Some Aspects of the Algebraic Theory of Quadratic Forms

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Abstract This article, based on the lectures at the Arizona Winter School on "Quadratic forms", gives a quick introduction to the algebraic theory of quadratic forms. It discusses some invariants associated to quadratic forms like the Pythagoras number and the u-invariant and touches on some recent progress on these topics.

Key words and Phrases Quadratic forms • Galois cohomology • Invariants • Number fields • Function fields

Mathematics Subject Classification (2010): 11E81, 14G27

This text is based on the lectures given at the Arizona Winter School on "Quadratic forms". The aim of the text is to give a brief introduction to the algebraic theory of quadratic forms. We explain invariants associated to quadratic forms—invariants with values in Galois cohomology as well as numerical invariants. We explain some open questions concerning these invariants and recent progress related to these questions.

There are many good references for this material on the algebraic theory of quadratic forms including [EKM, K, L, Pf] and [S].

The author was partially supported by NSF grant DMS-1001872.

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K. Alladi et al. (eds.), *Quadratic and Higher Degree Forms*, Developments in Mathematics 31, DOI 10.1007/978-1-4614-7488-3_7,

1 Quadratic Forms

Let k be a field with $\operatorname{char} k \neq 2$.

Definition 1.1. A **quadratic form** $q\colon V \to k$ on a vector space V over k is a map satisfying:

(1) $q(\lambda v) = \lambda^2 q(v)$ for $v \in V$, $\lambda \in k$.
(2) The map $b_q\colon V \times V \to k$, defined by

$$b_q(v,w) = \frac{1}{2}[q(v+w) - q(v) - q(w)]$$

is bilinear.

We denote a quadratic form by (V, q), or simply by q. Throughout, we restrict ourselves to the study of quadratic forms on finite-dimensional vector spaces.

The bilinear form b_q is symmetric; q determines b_q and for all $v \in V$, $q(v) = b_q(v,v)$.

For a choice of basis $\{e_1, \ldots, e_n\}$ of V, b_q is represented by a symmetric matrix $A(q) = (a_{ij})$ with $a_{ij} = b_q(e_i, e_j)$. If $v = \sum_{1 \le i \le n} X_i e_i \in V$, $X_i \in k$, then

$$q(v) = \sum_{1 \le i,j \le n} a_{ij} X_i X_j = \sum_{1 \le i \le n} a_{ii} X_i^2 + 2 \sum_{i<j} a_{ij} X_i X_j.$$

Thus q is represented by a homogeneous polynomial of degree 2. Clearly, every homogeneous polynomial of degree 2 corresponds to a quadratic form on V with respect to the chosen basis.

Definition 1.2. Two quadratic forms (V_1, q_1), (V_2, q_2) are **isometric** if there is an isomorphism $\phi\colon V_1 \xrightarrow{\sim} V_2$ such that $q_2(\phi(v)) = q_1(v)$ for all $v \in V_1$.

If $A(q_1)$, $A(q_2)$ are the matrices representing q_1 and q_2 with respect to bases B_1 and B_2 of V_1 and V_2 respectively, ϕ yields a matrix $T \in GL_n(k)$, $n = \dim V$, such that

$$T A(q_2)\, T^t = A(q_1).$$

In other words, the symmetric matrices $A(q_1)$ and $A(q_2)$ are congruent. Thus isometry classes of quadratic forms yield congruence classes of symmetric matrices.

Definition 1.3. The form $q\colon V \to k$ is said to be **regular** if $b_q\colon V \times V \to k$ is nondegenerate.

Thus q is regular if and only if the map $V \to V^* = \operatorname{Hom}(V, k)$, defined by $v \mapsto (w \mapsto b_q(v,w))$, is an isomorphism. This is the case if $A(q)$ is invertible.

Let (V, q) be a quadratic form. Then

$$V_0 = \{v \in V \,:\, b_q(v,w) = 0 \text{ for all } w \in V\}$$

is called the **radical** of V. If V_1 is any complementary subspace of V_0 in V, then $q|_{V_1}$ is regular and $(V, q) = (V_0, 0) \perp (V_1, q|_{V_1})$. Note that V is regular if and only if the radical of V is zero.

Henceforth, we shall only be concerned with regular quadratic forms.

Definition 1.4. Let W be a subspace of V and $q\colon V \to k$ be a quadratic form. The **orthogonal complement** of W denoted $W^\perp$ is the subspace

$$W^\perp = \{v \in V : b_q(v, w) = 0 \text{ for all } w \in W\}.$$

Exercise 1.5. Let (V, q) be a regular quadratic form and W a subspace of V. Then

(1) $\dim(W) + \dim(W^\perp) = \dim(V)$.
(2) $(W^\perp)^\perp = W$.

1.1 Orthogonal Sums

Let (V_1, q_1), (V_2, q_2) be quadratic forms. The form

$$(V_1, q_1) \perp (V_2, q_2) = (V_1 \oplus V_2, q_1 \perp q_2),$$

with $q_1 \perp q_2$ defined by

$$(q_1 \perp q_2)(v_1, v_2) = q_1(v_1) + q_2(v_2),\ v_1 \in V_1,\ v_2 \in V_2$$

is called the *orthogonal sum* of (V_1, q_1) and (V_2, q_2).

1.2 Diagonalization

Let (V, q) be a quadratic form. There exists a basis $\{e_1, \ldots, e_n\}$ of V such that $b_q(e_i, e_j) = 0$ for $i \neq j$. Such a basis is called an *orthogonal basis* for q. With respect to an orthogonal basis, b_q is represented by a diagonal matrix.

If $\{e_1, \ldots, e_n\}$ is an orthogonal basis of q and $q(e_i) = d_i$, we write $q = \langle d_1, \ldots, d_n\rangle$. In this case, $V = ke_1 \oplus \cdots \oplus ke_n$ is an orthogonal sum and $q|_{ke_i}$ is represented by $\langle d_i\rangle$. Thus every quadratic form is diagonalizable.

1.3 Hyperbolic Forms

Definition 1.6. A quadratic form (V, q) is said to be **isotropic** if there is a nonzero $v \in V$ such that $q(v) = 0$. It is **anisotropic** if q is not isotropic. A quadratic form (V, q) is said to be **universal** if it represents every element of k; i.e., given $\lambda \in k$, there is a vector $v \in V$ such that $q(v) = \lambda$.

Example 1.7. The quadratic form $X^2 - Y^2$ is isotropic over k. Suppose (V, q) is a regular form which is isotropic. Let $v \in V$ be such that $q(v) = 0$, $v \neq 0$. Since q is regular, there exists $w \in V$ such that $b_q(v, w) \neq 0$. After scaling we may assume $b_q(v, w) = 1$. If $q(w) \neq 0$, we may replace w by $w + \lambda v$, $\lambda = -\frac{1}{2}q(w)$, and assume that $q(w) = 0$. Thus $W = kv \oplus kw$ is a 2-dimensional subspace of V and $q|_W$ is represented by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ with respect to $\{v, w\}$.

Definition 1.8. A binary quadratic form isometric to $(k^2, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))$ is called a **hyperbolic plane**. A quadratic form (V, q) is **hyperbolic** if it is isometric to an orthogonal sum of hyperbolic planes. A subspace W of V such that q restricts to zero on W and $\dim W = \frac{1}{2} \dim V$ is called a **Lagrangian**.

Every regular quadratic form which admits a Lagrangian can easily be seen to be hyperbolic.

Exercise 1.9. Let (V, q) be a regular quadratic form and $(W, q|_W)$ a regular form on the subspace W. Then $(V, q) = (W, q|_W) \perp (W^\perp, q|_{W^\perp})$.

Theorem 1.10 (Witt's Cancellation Theorem). *Let* (V_1, q_1), (V_2, q_2), (V, q) *be quadratic forms over* k. *Suppose*

$$(V_1, q_1) \perp (V, q) \cong (V_2, q_2) \perp (V, q).$$

Then $(V_1, q_1) \cong (V_2, q_2)$.

The key ingredient of Witt's cancellation theorem is the following.

Proposition 1.11. *Let* (V, q) *be a quadratic form and* $v, w \in V$ *with* $q(v) = q(w) \neq 0$. *Then there is an isometry* $\tau\colon (V, q) \xrightarrow{\sim} (V, q)$ *such that* $\tau(v) = w$.

Proof. Let $q(v) = q(w) = d \neq 0$. Then

$$q(v + w) + q(v - w) = 2q(v) + 2q(w) = 4d \neq 0.$$

Thus $q(v + w) \neq 0$ or $q(v - w) \neq 0$. For any vector $u \in V$ with $q(u) \neq 0$, define $\tau_u\colon V \to V$ by

$$\tau_u(z) = z - \frac{2b_q(z, u)u}{q(u)}.$$

τ_u is an isometry called the *reflection with respect to* u.

Suppose $q(v - w) \neq 0$. Then $\tau_{v-w}\colon V \to V$ is an isometry of V which sends v to w. Suppose $q(v + w) \neq 0$. Then $\tau_w \circ \tau_{v+w}$ sends v to w. □

Remark 1.12. The orthogonal group of (V, q) denoted by $O(q)$ is the set of isometries of V onto itself. This group is generated by reflections. This is seen by an inductive argument on $\dim(q)$, using the above proposition.

Theorem 1.13 (Witt's decomposition). *Let* (V, q) *be a quadratic form (not necessarily regular). Then there is a decomposition*

$$(V, q) = (V_0, 0) \perp (V_1, q_1) \perp (V_2, q_2)$$

where V_0 is the radical of q, $q_1 = q|_{V_1}$ is anisotropic and $q_2 = q|_{V_2}$ is hyperbolic. If $(V, q) = (V_0, 0) \perp (W_1, f_1) \perp (W_2, f_2)$ with f_1 anisotropic and f_2 hyperbolic, then

$$(V_1, q_1) \cong (W_1, f_1),\ (V_2, q_2) \cong (W_2, f_2).$$

Remark 1.14. A hyperbolic form (W, f) is determined by $\dim(W)$; for if $\dim(W) = 2n$, $(W, f) \cong nH$, where $H = (k^2, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))$ is the hyperbolic plane.

From now on, we shall assume (V, q) is a regular quadratic form. We denote by q_{an} the quadratic form (V_1, q_1) in Witt's decomposition which is determined by q up to isometry. We call $\frac{1}{2}\dim(V_2)$ the *Witt index* of q. Thus any regular quadratic form q admits a decomposition $q \cong q_{an} \perp (nH)$, with q_{an} anisotropic and H denoting the hyperbolic plane. We also sometimes denote by H^n the sum of n hyperbolic planes.

2 Witt Group of Forms

2.1 Witt Groups

We set

$$W(k) = \{\text{isomorphism classes of regular quadratic forms over } k\}/\sim$$

where the Witt equivalence $\sim$ is given by:

$$(V_1, q_1) \sim (V_2, q_2) \iff \begin{array}{l}\text{there exist } r,\ s \in \mathbb{Z} \text{ such that} \\ (V_1, q_1) \perp H^r \cong (V_2, q_2) \perp H^s\end{array}.$$

$W(k)$ is a group under orthogonal sum:

$$[(V_1, q_1)] \perp [(V_2, q_2)] = [(V_1, q_1) \perp (V_2, q_2)].$$

The zero element in $W(k)$ is represented by the class of hyperbolic forms. For a regular quadratic form (V, q), $(V, q) \perp (V, -q)$ has Lagrangian

$$W = \{(v, v) \,:\, v \in V\}$$

so that $(V, q) \perp (V, -q) \cong H^n$, $n = \dim(V)$. Thus, $[(V, -q)] = -[(V, q)]$ in $W(k)$.

It follows from Witt's decomposition theorem that every element in $W(k)$ is represented by a unique anisotropic quadratic form up to isometry. Thus $W(k)$ may be thought of as a group made out of isometry classes of anisotropic quadratic forms over k.

The abelian group $W(k)$ admits a ring structure induced by tensor product on the associated bilinear forms. For example, if $q_1 \cong \langle a_1, \ldots, a_n\rangle$ and q_2 is a quadratic form, then $q_1 \otimes q_2 \cong a_1q_2 \perp a_2q_2 \perp \cdots \perp a_nq_2$.

Definition 2.1. Let $I(k)$ denote the ideal of classes of even-dimensional quadratic forms in $W(k)$. The ideal $I(k)$ is called the **fundamental ideal**. $I^n(k)$ stands for the nth power of the ideal $I(k)$.

Definition 2.2. Let $P_n(k)$ denote the set of isomorphism classes of forms of the type

$$\langle\langle a_1, \ldots, a_n\rangle\rangle := \langle 1, a_1\rangle \otimes \cdots \otimes \langle 1, a_n\rangle.$$

Elements in $P_n(k)$ are called n**-fold Pfister forms**.

The ideal $I(k)$ is generated additively by the forms $\langle 1, a\rangle$, $a \in k^*$. Moreover, the ideal $I^n(k)$ is generated additively by n-fold Pfister forms. For instance, for $n = 2$, the generators of $I^2(k)$ are of the form

$$\langle a, b\rangle \otimes \langle c, d\rangle \cong \langle 1, ac, ad, cd\rangle - \langle 1, cd, -bc, -bd\rangle = \langle\langle ac, ad\rangle\rangle - \langle\langle cd, -bc\rangle\rangle$$

Example 2.3. If $k = \mathbb{C}$, every 2-dimensional quadratic form over k is isotropic.

$$W(k) \cong \mathbb{Z}/2\mathbb{Z}$$

$$[(V, q)] \mapsto \dim(V) \pmod 2$$

is an isomorphism.

Example 2.4. Let $k = \mathbb{F}_{p^n}$, $p \neq 2$, be a finite field. Then $k^* = k \setminus \{0\}$ has two square classes, $\{1, u\}$. Every 3-dimensional quadratic form over k is isotropic. Further, $W(k) \cong \mathbb{Z}/4\mathbb{Z}$ if -1 is not a square in $\mathbb{F}_{p^n}$ and $W(k) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ if -1 is a square in $\mathbb{F}_{p^n}$ (cf. [L], Corollary 3.6).

Example 2.5. If $k = \mathbb{R}$, every quadratic form q is represented by

$$\langle 1, \ldots, 1, -1, \ldots, -1\rangle$$

with respect to an orthogonal basis. The number r of $+1$'s and the number s of -1's in the diagonalization above are uniquely determined by the isomorphism class of q. The *signature* of q is defined as $r - s$. The signature yields a homomorphism $\mathrm{sgn}\colon W(\mathbb{R}) \to \mathbb{Z}$ which is an isomorphism.

2.2 Quadratic Forms Over p-Adic Fields

Let k be a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers. We call k a *non-dyadic* p-adic field if $p \neq 2$. The field k has a discrete valuation v *extending* the p-adic valuation on $\mathbb{Q}_p$. Let π be a uniformizing parameter for v and κ the residue field for v. The field κ is a finite field of characteristic $p \neq 2$. Let u be a unit in k^* such that $\overline{u} \in \kappa$ is not a square. Then

$$k^*/{k^*}^2 = \{1, u, \pi, u\pi\}.$$

Since κ is finite, every 3-dimensional quadratic form over κ is isotropic. By Hensel's lemma, every 3-dimensional form $\langle u_1, u_2, u_3 \rangle$ over k, with u_i units in k is isotropic. Since every form q in k has a diagonal representation

$$\langle u_1, \ldots, u_r \rangle \perp \pi \langle v_1, \ldots, v_s \rangle,$$

if r or s exceeds 3, q is isotropic. In particular every 5-dimensional quadratic form over k is isotropic. Further, up to isometry, there is a unique quadratic form in dimension 4 which is anisotropic, namely,

$$\langle 1, -u, -\pi, u\pi \rangle.$$

This is the norm form of the unique quaternion division algebra $H(u, \pi)$ over k (cf. Sect. 2.3).

2.3 Central Simple Algebras and the Brauer Group

Recall that a finite-dimensional algebra A over a field k is a *central simple algebra* over k if A is simple (has no two-sided ideals) and the center of A is k. Recall also that for a field k,

$$\mathrm{Br}(k) = \{\text{Isomorphism classes of central simple algebras over } k\} \,/ \sim$$

where the Brauer equivalence $\sim$ is given by: $A \sim B$ if and only if $M_n(A) \cong M_m(B)$ for some integers m, n. The pair $(\mathrm{Br}(k), \otimes)$ is a group. The inverse of $[A]$ is $[A^{\mathrm{op}}]$ where A^{op} is the *opposite algebra* of A: the multiplication structure, $*$, on A^{op} is given by $a * b = ba$. We have a k-algebra isomorphism $\phi\colon A \otimes A^{\mathrm{op}} \xrightarrow{\sim} \mathrm{End}_k(A)$ induced by $\phi(a \otimes b)(c) = acb$. The identity element in $\mathrm{Br}(k)$ is given by $[k]$. By Wedderburn's theorem on central simple algebras, the elements of $\mathrm{Br}(k)$ parametrize the isomorphism classes of finite-dimensional central division algebras over k.

For elements $a, b \in k^*$, we define the **quaternion algebra** $H(a, b)$ to be the 4-dimensional central simple algebra over k generated by $\{i, j\}$ with the relations $i^2 = a$, $j^2 = b$, $ij = -ji$. This is a generalization of Hamilton's quaternion algebra

$H(-1,-1)$ over the field of real numbers. The algebra $H(a,b)$ admits a canonical involution$^-$: $H(a,b) \to H(a,b)$ given by

$$\overline{\alpha + i\beta + j\gamma + ij\delta} = \alpha - i\beta - j\gamma - ij\delta$$

This involution gives an isomorphism $H(a,b) \cong H(a,b)^{\mathrm{op}}$; in particular, $H(a,b)$ has order 2 in $\mathrm{Br}(k)$. Let ${}_2\mathrm{Br}(k)$ denote the 2-torsion subgroup of the Brauer group of k. The norm form for this algebra is given by $N(x) = x\overline{x}$, which is a quadratic form on $H(a,b)$ represented with respect to the orthogonal basis $\{1, i, j, ij\}$ by $\langle 1, -a, -b, ab\rangle = \langle\langle -a, -b\rangle\rangle$.

2.4 Classical Invariants for Quadratic Forms

Let (V,q) be a regular quadratic form. We define $\dim(q) = \dim(V)$ and $\dim_2(q) = \dim(V)$ modulo 2. We have a ring homomorphism $\dim_2\colon W(k) \to \mathbb{Z}/2\mathbb{Z}$. We note that $I(k)$ is the kernel of $\dim_2$. This gives an isomorphism

$$\dim_2\colon W(k)/I(k) \xrightarrow{\sim} \mathbb{Z}/2\mathbb{Z}.$$

Let $\mathrm{disc}(q) = (-1)^{n(n-1)/2}[\det(A(q))] \in k^*/k^{*2}$. Since $A(q)$ is determined up to congruence, $\det(A(q))$ is determined modulo squares. We have $\mathrm{disc}(H) = 1$, where H is the hyperbolic plane. The discriminant induces a group homomorphism

$$\mathrm{disc}\colon I(k) \to k^*/k^{*2}$$

which is clearly onto. It is easy to verify that $\ker(\mathrm{disc}) = I^2(k)$. Thus the discriminant homomorphism induces an isomorphism $I(k)/I^2(k) \to k^*/k^{*2}$.

Example 2.6. Let $\langle a,b\rangle$ be a binary quadratic form. Then $\mathrm{disc}\langle a,b\rangle = -ab$. The discriminant is trivial if and only if $\langle a,b\rangle \cong \langle 1,-1\rangle$ is a hyperbolic plane. Further, if $\langle a,b\rangle$ represents a value $c \in k^*$, then $\langle a,b\rangle \cong \langle c, abc\rangle$.

The next invariant for quadratic forms is the Clifford invariant. To each quadratic form (V,q) we wish to construct a central simple algebra containing V whose multiplication on elements of V satisfies $v \cdot v = q(v)$. The smallest such algebra (defined by a universal property) will be the Clifford algebra.

Definition 2.7. The **Clifford algebra** $C(q)$ of the quadratic form (V,q) is $T(V)/I_q$, where I_q is the two-sided ideal in the tensor algebra $T(V)$ generated by $\{v \otimes v - q(v) \mid v \in V\}$.

The algebra $C(q)$ has a $\mathbb{Z}/2\mathbb{Z}$ gradation $C(q) = C_0(q) \oplus C_1(q)$ induced by the gradation $T(V) = T_0(V) \oplus T_1(V)$, where

$$T_0(V) = \bigoplus_{i\geq 0,\ i \text{ even}} V^{\otimes i} \qquad \text{and} \qquad T_1(V) = \bigoplus_{i\geq 1,\ i\text{odd}} V^{\otimes i}.$$

If $\dim(q)$ is even, then $C(q)$ is a central simple algebra over k. If $\dim(q)$ is odd, $C_0(q)$ is a central simple algebra over k. The Clifford algebra $C(q)$ comes equipped with an involution τ defined by $\tau(v) = -v$ for $v \in V$. Thus, if $\dim(q)$ is even, $C(q)$ determines a 2-torsion element in $\mathrm{Br}(k)$.

Definition 2.8. The **Clifford invariant** $c(q)$ of (V, q) in $\mathrm{Br}(k)$ is defined as

$$c(q) = \begin{cases} [C(q)], & \text{if } \dim(q) \text{ is even} \\ [C_0(q)], & \text{if } \dim(q) \text{ is odd} \end{cases}$$

Example 2.9. Let $q \cong \otimes_{i=1}^{n} \langle\langle -a_i, -b_i \rangle\rangle \in I^2(k)$. Then

$$c(q) = [\otimes_{1 \leq i \leq n} H_i]$$

where $H_i = H(a_i, b_i)$.

Exercise 2.10. Given $\bigotimes_{1 \leq i \leq n} H_i$, a tensor product of n quaternion algebras over k, show that there is a quadratic form q over k **of dimension $\mathbf{2n + 2}$** such that $c(q) = [\bigotimes_{1 \leq i \leq n} H_i]$.

The Clifford invariant induces a homomorphism $c\colon I^2(k) \to_2 \mathrm{Br}(k)$, ${}_2\mathrm{Br}(k)$ denoting the 2-torsion in the Brauer group of k. The very first case of the Milnor conjecture (see Sect. 3) states: c is surjective and $\ker(c) = I^3(k)$.

Theorem 2.11 (Merkurjev [M1]). *The map c induces an isomorphism*

$$I^2(k)/I^3(k) \cong_2 \mathrm{Br}(k)$$

Thus the image of $I^2(q)$ in ${}_2\mathrm{Br}(k)$ is spanned by quaternion algebras. It was a longstanding question whether ${}_2\mathrm{Br}(k)$ is spanned by quaternion algebras. Merkurjev's theorem answers this question in the affirmative; further, it gives precise relations between quaternion algebras in ${}_2\mathrm{Br}(k)$.

3 Galois Cohomology and the Milnor Conjecture

Let $\bar{k}$ be a separable closure of k. Let $\Gamma_k = \mathrm{Gal}(\bar{k}|k)$ be the absolute Galois group of k. The group Γ_k is a profinite group:

$$\Gamma_k = \varprojlim_{L \subset \bar{k},\ L/k \text{ finite Galois}} \mathrm{Gal}(L/k).$$

A *discrete Γ_k-module M* is a continuous Γ_k-module for the discrete topology on M and the profinite topology on Γ_k. A Γ_k-module M is discrete if and only if the stabilizer of each $m \in M$ is an open subgroup, in particular, of finite index in Γ_k. For a discrete Γ_k-module M, we define $H^n(k, M)$ as the direct limit of the cohomology of the finite quotients

$$H^n(k, M) = \varinjlim_{L \subset \bar{k},\ L/k \text{ finite Galois}} H^n(\mathrm{Gal}(L/k), M^{\Gamma_L}).$$

Suppose $\mathrm{char}(k) \neq 2$ and $M = \mu_2$. The module μ_2 has trivial Γ_k action and is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. We have

$$H^0(k, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$$
$$H^1(k, \mathbb{Z}/2\mathbb{Z}) \cong k^*/k^{*2}$$
$$H^2(k, \mathbb{Z}/2\mathbb{Z}) \cong_2 \mathrm{Br}(k)$$

These can be seen from the Kummer exact sequence of Γ_k-modules:

$$0 \longrightarrow \mu_2 \longrightarrow \bar{k}^* \xrightarrow{\cdot 2} \bar{k}^* \longrightarrow 0$$

and noting that $H^1(\Gamma_k, \bar{k}^*) = 0$ (Hilbert's Theorem 90) and $H^2(\Gamma_k, \bar{k}^*) = \mathrm{Br}(k)$.

For an element $a \in k^*$, we denote by (a) its class in $H^1(k, \mathbb{Z}/2\mathbb{Z})$ and for $a_1, \dots, a_n \in k^*$, the cup product $(a_1) \cup \cdots \cup (a_n) \in H^n(k, \mathbb{Z}/2\mathbb{Z})$ is denoted by $(a_1) \cdot \cdots \cdot (a_n)$.

For $a, b \in k^*$, the element $(a).(b)$ represents the class of $H(a, b)$ in ${}_2\mathrm{Br}(k)$. The map

$$c\colon I^2(k) \to H^2(k, \mathbb{Z}/2\mathbb{Z})$$

sends $\langle 1, -a, -b, ab\rangle$ to the class of $H(a, b)$ in $H^2(k, \mathbb{Z}/2\mathbb{Z})$. The forms $\langle 1, -a, -b, ab\rangle$ additively generate $I^2(k)$. Merkurjev's theorem asserts that $H^2(k, \mathbb{Z}/2\mathbb{Z})$ is generated by $(a).(b)$, with $a, b \in k^*$. The Milnor conjecture (quadratic form version) proposes higher invariants $I^n(k) \to H^n(k, \mathbb{Z}/2\mathbb{Z})$ extending the classical invariants.

Milnor Conjecture. *The assignment*

$$\langle 1, a_1\rangle \otimes \cdots \otimes \langle 1, a_n\rangle \mapsto (a_1) \cdot \cdots \cdot (a_n)$$

yields a map $e_n\colon P_n(k) \to H^n(k, \mathbb{Z}/2\mathbb{Z})$. *This map extends to a homomorphism* $e_n\colon I^n(k) \to H^n(k, \mathbb{Z}/2\mathbb{Z})$ *which is onto and* $\ker(e_n) = I^{n+1}(k)$.

The maps *dimension mod* 2, *discriminant* and *Clifford invariant* coincide with e_0, e_1 and e_2. Unlike these classical invariants, which are defined on all quadratic forms, conjecturally e_n, $n \geq 3$, are defined only on elements in $I^n(k)$ on which the invariants e_i, $i \leq n-1$, vanish. In 1975, Arason [Ar] proved that $e_3\colon I^3(k) \to H^3(k, \mathbb{Z}/2\mathbb{Z})$ is well defined and is one-one on $P_3(k)$. As we mentioned earlier, the first nontrivial case of the Milnor conjecture was proved by Merkurjev for $n = 2$. The Milnor conjecture (quadratic form version) is now a theorem due to Orlov–Vishik–Voevodsky [OVV].

The Milnor conjecture gives a classification of quadratic forms by their Galois cohomology invariants: Given anisotropic quadratic forms q_1 and q_2, suppose $e_i(q_1 \perp -q_2) = 0$ for $i \geq 0$. Then $q_1 = q_2$ in $W(k)$. We need only to verify $e_i(q_1 \perp -q_2) = 0$ for $i \leq N$ where $N \leq 2^n$ and $\dim(q_1 \perp -q_2) \leq 2^n$, by the following theorem of Arason and Pfister.

Theorem 3.1 (Arason–Pfister Hauptsatz). *Let k be a field. The dimension of an anisotropic quadratic form in $I^n(k)$ is at least 2^n.*

4 Pfister Forms

The theory of Pfister forms (or multiplicative forms, as Pfister called them) evolved from questions on classification of quadratic forms whose nonzero values form a group (hereditarily).

Definition 4.1. A regular quadratic form q over k is called **multiplicative** if the nonzero values of q over any extension field L over k form a group.

We have the following examples of quadratic forms which are multiplicative.

Example 4.2. $\langle 1 \rangle$: nonzero squares are multiplicatively closed in k^*.

Example 4.3. $\langle 1, -a \rangle$: $x^2 - ay^2$, $a \in k^*$ is the norm from the quadratic algebra $k[t]/(t^2 - a)$ over k and the norm is multiplicative.

Example 4.4. $\langle 1, -a \rangle \otimes \langle 1, -b \rangle$: $x^2 - ay^2 - bz^2 + abt^2$ is a norm form from the quaternion algebra $H(a, b)$: $N(\alpha + i\beta + j\gamma + ij\delta) = \alpha^2 - a\beta^2 - b\gamma^2 + ab\delta^2$. The norm once again is multiplicative.

Example 4.5. $\langle 1, -a \rangle \otimes \langle 1, -b \rangle \otimes \langle 1, -c \rangle$: $(x^2 - ay^2 - bz^2 + abt^2) - c(u^2 - av^2 - bw^2 + abs^2)$ is the norm form from an octonion algebra associated to the triple (a, b, c); it is a non-associative algebra obtained from the quaternion algebra $H(a, b)$ by a doubling process (see [J, Sect. 7.6]). The norm is once again multiplicative.

Theorem 4.6 (Pfister). *An anisotropic quadratic form q over k is multiplicative if and only if q is isomorphic to a Pfister form.*

We shall sketch a proof of this theorem. The main ingredients are the Cassels–Pfister Theorem 4.7 and the Subform Theorem 4.10, which will not be proved in the text. We refer to [L, Chap. IX, Theorems 1.3 and 2.8] for the proofs.

Theorem 4.7 (Cassels–Pfister). *Let $q = \langle a_1, \ldots, a_n \rangle$ be a regular quadratic form over k and $f(X) \in k[X]$, a polynomial over k which is a value of q over $k(X)$. Then there exist polynomials $g_1, \ldots, g_n \in k[X]$ such that $f(X) = a_1 g_1^2 + \cdots + a_n g_n^2$.*

Corollary 4.8 (Specialization Lemma). *Let $q = \langle a_1, \ldots, a_n \rangle$ be a quadratic form over k, $X = \{X_1, \ldots, X_n\}$, $p(X) \in k(X)$ a rational function represented by q over $k(X)$. Then for any $v \in k^n$ where $p(v)$ is defined, $p(v)$ is represented by q over k.*

Proof. We may assume, by multiplying $p(X)$ by a square, that $p(X) \in k[X]$. Let $p(X) = p_1(X_n)$, where p_1 is a polynomial in X_n with coefficients in $k[X_1, \ldots, X_{n-1}]$. By the Cassels–Pfister theorem, $p_1(X_n)$ is represented by q over $k(X_1, \ldots, X_{n-1})[X_n]$. Let $v = (v_1, \ldots, v_n)$. Then specializing X_n to v_n, we have $p_1(v_n) \in k[X_1, \ldots, X_{n-1}]$ is represented by q over $k(X_1, \ldots, X_{n-1})$. By an induction argument, one concludes that $p(v_1, \ldots, v_n)$ is a value of q over k. □

Corollary 4.9. *Let q be an anisotropic quadratic form over k of dimension n. Then q is multiplicative if and only if, for indeterminates $X = (X_1, \ldots, X_n)$, $Y = (Y_1, \ldots, Y_n)$, $q(X)\,q(Y)$ is a value of q over $k(X_1, \ldots, X_n, Y_1, \ldots, Y_n)$.*

Proof. The only non-obvious part is "if". Suppose L/k is a field extension and $v, w \in L^n$. Let $q(v) = c$ and $q(w) = d$. Since $q(X)\,q(Y)$ is a value of q over $k(X, Y)$, by the Specialization lemma, $q(X)\,q(w)$ is a value of q over $L(X)$ and by the same lemma, $q(v)\,q(w)$ is a value of q over L. □

Theorem 4.10 (Subform Theorem). *Let $q = \langle a_1, \ldots, a_n \rangle$, $\gamma = \langle b_1, \ldots, b_m \rangle$ be quadratic forms over k with q anisotropic. Then γ is a subform of q (i.e., $q \cong \gamma \perp \gamma'$ for some form γ' over k) if and only if $b_1 X_1^2 + \cdots + b_m X_m^2$ is a value of q over $k(X_1, \ldots, X_m)$.*

Corollary 4.11. *Let q be an anisotropic quadratic form over k of dimension n. Let $X = \{X_1, \ldots, X_n\}$ be a list of n indeterminates. Then q is multiplicative if and only if $q \cong q(X)\,q$ over $k(X)$.*

Proof. Suppose $q \cong q(X)\,q$ over $k(X)$. Let A be the matrix representing q over k. There exists $W \in \mathrm{GL}_n(k(X))$ such that $q(X)A = WAW^t$. Let $Y = \{Y_1, \ldots, Y_n\}$ be a list of n indeterminates. Over $k(X, Y)$,

$$q(X)\,q(Y) = Y(q(X)A)Y^t = (YW)A(YW)^t = q(Z)$$

where $Z = YW$. Thus $q(X)\,q(Y)$ is a value of q over $k(X, Y)$ and by Corollary 4.9, q is multiplicative.

Suppose conversely that q is multiplicative. Then $q(X)\,q(Y)$ is a value of q over $k(X, Y)$. By the Subform theorem, $q(X)\,q$ is a subform of q. A dimension count yields $q \cong q(X)\,q$. □

Proof of Pfister's Theorem 4.6. Let $q = \langle 1, a_1 \rangle \otimes \cdots \otimes \langle 1, a_n \rangle$ be an anisotropic quadratic form over k. Over any field extension L/k, either q is an anisotropic Pfister form or isotropic in which case it is universal. Thus it suffices to show that the nonzero values of q form a subgroup of k^* for any anisotropic n-fold Pfister form q. The proof is by induction on n; for $n = 1$, q is the norm form from a quadratic extension of k (see Example 4.3) and we are done. Let $n \geq 2$. We have $q \cong q_1 \perp a_n q_1$, where $q_1 = \langle 1, a_1 \rangle \otimes \cdots \otimes \langle 1, a_{n-1} \rangle$ is an anisotropic $(n-1)$-fold Pfister form. Let $X = \{X_1, \ldots, X_{2^{n-1}}\}$, $Y = \{Y_1, \ldots, Y_{2^{n-1}}\}$ be two lists of 2^{n-1} indeterminates. Since q_1 is multiplicative, by Corollary 4.11, $q_1(X)\,q_1 \cong q_1$ over $k(X)$ and $q_1(Y)\,q_1 \cong q_1$ over $k(Y)$. We have, over $k(X, Y)$,

$$q \cong q_1(X)\, q_1 \perp a_n q_1(Y)\, q_1 \cong \langle q_1(X), a_n q_1(Y)\rangle \otimes q_1.$$

Since $q(X,Y) = q_1(X) + a_n q_1(Y)$, $\langle q_1(X), a_n q_1(Y)\rangle$ represents $q(X,Y)$. Therefore, by a comparison of discriminants,

$$\begin{aligned}\langle q_1(X), a_n q_1(Y)\rangle &\cong \langle q(X,Y), a_n q(X,Y) q_1(X) q_1(Y)\rangle \\ &\cong q(X,Y)(1 \perp a_n q_1(X) q_1(Y))\end{aligned}$$

In particular,

$$\begin{aligned} q &\cong q(X,Y)\langle 1, a_n q_1(X) q_1(Y)\rangle \otimes q_1 \\ &\cong q(X,Y)(q_1 \perp a_n q_1) \\ &\cong q(X,Y)\, q\end{aligned}$$

Thus by Corollary 4.11, q is multiplicative.

Conversely, let q be an anisotropic quadratic form over k which is multiplicative. Let n be the largest integer such that q contains an n-fold Pfister form $q_1 = \langle 1, a_1\rangle \otimes \cdots \otimes \langle 1, a_n\rangle$ as a subform. Suppose $q \cong q_1 \perp \gamma$, $\gamma = \langle b_1, \ldots, b_m\rangle$, with $m \geq 1$. Let $Z = \{Z_1, \ldots, Z_{2^n}\}$. Over $k(Z)$,

$$q \cong q(Z,0)\, q \cong q_1(Z)(q_1 \perp \gamma) \cong q_1(Z)\, q_1 \perp q_1(Z)\, \gamma \cong q_1 \perp q_1(Z)\, \gamma.$$

By Witt's cancellation, $\gamma \cong q_1(Z)\,\gamma$ over $k(Z)$. Thus γ represents $b_1 q_1(Z)$ over $k(Z)$ and by the Subform theorem, $\gamma \cong b_1\, q_1 \perp \gamma_1$. Then $q \cong q_1 \perp b_1\, q_1 \perp \gamma_1 \cong \langle 1, b_1\rangle \otimes q_1 \perp \gamma_1$ contains an $(n+1)$-fold Pfister form $\langle 1, b_1\rangle \otimes q_1$, leading to a contradiction to the maximality of n. Thus $q \cong q_1$. □

An important property of Pfister forms is stated in the following.

Proposition 4.12. *Let ϕ be an n-fold Pfister form. If ϕ is isotropic then ϕ is hyperbolic.*

Proof. Let $\phi = r\,\langle 1, -1\rangle \perp \phi_0$, with ϕ_0 anisotropic, $\dim(\phi_0) \geq 1$ and $r \geq 1$. Let $\dim(\phi) = m$ and $X = \{X_1, \ldots, X_m\}$ be a list of m indeterminates. Over $k(X_1, \ldots, X_m)$

$$r\,\langle 1, -1\rangle \perp \phi_0 = \phi \cong \phi(X_1, \ldots, X_m)\, \phi \cong r\,\langle 1, -1\rangle \perp \phi(X_1, \ldots, X_m)\, \phi_0.$$

By Witt's cancellation theorem

$$\phi_0 \cong \phi(X_1, \ldots, X_m)\, \phi_0.$$

If b is a value of ϕ_0, $b\phi(X_1, \ldots, X_m)$ is a value of ϕ_0 and by the Subform theorem, $b\phi$ is a subform of ϕ_0 contradicting $\dim(\phi_0) < \dim(\phi)$. Thus $\phi \cong r\,\langle 1, -1\rangle$ is hyperbolic. □

Corollary 4.13. *The only integers n such that a product of sums of n squares is again a sum of n squares over every field of characteristic zero are $n = 2^m$ for all $m \geq 0$.*

Proof. Consider the quadratic form $\phi_n = x_1^2 + x_2^2 + \cdots + x_n^2$ over $\mathbb{Q}$. The form ϕ_n is anisotropic. The condition that a product of sums of n squares is again a sum of n squares over any field of characteristic zero is equivalent to ϕ_n being a Pfister form. Thus $\dim(\phi_n) = n = 2^m$ for some m. □

5 Level of a Field

Definition 5.1. The **level** of a field k is the least positive integer n such that -1 is a sum of n squares in k. We denote the level of k by $s(k)$.

If the field is formally real (i.e., -1 is not a sum of squares), then the level is defined to be infinite. It was a longstanding open question since the 1950s whether the level of a field, if finite, is always a power of 2. Pfister's theory of quadratic forms leads to an affirmative answer to this question.

Theorem 5.2 ([Pf1]). *The level of a field is a power of 2 if it is finite.*

Proof. Let $n = s(k)$. We choose an integer m such that $2^m \leq n < 2^{m+1}$. Suppose

$$-1 = (u_1^2 + u_2^2 + \cdots + u_{2^m}^2) + (u_{2^m+1}^2 + \cdots + u_n^2) \tag{5.3}$$

The element $u_1^2 + u_2^2 + \cdots + u_{2^m}^2 \neq 0$ since $s(k) \geq 2^m$. Every ratio of sums of 2^m squares is again a sum of 2^m squares since $\langle 1, 1\rangle^{\otimes m}$ is a multiplicative form. Thus, from (5.3) we see that

$$\begin{aligned} 0 &= 1 + \frac{u_{2^m+1}^2 + \cdots + u_n^2 + 1}{u_1^2 + \cdots + u_{2^m}^2} \\ &= 1 + (v_1^2 + \cdots + v_{2^m}^2) \end{aligned}$$

Therefore, $-1 = v_1^2 + \cdots + v_{2^m}^2$ and $s(k) = 2^m$. □

Remark 5.4. There exist fields with level 2^n for any $n \geq 1$. For instance, $\mathbb{R}(X_1, \ldots, X_{2^n})(\sqrt{-(X_1^2 + \cdots + X_{2^n}^2)})$ is a field of level 2^n (cf. [L], Sect. XI.2).

Exercise 5.5. Let k be a p-adic field with $p \neq 2$ and with residue field $\mathbb{F}_q$. Prove the following:

(1) $s(k) = 1$ if $q \equiv 1 \pmod 4$.
(2) $s(k) = 2$ if $q \equiv -1 \pmod 4$.

6 The u-Invariant

Definition 6.1. The u**-invariant** of a field k, denoted by $u(k)$, is defined to be the largest integer n such that every $(n+1)$-dimensional quadratic form over k is isotropic and there is an anisotropic form in dimension n over k; if no such integer exists, the u-invariant is said to be infinite. In other words,

$$u(k) = \max\{\dim(q) \,:\, q \text{ anisotropic form over } k\}.$$

If k admits an ordering, then sums of nonzero squares are never zero and there is a refined u-invariant for fields with orderings, due to Elman–Lam [EL]. In this article, we do not discuss this refined invariant.

Example 6.2. (1) $u(\mathbb{F}_q) = 2$, if q is odd.

(2) $u(k(X)) = 2$, if k is algebraically closed and X is an integral curve over k (Tsen's theorem).

(3) $u(k) = 4$ for k a p-adic field. For $p \neq 2$, see Sect. 2.2. For $p = 2$, see [L, Sect. XI.6].

(4) $u(k) = 4$ for k a totally imaginary number field. This follows from the Hasse–Minkowski theorem.

(5) Suppose $u(k) = n < \infty$. Let $k((t))$ denote the field of Laurent series over k. Then $u(k((t))) = 2n$. In fact, the square classes in $k((t))^*$ are $\{u_\alpha, tu_\alpha\}_{\alpha\in I}$ where $\{u_\alpha\}_{\alpha\in I}$ are the square classes in k^*. As in the p-adic field case, every form over $k((t))$ is isometric to $\langle u_1,\ldots,u_r\rangle \perp t\langle v_1,\ldots,v_s\rangle$, $u_i, v_i \in k^*$ and this form is anisotropic if and only if $\langle u_1,\ldots,u_r\rangle$ and $\langle v_1,\ldots,v_s\rangle$ are anisotropic.

(6) More generally, if K is a complete discrete valuated field with residue field κ of u-invariant n, then $u(K) = 2n$. For the case $\mathrm{char}(\kappa) = 2$, we refer to [Sp].

Definition 6.3. A field k is $\boldsymbol{C_i}$ if every homogeneous polynomial in N variables of degree d with $N > d^i$ has a nontrivial zero.

Example 6.4. Finite fields and function fields in one variable over algebraically closed fields are C_1.

If k is a C_i field, $u(k) \leq 2^i$. Further, the property C_i behaves well with respect to function field extensions. If l/k is finite and k is C_i then l is C_i; further, if $t_1,\ldots,t_n$ are indeterminates, $k(t_1,\ldots,t_n)$ is C_{i+n}.

Example 6.5. The u-invariant of transcendental extensions:

(1) $u(k(t_1,\ldots,t_n)) = 2^n$ if k is algebraically closed. In fact,

$$u(k(t_1,\ldots,t_n)) \leq 2^n$$

since $k(t_1, \dots, t_n)$ is a C_n field. Further, the form

$$\langle\langle t_1, \dots, t_n \rangle\rangle = \langle 1, t_1 \rangle \otimes \cdots \otimes \langle 1, t_n \rangle$$

is anisotropic over $k((t_1))((t_2))\dots((t_n))$ and hence also over $k(t_1, \dots, t_n)$.

(2) $u(\mathbb{F}_q(t_1, \dots, t_n)) = 2^{n+1}$ if q is odd.

All fields of known u-invariant in the 1950s happened to have u-invariant a power of 2. Kaplansky raised the question whether the u-invariant of a field is always a power of 2.

Proposition 6.6. *The u-invariant does not take the values* $3, 5, 7$.

Proof. Let q be an anisotropic form of dimension 3. By scaling, we may assume that $q \cong \langle 1, a, b \rangle$. Then the form $\langle 1, a, b, ab \rangle$ is anisotropic; if $\langle 1, a, b, ab \rangle$ is isotropic, it is hyperbolic and Witt's cancellation yields $\langle a, b, ab \rangle \cong \langle 1, -1, -1 \rangle$ which is isotropic and $q \cong a\langle a, b, ab \rangle$ is isotropic leading to a contradiction. Thus $u(k) \neq 3$.

Let $u(k) < 8$. Every three-fold Pfister form (which has dimension 8) is isotropic and hence hyperbolic. Thus $I^3(k)$ which is generated by three-fold Pfister forms is zero. Let $q \in I^2(k)$ be any quadratic form. For any $c \in k^*$, $\langle 1, -c \rangle\, q \in I^3(k)$ is zero and cq is Witt equivalent to q, hence isometric to q by Witt's cancellation. We conclude that every quadratic form whose class is in $I^2(k)$ is universal.

Suppose $u(k) = 5$ or 7. Let q be an anisotropic form of dimension $u(k)$. Since every form in dimension $u(k) + 1$ is isotropic, if $\operatorname{disc}(q) = d$, $q \perp -d$ is isotropic and therefore q represents d. We may write $q \cong q_1 \perp \langle d \rangle$ where q_1 is even-dimensional with trivial discriminant. Hence $[q_1] \in I^2(k)$ so that q_1 is universal. This in turn implies that $q_1 \perp \langle d \rangle \cong q$ is isotropic, leading to a contradiction. □

In the 1990s Merkurjev [M2] constructed examples of fields k with $u(k) = 2n$ for any $n \geq 1$, $n = 3$ being the first open case, answering Kaplansky's question in the negative. Since then, it has been shown that the u-invariant could be odd. In [I], Izhboldin proves that there exist fields k with $u(k) = 9$ and in [V] Vishik has shown that there exist fields k with $u(k) = 2^r + 1$ for all $r \geq 3$.

Merkurjev's construction yields fields k which are not of arithmetic type, i.e., not finitely generated over a number field or a p-adic field. It is still an interesting question whether $u(k)$ is a power of 2 if k is of arithmetic type.

The behavior of the u-invariant is very little understood under rational function field extensions. For instance, it is an open question if $u(k) < \infty$ implies $u(k(t)) < \infty$ for the rational function field in one variable over k. This was unknown for $k = \mathbb{Q}_p$ until the late 1990s. Conjecturally, $u(\mathbb{Q}_p(t)) = 8$, in analogy with the positive characteristic local field case; the field $\mathbb{F}_p((X))(t)$ is C_3 (see [G]) so that $u(\mathbb{F}_p((X))(t)) \leq 8$ for p odd. If u is a nonsquare in $\mathbb{F}_p$, $\langle 1, -u \rangle \otimes \langle 1, -X \rangle \otimes \langle 1, -t \rangle$ is anisotropic over $\mathbb{F}_p((X))(t)$, so that $u(\mathbb{F}_p((X))(t)) = 8$.

We indicate some ways of bounding the u-invariant of a field k once we know how efficiently the Galois cohomology groups $H^n(k, \mathbb{Z}/2\mathbb{Z})$ are generated by symbols for all n.

We set

$$H^n_{\text{dec}}(k, \mathbb{Z}/2\mathbb{Z}) = \{(a_1) \cdot \cdots \cdot (a_n) : a_i \in k^*\}$$

and call elements in this set *symbols*. By Voevodsky's theorem on the Milnor conjecture, $H^n(k, \mathbb{Z}/2\mathbb{Z})$ is additively generated by $H^n_{\text{dec}}(k, \mathbb{Z}/2\mathbb{Z})$.

Proposition 6.7. *Let k be a field such that $H^{n+1}(k, \mathbb{Z}/2\mathbb{Z}) = 0$ and for $2 \leq i \leq n$, there exist integers N_i such that every element in $H^i(k, \mathbb{Z}/2\mathbb{Z})$ is a sum of N_i symbols. Then $u(k)$ is finite.*

Proof. Let q be a quadratic form over k of dimension m and discriminant d. Let $q_1 = \langle d \rangle$ if m is odd and $\langle 1, -d \rangle$ if m is even. Then $q \perp -q_1$ has even dimension and trivial discriminant. Hence $q \perp -q_1 \in I^2(k)$. Let $e_2(q \perp -q_1) = \sum_{j \leq N_2} \xi_{2j}$ where $\xi_{2j} \in H^2_{\text{dec}}(k, \mathbb{Z}/2\mathbb{Z})$. Let ϕ_{2j} be two-fold Pfister forms such that $e_2(\phi_{2j}) = \xi_{2j}$. Then $q_2 = \sum_{j \leq N_2} \phi_{2j}$ has dimension at most $4N_2$ and $e_2(q \perp -q_1 \perp -q_2) = 0$ and $q \perp -q_1 \perp -q_2 \in I^3(k)$, by Merkurjev's theorem. Repeating this process and using the Milnor conjecture, we get $q_i \in I^i(k)$ which is a sum of N_i i-fold Pfister forms and $q - \sum_{1 \leq i \leq n} q_i \in I^{n+1}(k) = 0$, since $H^{n+1}(k, \mathbb{Z}/2\mathbb{Z}) = 0$. Thus $[q] = \sum_{1 \leq i \leq n} q_i$ and $\dim(q_{an}) \leq \sum_{1 \leq i \leq n} 2^i N_i$. Thus $u(k) \leq \sum_{1 \leq i \leq n} 2^i N_i$. □

Definition 6.8. A field k is said to have **cohomological dimension at most n** (in symbols, $\text{cd}(k) \leq n$) if $H^i(k, M) = 0$ for $i \geq n + 1$ for all finite discrete Γ_k-modules M (cf. [Se, §3]).

Example 6.9. Finite fields and function fields in one variable over algebraically closed fields have cohomological dimension 1. Totally imaginary number fields and p-adic fields are of cohomological dimension 2. If k is a p-adic field, and $k(X)$ a function field in one variable over k, $\text{cd}(k(X)) \leq 3$. In particular, $H^4(k(X), \mathbb{Z}/2\mathbb{Z}) = 0$.

Theorem 6.10 (Saltman [Sa]). *Let k be a non-dyadic p-adic field and $k(X)$ a function field in one variable over k. Every element in $H^2(k(X), \mathbb{Z}/2\mathbb{Z})$ is a sum of two symbols.*

Theorem 6.11 (Parimala–Suresh [PS1]). *Let $k(X)$ be as in the previous theorem. Then every element in $H^3(k(X), \mathbb{Z}/2\mathbb{Z})$ is a symbol.*

Corollary 6.12. *For $k(X)$ as above, $u(k(X)) \leq 2 + 8 + 8 = 18$.*

It is not hard to show from the above theorems that $u(k(X)) \leq 12$. With some further work it was proved in [PS1] that $u(k(X)) \leq 10$. More recently in [PS2] the estimated value $u(k(X)) = 8$ was proved. For an alternate approach to $u(k(X)) = 8$, we refer to [HH, HHK, CTPS]. More recently, Heath-Brown and Leep [HB] have proved the following spectacular theorem: If k is *any* p-adic field and $k(X)$ the function field in n variables over k, then $u(k(X)) = 2^{n+2}$.

7 Hilbert's Seventeenth Problem

An additional reference for sums of squares is [C].

Definition 7.1. An element $f \in \mathbb{R}(X_1, \dots, X_n)$ is called **positive semi-definite** if $f(a) \geq 0$ for all $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ where f is defined.

Hilbert's seventeenth problem:
Let $\mathbb{R}(X_1, \dots, X_n)$ be the rational function field in n variables over the field $\mathbb{R}$ of real numbers. Hilbert's seventeenth problem asks whether every positive semi-definite $f \in \mathbb{R}(X_1, \dots, X_n)$ is a sum of squares in $\mathbb{R}(X_1, \dots, X_n)$. E. Artin settled this question in the affirmative and Pfister gave an effective version of Artin's result (cf. [Pf, Chap. 6]).

Theorem 7.2 (Artin, Pfister). *Every positive semi-definite function $f \in \mathbb{R}(X_1, \dots, X_n)$ can be written as a sum of 2^n squares in $\mathbb{R}(X_1, \dots, X_n)$.*

For $n \leq 2$ the above was due to Hilbert himself. If one asks for expressions of positive definite polynomials in $\mathbb{R}[X_1, \dots, X_n]$ as sums of 2^n squares in $\mathbb{R}[X_1, \dots, X_n]$, there are counterexamples for $n = 2$; the Motzkin polynomial

$$f(X_1, X_2) = 1 - 3X_1^2X_2^2 + X_1^4X_2^2 + X_1^2X_2^4$$

is positive semi-definite but not a sum of squares in $\mathbb{R}[X_1, X_2]$. In fact, Pfister's result has the following precise formulation.

Theorem 7.3 (Pfister). *Let $\mathbb{R}(X)$ be a function field in n variables over $\mathbb{R}$. Then every n-fold Pfister form in $\mathbb{R}(X)$ represents every sum of squares in $\mathbb{R}(X)$.*

We sketch a proof of this theorem below.

Definition 7.4. Let ϕ be an n-fold Pfister form with $\phi = 1 \perp \phi'$. The form ϕ' is called the **pure subform** of ϕ.

Proposition 7.5 (Pure Subform Theorem). *Let k be any field of characteristic not 2, ϕ an anisotropic n-fold Pfister form over k and ϕ' its pure subform. If b_1 is any value of ϕ', then $\phi \cong \langle\langle b_1, \dots, b_n\rangle\rangle$ for some $b_2, \dots, b_n \in k^*$.*

Proof. The proof is by induction on n; for $n = 1$ the statement is clear. Let $n > 1$. We assume the statement holds for all $(n-1)$-fold Pfister forms. Let $\phi = \langle\langle a_1, \dots, a_n\rangle\rangle$, $\psi = \langle\langle a_1, \dots, a_{n-1}\rangle\rangle$, and let ϕ', ψ' denote the pure subforms of ϕ and ψ respectively. We have $\phi = \psi \perp a_n\psi$, $\phi' = \psi' \perp a_n\psi$. Let b_1 be a value of ϕ'. We may write $b_1 = b_1' + a_n b$, with b_1' a value of ψ' and b a value of ψ. The only nontrivial case to discuss is when $b \neq 0$ and $b_1' \neq 0$. By induction, $\psi \cong \langle\langle b_1', b_2, \dots, b_{n-1}\rangle\rangle$ and $b\psi \cong \psi$. We thus have

$$\begin{aligned}\phi &\cong \langle\langle b_1', b_2, \dots, b_{n-1}, a_n\rangle\rangle \cong \langle\langle b_1', b_2, \dots, b_{n-1}, a_n b\rangle\rangle \\ &\cong \langle\langle b_1', a_n b\rangle\rangle \otimes \langle\langle b_2, \dots, b_{n-1}\rangle\rangle\end{aligned}$$

Since $b_1 = b_1' + a_n b$, $\langle b_1', a_n b\rangle \cong \langle b_1, b_1 b_1' a_n b\rangle$ and we have

$$\begin{aligned}\langle\langle b_1', a_n b\rangle\rangle &= \langle 1, b_1', a_n b, a_n b b_1'\rangle \\ &= \langle 1, b_1, b_1 b_1' a_n b, a_n b b_1'\rangle \\ &= \langle\langle b_1, c_1\rangle\rangle,\end{aligned}$$

where $c_1 = b_1 b_1' a_n b$. Thus,

$$\phi \cong \langle\langle b_1, c_1, b_2, \cdots, b_{n-1}\rangle\rangle. \qquad \square$$

Proof of Pfister's Theorem 7.3. Let ϕ be an anisotropic n-fold Pfister form over $K = \mathbb{R}(X)$. Let $b = b_1^2 + \cdots + b_m^2$, $b_i \in K^*$. We show that ϕ represents b by induction on m. For $m = 1$, b is a square and is represented by ϕ. Suppose $m = 2$, $b = b_1^2 + b_2^2$, $b_1 \neq 0$, $b_2 \neq 0$. The field $K(\sqrt{-1})$ is a function field in n variables over $\mathbb{C}$ and is C_n. Then ϕ is universal over $K(\sqrt{-1})$ and hence represents $\beta = b_1 + ib_2$. Let $v, w \in K^{2^n}$ such that $\phi_{K(\sqrt{-1})}(v + \beta w) = \beta$. Hence

$$\phi(v) + \beta^2\phi(w) + \beta(2\phi(v,w) - 1) = 0.$$

The irreducible polynomial of β over K is

$$\phi(w)X^2 + (2\phi(v,w) - 1)X + \phi(v)$$

and hence $N(\beta) = b = \frac{\phi(v)}{\phi(w)}$ is a value of ϕ since ϕ is multiplicative.

Suppose $m > 2$. We argue by induction on m. Suppose ϕ represents all sums of $m - 1$ squares. Let b be a sum of m squares. After scaling b by a square, we may assume that $b = 1 + c$, $c = c_1^2 + \cdots + c_{m-1}^2$, $c \neq 0$. Let $\phi \cong 1 \perp \phi'$. By induction hypothesis, ϕ represents c. Let $c = c_0^2 + c'$, c' a value of ϕ'. Let $\psi = \phi \otimes \langle 1, -b\rangle$ and $\psi = 1 \perp \psi'$ with $\psi' = \langle -b\rangle \perp \phi' \perp -b\phi'$. The form ψ' represents $c' - b = (c - c_0^2) - (1 + c) = -1 - c_0^2$. Thus, by the Pure Subform theorem,

$$\psi \cong \langle\langle -1 - c_0^2, d_1, \ldots, d_n\rangle\rangle = \langle 1, -1 - c_0^2\rangle \otimes \langle\langle d_1, \ldots, d_n\rangle\rangle.$$

By induction, the n-fold Pfister form $\langle\langle d_1, \ldots, d_n\rangle\rangle$ represents $1 + c_0^2$ which is a sum of 2 squares; thus ψ is isotropic, hence hyperbolic. Thus $\phi \cong b\phi$ represents b. $\square$

Corollary 7.6. *Let $K = \mathbb{R}(X)$ be a function field in n variables over $\mathbb{R}$. Then every sum of squares in K is a sum of 2^n squares.*

Proof. Set $\phi = \langle 1, 1\rangle^{\otimes n}$ in the above theorem. $\square$

8 Pythagoras Number

Definition 8.1. The **Pythagoras number** $p(k)$ of a field k is the least positive integer n such that every sum of squares in k^* is a sum of at most n squares; if no such n exists, $p(k)$ is defined to be infinity.

Example 8.2. If $\mathbb{R}$ is the field of real numbers, $p(\mathbb{R}) = 1$.

Example 8.3. If $\mathbb{R}(X_1, \ldots, X_n)$ is a function field in n variables over $\mathbb{R}$, by Pfister's theorem (Corollary 7.6), $p(\mathbb{R}(X_1, \ldots, X_n)) \leq 2^n$.

Let $K = \mathbb{R}(X_1, \ldots, X_n)$ be the rational function field in n variables over $\mathbb{R}$. We discuss the effectiveness of the bound $p(K) \leq 2^n$. For $n = 1$ the bound is sharp. For $n = 2$ the Motzkin polynomial

$$f(X_1, X_2) = 1 - 3X_1^2X_2^2 + X_1^4X_2^2 + X_1^2X_2^4$$

is positive semi-definite; Cassels–Ellison–Pfister [CEP] show that this polynomial is not a sum of three squares in $\mathbb{R}(X_1, X_2)$ (see also [CT]). Therefore $p(\mathbb{R}(X_1, X_2)) = 4$.

Lemma 8.4 (Key Lemma). *Let k be a field and $n = 2^m$. Let $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n) \in k^n$ be such that $u \cdot v = \sum_{1 \leq i \leq n} u_i v_i = 0$. Then there exist $w_j \in k$, $1 \leq j \leq n-1$ such that*

$$\Bigg(\sum_{1 \leq i \leq n} u_i^2\Bigg)\Bigg(\sum_{1 \leq i \leq n} v_i^2\Bigg) = \sum_{1 \leq j \leq n-1} w_j^2.$$

Proof. Let $\lambda = \sum_{1 \leq i \leq n} u_i^2$, $\mu = \sum_{1 \leq i \leq n} v_i^2$. We may assume without loss of generality that $u \neq 0$ and $v \neq 0$. The elements λ and μ are values of $\phi_m = \langle 1, 1\rangle^{\otimes m}$ and $\lambda\phi_m \cong \phi_m$, $\mu\phi_m \cong \phi_m$. We choose isometries $f\colon \lambda\phi_m \cong \phi_m$, $g\colon \mu\phi_m \cong \phi_m$ such that $f(1, 0, \ldots, 0) = u$ and $g(1, 0, \ldots, 0) = v$. If U and V are matrices representing f, g respectively, we have

$$UU^t = \lambda^{-1}, \;\; VV^t = \mu^{-1}, \;\; \lambda^{-1}\mu^{-1} = \lambda^{-1}VV^t = (VU^t)(VU^t)^t.$$

The first row of VU^t is the form $(0, w_2, \ldots, w_n)$ since $u \cdot v = 0$. Thus $\lambda^{-1}\mu^{-1} = \sum_{2 \leq i \leq n} w_i^2$. □

Corollary 8.5. *Let k be an ordered field with $p(k) = n$. Then $p(k(t)) \geq n + 1$.*

Proof. Let $\lambda \in k^*$ be such that λ is a sum of n squares and not a sum of less than n squares. Suppose $\lambda + t^2$ is a sum of n squares in $k(t)$. By the Cassels–Pfister theorem,

$$\lambda + t^2 = (\mu_1 + \nu_1 t)^2 + \cdots + (\mu_n + \nu_n t)^2$$

with $\mu_i, \nu_i \in k^*$. If $u = (\mu_1, \dots, \mu_n)$, $v = (\nu_1, \dots, \nu_n)$, then $u \cdot v = 0$, $\sum_{1 \le i \le n} \mu_i^2 = \lambda$, $\sum_{1 \le i \le n} \nu_i^2 = 1$. Thus $\lambda = (\sum_{1 \le i \le n} \mu_i^2)(\sum_{1 \le i \le n} \nu_i^2)$ is a sum of $n-1$ squares by the Key Lemma 8.4, contradicting the choice of λ. □

Corollary 8.6. *For* $n \ge 2$,

$$n + 2 \le p(\mathbb{R}(X_1, \dots, X_n)) \le 2^n.$$

Proof. By [CEP], we know that $p(\mathbb{R}(X_1, X_2)) = 4$. The fact that $n + 2 \le p(\mathbb{R}(X_1, \dots, X_n))$ now follows by Corollary 8.5 and induction. □

Remark 8.7. It is open whether $p(\mathbb{R}(X_1, X_2, X_3)) = 5, 6, 7$ or 8.

Remark 8.8. The possible values of the Pythagoras number of a field have all been listed ([H], [Pf, p. 97]).

Proposition 8.9. *If k is a non-formally real field, $p(k) = s(k)$ or $s(k) + 1$.*

Proof. If $s(k) = n$, then -1 is not a sum of less than n squares, so that $p(k) \ge s(k)$. For $a \in k^*$,

$$a = \left(\frac{a+1}{2}\right)^2 + (-1)\left(\frac{a-1}{2}\right)^2$$

is a sum of $n+1$ squares if -1 is a sum of n squares. Thus $p(k) \le s(k) + 1$. □

Let k be a p-adic field and $K = k(X_1, \dots, X_n)$ a rational function field in n variables over k. Then $s(k) = 1, 2$ or 4 so that $s(K) = 1, 2$ or 4. Thus $p(K) \le 5$. (In fact it is easy to see that if $s(k) = s$, $p(K) = s + 1$.)

Thus we have bounds for $p(k(X_1, \dots, X_n))$ if k is the field of real or complex numbers or the field of p-adic numbers. The natural questions concern a number field k.

9 Function Fields Over Number Fields

Let k be a number field and $F = k(t)$ the rational function field in one variable over k. In this case $p(k(t)) = 5$ is a theorem [La]. The fact that $p(k(t)) \le 8$ can be easily deduced from the following injectivity in the Witt groups [CTCS, Proposition 1.1]:

$$W(k(t)) \longrightarrow \prod_{w \in \Omega(k)} W(k_w(t)),$$

with $\Omega(k)$ denoting the set of places of k. In fact, if $f \in k(t)$ is a sum of squares, f is a sum of at most two squares in $k_w(t)$ for a real place w, by Pfister's theorem (which in the case of function fields of curves goes back to Witt). Further, for a finite place w of k or a complex place, $\langle 1, 1 \rangle^{\otimes 3} = 0$ in $W(k_w)$. Thus $\langle 1, 1 \rangle^{\otimes 3} \otimes \langle 1, -f \rangle$ is hyperbolic over $k_w(t)$ for all $w \in \Omega(k)$.

By the above injectivity, this form is hyperbolic over $k(t)$, leading to the fact that f is a sum of at most eight squares in $k(t)$.

We have the following conjecture due to Pfister for function fields over number fields.

Conjecture (Pfister). *Let k be a number field and $F = k(X)$ a function field in d variables over k. Then*

(1) *For $d = 1$, $p(F) \leq 5$.*
(2) *For $d \geq 2$, $p(F) \leq 2^{d+1}$.*

For a function field $k(X)$ in one variable over k, ($d = 1$), the best known result is due to F. Pop, $p(F) \leq 6$ [P]. For $d = 2$, the conjecture is settled in [CTJ]. We sketch some results and conjectures from the arithmetic side which imply Pfister's conjecture for $d \geq 3$ (see Colliot-Thélène and Jannsen [CTJ] for more details).

For any field k, by Voevodsky's theorem, we have an injection

$$e_n \colon P_n(k) \to H^n(k, \mathbb{Z}/2\mathbb{Z}).$$

In fact, for any field k, if $\phi_1, \phi_2 \in P_n(k)$ have the same image under e_n then $\phi_1 \perp -\phi_2 \in \ker(e_n) = I^{n+1}(k)$. In $W(k)$, $\phi_1 \perp -\phi_2 = \phi_1' \perp -\phi_2'$ where ϕ_1' and ϕ_2' are the pure subforms of ϕ_1 and ϕ_2. Moreover, $\dim(\phi_1' \perp -\phi_2')_{\text{an}} \leq 2^{n+1} - 2 < 2^{n+1}$. By the Arason–Pfister Hauptsatz, (Theorem 3.1), anisotropic forms in $I^{n+1}(k)$ must have dimension at least 2^{n+1}. Therefore $\phi_1 = \phi_2$.

Let k be a number field and $F = k(X)$ be a function field in d variables over k. Let $f \in F$ be a function which is a sum of squares in F. One would like to show that f is a sum of 2^{d+1} squares. Let $\phi_{d+1} = \langle 1, 1\rangle^{\otimes(d+1)}$ and $q = \phi_{d+1} \otimes \langle 1, -f\rangle$. This is a $(d+2)$-fold Pfister form and ϕ_{d+1} represents f if and only if q is hyperbolic or equivalently, by the injectivity of e_n above, $e_{d+2}(\phi_{d+1} \otimes \langle 1, -f\rangle) = 0$.

We look at this condition locally at all completions k_v at places v of k. Let $k_v(X)$ denote the function field of X over k_v. (We may assume that X is geometrically integral.) Let v be a complex place. The field $k_v(X)$ has cohomological dimension d so that $H^m(k_v(X), \mathbb{Z}/2\mathbb{Z}) = 0$ for $m \geq d+1$. Hence $e_{d+2}(\phi_{d+1} \otimes \langle 1, -f\rangle) = 0$ over $k_v(X)$. Let v be a real place. Over $k_v(X)$, f is a sum of squares, hence a sum of at most 2^d squares (by Pfister's Theorem 7.3) so that $\phi_{d+1} \otimes \langle 1, -f\rangle$ is hyperbolic over $k_v(X)$. Hence $e_{d+2}(\phi_{d+1} \otimes \langle 1, -f\rangle) = 0$.

Let v be a non-dyadic p-adic place of k. Then ϕ_2 is hyperbolic over k_v so that $\phi_{d+1} \otimes \langle 1, -f\rangle = 0$ and $e_{d+2}(\phi_{d+1} \otimes \langle 1, -f\rangle) = 0$.

Let v be a dyadic place of k. Over k_v, ϕ_3 is hyperbolic so that $e_{d+2}(\phi_{d+1} \otimes \langle 1, -f\rangle) = 0$. Thus for all completions v of k, $e_{d+2}(\phi_{d+1} \otimes \langle 1, -f\rangle)$ is zero. The following conjecture of Kato implies Pfister's conjecture for $d \geq 2$.

Conjecture (Kato). *Let k be a number field, X a geometrically integral variety over k of dimension d. Then the map*

$$H^{d+2}(k(X), \mathbb{Z}/2\mathbb{Z}) \to \prod_{v \in \Omega_k} H^{d+2}(k_v(X), \mathbb{Z}/2\mathbb{Z})$$

has trivial kernel.

The above conjecture is the classical Hasse–Brauer–Noether theorem if the dimension of X is zero, i.e., the injectivity of the Brauer group map:

$$\mathrm{Br}(k) \hookrightarrow \bigoplus_{v \in \Omega_k} \mathrm{Br}(k_v).$$

For $\dim X = 1$, the conjecture is a theorem of Kato [Ka]. Kato's conjecture is now a theorem due to Jannsen [Ja1, Ja2] for $\dim X \geq 2$. Thus for every function field $k(X)$ in d variables over a number field k, $d \geq 2$, we have $p(k(X)) \leq 2^{d+1}$.

We now explain how Kato's theorem was used by Colliot-Thélène to derive $p(k(X)) \leq 7$ for a curve X over a number field. We note that this bound is weaker than the bound established by F. Pop.

Suppose $K = k(X)$ has no ordering. We claim that $s(K) \leq 4$. To show this it suffices to show that $\langle 1,1\rangle^{\otimes 3}$ is zero over $k_v(X)$ for every place v of k. At finite places v, $\langle 1,1\rangle^{\otimes 3}$ is already zero in k_v. If v is a real place of k, $k_v(X)$ is the function field of a real curve over the field of real numbers which has no orderings. By a theorem of Witt, $\mathrm{Br}(k_v(X)) = 0$ and every sum of squares is a sum of two squares in $k_v(X)$. Thus -1 is a sum of two squares in $k_v(X)$ and $\langle 1,1\rangle^{\otimes 3} = 0$ over $k_v(X)$. Since $H^3(k(X), \mathbb{Z}/2\mathbb{Z}) \to \prod_{v \in \Omega_k} H^3(k_v(X), \mathbb{Z}/2\mathbb{Z})$ is injective by Kato's theorem, $e_3(\langle 1,1\rangle^{\otimes 3}) = 0$ in $H^3(k(X), \mathbb{Z}/2\mathbb{Z})$. Since e_3 is injective on three-fold Pfister forms, $\langle 1,1\rangle^{\otimes 3} = 0$ in $k(X)$. Thus $s(k(X)) \leq 4$. In this case, $p(k(X)) \leq 5$.

Suppose K has an ordering. Let $f \in K^*$ be a sum of squares in K. Then $K(\sqrt{-f})$ has no orderings and hence -1 is a sum of 4 squares in $K(\sqrt{-f})$. Let $a_i, b_i \in K$ be such that

$$-1 = \sum_{1 \leq i \leq 4} (a_i + b_i\sqrt{-f})^2, \quad a_i, b_i \in K.$$

Then

$$1 + \sum_{1 \leq i \leq 4} a_i^2 = f\Big(\sum_{1 \leq i \leq 4} b_i^2\Big), \quad \sum_{1 \leq i \leq 4} a_i b_i = 0.$$

By the Key Lemma 8.4, $(1 + \sum_{1 \leq i \leq 4} a_i^2)\sum_{1 \leq i \leq 4} b_i^2$ is a sum of at most 7 squares.

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