3

American Style Derivatives

In this Chapter, we will discuss that in order to find the value of an American style derivative, what kind of mathematical problems needs to be solved. When we have such discussions, we mainly take American options as examples. However the methods can be used for other American style derivatives. In the first section, we will derive the additional constraints on American style derivatives and discuss how the constraints affect the way the price is determined. In Sect. 3.2, we formulate the American call and put problems as linear complementarity (LC) problems and point out how to get the formulation for an American style derivative from the formulation for the corresponding European style derivative. In Sect. 3.3, we will discuss how to formulate an American option problem as a free-boundary problem (FBP) from a linear complementarity problem. For other American style derivatives, the method is similar. Finally we discuss some properties of options, including the relations between European and American options, by the arbitrage theory in the last section.

3.1 Constraints on American Style Derivatives

3.1.1 Constraints on American Options

Let $C(S,t)$ and $P(S,t)$ denote the prices of American call and put options, respectively. As we know from Sect. 1.2, an American option has the additional feature that it may be exercised at any time during the life of the option. What does this additional feature mean in mathematics? It means that the value of an American call option must satisfy the condition

$$C(S,t) \geq \max(S-E,0), \tag{3.1}$$

and that the value of an American put option must fulfill the inequality

$$P(S,t) \geq \max(E-S,0). \tag{3.2}$$

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Usually, $\max(S-E,0)$ and $\max(E-S,0)$ are called the intrinsic values of call and put options, respectively. Thus, satisfying the two inequalities above means that the value of an option must be at least equal to its intrinsic value. Conditions (3.1) and (3.2) are usually referred to as the constraints on American vanilla options. These conclusions can be proved by arbitrage arguments as follows.

First, let us consider an American call option. For $S \le E$, the condition (3.1) means $C(S,t) \ge 0$. This is always true because a solution of the Black–Scholes equation with a nonnegative payoff function as a final condition is always nonnegative. From the financial point of view, it is also clear that a holder of an option has only rights, no obligation, so he/she needs to pay something in order to get it, i.e., the option price should not be negative. Thus, the condition (3.1) always holds for any $S \in [0,E]$. Suppose that for a price $S > E$, the condition (3.1) is not fulfilled, i.e., $C(S,t) < S-E$. Then, an obvious arbitrage opportunity arises: by short selling the asset on the market for S, purchasing the option for C, and exercising the call option, a risk-free profit of $S-C-E$ is made. Of course, such an opportunity would not last long before the value of the option was pushed up by the demand of arbitrageurs. We conclude that on a value of an American call, we must impose the constraint (3.1). For an American put option the situation is similar. For any $S \ge E$, the condition (3.2) holds naturally. Suppose the option price satisfies $P(S,t) < E-S$ for a price $S < E$. Then, by purchasing the option for P, purchasing the asset from the market for S, and exercising the put option, an immediate risk-free profit of $E-P-S$ is made, and the demand will push the option price up so that condition (3.2) holds.

Bermudan options are similar to American options but can be exercised only at several predetermined dates, instead of the entire period $[0,T]$. This means that for a Bermudan option, condition (3.1) or condition (3.2) should be required at several predetermined dates but not on the entire period $[0,T]$, which is the only difference between American and Bermudan options.

How does a constraint affect the way of determining the price of an option? Let us take an American put option as an example. As we easily see, at $S=0$ the Black–Scholes equation degenerates to an ordinary differential equation

$$\frac{\partial V(0,t)}{\partial t} - rV(0,t) = 0$$

and its solution is

$$V(0,t) = V(0,T)\mathrm{e}^{-r(T-t)}.$$

For a put, $V(0,T) = E$. Therefore, the price of a European put option at $S=0$ is

$$p(0,t) = E\mathrm{e}^{-r(T-t)} < E$$

for any $t < T$ if $r > 0$. Consequently, the price of a European put option will not satisfy the constraint (3.2). Thus, in order to price an American put, we must modify the method for determining the price of an option if $r > 0$.

Roughly speaking, the way of determining the price of an American style derivative is as follows. Let $V(S,t)$ be the price of an American style derivative and $G_v(S,t)$ be the constraint. Suppose that for a time t, $V(S,t)$ is known for any S. Based on $V(S,t)$ and using the Black–Scholes equation, we can obtain the price of a derivative security at time $t-\Delta t$ for a small positive Δt. If the value satisfies the constraint condition $V(S,t-\Delta t) \geq G_v(S,t-\Delta t)$, it gives the price of the American style derivative; if not, the constraint is the value of the American style derivative, i.e., the Black–Scholes equation cannot be used for determining the price of the American style derivative in this case.

Let us explain why the price of the American style derivative is determined in this way. If $V(S,t) > G_v(S,t)$ in a neighborhood of a point $S = S^*$ at time t, then the solution $V(S,t-\Delta t)$ obtained by using the Black–Scholes equation must still satisfy the condition $V(S,t-\Delta t) > G_v(S,t-\Delta t)$ at that point if Δt is small enough. Therefore the event "the Black–Scholes equation cannot be used" only occurs at a point $S = S^*$ where $V(S,t) = G_v(S,t)$. Thus we need to discuss when the Black–Scholes equation can be used and when the Black–Scholes equation cannot be used only if $V(S,t) = G_v(S,t)$. On this question, we have the following theorem. In the future we will also consider other problems besides option problems, thus in the theorem, we consider a general partial differential equation (PDE) similar to the Black–Scholes equation. The theorem is described as follows.

Theorem 3.1 *Let* $\mathbf{L_{S,t}}$ *be an operator in a derivative security problem in the form:*

$$\mathbf{L_{S,t}} = a(S,t)\frac{\partial^2}{\partial S^2} + b(S,t)\frac{\partial}{\partial S} + c(S,t)$$

and $G_v(S,t)$ *be the constraint function for an American style derivative. Furthermore, we assume that* $\dfrac{\partial G_v}{\partial t} + \mathbf{L_{S,t}} G_v$ *exists. Suppose* $V(S,t^*) = G_v(S,t^*)$ *on an open interval* (A,B) *on the* S*-axis. Let* $t = t^* - \Delta t$*, where* Δt *is a sufficiently small positive number. For this case we have the following conclusions: If for any* $S \in (A,B)$,

$$\frac{\partial G_v}{\partial t}(S,t^*) + \mathbf{L_{S,t^*}} G_v(S,t^*) \geq 0,$$

then the value $V(S,t)$ *determined by the equation*

$$\frac{\partial V}{\partial t}(S,t) + \mathbf{L_{S,t}} V(S,t) = 0$$

satisfies the condition $V(S,t) - G_v(S,t) \geq 0$ *on* (A,B)*, which means the PDE can be used for determining the price of the American style derivative; and if for any* $S \in (A,B)$,

$$\frac{\partial G_v}{\partial t}(S,t^*) + \mathbf{L_{S,t^*}} G_v(S,t^*) < 0,$$

then the equation

$$\frac{\partial V}{\partial t}(S,t) + \mathbf{L_{S,t}} V(S,t) = 0$$

cannot give a solution satisfying the condition $V(S,t) - G_v(S,t) \geq 0$ *for any* $S \in (A, B)$*, which means the PDE cannot be used for determining the price of the American style derivative.*

Proof. Because $V(S,t^*) = G_v(S,t^*)$, the fact that $V(S,t) - G_v(S,t) > 0$ holds for any $t = t^* - \Delta t$, Δt being a sufficiently small positive number, is equivalent to that at time t^*, $V(S,t) - G_v(S,t)$ is a decreasing function with respect to t, that is,

$$\frac{\partial V}{\partial t}(S,t^*) - \frac{\partial G_v}{\partial t}(S,t^*) < 0.$$

If

$$\frac{\partial G_v}{\partial t}(S,t^*) + \mathbf{L_{S,t^*}} G_v(S,t^*) > 0$$

and

$$\frac{\partial V}{\partial t}(S,t^*) + \mathbf{L_{S,t^*}} V(S,t^*) = \frac{\partial V}{\partial t}(S,t^*) + \mathbf{L_{S,t^*}} G_v(S,t^*) = 0,$$

then

$$\frac{\partial G_v}{\partial t}(S,t^*) > -\mathbf{L_{S,t^*}} G_v(S,t^*) = \frac{\partial V}{\partial t}(S,t^*)$$

or

$$\frac{\partial V}{\partial t}(S,t^*) - \frac{\partial G_v}{\partial t}(S,t^*) < 0.$$

Therefore in this case we can use the equation

$$\frac{\partial V}{\partial t}(S,t) + \mathbf{L_{S,t}} V(S,t) = 0$$

to get a solution satisfying the condition $V(S,t) - G_v(S,t) > 0$, which means the PDE can be used for determining the price of the American style derivative.

If on a point (S,t^*) with $S \in (A, B)$

$$\frac{\partial G_v}{\partial t}(S,t^*) + \mathbf{L_{S,t^*}} G_v(S,t^*) = 0,$$

then $G_v(S,t)$ is the solution $V(S,t)$ in a sufficiently small neighborhood of the point (S,t^*). Putting this result and the result above together, we know that if

$$\frac{\partial G_v}{\partial t}(S,t^*) + \mathbf{L_{S,t^*}} G_v(S,t^*) \geq 0$$

then the PDE can be used for determining the price of the American style derivative.

If

$$\frac{\partial G_v}{\partial t}(S,t^*) + \mathbf{L}_{\mathbf{S},\mathbf{t}^*} G_v(S,t^*) < 0$$

and

$$\frac{\partial V}{\partial t}(S,t^*) + \mathbf{L}_{\mathbf{S},\mathbf{t}^*} V(S,t^*) = \frac{\partial V}{\partial t}(S,t^*) + \mathbf{L}_{\mathbf{S},\mathbf{t}^*} G_v(S,t^*) = 0,$$

then

$$\frac{\partial G_v}{\partial t}(S,t^*) < -\mathbf{L}_{\mathbf{S},\mathbf{t}^*} G_v(S,t^*) = \frac{\partial V}{\partial t}(S,t^*)$$

or

$$\frac{\partial V}{\partial t}(S,t^*) - \frac{\partial G_v}{\partial t}(S,t^*) > 0,$$

which will cause $V(S,t) - G_v(S,t) < 0$ for any $t = t^* - \Delta t$. Therefore, we cannot get the solution by using the equation

$$\frac{\partial V}{\partial t}(S,t) + \mathbf{L}_{\mathbf{S},\mathbf{t}} V(S,t) = 0,$$

which means the PDE cannot be used for determining the price of the American style derivative. □

About this theorem, we would like to make the following remark.

- Let us adopt $\tau = T - t$ instead of t and we want to have the solution at $\tau = \tau^* + \Delta\tau$ from the solution at $\tau = \tau^*$, where $\Delta\tau > 0$. Then the theorem is still true if the condition

$$\frac{\partial G_v}{\partial t}(S,t) + \mathbf{L}_{\mathbf{S},\mathbf{t}} G_v(S,t) \geq 0$$

is changed into

$$\frac{\partial \overline{G}_v}{\partial \tau}(S,\tau) - \mathbf{L}_{\mathbf{S},\tau} \overline{G}_v(S,\tau) \leq 0$$

and the condition

$$\frac{\partial G_v}{\partial t}(S,t) + \mathbf{L}_{\mathbf{S},\mathbf{t}} G_v(S,t) < 0$$

is changed into

$$\frac{\partial \overline{G}_v}{\partial \tau}(S,\tau) - \mathbf{L}_{\mathbf{S},\tau} \overline{G}_v(S,\tau) > 0,$$

where $\overline{G}_v(S,\tau) = G_v(S,t)$ and $\mathbf{L}_{\mathbf{S},\tau} \overline{\mathbf{G}}_\mathbf{v}(\mathbf{S},\tau) = \mathbf{L}_{\mathbf{S},\mathbf{t}} \mathbf{G}_\mathbf{v}(\mathbf{S},\mathbf{t})$.

From Theorem 3.1, we know that when $V(S,t) = G_v(S,t)$ and

$$\left[\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0)S\frac{\partial}{\partial S} - r\right] G_v < 0,$$

we cannot use the Black–Scholes equation to determine $V(S,t-\Delta t)$. What $V(S,t-\Delta t)$ should be in this case? In the above, we have pointed that in this case, $V(S, t-\varDelta t) = G_v(S, t-\varDelta t)$. Here let us explain why it should be. It is clear that a buyer of a derivative security wants the price to be as low as possible and that the price of an American style derivative cannot be less than the constraint as we discussed above. Thus for $V(S, t-\varDelta t)$ the constraint $G_v(S, t-\varDelta t)$ is the lowest price the buyer can expect. A seller wants the price to be as high as possible. Can the seller accept that the constraint is the price in this case? The answer is "yes", so the constraint is the price both the buyer and the seller accept. Let us explain why the seller accepts this price. Suppose that the seller sells the derivative security for $V(S, t-\varDelta t) = G_v(S, t-\varDelta t)$. After the derivative security is sold, using the money obtained, the seller buys $\dfrac{\partial V}{\partial S}$ shares and deposits the remains $V - \dfrac{\partial V}{\partial S} S$ into a money market account.[1] Because $V(S, t-\varDelta t) = G_v(S, t-\varDelta t)$, we will have

$$\begin{aligned}
&\left[\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0) S \frac{\partial}{\partial S} - r\right] V \\
&= \left[\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0) S \frac{\partial}{\partial S} - r\right] G_v < 0,
\end{aligned}$$

i.e.,

$$\begin{aligned}
\frac{\partial V}{\partial S} dS + \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt &< \frac{\partial V}{\partial S} dS - \left[(r - D_0) S \frac{\partial V}{\partial S} - rV\right] dt \\
&= r\left(V - \frac{\partial V}{\partial S} S\right) dt + \frac{\partial V}{\partial S}(dS + D_0 S dt).
\end{aligned}$$

This means that the return from the derivative security during a time step dt,

$$\frac{\partial V}{\partial S} dS + \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt,$$

will be less than the return from the money market account with a value of $V - \dfrac{\partial V}{\partial S} S$ and $\dfrac{\partial V}{\partial S}$ shares,

$$r\left(V - \frac{\partial V}{\partial S} S\right) dt + \frac{\partial V}{\partial S}(dS + D_0 S dt).$$

Thus the amount of money the seller obtains from the money market account and shares is more than the change of the derivative value, which means the seller will earn money. Hence the seller can accept this price.

[1] If $V - \dfrac{\partial V}{\partial S} S < 0$, the seller indeed borrows $-\left(V - \dfrac{\partial V}{\partial S} S\right)$, the money needed to buy $\dfrac{\partial V}{\partial S}$ shares, from somewhere.

3.1.2 Some Properties of American Style Derivatives

Consider a European style derivative and an American style derivative with identical payoffs and identical operators. Let $V(S,t)$, $v(S,t)$ denote the prices of the American and European style derivatives, respectively, let $G_v(S,t)$ be the constraint for the American style derivative, and the operator for the two derivatives is

$$\frac{\partial}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2}{\partial S^2}+(r-D_0)S\frac{\partial}{\partial S}-r$$

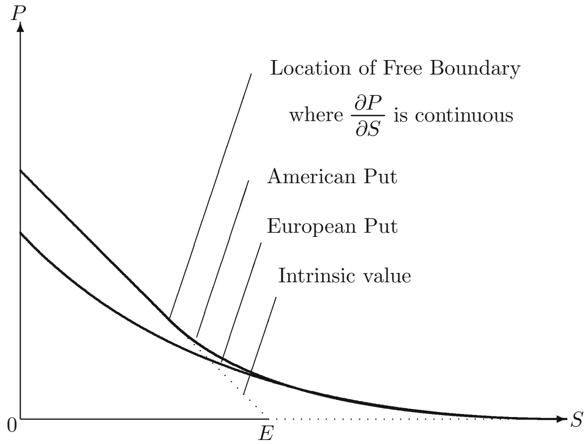

Fig. 3.1. The price of an American put option before expiry

with constant σ, r, and D_0. Using the results we have obtained, we can prove that the price of the European style derivative is never higher than the price of the American style derivative, i.e., $V(S,t)\ge v(S,t)$ holds. Let us prove this conclusion. Suppose that

$$V(S,T)=v(S,T)=G_v(S,T).$$

Set $\Delta t=T/N$, N being a positive integer and define $t_n=n\Delta t$, $n=N,N-1,\cdots,0$. For the European style derivative, from the formula (2.84) we have the relation between $v(S,t_n)$ and $v(S,t_{n+1})$

$$v(S,t_n)=\mathrm{e}^{-r\Delta t}\int_0^\infty v(S',t_{n+1})G(S',t_{n+1};S,t_n)dS'$$

for $n=N-1,N-2,\cdots,0$. Let

$$\tilde{V}(S,t_N)=G_v(S,T)$$

and for $n = N-1, N-2, \cdots, 0$, define

$$\tilde{V}(S,t_n) = \max\left(\mathrm{e}^{-r\Delta t}\int_0^\infty \tilde{V}(S',t_{n+1})G(S',t_{n+1};S,t_n)dS', G_v(S,t_n)\right), \tag{3.3}$$

where $G(S',t_{n+1};S,t_n)$ is given by the formula (2.85). Suppose $\tilde{V}(S,t_{n+1}) \geq v(S,t_{n+1})$, then we know

$$\begin{aligned} v(S,t_n) &= \mathrm{e}^{-r\Delta t}\int_0^\infty v(S',t_{n+1})G(S',t_{n+1};S,t_n)dS' \\ &\leq \mathrm{e}^{-r\Delta t}\int_0^\infty \tilde{V}(S',t_{n+1})G(S',t_{n+1};S,t_n)dS' \\ &\leq \tilde{V}(S,t_n). \end{aligned}$$

At $t = t_N = T$, the condition

$$\tilde{V}(S,t_{\scriptscriptstyle N}) = G_v(S,T) \geq v(S,T) = G_v(S,T)$$

holds. Therefore, using the induction method, we can prove $\tilde{V}(S,t_n) \geq v(S,t_n)$ for $n = N-1, N-2, \cdots, 0$ successively. Letting $N \to \infty$ and noticing that $\tilde{V}\left(S,\ \mathrm{int}\left(\dfrac{tN}{T}\right)\cdot\dfrac{T}{N}\right)$ generates $V(S,t)$ as $N \to \infty$, where $\mathrm{int}\left(\dfrac{tN}{T}\right)$ is the integer part of $\dfrac{tN}{T}$, we can have the conclusion:

$$V(S,t) \geq v(S,t) \quad \text{for any } S \text{ and } t.$$

The put and call options are such type of derivatives. Thus $C(S,t) \geq c(S,t)$ and $P(S,t) \geq p(S,t)$. This result has the following financial meaning. Because an American option can be exercised at any time by expiry, a holder of an American option has more rights than does a holder of a European option. Thus, the holder of an American option needs to pay at least as much premium as does the holder of a European option with the same parameters. Figure 3.1 shows this fact and other related facts for put options. From the figure, we can see that the price of America put option is always greater than the price of European put option and the intrinsic value, but the price of the European put option is greater than the intrinsic value for some S and less than the intrinsic value for other S. It can also be proved that the price of a Bermudan option should be between these of European and American options and the financial meaning can be expressed as follows. The Bermudan option can be exercised at several predetermined dates including the expiration date, its holder has less rights than does the holder of an American option and more rights than does the holder of a European option. Thus, its premium should be between the premiums of the American and European options with the same parameters.

The price of an American style derivative has another property: $V(S,t^*) \geq V(S,t^{**})$ for $t^* \leq t^{**}$ if $G_v(S,t) = G_v(S)$ or, more generally, the condition $G_v(S,t^*) \geq G_v(S,t^{**})$ for $t^* \leq t^{**}$ holds. Let us explain this fact by using mathematical tools. Suppose $\tilde{V}(S,t_n) \geq \tilde{V}(S,t_{n+1})$. According to the definition of $\tilde{V}(S,t_n)$, we have

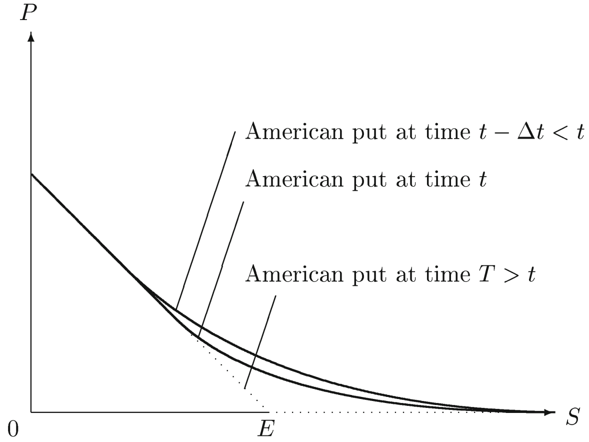

Fig. 3.2. $P(S,t-\varDelta t) \geq P(S,t)$ for any positive $\varDelta t$

$$\begin{aligned}\tilde{V}(S,t_n) &= \max\left(\mathrm{e}^{-r\varDelta t}\int_0^\infty \tilde{V}(S',t_{n+1})G(S',t_{n+1};S,t_n)dS', G_v(S,t_n)\right)\\ &\leq \max\left(\mathrm{e}^{-r\varDelta t}\int_0^\infty \tilde{V}(S',t_n)G(S',t_n;S,t_{n-1})dS', G_v(S,t_{n-1})\right)\\ &= \tilde{V}(S,t_{n-1}).\end{aligned}$$

Here we have used the facts

$$G(S',t_{n+1};S,t_n) = G(S',t_n;S,t_{n-1})$$

and

$$G_v(S,t_n) \leq G_v(S,t_{n-1}).$$

Because

$$\begin{aligned}\tilde{V}(S,t_{N-1}) &= \max\left(\mathrm{e}^{-r\varDelta t}\int_0^\infty \tilde{V}(S',t_N)G(S',t_N;S,t_{N-1})dS', G_v(S,t_{N-1})\right)\\ &\geq G_v(S,t_{N-1}) \geq G_v(S,t_N) = \tilde{V}(S,t_N),\end{aligned}$$

we can prove

$$\tilde{V}(S,t_n) \geq \tilde{V}(S,t_{n+1}) \quad \text{for} \quad n = N-2, N-3, \cdots, 0$$

successively. This means

$$\tilde{V}(S, t_n) \geq \tilde{V}(S, t_m) \quad \text{for} \quad n \leq m \leq N.$$

Letting $N \to \infty$ and noticing that $\tilde{V}(S,t)$ generates $V(S,t)$ as $N \to \infty$, we arrive at the conclusion

$$V(S, t^*) \geq V(S, t^{**}) \quad \text{if} \quad t^* \leq t^{**}.$$

For the American call/put option, $G_v(S,t) = G_v(S)$, so we have $C(S,t^*) \geq C(S,t^{**})$ and $P(S,t^*) \geq P(S,t^{**})$ if $t^* \leq t^{**}$. Figure 3.2 shows this fact graphically for an American put option. From the point of financial view, when $t^* < t^{**}$, a holder of an American call/put option at time t^* has more rights than does a holder at time t^{**}, so the premium of the option at time t^* should be higher than the premium of the option at time t^{**}.

As we have pointed out, $C(S,t) \geq \max(S-E, 0)$ and $P(S,t) \geq \max(E-S, 0)$, which means that $C(S,t) - \max(S-E,0)$ and $P(S,t) - \max(E-S,0)$ must be nonnegative. Because these two functions are usually called the time values of the American call and put options, respectively, this fact can be expressed as that the time values must be nonnegative. Using the result here $V(S,t^*) \geq V(S,t^{**})$ for $t^* \leq t^{**}$, we can have another conclusion: the time values $C(S,t) - \max(S-E,0)$ and $P(S,t) - \max(E-S,0)$ are non-increasing functions in time because

$$V(S, t^*) \geq V(S, t^{**}) \quad \text{for} \quad t^* \leq t^{**}$$

is equivalent to

$$V(S, t^*) - G_v(S) \geq V(S, t^{**}) - G_v(S) \quad \text{for} \quad t^* \leq t^{**}.$$

However not all American style derivatives have such a property. Here we give an example. Consider the following derivative security. It is a bond with a face value Z and it can be converted into n shares at any time. We assume that the price of the stock is a random variable, the interest rate is a constant and the bond pays no coupon. This problem is referred to as the problem of one-factor convertible bond paying no coupon. Let $B_c(S,t)$ stand for its value. It is clear that $B_c(S,T) = \max(Z, nS)$, $B_c(S,t) \geq nS$ for $t < T$, and the basic PDE for this problem is the Black-Scholes equation. Thus this derivative security problem is close to the American option problem and its some properties can be studied by using a similar way given in this subsection. For example, using the method given here, it can be shown that

$$B_c(S, t^*) - Z\mathrm{e}^{-r(T-t^*)} \geq B_c(S, t^{**}) - Z\mathrm{e}^{-r(T-t^{**})} \quad \text{if} \quad t^* \leq t^{**}$$

holds and

$$B_c(S, t^*) \geq B_c(S, t^{**}) \quad \text{if} \quad t^* \leq t^{**}$$

does not hold at least for $S = 0$. These results are left for readers to prove as Problem 6. Here we give an explanation for such results. The final condition

$B_c(S,T) = \max(Z, nS)$ can be rewritten as $B_c(S,T) = Z + \max(nS - Z, 0)$, so it consists of two problems, one is a bond problem with a solution of $Z\mathrm{e}^{-r(T-t)}$ and the other is a special American call problem with a payoff of $\max(nS - Z, 0)$ and a constraint $nS - Z\mathrm{e}^{-r(T-t)}$. For this special American call option, the price is a non-increasing function like the American option. However for the bond problem, the price is an increasing function. Thus the total is not a non-increasing function. Consequently, even the holder of this American derivative at t^* has "more rights" than does the holder at t^{**} if $t^* \leq t^{**}$, but the price at t^* is not always greater than or equal to the price at t^{**}.

3.2 American Options Problems as Linear Complementarity Problems

3.2.1 Formulation of the Linear Complementarity Problem in (S, t)-Plane

From Sect. 3.1.1, we know that the price of an American option usually is not a solution of the problem (2.73) anymore because usually in some regions the solution satisfies the PDE and in other regions it is not determined by the PDE. For American option problems, the price is given by a solution of a so-called linear complementarity (LC) problem.

Now let us formulate the LC problem the price of an American option should satisfy. Let us take an American put option as an example. Assume that at time t we have obtained $P(S,t)$ satisfying (3.2) and we need to determine $P(S, t - \Delta t)$ satisfying (3.2), where Δt is a sufficiently small positive number. Define $G_p(S,t) = \max(E - S, 0)$. For simplicity, we assume that the entire interval consists of three open intervals plus their boundaries. On the first open interval, $P(S,t) > G_p(S,t)$. For any point in this interval, we can use the Black–Scholes equation to determine $P(S, t - \Delta t)$ and $P(S, t - \Delta t)$ must be still greater than $G_p(S, t - \Delta t)$ if Δt is small enough. Therefore, at any point in this open interval

$$\begin{cases} \dfrac{\partial P}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 P}{\partial S^2} + (r - D_0) S \dfrac{\partial P}{\partial S} - rP = 0, \\ P(S,t) > G_p(S,t). \end{cases}$$

On the second open interval $P(S,t) = G_p(S,t)$ and

$$\left[\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0) S \frac{\partial}{\partial S} - r\right] G_p(S,t) \geq 0$$

and on the third open interval $P(S,t) = G_p(S,t)$ and

$$\left[\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0) S \frac{\partial}{\partial S} - r\right] G_p(S,t) < 0.$$

According to Theorem 3.1, for a point (S,t) in the second open interval the Black–Scholes equation can be used to determine $P(S,t-\Delta t)$ and the following is true:

$$\begin{cases} \dfrac{\partial P}{\partial t}+\dfrac{1}{2}\sigma^2 S^2\dfrac{\partial^2 P}{\partial S^2}+(r-D_0)S\dfrac{\partial P}{\partial S}-rP=0, \\ P(S,t)=G_p(S,t). \end{cases}$$

On the third interval, the Black–Scholes equation cannot be used to determine $P(S,t-\Delta t)$. Instead, $P(S,t-\Delta t)$ should equal $G_p(S,t-\Delta t)$. In this situation

$$\frac{P(S,t)-P(S,t-\Delta t)}{\Delta t}=\frac{G_p(S,t)-G_p(S,t-\Delta t)}{\Delta t}\rightarrow\frac{\partial G_p(S,t)}{\partial t}$$

as $\Delta t\rightarrow 0$ and we have

$$\begin{cases} \left[\dfrac{\partial}{\partial t}+\dfrac{1}{2}\sigma^2 S^2\dfrac{\partial^2}{\partial S^2}+(r-D_0)S\dfrac{\partial}{\partial S}-r\right]P \\ =\left[\dfrac{\partial}{\partial t}+\dfrac{1}{2}\sigma^2 S^2\dfrac{\partial^2}{\partial S^2}+(r-D_0)S\dfrac{\partial}{\partial S}-r\right]G_p(S,t)<0, \\ P(S,t)=G_p(S,t). \end{cases}$$

Because $P(S,T)=G_p(S,T)$, we can use this argument from T to 0. Putting all the cases together, for $S\in[0,\infty)$ and $t\le T$ we have

$$\begin{cases} \left[\dfrac{\partial P}{\partial t}+\dfrac{1}{2}\sigma^2 S^2\dfrac{\partial^2 P}{\partial S^2}+(r-D_0)S\dfrac{\partial P}{\partial S}-rP\right](P-G_p)=0, \\ \dfrac{\partial P}{\partial t}+\dfrac{1}{2}\sigma^2 S^2\dfrac{\partial^2 P}{\partial S^2}+(r-D_0)S\dfrac{\partial P}{\partial S}-rP\le 0, \\ P(S,t)-G_p(S,t)\ge 0, \\ P(S,T)=G_p(S,T), \end{cases}$$

where $G_p(S,t)=\max(E-S,0)$. Here, we use the fact that these relations in the formulation are also true in some sense at the boundary points of these open intervals because these relations are true on the two sides of a boundary point. It is clear that the formulation above can also be written in the following short form:

$$\begin{cases} \min\left(-\dfrac{\partial P}{\partial t}-\mathbf{L}_S P,\ P(S,t)-G_p(S,t)\right)=0, & 0\le S,\ t\le T, \\ P(S,T)=G_p(S,T), & 0\le S, \end{cases} \tag{3.4}$$

where

$$\mathbf{L}_S=\frac{1}{2}\sigma^2 S^2\frac{\partial^2}{\partial S^2}+(r-D_0)S\frac{\partial}{\partial S}-r$$

and

$$G_p(S,t) = \max(E - S, 0).$$

This problem is called the linear complementarity problem for an American put option. In order to determine the price of an American put option, we need to solve this problem.

Similarly, for an American call option, the corresponding linear complementarity problem is

$$\begin{cases} \min\left(-\dfrac{\partial C}{\partial t} - \mathbf{L}_S C,\ C(S,t) - G_c(S,t)\right) = 0, & 0 \le S,\ t \le T, \\ C(S,T) = G_c(S,T), & 0 \le S, \end{cases} \tag{3.5}$$

where $G_c(S,t) = \max(S - E, 0)$. From the derivation of the problem (3.4), we can see that the formulations are still correct when σ, r, D_0 depend on S and t.

3.2.2 Formulation of the Linear Complementarity Problem in $(x, \bar{\tau})$-Plane

As we know from Sect. 2.6.1, if we set

$$\begin{cases} x = \ln S + \left(r - D_0 - \dfrac{1}{2}\sigma^2\right)(T - t), \\ \bar{\tau} = \dfrac{1}{2}\sigma^2 (T - t), \\ V(S,t) = \mathrm{e}^{-r(T-t)} u(x, \bar{\tau}), \end{cases}$$

then

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0) S \frac{\partial V}{\partial S} - rV$$

becomes

$$-\frac{1}{2}\sigma^2 \mathrm{e}^{-r(T-t)} \left(\frac{\partial u}{\partial \bar{\tau}} - \frac{\partial^2 u}{\partial x^2}\right).$$

Thus,

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial P}{\partial S^2} + (r - D_0) S \frac{\partial P}{\partial S} - rP < 0$$

is equivalent to

$$\frac{\partial u}{\partial \bar{\tau}} - \frac{\partial^2 u}{\partial x^2} > 0$$

and the Black–Scholes equation holds if and only if

$$\frac{\partial u}{\partial \bar{\tau}} - \frac{\partial^2 u}{\partial x^2} = 0.$$

Let us define

$$g_p(x,\bar{\tau}) = \max\left(e^{2r\bar{\tau}/\sigma^2} - e^{x+(2D_0/\sigma^2+1)\bar{\tau}}, 0\right),$$

then

$$\begin{aligned} P - G_p &= P(S,t) - \max(1-S,0) \\ &= e^{-r(T-t)}u(x,\bar{\tau}) - \max\left(1 - e^{x-(r-D_0-\sigma^2/2)(T-t)}, 0\right) \\ &= e^{-r(T-t)}\left[u(x,\bar{\tau}) - \max\left(e^{r(T-t)} - e^{x+(D_0+\sigma^2/2)(T-t)}, 0\right)\right] \\ &= e^{-r(T-t)}[u(x,\bar{\tau}) - g_p(x,\bar{\tau})], \end{aligned}$$

where we suppose $E = 1$ for simplicity. Thus, $P - G_p > 0$ is equivalent to

$$u(x,\bar{\tau}) - g_p(x,\bar{\tau}) > 0$$

and $P - G_p = 0$ if and only if

$$u(x,\bar{\tau}) - g_p(x,\bar{\tau}) = 0.$$

Therefore, the American put option is the solution of the following problem:

$$\begin{cases} \min\left(\dfrac{\partial u}{\partial \bar{\tau}} - \dfrac{\partial^2 u}{\partial x^2},\ u(x,\bar{\tau}) - g_p(x,\bar{\tau})\right) = 0, & -\infty < x < \infty,\ 0 \le \bar{\tau}, \\ u(x,0) = g_p(x,0), & -\infty < x < \infty. \end{cases} \tag{3.6}$$

Similarly, for American call options we have

$$\begin{cases} \min\left(\dfrac{\partial u}{\partial \bar{\tau}} - \dfrac{\partial^2 u}{\partial x^2},\ u(x,\bar{\tau}) - g_c(x,\bar{\tau})\right) = 0, & -\infty < x < \infty,\ 0 \le \bar{\tau}, \\ u(x,0) = g_c(x,0), & -\infty < x < \infty, \end{cases} \tag{3.7}$$

where

$$g_c(x,\bar{\tau}) = \max\left(e^{x+(2D_0/\sigma^2+1)\bar{\tau}} - e^{2r\bar{\tau}/\sigma^2}, 0\right).$$

The derivation of the problem (3.7) is almost identical to the American put. The only difference is that instead of using $P - G_p = e^{-r(T-t)}\left[u(x,\bar{\tau}) - g_p(x,\bar{\tau})\right]$, we need to use the relation

$$\begin{aligned} C - G_c &= C(S,t) - \max(S-1,0) \\ &= e^{-r(T-t)}u(x,\bar{\tau}) - \max\left(e^{x-(r-D_0-\sigma^2/2)(T-t)} - 1, 0\right) \\ &= e^{-r(T-t)}\left[u(x,\bar{\tau}) - \max\left(e^{x+(D_0+\sigma^2/2)(T-t)} - e^{r(T-t)}, 0\right)\right] \\ &= e^{-r(T-t)}\left[u(x,\bar{\tau}) - g_c(x,\bar{\tau})\right], \end{aligned}$$

where we also assume $E = 1$.

It is clear that if r, D_0, and σ depend on t, then similar results hold. However, if σ depends on S, then we may not be able to convert the problems(3.4) and (3.5) into (3.6) and (3.7) by a simple transformation.

3.2.3 Formulation of the Linear Complementarity Problem on a Finite Domain

Generally speaking, r, D_0, and σ are not constants. For simplicity, we assume that σ depends on S in this subsection even though the derivation is almost the same when r, D_0, and σ all depend on S and t.

From Sect. (2.2.5), we know that through the transformation

$$\begin{cases} \xi = \dfrac{S}{S+E}, \\ \tau = T - t, \\ V(S,t) = (S+E)\overline{V}(\xi,\tau) = \dfrac{E}{1-\xi}\overline{V}(\xi,\tau), \end{cases}$$

the operator

$$\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2(S)S^2\frac{\partial^2}{\partial S^2} + (r - D_0)S\frac{\partial}{\partial S} - r$$

is converted into

$$\frac{-E}{1-\xi}\left\{\frac{\partial}{\partial \tau} - \frac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1-\xi)^2\frac{\partial^2}{\partial \xi^2} - (r - D_0)\xi(1-\xi)\frac{\partial}{\partial \xi} + [r(1-\xi) + D_0\xi]\right\},$$

where $\bar{\sigma}(\xi) = \sigma(E\xi/(1-\xi))$, and the function $\max(\pm(S-E), 0)$ becomes

$$\frac{E}{1-\xi}\max(\pm(2\xi - 1), 0).$$

Therefore, problem (3.4) can be rewritten as

$$\begin{cases} \min\left(\dfrac{\partial \overline{V}}{\partial \tau} - \mathbf{L}_\xi \overline{V},\ \overline{V}(\xi,\tau) - \max(1-2\xi, 0)\right) = 0, & 0 \le \xi \le 1, \\ & 0 \le \tau, \\ \overline{V}(\xi, 0) = \max(1 - 2\xi, 0), & 0 \le \xi \le 1, \end{cases} \tag{3.8}$$

where

$$\mathbf{L}_\xi = \frac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1-\xi)^2\frac{\partial^2}{\partial \xi^2} + (r - D_0)\xi(1-\xi)\frac{\partial}{\partial \xi} - [r(1-\xi) + D_0\xi].$$

This is the American put option problem reformulated as a linear complementarity problem on a finite domain. Similarly, from the problem (3.5) we know that the American call option problem can be reformulated as the following linear complementarity problem:

$$\begin{cases} \min\left(\dfrac{\partial \overline{V}}{\partial \tau} - \mathbf{L}_\xi \overline{V},\ \overline{V}(\xi,\tau) - \max(2\xi - 1, 0)\right) = 0, & 0 \le \xi \le 1, \\ & 0 \le \tau, \\ \overline{V}(\xi, 0) = \max(2\xi - 1, 0), & 0 \le \xi \le 1. \end{cases} \tag{3.9}$$

In this section, an American option is reduced to a linear complementarity problem. Such a problem usually needs to be solved numerically. Here, we need to point out that the version given in Sect. 3.2.2 can be applied only if σ does not depend on S and that the other two versions can be applied for any case. However, the version given in Sect. 3.2.2 has the simplest equation. Also, if an implicit scheme is used, then for the versions given in Sects. 3.2.1 and 3.2.2, artificial boundary conditions are needed at the boundaries because numerical methods have to be performed on a finite domain. However, the version given in this subsection does not have such a problem.

3.2.4 More General Form of the Linear Complementarity Problems

From the three previous subsections, we see that a linear complementarity problem could be in the form:

$$\begin{cases} \min\left(-\dfrac{\partial V(S,t)}{\partial t} - \mathbf{L_{S,t}}V(S,t),\ V(S,t) - G_v(S,t)\right) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad S_l \le S \le S_u,\ t \le T, \\ V(S,T) = G_v(S,T), \qquad\qquad S_l \le S \le S_u, \end{cases}$$

where[2]

$$\mathbf{L_{S,t}} = a(S,t)\frac{\partial^2}{\partial S^2} + b(S,t)\frac{\partial}{\partial S} + c(S,t).$$

However, a linear complementarity problem could have a more general form such as

$$\begin{cases} \min\left(-\dfrac{\partial V(S,t)}{\partial t} - \mathbf{L_{S,t}}V(S,t) - d(S,t),\ V(S,t) - G_v(S,t)\right) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad S_l \le S \le S_u,\ t \le T, \\ V(S,T) = G_1(S) \ge G_v(S,T), \qquad\qquad S_l \le S \le S_u. \end{cases} \tag{3.10}$$

In this problem there are two new features. There is a new function $d(S,t)$ called the nonhomogeneous term of the problem and the payoff $G_1(S)$ is not equal to $G_v(S,T)$. The linear complementarity problem for one-factor convertible bonds has such a form. For two-factor convertible bonds, the form of the linear complementarity problem is similar, but the operator $\mathbf{L_{S,t}}$ is replaced by a two-dimensional one (see Chap. 5).

From what we have done in this section, we know the following. Consider a European style derivative and an American style derivative with identical payoffs $G_1(S)$, identical operators, and identical nonhomogeneous terms.

[2] If $S_l = -\infty$, then the first "$\le$" needs to be changed into "$<$," and if $S_u = \infty$, then the second "$\le$" needs to be changed into "$<$." In what follows, the same notation is used.

Suppose that the American style derivative has a constraint $G_v(S,t)$ satisfying $G_v(S,T) \le G_1(S)$. If the price of the European style derivative is the solution of the PDE problem

$$\begin{cases} \dfrac{\partial v(S,t)}{\partial t} + \mathbf{L_{s,t}} v(S,t) + d(S,t) = 0, & S_l \le S \le S_u,\ t \le T, \\ v(S,T) = G_1(S) & S_l \le S \le S_u, \end{cases}$$

then the price of the American style derivative with a constraint $G_v(S,t)$ satisfying $G_v(S,T) \le G_1(S)$ is the solution of LC problem (3.10).

3.3 American Option Problems as Free-Boundary Problems

3.3.1 Free Boundaries

From the past two sections, we discovered that there are some regions where the Black–Scholes equation cannot be used. Therefore, there exist two different types of regions: one where the Black–Scholes equation is valid, and the other where the Black–Scholes equation cannot be used and the solution is equal to the constraint. Because we do not know *a priori* the location of the boundaries between the two types of different regions, these boundaries are called free boundaries. Because in some regions the solution is known, we only need to determine the price in other regions and the locations of these free boundaries. In order to do that, we reformulate the American option problems as so-called free-boundary problems (FBPs).

Let us first discuss how to find the locations of the free boundaries at time T. Using Theorem 3.1, we can easily determine the locations of free boundaries at time T, namely, the starting points of free boundaries. We will show that for an American put option with $r > 0$, there is a free boundary starting from the point $(\min(E, rE/D_0), T)$ on the (S,t)-plane. If $r = 0$, then there is no free boundary. This implies that the Black–Scholes equation is valid everywhere and that the prices of the American and European put options are the same if $r = 0$. For an American call option, the situation is similar. If $D_0 > 0$, then there is a free boundary starting from the point $(\max(E, rE/D_0), T)$ on the (S,t)-plane. If $D_0 = 0$, then there is no free boundary, implying that an American call option is the same as a European call option.

First, let us consider an American put option and let $P(S,t)$ denote its value as we did in Sect. 3.1.1. In this case

$$G_p(S,t) = \max(E - S, 0) = \begin{cases} E - S, & \text{for } S < E, \\ 0, & \text{for } S \ge E \end{cases}$$

and the operator $\mathbf{L_{s,t}}$ in this case does not depend on t and is equal to

$$\mathbf{L_s} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0)S\frac{\partial}{\partial S} - r.$$

For $S \in (E, \infty)$, we have $G_p(S, t) = 0$ and

$$\frac{\partial G_p}{\partial t}(S, T) + \mathbf{L_s} G_p(S, T) = 0,$$

which means that the PDE can be used on (E, ∞). For $S \in (0, E)$, we have $G_p(S, t) = E - S$ and

$$\begin{aligned} &\frac{\partial G_p}{\partial t}(S, T) + \mathbf{L_s} G_p(S, T) \\ &= \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 G_p}{\partial S^2} + (r - D_0)S\frac{\partial G_p}{\partial S} - rG_p \\ &= \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2}(E - S) + (r - D_0)S\frac{\partial}{\partial S}(E - S) - r(E - S) \\ &= D_0 S - rE. \end{aligned}$$

The root of the equation $D_0 S - rE = 0$ is $S^* = rE/D_0$. If $E > rE/D_0$, then there are two situations: $S \in (0, rE/D_0)$ and $S \in (rE/D_0, E)$. On $(0, rE/D_0)$

$$\frac{\partial G_p}{\partial t}(S, T) + \mathbf{L_s} G_p(S, T) = D_0 S - rE < 0$$

and on $(rE/D_0, E)$

$$\frac{\partial G_p}{\partial t}(S, T) + \mathbf{L_s} G_p(S, T) = D_0 S - rE > 0.$$

Thus in this case, the entire S-axis is divided into two parts: $(0, rE/D_0)$ where the Black–Scholes equation cannot be used and $(rE/D_0, \infty)$ where the Black–Scholes equation gives the price of the American put option. Consequently, if $E > rE/D_0$, there is only one free boundary at time T when $r > 0$ and the location of the free boundary is $S = rE/D_0$. If $E < rE/D_0$, then on the entire interval $(0, E)$

$$\frac{\partial G_p}{\partial t}(S, T) + \mathbf{L_s} G_p(S, T) = D_0 S - rE < 0.$$

Thus in this case, the entire S-axis is divided into two parts: $(0, E)$ where the Black–Scholes equation cannot be used and (E, ∞) where the Black–Scholes equation gives the price of the American put option. Consequently, if $E < rE/D_0$, then there is also only one free boundary at time T when $r > 0$ and the location of the free boundary is $S = E$. Put them together, we have that there is only one free boundary at time T when $r > 0$ and the location of the free boundary is $S = min(E, rE/D_0)$. Let $S_f(t)$ denote this free boundary. Because it starts from the point $(\min(E, rE/D_0), T)$, we have

$$S_f(T) = \min\left(E, \frac{rE}{D_0}\right). \tag{3.11}$$

If $r = 0$, then $\min(E, rE/D_0) = 0$, so in the entire interval $(0, \infty)$, the Black–Scholes equation can be used, and there is no free boundary.

Now let us explain that in the case $r > 0$, no new free boundary can appear at any time $t < T$, so $S_f(t)$ is the only free boundary in this problem, and that $S_f(t)$ is not a constant, but an increasing function in t (see Fig. 3.3). First let us explain this when t is discrete. Similarly to what we did in Sect. 3.1.2, set $\Delta t = T/N$ and $t_n = n\Delta t, n = 0, 1, \cdots, N, N$ being a large integer, let $\tilde{P}(S, t_N) = G_p(S)$ and $\tilde{S}_f(t_N) = S_f(T)$, and for $n = N-1, N-2, \cdots, 0$, successively, define $\tilde{P}(S, t_n)$ by

$$\tilde{P}(S, t_n) = \max\left(\tilde{p}(S, t_n), G_p(S)\right),$$

where

$$\tilde{p}(S, t_n) = \mathrm{e}^{-r\Delta t} \int_0^\infty \tilde{P}(S', t_{n+1}) G(S', t_{n+1}; S, t_n) dS'.$$

At $S = 0$, $G_p(0) = E > \tilde{p}(0, t_n) = \mathrm{e}^{-r\Delta t}E$ and at $S = S^* \approx \infty$, $G_p(S^*) = 0$ and $\tilde{p}(S^*, t_n) > 0$, the two continuous curves $\tilde{p}(S, t_n)$ and $G_p(S)$ must have at least one intersection point and let us denote the location of the intersection point with the largest S value by $\tilde{S}(t_n)$. Thus for any $S \in (\tilde{S}(t_n), \infty)$, $\tilde{P}(S, t_n) = \tilde{p}(S, t_n)$. If $S \in (E, \infty)$, for $\tilde{P}(S, t_{N-1})$ we have

$$\begin{aligned}
\tilde{P}(S, t_{N-1}) &= \max\left(\tilde{p}(S, t_{N-1}), G_p(S)\right) \\
&= \max\left(\tilde{p}(S, t_{N-1}), \max(E - S, 0)\right) \\
&= \tilde{p}(S, t_{N-1}) \\
&= \mathrm{e}^{-r\Delta t} \int_0^\infty \max(E - S', 0) G(S', t_N; S, t_{N-1}) dS' > 0.
\end{aligned}$$

Thus for the case $S_f(t_N) = E$, then $\tilde{P}(S, t_{N-1}) > G_p(S) = \max(E - S, 0) = 0$ for $S \in (E, \infty)$; for the case $S_f(t_N) = rE/D_0$, for any point in $(rE/D_0, E)$,

$$\frac{\partial G_p(S)}{\partial t} + \mathbf{L_s} G_p(S) = D_0 S - rE > 0,$$

so $\tilde{P}(S, t_{N-1}) > G_p(S)$ also holds for $S \in (rE/D_0, E)$. Consequently, put them together, we have that for any $S \in (S_f(t_N), \infty)$, $\tilde{P}(S, t_{N-1}) > G_p(S)$, from which we know $\tilde{P}(S_f(t_N), t_{N-1}) > G_p(S_f(t_N))$ holds also. Thus we have $\tilde{S}_f(t_{N-1}) < \tilde{S}_f(t_N)$ and $\tilde{P}(S, t_{N-1}) > G_p(S)$ on $(\tilde{S}_f(t_{N-1}), \tilde{S}_f(t_N))$.

Now let us assume that for certain n we have $\tilde{P}(S, t_{n+1}) > \tilde{P}(S, t_{n+2})$ for $S \in (\tilde{S}_f(t_{n+1}), \infty)$, and show $\tilde{P}(S, t_n) > \tilde{P}(S, t_{n+1})$ on $(\tilde{S}_f(t_n), \infty)$ and $\tilde{S}_f(t_n) < \tilde{S}_f(t_{n+1})$. In order to show this result, we only need to show $\tilde{p}(S, t_n) > \tilde{p}(S, t_{n+1})$ for $S \in (0, \infty)$. This is easy to see: for $S \in (0, \infty)$

$$\begin{aligned}\tilde{p}(S, t_n) &= \mathrm{e}^{-r\Delta t}\int_0^\infty \tilde{P}(S', t_{n+1})G(S', t_{n+1}; S, t_n)dS' \\ &> \mathrm{e}^{-r\Delta t}\int_0^\infty \tilde{P}(S', t_{n+2})G(S', t_{n+2}; S, t_{n+1})dS' \\ &= \tilde{p}(S, t_{n+1}) \end{aligned} \tag{3.12}$$

because from Sect. 3.1.2 we have $\tilde{P}(S, t_{n+1}) \geq \tilde{P}(S, t_{n+2})$ for any $S \in (0, \infty)$ and it is given that $\tilde{P}(S, t_{n+1}) > \tilde{P}(S, t_{n+2})$ on $(\tilde{S}_f(t_{n+1}), \infty)$. Here we also have used the fact that $G(S', t_{n+1}; S, t_n) = G(S', t_{n+2}; S, t_{n+1}) > 0$ for $S \in (0, \infty)$ and $S' \in (0, \infty)$. From the relation (3.12) we know $\tilde{P}(S, t_n) > \tilde{P}(S, t_{n+1})$ on $(\tilde{S}_f(t_{n+1}), \infty)$ because on this interval $\tilde{P}(S, t_n) = \tilde{p}(S, t_n)$ and $\tilde{P}(S, t_{n+1}) = \tilde{p}(S, t_{n+1})$, which means that we can have $\tilde{S}_f(t_n) < \tilde{S}_f(t_{n+1})$. From the definition of $\tilde{S}_f(t_n)$, we further know $\tilde{P}(S, t_n) > \tilde{P}(S, t_{n+1})$ on $(\tilde{S}_f(t_n), \tilde{S}_f(t_{n+1}))$. For $n = N - 1$, we already have $\tilde{P}(S, t_{N-1}) > G_p(S) = \tilde{P}(S, t_N)$ for $S \in (S_f(t_N), \infty)$. Thus this procedure can be done for $n = N-2, N-3, \cdots, 0$, successively.

On $(0, S_f(t_N))$, $\tilde{P}(S, t_N) = G_p(S)$ and the following inequality

$$\frac{\partial G_p}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 G_p}{\partial S^2} + (r - D_0)S\frac{\partial G_p}{\partial S} - rG_p < 0$$

holds, which means that $\tilde{p}(S, t_{N-1}) < G_p(S)$ on that interval if Δt is small enough. Therefore the inequality $\tilde{p}(S, t_{N-1}) < G_p(S)$ must hold on $(0, \tilde{S}_f(t_{N-1}))$ at least for a very small Δt. Consequently, no more intersection points exist. This procedure can also be done for $n = N-2, N-3, \cdots, 0$, successively. Consequently no new free boundary will appear during the entire procedure if Δt is small enough. Let $N \to \infty$, we will have the conclusion we need to explain.

Consequently, if $r > 0$, then there is a unique free boundary, and the entire domain is divided into two regions by the free boundary (see Fig. 3.3): one region is $[0, S_f(t)) \times [0, T]$, where

$$\begin{cases} P = \max(E - S, 0) = E - S, \\ \dfrac{\partial P}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 P}{\partial S^2} + (r - D_0)S\dfrac{\partial P}{\partial S} - rP < 0 \end{cases}$$

and the other is $(S_f(t),\ \infty) \times [0, T]$, where

$$\begin{cases} P > \max(E - S, 0), \\ \dfrac{\partial P}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 P}{\partial S^2} + (r - D_0)S\dfrac{\partial P}{\partial S} - rP = 0 \end{cases}$$

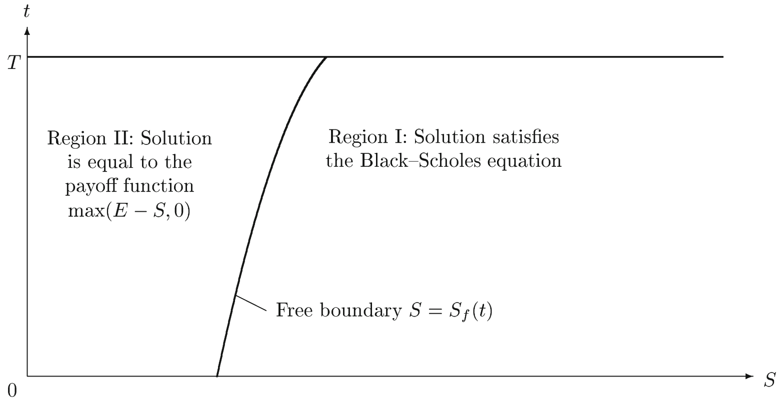

Fig. 3.3. Structure of solution to American put options ($r > 0$)

if $t < T$. Also if at a point (S, t), $P(S, t) > \max(E - S, 0)$, then $P(S, t - \Delta t) > P(S, t)$ for any positive Δt, and the location of the free boundary has the following property (see Fig. 3.2):

$$S_f(t) > S_f(t - \Delta t), \quad \Delta t > 0,$$

implying that $S_f(t)$ is an increasing function of t (see Fig. 3.3).

Before going further, we would like to give some remarks.

- What is the meaning of the inequality

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r - D_0) S \frac{\partial P}{\partial S} - rP < 0?$$

  As pointed out in Sect. 3.1.1, this means that holding the money market account with a value of $P - \frac{\partial P}{\partial S} S$ and $\frac{\partial P}{\partial S}$ shares will be better than holding the option. In this case exercising the option and holding a money market account with a value of $P - \frac{\partial P}{\partial S} S$ and $\frac{\partial P}{\partial S}$ shares will have better return than holding the option. Therefore the option should be exercised. If $P(S, t) > \max(E - S, 0)$, one should hold the option, as one should not give up a higher value (the option) for a lower value (the intrinsic value). Therefore, the free boundary is the optimal exercise price that divides the exercise region and the non-exercise region.
- Let $\mathbb{D}_{ge}$ denote the open domain where $\frac{\partial G_v}{\partial t}(S, t) + \mathbf{L}_{\mathbf{S},\mathbf{t}} G_v(S, t) \geq 0$ and $\mathbb{D}_l$ the open domain where $\frac{\partial G_v}{\partial t}(S, t) + \mathbf{L}_{\mathbf{S},\mathbf{t}} G_v(S, t) < 0$. For the put option case, $G_v(S, t) = \max(E - S, 0)$ and the open domain $\mathbb{D}_{ge}$ is $(\min(E, rE/D_0), \infty) \times [0, T]$ and $\mathbb{D}_l$ is $(0, \min(E, rE/D_0)) \times [0, T]$. In a neighborhood of a point in the open domain $\mathbb{D}_{ge}$, if $V(S, t) > G_v(S, t)$, then

the PDE can be used because we can let a positive Δt be small enough to guarantee $V(S, t-\Delta t) > G_v(S, t-\Delta t)$, and if $V(S,T) = G_v(S,T)$, then the PDE can also be used because $\dfrac{\partial G_v}{\partial t}(S,t) + \mathbf{L_{S,t}} G_v(S,t) \geq 0$. Thus a point on a free boundary cannot appear in the open domain $\mathbb{D}_{ge}$. In a neighborhood of a point in the open domain $\mathbb{D}_l$, if $V(S,t) > G_v(S,t)$, then the PDE can be used, and if $V(S,T) = G_v(S,T)$, then the PDE cannot be used because $\dfrac{\partial G_v}{\partial t}(S,t) + \mathbf{L_{S,t}} G_v(S,t) < 0$. Thus a point on a free boundary may appear in the open domain $\mathbb{D}_l$.

- From theorem 3.1, we can find that there are two types of points on free boundaries. The first type of points is: in a neighborhood of the point, $V(S,t) = G_v(S,t)$ and some portion of the neighborhood belongs to $\mathbb{D}_{ge}$ and another portion of the neighborhood belongs to $\mathbb{D}_l$. The second type of points is: in some portion of a neighborhood of the point, $V(S,t) > G_v(S,t)$ and in another portion of the neighborhood, $V(S,t) = G_v(S,t)$ and this portion belongs to $\mathbb{D}_l$. Thus a free boundary will appear only in the open domain $\mathbb{D}_l$ and on the boundary between the open domains $\mathbb{D}_{ge}$ and $\mathbb{D}_l$. If $V(S,T) = G_v(S,T)$, then a free boundary will start at a point between the open domains $\mathbb{D}_{ge}$ and $\mathbb{D}_l$. For example, the free boundary of an American put option starts at such a point. If $V(S,T) > G_v(S,T)$ on some portion of the entire domain and $V(S,T) = G_v(S,T)$ on another portion, then a free boundary might also start at a boundary between an open interval belonging to $\mathbb{D}_l$ and an open interval where $V(S,T) > G_v(S,T)$. As we will see in Sect. 5.7, the free boundary of a one-factor convertible bond can start from a point of the first type of points or a point of the second type of points. Later, a free boundary may move but never move into the open domain $\mathbb{D}_{ge}$.

Now let us consider an American call option. From Sect. 2.2.5 we know, at very large S, the solution of the Black–Scholes equation with final condition $V(S,t) = \max(S-E, 0)$ has the following asymptotic expression

$$V(S,t) \approx V(S,T)\mathrm{e}^{-D_0(T-t)} = \max(S-E,0)\mathrm{e}^{-D_0(T-t)},$$

so if $D_0 > 0$, then $V(S,t) < \max(S-E,0)$ for any $t < T$. Therefore, if $D_0 > 0$, the American call problem is a free-boundary problem. Now let us show that the free-boundary problem has only one free boundary, which is also denoted by $S_f(t)$ in what follows, and determine the location of the free boundary at $t = T$ from the constraint condition $C(S,t) \geq G_c(S,t)$.

In the case of an American call option,

$$G_c(S,t) = \max(S-E,0) = \begin{cases} S-E, & S > E, \\ 0, & S \leq E. \end{cases}$$

Let $S > \max\left(E, \frac{rE}{D_0}\right)$. In this case

$$G_c(S,t) = S - E$$

and

$$\begin{aligned}
&\frac{\partial G_c}{\partial t}(S,T) + \mathbf{L_s} G_c(S,T) \\
&= \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 G_c}{\partial S^2} + (r - D_0)S\frac{\partial G_c}{\partial S} - rG_c \\
&= \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 (S-E)}{\partial S^2} + (r - D_0)S\frac{\partial (S-E)}{\partial S} - r(S-E) \\
&= rS - D_0 S - rS + rE = -D_0 S + rE < 0
\end{aligned}$$

because $S > \frac{rE}{D_0}$. Therefore, the Black–Scholes equation cannot hold in this case, and $C(S, T-\Delta t)$ should be equal to $S-E$ for $S > \max\left(E, \frac{rE}{D_0}\right)$. Just like the case of the American put option, we can know that for $S < \max\left(E, \frac{rE}{D_0}\right)$, the Black–Scholes equation can hold. Thus, a free boundary starts at $S = \max\left(E, \frac{rE}{D_0}\right)$, i.e.,

$$S_f(T) = \max\left(E, \frac{rE}{D_0}\right). \tag{3.13}$$

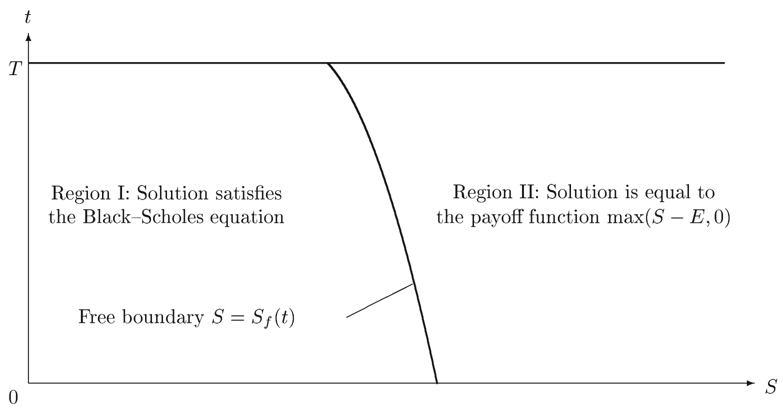

Fig. 3.4. Structure of solution to American call options ($D_0 > 0$)

Using the same argument we have used for an American put option, we can show that the free boundary starting from the point $(\max(E, rE/D_0), T)$ is the only free boundary because no new free boundary can appear at time $t < T$. Just like the put case, the entire domain is divided into two parts by the free boundary. However, the situation is a little different from the American put. Here in the region $[0, S_f(t)) \times [0, T]$, the Black–Scholes equation holds, whereas in the region $(S_f(t), \infty) \times [0, T]$, the Black–Scholes equation cannot be used. In other words, for $S \in [0, S_f(t))$ and $t < T$,

$$\begin{cases} C(S,t) \geq \max(S-E, 0), \\ \dfrac{\partial C}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 C}{\partial S^2} + (r - D_0)S\dfrac{\partial C}{\partial S} - rC = 0, \end{cases}$$

where the equal sign in $C(S,t) \geq \max(S-E, 0)$ holds only at $S = 0$; whereas for $S \in (S_f(t), \infty)$,

$$\begin{cases} C(S,t) = \max(S-E, 0) = S - E, \\ \dfrac{\partial C}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 C}{\partial S^2} + (r - D_0)S\dfrac{\partial C}{\partial S} - rC < 0 \end{cases}$$

and the option should be exercised. It can also be shown that for an American call option, the free boundary $S_f(t)$ is a decreasing function of t, as graphed in Fig. 3.4, and that the price of an American call option is the same as a European call if $D_0 = 0$.

3.3.2 Free-Boundary Problems

In this subsection, we will describe the formulation of American option problems as free-boundary problems. In order to give a complete formulation, we need to give the conditions on the free boundary. For an initial-boundary value problem of a parabolic equation on a finite interval, if the locations of the boundaries are given and if the coefficient of the second derivative at the boundaries is not equal to zero, one boundary condition at each boundary is needed in order for the problem to have a unique solution. However, the location of the free boundary is unknown, so two conditions are needed at the free boundary in order for the problem to have a unique solution. One boundary condition determines the option value on the free boundary and the other boundary condition determines the location of the free boundary. Now the question is what the two conditions should be. For some other linear complementarity problems, it has been proved that on the free boundary the value and the first derivative are continuous (see [31]). For this problem, from the proof given by Badea and Wang (see [4] and [5]), the situation is still the same. Therefore, the two conditions on the free boundary are: both the value and the derivative with respect to S are continuous.

For an American put option, in the region $[0,\ S_f(t))$,

$$P(S,t) = E - S$$

and

$$\frac{\partial P}{\partial S} = -1.$$

Therefore, the boundary conditions on the free boundary $S_f(t)$ are

$$P(S_f(t), t) = E - S_f(t) \tag{3.14}$$

and

$$\frac{\partial P}{\partial S}(S_f(t), t) = -1. \tag{3.15}$$

It is clear that when the boundary condition (3.15) holds, the gradient $\dfrac{\partial P}{\partial S}$ must be continuous at $S = S_f$, which is shown in Fig. 3.1.

Now we can formulate the American put option problem. On the domain $[0, S_f(t)) \times [0, T]$,

$$P(S, t) = E - S,$$

while on the domain $[S_f(t), \infty) \times [0, T]$, $P(S, t)$ is the solution of the free-boundary problem[3] for American put options

$$\begin{cases} \dfrac{\partial P}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 P}{\partial S^2} + (r - D_0) S \dfrac{\partial P}{\partial S} - rP = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad S_f(t) \le S, \quad 0 \le t \le T, \\ P(S, T) = \max(E - S, 0), \qquad S_f(T) \le S, \\ P(S_f(t), t) = E - S_f(t), \qquad t \le T, \\ \dfrac{\partial P(S_f(t), t)}{\partial S} = -1, \qquad t \le T, \\ S_f(T) = \min\left(E, \dfrac{rE}{D_0}\right). \end{cases} \tag{3.16}$$

Similarly, for call options we need two boundary conditions on the free boundary. One is

$$C(S_f(t), t) = S_f(t) - E \tag{3.17}$$

and the other still can be obtained by requiring the continuity of the slope of the solution at $S = S_f(t)$. In this case, the condition is

$$\frac{\partial C(S_f(t), t)}{\partial S} = 1. \tag{3.18}$$

[3] In this book we call this problem and the like a free-boundary problem. An LC problem usually involves free boundaries. Thus it is not strange that some people call an LC problem a free-boundary problem.

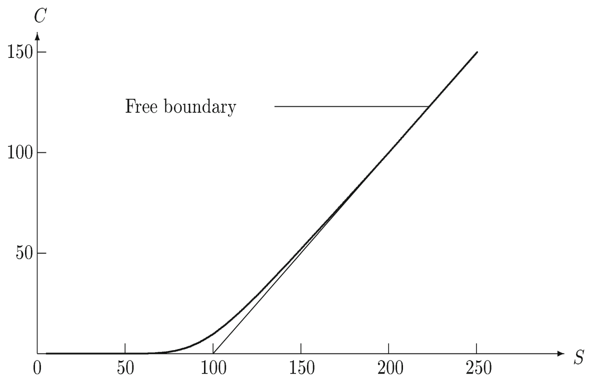

Fig. 3.5. Numerically calculated solution of an American call problem with $E = 100$, $r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$, and $T = 1$ year

Therefore for the American call option, the formulation is as follows. On the domain $[0, S_f(t)] \times [0, T]$, $C(S, t)$ is the solution of the free-boundary problem for American call options

$$\begin{cases} \dfrac{\partial C}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 C}{\partial S^2} + (r - D_0)S\dfrac{\partial C}{\partial S} - rC = 0, \\ \qquad\qquad\qquad\qquad 0 \le S \le S_f(t), \quad 0 \le t \le T, \\ C(S, T) = \max(S - E, 0), \quad 0 \le S \le S_f(T), \\ C(S_f(t), t) = S_f(t) - E, \quad 0 \le t \le T, \\ \dfrac{\partial C}{\partial S}(S_f(t), t) = 1, \quad 0 \le t \le T, \\ S_f(T) = \max\left(E, \dfrac{rE}{D_0}\right); \end{cases} \tag{3.19}$$

whereas on the domain $(S_f(t), \infty) \times [0, T]$, $C(S, t) = S - E$. In Fig. 3.5, the value of an American call option is plotted, from which we can see that the two parts of solution are connected smoothly. The parameters of the problem are $E = 100$, $r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$, and $T = 1$ year.

Here we need to point out that $S_f(T)$ is determined by the partial differential operator and the final condition. Therefore, in a free-boundary problem, the starting location of the free boundary is not arbitrary and should be consistent with the partial differential operator and the final condition.

As has been pointed, there are two formulations for American option problems. It is clear that the solutions obtained from the two formulations should be the same. In this book, we will not carefully study this problem. However in Sect. 3.3.5 for the perpetual American call option, we will prove that the solution obtained by solving the free-boundary problem is the solution of the

LC problem. Here we just show the following. For the American put problem, if the solution of the problem (3.16) satisfies the conditions $P(S,t) \geq 0$ and $\dfrac{\partial P^2(S,t)}{\partial S^2} \geq 0$ for $S_f(t) < S$, then the solution, including the part on the domain $[0, S_f(t)) \times [0,T]$ and the part on the domain $[S_f(t), \infty) \times [0,T]$, satisfies the LC relation:

$$\min\left(-\frac{\partial P}{\partial t} - \mathbf{L_s}P,\; P(S,t) - \max(E-S,0)\right) = 0, \quad 0 < S, \quad t \leq T,$$

where

$$\mathbf{L_s} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0)S\frac{\partial}{\partial S} - r.$$

The proof is as follows. On the interval $(0, S_f(t))$, $P(S,t) = G_p(S)$ and this interval belongs to the domain $\mathbb{D}_l$, which means

$$-\frac{\partial G_p(S)}{\partial t} - \mathbf{L}_S G_p(S) > 0.$$

Consequently, the LC relation holds for this case. For the case $S_f(t) < S \leq E$, we have $-\dfrac{\partial P}{\partial t} - \mathbf{L}_S P = 0$, and we need to show $P(S,t) - (E-S) \geq 0$ in order to prove our conclusion. Define $f(S,t) = P(S,t) - (E-S)$. We know that $f(S_f(t),t) = P(S_f(t),t) - (E - S_f(t)) = 0$ and $\dfrac{\partial f(S_f(t),t)}{\partial S} = \dfrac{\partial P(S_f(t),t)}{\partial S} + 1 = 0$. Thus for a fixed t, we have

$$\begin{aligned} f(S,t) &= f(S_f(t),t) + \frac{\partial f(S_f(t),t)}{\partial S}\left[S - S_f(t)\right] + \frac{1}{2}\frac{\partial^2 f(S^*,t)}{\partial S^2}\left[S - S_f(t)\right]^2 \\ &= \frac{1}{2}\frac{\partial^2 P(S^*,t)}{\partial S^2}\left[S - S_f(t)\right]^2 \geq 0, \end{aligned}$$

where $S^* \in (S_f(t), S)$ and we have used the condition $\dfrac{\partial P^2(S,t)}{\partial S^2} \geq 0$ for $S_f(t) < S$. For the case $E < S$, we have $-\dfrac{\partial P}{\partial t} - \mathbf{L}_S P = 0$ and $P(S,t) - \max(E-S,0) = P(S,t) \geq 0$, and thus the LC relation holds on $(S_f(t), \infty)$. Because the LC relation holds on $(0, S_f(t))$, $(S_f(t), \infty)$, the LC relation at the points 0 and $S_f(t)$ also holds, which can be shown by letting S go to these points. Consequently the LC relation holds for all the cases and the proof is completed.

As is proved in Problem 41 of Chap. 2, if $D_0 = 0$, then the value of an American call option is equal to the value of a European call option. Thus in this case there is no free boundary, that is, there is no optimal exercise price. A new question is: does the optimal exercise price exist when the dividends are paid discretely? The answer is that when there are discrete dividends, the American call option can only be optimal to exercise at a time immediately before the stock goes ex-dividend and that an optimal exercise price does not always exist even at those moments. Readers are asked to prove these conclusions as Problem 15.

3.3.3 Put–Call Symmetry Relations

As we know, the price of an American put option is the solution of the following LC problem:

$$\begin{cases} \min\left(-\dfrac{\partial P}{\partial t} - \mathbf{L}_S P,\ P(S,t) - \max(E-S,0)\right) = 0, & 0 \le S,\ t \le T, \\ P(S,T) = \max(E-S,0), & 0 \le S, \end{cases}$$

where

$$\mathbf{L}_S = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0)S\frac{\partial}{\partial S} - r.$$

Let

$$\begin{cases} \zeta = \dfrac{E^2}{S}, \\ C(\zeta,t) = \dfrac{EP(S,t)}{S}. \end{cases} \tag{3.20}$$

Because

$$\frac{E}{S}\max(E-S,0) = \max(\zeta - E, 0),$$

for $C(\zeta,t)$ the payoff and constraint are $\max(\zeta - E, 0)$. Noticing

$$\begin{aligned} \frac{\partial P}{\partial t} &= \frac{S}{E}\frac{\partial C}{\partial t}, \\ \frac{\partial P}{\partial S} &= \frac{1}{E}\left[C + S\frac{\partial C}{\partial \zeta}\left(-\frac{E^2}{S^2}\right)\right] = \frac{1}{E}\left(C - \zeta\frac{\partial C}{\partial \zeta}\right), \\ \frac{\partial^2 P}{\partial S^2} &= \frac{\zeta^3}{E^3}\frac{\partial^2 C}{\partial \zeta^2}, \end{aligned}$$

we have

$$\begin{aligned} &\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r - D_0)S\frac{\partial P}{\partial S} - rP \\ &= \frac{S}{E}\left\{\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2\zeta^2\frac{\partial^2 C}{\partial \zeta^2} + (D_0 - r)\zeta\frac{\partial C}{\partial \zeta} - D_0 C\right\}. \end{aligned}$$

Therefore the function $C(\zeta,t)$ is the solution of the following American call option problem:

$$\begin{cases} \min\left(-\dfrac{\partial C}{\partial t} - \mathbf{L}_\zeta C,\ C(\zeta, t) - \max(\zeta - E, 0)\right) = 0, & 0 \le \zeta,\ t \le T, \\ C(\zeta, T) = \max(\zeta - E, 0), & 0 \le \zeta, \end{cases} \tag{3.21}$$

where

$$\mathbf{L}_\zeta = \frac{1}{2}\sigma^2\zeta^2\frac{\partial^2}{\partial\zeta^2} + (D_0 - r)\zeta\frac{\partial}{\partial\zeta} - D_0.$$

Consequently, an American put problem can be converted into an American call problem. However in the two problems, the state variable and the parameters are different. From the definitions of $\mathbf{L}_S$ and $\mathbf{L}_\zeta$, we know that the volatilities of the put and call problems are the same, but the interest rate and the dividend yield of the call problem are equal to the dividend yield and the interest rate of the put problem, respectively. In order to explain these facts, we express the dependency of the options on interest rate and dividend yield explicitly. Let $P(S, t; b, a)$ denote the price of the put option and $C(\zeta, t; a, b)$ the price of the call option, where the first and second parameters after the semicolon are the interest rate and the dividend yield, respectively. From the definition of ζ and $C(\zeta, t; a, b)$, we know

$$P(S, t; b, a) = C\,(\zeta, t; a, b)\, S/E,$$

where $\zeta = \dfrac{E^2}{S}$. This can also be rewritten as

$$P(S, t; b, a) = C\left(E^2/S, t; a, b\right) S/E,$$

or

$$C\,(\zeta, t; a, b) = P\left(E^2/\zeta, t; b, a\right) \zeta/E,$$

where we used the relation $E/S = \zeta/E$. In the last relation, we can use S, instead of ζ, as the state variable. That is, we can write this relation as

$$C\,(S, t; a, b) = P\left(E^2/S, t; b, a\right) S/E.$$

Finally, putting them together, we have

$$\begin{cases} C(S, t; a, b) = P\left(E^2/S, t; b, a\right) S/E, \quad \text{or} \\ P(S, t; b, a) = C\left(E^2/S, t; a, b\right) S/E. \end{cases} \tag{3.22}$$

For the special case $S = E$, it becomes

$$P(E, t; b, a) = C(E, t; a, b).$$

Also, the location of free boundary in the latter problem, $\zeta_{cf}(t; a, b)$, must be equal to E^2 divided by the location of free boundary of the former problem, $E^2/S_{pf}(t; b, a)$, because $\zeta = E^2/S$, i.e.,

$$\zeta_{cf}(t;a,b) = E^2/S_{pf}(t;b,a)$$

or

$$S_{cf}(t;a,b) \times S_{pf}(t;b,a) = E^2, \tag{3.23}$$

where in the last relation, instead of ζ_{cf}, we use S_{cf} as the name of the function representing the location of the free boundary. From the derivation above we know that for European options, the following relations also hold:

$$\begin{cases} c(S,t;a,b) = p\left(E^2/S,t;b,a\right) S/E, \quad \text{or} \\ p(S,t;b,a) = c\left(E^2/S,t;a,b\right) S/E. \end{cases} \tag{3.24}$$

The relations (3.22)–(3.24) are called the put–call symmetry relations.

Now let us discuss the financial meaning of the put–call symmetry relations. Suppose that one British pound is worth S U.S. dollars and that E^2 U.S. dollars are worth ζ British pounds. It is clear that $\zeta = E^2/S$. Let P be the value of a put option whose holder can always sell one pound for E dollars if the holder wants. This means that the payoff and constraint of the put option is $\max(E-S,0)$ in dollars. Let C be the value of a call option whose holder can buy E^2 dollars by paying E pounds if the holder wants. This means that the payoff and constraint of the call option are $\max(E^2/S-E,0) = \max(\zeta-E,0)$ in pounds. The holder of the put option has the right to sell one pound for E U.S. dollars even if $S \le E$. The holder of $1/E$ units of the call option has the right to buy E dollars by paying one British pound even if $\zeta \ge E$. The condition $S \le E$ is equivalent to $E^2/S = \zeta \ge E$. Thus, both the holder of one unit of the put option and the holder of $1/E$ units of the call option have the right to exchange one pound for E dollars even if $S < E$. The two holders have the same rights, so the value of one unit of the put option and the value of $1/E$ units of the call option in U.S. dollars, which is equal to $S \cdot C/E$, should be equal, i.e.,

$$P = S \cdot C/E.$$

Here, we need to notice that P and C have different but related volatilities, interest rates, and dividend yields. According to Itô's lemma, if

$$dS = \mu S dt + \sigma S dX,$$

then

$$d\zeta = (-\mu + \sigma^2)\zeta dt - \sigma\zeta dX.$$

Hence, the volatilities of S and $\zeta = E^2/S$ are the same if the volatilities are constants. Suppose that σ, r, and D_0 are constant and that the interest rates of the British pound and the U.S. dollar are a and b, respectively. Then $r = a$ and $D_0 = b$ for the call and $r = b$ and $D_0 = a$ for the put, and the volatilities are the same. In this case, the relation above can be written as

$$P(S,t;b,a) = C\left(E^2/S,t;a,b\right) S/E.$$

The first relation in the set of relations (3.22) (or (3.24)) actually is another form of the second relation in the set of relations (3.22) (or (3.24)). Thus from the financial reasoning here, we know that all the relations in the sets of relations (3.22) and (3.24) hold. Because the state variable ζ for the call with $r = a$ and $D_0 = b$ and the state variable S for the put with $r = b$ and $D_0 = a$ have the relation $\zeta = E^2/S$, the argument above to obtain the relation (3.23) can still be used here. Hence from the financial reasoning above, we can also have the relation (3.23).

Actually such relations exist for more complicated cases. If σ depends upon S, then the following relations hold:

$$\begin{cases} C(S, t; a, b, \sigma(S)) = P\left(\dfrac{E^2}{S}, t; b, a, \sigma(S)\right) S/E, \quad \text{or} \\ P(S, t; b, a, \sigma(S)) = C\left(\dfrac{E^2}{S}, t; a, b, \sigma(S)\right) S/E \end{cases}$$

and

$$S_{cf}(t; a, b, \sigma(S)) \times S_{pf}(t; b, a, \sigma(E^2/S)) = E^2.$$

Here, the third argument after the semicolon is the function for the volatility. The proof is left for readers as an exercise (Problem 17).

The symmetry relations can be used when we write codes for pricing American options or calculate prices of options. Suppose that we need codes for pricing American call and put options and that we already have a code for pricing American call options. If it is very easy for the code to be modified to a code for pricing American put options, then we can have another code for put options by modifying the code we already have. If the code for put options will be quite a different from the code for call options, then we can use the code for call options to find $C(E^2/S, t; a, b)$ first and then obtain $P(S, t; b, a)$ by using the relation $P(S, t; b, a) = C(E^2/S, t; a, b) \cdot S/E$. If one already has a code that can deal with both American call and put options, then the symmetry relations can be used for checking the accuracy of the numerical results. Because the numerical results have errors, they will not exactly satisfy the symmetry relation and can be used as indicators to show how accurate the numerical results are if the values of a call and the corresponding put option have been obtained. For details, see the paper [98] by Zhu, Ren, and Xu. For more about symmetry relations and similar results, see [53, 54, 62] and [24].

3.3.4 Equations for Some Greeks

Here, for American options we would like to derive the equations and boundary conditions that $\mathcal{V} = \dfrac{\partial \Pi}{\partial \sigma}$, $\rho = \dfrac{\partial \Pi}{\partial r}$, and $\rho_d = \dfrac{\partial \Pi}{\partial D_0}$ should satisfy. Let us first consider American call options and write the dependence of C and S_f on r, D_0, and σ explicitly, that is, instead of $C(S, t)$ and $S_f(t)$, we use $C(S, t; r, D_0, \sigma)$ and $S_f(t; r, D_0, \sigma)$ to denote the price of American call options and the free boundary in what follows. Differentiating the partial differential equation in the problem (3.19) with respect to r, D_0, or σ yields the equations for $\dfrac{\partial C}{\partial r}$, $\dfrac{\partial C}{\partial D_0}$ or $\dfrac{\partial C}{\partial \sigma}$. For example, for $\dfrac{\partial C}{\partial \sigma}$ we have

$$\frac{\partial C_\sigma}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_\sigma}{\partial S^2} + (r - D_0)S\frac{\partial C_\sigma}{\partial S} - rC_\sigma + \sigma S^2 \frac{\partial^2 C}{\partial S^2} = 0,$$

where C_σ denotes the partial derivative of the call option with respect to σ. The final condition for the price of American call options is

$$C(S, T; r, D_0, \sigma) = \max(S - E, 0).$$

Therefore $\dfrac{\partial C}{\partial \sigma} = 0$ at $t = T$. The boundary conditions on the free boundary are

$$C(S_f(t; r, D_0, \sigma), t; r, D_0, \sigma) = S_f(t; r, D_0, \sigma) - E \tag{3.25}$$

and

$$\frac{\partial C(S_f(t; r, D_0, \sigma), t; r, D_0, \sigma)}{\partial S} = 1. \tag{3.26}$$

From the relation (3.25) we have

$$\frac{\partial C}{\partial S}\frac{\partial S_f}{\partial \sigma} + \frac{\partial C}{\partial \sigma} = \frac{\partial S_f}{\partial \sigma}$$

on the free boundary. Noticing (3.26), we have $\dfrac{\partial C}{\partial \sigma} = 0$ at the free boundary. Consequently, $\dfrac{\partial C}{\partial \sigma}$ is the solution of the following final-boundary value problem

$$\begin{cases} \dfrac{\partial C_\sigma}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 C_\sigma}{\partial S^2} + (r - D_0)S\dfrac{\partial C_\sigma}{\partial S} - rC_\sigma + \sigma S^2 \dfrac{\partial^2 C}{\partial S^2} = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 \le S \le S_f(t), \quad 0 \le t \le T, \\ C_\sigma(S, T) = 0, \qquad\qquad\qquad\quad 0 \le S \le S_f(T), \\ C_\sigma(S_f(t), t) = 0, \qquad\qquad\quad\;\; 0 \le t \le T, \end{cases} \tag{3.27}$$

where $\dfrac{\partial^2 C}{\partial S^2}$ and $S_f(t)$ are known functions obtained from the solution of problem (3.19).

For $\dfrac{\partial C}{\partial r}$ and $\dfrac{\partial C}{\partial D_0}$, we can derive the same final and boundary conditions as $\dfrac{\partial C}{\partial \sigma}$, namely,

$$\frac{\partial C}{\partial r} = \frac{\partial C}{\partial D_0} = 0 \tag{3.28}$$

at $t = T$ and

$$\frac{\partial C}{\partial r} = \frac{\partial C}{\partial D_0} = 0 \tag{3.29}$$

at the free boundary. The only difference is the equation. Differentiating the partial differential equation in the problem (3.19) with respect to r and D_0 yields

$$\frac{\partial C_r}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_r}{\partial S^2} + (r - D_0) S \frac{\partial C_r}{\partial S} - rC_r + S\frac{\partial C}{\partial S} - C = 0 \tag{3.30}$$

and

$$\frac{\partial C_{D_0}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_{D_0}}{\partial S^2} + (r - D_0) S \frac{\partial C_{D_0}}{\partial S} - rC_{D_0} - S\frac{\partial C}{\partial S} = 0 \tag{3.31}$$

respectively, where C_r stands for $\dfrac{\partial C}{\partial r}$ and C_{D_0} for $\dfrac{\partial C}{\partial D_0}$.

For American put options, the Greeks are solutions of similar problems. This is left for readers to show as Problem 19 of this chapter.

3.3.5 Solutions for Perpetual American Call Options

If an option does not have an expiry date but rather an infinite time zone, then the option is called a perpetual option. Let $C(S, 0; T)$ be the today's price of an American call option with expiry T, and let $C_\infty(S)$ be the price of the corresponding perpetual American call option. Between them, there is the following relation:

$$C_\infty(S) = \lim_{T\to\infty} C(S, 0; T).$$

Since $\left.\dfrac{\partial C(S, t; T)}{\partial t}\right|_{t=0} = 0$ as $T \to \infty$, we know from the problem (3.19) that for $S \in [0, S_f]$, S_f standing for the location of the corresponding free boundary, $C_\infty(S)$ is the solution of the following problem

$$\begin{cases} \dfrac{1}{2}\sigma^2 S^2 \dfrac{d^2 C_\infty}{dS^2} + (r - D_0) S \dfrac{dC_\infty}{dS} - rC_\infty = 0, \quad 0 \le S \le S_f, \\ C_\infty(S_f) = S_f - E, \\ \dfrac{dC_\infty(S_f)}{dS} = 1. \end{cases} \tag{3.32}$$

Let

$$C_\infty(S) = S^\alpha,$$

then

$$\frac{dC_\infty}{dS} = \alpha S^{\alpha-1}$$

and

$$\frac{d^2C_\infty}{dS^2} = \alpha(\alpha-1)S^{\alpha-2}.$$

Substituting these into the ordinary differential equation in the problem (3.32), we get

$$\frac{1}{2}\sigma^2\alpha^2 + \left(r - D_0 - \frac{1}{2}\sigma^2\right)\alpha - r = 0.$$

The two roots of this equation are

$$\alpha_\pm = \frac{1}{\sigma^2}\left[-\left(r - D_0 - \frac{1}{2}\sigma^2\right) \pm \sqrt{\left(r - D_0 - \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 r}\,\right].$$

Thus

$$C_\infty(S) = C_+(S/S_f)^{\alpha_+} + C_-(S/S_f)^{\alpha_-}.$$

It is clear that $\alpha_+ > 0$ and $\alpha_- < 0$. In order to guarantee the solution to be bounded at $S = 0$, C_- should equal zero. Consequently, we arrive at

$$C_\infty(S) = C_+(S/S_f)^{\alpha_+}.$$

From the free-boundary conditions in the problem (3.32) we obtain

$$C_+ = S_f - E,$$
$$C_+\alpha_+S_f^{-1} = 1.$$

Solving these two equations we get

$$S_f = \frac{E}{1 - 1/\alpha_+} \quad \text{and} \quad C_+ = \frac{1}{\alpha_+S_f^{-1}}.$$

Thus, the solution of problem (3.32) is

$$C_\infty(S) = \frac{S_f}{\alpha_+}\left(\frac{S}{S_f}\right)^{\alpha_+}. \tag{3.33}$$

On $[0, \infty)$, the solution of the perpetual American call option is

$$C_\infty(S) = \begin{cases} \text{the solution of the free-boundary problem}, & 0 \le S \le S_f, \\ S - E, & S_f < S. \end{cases}$$

$C_\infty(S)$ should satisfy the following LC relation of the perpetual American call option for any S:

$$\min\left(-\left[\frac{1}{2}\sigma^2 S^2\frac{d^2C_\infty}{dS^2}+(r-D_0)S\frac{dC_\infty}{dS}-rC_\infty\right],\right.$$
$$\left.C_\infty-\max(S-E,0)\right)=0.$$

Here let us verify this conclusion by direct computation. Before doing that, we point out that the following is true: $S_f = E/(1-1/\alpha_+) \geq E\max(1, r/D_0)$. As we know, for a vanilla call option, $S_f(T) = E\max(1, r/D_0)$ and $S_f(0) \geq S_f(T) = E\max(1, r/D_0)$. This still holds as $T \to \infty$. Thus[4]

$$S_f \geq E\max(1, r/D_0).$$

For $S \in (0, E)$, C_∞ satisfies the ODE and is greater than 0, and $\max(S-E, 0) = 0$. Thus the LC relation

$$\min\left(-\left[\frac{1}{2}\sigma^2 S^2\frac{d^2C_\infty}{dS^2}+(r-D_0)S\frac{dC_\infty}{dS}-rC_\infty\right],\right.$$
$$\left.C_\infty-\max(E-S,0)\right)=0$$

holds. Now let us check if the LC relation holds for $S \in (E, S_f)$. Suppose that $f(x)$, $f'(x)$, and $f''(x)$ are continuous functions on $[a, b]$. As we know, if $f(b) = 0$ and $f'(b) = 0$, then the following relation is true: $f(x) = \frac{1}{2}f''(\xi)(x-b)^2$, where $x \in [a, b]$ and $\xi \in [x, b]$. Using this fact, we know that because $C_\infty(S_f) - (S_f - E) = 0$ and $\dfrac{dC_\infty(S_f)}{dS} - 1 = 0$, $C_\infty(S) - (S-E) \geq 0$ on (E, S_f) if $\dfrac{d^2C_\infty(S)}{dS^2} \geq 0$ on (E, S_f). From the expression of C_∞, we have

$$\frac{d^2C_\infty}{dS^2} = \frac{\alpha_+-1}{S_f}\left(\frac{S}{S_f}\right)^{\alpha_+-2}.$$

Because $\dfrac{\alpha_+-1}{S_f} = \dfrac{(\alpha_+-1)^2}{E\alpha_+} > 0$, we know $\dfrac{d^2C_\infty}{dS^2} \geq 0$ and the LC relation holds on (E, S_f). For $S \in (S_f, \infty)$, because $S_f \geq E\max(1, r/D_0)$, we have $C_\infty(S) = S-E = \max(S-E, 0)$, which means $C_\infty(S) - \max(S-E, 0) = 0$, and

$$\begin{aligned}&-\frac{\sigma^2S^2}{2}\frac{d^2C_\infty}{dS^2}-(r-D_0)\,S\frac{dC_\infty}{dS}+rC_\infty\\&=D_0S-rE=D_0(S-rE/D_0)\geq 0.\end{aligned}$$

[4] This result can also be obtained from direct calculation, which is left for readers as Problem 20.

Thus the LC relation also holds for $S \in (S_f, \infty)$. Consequently, we have proved our conclusion for all the cases.

For an American put option, as $T \to \infty$, the free-boundary problem (3.16) becomes

$$\begin{cases} \dfrac{1}{2}\sigma^2 S^2 \dfrac{d^2 P_\infty}{dS^2} + (r - D_0) S \dfrac{dP_\infty}{dS} - rP_\infty = 0, \quad S_f \le S, \\ P_\infty(S_f) = E - S_f, \\ \dfrac{dP_\infty(S_f)}{dS} = -1. \end{cases}$$

Similar to the call option, for $S \ge S_f$ the price of a perpetual American put option is

$$P_\infty(S) = \frac{-S_f}{\alpha_-}\left(\frac{S}{S_f}\right)^{\alpha_-}, \tag{3.34}$$

where

$$S_f = \frac{E}{1 - 1/\alpha_-}.$$

3.4 Some Conclusion from Arbitrage Theory

In Sect. 2.2, we derived the Black–Scholes equation by using arbitrage arguments. Here, we will further use arbitrage arguments to obtain some properties of option prices. Similar materials can be found in the book [43] by Hull.

3.4.1 Three Conclusions and Some Portfolios

Consider two portfolios $\mathbf{X}$ and $\mathbf{Y}$, whose values depend on a stock price S and time t. Let $\mathbf{X}(S,t)$ and $\mathbf{Y}(S,t)$ denote the values of portfolios $\mathbf{X}$ and $\mathbf{Y}$, respectively. $\mathbf{X}$ and $\mathbf{Y}$ could involve options, and all their expiries are T. By using arbitrage arguments, we can have three conclusions, which are written in the form of theorems.

Theorem 3.2 *If only European options are involved and* $\mathbf{X}(S,T) \ge \mathbf{Y}(S,T)$ *for any* S*, then for any* $t \le T$*,* $\mathbf{X}(S,t)$ *must be greater than or equal to* $\mathbf{Y}(S,t)$*.*

Proof. Suppose that at time $\bar{t}$ the value of portfolio $\mathbf{X}$ is less than the value of portfolio $\mathbf{Y}$ and that the latter is higher than the former by an amount of $Z(\bar{t})$. In this case, an arbitrageur can earn at least $Z(\bar{t})e^{r(T-\bar{t})}$ at time T by doing the following: sell $\mathbf{Y}$, buy $\mathbf{X}$, and invest $Z(\bar{t})$ into a bank at an interest rate r at time $\bar{t}$, and get $\mathbf{X}(S,T)$ from portfolio $\mathbf{X}$, pay $\mathbf{Y}(S,T)$ for portfolio $\mathbf{Y}$, and obtain $Z(\bar{t})e^{r(T-\bar{t})}$ from the risk-free investment at time T. Because $\mathbf{X}(S,T) \ge \mathbf{Y}(S,T)$ for any S, the arbitrageur will always earn at least

$Z(\bar{t})e^{r(T-\bar{t})}$ at the time T, which means that the earning is risk-free. Thus, everyone will do such a thing. Because so many people sell $\mathbf{Y}$ and buy $\mathbf{X}$, the price of $\mathbf{Y}$ will drop and the price of $\mathbf{X}$ will rise and will be immediately equal to or greater than the price of $\mathbf{Y}$. Therefore, Theorem 3.2 holds. □

From this result, assuming $\mathbf{X}(S,T) \le \mathbf{Y}(S,T)$, we can immediately get that for any time $t \le T$, $\mathbf{X}(S,t) \le \mathbf{Y}(S,t)$ and furthermore we can have

Theorem 3.3 *If $\mathbf{X}(S,T) = \mathbf{Y}(S,T)$ for any S, then for any $t \le T$, $\mathbf{X}(S,t)$ must be equal to $\mathbf{Y}(S,t)$ for any S.*

Proof. Because $\mathbf{X}(S,T) = \mathbf{Y}(S,T)$ means $\mathbf{X}(S,T) \ge \mathbf{Y}(S,T)$ and $\mathbf{X}(S,T) \le \mathbf{Y}(S,T)$, from the conclusion above we have for any t

$$\mathbf{X}(S,t) \ge \mathbf{Y}(S,t) \quad \text{and} \quad \mathbf{X}(S,t) \le \mathbf{Y}(S,t),$$

which means

$$\mathbf{X}(S,t) = \mathbf{Y}(S,t).$$

Thus we have Theorem 3.3. □

We can also have the following conclusion.

Theorem 3.4 *Suppose that portfolio $\mathbf{Y}$ involves only one American option and no European option and that portfolio $\mathbf{X}$ involves only European options. If $\mathbf{X}(S,T) \ge \mathbf{Y}(S,T)$ at time T and if the amount of cash and the number of stocks in $\mathbf{X}$ is greater than or equal to the amount of cash and the number of stocks the holder of $\mathbf{Y}$ has when the American option is exercised at time $\bar{t} < T$, then $\mathbf{X}(S,t) \ge \mathbf{Y}(S,t)$ for any time t.*

Proof. The argument is similar to the argument for proving Theorem 3.2. Suppose $\mathbf{X}(S,t) < \mathbf{Y}(S,t)$ at time $t < T$. Then, an arbitrageur can purchase $\mathbf{X}$, sell $\mathbf{Y}$, and earn some money. Later, when the American option is exercised early at time $\bar{t} < T$, the arbitrageur will never lose money because the amount of cash and the number of stocks in $\mathbf{X}$ are greater than or equal to the amount of cash and the number of stocks the holder of $\mathbf{Y}$ has. When the American option is not exercised before time T, the arbitrageur will also never lose any money because the value of $\mathbf{X}$ is greater than or equal to the value of $\mathbf{Y}$ at time T. Therefore, the earning is risk-free, which means $\mathbf{X}(S,t)$ should not be less than $\mathbf{Y}(S,t)$ at any time. □

Before applying these conclusions, we define some portfolios and find their values at time T along with what their holders will have if American options are exercised at time $\bar{t} < T$.

Portfolio A: An amount of cash equal to $Ee^{-r(T-t)}$ invested at an interest rate r. It is clear that its value at time T is E.

Portfolio B: $e^{-D_0(T-t)}$ shares of a stock with dividends being reinvested in the stock if the stock pays the dividend continuously or one share of a stock plus a loan $D_p(S,t)$[5] if the stock pays cash dividends discretely. Here, $D_p(S,t)$ is equal to the present value of these dividends to be paid from time t to time T, and the money will be returned to the loaner as soon as the stock pays a dividend. Obviously, its value at time T is the price of the stock S.

Portfolio C: One European call option plus portfolio **A**. The value of this portfolio at time T is $\max(S-E,0)+E=\max(S,E)$.

Portfolio D: One European put option plus portfolio **B**. Its value at time T is $\max(E-S,0)+S=\max(S,E)$.

Portfolio E: One American call option plus portfolio **A**. If the American call option is not exercised before time T, its value at time T is $\max(S-E,0)+E=\max(S,E)$. If at some time $\bar{t}<T$, the stock price S is greater than E and the American option is exercised, then the holder of the portfolio has one share plus a loan of $(1-\mathrm{e}^{-r(T-\bar{t})})E$.

Portfolio F: One American put option plus portfolio **B**. $\max(S,E)$ is its value at time T if the put option is not exercised before time T; while its holder has an amount of cash E minus $(1-\mathrm{e}^{-D_0(T-\bar{t})})$ shares or an amount of cash $E-D_p(S,\bar{t})$ if the stock price S is less than E and the put option is exercised at some time $\bar{t}<T$.

Portfolio G: One European call option plus E. Its value at time T is equal to $\max(S,E)$.

Portfolio H: One European put option plus one share. Its value is equal to $\max(S,E)$ at expiry.

3.4.2 Bounds of Option Prices

Consider a European call option and portfolio **B**. At time T, $c(S,T)=\max(S-E,0)\le \mathbf{B}(S,T)=S$. From Theorem 3.2, we have

$$c(S,t)\le S\mathrm{e}^{-D_0(T-t)}$$

or

$$c(S,t)\le S-D_p(S,t).$$

Now let us compare portfolio **C** with portfolio **B**. Because at time T

$$\mathbf{C}(S,T)=\max(S,E)\ge \mathbf{B}(S,T)=S,$$

we have

$$c(S,t)+E\mathrm{e}^{-r(T-t)}\ge S\mathrm{e}^{-D_0(T-t)}$$

or

$$c(S,t)+E\mathrm{e}^{-r(T-t)}\ge S-D_p(S,t).$$

[5]Here we assume that the value of the dividends depends on S, just like what we did in Sect. 2.2.2.

Clearly, $c(S,t) \geq 0$ for any case. Therefore, for a European call option we have

$$\max\left(Se^{-D_0(T-t)} - Ee^{-r(T-t)}, 0\right) \leq c(S,t) \leq Se^{-D_0(T-t)} \tag{3.35}$$

or

$$\max\left(S - D_p(S,t) - Ee^{-r(T-t)}, 0\right) \leq c(S,t) \leq S - D_p(S,t). \tag{3.36}$$

Consequently, the lower bound of $c(S,t)$ is $\max\left(Se^{-D_0(T-t)} - Ee^{-r(T-t)}, 0\right)$ or $\max\left(S - D_p(S,t) - Ee^{-r(T-t)}, 0\right)$ and the upper bound is $Se^{-D_0(T-t)}$ or $S - D_p(S,t)$. Here, we assume that $S - D_p(S,t)$ is always greater than zero. If $S < D_p(S,t)$ at time t, then any person will buy one share of the stock by finding a loan of amount S at time t and returning the loan as soon as the stock pays a dividend. In this way, the person gets one share and some cash free at time T. Therefore, the price must rise until $S \geq D_p(S,t)$.

Because $C(S,t) \geq c(S,t)$, we require that $C(S,t)$ is greater than or equal to the lower bound of $c(S,t)$. Also, $C(S,t)$ needs to be greater than or equal to the constraint $\max(S - E, 0)$. Thus

$$\max\left(Se^{-D_0(T-t)} - Ee^{-r(T-t)}, S - E, 0\right)$$

or

$$\max\left(S - D_p(S,t) - Ee^{-r(T-t)}, S - E, 0\right)$$

is a lower bound. Clearly, S is an upper bound for an American call option. Consequently, for the price of an American call option, we have

$$\max\left(Se^{-D_0(T-t)} - Ee^{-r(T-t)}, S - E, 0\right) \leq C(S,t) \leq S \tag{3.37}$$

or

$$\max\left(S - D_p(S,t) - Ee^{-r(T-t)}, S - E, 0\right) \leq C(S,t) \leq S. \tag{3.38}$$

Now let us compare a European put option with portfolio **A**. At time T,

$$p(S,T) = \max(E - S, 0) \leq \mathbf{A}(S,T) = E.$$

Thus

$$p(S,t) \leq Ee^{-r(T-t)}.$$

In order to get a lower bound of $p(S,t)$, let us look at portfolios **D** and **A**. Because at time T,

$$\mathbf{D}(S,T) = \max(S, E) \geq \mathbf{A}(S,T) = E,$$

we arrive at

$$p(S,t) + Se^{-D_0(T-t)} \geq Ee^{-r(T-t)}$$

or

$$p(S,t) + S - D_p(S,t) \geq E\mathrm{e}^{-r(T-t)}.$$

Also, $p(S,t)$ must be nonnegative. Therefore, we have

$$\max\left(E\mathrm{e}^{-r(T-t)} - S\mathrm{e}^{-D_0(T-t)}, 0\right) \leq p(S,t) \leq E\mathrm{e}^{-r(T-t)} \tag{3.39}$$

or

$$\max\left(E\mathrm{e}^{-r(T-t)} - S + D_p(S,t), 0\right) \leq p(S,t) \leq E\mathrm{e}^{-r(T-t)}. \tag{3.40}$$

These give the lower and upper bounds of European put options.

For an American put option, we can also get the lower and upper bounds. Because $P(S,t) \geq p(S,t)$, we have

$$P(S,t) \geq \max\left(E\mathrm{e}^{-r(T-t)} - S\mathrm{e}^{-D_0(T-t)}, 0\right)$$

or

$$P(S,t) \geq \max\left(E\mathrm{e}^{-r(T-t)} - S + D_p(S,t), 0\right).$$

Also, $P(S,t)$ must be greater than or equal to $\max(E-S,0)$. Therefore, we further obtain

$$P(S,t) \geq \max\left(E\mathrm{e}^{-r(T-t)} - S\mathrm{e}^{-D_0(T-t)}, E-S, 0\right)$$

or

$$P(S,t) \geq \max\left(E\mathrm{e}^{-r(T-t)} - S + D_p(S,t), E-S, 0\right).$$

E is a upper bound of $P(S,t)$, consequently we have

$$\max\left(E\mathrm{e}^{-r(T-t)} - S\mathrm{e}^{-D_0(T-t)}, E-S, 0\right) \leq P(S,t) \leq E \tag{3.41}$$

or

$$\max\left(E\mathrm{e}^{-r(T-t)} - S + D_p(S,t), E-S, 0\right) \leq P(S,t) \leq E. \tag{3.42}$$

From the proofs we know that if one of these relations is not true, then we can find an arbitrage opportunity to earn some money. This means that the lower bound is the greatest lower bound and that the upper bound is the least upper bound. From Sect. 1.2.4, we know that the price of an option is an increasing function of the volatility. Therefore, if the lower bound is the greatest lower bound, then as the volatility approaches zero, the limit of option should be the lower bound. Similarly, if the upper bound is the least upper bound, then as the volatility approaches infinity, the limit of the option should be the upper bound. When r, D_0, and σ are constant, the European option price is given by the Black–Scholes formulae in Sect. 2.6.5:

$$c(S,t) = S\mathrm{e}^{-D_0(T-t)}N(d_1) - E\mathrm{e}^{-r(T-t)}N(d_2)$$

and

$$p(S,t) = E\mathrm{e}^{-r(T-t)}N(-d_2) - S\mathrm{e}^{-D_0(T-t)}N(-d_1),$$

where

$$d_1 = \left[\ln \frac{S\mathrm{e}^{-D_0(T-t)}}{E\mathrm{e}^{-r(T-t)}} + \frac{1}{2}\sigma^2(T-t)\right] \Bigg/ \left(\sigma\sqrt{T-t}\right)$$

and

$$d_2 = \left[\ln \frac{S\mathrm{e}^{-D_0(T-t)}}{E\mathrm{e}^{-r(T-t)}} - \frac{1}{2}\sigma^2(T-t)\right] \Bigg/ \left(\sigma\sqrt{T-t}\right).$$

Therefore we have

$$\begin{cases} \lim\limits_{\sigma\to 0} c(S,t) &= \begin{cases} 0, & \text{if } S\mathrm{e}^{-D_0(T-t)} < E\mathrm{e}^{-r(T-t)}, \\ S\mathrm{e}^{-D_0(T-t)} - E\mathrm{e}^{-r(T-t)}, & \text{if } S\mathrm{e}^{-D_0(T-t)} > E\mathrm{e}^{-r(T-t)}, \end{cases} \\ \lim\limits_{\sigma\to\infty} c(S,t) &= S\mathrm{e}^{-D_0(T-t)}, \\ \lim\limits_{\sigma\to 0} p(S,t) &= \begin{cases} 0, & \text{if } E\mathrm{e}^{-r(T-t)} < S\mathrm{e}^{-D_0(T-t)}, \\ E\mathrm{e}^{-r(T-t)} - S\mathrm{e}^{-D_0(T-t)}, & \text{if } E\mathrm{e}^{-r(T-t)} > S\mathrm{e}^{-D_0(T-t)}, \end{cases} \\ \lim\limits_{\sigma\to\infty} p(S,t) &= E\mathrm{e}^{-r(T-t)}. \end{cases}$$

That is, the inequalities (3.35) and (3.39) truly provide the least upper and greatest lower bounds of European options, respectively.

Here, we give an example to show that if the price of an option does not satisfy a related condition, then there exists an arbitrage opportunity. More examples are given as problems for readers to study.

Example 1. Consider a European call option on a dividend-paying stock. Suppose the following: $S = \$102, E = \$100, c = \$8.50, r = 0.1$, the time to maturity is 9 months, and the present value of the dividend $D_p(102,t)$ is \$0.50. Is there any arbitrage opportunity?

Solution: As we know, the price of a call option has to satisfy the condition (3.36):

$$\max\left(S - D_p(102,t) - E\mathrm{e}^{-r(T-t)}, 0\right) \le c(S,t) \le S - D_p(102,t).$$

In this case

$$\begin{aligned} \max\left(S - D_p(102,t) - E\mathrm{e}^{-r(T-t)}, 0\right) &= \max\left(102 - 0.5 - 100\mathrm{e}^{-0.9/12}, 0\right) \\ &= 8.73. \end{aligned}$$

Therefore, the price of the call option is less than the lower bound. In this case, if we own one share of the stock or if you can borrow one share of the stock for the period $[t,T]$, then we should take a long position in a portfolio **C** and a short position in a portfolio **B**. In other words, buy one call option, sell one share, and deposit $E\mathrm{e}^{-r(T-t)}+D_p(102,t)$ in a bank at time t. In this case we will get $-8.5+102-100\mathrm{e}^{-0.9/12}-0.5=\$0.23$ at time t, and this is a risk-free earning. This is because we can get the money from the bank to pay the dividends on the stock during the time interval $[t,T]$ and get E from the bank at time T. If $S\ge E$ at time T, we can exercise the call option and get one share. If $S<E$, we can have one share of the stock that is bought from the market and an amount of cash $E-S$. In any case, we have one share plus at least \$0.23. That is, we can get one share back or return one share to the borrower and earn at least \$0.23 free at time T.

3.4.3 Relations Between Call and Put Prices

Let us look at portfolios **C** and **D**. Because $\mathbf{C}(S,T)=\mathbf{D}(S,T)$, we have

$$c(S,t)+E\mathrm{e}^{-r(T-t)}=p(S,t)+S\mathrm{e}^{-D_0(T-t)} \tag{3.43}$$

or

$$c(S,t)+E\mathrm{e}^{-r(T-t)}=p(S,t)+S-D_p(S,t) \tag{3.44}$$

according to Theorem 3.3. These are called put–call parities of European options. For stocks with continuous dividends, we obtained such a relation through a very long procedure in Sect. 2.6. However, the derivation here is so simple. This shows that arbitrage theory is a very powerful tool.

The put–call parity relations hold only for European options. For American options they are not true, but the following inequalities on the difference between the American call and put option prices are fulfilled

$$S\mathrm{e}^{-D_0(T-t)}-E\le C(S,t)-P(S,t)\le S-E\mathrm{e}^{-r(T-t)} \tag{3.45}$$

or

$$S-D_p(S,t)-E\le C(S,t)-P(S,t)\le S-E\mathrm{e}^{-r(T-t)}. \tag{3.46}$$

The two inequalities can also be written as

$$\begin{cases} S\mathrm{e}^{-D_0(T-t)}-E+P(S,t)\le C(S,t)\le S-E\mathrm{e}^{-r(T-t)}+P(S,t), \\ C(S,t)-S+E\mathrm{e}^{-r(T-t)}\le P(S,t)\le C(S,t)-S\mathrm{e}^{-D_0(T-t)}+E \end{cases}$$

or

$$\begin{cases} S-D_p(S,t)-E+P(S,t)\le C(S,t)\le S-E\mathrm{e}^{-r(T-t)}+P(S,t), \\ C(S,t)-S+E\mathrm{e}^{-r(T-t)}\le P(S,t)\le C(S,t)-S+D_p(S,t)+E, \end{cases}$$

which gives the lower and upper bounds of an American call (put) option if the price of the corresponding American put (call) option is known.

First, let us prove the left portions of the inequalities (3.45) and (3.46). Consider portfolios **G** and **F**. Because **G** contains European options only and **F** contains only one American option, it is possible to use Theorem 3.4. According to Theorem 3.4, the value of **G** is always greater than or equal to the value of **F** if we can prove two things:

1. The value of **G** is greater than or equal to the value of **F** at time T;
2. The amount of cash and the number of stocks in **G** is greater than or equal to the amount of cash and the number of stocks in **F** when the American option is exercised at time $\bar{t} < T$.

At time T, the value of **G** is equal to the value of **F**. At any time $\bar{t} < T$, there is an amount of cash E and no stock in **G**. If the American put option in **F** is exercised before time T, **F** contains an amount of cash E and $-(1-\mathrm{e}^{-D_0(T-\bar{t})})$ shares or an amount of cash $E - D_p(S,t)$. Therefore, both the amount of cash and the number of stocks in **G** is greater than or equal to those in **F** if the American option in **F** is exercised at some time $\bar{t} < T$. Consequently, according to Theorem 3.4, the value of **G** is greater than or equal to the value of **F** for any case, that is,

$$P(S,t) + S\mathrm{e}^{-D_0(T-t)} \leq c(S,t) + E$$

or

$$P(S,t) + S - D_p(S,t) \leq c(S,t) + E.$$

Because $C(S,t) \geq c(S,t)$, we further have

$$S\mathrm{e}^{-D_0(T-t)} - E \leq C(S,t) - P(S,t)$$

or

$$S - D_p(S,t) - E \leq C(S,t) - P(S,t).$$

In order to prove the right portions of the relations, we need to look at portfolios **H** and **E**. In **H** there is only one European option and in **E** the American option is the only option, so we can use Theorem 3.4 again. When the American call option in **E** is exercised before time T, the amount of cash and the number of stocks in **H** is greater than or equal to those in **E**. When it is not exercised before expiry, the value of **H** is equal to the value of **E** at time T. Therefore

$$C(S,t) + E\mathrm{e}^{-r(T-t)} \leq p(S,t) + S.$$

Noticing $P(S,t) \geq p(S,t)$, we have

$$C(S,t) - P(S,t) \leq S - E\mathrm{e}^{-r(T-t)}.$$

This completes our proof.

Example 2. Suppose there are an American call option and an American put option on the same stock. The stock pays dividends continuously, and $D_0 = 0.05$. For both options, $E = \$100$ and $T = 1\,\text{month}$. At present, $r = 0.1, S = \$103$, and $C = \$5.50$. Find the upper and lower bounds for the price of the American put option by using the relation (3.45). How do we take the arbitrage opportunity if the price of the American put option is greater than the calculated upper bound?

Solution: According to the relation (3.45), the lower bound of $P(S,t)$ is

$$C(S,t) - S + E\mathrm{e}^{-r(T-t)} = 5.5 - 103 + 100\mathrm{e}^{-0.1/12} = 1.67$$

and the upper bound is

$$C(S,t) - S\mathrm{e}^{-D_0(T-t)} + E = 5.5 - 103\mathrm{e}^{-0.05/12} + 100 = 2.93.$$

Suppose that on the market $P(103,t) = \$3.50$. Now we describe how to take advantage of the arbitrage opportunity. At time t, we can sell the American put option and short-sell $\mathrm{e}^{-0.05/12}$ shares, purchase one European call option that is less than or equal to \$5.50, and hold at least an amount of cash $3.5{+}103\mathrm{e}^{-0.05/12} - 5.5 = \$100.57$. If we want, it can be deposited into a bank. At any time $\bar{t} \in [t,T)$, the holder of the American put option wants to exercise the option, we pay \$100 and get one share. In this case, we have at least one share of stock and at least an amount of cash equal to \$0.57 at time T. If the holder of the American put option does not exercise the option before time T, we also will always have at least \$0.57 in cash plus one share of stock at time T. The reason is that we can exercise the European call option and get one share if $S > E$, whereas we can purchase one share from the market if $S \le E$. Because we need to return only one share to the borrower at time T, we always have enough shares of stocks. Therefore, the risk-free earning in this case is at least \$0.57.

Problems

Table 3.1. Problems and subsections

Problems	Subsections	Problems	Subsections	Problems	Subsections
1–2	3.1.1	3–7	3.1.2	8	3.2.1
9–15	3.3.2	16–18	3.3.3	19	3.3.4
20–23	3.3.5	24–25	3.4.1	26–27	3.4.2
28–30	3.4.3				

1. Let $\mathbf{L}_{\mathbf{S},\mathbf{t}}$ be an operator in an option problem in the form:

$$\mathbf{L}_{\mathbf{S},\mathbf{t}} = a(S,t)\frac{\partial^2}{\partial S^2} + b(S,t)\frac{\partial}{\partial S} + c(S,t)$$

and $G_v(S,t)$ be the constraint function for an American option. Furthermore we assume that $\dfrac{\partial G_v}{\partial t} + \mathbf{L}_{\mathbf{S},\mathbf{t}} G_v$ exists. Suppose $V(S,t^*) = G_v(S,t^*)$ on an open interval (A,B) on the S-axis. Let $t = t^* - \Delta t$, where Δt is a sufficiently small positive number. Show the following conclusions: If for any $S \in (A,B)$,

$$\frac{\partial G_v}{\partial t}(S,t^*) + \mathbf{L}_{\mathbf{S},\mathbf{t}^*} G_v(S,t^*) + d(S,t^*) \geq 0,$$

then the value $V(S,t)$ determined by the equation

$$\frac{\partial V}{\partial t}(S,t) + \mathbf{L}_{\mathbf{S},\mathbf{t}} V(S,t) + d(S,t) = 0$$

satisfies the condition $V(S,t) - G_v(S,t) \geq 0$ on (A,B); and if for any $S \in (A,B)$,

$$\frac{\partial G_v}{\partial t}(S,t^*) + \mathbf{L}_{\mathbf{S},\mathbf{t}^*} G_v(S,t^*) + d(S,t^*) < 0,$$

then the equation

$$\frac{\partial V}{\partial t}(S,t) + \mathbf{L}_{\mathbf{S},\mathbf{t}} V(S,t) + d(S,t) = 0$$

cannot give a solution satisfying the condition $V(S,t) - G_v(S,t) \geq 0$ for any $S \in (A,B)$.
2. *Suppose that for an American option, the constraint is $G_v(S,t)$, its value at time t is $V(S,t)$, and $V(S,t) = G_v(S,t)$ on (A,B). Assume that when $V(S,t)$ were given as the value of a European option at t, the value of the European option at $t - \Delta t$ for a positive and very small Δt is $v(S, t - \Delta t)$. Explain that if in an open interval containing $S^* \in (A,B)$, $v(S,t-\Delta t) < G_v(S, t - \Delta t)$, then for the American option a fair value at the point $(S^*, t - \Delta t)$ should be $G_v(S^*, t - \Delta t)$.
3. *Show that an American option is always worth at least as much as a European option on the same asset with the same strike price and exercise date if r, D_0, σ are constant, and give a financial explanation.
4. Show that a Bermudan option is always worth at least as much as a European option on the same asset with the same strike price and exercise date if r, D_0, σ are constant, and give a financial explanation of this fact. (Hint: For a Bermudan option, the approximate relation between the price at t_n and the price at t_{n+1} is the same as for a European option if at $t = t_n$ the option cannot be exercised, and the same as for an American option if at $t = t_n$ the option can be exercised.)

5. (a) *Explain why an American option is always worth at least as much as its intrinsic value. What is the definition of the time value of an American option?
   (b) *Let $V(S,t)$ be the price of a vanilla American option. Show that $V(S,t^*) \geq V(S,t^{**})$ is always true, where $t^* \leq t^{**}$. This means that the time value of a vanilla American option for a fixed S is decreasing as $t \to T$, and give a financial explanation of this fact.
6. (a) The price of a one-factor convertible bond paying no coupon is the solution of the following linear complementarity problem

$$\begin{cases} \min\left(-\dfrac{\partial V}{\partial t} - \mathbf{L}_S V,\ V(S,t) - nS\right) = 0, & 0 \leq S,\ 0 \leq t \leq T, \\ V(S,T) = \max(Z, nS) \geq nS, & 0 \leq S, \end{cases}$$

   where

$$\mathbf{L}_S = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0)\, S \frac{\partial}{\partial S} - r$$

   and n, Z, σ, r, and D_0 are positive constants. Show

$$V(S,t^*) - Z\mathrm{e}^{-r(T-t^*)} \geq V(S,t^{**}) - Z\mathrm{e}^{-r(T-t^{**})} \quad \text{if} \quad t^* \leq t^{**}.$$

   (Hint: Define $\overline{V}(S,t) = V(S,t) - Z\mathrm{e}^{-r(T-t)}$ and show $\overline{V}(S,t^*) \geq \overline{V}(S,t^{**})$ if $t^* \leq t^{**}$.)
   (b) Can you prove that $V(S,t^*) \geq V(S,t^{**})$ for $t^* \leq t^{**}$ by using the method used in part (a)? If your answer is "Yes", give a proof; otherwise explain why you cannot.
   (c) "A holder of a convertible bond at time t^* has "more rights" than a holder of a convertible bond at time t^{**} does if $t^* \leq t^{**}$, so the premium at t^* should be higher than the premium at t^{**}, i.e., the inequality $V(S,t^*) \geq V(S,t^{**})$ should hold for any $t^* \leq t^{**}$." Do you think that this statement is true and why?
7. The price of a one-factor convertible bond paying constant coupon is the solution of the following linear complementarity problem

$$\begin{cases} \min\left(-\dfrac{\partial V}{\partial t} - \mathbf{L}_S V - kZ,\ V(S,t) - nS\right) = 0, & 0 \leq S,\ 0 \leq t \leq T, \\ V(S,T) = \max(Z, nS) \geq nS, & 0 \leq S, \end{cases}$$

   where

$$\mathbf{L}_S = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0)\, S \frac{\partial}{\partial S} - r$$

   and k, Z, n, σ, r, and D_0 are positive constants. Study whether or not $V(S,t^*) \geq V(S,t^{**})$ for $t^* \leq t^{**}$ holds in the cases $r > k$ and $r = k$, and if not, try to find a relation between $V(S,t^*)$ and $V(S,t^{**})$.

8. A European option is the solution of the problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \mathbf{L}_S V = 0, & 0 \le S, \quad t \le T, \\ V(S,T) = V_T(S), & 0 \le S, \end{cases}$$

where

$$\mathbf{L}_S = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0)S\frac{\partial}{\partial S} - r.$$

For an American option, the constraint is that the inequality

$$V(S,t) \ge G(S,t)$$

holds for any S and t, where $G(S,T) = V_T(S)$. Derive the linear complementarity problem for the American option.

9. The American call option is the solution of the following linear complementarity problem on a finite domain:

$$\begin{cases} \min\left(\dfrac{\partial \overline{V}}{\partial \tau} - \mathbf{L}_\xi \overline{V},\ \overline{V}(\xi,\tau) - \max(2\xi - 1, 0)\right) = 0, & 0 \le \xi \le 1,\ 0 \le \tau, \\ \overline{V}(\xi,0) = \max(2\xi - 1, 0), & 0 \le \xi \le 1, \end{cases}$$

where

$$\mathbf{L}_\xi = \frac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1-\xi)^2\frac{\partial^2}{\partial \xi^2} + (r - D_0)\xi(1-\xi)\frac{\partial}{\partial \xi} - [r(1-\xi) + D_0\xi].$$

Reformulate this problem as a free-boundary problem if $D_0 > 0$.

10. The American put option is the solution of the following linear complementarity problem:

$$\begin{cases} \min\left(\dfrac{\partial u}{\partial \bar{\tau}} - \dfrac{\partial^2 u}{\partial x^2},\ u(x,\bar{\tau}) - g_p(x,\bar{\tau})\right) = 0, & -\infty < x < \infty,\ 0 \le \bar{\tau}, \\ u(x,0) = g_p(x,0), & -\infty < x < \infty, \end{cases}$$

where

$$g_p(x,\bar{\tau}) = \max\left(e^{2r\bar{\tau}/\sigma^2} - e^{x + (2D_0/\sigma^2 + 1)\bar{\tau}}, 0\right).$$

Find the domain where a free boundary may appear and the domain where it is impossible for a free boundary to appear, show that there is only one free boundary at $\bar{\tau} = 0$, and give the starting location of this free boundary.

11. The price of a one-factor convertible bond is the solution of the linear complementarity problem

$$\begin{cases} \min\left(-\dfrac{\partial V}{\partial t} - \mathbf{L}_S V - kZ,\ V(S,t) - nS\right) = 0, & 0 \le S,\ 0 \le t \le T, \\ V(S,T) = \max(Z, nS) \ge nS, & 0 \le S, \end{cases}$$

where

$$\mathbf{L}_S = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0)S\frac{\partial}{\partial S} - r,$$

and k, Z, n, σ, r and D_0 are constants. Show that if $D_0 > 0$, then the solution of a one-factor convertible bond must involve a free boundary and its location at $t = T$ is $S = \max\left(\frac{Z}{n}, \frac{kZ}{D_0 n}\right)$. Also, derive the corresponding free-boundary problem if this problem has only one free boundary.

12. Consider the following LC problem:

$$\begin{cases} \min\left(-\dfrac{\partial W}{\partial t} - \mathbf{L}_{\alpha,t}W,\ W(\eta,t) - \max(\alpha - \eta, 0)\right) = 0, & 0 \le \eta,\ t \le T, \\ W(\eta,T) = \max(\alpha - \eta, 0), & 0 \le \eta, \end{cases}$$

where the operator $\mathbf{L}_{a,t}$ is defined by

$$\mathbf{L}_{a,t} = \frac{1}{2}\sigma^2\eta^2\frac{\partial^2}{\partial\eta^2} + \left[(D_0 - r)\eta + \frac{1-\eta}{t}\right]\frac{\partial}{\partial\eta} - D_0.$$

Suppose that there is only one free-boundary for this problem, reformulate this problem as a free-boundary problem.

13. Consider the following LC problem:

$$\begin{cases} \min\left(-\dfrac{\partial W}{\partial t} - \mathbf{L}_\eta W,\ W(\eta,t) - G_{lsp}(\eta,t)\right) = 0, & 1 \le \eta,\ t \le T, \\ W(\eta,T) = G_{lsp}(\eta,T), & 1 \le \eta, \\ \dfrac{\partial W}{\partial \eta}(1,t) = 0, & t \le T, \end{cases}$$

where $G_{lsp}(\eta,t) = \max(\eta - \beta, 0)$ with $\beta \ge 1$ and $\mathbf{L}_\eta = \frac{1}{2}\sigma^2\eta^2\frac{\partial^2}{\partial\eta^2} + (D_0 - r)\eta\frac{\partial}{\partial\eta} - D_0$. Find the domain where it is impossible for a free boundary to appear and the domain where a free boundary may appear.

14. As we know, when the LC problem of an American call option is formulated as a free-boundary problem, on the free boundary $S = S_f(t) \ge \max(E, rE/D_0)$, we need to require $C\left(S_f(t), t\right) = \max\left(S_f(t) - E, 0\right) = S_f(t) - E$ and $\dfrac{\partial C\left(S_f(t), t\right)}{\partial S} = 1$, where $C\left(S, t\right)$ and $\max\left(S - E, 0\right)$ are the solution of the free-boundary problem and the constraint. Show that if $C(S,t) \ge 0$ and $\dfrac{\partial C^2\left(S,t\right)}{\partial S^2} \ge 0$ for $S < S_f(t)$, then the solution of the free-boundary problem satisfies the LC condition

$$\min\left(-\frac{\partial C}{\partial t} - \mathbf{L}_S C,\ C - \max(S - E, 0)\right) = 0,$$

where $\mathbf{L}_S = \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2}{\partial S^2} + (r - D_0) S \dfrac{\partial}{\partial S} - r$, that is, $C(S,t)$ truly is the solution of the LC problem for $S \in [0, S_f(t))$.

15. Consider an American call option on a stock paying discrete dividends.
   (a) Show that in this case, the optimal exercise price cannot appear for t between two successive ex-dividend dates.
   (b) Suppose that t_n, t_{n+1} are two successive ex-dividend dates with $t_n < t_{n+1}$. Assume $D_n(S)$ be the dividend payment at time t_n. Show that if $D_n(S) \le E\left(1 - \mathrm{e}^{-r(t_{n+1}-t_n)}\right)$, then there is no chance for an optimal exercise price to appear at time t_n^-; if $D_n(S) > E\left(1 - \mathrm{e}^{-r(t_{n+1}-t_n)}\right)$, it is possible for an optimal exercise price to appear at time t_n^-.
16. *Suppose r, D_0, and σ are constant.
   (a) Derive the put–call symmetry relations.
   (b) Explain the financial meaning of the symmetry relation.
   (c) Explain how to use these relations when we write codes if a code for put options is quite a different from a code for call options.
17. (a) Suppose $\sigma = \sigma(S,t)$, $r = r(t)$, and $D_0 = D_0(S,t)$. Show that the problem of pricing a put option can always be converted into a problem of pricing a call option. Also explain how to use this conclusion when we write codes if a code for put options is quite a different from a code for call options.
   (b) Let the exercise price be E. Suppose that r, D_0 are constants and $\sigma = \sigma(S)$. Show

$$P(S,t;b,\ a,\ \sigma(S)) = C\left(E^2/S, t; a,\ b,\ \sigma(S)\right) S/E,$$
$$C\left(S,t;a,\ b,\ \sigma(S)\right) = P\left(E^2/S, t; b,\ a,\ \sigma(S)\right) S/E$$

   and

$$S_{cf}(t;a,\ b,\ \sigma(S)) \times S_{pf}(t;b,\ a,\ \sigma(E^2/S)) = E^2.$$

   Here, the first, second, and third parameters after the semicolon in P, C, S_{pf}, and S_{cf} are the interest rate, the dividend yield and the volatility function, respectively.
   (c) Show that for Bermudan options the symmetry relation is still true.
18. Suppose that σ, r, D_0 are constants. In this case we have the following symmetry relation for European options

$$p(S,t;b,\ a) = c\left(\frac{E^2}{S}, t; a,\ b\right) S/E,$$

where the first and second arguments after the semicolon in p and c are the values of the interest rate and the dividend yield, respectively. For a European call option, the price is

$$c(S,t) = S\mathrm{e}^{-D_0(T-t)}N(d_1) - E\mathrm{e}^{-r(T-t)}N(d_2),$$

where

$$d_1 = \frac{\ln \dfrac{Se^{-D_0(T-t)}}{Ee^{-r(T-t)}} + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\ln \dfrac{Se^{-D_0(T-t)}}{Ee^{-r(T-t)}} - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}.$$

Find the price of a European put option by using the symmetry relation.

19. Derive the formulation of the problem for $\dfrac{\partial P}{\partial r}$ and write down the formulation of the problems for $\dfrac{\partial P}{\partial \sigma}$ and $\dfrac{\partial P}{\partial D_0}$, where P is the price of an American put option.
20. Define

$$\alpha_{\pm} = \frac{1}{\sigma^2}\left[-\left(r - D_0 - \frac{1}{2}\sigma^2\right) \pm \sqrt{\left(r - D_0 - \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 r}\,\right],$$

where $r \geq 0$ and $D_0 \geq 0$.

(a) Show that $\alpha_+ \geq 1$, $\alpha_- \leq 0$, and $-(r - D_0)\alpha_{\pm} + r \geq 0$.

(b) Based on the results in part (a), show that $1/(1-1/\alpha_+) \geq \max(1, r/D_0)$ and $1/(1 - 1/\alpha_-) \leq \min(1, r/D_0)$.

21. (a) Find the solution of the following free-boundary problem:

$$\begin{cases} \dfrac{1}{2}\sigma^2 S^2 \dfrac{d^2 P_\infty}{dS^2} + (r - D_0)S\dfrac{dP_\infty}{dS} - rP_\infty = 0, \quad S_f \leq S, \\ P_\infty(S_f) = E - S_f, \\ \dfrac{dP_\infty(S_f)}{dS} = -1. \end{cases}$$

(b) Define

$$P_\infty(S) = \begin{cases} E - S, & 0 \leq S < S_f, \\ \text{the solution of the free-boundary problem}, & S_f \leq S. \end{cases}$$

Show that $P_\infty(S)$ satisfies

$$\min\left(-\left[\frac{1}{2}\sigma^2 S^2 \frac{d^2 P_\infty}{dS^2} + (r - D_0)S\frac{dP_\infty}{dS} - rP_\infty\right],\right.$$
$$\left. P_\infty - \max(E - S, 0)\right) = 0,$$

that is, $P_\infty(S)$ is a solution of the perpetual American put option.

22. (a) Find the solution of the following free-boundary problem:

$$\begin{cases} \dfrac{1}{2}\sigma^2\eta^2\dfrac{d^2W_\infty}{d\eta^2} + (D_0 - r)\,\eta\dfrac{dW_\infty}{d\eta} - D_0W_\infty = 0, & 1 \le \eta \le \eta_f, \\ \dfrac{dW_\infty(1)}{d\eta} = 0, \\ W_\infty(\eta_f) = \eta_f, \\ \dfrac{dW_\infty(\eta_f)}{d\eta} = 1, \end{cases}$$

where η_f is a number representing the location of this free boundary.

(b) Define

$$W_\infty(\eta) = \begin{cases} \text{the solution of the free-boundary problem,} & 1 \le \eta \le \eta_f, \\ \eta, & \eta_f < \eta. \end{cases}$$

Show that $W_\infty(\eta)$ is a solution of the following LC problem

$$\begin{cases} \min\left(-\dfrac{\sigma^2\eta^2}{2}\dfrac{d^2W_\infty}{d\eta^2} - (D_0 - r)\,\eta\dfrac{dW_\infty}{d\eta} + D_0W_\infty,\ W_\infty - \eta\right) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1 \le \eta, \\ \dfrac{dW_\infty(1)}{d\eta} = 0. \end{cases}$$

(This problem is related to the Russian option.)

23. Find the solution of the problem:

$$\begin{cases} \dfrac{1}{2}\sigma^2\xi^2\dfrac{d^2W_\infty}{d\xi^2} + (D_{02} - D_{01})\,\xi\dfrac{dW_\infty}{d\xi} - D_{02}W_\infty = 0, & \xi_{f_1} \le \xi \le \xi_{f_2}, \\ W_\infty(\xi_{f_1}) = 1, \\ \dfrac{dW_\infty}{d\xi}(\xi_{f_1}) = 0, \\ W_\infty(\xi_{f_2}) = \xi_{f_2}, \\ \dfrac{dW_\infty}{d\xi}(\xi_{f_2}) = 1, \end{cases}$$

where $\xi_{f_1} < \xi_{f_2}$. (This problem is related to the perpetual American better-of option.)

24. Suppose that $c_1(S,t)$ and $c_2(S,t)$ are the prices of European call options with strikes E_1 and E_2, respectively, where $E_1 < E_2$. Also assume that the two options have the same maturity T and that the interest rate r is a constant. Show

$$0 \le c_1(S,t) - c_2(S,t) \le (E_2 - E_1)\mathrm{e}^{-r(T-t)}.$$

25. Suppose that p_1, p_2, and p_3 are the prices of European put options with strike prices E_1, E_2, and E_3, respectively, where $E_2 = \frac{1}{2}(E_1 + E_3)$. All the options have the same maturity. Show

$$p_2 \le \frac{1}{2}(p_1 + p_3).$$

26. Consider a European call option with $T = 6$ months and $E = \$80$ on a dividend-paying stock. The dividend is paid continuously with a dividend yield $D_0 = 0.05$. Today, $t = 0$, $r = 0.1$ and $S = \$82$.
    (a) Find the lower bound of the call option.
    (b) What are the least profits we could make at time T by arbitrage if the call option price today is \$0.10 less than the lower bound and why?
27. Consider a European put option with $T = 3$ months and $E = \$60$ on a dividend-paying stock. Today $t = 0$, $r = 0.05$, and $S = \$55$. The dividends are paid discretely, and the total present value of them is $D_p(55, 0) = \$0.30$.
    (a) Find the lower bound of the put option.
    (b) What are the least profits we could make at time T by arbitrage if the put option price today is \$0.20 less than the lower bound and why?
28. *Use arbitrage arguments to show the put–call parity of European options for the following two cases.
    (a) When the dividend is paid continuously, the put–call parity is

$$c(S,t) - p(S,t) = S\mathrm{e}^{-D_0(T-t)} - E\mathrm{e}^{-r(T-t)};$$

    (b) when the dividend is paid discretely, the put–call parity is

$$c(S,t) - p(S,t) = S - D_p(S,t) - E\mathrm{e}^{-r(T-t)},$$

    where $D_p(S,t)$ is the value of "will-be-paid" dividends at time t.
29. *Use arbitrage arguments to show the inequalities of American options for the following two cases.
    (a) When the dividend is paid continuously, there is the inequality

$$S\mathrm{e}^{-D_0(T-t)} - E \le C(S,t) - P(S,t) \le S - E\mathrm{e}^{-r(T-t)}$$

    between American put option $P(S,t)$ and American call option $C(S,t)$ with the same parameters.
    (b) When the dividend is paid discretely, there is the inequality

$$S - D_p(S,t) - E \le C(S,t) - P(S,t) \le S - E\mathrm{e}^{-r(T-t)}$$

    between American put option $P(S,t)$ and American call option $C(S,t)$ with the same parameters.

30. Suppose that there are an American call option and an American put option on the same stock that pays dividends discretely. For both of them, $E = \$90$ and $T = 3\,\text{months}$. At time $t = 0$, the stock price is \$93 and the present value of dividend payments during the period $[0, T]$ is $D_p(93, 0) = \$0.50$. Assume that $r = 0.10$ and $P(93, 0) = \$2.50$.
    (a) Find the upper and lower bounds of the price of the American call option.
    (b) What are the risk-free profits we could make today by arbitrage if the price of the call option today is \$0.10 greater than the calculated upper bound and why?