbert space and the results of Sects. 17.4 and 17.5 hold in their strongest form, as they do for all linear operators.

Let  $\tilde{A} : \text{dom}A \rightarrow 2^{\text{dom}A}$  be the single-valued selection of  $A$  generated in the manner of Proposition 17.24 with  $V = \text{dom}A$ . By Proposition 17.19,  $\tilde{A}$  is a monotone linear operator. As the only difference between  $A$  and  $\tilde{A}$  are elements perpendicular to the domain, for any  $(x, x^*), (y, y^*) \in \text{gra}A$ ,

$$\langle x - y, x^* - y^* \rangle = \langle x - y, \tilde{A}x - \tilde{A}y \rangle,$$

and so the monotone classes of each, while not necessarily equivalent, are highly correlated.

Below, we consider linear operators operating on  $\mathbb{R}^2$ ,  $\mathbb{R}^n$ , and on Hilbert spaces of infinite dimension. Note that linear operators acting on  $\mathbb{R}^n$  will be identified with their matrix representation in the standard basis, and recall from Proposition 17.29 that symmetric linear operators are the subdifferentials of a lower semicontinuous convex function.

### 17.6.1 Monotone Linear Operators on $\mathbb{R}^2$

In this section we consider linear operators  $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ , which can be represented by the matrix

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

The operator  $A$  so defined is monotone if and only if  $a + d \geq 0$  and  $4ad \geq (b + c)^2$ . We consider some simple examples, examine their properties, and provide some sufficient and necessary conditions for inclusion within various monotone classes.

**Proposition 17.39** (3-cyclic monotone linear operators on  $\mathbb{R}^2$ ). *If  $A$  is 3-cyclic monotone, then*

$$\max\{|b|, |c|\} - a - d \leq 0. \quad (17.24)$$

*Proof.* Choose  $x = (0, 0)$ ,  $y = (1, 0)$ , and  $z = (0, 1)$ ; let  $x^* = Ax = (0, 0)$ ,  $y^* = Ay = (a, b)$ , and  $z^* = Az = (c, d)$ . If the mapping associated with  $A$  is 3-cyclic monotone, then

$$\begin{aligned} 0 &\leq \langle x - y, x^* \rangle + \langle y - z, y^* \rangle + \langle z - x, z^* \rangle \\ &= \langle (1, -1), (a, b) \rangle + \langle (0, 1), (c, d) \rangle \\ &= a + d - b. \end{aligned}$$

Similarly, by choosing different  $y$  and  $z$ , the following conditions are also necessary for any matrix  $A$  as defined above:

$$0 \geq \begin{cases} b - a - d, & y = (1, 0), z = (0, 1), \\ -b - a - d, & y = (-1, 0), z = (0, 1), \\ c - a - d, & y = (0, 1), z = (1, 0), \\ -c - a - d, & y = (0, -1), z = (1, 0). \end{cases} \quad (17.25)$$

In all cases,  $x = (0, 0)$ . ■

There are many monotone linear operators in  $\mathbb{R}^2$  that are not 3-cyclic monotone, and furthermore Examples 17.40 and 17.41 below demonstrate that 3-cyclic monotonicity does not follow from strict and maximal monotonicity.

*Example 17.40.* Consider the monotone linear operator  $\tilde{R} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by

$$\tilde{R} = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}. \quad (17.26)$$

The operator  $\tilde{R}$  violates the necessary condition (17.24) for 3-cyclic monotonicity since  $b - a - d > 0$  and  $\tilde{R}$  satisfies the monotonicity conditions  $(a + d) \geq 0$  and  $4ad \geq (b + c)^2$ , using the format  $\tilde{R} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$  above. Note that  $\langle x, \tilde{R}x \rangle = 0$  implies that  $x = 0$ , so  $\tilde{R}$  is strictly monotone and therefore paramonotone. Hence, by Proposition 17.47,  $\tilde{R}$  is also 3\*-monotone. Finally,  $\tilde{R}$  is maximal monotone by Proposition 17.19.

*Example 17.41.* Consider the rotation operator  $R_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  with matrix representation

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}. \tag{17.27}$$

Note that  $R_\theta$  is monotone if and only if  $|\theta| \leq \pi/2$ , since this is precisely when  $\cos(\theta) \geq 0$ . In this range,  $R_\theta$  is maximal monotone by Proposition 17.19.

Now,  $R_\theta$  is 3-cyclic monotone if and only if  $|\theta| < \pi/3$  by Fact 17.42 below.

Therefore, for any  $\theta \in ]\pi/3, \pi/2[$ ,  $R_\theta$  is maximal monotone and strictly monotone, but not 3-cyclic monotone.

Now,  $\langle x, R_\theta x \rangle = 0$  implies that  $x = 0$  unless  $\theta = \pi/2$ . Therefore,  $R_\theta$  is strictly monotone and hence paramonotone when  $|\theta| < \pi/2$ . By Proposition 17.47,  $R_\theta$  is  $3^*$ -monotone as well when  $|\theta| < \pi/2$ . When  $\theta = \pi/2$ ,  $R_\theta$  is not paramonotone, and therefore neither is it strictly monotone nor, by Proposition 17.33, is it  $3^*$ -monotone.

By the following fact (Proposition 7.1 in [3]),  $\mathbb{R}^2$  is large enough to contain distinct instances of  $n$ -cyclic monotone operators for  $n \geq 2$ .

**Fact 17.42 ([3]).** Let  $n \in \{2, 3, \dots\}$ . Then  $R_\theta$  is  $n$ -cyclic monotone if and only if  $|\theta| \in [0, \pi/n]$ .

*Proof.* See Example 4.6 in [3] for a detailed proof. ■

The zero operator yields trivial solutions to any associated variational inequality problem, and so the following, which shares the monotone classes of  $\mathbf{0}$ , is introduced in its stead.

*Example 17.43.* The orthogonal projection  $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by  $A(x_1, x_2) := (x_1, 0)$  is maximal monotone, paramonotone, 3-cyclic monotone, and  $3^*$ -monotone.

*Proof.* Using the notation of Sect. 17.3, we have that  $A = Id \times \mathbf{0}$ , where  $\mathbf{0} : \mathbb{R} \rightarrow \mathbb{R}$  is the zero operator, and  $Id : \mathbb{R} \rightarrow \mathbb{R}$  is the identity. The  $\mathbf{0}$  operator is maximal monotone, paramonotone, 3-cyclic monotone, and  $3^*$ -monotone, as is  $Id$ , which is also strictly monotone, while  $\mathbf{0}$  is not. The properties of  $A$  follow directly from the results in Sect. 17.3. ■

Finally, paramonotone linear operators in  $\mathbb{R}^2$  are further restricted to be either strictly monotone or symmetric.

**Proposition 17.44.** A linear operator  $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is paramonotone if and only if it is strictly monotone or symmetric.

*Proof.* Strictly monotone operators and symmetric linear operators are paramonotone by Facts 17.8 and 17.48, respectively. It remains to show that these are the only two possibilities. Assuming then that  $A$  is paramonotone, consider the general case,

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \text{ and } A_+ = \begin{bmatrix} a & \frac{b+c}{2} \\ \frac{b+c}{2} & d \end{bmatrix}. \text{ If } \ker(A_+) = \{0\}, \text{ then } A \text{ is strictly monotone}$$

by Fact 17.48. If  $\ker(A_+) \neq \{0\}$ , then by Fact 17.48  $\ker(A_+) \subseteq \ker(A)$ , and so  $\ker(A) \neq \{0\}$ , from which  $\det(A) = 0$  and  $ad = bc$ . Hence, since  $\det(A_+) = 0$ ,  $4bc = (b+c)^2$ , so  $(b-c)^2 = 0$  and  $b = c$ . Therefore  $A$  is symmetric. ■

*Remark 17.45.* The only paramonotone linear operators in  $\mathbb{R}^2$  that are not strictly monotone are the symmetric linear operators  $A := \begin{bmatrix} a & b \\ b & \frac{b^2}{a} \end{bmatrix}$  for  $a > 0$  and  $b \in \mathbb{R}$  and the zero operator  $x \mapsto (0, 0)$ . By Proposition 17.29, since both examples of  $A$  are symmetric linear operators, they are also maximal monotone and maximal cyclical monotone, as they are subdifferentials of proper lower semicontinuous convex functions.

All relationships among the classes of monotone linear operators in  $\mathbb{R}^2$  are now known completely and are summarized in Table 17.1. Recall that all monotone linear operators are assumed to have full domain and are therefore maximal monotone by Proposition 17.19.

### 17.6.2 Linear Operators on $\mathbb{R}^n$

On  $\mathbb{R}^n$  the restriction that linear operators are single-valued is redundant as this also follows from having full domain.

**Proposition 17.46.** *A single-valued monotone linear relation  $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is maximal monotone if and only if  $\text{dom}A = \mathbb{R}^n$ .*

*Proof.* In  $\mathbb{R}^n$ , all subspaces are closed, and so by Corollary 17.21, any maximal monotone single-valued linear relations have full domain. The converse follows from Proposition 17.19. ■

Since linear operators are maximal monotone, the following result is a consequence of Proposition 17.30 and appears in Remark 4.11 in [3].

**Proposition 17.47 ([3]).** *Given a monotone linear operator  $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $A$  is  $3^*$ -monotone if and only if  $A$  is paramonotone.*

In the following fact (from Proposition 3.2 in [24]), we denote by  $A_+ := \frac{1}{2}(A + A^*)$  the symmetric part of a linear operator  $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and by  $\ker A := \{x \in \mathbb{R}^n : Ax = 0\}$  the kernel of  $A$ .

**Fact 17.48 ([24]).** Let  $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear operator. Then  $A$  is paramonotone if and only if  $A$  is monotone and  $\ker(A_+) \subseteq \ker(A)$ .

In Remark 17.45 we noted that the converse of Proposition 17.10 holds for monotone linear operators that are not strictly monotone operators on  $\mathbb{R}^2$ . We now demonstrate that this result does not generalize to  $\mathbb{R}^3$ .

*Example 17.49.* Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the linear operator defined by

$$Tx := \begin{bmatrix} 1 & -2 & 1 \\ 3 & 1 & 3 \\ 1 & -2 & 1 \end{bmatrix} x. \tag{17.28}$$

The operator  $T$  is paramonotone and maximal monotone, but not strictly monotone. Further,  $T$  is not 3-cyclic monotone, but is  $3^*$ -monotone.

*Proof.* The symmetric part of  $T$  is

$$T_+ := \begin{bmatrix} 1 & 1/2 & 1 \\ 1/2 & 1 & 1/2 \\ 1 & 1/2 & 1 \end{bmatrix}.$$

Since the eigenvalues of  $T_+$ , consisting of  $\{0, \frac{1}{2}(3 + \sqrt{3}), \frac{1}{2}(3 - \sqrt{3})\}$ , are nonnegative,  $T_+$  is positive semidefinite, hence monotone, and so  $T$  is monotone.

An elementary calculation yields that  $\ker T_+ = \{t(-1, 0, 1) : t \in \mathbb{R}\}$ . Clearly,  $\ker T = \ker T_+$ , so by Fact 17.48,  $T$  is paramonotone. However,  $T$  is not strictly monotone since the kernel contains more than the zero element.

Furthermore,  $T$  is maximal monotone since it is linear and has full domain (Proposition 17.19). The operator  $T$  is not 3-cyclic monotone since the points  $(0, 0, 0)$ ,  $(1, 0, 0)$ , and  $(0, 1, 0)$  do not satisfy the defining condition (17.6). (For a shortcut, call to mind Example 17.40 and Proposition 17.39.) Finally, since  $T$  is a linear operator in  $\mathbb{R}^3$  that is paramonotone, it is  $3^*$ -monotone by Proposition 17.47. ■

### 17.6.3 Monotone Linear Operators in Infinite Dimensions

Recall from Proposition 17.47 that linear paramonotone operators on  $\mathbb{R}^n$  are  $3^*$  monotone. Example 17.50 below demonstrates that larger spaces are more permissive. A similar example appears in [7].

*Example 17.50.* Let  $\theta_k := \pi/2 - 1/k^4$  and let  $A : \ell^2 \rightarrow \ell^2$  be the linear operator defined by

$$A\mathbf{x} \mapsto \sum_{k=1}^{+\infty} (\cos(\theta_k)x_{2k-1} - \sin(\theta_k)x_{2k})\mathbf{e}_{2k-1} + (\sin(\theta_k)x_{2k-1} + \cos(\theta_k)x_{2k})\mathbf{e}_{2k}. \tag{17.29}$$

The structure of  $A$  is such that every  $\mathbf{x}^* = A\mathbf{x}$  obeys

$$\begin{bmatrix} x_{2k-1}^* \\ x_{2k}^* \end{bmatrix} = R_{\theta_k} \begin{bmatrix} x_{2k-1} \\ x_{2k} \end{bmatrix} \tag{17.30}$$

for all  $\mathbf{x} \in \ell^2$  and  $k \in \mathbb{N}$ , where  $R_{\theta_k}$  is the rotation matrix as introduced in Example 17.41.  $A$  is strictly monotone and maximal monotone, but not  $3^*$ -monotone. It follows that  $A$  is also paramonotone but not 3-cyclic monotone.

*Proof.* The monotonicity of  $T$  is evident from (17.30). Suppose that  $\mathbf{x} \in \ell^2$  is such that  $\langle \mathbf{x}, A\mathbf{x} \rangle = 0$ . Now,

$$\langle \mathbf{x}, A\mathbf{x} \rangle = \sum_{k=1}^{+\infty} \cos(\theta_k)(x_{2k-1}^2 + x_{2k}^2)$$

is equal to zero if and only if  $\mathbf{x} = \mathbf{0}$ , and so  $A$  is strictly monotone.

By Proposition 17.19,  $A$  is maximal monotone since it is linear and has full domain.

Let  $\mathbf{x} = \mathbf{0}$ , so that  $A\mathbf{x} = \mathbf{0}$ , and let  $\mathbf{z} = \sum_{k=1}^{+\infty} \frac{1}{k}(\mathbf{e}_{2k-1} + \mathbf{e}_{2k})$ . Define a sequence  $\mathbf{y}_n \in \ell^2$  by  $\mathbf{y}_n := n^2 \mathbf{e}_{2n-1}$ , and so  $A\mathbf{y}_n = n^2 \cos(\theta_n) \mathbf{e}_{2n-1} + n^2 \sin(\theta_n) \mathbf{e}_{2n}$ . For all  $n$ ,  $0 < \cos(\theta_n) \leq 1/n^4$ , and from the Taylor’s series  $\sin(\theta_n) \geq 1 - 1/(2n^8)$  for all large  $n$ . Considering the inequality (17.3) for  $3^*$ -monotonicity, we have

$$\begin{aligned} \langle \mathbf{z} - \mathbf{y}_n, A\mathbf{y}_n - A\mathbf{x} \rangle &= n(\cos(\theta_n) + \sin(\theta_n)) - n^4 \cos(\theta_n) \\ &\geq n(0 + 1 - 1/(2n^8)) - 1 \\ &\rightarrow +\infty, \quad \text{as } n \rightarrow +\infty, \end{aligned} \tag{17.31}$$

and so  $A$  fails to be  $3^*$ -monotone. ■

*Remark 17.51.* The operator  $A$  from Example 17.50 can be modified to lose its strict monotonicity property by using the zero function  $\mathbf{0} : \mathbb{R} \rightarrow \mathbb{R}$  as a prefactor in the product space, yielding  $T = \mathbf{0} \times A$ . In this manner,

$$T\mathbf{x} := \sum_{k=1}^{+\infty} \begin{bmatrix} (\cos(\theta_k)x_{2k} - \sin(\theta_k)x_{2k+1}) \mathbf{e}_{2k} \\ + (\sin(\theta_k)x_{2k} + \cos(\theta_k)x_{2k+1}) \mathbf{e}_{2k+1} \end{bmatrix}. \tag{17.32}$$

*Proof.* The Hilbert space  $\ell^2$  can be written as a product space  $\ell^2 = \mathbb{R} \times \ell^2$ . More precisely, all of these spaces can be embedded in the larger space  $\ell^2(\mathcal{Z})$  with standard unit vectors  $e_i$  for  $i$  in  $\mathcal{Z}$ , the set of integers. In this setting  $\ell^2 = \text{span}\{e_i : i \in \mathbb{N}\}$ , and let  $V_0 = \text{span}\{e_0\}$  so that  $\ell^2(\mathbb{N} \cup \{0\}) = V_0 \times \ell^2$ . Let  $T = \mathbf{0} \times A$ , where  $A$  is the linear operator from Example 17.50. The operator  $\mathbf{0} : V_0 \rightarrow V_0$  is paramonotone, maximal monotone, 3-cyclic monotone, and  $3^*$ -monotone, but not strictly monotone on  $\mathbb{R}$ . The operator  $A : \ell^2 \rightarrow \ell^2$  from Example 17.50 is strictly monotone and maximal monotone, but not  $3^*$ -monotone. Therefore, by the results of Sect. 17.3,  $T := \mathbf{0} \times A$  is paramonotone and maximal monotone and fails to be strictly monotone or  $3^*$ -monotone. ■

Note that all linear operators are assumed to have full domain and are therefore maximal monotone by Proposition 17.19. Also, if a linear operator fails to be paramonotone, it fails to be  $3^*$ -monotone and 3-cyclic monotone as well. The monotone class characterizations for linear operators in a Hilbert space are now known completely, as summarized in Table 17.2 below.

Since the only linear operators on  $\mathbb{R}$  are the constant operators, by the results shown in Table 17.1 and by Proposition 17.47, each example in Table 17.2 operates on a space with the lowest dimension for which its monotone class combination is possible. In particular, note how examples with binary label 1000, 1001, and 1100, although absent in Table 17.1, exist for spaces of higher dimension. Finally,



for every operator  $T$  in Table 17.2, an operator with the same monotone class combination on any higher dimension can be constructed by a product space composition with  $\text{Id}$  ( $T \times \text{Id}$ ).

## 17.7 Summary

The relationships among the five classes of monotone linear operator considered, that is, maximal, para-,  $3^*$ -, 3-cyclic, and strictly monotone operators, are now fully understood in  $\mathbb{R}^2$ ,  $\mathbb{R}^n$ , and in general Hilbert spaces. They are depicted in each case with the Venn diagrams below. Further, a sample linear monotone operator has been provided for every possible combination of monotone class. In Sect. 17.3, a method by which these examples can be combined and extended to create linear operators in higher dimension with a known monotone class configuration has been described. Various properties and monotone class relationships of linear relations have been explored above. Furthermore, only two monotone class relationships do not apply for linear relations: a linear relation may be  $3^*$ -monotone and not paramonotone (as in Example 17.38), and 3-cyclic monotone linear relations that are not paramonotone could exist, but they must not be maximal monotone.

Some of the results and examples in this paper were presented at a meeting of the Canadian Mathematical Society in Vancouver, Canada, on December 4, 2010, and a similar and complete characterization of these same monotone class relationships for nonlinear operators will appear shortly.

**Acknowledgements** The author would like to thank the professional and editorial support of Philip Loewen.

## References

1. Attouch, H., Damlamian, A.: On multivalued evolution equations in Hilbert spaces. *Israel J. Math.* **12**(4), 373–390 (1972)
2. Auslender, A.A., Haddou, M.: An interior-proximal method for convex linearly constrained problems and its extension to variational inequalities. *Math. Program.* **71**, 77–100 (1995)
3. Bauschke, H.H., Borwein, J.M., Wang, X.: Fitzpatrick functions and continuous linear monotone operators. *SIAM J. Optim.* **18**, 789–809 (2007)
4. Bauschke, H.H., Combettes, P.L.: *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*. Springer, New York (2011)
5. Bauschke, H.H., Wang, W., Yao, L.: Monotone linear relations: maximality and Fitzpatrick functions. *J. Convex Anal.* **25**, 673–686 (2009)
6. Bauschke, H.H., Wang, X., Yao, L.: On Borwein-Wiersma decompositions of monotone linear relations. *SIAM J. Optim.* **20**, 2636–2652 (2010)
7. Bauschke, H.H., Wang, X., Yao, L.: Rectangularity and paramonotonicity of maximally monotone operators. *Optimization*. In press

8. Bello Cruz, J.Y., Iusem, A.N.: Convergence of direct methods for paramonotone variational inequalities. *Comput. Optim. Appl.* **46**(2), 247–263 (2010)
9. Borwein, J.M.: Maximal monotonicity via convex analysis. *J. Convex Anal.* **13**(3), 561–586 (2006)
10. Bressan, A., Staicu, V.: On nonconvex perturbations of maximal monotone differential inclusions. *Set-Valued Var. Anal.* **2**, 415–437 (1994)
11. Brézis, H., Haraux, A.: Image d'une somme d'opérateurs monotones et applications. *Israel J. Math.* **23**(2), 165–186 (1976)
12. Bruck, R.E., Jr.: An iterative solution of a variational inequality for certain monotone operators in Hilbert space. *Bull. Amer. Math. Soc.* **81**(5), 890–892 (1975)
13. Burachik, R.S., Iusem, A.N.: An iterative solution of a variational inequality for certain monotone operators in a Hilbert space. *SIAM J. Optim.* **8**, 197–216 (1998)
14. Burachik, R.S., Lopes, J.O., Svaiter, B.F.: An outer approximation method for the variational inequality problem. *SIAM J. Control Optim.* **43**, 2071–2088 (2005)
15. Censor, Y., Iusem, A.N., Zenios, S.A.: An interior point method with Bregman functions for the variational inequality problem with paramonotone operators. *Math. Program.* **81**(3), 373–400 (1998)
16. Chu, L.-J.: On the sum of monotone operators. *Michigan Math. J.* **43**, 273–289 (1996)
17. Cross, R.: *Monotone Linear Relations*. M. Dekker, New York (1998)
18. Eckstein, J., Bertsekas, D.P.: On the Douglas-Rachford splitting method and the proximal point algorithm for maximal monotone operators. *Math. Program.* **55**, 293–318 (1992)
19. Facchinei, F., Pang, J.S.: *Finite-dimensional Variational Inequalities and Complementarity Problems*. Springer, New York (2003)
20. Ferris, M.C., Pang, J.S.: Engineering and economic applications of complementarity problems. *SIAM Rev.* **39**(4), 669–713 (December 1997)
21. Ghoussoub, N.: *Selddual partial differential systems and their variational principles*. Springer Monographs in Mathematics, vol. 14. Springer, New York (2010)
22. Hadjisavvas, N., Schaible, S.: On a generalization of paramonotone maps and its application to solving the stampacchia variational inequality. *Optimization* **55**(5-6), 593–604 (October-December 2006)
23. Hartmann, P., Stampacchia, G.: On some non-linear elliptic differential-functional equations. *Acta Math.* **115**(1), 271–310 (1966)
24. Iusem, A.N.: On some properties of paramonotone operators. *J. Convex Anal.* **5**(2), 269–278 (1998)
25. Minty, G.J.: Monotone (nonlinear) operators in a Hilbert space. *Duke Math. J.* **29**, 341–346 (1962)
26. Papageorgiou, N.S., Shahzad, N.: On maximal monotone differential inclusions in  $\mathbb{R}^n$ . *Acta Math. Hungar.* **78**(3), 175–197 (1998)
27. Pennanen, T.: *Dualization of monotone generalized equations*, PhD thesis. University of Washington, Seattle, Washington (1999)
28. Rockafellar, R.T., Wets, R.J.-B.: *Variational analysis*. Grundlehren der Mathematischen Wissenschaften, vol. 317, 2nd edn. Springer, Berlin (2004)
29. Simons, S.: LC functions and maximal monotonicity. *J. Nonlinear Convex Anal.* **7**, 123–138 (2006)
30. Yagi, A.: Generation theorem of semigroup for multivalued linear operators. *Osaka J. Math.* **28**, 385–410 (1991)
31. Yamada, I., Ogura, N.: Hybrid steepest descent method for variational inequality problems over the fixed point set of certain quasi-nonexpansive mappings. *Numer. Funct. Anal. Optim.* **25**, 619–655 (2004)
32. Zeidler, E.: *Nonlinear Functional Analysis and Its Applications, II/B - Nonlinear Monotone Operators*. Springer, New York (1990)