A Double Complex Construction and Discrete Bogomolny Equations

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Abstract We study discrete models which are generated by the self-dual Yang–Mills equations. Using a double complex construction, we construct a new discrete analog of the Bogomolny equations. Discrete Bogomolny equations, a system of matrix-valued difference equations, are obtained from discrete self-dual equations. The gauge invariance of the discrete model is established.

Keywords Discrete model • Difference equations • Bogomolny equations • Yang-Mills equations

1 Introduction

This work is concerned with discrete model of the $SU(2)$ self-dual Yang–Mills equations described in [11]. It is well known that the self-dual Yang–Mills equations admit reduction to the Bogomolny equations [1]. Let A be an $SU(2)$-connection on $\mathbb{R}^3$. This means that A is an $su(2)$-valued 1-form and we can write

$$A = \sum_{i=1}^{3} A_i(x)\mathrm{d}x^i, \tag{1}$$

where $A_i : \mathbb{R}^3 \to su(2)$. Here $su(2)$ is the Lie algebra of $SU(2)$. The connection A is also called a gauge potential with the gauge group $SU(2)$ (see [8] for more details). Given the connection A, we define the curvature 2-form F by

$$F = \mathrm{d}A + A \wedge A, \tag{2}$$

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where $\wedge$ denotes the exterior multiplication of differential forms. Let $\Phi : \mathbb{R}^3 \to su(2)$ be a scalar field (a Higgs field). The Bogomolny equations are a set of nonlinear partial differential equations, where unknown is a pair (A, Φ). These equations can be written as

$$F = *\mathrm{d}_A\Phi, \tag{3}$$

where $*$ is the Hodge star operator on $\mathbb{R}^3$ and d_A is the covariant exterior differential operator. This operator is defined by the formula

$$\mathrm{d}_A\Omega = \mathrm{d}\Omega + A \wedge \Omega + (-1)^{r+1}\Omega \wedge A,$$

where Ω is an arbitrary $su(2)$-valued r-form.

Let us now consider the connection A on $\mathbb{R}^4$. We define A to be

$$A = \sum_{i=1}^{3} A_i(x)\mathrm{d}x^i + \Phi(x)\mathrm{d}x^4, \tag{4}$$

where A_i and Φ are independent of x^4. In other words, the scalar field Φ is identified with a fourth component A_4 of the connection A. It is easy to check that if the pair (A, Φ) satisfies Eq. (3), then the connection (4) is a solution of the self-dual equation

$$F = *F. \tag{5}$$

In fact, the Bogomolny equations can be obtained from the self-dual equations by using dimensional reduction from $\mathbb{R}^4$ to $\mathbb{R}^3$ [1].

The aim of this paper is to construct a discrete model of Eq. (3) that preserves the geometric structure of the original continual object. This means that speaking of a discrete model, we mean not only the direct replacement of differential operators by difference ones but also a discrete analog of the Riemannian structure over a properly introduced combinatorial object. The idea presented here is strongly influenced by the book by Dezin [3]. Using a double complex construction, we construct a new discrete analog of the Bogomolny equations. In much the same way as in the continual case, these discrete equations are obtained from discrete self-dual equations. The gauge invariance of the discrete model is proved. We continue the investigations [10, 11], where discrete analogs of the self-dual and anti-self-dual equations on a double complex are studied. It should be noted that there are many other approaches to discretization of Yang–Mills theories. As the list of papers on the subject is very large, we content ourselves by referencing the works [2, 4–7, 9]. In these papers some other discrete versions of the Bogomolny equations are studied.

2 Double Complex Construction

The double complex construction is described in [10]. For the convenience of the reader we briefly repeat the relevant material from [10] without proofs. Let the tensor product $C(n) = C \otimes \ldots \otimes C$ of a 1-dimensional complex C be a combinatorial model of the Euclidean space $\mathbb{R}^n$. The 1-dimensional complex C is defined in the following way. Let C^0 denote the real linear space of 0-dimensional chains generated by basis elements x_i (points), $i \in \mathbb{Z}$. It is convenient to introduce the shift operator τ in the set of indices by

$$\tau i = i + 1.$$

We denote the open interval $(x_i,\ x_{\tau i})$ by e_i. We regard the set $\{e_i\}$ as a set of basis elements of the real linear space C^1 of 1-dimensional chains. Then the 1-dimensional complex (combinatorial real line) is the direct sum of the spaces introduced above: $C = C^0 \oplus C^1$. The boundary operator ∂ on the basis elements of C is given by

$$\partial x_i = 0, \qquad \partial e_i = x_{\tau i} - x_i. \tag{6}$$

The definition is extended to arbitrary chains by linearity.

Multiplying the basis elements x_i and e_i of C in various ways, we obtain the basis elements of $C(n)$. Let $s_k^{(r)} = s_{k_1} \otimes \ldots \otimes s_{k_n}$, where $k = (k_1, \ldots, k_n)$ and $k_i \in \mathbb{Z}$, be an arbitrary r-dimensional basis element of $C(n)$. The product contains exactly r of 1-dimensional elements e_{k_i} and $n - r$ of 0-dimensional elements x_{k_i}. The superscript (r) also uniquely determines an r-dimensional basis element of $C(n)$. For example, the 1-dimensional e_k^i and 2-dimensional ε_k^{ij} basis elements of $C(3)$ can be written as

$$e_k^1 = e_{k_1} \otimes x_{k_2} \otimes x_{k_3}, \quad e_k^2 = x_{k_1} \otimes e_{k_2} \otimes x_{k_3}, \quad e_k^3 = x_{k_1} \otimes x_{k_2} \otimes e_{k_3},$$
$$\varepsilon_k^{12} = e_{k_1} \otimes e_{k_2} \otimes x_{k_3}, \quad \varepsilon_k^{13} = e_{k_1} \otimes x_{k_2} \otimes e_{k_3}, \quad \varepsilon_k^{23} = x_{k_1} \otimes e_{k_2} \otimes e_{k_3},$$

where $k = (k_1, k_2, k_3)$ and $k_i \in \mathbb{Z}$.

Now we consider a dual object of the complex $C(n)$. Let $K(n)$ be a cochain complex with $gl(2,\mathbb{C})$-valued coefficients, where $gl(2,\mathbb{C})$ is the Lie algebra of the group $GL(2,\mathbb{C})$. We suppose that the complex $K(n)$, which is a conjugate of $C(n)$, has a similar structure: $K(n) = K \otimes \ldots \otimes K$, where K is a dual of the 1-dimensional complex C. We will write the basis elements of K as x^i, e^i. Then an arbitrary basis element of $K(n)$ is given by $s^k = s^{k_1} \otimes \ldots \otimes s^{k_n}$, where s^{k_i} is either x^{k_i} or e^{k_i}. For an r-dimensional cochain $\varphi \in K(n)$, we have

$$\varphi = \sum_k \sum_r \varphi_k^{(r)} s_{(r)}^k, \tag{7}$$

where $\varphi_k^{(r)} \in gl(2,\mathbb{C})$. We will call cochains forms, emphasizing their relationship with the corresponding continual objects, differential forms.

We define the pairing operation for arbitrary basis elements $\varepsilon_k \in C(n)$, $s^k \in K(n)$ by the rule

$$< \varepsilon_k,\, as^k >= \begin{cases} 0, & \varepsilon_k \neq s_k \\ a, & \varepsilon_k = s_k, \quad a \in gl(2,\mathbb{C}). \end{cases} \tag{8}$$

Here for simplicity the superscript (r) is omitted. The operation (8) is linearly extended to cochains.

The operation ∂ induces the dual operation d^c on $K(n)$ in the following way:

$$< \partial\varepsilon_k,\, as^k >=< \varepsilon_k,\, a\mathrm{d}^c s^k > . \tag{9}$$

For example, if φ is a 0-form, i.e., $\varphi = \sum_k \varphi_k x^k$, where $x^k = x^{k_1} \otimes \ldots \otimes x^{k_n}$, then

$$\mathrm{d}^c\varphi = \sum_k \sum_{i=1}^{n} (\Delta_i \varphi_k) e_i^k, \tag{10}$$

where e_i^k is the 1-dimensional basis elements of $K(n)$ and

$$\Delta_i \varphi_k = \varphi_{\tau_i k} - \varphi_k. \tag{11}$$

Here the shift operator τ_i acts as

$$\tau_i k = (k_1, \ldots, \tau k_i, \ldots, k_n).$$

The coboundary operator d^c is an analog of the exterior differentiation operator d.

Introduce a cochain product on $K(n)$. We denote this product by $\cup$. In terms of the homology theory this is the so-called Whitney product. For the basis elements of 1-dimensional complex K, the $\cup$-product is defined as follows:

$$x^i \cup x^i = x^i, \quad e^i \cup x^{\tau i} = e^i, \quad x^i \cup e^i = e^i, \quad i \in \mathbb{Z},$$

supposing the product to be zero in all other cases. By induction we extend this definition to basis elements of $K(n)$ (see [10] for details). For example, for the 1-dimensional basis elements $e_i^k \in K(3)$ we have

$$\begin{aligned} &e_1^k \cup e_2^{\tau_1 k} = \varepsilon_{12}^k, \qquad e_1^k \cup e_3^{\tau_1 k} = \varepsilon_{13}^k, \qquad e_2^k \cup e_3^{\tau_2 k} = \varepsilon_{23}^k, \\ &e_2^k \cup e_1^{\tau_2 k} = -\varepsilon_{12}^k, \quad e_3^k \cup e_1^{\tau_3 k} = -\varepsilon_{13}^k, \quad e_3^k \cup e_2^{\tau_3 k} = -\varepsilon_{23}^k. \end{aligned} \tag{12}$$

To arbitrary forms the $\cup$-product be extended linearly. Note that the components of forms multiply as matrices. It is worth pointing out that for any forms $\varphi, \psi \in K(n)$, the following relation holds:

$$\mathrm{d}^c(\varphi \cup \psi) = \mathrm{d}^c\varphi \cup \psi + (-1)^r \varphi \cup \mathrm{d}^c\psi, \tag{13}$$

where r is the dimension of a form φ. For the proof we refer the reader to [3]. Relation (13) is a discrete analog of the Leibniz rule for differential forms.

Let us now together with the complex $C(n)$ consider its "double," namely, the complex $\tilde{C}(n)$ of exactly the same structure. Define the one-to-one correspondence

$$* : C(n) \to \tilde{C}(n), \qquad * : \tilde{C}(n) \to C(n) \tag{14}$$

in the following way:

$$* : s_k^{(r)} \to \pm \tilde{s}_k^{(n-r)}, \qquad * : \tilde{s}_k^{(r)} \to \pm s_k^{(n-r)}, \tag{15}$$

where $\tilde{s}_k^{(n-r)} = *s_{k_1} \otimes \ldots \otimes *s_{k_n}$ and $*s_{k_i} = \tilde{e}_{k_i}$ if $s_{k_i} = x_{k_i}$ and $*s_{k_i} = \tilde{x}_{k_i}$ if $s_{k_i} = e_{k_i}$. We let the plus sign in (15) if a permutation of $(1,\ldots,n)$ with $(1,\ldots,n) \to ((r),\ldots,(n-r))$ is representable as the product of an even number of transpositions and the minus sign otherwise.

The complex of the cochains $\tilde{K}(n)$ over the double complex $\tilde{C}(n)$ has the same structure as $K(n)$. Note that forms $\varphi \in K(n)$ and $\tilde{\varphi} \in \tilde{K}(n)$ have both the same components. The operation (14) induces the respective mapping

$$* : K(n) \to \tilde{K}(n), \qquad * : \tilde{K}(n) \to K(n) \tag{16}$$

by the rule: $< \tilde{c},\ *\varphi > = < *\tilde{c},\ \varphi >$, $< c,\ *\tilde{\psi} > = < *c,\ \tilde{\psi} >$, where $c \in C(n)$, $\tilde{c} \in \tilde{C}(n)$, $\varphi \in K(n)$, $\tilde{\psi} \in \tilde{K}(n)$. For example, for the 2-dimensional basis elements $\varepsilon_{ij}^k \in K(3)$ we have

$$*\varepsilon_{12}^k = \tilde{e}_3^k, \quad *\varepsilon_{13}^k = -\tilde{e}_2^k, \quad *\varepsilon_{23}^k = \tilde{e}_1^k. \tag{17}$$

This operation is a discrete analog of the Hodge star operation. Similarly to the continual case, we have $**\varphi = (-1)^{r(n-r)}\varphi$ for any discrete r-form $\varphi \in K(n)$.

Finally, for convenience we introduce the operation

$$\tilde{\iota} : K(n) \to \tilde{K}(n), \qquad \tilde{\iota} : \tilde{K}(n) \to K(n) \tag{18}$$

by setting $\tilde{\iota} s_{(r)}^k = \tilde{s}_{(r)}^k, \quad \tilde{\iota}\tilde{s}_{(r)}^k = s_{(r)}^k$. It is easy to check that the following hold:

$$\tilde{\iota}* = *\tilde{\iota}, \quad \tilde{\iota} d^c = d^c \tilde{\iota}, \quad \tilde{\iota}\varphi = \tilde{\varphi}, \quad \tilde{\iota}\tilde{\iota}\varphi = \varphi, \quad \tilde{\iota}(\varphi \cup \psi) = \tilde{\iota}\varphi \cup \tilde{\iota}\psi,$$

where $\varphi, \psi \in K(n)$.

3 Discrete Bogomolny Equations

Let us consider a discrete $su(2)$-valued 0-form $\Phi \in K(3)$. We put

$$\Phi = \sum_k \Phi_k x^k, \tag{19}$$

where $\Phi_k \in su(2)$ and $x^k = x^{k_1} \otimes x^{k_2} \otimes x^{k_3}$ is the 0-dimensional basis element of $K(3)$, $k = (k_1, k_2, k_3)$, $k_i \in \mathbb{Z}$. We define a discrete $SU(2)$-connection A to be

$$A = \sum_k \sum_{i=1}^{3} A_k^i e_i^k, \tag{20}$$

where $A_k^i \in su(2)$ and e_i^k is the 1-dimensional basis element of $K(3)$.

On account of (7), an arbitrary discrete 2-form $F \in K(3)$ can be written as follows:

$$F = \sum_k \sum_{i<j} F_k^{ij} \varepsilon_{ij}^k = \sum_k \left(F_k^{12} \varepsilon_{12}^k + F_k^{13} \varepsilon_{13}^k + F_k^{23} \varepsilon_{23}^k \right), \tag{21}$$

where $F_k^{ij} \in gl(2, \mathbb{C})$ and ε_{ij}^k is the 2-dimensional basis element of $K(3)$. Define a discrete analog of the curvature form (2) by

$$F = \mathrm{d}^c A + A \cup A. \tag{22}$$

By the definition of d^c (9) and using (12) we have

$$\mathrm{d}^c A = \sum_k \sum_{i<j} (\Delta_i A_k^j - \Delta_j A_k^i) \varepsilon_{ij}^k, \tag{23}$$

$$A \cup A = \sum_k \sum_{i<j} (A_k^i A_{\tau_i k}^j - A_k^j A_{\tau_j k}^i) \varepsilon_{ij}^k. \tag{24}$$

Recall that Δ_i is the difference operator (11). Combining (23) and (24) with (21), we obtain

$$F_k^{ij} = \Delta_i A_k^j - \Delta_j A_k^i + A_k^i A_{\tau_i k}^j - A_k^j A_{\tau_j k}^i. \tag{25}$$

It should be noted that in the continual case the curvature form F takes values in the algebra $su(2)$ for any $su(2)$-valued connection form A. Unfortunately, this is not true in the discrete case because, generally speaking, the components $A_k^i A_{\tau_i k}^j - A_k^j A_{\tau_j k}^i$ of the form $A \cup A$ in (22) do not belong to $su(2)$. For a definition of the $su(2)$-valued discrete curvature form, we refer the reader to [11].

Define a discrete analog of the exterior covariant differential operator d_A as

$$\mathrm{d}_A^c \varphi = \mathrm{d}^c \varphi + A \cup \varphi + (-1)^{r+1} \varphi \cup A,$$

where φ is an arbitrary r-form (7) and A is given by (20) . Then for the 0-form (19), we obtain

$$\mathrm{d}_A^c \Phi = \mathrm{d}^c \Phi + A \cup \Phi - \Phi \cup A. \tag{26}$$

Using (10) and the definition of $\cup$, we can rewritten (26) as follows:

$$\mathrm{d}_A^c\Phi = \sum_k \sum_{i=1}^{3} (\Delta_i\Phi_k + A_k^i\Phi_{\tau_i k} - \Phi_k A_k^i)e_i^k. \tag{27}$$

Applying the operation $*$ (16) to this expression and by (17) we find

$$\begin{aligned} *\,\mathrm{d}_A^c\Phi = & \sum_k (\Delta_1\Phi_k + A_k^1\Phi_{\tau_1 k} - \Phi_k A_k^1)\tilde{\varepsilon}_{23}^k \\ & - \sum_k (\Delta_2\Phi_k + A_k^2\Phi_{\tau_2 k} - \Phi_k A_k^2)\tilde{\varepsilon}_{13}^k \\ & + \sum_k (\Delta_3\Phi_k + A_k^3\Phi_{\tau_3 k} - \Phi_k A_k^3)\tilde{\varepsilon}_{12}^k. \end{aligned} \tag{28}$$

Now suppose that Φ in the form (19) is a discrete analog of the Higgs field. Then the discrete analog of the Bogomolny equation (3) is given by the formula

$$F = \tilde{\iota} * \mathrm{d}_A^c\Phi, \tag{29}$$

where $\tilde{\iota}$ is the operation (17). From (21) and (28) it follows immediately that Eq. (29) is equivalent to the following difference equations:

$$\begin{aligned} F_k^{12} &= \Delta_3\Phi_k + A_k^3\Phi_{\tau_3 k} - \Phi_k A_k^3, \\ F_k^{13} &= -\Delta_2\Phi_k - A_k^2\Phi_{\tau_2 k} + \Phi_k A_k^2, \\ F_k^{23} &= \Delta_1\Phi_k + A_k^1\Phi_{\tau_1 k} - \Phi_k A_k^1. \end{aligned} \tag{30}$$

Consider now the discrete curvature form (22) in the 4-dimensional case, i. e., $F \in K(4)$. The discrete analog of the self-dual Eq. (5) can be written as follows:

$$F = \tilde{\iota} * F. \tag{31}$$

By the definition of $*$ for the 2-dimensional basis elements $\varepsilon_{ij}^k \in K(4)$, we have

$$\begin{aligned} *\varepsilon_{12}^k &= \tilde{\varepsilon}_{34}^k, \quad *\varepsilon_{13}^k = -\tilde{\varepsilon}_{24}^k, \quad *\varepsilon_{14}^k = \tilde{\varepsilon}_{23}^k, \\ *\varepsilon_{23}^k &= \tilde{\varepsilon}_{14}^k, \quad *\varepsilon_{24}^k = -\tilde{\varepsilon}_{13}^k, \quad *\varepsilon_{34}^k = \tilde{\varepsilon}_{12}^k. \end{aligned}$$

Using this we may compute $*F$:

$$*F = \sum_k \left(F_k^{12}\tilde{\varepsilon}_{34}^k - F_k^{13}\tilde{\varepsilon}_{24}^k + F_k^{14}\tilde{\varepsilon}_{23}^k + F_k^{23}\tilde{\varepsilon}_{14}^k - F_k^{24}\tilde{\varepsilon}_{13}^k + F_k^{34}\tilde{\varepsilon}_{12}^k\right).$$

Then Eq. (31) becomes

$$F_k^{12} = F_k^{34}, \qquad F_k^{13} = -F_k^{24}, \qquad F_k^{14} = F_k^{23}. \tag{32}$$

Let the discrete connection 1-form $A \in K(4)$ be given by

$$A = \sum_k \sum_{i=1}^{3} A_k^i e_i^k + \sum_k \Phi_k e_4^k, \tag{33}$$

where $A_k^i \in su(2)$, $\Phi_k \in su(2)$ and $k = (k_1, k_2, k_3, k_4)$, $k_i \in \mathbb{Z}$. Note that here we put $A_k^4 = \Phi_k$ and Φ_k are the components of the discrete Higgs field. Suppose that the connection form (33) is independent of k_4, i.e.,

$$\Delta_4 A_k^i = 0, \qquad \Delta_4 \Phi_k = 0 \tag{34}$$

for any $i = 1,2,3$ and $k = (k_1, k_2, k_3, k_4)$. Substituting (34) into (25) yields

$$F_k^{i4} = \Delta_i \Phi_k + A_k^i \Phi_{\tau_i k} - \Phi_k A_k^i, \qquad i = 1,2,3.$$

Putting these expressions in Eq. (32) we obtain Eq. (30).

Thus, we have the following:

Theorem 1. *The discrete Bogomolny equation (29) and the discrete self-dual Eq. (31) are equivalent.*

Let us consider the $SU(2)$-valued 0-form

$$h = \sum_k h_k x^k, \tag{35}$$

where $h_k \in SU(2)$ and $x^k = x^{k_1} \otimes x^{k_2} \otimes x^{k_3}$ is the 0-dimensional basis element of $K(3)$. By analogy with classical Yang–Mills theories, we define a gauge transformation for the discrete potential $A \in K(3)$ and discrete field $\Phi \in K(3)$ as

$$A' = h \cup d^c h^{-1} + h \cup A \cup h^{-1}, \tag{36}$$

$$\Phi' = h \cup \Phi \cup h^{-1}, \tag{37}$$

where h^{-1} is the 0-form with inverse components (inverse matrices) of h. Suppose that the components $h_k \in SU(2)$ of (35) satisfy the following conditions:

$$h_{\tau_1 \tau_2 k} = h_{\tau_3 k}, \qquad h_{\tau_1 \tau_3 k} = h_{\tau_2 k}, \qquad h_{\tau_2 \tau_3 k} = h_{\tau_1 k} \tag{38}$$

for all $k = (k_1, k_2, k_3)$, $k_i \in \mathbb{Z}$. It is easy to check that the set of forms (35) satisfying conditions (38) is a group under $\cup$-product.

Theorem 2. *The discrete Bogomolny equation (29) is invariant under the gauge transformation (36) and (37), where h satisfies condition (38).*

Proof. Rewrite Eq. (29) in the form

$$\tilde{\iota} * F - d^c_A \Phi = 0. \tag{39}$$

The proof is based on Theorem 4.3 and Lemma 4.6 in [11]. Under the transformation (36) the curvature form (22) changes as

$$F' = h \cup F \cup h^{-1}.$$

Using conditions (38) and Lemma 4.6 of [11] we have

$$\tilde{\iota} * F' = \tilde{\iota} * (h \cup F \cup h^{-1}) = h \cup \tilde{\iota} * F \cup h^{-1}. \tag{40}$$

Since $d^c h \cup h^{-1} = -h \cup d^c h^{-1}$ by (13), (26), (36), and (37), we compute

$$\begin{aligned} d^c_{A'} \Phi' &= d^c_{A'} (h \cup \Phi \cup h^{-1}) = h \cup d^c \Phi \cup h^{-1} \\ &+ h \cup A \cup \Phi \cup h^{-1} - h \cup \Phi \cup A \cup h^{-1} = h \cup d^c_A \Phi \cup h^{-1}. \end{aligned} \tag{41}$$

Comparing (40) and (41) we obtain

$$\tilde{\iota} * F' - d^c_{A'} \Phi' = h \cup (\tilde{\iota} * F - d^c_A \Phi) \cup h^{-1}.$$

Thus, if the pair (A, Φ) is a solution of Eq. (29), then (A', Φ') is also a solution of (29). □

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