

Chapter 8

Scale Interactions

8.1 Introduction

The tropical atmospheric circulation contains a number of spatial and temporal scales of motion. In terms of temporal scales – or frequency domain – these range from scales of a few seconds (for turbulent motions), through semi-diurnal and diurnal scales, synoptic (2–7 day) scales, semi-biweekly scales, Madden-Julian oscillation scales (30–60 days), annual cycle, ENSO, PDO, etc. In terms of spatial scales – or wave number domain – the scales range from a few mm for turbulent processes, few 100 m for cloud-scale motions, few 100 km for tropical disturbances, few 1,000 km for the African easterly waves, and so on.

These scales do not exist separately from one another, but instead are in constant interplay. The formalism describing this communication is called ‘scale interaction’. The mathematical formulation of scale interactions is somewhat complicated, but the physical interpretations are elegant. Scale interaction is a central topic in the atmosphere–ocean dynamics and thermodynamics. It is possible to calculate the rate of change of kinetic or potential energy of a given scale (either spatial or temporal) as a result of its interactions with any other scales based on either observations or numerical model data sets.

An interesting example of scale interactions is the following. It is well known that deep cumulus convection drives a hurricane/typhoon/tropical cyclone. The scale of deep cumulus convection (the cumulonimbus clouds) is only on the order of a few km, whereas the scale of the hurricane is on the order of 100 km and larger. How do these diverse spatial scales communicate with each other? This is the type of question that is addressed by the scale interaction approach.

8.2 Wave-Number Domain

The only way energy can be transferred between scales (excluding the mean) is via triad interactions; a triad of waves with wave numbers n , m and k interact if and only if $k + m = n$, or $-k + m = n$, or $k - m = n$. These are the so called trigonometric selection rules (see Saltzman 1957).

Symbolically, the changes in kinetic and potential energy for wave number n and for the zonal mean are given by

$$\frac{\partial K_n}{\partial t} = \langle K_0 \rightarrow K_n \rangle + \langle K_{n,m} \rightarrow K_k \rangle + \langle P_n \rightarrow K_n \rangle + \langle F_n \rightarrow K_n \rangle$$

$$\frac{\partial K_0}{\partial t} = - \sum_n \langle K_0 \rightarrow K_n \rangle + \langle P_0 \rightarrow K_0 \rangle + \langle F_0 \rightarrow K_0 \rangle$$

$$\frac{\partial P_n}{\partial t} = \langle P_0 \rightarrow P_n \rangle + \langle P_{k,m} \rightarrow P_n \rangle - \langle P_n \rightarrow K_n \rangle + \langle H_n \rightarrow P_n \rangle$$

$$\frac{\partial P_0}{\partial t} = - \sum_n \langle P_0 \rightarrow P_n \rangle - \langle P_0 \rightarrow K_0 \rangle + \langle H_0 \rightarrow P_0 \rangle$$

where K_n and K_0 represent the kinetic energy of wave number n and wave number 0 (the zonal mean) respectively; similarly, P , F and H are the available potential energy, frictional dissipation and the heating, respectively, at the wave number indicated by the subscript. The terms in triangular brackets indicate exchanges (positive in the direction of the arrow). These exchanges can be analyzed as follows:

(a) Zonal to wave exchange of kinetic energy

$$\begin{aligned} \langle K_0 \rightarrow K_n \rangle = & - \left[\Phi_{uv}(n) \frac{\cos \varphi}{a} \frac{\partial}{\partial \varphi} \left(\frac{U(0)}{\cos \varphi} \right) - \Phi_{vz}(n) \frac{1}{a} \frac{\partial V(0)}{\partial \varphi} \right. \\ & \left. + \Phi_{u\omega}(n) \frac{\partial U(0)}{\partial p} + \Phi_{v\omega}(n) \frac{\partial V(0)}{\partial p} - \Phi_{uv}(n) V(0) \frac{\tan \varphi}{a} \right] \end{aligned}$$

is the conversion of the kinetic energy of the zonal mean to kinetic energy of wave number n (see Appendix 1 for the meaning of symbols and for the derivation of the conversion equations). This is more or less a barotropic energy exchange that invokes the covariance among the zonally averaged motion and the eddy flux of momentum.

(b) Waves to waves exchange of kinetic energy

$$\begin{aligned}
\langle K_0 \rightarrow K_n \rangle = & \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \left\{ U(m) \left(\frac{1}{a \cos \varphi} \Psi_{uu_\lambda}(m, n) + \frac{1}{a} \Psi_{vu_\varphi}(m, n) \right. \right. \\
& + \left. \Psi_{\omega u_p}(m, n) - \frac{\tan \varphi}{a} \Psi_{uv}(m, n) \right) \\
& + V(m) \left(\frac{1}{a \cos \varphi} \Psi_{uv_\lambda}(m, n) + \frac{1}{a} \Psi_{vv_\varphi}(m, n) \right. \\
& + \left. \Psi_{\omega v_p}(m, n) + \frac{\tan \varphi}{a} \Psi_{vu}(m, n) \right) \\
& - \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} [\cos \varphi (U(m) \Psi_{vu}(m, n) + V(m) \Psi_{vv}(m, n))] \\
& \left. - \frac{\partial}{\partial p} (U(m) \Psi_{\omega u}(m, n) + V(m) \Psi_{\omega v}(m, n)) \right\}
\end{aligned}$$

is the non-linear exchange of kinetic energy among different scales. This is a triplet interaction, since a triple product is involved. The three scales (m , k and n) have to be related by the requirement that $m + k = n$ or $m - k = n$ or $-m + k = n$, or else the energy exchange is zero.

(c) Wave to wave conversion between available potential and kinetic energy

$\langle P_n \rightarrow K_n \rangle = -\frac{R}{p} \Phi_{T\omega}(n)$ is the conversion of the potential energy of the wave number n to the kinetic energy of the same wave number. This is a quadratic (in-scale) interaction.

(d) Wave to wave conversion between friction and kinetic energy

$\langle F_n \rightarrow K_n \rangle$ is the conversion between frictional dissipation and kinetic energy. This also is a quadratic (in-scale) interaction. It would be expected to be negative, since friction is a sink of kinetic energy.

(e) Zonal conversion between available potential and kinetic energy

$\langle P_0 \rightarrow K_0 \rangle = -\frac{R}{p} \Phi_{T\omega}(0)$ is the conversion of zonally averaged potential energy to zonally averaged kinetic energy. This is the Hadley cell vertical overturning, with warm air rising and cold air sinking.

(f) Zonal to wave exchange of available potential energy

$\langle P_0 \rightarrow P_n \rangle = - \left[\frac{C_p \gamma}{a} \Phi_{T_V}(n) \frac{\partial Y(0)}{\partial \phi} + \frac{\gamma p^\mu}{\mu} \left\{ \Phi_{T\omega}(n)'' \frac{\partial \bar{\theta}''}{\partial p} \right\} \right]$ is the potential energy exchange between the zonally averaged flow and wave number n . It relates to the heat flux from the zonal mean to the scales of the waves.

(g) Waves to waves exchange of available potential energy

$$\begin{aligned} \langle P_{m,n} \rightarrow P_k \rangle = & C_p \gamma \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} Y(m) \left(\frac{1}{a \cos \varphi} \Psi_{uT_\lambda}(m, n) + \frac{1}{a} \Psi_{vT_\phi}(m, n) \right. \\ & \left. + \Psi_{\omega T_p}(m, n) + \frac{R}{C_p p} \Psi_{\omega T}(m, n) \right) \\ & - \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} \cos \varphi Y(m) \Psi_{vT}(n, m) - \frac{\partial}{\partial p} Y(m) \Psi_{\omega T} \end{aligned}$$

is the non-linear potential energy exchange between scales. This is a triad interaction among a triplet of waves that satisfies the selection rule.

(h) Generation of wave available potential energy

$\langle H_n \rightarrow P_n \rangle = \gamma \Phi_{Th}(n)$ is the generation of available potential energy at wave number n from heating. This involves a covariance among heating and temperature on the scale of wave number n . This term is quadratic.

(i) Generation of zonal available potential energy

$\langle H_0 \rightarrow P_0 \rangle = \gamma \Phi_{Th}(0)$ is the generation of available potential energy at the scale of the zonal mean from heating. This generation is achieved by the covariance of zonally averaged heating and temperature.

The totality of interactions is illustrated schematically in Fig. 8.1.

The total energy budget in the wave number domain for a selected scale n has to include the following components:

(a) Exchanges involving quadratic non-linearities (i.e. in-scale exchanges):

- Generation or loss of available potential energy from heat sources or sinks on the scale of n ;
- Conversion between kinetic and available potential energy on the scale of n ,
- Conversion between the kinetic energy of the zonal mean and the kinetic energy of scale n , and
- Conversion between the potential energy of the zonal mean and the potential energy of scale n .

(b) Exchanges involving triple product non-linearities (i.e. cross-scale exchanges):

- Triplet potential energy exchanges with wave numbers m and k such that $n = \pm(m \pm k)$, and
- Triplet kinetic energy exchanges with wave numbers m and k such that $n = \pm(m \pm k)$.

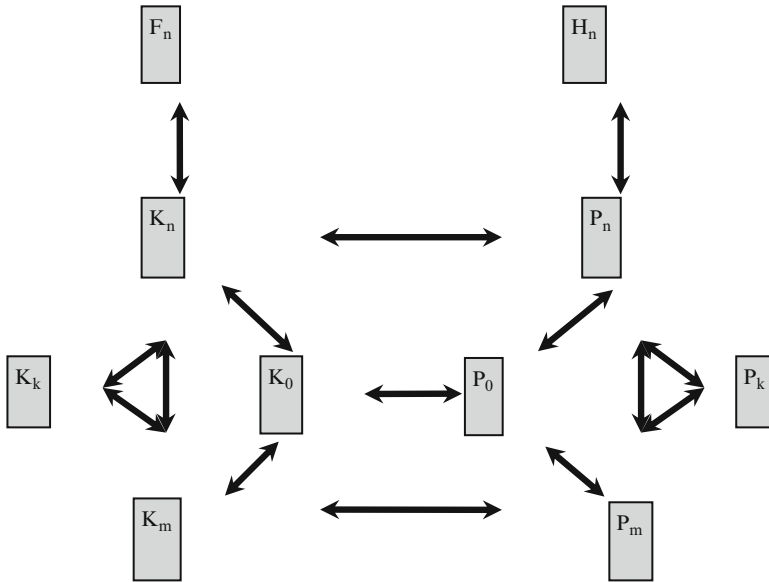


Fig. 8.1 Schematic illustration of the possible energy exchanges among the mean (subscript 0) and different scales (subscripts k, m, n)

8.3 Frequency Domain

A similar approach can be used to derive equations for the energy exchanges in a frequency domain. In that case, the Fourier decomposition of variables would be in time, instead of space, i.e., $f(t) = \sum_{n=-\infty}^{\infty} F(n)e^{i2\pi nt/T}$, where T is the length of the data set. In this case, n would denote the wave frequency (instead of wave number). Oscillations in time can be separated into a long term mean ($n = 0$), low-frequency modes (small n), and high frequency modes (large n). For an energy exchange among oscillations with frequencies n, m and k to take place, it is necessary that the selection rule $k = \pm(m \pm n)$ be satisfied. For example, if one uses a year-long data set ($T = 365$ days), the synoptic scale oscillations (2.5–5 days) have frequencies m , $n = (146-73)$. Madden-Julian oscillations, on the other hand have typical periods of 30–50 days, or $k = (12-7)$. The allowable interactions between MJO and the synoptic time scales then are:

- $m = 146, n = 139, k = 7; m = 146, n = 138, k = 8; \dots; m = 146, n = 134, k = 12;$
- $m = 145, n = 138, k = 7; m = 145, n = 137, k = 8; \dots; m = 146, n = 133, k = 12;$
- $m = 85, n = 78, k = 7; m = 85, n = 77, k = 8; \dots; m = 85, n = 73, k = 12$

8.4 Frequency Domain Examples

The maintenance of the MJO is an attractive problem for scale interactions. The time scale of the MJO (roughly 20–60 days) can in principle interact with several other time scales such as:

- (i) The tropical disturbance time scales (4–7 days); this would include tropical waves and depressions.
- (ii) The annual cycle.
- (iii) The El Nino-Southern Oscillation time scales.
- (iv) The long-term mean state, i.e., the climatology.

Sheng and Hayashi (1990a, b) examined this problem using 30 years of past global analysis data sets. The most important result they obtained was that the synoptic scale typical waves (on the time scales of 4–7 days) supplied kinetic energy towards the maintenance of the MJO time scale (20–60 days). This kinetic energy exchange from the synoptic to the MJO time scale invokes a number of triads of frequencies of which two frequencies generally reside in the synoptic time scales and one in the MJO time scale (the time scale is the inverse of the frequency). One such example is composed of 5 day and 6 day times scales interacting with the 30 day time scale, i.e., $1/30 = 1/5 - 1/6$. This satisfies the selection rule $n_{\text{MJO}} = n_{\text{synoptic1}} - n_{\text{synoptic2}}$.

Some of the other findings of the Shen and Hayashi studies are

- There is another family of triads that can provide potential energy to the scale of the MJO. These exchanges were generally quite small.
- The potential energy for the MJO time scale is largely received from in-scale heating on the MJO time scale.
- Kinetic energy on the MJO time scale also builds from overturnings (potential to kinetic energy conversion via the net ascent of warmer air and a descent of colder air on the same time scale). This is a smaller contribution compared to the triad exchange of kinetic energy for the synoptic to the MJO time scale.

This brings out interactions among triplets of waves. In a recent study, Krishnamurti et al. (2003) asked what scales were most important for the latent heat fluxes from the ocean to the atmosphere over the tropics on the time scale of MJO. They found again those triplet interactions of waves where two of the waves are in the synoptic range and one is in the MJO range stood out. A large number of such triplets of waves contribute to these latent heat fluxes. Recent results at the surface and at the top of the planetary boundary layer are shown in Figs. 8.2 (a through h) and 8.3 (a through h). These two illustrations show the respective annual fluxes for the tropical Indian and Western Pacific Oceans.

In these two illustrations, panels (a) through (h) show the following:

- (a) Map of the total fluxes on the time scale of the MJO.
- (b) Map of the total fluxes on the time scale of the MJO arising solely from the interactions of the MJO time scale with the synoptic time scale of 3–7 days.

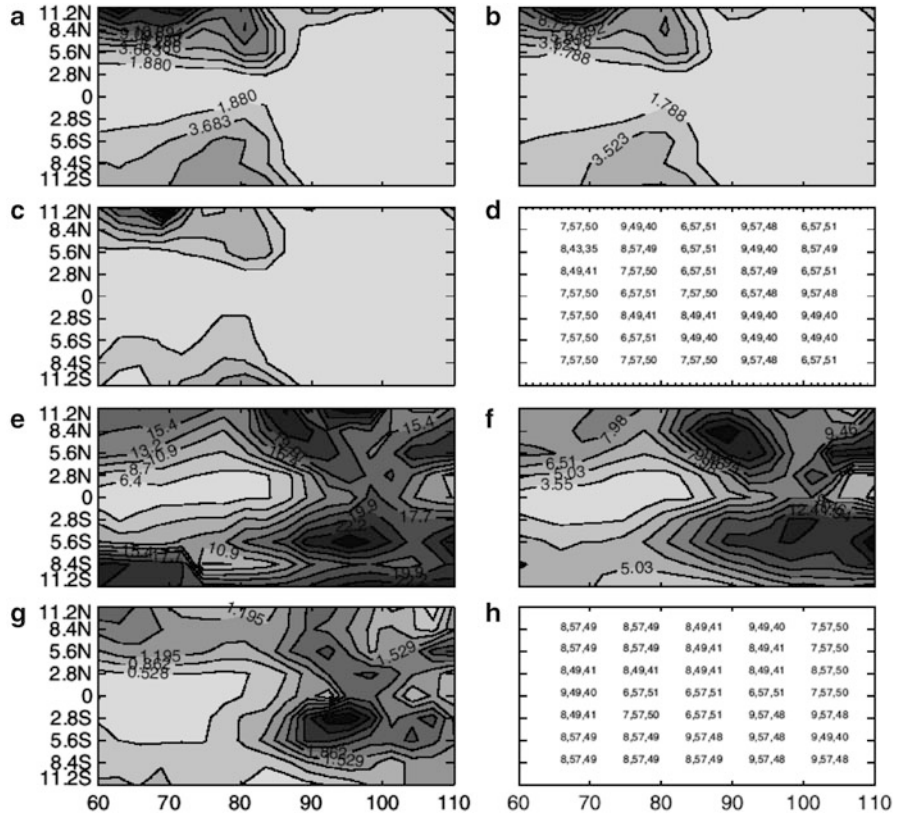


Fig. 8.2 Latent-heat fluxes (Wm^{-2}) over the Indian Ocean region. (a) Total latent-heat fluxes on the time-scale of the MJO across the constant-flux layer. (b) Total fluxes of latent heat across the constant-flux layer on the time-scale of the MJO arising from interaction of the MJO with the synoptic time-scale of 2–7 days. (c) Fluxes of latent heat contributed by salient (strongest contributing) triad interactions in the surface layer. (d) Salient triad interaction frequencies contributing to latent-heat fluxes on the time-scale of the MJO across the constant-flux layer. (e) Total latent-heat fluxes on the time-scale of the MJO in the planetary boundary layer (PBL) at 850 hPa. (f) Total latent-heat fluxes in the PBL on the time-scale of the MJO arising from interaction of the MJO time-scale with the synoptic time-scale of 2–7 days. (g) Latent-heat fluxes contributed by the salient triad interactions in the PBL. (h) Salient triad interaction frequencies contributing to latent-heat fluxes on the time-scale of the MJO in the PBL at 850 hPa (After Krishnamurti et al. 2003)

- (c) Contribution to the total fluxes in Panel b that arise from triads (MJO time scale interacting with pairs of synoptic time scale frequencies).
- (d) Salient triads of Panel c. A triad denoted as 7, 57, 50 is interpreted as follows. The time scales are 364/7, 364/57, and 364/50 days respectively. Here the triplet interactions involve a triple product of the surface wind V , the saturation specific humidity q_s at the ocean temperature T , and the stability-dependent drag coefficient CD .

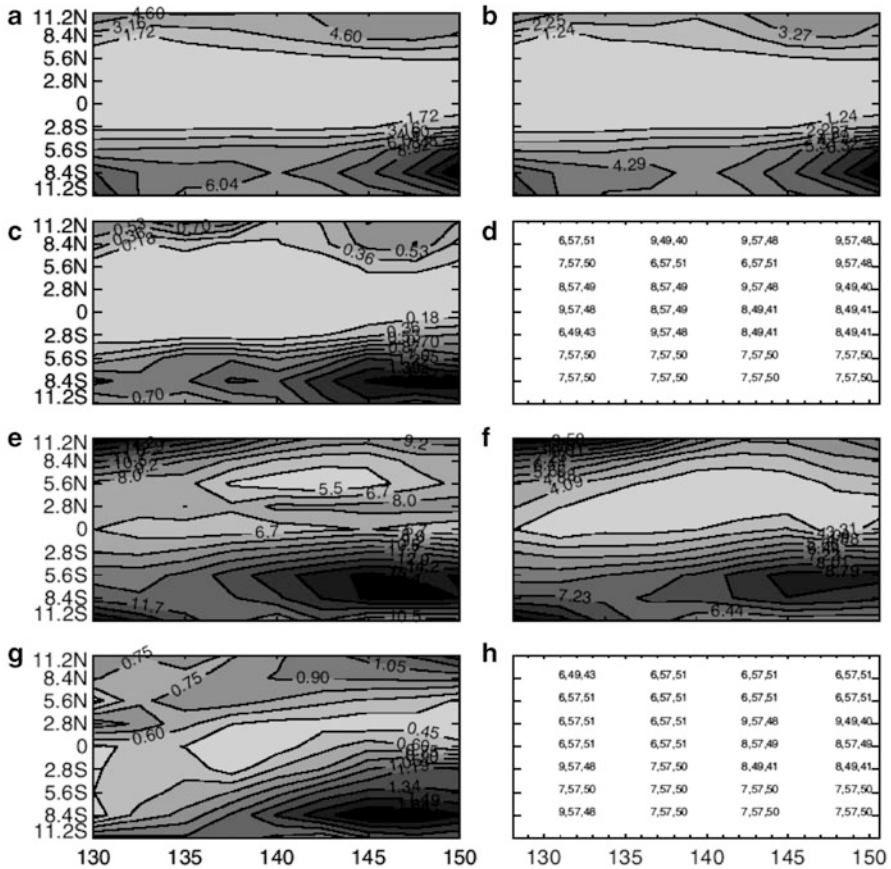


Fig. 8.3 As in Fig. 8.2, but for the West Pacific Ocean (After Krishnamurti et al. 2003)

Panels (a) through (d) refer to the constant flux layer, i.e., the surface level. Panels (e) through (h) show the same respective fields as panels (a) through (d) but at the top of the planetary boundary layer. All fluxes are in units of Wm^{-2} . The results over the Indian Ocean are quite similar to those over the Pacific. In summary, a substantial fraction of the total annual fluxes (nearly 15 %) occur on the time scale of MJO. Such fluxes on the MJO time scale from the ocean to the atmosphere require SST variability on the same scale. The synoptic disturbances (4–7 day time scales) seem to tap this energy from the ocean at the MJO time scale. If SST variability were very small on the MJO time scale, then the latent heat flux on the MJO time scale would be very small as well. It was found that there are some salient triads (illustrated in panels (d) and (h)) that contribute to a substantial portion of the total fluxes on the MJO time scale. These seem to involve MJO and the synoptic time scales.

The mechanism of the MJO that emerges from computations of scale interactions are as follows: Organized clouds are not found on the entire MJO scale encircling the global tropics. Clouds are thus not organized on the scale of the MJO. Where we do see clouds seemingly relating to an MJO wave, such as over the monsoon belt and the western tropical Pacific Ocean, are in fact clouds that relate to synoptic and mesoscale disturbances over those regions. The magnitude of convergence and divergence on the scale of the MJO is of the order of 10^{-6} to 10^{-7} is too weak to organize deep convection on the scale of the MJO. There is sufficient evidence to state that clouds are organized on the scales of tropical synoptic waves and imbedded mesoscale motions within the synoptic scales. In this context the term “organization of convection” implies organization of clouds around the scale of a synoptic disturbance, organization of the large centers contributing to the covariance of convective heating and temperature for the generation of available potential energy on the scale of organized convection and the synoptic scale disturbances. Furthermore the covariance of vertical velocity and temperatures also occurs on these same organized scales to provide eddy kinetic energy to the scale of that synoptic scale disturbance. This is a mechanism for the maintenance of the synoptic scales over the Asian monsoon and the western tropical pacific belts. These quadratic nonlinearities (these covariances) support inscale processes. The clouds not organized about the synoptic scale fail to meet the trigonometric selection rules and the entire scenario weakens. The next component of this scenario is on the maintenance of the MJO. That invokes an organization of dynamics. Here a pair of synoptic scales (such as 5 and 6 day time scale disturbances) can interact with a member of the MJO time scale (such as the 30 day time scale, because $1/5 - 1/6 = 1/30$ meets the triad selection rule) and provides energy to the MJO time scale. This energy exchange dictated by triads largely arises from the non linear advective dynamics terms and hence calls for an organization of dynamics to meet the energy exchange and the maintenance of the MJO.

8.5 Wave-Number Domain Examples

8.5.1 *Global Tropics*

In the context of the Asian summer monsoon the scale interactions are very revealing of the mechanisms for its maintenance. As the monsoon onset occurs, the planetary scale monsoon flow (zonal wave numbers 1 and 2) builds up very fast. Figure 8.4 shows the kinetic energy spectra before, during and after onset, as a function of zonal wave number. This kinetic energy is a mass averaged quantity between 50°E and 150°E , 30°S and 40°N , and between the surface and 100 hPa level. This clearly shows that soon after the onset in the month of June the planetary scale (i.e. wave numbers 1 and 2) show an explosive gain of energy. This planetary

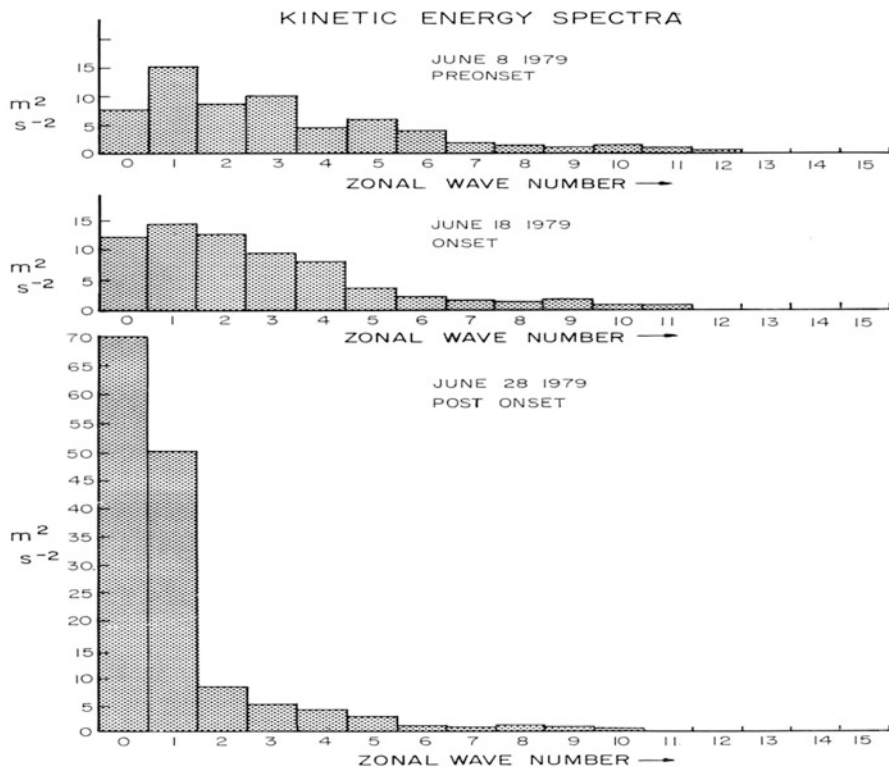


Fig. 8.4 Kinetic energy spectra (averaged between 50°E and 150°E, 30°S and 40°N, and surface to 100 hPa) before (*top*), during (*middle*) and after (*bottom*) the onset of the 1979 monsoon (Computed from NCEP-NCAR reanalysis data)

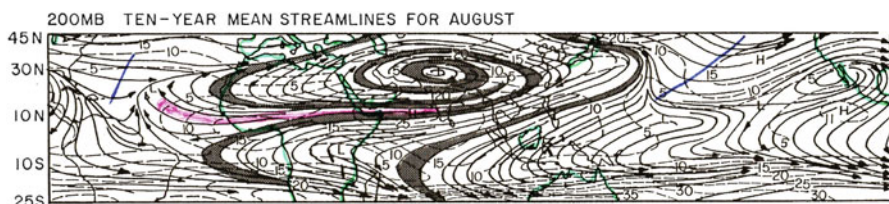


Fig. 8.5 Mean August streamlines at 200 hPa (Computed from NCEP-NCAR reanalysis)

scale can be seen in Fig. 8.5 where the time mean flows at 200 hPa are shown. This shows the geographical extent of the planetary scale monsoon as revealed by the buildup of the Tibetan High after the onset of the monsoon. This stretch of the monsoon extends for roughly 20°W to the date line in the Pacific Ocean. The troposphere below this upper anticyclonic complex is warm largely maintained by cumulus convection over Asia. The time averaged thermal field is illustrated in Fig. 8.6 that shows the 300 hPa temperatures. An east–west thermal asymmetry is

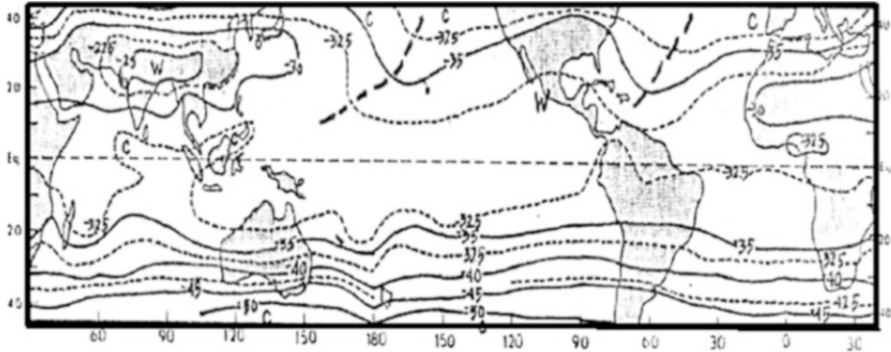


Fig. 8.6 Mean summer temperature (°C) at 300 hPa (Computed from NCEP-NCAR reanalysis)

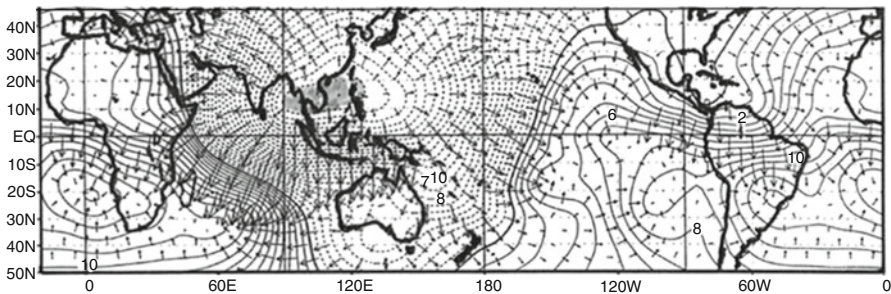


Fig. 8.7 Mean 200 hPa velocity potential ($10^6\text{m}^2\text{s}^{-2}$) and divergent wind (CDAS/reanalysis) for August 1996

also a part of this planetary scale feature. Cold lobes are found in the subtropical middle Atlantic and the middle Pacific. These are the places where the mid-oceanic upper troughs reside. Another related feature of importance in the context of scale interactions are the divergent east–west circulations that show slow ascent of mass in the region of the upper anticyclone and a slow descent in the region of the upper trough. This is best seen from the geometry of the velocity potential and the streamlines of the divergent wind (Fig. 8.7).

A schematic diagram (from Kanamitsu et al. 1972) of the gross energetics of the tropics for the summer season, derived using a scale interaction approach, is presented in Fig. 8.8. This illustrates the following scenario: There is a pronounced differential heating (H_L) in the east–west direction on the scale of long waves. This builds the available potential energy (P_L) on the scale of the long waves. The divergent east–west circulations carry a net covariance of vertical velocity and temperature, i.e., $-\frac{\omega_L T_L}{p} > 0$ and generate kinetic energy of the long waves. In the zonally averaged sense the summer Hadley cell energetics is also an important component of this scenario. Zonally averaged heating H_Z near 10°N and net cooling

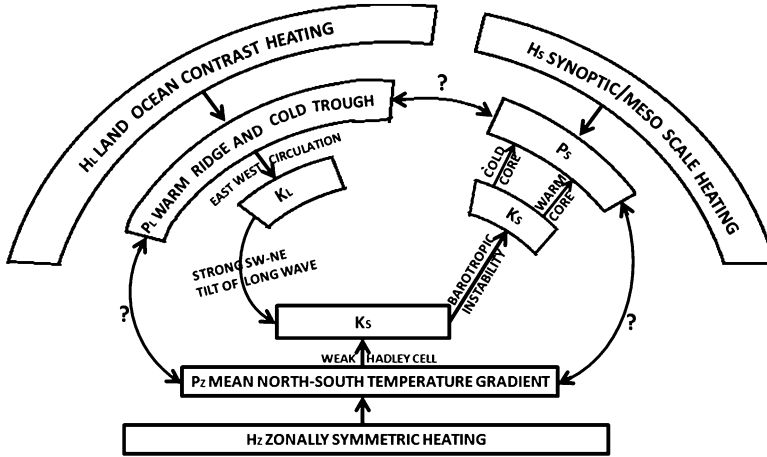


Fig. 8.8 Diagram of the energy exchanges in the tropical atmosphere for the zonal mean, short, medium and long waves (After Krishnamurti 1979). Units of $10^{-6} \text{m}^2 \text{s}^{-3}$

near 25°S generate zonal available potential energy P_z . The Hadley overturning of mass generates kinetic energy K_z from the ascent of warmer air near 10°N and a descent in the southern hemisphere near 25°S .

Shorter time and space scale disturbances, indicated by the subscript S, include tropical waves, depressions and storms. Some are cold core disturbances while others are characterized by a warm core. These can have a conversion from the available potential to the eddy kinetic energy or vice versa, depending on the structure and thermal status.

A number of other arms of the schematic diagram shown in Fig. 8.8 show the lateral energy exchanges among scales that signify scale interactions. Non-linear kinetic energy exchanges from the longer to the shorter waves of the tropical have been demonstrated. There is also considerable evidence of the transfer of kinetic energy from the zonal scales to these shorter scales (Krishnamurti et al. 1975b). The non-linear exchange of potential energy among scales is somewhat uncertain; however current data suggest a down-the-scale transfer from long waves P_L to shorter waves P_S . Over the monsoon complex, a long wave giant sea breeze line system sets up. This large scale system supplies energy to other scales via the mechanism of scale interaction.

8.5.2 Hurricanes

Another application of the scale interaction approach is towards understanding the scale interactions within a hurricane. There a cylindrical coordinate system is used, and the Fourier decomposition is done in the azimuthal direction. Azimuthal wave

number zero corresponds to the hurricane mean circulation, wave numbers one and two correspond to the hurricane asymmetry, and higher wave numbers can be related to contributions on the cloud scales. There are energy exchanges among various forms of energy such as zonal and eddy energies for kinetic and potential and nonlinear exchanges among the same types such as kinetic to kinetic, and eddy available to eddy available, these are exchanges among different scales. Appendix 1 provides the mathematical formulations.

- (i) $\langle P_o \cdot K_o \rangle$ is the generation of eddy available potential energy and its transformation to eddy kinetic energy, which is the storm's kinetic energy, relies on the magnitudes of two covariances. These are the products of convective heating and temperature, and of the vertical velocity and temperature. The eye wall warming compared to the surroundings and the descent around the eye wall and the rain bands plus the convective heating in heavy rain areas contribute to these processes.
- (ii) $\langle P_l \cdot K_l \rangle$ The scale of a hurricane can be assessed from a Fourier analysis of winds along the azimuthal coordinate for inner radii that includes the maximum wind belt. Such an analysis reveals that most of the azimuthal variance resides in the azimuthal mean, i.e. wave number zero, and the first two or three long waves around the azimuthal coordinate (here expressed in a local cylindrical coordinate about the storm's center as the origin). These long waves and the wave number zero are the ones that contribute the most to the conversion of eddy available potential energy to eddy kinetic energy for the hurricane scale. That arises from the vertical overturning, the Hadley type overturning for wave number zero. This happens also on the long wave scales where the rising of warm air the descent of relatively cooler air contributes this covariance and hence to a strengthening of the hurricane. The end result of such energy conversions is the increase of eddy kinetic energy and the strengthening of the hurricane. Disruption of the organization of convection can also, in a reverse manner lead to a weakening of a hurricane.
- (iii) $\langle P_c \cdot K_c \rangle$ is the energy conversion from eddy available potential to eddy kinetic energy P_c to K_c on cloud scales can only generate kinetic energy on the cloud scales and not on the hurricane scale. The reason for that is this conversion invokes a quadratic nonlinearity among vertical motion and temperature and all such quadratic processes can only do in-scale conversions.
- (iv) $\langle H_o \cdot P_o \rangle$ is the generation of available potential energy on the cloud scale H_o to P_o defines the generation from heating on the cloud scale where it is already warm and cooling where it is relatively less war. This is a quadratic term and will only describe an in-scale process hence this does not address directly the hurricane scales. The covariance of cloud scale convective heating and the cloud scale temperatures are involved in this computation.
- (v) $\langle H_l \cdot P_l \rangle$ is an important component of hurricane energetics. Here the heating generates available potential energy on the long azimuthal hurricane scales by heating where it warm and cooling where it is less warm.
- (vi) $\langle H_c \cdot P_c \rangle$ generate available potential energy on small scales.

- (vii) $\langle P_s \cdot P_l \rangle$ P_s to P_l denotes a transfer of available potential energy among the short and the long wave scales. This computation invokes triple product nonlinearity and calls for sensible heat transfer among the short and the long waves via heat transfer that can be up or down the thermal gradient of the long waves. That transfer is from long to short waves if the short waves transfer sensible heat from the long to the short waves (thus building the latter) or vice versa. This is not found to be a major contributor to the overall driving of a hurricane energetics.
- (viii) $\langle K_s \cdot K_l \rangle$ is the direct energy exchange of kinetic energy among short and long waves. This invokes a triple product non linearity, such that generally two short waves interact with a long wave resulting in a gain or a loss of kinetic energy for the long wave. That sign of exchange is dictated by the covariance of the local long wave flows and the convergence of eddy momentum flux by the short waves. If the planetary scale monsoon builds the long scales kinetic energy from the available potential energy of long waves that energy is generally dispersed to shorter waves by this triad of wave mechanism. In a hurricane this nonlinear exchange is not a very important contributor for the mutual exchanges among azimuthal long and short waves.
- (ix) $\langle K_0 \cdot K_l \rangle$ – The exchange from K_0 to K_l denotes the conventional barotropic process from zonal to long waves scales. This invokes the covariance of azimuthally averaged flows and the convergence of eddy flux of momentum by the long wave scales. In hurricane dynamics, and also for the planetary scale monsoon, this is an important contributor.
- (x) $\langle K_0 \cdot K_s \rangle$ – The exchange of kinetic energy from the azimuthally averaged zonal kinetic energy to short wave scales is simply another component of the barotropic stability/instability problem. This invokes the covariance of zonal flows and the eddy convergence of flux of kinetic energy by the shorter waves. This is a nontrivial effect but does not directly seem to be important for the hurricane scales.
- (xi) $\langle P_o \cdot P_l \rangle$ – The exchange P_0 to P_l of available potential energy from the azimuthally averaged motions to the scale of long azimuthal waves $\langle P_0, P_l \rangle$ is an important ingredient of hurricane dynamics. The wave number zero, i.e. the azimuthal mean carries the warm core and it exchanges available potential
- (xii) $\langle P_o \cdot P_c \rangle$ – This denotes the exchange of potential energy from the zonal available potential energy to that of long waves. This relates to sensible heat flux down or up the thermal gradient of the azimuthally averaged flows.

Organization of convection occurs when individual cumulonimbus clouds or cloud clusters evolve into a large-scale cluster over a period of time (Moncrieff 2004). In a simple linear model it is a positive feedback loop, whereby intense latent heating generates lifting, creates divergence aloft, and lowers the pressure at the surface. Lower surface pressure increases the pressure gradient at low levels and creates stronger low level winds. This leads to stronger convergence into the center of the storm and enhanced convection. The whole process then repeats itself.

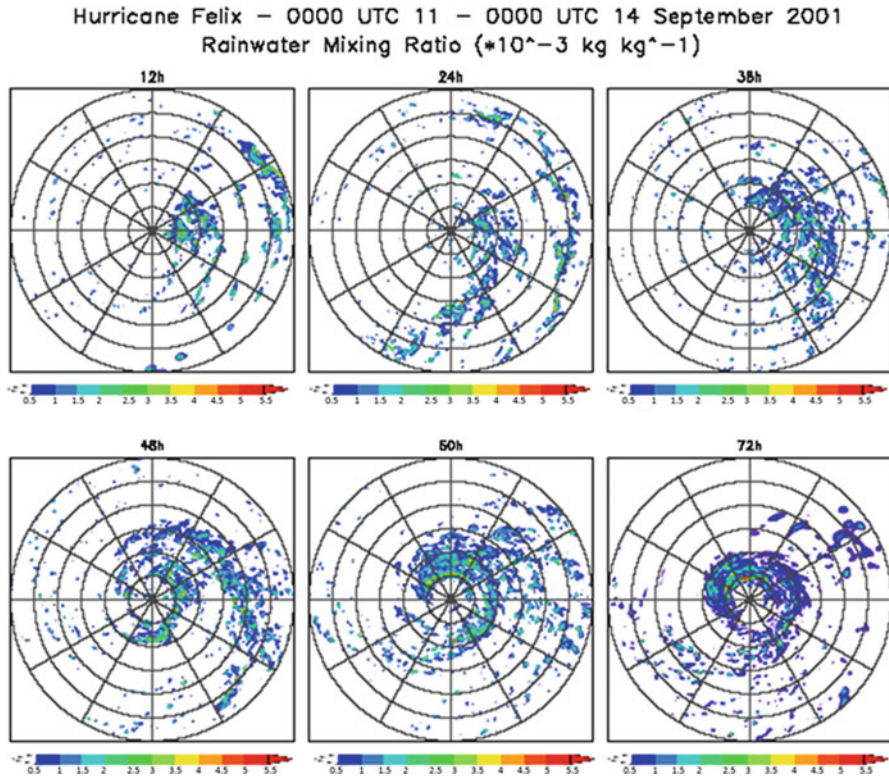


Fig. 8.9 Distribution of rainwater mixing ratio ($\times 10^{-3} \text{ kg kg}^{-1}$) at 900 mb for hours 12, 24, 36, 48, 60, and 72 of the forecast for Hurricane Felix shown in storm centered coordinates. Radial increments are 0.5° ($\sim 56 \text{ km}$)

To show organization of convection in the simulated hurricanes, the amplitude spectra of the rainwater mixing ratio at various wavenumbers was examined. Rainwater mixing ratio was chosen as it can indicate the features of deep convective cloud elements in a hurricane (Krishnamurti et al. 2005). Organization of convection was illustrated in this way for three reasons: (1) amplitude is the measure of the strength of the signal, (2) it can be shown that as the storm is forming, the strength of the signal of certain wavenumbers increases, and (3) the part of the storm in which organization is occurring can be ascertained, i.e., determine which radii have the strongest signals. Each rainwater mixing ratio data point in cylindrical coordinates at the 900 mb level for all hurricane simulations was Fourier transformed and time-radius plots of the amplitude spectra were then plot for wavenumbers 0, 1, 2, and 25. Because the energy falls off very rapidly beyond wavenumber 3, we have lumped all other wavenumbers beyond this together and refer to them collectively as the “cloud-scale”. Hence, wavenumber 25 was randomly chosen as a representative for any cloud-scale harmonic.

Figures 8.9 and 8.10 show the predicted rainwater mixing ratio (kg kg^{-1}) at 900 mb and winds at 900 mb, respectively, for the 72 h forecast for Hurricane Felix.

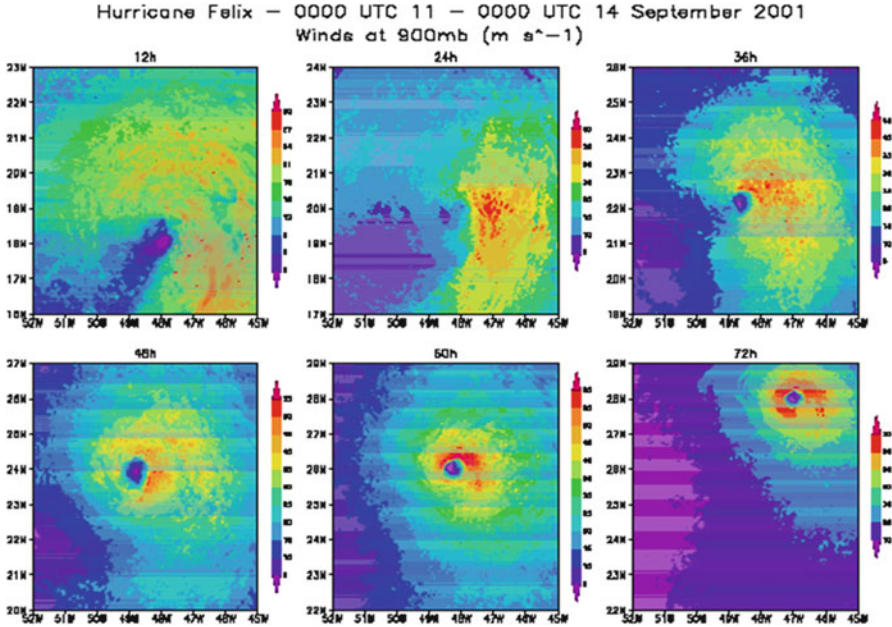


Fig. 8.10 Winds ($m s^{-1}$) at 900 mb for hours 12, 24, 36, 48, 60, and 72 of the forecast for Hurricane Felix. Partial 1.67 km domain is shown

From 120 to 36 h, the storm was very asymmetrical with maximum values of rainwater mixing ratio occurring to the southeast and east of the storm center and the maximum winds at 900 mb occurring to the east of the storm center. Hence, a stronger signal from wavenumber 1 during these hours would be expected. From 48 to 72 h, the storm becomes increasingly more symmetric and the largest values of amplitude can be expected to occur in wavenumber 0. However, the storm is still somewhat asymmetrical.

The central issue in the energetics is that deep cumulus convection drives a hurricane only from an azimuthal organization of clouds on the scale of the hurricane. Plethora of single clouds, if they are not organized, then they cannot contribute to the generation of available potential energy and its disposition to the kinetic energy of a hurricane. Intensity changes can be expected as the organization evolves. In the early stages of a hurricane formation dynamical instabilities play an important role in organizing convection by providing a steering of clouds along a circular geometry. That happens in barotropic processes when shear flow vorticity is transformed into curvature flow.

The hurricane scales are largely described by azimuthal wave numbers 0, 1, 2 and 3. The mechanism of a mature hurricane energetics relies on the organization of convection on the hurricane scale. On these scales convective heating and temperature carry a large positive value for their covariance; this generates available potential energy on the scale of the hurricane. That energy is disposed off to

kinetic energy for all those respective azimuthal scales 0, 1, 2 and 3 by the individual scale vertical overturning. This is largely the warm air rising and the relatively cooler air descending. That is the conversion of available potential energy to the eddy kinetic energy for each scale that happens separately mathematically since this is an inscale process. Both of the above energy processes are described by quadratic non linearities. The other energy exchanges to other shorter scales carry much smaller magnitudes. The barotropic exchanges from azimuthally averaged flow kinetic energy to hurricane scales can also be somewhat important. The nonlinear triad interactions are generally small but are non trivial.

The total available potential energy for all scales summed together is shown in Fig. 8.11. The histograms show these results for wave numbers in groups. These results also grouped for inner radii less than 40 km and for larger outer radii greater than 40 km. Overall the results shows the largest contributions from the wave number 0 which are the azimuthally averaged modes.

The conversions for wave numbers 1 and 2 are about half as large as those for wave number domain for the middle latitude zonally averaged jet, wave number 180 was in fact small and even fluctuating in sign for different smaller scales. The energetics in the wave number domain for the middle latitude zonally averaged jet, wave number 0, is opposite to that for a hurricanes azimuthal wave number 0. Waves provide energy to the wave number 0 in the former case whereas they seem to remove the energy from the hurricane circulation. The former is generally regarded as a low Rossby number phenomenon whereas the latter is clearly one where the Rossby number exceeds one. Although the cloud scales are very small close to wave number 80, the organization of convection makes the available potential energy at wave number 0 and also for the long waves as the primary contribution (Fig. 8.12).

Figure 8.13 is a schematic diagram of the directions of energy transfer for the mature stage of hurricane Katrina of 2005. It illustrates the difference between the energy exchanges in the inner radii of a hurricane and the exchanges near the area of maximum winds. At the inner radii the sinking of warm air (thermally indirect circulation) leads to a conversion of kinetic to potential energy. The converse is true outside the inner radii. In the inner radii, both the kinetic energy transfer and the potential energy transfer is generally from the scales of the asymmetry and clouds to the azimuthal mean. Again, the converse is true outside of this region.

Appendix 1: Derivation of Equations in Wave-Number Domain

The classical Lorenz approach separates the motion into zonal mean and eddy, and stationary and transient. Here we shall show an approach developed by Saltzman (1957) for the wave number domain and expanded by Hayashi (1980) for use in the frequency domain.

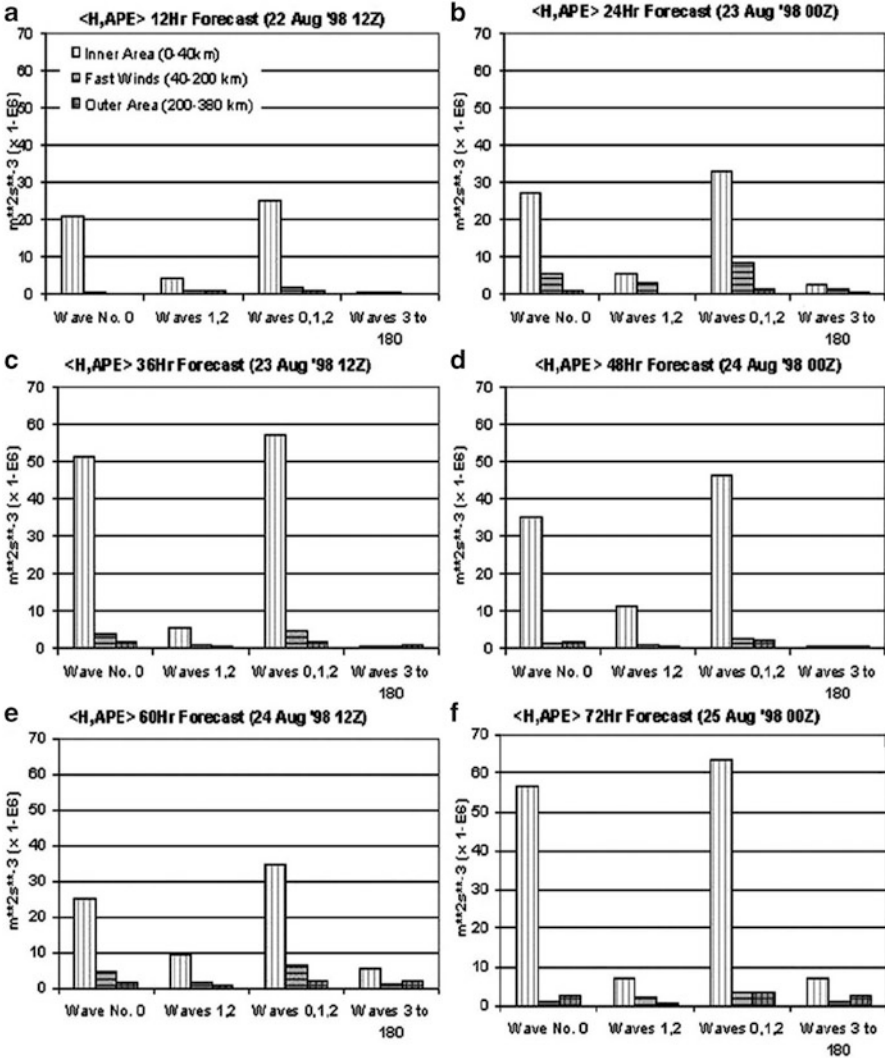


Fig. 8.11 Generation of available potential energy for Hurricane Bonnie. Three different histograms in each panel represent three regions—inner area (0–40 km), fast winds (40–200 km), and outer area (200–380 km). Forecast times are (a) 12, (b) 24, (c) 36, (d) 48, (e) 60, and (f) 72 h of the model output. Units are $m^2 s^{-3} (\times 10^{-6})$ (After Krishnamurti et al. 2005)

We shall follow the procedure adopted by Salzman (1957) to derive the equations for the rate of growth of a scale (wavenumber n) as it interacts with the zonal mean flow or with any other scales. This requires some preliminaries on Fourier transforms.

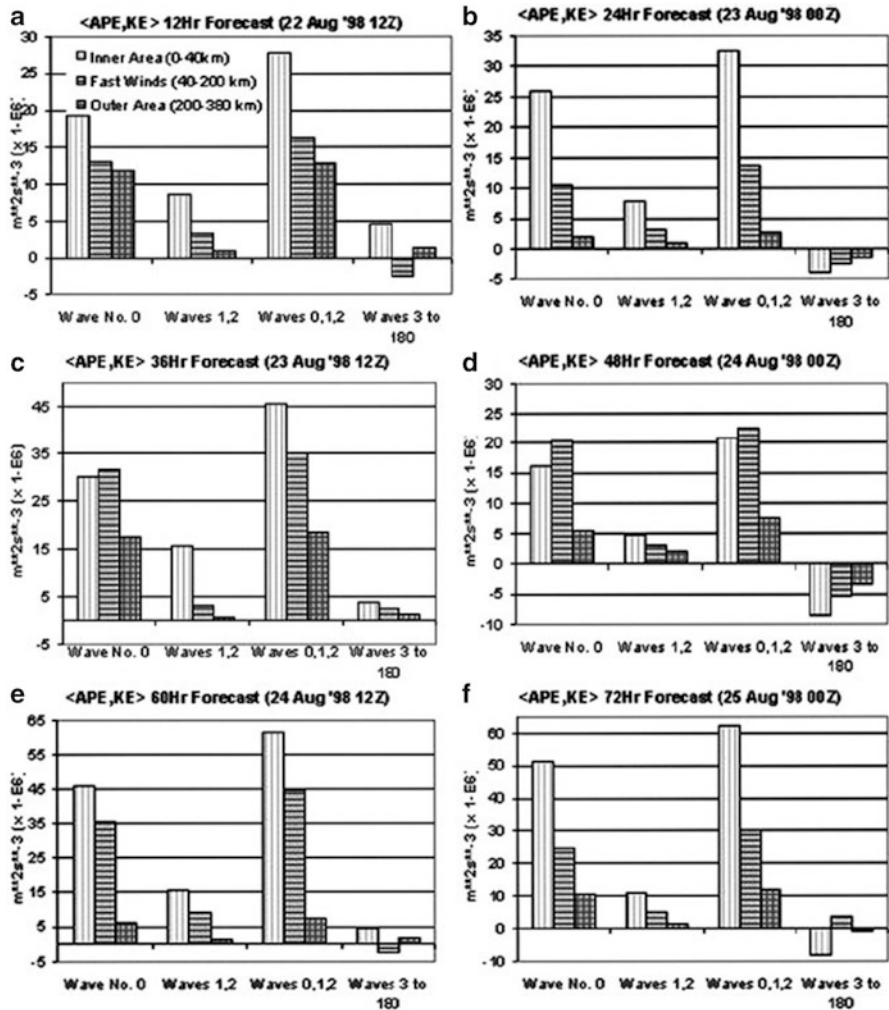
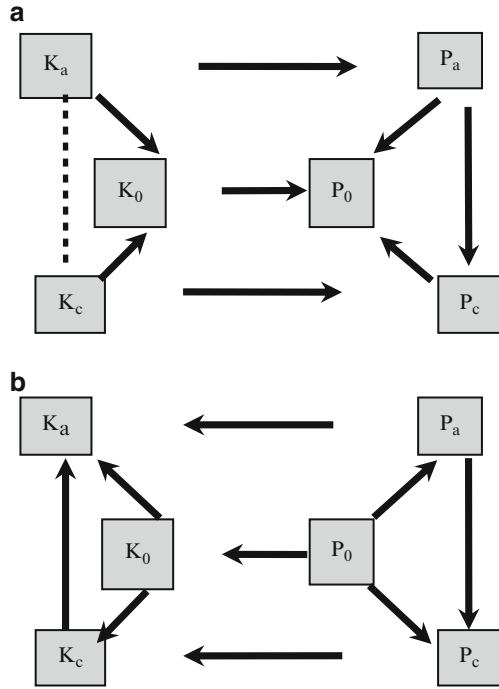


Fig. 8.12 Generation of kinetic energy by vertical overturning for Hurricane Bonnie. Three different histograms in each panel represent three regions—inner area (0–40 km), fast winds (40–200 km), and outer area (200–380 km). Forecast times are (a) 12, (b) 24, (c) 36, (d) 48, (e) 60, and (f) 72 h of the model output. Units are $m^2 s^{-3} (\times 10^{-6})$ (After Krishnamurti et al. 2005)

Any smooth variable can be subjected to Fourier transform along a latitude circle:

$$f(\lambda) = \sum_{n=-\infty}^{\infty} F(n)e^{in\lambda}, \tag{8.1}$$

Fig. 8.13 Schematic diagram of the directions of energy transfer for the mature stage of hurricane Katrina (2005) (a) inwards of the area of maximum winds, and (b) outside of this area



where the complex Fourier coefficients $F(n)$ are given by

$$F(n) = \frac{1}{2\pi} \int_0^{2\pi} f(\lambda) e^{-in\lambda} d\lambda. \tag{8.2}$$

The Fourier transform of the product of two functions can be expressed as:

$$\frac{1}{2\pi} \int_0^{2\pi} f(\lambda)g(\lambda)e^{-in\lambda} d\lambda = \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=-\infty}^{\infty} F(k)e^{ik\lambda} \sum_{m=-\infty}^{\infty} G(m)e^{im\lambda} e^{-in\lambda} d\lambda. \tag{8.3}$$

From the orthogonality property of the exponents it follows that the only non-zero contribution from this integral arises when $k + m - n = 0$, i.e., $k = n - m$.

The expression for the Fourier transform of the product of two functions then becomes

$$\frac{1}{2\pi} \int_0^{2\pi} f(\lambda)g(\lambda)e^{-in\lambda} d\lambda = \sum_{m=-\infty}^{\infty} F(n - m)G(m) \tag{8.4}$$

This relationship relating the product of two functions to their Fourier transforms is known as Parseval's Theorem. Since the Fourier coefficients are independent of λ , the Fourier expansion of the derivative of a function with respect to λ is

$$\frac{\partial f(\lambda)}{\partial \lambda} = \sum_{n=-\infty}^{\infty} F(n) \frac{\partial e^{in\lambda}}{\partial \lambda} = \sum_{n=-\infty}^{\infty} inF(n)e^{in\lambda} \quad (8.5)$$

The derivative of a function with respect to any other independent variable α is simply

$$\frac{\partial f(\lambda)}{\partial \alpha} = \sum_{n=-\infty}^{\infty} \frac{\partial F(n)}{\partial \alpha} e^{in\lambda} \quad (8.6)$$

Here we will start with the equations of motion, mass continuity, the hydrostatic law and the first law of thermodynamics. Next, we shall perform a Fourier transform of all the above equations. Using these, we shall then derive an eddy kinetic energy equation K_n for the single zonal harmonic wave n . We will also derive a similar equation for the available potential energy of a single harmonic P_n . We will next sort these equations to examine the rates of growth (or decay) of K_n and P_n as they interact with other waves or as they interact with the zonal flows and vice versa.

The equations of motion in spherical coordinate system with hydrostatic assumption are:

– The zonal equation of motion:

$$\frac{\partial u}{\partial t} = -\frac{u}{a \cos \varphi} \frac{\partial u}{\partial \lambda} - \frac{v}{a} \frac{\partial u}{\partial \varphi} - \omega \frac{\partial u}{\partial p} + v \left(f + \frac{u \tan \varphi}{a} \right) - \frac{g}{a \cos \varphi} \frac{\partial z}{\partial \lambda} - F_1, \quad (8.7)$$

– The meridional equation of motion:

$$\frac{\partial v}{\partial t} = -\frac{u}{a \cos \varphi} \frac{\partial v}{\partial \lambda} - \frac{v}{a} \frac{\partial v}{\partial \varphi} - \omega \frac{\partial v}{\partial p} - u \left(f + \frac{u \tan \varphi}{a} \right) - \frac{g}{a} \frac{\partial z}{\partial \varphi} - F_2 \quad (8.8)$$

– The continuity equation:

$$\frac{\partial \omega}{\partial p} = - \left(\frac{1}{a \cos \varphi} \frac{\partial u}{\partial \lambda} + \frac{1}{a} \frac{\partial v}{\partial \varphi} - \frac{v \tan \varphi}{a} \right) \quad (8.9)$$

– The hydrostatic equation:

$$\frac{\partial z}{\partial p} = - \frac{RT}{gp} \quad (8.10)$$

– The First law of thermodynamics:

$$\frac{dT}{dt} = \frac{h}{C_p} + \frac{\omega RT}{C_p p} \quad (8.11)$$

where λ is the longitude, φ is the latitude, p is the pressure, u and v are the eastward and northward wind components, ω is the pressure vertical velocity, z is the pressure surface height, a is the earth's radius, T is the temperature, f is the Coriolis parameter, R is the ideal gas constant, C_p is the specific heat at constant pressure, F_1 and F_2 are the u and v components of the frictional force and h is the heating.

Equations 8.7 through 8.11 form a closed system for the dependent variables (u , v , ω , z , T) provided the heating h is known. Based on this system, in the following section we shall derive the equations for the scale interactions involving the kinetic and potential energy.

Integrating (8.7) through (8.11) from 0 to 2π , and using the notation listed in the table below, one can write down the Fourier transforms of these variables:

Variable	$u(\lambda)$	$v(\lambda)$	$\omega(\lambda)$	$z(\lambda)$	$T(\lambda)$
Fourier transform	$U(n)$	$V(n)$	$\Omega(n)$	$Z(n)$	$Y(n)$

We shall next do the Fourier transform of the closed set of Eqs. 8.7 through 8.11 as follows:

– The zonal equation of motion:

$$\begin{aligned} \frac{\partial U(n)}{\partial t} = & - \sum_{m=-\infty}^{\infty} \left[\frac{im}{a \cos \varphi} U(m)U(n-m) + \frac{1}{a} U_{\varphi}(m)V(n-m) \right. \\ & \left. + U_p(m)\Omega(n-m) - \frac{\tan \varphi}{a} U(m)V(n-m) \right] \\ & - \frac{ing}{a \cos \varphi} Z(n) + fV(n) - F_1(n) \end{aligned} \quad (8.12)$$

– The meridional equation of motion:

$$\begin{aligned} \frac{\partial V(n)}{\partial t} = & - \sum_{m=-\infty}^{\infty} \left[\frac{im}{a \cos \varphi} V(m)U(n-m) + \frac{1}{a} V_{\varphi}(m)V(n-m) \right. \\ & \left. + V_p(m)\Omega(n-m) + \frac{\tan \varphi}{a} U(m)U(n-m) \right] \\ & - \frac{g}{a} Z(n) - fU(n) - F_2(n) \end{aligned} \quad (8.13)$$

– The mass continuity equation:

$$\Omega_p(n) = - \left[\frac{in}{a \cos \varphi} U(n) + \frac{1}{a} V_\varphi(n) - \frac{\tan \varphi}{a} V(n) \right] \quad (8.14)$$

– The hydrostatic equation

$$Z_p(n) = - \frac{R}{gp} Y(n) \quad (8.15)$$

– The first law of thermodynamics:

$$\begin{aligned} \frac{\partial Y(n)}{\partial t} = & - \sum_{m=-\infty}^{\infty} \left[\frac{im}{a \cos \varphi} Y(m) U(n-m) + \frac{1}{a} Y_\varphi(m) V(n-m) \right. \\ & \left. + Y_p(m) \Omega(n-m) - \frac{R}{C_p p} Y(m) \Omega(n-m) \right] \\ & - \frac{1}{C_p} H(n) \end{aligned} \quad (8.16)$$

Equations 8.12, 8.13, 8.14, 8.15, and 8.16 form a closed system for the Fourier amplitudes of the dependent variables. The zonally averaged kinetic energy is given by

$$\bar{k} = \frac{1}{2\pi} \int_0^{2\pi} \frac{u^2 + v^2}{2} d\lambda = \frac{U(0)^2 + V(0)^2}{2} + \sum_{n=1}^{\infty} \left[|U(n)|^2 + |V(n)|^2 \right]. \quad (8.17)$$

The first term on the right hand side is the kinetic energy of the zonal mean. The second term is the sum of the spectral eddy kinetic energy corresponding to all wavenumber n , given by $K(n) = |U(n)|^2 + |V(n)|^2$.

Similarly to (8.12) and (8.13), equations can be written for $\frac{\partial U(-n)}{\partial t}$ and $\frac{\partial V(-n)}{\partial t}$. Multiplying (8.12) with $U(-n)$ and (8.13) with $V(-n)$ and the expressions for $\frac{\partial U(-n)}{\partial t}$ and $\frac{\partial V(-n)}{\partial t}$ by $U(n)$ and $V(n)$ respectively, and adding the resulting equations together, one obtains an equation for the rate of change of the kinetic energy of wave number n :

$$\begin{aligned}
\frac{\partial K(n)}{\partial t} = & \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \left\{ U(m) \left(\frac{1}{a \cos \varphi} \Psi_{uu_\lambda}(m, n) + \frac{1}{a} \Psi_{vu_\varphi}(m, n) + \Psi_{\omega u_p}(m, n) - \frac{\tan \varphi}{a} \Psi_{uv}(m, n) \right) \right. \\
& + V(m) \left(\frac{1}{a \cos \varphi} \Psi_{uv_\lambda}(m, n) + \frac{1}{a} \Psi_{vv_\varphi}(m, n) + \Psi_{\omega v_p}(m, n) + \frac{\tan \varphi}{a} \Psi_{uu}(m, n) \right) \\
& - \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} [\cos \varphi (U(m) \Psi_{vu}(m, n) + V(m) \Psi_{vv}(m, n))] \\
& \left. - \frac{\partial}{\partial p} (U(m) \Psi_{ou}(m, n) + V(m) \Psi_{ov}(m, n)) \right\} \\
& - \left[\Phi_{uv}(n) \frac{\cos \varphi}{a} \frac{\partial}{\partial \varphi} \left(\frac{U(0)}{\cos \varphi} \right) - \Phi_{vv}(n) \frac{1}{a} \frac{\partial V(0)}{\partial \varphi} + \Phi_{\omega v}(n) \frac{\partial U(0)}{\partial p} \right. \\
& \left. + \Phi_{vo}(n) \frac{\partial V(0)}{\partial p} - \Phi_{uu}(n) V(0) \frac{\tan \varphi}{a} \right] \\
& - \left[\frac{1}{a \cos \varphi} \Phi_{uz_\lambda}(n) + \frac{1}{a} \Phi_{vz_\varphi}(n) \right] - [\Phi_{uF_1}(n) + \Phi_{uF_2}(n)]
\end{aligned} \tag{8.18}$$

Where

$$\begin{aligned}
\Psi_{ab}(m, n) &= A(n-m)B(-n) + A(-n-m)B(n), \text{ and} \\
\Phi_{ab}(n) &= A(n)B(-n) + A(-n)B(n),
\end{aligned}$$

where a and b are any two dependent variables, and A and B are their respective Fourier transforms. Subscripts in (8.19) indicate derivatives, i.e., for example, $u_\lambda \equiv \frac{\partial u}{\partial \lambda}$. (8.18) is the equation for the rate of change of kinetic energy $K(n)$ for a particular wave number n . On the face of it this looks like a complex mess. However it can be sorted out to make it physically attractive and interpretable.

In a symbolic form Eq. 8.18 can be expressed as

$$\frac{\partial K_n}{\partial t} = \langle (K_m, K_p) \cdot K_n \rangle + \langle K_0 \cdot K_n \rangle + \langle P_n \cdot K_n \rangle + F_n$$

The terms on the right hand side are as follows:

- (i) The first term denotes the transfer of kinetic energy to the scale of n from pairs of other wave numbers m and p . This follows a trigonometric selection rule, i.e. $n = m + p$ or $n = |m - p|$ for an exchange; if the three wave numbers n , m and p do not satisfy the selection rule, the exchanges are zero. This is an important dynamical term in scale interactions since it allows three different scales to interact resulting in gain or loss of energy from one to the other two. This permits 'out of scale' energy exchanges.

- (ii) The second term denotes the gain of energy by a wave number n when it interacts with the zonal flow 0. This is also identified as barotropic energy exchange and can be related to horizontal shear flow instabilities.
- (iii) The third term denotes a growth of kinetic energy K_n from the eddy available potential energy on the same scale P_n . This is an in-scale (n) energy transfer. This is largely defined by the domain integral of $-\frac{\Phi_n(\omega, T)}{p}$, i.e. it largely involves the covariance of vertical velocity ω and the temperature T on the scale of n . If within a domain of interest warm air is rising and relatively colder air is sinking, this covariance would be positive and the net contribution to $\frac{\partial K_n}{\partial t}$ would be positive, and vice versa.
- (iv) The last term denotes the net gain or loss of kinetic energy K_n for a scale n from frictional dissipation.

The eddy available potential energy of a wave number n is defined as $P(n) = C_p \gamma |Y(n)|^2$, where $\gamma = -\frac{R^2}{C_p p^{R/c_p}} \left(\frac{\partial \theta}{\partial p}\right)^{-1}$. In a similar manner, we obtain

$$\begin{aligned}
 \frac{\partial P(n)}{\partial t} &= C_p \gamma \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} Y(m) \left(\frac{1}{a \cos \varphi} \Psi_{uT_\lambda}(m, n) + \frac{1}{a} \Psi_{vT_\varphi}(m, n) + \Psi_{\omega T_p}(m, n) + \frac{R}{C_p p} \Psi_{\omega T}(m, n) \right) \\
 &\quad - \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} \cos \varphi Y(m) \Psi_{vT}(n, m) - \frac{\partial}{\partial p} Y(m) \Psi_{\omega T} \\
 &\quad - \left[\frac{C_p \gamma}{a} \Phi_{Tv}(n) \frac{\partial Y(0)}{\partial \varphi} + \frac{\gamma p^\mu}{\mu} \left\{ \Phi_{T\omega}(n)'' \frac{\partial \bar{\theta}''}{\partial p} \right\} \right] \\
 &\quad + \frac{R}{p} \Phi_{T\omega}(n) + \gamma \Phi_{Th}(n)
 \end{aligned} \tag{8.19}$$

as the expression for the local rate of change of the available potential energy of wave number n due to non-linear interactions with wave numbers m and $n \pm m$, interactions with the mean flow (wave number zero), and quadratic (in-scale) interactions (wave number n).

Appendix 2: A Simple Example

Let us assume for simplicity that we are given an equation of motion

$$\frac{\partial u}{\partial t} = -\frac{u}{a \cos \varphi} \frac{\partial u}{\partial \lambda} \tag{8.20}$$

The zonal wind $u(\lambda, t)$ can be expanded into Fourier series, following Eq. 8.1 as

$$u(\lambda) = \sum_{n=-\infty}^{\infty} U(n)e^{in\lambda}, \quad (8.21)$$

where the complex Fourier coefficients $U(n)$ are given, following (8.2), by

$$U(n) = \frac{1}{2\pi} \int_0^{2\pi} u(\lambda)e^{-in\lambda} d\lambda. \quad (8.22)$$

Also, by the Parseval's theorem (from Eq. 8.3),

$$\frac{1}{2\pi} \int_0^{2\pi} (u(\lambda))^2 e^{-in\lambda} d\lambda = \sum_{m=-\infty}^{\infty} U(n)U(n-m). \quad (8.23)$$

Now if both sides of (8.20) are multiplied by $1/(2\pi)$ and integrated over a longitude circle, the resulting tendency equation for the n th complex Fourier coefficient of the zonal wind can be written as

$$\frac{\partial U(n)}{\partial t} = - \sum_{m=-\infty}^{\infty} \frac{i(n-m)}{a \cos \varphi} U(m)U(n-m) \quad (8.24)$$

A similar equation can be written for $U(-n)$, i.e.,

$$\frac{\partial U(-n)}{\partial t} = - \sum_{m=-\infty}^{\infty} \frac{i(-n-m)}{a \cos \varphi} U(m)U(-n-m) \quad (8.25)$$

Multiplying (8.25) by $U(n)$ and (8.24) by $U(-n)$ and adding the results together, after noting that $K(n) = |U(n)|^2$ and thus

$$\frac{\partial K(n)}{\partial t} = U(n) \frac{\partial U(-n)}{\partial t} + U(-n) \frac{\partial U(n)}{\partial t}, \quad (8.26)$$

yields the following expression for the rate of change of kinetic energy of the n th wave number:

$$\begin{aligned} \frac{\partial K(n)}{\partial t} = & - \sum_{m=-\infty}^{\infty} \frac{i}{a \cos \varphi} U(m)[(n-m)U(n-m)U(-n) \\ & + (-n-m)U(-n-m)U(n)] \end{aligned} \quad (8.27)$$

In the case of $m = 0$, the right-hand side of (25) becomes

$$-\frac{i}{a \cos \varphi} U(0)[nU(n)U(-n) - nU(-n)U(n)] \equiv 0 \quad (8.28)$$

It can also be easily shown that the contributions of $m = -n$ and $m = n$ cancel each other out, so that the final form of (8.25) becomes

$$\frac{\partial K(n)}{\partial t} = - \sum_{\substack{m=-\infty \\ m \neq 0 \\ m \neq \pm n}}^{\infty} \frac{i}{a \cos \varphi} U(m)[(n-m)U(n-m)U(-n) + (-n-m)U(-n-m)U(n)] \quad (8.29)$$

Since $m \neq 0$ and $m \neq \pm n$, there are no quadratic interactions taking place in this problem. The calculation of the triplet interactions can be illustrated with a simplified example where only waves with wave numbers ± 1 , ± 2 and ± 3 are considered. For such a case, the kinetic energy of wave number 1 can be expressed, using (8.29), as

$$\begin{aligned} \frac{\partial K(1)}{\partial t} = & -\frac{i}{a \cos \varphi} U(2)[-U(-1)U(-1) - 3U(-3)U(1)] \\ & -\frac{i}{a \cos \varphi} U(-2)[3U(3)U(-1) + U(1)U(1)] \\ & -\frac{i}{a \cos \varphi} U(3)[-2U(-2)U(-1)] \\ & -\frac{i}{a \cos \varphi} U(-3)[2U(2)U(1)] \end{aligned} \quad (8.30)$$

Or, after rearranging,

$$\begin{aligned} \frac{\partial K(1)}{\partial t} = & -\frac{i}{a \cos \varphi} [-U(1)U(2)U(-3) + U(-1)U(-2)U(3) \\ & - U(-1)U(-1)U(2) + U(1)U(1)U(-2)] \end{aligned} \quad (8.31)$$

Each of the complex Fourier coefficients has both a real and an imaginary component, i.e., for any wave number n ,

$$U(n) = U_R(n) + iU_I(n) \quad (8.32)$$

where subscript R indicates the real component, and subscript I the imaginary one. If we use (8.32) to express all the Fourier coefficients in (8.31) and perform the multiplication, we obtain

$$\begin{aligned}
\frac{\partial K(1)}{\partial t} = & -\frac{2}{a \cos \varphi} [U_I(1)U_R(2)U_R(3) + U_R(1)U_I(2)U_R(3) \\
& + U_R(1)U_R(2)U_I(3) - U_I(1)U_I(2)U_I(3) \\
& + U_I(1)U_R(1)U_R(2) + U_R(1)U_I(1)U_R(2) \\
& + U_R(1)U_R(1)U_I(2) - U_I(1)U_I(1)U_I(2)] \quad (8.33)
\end{aligned}$$

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