8

Initial-Boundary Value and LC Problems

Evaluation of European-style derivatives can be reduced to solving initial value or initial-boundary value problems of parabolic partial differential equations. This chapter discusses numerical methods for such problems. If an American option problem is formulated as a linear complementarity problem, then the only difference between solving a European option and an American option is that if the solution obtained by the partial differential equation does not satisfy the constraint at some point, then the solution of the PDE at the point should be replaced by the value determined from the constraint condition. Such methods are usually referred to as projected methods for American-style derivatives. Therefore, the two methods are very close, and we also study the projected methods in this chapter.

In this chapter, there are four sections. The first two sections are devoted to explicit and implicit schemes, respectively. As we know, the derivative of the function representing the payoff of an option usually is discontinuous. This fact makes numerical methods inefficient. In many cases, an option problem can be reduced to another problem that has either a smooth solution or a solution with a weaker singularity than the solution of the option problem itself, and the numerical solution of the new problem can be obtained efficiently. We call such a method the singularity-separating method. In Sect. 8.3, we give several examples to illustrate how such a method works. In the final section, we discuss the pseudo-spectral method, which is very efficient if the solution is smooth. Examples are given to explain this fact.

8.1 Explicit Methods

8.1.1 Pricing European Options by Using $\overline{V}$, ξ, τ or u, x, $\bar{\tau}$ Variables

In Sect. 2.2.5, we obtained the formulation of the problem satisfied by a call/put option on a finite domain:

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$$\begin{cases} \dfrac{\partial \overline{V}}{\partial \tau} = \dfrac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1-\xi)^2\dfrac{\partial^2 \overline{V}}{\partial \xi^2} + (r-D_0)\xi(1-\xi)\dfrac{\partial \overline{V}}{\partial \xi} \\ \qquad -[r(1-\xi)+D_0\xi]\overline{V}, \qquad 0 \le \xi \le 1, \quad 0 \le \tau \le T, \\ \overline{V}(\xi,0) = \max(\pm(2\xi-1),0), \qquad 0 \le \xi \le 1, \end{cases} \tag{8.1}$$

where the sign "+" in $\pm$ corresponds to the call option and the sign "−" in $\pm$ the put option. Here, we assume that the volatility depends on S, so $\bar{\sigma}$ is a function of ξ in the equation. Let

$$\begin{cases} \xi_m = m\Delta\xi, \; m = 0,1,\cdots,M, \\ \tau^n = n\Delta\tau, \;\; n = 0,1,\cdots,N, \end{cases} \tag{8.2}$$

where M and N are given integers, and $\Delta\xi = 1/M$ and $\Delta\tau = T/N$. This means that we use an $M \times N$ equidistant mesh on the domain $[0,1] \times [0,T]$. Let v_m^n denote the approximate value of $\overline{V}(\xi,\tau)$ at $\xi = \xi_m$ and $\tau = \tau^n$, and $\{v_m^n\}$ represent the set v_m^n, $m = 0,1,\cdots,M$. Discretizing the partial differential equation in the problem (8.1) at the point $(\xi_m,\ \tau^n)$ by scheme (7.5), i.e., by using the forward difference for $\dfrac{\partial \overline{V}}{\partial \tau}$ and the central difference for $\dfrac{\partial^2 \overline{V}}{\partial \xi^2}$ and $\dfrac{\partial \overline{V}}{\partial \xi}$, we get

$$\begin{aligned} \frac{v_m^{n+1}-v_m^n}{\Delta\tau} &= \frac{1}{2}\bar{\sigma}_m^2\xi_m^2(1-\xi_m)^2\frac{v_{m+1}^n-2v_m^n+v_{m-1}^n}{\Delta\xi^2} \\ &\quad +(r-D_0)\xi_m(1-\xi_m)\frac{v_{m+1}^n-v_{m-1}^n}{2\Delta\xi} \\ &\quad -[r(1-\xi_m)+D_0\xi_m]v_m^n \end{aligned}$$

or

$$\begin{aligned} v_m^{n+1} &= \frac{1}{2}\left[\bar{\sigma}_m^2\xi_m^2(1-\xi_m)^2-(r-D_0)\xi_m(1-\xi_m)\Delta\xi\right]\alpha v_{m-1}^n \\ &\quad +[1-\bar{\sigma}_m^2\xi_m^2(1-\xi_m)^2\alpha-(r(1-\xi_m)+D_0\xi_m)\Delta\tau]v_m^n \\ &\quad +\frac{1}{2}\left[\bar{\sigma}_m^2\xi_m^2(1-\xi_m)^2+(r-D_0)\xi_m(1-\xi_m)\Delta\xi\right]\alpha v_{m+1}^n, \\ &\quad m = 0,1,\cdots,M, \quad n = 0,1,\cdots,N-1, \end{aligned} \tag{8.3}$$

where

$$\alpha = \frac{\Delta\tau}{\Delta\xi^2}.$$

In order for scheme (8.3) to be stable, we require

$$\max_{0\le m\le M}[\bar{\sigma}_m^2\xi_m^2(1-\xi_m)^2]\frac{\Delta\tau}{2\Delta\xi^2} \le \frac{1}{2}$$

because if this is true, then $|\lambda_\theta| \le 1 + O(\varDelta\tau)$ (see Problem 14 in Chap. 7). In practice, we can replace this condition by a slightly stronger condition as follows:

$$\varDelta\tau \le \frac{16\varDelta\xi^2}{\max\limits_{0\le m\le M} \bar{\sigma}_m^2}. \tag{8.4}$$

Sometimes, for example, when a lookback option needs to be priced, the value at a boundary is determined by a boundary condition which involves a derivative. In such cases, $\dfrac{\partial}{\partial\xi}$ needs to be discretized by a one-sided first or second order scheme.

From the difference scheme (8.3), we know that when the values v_{m-1}^n, v_m^n, and v_{m+1}^n are given, v_m^{n+1} can be obtained immediately. At a glance, it appears that v_{-1}^n and v_{M+1}^n are needed when v_0^{n+1} and v_M^{n+1} are calculated. As pointed out in Sect. 7.1, because the coefficients of v_{-1}^n and v_{M+1}^n equal zero, the values of v_{-1}^n and v_{M+1}^n will not be used. Therefore, if $\{v_m^n\}$ is given, then $\{v_m^{n+1}\}$ can be obtained by the difference scheme (8.3). According to the initial condition given in the problem (8.1), we have

$$v_m^0 = \max(\pm(2\xi_m - 1), 0).$$

Therefore, from $\{v_m^0\}$, we can get $\{v_m^n\}$, $n = 1, 2, \cdots, N$ successively. Usually, we need the value of V at a certain point S^* at time zero. After $\{v_m^N\}$ have been obtained, $V(S^*, 0)$ can be found in the following way. First, we need to find $v(\xi^*, T)$ by using the quadratic interpolation given in Sect. 6.1, where $\xi^* = \dfrac{S^*}{S^* + E}$. Then, we can obtain $V(S^*, 0)$ from $v(\xi^*, T)$ by

$$V(S^*, 0) = (S^* + E)v(\xi^*, T).$$

This method works not only for a constant σ but also for a variable σ, namely, $\sigma = \sigma(S)$, even $\sigma = \sigma(S, t)$. In what follows, this scheme is referred to as the explicit finite-difference method I, and its abbreviation is EFDI.

If σ is a constant, then an alternative way to find the approximate solution of the European options is to use u, x, $\bar{\tau}$ variables. From Sect. 2.6.1, we know that if $E = 1$, i.e., if the stock price and the option price has been divided by the exercise price, then pricing a call/put option can be reduced to finding $u(x, \bar{\tau})$, which is the solution of the problem:

$$\begin{cases} \dfrac{\partial u}{\partial \bar{\tau}} = \dfrac{\partial^2 u}{\partial x^2}, & -\infty < x < \infty, \quad 0 \le \bar{\tau} \le \frac{1}{2}\sigma^2 T, \\ u(x, 0) = \max(\pm(\mathrm{e}^x - 1), 0), & -\infty < x < \infty. \end{cases} \tag{8.5}$$

Here,

$$x = \ln S + (r - D_0 - \sigma^2/2)(T - t), \quad \bar{\tau} = \sigma^2(T - t)/2$$

and

$$u(x, \bar{\tau}) = \mathrm{e}^{r(T-t)} V(S, t).$$

Let $x_m = a + m\Delta x$, a being a given number and $\bar{\tau}^n = n\Delta\bar{\tau}$, and let u_m^n denote the approximate value of $u(x_m, \bar{\tau}^n)$. Then, the partial differential equation can be discretized by the difference scheme (7.8):

$$u_m^{n+1} = \bar{\alpha} u_{m+1}^n + (1 - 2\bar{\alpha}) u_m^n + \bar{\alpha} u_{m-1}^n, \tag{8.6}$$

where

$$\bar{\alpha} = \frac{\Delta\bar{\tau}}{\Delta x^2}.$$

From Sect. 7.2.1, we know that in order for the scheme to be stable, we need to require

$$\bar{\alpha} = \frac{\Delta\bar{\tau}}{\Delta x^2} \leq \frac{1}{2}. \tag{8.7}$$

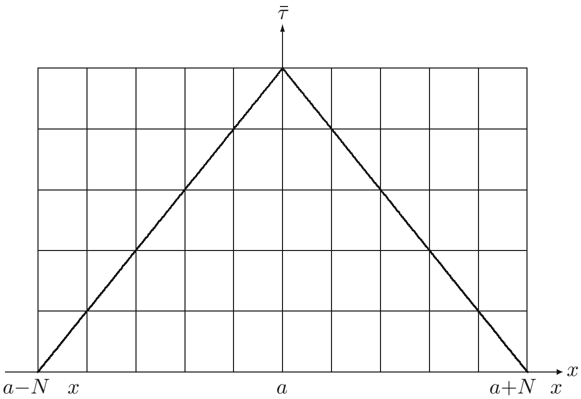

Fig. 8.1. A triangle mesh ($N = 5$)

Suppose again that we need to find $V(S^*, 0)$, i.e., we need to know $u\left(\ln S^* + (r - D_0 - \sigma^2/2)T, \sigma^2 T/2\right)$. Assume that we will use N steps in $\bar{\tau}$ direction, i.e., $\Delta\bar{\tau} = \dfrac{\sigma^2 T}{2N}$. In order to find $u\left(\ln S^* + (r - D_0 - \sigma^2/2)T, \sigma^2 T/2\right)$, we can use a triangle mesh (see Fig. 8.1): $\bar{\tau}^n = n\Delta\bar{\tau}, n = 0, 1, \cdots, N$ and for each n, $x_m = \ln S^* + (r - D_0 - \sigma^2/2)T + m\Delta x$, $m = -N + n, -N + n + 1, \cdots, N - n - 1, N - n$. From the initial condition at $\bar{\tau} = 0$, we have

$$u_m^0 = \max\left(\pm(\mathrm{e}^{x_m} - 1), 0\right), \quad m = -N, -N + 1, \cdots, N - 1, N.$$

It is clear that when $u_m^n, m = -N+n, -N+n+1, \cdots, N-n-1, N-n$ are given, we can obtain $u_m^{n+1}, m = -N + n + 1, -N + n + 2, \cdots, N - n - 2, N - n - 1$.

Therefore, starting from $u_m^0, m = -N, -N+1, \cdots, N-1, N$, we can find $u_m^n, m = -N+n, -N+n+1, \cdots, N-n-1, N-n$ for $n = 1, 2, \cdots, N$ successively. When we get u_0^N, $V(S^*, 0)$ can be calculated by

$$V(S^*, 0) = \mathrm{e}^{-rT} u_0^N$$

because $V(S,t) = \mathrm{e}^{-r(T-t)} u(\ln S + (r - D_0 - \sigma^2/2)(T-t), \sigma^2(T-t)/2)$.

Table 8.1. Values of European put options (EFDI)

($E = 50$, $S = 48$, $r = 0.05$, $\sigma = 0.20$, and $D_0 = 0$)

$\Delta\tau$	$T = 0.25$	$T = 0.50$	$T = 0.75$	$T = 1.00$
0.01	2.7220	3.1163	3.4045	3.5852
0.001	2.7087	3.1275	3.3989	3.5910
0.0001	2.7083	3.1272	3.3986	3.5907
Exact	$2.708349\cdots$	$3.127199\cdots$	$3.398586\cdots$	$3.590738\cdots$

Assume that we want to calculate the value of an option on a stock when the stock price is \$100 and the exercise price is \$90. In this method above, the stock price and the option price has been divided by E, so S^* should be 100/90, and the real option price should be obtained by $90 \times V(S^*, 0)$. This method is referred to as the explicit finite-difference method II, and its abbreviation is EFDII.

Example 1: Using EFDI with

$$\Delta\xi \approx \sqrt{\max_{0 \le m \le M} \bar{\sigma}_m^2 \Delta\tau}/4,$$

we have solved European put problems using different $\Delta\tau$. Numerical results for $T = 0.25$, 0.5, 0.75, and 1.00 are listed in Table 8.1, and the other problem parameters are also shown there. From the table we see that for $\Delta\tau = 0.01, 0.001$, and 0.0001, the error is about on the second, third, and fourth decimal places.

8.1.2 Projected Methods for LC Problems

In Sect. 3.2, we saw that an American option problem could be formulated as a linear complementarity problem. When the variables $\overline{V}, \xi, \tau$ are adopted, the linear complementarity problem is

$$\begin{cases} \min\left(\dfrac{\partial \overline{V}}{\partial \tau} - \mathbf{L}_\xi \overline{V}, \overline{V}(\xi, \tau) - \max(\pm(2\xi - 1), 0)\right) = 0, \\ \overline{V}(\xi, 0) = \max(\pm(2\xi - 1), 0), \end{cases} \tag{8.8}$$

where

$$\mathbf{L}_\xi = \frac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1-\xi)^2\frac{\partial^2}{\partial\xi^2} + (r-D_0)\xi(1-\xi)\frac{\partial}{\partial\xi} - [r(1-\xi)+D_0\xi];$$

whereas if the variables $u, x, \bar{\tau}$ are used, the linear complementarity problem is

$$\begin{cases} \min\left(\dfrac{\partial u}{\partial\bar{\tau}} - \dfrac{\partial^2 u}{\partial x^2},\ u(x,\bar{\tau}) - g(x,\bar{\tau})\right) = 0, \\ u(x,0) = g(x,0), \end{cases} \tag{8.9}$$

where

$$g(x,\bar{\tau}) = \max\left(\pm(\mathrm{e}^{x+(2D_0/\sigma^2+1)\bar{\tau}} - \mathrm{e}^{2r\bar{\tau}/\sigma^2}), 0\right).$$

Such a formulation can be described in another way. Let us take the problem (8.9) as an example in order to explain it. Suppose that we have obtained the solution at $\bar{\tau} = \bar{\tau}^*$, $u(x,\bar{\tau}^*)$. Starting from $u(x,\bar{\tau}^*)$, we can find the solution $u(x,\bar{\tau}^* + \Delta\bar{\tau})$ in the following way. Let $\tilde{u}(x,\bar{\tau}^* + \Delta\bar{\tau})$ be the solution determined by an approximation to the equation

$$\frac{\partial\tilde{u}}{\partial\bar{\tau}} - \frac{\partial^2\tilde{u}}{\partial x^2} = 0.$$

If

$$\tilde{u}(x,\bar{\tau}^* + \Delta\bar{\tau}) \ge g(x,\bar{\tau}^* + \Delta\bar{\tau})$$

at a point, then

$$u(x,\bar{\tau}^* + \Delta\bar{\tau}) = \tilde{u}(x,\bar{\tau}^* + \Delta\bar{\tau});$$

otherwise

$$u(x,\bar{\tau}^* + \Delta\bar{\tau}) = g(x,\bar{\tau}^* + \Delta\bar{\tau}).$$

That is, for each x,

$$u(x,\bar{\tau}^* + \Delta\bar{\tau}) = \max\left(\tilde{u}(x,\bar{\tau}^* + \Delta\bar{\tau}),\ g(x,\bar{\tau}^* + \Delta\bar{\tau})\right).$$

Does the solution determined in this way satisfy all the requirements in the problem (8.9)? When $\tilde{u}(x,\bar{\tau}^* + \Delta\bar{\tau}) \ge g(x,\bar{\tau}^* + \Delta\bar{\tau})$, we have $u(x,\bar{\tau}^* + \Delta\bar{\tau}) = \tilde{u}(x,\bar{\tau}^* + \Delta\bar{\tau})$, $u(x,\bar{\tau}^* + \Delta\bar{\tau}) \ge g(x,\bar{\tau}^* + \Delta\bar{\tau})$ and $\dfrac{\partial u}{\partial\bar{\tau}} - \dfrac{\partial^2 u}{\partial x^2} = 0$, so the first relation in the problem (8.9) holds; when $\tilde{u}(x,\bar{\tau}^* + \Delta\bar{\tau}) < g(x,\bar{\tau}^* + \Delta\bar{\tau})$, we have $u(x,\bar{\tau}^* + \Delta\bar{\tau}) = g(x,\bar{\tau}^* + \Delta\bar{\tau})$ and $\dfrac{\partial u}{\partial\bar{\tau}} - \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial g}{\partial\bar{\tau}} - \dfrac{\partial^2 g}{\partial x^2} > 0$, so the first relation in the problem (8.9) also holds. Thus the first relation in the problem (8.9) holds at any point. If the problem is formulated in the form (8.8), the situation is the same.

Therefore, if an American option is formulated as a linear complementarity problem, the difference between the numerical methods for European options and American options is not big. In fact, if the formulation (8.8) is used, then we can compute the value of American options by

$$v_m^{n+1} = \max(\tilde{v}_m^{n+1},\ \pm(2\xi - 1),\ 0), \tag{8.10}$$

where

$$\begin{aligned}\tilde{v}_m^{n+1} = & \frac{1}{2}\left[\bar{\sigma}_m^2 \xi_m^2 (1-\xi_m)^2 - (r - D_0)\xi_m(1-\xi_m)\Delta\xi\right] \alpha v_{m-1}^n \\ & + [1 - \bar{\sigma}_m^2 \xi_m^2 (1-\xi_m)^2 \alpha - (r(1-\xi_m) + D_0\xi)\Delta\tau] v_m^n \\ & + \frac{1}{2}\left[\bar{\sigma}_m^2 \xi_m^2 (1-\xi_m)^2 + (r - D_0)\xi_m(1-\xi_m)\Delta\xi\right] \alpha v_{m+1}^n .\end{aligned}$$

If the formulation (8.9) is adopted, then the computation is done by

$$u_m^{n+1} = \max\left(\bar{\alpha} u_{m+1}^n + (1-2\bar{\alpha}) u_m^n + \bar{\alpha} u_{m-1}^n,\ g(x_m, \bar{\tau}^{n+1})\right). \tag{8.11}$$

Table 8.2. American call option (PEFDII)

($r = 0.1$, $\sigma = 0.2$, $D_0 = 0.05$, $S = E = 100$, $T = 1$, and the exact solution is $C = 9.94092345\cdots$)

Numbers of time steps	Results	|Errors|	CPU(s)
50	9.902768	0.038156	0.0003
100	9.921822	0.019102	0.0013
200	9.931367	0.009557	0.0053
400	9.936144	0.004780	0.0220
800	9.938533	0.002390	0.0880

Table 8.3. American put option (PEFDII)

($r = 0.1$, $\sigma = 0.2$, $D_0 = 0.05$, $S = E = 100$, $T = 1$, and the exact solution is $P = 5.92827717\cdots$)

Numbers of time steps	Results	|Errors|	CPU(s)
50	5.911829	0.016448	0.0003
100	5.920472	0.007805	0.0013
200	5.924476	0.003801	0.0054
400	5.926424	0.001853	0.0220
800	5.927360	0.000917	0.0880

Finding the prices of American options in such a way is referred to as a projected method in the book [84] by Wilmott, Dewynne, and Howison. We call Eqs. (8.10) and (8.11) projected explicit finite-difference methods I and II, respectively, and their abbreviations are PEFDI and PEFDII. Clearly, PEFDI can be applied to the cases with both a constant σ and a variable σ, and PEFDII is suitable only for the case that σ is a constant. In Tables 8.2 and 8.3, the results of call and put options on several meshes are given. The method used is PEFDII. The error and the CPU time needed are also shown. In order to have an error, we must have the exact solutions. The exact solution for the American call and put option problems with these parameters are $C = 9.94092345\cdots$ and $P = 5.92827717\cdots$, which are obtained by the SSM given in Chap. 9. Here, the first nine digits are given, and it is enough to determine the first few digits of the errors given in these tables. Computation is done on a Space Ultra 10 computer. In this book, when a CPU time is mentioned, the computation is done on such a computer if no other explanation is given.

8.1.3 Binomial and Trinomial Methods

This subsection is devoted to the binomial and trinomial methods. In these methods, there is a lattice of possible asset prices. Thus, such methods are also called lattice methods.

Binomial Methods. The binomial method is a simple and very effective method for computing the option prices.

When the Black–Scholes equation is derived, a risk-free portfolio is established. This idea can also be used to design numerical methods. Let S_n be the given stock price at time t^n, S_{n+1} be the stock price at time $t^{n+1} = t^n + \Delta t$, and the possible values of S_{n+1} be $S_{n+1,0}$ and $S_{n+1,1}$. Assume that the stock pays dividends continuously and the dividend yield is D_0. Therefore one share of stock at time t^n becomes $\mathrm{e}^{D_0 \Delta t}$ shares at time t^{n+1}. Let V_n be the price of a derivative at time t^n, and $V_{n+1,i}$ be the price of the derivative at time t^{n+1} if the stock price is $S_{n+1,i}$, $i = 0$ and 1. That the portfolio

$$V - \Delta S$$

is risk-free means that

$$V_{n+1,0} - \Delta \mathrm{e}^{D_0 \Delta t} S_{n+1,0} = V_{n+1,1} - \Delta \mathrm{e}^{D_0 \Delta t} S_{n+1,1} = (V_n - \Delta S_n)\, \mathrm{e}^{r \Delta t}.$$

Therefore

$$\Delta = \frac{V_{n+1,1} - V_{n+1,0}}{S_{n+1,1} - S_{n+1,0}} \mathrm{e}^{-D_0 \Delta t}$$

and

$$\begin{aligned} V_n &= \mathrm{e}^{-r\Delta t} \left(V_{n+1,0} - \Delta \mathrm{e}^{D_0 \Delta t} S_{n+1,0}\right) + \Delta S_n \\ &= \mathrm{e}^{-r\Delta t} \left(V_{n+1,0} - \frac{V_{n+1,1} - V_{n+1,0}}{S_{n+1,1} - S_{n+1,0}} S_{n+1,0}\right) + \frac{V_{n+1,1} - V_{n+1,0}}{S_{n+1,1} - S_{n+1,0}} \mathrm{e}^{-D_0 \Delta t} S_n \\ &= \mathrm{e}^{-r\Delta t} \left[\frac{S_n \mathrm{e}^{(r-D_0)\Delta t} - S_{n+1,0}}{S_{n+1,1} - S_{n+1,0}} V_{n+1,1} \right. \\ &\qquad \left. + \left(1 - \frac{S_n \mathrm{e}^{(r-D_0)\Delta t} - S_{n+1,0}}{S_{n+1,1} - S_{n+1,0}}\right) V_{n+1,0}\right]. \end{aligned}$$

Let

$$p = \frac{S_n e^{(r-D_0)\Delta t} - S_{n+1,0}}{S_{n+1,1} - S_{n+1,0}}, \tag{8.12}$$

then we have

$$V_n = e^{-r\Delta t}\left[pV_{n+1,1} + (1-p)\,V_{n+1,0}\right]. \tag{8.13}$$

Suppose that in the real world, the stock price satisfies

$$dS = \mu S dt + \sigma S dX = \mu S dt + \sigma S \phi \sqrt{dt},$$

or

$$S_{n+1} - S_n = \mu S_n \Delta t + \sigma S_n \phi \sqrt{\Delta t},$$

where ϕ is the standardized normal random variable. Using Itô's lemma, this model can be rewritten as

$$d\ln S = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dX = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma\phi\sqrt{dt},$$

or

$$\ln S_{n+1} - \ln S_n = \left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\phi\sqrt{\Delta t}. \tag{8.14}$$

According to this model, the number of possible prices of the stock at time t_{n+1} is infinity. In the derivation above, we think that there are only two possible values the price of the stock can take at time t_{n+1}. Thus the random variable ϕ is approximated by a binomial random variable. Let ψ denote this binomial random variable. Because $E[\phi] = 0$ and $E[\phi^2] = \mathrm{Var}[\phi] + E^2[\phi] = 1$, it is natural to require $E[\psi] = 0$ and $E[\psi^2] = 1$. Suppose that the two values of ψ are ψ_0 and ψ_1 and that the probabilities of taking ψ_0 and ψ_1 are $1-q$ and q, respectively. Then the two conditions can be written as

$$\begin{cases} (1-q)\,\psi_0 + q\psi_1 = 0, \\ (1-q)\,\psi_0^2 + q\psi_1^2 = 1. \end{cases}$$

From these two equations we can have

$$\begin{cases} q = \dfrac{-\psi_0}{\psi_1 - \psi_0}, \\ q = \dfrac{1-\psi_0^2}{\psi_1^2 - \psi_0^2}. \end{cases}$$

Hence

$$-\psi_0 = \frac{1-\psi_0^2}{\psi_1 + \psi_0}$$

or

$$\psi_0\psi_1 = -1.$$

Therefore $\psi_0\psi_1 = -1$ is a necessary condition for $\mathrm{E}[\psi^2] = 1$ and $\mathrm{E}[\psi] = 0$. From the procedure of deriving this condition, it is easy to see that this condition is also a sufficient condition for $\mathrm{E}[\psi^2] = 1$ if $\mathrm{E}[\psi] = 0$. It is clear, if we choose ψ_0 and ψ_1 so that

$$\psi_0\psi_1 = -1 + O(\Delta t)$$

and require $\mathrm{E}[\psi] = 0$, then ψ is still a good approximate to ϕ.

Suppose that ψ_i is related to $S_{n+1,i}, i = 0, 1$. Thus we have

$$\begin{cases} \ln S_{n+1,0} = \ln S_n + \left(\mu - \dfrac{\sigma^2}{2}\right)\Delta t + \sigma\psi_0\sqrt{\Delta t}, \\ \ln S_{n+1,1} = \ln S_n + \left(\mu - \dfrac{\sigma^2}{2}\right)\Delta t + \sigma\psi_1\sqrt{\Delta t}. \end{cases}$$

Let us choose

$$\begin{cases} \psi_0 = -1 - \left(\mu - \dfrac{\sigma^2}{2}\right)\sqrt{\Delta t}/\sigma, \\ \psi_1 = 1 - \left(\mu - \dfrac{\sigma^2}{2}\right)\sqrt{\Delta t}/\sigma. \end{cases} \tag{8.15}$$

Because $\psi_0\psi_1 = -1 + \left(\mu - \dfrac{\sigma^2}{2}\right)^2 \Delta t/\sigma^2$, ψ is an approximate to ϕ. In this case

$$\begin{cases} \ln S_{n+1,0} = \ln S_n - \sigma\sqrt{\Delta t}, \\ \ln S_{n+1,1} = \ln S_n + \sigma\sqrt{\Delta t}, \end{cases}$$

or

$$\begin{cases} S_{n+1,0} = S_n \mathrm{e}^{-\sigma\sqrt{\Delta t}}, \\ S_{n+1,1} = S_n \mathrm{e}^{\sigma\sqrt{\Delta t}}. \end{cases} \tag{8.16}$$

Using the formulae (8.12), (8.13) and (8.16), we can evaluate the price of a derivative if the stock price satisfies Eq. (8.14). This is called the binomial method which was proposed by Cox, Ross, and Rubinstein in 1979 [22].

For ψ_0 and ψ_1, we can choose other expressions. For example (see the book by McDonald [61]), let

$$\begin{cases} \psi_0 = -1 - \left(\mu - r + D_0 - \dfrac{\sigma^2}{2}\right)\sqrt{\Delta t}/\sigma, \\ \psi_1 = 1 - \left(\mu - r + D_0 - \dfrac{\sigma^2}{2}\right)\sqrt{\Delta t}/\sigma. \end{cases} \tag{8.17}$$

Because $\psi_0\psi_1 = -1 + \left(\mu - r - D_0 - \frac{\sigma^2}{2}\right)^2 \Delta t/\sigma^2$, ψ is an approximate to ϕ. In this case

$$\begin{cases} S_{n+1,0} = S_n \mathrm{e}^{(r-D_0)\Delta t - \sigma\sqrt{\Delta t}}, \\ S_{n+1,1} = S_n \mathrm{e}^{(r-D_0)\Delta t + \sigma\sqrt{\Delta t}}. \end{cases} \tag{8.18}$$

Generally, we can choose

$$\begin{cases} \psi_0 = -1 - \left(\mu - c - \frac{\sigma^2}{2}\right)\sqrt{\Delta t}/\sigma, \\ \psi_1 = 1 - \left(\mu - c - \frac{\sigma^2}{2}\right)\sqrt{\Delta t}/\sigma. \end{cases} \tag{8.19}$$

In this case

$$\begin{cases} S_{n+1,0} = S_n \mathrm{e}^{c\Delta t - \sigma\sqrt{\Delta t}}, \\ S_{n+1,1} = S_n \mathrm{e}^{c\Delta t + \sigma\sqrt{\Delta t}}, \end{cases} \tag{8.20}$$

and both the formulae (8.16) and (8.18) are in this form.

If p is determined by the formula (8.12), then we have

$$\begin{aligned} &pS_{n+1,1} + (1-p)\,S_{n+1,0} \\ &= \frac{S_n \mathrm{e}^{(r-D_0)\Delta t} - S_{n+1,0}}{S_{n+1,1} - S_{n+1,0}} S_{n+1,1} + \frac{S_{n+1,1} - S_n \mathrm{e}^{(r-D_0)\Delta t}}{S_{n+1,1} - S_{n+1,0}} S_{n+1,0} \\ &= \mathrm{e}^{(r-D_0)\Delta t} S_n. \end{aligned}$$

When $0 \le p \le 1$, this relation can be interpreted as follows. When a derivative is priced, the probability of the price at t^{n+1} being $S_{n+1,1}$ is p and the probability of the price at t^{n+1} being $S_{n+1,0}$ is $1-p$, and the expectation of the stock price at t^{n+1} is $\mathrm{e}^{(r-D_0)\Delta t} S_n$:

$$\mathrm{E}_D\left[S_{n+1}\right] = pS_{n+1,1} + (1-p)\,S_{n+1,0} = \mathrm{e}^{(r-D_0)\Delta t} S_n = \mathrm{e}^{r\Delta t}\mathrm{e}^{-D_0\Delta t} S_n, \tag{8.21}$$

where we use E_D as the notation for expectation in the case a derivative is priced. In the front of S_n there is a factor $\mathrm{e}^{-D_0\Delta t}$ because the expectation of the stock price should go down by a factor of $\mathrm{e}^{-D_0\Delta t}$ as one share of stock at time t^n becomes $\mathrm{e}^{D_0\Delta t}$ shares of stock at time t^{n+1}, and there is another factor $\mathrm{e}^{r\Delta t}$ because the expectation of the stock price should go up by a factor of $\mathrm{e}^{r\Delta t}$ just like any risk-free investment. Because of this, we usually say that $\mathrm{E}_D\left[S_{n+1}\right]$ is the expectation of S_{n+1} in the "risk-neutral" world. According to the model for the stock price, we have

$$\mathrm{E}[S_{n+1}] = S_n + \mu S_n \Delta t = \left(\mathrm{e}^{\mu\Delta t} + O(\Delta t^2)\right) S_n.$$

That is, in the expression for the expectation of the stock price at time t_{n+1} in the real world, there is a factor about $\mathrm{e}^{\mu\Delta t}$ in the front of S_n, which is completely different from the case when we price derivatives.

When $S_{n+1,0}$ and $S_{n+1,1}$ are given by Eq. (8.16), then

$$p = \frac{S_n e^{(r-D_0)\Delta t} - S_n e^{-\sigma\sqrt{\Delta t}}}{S_n e^{\sigma\sqrt{\Delta t}} - S_n e^{-\sigma\sqrt{\Delta t}}} = \frac{e^{(r-D_0)\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} \tag{8.22}$$

and $0 \le p \le 1$ is equivalent to $e^{-\sigma\sqrt{\Delta t}} \le e^{(r-D_0)\Delta t} \le e^{\sigma\sqrt{\Delta t}}$. The inequality $e^{(r-D_0)\Delta t} \le e^{\sigma\sqrt{\Delta t}}$ might not hold for large Δt and p does not represent a probability in this case. However this case usually does not occur in practice because Δt would be small in real computation. When $S_{n+1,0}$ and $S_{n+1,1}$ are given by the formula (8.18), then

$$p = \frac{S_n e^{(r-D_0)\Delta t} - S_n e^{(r-D_0)\Delta t-\sigma\sqrt{\Delta t}}}{S_n e^{(r-D_0)\Delta t+\sigma\sqrt{\Delta t}} - S_n e^{(r-D_0)\Delta t-\sigma\sqrt{\Delta t}}} = \frac{1 - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} \tag{8.23}$$

and $0 \le p \le 1$ always holds. Hence in this case p can always be interpreted as the probability of the price being $S_{n+1,1}$ at t_{n+1}.

In the "risk-neutral" world, the variance of S_{n+1} is

$$\begin{aligned}
&\mathrm{Var}_D\left[S_{n+1}\right] \\
&= \frac{S_n e^{(r-D_0)\Delta t} - S_{n+1,0}}{S_{n+1,1} - S_{n+1,0}} \left(S_{n+1,1} - e^{(r-D_0)\Delta t} S_n\right)^2 \\
&\quad + \frac{S_{n+1,1} - S_n e^{(r-D_0)\Delta t}}{S_{n+1,1} - S_{n+1,0}} \left(S_{n+1,0} - e^{(r-D_0)\Delta t} S_n\right)^2 \\
&= \left(S_n e^{(r-D_0)\Delta t} - S_{n+1,0}\right)\left(S_{n+1,1} - S_n e^{(r-D_0)\Delta t}\right) \\
&= S_n^2 e^{2(r-D_0)\Delta t} \cdot \left(1 - \frac{S_{n+1,0}}{S_n e^{(r-D_0)\Delta t}}\right)\left(\frac{S_{n+1,1}}{S_n e^{(r-D_0)\Delta t}} - 1\right) \\
&= S_n^2 e^{2(r-D_0)\Delta t} \cdot \left(\frac{S_{n+1,0}}{S_n e^{(r-D_0)\Delta t}} + \frac{S_{n+1,1}}{S_n e^{(r-D_0)\Delta t}} - \frac{S_{n+1,0}S_{n+1,1}}{S_n^2 e^{2(r-D_0)\Delta t}} - 1\right).
\end{aligned}$$

When $S_{n+1,0}$ and $S_{n+1,1}$ are given by the expression (8.20), both the formulae (8.16) and (8.18) being in this form, the expression above can further be written as:

$$\begin{aligned}
&\mathrm{Var}_D\left[S_{n+1}\right] \\
&= S_n^2 e^{2(r-D_0)\Delta t}\left(e^{-(r-D_0-c)\Delta t-\sigma\sqrt{\Delta t}} + e^{-(r-Dt_0-c)\Delta t+\sigma\sqrt{\Delta t}}\right. \\
&\qquad\qquad \left. - e^{-2(r-D_0-c)\Delta t} - 1\right) \\
&= S_n^2 e^{(r-D_0+c)\Delta t}\left(e^{-\sigma\sqrt{\Delta t}} + e^{\sigma\sqrt{\Delta t}} - e^{-(r-D_0-c)\Delta t} - e^{(r-D_0-c)\Delta t}\right) \\
&= S_n^2 e^{(r-D_0+c)\Delta t}\left[1 - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t - \frac{1}{6}\sigma^3\Delta t^{3/2} + 1 + \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t\right. \\
&\qquad \left. + \frac{1}{6}\sigma^3\Delta t^{3/2} - 1 + (r - D_0 - c)\Delta t - 1 - (r - D_0 - c)\Delta t + O\left(\Delta t^2\right)\right] \\
&= S_n^2 e^{(r-D_0+c)\Delta t}\left[\sigma^2\Delta t + O\left(\Delta t^2\right)\right] \\
&= S_n^2\sigma^2\Delta t + O\left(\Delta t^2\right).
\end{aligned} \tag{8.24}$$

In the real world,

$$\mathrm{Var}[S_{n+1}] = \mathrm{Var}\left[S_n + \mu S_n \Delta t + \sigma S_n \phi \sqrt{\Delta t}\right] = \sigma^2 S_n^2 \Delta t.$$

Therefore as $\Delta t \to 0$ the variance of S_{n+1} in the "risk-neutral" world will tend to the variance of S_{n+1} in the real world.

Now let us describe the complete method proposed by Cox, Ross, and Rubinstein [22]. Define

$$d = \mathrm{e}^{-\sigma\sqrt{\Delta t}} \tag{8.25}$$

and

$$u = \frac{1}{d} = \mathrm{e}^{\sigma\sqrt{\Delta t}}, \tag{8.26}$$

then $S_{n+1,1} = S_n u$, $S_{n+1,0} = S_n d$, and Eqs. (8.22) and (8.13) can be rewritten as

$$p = \frac{\mathrm{e}^{(r-D_0)\Delta t} - d}{u - d} \tag{8.27}$$

and

$$\begin{aligned} &V(S_n, n\Delta t) \\ &= \mathrm{e}^{-r\Delta t}\left[pV\left(S_{n+1,1}, (n+1)\Delta t\right) + (1-p)V\left(S_{n+1,0}, (n+1)\Delta t\right)\right]. \end{aligned} \tag{8.28}$$

Here $V(S,t)$ is the value of an option.

Suppose the asset price at the current time t to be S, and we divide the remaining life of the derivative security into N equal time subintervals with time step $\Delta t = (T-t)/N$. At the first time level $t + \Delta t$, there are two possible asset prices Su and $Sd = Su^{-1}$. At the second time level $t + 2\Delta t$, there are three possible asset prices, Su^2, $Sud = Sdu = S$, and $Sd^2 = Su^{-2}$. At the third time level $t + 3\Delta t$, there are four possible asset prices, Su^3, $Su^2d = Su$, $Sud^2 = Su^{-1}$, and $Sd^3 = Su^{-3}$. In general, at the n-th time level $t + n\Delta t$, there are $n+1$ possible values of the asset price. Originally, at the n-th time level, there should be 2^n possible values of the asset price. However since $d = 1/u$ is required, many points are the same. For example, S, Su^2d^2, Su^4d^4, $\cdots$ are the same point because $d = 1/u$. Hence the number of possible values is greatly reduced. Let $S_{n,m}, m = 0, 1, \cdots, n$, denote the $n+1$ possible values of the asset price at the n-th time level from the smallest to the largest. Then

$$S_{n,m} = Su^{2m-n},\ m = 0, 1, \cdots, n. \tag{8.29}$$

For $N = 4$, all the possible prices for each n are given in Fig. 8.2. This plot is usually referred to as a tree or lattice of possible asset prices.

Assuming that we know the payoff function for our derivative security and that it depends only on the values of the underlying asset at expiry, this

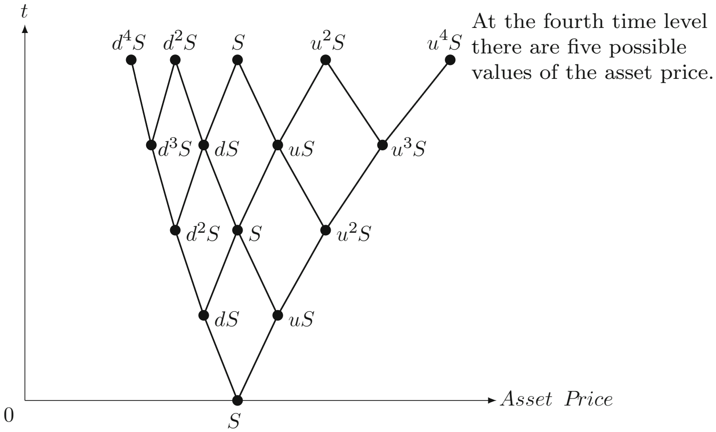

Fig. 8.2. Tree of asset prices for a binomial model

enables us to value it at expiry, the N-th time level. If we are considering a call, for example, we find

$$c_{N,m} = \max(S_{N,m} - E, 0), \quad m = 0, 1, \cdots, N, \tag{8.30}$$

where E is the exercise price and $c_{N,m}$ denotes the value of the call for the m-th possible asset value $S_{N,m}$ at time-step N. For a put, we know that

$$p_{N,m} = \max(E - S_{N,m}, 0), \quad m = 0, 1, \cdots, N, \tag{8.31}$$

where $p_{N,m}$ denotes the value of the put for the m-th possible asset value $S_{N,m}$ at expiry.

We can now find the expected value of the derivative security at the $(N-1)$-th time level and for possible asset prices $S_{N-1,m}$, $m = 0, 1, \ldots, N-1$ because we know that the probability of an asset price moving from $S_{N-1,m}$ to $S_{N,m+1}$ during a time step is p and that the probability of it moving to $S_{N,m}$ is $(1-p)$. Using the discounting factor $\mathrm{e}^{-r\Delta t}$, we can obtain the value of the security at each possible asset price for the $(N-1)$-th time level. This procedure can be applied to the n-th time level if the values of the option for the $(n+1)$-th time level have been obtained, and the computational formula is Eq. (8.28) or, in a general form,

$$V_{n,m} = \mathrm{e}^{-r\Delta t}(pV_{n+1,m+1} + (1-p)V_{n+1,m}), \quad m = 0, 1, \cdots, n. \tag{8.32}$$

Here, $V_{n,m}$ denotes the value of a European option at the n-th time level and corresponding to asset price $S_{n,m}$. According to this formula, starting from the payoff function, $V_{N,m}$, $m = 0, 1, \cdots, N$, we can recursively determine $V_{n,m}, m = 0, 1, \cdots, n$ for $n = N-1, N-2, \cdots, 0$, and the final value $V_{0,0}$ is the current value of the option.

For American options, we can easily incorporate the possibility of early exercise of an option into the binomial model. Because the price of an American call option must be greater than or equal to

$$\max(S_{n,m} - E, 0), \tag{8.33}$$

when calculating the price of an American call option, we need to replace the formula (8.32) by

$$C_{n,m} = \max\left(\mathrm{e}^{-r\Delta t}\left[pC_{n+1,m+1} + (1-p)C_{n+1,m}\right],\ S_{n,m} - E,\ 0\right) \tag{8.34}$$

at each point. Similarly, for an American put option, the formula is

$$P_{n,m} = \max\left(\mathrm{e}^{-r\Delta t}\left[pP_{n+1,m+1} + (1-p)P_{n+1,m}\right],\ E - S_{n,m},\ 0\right) \tag{8.35}$$

because the price of an American put option has to be at least

$$\max(E - S_{n,m}, 0). \tag{8.36}$$

From what has been described, we see that the entire computation can be done in two steps. In the first step, we calculate all the $S_{n,m}$ to be used. Then, we find $V_{N,m}, m = 0, 1, \cdots, N$ and calculate $V_{n,m}, m = 0, 1, \cdots, n$ for $n = N-1, N-2, \cdots, 0$ successively. When a European option is calculated, only the $S_{N,m}, m = 0, 1, \cdots, N$, are used in order to find $V_{N,m}$. When an American option is evaluated, all the $S_{n,m}$ are needed. However, because $S_{n,m} = Su^{2m-n} = Su^{2(m-1)-(n-2)} = S_{n-2,m-1}$, we indeed only need to calculate $S_{N,m}, m = 0, 1, \cdots, N$ and $S_{N-1,m}, m = 0, 1, \cdots, N-1$, i.e., $Su^m, m = -N, -N+1, \ldots, N$. For this method, the total number of nodes is $(N+2)(N+1)/2$, so the execution time for computing all the $V_{n,m}$ is $O(N^2)$.

If the method given in the book by McDonald [61] wants to be adopted, instead of the formulae (8.25)–(8.27), (8.18) and (8.23) should be used. Also the tree of asset prices is different. In this case we should define

$$S_{n,m} = S\mathrm{e}^{n(r-D_0)\Delta t}u^{2m-n},\ m = 0, 1, \cdots, n$$

with $u = \mathrm{e}^{\sigma\sqrt{\Delta t}}$.

Trinomial Methods. If σ depends on S, then u is not a constant. In this case, generally speaking, at the n-th time level, there are 2^n possible values of the asset prices that need to be considered, and the total nodes and the execution time will be very large if a binomial method is used. In order to reduce the nodes for a problem with variable σ, we can use trinomial methods. In a trinomial method, given a current asset value S, the asset value after a time-step Δt can take any of the three values

$$Su,\ Sq,\ Sd,$$

where $0 \le d < q < u$. Let p_u be the probability of the value of the asset after a time-step Δt being Su, p_q be the probability of the value being Sq, and p_d

be the probability of the value being Sd. Because there are only three possible cases, we must have

$$p_u + p_q + p_d = 1, \quad 0 \le p_u \le 1, \quad 0 \le p_q \le 1, \quad 0 \le p_u \le 1.$$

When the binomial method is used for pricing call/put options, from the expressions (8.21) and (8.24) we have

$$\mathrm{E}_D\left[S_{n+1}\right] = \mathrm{e}^{(r-D_0)\Delta t} S_n$$

and

$$\begin{aligned}
\mathrm{E}_D\left[S_{n+1}^2\right] &= \mathrm{Var}_D\left[S_{n+1}\right] + \left(\mathrm{E}_D\left[S_{n+1}\right]\right)^2 \\
&= S_n^2 \sigma^2 \Delta t + O(\Delta t^2) + \mathrm{e}^{2(r-D_0)\Delta t} S_n^2 \\
&= \mathrm{e}^{[2(r-D_0)+\sigma^2]\Delta t} S_n^2 + O(\Delta t^2).
\end{aligned}$$

Thus for p_u, p_q and p_d, we require[1]

$$\begin{aligned}
p_u u + p_q q + p_d d &= \mathrm{e}^{(r-D_0)\Delta t}, \\
p_u u^2 + p_q q^2 + p_d d^2 &= \mathrm{e}^{(2(r-D_0)+\sigma^2)\Delta t}.
\end{aligned}$$

Because there are three equations above for six unknowns, u, q, d, p_u, p_q, p_d, we can choose three parameters. In order that the number of the possible asset prices is not 3^n at the n-th time level, we can choose

$$d = 1/u \quad \text{and} \quad q = 1. \tag{8.37}$$

Now there are only four parameters u, p_u, p_q, p_d left. They should satisfy the three conditions above. If u is given, then this is a linear system for p_u, p_q, p_d and can be solved for them easily. Its solution is

$$\left\{\begin{aligned}
p_u &= \frac{\mathrm{e}^{(2(r-D_0)+\sigma^2)\Delta t} - \mathrm{e}^{(r-D_0)\Delta t}(q+d) + qd}{(u-q)(u-d)}, \\
p_q &= \frac{\mathrm{e}^{(2(r-D_0)+\sigma^2)\Delta t} - \mathrm{e}^{(r-D_0)\Delta t}(d+u) + du}{(q-d)(q-u)}, \\
p_d &= \frac{\mathrm{e}^{(2(r-D_0)+\sigma^2)\Delta t} - \mathrm{e}^{(r-D_0)\Delta t}(u+q) + uq}{(d-u)(d-q)}.
\end{aligned}\right. \tag{8.38}$$

Because they represent probabilities, we need to choose such a u that p_u, p_q and p_d all are nonnegative. If σ depends on S and t, then p_u, p_q and p_d will be different at different points. In this case, we need to choose such a u that at all the points p_u, p_q and p_d are nonnegative and the set of formulae (8.38) can still be used.

[1] We also know that because the Black–Scholes equation holds, $\mathrm{E}_D\left[S_{n+1}\right] = \mathrm{e}^{(r-D_0)\Delta t} S_n$ and $\mathrm{E}_D\left[S_{n+1}^2\right] = \mathrm{e}^{[2(r-D_0)+\sigma^2]\Delta t} S_n^2$ should be true (see Problem 39 of Chap. 2).

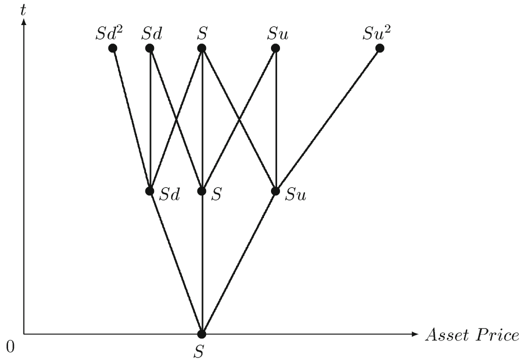

Fig. 8.3. Lattice generated by a trinomial model

The details for evaluating derivative securities using a trinomial method are nearly identical to the binomial method. The only major difference is that the expected value of the security at the n-th time level depends on the three possible values at the $(n+1)$-th time level, and that at the n-th time level, there are $2n+1$ possible asset prices, which are

$$S_{n,m} = Su^m, m = -n, -n+1, \cdots, n.$$

In this case, the corresponding lattice is illustrated in Fig. 8.3. Let $V_{n,m}$ be the security price at $S_{n,m}$. Then, the formula for finding the expected value of a security at time level $n+1$ is

$$\mathrm{E}_D\left[V_{n+1,m}\right] = p_u V_{n+1,m+1} + p_q V_{n+1,m} + p_d V_{n+1,m-1}$$

and the value of a European derivative security for $S_{n,m}$ is

$$V_{n,m} = \mathrm{e}^{-r\Delta t}(p_u V_{n+1,m+1} + p_q V_{n+1,m} + p_d V_{n+1,m-1}),$$

and for American puts and calls we have

$$P_{n,m} = \max\left(\mathrm{e}^{-r\Delta t}\left[p_u P_{n+1,m+1} + p_q P_{n+1,m} + p_d P_{n+1,m-1}\right],\ E - S_{n,m},\ 0\right), \tag{8.39}$$

$$C_{n,m} = \max\left(\mathrm{e}^{-r\Delta t}\left[p_u C_{n+1,m+1} + p_q C_{n+1,m} + p_d C_{n+1,m-1}\right],\ S_{n,m} - E,\ 0\right). \tag{8.40}$$

In Tables 8.4 and 8.5, we give binomial lattice approximations to American call and put options when the formulae (8.25)–(8.28) are used. The errors and the CPU times on a computer are also shown.

Table 8.4. American call option [binomial method (8.25)–(8.28)]

($r = 0.1$, $\sigma = 0.2$, $D_0 = 0.05$, $T = 1\,\text{year}$, $S = E = 100$, and the exact solution is $C = 9.94092345\cdots$)

Numbers of time steps	Results	|Errors|	CPU(s)
50	9.902969	0.037955	0.0004
100	9.921921	0.019002	0.0013
200	9.931416	0.009507	0.0053
400	9.936168	0.004755	0.0220
800	9.938546	0.002378	0.0890

Table 8.5. American put option [binomial method (8.25)–(8.28)]

($r = 0.1$, $\sigma = 0.2$, $D_0 = 0.05$, $T = 1\,\text{year}$, $S = E = 100$, and the exact solution is $P = 5.92827717\cdots$)

Numbers of time steps	Results	|Errors|	CPU(s)
50	5.911020	0.017257	0.0004
100	5.920066	0.008211	0.0014
200	5.924273	0.004005	0.0053
400	5.926323	0.001955	0.0210
800	5.927309	0.000968	0.0880

8.1.4 Relations Between the Lattice Methods and the Explicit Finite-Difference Methods

From the view point of PDEs, the procedure given by the formulae (8.12), (8.13), and (8.20) can be understood in the following way. The value of any derivative, V, satisfies

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0)\, S \frac{\partial V}{\partial S} - rV = 0.$$

Let $\bar{S} = S\mathrm{e}^{-ct}$ and $\overline{V}\left(\bar{S}, t\right) = V(S, t)$. Since

$$\frac{\partial V}{\partial S} = \frac{\partial \overline{V}}{\partial \bar{S}} \mathrm{e}^{-ct},$$
$$\frac{\partial^2 V}{\partial S^2} = \frac{\partial^2 \overline{V}}{\partial \bar{S}^2} \mathrm{e}^{-2ct},$$

and

$$\frac{\partial V}{\partial t} = \frac{\partial \overline{V}}{\partial t} + \frac{\partial \overline{V}}{\partial \bar{S}} S\mathrm{e}^{-ct} \cdot (-c),$$

we have

$$\frac{\partial \overline{V}}{\partial t} + \frac{1}{2}\sigma^2 \bar{S}^2 \frac{\partial^2 \overline{V}}{\partial \bar{S}^2} + (r - D_0 - c)\, \bar{S} \frac{\partial \overline{V}}{\partial \bar{S}} - r\overline{V} = 0.$$

Furthermore let us set $x = \ln \bar{S}$ and $\tilde{V}(x,t) = \overline{V}(\bar{S},t)$. Noticing

$$\frac{\partial \overline{V}}{\partial \bar{S}} = \frac{\partial \tilde{V}}{\partial x}\frac{1}{\bar{S}},$$
$$\frac{\partial^2 \overline{V}}{\partial \bar{S}^2} = \frac{\partial^2 \tilde{V}}{\partial x^2}\frac{1}{\bar{S}^2} - \frac{1}{\bar{S}^2}\frac{\partial \tilde{V}}{\partial x},$$

and

$$\frac{\partial \overline{V}}{\partial t} = \frac{\partial \tilde{V}}{\partial t},$$

we arrive at

$$\frac{\partial \tilde{V}}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 \tilde{V}}{\partial x^2} + \left(r - D_0 - c - \sigma^2/2\right)\frac{\partial \tilde{V}}{\partial x} - r\tilde{V} = 0. \tag{8.41}$$

For this equation, we can have the following finite-difference scheme

$$\frac{\tilde{V}_m^{n+1} - \tilde{V}_m^n}{\Delta t} + \frac{1}{2}\sigma^2 \frac{\tilde{V}_{m+1}^{n+1} - 2\tilde{V}_m^{n+1} + \tilde{V}_{m-1}^{n+1}}{\Delta x^2}$$
$$+ \left(r - D_0 - c - \sigma^2/2\right)\frac{\tilde{V}_{m+1}^{n+1} - \tilde{V}_{m-1}^{n+1}}{2\Delta x} - r\tilde{V}_m^n = 0,$$

or

$$\begin{aligned}\tilde{V}_m^n = \frac{1}{1 + r\Delta t}\Bigg[&\left(\frac{\sigma^2}{2}\frac{\Delta t}{\Delta x^2} + \frac{r - D_0 - c - \sigma^2/2}{2}\frac{\Delta t}{\Delta x}\right)\tilde{V}_{m+1}^{n+1} \\ &+ \left(1 - \frac{\sigma^2 \Delta t}{\Delta x^2}\right)\tilde{V}_m^{n+1} \\ &+ \left(\frac{\sigma^2}{2}\frac{\Delta t}{\Delta x^2} - \frac{r - D_0 - c - \sigma^2/2}{2}\frac{\Delta t}{\Delta x}\right)\tilde{V}_{m-1}^{n+1}\Bigg].\end{aligned} \tag{8.42}$$

Here $\tilde{V}_m^n$ denotes the value of $\tilde{V}$ at $x_m = \bar{x} + m\Delta x$ and $t^n = n\Delta t$. If we choose

$$\Delta x = \sigma\sqrt{\Delta t}, \tag{8.43}$$

then we have

$$\begin{aligned}\tilde{V}_m^n = \frac{1}{1 + r\Delta t}\Bigg[&\left(\frac{1}{2} + \frac{r - D_0 - c - \frac{1}{2}\sigma^2}{2\sigma}\sqrt{\Delta t}\right)\tilde{V}_{m+1}^{n+1} \\ &+ \left(\frac{1}{2} - \frac{r - D_0 - c - \frac{1}{2}\sigma^2}{2\sigma}\sqrt{\Delta t}\right)\tilde{V}_{m-1}^{n+1}\Bigg].\end{aligned} \tag{8.44}$$

Now we show that a trinomial method (a binomial method) is close to an explicit method (8.42) [an explicit method (8.44)]. First we will show that the mesh here can overlap the lattices of trinomial and binomial methods. Consider the case $c = 0$ and let $\bar{x} = \ln S^*$, S^* being the asset price at the current time. In this case

$$S(x_m) = \mathrm{e}^{\bar{x}+m\Delta x} = S^* \left(\mathrm{e}^{\Delta x}\right)^m .$$

Therefore, a uniform mesh on (x,t)-plane (see Fig. 8.4) corresponds to a non-uniform mesh on (S,t)-plane (see Fig. 8.5), which overlaps the lattices in Figs. 8.2 and 8.3 with $u = \mathrm{e}^{\Delta x}$ and $S = S^*$. Consequently, this explicit difference method can be understood as a trinomial method with a lattice in Fig. 8.3 and as a binomial method with a lattice in Fig. 8.2 if the expression (8.43) holds.

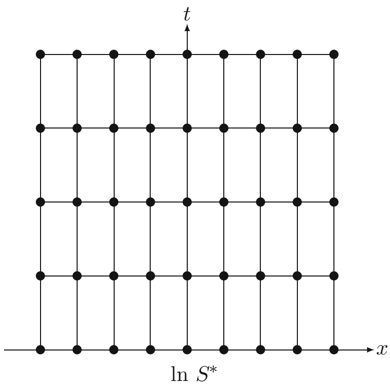

Fig. 8.4. A uniform mesh on (x,t)-plane

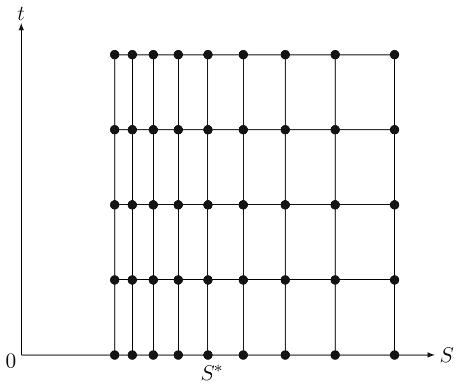

Fig. 8.5. The mesh on (S,t)-plane corresponding to a uniform mesh on (x,t)-plane

Now let show that the difference between the formulae (8.13) and (8.44) is very small. Let x_m, S_m^n, and $\bar{S}_m^n$ denote the x-coordinates, S-coordinates, and $\bar{S}$-coordinates of the m-point at time t^n, respectively. Because

$$x_{m+1} = x_m + \Delta x = x_m + \sigma\sqrt{\Delta t},$$

which means

$$\ln \bar{S}_{m+1}^{n+1} = \ln \bar{S}_m^n + \sigma\sqrt{\Delta t}$$

or

$$\ln\left(S_{m+1}^{n+1}\mathrm{e}^{-ct^{n+1}}\right) = \ln\left(S_m^n\mathrm{e}^{-ct^n}\right) + \sigma\sqrt{\Delta t},$$

we have

$$S_{m+1}^{n+1} = S_m^n\mathrm{e}^{c\left(t^{n+1}-t^n\right)+\sigma\sqrt{\Delta t}} = S_m^n\mathrm{e}^{c\Delta t+\sigma\sqrt{\Delta t}}. \tag{8.45}$$

Similarly,

$$S_{m-1}^{n+1} = S_m^n\mathrm{e}^{c\Delta t-\sigma\sqrt{\Delta t}}. \tag{8.46}$$

Noticing that $S_{m+1}^{n+1}, S_{m-1}^{n+1}$ and S_m^n correspond to $S_{n+1,1}, S_{n+1,0}$ and S_n, we have the relations (8.20). Therefore from the expression (8.12), we have

$$\begin{aligned}
p &= \frac{S_n\mathrm{e}^{(r-D_0)\Delta t} - S_n\mathrm{e}^{c\Delta t-\sigma\sqrt{\Delta t}}}{S_n\mathrm{e}^{c\Delta t+\sigma\sqrt{\Delta t}} - S_n\mathrm{e}^{c\Delta t-\sigma\sqrt{\Delta t}}} = \frac{\mathrm{e}^{(r-D_0-c)\Delta t} - \mathrm{e}^{-\sigma\sqrt{\Delta t}}}{\mathrm{e}^{\sigma\sqrt{\Delta t}} - \mathrm{e}^{-\sigma\sqrt{\Delta t}}} \\
&= \frac{1 + (r - D_0 - c)\,\Delta t - \left(1 - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t - \frac{1}{6}\sigma^3\Delta t^{3/2}\right) + O\left(\Delta t^2\right)}{2\sigma\sqrt{\Delta t} + \frac{1}{3}\sigma^3\Delta t^{3/2} + O\left(\Delta t^2\right)} \\
&= \frac{\sigma\sqrt{\Delta t}\left[1 + \left(r - D_0 - c - \sigma^2/2\right)\sqrt{\Delta t}/\sigma + \frac{1}{6}\sigma^2\Delta t + O\left(\Delta t^{3/2}\right)\right]}{2\sigma\sqrt{\Delta t}\left[1 + \frac{1}{6}\sigma^2\Delta t + O\left(\Delta t^{3/2}\right)\right]} \\
&= \frac{1}{2}\left[1 + \left(r - D_0 - c - \sigma^2/2\right)\sqrt{\Delta t}/\sigma + \frac{1}{6}\sigma^2\Delta t + O\left(\Delta t^{3/2}\right)\right] \\
&\quad \times \left[1 - \frac{1}{6}\sigma^2\Delta t + O\left(\Delta t^{3/2}\right)\right] \\
&= \frac{1}{2}\left[1 + \frac{r - D_0 - c - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right] + O\left(\Delta t^{3/2}\right).
\end{aligned}$$

Also the difference between $\mathrm{e}^{-r\Delta t}$ and $\frac{1}{1+r\Delta t}$ is $O(\Delta t^2)$. Thus the formula (8.13) is almost the same as the formula (8.44). Consequently, the method given by the formulae (8.12), (8.13), and (8.20) is almost an explicit scheme (8.44). Therefore, the binomial method and the trinomial method can be understood as explicit finite-difference methods in some sense.

Finally we point out that because the convergence of the explicit scheme here with $\Delta t/\Delta x^2 = \sigma^{-2}$ can be easily proved, the difference between $\frac{1}{2}\left[1 + \frac{r-D_0-c-\frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right]$ and p is $O\left(\Delta t^{3/2}\right)$, and the difference between $\mathrm{e}^{-r\Delta t}$ and $\frac{1}{1+r\Delta t}$ is $O(\Delta t^2)$, the convergence of the binomial method can also be proved.

The formulae (8.45) and (8.46) actually are the formula (8.20), so the conclusion given here can be used for both the Cox–Ross–Rubinstein method (See [22]) and the McDonald method (See [61]).

8.1.5 Examples of Unstable Schemes

As has been pointed out in Sect. 8.1.1, when the scheme (8.3) or (8.6) is used, stability condition (8.4) or (8.7) is required. What will happen if these conditions are violated?

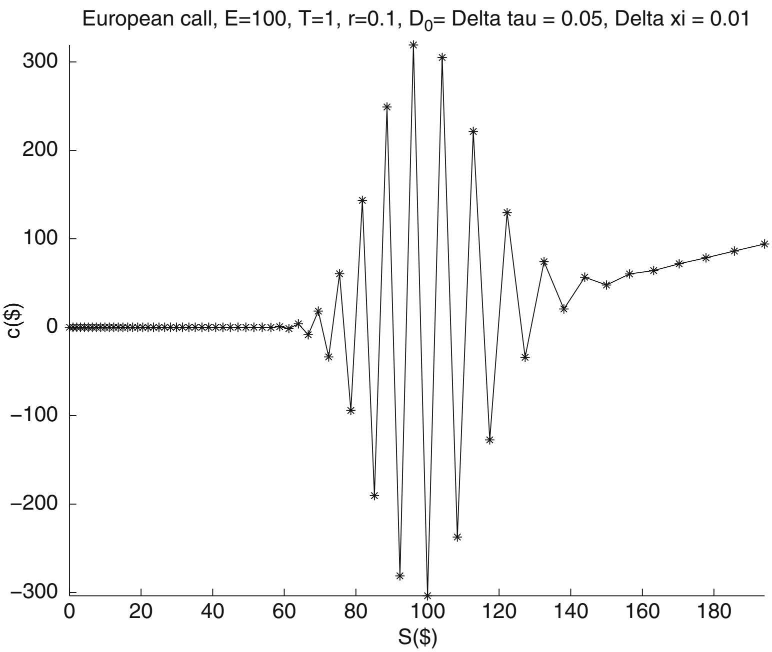

Fig. 8.6. A unstable solution of EFDI
(The solution appears when Eq. (8.4) is violated. $E = 100$, $T = 1$, $r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$, $\Delta\tau = 0.05$, and $\Delta\xi = 0.01$.)

Let us try scheme (8.3) for a European call option with parameters $E = 100$, $T = 1$, $r = 0.1$, $D_0 = 0.05$, and $\sigma = 0.2$. Take $\Delta\tau = 0.05$ and $\Delta\xi = 0.01$. The solution at $t = 0$ is shown in Fig. 8.6, where we see that rather large oscillations develop. In this case, $\Delta\tau = 0.05$ and $16\Delta\xi^2/\sigma^2 = 0.04$, so condition (8.4) does not hold, and the scheme is unstable. We cannot get a useful solution if such a set of $\Delta\tau$ and $\Delta\xi$ is adopted.

8.2 Implicit Methods

8.2.1 Pricing European Options by Using $\overline{V}$, ξ, τ Variables

The problem (8.1) can also be discretized by an implicit scheme, for example, by scheme (7.6). In this case, the difference equations are

$$\begin{aligned}
&\frac{v_m^{n+1}-v_m^n}{\Delta\tau}\\
&=\frac{1}{4}\bar{\sigma}_m^2\xi_m^2(1-\xi_m)^2\left(\frac{v_{m+1}^{n+1}-2v_m^{n+1}+v_{m-1}^{n+1}}{\Delta\xi^2}+\frac{v_{m+1}^n-2v_m^n+v_{m-1}^n}{\Delta\xi^2}\right)\\
&\quad+\frac{1}{2}(r-D_0)\xi_m(1-\xi_m)\left(\frac{v_{m+1}^{n+1}-v_{m-1}^{n+1}}{2\Delta\xi}+\frac{v_{m+1}^n-v_{m-1}^n}{2\Delta\xi}\right)\\
&\quad-\frac{1}{2}[r(1-\xi_m)+D_0\xi_m](v_m^{n+1}+v_m^n),\\
&\qquad\qquad m=0,1,\cdots,M,\quad n=0,1,\cdots,N-1.
\end{aligned}\tag{8.47}$$

Like the explicit scheme case, v_{-1}^n and v_{M+1}^n do not appear in the equation. For a fixed n, there are $M+1$ equations in the following form:

$$\begin{aligned}
a_m v_{m-1}^{n+1}+b_m v_m^{n+1}+c_m v_{m+1}^{n+1}&=-a_m v_{m-1}^n+(2-b_m)v_m^n-c_m v_{m+1}^n,\\
m&=0,1,\cdots,M,
\end{aligned}$$

where

$$\begin{aligned}
a_m&=\left[(r-D_0)m(1-\xi_m)-\bar{\sigma}_m^2m^2(1-\xi_m)^2\right]\Delta\tau/4,\\
b_m&=1+[\bar{\sigma}_m^2m^2(1-\xi_m)^2+r(1-\xi_m)+D_0\xi_m]\Delta\tau/2,\\
c_m&=\left[-(r-D_0)m(1-\xi_m)-\bar{\sigma}_m^2m^2(1-\xi_m)^2\right]\Delta\tau/4,\\
&\qquad m=0,1,\cdots,M.
\end{aligned}$$

In matrix form, we have

$$A\mathbf{v}^{n+1}=\mathbf{q}^n,$$

where

$$A=\begin{bmatrix}
b_0 & c_0 & & & & \\
a_1 & b_1 & c_1 & & 0 & \\
 & \ddots & \ddots & \ddots & & \\
 & & \ddots & \ddots & \ddots & \\
 & 0 & & a_{M-1} & b_{M-1} & c_{M-1}\\
 & & & & a_M & b_M
\end{bmatrix},$$

$$\mathbf{v}^n=\begin{bmatrix} v_0^n\\ v_1^n\\ \vdots\\ v_M^n\end{bmatrix},\quad
\mathbf{q}^n=\begin{bmatrix} (2-b_0)v_0^n-c_0v_1^n\\ -a_1v_0^n+(2-b_1)v_1^n-c_1v_2^n\\ \vdots\\ -a_Mv_{M-1}^n+(2-b_M)v_M^n\end{bmatrix}.$$

The difference between an implicit method and an explicit method is that for an implicit method, a linear system needs to be solved in order to get $\mathbf{v}^{n+1}$ from $\mathbf{v}^n$. This can be done by the LU decomposition or an iteration method given in Sects. 6.2.1 and 6.2.2. The linear system here has a variable coefficient matrix, however, it does not depend on time if σ does not depend on t. Thus, the linear system can be solved with only slightly more cost compared to a linear system with a constant coefficient matrix. It is clear that scheme (8.47) can even be applied to the case when σ depends on S and t. We will refer to this scheme as the implicit finite-difference scheme. From Problem 15 in Chap. 7, we can expect this scheme to be stable without any condition on the ratio $\Delta\tau/\Delta\xi$. In fact, in the paper by Sun, Yan, and Zhu [79], it is rigorously proved that this scheme with variable coefficients is unconditionally stable.

When σ is a constant, we can also use the variables u, x and $\bar{\tau}$. In this case, the difference scheme (7.9) can be applied to the equation in problem (8.5). However, when the scheme (7.9) is used for problem (8.5), we have to modify the problem formulation slightly. Let the problem be defined on a finite domain $[x_l, x_u]$ and give an artificial boundary condition on each boundary. From the expressions (2.19) and (2.23) in Sect. 2.2.5, we know at $S = 0, V(0,t) = V(0,T)\mathrm{e}^{-r(T-t)}$ and for $S \approx \infty$, $V(S,t) \approx V(S,T)\mathrm{e}^{-D_0(T-t)}$. Therefore, noticing $u(x,\bar{\tau}) = \mathrm{e}^{r(T-t)}V(S,t)$, for $S \approx 0$, i.e., $x \approx -\infty$ we have

$$u(x,\bar{\tau}) \approx V(S,T)$$

and for $S \approx \infty$, i.e., $x \approx \infty$,

$$u(x,\bar{\tau}) \approx V(S,T)\mathrm{e}^{(r-D_0)(T-t)},$$

where $x = \ln S + (r - D_0 - \sigma^2/2)(T-t)$ and $\bar{\tau} = \sigma^2(T-t)/2$. These two relations can be taken as artificial boundary conditions at $x = x_l$ and $x = x_u$, respectively, if x_l is small enough and x_u is large enough. For example, in order to calculate a call option,

$$u(x_l,\bar{\tau}) = 0 \quad \text{and} \quad u(x_u,\bar{\tau}) = (\mathrm{e}^{x_u-(2(r-D_0)/\sigma^2-1)\bar{\tau}} - E)\mathrm{e}^{2(r-D_0)\bar{\tau}/\sigma^2}$$

can be adopted as artificial boundary conditions. If the call option has parameters $r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$, $E = 1$, and $T = 1$, we can let $x_l = \ln 0.2$ and $x_u = \ln 2.3$.

The method for solving European average strike and double average options with continuous sampling is similar. However the transformations will be different for the two different cases.

8.2.2 European Options with Discrete Dividends and Asian and Lookback Options with Discrete Sampling

A holder of a stock usually obtains dividends on certain days, not continuously. Thus, in practice, it is important to know how to price options on stocks with discrete dividends. For Asian and lookback options, sampling is usually done discretely even though the time interval between two samples is very small so it can be seen as being continuously. This subsection is devoted to discussing how to evaluate European options with discrete dividends and European-style Asian and lookback options with discrete sampling. We give details here only for European options with discrete dividends and European average price options with discrete sampling. For other cases, the prices can be obtained in a similar way. Some results on such options are also given here.

European Options with Discrete Dividends. First, we work on options on stocks with discrete dividends. Let $V(S,t)$ be the price of an option on stocks with discrete dividends. From Sect. 2.2.2, we know that $V(S,t)$ is the solution to the following problem:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2(S)S^2\dfrac{\partial^2 V}{\partial S^2} + [rS - D(S,t)]\dfrac{\partial V}{\partial S} - rV = 0, \quad 0 \le S, \quad t \le T, \\ V(S,T) = V_T(S), \qquad 0 \le S, \end{cases} \tag{8.48}$$

where $D(S,t) = \sum_{i=1}^{I} D_i(S)\delta(t-t_i)$ and $D_i(S) \le S$ for any S. The meaning of the condition $D_i(S) \le S$ here is that the price of a stock at any time should be greater than or equal to the dividend paid at that time. From the problem (8.48), we know the following: At $t \ne t_i, i = 1, 2, \cdots, I$, V satisfies

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2(S)S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \quad 0 \le S \tag{8.49}$$

and at $t = t_i, i = 1, 2, \cdots,$ or I, the equation

$$\frac{\partial V}{\partial t} - D_i(S)\delta(t-t_i)\frac{\partial V}{\partial S} = 0, \quad 0 \le S$$

holds. From Sect. 2.5.2, we see that this equation gives

$$V(S,t_i^-) = V\left(S - D_i(S), t_i^+\right). \tag{8.50}$$

As we know from Sect. 2.2.5, through the transformation

$$\begin{cases} \xi = \dfrac{S}{S+P_m}, \\ \tau = T - t, \\ \overline{V}(\xi,\tau) = \dfrac{V(S,t)}{S+P_m}, \end{cases} \tag{8.51}$$

Eq. (8.49) becomes

$$\frac{\partial \overline{V}}{\partial \tau} = \frac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1-\xi)^2\frac{\partial^2 \overline{V}}{\partial \xi^2} + r\xi(1-\xi)\frac{\partial \overline{V}}{\partial \xi} - r(1-\xi)\overline{V}, \quad 0 \le \xi \le 1, \tag{8.52}$$

where $\bar{\sigma}(\xi) = \sigma\left(\dfrac{P_m\xi}{1-\xi}\right)$, the final condition in the problem (8.48) is converted into an initial condition of the form

$$\overline{V}(\xi, 0) = \frac{1-\xi}{P_m} V_T\left(\frac{P_m \xi}{1-\xi}\right), \quad 0 \le \xi \le 1 \tag{8.53}$$

and the condition (8.50) is transferred to

$$\overline{V}(\xi,\ \tau_i^+) = \left[1 - D_i\left(\frac{\xi P_m}{1-\xi}\right)\frac{1-\xi}{P_m}\right]\overline{V}\left(\frac{P_m\xi - D_i\left(\dfrac{\xi P_m}{1-\xi}\right)(1-\xi)}{P_m - D_i\left(\dfrac{\xi P_m}{1-\xi}\right)(1-\xi)},\ \tau_i^-\right). \tag{8.54}$$

Table 8.6. European and American options on stocks with discrete dividends

($r = 0.09$, $\sigma = 0.3$, $T = 0.5$ year and $E = 40$.
There are two dividend payments and each pays 0.50.)

		$t_1 = 1/12$ and $t_2 = 4/12$		$t_1 = 2/12$ and $t_2 = 5/12$	
	S	European	American	European	American
Call	38	2.64	2.64	2.66	2.69
	40	3.70	3.70	3.72	3.77
	42	4.95	4.95	4.97	5.03
Put	38	3.86	4.08	3.87	4.02
	40	2.92	3.08	2.93	3.04
	42	2.17	2.28	2.18	2.26

We solve the problem here using the following mesh. The mesh is still uniform in ξ with $\Delta\xi = 1/M$, but in the τ direction, the interval $[0, T]$ is divided into N subintervals with $\tau = \tau_n,\ n = 0, 1, \cdots, N$, where $\tau_0 = 0 < \tau_1 < \cdots < \tau_{N-1} < \tau_N = T$, and suppose t_i corresponds to τ_{n_i}, $i = 1, 2, \cdots, I$. Furthermore, define $n_0 = 0$ and $n_{I+1} = N$. Just like before, let v_m^n be an approximate value of $\overline{V}$ at $\xi = \xi_m$ and $\tau = \tau_n$ and $\{v_m^n\}$ denote v_m^n, $m = 0, 1, \cdots, M$. The problem can be solved in the following way. When $\{v_m^{n_i^+}\}$ are known at $\tau_{n_i}^+$, we can obtain $\{v_m^{n_{i+1}^-}\}$ at $\tau_{n_{i+1}}^-$ by a scheme approximating Eq. (8.52), for example, the scheme (8.47). Then we use condition (8.54) to interpolate $\{v_m^{n_{i+1}^+}\}$ from $\{v_m^{n_{i+1}^-}\}$. At $t = 0$, the option values are the same for $t = 0^-$ and $t = 0^+$. Thus, from the initial condition (8.53), we can have $\{v_m^{n_0^+}\}$. Consequently, we can do the procedure of getting $\{v_m^{n_{i+1}^+}\}$ from $\{v_m^{n_i^+}\}$ for $i = 0, 1, \cdots, I-1$ successively. As soon as we have $\{v_m^{n_I^+}\}$, we can find $\{v_m^{n_{I+1}^-}\}$, that is, $\{v_m^N\}$ by scheme (8.47). For American options, the maximum between $v_m^{n_{i+1}^+}$ and the constraint condition should be taken as the value of the American option at $\tau = \tau_{n_{i+1}}^+$, $i = 0, 1, \cdots, I-1$.

In Table 8.6, we give some values of half-year European and American options with two dividend payments. Each time, the dividend payment is 0.50 if the price of stock is greater than or equal to 0.50. If $S < 0.50$, we let

$D_i(S) = S$ in the computation. The payments are given at times 1/12 and 4/12 or 2/12 and 5/12. In order to check if the results of European options are correct, we can check if the put–call parity relation holds. For European options on stocks with discrete dividends, the put–call parity relation is in the form (3.44) in Chap. 3. For the case with $S = 40$ and the payment dates $t_1 = 2/12$ and $t_2 = 5/12$, this relation is $c(40,0) + E\mathrm{e}^{-rT} = p(40,0) + 40 - 0.5\left(\mathrm{e}^{-r\cdot 2/12} + \mathrm{e}^{-r\cdot 5/12}\right)$. From the data given in Table 8.6, we have

$$\begin{aligned} c(40,0) + E\mathrm{e}^{-rT} &= 3.72 + 40 \cdot \mathrm{e}^{-0.09\cdot 0.5} = 3.72 + 38.24 = 41.96, \\ p(40,0) + 40 - 0.5\left(\mathrm{e}^{-r\cdot 2/12} + \mathrm{e}^{-r\cdot 5/12}\right) &= 2.93 + 40 \\ -0.5\left(\mathrm{e}^{-0.09\cdot 2/12} + \mathrm{e}^{-0.09\cdot 5/12}\right) &= 42.93 - 0.97 = 41.96. \end{aligned}$$

Thus, the put–call parity relation holds. In Hull's book [43], an approximate method to get $c(S,t)$ is provided. It gives $c(40,0) = 3.67$ for this case. The numerical result here is 3.72, so it gives a very good estimate. From Table 8.6, we know that for the case $t_1 = 1/12$ and $t_2 = 4/12$, the values of European and American call options are the same. This is because $E(1 - \mathrm{e}^{-r(T-t_2)}) = 40 \cdot (1 - \mathrm{e}^{-0.09/6}) = 0.60 > 0.5$ and $E(1 - \mathrm{e}^{-r(t_2-t_1)}) = 40 \cdot (1 - \mathrm{e}^{-0.09/4}) = 0.89 > 0.5$, where 0.5 is the dividend payment. When such inequalities hold, it is impossible to have an optimal exercise price and the value of the American option must be equal to the value of the European option (see Problem 15 in Chap. 3 or the book [43] by Hull).

European Average Price Options with Discrete Sampling. Now we give some details on how to price European average price options. Suppose that sampling is done at $t = t_1, t_2, \cdots, t_K$, where $0 \le t_1 < t_2 < \cdots < t_K \le T$. Define

$$I = \frac{1}{K}\int_0^t S(\tau) f(\tau)\, d\tau,$$

where $f(\tau) = \sum\limits_{i=1}^{K} \delta(\tau - t_i)$. It is clear that at $t = T$, $I = A$. Let the price of a European average price option be $V(S,I,t)$ and let E be the exercise price. Then $V(S,I,t)$ is the solution of the problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r - D_0) S \dfrac{\partial V}{\partial S} \\ \quad + \dfrac{S}{K}\sum\limits_{i=1}^{K} \delta(t - t_i) \dfrac{\partial V}{\partial I} - rV = 0, \quad 0 \le S < \infty, \quad 0 \le I < \infty, \quad t \le T, \\ V(S,I,T) = \max(\pm(A - E), 0) \\ \quad = \max(\pm(I - E), 0), \quad 0 \le S < \infty, \quad 0 \le I < \infty, \end{cases}$$

where the "+" and "−" in $\pm$ correspond to the call and put options, respectively. Let $\eta = \dfrac{I - E}{S}$, $W = \dfrac{V}{S}$. In this case, the first three relations in the set of expressions (4.24) are still true and $\dfrac{\partial V}{\partial I} = \dfrac{\partial W}{\partial \eta}$. Also, from $V(S, I, T) = \max(\pm(I - E), 0)$, we have

$$W(\eta, T) = \max(\pm\eta, 0).$$

Therefore, $W(\eta, t)$ satisfies

$$\begin{cases} \dfrac{\partial W}{\partial t} + \dfrac{1}{2}\sigma^2\eta^2\dfrac{\partial^2 W}{\partial \eta^2} + \left[(D_0 - r)\eta + \dfrac{1}{K}\sum_{i=1}^{K}\delta(t - t_i)\right]\dfrac{\partial W}{\partial \eta} \\ -D_0 W = 0, \qquad -\infty < \eta < \infty, \quad t \le T, \\ W(\eta, T) = \max(\pm\eta, 0), \qquad -\infty < \eta < \infty. \end{cases} \tag{8.55}$$

Suppose $t_1 = 0$ and let $t_{K+1} = T > t_K$; then the problem can be solved as follows. Starting with $f_{K+1,w} = \max(\pm\eta, 0)$, for $i = K + 1, K, \cdots, 2$, successively, solve the following problem:

$$\begin{cases} \dfrac{\partial W}{\partial t} + \frac{1}{2}\sigma^2\eta^2\dfrac{\partial^2 W}{\partial \eta^2} + (D_0 - r)\eta\dfrac{\partial W}{\partial \eta} - D_0 W = 0, \\ \qquad -\infty < \eta < \infty, \quad t_{i-1}^+ < t < t_i^-, \\ W(\eta, t_i^-) = f_{i,w}(\eta), \qquad -\infty < \eta < \infty \end{cases} \tag{8.56}$$

and obtain $W(\eta, t_{i-1}^-)$ from $W(\eta, t_{i-1}^+)$ by the jump condition

$$W\left(\eta, t_i^-\right) = W\left(\eta + \frac{1}{K}, t_i^+\right). \tag{8.57}$$

We want to solve this problem as an initial-value problem on a finite domain. Thus, we introduce the following transformation:

$$\begin{cases} \xi = \dfrac{\eta}{|\eta| + P_m}, \\ \tau = T - t, \\ W(\eta, t) = (|\eta| + P_m)\overline{u}(\xi, \tau), \end{cases} \tag{8.58}$$

where $P_m > 0$. From the expression (8.58), we have

$$\operatorname{sign}(\xi) = \operatorname{sign}(\eta),\ |\xi| \le 1,\ |\eta| = \frac{P_m|\xi|}{1 - |\xi|},\ \eta = \frac{P_m\xi}{1 - |\xi|},\ |\eta| + P_m = \frac{P_m}{1 - |\xi|},$$

and

$$\frac{d\xi}{d\eta} = \frac{|\eta| + P_m - \eta \cdot \operatorname{sign}(\eta)}{(|\eta| + P_m)^2} = \frac{P_m}{(|\eta| + P_m)^2} = \frac{(1 - |\xi|)^2}{P_m}.$$

Because

$$\begin{aligned}
\frac{\partial W}{\partial t} &= -(|\eta| + P_m)\frac{\partial \overline{u}}{\partial \tau} = -\frac{P_m}{1-|\xi|}\frac{\partial \overline{u}}{\partial \tau},\\
\frac{\partial W}{\partial \eta} &= \frac{\partial}{\partial \eta}\left[(|\eta| + P_m)\overline{u}\right] = \operatorname{sign}(\eta)\overline{u} + (\eta + |P_m|)\frac{\partial \overline{u}}{\partial \xi}\frac{d\xi}{d\eta}\\
&= \operatorname{sign}(\xi)\overline{u} + (1-|\xi|)\frac{\partial \overline{u}}{\partial \xi},\\
\frac{\partial^2 W}{\partial \eta^2} &= \frac{\partial}{\partial \xi}\left[(1-|\xi|)\frac{\partial \overline{u}}{\partial \xi} + \operatorname{sign}(\xi)\overline{u}\right]\frac{d\xi}{d\eta} = \frac{(1-|\xi|)^3}{P_m}\frac{\partial^2 \overline{u}}{\partial \xi^2},
\end{aligned}$$

from the PDE for W we have

$$\begin{aligned}
\frac{P_m}{1-|\xi|}\frac{\partial \overline{u}}{\partial \tau} &= \frac{\sigma^2 P_m \xi^2 (1-|\xi|)}{2}\frac{\partial^2 \overline{u}}{\partial \xi^2}\\
&\quad + \left[(D_0 - r)\frac{P_m \xi}{1-|\xi|}\right]\left[\operatorname{sign}(\xi)\overline{u} + (1-|\xi|)\frac{\partial \overline{u}}{\partial \xi}\right] - D_0 \frac{P_m}{1-|\xi|}\overline{u}
\end{aligned}$$

or

$$\begin{aligned}
\frac{\partial \overline{u}}{\partial \tau} &= \frac{\sigma^2 \xi^2 (1-|\xi|)^2}{2}\frac{\partial^2 \overline{u}}{\partial \xi^2} + (D_0 - r)\xi(1-|\xi|)\frac{\partial \overline{u}}{\partial \xi}\\
&\quad + \left[(D_0 - r)|\xi| - D_0\right]\overline{u}, \quad -1 < \xi < 1, \quad 0 \le \tau.
\end{aligned}$$

Thus, under this transformation, the problem (8.56) becomes

$$\begin{cases}
\dfrac{\partial \overline{u}}{\partial \tau} = \dfrac{1}{2}\sigma^2 \xi^2 (1-|\xi|)^2 \dfrac{\partial^2 \overline{u}}{\partial \xi^2} + (D_0 - r)\xi(1-|\xi|)\dfrac{\partial \overline{u}}{\partial \xi}\\
-\left[r|\xi| + D_0(1-|\xi|)\right]\overline{u}, \qquad -1 \le \xi \le 1, \quad \tau_i^+ < \tau < \tau_{i-1}^-,\\
\overline{u}(\xi, \tau_i^+) = \dfrac{1-|\xi|}{P_m} f_{i,w}\left(\dfrac{P_m \xi}{1-|\xi|}\right), -1 \le \xi \le 1.
\end{cases} \tag{8.59}$$

Here we have used the following relation:

$$\overline{u}(\xi, \tau_i^+) = \frac{W\left(\eta, t_i^-\right)}{|\eta| + P_m} = \frac{f_{i,w}(\eta)}{|\eta| + P_m} = \frac{1-|\xi|}{P_m} f_{i,w}\left(\frac{P_m \xi}{1-|\xi|}\right).$$

At $\xi = 0$, the PDE in the problem (8.59) degenerates into

$$\frac{\partial \overline{u}}{\partial \tau} = -D_0 \overline{u}.$$

Thus, the solution at $\xi = 0$ can be determined alone. Therefore, the problem (8.59) can be divided into two problems:

$$\begin{cases} \dfrac{\partial \overline{u}}{\partial \tau} = \dfrac{1}{2}\sigma^2\xi^2(1-\xi)^2\dfrac{\partial^2 \overline{u}}{\partial \xi^2} + (D_0 - r)\xi(1-\xi)\dfrac{\partial \overline{u}}{\partial \xi} \\ -\left[r\xi + D_0(1-\xi)\right]\overline{u}, \qquad 0 \le \xi \le 1, \quad \tau_i^+ < \tau < \tau_{i-1}^-, \\ \overline{u}(\xi, \tau_i^+) = \dfrac{1-\xi}{P_m} f_{i,w}\left(\dfrac{P_m\xi}{1-\xi}\right), 0 \le \xi \le 1 \end{cases} \tag{8.60}$$

and

$$\begin{cases} \dfrac{\partial \overline{u}}{\partial \tau} = \dfrac{1}{2}\sigma^2\xi^2(1-|\xi|)^2\dfrac{\partial^2 \overline{u}}{\partial \xi^2} + (D_0 - r)\xi(1-|\xi|)\dfrac{\partial \overline{u}}{\partial \xi} \\ -\left[r|\xi| + D_0(1-|\xi|)\right]\overline{u}, \qquad -1 \le \xi \le 0, \quad \tau_i^+ < \tau < \tau_{i-1}^-, \\ \overline{u}(\xi, \tau_i^+) = \dfrac{1-|\xi|}{P_m} f_{i,w}\left(\dfrac{P_m\xi}{1-|\xi|}\right), -1 \le \xi \le 0. \end{cases} \tag{8.61}$$

Letting $\xi_1 = -\xi$ and $\overline{u}_1(\xi_1, \tau) = \overline{u}(\xi, \tau)$, we have $|\xi| = \xi_1$ for any $\xi \in [-1, 0]$ and $\xi\dfrac{\partial \overline{u}}{\partial \xi} = \xi_1\dfrac{\partial \overline{u}_1}{\partial \xi_1}$. Thus, the problem (8.61) can be rewritten as

$$\begin{cases} \dfrac{\partial \overline{u}_1}{\partial \tau} = \dfrac{1}{2}\sigma^2\xi_1^2(1-\xi_1)^2\dfrac{\partial^2 \overline{u}_1}{\partial \xi_1^2} + (D_0 - r)\xi_1(1-\xi_1)\dfrac{\partial \overline{u}_1}{\partial \xi_1} \\ -\left[r\xi_1 + D_0(1-\xi_1)\right]\overline{u}_1, \qquad 0 \le \xi_1 \le 1, \quad \tau_i^+ < \tau < \tau_{i-1}^-, \\ \overline{u}_1(\xi_1, \tau_i^+) = \dfrac{1-\xi_1}{P_m} f_{i,w}\left(\dfrac{-P_m\xi_1}{1-\xi_1}\right), \ \ 0 \le \xi_1 \le 1. \end{cases} \tag{8.62}$$

The formulation of the two problems are the same as the problem (8.1). Thus, using the scheme (8.47), we can obtain $\overline{u}(\xi, \tau_{i-1}^-)$ from $\overline{u}(\xi, \tau_i^+)$ for $-1 \le \xi \le 1$. In order to have $\overline{u}(\xi, \tau_{i-1}^+)$ from $\overline{u}(\xi, \tau_{i-1}^-)$ for $-1 \le \xi \le 1$, we need to use the jump condition:

$$\overline{u}\left(\xi,\ \tau_i^+\right) = \frac{\left|\frac{P_m\xi}{1-|\xi|} + \frac{1}{K}\right| + P_m}{\left|\frac{P_m\xi}{1-|\xi|}\right| + P_m}\overline{u}\left(\frac{\frac{P_m\xi}{1-|\xi|} + \frac{1}{K}}{\left|\frac{P_m\xi}{1-|\xi|} + \frac{1}{K}\right| + P_m}, \quad \tau_i^-\right), \tag{8.63}$$

which is another version of the jump condition (8.57) if the function $\overline{u}(\xi, \tau)$ is used instead of the function $W(\eta, t)$. It is not difficult to rewrite the jump condition (8.57) into the jump condition (8.63), which is left as a portion of Problem 9. As soon as we have $\overline{u}(\xi, T^+)$ when $\tau_1 = T$, that is, $t_1 = 0$, we can find

$$V(S,0,0) = SW\left(-E/S, 0\right) = S\left(E/S + P_m\right)\overline{u}\left(\frac{-E/S}{E/S + P_m}, T\right).$$

Because $\dfrac{d\xi}{d\eta} = \dfrac{(1-|\xi|)^2}{P_m} > 0$, when η varies from $-\infty$ to ∞, ξ varies from -1 to 1 monotonically. Thus, $\xi(\eta) < \xi(\eta + 1/K)$ for any η; that is,

$$\xi(\eta)=\frac{\eta}{|\eta|+P_m}<\xi\left(\eta+\frac{1}{K}\right)=\frac{\eta+\dfrac{1}{K}}{|\eta+\dfrac{1}{K}|+P_m}=\frac{\frac{P_m\xi}{1-|\xi|}+\frac{1}{K}}{\left|\frac{P_m\xi}{1-|\xi|}+\frac{1}{K}\right|+P_m}.$$

Consequently, when ξ varies from 0 to 1, $\frac{\frac{P_m\xi}{1-|\xi|}+\frac{1}{K}}{\left|\frac{P_m\xi}{1-|\xi|}+\frac{1}{K}\right|+P_m}$ varies from $\dfrac{1/K}{1/K+P_m}$ to 1, and when ξ varies from -1 to 0, $\frac{\frac{P_m\xi}{1-|\xi|}+\frac{1}{K}}{\left|\frac{P_m\xi}{1-|\xi|}+\frac{1}{K}\right|+P_m}$ varies from -1 to $\dfrac{1/K}{1/K+P_m}$. For an average price call option, we need to solve problems (8.60) and (8.62) from τ_i^+ to τ_{i-1}^- and then use condition (8.63) for $\xi\in[-1,1]$, $i=K+1,K,\cdots,2$, successively.[2] For an average rate put option, $\overline{u}(\xi,0)=0$ for $\xi\in[0,1]$, and so the solution of the problem (8.60) with the jump condition

Table 8.7. Prices of average price put options with discrete sampling

($T=1$, $S=100$, $r=0.05$, $D_0=0$, $\sigma=0.2$)

E	Monthly	Weekly	Daily
90.0000	0.7861	0.6929	0.6694
92.5000	1.2239	1.1092	1.0800
95.0000	1.8162	1.6840	1.6501
97.5000	2.5823	2.4392	2.4023
100.0000	3.5345	3.3888	3.3512
102.5000	4.6771	4.5378	4.5020
105.0000	6.0068	5.8823	5.8506
107.5000	7.5132	7.4107	7.3850
110.0000	9.1810	9.1055	9.0871

Table 8.8. Prices of average price call options with discrete sampling

($T=1$, $S=100$, $r=0$, $D_0=0$, $\sigma=0.2$)

E	Monthly	Weekly	Daily	Continuously
90.0000	11.2304	11.0853	11.0487	11.0426
92.5000	9.3506	9.1760	9.1315	9.1240
95.0000	7.6595	7.4610	7.4102	7.4016
97.5000	6.1708	5.9566	5.9015	5.8922
100.0000	4.8888	4.6685	4.6118	4.6022
102.5000	3.8091	3.5922	3.5365	3.5271
105.0000	2.9194	2.7143	2.6618	2.6529
107.5000	2.2019	2.0148	1.9672	1.9592
110.0000	1.6350	1.4701	1.4284	1.4215

[2] In this case, the problem (8.60) with the jump condition (8.63) can be solved independently and have an analytic solution (see Andreasen [3], Zhu [90], or Problem 32 in Chap. 2).

Table 8.9. Comparison between two sampling-daily-average price call options

($T = 1$, $S = 100$, $r = 0$, $D_0 = 0$, $\sigma = 0.2$)

	Money spent in the case with $E = 100$		Money spent in the case with $E = 90$
$A \geq 100$	104.61	$>$	101.05
$100 > A > 96.44$	$4.61 + A$	$>$	101.05
$A = 96.44$	101.05	$=$	101.05
$96.44 > A \geq 90$	$4.61 + A$	$<$	101.05
$90 > A$	$4.61 + A$	$<$	$11.05 + A$

(8.63) is zero. Thus, in order to obtain $\overline{u}\left(\dfrac{-E/S}{E/S + P_m}, T\right)$, we only need to solve the problem (8.62) and to use the jump condition (8.63) alternatively.

In Sect. 4.3.7, we have given some results on European average price options with discrete sampling. Here we give more results for the European average rate call and put options obtained by the method described here. In Table 8.7 for the cases with sampling monthly, weekly, or daily, for $S = 100$, the values of the average price put options with $T = 1, r = 0.05, D_0 = 0, \sigma = 0.2$ are listed. In Table 8.8 for the cases with sampling monthly, weekly, or daily, for $S = 100$, the values of the average price call options with $T = 1, r = 0, D_0 = 0, \sigma = 0.2$ are given. Here we assume that there are 12 months, 52 weeks, 360 days per year, which are not real. The error of the results given in the table should be around 0.0001 because when a finer mesh is used, the difference between the new value and the value given here is less than 0.0001. In Table 8.8, the results of options with continuous sampling are also given. From that table, we can see that the difference between the option price with sampling daily and the option price with sampling continuously is about 0.01.

Suppose that a company will buy a certain amount of some raw material every day during the next year. Let A be the average price of the raw material the company paid during this period. Usually, the company does not want A to be much higher than the price today S. It is clear that the company cannot control the price on the market. However, if the company purchases certain units of sampling-daily-average price call options on such a raw material, then the company will get some money from exercising these call options when A is higher than E, so it will be guaranteed that the money spent on this raw material will be less than a certain level. From Table 8.8, we can see that when today's price of the raw material is \$100, the company needs to pay \$4.61 in order to buy a sampling-daily-average price call option with $E = 100$. Thus, the money spent on each unit of the raw material is \$ $4.61 + 100 =$ \$104.61 if $A \geq 100$ or \$ $4.61 + A$ if $A < 100$, which means that the money spent on each unit of the raw material is not greater than \$104.61. When an option with $E = 90$ is purchased, the money spent on each unit of the raw material is not greater than \$$11.05 + 90 =$\$101.05 because the premium for the call option for this case is \$11.05 (see Table 8.8). Which choice is better? This is determined by what you want. When the option with $E = 90$ is purchased, the maximum money spent is lower than that for the case with $E = 100$, but the money spent for lower A is higher than that for the case with $E = 100$. Table 8.9 shows you this fact.

Table 8.10. Double average call option prices on four meshes ($D_0 < r$)

($T = 1$, $S = 100$, $r = 0.05$, $D_0 = 0$, $\sigma = 0.2$, $T_{1s} = 0.1$, $T_{1e} = 0.5$, $K_1 = 5$, $T_s = 0.6$, $T_e = 1.0$, $K = 5$, $P_m = 0.4$, the payoff $= \max\left(\frac{I}{K} - \frac{I_1}{K_1}, 0\right)$, and the exact solution $= 5.872133\cdots$)

Mesh sizes	Results	|Errors|	CPU times	Results without extrapolation	|Errors|	CPU times
200×20	5.870320	0.001813	0.0042	5.869883	0.002250	0.0020
400×40	5.871861	0.000272	0.0094	5.871367	0.000766	0.0077
800×80	5.872133	0.000000	0.0282	5.871942	0.000191	0.0203
$1,600 \times 160$	5.872126	0.000007	0.0928	5.872080	0.000053	0.0745

Table 8.11. Double average call option prices on four meshes ($D_0 > r$)

($T = 1$, $S = 100$, $r = 0.05$, $D_0 = 0.1$, $\sigma = 0.2$, $T_{1s} = 0.1$, $T_{1e} = 0.5$, $K_1 = 5$, $T_s = 0.6$, $T_e = 1.0$, $K = 5$, $P_m = 0.2$, the payoff $= \max\left(\frac{I}{K} - \frac{I_1}{K_1}, 0\right)$, and the exact solution $= 3.244201\cdots$)

Mesh sizes	Results	|Errors|	CPU times	Results without extrapolation	|Errors|	CPU times
200×20	3.241122	0.003079	0.0052	3.235091	0.009110	0.0030
400×40	3.244162	0.000039	0.0116	3.241894	0.002307	0.0084
800×80	3.244263	0.000062	0.0321	3.243671	0.000530	0.0217
$1,600 \times 160$	3.244196	0.000005	0.1009	3.244064	0.000137	0.0813

Some Results of Double Average Call Options. For European-style other Asian and lookback options with discrete sampling, the method is similar. That is, the problem is solved by numerical schemes for partial differential equations and interpolation alternately. For details of the methods, see the papers by Andreasen [3] and Zhu [90]. Some results for such options are given in Sects. 4.3.7 and 4.4.7. Here we give some results for two double average call options, to show the effect of the extrapolation technique and how the approximate solutions converge to exact solutions in Tables 8.10 and 8.11. In Table 8.10 $D_0 = 0.1 > r = 0.05$, and in Table 8.11 $D_0 = 0 < r = 0.05$. There are 10 samplings at $t = 0.1, 0.2, \cdots, 1.0$. From these tables, we can see that the extrapolation technique greatly improves the rate of convergence and the accuracy, with about 25 % extra CPU time.

8.2.3 Projected Direct Methods for the LC Problem

As seen in Sect. 8.2.1, using implicit finite-difference methods for European options is straightforward. From Sect. 8.1.2, if an American option is formulated as a linear complementarity problem, then there is not a big difference between explicit finite-difference methods for European and American options. The implicit methods for American options are also only a little more complicated than the methods for European options.

Suppose we use a direct method to solve the system related to an American call option, which is formulated as the problem (8.8). Assuming that the partial differential equation holds everywhere and using scheme (8.47), we have a system in the form:

$$a_m \bar{v}_{m-1}^{n+1} + b_m \bar{v}_m^{n+1} + c_m \bar{v}_{m+1}^{n+1} = q_m^n, \quad m = 0, 1, 2, \cdots, M. \tag{8.64}$$

Actually, $\bar{v}_{-1}^{n+1}$ and $\bar{v}_{M+1}^{n+1}$ do not appear in the system because

$$a_0 = c_M = 0.$$

It is clear that the solution of the system (8.64) may not be the solution of the American option. However, we can find the solution of the American option with the aid of the system (8.64).

Similar to what we did in Sect. 6.2.1, if we let

$$u_0 = b_0, \quad y_0 = q_0^n, \tag{8.65}$$

and

$$u_m = b_m - \frac{c_{m-1} a_m}{u_{m-1}}, \quad y_m = q_m^n - \frac{y_{m-1} a_m}{u_{m-1}}, \quad m = 1, 2, \cdots, M, \tag{8.66}$$

then the equations in system (8.64) can be rewritten as

$$\bar{v}_m^{n+1} = \frac{y_m - c_m \bar{v}_{m+1}^{n+1}}{u_m}, \quad m = M, M-1, \cdots, 0, \tag{8.67}$$

where the relation with $m = M$ actually is

$$\bar{v}_M^{n+1} = \frac{y_M}{u_M}$$

because $c_M = 0$. From the derivation, we know that the relations in the system (8.67) with $m = 0, 1, \cdots, M_f$ are equivalent to the equations in the system (8.64) with $m = 0, 1, \cdots, M_f$, where M_f is any positive integer less than or equal to M. Obviously, $\bar{v}_m^{n+1}$ may not be greater than or equal to $\max(2\xi_m - 1, 0)$. Therefore, we need to find the value of the American option by

$$v_m^{n+1} = \max\left(\bar{v}_m^{n+1}, 2\xi_m - 1, 0\right), \quad m = 0, 1, \cdots, M \tag{8.68}$$

Table 8.12. American call option (PIFDI)

($r = 0.05$, $\sigma = 0.2$, $D_0 = 0.1$, $S = E = 100$, $T = 1$, and the exact solution is $C = 5.92827717\cdots$)

Meshes	Results by Eq. (8.68)	|Errors|	Results by Eq. (8.69)	|Errors|
50×50	5.752424	0.175853	5.760096	0.168181
100×100	5.878708	0.049569	5.884210	0.044067
200×200	5.914582	0.013695	5.917403	0.010874
400×400	5.924045	0.004132	5.925541	0.002736
800×800	5.926810	0.001467	5.927574	0.000703
$1,600 \times 1,600$	5.927706	0.000571	5.928097	0.000180
$3,200 \times 3,200$	5.928032	0.000245	5.928230	0.000047

or by

$$v_m^{n+1} = \max\left(\frac{y_m - c_m v_{m+1}^{n+1}}{u_m}, 2\xi_m - 1, 0\right), \quad m = M, M-1, \cdots, 0, \quad (8.69)$$

successively. This method is referred to as the projected implicit finite-difference method I (PIFDI).

Is there any difference between the formulae (8.68) and (8.69)? The answer is yes. Let us explain this. As we know from Sect. 3.3.1, there is only one free boundary for a call option. It is natural to expect that when the formula (8.68) is used, there exists an M_f so that $v_m^{n+1} = \bar{v}_m^{n+1}$ for $m = 0, 1, \cdots, M_f$ and $v_m^{n+1} = \max(2\xi_m - 1, 0)$ for $m = M_f + 1, M_f + 2, \cdots, M$. When $\bar{v}_m^{n+1}$ are determined, we assume all the equations in the system (8.64) to hold. Even though for $m = M_f+1, M_f+2, \cdots, M$ we do not take $\bar{v}_m^{n+1}$ as solutions so that the constraint condition is satisfied, v_m^{n+1}, $m = 0, 1, \cdots, M_f$ are determined under the assumption of all the equations in the system (8.64) holding. For the formula (8.69), the situation is different. We assume that for $m = M, M-1, \cdots, M_f + 1$, $v_m^{n+1} = \max(2\xi_m - 1, 0)$ and for $m = M_f, M_f - 1, \cdots, 0$,

$$v_m^{n+1} = \frac{y_m - c_m v_{m+1}^{n+1}}{u_m}.$$

In this case, we only use the relations in the expression (8.67) with $m = M_f, M_f - 1, \cdots, 0$, which are equivalent to the equations in the system (8.64) with $m = M_f, M_f - 1, \cdots, 0$. Therefore, we only assume that the equations in the system (8.64) hold for $m = M_f, M_f - 1, \cdots, 0$. Consequently, this is closer to what the situation should be. In Table 8.12, results obtained by the formulae (8.68) and (8.69) and their errors are listed. You can see that on the same mesh, the error of the results obtained by the formula (8.68) is greater than the formula (8.69) and that the smaller the mesh size, the greater the difference. Even though the formula (8.68) can be used to obtain the price of American options, it brings some error that can be avoided if the formula (8.69) is used. However, if the free boundary is far away from $S = E$, then in the region $S \approx E$, the difference of the solutions obtained by the two direct methods is very small.

When an implicit scheme is used to solve problem (8.9), we need to choose the lower and upper bounds of the computational domain and give some artificial boundary conditions at these two boundaries because we cannot do computation on an infinite domain. Let the lower and upper bounds be x_l and x_u. For a call option, we assume $u(x_l, \bar{\tau}) = 0$ and $u(x_u, \bar{\tau}) = g(x_u, \bar{\tau})$, and for a put option, $u(x_l, \bar{\tau}) = g(x_l, \bar{\tau})$ and $u(x_u, \bar{\tau}) = 0$. As soon as we set these conditions, the problem (8.9) can be discretized and solved in the same way as described above for the problem (8.8). This method is referred to as the projected implicit finite-difference method II (PIFDII).

In Tables 8.13 and 8.14, the values of American call and put options obtained by PIFDII are given. When we do computation, we take

$$x_l = \ln(S_l/E) - |(r - D_0 - \sigma^2/2)T|$$

and

$$x_u = \ln(S_u/E) + |(r - D_0 - \sigma^2/2)T|.$$

For the call option, $S_l = 20$ and $S_u = 230$, and for the put, $S_l = 80$ and $S_u = 350$. There, we also give a solution with an error less than 10^{-8} in each table, which is obtained by the SSM given in Chap. 9. Therefore, we can have the errors of the solutions on different meshes. The CPU time used is also given, so you can have a notion about the performance of the method.

Table 8.13. American call option (PIFDII)

($r = 0.1$, $\sigma = 0.2$, $D_0 = 0.05$, $S = E = 100$, $T = 1$, and the exact solution is $C = 9.94092345\cdots$)

Meshes	Results	|Errors|	CPU(s)
100×25	9.928528	0.012396	0.0025
200×50	9.937831	0.003093	0.0096
400×100	9.940151	0.000773	0.0400
800×200	9.940729	0.000194	0.1700
$1,600 \times 400$	9.940875	0.000048	0.6700

Table 8.14. American put option (PIFDII)

($r = 0.1$, $\sigma = 0.2$, $D_0 = 0.05$, $S = E = 100$, $T = 1$, and the exact solution is $P = 5.92827717\cdots$)

Meshes	Results	|Errors|	CPU(s)
100×25	5.922275	0.006002	0.0025
200×50	5.926394	0.001883	0.0094
400×100	5.927654	0.000623	0.0400
800×200	5.928050	0.000227	0.1700
$1,600 \times 400$	5.928188	0.000089	0.6700

8.2.4 Projected Iteration Methods for the LC Problem

As we know, there are two types of methods to solve a linear system: iteration methods and direct methods. Similarly, there are two ways to solve the system related to American options. We discussed direct methods in the last subsection. Now let us study an iteration method. We still consider call options and use the system (8.64). This problem can be solved by a method similar to the SOR method for a system of linear equations given in Sect. 6.2.2. Any equation in the system (8.64) can be rewritten as

$$\bar{v}_m^{n+1} = (1-\omega)\bar{v}_m^{n+1} + \frac{\omega}{b_m}\left(q_m^n - a_m \bar{v}_{m-1}^{n+1} - c_m \bar{v}_{m+1}^{n+1}\right),$$

where ω is a constant. The value of the American option v_m^{n+1} satisfies the relation above if $\bar{v}_m^{n+1} > \max(2\xi_m - 1, 0)$ or equal to $\max(2\xi_m - 1, 0)$ otherwise. Therefore, for v_m^{n+1} we have the following relations:

$$v_m^{n+1} = \max\left((1-\omega)v_m^{n+1} + \frac{\omega}{b_m}\left(q_m^n - a_m v_{m-1}^{n+1} - c_m v_{m+1}^{n+1}\right), 2\xi_m - 1, 0\right),$$
$$m = 0, 1, \cdots, M.$$

We use an iteration method for finding its solution. Let $v_m^{(k)}$ be the k-th iteration of v_m^{n+1}, and the relation above can be rewritten in the following iteration form:

$$v_m^{(k+1)} = \max\left((1-\omega)v_m^{(k)} + \frac{\omega}{b_m}\left(q_m^n - a_m v_{m-1}^{(k+1)} - c_m v_{m+1}^{(k)}\right), 2\xi_m - 1, 0\right), \tag{8.70}$$

where $\omega \in (0, 2)$. Let $v_m^{(0)} = v_m^n$ for $m = 0, 1, \cdots, M$. As soon as we have $v_m^{(k)}$ for all m, the $(k+1)$-th iterative value of v_m^{n+1} can be obtained by equality (8.70) for $m = 0, 1, \cdots, M$ successively, starting from $k = 0$. When

$$\frac{1}{M+1}\sum_{m=0}^{M}\left(v_m^{(k)} - v_m^{(k+1)}\right)^2 \le \epsilon^2,$$

where ϵ^2 is a small number given according to the required accuracy, we can stop the iteration because for any m, $v_m^{(k)}$ and $v_m^{(k+1)}$ are very close to each other. This method is referred to as the projected successive over relaxation method I (PSORI). If the formulation (8.9) is adopted, after setting the values of x_l, x_u and the artificial boundary conditions, we can have a similar method and the corresponding method is referred to as PSORII. The details of the PSORII are left for readers to write as Problem 14.

In Tables 8.15 and 8.16, the prices of American call and put options on several meshes obtained by PSORII are given. The corresponding errors, CPU times, and ϵ^2 are also listed. All the parameters are the same as those given in

Table 8.15. American call option (PSORII)

($r = 0.1$, $\sigma = 0.2$, $D_0 = 0.05$, $S = E = 100$,
and the exact solution is $C = 9.94092345\cdots$)

Meshes	Results	|Errors|	CPU(s)	ϵ^2
100×25	9.929351	0.011573	0.0240	10^{-8}
200×50	9.938037	0.002887	0.1100	$0.5 \cdot 10^{-9}$
400×100	9.940202	0.000721	0.5300	$0.25 \cdot 10^{-10}$
800×200	9.940743	0.000181	2.7500	$0.125 \cdot 10^{-11}$
$1,600 \times 400$	9.940878	0.000046	20.000	$0.6125 \cdot 10^{-13}$

Table 8.16. American put option (PSORII)

($r = 0.1$, $\sigma = 0.2$, $D_0 = 0.05$, $S = E = 100$,
and the exact solution is $P = 5.92827717\cdots$)

Meshes	Results	|Errors|	CPU(s)	ϵ^2
100×25	5.922349	0.005928	0.0180	10^{-8}
200×50	5.926410	0.001867	0.0960	$0.5 \cdot 10^{-9}$
400×100	5.927651	0.000626	0.6100	$0.25 \cdot 10^{-10}$
800×200	5.928048	0.000230	5.2200	$0.125 \cdot 10^{-11}$
$1,600 \times 400$	5.928188	0.000089	46.300	$0.6125 \cdot 10^{-13}$

Tables 8.13 and 8.14. The only difference between the results here and there is the way we solved the system.

Comparing Tables 8.13 and 8.14 with Tables 8.15 and 8.16 shows that the CPU time here is longer. This implies that the cost of PSORII method is greater than the PIFDII method for this case. However, we need to point out that for most of multi-dimensional problems, the iteration methods may be better than the direct methods even though here we show that the direct method is better than the iteration method for one-dimensional problems.

8.2.5 Comparison with Explicit Methods

Explicit methods are usually very simple and very easy to use. The main problem of explicit methods is the stability requirement. For the explicit method (8.6), the stability requirement is

$$\bar{\alpha} \leq \frac{1}{2} \quad \text{or} \quad \Delta\bar{\tau} \leq \frac{1}{2}\Delta x^2.$$

Thus, if the accuracy of the solution requires a small Δx, then a much smaller $\Delta\bar{\tau}$ must be taken in order to satisfy the stability condition, which slows down the computation. For implicit methods, no such restrictions are needed, and we can let $\Delta\bar{\tau}/\Delta x = constant$. Therefore, if we require higher accuracy, an implicit scheme will give a better performance. This can be seen in the following way.

Suppose we solve the problem (8.5) by the explicit scheme (8.6) and the implicit scheme (7.9). Assume that for the scheme (8.6) $\Delta\bar{\tau} = \alpha\Delta x^2$, where

α is a constant not greater than 1/2 and that for the scheme (7.9), $\Delta\bar{\tau} = \beta\Delta x$, where β is a constant. For the explicit scheme (8.6), the amount of computational work is

$$W_e = \frac{a_e}{\Delta\bar{\tau}\Delta x} = \frac{a_e}{\alpha\Delta x^3},$$

and the error is

$$E = b_{e\bar{\tau}}\Delta\bar{\tau} + b_{ex}\Delta x^2 = (b_{e\bar{\tau}}\alpha + b_{ex})\Delta x^2,$$

where a_e, $b_{e\bar{\tau}}$, and b_{ex} are three parameters related to scheme (8.6) and the solution. From these two relations for the scheme (8.6), we have the relation between the amount of work and the error required:

$$W_e = \frac{a_e[b_{e\bar{\tau}}\alpha + b_{ex}]^{3/2}}{\alpha}E^{-3/2}.$$

For the scheme (7.9),

$$W_i = \frac{a_i}{\Delta\bar{\tau}\Delta x} = \frac{a_i}{\beta\Delta x^2}$$

and

$$E = b_{i\bar{\tau}}\Delta\bar{\tau}^2 + b_{ix}\Delta x^2 = (b_{i\bar{\tau}}\beta^2 + b_{ix})\Delta x^2,$$

where a_i, $b_{i\bar{\tau}}$, and b_{ix} are three parameters related to scheme (7.9) and the solution. Here, we assume that a direct method is used for solving the linear system. Therefore, the relation between the amount of work and the error required is

$$W_i = \frac{a_i(b_{i\bar{\tau}}\beta^2 + b_{ix})}{\beta}E^{-1}.$$

Usually, a_i is greater than a_e because for the scheme (7.9) a linear system needs to be solved at each time step. Consequently, when E is not too small, it is possible that W_i is greater than W_e for the same E, which means that the scheme (8.6) is better than the scheme (7.9). When the solution is much smoother in the $\bar{\tau}$-direction than in the x-direction, the scheme (7.9) might be better than the scheme (8.6) even if E is not very small. This is because in this case for the scheme (7.9) we can choose a big β such that $b_{i\bar{\tau}}\beta^2$ is close to b_{ix}, which makes W_i smaller, but for the scheme (8.6) we cannot take this advantage because of the stability requirement. However, when E is small enough, then W_i must be less than W_e. This can be seen from comparing Tables 8.2 and 8.3 with Tables 8.13 and 8.14. The tables show that for the American call problem with the parameters given there, in order to reach an error about 0.003, the CPU time for the scheme (8.6) is about 0.06 and the CPU time for the scheme (7.9) is about 0.01.

8.2.6 Two-Asset Options

Sometimes two assets are involved in an option problem. In this case, usually a two-dimensional problem needs to be solved. As shown in Sect. 4.5.4, pricing a two-asset option can be reduced to solving Eq. (4.79) with final condition (4.80). This problem is a final-value problem. In order to use the scheme (7.46), we need to introduce a new variable $\tau = T - t$ and modify Eq. (4.79) into an equation with independent variables ξ, θ and τ. Let us call the new equation the modified Eq. (4.79). The modified Eq. (4.79) can be discretized by scheme (7.46). For a two-asset call option, the final condition is

$$V(S_1, S_2, T) = \max(E_1 - S_1, E_2 - S_2, 0),$$

and for a two-asset put option, the final condition is

$$V(S_1, S_2, T) = \max(S_1 - E_1, S_2 - E_2, 0).$$

Under the coordinate system (ξ, θ, t) introduced in Sect. 4.5.4, letting $\tau = T - t$, and instead of V, using $w = \dfrac{V}{S + P_m}$ as a dependent variable, S being $\dfrac{\xi P_m}{1-\xi}$, these two conditions become

$$w(\xi, \theta, 0) = \frac{1}{\sqrt{\left(\dfrac{S_1}{P_1}\right)^2 + \left(\dfrac{S_2}{P_2}\right)^2} + P_m} \max\left(E_1 - S_1, E_2 - S_2, 0\right) \qquad (8.71)$$

Table 8.17. Prices of a European two-asset call option

($r = 0.02, D_{01} = 0.01, D_{02} = 0.01, \sigma_1 = 0.2,$
$\sigma_2 = 0.15, \rho = 0.8, E_1 = 100, E_2 = 95$, and $T = 1$)

S_1	S_2	Price
95.0	90.0	6.76
97.5	92.5	8.22
100.0	95.0	9.84
102.5	97.5	11.61
105.0	100.0	13.52

Table 8.18. Prices of a European two-asset put option

($r = 0.02, D_{01} = 0.01, D_{02} = 0.01, \sigma_1 = 0.2,$
$\sigma_2 = 0.15, \rho = 0.8, E_1 = 100, E_2 = 95$, and $T = 1$)

S_1	S_2	Price
95.0	90.0	11.29
97.5	92.5	9.78
100.0	95.0	8.41
102.5	97.5	7.19
105.0	100.0	6.11

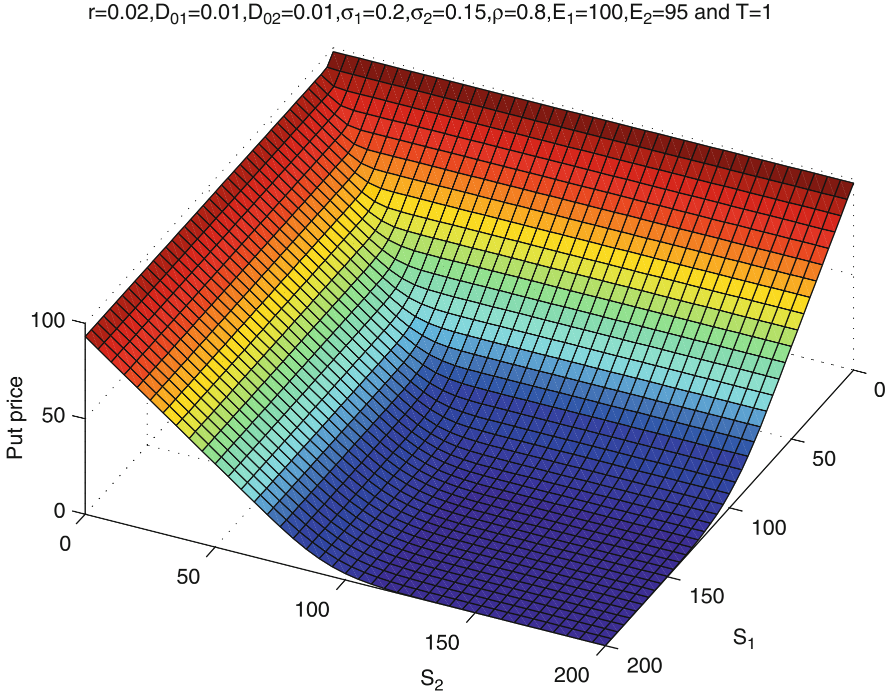

Fig. 8.7. Values of a European two-asset put option ($r = 0.02$, $D_{01} = 0.01$, $D_{02} = 0.01$, $\sigma_1 = 0.2$, $\sigma_2 = 0.15$, $\rho = 0.8$, $E_1 = 100$, $E_2 = 95$, and $T = 1$)

and

$$w(\xi,\theta,0) = \frac{1}{\sqrt{\left(\dfrac{S_1}{P_1}\right)^2 + \left(\dfrac{S_2}{P_2}\right)^2} + P_m} \max\left(S_1 - E_1, S_2 - E_2, 0\right), \quad (8.72)$$

respectively. Here P_1, P_2, and P_m are parameters, and

$$S_1 = P_1 \frac{\xi P_m}{1-\xi}\cos\theta,$$

$$S_2 = P_2 \frac{\xi P_m}{1-\xi}\sin\theta.$$

About the value of the parameters P_1, P_2, P_m, we can let

$$P_1 = E_1, \quad P_2 = E_2, \quad P_m = 1.$$

Using the initial condition (8.71) or (8.72) and scheme (7.46) obtained by discretizing the modified Eq. (4.79), we can get the price of a European two-asset call or put option. Some values of such options are listed in Tables 8.17

and 8.18. These results are obtained by a $400 \times 600 \times 400$ mesh, which means that

$$\Delta\xi = 1/400, \quad \Delta\theta = 1/600, \quad \Delta\tau = 1/400.$$

Computation is also done on the $800 \times 1,200 \times 800$ mesh; the results to two decimal places are the same except for the case of the put option with $S_1 = 95$ and $S_2 = 90$. For this case, on the $800 \times 1,200 \times 800$ mesh, the result is 11.28, and on the $400 \times 600 \times 400$ mesh, the result is 11.29. In order to give readers an idea as to what solutions of two-asset put options look like, the value of a two-asset put option for $(S_1, S_2) \in [0, 200] \times [0, 200]$ is shown in Fig. 8.7.

8.3 Singularity-Separating Method

In this section, we will discuss how to make numerical methods more efficient. Generally speaking, the smoother the solution, the smaller the truncation error. Therefore, if the solution is smooth, even on a coarse mesh, the numerical result is still quite good. Suppose that the solution we need to find is not very smooth but has a certain type of singularity caused by the final condition. Also, we assume that there is an analytic expression that satisfies the same final condition and the same equation or a similar equation. If both the final conditions and the equations are the same, their singularities caused by the final conditions are the same, and the difference between them is a smooth function; if only the final conditions are the same, they possess similar singularities, and the difference between them is usually smoother than the solution we need to find. In both cases, we can first compute the difference using numerical methods and then have our solution by adding the analytic expression and the difference together. Such a method or technique will be referred to as singularity-separating method (SSM), or singularity-separating technique, in this book. Because computing the difference is quite efficient, we can have the solution quite efficiently. Of course, there is some extra work in order to compute the difference. However, from the examples we are going to show, such a way can truly make numerical methods more efficient. In this section, we will give some details of the method for European double moving barrier options, European vanilla option with variable volatilities, Bermudan options, European Parisian options, European average price options, two-factor vanilla options, and two-factor convertible bonds with $D_0 = 0$. Indeed, the method can be used for many more cases, including multi-factor derivative securities.

8.3.1 Barrier Options

If the option has a fixed barrier and σ, r, and D_0 are constants, we can find analytic solutions of barrier options (see Sect. 4.2). However, if the option has two moving barriers, analytic solutions may not exist even if σ, r, and D_0 are constants, and we may need to rely on numerical methods for pricing such an option. Here, we discuss how to make numerical methods more efficient.

The price $V(S,t)$ of a double moving barrier call option with rebates satisfies the equation

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r - D_0)S\dfrac{\partial V}{\partial S} - rV = 0, \\ \qquad\qquad\qquad\qquad f(t) \le S \le g(t), \quad 0 \le t \le T, \\ V(S,T) = \max(S-E,0), \qquad f(T) \le S \le g(T), \\ V(f(t),t) = 0, \qquad 0 \le t \le T, \\ V(g(t),t) = g(t) - E, \qquad 0 \le t \le T, \end{cases} \tag{8.73}$$

where $f(t)$ and $g(t)$ are the locations of the lower and upper barriers with

$$f(t) < E \quad \text{and} \quad g(t) > E,$$

and we assume that at the lower barrier, there is no rebate and at the upper barrier, the rebate is

$$g(t) - E.$$

Because the derivative of the payoff function $\max(S-E,0)$ is discontinuous at $S = E$, the solution $V(S,t)$ at $t \approx T$ and $S \approx E$ is not very smooth. Therefore, the error of numerical solutions in the region around $t = T$ and $S = E$ is relatively large compared with that in the region far away from this point. In order to make the numerical solution better, we introduce a new function

$$\overline{V}(S,t) = V(S,t) - c(S,t),$$

where $c(S,t)$ is the price of the vanilla call option. Because $c(S,t)$ also satisfies the partial differential equation and the final condition in problem (8.73), $\overline{V}(S,t)$ satisfies

$$\begin{cases} \dfrac{\partial \overline{V}}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 \overline{V}}{\partial S^2} + (r - D_0)S\dfrac{\partial \overline{V}}{\partial S} - r\overline{V} = 0, \\ \qquad\qquad\qquad\qquad f(t) \le S \le g(t), \quad 0 \le t \le T, \\ \overline{V}(S,T) = 0, \qquad f(T) \le S \le g(T), \\ \overline{V}(f(t),t) = -c(f(t),t), \qquad 0 \le t \le T, \\ \overline{V}(g(t),t) = g(t) - E - c(g(t),t), \qquad 0 \le t \le T. \end{cases} \tag{8.74}$$

The derivative of $\overline{V}(S,t)$ at $t \approx T$ and $S \approx E$ is very smooth, so the error of the numerical solution of $\overline{V}(S,t)$ is usually smaller than that of $V(S,t)$.

Therefore, in order to get a better $V(S,t)$, we can first obtain the numerical solution of $\overline{V}(S,t)$ and then have $V(S,t)$ by adding $\overline{V}(S,t)$ and $c(S,t)$ together. We refer to this procedure as the singularity-separating method (SSM) or the singularity-separating technique for European barrier options. The reason is as follows. The derivative of $V(S,t)$ is discontinuous at $t=T$ and $S=E$. Thus, we say that $V(S,t)$ has some weak singularity. The function $\overline{V}(S,t)$, which will be determined numerically, is smooth. Therefore, the weak singularity has been "separated" from the numerical computation. The CPU time of getting $\overline{V}(S,t)$ is slightly longer than that of getting $V(S,t)$ directly because $c(f(t),t)$ and $c(g(t),t)$ need to be computed in order to get $\overline{V}(S,t)$. Because the error is smaller, we can usually expect better performance, i.e., we can usually expect to have the same accuracy by spending less CPU time or to spend the same CPU time for a better accuracy. Consequently, the singularity-separating technique can usually improve the performance.

Both $V(S,t)$ and $\overline{V}(S,t)$ are solutions of the following problem

$$\begin{cases} \dfrac{\partial \bar{u}}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 \bar{u}}{\partial S^2} + (r - D_0) S \dfrac{\partial \bar{u}}{\partial S} - r\bar{u} = 0, & \\ & f(t) \le S \le g(t), \quad 0 \le t \le T, \\ \bar{u}(S,T) = \bar{f}_1(S), & f(T) \le S \le g(T), \\ \bar{u}(f(t),t) = \bar{b}_l(t), & 0 \le t \le T, \\ \bar{u}(g(t),t) = \bar{b}_u(t), & 0 \le t \le T. \end{cases} \tag{8.75}$$

The only difference between the two cases is the functions in the final condition and in the boundary conditions. Thus, no matter whether the singularity-separating technique is used, we need a numerical method for problem (8.75) in order to have $V(S,t)$.

Problem (8.75) is a typical moving boundary problem. In order to convert it into a problem with fixed boundaries and transfer the final condition to an initial condition, we use the following transformation:

$$\begin{cases} \eta = \dfrac{S - f(t)}{g(t) - f(t)}, \\ \tau = T - t. \end{cases} \tag{8.76}$$

Let

$$\begin{aligned} &u(\eta,\ \tau) = u(\eta(S,t),\ T-t) = \bar{u}(S,\ t), \\ &F(\tau) = F(T-t) = f(t), \\ &G(\tau) = G(T-t) = g(t). \end{aligned}$$

Because

$$
\begin{aligned}
\frac{\partial \bar{u}}{\partial t} &= \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial \tau}\frac{\partial \tau}{\partial t} \\
&= -\frac{1}{g-f}\left[\frac{df}{dt} + \eta\left(\frac{dg}{dt} - \frac{df}{dt}\right)\right]\frac{\partial u}{\partial \eta} - \frac{\partial u}{\partial \tau} \\
&= \frac{1}{G-F}\left[\frac{dF}{d\tau} + \eta\left(\frac{dG}{d\tau} - \frac{dF}{d\tau}\right)\right]\frac{\partial u}{\partial \eta} - \frac{\partial u}{\partial \tau}, \\
\frac{\partial \bar{u}}{\partial S} &= \frac{1}{G(\tau)-F(\tau)}\frac{\partial u}{\partial \eta}, \\
\frac{\partial^2 \bar{u}}{\partial S^2} &= \frac{1}{[G(\tau)-F(\tau)]^2}\frac{\partial^2 u}{\partial \eta^2},
\end{aligned}
$$

$u(\eta, \tau)$ is the solution of the problem

$$
\begin{cases}
\dfrac{\partial u}{\partial \tau} = \mathbf{L}_{\eta 1} u, & 0 \le \eta \le 1 \quad 0 \le \tau \le T, \\
u(\eta, 0) = f_1(\eta), & 0 \le \eta \le 1, \\
u(0, \tau) = b_l(\tau), & 0 \le \tau \le T, \\
u(1, \tau) = b_u(\tau), & 0 \le \tau \le T,
\end{cases}
\tag{8.77}
$$

where

$$
\begin{aligned}
\mathbf{L}_{\eta 1} &= \frac{1}{2}\left(\frac{S\sigma}{G-F}\right)^2 \frac{\partial^2}{\partial \eta^2} + \left\{\frac{S}{G-F}(r - D_0)\right. \\
&\quad \left. + \frac{1}{G-F}\left[\frac{dF}{d\tau} + \eta\left(\frac{dG}{d\tau} - \frac{dF}{d\tau}\right)\right]\right\}\frac{\partial}{\partial \eta} - r, \\
f_1(\eta) &= \bar{f}_1\left(F(0) + \eta[G(0) - F(0)]\right), \\
b_l(\tau) &= \bar{b}_l(T - \tau), \\
b_u(\tau) &= \bar{b}_u(T - \tau).
\end{aligned}
$$

The problem (8.77) can be solved by explicit finite-difference schemes or implicit finite-difference schemes and even by pseudo-spectral methods. Here, we give some results to explain the effect of this technique if implicit finite-difference methods are used.

We have solved an identical problem by scheme (7.6) in two different ways: with and without SSM. In Table 8.19, the results, the errors, and the CPU time in seconds for four meshes are given. There, $N \times M$ in the column "Meshes" stands for a mesh that has $N+1$ nodes in the t-direction (the τ-direction) and $M+1$ nodes in the S-direction (the η-direction). The lower and upper knock-out boundaries are

$$
f(t) = 0.9E\mathrm{e}^{-0.1t} \quad \text{and} \quad g(t) = 1.6E\mathrm{e}^{0.1t}.
$$

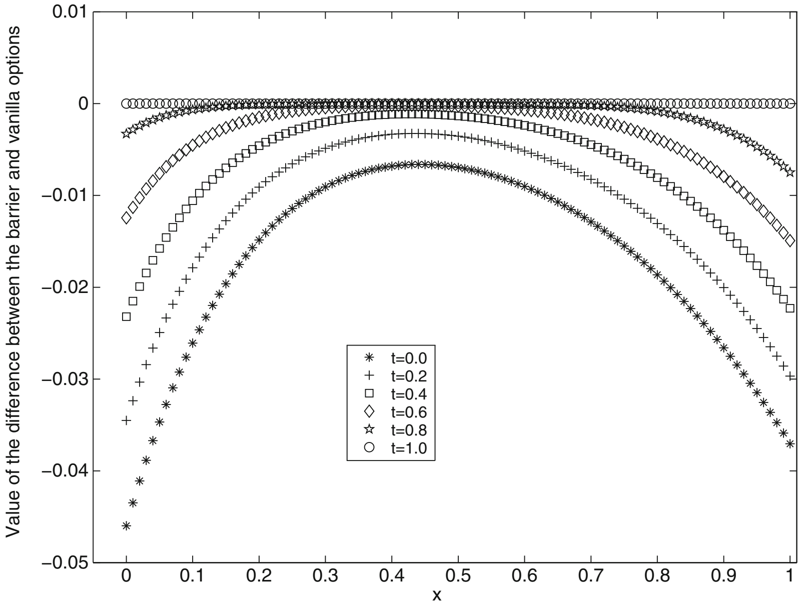

Fig. 8.8. Variation of the difference between the barrier and vanilla option values

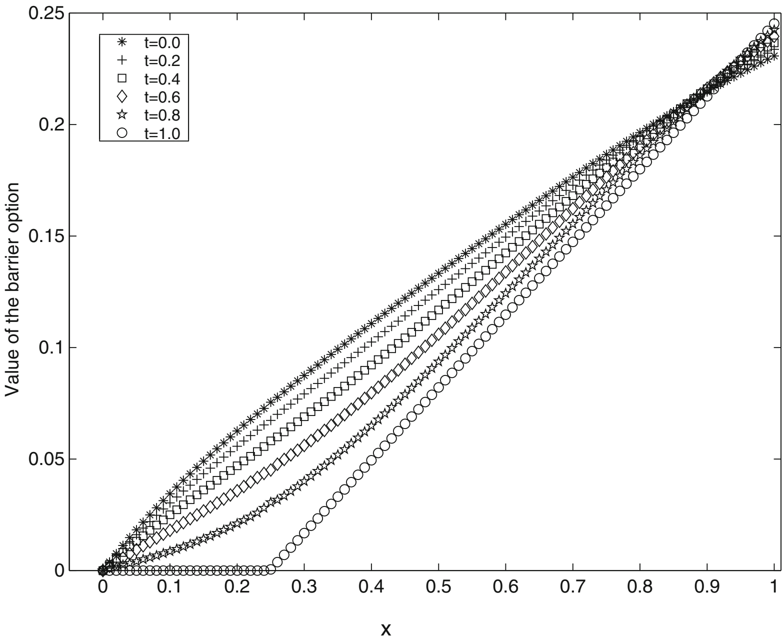

Fig. 8.9. Variation of the barrier option value

Table 8.19. Implicit methods with and without the SSM

($S = 95$, $T = 1$, $E = 100$, $\sigma = 0.25$, $r = 0.1$, $D_0 = 0$,
$f(t) = 0.9E\mathrm{e}^{-0.1t}$, $g(t) = 1.6E\mathrm{e}^{0.1t}$, the rebate $= g(t) - E$,
and the exact solution is $6.8441468\cdots$)

	Without SSM			With SSM		
Meshes	Solution	|Errors|	CPU	Solution	|Errors|	CPU
12×48	6.845973	0.001826	0.00039	6.843292	0.000855	0.00049
25×100	6.844623	0.000476	0.0019	6.844205	0.000058	0.0019
50×200	6.844187	0.000040	0.0062	6.844163	0.000016	0.0063
100×400	6.844167	0.000020	0.0221	6.844150	0.000003	0.0221

There, the results both with and without SSM are given. In order to give errors, we have to find the exact solution. To our knowledge, no analytic solution for such a problem has been found. Therefore, we take a very accurate approximate solution as an exact solution. For this case, the exact solution is $6.8441468\cdots$ (here the eight digits are correct). From there, we can see that the result with SSM is clearly better than without SSM on the same mesh whereas the CPU time difference between the two cases is very small. Therefore, the advantage of the singularity-separating technique is obvious for this case. As we know, if the error $\approx a\Delta\tau^{\alpha} = a(T/N)^{\alpha}$ (suppose $\Delta\tau/\Delta\eta = constant$), then we say that the convergence rate is $O(\Delta\tau^{\alpha})$. From Table 8.19, we can see that when N is doubled, the error of the implicit finite-difference method with the singularity-separating technique decreases by a factor of about 4. This implies that the convergence rate of this method is $O(\Delta\tau^{2})$.

In what follows, we give an intuitive explanation on why the singularity-separating method can improve the numerical results. The functions computed numerically for the methods with and without the singularity-separating technique are plotted in Figs. 8.8 and 8.9 respectively. In each figure, there are six curves, which correspond to $t = 0$, 0.2, 0.4, 0.6, 0.8, 1.0. In Fig. 8.9, the functions are not as smooth as those in Fig. 8.8, especially, the derivative of the function for $t = 1$ in Fig. 8.9 is discontinuous. Therefore, when the singularity-separating technique is used, the truncation is smaller.

When there is no rebate at the upper barrier, such a method can still improve the performance. This is left for the reader to study (see Problem 16). For the case discussed in this subsection, the singularity is removed completely. For the European options with discrete dividends and some other cases, the singularity can also be completely removed in the same way. In many other cases, the singularity cannot be completely separated but can be made much weaker. In the next several subsections, we will discuss how the SSM works for other cases.

8.3.2 European Vanilla Options with Variable Volatilities

When σ is a constant, for European vanilla options we can get their prices by the Black–Scholes formulae. However, it seems that the assumption of σ being a constant needs to be modified. One of the modifications is to let σ be a function of S. In this case, in order to evaluate an option, we usually need to solve a partial differential equation problem numerically. In order to overcome the problem caused by the discontinuous derivative in the payoff, we can do the following.

Let us consider call options. Their prices $c(S,t)$ are solutions of the problem:

$$\begin{cases} \dfrac{\partial c}{\partial t} + \dfrac{1}{2}\sigma^2(S)S^2\dfrac{\partial^2 c}{\partial S^2} + (r-D_0)S\dfrac{\partial c}{\partial S} - rc = 0, \quad 0 \le S, \quad t \le T, \\ c(S,T) = \max(S-E,0), \quad 0 \le S. \end{cases}$$

Suppose that $c_E(S,t;\sigma(E))$ is the price of the option with the volatility at $S=E$, $\sigma(E)$, i.e., $c_E(S,t;\sigma(E))$ satisfies

$$\begin{cases} \dfrac{\partial c_E}{\partial t} + \dfrac{1}{2}\sigma^2(E)S^2\dfrac{\partial^2 c_E}{\partial S^2} + (r-D_0)S\dfrac{\partial c_E}{\partial S} - rc_E = 0, \quad 0 \le S, \quad t \le T, \\ c_E(S,T) = \max(S-E,0), \quad 0 \le S. \end{cases}$$

Let $\bar{c}(S,t) = c(S,t) - c_E(S,t;\sigma(E))$. Then, $\bar{c}(S,t)$ is the solution of the problem

$$\begin{cases} \dfrac{\partial \bar{c}}{\partial t} + \dfrac{1}{2}\sigma^2(S)S^2\dfrac{\partial^2 \bar{c}}{\partial S^2} + (r-D_0)S\dfrac{\partial \bar{c}}{\partial S} - r\bar{c} = f(S,t), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le S, \quad t \le T, \\ \bar{c}(S,T) = 0, \qquad\qquad\qquad\qquad\qquad 0 \le S, \end{cases} \tag{8.78}$$

where

$$\begin{aligned} f(S,t) &= \frac{1}{2}\left[\sigma^2(E) - \sigma^2(S)\right] S^2 \frac{\partial^2 c_E}{\partial S^2} \\ &= \frac{1}{2\sigma(E)\sqrt{2\pi(T-t)}}\left[\sigma^2(E) - \sigma^2(S)\right] S\mathrm{e}^{-(D_0(T-t)+d_1^2/2)} \end{aligned} \tag{8.79}$$

and

$$d_1 = \left\{\ln(S/E) + \left[r - D_0 + \frac{1}{2}\sigma^2(E)\right](T-t)\right\} \Big/ \left[\sigma(E)\sqrt{T-t}\right].$$

This problem is defined on an infinite domain. In order to convert it into a problem on a finite domain with a bounded solution, we use the following transformation:

$$\begin{cases} \xi = \dfrac{S}{S+E}, \\ \tau = T - t, \\ \overline{c}(S,t) = (S+E)\overline{V}(\xi,\tau). \end{cases}$$

Finally, we have

$$\begin{cases} \dfrac{\partial \overline{V}}{\partial \tau} = \dfrac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1-\xi)^2\dfrac{\partial^2 \overline{V}}{\partial \xi^2} + (r - D_0)\xi(1-\xi)\dfrac{\partial \overline{V}}{\partial \xi} \\ \qquad - [r(1-\xi) + D_0\xi]\overline{V} + \bar{f}(\xi,\tau), \quad 0 \le \xi \le 1, \quad 0 \le \tau, \\ \overline{V}(\xi, 0) = 0, \qquad\qquad 0 \le \xi \le 1, \end{cases} \tag{8.80}$$

where

$$\bar{\sigma}(\xi) = \sigma(\xi E/(1-\xi)),$$

$$\bar{f}(\xi,\tau) = \frac{-f(S,t)}{S+E} = \frac{1}{2\sigma(E)\sqrt{2\pi\tau}}\left[\bar{\sigma}^2(\xi) - \sigma^2(E)\right]\xi \mathrm{e}^{-(D_0\tau + d_1^2/2)}$$

and

$$d_1 = \left\{\ln\frac{\xi}{1-\xi} + \left[r - D_0 + \frac{1}{2}\sigma^2(E)\right]\tau\right\} \Big/ \left[\sigma(E)\sqrt{\tau}\right].$$

In order to do some computation, we need the function $\sigma(S)$ or $\bar{\sigma}(\xi)$. For the Japanese yen–U.S. dollar exchange rate, we determine the function by the method in Sect. 6.3.2. In order to avoid approximating a function on an infinite domain, a new variable $\xi = S/(S + P_m)$ is introduced. Because the exchange rate is around 0.01, we set $P_m = 0.01$. Using the data of 1990–2000 from the market (see the curve in Fig. 1.5), we find the maximum and minimum values, $S_{\max} = 0.01232741616$ and $S_{\min} = 0.00625390870$. The corresponding values of ξ are

$$\xi_l = \frac{S_{\min}}{S_{\min} + P_m} = 0.384763371, \quad \xi_u = \frac{S_{\max}}{S_{\max} + P_m} = 0.552120141.$$

Assume that the function $\bar{\sigma}(\xi)$ is in the form:

$$\bar{\sigma}(\xi) = \begin{cases} c_l + a_l\left[1 - \left(\dfrac{\xi}{\xi_l}\right)^{200}\right], & 0 \le \xi < \xi_l, \\ a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3, & \xi_l \le \xi \le \xi_u, \\ c_u + a_u\left[1 - \left(\dfrac{1-\xi}{1-\xi_u}\right)^{200}\right], & \xi_u \le \xi \le 1, \end{cases}$$

where c_l, a_l, a_0, a_1, a_2, a_3, c_u, a_u are eight parameters to be determined. Taking the data of 1990–2000 from the market, using the method described in Sect. 6.3.2 with $g(\xi) \equiv 1$ and setting $M = 7$, we find the values of a_0, a_1, a_2, a_3:

$$a_0 = -10.7848, \quad a_1 = 72.8005, \quad a_2 = -161.134, \quad a_3 = 118.208.$$

Then, requiring the continuity of the function at $\xi = \xi_l$ and $\xi = \xi_u$ up to the first derivative yields

$$c_l = 0.104667, \quad a_l = -0.00250664, \quad c_u = 0.185335, \quad a_u = 0.00665520.$$

In Fig. 8.10, this function is plotted as a solid line, and the circles are the volatilities for different S obtained by statistics.

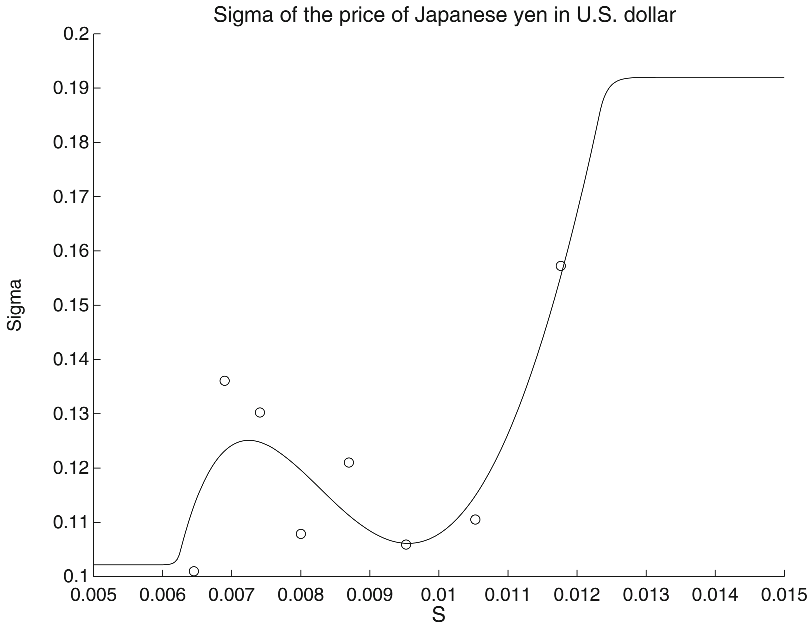

Fig. 8.10. The volatility function for Japanese yen–U.S. dollar exchange rate

As soon as we have this function, we can evaluate the price of options on the Japanese yen–U.S. dollar exchange rate. Discretizing problem (8.80) by the difference scheme (7.6) and solving the linear system by the LU decomposition, we can find the price. In Fig. 8.11, the solid line gives the value of the European call option. There, we also compare different models. Another model is to let the volatility be a constant. Using the same data, we find $\sigma = 0.1165$. The dashed line in Fig. 8.11 gives the option price for this model obtained by the Black–Scholes formula. The maximum difference of the results between

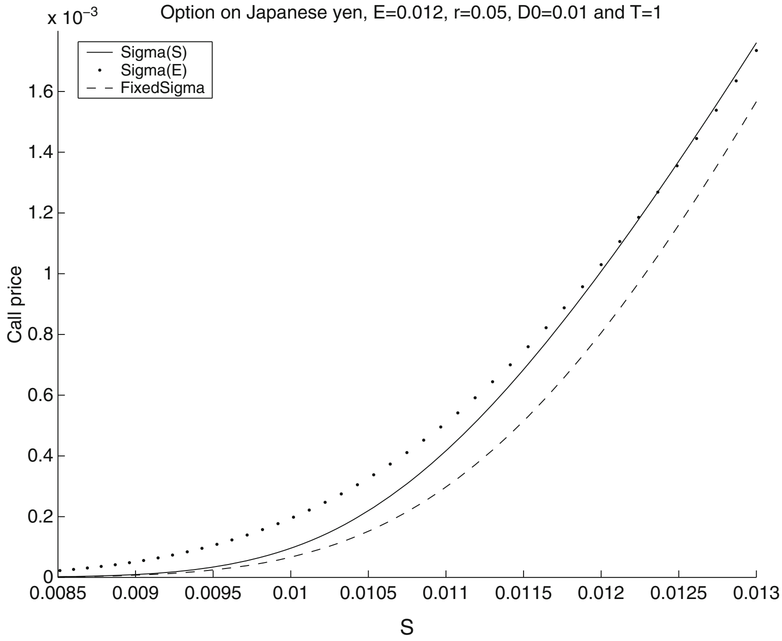

Fig. 8.11. The value of a European call option with a variable volatility with $E = 0.012$, $r = 0.05$, $D_0 = 0.01$, and $T = 1\,$year

the two models is more than 30 % if $S \in [0.0115, 0.0125]$. If we assume σ to take the value of $\sigma(E)$ (the result for this case is given by the dotted line in Fig. 8.11), the maximum difference is more than 8 % for $S \in [0.0115, 0.0125]$. Therefore, among the results obtained by using different models, there is quite a big difference. In our computation for the model with variable volatility, the numerical method is quite efficient because we are calculating the difference numerically. For this example problem, on a 60×4 mesh for the option price at $S = E$, the error is $6 \times 10^{-5}E$ when the SSM is used and $1 \times 10^{-3}E$ when the SSM is not used.

Finally, we would like to point out that unlike the barrier options, in this case the weak singularity is not removed completely. However, the singularity is weakened so the SSM still succeeds as shown above. Let us explain this matter as follows. Because $\dfrac{\partial^2 c_E}{\partial S^2}$ has some singularity at the point $T = t$ and $S = E$, the function $f(S,t) = \dfrac{1}{2}\left[\sigma^2(E) - \sigma^2(S)\right] S^2 \dfrac{\partial^2 c_E}{\partial S^2}$ also has some singularity. However, because the term $\sigma^2(E) - \sigma^2(S)$ is equal to zero at $S = E$, the singularity of $f(S,t)$ at that point is much weaker than that of $\dfrac{\partial^2 c_E}{\partial S^2}$. In Figs. 8.12 and 8.13, $f(S,t)$ used in this example and $\dfrac{\partial^2 c_E}{\partial S^2}$ for $t = T - 0.01$, $T - 0.001$, $T - 0.0001$ are plotted, respectively. Noticing the maximum value

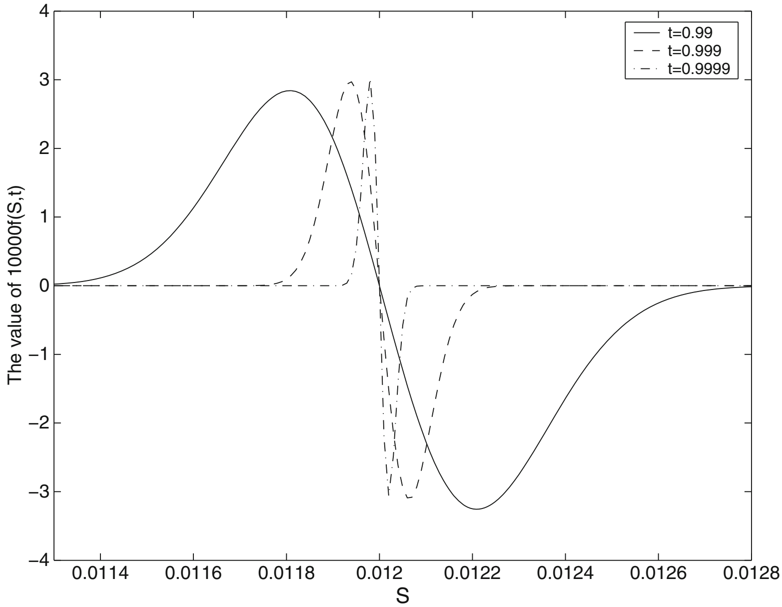

Fig. 8.12. The value of the function $10,000f(S,t)$ at $t \approx T$ $(T = 1)$

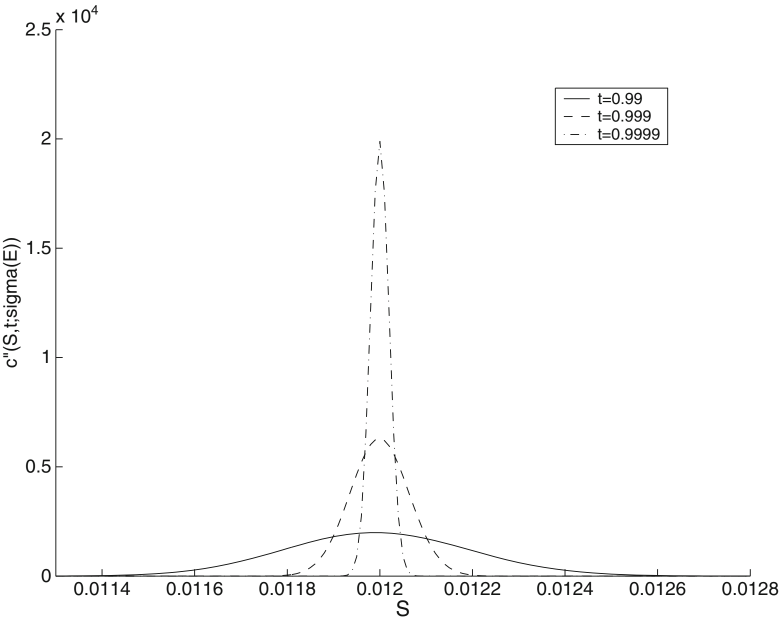

Fig. 8.13. The value of $\dfrac{\partial^2 c_E}{\partial S^2}$ at $t \approx T$ $(T = 1)$

of $|f(S,t)|$ is about 3.5×10^{-4} and the value of $\dfrac{\partial^2 c_E}{\partial S^2}$ could be very large, reaching 2×10^4 at $t = T - 0.0001$, we can see that the singularity of $f(S,t)$ at that point is truly weaker than that of $\dfrac{\partial^2 c_E}{\partial S^2}$. Because the singularity of $f(S,t)$ is quite weak and the singularity of $\bar{c}(S,t)$ is weaker than $f(S,t)$, the function $\bar{c}(S,t)$ is quite smooth. This is an important reason to guarantee the success of the SSM.

8.3.3 Bermudan Options

A Bermudan option is an option that can be exercised early, but only on predetermined dates. It is clear that the holder of a Bermudan option has more rights than the holder of a European option and less rights than the holder of an American option, just like the fact that Bermuda is situated between America and Europe. This is how the option got its name. If we use projected methods, it is easy to price. Here, we suggest some more efficient methods. Assume the expiry of the option to be T and suppose the option can be exercised at time $t = T_1, T_2, \cdots, T_K = T$, where $T_k = kT/K$, $k = 1, 2, \cdots, K$.

Let us consider a Bermudan call option with $D_0 > 0$ and a variable $\sigma(S)$, and denote its value by $C_b(S,t)$. Define $T_0 = 0$ and assume $T_0 < T_1 < \cdots < T_K$. Then, $C_b(S,t)$ is a solution of K successive problems:

$$\begin{cases} \dfrac{\partial C_b}{\partial t} + \dfrac{1}{2}\sigma^2(S)S^2\dfrac{\partial^2 C_b}{\partial S^2} + (r - D_0)S\dfrac{\partial C_b}{\partial S} - rC_b = 0, \\ \qquad\qquad 0 \le S, \quad T_{k-1} < t < T_k, \\ C_b(S, T_k^-) = \max(C_b(S, T_k^+), \max(S - E, 0)), \quad 0 \le S, \end{cases} \tag{8.81}$$

$$k = K, K-1, \cdots, 1$$

with $C_b(S, T_K^+) = \max(S - E, 0)$. Clearly, at $t = T_K$, $C_b(S, T_K^-) = \max(S - E, 0)$ for $S \in [0, \infty)$. At $t = T_k$, $k = K-1, K-2, \cdots, 1$, the whole interval $[0, \infty)$ is divided into two parts $[0, S_k^*]$ and (S_k^*, ∞). On $[0, S_k^*]$, $C_b(S, T_k^+) \ge \max(S - E, 0)$ and on (S_k^*, ∞), $C_b(S, T_k^+) < \max(S - E, 0)$. Because these functions are nonnegative and continuous, $S_k^* \ge E$ and $C_b(S_k^*, T_k^+) = S_k^* - E$. Therefore, the final condition of each problem above can be written as

$$C_b(S, T_k^-) = \begin{cases} C_b(S, T_k^+), \text{ if } 0 \le S \le S_k^*, \\ S - E, \qquad \text{if } S_k^* < S. \end{cases}$$

Because a European call option with a constant volatility has a closed-form solution, just like what we did in the last subsection, we consider the difference between the Bermudan call option and the European call option with a constant volatility $\sigma(E)$ and denote the difference by

$$\tilde{C}_b = C_b - c_{\scriptscriptstyle E}(S, t; \sigma(E)).$$

It is clear that $\tilde{C}_b$ satisfies the partial differential equation in problem (8.78). At $t = T_k$, we have

$$\tilde{C}_b(S, T_k^-) = \begin{cases} \tilde{C}_b(S, T_k^+), & \text{if } 0 \le S \le S_k^*, \\[2ex] S - E - c_{\scriptscriptstyle E}\left(S, T_k; \sigma(E)\right), & \text{if } S_k^* < S. \end{cases}$$

Therefore, $\tilde{C}_b$ is the solution of the following K successive problems:

$$\begin{cases} \dfrac{\partial \tilde{C}_b}{\partial t} + \dfrac{1}{2}\sigma^2(S)S^2 \dfrac{\partial^2 \tilde{C}_b}{\partial S^2} + (r - D_0)S\dfrac{\partial \tilde{C}_b}{\partial S} - r\tilde{C}_b = f(S, t), \\[1ex] \qquad\qquad\qquad\qquad\qquad\qquad 0 \le S, \quad T_{k-1} < t < T_k, \\[1ex] \tilde{C}_b(S, T_k^-) = \max\left(\tilde{C}_b(S, T_k^+), \max(S - E, 0) - c_{\scriptscriptstyle E}(S, T_k; \sigma(E))\right), \\[1ex] \qquad\qquad\qquad\qquad\qquad\qquad 0 \le S, \end{cases} \tag{8.82}$$

$$k = K, K - 1, \cdots, 1$$

with $\tilde{C}_b(S, T_{\scriptscriptstyle K}^+) = 0$. This problem can be solved in a way similar to what we have used to find the solution of a European option with discrete dividends in Sect. 8.2.2. The only difference is that using jump conditions should be replaced by taking the maximum between $\tilde{C}_b(S, T_k^+)$ and $\max(S - E, 0) - c_{\scriptscriptstyle E}(S, t; \sigma(E))$ at these specified times.

In many cases, this method can be further improved by doing the following. For $k = K - 1, K - 2, \cdots, 1$, let us define $K - 1$ polynomials of degree J: $f_k(S) = a_{0,k} + a_{1,k}S + \cdots + a_{\scriptscriptstyle J,k}S^J$ on $[S_k^{**}, S_k^*]$, which satisfies the conditions $f_k(S_k^*) = S_k^* - E$ and $f_k(S_k^{**}) = 0$. Besides satisfying these two conditions, we choose these coefficients $a_{0,k}$, $a_{1,k}$, $\cdots$, $a_{\scriptscriptstyle J,k}$ and $S_k^{**} \in [0, S_k^*]$ such that the norm of the function

$$\begin{cases} C_b(S, T_k^-), & \text{if } 0 \le S < S_k^{**}, \\[2ex] C_b(S, T_k^-) - f_k(S), & \text{if } S_k^{**} \le S < S_k^* \end{cases}$$

is as small as possible. It is clear that the function

$$\begin{cases} 0, & \text{if } 0 \le S < S_k^{**}, \\[2ex] f_k(S), & \text{if } S_k^{**} \le S < S_k^*, \\[2ex] S - E, & \text{if } S_k^* \le S \end{cases}$$

is a good approximation to $C_b(S, T_k^-)$. For $k = K$, if we define $S_k^* = S_k^{**} = E$, then the function defined above is equal to $C_b(S, T_{\scriptscriptstyle K}^-)$. Therefore, we assume the function above to be defined for $k = K, K - 1, \cdots, 1$.

Consider the problems

$$\begin{cases} \dfrac{\partial c_b}{\partial t} + \dfrac{1}{2}\sigma^2(S_k^*)S^2\dfrac{\partial^2 c_b}{\partial S^2} + (r-D_0)S\dfrac{\partial c_b}{\partial S} - rc_b = 0, \\ \qquad\qquad 0 \le S, \ \ T_{k-1} < t < T_k, \\ c_b(S, T_k^-) = \begin{cases} 0, & \text{if } 0 \le S < S_k^{**}, \\ f_k(S), & \text{if } S_k^{**} \le S < S_k^*, \\ S - E, & \text{if } S_k^* \le S. \end{cases} \end{cases} \tag{8.83}$$

Noticing that for any integer n, we have (see Problem 39 in Chap. 2)

$$\frac{1}{\sqrt{2\pi}b}\int_c^d S^n \mathrm{e}^{-\left(\ln(S/a)+b^2/2\right)^2/2b^2}\frac{dS}{S}$$
$$= a^n \mathrm{e}^{(n^2-n)b^2/2}\left[N\left(\frac{\ln(d/a)+(1/2-n)b^2}{b}\right) - N\left(\frac{\ln(c/a)+(1/2-n)b^2}{b}\right)\right],$$

we can find a closed-form solution of problem (8.83) (see Problem 48 in Chap. 2)

$$\begin{aligned} c_b(S,t) = \sum_{n=0}^{J} \Big\{ & a_{n,k}S^n \mathrm{e}^{[(n-1)r - nD_0 + (n-1)n\sigma^2(S_k^*)/2](T_k - t)} \\ & \times \left[N\left(d_k^* - n\sigma(S_k^*)\sqrt{T_k - t}\right) - N\left(d_k^{**} - n\sigma(S_k^*)\sqrt{T_k - t}\right)\right]\Big\} \\ +S\mathrm{e}^{-D_0(T_k - t)} & \left[1 - N\left(d_k^* - \sigma(S_k^*)\sqrt{T_k - t}\right)\right] - E\mathrm{e}^{-r(T_k - t)}[1 - N(d_k^*)], \end{aligned} \tag{8.84}$$

where $t \in (T_{k-1}, T_k)$ and

$$d_k^* = \left[\ln(S_k^*/S) - \left(r - D_0 - \frac{1}{2}\sigma^2(S_k^*)\right)(T_k - t)\right] \Big/ \left(\sigma(S_k^*)\sqrt{T_k - t}\right),$$
$$d_k^{**} = \left[\ln(S_k^{**}/S) - \left(r - D_0 - \frac{1}{2}\sigma^2(S_k^*)\right)(T_k - t)\right] \Big/ \left(\sigma(S_k^*)\sqrt{T_k - t}\right).$$

It is easy to see that for $t \in (T_{K-1}, T_K]$, c_b represents the price of the European option with a constant volatility $\sigma(S^*) = \sigma(E)$ because $S^* = E$ at time $t = T$, that is, $c_b(S,t)$ is equal to $c_E(S,t;\sigma(E))$ for this period.

At the point $S = S_k^*$ and $t = T_k$, the singularity of the solution of the problem (8.83) is very close to that of the problem (8.81). Therefore, the function

$$\overline{C}_b = C_b - c_b$$

is smooth near this point for $t \in (T_{k-1}, T_k)$ and its value is quite small if $T_k - T_{k-1}$ is not big. This function satisfies the following equation and condition:

$$
\begin{cases}
\dfrac{\partial \overline{C}_b}{\partial t} + \dfrac{1}{2}\sigma^2(S)S^2 \dfrac{\partial^2 \overline{C}_b}{\partial S^2} + (r - D_0)S\dfrac{\partial \overline{C}_b}{\partial S} - r\overline{C}_b = \\
\qquad \dfrac{1}{2}(\sigma^2(S_k^*) - \sigma^2(S))S^2 \dfrac{\partial^2 c_b}{\partial S^2}, \quad 0 \le S, \quad T_{k-1} < t < T_k, \\
\overline{C}_b(S, T_k^-) = \begin{cases} C_b(S, T_k^-), & \text{if } 0 \le S < S_k^{**}, \\ C_b(S, T_k^-) - f_k(S), & \text{if } S_k^{**} \le S < S_k^*, \\ 0, & \text{if } S_k^* \le S. \end{cases}
\end{cases}
\tag{8.85}
$$

Therefore, in order to have $C_b(S, T_{k-1}^+)$, we can first find $\overline{C}_b(S, T_{k-1}^+)$ by solving the problem (8.85) from $t = T_k$ to T_{k-1} and then obtain $C_b(S, T_{k-1}^+)$ by

$$
C_b(S, T_{k-1}^+) = \overline{C}_b(S, T_{k-1}^+) + c_b(S, T_{k-1}^+).
$$

Because for a variable σ the partial differential equation in the problem (8.85) is nonhomogeneous and the right-hand side is quite complicated, the amount of computation of solving the problem (8.85) is greater than solving the problem (8.81) on the same mesh. However, in order to have a solution with the same accuracy, the number of mesh points needed for the problem (8.85) is much smaller than the problem (8.81). It is expected that in order to reach the same accuracy, solving the problem (8.85) is better. If $\sigma = constant$, then the problem (8.85) becomes

$$
\begin{cases}
\dfrac{\partial \overline{C}_b}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 \overline{C}_b}{\partial S^2} + (r - D_0)S\dfrac{\partial \overline{C}_b}{\partial S} - r\overline{C}_b = 0, \\
\qquad 0 \le S, \quad T_{k-1} < t < T_k, \\
\overline{C}_b(S, T_k^-) = \begin{cases} C_b(S, T_k^-), & \text{if } 0 \le S < S_k^{**}, \\ C_b(S, T_k^-) - f_k(S), & \text{if } S_k^{**} \le S < S_k^*, \\ 0, & \text{if } S_k^* \le S. \end{cases}
\end{cases}
\tag{8.86}
$$

The partial differential equation in the problem (8.86) is a homogeneous equation. Hence, the amount of computation of solving the problem (8.86) is very close to that of solving the original problem (8.81).

Sometimes, the singularities at the points $S = S_k^*$ and $t = T_k$, $k = K-1, K-2, \cdots, 1$, are quite weak and far away from the region $S \approx E$. Therefore, these singularities only cause small errors in the region $S \approx E$. Also, $[S_k^{**}, S_k^*]$ is not a small interval, so $f_k(S)$ may not be a good approximation to $C_b(S, T_k^-)$. In this case, using the method described at the beginning of this subsection might be better.

Table 8.20. Bermudan call option prices ($r < D_0$)

($S = 100$, $E = 100$, $T = 1$, $r = 0.05$, $D_0 = 0.1$, $\sigma = 0.2$, $K = 4$, and the exact solution $= 5.77654\cdots$)

	Implicit method			SSM		
Mesh sizes	Results	|Errors|	CPU(s)	Results	|Errors|	CPU(s)
24×24	5.0474	0.7291	0.0002	5.8564	0.0799	0.0014
36×36	5.4507	0.3258	0.0005	5.7788	0.0023	0.0019
48×48	5.6143	0.1622	0.0008	5.7881	0.0116	0.0028
60×60	5.6732	0.1033	0.0013	5.7845	0.0080	0.0037
72×72	5.7069	0.0696	0.0018	5.7833	0.0068	0.0048
84×84	5.7332	0.0433	0.0024	5.7833	0.0068	0.0061
96×96	5.7362	0.0403	0.0032	5.7809	0.0044	0.0073
108×108	5.7479	0.0286	0.0039	5.7807	0.0042	0.0086
120×120	5.7543	0.0222	0.0049	5.7797	0.0032	0.0101
132×132	5.7599	0.0166	0.0059	5.7804	0.0039	0.0119
144×144	5.7592	0:0173	0.0073	5.7800	0.0035	0.0134
156×156	5.7649	0.0116	0.0082	5.7799	0.0034	0.0152
168×168	5.7674	0.0091	0.0096	5.7790	0.0025	0.0172
180×180	5.7659	0.0106	0.0109	5.7784	0.0019	0.0190

Table 8.21. Bermudan call option prices ($r > D_0$)

($S = 100$, $E = 100$, $T = 1$, $r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$, $K = 12$, and the exact solution $= 9.940918\cdots$)

	Implicit method			SSM		
Mesh sizes	Results	|Errors|	CPU(s)	Results	|Errors|	CPU(s)
24×24	9.1488	0.7922	0.0003	9.9411	0.0002	0.0017
36×36	9.6261	0.3148	0.0006	9.9410	0.0001	0.0026
48×48	9.7704	0.1705	0.0011	9.9410	0.0001	0.0037
60×60	9.8333	0.1076	0.0015	9.9409	0.0000	0.0049
72×72	9.8667	0.0742	0.0020	9.9409	0.0000	0.0062
84×84	9.8866	0.0543	0.0027	9.9409	0.0000	0.0075
96×96	9.8995	0.0414	0.0034	9.9409	0.0000	0.0090
108×108	9.9082	0.0327	0.0043	9.9409	0.0000	0.0105
120×120	9.9145	0.0264	0.0052	9.9409	0.0000	0.0121

In what follows, we give some examples. Consider a Bermudan call option with $r = 0.05$, $D_0 = 0.1$, and $T_k = k/4$, $k = 1, 2, 3, 4$. The price of the option is evaluated by two different ways. One is to solve problem (8.81) by the implicit method (7.6) and the other is to take $J = 6$ and solve problem (8.86) by difference scheme (7.6). For $S = 100$, the results obtained by the two ways, the errors, and CPU times needed on a Pentium III 800 MHz computer are given in Table 8.20. From there, we can see that for this case in order to have a result with an error about 10^{-2} (the corresponding relative error to E is 10^{-4}), CPU time needed is about 0.003 s if the singularity-separating method described here is used, and CPU time needed is about 0.01 s if the singularity-separating method is not used. Therefore, even though on an identical mesh, the CPU time needed for the SSM is much longer, overall the SSM is still fast for a fixed accuracy.

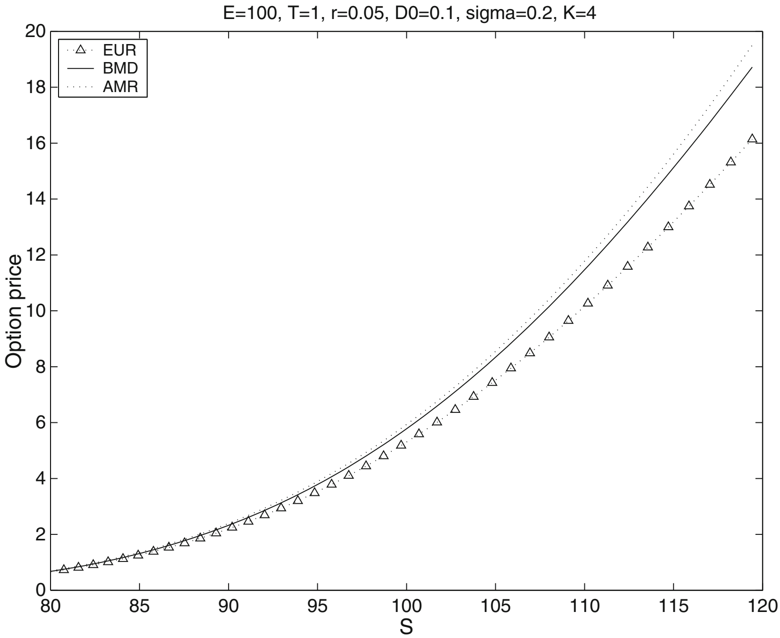

Fig. 8.14. Prices of American, Bermudan, and European call options ($E = 100$, $T = 1$, $r = 0.05$, $D_0 = 0.1$, $\sigma = 0.2$, and $K = 4$)

The next example is a Bermudan option with $r = 0.1$, $D_0 = 0.05$, and $T_k = k/12$, $k = 1, 2, \cdots, 12$. The other parameters are the same as those for the first example. In this case, the singularities at the points $S = S_k^*$ and $t = T_k$, $k = K-1, K-2, \cdots, 1$, are weak and we choose $c_E(S, t; \sigma(E))$ as c_b and solve problem (8.82) by the difference scheme (7.6). The results for $S = 100$ are given in Table 8.21. When the SSM is not used, the errors are close to those in the first example. However, when the SSM is used, the errors are even much less than those in the first example due to the very small value of $\overline{C}_b$.

In Fig. 8.14, the price of the first Bermudan call option is given as a function of S. The prices of the American and European call options are also given there. The figure shows that the price of the Bermudan option is less than the price of the American option and greater than the price of the European option, and it is quite close to the price of the corresponding American option. The financial reason of this fact is as follows. As has been pointed out at the beginning of this subsection, the holder of a Bermudan option has more rights than a holder of a European option and less rights than a holder of an American option. Thus, the money paid by the holder of the Bermudan option should be greater than the price of a European option and less than the price of an American option.

The symmetry relations also hold for Bermudan options, which is left for readers to prove. Therefore, we only need to study numerical methods for Bermudan call options. In order to obtain the price of a put option, we first solve a corresponding call option problem and then find the price of the put option by the symmetry relation.

8.3.4 European Parisian Options

Let us take a European Parisian up-and-out call option with continuous sampling as an example to show how the singularity-separating method works for Parisian options.

Suppose c_p is the price of the Parisian up-and-out call option. From Sect. 4.2.4, we see that $c_p(S, t_d, t)$ is the solution of problem (4.6):

$$\begin{cases} \dfrac{\partial c_p}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 c_p}{\partial S^2} + (r - D_0)\, S \dfrac{\partial c_p}{\partial S} + H(S - B_u)\dfrac{\partial c_p}{\partial t_d} - rc_p = 0, \\ \quad 0 \le S, \quad t_d = 0 \quad \text{and} \quad B_u \le S, \quad 0 < t_d \le T_d, \quad 0 \le t \le T, \\ c_p(S, t_d, T) = \begin{cases} \max(S - E, 0), & 0 \le S < B_u, \quad t_d = 0, \\ S - E, & B_u \le S, \quad 0 \le t_d < T_d, \\ 0, & B_u \le S, \quad t_d = T_d, \end{cases} \\ c_p(B_u, t_d, t) = c_p(B_u, 0, t), \qquad t_d \in (0, T_d), \quad 0 \le t \le T, \\ c_p(S, T_d, t) = 0, \qquad B_u \le S, \quad 0 \le t \le T. \end{cases}$$

Let $c(S, t)$ be the price of the European vanilla call option and define

$$\bar{c}_p(S, t_d, t) = c_p(S, t_d, t) - c(S, t).$$

Because $c(S, t)$ does not depend on t_d, it is clear that $c(S, t)$ also satisfies the partial differential equation in the problem (4.6). Therefore, $\bar{c}_p(S, t_d, t)$ is the solution of the following problem:

$$\begin{cases} \dfrac{\partial \bar{c}_p}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 \bar{c}_p}{\partial S^2} + (r - D_0)\, S \dfrac{\partial \bar{c}_p}{\partial S} + H(S - B_u) \dfrac{\partial \bar{c}_p}{\partial t_d} - r\bar{c}_p = 0, \\ \quad 0 \le S, \quad t_d = 0 \quad \text{and} \quad B_u \le S, \quad 0 < t_d \le T_d, \quad 0 \le t \le T, \\ \bar{c}_p(S, t_d, T) = \begin{cases} 0, & 0 \le S < B_u, \quad t_d = 0, \\ 0, & B_u \le S, \quad 0 \le t_d < T_d, \\ E - S, & B_u \le S, \quad t_d = T_d, \end{cases} \\ \bar{c}_p(B_u, t_d, t) = \bar{c}_p(B_u, 0, t), \quad t_d \in (0, T_d), \quad 0 \le t \le T, \\ \bar{c}_p(S, T_d, t) = -c(S, t), \qquad B_u \le S, \quad 0 \le t \le T. \end{cases} \tag{8.87}$$

Because $c_p(S, t_d, t)$ and $c(S, t)$ have the same singularity at the point $S = E$ and $t = T$, $\bar{c}_p$ is quite smooth near $S = E$ and $t = T$, that is, the singularity has been separated. Therefore, it is expected that on the same mesh, the error of the numerical results obtained by solving the problem (8.87) is smaller than that obtained by solving the problem (4.6). Tables 8.22 and 8.23 (see [58]) give the results and the relative errors when the SSM is not used and when it is used, respectively. From there, we can see that the results with the SSM are much better than the results without the SSM.

Problem (8.87) is a two-dimensional problem. However, it can be solved by a modified one-dimensional method. Let us explain why this problem can be solved like a one-dimensional problem. Because there is no second derivative in the t_d-direction, the coefficient of $\dfrac{\partial \bar{c}_p}{\partial t_d}$ is positive or zero, and the boundary condition is given at $t_d = T_d$, for a fixed time t^* the solution of the problem can be obtained from $t_d = T_d$ to $t_d = 0$ successively. Suppose the value of $\bar{c}_p$ for $t = t^*$ and $t_d \ge t_d^*$ has been obtained. We want to find the value of $\bar{c}_p$ for $t = t^*$ and $t_d = t_d^* - \Delta t_d$ with a positive Δt_d. Because the value at $t = t^*$ and $t_d = t_d^*$ is known, the value at $t = t^*$ and $t_d = t_d^* - \Delta t_d$ can be found by solving a one-dimensional problem on an (S, t)-plane. This can be done by various methods. After transforming the problem to one defined on

Table 8.22. Numerical solutions for Parisian up-and-out call options

($r = 0.1$, $D_0 = 0.05$, $\sigma = 0.25$, $E = 100$, $T = 0.5$, $B_u = 150$, and $T_d = 0.02$)

	$S = 100$		$S = 120$		$S = 150$	
Meshes	Solutions	|Errors|	Solution	|Errors|	Solution	|Errors|
200×100	7.4139	$1.08 \cdot 10^{-3}$	15.3107	$7.79 \cdot 10^{-3}$	5.0574	$3.73 \cdot 10^{-2}$
300×150	7.4067	$1.08 \cdot 10^{-4}$	15.2886	$6.33 \cdot 10^{-3}$	4.9389	$1.30 \cdot 10^{-2}$
400×200	7.4059	–	15.1924	–	4.8754	–

Table 8.23. Numerical solutions for Parisian up-and-out call options (with SSM) ($r = 0.1$, $D_0 = 0.05$, $\sigma = 0.25$, $E = 100$, $T = 0.5$, $B_u = 150$, and $T_d = 0.02$)

	$S = 100$		$S = 120$		$S = 150$	
Meshes	Solutions	|Errors|	Solution	|Errors|	Solution	|Errors|
200×100	7.3943	$1.76 \cdot 10^{-4}$	15.2016	$5.13 \cdot 10^{-4}$	4.9232	$2.09 \cdot 10^{-2}$
300×150	7.3936	$8.16 \cdot 10^{-5}$	15.1947	$5.92 \cdot 10^{-5}$	4.8251	$5.18 \cdot 10^{-4}$
400×200	7.3930	–	15.1938	–	4.8226	–

a finite domain by the transformation (2.17), the partial differential equation can be discretized by scheme (7.6) at interior points, and the right boundary point and the solution can be found from these finite-difference equations. The results given in this subsection are obtained by using a method that is a little different from what we have described here. For details, see the paper [58] by Luo and Wu.

When σ is a function of S, the SSM method can still be used. However, a European vanilla call option has a closed-form solution only when σ is a constant. Therefore, we do not have a closed-form solution for the corresponding European vanilla call option. In this case, we can consider the difference between the Parisian call option and the vanilla call option with a constant volatility $\sigma(E)$. This difference satisfies a nonhomogeneous equation (for details, see Sect. 8.3.2), but we still can expect that the SSM will make the computation more efficient.

8.3.5 European Average Price Options

In the last few subsections, we always computed the difference between an option and the corresponding vanilla option with a constant volatility. However, other functions can also be used as long as they have a similar singularity, and even they may be better. In this subsection, we give such an example.

From Eq. (4.20), we know that if sampling is done continuously, then the European-style Asian option may be modeled by the following partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0)S\frac{\partial V}{\partial S} + \frac{S}{T}\frac{\partial V}{\partial I} - rV = 0, \tag{8.88}$$

where

$$I = \frac{1}{T}\int_0^t S(\tau)d\tau.$$

Let us consider an average price call option whose final condition is

$$V(S, I, T) = \max(I - E, 0). \tag{8.89}$$

Zhang[3] in his paper [88] proposed to solve the problem in the following way. By letting (see Sect. 4.3.4)

$$\eta = \frac{I-E}{S} \quad \text{and} \quad W(\eta,t) = \frac{V(S,I,t)}{S},$$

the two-dimensional equation (8.88) can be converted into a one-dimensional equation:

$$\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2\eta^2\frac{\partial^2 W}{\partial \eta^2} + \left[(D_0 - r)\eta + \frac{1}{T}\right]\frac{\partial W}{\partial \eta} - D_0 W = 0$$

and the final condition becomes

$$W(\eta, T) = \max(\eta, 0).$$

Because the equation

$$\frac{d\eta}{dt} = (D_0 - r)\eta + \frac{1}{T}$$

has solutions in the form

$$\eta e^{-(r-D_0)(T-t)} + \frac{1}{(r-D_0)T}\left(1 - e^{-(r-D_0)(T-t)}\right) = constant,$$

introducing the transformation

$$\begin{cases} \xi = \eta e^{-(r-D_0)(T-t)} + \dfrac{1}{(r-D_0)T}\left(1 - e^{-(r-D_0)(T-t)}\right), \\ \tau = T - t, \\ W(\eta, t) = e^{-D_0\tau} f(\xi, \tau), \end{cases} \tag{8.90}$$

we can get rid of the first derivative of W and the function W, and we arrive at an initial value problem of a heat equation

$$\begin{cases} \dfrac{\partial f}{\partial \tau} - \dfrac{1}{2}\sigma^2\left[\xi - \dfrac{1}{(r-D_0)T}(1 - e^{-(r-D_0)\tau})\right]^2 \dfrac{\partial^2 f}{\partial \xi^2} = 0, \\ \qquad\qquad\qquad -\infty < \xi < \infty, \quad 0 \le \tau \le T, \\ f(\xi, 0) = \max(\xi, 0), \qquad -\infty < \xi < \infty. \end{cases} \tag{8.91}$$

The initial condition $f(\xi, 0) = \max(\xi, 0)$ is not smooth at the point $\xi = 0$. To separate the singularity, the problem that is obtained by setting $\xi = 0$ in the above equation

$$\begin{cases} \dfrac{\partial \tilde{f}_0}{\partial \tau} - \dfrac{\sigma^2}{2(r-D_0)^2T^2}\left(1 - e^{-(r-D_0)\tau}\right)^2 \dfrac{\partial^2 \tilde{f}_0}{\partial \xi^2} = 0, \\ \qquad\qquad\qquad -\infty < \xi < \infty, \quad 0 \le \tau \le T, \\ \tilde{f}_0(\xi, 0) = \max(\xi, 0), \qquad -\infty < \xi < \infty \end{cases} \tag{8.92}$$

[3] In his paper, he assumes $D_0 = 0$. However, it is not difficult to generalize that result to the case with $D_0 \neq 0$.

is considered. Introducing a new variable $\tau_1(\tau)$ by

$$d\tau_1 = \frac{\sigma^2}{2(r-D_0)^2T^2}\left(1-\mathrm{e}^{-(r-D_0)\tau}\right)^2 d\tau \quad \text{with} \quad \tau_1(0)=0,$$

which gives

$$\begin{aligned}\tau_1(\tau) &= \int_0^\tau \frac{\sigma^2}{2(r-D_0)^2T^2}\left(1-\mathrm{e}^{-(r-D_0)\tau}\right)^2 d\tau \\ &= \frac{\sigma^2}{4(r-D_0)^3T^2}\left[2(r-D_0)\tau+4\mathrm{e}^{-(r-D_0)\tau}-\mathrm{e}^{-2(r-D_0)\tau}-3\right],\end{aligned} \tag{8.93}$$

and letting $f_0(\xi,\tau_1)=\tilde{f}_0\left(\xi,\tau(\tau_1)\right)$, we obtain the following parabolic problem

$$\begin{cases}\dfrac{\partial f_0}{\partial \tau_1}-\dfrac{\partial^2 f_0}{\partial \xi^2}=0, & -\infty<\xi<\infty, \quad 0\le\tau_1\le\tau_1(T), \\ f_0(\xi,0)=\max(\xi,0), & -\infty<\xi<\infty.\end{cases} \tag{8.94}$$

The solution of this problem is given by

$$f_0(\xi,\tau_1)=\int_0^\infty \frac{\xi_T}{2\sqrt{\pi\tau_1}}\mathrm{e}^{-(\xi_T-\xi)^2/4\tau_1}d\xi_T=\xi N\left(\frac{\xi}{\sqrt{2\tau_1}}\right)+\sqrt{\frac{\tau_1}{\pi}}\mathrm{e}^{-\xi^2/4\tau_1}. \tag{8.95}$$

This analytic formula gives quite a good approximation to the prices of European average price call options. That is, the value of the difference between $f(\xi,\tau)$ and $f_0\left(\xi,\tau_1(\tau)\right)$,

$$f_1(\xi,\tau)=f(\xi,\tau)-f_0\left(\xi,\tau_1(\tau)\right), \tag{8.96}$$

is quite small. If we want to have more accurate results, we need to find $f_1(\xi,\tau)$. This function satisfies the following equation and initial condition:

$$\begin{cases}\dfrac{\partial f_1}{\partial \tau}-\dfrac{1}{2}\sigma^2\left[\xi-\dfrac{1}{(r-D_0)T}\left(1-\mathrm{e}^{-(r-D_0)\tau}\right)\right]^2\dfrac{\partial^2 f_1}{\partial \xi^2}=\dfrac{\sigma^2\xi\,\mathrm{e}^{-\xi^2/4\tau_1}}{4\sqrt{\pi\tau_1}} \\ \times\left[\xi-\dfrac{2}{(r-D_0)T}\left(1-\mathrm{e}^{-(r-D_0)\tau}\right)\right], \quad -\infty<\xi<\infty, \quad 0\le\tau\le T, \\ f_1(\xi,0)=0, \quad -\infty<\xi<\infty.\end{cases} \tag{8.97}$$

The function $f_1(\xi,\tau)$ is smooth, and its value is quite small compared with $f(\xi,\tau)$, so in order to get a very good numerical solution, we need only a very coarse mesh. In this way, we can find quite accurate solutions very fast. The problem (8.97) is defined on an infinite domain. In order to convert the infinite domain into a finite domain, we can introduce the following transformation:

$$\xi_1 = \frac{1}{2}\left(\frac{\xi}{|\xi| + P_m} + 1\right) \quad \text{and} \quad u(\xi_1, \tau) = \frac{f_1(\xi, \tau)}{|\xi| + P_m}.$$

After this transformation, the problem for $u(\xi_1, \tau)$ is defined on $[0, 1] \times [0, T]$ in the (ξ_1, τ)-space and can be solved by scheme (7.6).

We can also take the difference between the price of a European-style Asian option and the price of a European vanilla option and do the numerical computation. However, the performance might not be as good as the method here. The reason is that the difference in the method given here is smaller than the difference between the price of a European-style Asian option and the price of a European vanilla option. This can be roughly explained as follows. Consider the following linear parabolic problem:

$$\begin{cases} \dfrac{\partial u}{\partial \tau} = a_2 \dfrac{\partial^2 u}{\partial \xi^2} + a_1 \dfrac{\partial u}{\partial \xi} + a_0 u + g(\xi, \tau), & -\infty < \xi < \infty, \quad 0 \le \tau \le T, \\ u(\xi, 0) = f(\xi), & -\infty < \xi < \infty. \end{cases}$$

Suppose that $\tilde{u}$ is an approximate solution by a numerical method on a certain mesh. It is clear that $v = u/10$ is the solution of the problem:

$$\begin{cases} \dfrac{\partial v}{\partial \tau} = a_2 \dfrac{\partial^2 v}{\partial \xi^2} + a_1 \dfrac{\partial v}{\partial \xi} + a_0 v + g(\xi, \tau)/10, & -\infty < \xi < \infty, \quad 0 \le \tau \le T, \\ v(\xi, 0) = f(\xi)/10, & -\infty < \xi < \infty. \end{cases}$$

Let $\tilde{v}$ be the approximate solution of this problem by using the same method on the same mesh. Just like the relation between u and v, we have $\tilde{v} = \tilde{u}/10$. Thus, $v - \tilde{v} = (u - \tilde{u})/10$, which means that the smaller the solution, the smaller the error of approximate solutions. Therefore, when we choose an analytic solution, we should let the analytic solution be as close to the desired solution as possible. In this way, we can have a better performance.

8.3.6 European Two-Factor Options

In Sect. 8.3.2, we pointed out that the assumption of the volatility being constant might need to be modified. One possible modification is to let the volatility be a given function of S. In Sect. 8.3.2, we discussed how to solve such a problem. Another possible modification is to allow the volatility to be a random variable, i.e., the volatility is stochastic. This subsection is devoted to studying how to solve this problem. In this case, option prices depend on two random variables. In what follows, such an option will be referred to as a two-factor option, and we will call an option a one-factor option if only the stock price is considered as a random variable.

Now let us discuss how to evaluate quickly such a European vanilla option or American vanilla option with $D_0 = 0$. We assume that the asset price S and the stochastic volatility are governed by the following two stochastic processes

$$\begin{cases} dS = \mu S dt + \sigma S dX_1, \quad 0 \le S, \\ d\sigma = p(\sigma, t)dt + q(\sigma, t)dX_2, \quad \sigma_l \le \sigma \le \sigma_u, \end{cases} \tag{8.98}$$

where dX_1 and dX_2 are two Wiener processes. These two random variables could be correlated and $\mathrm{E}[dX_1 dX_2] = \rho dt$.

As we have seen in Sect. 2.4.1, in order to guarantee $\sigma \in [\sigma_l, \sigma_u]$, p and q in the model for the volatility need to satisfy the following reversion conditions:

$$\begin{cases} p(\sigma_l, t) - q(\sigma_l, t)\dfrac{\partial q(\sigma_l, t)}{\partial \sigma} \ge 0, \\ q(\sigma_l, t) = 0 \end{cases} \tag{8.99}$$

and

$$\begin{cases} p(\sigma_u, t) - q(\sigma_u, t)\dfrac{\partial q(\sigma_u, t)}{\partial \sigma} \le 0, \\ q(\sigma_u, t) = 0. \end{cases} \tag{8.100}$$

It is clear that if $\dfrac{\partial q(\sigma_l, t)}{\partial \sigma}$ and $\dfrac{\partial q(\sigma_u, t)}{\partial \sigma}$ are bounded, then the conditions (8.99) and (8.100) are simplified into

$$\begin{cases} p(\sigma_l, t) \ge 0, \\ q(\sigma_l, t) = 0 \end{cases} \tag{8.101}$$

and

$$\begin{cases} p(\sigma_u, t) \le 0, \\ q(\sigma_u, t) = 0. \end{cases} \tag{8.102}$$

Suppose $V(S, \sigma, t)$ is the value of an option depending on two random variables S and σ. From Sect. 2.3, such an option satisfies the following equation:

$$\frac{\partial V}{\partial t} + \mathbf{L}_{\mathbf{s},\sigma} V = 0, \tag{8.103}$$

where $\mathbf{L}_{S,\sigma}$ is an operator defined by

$$\begin{aligned} \mathbf{L}_{\mathbf{s},\sigma} = & \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + \rho\sigma S q \frac{\partial^2}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2}{\partial \sigma^2} \\ & + (r - D_0) S \frac{\partial}{\partial S} + (p - \lambda q)\frac{\partial}{\partial \sigma} - r. \end{aligned} \tag{8.104}$$

Consider a two-factor European vanilla call option problem, and let its value be $c(S, \sigma, t)$. Because the volatility model satisfies the reversion conditions, no boundary conditions need to be given at the boundaries $\sigma = \sigma_l$ and $\sigma = \sigma_u$. Therefore, the value of the two-factor European vanilla call option is the solution of the following final-value problem:

$$\begin{cases} \dfrac{\partial c}{\partial t} + \mathbf{L}_{S,\sigma} c = 0, & 0 \le S, \quad \sigma_l \le \sigma \le \sigma_u, \quad t \le T, \\ c(S, \sigma, T) = \max(S - E, 0), & 0 \le S, \quad \sigma_l \le \sigma \le \sigma_u. \end{cases} \tag{8.105}$$

In order to make the computed solution smoother, which will make numerical methods more efficient, we let

$$\overline{c}(S, \sigma, t) = c(S, \sigma, t) - c_1(S, \sigma, t) \tag{8.106}$$

on the entire computational domain. $c_1(S, \sigma, t)$ is the price of the one-factor European vanilla call option, that is, the price of the European vanilla call option with a parameter σ. Here, we denote the value of this option by $c_1(S, \sigma, t)$ instead of $c(S, t)$ in order to indicate explicitly its dependence on σ and to explain that it is the price of the one-factor model. From Sect. 2.6.5, we know that its expression is given by

$$c_1(S, \sigma, t) = S\mathrm{e}^{-D_0(T-t)} N(d_1) - E\mathrm{e}^{-r(T-t)} N(d_2),$$

where

$$\begin{aligned} N(z) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \mathrm{e}^{-\xi^2/2} d\xi, \\ d_1 &= \left[\ln \frac{S\mathrm{e}^{-D_0(T-t)}}{E\mathrm{e}^{-r(T-t)}} + \frac{1}{2}\sigma^2 (T - t) \right] \Big/ \left(\sigma\sqrt{T - t} \right), \\ d_2 &= d_1 - \sigma\sqrt{T - t}. \end{aligned}$$

Because $c_1(S, \sigma, t)$ satisfies the Black–Scholes equation, the difference $\overline{c}$ is the solution of the following final-value problem:

$$\begin{cases} \dfrac{\partial \overline{c}}{\partial t} + \mathbf{L}_{\mathbf{S},\sigma} \overline{c} = f(S, \sigma, t), & 0 \le S, \quad \sigma_l \le \sigma \le \sigma_u, \quad 0 \le t \le T, \\ \overline{c}(S, \sigma, T) = 0, & 0 \le S, \quad \sigma_l \le \sigma \le \sigma_u, \end{cases} \tag{8.107}$$

where

$$f(S, \sigma, t) = -\rho\sigma S q \frac{\partial^2 c_1}{\partial S \partial \sigma} - \frac{1}{2} q^2 \frac{\partial^2 c_1}{\partial \sigma^2} - (p - \lambda q) \frac{\partial c_1}{\partial \sigma}.$$

From the expressions of $c_1(S, \sigma, t)$, noticing

$$\begin{aligned} \frac{\partial c_1}{\partial S} &= \mathrm{e}^{-D_0(T-t)} N(d_1), \\ \frac{\partial d_1}{\partial \sigma} &= \sqrt{T - t} - \left[\ln \frac{S\mathrm{e}^{-D_0(T-t)}}{E\mathrm{e}^{-r(T-t)}} + \frac{1}{2}\sigma^2 (T - t) \right] \Big/ (\sigma^2 \sqrt{T - t}) \\ &= \sqrt{T - t} - \frac{d_1}{\sigma}, \\ \frac{\partial d_2}{\partial \sigma} &= \frac{\partial d_1}{\partial \sigma} - \sqrt{T - t} = -\frac{d_1}{\sigma}, \\ N'(z) &= \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-z^2/2}, \end{aligned}$$

we can easily find

$$
\begin{cases}
\dfrac{\partial c_1}{\partial \sigma} &= S\mathrm{e}^{-D_0(T-t)}N'(d_1)\dfrac{\partial d_1}{\partial \sigma} - E\mathrm{e}^{-r(T-t)}N'(d_2)\dfrac{\partial d_2}{\partial \sigma} \\
&= S\sqrt{T-t}\,\mathrm{e}^{-D_0(T-t)}N'(d_1), \\
\dfrac{\partial^2 c_1}{\partial \sigma^2} &= S\sqrt{T-t}\,\mathrm{e}^{-D_0(T-t)}N''(d_1)\dfrac{\partial d_1}{\partial \sigma} \\
&= -S\sqrt{T-t}\,\mathrm{e}^{-D_0(T-t)}d_1 N'(d_1)\dfrac{\partial d_1}{\partial \sigma}, \\
\dfrac{\partial^2 c_1}{\partial S \partial \sigma} &= \mathrm{e}^{-D_0(T-t)}N'(d_1)\dfrac{\partial d_1}{\partial \sigma}.
\end{cases}
\tag{8.108}
$$

As we see from the problem (8.105), the derivative of $c(S, \sigma, t)$ with respect to S is discontinuous at $t = T$ and $S = E$. However, the problem (8.107) shows the derivative of $\overline{c}(S, \sigma, t)$ with respect to S to be identically equal to zero at $t = T$. Therefore, when a numerical method is used, the truncation error for the problem (8.107) will be much smaller than the problem (8.105). This is why we consider the formulation (8.107) instead of the formulation (8.105).

The final-value problem (8.107) is defined on an infinite domain. In order to convert it into a problem on a finite domain, we introduce the following transformation

$$
\begin{cases}
\xi = \dfrac{S}{S + P_m}, \\
\sigma = \sigma, \\
\tau = T - t, \\
u(\xi, \sigma, \tau) = \dfrac{\overline{c}(S, \sigma, t)}{S + P_m}.
\end{cases}
\tag{8.109}
$$

In the $\{\xi, \sigma, \tau\}$-space, we need to solve a problem on the domain $[0, 1] \times [\sigma_l, \sigma_u] \times [0, T]$. This is a finite domain, and it is easy to construct numerical methods to solve the problem on this domain. From the expression (8.109), we have

$$
\overline{c}(S, \sigma, t) = (S + P_m)u(\xi, \sigma, \tau) = \frac{P_m}{1-\xi}u(\xi, \sigma, \tau) \quad \text{and} \quad \frac{d\xi}{dS} = \frac{(1-\xi)^2}{P_m}.
$$

Therefore, among the derivatives of $\overline{c}$ and u, there are the following relations:

$$
\begin{aligned}
\frac{\partial \overline{c}}{\partial t} &= -\frac{P_m}{1-\xi}\frac{\partial u}{\partial \tau}, \\
\frac{\partial \overline{c}}{\partial S} &= (1-\xi)\frac{\partial u}{\partial \xi} + u, \\
\frac{\partial \overline{c}}{\partial \sigma} &= \frac{P_m}{1-\xi}\frac{\partial u}{\partial \sigma},
\end{aligned}
$$

$$\frac{\partial^2 \overline{c}}{\partial S^2} = \frac{(1-\xi)^3}{P_m}\frac{\partial^2 u}{\partial \xi^2},$$
$$\frac{\partial^2 \overline{c}}{\partial S \partial \sigma} = (1-\xi)\frac{\partial^2 u}{\partial \xi \partial \sigma} + \frac{\partial u}{\partial \sigma},$$
$$\frac{\partial^2 \overline{c}}{\partial \sigma^2} = \frac{P_m}{1-\xi}\frac{\partial^2 u}{\partial \sigma^2}.$$

Substituting them into the partial differential equation in the problem (8.107) yields

$$\frac{\partial u}{\partial \tau} = a_1\frac{\partial^2 u}{\partial \xi^2} + a_2\frac{\partial^2 u}{\partial \xi \partial \sigma} + a_3\frac{\partial^2 u}{\partial \sigma^2} + a_4\frac{\partial u}{\partial \xi} + a_5\frac{\partial u}{\partial \sigma} + a_6 u + a_7,$$

where

$$\begin{aligned}
a_1 &= \frac{1}{2}\sigma^2\xi^2(1-\xi)^2, & a_2 &= \rho\sigma q\xi(1-\xi),\\
a_3 &= \frac{1}{2}q^2, & a_4 &= (r-D_0)\xi(1-\xi),\\
a_5 &= p-(\lambda-\rho\sigma\xi)q, & a_6 &= -[r(1-\xi)+D_0\xi],
\end{aligned}$$
$$\begin{aligned}
a_7 &= -f(\xi P_m/(1-\xi),\sigma,T-\tau)(1-\xi)/P_m\\
&= \rho\sigma\xi q\,\mathrm{e}^{-D_0(T-t)}N'(d_1)\frac{\partial d_1}{\partial \sigma} - \frac{1}{2}q^2\xi\sqrt{T-t}\,\mathrm{e}^{-D_0(T-t)}d_1N'(d_1)\frac{\partial d_1}{\partial \sigma}\\
&\quad +(p-\lambda q)\xi\sqrt{T-t}\,\mathrm{e}^{-D_0(T-t)}N'(d_1)\\
&= \frac{1}{\sqrt{2\pi}}\xi\mathrm{e}^{-D_0\tau-d_1^2/2}\left[q(\sqrt{\tau}-d_1/\sigma)(\rho\sigma-q\sqrt{\tau}d_1/2)+(p-\lambda q)\sqrt{\tau}\right].
\end{aligned}$$

Therefore, the problem (8.107) becomes

$$\begin{cases}
\dfrac{\partial u}{\partial \tau} = a_1\dfrac{\partial^2 u}{\partial \xi^2} + a_2\dfrac{\partial^2 u}{\partial \xi \partial \sigma} + a_3\dfrac{\partial^2 u}{\partial \sigma^2} + a_4\dfrac{\partial u}{\partial \xi} + a_5\dfrac{\partial u}{\partial \sigma} + a_6 u + a_7,\\
\qquad\qquad\qquad 0 \le \xi \le 1, \quad \sigma_l \le \sigma \le \sigma_u, \quad 0 \le \tau \le T,\\
u(\xi,\sigma,0) = 0, \qquad 0 \le \xi \le 1, \quad \sigma_l \le \sigma \le \sigma_u.
\end{cases} \tag{8.110}$$

Once we have $u(\xi,\sigma,\tau)$, we can get the value of the two-factor European call option by

$$\begin{aligned}
c(S,\sigma,t) &= \overline{c}(S,\ \sigma,\ t) + c_1(S,\ \sigma,\ t)\\
&= (S+P_m)u\left(\frac{S}{S+P_m},\ \sigma,\ T-t\right) + c_1(S,\ \sigma,\ t).
\end{aligned} \tag{8.111}$$

As we have pointed out in Sect. 2.4.4, when the reversion conditions (8.99) and (8.100), and conditions (ii) and (iii) in Theorem 2.2 hold, it has been proved that the final value problem (8.110) has a unique solution. In this case it is not difficult to design a well-posed numerical method to solve this problem.

The following is such a numerical method for problem (8.110). Let $u^n_{m,i}$ be the approximate value of u at $\xi = m\Delta\xi$, $\sigma = \sigma_l + i\Delta\sigma$, and $\tau = n\Delta\tau$, where $\Delta\xi = 1/M$, $\Delta\sigma = (\sigma_u - \sigma_l)/I$, and $\Delta\tau = 1/N$, M, I, N being integers. This partial differential equation can be discretized by the following scheme. If $\sigma \neq \sigma_l$ and $\sigma \neq \sigma_u$, at a point $(\xi_m, \sigma_i, \tau^{n+1/2})$ the partial differential equation in the problem (8.110) can be discretized by the following second-order approximation:

$$\begin{aligned}
&\frac{u^{n+1}_{m,i} - u^n_{m,i}}{\Delta\tau} \\
&= \frac{a_1}{2\Delta\xi^2}\left(u^{n+1}_{m+1,i} - 2u^{n+1}_{m,i} + u^{n+1}_{m-1,i} + u^n_{m+1,i} - 2u^n_{m,i} + u^n_{m-1,i}\right) \\
&\quad + \frac{a_2}{8\Delta\sigma\Delta\xi}(u^{n+1}_{m+1,i+1} - u^{n+1}_{m+1,i-1} - u^{n+1}_{m-1,i+1} + u^{n+1}_{m-1,i-1} \\
&\qquad\qquad + u^n_{m+1,i+1} - u^n_{m+1,i-1} - u^n_{m-1,i+1} + u^n_{m-1,i-1}) \\
&\quad + \frac{a_3}{2\Delta\sigma^2}(u^{n+1}_{m,i+1} - 2u^{n+1}_{m,i} + u^{n+1}_{m,i-1} \\
&\qquad\qquad + u^n_{m,i+1} - 2u^n_{m,i} + u^n_{m,i-1}) \\
&\quad + \frac{a_4}{4\Delta\xi}(u^{n+1}_{m+1,i} - u^{n+1}_{m-1,i} + u^n_{m+1,i} - u^n_{m-1,i}) \\
&\quad + \frac{a_5}{4\Delta\sigma}(u^{n+1}_{m,i+1} - u^{n+1}_{m,i-1} + u^n_{m,i+1} - u^n_{m,i-1}) \\
&\quad + \frac{a_6}{2}(u^{n+1}_{m,i} + u^n_{m,i}) + a_7, \qquad m = 0, 1, \cdots, M, \quad i = 1, 2, \cdots, I-1.
\end{aligned} \tag{8.112}$$

Here, all the coefficients a_1–a_7 should be evaluated at the point $(\xi_m, \sigma_i, \tau^{n+1/2})$ in order to guarantee second-order accuracy.

At the boundaries $\sigma = \sigma_l$ and $\sigma = \sigma_u$, due to $q = 0$ the partial differential equation in the problem (8.110) becomes

$$\frac{\partial u}{\partial \tau} = a_1\frac{\partial^2 u}{\partial \xi^2} + a_4\frac{\partial u}{\partial \xi} + a_5\frac{\partial u}{\partial \sigma} + a_6 u + a_7,$$

which possesses hyperbolic properties in the σ-direction. From the reversion conditions, we see $a_5 = p - (\lambda - \rho\sigma\xi)q = p \geq p - q\dfrac{\partial q}{\partial \sigma} \geq 0$ at the boundary $\sigma = \sigma_l$ and $a_5 = p - (\lambda - \rho\sigma\xi)q = p \leq p - q\dfrac{\partial q}{\partial \sigma} \leq 0$ at $\sigma = \sigma_u$. These facts tell us that the value of u on the boundaries $\sigma = \sigma_l$ and $\sigma = \sigma_u$ can be determined by the value of u inside the domain. Hence, we can approximate the partial differential equation in the problem (8.110) at the boundaries $\sigma = \sigma_l$ and $\sigma = \sigma_u$ by

$$\begin{aligned}
&\frac{u_{m,0}^{n+1}-u_{m,0}^{n}}{\Delta\tau}\\
=&\frac{a_1}{2\Delta\xi^2}(u_{m+1,0}^{n+1}-2u_{m,0}^{n+1}+u_{m-1,0}^{n+1}+u_{m+1,0}^{n}-2u_{m,0}^{n}+u_{m-1,0}^{n})\\
&+\frac{a_4}{4\Delta\xi}(u_{m+1,0}^{n+1}-u_{m-1,0}^{n+1}+u_{m+1,0}^{n}-u_{m-1,0}^{n})\\
&+\frac{a_5}{4\Delta\sigma}(-u_{m,2}^{n+1}+4u_{m,1}^{n+1}-3u_{m,0}^{n+1}-u_{m,2}^{n}+4u_{m,1}^{n}-3u_{m,0}^{n})\\
&+\frac{a_6}{2}(u_{m,0}^{n+1}+u_{m,0}^{n})+a_7,\\
&\qquad\qquad m=0,1,\cdots,M
\end{aligned} \tag{8.113}$$

and

$$\begin{aligned}
&\frac{u_{m,I}^{n+1}-u_{m,I}^{n}}{\Delta\tau}\\
=&\frac{a_1}{2\Delta\xi^2}(u_{m+1,I}^{n+1}-2u_{m,I}^{n+1}+u_{m-1,I}^{n+1}+u_{m+1,I}^{n}-2u_{m,I}^{n}+u_{m-1,I}^{n})\\
&+\frac{a_4}{4\Delta\xi}(u_{m+1,I}^{n+1}-u_{m-1,I}^{n+1}+u_{m+1,I}^{n}-u_{m-1,I}^{n})\\
&+\frac{a_5}{4\Delta\sigma}(3u_{m,I}^{n+1}-4u_{m,I-1}^{n+1}+u_{m,I-2}^{n+1}+3u_{m,I}^{n}-4u_{m,I-1}^{n}+u_{m,I-2}^{n})\\
&+\frac{a_6}{2}(u_{m,I}^{n+1}+u_{m,I}^{n})+a_7,\\
&\qquad\qquad m=0,1,\cdots,M
\end{aligned} \tag{8.114}$$

respectively. Here, $\frac{\partial u}{\partial \sigma}$ is discretized by one-sided second-order scheme in order for all the node points involved to be in the computational domain. a_1 and a_4–a_7 are also evaluated at the point $(\xi_m, \sigma_i, \tau^{n+1/2})$, $i = 0$ or I. When $u_{m,i}^n$, $m = 0, 1, \cdots, M$, $i = 0, 1, \cdots, I$ are known, from the difference scheme (8.112)–(8.114) we can determine $u_{m,i}^{n+1}$, $m = 0, 1, \cdots, M$, $i = 0, 1, \cdots, I$. The initial condition gives $u_{m,i}^0$, $m = 0, 1, \cdots, M$, $i = 0, 1, \cdots, I$. Therefore, we can do this procedure for $n = 0, 1, \cdots, N-1$ successively and finally find $u_{m,i}^N$, $m = 0, 1, \cdots, M$, $i = 0, 1, \cdots, I$.

In Fig. 8.15, the price of a European call option obtained in this way is given. The mesh used is $20\times20\times20$, where the first, second, and third numbers are M, I, and N, respectively. The parameters of the problem are given in the figure and the parameter functions are

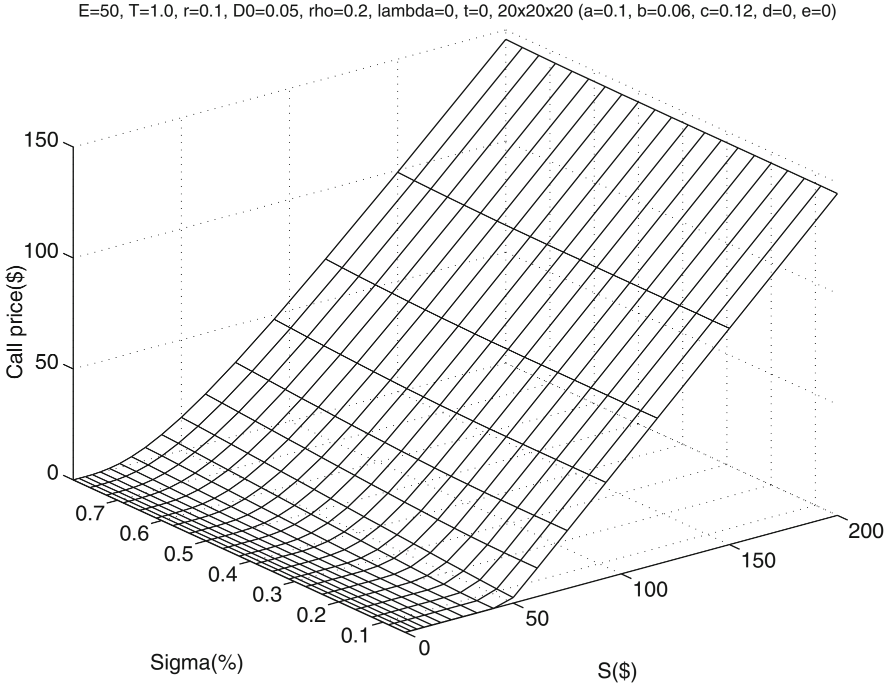

Fig. 8.15. The price of a two-factor European call option

$$\begin{cases} p = a(b-\sigma), & \sigma_l \le \sigma \le \sigma_u, \\ q = c\dfrac{1-\left(1-2\dfrac{\sigma-\sigma_l}{\sigma_u-\sigma_l}\right)^2}{1-0.975\left(1-2\dfrac{\sigma-\sigma_l}{\sigma_u-\sigma_l}\right)^2}\sigma, & \sigma_l \le \sigma \le \sigma_u, \\ \rho = 0.2, & \\ \lambda = d + e\sigma, & \sigma_l \le \sigma \le \sigma_u, \end{cases}$$

where $a = 0.1$, $b = 0.06$, $c = 0.12$, $d = 0$, $e = 0$, $\sigma_l = 0.05$, and $\sigma_u = 0.8$.

When the singularity-separating technique is not adopted, the scheme above can also be used. In that case,

$$a_7 = 0 \quad \text{and} \quad u(\xi, \sigma, 0) = \max(2\xi - 1, 0).$$

In order to give some idea about the performance of the method described in this subsection, we list the values of the option obtained by the method here with and without using extrapolation technique in Tables 8.24 and 8.25 for $S = 50$ and $\sigma = 0.2$. When these results were computed, for the first five coarser meshes, the linear systems were solved by the LU decomposition method and for the last three finer meshes, the Gauss–Seidel iteration was

Table 8.24. SSM with and without extrapolation technique

($S = 50$, $E = 50$, $T = 1$, $\sigma = 0.2$, $r = 0.1$, $D_0 = 0.05$,
$a = 0.1$, $b = 0.06$, $c = 0.12$, $d = 0$, and $e = 0$.
The exact solution is $4.848069\cdots$.)

	Without extrapolation		With extrapolation	
Meshes	Solution	|Errors|	Solution	|Errors|
$10 \times 10 \times 10$	4.8143085	0.033761	–	–
$20 \times 20 \times 20$	4.8361039	0.011966	4.8433691	0.004700
$40 \times 40 \times 40$	4.8460151	0.002054	4.8493188	0.001249
$80 \times 80 \times 80$	4.8476154	0.000454	4.8481488	0.000079
$160 \times 160 \times 160$	4.8479592	0.000110	4.8480738	0.000004
$320 \times 320 \times 320$	4.8480421	0.000027	4.8480697	Less than 10^{-6}
$640 \times 640 \times 640$	4.8480626	0.000007	4.8480694	Less than 10^{-6}
$960 \times 960 \times 960$	4.8480664	0.000003	4.8480694	Less than 10^{-6}

Table 8.25. Implicit method with and without extrapolation technique

($S = 50$, $E = 50$, $T = 1$, $\sigma = 0.2$, $r = 0.1$, $D_0 = 0.05$,
$a = 0.1$, $b = 0.06$, $c = 0.12$, $d = 0$, and $e = 0$.
The exact solution is $4.848069\cdots$.)

	Without extrapolation		With extrapolation	
Meshes	Solution	|Errors|	Solution	|Errors|
$10 \times 10 \times 10$	3.1774889	1.670580	–	–
$20 \times 20 \times 20$	4.2406270	0.607442	4.5950063	0.253063
$40 \times 40 \times 40$	4.7179697	0.130100	4.8770840	0.029015
$80 \times 80 \times 80$	4.8171183	0.030951	4.8501678	0.002098
$160 \times 160 \times 160$	4.8404088	0.007661	4.8481722	0.000103
$320 \times 320 \times 320$	4.8461590	0.001910	4.8480758	0.000006
$640 \times 640 \times 640$	4.8475923	0.000477	4.8480700	0.000001
$960 \times 960 \times 960$	4.8478575	0.000212	4.8480697	Less than 10^{-6}

used in order to solve the linear systems. From the tables, we see that the exact solution up to the sixth decimal place is 4.848069, which we obtained by a very fine mesh. Therefore, we can find out the errors of the results up to the sixth decimal place, which are also listed there. From the results without extrapolation in Table 8.24, it can be seen that this method has a second order accuracy because the error is reduced by a factor of about 1/4 when the mesh size is reduced by a factor of 1/2 (see the errors for the meshes $20 \times 20 \times 20, \cdots, 640 \times 640 \times 640$). Table 8.24 also shows that for a $20 \times 20 \times 20$ mesh with extrapolation, the error relative to E is $0.0047/50 \approx 10^{-4}$ and that the error relative to the option value is $0.0047/4.848069 \approx 10^{-3}$. In practice, requiring such accuracy is reasonable. The CPU time on a Pentium III 800 MHz computer is 0.07 s. If the singularity-separating technique is not used, in order to reach a similar accuracy, the mesh is between $40\times40\times40$ and $80\times80\times80$ and the CPU time is between 1 to 8 s and close to 8 s, respectively.

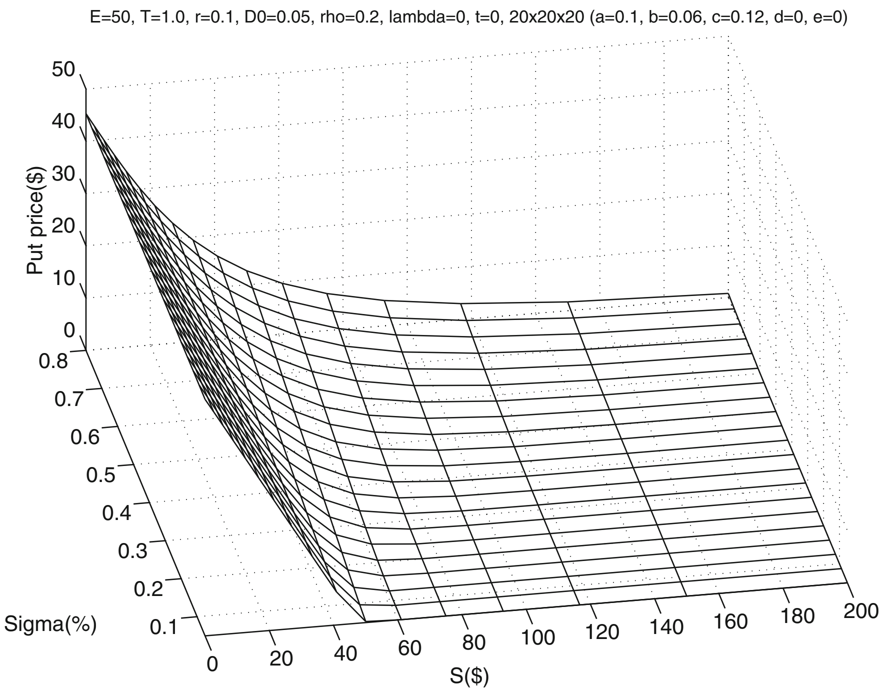

Fig. 8.16. The price of a two-factor European put option

Noticing

$$\frac{\partial p_1}{\partial \sigma} = \frac{\partial c_1}{\partial \sigma}, \quad \frac{\partial^2 p_1}{\partial \sigma^2} = \frac{\partial^2 c_1}{\partial \sigma^2}, \quad \frac{\partial^2 p_1}{\partial S \partial \sigma} = \frac{\partial^2 c_1}{\partial S \partial \sigma},$$

where p_1 is the price of the one-factor put option, we see that the difference between the two-factor and one-factor put options is also the solution of the problem (8.110). Therefore, in order to have the price of a European put option, we proceed as follows. First solving problem (8.110), then we can have the put price by

$$p(S, \sigma, t) = (S + P_m)u\left(\frac{S}{S + P_m},\ \sigma,\ T - t\right) + p_0(S,\ \sigma,\ t).$$

In Fig. 8.16, the price of a two-factor European put option obtained by this way is shown. The parameters of the problem and the parameter functions are the same as these for the two-factor European call option. Also, for European vanilla options, both the put–call parity relation and the put–call symmetric relation exist. The put–call parity relation still is

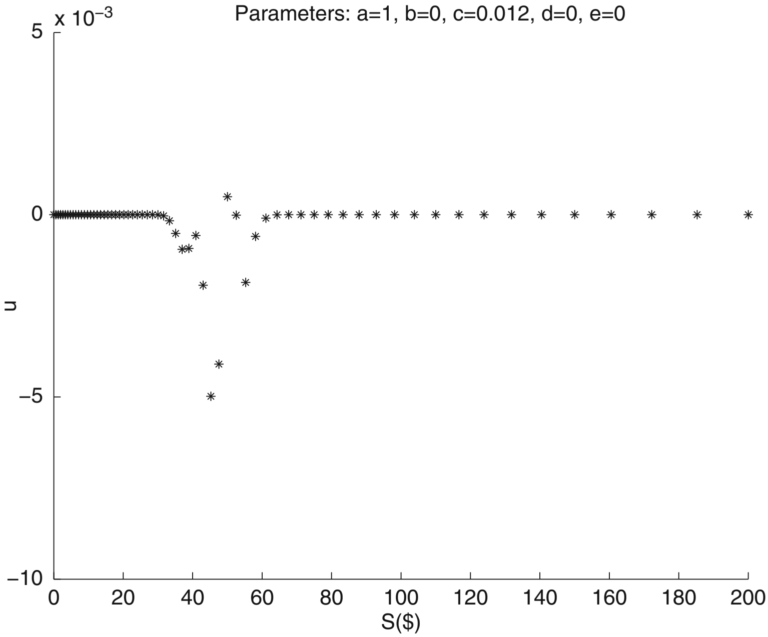

Fig. 8.17. An unstable solution of implicit schemes (Variation of u with respect to S on the line $\sigma = 0.05$ at $t = 0$. $E = 50$, $T = 2$, $r = 0.1$, and $D_0 = 0.05$. The solution is on a $80 \times 40 \times 80$ mesh.)

$$p(S, \sigma, t) = c(S, \sigma, t) - S\mathrm{e}^{-D_0(T-t)} + E\mathrm{e}^{-r(T-t)}. \tag{8.115}$$

When we calculate put option prices without using SSM, this relation can be used to check the correctness of the code to some extent. First, we compute the prices of a call option and a put option with the same parameters. Then, the results are substituted into the put–call parity relation to see if it holds. If it holds with a small error, then the code most likely gives correct results; if the relation does not hold, then the code must have some problems.

Finally, we give an example to explain that if the reversion conditions are not satisfied, then the final-value problem (8.110) is not well-posed and we cannot determine the solution using only the partial differential equation and the final condition in the problem (8.110). Consider a problem with $a = 1$, $b = 0$, $c = 0.012$, $d = 0$, $e = 0$, and $T = 2$. The other parameters are the same as before. We still use the numerical method above to find the numerical solution. In Fig. 8.17, we give the variation of u with S on the line $\sigma = \sigma_l$ at time $t = 0$. From there, we can see some "nonphysical" oscillations, which means that the computation is unstable even though an implicit scheme is used. This indicates that for this case, the solution is not determined only by the partial differential equation and final condition. The reason is that a proper boundary condition is needed at the boundary $\sigma = \sigma_l$ because the inequality condition in the condition (8.101) is not satisfied at $\sigma = \sigma_l$ due to $b = 0 < \sigma_l = 0.05$. If a reasonable condition cannot be given, then an artificial boundary condition has to be added. If the artificial boundary condition is not proper, then one will encounter some difficulty during computation.

8.3.7 Two-Factor Convertible Bonds with $D_0 = 0$

If $D_0 = 0$, then the convertible bond problem has no free boundary, and the problem has the same form as a European-style two-factor derivative problem does. The only difference is that the another random variable is the spot interest rate instead of the volatility. In order to make numerical methods more efficient, there are also two things we need to deal with. The first thing is the weak singularity generated by a discontinuous derivative of the payoff function. In order to separate this singularity, we can calculate numerically the difference between the values of two-factor and one-factor convertible bonds for the case $D_0 = 0$. We will not give the method here because it is similar to the method for two-factor options and the method for two-factor convertible bonds with $D_0 \neq 0$, which will be given in Sect. 9.1.2. The second thing is that the problem is defined on an infinite domain. Through a transformation similar to expression (8.109), the problem can be converted into a problem similar to problem (8.110) and the solution can be obtained by numerical methods efficiently. The details are similar to what we have done for two-factor options and left for readers to complete (Problem 26).

8.4 Pseudo-Spectral Methods

After the singularity-separating method is used, the solution to be computed numerically (the difference between the original unknown solution and a closed-form solution) is quite smooth. In this case, the pseudo-spectral method might be another good choice for computing the difference numerically. The basic principle of the method was discussed in Chap. 6. In this subsection, we give some details when the pseudo-spectral method is applied to problems (7.1) and (7.2).

Let us take $M + 1$ grid points x_m, $m = 0, 1, \cdots, M$, on $[0, 1]$ and assume that the values of a function $u(x)$ for any x_m are given. Then, the values of the derivatives of $u(x)$ can be expressed as linear combinations of $u(x_m)$. Especially, if x_m is given by the expression (6.6), then the first derivative is approximated by the formula (6.7):

$$\frac{\partial u}{\partial x}(x_m) = \sum_{i=0}^{M} D_{x,m,i} u(x_i)$$

and the second derivative by expression (6.9):

$$\frac{\partial^2 u}{\partial x^2}(x_m) = \sum_{i=0}^{M} D_{xx,m,i} u(x_i),$$

where $D_{x,m,i}$ and $D_{xx,m,i}$ are given by the formulae (6.8) and (6.10), respectively. Consequently, the PDE in the problem (7.2) can be approximated by

$$\begin{aligned}&\frac{u^{n+1}(x_m) - u^n(x_m)}{\Delta\tau}\\ =&\frac{1}{2}\left[a_m^{n+\frac{1}{2}} \sum_{i=0}^{M} D_{xx,m,i} u^{n+1}(x_i) + b_m^{n+\frac{1}{2}} \sum_{i=0}^{M} D_{x,m,i} u^{n+1}(x_i) + c_m^{n+\frac{1}{2}} u^{n+1}(x_m)\right]\\ &+\frac{1}{2}\left[a_m^{n+\frac{1}{2}} \sum_{i=0}^{M} D_{xx,m,i} u^n(x_i) + b_m^{n+\frac{1}{2}} \sum_{i=0}^{M} D_{x,m,i} u^n(x_i) + c_m^{n+\frac{1}{2}} u^n(x_m)\right]\\ &+g_m^{n+\frac{1}{2}},\\ &\qquad m = 0, 1, \cdots, M, \end{aligned} \tag{8.116}$$

where $u^{n+1}(x_m) = u\left(x_m, (n+1)\Delta\tau\right)$. Just like the implicit finite-difference method, if $u^n(x_m)$, $m = 0, 1, \cdots, M$ are given, we can determine $u^{n+1}(x_m)$, $m = 0, 1, \cdots, M$ by the linear system (8.116). However, the matrix of the current system is a full matrix, and the CPU time needed for solving this system is longer than the implicit finite-difference method if M is the same. When the solution is very smooth, only a small M might be needed in order to get a satisfying result. In such a case, its performance could be better than the implicit finite-difference method. This numerical method is referred to as the implicit pseudo-spectral method for one-dimensional problems.

Table 8.26. Pseudo-spectral methods

($S = 95$, $T = 1$, $E = 100$, $\sigma = 0.25$, $r = 0.1$, $D_0 = 0$,
$f(t) = (0.9 - 0.05t)E$, $g(t) = (1.6 + 0.05t)E$, and
the rebate $= g(t) - E$. The exact solution is $6.43129316\cdots$.)

	Without SSM			With SSM		
Meshes	Solutions	|Errors|	CPU	Solution	|Errors|	CPU
7 × 50	6.454922	0.023629	0.0007	6.431842	0.000549	0.0014
7 × 100	6.454789	0.023596	0.0015	6.431438	0.000145	0.0022
7 × 200	6.454755	0.023462	0.0028	6.431426	0.000133	0.0043
8 × 50	6.438364	0.007071	0.0010	6.431351	0.000058	0.0014
8 × 100	6.438227	0.006934	0.0019	6.431305	0.000012	0.0028
8 × 200	6.438193	0.006900	0.0038	6.431293	0.0000005	0.0058
9 × 50	6.404701	0.026592	0.0013	6.431350	0.000057	0.0021
9 × 100	6.404555	0.026738	0.0024	6.431304	0.000011	0.0036
9 × 200	6.404518	0.002678	0.0044	6.431292	0.000001	0.0065

If we consider problem (7.1), the only difference is that instead of the partial differential equation being discretized at x_m, $m = 0, 1, \cdots, M$, now it is discretized at x_m, $m = 1, 2, \cdots, M-1$, and these equations and the boundary conditions given in the problem (7.1) form the entire system we need.

Table 8.26 gives some results obtained by the implicit pseudo-spectral method described above with $M = 7$, 8, 9. The corresponding time steps used are $\Delta\tau = 1/N$, $N = 50, 100, 200$, respectively. In the column "Meshes," $M \times N$ is given. The problem is a double barrier call option whose lower and upper knock-out boundaries are $f(t) = (0.9 - 0.05t)E$ and $g(t) = (1.6 + 0.05t)E$. The other parameters are given in the table. When the computation is done, the independent variable x is defined by

$$x = \frac{\dfrac{S}{E+S} - \dfrac{f(t)}{E+f(t)}}{\dfrac{g(t)}{E+g(t)} - \dfrac{f(t)}{E+f(t)}}.$$

The exact solution for this case is $6.43129316\cdots$, where the nine digits given are correct. When we have the exact solution, we can have the error of the solution, which is also given. The CPU time in seconds is also shown in order to see the performance.

In Table 8.26, both the results with and without the SSM are listed. From there, we can see that if the SSM is not used, the result obtained by using higher order polynomials might be worse than the results obtained by using lower order polynomials. However, it shows that when the pseudo-spectral method is combined with the singularity-separating technique, the higher the polynomial order, the better the result. Hence, the result of the pseudo-spectral method with the singularity-separating technique is much better than without it. Consequently, if the pseudo-spectral method is adopted, then combining it with SSM is essential. In Figs. 8.8 and 8.9, the functions computed when SSM is used and not used are shown, respectively. As pointed out, the functions in Fig. 8.9 are not as smooth as those in Fig. 8.8, especially, the derivative of the function for $t = 1$ in Fig. 8.9 is discontinuous. Therefore, the pseudo-spectral method does not provide a good performance for this case. However, if the singularity-separating technique is used, then the functions determined numerically are always very smooth, which can be seen from Fig. 8.8. In this case, the performance of the pseudo-spectral method is very good, and in certain cases it may even be better than the second-order implicit finite-difference methods because a pseudo-spectral method can be understood as a high-order difference method. In Fig. 8.18, the performances of the implicit finite-difference method and the implicit pseudo-spectral method with the singularity-separating technique are compared, which confirms this conclusion.

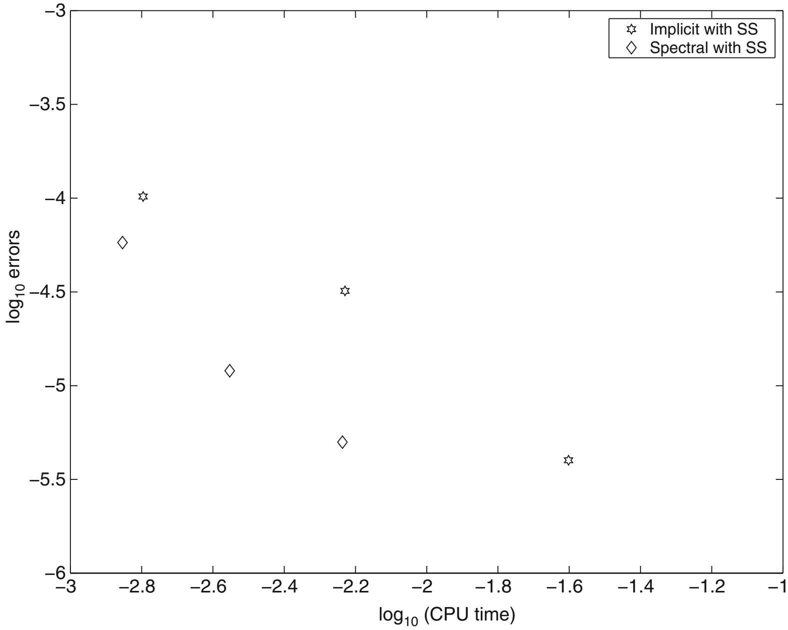

Fig. 8.18. Comparison between an implicit scheme and a pseudo-spectral method

The idea described here also works for double moving barrier put options with rebates and many other cases. For details, see the paper [92] by Zhu and Abifaker.

In Sect. 8.3, we pointed out that two-dimensional European-style derivative problems and American-style derivative problems that do not have free boundaries could be written in the form (8.110). The pseudo-spectral method can also be applied to such a problem. When this method is combined with the singularity-separating method, a good performance can be expected. For details of this method for two-dimensional case, see Chap. 9.

Problems

Table 8.27. Problems and sections

Problems	Sections	Problems	Sections	Problems	Sections
1–7	8.1	8–15	8.2	16–27	8.3
28–29	8.4				

1. *Suppose that we determine the price of an American vanilla call/put option through solving the following problem:

$$\begin{cases} \min\left(\dfrac{\partial u}{\partial \bar{\tau}} - \dfrac{\partial^2 u}{\partial x^2},\ u(x,\bar{\tau}) - g(x,\bar{\tau})\right) = 0, & -\infty < x < \infty,\ \bar{\tau} \ge 0, \\ u(x,0) = g(x,0), & -\infty < x < \infty, \end{cases}$$

where

$$g(x,\bar{\tau}) = \max\left(\pm(\mathrm{e}^{x+(2D_0/\sigma^2+1)\bar{\tau}} - \mathrm{e}^{2r\bar{\tau}/\sigma^2}), 0\right).$$

Describe a numerical method for solving this problem by using an explicit scheme.

2. As we know, an American lookback strike put option is the solution of the following linear complementarity problem:

$$\begin{cases} \min\left(-\dfrac{\partial W}{\partial t} - \mathbf{L}_\eta W,\ W - \max(\eta-\beta,0)\right) = 0, & 1 \le \eta, \quad t \le T, \\ W(\eta,T) = \max(\eta-\beta,0), & 1 \le \eta, \\ \dfrac{\partial W}{\partial \eta}(1,t) = 0, & t \le T, \end{cases}$$

where we assume that $\beta \ge 1$ and the operator $\mathbf{L}_\eta$ to be defined by

$$\mathbf{L}_\eta \equiv \frac{1}{2}\sigma^2\eta^2\frac{\partial^2}{\partial \eta^2} + (D_0 - r)\,\eta\frac{\partial}{\partial \eta} - D_0.$$

Convert this problem into a problem on $[0,1]$ and with an initial condition, and design an explicit method with a first-order accuracy in time and a second-order accuracy in space for solving the new problem.

3. Suppose that ψ is a binomial random variable and its two values are ψ_0 and ψ_1. Show the following:
   (a) If $\mathrm{E}[\psi] = 0$ and $\mathrm{E}[\psi^2] = 1$, then $\psi_0\psi_1 = -1$.
   (b) If $\mathrm{E}[\psi] = 0$ and $\psi_0\psi_1 = -1$, then $\mathrm{E}[\psi^2] = 1$.
   (c) If $\mathrm{E}[\psi] = 0$ and $\psi_0\psi_1 = -1 + O(\Delta t)$, then $\mathrm{E}[\psi^2] = 1 + O(\Delta t)$.
4. (a) *Derive the binomial methods proposed by Cox, Ross, and Rubinstein and by McDonald.
   (b) *Can the parameter p in the Cox–Ross–Rubinstein binomial method always represent a probability? Find out when it can and when it cannot. Can the parameter p given in the book by McDonald always represent a probability? Find out when it can and when it cannot.
5. From the Black–Scholes equation, we know that when a derivative security is priced, the value of the stock price at time t^n, S_n, and the value at time t^{n+1}, S_{n+1}, have the following relations:

$$\mathrm{E}_D[S_{n+1}] = \mathrm{e}^{(r-D_0)\Delta t}S_n$$

and

$$\mathrm{E}_D\left[S_{n+1}^2\right] = \mathrm{e}^{[2(r-D_0)+\sigma^2]\Delta t}S_n^2,$$

where $\Delta t = t^{n+1} - t^n$ (see Problem 39 of Chap. 2). Thus if the possible values of S_{n+1} are $S_{n+1,0} = S_n/u$ and $S_{n+1,1} = uS_n$, and the probabilities of S_{n+1} being $S_{n+1,0}$ and the probabilities of S_{n+1} being $S_{n+1,1}$ are $1-p$ and p, respectively, then in a binomial method, u and p should be determined by

$$\begin{cases} pu + (1-p)\,u^{-1} = \mathrm{e}^{(r-D_0)\Delta t}, \\ pu^2 + (1-p)\,u^{-2} = \mathrm{e}^{[2(r-D_0)+\sigma^2]\Delta t}. \end{cases}$$

If Δt is very small, this problem can be approximated by

$$\begin{cases} pu + (1-p)\,u^{-1} = 1 + (r-D_0)\Delta t, \\ pu^2 + (1-p)\,u^{-2} = 1 + [2(r-D_0)+\sigma^2]\Delta t. \end{cases}$$

(a) Find u and p for both cases (suppose $u > 1$).
(b) Consider a more general system

$$\begin{cases} pu + (1-p)\,u^{-1} = 1 + (r-D_0)\Delta t + O(\Delta t^2), \\ pu^2 + (1-p)\,u^{-2} = 1 + [2(r-D_0)+\sigma^2]\Delta t + O(\Delta t^2). \end{cases}$$

Show that if u and p are determined by such a system, then we always have

$$u = \mathrm{e}^{\sigma\sqrt{\Delta t}} + O\left(\Delta t^{3/2}\right)$$

and

$$p = \frac{1}{2}\left[1 + \frac{\sqrt{\Delta t}}{\sigma}\left(r - D_0 - \frac{1}{2}\sigma^2\right)\right] + O\left(\Delta t^{3/2}\right).$$

(Hint: When you derive the expression for p, write u as $1 + \sigma\sqrt{\Delta t} + \sigma^2\Delta t/2 + c\Delta t^{3/2} + O(\Delta t^2)$, c being a constant.)

6. *Describe the binomial methods for solving American vanilla call/put options
7. *Show that the Cox–Ross–Rubinstein binomial method for European options almost is an explicit difference scheme for the following problem:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2\dfrac{\partial^2 V}{\partial y^2} + \left(r - D_0 - \dfrac{1}{2}\sigma^2\right)\dfrac{\partial V}{\partial y} - rV = 0, \\ \qquad\qquad -\infty < y < \infty, \quad t \le T, \\ V(y,T) = \max(\pm(\mathrm{e}^y - 1), 0), \qquad -\infty < y < \infty, \end{cases}$$

where $y = \ln S$, S being the price of the stock and assume $E = 1$.

8. Show that the relation

$$V(S, t_i^-) = V(S - D_i(S), t_i^+)$$

becomes

$$\overline{V}(\xi,\ \tau_i^+)$$
$$= \left[1 - D_i\left(\frac{P_m\xi}{1-\xi}\right)\frac{1-\xi}{P_m}\right]\overline{V}\left(\frac{P_m\xi - D_i\left(\frac{P_m\xi}{1-\xi}\right)(1-\xi)}{P_m - D_i\left(\frac{P_m\xi}{1-\xi}\right)(1-\xi)},\ \tau_i^-\right)$$

under the transformation

$$\begin{cases} \xi = \dfrac{S}{S+P_m}, \\ \tau = T - t, \\ \overline{V}(\xi, \tau) = \dfrac{V(S,t)}{S+P_m}. \end{cases}$$

9. (a) Show that the jump condition

$$W\left(\eta, t_i^-\right) = W\left(\eta + J, t_i^+\right)$$

becomes

$$\overline{u}\left(\xi,\ \tau_i^+\right) = \frac{\left|\frac{P_m\xi}{1-|\xi|} + J\right| + P_m}{\left|\frac{P_m\xi}{1-|\xi|}\right| + P_m}\overline{u}\left(\frac{\frac{P_m\xi}{1-|\xi|} + J}{\left|\frac{P_m\xi}{1-|\xi|} + J\right| + P_m},\qquad \tau_i^-\right)$$

under the transformation

$$\begin{cases} \xi = \dfrac{\eta}{|\eta| + P_m}, \\ \tau = T - t, \\ W(\eta, t) = (|\eta| + P_m)\overline{u}(\xi, \tau), \end{cases}$$

where $P_m > 0$.

(b) Suppose that the jump condition for $W(\eta, t)$ is

$$W\left(\eta, t_i^-\right) = W\left(\eta + J, t_i^+\right)$$

and introduce the transformation

$$\begin{cases} \xi = \dfrac{\eta}{|\eta| + P_m(\eta)}, \\ \tau = T - t, \\ W(\eta, t) = (|\eta| + P_m(\eta))\overline{u}(\xi, \tau), \end{cases}$$

where

$$P_m(\eta) = \begin{cases} P_{mr}, & \text{if } \eta > 0, \\ \\ P_{ml}, & \text{if } \eta < 0. \end{cases}$$

Here $P_{mr} > 0$ and $P_{ml} > 0$. Find the jump condition for $\overline{u}(\xi, \tau)$.

10. *Suppose that we determine the price of an American vanilla call/put option through solving the following problem:

$$\begin{cases} \min\left(\dfrac{\partial \overline{V}}{\partial \tau} - \mathbf{L}_\xi \overline{V},\ \overline{V}(\xi,\tau) - \max(\pm(2\xi-1),0)\right) = 0, & 0 \le \xi \le 1, \\ & \tau \ge 0, \\ \overline{V}(\xi,0) = \max(\pm(2\xi-1),0), & 0 \le \xi \le 1, \end{cases}$$

where

$$\mathbf{L}_\xi = \frac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1-\xi)^2\frac{\partial^2}{\partial \xi^2} + (r - D_0)\xi(1-\xi)\frac{\partial}{\partial \xi} - [r(1-\xi) + D_0\xi].$$

Describe a numerical method for solving this problem by using a second order implicit scheme. (Discuss the discretization of the problem only.)

11. As we know, an American average strike call option is the solution of the following linear complementarity problem:

$$\begin{cases} \min\left(-\dfrac{\partial W}{\partial t} - \mathbf{L}_{a,t}W,\ W(\eta,t) - \max(\alpha - \eta, 0)\right) = 0, & 0 \le \eta,\ t \le T, \\ W(\eta, T) = \max(\alpha - \eta, 0), & 0 \le \eta, \end{cases}$$

where $\alpha \approx 1$ and

$$\mathbf{L}_{a,t} = \frac{1}{2}\sigma^2\eta^2\frac{\partial^2}{\partial \eta^2} + \left[(D_0 - r)\eta + \frac{1-\eta}{t}\right]\frac{\partial}{\partial \eta} - D_0.$$

Convert this problem into a problem on a finite domain and with an initial condition, and design an implicit second-order method for solving this new problem. (Discuss the discretization of the problem only.)

12. Based on the partial differential equation

$$\frac{\partial \overline{V}}{\partial \tau} = \frac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1-\xi)^2\frac{\partial^2 \overline{V}}{\partial \xi^2} + r\xi(1-\xi)\frac{\partial \overline{V}}{\partial \xi} - r(1-\xi)\overline{V},$$

design an implicit method for the LC problem of American options with discrete dividends.

13. *Suppose that the scheme

$$\begin{aligned}
&\frac{v_m^{n+1} - v_m^n}{\Delta\tau} \\
&= \frac{1}{4}\bar{\sigma}_m^2 \xi_m^2 (1-\xi_m)^2 \left(\frac{v_{m+1}^{n+1} - 2v_m^{n+1} + v_{m-1}^{n+1}}{\Delta\xi^2} + \frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{\Delta\xi^2} \right) \\
&\quad + \frac{1}{2}(r - D_0)\xi_m (1-\xi_m) \left(\frac{v_{m+1}^{n+1} - v_{m-1}^{n+1}}{2\Delta\xi} + \frac{v_{m+1}^n - v_{m-1}^n}{2\Delta\xi} \right) \\
&\quad - \frac{1}{2}[r(1-\xi_m) + D_0\xi_m](v_m^{n+1} + v_m^n)
\end{aligned}$$

is used for solving an American call option problem. Design a projected direct method, which you think is most accurate, to find the solution at each time step.

14. *Consider the following LC problem:

$$\begin{cases} \min\left(\dfrac{\partial u}{\partial \bar{\tau}} - \dfrac{\partial^2 u}{\partial x^2},\ u(x,\bar{\tau}) - g(x,\bar{\tau}) \right) = 0, & -\infty < x < \infty,\ \bar{\tau} \ge 0, \\ u(x,0) = g(x,0), & -\infty < x < \infty, \end{cases}$$

where

$$g(x,\bar{\tau}) = \max\left(\pm\left(\mathrm{e}^{x+(2D_0/\sigma^2+1)\bar{\tau}} - \mathrm{e}^{2r\bar{\tau}/\sigma^2}\right), 0 \right).$$

Suppose an implicit finite-difference method based on such a formulation is used for solving an American option problem. Design an iteration method similar to the SOR method for a linear system to find the solution of the problem at each time step.

15. *The heat equation

$$\frac{\partial u}{\partial \tau} = a\frac{\partial^2 u}{\partial x^2}$$

can be approximated by the explicit first-order scheme

$$\frac{u_m^{n+1} - u_m^n}{\Delta\tau} = a\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2}$$

or the implicit second-order scheme (the Crank–Nicolson scheme)

$$\frac{u_m^{n+1} - u_m^n}{\Delta\tau} = \frac{a}{2}\left(\frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{\Delta x^2} + \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2} \right).$$

When do we choose the explicit first-order scheme and when do we use the implicit second-order scheme? Why should we choose the implicit second-order scheme if we need highly accurate results?

16. (a) Find a closed-form solution of the problem:

$$\begin{cases} \dfrac{\partial c_u}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 c_u}{\partial S^2} + (r - D_0) S \dfrac{\partial c_u}{\partial S} - r c_u = 0, \quad 0 \le S, \quad 0 \le t \le T, \\ c_u(S,T) = \begin{cases} \max(S-E,0), \text{ if } & 0 \le S < g(T), \\ 0, \qquad\qquad\quad \text{ if } & g(T) \le S. \end{cases} \end{cases}$$

Here we assume $g(T) > E$.

(b) Consider the following European barrier option problem:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r - D_0) S \dfrac{\partial V}{\partial S} - rV = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad f(t) \le S \le g(t), \quad 0 \le t \le T, \\ V(S,T) = \max(S-E,0), \qquad f(T) \le S \le g(T), \\ V(f(t),t) = 0, \qquad\qquad\qquad 0 \le t \le T, \\ V(g(t),t) = 0, \qquad\qquad\qquad 0 \le t \le T, \end{cases}$$

where $S = f(t)$ and $S = g(t)$ are the locations of the lower and upper barriers with $f(t) < E$ and $g(t) > E$. Assume that we need to find the solution by numerical methods. Design a SSM for this problem based on the result given in part (a) . (Here the problem can be defined on a non-rectangular domain.)

17. Suppose that η_1, η_2, and p are given, where $0 < \eta_1 < \eta_2$, $2\eta_2 - \eta_1 < 1$, and $p > 1$. Set $\eta_3 = 2\eta_2 - \eta_1$ and let $f(\eta)$ be a function on $[0,1]$ satisfying the condition $f(0) = 0$ and its derivative be equal to

$$f_\eta(\eta) = \begin{cases} d, & 0 \le \eta < \eta_1, \\ a(\eta - \eta_2)^4 + b(\eta - \eta_2)^2 + c, & \eta_1 \le \eta < \eta_3, \\ d, & \eta_3 \le \eta \le 1. \end{cases}$$

Here $d = a(\eta_2 - \eta_1)^4 + b(\eta_2 - \eta_1)^2 + c$, which guarantees $f_\eta(\eta)$ is continuous at $\eta = \eta_1$. From the definition of η_3, we know $\eta_2 - \eta_1 = \eta_3 - \eta_2$, so $f_\eta(\eta)$ is also continuous at $\eta = \eta_3$.

(a) Assume that the following three conditions hold:

(i) $\dfrac{f_\eta(\eta_2)}{f_\eta(\eta_1)} = \dfrac{c}{a(\eta_2 - \eta_1)^4 + b(\eta_2 - \eta_1)^2 + c} = p,$

(ii) $f_{\eta\eta}(\eta_1) = 4a(\eta_1 - \eta_2)^3 + 2b(\eta_1 - \eta_2) = 0,$

(iii) $f(1) = 1.$

Find the expressions of a, b, and c as functions of η_1, η_2, η_3, and p and show that $f(\eta)$ is an increasing function on $[0,1]$ in this case.

(b) Find the expression of $f(\eta)$.
(c) When solving a PDE/OPE problem, a variable mesh can be realized by using transformation. Suppose that the independent variable in a PDE/ODE problem is η and a new variable is introduced by setting $\xi = f(\eta)$. How should we choose the parameters in the function $f(\eta)$ if we want to let the mesh size in the region near the point $\eta = 0.4$ is about 1/10 of the mesh size in the regions $[0, 0.2]$ and $[0.6, 1]$?

18. Let $\bar{c}(\xi, \tau) = c(S, t)/(S + P_m)$ and $\bar{p}(\xi, \tau) = p(S, t)/(S + P_m)$, where $\xi = S/(S + P_m)$ and $\tau = T - t$. Derive the expressions of $\bar{c}(\xi, \tau)$ and $\bar{p}(\xi, \tau)$ and find the limits of $\bar{c}(\xi, \tau)$ and $\bar{p}(\xi, \tau)$ as ξ tends to 0 and 1. Also write down the formulae for the case $P_m = E$.
19. Suppose that $V(S, t)$ satisfies the following jump condition at $t = t_i$:

$$V(S, t_i^-) = V(S - D_i(S), t_i^+)$$

and that $V_0(S, t)$ is continuous at $t = t_i$. Define

$$\begin{cases} \xi = \dfrac{S}{S + P_m}, \\ \tau = T - t, \\ u(\xi, \tau) = \dfrac{V(S, t) - V_0(S, t)}{S + P_m}, \\ u_0(\xi, \tau) = \dfrac{V_0(S, t)}{S + P_m}, \end{cases}$$

where P_m is a positive number. Show that the following jump condition for $u(\xi, \tau)$ holds:

$$\begin{aligned} & u\left(\xi, \tau_i^+\right) \\ = & \left[1 - \frac{1-\xi}{P_m} D_i\left(\frac{\xi P_m}{1-\xi}\right)\right] \left[u\left(\frac{P_m\xi - D_i\left(\dfrac{\xi P_m}{1-\xi}\right)(1-\xi)}{P_m - D_i\left(\dfrac{\xi P_m}{1-\xi}\right)(1-\xi)}, \tau_i^-\right)\right. \\ & \left. + u_0\left(\frac{P_m\xi - D_i\left(\dfrac{\xi P_m}{1-\xi}\right)(1-\xi)}{P_m - D_i\left(\dfrac{\xi P_m}{1-\xi}\right)(1-\xi)}, \tau_i\right)\right] - u_0\left(\xi, \tau_i\right). \end{aligned}$$

20. Design a SSM for European vanilla options with discrete dividends and a constant volatility, and formulate the problem as a problem defined on a finite domain and with an initial condition.
21. *Design a SSM for Bermudan options with variable volatilities and formulate the problem as a problem defined on a finite domain and with an initial condition.

22. Suppose r and D_0 are constant and $\sigma = \sigma(S)$. Derive the symmetry relations for Bermudan options.
23. *Find a transformation to convert an average price call option problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r - D_0)S\dfrac{\partial V}{\partial S} + \dfrac{S}{T}\dfrac{\partial V}{\partial I} - rV = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 \le S, \quad 0 \le I, \quad t \le T, \\ V(S, I, T) = \max(I - E, 0), \qquad\qquad 0 \le S, \quad 0 \le I, \end{cases}$$

where

$$I = \frac{1}{T}\int_0^t S(\tau)d\tau,$$

into the problem

$$\begin{cases} \dfrac{\partial f}{\partial \tau} - \dfrac{1}{2}\sigma^2 \left[\xi - \dfrac{1}{(r - D_0)T}(1 - \mathrm{e}^{-(r-D_0)\tau})\right]^2 \dfrac{\partial^2 f}{\partial \xi^2} = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad -\infty < \xi < \infty, \quad 0 \le \tau \le T, \\ f(\xi, 0) = \max(\xi, 0), \qquad\qquad -\infty < \xi < \infty. \end{cases}$$

24. Find a closed-form solution of the problem:

$$\begin{cases} \dfrac{\partial \tilde{f}_0}{\partial \tau} - \dfrac{\sigma^2}{2(r - D_0)^2 T^2}\left(1 - \mathrm{e}^{-(r-D_0)\tau}\right)^2 \dfrac{\partial^2 \tilde{f}_0}{\partial \xi^2} = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad -\infty < \xi < \infty, \quad 0 \le \tau \le T, \\ \tilde{f}_0(\xi, 0) = \max(\xi, 0), \qquad\qquad -\infty < \xi < \infty. \end{cases}$$

25. Convert the problem

$$\begin{cases} \dfrac{\partial f_1}{\partial \tau} - \dfrac{1}{2}\sigma^2 \left[\xi - \dfrac{1}{(r - D_0)T}\left(1 - \mathrm{e}^{-(r-D_0)\tau}\right)\right]^2 \dfrac{\partial^2 f_1}{\partial \xi^2} = \dfrac{\sigma^2 \xi\, \mathrm{e}^{-\xi^2/4\tau_1}}{4\sqrt{\pi\tau_1}} \\ \quad \times \left[\xi - \dfrac{2}{(r - D_0)T}\left(1 - \mathrm{e}^{-(r-D_0)\tau}\right)\right], \qquad -\infty < \xi < \infty, \quad 0 \le \tau \le T, \\ f_1(\xi, 0) = 0, \qquad\qquad -\infty < \xi < \infty. \end{cases}$$

into a problem defined on $[0, 1]$ and with an initial condition, and design an implicit second-order scheme for the new problem.
26. By using the transformation

$$\begin{cases} \xi = \dfrac{S}{S + P_m}, \\ r = r, \\ \tau = T - t, \\ u(\xi, r, \tau) = \dfrac{B_c(S, r, t)}{n\,(S + P_m)}, \end{cases}$$

the two-factor convertible bond problem for non-dividend stocks

$$\begin{cases} \dfrac{\partial B_c}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 B_c}{\partial S^2} + \rho\sigma S w \dfrac{\partial^2 B_c}{\partial S \partial r} + \dfrac{1}{2} w^2 \dfrac{\partial^2 B_c}{\partial r^2} + rS\dfrac{\partial B_c}{\partial S} \\ \qquad + (u - \lambda w)\dfrac{\partial B_c}{\partial r} - rB_c + kZ = 0, \\ \qquad\qquad\qquad 0 \le S, \quad r_l \le r \le r_u, \quad 0 \le t \le T, \\ B_c(S, r, T) = \max(Z, nS), \qquad 0 \le S, \quad r_l \le r \le r_u \end{cases}$$

can be converted into a problem on a finite domain with a bounded final condition. The one-factor convertible zero-coupon bond problem for non-dividend stocks

$$\begin{cases} \dfrac{\partial b_c}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 b_c}{\partial S^2} + rS\dfrac{\partial b_c}{\partial S} - rb_c = 0, & 0 \le S, \quad 0 \le t \le T, \\ b_c(S, r, T) = \max(Z, nS), & 0 \le S \end{cases}$$

has the following solution:

$$nc(S, t; Z/n) + \mathrm{e}^{-r(T-t)}Z,$$

where $c(S, t; Z/n)$ is the price of a call option with an exercise price Z/n. Find the partial differential equation and the final condition the difference between the two bonds should satisfy. Convert the derived problem into a problem on a finite domain and with an initial condition by using the transformation above, and briefly describe a second-order implicit scheme for the new problem.

27. Suppose that $c(S, \sigma, t)$ and $p(S, \sigma, t)$ are solutions of the following problems

$$\begin{cases} \dfrac{\partial c}{\partial t} + \mathbf{L}_{S,\sigma} c = 0, & 0 \le S, \quad \sigma_l \le \sigma \le \sigma_u, \quad t \le T, \\ c(S, \sigma, T) = \max(S - E, 0), & 0 \le S, \quad \sigma_l \le \sigma \le \sigma_u \end{cases}$$

and

$$\begin{cases} \dfrac{\partial p}{\partial t} + \mathbf{L}_{S,\sigma} p = 0, & 0 \le S, \quad \sigma_l \le \sigma \le \sigma_u, \quad t \le T, \\ p(S, \sigma, T) = \max(E - S, 0), & 0 \le S, \quad \sigma_l \le \sigma \le \sigma_u, \end{cases}$$

where $\mathbf{L}_{S,\sigma}$ is an operator defined by

$$\mathbf{L}_{S,\sigma} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + \rho\sigma S q \frac{\partial^2}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2}{\partial \sigma^2} + (r - D_0)S\frac{\partial}{\partial S} + (p - \lambda q)\frac{\partial}{\partial \sigma} - r.$$

Show that the following put–call parity relation

$$c(S, \sigma, t) - p(S, \sigma, t) = S\mathrm{e}^{-D_0(T-t)} - E\mathrm{e}^{-r(T-t)}$$

holds by the superposition principle. (Hint: Let u denote $c(S,\sigma,t) - p(S,\sigma,t)$. Show that u is the solution of the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathbf{L}_{S,\sigma} u = 0, & 0 \le S, \quad \sigma_l \le \sigma \le \sigma_u, \quad t \le T, \\ u(S,\sigma,T) = S - E, & 0 \le S, \quad \sigma_l \le \sigma \le \sigma_u \end{cases}$$

and that $S\mathrm{e}^{-D_0(T-t)} - E\mathrm{e}^{-r(T-t)}$ is also the solution of this problem.)

28. *Convert the following double moving barrier call option problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r - D_0) S \dfrac{\partial V}{\partial S} - rV = 0, & \\ & f(t) \le S \le g(t), \quad 0 \le t \le T, \\ V(S,T) = \max(S - E, 0), & f(T) \le S \le g(T), \\ V(f(t),t) = 0, & 0 \le t \le T, \\ V(g(t),t) = g(t) - E, & 0 \le t \le T \end{cases}$$

into a problem that has a smooth solution and an initial condition, and design an implicit pseudo-spectral method for the new problem.

29. For the new problem obtained in Problem 26, design an implicit pseudo-spectral method.

Projects

General Requirements

(A) Submit a code or codes in C or C^{++} that will work on a computer the instructor can get access to. At the beginning of the code, write down the name of the student and indicate on which computer it works and the procedure to make it work.

(B) Each code should use an input file to specify all the problem parameters and the computational parameters for each computation and an output file to store all the results. In an output file, the name of the problem, all the problem parameters, and the computational parameters should be given, so that one can know what the results are and how they were obtained. The input file should be submitted with the code.

(C) If not specified, for each case two results are required. For the first result, a 20×12 mesh should be used.(In this case, the error of the solution might be quite large.) For the second result, the accuracy required is 0.01, and the mesh used should be as coarse as possible.

(D) Submit results in form of tables or figures. When a result is given, always provide the problem parameters and the computational parameters.

1. **Explicit Method (8.3).** Suppose that σ, r are constants and the dividends are given discretely or continuously. Write a code for European, Bermudan, and American calls and puts.
   - For American call and put options, give the results for the case: $S = 100$, $E = 100$, $T = 0.75$, $r = 0.1$, $D_0 = 0.05$, and $\sigma = 0.3$.
   - For Bermudan call and put options, give the results for the case: $S = 100$, $E = 100$, $T = 1$, $r = 0.05$, $D_0 = 0.1$, $\sigma = 0.2$, and $K = 4$ (see Sect. 8.3.3).
   - For European call and put options, give the results for the cases: $S = 100$, $E = 95, 100, 105$, $T = 1$, $r = 0.1$, $\sigma = 0.2$, and two dividend payments of \$1.25 paid at $t = 2$ months and $t = 8$ months. $D(S)$ is defined by
   $$D(S) = \begin{cases} S & \text{if } S \le d, \\ d & \text{if } S > d, \end{cases}$$
   where d is the dividend payment.
   - Taking the European call option with $E = 100$, $T = 1$, $r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$ as an example, show that the explicit method (8.3) is unstable if $\varDelta\tau$ is too large. For this problem, only one example is required. Plot the S-c curve with $t = 0$.
2. **Binomial Methods (8.28) with the formulae (8.25)–(8.27) and Eq. (8.28) with the formulae (8.18) and (8.23).** Suppose that σ, r, D_0 are constants. Write a code for European, Bermudan, and American calls and puts. For this problem, instead of the result on a 20×12 mesh, a result with $\varDelta t = T/12$ is required.
   - For European call and put options, give the results for the cases: $S = 100$, $E = 95, 100, 105$, $T = 1$, $r = 0.1$, $D_0 = 0.025$, and $\sigma = 0.2$.
   - For Bermudan call and put options, give the results for the case: $S = 100$, $E = 100$, $T = 1$, $r = 0.05$, $D_0 = 0.1$, $\sigma = 0.2$, and $K = 4$.
   - For American call and put options, give the results for the case: $S = 100$, $E = 100$, $T = 0.75$, $r = 0.1$, $D_0 = 0.05$, and $\sigma = 0.3$.
3. **Implicit Method (8.47) for Vanilla Options (Solving the Corresponding System by Direct Methods).** Suppose that σ, r, and D_0 are constants. Write a code for European, Bermudan, and American calls and puts.
   - For European call and put options, give the results for the cases: $S = 100$, $E = 95, 100, 105$, $T = 1$, $r = 0.1$, $D_0 = 0.025$, and $\sigma = 0.2$.
   - For Bermudan call and put options, give the results for the case: $S = 100$, $E = 100$, $T = 1$, $r = 0.05$, $D_0 = 0.1$, $\sigma = 0.2$, and $K = 4$.
   - For American call and put options, give the results for the case: $S = 100$, $E = 100$, $T = 0.75$, $r = 0.1$, $D_0 = 0.05$, and $\sigma = 0.3$.
4. **Implicit Method (8.47) for European Average Price Options with Discrete Sampling (Solving the Corresponding System by Direct Methods).** Suppose that σ, r, and D_0 are constants. Write a

code for European average price call and put options with various discrete samplings.

- For European average price call and put options with sampling daily, give the results for the cases: $S = 100$, $E = 90, 95, 100, 105, 110$, $T = 1$, $r = 0.05$, $D_0 = 0.025$, and $\sigma = 0.2$. (The results on a 20×12 mesh are not required.)
- For European average price call and put options with sampling weekly, give the results for the cases: $S = 100$, $E = 90, 95, 100, 105, 110$, $T = 0.5$, $r = 0.05$, $D_0 = 0.0$, and $\sigma = 0.2$. (The results on a 20×12 mesh are not required.)
- For European average price call and put options with sampling monthly, give the results for the cases: $S = 100$, $E = 90, 95, 100, 105, 110$, $T = 1$, $r = 0.0$, $D_0 = 0.0$, and $\sigma = 0.3$.

5. **Singularity-Separating Implicit Method with Scheme (8.47).** Suppose that σ, r are constants and the dividends are given discretely or continuously. Write a code for Bermudan calls and puts with continuous dividends and a code for European vanilla calls and puts with discrete dividends. Calculate the difference between the value of the option and the closed-form solution of a corresponding European vanilla option numerically. In order to calculate the price of a Bermudan put, Compute a corresponding call first and then obtain the value of the Bermudan put by using the symmetry relation.

- For Bermudan call options, give the results for the case: $S = 100$, $E = 100$, $T = 1$, $r = 0.05$, $D_0 = 0.1$, $\sigma = 0.2$, and $K = 4$. For Bermudan put options, give the results for the case: $S = 100$, $E = 100$, $T = 1$, $r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$, and $K = 12$.
- For European call and put options, give the results for the cases: $S = 100$, $E = 95, 100, 105$, $T = 1$, $r = 0.1$, $\sigma = 0.2$, and two dividend payments of \$1.25 paid at $t = 2\,\text{months}$ and $t = 8\,\text{months}$. $D(S)$ is defined by

$$D(S) = \begin{cases} S & \text{if } S \le d, \\ d & \text{if } S > d, \end{cases}$$

where d is the dividend payment.