2

European Style Derivatives

2.1 Asset Price Models and Itô's Lemma

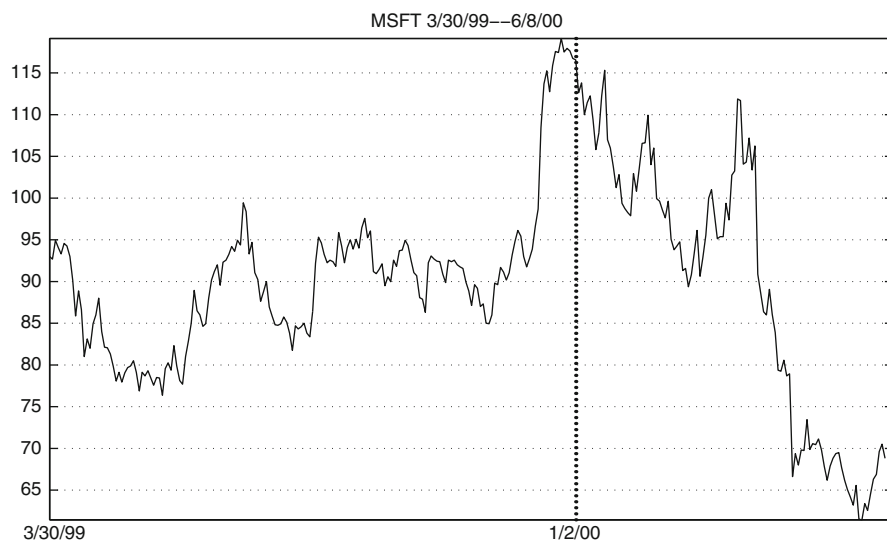### 2.1.1 Models for Asset Prices

MSFT 3/30/99--6/8/00

Fig. 2.1. Stock price of Microsoft Inc.

As examples, in Figs. 1.1–1.7 we showed how the prices of assets vary with time t. Figure 2.1 shows the stock price of Microsoft Inc. in the period March 30, 1999, to June 8, 2000. From these figures, we can see the following: the graphs are jagged, and the size of the jags changes all the time. This means that we cannot express S as a smooth function of t, and it is difficult to predict exactly the price at time t from the price before time t. It is natural to think

of the price at time t as a random variable. Now let us give a model for such a random variable.

Suppose that at time t the asset price is S. Let us consider a small subsequent time interval dt, during which S changes to $S+dS$. (We use the notation df for the small change in any quantity f over this time interval.) How might we model the corresponding return rate on the asset, dS/S?

Assume that the return rate on the asset can be described by the following stochastic differential equation:

$$\frac{dS}{S} = \mu(S,t)dt + \sigma(S,t)dX, \tag{2.1}$$

where μ and σ are called the **drift** and the **volatility**, respectively, and dX is known as a Wiener process defined by

$$\begin{cases} dX = \phi\sqrt{dt}, \\ \phi \text{ being a standardized normal random variable.} \end{cases}$$

In this model, the first part is an anticipated and deterministic return rate, and the second part is the random return rate of the asset price in response to unexpected events. As we can see, the random increment dS depends solely on today's price. This independence from the past is known as the Markov property. In many situations, it is assumed that μ and σ are constants. Due to its simplicity, this is a popular model for asset prices

For a random variable ψ with a probability density function $f(\psi)$ defined on $(-\infty, \infty)$, the expectation of any function $F(\psi)$, $\mathrm{E}\left[F\left(\psi\right)\right]$, is given by

$$\mathrm{E}\left[F\left(\psi\right)\right] = \int_{-\infty}^{\infty} F(\psi)f(\psi)d\psi.$$

The variance of $F(\psi)$, $\mathrm{Var}\left[F(\psi)\right]$, is defined by

$$\mathrm{Var}\left[F(\psi)\right] = \mathrm{E}\left[(F(\psi) - \mathrm{E}\left[F(\psi)\right])^2\right].$$

According to these definitions, for any constants a, b, c, and random variable W, we have

$$\mathrm{E}\left[aW - b\right] = a\mathrm{E}\left[W\right] - b,$$
$$\mathrm{Var}\left[W\right] = \mathrm{E}\left[(W - \mathrm{E}\left[W\right])^2\right]$$
$$= \mathrm{E}\left[W^2\right] - (\mathrm{E}\left[W\right])^2$$

and

$$\mathrm{Var}\left[\frac{W}{c}\right] = \frac{1}{c^2}\mathrm{Var}\left[W\right].$$

For a standardized normal random variable ϕ, the probability density function is

$$\frac{1}{\sqrt{2\pi}}e^{-\phi^2/2}, \qquad -\infty < \phi < \infty.$$

As a probability density function, this function satisfies[1]

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\phi^2/2}d\phi = 1.$$

Therefore we have

$$\mathrm{E}\left[\phi\right] = \int_{-\infty}^{\infty} \phi\frac{1}{\sqrt{2\pi}}e^{-\phi^2/2}d\phi = 0$$

and

$$\begin{aligned}
\mathrm{Var}\left[\phi\right] &= \mathrm{E}\left[\phi^2\right] \\
&= \int_{-\infty}^{\infty} \phi^2\frac{1}{\sqrt{2\pi}}e^{-\phi^2/2}d\phi \\
&= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \phi d\left(e^{-\phi^2/2}\right) \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\phi^2/2}d\phi \\
&= 1.
\end{aligned}$$

From these we obtain

$$\mathrm{E}\left[dX\right] = \mathrm{E}\left[\phi\right]\sqrt{dt} = 0$$

and

$$\mathrm{Var}\left[dX\right] = \mathrm{E}\left[dX^2\right] = \mathrm{E}\left[\phi^2\right]dt = dt.$$

Consequently[2]

$$\mathrm{E}\left[dS\right] = \mathrm{E}\left[\sigma S\,dX + \mu S\,dt\right] = \mu S\,dt,$$

and

$$\begin{aligned}
\mathrm{Var}\left[dS\right] &= \mathrm{E}\left[dS^2\right] - \left(\mathrm{E}\left[dS\right]\right)^2 \\
&= \mathrm{E}\left[\sigma^2 S^2 dX^2 + 2\sigma S^2\mu\,dt\,dX + \mu^2 S^2 dt^2\right] - \mu^2 S^2 dt^2 \\
&= \sigma^2 S^2 dt.
\end{aligned}$$

The square root of the variance is known as the standard deviation. Thus, the deviation of the return on the asset is proportional to σ. This means

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[1]Because $\int_0^\infty e^{-x^2/2}dx \times \int_0^\infty e^{-y^2/2}dy = \int_0^{\pi/2}\int_0^\infty e^{-r^2/2}r\,dr\,d\theta = \pi/2$, we have $\int_0^\infty e^{-\phi^2/2}d\phi = \sqrt{\pi/2}$.

[2]Here, dX is a random variable and S is unchanged. In stochastic calculus, it is called conditional expectation (see [51, 6]).

that an asset price with a larger σ would appear more jagged. Typically, for stocks, indices, exchange rates, and bonds, the value of σ is in the range 0.02–0.4. Usually, the volatility of stocks is greater than indices, exchange rates, and bonds, and government bonds have the smallest volatility among these. Among shares, high-tech companies tend to have higher volatility than other companies. For example, assume that the volatility of the price of IBM stock is a constant during 1990–2000, then its value is 0.31. Under the same assumption, for the price of GE stock, $\sigma = 0.23$. For S&P 500, British pound—U.S. dollar exchange rate, Japanese yen—U.S. dollar exchange rate, and a five-year government bond with coupon 6.5 % and maturing on May 31, 2001, $\sigma = 0.10,\ 0.11,\ 0.12$, and 0.03, respectively. For the bond, we assume that σ depends on the time to maturity. Clearly, at maturity σ is zero. The value 0.03 means that the maximum value of σ is 0.03. In practice, the volatility is often quoted as a percentage so that $\sigma = 0.2$ would be 20 % volatility.

If $\sigma = 0$, then

$$\frac{dS}{S} = \mu dt \quad \text{and} \quad S(t) = S_0 e^{\mu(t-t_0)},$$

where S_0 is the value of the asset at $t = t_0$.

In this asset price model, μ and σ are two parameters. In general, these parameters depend on the asset price S and time t, i.e., $\mu = \mu(S,t)$, $\sigma = \sigma(S,t)$. According to the historical data, we can determine these parameters (or parameter functions) for the past by statistical analysis. If we assume that μ and σ depend on S only, then the functions $\mu(S)$ and $\sigma(S)$ determined by the historical data can be used for the future.

A Wiener process is also referred to as a Brownian motion. There are many excellent books on the Brownian motion. Readers interested in this subject can read, for example, [51]. A basic and very important feature of the Wiener process is that the sum of two independent Wiener processes is also a Wiener process, and the variance of the sum is the sum of the two original variances. That is, if $dX_1 = \phi_1 \sqrt{dt_1}$ and $dX_2 = \phi_2 \sqrt{dt_2}$ are two Wiener processes and they are independent, namely, $\mathrm{E}\,[\phi_1 \phi_2] = 0$, then

$$dX_3 = dX_1 + dX_2 = \phi_1 \sqrt{dt_1} + \phi_2 \sqrt{dt_2} = \phi_3 \sqrt{dt_1 + dt_2}, \qquad (2.2)$$

where ϕ_3 is also a standardized normal random variable. Readers are asked to prove a similar conclusion as a portion of Problem 4.

2.1.2 Itô's Lemma

There is a practical lower bound for the basic time-step dt of the random walk of an asset price. Thus, an asset price is a discrete random variable. However, sometimes the lower bound is so small that we consider an asset price as a continuous random variable. Also, because it is much more efficient to solve the resulting differential equations than to evaluate options by direct simulation

of the random walk on a practical time scale, we will assume that an asset price is a continuous random variable even if the basic time-step is not very small.

Before coming to Itô's lemma, we need one result, which we do not prove. This result is, with probability one,

$$dX^2 = \phi^2 dt \to dt \quad \text{as} \quad dt \to 0.$$

This can be explained as follows. Because

$$\mathrm{E}\left[dX^2\right] = \mathrm{E}\left[\phi^2\right] dt = dt$$

and

$$\mathrm{Var}\left[dX^2\right] = \mathrm{E}\left[dX^4\right] - (\mathrm{E}\left[dX^2\right])^2 = O(dt^2),$$

the variance of dX^2 is very small and the smaller dt becomes, the closer dX^2 comes to being equal to dt.

Assume

$$dS = a(S,t)dt + b(S,t)dX$$

and suppose $f(S,t)$ is a smooth function of a random variable S and time t. We need to find a stochastic differential equation for f. If we vary S and t by a small amount dS and dt, then f also varies by a small amount. From the Taylor series expansion we can write

$$df = \frac{\partial f}{\partial S}dS + \frac{\partial f}{\partial t}dt + \frac{1}{2}\left(\frac{\partial^2 f}{\partial S^2}dS^2 + 2\frac{\partial^2 f}{\partial t \partial S}dt\,dS + \frac{\partial^2 f}{\partial t^2}dt^2\right) + \cdots.$$

Because

$$dS^2 = [a(S,t)dt + b(S,t)dX]^2 = \left(adt + b\phi\sqrt{dt}\right)^2$$
$$= a^2(dt)^2 + 2ab\phi(dt)^{3/2} + b^2\phi^2 dt \to b^2 dt \quad \text{as} \quad dt \to 0,$$

we have[3]

$$df = \frac{\partial f}{\partial S}dS + \left(\frac{\partial f}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 f}{\partial S^2}\right)dt \quad \text{as} \quad dt \to 0 \qquad (2.3)$$

or in the form of a stochastic differential equation

$$df = b\frac{\partial f}{\partial S}dX + \left(\frac{\partial f}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 f}{\partial S^2} + a\frac{\partial f}{\partial S}\right)dt.$$

This is Itô's lemma. If in the asset price model (2.1), μ and σ are constants, i.e.,

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[3] As we know, in calculus we have $df(S,t) = \frac{\partial f}{\partial S}dS + \frac{\partial f}{\partial t}dt$. Thus this relation is the same as the relation in calculus only if $f(S,t)$ is a linear function in S.

$$dS = \mu S dt + \sigma S dX,$$

then Itô's lemma is in the form:

$$
\begin{aligned}
df &= \frac{\partial f}{\partial S} dS + \left(\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) dt \\
&= \sigma S \frac{\partial f}{\partial S} dX + \left(\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \mu S \frac{\partial f}{\partial S} \right) dt.
\end{aligned}
$$

2.1.3 Expectation and Variance of Lognormal Random Variables

As a simple example, consider the function $f(S) = \ln S$. Differentiation of this function gives

$$\frac{df}{dS} = \frac{1}{S} \quad \text{and} \quad \frac{d^2 f}{dS^2} = -\frac{1}{S^2}.$$

Suppose that S satisfies Eq. (2.1) with constant μ and σ, i.e., $dS = \mu S dt + \sigma S dX$. Using Itô's lemma, for $\ln S$ we have

$$d \ln S = \sigma dX + \left(\mu - \frac{\sigma^2}{2} \right) dt = m dt + \sigma dX, \tag{2.4}$$

where

$$m = \mu - \frac{\sigma^2}{2}. \tag{2.5}$$

It is clear that

$$\mathrm{E}\left[d \ln S \right] = \mathrm{E}\left[m dt + \sigma dX \right] = m dt$$

and

$$
\begin{aligned}
\mathrm{Var}\left[d \ln S \right] &= \mathrm{E}\left[(d \ln S)^2 \right] - (\mathrm{E}\left[d \ln S \right])^2 \\
&= \mathrm{E}\left[\sigma^2 dX^2 + 2 \sigma m \, dt \, dX + m^2 dt^2 \right] - m^2 dt^2 \\
&= \sigma^2 \mathrm{E}\left[\phi^2 dt \right] = \sigma^2 dt.
\end{aligned}
$$

From Eq. (2.4), the probability density function for $d \ln S$ is[4]

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[4] • Here $\mathrm{e}^{-(d \ln S - m dt)^2 / 2\sigma^2 dt}$ means $\mathrm{e}^{-(d \ln S - m dt)^2 / (2\sigma^2 dt)}$. That is, in the expression $(d \ln S - m dt)^2 / 2\sigma^2 dt$, the division between $(d \ln S - m dt)^2$ and $2\sigma^2 dt$ should be done after $2 \times \sigma^2 \times dt$ is obtained. Throughout the entire book we use such a notation.

• If x is a normal random variable and its mean and variance are a and b^2, then its probability density function is

$$\frac{1}{b\sqrt{2\pi}} \mathrm{e}^{-(x-a)^2 / 2b^2}.$$

$$\frac{1}{\sigma\sqrt{2\pi dt}}e^{-(d\ln S-mdt)^2/2\sigma^2 dt}.$$

Let $d\ln S=\ln S'-\ln S$. Then for $\ln S'$, the probability density function is

$$G_1\left(\ln S'\right)=\frac{1}{\sigma\sqrt{2\pi dt}}e^{-\left[\ln S'-\ln S-mdt\right]^2/2\sigma^2 dt}.$$

Here, S is the value of the asset at time t and S' is the value of the asset at time $t+dt$ which is a random variable. In Fig. 2.2, the curve of $G_1\left(\ln S'\right)$ with $\ln S+mdt=0$ and $\sigma\sqrt{dt}=0.2$ is shown.
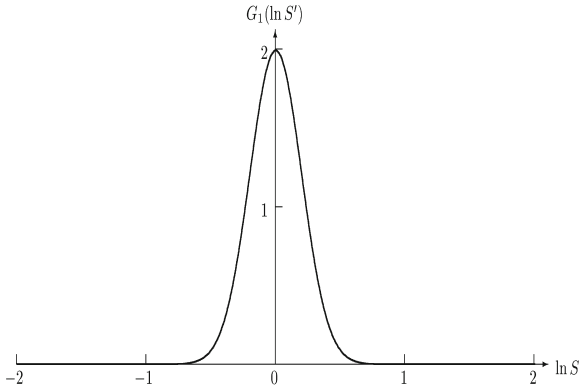
Fig. 2.2. The probability density function for $\ln S'$
with $\ln S+mdt=0$ and $\sigma\sqrt{dt}=0.2$

Suppose that for S' the probability density function is $G\left(S'\right)$. Because[5]

$$G\left(S'\right)dS'=\frac{1}{\sigma\sqrt{2\pi dt}}e^{-\left(\ln S'-\ln S-mdt\right)^2/2\sigma^2 dt}d\ln S',$$

we have

$$G(S')=\frac{1}{S'\sigma\sqrt{2\pi dt}}e^{-(\ln S'-\ln S-mdt)^2/2\sigma^2 dt}.$$

---

[5] If for x the probability density function is $f(x)$, then the probability of $x\in[x,x+dx]$ is $f(x)dx$. If $y=y(x)$ and $y(x)$ is a nondecreasing function, then $x\in[x,x+dx]$ if and only if $y\in[y(x),y(x+dx)]\approx\left[y(x),y(x)+\frac{dy}{dx}dx\right]$. Thus, the probability of the event $y\in\left[y(x),y(x)+\frac{dy}{dx}dx\right]$ is also $f(x)dx$. If for y the probability density function is $f_1(y)$, then $f_1(y)dy=f(x)dx$, from which we have $f_1(y)=f(x(y))\frac{dx}{dy}$. If $x=\ln S'$ and $y=S'$, then $f_1(S')=f(x(y))\frac{dx}{dy}=f(\ln S')\frac{1}{S'}$.
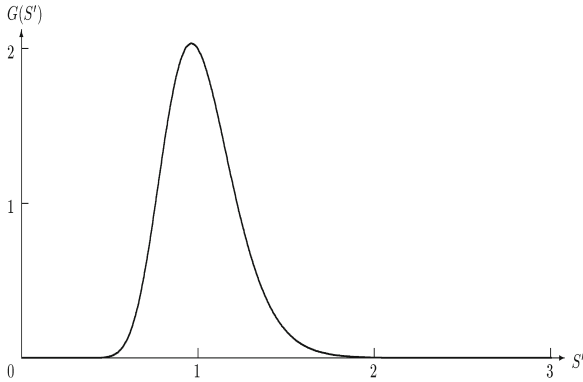
Fig. 2.3. The probability density function for S'
with $\ln S + mdt = 0$ and $\sigma\sqrt{dt} = 0.2$

In Fig. 2.3, the corresponding curve of $G(S')$ is given. This is called a lognormal because the corresponding distribution for $\ln S'$ is normal.

Now the question is what are $\mathrm{E}\,[S']$ and $\mathrm{Var}\,[S']$. Because we have the probability density function, let

$$y = \frac{\ln S' - \ln S - mdt}{\sigma\sqrt{dt}}$$

and we have

$$
\begin{aligned}
\mathrm{E}\,[S'] &= \int_0^\infty G(S')S'dS' \\
&= \frac{1}{\sigma\sqrt{2\pi dt}} \int_0^\infty e^{-(\ln S' - \ln S - mdt)^2/2\sigma^2 dt}\frac{1}{S'} \times S'dS' \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-y^2/2}e^{y\sigma\sqrt{dt}+\ln S+mdt}dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\left(y-\sigma\sqrt{dt}\right)^2/2} \times e^{\sigma^2 dt/2+\ln S+mdt}dy \\
&= e^{\sigma^2 dt/2+\ln S+mdt} = Se^{\mu dt},
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}\,[S'^2] &= \int_0^\infty G(S')S'^2 dS' \\
&= \frac{1}{\sigma\sqrt{2\pi dt}} \int_0^\infty e^{-(\ln S' - \ln S - mdt)^2/2\sigma^2 dt}\frac{1}{S'}S'^2 dS' \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-y^2/2} \times e^{2(y\sigma\sqrt{dt}+\ln S+mdt)}dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\left(y-2\sigma\sqrt{dt}\right)^2/2}e^{2\sigma^2 dt+2(\ln S+mdt)}dy \\
&= e^{2\sigma^2 dt+\ln S^2+2mdt} = S^2 e^{2\mu dt+\sigma^2 dt}
\end{aligned}
$$

and

$$\text{Var}\,[S'] = S^2 \mathrm{e}^{2\mu dt + \sigma^2 dt} - S^2 \mathrm{e}^{2\mu dt}$$
$$= S^2 \mathrm{e}^{2\mu dt} \left(\mathrm{e}^{\sigma^2 dt} - 1 \right),$$

where we have used the relation (2.5).

If m and σ in the expression (2.4) are constants, then for a large time period $T - t$, we can have

$$\ln S' - \ln S = \int_t^T d\ln S = m \int_t^T dt + \sigma \int_t^T dX(t) = m(T - t) + \sigma \phi \sqrt{T - t},$$

where S' is the stock price at time T, S is the stock price at time t, and ϕ is a standardized normal random variable. Here we used the relation $\int_t^T dX(t) = \phi \sqrt{T - t}$, which can be obtained from the relation (2.2). Therefore, in this case, the probability density function for S' is

$$G(S') = \frac{1}{S' \sigma \sqrt{2\pi (T - t)}} \mathrm{e}^{-[\ln S' - \ln S - m(T-t)]^2 / 2\sigma^2 (T - t)}$$

and

$$\begin{cases} \mathrm{E}\,[S'] = S \mathrm{e}^{\mu(T-t)}, \\[2mm] \text{Var}\,[S'] = S^2 \mathrm{e}^{2\mu(T-t)} \left[\mathrm{e}^{\sigma^2(T-t)} - 1 \right], \end{cases} \tag{2.6}$$

where μ is given by the relation (2.5):

$$\mu = m + \frac{\sigma^2}{2}.$$

2.2 Derivation of the Black–Scholes Equation

2.2.1 Arbitrage Arguments

In the modern world, financial transactions may be done simultaneously in more than one market. Suppose the price of gold is \$324 per ounce in New York and 37,275 Japanese Yen in Tokyo, while the exchange rate is 1 U.S. dollar for 115 Japanese Yen. An arbitrageur, who is always looking for any arbitrage opportunities to make money, could simultaneously buy 1,000 ounces in New York, sell them in Tokyo to obtain a risk-free profit of

$$37{,}275 \times 1{,}000/115 - 324 \times 1{,}000 = \$130.43$$

if the transaction costs can be ignored. Small investors may not profit from such opportunity due to the transaction costs. However, the transaction costs for large investors might be a small portion of the profit, which makes the arbitrage opportunity very attractive.

Arbitrage opportunities usually cannot last long. As arbitrageurs buy the gold in New York, the price of the gold will rise. Similarly, as they sell the gold in Tokyo, the price of the gold will be driven down. Very quickly, the ratio between the two prices will become closer to the current exchange rate. In practice, due to the existence of arbitrageurs, very few arbitrage opportunities can be observed. Therefore, throughout this book we will assume that there are no arbitrage opportunities for financial transactions.

Let us make the following assumptions: both the borrowing short-term interest rate and the lending short-term interest rate are equal to r, short selling is permitted, the assets and options are divisible, and there is no transaction cost. Then, we can conclude that the absence of arbitrage opportunities is equivalent to all risk-free portfolios having the same return rate r.

Let us show this point. Suppose that Π is the value of a portfolio and that during a time step dt the return of the portfolio $d\Pi$ is risk-free. If

$$d\Pi > r\Pi dt,$$

then an arbitrageur could make a risk-free profit $d\Pi - r\Pi dt$ during the time step dt by borrowing an amount Π from a bank to invest in the portfolio. Conversely, if

$$d\Pi < r\Pi dt,$$

then the arbitrageur would short the portfolio and invest Π in a bank and get a net income $r\Pi dt - d\Pi$ during the time step dt without taking any risk. Only when

$$d\Pi = r\Pi dt$$

holds, is it guaranteed that there are no arbitrage opportunities.

In the next subsection, we will derive the equation the prices of derivative securities should satisfy by using the conclusion that all risk-free portfolios have the same return rate r.

2.2.2 The Black–Scholes Equation

Let V denote the value of an option that depends on the value of the underlying asset S and time t, i.e., $V = V(S, t)$. It is not necessary at this stage to specify whether V is a call or a put; indeed, V can even be the value of a whole portfolio of various options. For simplicity, readers may think of a simple call or put.

Assume that in a time step dt, the underlying asset pays out a dividend $SD_0 dt$, where D_0 is a constant known as the dividend yield.[6] Suppose S satisfies Eq. (2.1):

_____
[6]This dividend structure is a good model for an index. In this case, many discrete dividend payments are paid at many different times, and the dividend payment can be approximated by a continuous yield without serious error. Also, if the asset is a foreign currency, then the interest rate for the foreign currency plays the role of D_0.

$$\frac{dS}{S} = \mu(S,t)dt + \sigma(S,t)dX.$$

According to Itô's lemma (2.3), the random walk followed by V is given by

$$dV = \frac{\partial V}{\partial S}dS + \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt. \tag{2.7}$$

Here we require V to have at least one t derivative and two S derivatives.

Now construct a portfolio consisting of one option and a number $-\Delta$ of the underlying asset. This number is as yet unspecified. The value of this portfolio is

$$\Pi = V - \Delta S. \tag{2.8}$$

Because the portfolio contains one option and a number $-\Delta$ of the underlying asset, and the owner of the portfolio receives $SD_0 dt$ for every asset held, the earnings for the owner of the portfolio during the time step dt is

$$d\Pi = dV - \Delta\left(dS + SD_0 dt\right).$$

Using the relation (2.7), we find that Π follows the random walk

$$d\Pi = \left(\frac{\partial V}{\partial S} - \Delta\right)dS + \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \Delta S D_0\right)dt.$$

The random component in this random walk can be eliminated by choosing

$$\Delta = \frac{\partial V}{\partial S}. \tag{2.9}$$

This results in a portfolio whose increment is wholly deterministic:

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \Delta S D_0\right)dt. \tag{2.10}$$

Because the return for any risk-free portfolio should be r, we have

$$r\Pi dt = d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \Delta S D_0\right)dt. \tag{2.11}$$

Substituting the relations (2.8) and (2.9) into Eq. (2.11) and dividing by dt, we arrive at

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0)S\frac{\partial V}{\partial S} - rV = 0. \tag{2.12}$$

When we take different Π for different S and t, we can conclude that Eq. (2.12) holds on a domain. In this book, Eq. (2.12) is called the Black–Scholes partial differential equation, or the Black–Scholes equation,[7] even though $D_0 = 0$ in the equation originally given by Black and Scholes (see [11]). With its extensions and variants, it plays the major role in the rest of the book.

About the derivation of this equation and the equation itself, we give more explanation here.

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[7]It is also called Black–Scholes–Merton differential equation (see [43]).

- The key idea of deriving this equation is to eliminate the uncertainty or the risk. $d\Pi$ is not a differential in the usual sense. It is the earning of the holder of the portfolio during the time step dt. Therefore, $\Delta S D_0 dt$ appear. In the derivation, in order to eliminate any small risk, Δ is chosen before an uncertainty appears and does not depend on the coming risk. Therefore, no differential of Δ is needed.
- The linear differential operator given by

$$\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0)S\frac{\partial}{\partial S} - r$$

has a financial interpretation as a measure of the difference between the return on a hedged option portfolio

$$\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} - D_0 S\frac{\partial}{\partial S}$$

and the return on a bank deposit

$$r\left(1 - S\frac{\partial}{\partial S}\right).$$

Although the difference between the two returns is identically zero for European options, we will later see that the difference between the two returns may be nonzero for American options.
- From the Black–Scholes equation (2.12), we know that the parameter μ in Eq. (2.1) does not affect the option price, i.e., the option price determined by this equation is independent of the average return rate of an asset price per unit time.
- From the derivation procedure of the Black-Scholes equation we know that the Black-Scholes equation still holds if r and D_0 are functions of S and t.
- If dividends are paid only on certain dates, then the money the owner of the portfolio will get during the time period $[t, t + dt]$ is

$$dV - \Delta dS - \Delta D(S,t)dt,$$

where $D(S,t)$ is a sum of several Dirac delta functions. Suppose that a stock pays dividend $D_1(S)$ at time t_1 and $D_2(S)$ at time t_2 for a share, where $D_1(S) \leq S$ and $D_2(S) \leq S$. Then

$$D(S,t) = D_1(S)\delta(t - t_1) + D_2(S)\delta(t - t_2),$$

where the Dirac delta function[8] $\delta(t)$ is defined as follows:

---

[8]It is the limit as $\varepsilon \to 0$ of the one-parameter family of functions:

$$\delta_\varepsilon(x) = \begin{cases} \dfrac{1}{2\varepsilon}, & -\varepsilon \leq x \leq \varepsilon, \\ 0, & |x| > \varepsilon. \end{cases}$$

$$\delta(t) = \begin{cases} 0, & \text{if } t \neq 0, \\ \infty, & \text{if } t = 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t) = 1.$$

In this case, the modified Black–Scholes equation is in the form

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + [rS - D(S,t)]\frac{\partial V}{\partial S} - rV = 0. \qquad (2.13)$$

2.2.3 Final Conditions for the Black–Scholes Equation

From the derivation of the Black–Scholes equation (2.12), we know that this partial differential equation holds for any option (or portfolio of options) whose value depends only on S and t. In order to determine a unique solution of the Black–Scholes equation, the solution at the expiry, $t = T$, needs to be given. This condition is called the final condition for the partial differential equation. Different options satisfy the same partial differential equation, but different final conditions. Therefore, in order to determine the price of an option, we need to know the value of the option at time T. In what follows, we will derive the final conditions for call and put options.

Final Condition for Call Options. Let us examine what a holder of a call option will do just at the moment of expiry. If $S > E$ at expiry, it makes financial sense for the holder to exercise the call option, handing over an amount E for an asset worth S. The money earned by the holder from such a transaction is then $S - E$. On the other hand, if $S < E$ at expiry, the holder should not exercise the option because the holder would lose an amount of $E - S$. In this case, the option expires valueless. Thus, the value of the call option at expiry can be written as

$$C(S,T) = \max(S - E, 0). \qquad (2.14)$$

This function giving the value of a call option at expiry is usually called the payoff function of a call option. In Fig. 1.9, we plot $\max(S-E, 0)$ as a function of S, which is usually known as a payoff diagram. A call option with such a payoff is the simplest call option and is known as a vanilla call option.

Final Condition for Put Options. Each option or each portfolio of options has its own payoff at expiry. An argument similar to that given above for the value of a call at expiry leads to the payoff for a put option. At expiry, the put option is worthless if $S > E$ but has the value $E - S$ for $S < E$. Thus, the payoff function of a put option is

$$P(S,T) = \max(E - S, 0). \qquad (2.15)$$

The payoff diagram for a put is shown in Fig. 1.10 where the line shows the payoff function $\max(E-S, 0)$. In order to distinguish this put option from other more complicated put options, sometimes it is referred to as the vanilla put option.

2.2.4 Hedging and Greeks

The way to reduce the sensitivity of a portfolio to the movement of something by taking opposite positions in different financial instruments is called hedging. Hedging is a basic concept in finance. When we derived the Black–Scholes equation in Sect. 2.2.2, we chose the delta to be $\dfrac{\partial V}{\partial S}$, so that the portfolio Π became risk-free. This gives an important example on how hedging is applied. Let us see another example of hedging that is similar to what we have used in deriving the Black–Scholes equation.

Consider a call option on a stock. Figure 2.4 shows the relation between the call price and the underlying stock price. When the stock price corresponds to point A, the option price corresponds to point B and the Δ of the call is the slope of the line indicated. As an approximation

$$\Delta = \frac{\delta c}{\delta S},$$

where δS is a small change in the stock price and δc is the corresponding change in the call price.
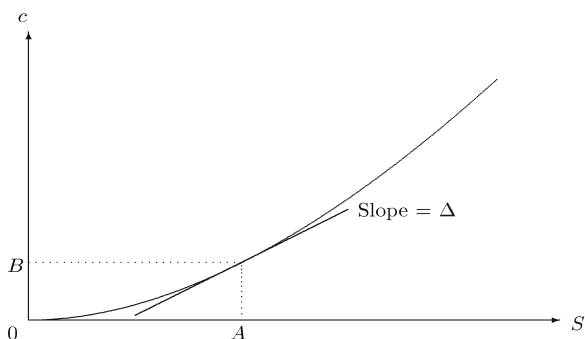

Fig. 2.4. $\Delta =$ the slope of a curve

Assume that the delta of the call option is 0.7 and a writer sold 10,000 units of call options. Then, the writer's position could be hedged by buying $0.7 \times 10,000 = 7,000$ shares of stocks. If the stock price goes up by $0.50, the writer will earn $3,500 from the 7,000 shares of stocks held. At the same time, the price of call options will go up approximately $0.7 \times 0.5 = \$0.35$, and he will lose $10,000 \times \$0.35 = \$3,500$ from 10,000 shares of option he sold. Therefore, the net profit or loss is about zero. If the price falls down by a small amount, the situation is similar. Consequently, the writer's position has been hedged quite well as long as the movement of the price is small. This is called delta hedging.

In the example above, we have considered only a call option. Actually, any portfolio can be hedged in this way. If Π denotes the price of option, then the slope is

$$\Delta = \frac{\partial \Pi}{\partial S}.$$

If the movement of the price is not very small, then it might be necessary to use the value of the second derivative of the portfolio with respect to S in order to eliminate most of the risk. The second derivative is known as gamma

$$\Gamma = \frac{\partial^2 \Pi}{\partial S^2}.$$

When hedging in practice, some other values, for example, $\frac{\partial \Pi}{\partial t}, \frac{\partial \Pi}{\partial \sigma}, \frac{\partial \Pi}{\partial r},$ $\frac{\partial \Pi}{\partial D_0}$, may need to be known. Usually, $\frac{\partial \Pi}{\partial t}, \frac{\partial \Pi}{\partial \sigma}$, and $\frac{\partial \Pi}{\partial r}$ are called theta, vega, and rho, respectively; namely, the following notation is used:

$$\Theta = \frac{\partial \Pi}{\partial t}, \quad \mathcal{V} = \frac{\partial \Pi}{\partial \sigma},$$

and

$$\rho = \frac{\partial \Pi}{\partial r}.$$

In currency options, D_0 is the interest rate in the foreign country. Thus, $\frac{\partial \Pi}{\partial D_0}$ is also known as rho. In order to distinguish $\frac{\partial \Pi}{\partial r}$ and $\frac{\partial \Pi}{\partial D_0}$, here we define

$$\rho_d = \frac{\partial \Pi}{\partial D_0}.$$

These quantities are usually referred to as Greeks.

When σ depends on S, or the coefficient of $\frac{\partial V}{\partial S}$ is more complicated, analytic expressions of option prices may not exist. In this case, we have to use numerical methods. Also sometimes (for example, for a call option), the solution is unbounded. It is not convenient to solve a problem numerically on an infinite domain with an unbounded solution. Therefore in Sect. 2.2.5, we also provide a transformation under which the Black–Scholes equation on $[0, \infty)$ becomes an equation on $[0, 1)$ with a bounded solution.

2.2.5 Transforming the Black–Scholes Equation into an Equation Defined on a Finite Domain

Let us consider the following option problem:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2(S)S^2\dfrac{\partial^2 V}{\partial S^2} + (r - D_0)S\dfrac{\partial V}{\partial S} - rV = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 \leq S < \infty, \quad t \leq T, \\ V(S, T) = V_T(S), \qquad\qquad\qquad 0 \leq S < \infty. \end{cases} \quad (2.16)$$

The transformation to be described in this subsection works even when σ, r, or D_0 depends on S and t. For simplicity, we assume in the derivation that σ depends on S and that r, D_0 are constant. In this case, an analytic expression of the solution $V(S,t)$ may not exist, and numerical methods may be necessary. Also for a call option, the solution $V(S,t)$ is not bounded. Therefore, we introduce new independent variables and dependent variable through the following transformation:

$$
\begin{cases}
\xi = \dfrac{S}{S + P_m}, \\[2mm]
\tau = T - t, \\[2mm]
V(S,t) = (S + P_m)\overline{V}(\xi, \tau).
\end{cases}
\tag{2.17}
$$

From Eq. (2.17) we have

$$
S = \frac{P_m \xi}{1 - \xi}, \quad S + P_m = \frac{P_m}{1 - \xi}
$$

and

$$
\frac{d\xi}{dS} = \frac{P_m}{(S + P_m)^2} = \frac{(1 - \xi)^2}{P_m}.
$$

Because

$$
\frac{\partial V}{\partial t} = \frac{\partial}{\partial t}\left[(S + P_m)\overline{V}(\xi, \tau)\right] = -(S + P_m)\frac{\partial \overline{V}}{\partial \tau} = -\frac{P_m}{1 - \xi}\frac{\partial \overline{V}}{\partial \tau},
$$

$$
\frac{\partial V}{\partial S} = \frac{\partial}{\partial S}\left[(S + P_m)\overline{V}(\xi, \tau)\right] = (S + P_m)\frac{\partial \overline{V}}{\partial \xi}\frac{d\xi}{dS} + \overline{V} = (1 - \xi)\frac{\partial \overline{V}}{\partial \xi} + \overline{V},
$$

$$
\frac{\partial^2 V}{\partial S^2} = \frac{\partial}{\partial \xi}\left[(1 - \xi)\frac{\partial \overline{V}}{\partial \xi} + \overline{V}\right]\frac{d\xi}{dS} = \frac{(1 - \xi)^3}{P_m}\frac{\partial^2 \overline{V}}{\partial \xi^2},
$$

and let

$$
\bar{\sigma}(\xi) = \sigma(S(\xi)) = \sigma\left(\frac{P_m \xi}{1 - \xi}\right),
$$

the original equation becomes[9]

$$
\frac{P_m}{1 - \xi}\frac{\partial \overline{V}}{\partial \tau} = \frac{\bar{\sigma}^2(\xi)P_m\xi^2(1 - \xi)}{2}\frac{\partial^2 \overline{V}}{\partial \xi^2} + (r - D_0)P_m\xi\frac{\partial \overline{V}}{\partial \xi} + \frac{(r - D_0)\xi - r}{1 - \xi}P_m\overline{V}
$$

or

$$
\frac{\partial \overline{V}}{\partial \tau} = \frac{\bar{\sigma}^2(\xi)\xi^2(1 - \xi)^2}{2}\frac{\partial^2 \overline{V}}{\partial \xi^2} + (r - D_0)\xi(1 - \xi)\frac{\partial \overline{V}}{\partial \xi} - [r(1 - \xi) + D_0\xi]\overline{V},
$$

$$
0 \le \xi < 1, \quad 0 \le \tau.
$$

_____

[9] Actually, the same equation can be directly obtained by constructing a portfolio and using Itô lemma (see Problem 23).

Assume that $\overline{V}$ is a smooth function of ξ, then the equation also holds at $\xi = 1$. Because $V(S,T) = (S + P_m)\overline{V}(\xi,0) = \overline{V}(\xi,0)\dfrac{P_m}{1-\xi}$, the condition $V(S,T) = V_T(S)$ can be rewritten as $\overline{V}(\xi,0) = V_T\left(\dfrac{P_m\xi}{1-\xi}\right)\dfrac{1-\xi}{P_m}$. Consequently, the problem (2.16) becomes

$$
\begin{cases}
\dfrac{\partial \overline{V}}{\partial \tau} = \dfrac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1-\xi)^2\dfrac{\partial^2 \overline{V}}{\partial \xi^2} + (r - D_0)\xi(1-\xi)\dfrac{\partial \overline{V}}{\partial \xi} - [r(1-\xi) + D_0\xi]\overline{V}, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le \xi \le 1, \quad 0 \le \tau, \\
\overline{V}(\xi,0) = \dfrac{1-\xi}{P_m}V_T\left(\dfrac{P_m\xi}{1-\xi}\right), \qquad\qquad\qquad\qquad 0 \le \xi \le 1.
\end{cases}
$$

$$(2.18)$$

Thus, the transformation (2.17) converts a problem on an infinite domain into a problem on a finite domain. For a parabolic equation defined on a finite domain to have a unique solution, besides an initial condition, boundary conditions are usually needed. However, in this equation the coefficients of $\dfrac{\partial^2 \overline{V}}{\partial \xi^2}$ and $\dfrac{\partial \overline{V}}{\partial \xi}$ at $\xi = 0$ and at $\xi = 1$ are equal to zero, i.e., the equation degenerates to ordinary differential equations at the boundaries. Actually, at $\xi = 0$ the equation becomes

$$\frac{\partial \overline{V}(0,\tau)}{\partial \tau} = -r\overline{V}(0,\tau)$$

and the solution is

$$\overline{V}(0,\tau) = \overline{V}(0,0)\mathrm{e}^{-r\tau}. \tag{2.19}$$

Similarly, at $\xi = 1$ the equation reduces to

$$\frac{\partial \overline{V}(1,\tau)}{\partial \tau} = -D_0\overline{V}(1,\tau),$$

from which we have

$$\overline{V}(1,\tau) = \overline{V}(1,0)\mathrm{e}^{-D_0\tau}. \tag{2.20}$$

Therefore for this equation, the two solutions of the ordinary differential equations provide the values at the boundaries, and no boundary conditions are needed in order for the problem (2.18) to have a unique solution.

Consequently, in order to price an option, we need to solve a problem on a finite domain if this formulation is adopted. From the point view of numerical methods, such a formulation is better. This is its first advantage. Actually, the uniqueness of solution for problem (2.18) can easily be proved (see Sect. 2.4). Indeed, not only the uniqueness can be proved, but the stability of the solution with respect to the initial value can also be shown easily. That is, this formulation makes proof of some theoretical problems easy. This is its other advantage.

For a call option, the payoff is

$$V(S,T) = \max(S - E, 0),$$

so the initial condition in the problem (2.18) for a call is

$$\overline{V}(\xi, 0) = \max(S - E, 0)(1 - \xi)/P_m$$
$$= \max\left(\frac{P_m \xi}{1 - \xi} - E, 0\right)(1 - \xi)/P_m$$
$$= \max\left(\xi - \frac{E}{P_m}(1 - \xi), 0\right).$$

For a put option

$$V(S,T) = \max(E - S, 0).$$

Therefore

$$\overline{V}(\xi, 0) = \max\left(\frac{E}{P_m}(1 - \xi) - \xi, 0\right).$$

Let $P_m = E$, the two initial conditions become

$$\overline{V}(\xi, 0) = \max(2\xi - 1, 0) \quad \text{and} \quad \overline{V}(\xi, 0) = \max(1 - 2\xi, 0),$$

respectively. Therefore, a European call option is the solution of the following problem:

$$
\begin{cases}
\dfrac{\partial \overline{V}}{\partial \tau} = \dfrac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1 - \xi)^2\dfrac{\partial^2 \overline{V}}{\partial \xi^2} + (r - D_0)\xi(1 - \xi)\dfrac{\partial \overline{V}}{\partial \xi} - [r(1 - \xi) + D_0\xi]\overline{V}, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le \xi \le 1, \quad 0 \le \tau, \\
\overline{V}(\xi, 0) = \max(2\xi - 1, 0), \qquad\qquad\qquad 0 \le \xi \le 1
\end{cases}
$$

(2.21)

and the solution of the problem

$$
\begin{cases}
\dfrac{\partial \overline{V}}{\partial \tau} = \dfrac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1 - \xi)^2\dfrac{\partial^2 \overline{V}}{\partial \xi^2} + (r - D_0)\xi(1 - \xi)\dfrac{\partial \overline{V}}{\partial \xi} - [r(1 - \xi) + D_0\xi]\overline{V}, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le \xi \le 1, \quad 0 \le \tau, \\
\overline{V}(\xi, 0) = \max(1 - 2\xi, 0), \qquad\qquad\qquad 0 \le \xi \le 1
\end{cases}
$$

(2.22)

gives the price of a European put option. In the problem (2.21) the initial condition is bounded, so $\overline{V}(\xi, \tau)$, as a solution of a linear parabolic equation, is also bounded. Therefore in this case, the solution that needs to be found numerically is bounded.

So far, we assumed that σ depends only on S and that r and D_0 are constant. However, the result will be the same if σ depends on both S and t, and r and D_0 also depend on S and t.

Finally, we would like to point out that from the expression (2.20) we can have an asymptotic expression of the solution of the Black–Scholes equation at infinity. Because at $\xi = 1$ there is an analytic solution (2.20), noticing

$$V(S,t) = (S + P_m)\overline{V}(\xi, \tau),$$

for $S \approx \infty$ we have

$$
\begin{aligned}
V(S,t) = (S + P_m)\overline{V}(\xi, \tau) &\approx (S + P_m)\overline{V}(1, \tau) \\
&= (S + P_m)\overline{V}(1,0)\mathrm{e}^{-D_0\tau} \\
&\approx V(S,T)\mathrm{e}^{-D_0\tau} = V(S,T)\mathrm{e}^{-D_0(T-t)}.
\end{aligned}
\tag{2.23}
$$

This is an asymptotic expression of the solution of the Black–Scholes equation at infinity.

2.2.6 Derivation of the Equation for Options on Futures

As we know, a futures contract in finance is a standardized contract between two parties to exchange a specified asset of a standardized quantity and quality for a price K (the delivery price) agreed today with delivery occurring at a specified future date, while a forward contract in finance is a nonstandardized contract between two parties to buy or sell an asset at a specified future time at a price K agreed today. There are some differences between a futures contract and a forward contract, but both are a contract in which two parties agree to exchange a specified asset for a specified amount of cash at a specified future date. Here we derive the PDE for options on such a contract.

Suppose that the price of the underlying asset satisfies

$$dS = \mu S dt + \sigma S dX, \tag{2.24}$$

and it pays dividends continuously with a constant dividend yield D_0. We also assume that the interest rate r is a constant. Let T be the expiration date of the contract and t be the time today.

Before deriving the PDE, we point out that the value of a forward/futures contract at time t is

$$f = S\mathrm{e}^{-D_0(T-t)} - K\mathrm{e}^{-r(T-t)}, \tag{2.25}$$

from which we can have

$$S = \mathrm{e}^{D_0(T-t)}\left(f + K\mathrm{e}^{-r(T-t)}\right). \tag{2.26}$$

The reason is as follows. At time t, the seller of this contract, who gets f when the contract is sold, can borrow $K\mathrm{e}^{-r(T-t)}$ from a bank with an interest rate r and buy $\mathrm{e}^{-D_0(T-t)}$ units of the asset by spending $S\mathrm{e}^{-D_0(T-t)}$. At time T,

the seller will get K from the holder of the contract, which will be paid to the bank, and give a unit of the asset to the holder. Therefore, there is no risk for seller, and it is a reasonable price for the contract.

Now we consider an option on such a contract. When we consider options on stocks, we let its value be a function of the value of the stock, S, and t. Thus, it is natural to let the value of options on futures be a function of the value of futures contracts, f, and t. That is, let $V_1(f,t)$ denote the price of the option. The PDE for $V_1(f,t)$ can be derived in the following way. Consider a portfolio

$$\Pi = V_1(f,t) - \Delta f.$$

Because we assume that S is a lognormal variable,[10] using Itô's lemma, for f we have

$$
\begin{aligned}
df &= \frac{\partial f}{\partial S}dS + \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right)dt \\
&= e^{-D_0(T-t)}dS + \left(D_0 Se^{-D_0(T-t)} - rKe^{-r(T-t)}\right)dt \\
&= e^{-D_0(T-t)}(\mu Sdt + \sigma SdX) + \left(D_0 Se^{-D_0(T-t)} - rKe^{-r(T-t)}\right)dt \\
&= \left(e^{-D_0(T-t)}\mu S + D_0 Se^{-D_0(T-t)} - rKe^{-r(T-t)}\right)dt + e^{-D_0(T-t)}\sigma SdX \\
&= \left[(\mu + D_0)\left(f + Ke^{-r(T-t)}\right) - rKe^{-r(T-t)}\right]dt \\
&\quad + \sigma\left[f + Ke^{-r(T-t)}\right]dX.
\end{aligned}
$$

Using this relation and Itô's lemma again, we can further have

$$
\begin{aligned}
d\Pi &= \frac{\partial V_1}{\partial f}df + \left[\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2\left(f + Ke^{-r(T-t)}\right)^2 \frac{\partial^2 V_1}{\partial f^2}\right]dt - \Delta df \\
&= \left(\frac{\partial V_1}{\partial f} - \Delta\right)df + \left[\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2\left(f + Ke^{-r(T-t)}\right)^2 \frac{\partial^2 V_1}{\partial f^2}\right]dt.
\end{aligned}
$$

If we choose $\Delta = \dfrac{\partial V_1}{\partial f}$, then the portfolio $d\Pi$ is risk-free and

$$
\begin{aligned}
d\Pi &= \left[\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2\left(f + Ke^{-r(T-t)}\right)^2 \frac{\partial^2 V_1}{\partial f^2}\right]dt \\
&= r\Pi dt = r\left(V_1(f,t) - \frac{\partial V_1}{\partial f}f\right)dt.
\end{aligned}
$$

This relation can be rewritten as

$$
\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2\left(f + Ke^{-r(T-t)}\right)^2 \frac{\partial^2 V_1}{\partial f^2} + rf\frac{\partial V_1}{\partial f} - rV_1 = 0. \tag{2.27}
$$

---

[10]If we assume that f has a lognormal distribution, the PDE will be different.

Actually, if we use another independent variable, the PDE will become simple. This independent variable is the forward price F. What is the forward price? Consider a foreign currency. Let S be the current spot price in dollars of one unit of the foreign currency at time t and F be the forward price in dollars of one unit of the foreign currency in the forward contract issued at time t and expiring at time T. Let D_0 be the interest rate in the foreign country. Then for the forward price F, there is the following expression:

$$F = e^{(r-D_0)(T-t)} S. \tag{2.28}$$

This is because the seller of the forward contract can borrow $e^{-D_0(T-t)} S$ to buy $e^{-D_0(T-t)}$ units of the foreign currency at time t, and at time T he or she can have one unit of the foreign currency and can obtain an amount of $e^{(r-D_0)(T-t)} S$ from one unit of the foreign currency, which is what he or she needs in order to pay off the borrowing. It is clear that between F and f, there are the following relations:

$$f = e^{-r(T-t)} \left(Se^{(r-D_0)(T-t)} - K \right) = e^{-r(T-t)} (F - K) \tag{2.29}$$

and

$$F = e^{r(T-t)} f + K.$$

Let $V(F, t)$ denote the value of that option, and let us find the PDE for the function $V(F, t)$. Set

$$\begin{cases} F = e^{r(T-t)} f + K, \\ t = t, \\ V_1(f, t) = V(F(f, t), t) = V(e^{r(T-t)} f + K, t). \end{cases}$$

From these expressions, we have

$$\frac{\partial V_1}{\partial t} = \frac{\partial V}{\partial t} - re^{r(T-t)} f \frac{\partial V}{\partial F},$$

$$\frac{\partial V_1}{\partial f} = e^{r(T-t)} \frac{\partial V}{\partial F},$$

and

$$\frac{\partial^2 V_1}{\partial f^2} = e^{2r(T-t)} \frac{\partial^2 V}{\partial F^2}.$$

Using these relations, we can rewrite the PDE for V_1 as

$$\frac{\partial V}{\partial t} - re^{r(T-t)} f \frac{\partial V}{\partial F} + \frac{1}{2} \sigma^2 \left(f + Ke^{-r(T-t)} \right)^2 e^{2r(T-t)} \frac{\partial^2 V}{\partial F^2}$$
$$+ rfe^{r(T-t)} \frac{\partial V}{\partial F} - rV = 0$$

or

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 V}{\partial F^2} - rV = 0. \tag{2.30}$$

Usually, this equation is called the PDE for an option on a futures contract (see [8]). However, the PDE indeed is a variant of the Black–Scholes equation in Sect. 2.2.2. Because F is a function of S and t, we can define a function of S, t as follows: $V_2(S, t) = V(F(S, t), t)$. It is clear that $V_2(S, t)$ also gives the value of the option. The only difference is that it is a function of S, t, not F, t. As we know, any function of S, t, giving the value of a derivative security, should satisfy the Black–Scholes equation; that is, the equation

$$\frac{\partial V_2}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_2}{\partial S^2} + (r - D_0)S\frac{\partial^2 V_2}{\partial S} - rV_2 = 0$$

holds. Let us show by direct calculation that $V_2(S, t)$ satisfies the Black–Scholes equation. Because

$$V(F, t) = V_2(S(F, t), t)$$

and

$$S = e^{-(r-D_0)(T-t)}F,$$

we have

$$\frac{\partial V}{\partial t} = \frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial S}\frac{\partial S}{\partial t} = \frac{\partial V_2}{\partial t} + (r - D_0)S\frac{\partial V_2}{\partial S},$$
$$\frac{\partial V}{\partial F} = e^{-(r-D_0)(T-t)}\frac{\partial V_2}{\partial S},$$
$$\frac{\partial^2 V}{\partial F^2} = e^{-2(r-D_0)(T-t)}\frac{\partial^2 V_2}{\partial S^2}.$$

From Eq. (2.30) we can have

$$\frac{\partial V_2}{\partial t} + (r - D_0)S\frac{\partial V_2}{\partial S} + \frac{1}{2}\sigma^2 F^2 e^{-2(r-D_0)(T-t)}\frac{\partial^2 V_2}{\partial S^2} - rV_2 = 0$$

or

$$\frac{\partial V_2}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_2}{\partial S^2} + (r - D_0)S\frac{\partial V_2}{\partial S} - rV_2 = 0.$$

Thus, we have proved that if the value of an option on a futures contract is a function of S and t, then it satisfies the Black–Scholes equation. It can also be proved that if we let $V_3(S, t) = V_1(f(S, t), t)$, then $V_3(S, t)$ also satisfies the Black–Scholes equation. This means that Eq. (2.27) is also a variant of the Black–Scholes equation. The proof is left for readers as a part of Problem 16. When the Black–Scholes equation is derived, the randomness of the value of derivative securities is cancelled by the randomness of the value of the stock, S, and when Eq. (2.27) is derived, the randomness is cancelled by the randomness

of the value of the forward/futures contract, f. However, f is a function of S and t given by the expression (2.25). Thus, their randomnesses are related. Consequently, the Black–Scholes equation and the equation for options on futures contracts are the same essentially.

2.3 General Equations for Derivatives

Generally speaking, a financial derivative could depend on several random variables, and a random variable may not represent a price of an asset that can be traded on the market. For example, a derivative could depend on prices of several assets. Also interest rates and volatilities may need to be treated as random variables. As we know, both interest rates and volatilities are not prices of assets. In this section, we will derive the general partial differential equations satisfied by derivatives, where there exist several state variables and a state variable may not be a price of an asset traded on the market or even not be related to a price. The derivation of equations for derivatives with several state variables can be found from other books, for example, the books by Hull [42], and Wilmott, Dewynne, and Howison [84].

2.3.1 Generalization of Itô's Lemma

Suppose a financial derivative depends on time t and n random state variables, namely, $S_1, S_2, \cdots, S_n$. Each of them satisfies a stochastic differential equation

$$dS_i = a_i dt + b_i dX_i, \quad i = 1, 2, \cdots, n, \tag{2.31}$$

where a_i, b_i are functions of $S_1, S_2, \cdots, S_n$ and t, and $dX_i = \phi_i \sqrt{dt}$ are Wiener processes. In addition, $\phi_1, \phi_2, \cdots, \phi_n$ have a joint normal distribution and

$$E[\phi_i \phi_j] = \rho_{ij}, \tag{2.32}$$

where

$$-1 \le \rho_{ij} \le 1.$$

If $\rho_{ij} = 0$, then ϕ_i and ϕ_j are not correlated. If $\rho_{ij} = \pm 1$, then ϕ_i and ϕ_j are completely correlated. It is clear that $\rho_{ii} = 1$. In this book ρ_{ij} is referred to as the correlation coefficient between S_i and S_j.

Let $V = V(S_1, S_2, \cdots, S_n, t)$. According to the Taylor expansion, we have

$$dV = V(S_1 + dS_1, S_2 + dS_2, \cdots, S_n + dS_n, t + dt) - V(S_1, S_2, \cdots, S_n, t)$$

$$= \sum_{i=1}^{n} \frac{\partial V}{\partial S_i} dS_i + \frac{\partial V}{\partial t} dt + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 V}{\partial S_i \partial S_j} dS_i dS_j$$

$$+ \sum_{i=1}^{n} \frac{\partial^2 V}{\partial S_i \partial t} dS_i dt + \frac{1}{2} \frac{\partial^2 V}{\partial t^2} (dt)^2 + \cdots.$$

Because

$$\lim_{dt \to 0} dS_i dS_j / dt = b_i b_j \rho_{ij}$$

and $dS_i dt$ is a quantity of order $(dt)^{3/2}$, the relation above as $dt \to 0$ becomes

$$dV = f dt + \sum_{i=1}^{n} \frac{\partial V}{\partial S_i} dS_i, \qquad (2.33)$$

where

$$f = \frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 V}{\partial S_i \partial S_j} b_i b_j \rho_{ij}.$$

This is called the generalized Itô's lemma.

2.3.2 Derivation of Equations for Financial Derivatives

On the n random variables, we further assume that

$$S_1, S_2, \cdots, \text{ and } S_m, \quad m \le n,$$

are prices of some assets which can be traded on markets, and that the k-th asset pays a dividend payment $D_k dt$ during the time interval $[t, t + dt]$, D_k being a known function that may depend on $S_1, S_2, \cdots, S_n$ and t. In order to derive the general PDE for financial derivatives, we suppose that there are

$$n - m + 1$$

distinct financial derivatives dependent on $S_1, S_2, \cdots, S_n$ and t. Let V_k stand for the value of the k-th derivative, $k = 0, 1, \cdots, n - m$ and assume that the k-th derivative during the time interval $[t, t + dt]$ pays coupon payment $K_k dt$, K_k being a known function that may depend on $S_1, S_2, \cdots, S_n$ and t. They could have different expiries, different exercise prices, or different payoff functions. Even some of the derivatives may depend on only some of the random variables. According to the generalized Itô's lemma, for each derivative, we have

$$dV_k = f_k dt + \sum_{i=1}^{n} \nu_{i,k} dS_i,$$

where

$$f_k = \frac{\partial V_k}{\partial t} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 V_k}{\partial S_i \partial S_j} b_i b_j \rho_{ij}$$

and

$$\nu_{i,k} = \frac{\partial V_k}{\partial S_i}.$$

Consider a portfolio consisting of the $n - m + 1$ derivatives and the m assets, whose prices are $S_1, S_2, \cdots, S_m$:

$$\Pi = \sum_{k=0}^{n-m} \Delta_k V_k + \sum_{k=n-m+1}^{n} \Delta_k S_{k-n+m},$$

where Δ_k is the amount of the k-th derivative for $k = 0, 1, \cdots, n - m$ and the amount of the $(k - n + m)$-th asset, for $k = n - m + 1, n - m + 2, \cdots, n$. During the time interval $[t, t + dt]$, the holder of this portfolio will earn

$$\sum_{k=0}^{n-m} \Delta_k (dV_k + K_k dt) + \sum_{k=n-m+1}^{n} \Delta_k (dS_{k-n+m} + D_{k-n+m} dt)$$

$$= \sum_{k=0}^{n-m} \Delta_k \left(f_k dt + \sum_{i=1}^{n} \nu_{i,k} dS_i + K_k dt \right)$$

$$+ \sum_{k=n-m+1}^{n} \Delta_k (dS_{k-n+m} + D_{k-n+m} dt)$$

$$= \sum_{k=0}^{n-m} \Delta_k (f_k + K_k) \, dt + \sum_{i=1}^{n} \left(\sum_{k=0}^{n-m} \Delta_k \nu_{i,k} \right) dS_i$$

$$+ \sum_{i=1}^{m} \Delta_{i+n-m} dS_i + \sum_{k=n-m+1}^{n} \Delta_k D_{k-n+m} dt$$

$$= \sum_{k=0}^{n-m} \Delta_k (f_k + K_k) \, dt + \sum_{i=1}^{m} \left(\sum_{k=0}^{n-m} \Delta_k \nu_{i,k} + \Delta_{i+n-m} \right) dS_i$$

$$+ \sum_{i=m+1}^{n} \left(\sum_{k=0}^{n-m} \Delta_k \nu_{i,k} \right) dS_i + \sum_{k=n-m+1}^{n} \Delta_k D_{k-n+m} dt.$$

Let us choose Δ_k so that

$$\sum_{k=0}^{n-m} \Delta_k \nu_{i,k} + \Delta_{i+n-m} = 0, \quad i = 1, 2, \cdots, m$$

and

$$\sum_{k=0}^{n-m} \Delta_k \nu_{i,k} = 0, \quad i = m + 1, m + 2, \cdots, n.$$

In this case the portfolio is risk-free, so its return rate is r, i.e.,

$$\sum_{k=0}^{n-m} \Delta_k (f_k + K_k) \, dt + \sum_{k=n-m+1}^{n} \Delta_k D_{k-n+m} dt$$

$$= r \left[\sum_{k=0}^{n-m} \Delta_k V_k + \sum_{k=n-m+1}^{n} \Delta_k S_{k-n+m} \right] dt,$$

or

$$\sum_{k=0}^{n-m} \Delta_k \left(f_k + K_k - rV_k \right) + \sum_{k=n-m+1}^{n} \Delta_k \left(D_{k-n+m} - rS_{k-n+m} \right) = 0,$$

or

$$\sum_{k=0}^{n-m} \Delta_k \left(f_k + K_k - rV_k \right) + \sum_{k=1}^{m} \Delta_{n-m+k} \left(D_k - rS_k \right) = 0.$$

This relation and the relations the chosen Δ_k satisfy can be written together in a matrix form:

$$\begin{bmatrix} \nu_{1,0} & \nu_{1,1} & \cdots & \nu_{1,n-m} & 1 & 0 & \cdots & 0 \\ \nu_{2,0} & \nu_{2,1} & \cdots & \nu_{2,n-m} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \nu_{m,0} & \nu_{m,1} & \cdots & \nu_{m,n-m} & 0 & 0 & \cdots & 1 \\ \nu_{m+1,0} & \nu_{m+1,1} & \cdots & \nu_{m+1,n-m} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \nu_{n,0} & \nu_{n,1} & \cdots & \nu_{n,n-m} & 0 & 0 & \cdots & 0 \\ g_0 & g_1 & \cdots & g_{n-m} & h_1 & h_2 & \cdots & h_m \end{bmatrix} \begin{bmatrix} \Delta_0 \\ \Delta_1 \\ \vdots \\ \Delta_{n-m} \\ \Delta_{n-m+1} \\ \Delta_{n-m+2} \\ \vdots \\ \Delta_n \end{bmatrix} = 0,$$

where

$$g_k = f_k + K_k - rV_k, \ k = 0, 1, \cdots, n-m$$

and

$$h_k = D_k - rS_k, \ k = 1, 2, \cdots, m.$$

In order for the system to have a non-trivial solution, the determinant of the matrix must be zero, or the $n+1$ row vectors of the matrix must be linearly dependent. Therefore, it is expected that the last row can be expressed as a linear combination of the other rows with coefficients $\tilde{\lambda}_1, \tilde{\lambda}_2, \cdots, \tilde{\lambda}_n$:

$$g_k = \sum_{i=1}^{n} \tilde{\lambda}_i \nu_{i,k}, \quad k = 0, 1, \cdots, n-m$$

and

$$h_k = \tilde{\lambda}_k, \quad k = 1, 2, \cdots, m.$$

Using the last m relations, we can rewrite the first $n-m+1$ relations as

$$g_k - \sum_{i=1}^{m} h_i \nu_{i,k} - \sum_{i=m+1}^{n} \tilde{\lambda}_i \nu_{i,k} = 0, \quad k = 0, 1, \cdots, n-m,$$

which means that any derivative satisfies an equation in the form

$$f + K - rV - \sum_{i=1}^{m} h_i \frac{\partial V}{\partial S_i} - \sum_{i=m+1}^{n} \tilde{\lambda}_i \frac{\partial V}{\partial S_i} = 0,$$

or

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}b_ib_j\rho_{ij}\frac{\partial^2 V}{\partial S_i\partial S_j} + \sum_{i=1}^{m}(rS_i - D_i)\frac{\partial V}{\partial S_i}$$

$$- \sum_{i=m+1}^{n}\tilde{\lambda}_i\frac{\partial V}{\partial S_i} - rV + K = 0,$$

where b_i, ρ_{ij} are given functions in the models of S_i, $\tilde{\lambda}_i$ are unknown functions which are independent of $V_0, V_1, \cdots, V_{n-m}$ and could depend on $S_1, S_2, \cdots, S_n$ and t, and K depends on the individual derivative security. Usually $\tilde{\lambda}_i$ is written in the form:

$$\tilde{\lambda}_i = \lambda_i b_i - a_i$$

and λ_i is called the market price of risk for S_i. Using this notation, we finally arrive at

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}b_ib_j\rho_{ij}\frac{\partial^2 V}{\partial S_i\partial S_j} + \sum_{i=1}^{m}(rS_i - D_i)\frac{\partial V}{\partial S_i}$$

$$+ \sum_{i=m+1}^{n}(a_i - \lambda_i b_i)\frac{\partial V}{\partial S_i} - rV + K = 0. \tag{2.34}$$

It is clear that if $m = n = 1$, $b_1 = \sigma_1 S_1$, $D_1 = D_{01}S_1$, and $K = 0$, then this equation becomes the Black–Scholes equation (2.12) after ignoring the subscript 1.

In the last we give some explanation on why λ_i is called the market price of risk for S_i. For simplicity, assume that none of S_k, $k = 1, 2, \cdots, n$, is a price. In this case the PDE above becomes

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}b_ib_j\rho_{ij}\frac{\partial^2 V}{\partial S_i\partial S_j} + \sum_{i=1}^{n}(a_i - \lambda_i b_i)\frac{\partial V}{\partial S_i} - rV + K = 0.$$

According to Itô's lemma and using this PDE, we have

$$dV = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 V}{\partial S_i\partial S_j}b_ib_j\rho_{ij}\right)dt + \sum_{i=1}^{n}\frac{\partial V}{\partial S_i}dS_i$$

$$= \left[\sum_{i=1}^{n}(\lambda_i b_i - a_i)\frac{\partial V}{\partial S_i} + rV - K\right]dt + \sum_{i=1}^{n}\frac{\partial V}{\partial S_i}(a_idt + b_idX_i)$$

or

$$dV + Kdt - rVdt = \sum_{i=1}^{n}\frac{\partial V}{\partial S_i}b_i\left(dX_i + \lambda_idt\right).$$

Here, $dV + Kdt$ is the return for the derivative including the coupon payment and $rVdt$ is the return if the investment is risk-free. Therefore, $dV + Kdt - rVdt$

is the excess return above the risk-free rate during the time interval $[t, t + dt]$. This equals the right-hand side of the equation. Its expectation is $\sum\limits_{i=1}^{n} \dfrac{\partial V}{\partial S_i} b_i \lambda_i dt$ because $\mathrm{E}\,[dX_i] = 0$, $i = 1, 2, \cdots, n$. Therefore, the term $\dfrac{\partial V}{\partial S_i} b_i \lambda_i dt$ may be interpreted as an excess return above the risk-free return for taking the risk dX_i. Consequently, λ_i is a price of risk for S_i that is associated with dX_i and is often called the market price of risk for S_i.

2.3.3 Three Types of State Variables

When we talk about the market price of risk, we can think that there are three types of state variables.

The first type of state variable is a price of an asset. In this case the coefficient of $\dfrac{\partial V}{\partial S_i}$ in Eq. (2.34) is $rS_i - D_i$. Thus for such a state variable, there is no market price of risk. However, this fact can also be understood in another way: there still is a market price of risk and the market price of risk for an asset is determined by

$$a_i - \lambda_i b_i = rS_i - D_i(S_1, S_2, \cdots, S_n, t). \tag{2.35}$$

This can be explained as follows. Suppose that the $(m+1)$-th random variable actually is a price of an asset. In this case, let us consider a portfolio consisting of the $n - m$ derivatives and the $m + 1$ assets, and derive the PDE. In the new PDE obtained the coefficient of $\dfrac{\partial V}{\partial S_{m+1}}$ is $rS_{m+1} - D_{m+1}$. The price of any financial derivative dependent on $S_1, S_2, \cdots, S_n, t$ should satisfy the same equation. Thus $a_{m+1} - \lambda_{m+1} b_{m+1}$ should equal $rS_{m+1} - D_{m+1}(S_1, S_2, \cdots, S_n, t)$, which means that the relation (2.35) holds. If $D_i = D_{0i} S_i$, then the following should be true:

$$a_i - \lambda_i b_i = (r - D_{0i}) S_i. \tag{2.36}$$

This can be shown in another way, which is left for readers as Problem 22.

A state variable S_i with $b_i = 0$ in Eq. (2.31) is another type of state variable. From $b_i = 0$, we have

$$a_i - \lambda_i b_i = a_i, \tag{2.37}$$

so λ_i disappears in Eq. (2.34). As we will see from Chap. 4, if S_i' is the price of a stock and S_i is the maximum, minimum, or average price of the stock during a time period, and both of them are state variables, then $dS_i = a_i dt$.

If S_i is the short-term interest rate, then in order to determine λ_i, we have to solve an inverse problem. We will discuss this problem in detail in Chap. 5. This is an example of the third type of state variable. Besides the interest rate, the random volatility also falls into this type of state variable.

2.3.4 Random Variables Not Being But Related to Prices of Assets

In Sect. 2.3.2 we assume that a random variable either is a value of a derivative or is a price of an asset. However, sometimes a random variable is merely related to an asset price. The random variable ξ in Sect. 2.2.5 and the random variable F in Sect. 2.2.6 are such examples. In Sects. 2.2.5 and 2.2.6, the PDEs for $\overline{V}(\xi, \tau)$ and $V(F, t)$ are obtained from the known PDEs by using transformations of independent and dependent variables. However, the two PDEs can also be obtained by setting a portfolio and using Itô's lemma. In Problems 23 and 24, readers are asked to derive the two PDEs and some other PDEs in this way. Here we assume that there are m random variables that do not represent prices of assets, but there exist m known different functions dependent on the m random variables that represent asset prices. In this case, in the procedure of deriving a PDE, determining $\Delta_0, \cdots, \Delta_n$ in the portfolio will involve solving a linear system; the expressions of the coefficients of the first derivatives in the PDE are more complicated. Here we give an example with $m = 2$, and readers are asked to do Problem 26 with $m = 3$.

Suppose that ξ_1 and ξ_2 satisfy the system of stochastic differential equations

$$d\xi_i = \mu_i(\xi_1, \xi_2, t)dt + \sigma_i(\xi_1, \xi_2, t)dX_i, \quad i = 1, 2,$$

where dX_i are the Wiener processes and $\mathrm{E}[dX_i dX_j] = \rho_{ij}dt$ with $-1 \leq \rho_{ij} \leq 1$. The functions

$$
\begin{cases}
Z_1(\xi_1) & = Z_{1,l} + \xi_1(1 - Z_{1,l}), \\[2mm]
Z_2(\xi_1, \xi_2) = Z_{2,l} + \xi_2[Z_1(\xi_1) - Z_{2,l}] \\[2mm]
\qquad\qquad = Z_{2,l} + \xi_2[Z_{1,l} + \xi_1(1 - Z_{1,l}) - Z_{2,l}]
\end{cases}
$$

represent prices of two nondividend-paying assets, where $Z_{1,l}$ and $Z_{2,l}$ are two constants. Let $V(\xi_1, \xi_2, t)$ be the value of a noncoupon-paying derivative security. Because $Z_1(\xi_1)$ and $Z_2(\xi_1, \xi_2)$ are prices of two assets, we can set a portfolio

$$\Pi = V - \Delta_1 Z_1(\xi_1) - \Delta_2 Z_2(\xi_1, \xi_2)$$

when deriving the PDE for $V(\xi_1, \xi_2, t)$. According to Itô's lemma and noticing the form of functions $Z_1(\xi_1)$ and $Z_2(\xi_1, \xi_2)$, we have

$$dV = \sum_{i=1}^{2} \frac{\partial V}{\partial \xi_i} d\xi_i + \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \sigma_i \sigma_j \rho_{i,j} \frac{\partial^2 V}{\partial \xi_i \partial \xi_j} \right) dt,$$

$$dZ_1 = \frac{\partial Z_1}{\partial \xi_1} d\xi_1,$$

$$dZ_2 = \sum_{i=1}^{2} \frac{\partial Z_2}{\partial \xi_i} d\xi_i + \sigma_1 \sigma_2 \rho_{1,2} \frac{\partial^2 Z_2}{\partial \xi_1 \partial \xi_2} dt.$$

Using these expressions, we obtain

$$
d\Pi = \sum_{i=1}^{2} \frac{\partial V}{\partial \xi_i} d\xi_i + \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \sigma_i \sigma_j \rho_{i,j} \frac{\partial^2 V}{\partial \xi_i \partial \xi_j} \right) dt
$$

$$
- \Delta_1 \frac{\partial Z_1}{\partial \xi_1} d\xi_1 - \Delta_2 \left(\sum_{i=1}^{2} \frac{\partial Z_2}{\partial \xi_i} d\xi_i + \sigma_1 \sigma_2 \rho_{1,2} \frac{\partial^2 Z_2}{\partial \xi_1 \partial \xi_2} dt \right)
$$

$$
= \left(\frac{\partial V}{\partial \xi_1} - \Delta_1 \frac{\partial Z_1}{\partial \xi_1} - \Delta_2 \frac{\partial Z_2}{\partial \xi_1} \right) d\xi_1 + \left(\frac{\partial V}{\partial \xi_2} - \Delta_2 \frac{\partial Z_2}{\partial \xi_2} \right) d\xi_2
$$

$$
+ \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \sigma_i \sigma_j \rho_{i,j} \frac{\partial^2 V}{\partial \xi_i \partial \xi_j} \right) dt
$$

$$
- \Delta_2 \sigma_1 \sigma_2 \rho_{1,2} \frac{\partial^2 Z_2}{\partial \xi_1 \partial \xi_2} dt.
$$

Let us choose

$$
\Delta_2 = \frac{1}{\frac{\partial Z_2}{\partial \xi_2}} \frac{\partial V}{\partial \xi_2},
$$

$$
\Delta_1 = \frac{1}{\frac{\partial Z_1}{\partial \xi_1}} \left(\frac{\partial V}{\partial \xi_1} - \Delta_2 \frac{\partial Z_2}{\partial \xi_1} \right) = \frac{1}{\frac{\partial Z_1}{\partial \xi_1}} \frac{\partial V}{\partial \xi_1} - \frac{\frac{\partial Z_2}{\partial \xi_1}}{\frac{\partial Z_1}{\partial \xi_1} \frac{\partial Z_2}{\partial \xi_2}} \frac{\partial V}{\partial \xi_2},
$$

so that

$$
\frac{\partial V}{\partial \xi_1} - \Delta_1 \frac{\partial Z_1}{\partial \xi_1} - \Delta_2 \frac{\partial Z_2}{\partial \xi_1} = \frac{\partial V}{\partial \xi_2} - \Delta_2 \frac{\partial Z_2}{\partial \xi_2} = 0.
$$

In this case, the portfolio is risk-free and the return rate should be r:

$$
\left(\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \sigma_i \sigma_j \rho_{i,j} \frac{\partial^2 V}{\partial \xi_i \partial \xi_j} \right) dt - \Delta_2 \sigma_1 \sigma_2 \rho_{1,2} \frac{\partial^2 Z_2}{\partial \xi_1 \partial \xi_2} dt
$$

$$
= r \left(V - \Delta_1 Z_1 - \Delta_2 Z_2 \right) dt
$$

or

$$
\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \sigma_i \sigma_j \rho_{i,j} \frac{\partial^2 V}{\partial \xi_i \partial \xi_j} + \frac{r Z_1}{\frac{\partial Z_1}{\partial \xi_1}} \frac{\partial V}{\partial \xi_1}
$$

$$
+ \left(-\frac{r Z_1 \frac{\partial Z_2}{\partial \xi_1}}{\frac{\partial Z_1}{\partial \xi_1} \frac{\partial Z_2}{\partial \xi_2}} + \frac{r Z_2}{\frac{\partial Z_2}{\partial \xi_2}} - \frac{\sigma_1 \sigma_2 \rho_{1,2} \frac{\partial^2 Z_2}{\partial \xi_1 \partial \xi_2}}{\frac{\partial Z_2}{\partial \xi_2}} \right) \frac{\partial V}{\partial \xi_2} - r V = 0.
$$

Noticing

$$\frac{\partial Z_1}{\partial \xi_1} = 1 - Z_{1,l},$$

$$\frac{\partial Z_2}{\partial \xi_1} = \xi_2 \left(1 - Z_{1,l}\right), \quad \frac{\partial Z_2}{\partial \xi_2} = Z_1 - Z_{2,l}, \quad \frac{\partial^2 Z_2}{\partial \xi_1 \partial \xi_2} = 1 - Z_{1,l},$$

we can rewrite the PDE as

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \sigma_i \sigma_j \rho_{i,j} \frac{\partial^2 V}{\partial \xi_i \partial \xi_j} + \frac{r Z_1}{1 - Z_{1,l}} \frac{\partial V}{\partial \xi_1}$$

$$+ \left(- \frac{r Z_1 \xi_2 \left(1 - Z_{1,l}\right)}{\left(1 - Z_{1,l}\right)\left(Z_1 - Z_{2,l}\right)} + \frac{r Z_2}{Z_1 - Z_{2,l}} - \frac{\sigma_1 \sigma_2 \rho_{1,2}\left(1 - Z_{1,l}\right)}{Z_1 - Z_{2,l}}\right) \frac{\partial V}{\partial \xi_2}$$

$$- r V = 0,$$

which can be simplified to

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \sigma_i \sigma_j \rho_{i,j} \frac{\partial^2 V}{\partial \xi_i \partial \xi_j} + \frac{r Z_1}{1 - Z_{1,l}} \frac{\partial V}{\partial \xi_1}$$

$$+ \left[\frac{r \left(Z_2 - Z_1 \xi_2\right) - \sigma_1 \sigma_2 \rho_{1,2}\left(1 - Z_{1,l}\right)}{Z_1 - Z_{2,l}}\right] \frac{\partial V}{\partial \xi_2} - r V = 0.$$

From Sect. 5.6 you can see that it could be a PDE for a two-factor interest rate model.

2.4 Uniqueness of Initial-Value Problems for Degenerate Parabolic PDEs

2.4.1 Reversion Conditions for Stochastic Models

In many cases, a stochastic model in finance usually describes a random variable which can take its value on an infinite domain. For such a model, closed-form solutions can be found in many situations. This is an advantage of such a model. However it seems that assuming a random variable (such as interest rates, volatilities) to be defined on a finite domain and designing a model from market data are more realistic. How do we model a random variable with such a property? For simplicity, we consider problems with only one random variable S. Suppose that we want a random variable S to have a lower boundary S_l, i.e., if $S \geq S_l$ at time t, then we want to guarantee that S is still greater than or equal to S_l after time t even though the movement of S possesses some uncertainty. In this case, we need to require that $a\left(S, t\right)$ and $b\left(S, t\right)$ at $S = S_l$ satisfy the conditions

$$\begin{cases} a\left(S_l,t\right) \geq 0, & 0 \leq t \leq T, \\ b\left(S_l,t\right) = 0, & 0 \leq t \leq T. \end{cases} \tag{2.38}$$

This is a necessary condition because if either of the two conditions does not hold, then there is a chance for S to be lower than S_l at time $t + dt$ when $S = S_l$ at time t. In Sect. 2.4.2, we will see that if

$$\begin{cases} a\left(S_l,t\right) - b(S_l,t)\dfrac{\partial}{\partial S}b(S_l,t) \geq 0, & 0 \leq t \leq T, \\ b\left(S_l,t\right) = 0, & 0 \leq t \leq T \end{cases} \tag{2.39}$$

holds, then a unique solution of the corresponding partial differential equation can be determined by a final condition on $[S_l, \infty)$ without any boundary conditions at $S = S_l$. Therefore, what happens at $S = S_l$ will not affect the solution at $t = 0$ for any S. This fact can be interpreted as follows. If the condition (2.39) holds for any $t \in [t_0, T]$, then for any such time t, S will be greater than or equal to S_l if $S > S_l$ at $t = t_0$. That is, S is either reflected into the region $S > S_l$ or is absorbed by the boundary $S = S_l$ in the event S hits the lower bound S_l at some time $t \in [t_0, T]$. This is because if there are paths that pass through a point (S_l, t) and go to the outside of $[S_l, \infty)$, then the solution at the point $(S, 0)$ should depend on the value of the solution at the point (S_l, t). The solution is determined only by the final condition, so there is no path passing the boundary $S = S_l$. Consequently the condition (2.39) is a sufficient condition to guarantee $S \geq S_l$ for any t.

In the popular model

$$dS = \mu S dt + \sigma S dX,$$

we have $a = \mu S$ and $b = \sigma S$. Therefore, the condition (2.39) holds at $S = 0$, and S is always greater than or equal to zero. In the Cox–Ingersoll–Ross interest rate model (see [23])

$$dr = (\bar{\mu} - \bar{\gamma}r)dt + \sqrt{\alpha r}dX, \quad \bar{\mu}, \bar{\gamma}, \alpha > 0,$$

which will be discussed in Chap. 5, $a = \bar{\mu} - \bar{\gamma}r$, $b = \sqrt{\alpha r}$, and the condition (2.39) is reduced to $\bar{\mu} - \alpha/2 \geq 0$ if the lower bound is zero. This means that if $\bar{\mu} - \alpha/2 \geq 0$, then at $r = 0$, no boundary condition is needed. In fact, if $\bar{\mu} - \alpha/2 \geq 0$, the upward drift is sufficiently large to make the origin inaccessible (see [23]). Therefore, no boundary condition at $r = 0$ is related to inaccessibility to the origin.

Actually, S_l may not be zero, and a similar condition

$$\begin{cases} a\left(S_u,t\right) - b(S_u,t)\dfrac{\partial}{\partial S}b(S_u,t) \leq 0, & 0 \leq t \leq T, \\ b\left(S_u,t\right) = 0, & 0 \leq t \leq T \end{cases} \tag{2.40}$$

can also be required at $S = S_u > S_l$ so that S will always be in $[S_l, S_u]$. If $a(S_l, t) \geq 0$ and $a(S_u, t) \leq 0$, then it is usually said that the model has a mean reversion property. However, if $b(S_l, t) \neq 0$ or $b(S_u, t) \neq 0$, then there is still a chance for S to become less than S_l or greater than S_u. If the conditions (2.39) and (2.40) hold, then we say that the model really has a reversion property because S will always be in $[S_l, S_u]$. In this book, the conditions (2.39) and (2.40) will be referred to as the reversion conditions, and (2.38) and the like will be referred to as the weak-form reversion conditions. When $\frac{\partial}{\partial S} b(S, t)$ is bounded, the two types of reversion conditions are the same.

The two random variables given above as examples are defined on $[0, \infty)$. In what follows, we will show that they can be converted into new random variables whose domains are $[0, 1)$ and can be naturally extended to $[0, 1]$, and for them the reversion conditions hold at both the lower and upper boundaries.

Let us introduce a new random variable

$$\xi = \frac{S}{S + P_m},$$

where P_m is a positive parameter. From this relation, we can have

$$S = \frac{P_m \xi}{1 - \xi},$$

$$S + P_m = \frac{P_m}{1 - \xi},$$

$$\frac{d\xi}{dS} = \frac{P_m}{(S + P_m)^2} = \frac{(1 - \xi)^2}{P_m},$$

and

$$\frac{d^2\xi}{dS^2} = \frac{-2P_m}{(S + P_m)^3} = \frac{-2(1 - \xi)^3}{P_m^2}.$$

According to Itô's lemma, if S satisfies $dS = \mu S dt + \sigma S dX$, then for ξ the stochastic differential equation is

$$d\xi = \frac{(1 - \xi)^2}{P_m} dS - \frac{(1 - \xi)^3}{P_m^2} \sigma^2 S^2 dt$$
$$= \left[\mu \xi (1 - \xi) - \sigma^2 \xi^2 (1 - \xi) \right] dt + \sigma \xi (1 - \xi) dX.$$

Consequently for ξ, the conditions (2.39) and (2.40) are fulfilled at $\xi = 0$ and $\xi = 1$, respectively.

Similarly for the Cox–Ingersoll–Ross interest rate model, let

$$\xi = \frac{r}{r + P_m},$$

then we get

$$d\xi = \left[\frac{(1-\xi)^2}{P_m}\left(\bar{\mu} - \frac{\bar{\gamma}P_m\xi}{1-\xi}\right) - \frac{\alpha\xi(1-\xi)^2}{P_m}\right]dt + \frac{\sqrt{\alpha}\xi^{1/2}(1-\xi)^{3/2}}{P_m^{1/2}}dX.$$

In this case $\xi_l = 0$ and $\xi_u = 1$ and it is easy to show that both the conditions (2.39) and (2.40) hold if $\bar{\mu} - \alpha/2 \geq 0$. All the proofs here are left for readers as Problem 28. In this book we only talk these models satisfying conditions (2.39) and (2.40) or these models which can become models satisfying conditions (2.39) and (2.40) after introducing new random variables.

Suppose that a model defined on $[S_l, S_u]$ has the property of mean reverting, but it does not satisfy the reversion condition. The model can be modified as follows: the coefficient of dX is multiplied by a function, for example,

$$\Phi(x) = \frac{1 - (1 - 2x)^2}{1 - 0.975(1 - 2x)^2},$$

where $x = \frac{(S-S_l)}{(S_u-S_l)}$. Because $\Phi(x)$ are equal to zero at $S = S_l$ and $S = S_u$ and very close to one at $S \in (S_l + \varepsilon, S_u - \varepsilon)$, ε being a very small number, almost all the properties of the original model are kept in the modified model and the reversion conditions will hold after the modification is made.

Now we describe the reversion conditions for the case involving n random variables. Suppose that a financial derivative depends on the time t and n random variables $S_1, S_2, \cdots, S_n$ and that for $i = 1, 2, \cdots, n$, S_i satisfies the equation

$$dS_i = a_i(S_1, S_2, \cdots, S_n, t)dt + b_i(S_1, S_2, \cdots, S_n, t)dX_i \qquad (2.41)$$

in a rectangular domain $\Omega : [S_{1l}, S_{1u}] \times [S_{2l}, S_{2u}] \times \cdots \times [S_{nl}, S_{nu}]$. In this case we require that the following conditions hold:

$$\begin{cases} \left[a_i(S_1, \cdots, S_n, t) - b_i(S_1, \cdots, S_n, t)\frac{\partial b_i(S_1, \cdots, S_n, t)}{\partial S_i}\right]\Bigg|_{\substack{S_i = S_{il} \\ S_j \in [S_{jl}, S_{ju}] \\ j \neq i}} \geq 0, \\ b_i(S_1, S_2, \cdots, S_n, t)\Big|_{\substack{S_i = S_{il} \\ S_j \in [S_{jl}, S_{ju}] \\ j \neq i}} = 0 \end{cases}$$

$$(2.42)$$

and

$$\begin{cases} \left[a_i(S_1, \cdots, S_n, t) - b_i(S_1, \cdots, S_n, t)\frac{\partial b_i(S_1, \cdots, S_n, t)}{\partial S_i}\right]\Bigg|_{\substack{S_i = S_{iu} \\ S_j \in [S_{jl}, S_{ju}] \\ j \neq i}} \leq 0, \\ b_i(S_1, \cdots, S_n, t)\Big|_{\substack{S_i = S_{iu} \\ S_j \in [S_{jl}, S_{ju}] \\ j \neq i}} = 0. \end{cases}$$

$$(2.43)$$

These conditions are called the reversion conditions on a rectangular domain Ω. It is clear that if

$$\left.\frac{\partial b_i(S_1,\cdots,S_n,t)}{\partial S_i}\right|_{\substack{S_i=S_{il} \\ S_j\in[S_{jl},S_{ju}] \\ j\neq i}}$$

and

$$\left.\frac{\partial b_i(S_1,\cdots,S_n,t)}{\partial S_i}\right|_{\substack{S_i=S_{iu} \\ S_j\in[S_{jl},S_{ju}] \\ j\neq i}}$$

are bounded, then the two conditions (2.42) and (2.43) can be reduced to

$$
\begin{cases}
a_i(S_1,\cdots,S_n,t)\Big|_{\substack{S_i=S_{il} \\ S_j\in[S_{jl},S_{ju}] \\ j\neq i}} \geq 0, \\[2ex]
b_i(S_1,\cdots,S_n,t)\Big|_{\substack{S_i=S_{il} \\ S_j\in[S_{jl},S_{ju}] \\ j\neq i}} = 0
\end{cases}
\tag{2.44}
$$

and

$$
\begin{cases}
a_i(S_1,\cdots,S_n,t)\Big|_{\substack{S_i=S_{iu} \\ S_j\in[S_{jl},S_{ju}] \\ j\neq i}} \leq 0, \\[2ex]
b_i(S_1,\cdots,S_n,t)\Big|_{\substack{S_i=S_{iu} \\ S_j\in[S_{jl},S_{ju}] \\ j\neq i}} = 0.
\end{cases}
\tag{2.45}
$$

If the domain is not rectangular, the form of reversion conditions will be a little different. If all the coefficients in the models are differential, then the form is relatively simple. For example, consider the case of $n = 3$. Let the outer normal vector be $(n_1,n_2,n_3)^T$. Then the reversion conditions are that

$$
\begin{cases}
n_1a_1 + n_2a_2 + n_3a_3 \geq 0, \\[1ex]
\text{Var}(n_1b_1dX_1 + n_2b_2dX_2 + n_3b_3dX_3) = 0
\end{cases}
$$

hold on the boundary of the domain.

2.4.2 †Uniqueness of Solutions for One-Dimensional Case

Equation (2.34) is a parabolic equation. When S_i is defined on $[S_{il}, S_{iu}]$, $i = 1, 2, \cdots, n$, Eq. (2.34) is defined on the rectangular domain Ω. If $b_i = 0$ at $S_i = S_{i,l}$ and $S_i = S_{i,u}$, $i = 1, 2, \cdots, n$, then we say that the equation is a degenerate parabolic partial differential equation. In this subsection, we are going to discuss when a degenerate equation has a unique solution. The conclusion expected is that if for any i,

$$
\left[a_i(S_1,\cdots,S_n,t) - b_i(S_1,\cdots,S_n,t)\frac{\partial b_i(S_1,\cdots,S_n,t)}{\partial S_i}\right]\Bigg|_{\substack{S_i=S_{il} \\ S_j\in[S_{jl},S_{ju}] \\ j\neq i}} \geq 0
\tag{2.46}
$$

and

$$\left[a_i(S_1,\cdots,S_n,t) - b_i(S_1,\cdots,S_n,t)\frac{\partial b_i(S_1,\cdots,S_n,t)}{\partial S_i} \right]\Bigg|_{\substack{S_i=S_{iu} \\ S_j\in[S_{jl},S_{ju}] \\ j\neq i}} \leq 0$$

(2.47)

hold, the solution of the degenerate parabolic equation on a rectangular domain with a final condition at $t = T$ is unique.[11] If

$$\left[a_i(S_1,\cdots,S_n,t) - b_i(S_1,\cdots,S_n,t)\frac{\partial b_i(S_1,\cdots,S_n,t)}{\partial S_i} \right]\Bigg|_{\substack{S_i=S_{il} \\ S_j\in[S_{jl},S_{ju}] \\ j\neq i}} < 0$$

(2.48)

or

$$\left[a_i(S_1,\cdots,S_n,t) - b_i(S_1,\cdots,S_n,t)\frac{\partial b_i(S_1,\cdots,S_n,t)}{\partial S_i} \right]\Bigg|_{\substack{S_i=S_{iu} \\ S_j\in[S_{jl},S_{ju}] \\ j\neq i}} > 0,$$

(2.49)

then a boundary condition at $S_i = S_{i,l}$ or $S_i = S_{i,u}$ needs to be imposed besides the final condition in order to have a unique solution. In this subsection we now prove this conclusion for the one-dimensional case. In the next subsection we will prove that for a final-value problem the solution is unique if the reversion conditions hold.

In the case $m = 0$ and $n = 1$, after ignoring the subscript 1, Eq. (2.34) becomes

$$\frac{\partial V}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 V}{\partial S^2} + (a - \lambda b)\frac{\partial V}{\partial S} - rV + K = 0.$$

Here, the sign of the coefficient of the second derivative is opposite to that of the coefficient of the second derivative in the heat equation. We say that such a parabolic equation has an "anti-directional" time. For a heat equation, an initial condition is given at $t = 0$, and the solution for $t \geq 0$ needs to be determined. Therefore, for the equation with an "anti-directional" time, a final condition should be given at $t = T$, and the solution for $t \leq T$ needs to be determined. Consequently, we consider the following problem:

---

[11]For a parabolic equation defined on a non-rectangular domain, the conditions for a parabolic partial differential equation to be degenerate and the conditions for the solution of its initial-value problem to be unique, see the paper [91] by Zhu.

$$
\begin{cases}
\dfrac{\partial V}{\partial t} + \dfrac{1}{2}b^2\dfrac{\partial^2 V}{\partial S^2} + (a - \lambda b)\dfrac{\partial V}{\partial S} - rV + K = 0, \\
\qquad\qquad\qquad\qquad\qquad 0 \le t \le T, \quad S_l \le S \le S_u, \\
V(S,T) = f(S), \qquad\qquad\qquad S_l \le S \le S_u, \\
V(S_l,t)\begin{cases} \text{needs not to be given if the condition (2.46) holds,} \\ = f_l(t) \text{ if the condition (2.46) does not hold,} \end{cases} \\
V(S_u,t)\begin{cases} \text{needs not to be given if the condition (2.47) holds,} \\ = f_u(t) \text{ if the condition (2.47) does not hold.} \end{cases}
\end{cases}
\tag{2.50}
$$

Let $\tau = T - t$ and $x = (S - S_l)/(S_u - S_l)$, then the problem (2.50) is converted into a problem in the form:

$$
\begin{cases}
\dfrac{\partial u}{\partial \tau} = f_1(x,\tau)\dfrac{\partial^2 u}{\partial x^2} + f_2(x,\tau)\dfrac{\partial u}{\partial x} + f_3(x,\tau)u + g(x,\tau), \\
\qquad\qquad\qquad\qquad\qquad 0 \le x \le 1, \quad 0 \le \tau \le T, \\
u(x,0) = f(x), \qquad\qquad 0 \le x \le 1, \\
u(0,\tau)\begin{cases} \text{needs not to be given if } f_2(0,\tau) - \dfrac{\partial f_1(0,\tau)}{\partial x} \ge 0, \\ = f_l(\tau) \text{ if } f_2(0,\tau) - \dfrac{\partial f_1(0,\tau)}{\partial x} < 0, \end{cases} \\
u(1,\tau)\begin{cases} \text{needs not to be given if } f_2(1,\tau) - \dfrac{\partial f_1(1,\tau)}{\partial x} \le 0, \\ = f_u(\tau) \text{ if } f_2(1,\tau) - \dfrac{\partial f_1(1,\tau)}{\partial x} > 0, \end{cases}
\end{cases}
\tag{2.51}
$$

where $f_1(0,\tau) = f_1(1,\tau) = 0$ and $f_1(x,\tau) \ge 0$. Thus, if we can prove the uniqueness of the solution of the problem (2.51), then we have the uniqueness of the solution of the problem (2.50). The third and fourth relations in the problem (2.51) are the boundary conditions for degenerate parabolic equations. For parabolic equations, there is always a boundary condition at any boundary, that is, the number of boundary conditions for parabolic equations is always one. However, for degenerate parabolic equations, sometimes there is a boundary condition and sometimes there is not, depending on the value of $f_2(x,\tau) - \dfrac{\partial f_1(x,\tau)}{\partial x}$ at the boundary. For the problem (2.51), we have the following theorem (see [79]).

Theorem 2.1 *Suppose that the solution of the problem (2.51) exists and is bounded*[12] *and that there exist a constant c_1 and two bounded functions $c_2(\tau)$*

---

[12]This is proved in the paper [7] by Behboudi.

and $c_3(\tau)$ such that

$$1 + \max_{0 \le x \le 1,\, 0 \le \tau \le T} \left(\left| \frac{\partial^2 f_1(x,\tau)}{\partial x^2} - \frac{\partial f_2(x,\tau)}{\partial x} + 2f_3(x,\tau) \right| \right) \le c_1,$$

$$- \min \left(0, f_2(0,\tau) - \frac{\partial f_1(0,\tau)}{\partial x} \right) \le c_2(\tau),$$

and

$$\max \left(0, f_2(1,\tau) - \frac{\partial f_1(1,\tau)}{\partial x} \right) \le c_3(\tau).$$

In this case, its solution is unique and stable with respect to the initial value $f(x)$, inhomogeneous term $g(x,\tau)$, and the boundary values $f_l(\tau), f_u(\tau)$ if there are any.

Proof. Because the partial differential equation in the problem (2.51) can be rewritten as

$$\frac{\partial u}{\partial \tau} = \frac{\partial}{\partial x} \left[f_1(x,\tau) \frac{\partial u}{\partial x} \right] + \left[f_2(x,\tau) - \frac{\partial f_1(x,\tau)}{\partial x} \right] \frac{\partial u}{\partial x} + f_3(x,\tau)u + g(x,\tau),$$

multiplying that equation by $2u$, we have

$$\frac{\partial(u^2)}{\partial \tau} = 2\frac{\partial}{\partial x} \left(f_1 u \frac{\partial u}{\partial x} \right) + \left(f_2 - \frac{\partial f_1}{\partial x} \right) \frac{\partial(u^2)}{\partial x} - 2f_1 \left(\frac{\partial u}{\partial x} \right)^2 + 2f_3 u^2 + 2gu$$

$$= 2\frac{\partial}{\partial x} \left(f_1 u \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial x} \left[\left(f_2 - \frac{\partial f_1}{\partial x} \right) u^2 \right] - 2f_1 \left(\frac{\partial u}{\partial x} \right)^2$$

$$+ \left(\frac{\partial^2 f_1}{\partial x^2} - \frac{\partial f_2}{\partial x} + 2f_3 \right) u^2 + 2gu.$$

Integrating this equality with respect to x on the interval $[0,1]$, we obtain the second equality

$$\frac{d}{d\tau} \int_0^1 u^2(x,\tau)dx$$

$$= 2 \left(f_1 u \frac{\partial u}{\partial x} \right) \Big|_{x=0}^1 + \left[\left(f_2 - \frac{\partial f_1}{\partial x} \right) u^2 \right] \Big|_{x=0}^1 - 2 \int_0^1 f_1 \left(\frac{\partial u}{\partial x} \right)^2 dx$$

$$+ \int_0^1 \left(\frac{\partial^2 f_1}{\partial x^2} - \frac{\partial f_2}{\partial x} + 2f_3 \right) u^2 dx + 2 \int_0^1 gu dx.$$

Because

$$\left[\left(f_2 - \frac{\partial f_1}{\partial x} \right) u^2 \right] \Big|_{x=0}^1$$

$$= \left[f_2(1,\tau) - \frac{\partial f_1(1,\tau)}{\partial x} \right] u^2(1,\tau) - \left[f_2(0,\tau) - \frac{\partial f_1(0,\tau)}{\partial x} \right] u^2(0,\tau)$$

$$\le \max \left(0, f_2(1,\tau) - \frac{\partial f_1(1,\tau)}{\partial x} \right) f_u^2(\tau) - \min \left(0, f_2(0,\tau) - \frac{\partial f_1(0,\tau)}{\partial x} \right) f_l^2(\tau),$$

from the equality above and the relations $f_1(0,\tau) = f_1(1,\tau) = 0$ and $f_1(x,\tau) \geq 0$, we have

$$\frac{d}{d\tau}\int_0^1 u^2(x,\tau)dx \leq c_1\int_0^1 u^2(x,\tau)dx + \int_0^1 g^2(x,\tau)dx + c_2(\tau)f_l^2(\tau) + c_3(\tau)f_u^2(\tau).$$

Based on this inequality and by the Gronwall inequality,[13] we arrive at

$$\int_0^1 u^2(x,\tau)dx$$
$$\leq e^{c_1\tau}\left\{\int_0^1 f^2(x)dx + \int_0^\tau\left[\int_0^1 g^2(x,s)dx + c_2(s)f_l^2(s) + c_3(s)f_u^2(s)\right]ds\right\},$$
$$t \in [0,T].$$

From the last inequality, we know that the solution is stable with respect to $f(x)$ and $g(x,\tau)$. Also if

$$f_2(0,\tau) - \frac{\partial f_1(0,\tau)}{\partial x} \geq 0 \quad \text{and} \quad f_2(1,\tau) - \frac{\partial f_1(1,\tau)}{\partial x} \leq 0$$

hold and

$$f(x) \equiv 0, \quad g(x,\tau) \equiv 0,$$

then the solution of the problem (2.51) must be zero. Hence, the functions $f(x)$ and $g(x,\tau)$ determine the solution uniquely. If

$$f_2(0,\tau) - \frac{\partial f_1(0,\tau)}{\partial x} < 0 \quad \text{and} \quad f_2(1,\tau) - \frac{\partial f_1(1,\tau)}{\partial x} \leq 0$$

hold, then the solution is determined by $f(x)$, $g(x,\tau)$, and $f_l(\tau)$ uniquely. The situation for other cases are similar. Therefore, we may conclude that if the solution of the problem (2.51) exists, then it is unique and stable with respect to the initial value $f(x)$, the inhomogeneous term $g(x,\tau)$, and the boundary values $f_l(\tau), f_u(\tau)$ if there are any. This completes the proof and gives an explanation on when a boundary condition is necessary. $\square$

Here we give some remarks.

- From the probabilistic point of view, a boundary condition on a boundary is needed if and only if there are paths reaching the boundary from a point $x \in (0,1)$ and $t = 0$. Therefore, on whether or not a random variable can reach a boundary from the interior, there are similar conclusions (see [33]).
- This result indicates that a degenerate parabolic equation at boundaries is similar to a hyperbolic equation.[14] Due to this fact, roughly speaking, we might say that the parabolic equation degenerates into a hyperbolic equation at the boundaries. When conditions (2.46) and (2.47) hold,

---

[13]The inequality $dA(\tau)/d\tau \leq cA(\tau) + B(\tau)$ can be rewritten as $e^{-c\tau}[dA(\tau)/d\tau - cA(\tau)] \leq e^{-c\tau}B(\tau)$ or $d(e^{-c\tau}A(\tau))/d\tau \leq e^{-c\tau}B(\tau)$, so for positive $\tau, c, B(\tau)$ we have $A(\tau) \leq e^{c\tau}[A(0) + \int_0^\tau e^{-c\bar\tau}B(\bar\tau)d\bar\tau] \leq e^{c\tau}[A(0) + \int_0^\tau B(\bar\tau)d\bar\tau]$.

[14]When $f_1(x,t) \equiv 0$, the partial differential equation in Eq. (2.51) is called a hyperbolic equation.

incoming information is not needed at boundaries, that is, the value of V at the boundaries at $t = t^*$ is determined by the value V on the region: $S_l \leq S \leq S_u$ and $t^* \leq t \leq T$. Therefore, in this case, in order for a degenerate parabolic equation to have a unique solution, only the final condition is needed.[15]

- When the domain of S is not finite, a final condition is still enough for such an equation to have a unique solution if S can be converted into a random variable for which the reversion conditions hold. The reason is that a final condition can determine a unique solution if the new random variable is used. However, a transformation will not change the nature of the problem. If the problem has a unique solution as a function of a random variable, the problem will also have a unique solution as a function of another random variable associated by a transformation. Applying this theorem to problem (2.18), we know that its solution is unique and stable with respect to the initial value. Problem (2.18) is obtained through a transformation from the European option problem (2.16). Therefore, the European option problem (2.16) also has a unique solution. In Sect. 2.2.5 it is pointed that for problem (2.18) the values at $\xi = 0$ and $\xi = 1$ are given by the expressions (2.19) and (2.20), respectively. This means that when a solution of the problem (2.18) is determined, no boundary condition is needed. The result here points out not only that no boundary condition is needed when a solution of the problem (2.18) is determined, but also that it is impossible for problem (2.18) to have several solutions.

2.4.3 ‡Uniqueness of Solutions for Two-Dimensional Case

On a multidimensional rectangular domain, it can be proved that if the reversion conditions are satisfied, then the final-value problem for degenerate parabolic partial differential equations has a unique solution. In this subsection, we give a detailed proof only for the two-dimensional case; at the end of this subsection, we point out the key part of the proof for the multidimensional case.

Suppose that a financial derivative depends on the time t and two random variables S_1 and S_2, which satisfy Eq. (2.41) and the reversion conditions, and let $V(S_1, S_2, t)$ be the price of the financial derivative. By an arbitrage argument, it can be shown that $V(S_1, S_2, t)$ should satisfy the following equation (see Sect. 2.3.2):

$$\frac{\partial V}{\partial t} + \frac{1}{2}b_1^2\frac{\partial^2 V}{\partial S_1^2} + \rho b_1 b_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{1}{2}b_2^2\frac{\partial^2 V}{\partial S_2^2}$$
$$+(a_1 - \lambda_1 b_1)\frac{\partial V}{\partial S_1} + (a_2 - \lambda_2 b_2)\frac{\partial V}{\partial S_2} - rV + g(S_1, S_2, t) = 0,$$

---

[15]Oleĭnik and Radkevič in their book [65] discussed the uniqueness of solutions of this type of partial differential equations under different conditions.

where λ_1 and λ_2 are two bounded functions and called the market prices of risk on S_1 and S_2, respectively, and r is the short-term interest rate.[16] Also, many financial derivatives should be solutions of the final-value problem

$$
\begin{cases}
\dfrac{\partial V}{\partial t} + \tfrac{1}{2}b_1^2\dfrac{\partial^2 V}{\partial S_1^2} + \rho b_1 b_2 \dfrac{\partial^2 V}{\partial S_1 \partial S_2} + \tfrac{1}{2}b_2^2\dfrac{\partial^2 V}{\partial S_2^2} \\[2mm]
\quad + (a_1 - \lambda_1 b_1)\dfrac{\partial V}{\partial S_1} + (a_2 - \lambda_2 b_2)\dfrac{\partial V}{\partial S_2} - rV + g(S_1, S_2, t) = 0, \\[2mm]
\qquad\qquad\qquad\qquad S_1 \in [S_{1l}, S_{1u}], \quad S_2 \in [S_{2l}, S_{2u}], \quad t \in [0, T], \\[2mm]
V(S_1, S_2, T) = f(S_1, S_2), \quad S_1 \in [S_{1l}, S_{1u}], \quad S_2 \in [S_{2l}, S_{2u}].
\end{cases}
\tag{2.52}
$$

Now let us discuss when Problem (2.52) has a unique solution. For this question, we have the following theorem:

Theorem 2.2 *If*

(i) the reversion conditions (2.42) and (2.43) hold;
(ii) there exists a constant c_1 such that

$$
\max_{\substack{S_{1l} \le S_1 \le S_{1u} \\ S_{2l} \le S_2 \le S_{2u}}} \left| \frac{\partial}{\partial S_1}\left(a_1 - \lambda_1 b_1 - b_1 \frac{\partial b_1}{\partial S_1} - \frac{1}{2}\frac{\partial}{\partial S_2}(\rho b_1 b_2) \right) \right.
$$

$$
\left. + \frac{\partial}{\partial S_2}\left(a_2 - \lambda_2 b_2 - b_2 \frac{\partial b_2}{\partial S_2} - \frac{1}{2}\frac{\partial}{\partial S_1}(\rho b_1 b_2) \right) + 2r \right| + 1 \le c_1;
$$

(iii) solutions of Problem (2.52) exist and their first derivatives are bounded,

then the solution of Eq. (2.52) is unique.

Proof. Suppose that $u(S_1, S_2, t)$ is a solution of the problem (2.52). Let $\tau = T - t$ and define

$$
W(\tau) = \int_{S_{2l}}^{S_{2u}} \int_{S_{1l}}^{S_{1u}} u^2(S_1, S_2, T - \tau)dS_1 dS_2.
\tag{2.53}
$$

Since the partial differential equation in the problem (2.52) can be rewritten as

$$
\frac{\partial u}{\partial \tau} = \frac{1}{2}\frac{\partial}{\partial S_1}\left(b_1^2 \frac{\partial u}{\partial S_1} + \rho b_1 b_2 \frac{\partial u}{\partial S_2} \right) + \frac{1}{2}\frac{\partial}{\partial S_2}\left(\rho b_1 b_2 \frac{\partial u}{\partial S_1} + b_2^2 \frac{\partial u}{\partial S_2} \right)
$$

$$
+ \left(a_1 - \lambda_1 b_1 - b_1 \frac{\partial b_1}{\partial S_1} - \frac{1}{2}\frac{\partial}{\partial S_2}(\rho b_1 b_2) \right)\frac{\partial u}{\partial S_1}
$$

$$
+ \left(a_2 - \lambda_2 b_2 - b_2 \frac{\partial b_2}{\partial S_2} - \frac{1}{2}\frac{\partial}{\partial S_1}(\rho b_1 b_2) \right)\frac{\partial u}{\partial S_2} - ru + g,
$$

---

[16]If r is replaced by a bounded function, Theorem 2.2 still holds.

we have

$$
\frac{1}{2}\frac{dW(\tau)}{d\tau} = \int_{S_{2l}}^{S_{2u}}\int_{S_{1l}}^{S_{1u}} u\frac{\partial u}{\partial \tau}dS_1 dS_2
$$

$$
= \int_{S_{2l}}^{S_{2u}}\int_{S_{1l}}^{S_{1u}} \frac{u}{2}\frac{\partial}{\partial S_1}\left(b_1^2\frac{\partial u}{\partial S_1} + \rho b_1 b_2\frac{\partial u}{\partial S_2}\right)dS_1 dS_2
$$

$$
+ \int_{S_{2l}}^{S_{2u}}\int_{S_{1l}}^{S_{1u}} \frac{u}{2}\frac{\partial}{\partial S_2}\left(\rho b_1 b_2\frac{\partial u}{\partial S_1} + b_2^2\frac{\partial u}{\partial S_2}\right)dS_1 dS_2
$$

$$
+ \int_{S_{2l}}^{S_{2u}}\int_{S_{1l}}^{S_{1u}} u\left(a_1 - \lambda_1 b_1 - b_1\frac{\partial b_1}{\partial S_1} - \frac{1}{2}\frac{\partial}{\partial S_2}(\rho b_1 b_2)\right)\frac{\partial u}{\partial S_1}dS_1 dS_2
$$

$$
+ \int_{S_{2l}}^{S_{2u}}\int_{S_{1l}}^{S_{1u}} u\left(a_2 - \lambda_2 b_2 - b_2\frac{\partial b_2}{\partial S_2} - \frac{1}{2}\frac{\partial}{\partial S_1}(\rho b_1 b_2)\right)\frac{\partial u}{\partial S_2}dS_1 dS_2
$$

$$
- \int_{S_{2l}}^{S_{2u}}\int_{S_{1l}}^{S_{1u}} ru^2 dS_1 dS_2 + \int_{S_{2l}}^{S_{2u}}\int_{S_{1l}}^{S_{1u}} gu\, dS_1 dS_2. \tag{2.54}
$$

Now let us look at the first four terms in the right-hand side of the relation (2.54). Using integration by parts and the equality conditions in the conditions (2.42) and (2.43), we can rewrite the first and second terms as follows:

$$
\int_{S_{2l}}^{S_{2u}}\int_{S_{1l}}^{S_{1u}} \frac{u}{2}\frac{\partial}{\partial S_1}\left(b_1^2\frac{\partial u}{\partial S_1} + \rho b_1 b_2\frac{\partial u}{\partial S_2}\right)dS_1 dS_2
$$

$$
= \frac{1}{2}\int_{S_{2l}}^{S_{2u}}\left\{\left[u\left(b_1^2\frac{\partial u}{\partial S_1} + \rho b_1 b_2\frac{\partial u}{\partial S_2}\right)\right]\Big|_{S_{1l}}^{S_{1u}}\right.
$$

$$
\left. - \int_{S_{1l}}^{S_{1u}}\left(b_1^2\frac{\partial u}{\partial S_1} + \rho b_1 b_2\frac{\partial u}{\partial S_2}\right)\frac{\partial u}{\partial S_1}dS_1\right\}dS_2
$$

$$
= -\frac{1}{2}\int_{S_{2l}}^{S_{2u}}\int_{S_{1l}}^{S_{1u}}\left(b_1^2\frac{\partial u}{\partial S_1} + \rho b_1 b_2\frac{\partial u}{\partial S_2}\right)\frac{\partial u}{\partial S_1}dS_1 dS_2 \tag{2.55}
$$

and

$$
\int_{S_{2l}}^{S_{2u}}\int_{S_{1l}}^{S_{1u}} \frac{u}{2}\frac{\partial}{\partial S_2}\left(\rho b_1 b_2\frac{\partial u}{\partial S_1} + b_2^2\frac{\partial u}{\partial S_2}\right)dS_1 dS_2
$$

$$
= \frac{1}{2}\int_{S_{1l}}^{S_{1u}}\left\{\left[u\left(\rho b_1 b_2\frac{\partial u}{\partial S_1} + b_2^2\frac{\partial u}{\partial S_2}\right)\right]\Big|_{S_{2l}}^{S_{2u}}\right.
$$

$$
\left. - \int_{S_{2l}}^{S_{2u}}\left(\rho b_1 b_2\frac{\partial u}{\partial S_1} + b_2^2\frac{\partial u}{\partial S_2}\right)\frac{\partial u}{\partial S_2}dS_2\right\}dS_1
$$

$$
= -\frac{1}{2}\int_{S_{2l}}^{S_{2u}}\int_{S_{1l}}^{S_{1u}}\left(\rho b_1 b_2\frac{\partial u}{\partial S_1} + b_2^2\frac{\partial u}{\partial S_2}\right)\frac{\partial u}{\partial S_2}dS_1 dS_2. \tag{2.56}
$$

Also, according to the equality condition in the condition (2.42), $b_1(S_{1l}, S_2, t) = 0$ holds for any S_2, so

$$\frac{\partial}{\partial S_2}(\rho b_1 b_2)\bigg|_{S_1=S_{1l}} = 0.$$

Similarly, we have

$$\frac{\partial}{\partial S_2}(\rho b_1 b_2)\bigg|_{S_1=S_{1u}} = \frac{\partial}{\partial S_1}(\rho b_1 b_2)\bigg|_{S_2=S_{2l}} = \frac{\partial}{\partial S_1}(\rho b_1 b_2)\bigg|_{S_2=S_{2u}} = 0.$$

Noticing these facts and the inequality conditions in the conditions (2.42) and (2.43), for the third and fourth integrals in the right-hand side of the relation (2.54), we have

$$\int_{S_{2l}}^{S_{2u}} \int_{S_{1l}}^{S_{1u}} u \left(a_1 - \lambda_1 b_1 - b_1 \frac{\partial b_1}{\partial S_1} - \frac{1}{2}\frac{\partial}{\partial S_2}(\rho b_1 b_2) \right) \frac{\partial u}{\partial S_1} dS_1 dS_2$$

$$= \frac{1}{2} \int_{S_{2l}}^{S_{2u}} \left\{ \left[u^2 \left(a_1 - \lambda_1 b_1 - b_1 \frac{\partial b_1}{\partial S_1} - \frac{1}{2}\frac{\partial}{\partial S_2}(\rho b_1 b_2) \right) \right] \bigg|_{S_{1l}}^{S_{1u}} \right.$$

$$\left. - \int_{S_{1l}}^{S_{1u}} u^2 \frac{\partial}{\partial S_1} \left(a_1 - \lambda_1 b_1 - b_1 \frac{\partial b_1}{\partial S_1} - \frac{1}{2}\frac{\partial}{\partial S_2}(\rho b_1 b_2) \right) dS_1 \right\} dS_2$$

$$\leq -\frac{1}{2} \int_{S_{2l}}^{S_{2u}} \int_{S_{1l}}^{S_{1u}} u^2 \frac{\partial}{\partial S_1} \left(a_1 - \lambda_1 b_1 - b_1 \frac{\partial b_1}{\partial S_1} - \frac{1}{2}\frac{\partial}{\partial S_2}(\rho b_1 b_2) \right) dS_1 dS_2$$

$$(2.57)$$

and

$$\int_{S_{1l}}^{S_{1u}} \int_{S_{2l}}^{S_{2u}} u \left(a_2 - \lambda_2 b_2 - b_2 \frac{\partial b_2}{\partial S_2} - \frac{1}{2}\frac{\partial}{\partial S_1}(\rho b_1 b_2) \right) \frac{\partial u}{\partial S_2} dS_2 dS_1$$

$$= \frac{1}{2} \int_{S_{1l}}^{S_{1u}} \left\{ \left[u^2 \left(a_2 - \lambda_2 b_2 - b_2 \frac{\partial b_2}{\partial S_2} - \frac{1}{2}\frac{\partial}{\partial S_1}(\rho b_1 b_2) \right) \right] \bigg|_{S_{2l}}^{S_{2u}} \right.$$

$$\left. - \int_{S_{2l}}^{S_{2u}} u^2 \frac{\partial}{\partial S_2} \left(a_2 - \lambda_2 b_2 - b_2 \frac{\partial b_2}{\partial S_2} - \frac{1}{2}\frac{\partial}{\partial S_1}(\rho b_1 b_2) \right) dS_2 \right\} dS_1$$

$$\leq -\frac{1}{2} \int_{S_{2l}}^{S_{2u}} \int_{S_{1l}}^{S_{1u}} u^2 \frac{\partial}{\partial S_2} \left(a_2 - \lambda_2 b_2 - b_2 \frac{\partial b_2}{\partial S_2} - \frac{1}{2}\frac{\partial}{\partial S_1}(\rho b_1 b_2) \right) dS_1 dS_2.$$

$$(2.58)$$

Adding the relations (2.55) and (2.56) together, due to $|\rho| \leq 1$, we have

$$\int_{S_{2l}}^{S_{2u}} \int_{S_{1l}}^{S_{1u}} \frac{u}{2} \frac{\partial}{\partial S_1} \left(b_1^2 \frac{\partial u}{\partial S_1} + \rho b_1 b_2 \frac{\partial u}{\partial S_2} \right) dS_1 dS_2$$

$$+ \int_{S_{2l}}^{S_{2u}} \int_{S_{1l}}^{S_{1u}} \frac{u}{2} \frac{\partial}{\partial S_2} \left(\rho b_1 b_2 \frac{\partial u}{\partial S_1} + b_2^2 \frac{\partial u}{\partial S_2} \right) dS_1 dS_2$$

$$= -\frac{1}{2} \int_{S_{2l}}^{S_{2u}} \int_{S_{1l}}^{S_{1u}} \left[\left(b_1 \frac{\partial u}{\partial S_1} \right)^2 + 2\rho b_1 b_2 \frac{\partial u}{\partial S_1} \frac{\partial u}{\partial S_2} + \left(b_2 \frac{\partial u}{\partial S_2} \right)^2 \right] dS_1 dS_2 \leq 0. \tag{2.59}$$

Substituting the relations (2.55)–(2.56) and the inequalities (2.57)–(2.58) into the relation (2.54) and applying the inequality (2.59) and condition (ii), we have

$$\frac{1}{2} \frac{dW(\tau)}{d\tau} \leq -\frac{1}{2} \int_{S_{2l}}^{S_{2u}} \int_{S_{1l}}^{S_{1u}} u^2 \left\{ \frac{\partial}{\partial S_1} \left(a_1 - \lambda_1 b_1 - b_1 \frac{\partial b_1}{\partial S_1} - \frac{1}{2} \frac{\partial}{\partial S_2} (\rho b_1 b_2) \right) \right.$$

$$\left. + \frac{\partial}{\partial S_2} \left(a_2 - \lambda_2 b_2 - b_2 \frac{\partial b_2}{\partial S_2} - \frac{1}{2} \frac{\partial}{\partial S_1} (\rho b_1 b_2) \right) + 2r \right\} dS_1 dS_2$$

$$+ \int_{S_{2l}}^{S_{2u}} \int_{S_{1l}}^{S_{1u}} g(S_1, S_2, T - \tau) u \, dS_1 dS_2$$

$$\leq \frac{1}{2} c_1 W(\tau) + \frac{1}{2} \int_{S_{2l}}^{S_{2u}} \int_{S_{1l}}^{S_{1u}} g^2(S_1, S_2, T - \tau) dS_1 dS_2.$$

Therefore, according to the Gronwall inequality, we arrive at

$$0 \leq W(\tau) \leq e^{c_1 \tau} \left[W(0) + \int_0^\tau \int_{S_{2l}}^{S_{2u}} \int_{S_{1l}}^{S_{1u}} g^2(S_1, S_2, T - \tau) dS_1 dS_2 d\tau \right].$$

Suppose that u_1 and u_2 are two solutions of the problem (2.52) and let $u = u_1 - u_2$. It is clear that u is the solution of the problem (2.52) with $V(S_1, S_2, T) = f(S_1, S_2) \equiv 0$ and $g(S_1, S_2, t) \equiv 0$. In this case, we get $W(\tau) \equiv 0$. Then, $u \equiv 0$, or $u_1 \equiv u_2$; that is, the solution of the problem (2.52) is unique. □

Here we would like to make some remarks. The first one is about the conditions given in the theorem. If a_1, a_2, b_1, b_2, λ_1, λ_2, r, the first derivatives of a_1, a_2, λ_1, and λ_2, and the first and second derivatives of ρ, b_1, and b_2 are bounded, then conditions (2.42), (2.43) are reduced to the conditions (2.44), (2.45) respectively, and condition (ii) is always satisfied. The partial differential equation in the problem (2.52) is called a degenerate parabolic partial differential equation because of the equality conditions in the conditions (2.42) and (2.43). It is clear that the result can be applied to any degenerate parabolic problems from various fields.

When there are K random variables, $K \geq 3$, governed by

$$dS_i = a_i(S_1, S_2, \cdots, S_K, t)dt + b_i(S_1, S_2, \cdots, S_K, t)dX_i, \quad i = 1, 2, \cdots, K,$$

similar results can still be proved. For the proof above, a key fact we used is $|\rho| \leq 1$, which means that the correlation matrix is semi-positive. For multidimensional cases, the key fact we need is that the correlation matrix

$$\begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1K} \\ \rho_{21} & 1 & \cdots & \rho_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{K1} & \rho_{K2} & \cdots & 1 \end{pmatrix}$$

is semi-positive definite. Here $\rho_{i,j} = \mathrm{E}[dX_i dX_j]/dt$.

The meaning of the final-value problem (2.52) having a unique solution is that the solution of the problem (2.52) is completely determined by the PDE and the final condition. This also means that the random variables will never reach the boundaries if they are inside the domain at the beginning [33]. This is because if the random variables reach the boundaries, then the solution must also be affected by what happens at the boundaries. Therefore, if stochastic models satisfy the reversion conditions, then those random variables should be guaranteed on the finite domain $[S_{1l}, S_{1u}] \times [S_{2l}, S_{2u}]$. When $\left.\dfrac{\partial b_i(S_1, S_2, t)}{\partial S_i}\right|_{S_i=S_{il}}$ and $\left.\dfrac{\partial b_i(S_1, S_2, t)}{\partial S_i}\right|_{S_i=S_{iu}}$ are bounded, then conditions (2.42) and (2.43) are reduced to the conditions (2.44) and (2.45). Under conditions (2.44) and (2.45), the fact that the random variable will never reach the boundaries has been proved for the one-dimensional case in [33]. It can be expected that the same result is still true for multidimensional cases and when conditions (2.42) and (2.43) cannot be reduced to the conditions (2.44) and (2.45).

2.4.4 ‡Uniqueness of Solutions for European Options on Assets with Stochastic Volatilities

In this subsection, we consider a special two-factor financial derivative: options on assets with stochastic volatilities. We assume that the asset price S follows the following stochastic process:

$$dS = \mu S dt + \sigma S dX_1, \quad 0 \leq S \tag{2.60}$$

and that the volatility σ is also a random variable and its evolution is governed by

$$d\sigma = p(\sigma, t)dt + q(\sigma, t)dX_2, \quad \sigma_l \leq \sigma \leq \sigma_u, \tag{2.61}$$

where the two random increments dX_1 and dX_2 are two Wiener processes. dX_1 and dX_2 are correlated and $\mathrm{E}[dX_1 dX_2] = \rho dt$. Furthermore, we assume that the stochastic model for σ satisfies reversion conditions; that is, the following relations hold:

$$
\begin{cases}
p(\sigma_l, t) - q(\sigma_l, t)\dfrac{\partial q}{\partial \sigma}(\sigma_l, t) \geq 0, \\
q(\sigma_l, t) = 0
\end{cases}
\tag{2.62}
$$

and

$$
\begin{cases}
p(\sigma_u, t) - q(\sigma_u, t)\dfrac{\partial q}{\partial \sigma}(\sigma_u, t) \leq 0, \\
q(\sigma_u, t) = 0,
\end{cases}
\tag{2.63}
$$

or when $\dfrac{\partial q}{\partial \sigma}(\sigma_l, t)$ and $\dfrac{\partial q}{\partial \sigma}(\sigma_u, t)$ are bounded,

$$
\begin{cases}
p(\sigma_l, t) \geq 0, \\
q(\sigma_l, t) = 0
\end{cases}
\tag{2.64}
$$

and

$$
\begin{cases}
p(\sigma_u, t) \leq 0, \\
q(\sigma_u, t) = 0
\end{cases}
\tag{2.65}
$$

hold. Suppose that $V(S, \sigma, t)$ is the value of such an option. $V(S, \sigma, t)$ satisfies

$$
\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma q S \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2}q^2 \frac{\partial^2 V}{\partial \sigma^2} + (r - D_0)S\frac{\partial V}{\partial S} + (p - \lambda q)\frac{\partial V}{\partial \sigma} - rV = 0.
\tag{2.66}
$$

This equation holds for $S \in [0, \infty)$. In order to convert the problem on an infinite domain into one on a finite domain, we introduce the following transformation:

$$
\begin{cases}
\xi = \dfrac{S}{S + P_m}, \\[2mm]
\sigma = \sigma, \\[2mm]
t = t, \\[2mm]
\overline{V} = \dfrac{V}{S + P_m},
\end{cases}
\tag{2.67}
$$

where P_m is a positive constant. Since the following expressions exist:

$$S = \frac{\xi P_m}{1 - \xi}, \qquad\qquad S + P_m = \frac{P_m}{1 - \xi},$$

$$\frac{d\xi}{dS} = \frac{(1 - \xi)^2}{P_m}, \qquad\qquad \frac{\partial V}{\partial t} = \frac{P_m}{1 - \xi}\frac{\partial \overline{V}}{\partial t},$$

$$\frac{\partial V}{\partial S} = \overline{V} + (1 - \xi)\frac{\partial \overline{V}}{\partial \xi}, \qquad\qquad \frac{\partial V}{\partial \sigma} = \frac{P_m}{1 - \xi}\frac{\partial \overline{V}}{\partial \sigma},$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{(1 - \xi)^3}{P_m}\frac{\partial^2 \overline{V}}{\partial \xi^2}, \qquad\qquad \frac{\partial^2 V}{\partial S \partial \sigma} = \frac{\partial \overline{V}}{\partial \sigma} + (1 - \xi)\frac{\partial^2 \overline{V}}{\partial \xi \partial \sigma},$$

$$\frac{\partial^2 V}{\partial \sigma^2} = \frac{P_m}{1 - \xi}\frac{\partial^2 \overline{V}}{\partial \sigma^2},$$

we can rewrite Eq. (2.66) as

$$\frac{\partial \overline{V}}{\partial t} + \frac{1}{2}\sigma^2 \xi^2 (1 - \xi)^2 \frac{\partial^2 \overline{V}}{\partial \xi^2} + \rho\sigma\xi(1 - \xi)q\frac{\partial^2 \overline{V}}{\partial \xi \partial \sigma} + \frac{1}{2}q^2\frac{\partial^2 \overline{V}}{\partial \sigma^2}$$

$$+(r - D_0)\xi(1 - \xi)\frac{\partial \overline{V}}{\partial \xi} + [p - (\lambda - \rho\sigma\xi)q]\frac{\partial \overline{V}}{\partial \sigma} - [r - (r - D_0)\xi]\overline{V} = 0.$$

Since the transformation above converts a value of $S \in [0, \infty)$ into a value of $\xi \in [0, 1)$, $\overline{V}(\xi, \sigma, t)$ is defined on the domain $[0, 1] \times [\sigma_l, \sigma_u] \times [0, T]$. Therefore, the determination of European option prices in this case reduces to finding the solution of the following final-value problem:

$$\begin{cases} \dfrac{\partial \overline{V}}{\partial t} + \frac{1}{2}\sigma^2 \xi^2 (1 - \xi)^2 \dfrac{\partial^2 \overline{V}}{\partial \xi^2} + \rho\sigma\xi(1 - \xi)q\dfrac{\partial^2 \overline{V}}{\partial \xi \partial \sigma} + \frac{1}{2}q^2\dfrac{\partial^2 \overline{V}}{\partial \sigma^2} \\[2mm] \quad +(r - D_0)\xi(1 - \xi)\dfrac{\partial \overline{V}}{\partial \xi} + [p - (\lambda - \rho\sigma\xi)q]\dfrac{\partial \overline{V}}{\partial \sigma} - [r - (r - D_0)\xi]\overline{V} = 0, \\[2mm] \qquad\qquad\qquad \xi \in [0, 1], \quad \sigma \in [\sigma_l, \sigma_u], \quad t \in [0, T], \\[2mm] \overline{V}(\xi, \sigma, T) = f(\xi, \sigma), \quad \xi \in [0, 1], \quad \sigma \in [\sigma_l, \sigma_u]. \end{cases}$$

$$(2.68)$$

It is easy to see $d\xi = a_1(\xi)dt + b_1(\xi)dX_1$, where $a_1(\xi) = (\mu - \sigma\xi)\xi(1 - \xi)$ and $b_1(\xi) = \sigma\xi(1 - \xi)$. Thus, this problem is in the form of the problem (2.52) with

$$\lambda_1 = \frac{\mu - \sigma\xi - r + D_0}{\sigma}, \qquad a_2 = p(\sigma, t),$$

$$b_2 = q(\sigma, t), \qquad\qquad \lambda_2 = \lambda - \rho\sigma\xi,$$

and the coefficient of $\overline{V}$ here is $-[r - (r - D_0)\xi]$ instead of $-r$. In order to have a unique solution, the key is that a_1, b_1, a_2, and b_2 should satisfy the

reversion conditions (2.42) and (2.43). In this case, a_1 and b_1 always satisfy the conditions (2.42) and (2.43). That a_2 and b_2 satisfy the reversion conditions is equivalent to fulfillment of the conditions (2.62) and (2.63). Therefore, if the conditions (2.62), (2.63), conditions (ii) and (iii) of Theorem 2.2 are satisfied, then the problem (2.68) has a unique solution.

Suppose that a problem is defined on an infinite domain and its closed-form solution cannot be found. In order to get its solution, we need to solve the problem numerically on a finite domain. In this case, an artificial boundary condition will be needed, which causes some error and problems. The problem here is defined on a finite domain, so its numerical solution can be obtained without using any artificial boundary conditions; if the singularity-separating method and extrapolation techniques are used, then numerical solutions are very good even on quite coarse meshes.

2.5 Jump Conditions

2.5.1 Hyperbolic Equations with a Dirac Delta Function

Consider the following linear hyperbolic partial differential equation

$$\frac{\partial u}{\partial t} + f_1(x_1, x_2, \cdots, x_K, t)\frac{\partial u}{\partial x_1} + \cdots + f_K(x_1, x_2, \cdots, x_K, t)\frac{\partial u}{\partial x_K} = 0.$$

Let C be a curve defined by the system of ordinary differential equations

$$\frac{dx_1(t)}{dt} = f_1(x_1, x_2, \cdots, x_K, t),$$

$$\vdots$$

$$\frac{dx_K(t)}{dt} = f_K(x_1, x_2, \cdots, x_K, t)$$

with initial conditions

$$x_1(0) = \xi_1, \ x_2(0) = \xi_2, \ \cdots, \ x_K(0) = \xi_K.$$

Along the curve C we have

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x_1}\frac{dx_1}{dt} + \cdots + \frac{\partial u}{\partial x_K}\frac{dx_K}{dt} = 0.$$

Therefore, u is a constant along the curve:

$$u(x_1(t^*), x_2(t^*), \cdots, x_K(t^*), t^*) = u(x_1(t^{**}), x_2(t^{**}), \cdots, x_K(t^{**}), t^{**}),$$

where t^* and t^{**} are any two times. If

$$f_k(x_1, x_2, \cdots, x_K, t) = F_k(x_1, x_2, \cdots, x_K, t)\delta(t - t_i),$$

where $\delta(t - t_i)$ is the Dirac delta function, then [17]

$$
\begin{aligned}
x_k(t_i^+) - x_k(t_i^-) &= \int_{t_i^-}^{t_i^+} F_k(x_1(t), x_2(t), \cdots, x_K(t), t)\delta(t - t_i)dt \\
&= F_k\left(x_1(t_i^-), x_2(t_i^-), \cdots, x_K(t_i^-), t_i^-\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&u\left(x_1(t_i^-), x_2(t_i^-), \cdots, x_K(t_i^-), t_i^-\right) \\
&= u\left(x_1(t_i^+), x_2(t_i^+), \cdots, x_K(t_i^+), t_i^+\right) \\
&= u\left(x_1(t_i^-) + F_{1i}^-, \ x_2(t_i^-) + F_{2i}^-, \ \cdots, \ x_K(t_i^-) + F_{Ki}^-, \ t_i^+\right), \quad (2.69)
\end{aligned}
$$

where t_i^- and t_i^+ denote the time just before and after t_i, respectively, and

$$F_{ki}^- \equiv F_k\left(x_1(t_i^-), x_2(t_i^-), \cdots, x_K(t_i^-), t_i^-\right).$$

For such a jump condition, a similar derivation is given in the book [84] by Wilmott, Dewynne, and Howison.

2.5.2 Jump Conditions for Options on Stocks with Discrete Dividends and Discrete Sampling

From the relation (2.69), jump conditions of various options can be derived. Here, we give three examples. Two are simple and the other is quite complicated. Jump conditions for other options will be given when they are discussed.

Suppose $V(S, t)$ is the value of an option on a stock, which pays a dividend D_i at time t_i, $i = 1, 2, \cdots, I$. Here, we assume that $t_i \leq T$, T being expiry. From Sect. 2.2, we know that $V(S, t)$ satisfies Eq. (2.13):

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + [rS - D(S, t)]\frac{\partial V}{\partial S} - rV = 0,$$

---

[17]Here an integral is defined in the following way. Suppose that on $[0, T]$ we have a partition with $N + 1$ points: $0 = t_0 < t_1 < \cdots < T_N = T$. The definition of an integral is

$$\int_0^T f(t)dt = \lim_{dt \to 0} \sum_{n=0}^{n=N-1} f(t_n)(t_{n+1} - t_n),$$

where $dt = \max_{0 \leq n \leq N-1}(t_{n+1} - t_n)$. Let us call it an Itô integral. Such a definition is usually used in financial calculus.

where

$$D(S,t) = \sum_{i=1}^{I} D_i(S)\delta(t - t_i), \quad \text{with} \quad D_i(S) \leq S.$$

This means that at $t \neq t_i, i = 1, 2, \cdots, I$, V satisfies

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

and at $t = t_i, i = 1, 2, \cdots$, or I, the equation

$$\frac{\partial V}{\partial t} - D_i(S)\delta(t - t_i)\frac{\partial V}{\partial S} = 0$$

holds. According to Eq. (2.69), at $t = t_i$ we have

$$V(S, t_i^-) = V(S - D_i(S), t_i^+). \tag{2.70}$$

This is the jump condition for options on stocks with discrete dividends. We now explain the financial meaning of this relation. At $t = t_i$, the stock pays a dividend D_i, so the stock price will drop by D_i. If the price is S at t_i^-, then the price is $S - D_i$ at t_i^+. However, the price of the option is unchanged at time t_i because the holder of the option does not receive any money at time t_i.

The second example is similar to the first one. Suppose that $W(\eta, t)$ satisfies

$$\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2\eta^2\frac{\partial^2 W}{\partial \eta^2} + \left[(D_0 - r)\eta + \frac{1}{K}\sum_{i=1}^{K}\delta(t - t_i)\right]\frac{\partial W}{\partial \eta} - D_0 W = 0,$$

Then at $t = t_i, i = 1, 2, \cdots$, or K, W satisfies

$$\frac{\partial W}{\partial t} + \frac{1}{K}\delta(t - t_i)\frac{\partial W}{\partial \eta} = 0$$

Thus according to the relation (2.69), at $t = t_i$ we have

$$W(\eta, t_i^-) = W\left(\eta + \frac{1}{K}, t_i^+\right). \tag{2.71}$$

We will see in Chap. 4 that this jump condition will be often used when pricing Asian options because usually the average is measured discretely.

The third example involves several independent variables. Suppose the stock price is measured discretely and let $S_1, S_2, \cdots, S_N$ be the first N largest sampled stock prices until time t and $S_1 \geq S_2 \geq \cdots \geq S_N$. Assume that the value of option V depends on $S, S_1, \cdots, S_N, t$. From Sect. 4.4.6, we will see that if sampling occurs at $t = t_i$, then

$$\frac{dS_n}{dt} = \begin{cases} \left[\max(S, S_1(t_i^-)) - S_1(t_i^-)\right] \delta(t - t_i), & \text{if } n = 1, \\[2mm] \left[\max(\min(S, S_{n-1}(t_i^-)),\, S_n(t_i^-)) - S_n(t_i^-)\right] \delta(t - t_i), \\[2mm] \qquad\qquad\qquad\qquad\qquad\qquad \text{if } n = 2, 3, \cdots, N; \end{cases}$$

otherwise

$$\frac{dS_n}{dt} = 0.$$

According to Sect. 2.3, in this case, the option price is the solution of

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0)S\frac{\partial V}{\partial S}$$
$$+ \frac{\partial V}{\partial S_1}\frac{dS_1}{dt} + \frac{\partial V}{\partial S_2}\frac{dS_2}{dt} + \cdots + \frac{\partial V}{\partial S_N}\frac{dS_N}{dt} - rV = 0.$$

Consequently, at $t = t_i$, V satisfies

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S_1}\frac{dS_1}{dt} + \frac{\partial V}{\partial S_2}\frac{dS_2}{dt} + \cdots + \frac{\partial V}{\partial S_N}\frac{dS_N}{dt} = 0.$$

From the relation (2.69) we know when $t = t_i$, the jump condition

$$V(S, S_1^-, S_2^-, \cdots, S_N^-, t_i^-) = V(S,\ \max(S, S_1^-),\ \max(\min(S, S_1^-), S_2^-),$$
$$\cdots, \max(\min(S, S_{N-1}^-), S_N^-), t_i^+) \qquad (2.72)$$

holds, where S_n^- denotes $S_n(t_i^-)$ for brevity.

It is clear how to use such a jump condition when a European-style derivative is evaluated. When the price of an American-style derivative needs to be calculated, such a condition should be used on the solution obtained by the PDE. After that, taking the maximum between the new solution and the constraint yields the solution for the American derivative.

2.6 Solutions of European Options

A linear partial differential equation

$$A\frac{\partial^2 u}{\partial t^2} + 2B\frac{\partial^2 u}{\partial t \partial x} + C\frac{\partial^2 u}{\partial x^2} = F\left(x, t, u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}\right)$$

is called a parabolic partial differential equation if $AC - B^2 = 0$, where A, B, and C are not all equal to zero. The diffusion equation is the simplest parabolic equation. The Black–Scholes equation is another parabolic equation. In this section we mainly do two things. We reduce the Black–Scholes equation to a diffusion equation, and find out the analytic expression of the solution of the Black–Scholes equation and the Black–Scholes formulae for European options based on the analytic solution of the diffusion equation.

2.6.1 Converting the Black–Scholes Equation into a Heat Equation

In this subsection, we introduce one transformation that reduces the Black–Scholes equation to the heat equation. Because Green's function[18] of the heat equation has an analytic expression, we can obtain an analytic expression of Green's function for the Black–Scholes equation using the inverse transformation. Based on this, analytic expressions of European call and put option prices can be derived. These are the famous Black–Scholes formulae.

The price of a European option is a solution of the following problem:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r - D_0)S\dfrac{\partial V}{\partial S} - rV = 0, \ \ 0 \le S, \ \ t \le T, \\ V(S,T) = V_T(S), \ \ 0 \le S. \end{cases} \tag{2.73}$$

The payoff function $V_T(S)$ is determined by the feature of the option. For example, the payoffs of European calls and puts are given by

$$V_T(S) = \max(\pm(S - 1),\ 0), \ \ 0 \le S,$$

where $+$ and $-$ in $\pm$ correspond to call and put options, respectively. Here, the exercise price is 1 because we assume that both the price of the stock and the price of option have been divided by the exercise price. We call a problem with such a payoff a standard put/call problem. Let us set

$$\begin{cases} y = \ln S, \\ \tau = T - t, \\ V(S,t) = e^{-r(T-t)}v(y,\tau). \end{cases} \tag{2.74}$$

Because

$$\frac{\partial V}{\partial t} = e^{-r(T-t)}\left(rv - \frac{\partial v}{\partial \tau} \right),$$

$$\frac{\partial V}{\partial S} = e^{-r(T-t)}\frac{\partial v}{\partial y}\frac{dy}{dS} = e^{-r(T-t)}\frac{1}{S}\frac{\partial v}{\partial y},$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{\partial V}{\partial S} \right) = \frac{\partial}{\partial S}\left(e^{-r(T-t)}\frac{1}{S}\frac{\partial v}{\partial y} \right)$$

$$= e^{-r(T-t)}\left(-\frac{1}{S^2}\frac{\partial v}{\partial y} + \frac{1}{S^2}\frac{\partial^2 v}{\partial y^2} \right),$$

the Black–Scholes equation is converted into

$$-\frac{\partial v}{\partial \tau} + \frac{1}{2}\sigma^2\left(\frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial y} \right) + (r - D_0)\frac{\partial v}{\partial y} = 0,$$

---

[18]The definitions of Green's functions of the heat equation and the Black–Scholes equation are given in Sect. 2.6.

and the problem above becomes

$$
\begin{cases}
\dfrac{\partial v}{\partial \tau} = \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 v}{\partial y^2} + \left(r - D_0 - \dfrac{1}{2}\sigma^2\right)\dfrac{\partial v}{\partial y}, & -\infty < y < \infty, \ \ 0 \le \tau, \\
v(y,0) = V_T(e^y), & -\infty < y < \infty.
\end{cases}
\tag{2.75}
$$

Furthermore, we let

$$
\begin{cases}
x = y + \left(r - D_0 - \dfrac{1}{2}\sigma^2\right)\tau, \\
\bar{\tau} = \dfrac{1}{2}\sigma^2\tau, \\
v(y,\tau) = u(x,\bar{\tau}).
\end{cases}
\tag{2.76}
$$

Noticing the relations

$$
\frac{\partial v}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial u}{\partial \bar{\tau}} + \left(r - D_0 - \frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial x},
$$

$$
\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x},
$$

$$
\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial x^2},
$$

we finally arrive at

$$
\begin{cases}
\dfrac{\partial u}{\partial \bar{\tau}} = \dfrac{\partial^2 u}{\partial x^2}, & -\infty < x < \infty, \ \ 0 \le \bar{\tau}, \\
u(x,0) = V_T(e^x), & -\infty < x < \infty,
\end{cases}
\tag{2.77}
$$

where $V_T(e^x) = \max(\pm(e^x - 1), 0)$ for the European call and put options. The partial differential equation in this problem is usually called the heat or diffusion equation.

Before we go to the next subsection, we point out the following:

1. From the relations (2.74) and (2.76), we know

$$
V(S,t) = e^{-r(T-t)}u(\ln S + (r - D_0 - \sigma^2/2)(T - t), \ \sigma^2(T-t)/2)
$$

$$
= e^{-r(T-t)}u\left(\ln \frac{Se^{-D_0(T-t)}}{e^{-r(T-t)}} - \sigma^2(T-t)/2, \ \sigma^2(T-t)/2\right).
$$

Therefore, besides those parameters in the payoff function $V_T(S)$, $V(S,t)$ depends on only three parameters: $Se^{-D_0(T-t)}$, $e^{-r(T-t)}$, and $\sigma^2(T-t)/2$.

2. Actually, the transformations (2.74) and (2.76) can be combined into one transformation[19]

---

[19]The transform converting the Black-Scholes equation into a heat equation is not unique. For example, let $\bar{x} = \sqrt{2}\left[\ln S + \left(r - D_0 - \dfrac{1}{2}\sigma^2\right)(T - t)\right]/\sigma, \tau = T - t$, and

$$\begin{cases} x = \ln S + \left(r - D_0 - \frac{1}{2}\sigma^2\right)(T - t), \\ \bar{\tau} = \frac{1}{2}\sigma^2(T - t), \\ V(S, t) = e^{-r(T-t)}u(x, \bar{\tau}). \end{cases} \quad (2.78)$$

That is, through the transformation (2.78), the Black–Scholes equation can be directly converted into the heat equation. The reason we complete the transformation through two steps is to see the function of each single transformation. In fact, from the derivation we know the following:

- Through setting $\tau = T - t$, we change a problem with a final condition to a problem with an initial condition and let the initial time be zero.
- The transformation $y = \ln S$ is to reduce an equation with variable coefficients to one with constant coefficients. This is the transformation by which the Euler equation in ordinary differential equations becomes a differential equation with constant coefficients.
- Letting $V = e^{-r(T-t)}v(y, \tau)$, we eliminate the term rV in the equation. This is similar to the fact that an equation $\frac{dV}{d\tau} - rV = f$ can be written as $\frac{d(e^{-r\tau}V)}{d\tau} = e^{-r\tau}f$ after the equation is multiplied by $e^{-r\tau}$. The factor $e^{-r\tau}$ is called an integrating factor for the ordinary differential equation. If r depends on t, then the integrating factor is $e^{-\int_0^\tau r(T-s)ds} = e^{-\int_t^T r(s)ds}$ and the term rV can be eliminated in the same way.
- The transformation $x = y + (r - D_0 - \sigma^2/2)\tau$ is to eliminate the term $(r - D_0 - \sigma^2/2)\frac{\partial v}{\partial y}$. This is similar to reducing the simplest hyperbolic partial differential equation $\frac{\partial v}{\partial \tau} - a\frac{\partial v}{\partial y} = 0$ to an ordinary differential equation. For this case, the characteristic equation is $\frac{dy}{d\tau} = -a$ and its solution is $y = -a\tau + c$ or $y + a\tau = c$. Let $x = y + a\tau$ and $v(y, \tau) = u(x, \tau)$, then the hyperbolic partial differential equation becomes $\frac{\partial u(x, \tau)}{\partial \tau} = 0$. If a depends on t, then the solution of the characteristic equation is $y = -\int_0^\tau a(T - s)ds + c = -\int_t^T a(s)ds + c$. Letting $x = y + \int_t^T a(s)ds$ and $v(y, \tau) = u(x, \tau)$, we still have $\frac{\partial u(x, \tau)}{\partial \tau} = 0$.

$V(S, t) = e^{-r(T-t)}u(\bar{x}, \tau)$, then $u(\bar{x}, \tau)$ is a solution of the problem

$$\begin{cases} \dfrac{\partial u}{\partial \tau} = \dfrac{\partial^2 u}{\partial \bar{x}^2}, & -\infty < \bar{x} < \infty, \quad 0 \le \tau, \\ u(x, 0) = V_T(e^{\sigma\bar{x}/\sqrt{2}}), & -\infty < \bar{x} < \infty. \end{cases}$$

- In order for the coefficient of $\dfrac{\partial^2 u}{\partial x^2}$ to be one, we let $\bar\tau = \sigma^2\tau/2$. If σ depends on t, then letting $\bar\tau = \frac{1}{2}\int_0^\tau \sigma^2(T-s)ds = \frac{1}{2}\int_t^T \sigma^2(s)ds$ can still make the coefficient of $\dfrac{\partial^2 u}{\partial x^2}$ be one.

3. From the explanation on the function of each single transformation given above, we can see that if r, D_0, and σ are not constant, but depend on t only, then the Black–Scholes equation can still be converted into a heat equation by the following transformation

$$
\begin{cases}
x = \ln S + \int_t^T \left[r(s) - D_0(s) - \sigma^2(s)/2 \right] ds, \\[2mm]
\bar\tau = \dfrac{1}{2}\int_t^T \sigma^2(s)ds, \\[2mm]
V(S,t) = e^{-\int_t^T r(s)ds} u(x,\bar\tau)
\end{cases}
\tag{2.79}
$$

and the solution $V(S,t)$ possesses the following form:

$$
e^{-\int_t^T r(s)ds} u \left(\ln \frac{Se^{-\int_t^T D_0(s)ds}}{e^{-\int_t^T r(s)ds}} - \frac{1}{2}\int_t^T \sigma^2(s)ds, \ \frac{1}{2}\int_t^T \sigma^2(s)ds \right),
\tag{2.80}
$$

where $u(x,\bar\tau)$ is a solution of the heat equation (see [84]). This is left for readers as an exercise (Problem 36). There, in order to see the function of each part of the transformation, readers are asked to reduce the Black–Scholes equation with time-dependent parameters to a heat equation through two steps.

4. The transformation to convert the Black–Scholes equation into a heat equation is not unique. In fact, we can let $x = \ln S$, $\bar\tau = \dfrac{1}{2}\sigma^2(T-t)$, $V(S,t) = e^{\alpha x + \beta\bar\tau} u(x,\bar\tau)$, and choose constants α and β such that $u(x,\bar\tau)$ satisfies the heat equation (see [84]).

2.6.2 The Solutions of Parabolic Equations

In order for a parabolic differential equation to have a unique solution, one has to specify some conditions. For example, the initial value problem for a heat equation

$$
\frac{\partial u}{\partial \bar\tau} = \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad \bar\tau \ge 0
\tag{2.81}
$$

with

$$
u(x,0) = u_0(x)
\tag{2.82}
$$

has a unique solution under certain conditions that usually hold for cases considered in this book.

Let us find the solution of Eq. (2.81) with initial condition (2.82). The way to find the solution is not unique. Here, we use the following method (see [52]). We first try to find a special solution of Eq. (2.81) in the form

$$u(x, \bar{\tau}) = \bar{\tau}^{-1/2} U(\eta),$$

where

$$\eta = \frac{x - \xi}{\sqrt{\bar{\tau}}}, \qquad \xi \text{ being a parameter.}$$

Because

$$\frac{\partial u}{\partial \bar{\tau}} = -\frac{\bar{\tau}^{-3/2}}{2} \left(U + \eta \frac{dU}{d\eta} \right) = -\frac{\bar{\tau}^{-3/2}}{2} \frac{d}{d\eta} [\eta U(\eta)],$$

$$\frac{\partial u}{\partial x} = \bar{\tau}^{-1/2} \frac{dU}{d\eta} \frac{1}{\sqrt{\bar{\tau}}} = \bar{\tau}^{-1} \frac{dU}{d\eta},$$

$$\frac{\partial^2 u}{\partial x^2} = \bar{\tau}^{-3/2} \frac{d^2 U}{d\eta^2},$$

from Eq. (2.81) we have

$$-\frac{\bar{\tau}^{-3/2}}{2} \frac{d}{d\eta} (\eta U) = \bar{\tau}^{-3/2} \frac{d^2 U}{d\eta^2},$$

that is,

$$\frac{d^2 U}{d\eta^2} + \frac{1}{2} \frac{d}{d\eta} (\eta U) = 0.$$

Integrating this equation, we have

$$\frac{dU}{d\eta} + \frac{\eta}{2} U = c_1,$$

where c_1 is a constant. Let us choose $c_1 = 0$, so now we have a linear homogeneous equation. The solution of this equation is

$$U(\eta) = c e^{-\eta^2/4},$$

where c is a constant. Thus, for the diffusion equation we have a special solution in the form

$$c \bar{\tau}^{-1/2} e^{-(x-\xi)^2/4\bar{\tau}}.$$

If we further require

$$\int_{-\infty}^{\infty} c \bar{\tau}^{-1/2} e^{-(x-\xi)^2/4\bar{\tau}} d\xi = 1,$$

then

$$c = \frac{1}{\displaystyle\int_{-\infty}^{\infty} \bar{\tau}^{-1/2} e^{-(x-\xi)^2/4\bar{\tau}} d\xi} = \frac{1}{\sqrt{2}\displaystyle\int_{-\infty}^{\infty} e^{-\eta^2/2} d\eta} = \frac{1}{2\sqrt{\pi}}$$

and the special solution is

$$\frac{1}{2\sqrt{\pi\bar{\tau}}} e^{-(x-\xi)^2/4\bar{\tau}}.$$

This solution is called the fundamental solution, or Green's function, for the heat equation (2.81). Let $g(\xi; x, \bar{\tau})$ represent this class of functions with ξ as parameters. It is clear that the relation

$$\frac{\partial g(\xi; x, \bar{\tau})}{\partial \bar{\tau}} = \frac{\partial^2 g(\xi; x, \bar{\tau})}{\partial x^2}$$

holds for any ξ. Thus, for any $u_0(\xi)$ we have

$$\int_{-\infty}^{\infty} u_0(\xi) \frac{\partial g(\xi; x, \bar{\tau})}{\partial \bar{\tau}} d\xi = \int_{-\infty}^{\infty} u_0(\xi) \frac{\partial^2 g(\xi; x, \bar{\tau})}{\partial x^2} d\xi,$$

that is,

$$\frac{\partial \left[\int_{-\infty}^{\infty} u_0(\xi) g(\xi; x, \bar{\tau}) d\xi\right]}{\partial \bar{\tau}} = \frac{\partial^2 \left[\int_{-\infty}^{\infty} u_0(\xi) g(\xi; x, \bar{\tau}) d\xi\right]}{\partial x^2}.$$

Consequently,

$$u(x, \bar{\tau}) = \int_{-\infty}^{\infty} u_0(\xi) \times \frac{1}{2\sqrt{\pi\bar{\tau}}} e^{-(x-\xi)^2/4\bar{\tau}} d\xi \tag{2.83}$$

is also a solution of Eq. (2.81). Because

$$\lim_{\bar{\tau}\to 0} \frac{1}{2\sqrt{\pi\bar{\tau}}} e^{-(x-\xi)^2/4\bar{\tau}} = \begin{cases} 0, & x - \xi \neq 0, \\ \infty, & x - \xi = 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi\bar{\tau}}} e^{-(x-\xi)^2/4\bar{\tau}} d\xi = 1$$

is true for any $\bar{\tau}$, we have

$$\lim_{\bar{\tau}\to 0} \frac{1}{2\sqrt{\pi\bar{\tau}}} e^{-(x-\xi)^2/4\bar{\tau}} = \delta(x - \xi)$$

and

$$\lim_{\bar{\tau}\to 0} \int_{-\infty}^{\infty} u_0(\xi) \times \frac{1}{2\sqrt{\pi\bar{\tau}}} e^{-(x-\xi)^2/4\bar{\tau}} d\xi = u_0(x).$$

Consequently, Eq. (2.83) is the solution of the initial-value problem

$$\begin{cases} \dfrac{\partial u}{\partial \bar{\tau}} = \dfrac{\partial^2 u}{\partial x^2}, & -\infty < x < \infty, \quad \bar{\tau} > 0, \\ u(x, 0) = u_0(x), & -\infty < x < \infty. \end{cases}$$

2.6.3 Solutions of the Black–Scholes Equation

Because the solution of the problem (2.77) is the expression (2.83), from the relation (2.78) we know that the solution of the final value problem

$$
\begin{cases}
\dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r - D_0)S\dfrac{\partial V}{\partial S} - rV = 0, & 0 \le S, \quad 0 \le t \le T, \\
V(S,T) = V_T(S), & 0 \le S
\end{cases}
$$

is

$$
\begin{aligned}
V(S,t) &= \mathrm{e}^{-r(T-t)} \int_{-\infty}^{\infty} u_0(\xi)\frac{1}{2\sqrt{\pi\bar\tau}}\mathrm{e}^{-(x-\xi)^2/4\bar\tau}d\xi \\
&= \mathrm{e}^{-r(T-t)} \int_{-\infty}^{\infty} V_T\left(\mathrm{e}^{\xi}\right)\frac{1}{2\sqrt{\pi\bar\tau}}\mathrm{e}^{-(\xi-x)^2/4\bar\tau}d\xi \\
&= \mathrm{e}^{-r(T-t)} \frac{1}{\sigma\sqrt{2\pi(T-t)}} \\
&\quad \times \int_0^{\infty} V_T(S')\mathrm{e}^{-\left\{\ln S' - \left[\ln S + (r - D_0 - \sigma^2/2)(T-t)\right]\right\}^2/2\sigma^2(T-t)}\frac{dS'}{S'}.
\end{aligned}
$$

This result can be written as

$$
V(S,t) = \mathrm{e}^{-r(T-t)} \int_0^{\infty} V_T(S')G(S',T;S,t)dS', \tag{2.84}
$$

where

$$
\begin{aligned}
&G(S',T;S,t) \\
&= \frac{1}{\sigma\sqrt{2\pi(T-t)}S'}\mathrm{e}^{-\left\{\ln S' - \left[\ln S + (r - D_0 - \sigma^2/2)(T-t)\right]\right\}^2/2\sigma^2(T-t)}. \tag{2.85}
\end{aligned}
$$

Equations (2.84) and (2.85) are usually referred to as the general solution and Green's function of the Black–Scholes equation, respectively. From Sect. 2.1.3, we know that this function is also the probability density function for a log-normal distribution, that is, we can say that S' is a lognormal random variable and according to the result (2.6) its expectation is

$$
\mathrm{E}\left[S'\right] = S\mathrm{e}^{(r-D_0)(T-t)}. \tag{2.86}
$$

In order to make the expression of this function short, we rewrite it as

$$
G(S',T;S,t) = \frac{1}{\sqrt{2\pi}bS'}\mathrm{e}^{-\left[\ln(S'/a)+b^2/2\right]^2/2b^2},
$$

where

$$
a = S\mathrm{e}^{(r-D_0)(T-t)} \quad \text{and} \quad b = \sigma\sqrt{T-t}.
$$

For this function, there are the following useful formulae:

$$\int_c^\infty G(S', T; S, t)dS' = N\left(\frac{\ln(a/c) - b^2/2}{b}\right) \tag{2.87}$$

and

$$\int_c^\infty S'G(S', T; S, t)dS' = aN\left(\frac{\ln(a/c) + b^2/2}{b}\right), \tag{2.88}$$

where $N(z)$ is the cumulative distribution function for the standard normal distribution defined by[20]

$$N(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^z e^{-\xi^2/2}d\xi. \tag{2.89}$$

The proof of the two formulae is straightforward. Let

$$\eta(S') = \frac{\ln(S'/a) + b^2/2}{b},$$

that is,

$$S' = ae^{b\eta - b^2/2}.$$

Thus

$$dS' = ae^{b\eta - b^2/2}bd\eta = S'bd\eta.$$

Consequently, we have

$$\int_c^\infty \frac{1}{\sqrt{2\pi}bS'}e^{-\left[\ln(S'/a) + b^2/2\right]^2/2b^2}dS'$$

$$= \int_{\eta(c)}^\infty \frac{1}{\sqrt{2\pi}bS'}e^{-\eta^2/2}S'bd\eta$$

$$= N(-\eta(c))$$

$$= N\left(-\frac{\ln(c/a) + b^2/2}{b}\right)$$

$$= N\left(\frac{\ln(a/c) - b^2/2}{b}\right)$$

---

[20]The value of this function has to be obtained by numerical methods. If $z \leq 0$, this function can be approximated by

$$N(z) = 0.5t\exp(-x^2 - 1.26551223 + t(1.00002368 + t(0.37409196 + t(0.09678418$$
$$+t(-0.18628806 + t(0.27886807 + t(-1.13520398 + t(1.48851587$$
$$+t(-0.82215223 + t \times 0.17087277))))))))),$$

where $x = -z \times 0.707106781186550$ and $t = 1.0/(1.0 + 0.5x)$. If $z > 0$, then $N(z) = 1 - N(-z)$. The fractional error is less than 0.6×10^{-7} everywhere. See NUMERICAL RECIPES IN C: The Art of Scientific Computing. Cambridge University Press, Cambridge (1988–1992).

and

$$\int_c^\infty S' \frac{1}{\sqrt{2\pi b S'}} e^{-[\ln(S'/a)+b^2/2]^2/2b^2} dS'$$

$$= \int_{\eta(c)}^\infty \frac{1}{\sqrt{2\pi b}} e^{-\eta^2/2} a e^{b\eta-b^2/2} b \, d\eta$$

$$= \frac{a}{\sqrt{2\pi}} \int_{\eta(c)}^\infty e^{-(\eta-b)^2/2} d\eta$$

$$= \frac{a}{\sqrt{2\pi}} \int_{\eta(c)-b}^\infty e^{-\xi^2/2} d\xi$$

$$= aN\left(-\frac{\ln(c/a)+b^2/2}{b}+b\right)$$

$$= aN\left(\frac{\ln(a/c)+b^2/2}{b}\right).$$

2.6.4 Prices of Forward Contracts and Delivery Prices

From Sect. 1.2.1, we know that the payoff function for a forward contract is

$$V(S,T) = S - K.$$

Therefore, according to the formula (2.84) and using the result (2.86), we see that its price is

$$V(S,t) = e^{-r(T-t)} \int_0^\infty (S' - K)G(S',T;S,t)dS'$$

$$= e^{-r(T-t)}(Se^{(r-D_0)(T-t)} - K)$$

$$= Se^{-D_0(T-t)} - Ke^{-r(T-t)}.$$

Because for a forward contract the buyer does not need to pay any premium at $t = 0$, we have

$$V(S,0) = Se^{-D_0 T} - Ke^{-rT} = 0.$$

Consequently, the delivery price should be

$$K = e^{(r-D_0)T} S_0,$$

where in order to make it clear, we use S_0, instead of S, to denote the price of the underlying asset at the initiation of the contract.

2.6.5 Derivation of the Black–Scholes Formulae

At $t = T$, the value of a call option is

$$c(S,T) = \max(S - E, 0).$$

According to the formulae (2.84), (2.87), and (2.88), the value of a European call is

$$c(S,t) = e^{-r(T-t)} \int_0^\infty \max(S'-E,0)G(S',T;S,t)dS'$$

$$= e^{-r(T-t)} \int_E^\infty (S'-E)G(S',T;S,t)dS'$$

$$= e^{-r(T-t)} \left[\int_E^\infty S'G(S',T;S,t)dS' - \int_E^\infty EG(S',T;S,t)dS' \right]$$

$$= e^{-r(T-t)} \left[Se^{(r-D_0)(T-t)}N(d_1) - EN(d_2) \right]$$

$$= Se^{-D_0(T-t)}N(d_1) - Ee^{-r(T-t)}N(d_2), \tag{2.90}$$

where

$$d_1 = \left[\ln \frac{Se^{(r-D_0)(T-t)}}{E} + \frac{1}{2}\sigma^2(T-t) \right] \Big/ \left(\sigma\sqrt{T-t} \right)$$

$$= \left[\ln \frac{Se^{-D_0(T-t)}}{Ee^{-r(T-t)}} + \frac{1}{2}\sigma^2(T-t) \right] \Big/ \left(\sigma\sqrt{T-t} \right),$$

$$d_2 = \left[\ln \frac{Se^{(r-D_0)(T-t)}}{E} - \frac{1}{2}\sigma^2(T-t) \right] \Big/ \left(\sigma\sqrt{T-t} \right)$$

$$= \left[\ln \frac{Se^{-D_0(T-t)}}{Ee^{-r(T-t)}} - \frac{1}{2}\sigma^2(T-t) \right] \Big/ \left(\sigma\sqrt{T-t} \right)$$

$$= d_1 - \sigma\sqrt{T-t}.$$

For a put, the final value is

$$p(S,T) = \max(E-S,0).$$

Thus, the value of a European put is

$$p(S,t) = e^{-r(T-t)} \int_0^\infty \max(E-S',0)G(S',T;S,t)dS'$$

$$= e^{-r(T-t)} \int_0^E (E-S')G(S',T;S,t)dS'$$

$$= e^{-r(T-t)} \left[E \int_0^E G(S',T;S,t)dS' - \int_0^E S'G(S',T;S,t)dS' \right]$$

$$= e^{-r(T-t)} \left\{ E[1-N(d_2)] - Se^{(r-D_0)(T-t)}[1-N(d_1)] \right\}$$

$$= Ee^{-r(T-t)}N(-d_2) - Se^{-D_0(T-t)}N(-d_1). \tag{2.91}$$

It is interesting that the values of European call and put options can be expressed in terms of the cumulative distribution function for the standardized

normal random variable, $N(z)$. Expressions (2.90) and (2.91) give closed-form solutions for European vanilla options and are usually referred to as the Black–Scholes formulae.

When hedging is involved, we not only seek the value of options, but also the value of the first and second derivatives with respect to S, Δ, and Γ. For European call, $\Delta = \dfrac{\partial c}{\partial S}$ is

$$\frac{\partial c}{\partial S} = \mathrm{e}^{-D_0(T-t)}N(d_1) + S\mathrm{e}^{-D_0(T-t)}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-d_1^2/2}\frac{\partial d_1}{\partial S}$$
$$-E\mathrm{e}^{-r(T-t)}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-d_2^2/2}\frac{\partial d_2}{\partial S}$$
$$= \mathrm{e}^{-D_0(T-t)}N(d_1) + \frac{1}{\sqrt{2\pi}}\frac{\partial d_1}{\partial S}\left(S\mathrm{e}^{-D_0(T-t)-d_1^2/2} - E\mathrm{e}^{-r(T-t)-d_2^2/2}\right).$$

Noticing

$$-r(T-t) - d_2^2/2$$
$$= -r(T-t) - \left[d_1^2 - 2d_1\sigma\sqrt{T-t} + \sigma^2(T-t)\right]\Big/2$$
$$= -r(T-t) - \left[d_1^2 - 2\ln(S/E) - 2\left(r - D_0 + \sigma^2/2\right)(T-t) + \sigma^2(T-t)\right]\Big/2$$
$$= -d_1^2/2 - D_0(T-t) + \ln(S/E),$$

that is,

$$S\mathrm{e}^{-D_0(T-t)-d_1^2/2} = E\mathrm{e}^{-r(T-t)-d_2^2/2},$$

we have

$$\frac{\partial c}{\partial S} = \mathrm{e}^{-D_0(T-t)}N(d_1).$$

Taking the derivative with respect to S again yields

$$\frac{\partial^2 c}{\partial S^2} = \frac{1}{S\sigma\sqrt{2\pi(T-t)}}\mathrm{e}^{-D_0(T-t)-d_1^2/2}.$$

Similarly, for put options

$$\frac{\partial p}{\partial S} = -\mathrm{e}^{-D_0(T-t)}N(-d_1) \quad \text{and} \quad \frac{\partial^2 p}{\partial S^2} = \frac{\partial^2 c}{\partial S^2}.$$

We need to point out that if the value of an option and the price of the underlying asset are divided by E, then the dimensionless option value V/E and the derivatives of V/E can still be obtained by the same formulae. The only change is to let $E = 1$ and S should have dimensionless value.

What the values of $c(S,t)$ and $p(S,t)$ are? What do the functions $c(S,t)$ and $p(S,t)$ look like? For the case $S = E$, $r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$ and $T - t = 1$,
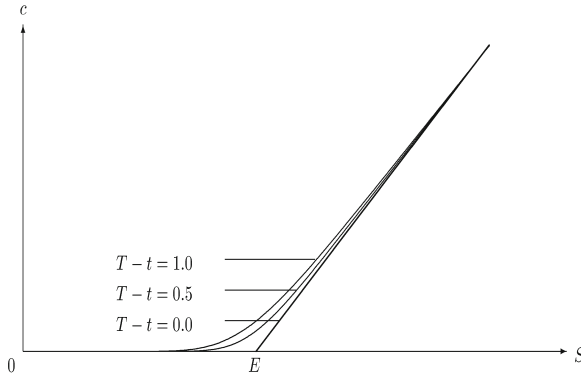
Fig. 2.5. The European call value $c(S,t)$ as a function of S
($r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$, and $T - t = 0$, 0.5, and 1.0)
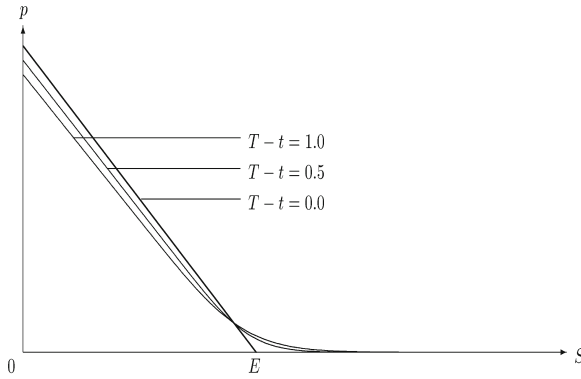

Fig. 2.6. The European put value $p(S,t)$ as a function of S
($r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$, and $T - t = 0$, 0.5, and 1.0)

$$c(S,t)/E = 0.0994 \quad \text{and} \quad p(S,t)/E = 0.0530$$

and for the case $S = E$, $r = 0.02$, $D_0 = 0.01$, $\sigma = 0.2$ and $T - t = 1$,

$$c(S,t)/E = 0.0835 \quad \text{and} \quad p(S,t)/E = 0.0736.$$

The functions of the European call and put options for the case $r = 0.1$,
$D_0 = 0.05$, $\sigma = 0.2$, $T - t = 0, 0.5, 1$ are shown in Figs. 2.6 and 2.5. Clearly,
the curves should approach the payoff functions as $t \to T$, which can be seen
from the two figures. From Fig. 2.6, we can also see that when S is close to
zero, the curves approach the payoff from the bottom and when S is large,
the curves tend to the payoff from the top. That is, $p(S,t)$ is less than the
payoff for small S and greater than the payoff for large S. In Sect. 3.1, we will
see that for American options, the price should always be at least the payoff.
Because of this, the Black–Scholes equation cannot be used to determine the
price of American options in some situations.

When σ, r, and D_0 depend on t, closed-form solutions can still be obtained (see [63, 84]). Actually, through the transformation (2.79), the Black–Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2(t)S^2\frac{\partial^2 V}{\partial S^2} + [r(t) - D_0(t)]S\frac{\partial V}{\partial S} - r(t)V = 0$$

can still be reduced to a diffusion equation. Let

$$\begin{cases} \alpha(t) = \frac{1}{2}\int_t^T \sigma^2(s)ds, \\ \delta(t) = \int_t^T D_0(s)ds, \\ \gamma(t) = \int_t^T r(s)ds, \end{cases}$$

then the solution of the Black–Scholes equation in this case is

$$V(S,t) = e^{-\gamma(t)}\int_{-\infty}^{\infty} V_T\left(e^\xi\right)\frac{1}{2\sqrt{\pi\bar{\tau}}}e^{-(\xi-x)^2/4\bar{\tau}}d\xi,$$

where $x = \ln S + \gamma(t) - \delta(t) - \alpha(t)$ and $\bar{\tau} = \alpha(t)$. Therefore, for a call with coefficients $r(t)$, $D_0(t)$, and $\sigma(t)$, the solution should be

$$c(S,t) = Se^{-\delta(t)}N(\bar{d}_1) - Ee^{-\gamma(t)}N(\bar{d}_2),$$

where

$$\bar{d}_1 = \left[\ln\frac{Se^{-\delta(t)}}{Ee^{-\gamma(t)}} + \alpha(t)\right] \bigg/ [2\alpha(t)]^{1/2},$$

$$\bar{d}_2 = \left[\ln\frac{Se^{-\delta(t)}}{Ee^{-\gamma(t)}} - \alpha(t)\right] \bigg/ [2\alpha(t)]^{1/2}.$$

2.6.6 Put–Call Parity Relation

Although call and put options are superficially different, they can be combined in such a way that they are perfectly correlated. In fact, there is the following relation:

$$c(S,t) - p(S,t) = Se^{-D_0(T-t)} - Ee^{-r(T-t)}, \tag{2.92}$$

which is usually called the put–call parity relation. It can be obtained in different ways. From the Black–Scholes formulae (2.90) and (2.91), we can have

$$\begin{aligned} c(S,t) - p(S,t) &= Se^{-D_0(T-t)}N(d_1) - Ee^{-r(T-t)}N(d_2) \\ &\quad - Ee^{-r(T-t)}N(-d_2) + Se^{-D_0(T-t)}N(-d_1) \\ &= Se^{-D_0(T-t)} - Ee^{-r(T-t)}. \end{aligned}$$

This is one way to get it.

We can also find this relation without finding the concrete expressions of $c(S,t)$ and $p(S,t)$. Let us look at a portfolio whose payoff is

$$\Pi(S,T) = S + \max(E - S, 0) - \max(S - E, 0) - E = 0.$$

According to the formula (2.84), $\Pi(S,t) = 0$ and we also have

$$\Pi(S,t)$$
$$= e^{-r(T-t)} \int_0^\infty [S' + \max(E - S', 0) - \max(S' - E, 0) - E]G(S',T;S,t)dS'$$
$$= Se^{-D_0(T-t)} + p(S,t) - c(S,t) - Ee^{-r(T-t)}.$$

Here, we are actually using the superposition principle of homogeneous linear partial differential equations. From these relations, we immediately have the put–call parity. In Sect. 3.4, we will derive this relation again without using a partial differential equation. Here, we need to point out that the put–call parity relation is true only for European options. For American options, the equality becomes an inequality, which will be discussed in Sect. 3.4.

2.6.7 An Explanation in Terms of Probability

The function $G(S',T;S,t)$ given by the expression (2.85) represents a probability density function of a random variable S', and S' can be interpreted as the random price of a stock at time T. Then, we can understand S as the price of the stock at time t because $G(S',T;S,t)$ goes to a Dirac delta function $\delta(S' - S)$ as $T \to t$. $V_T(S')$ is the value of an option at time T if the price is S'. Therefore

$$\int_0^\infty V_T(S')G(S',T;S,t)dS'$$

is the expectation of the value of the option at time T if the price is S at time t, and

$$e^{-r(T-t)} \int_0^\infty V_T(S')G(S',T;S,t)dS'$$

is the present (or discounted) value of the expectation at time T. That is, the price of an option at time t given by the formula (2.84) is the present value of the expectation of the option value at time T. This is the explanation of the solution given by the formula (2.84) in terms of probability.

Suppose that S and S' are the prices of a stock at time $T - \Delta t$ and time T, respectively, and that S' has the probability density function $G(S',T;S,T-\Delta t)$. According to the result (2.6) we have

$$E[S'] = Se^{(r-D_0)\Delta t}$$

and

$$\text{Var}\left[S'\right] = S^2 e^{2(r-D_0)\Delta t}\left(e^{\sigma^2 \Delta t} - 1\right) \approx S^2 \sigma^2 \Delta t.$$

Therefore[21]

$$\text{E}\left[\frac{S' - S}{S}\right] = \frac{Se^{(r-D_0)\Delta t} - S}{S} \approx (r - D_0)\Delta t$$

and

$$\text{Var}\left[\frac{S' - S}{S}\right] \approx \sigma^2 \Delta t.$$

Consequently

$$\frac{dS}{S} = (r - D_0)dt + \sigma dX.$$

However, in the real world

$$\frac{dS}{S} = \mu dt + \sigma dX.$$

Therefore, the random variable in the expression of the solution is a different random variable from that in the real world. Usually, we say that the random variable in the expression of the solution is in a "risk-neutral" world. In this case, the expected return rate per unit time on any asset is the difference between the riskless interest rate r and the dividend yield D_0.

It is clear that if we let

$$\overline{V}(S, t) = e^{r(T-t)}V(S, t),$$

then $\overline{V}$ is the solution of the problem

$$\begin{cases} \dfrac{\partial \overline{V}}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 \overline{V}}{\partial S^2} + (r - D_0)S\dfrac{\partial \overline{V}}{\partial S} = 0, 0 \le S, \quad t \le T, \\ \overline{V}(S, T) = \overline{V}_T(S), \qquad\qquad\qquad\qquad\quad 0 \le S \end{cases}$$

and

$$\overline{V}(S, t) = \int_0^\infty \overline{V}_T(S')G(S', T; S, t)dS' = \text{E}\left[\overline{V}_T(S')\right].$$

In probability theory, when this relation holds, it is said that $\overline{V}(S, t)$ is a martingale under the probability density function $G(S', T; S, t)$.

---

[21] Here we take a conditional expectation, i.e., S' is a random variable and S is fixed.

Problems

Table 2.1. Problems and subsections

Problems	Subsections	Problems	Subsections	Problems	Subsections
1–4	2.1.1	5–10	2.1.2	11(a)	2.2.1
11(b)	2.2.2	12–15	2.2.5	16	2.2.6
17–18	2.3.1	19–21	2.3.2	22	2.3.3
23–26	2.3.4	27–28	2.4.1	29–30	2.4.2
31(a)	2.5.1	31(b–d)–33	2.5.2	34–36	2.6.1
37	2.6.2	38–42	2.6.3	43–58	2.6.5
59–61	2.6.6				

1. (a) Show

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 1.$$

   (b) Show that

$$\int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} e^{-(x-a)^2/2b^2} dx = 1$$

   holds for any a and b. (Because this is true and the integrand is always positive, it can be a probability density function.)

   (c) If the probability density function of a random variable x is

$$\frac{1}{b\sqrt{2\pi}} e^{-(x-a)^2/2b^2},$$

   then it is called a normal random variable. Show $E[x] = a$ and $\text{Var}[x] = b^2$.

2. Define $dX = \phi\sqrt{dt}$, where ϕ is a standardized normal random variable and its probability density function is

$$\frac{1}{\sqrt{2\pi}} e^{-\phi^2/2}, \qquad -\infty < \phi < \infty.$$

   Find $E[dX]$, $\text{Var}[dX]$, $E[(dX)^2]$, and $\text{Var}[(dX)^2]$.

3. Suppose that S has the probability density function

$$G(S) = \frac{1}{\sqrt{2\pi}bS} e^{-\left[\ln(S/a)+b^2/2\right]^2/2b^2}.$$

   Find the probability density function for $\xi = \dfrac{1}{S}$, $E[\xi]$ and $\text{Var}[\xi]$.

4. (a) Suppose that S_1 and S_2 are two independent normal random variables. The mean and variance of S_1 are μ_1 and σ_1^2 and for S_2 they are μ_2 and σ_2^2. Find the probability density function of the random variable $S_1 + S_2$ and using this function, show that $S_1 + S_2$ is a normal random variable with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$.[22]

(b) Suppose that $\Delta t = t/n$ and ϕ_i, $i = 1, 2, \cdots, n$, are independent standardized normal random variables. Show that

$$X(t) = \lim_{n \to \infty} \left(\phi_1 \sqrt{\Delta t} + \phi_2 \sqrt{\Delta t} + \cdots + \phi_n \sqrt{\Delta t} \right)$$

is a normal random variable with mean zero and variance t.

(c) Define $dX = X(t + dt) - X(t)$. Show that it is a normal random variable with mean zero and variance dt.

(d) Suppose $S(t) = e^{\mu t + \sigma X(t)}$. Show that $d \ln S(t) \equiv \ln S(t + dt) - \ln S(t) = \mu dt + \sigma dX$ without using Itô's lemma. (This result shows that $S(t) = e^{\mu t + \sigma X(t)}$ is a solution of the equation $d \ln S(t) = \mu dt + \sigma dX$.)

5. *[23]Suppose

$$dS = a(S, t)dt + b(S, t)dX,$$

where dX is a Wiener process. Let f be a function of S and t. Show that

$$df = \frac{\partial f}{\partial S} dS + \left(\frac{\partial f}{\partial t} + \frac{1}{2} b^2 \frac{\partial^2 f}{\partial S^2} \right) dt$$

$$= b \frac{\partial f}{\partial S} dX + \left(\frac{\partial f}{\partial t} + \frac{1}{2} b^2 \frac{\partial^2 f}{\partial S^2} + a \frac{\partial f}{\partial S} \right) dt.$$

This result is usually referred to as Itô's lemma.

6. Suppose that a random variable satisfies

$$dS = \mu S dt + \sigma S dX,$$

where dX is a Wiener process. Find the stochastic equation for $\xi = \dfrac{1}{S}$ by using Itô's lemma and determine the mean and variance of $\dfrac{d\xi}{\xi}$.

7. Suppose that S satisfies

$$dS = \mu S dt + \sigma S dX.$$

---

[22]You have to show directly the relation

$$\frac{1}{\sqrt{2\pi}\sigma_1} \int_{-\infty}^{\infty} e^{tS_1} e^{-(S_1 - \mu_1)^2/2\sigma_1^2} dS_1 \cdot \frac{1}{\sqrt{2\pi}\sigma_2} \int_{-\infty}^{\infty} e^{tS_2} e^{-(S_2 - \mu_2)^2/2\sigma_2^2} dS_2$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{\sigma_1^2 + \sigma_1^2}} \int_{-\infty}^{\infty} e^{tS} e^{-(S - \mu_1 - \mu_2)^2/2(\sigma_1^2 + \sigma_2^2)} dS$$

if such a conclusion is used.

[23]A problem with * in this book means that you can find the answer in this book. It is suggested that a student should first read and understand the corresponding material and then do the problem without looking at the book.

(a) Let $F = e^{(r-D_0)(T-t)}S$, which is called the forward/futures price, and $f = Se^{-D_0(T-t)} - Ke^{-r(T-t)}$, which is the value of a forward/futures contract. Here K is a constant and we assume that r and D_0 are constant. By Itô's lemma, show that F and f satisfy

$$dF = (\mu - r + D_0)Fdt + \sigma FdX$$

and

$$df = \left[(\mu + D_0)\left(f + Ke^{-r(T-t)}\right) - rKe^{-r(T-t)}\right]dt$$
$$+\sigma \left[f + Ke^{-r(T-t)}\right]dX,$$

respectively.

(b) Define $\xi_{10} = \dfrac{Se^{-D_0(T-t)}}{Ee^{-r(T-t)}} = \dfrac{Se^{(r-D_0)(T-t)}}{E}$ and $\xi_{01} = \dfrac{Ee^{-r(T-t)}}{Se^{-D_0(T-t)}} = \dfrac{E}{Se^{(r-D_0)(T-t)}}$, where E is a constant. Show

$$d\xi_{10} = (\mu - r + D_0)\xi_{10}dt + \sigma\xi_{10}dX$$

and

$$d\xi_{01} = (-\mu + r - D_0 + \sigma^2)\xi_{01}dt - \sigma\xi_{01}dX.$$

8. Suppose that S satisfies

$$dS = a(S,t)dt + b(S,t)dX.$$

Show that for any functions $f_1(S,t)$ and $f_2(S,t)$, the following is true:

$$d\left(f_1 f_2\right) = f_1 df_2 + f_2 df_1 + b^2\frac{\partial f_1}{\partial S}\frac{\partial f_2}{\partial S}dt.$$

9. Suppose that S satisfies

$$dS = \mu Sdt + \sigma SdX, \quad 0 \le S < \infty,$$

where μ, σ are positive constants and dX is a Wiener process. Let

$$\xi = \frac{S}{S + P_m},$$

where P_m is a positive constant. The range of ξ is $[0, 1)$ and the stochastic differential equation for ξ is in the form:

$$d\xi = a(\xi)dt + b(\xi)dX.$$

Find the concrete expressions for $a(\xi)$ and $b(\xi)$ by Itô's lemma and show

$$\begin{cases} a(0) = 0, \\ b(0) = 0, \end{cases} \quad \text{and} \quad \begin{cases} a(1) = 0, \\ b(1) = 0. \end{cases}$$

10. Consider a random variable r satisfying the stochastic differential equation

$$dr = (\mu - \gamma r)dt + wdX, \quad -\infty < r < \infty,$$

where μ, γ, w are positive constants and dX is a Wiener process. Define

$$\xi = \frac{r}{|r| + P_m}, \quad P_m > 0,$$

which transforms the domain $(-\infty, \infty)$ for r into $(-1, 1)$ for ξ. Suppose the stochastic equation for the new random variable ξ is

$$d\xi = a(\xi)dt + b(\xi)dX.$$

Find the concrete expressions of $a(\xi)$ and $b(\xi)$ and show that $a(\xi)$ and $b(\xi)$ fulfill the conditions

$$\begin{cases} a(-1) = 0, \\ b(-1) = \dfrac{db(-1)}{d\xi} = 0, \end{cases} \quad \text{and} \quad \begin{cases} a(1) = 0, \\ b(1) = \dfrac{db(1)}{d\xi} = 0. \end{cases}$$

11. (a) *Show that if an investment is risk-free, then theoretically its return rate must be the short-term interest rate.
    (b) *Using this fact and Itô's lemma, derive the Black–Scholes equation.
12. *Suppose $V(S,t)$ is the solution of the problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2(S)S^2\dfrac{\partial^2 V}{\partial S^2} + (r - D_0)S\dfrac{\partial V}{\partial S} - rV = 0, & 0 \le S, \ t \le T, \\ V(S,T) = V_T(S), \ \ 0 \le S. \end{cases}$$

Let $\xi = \dfrac{S}{S + P_m}$, $\tau = T - t$, and $V(S,t) = (S + P_m)\overline{V}(\xi, \tau)$, where P_m is a positive constant.
(a) Show that $\overline{V}(\xi, \tau)$ is the solution of the problem

$$\begin{cases} \dfrac{\partial \overline{V}}{\partial \tau} = \dfrac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1 - \xi)^2\dfrac{\partial^2 \overline{V}}{\partial \xi^2} + (r - D_0)\xi(1 - \xi)\dfrac{\partial \overline{V}}{\partial \xi} \\ \qquad\quad - [r(1 - \xi) + D_0\xi]\overline{V}, & 0 \le \xi \le 1, \quad 0 \le \tau, \\ \overline{V}(\xi, 0) = \dfrac{1 - \xi}{P_m}V_T\left(\dfrac{P_m\xi}{1 - \xi}\right), & 0 \le \xi \le 1, \end{cases}$$

where $\bar{\sigma}(\xi) = \sigma\left(\dfrac{P_m\xi}{1 - \xi}\right)$.
(b) What are the advantages of reformulating the problem on a finite domain?

13. Consider the problem

$$\begin{cases} \dfrac{\partial W}{\partial t} + \dfrac{1}{2}\sigma^2\eta^2\dfrac{\partial^2 W}{\partial \eta^2} + \left[(D_0 - r)\,\eta + \dfrac{1}{T}\right]\dfrac{\partial W}{\partial \eta} - D_0 W = 0, \\ \qquad\qquad\qquad\qquad\qquad -\infty < \eta < \infty, \quad t \le T, \\ W(\eta, T) = W_T(\eta), \qquad\qquad -\infty < \eta < \infty, \end{cases}$$

which is related to the European average price options. Let us introduce the following transformation:

$$\begin{cases} \xi = \dfrac{\eta}{|\eta| + P_m}, \\ \tau = T - t, \\ W(\eta, t) = (|\eta| + P_m)\overline{u}(\xi, \tau), \end{cases}$$

where $P_m > 0$. Find the PDE and the initial condition $\overline{u}(\xi, \tau)$ should satisfy.

14. As we know, the prices of European call and put options are solutions of the problem

$$\begin{cases} \dfrac{\partial c}{\partial t} + \dfrac{1}{2}\sigma^2 S^2\dfrac{\partial^2 c}{\partial S^2} + (r - D_0)S\dfrac{\partial c}{\partial S} - rc = 0, \quad 0 \le S, \quad t \le T, \\ c(S, T) = \max(S - E, 0), \qquad\qquad\qquad 0 \le S, \end{cases}$$

and the problem

$$\begin{cases} \dfrac{\partial p}{\partial t} + \dfrac{1}{2}\sigma^2 S^2\dfrac{\partial^2 p}{\partial S^2} + (r - D_0)S\dfrac{\partial p}{\partial S} - rp = 0, \quad 0 \le S, \quad t \le T, \\ p(S, T) = \max(E - S, 0), \qquad\qquad\qquad 0 \le S, \end{cases}$$

respectively.

(a) Let $S_0^* = Ee^{-r(T-t)}$, $S_1^* = Se^{-D_0(T-t)}$, $\xi_{10} = S_1^*/S_0^*$, and $\xi_{01} = S_0^*/S_1^*$. Define $V_0(\xi_{10}, t) = c(S, t)/S_0^*$ and $V_1(\xi_{01}, t) = p(S, t)/S_1^*$. Find the PDEs and final conditions for $V_0(\xi_{10}, t)$ and $V_1(\xi_{01}, t)$.

(b) Based on the results in part (a), show that when S_1^* is replaced by S_0^* and S_0^* by S_1^* at the same time, the expression for $c(S, t) = S_0^* V_0(S_1^*/S_0^*, t)$ becomes the expression for $p(S, t) = S_1^* V_1(S_0^*/S_1^*, t)$.

15. Consider the following option problem:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2\dfrac{\partial^2 V}{\partial S^2} + (r - D_0)S\dfrac{\partial V}{\partial S} - rV = 0, \quad 0 \le S, \quad t \le T, \\ V(S, T) = \max(E, S), \qquad\qquad\qquad 0 \le S. \end{cases}$$

Suppose that the uniqueness of the solution has been proved.

a) Let $S_0^* = E e^{-r(T-t)}$, $S_1^* = S e^{-D_0(T-t)}$, $\xi_{10} = S_1^*/S_0^*$, and $\xi_{01} = S_0^*/S_1^*$. Define $V_0(\xi_{10}, t) = V(S, t)/S_0^*$ and $V_1(\xi_{01}, t) = V(S, t)/S_1^*$. Based on these relations, find the PDEs and final conditions for $V_0(\xi_{10}, t)$ and $V_1(\xi_{01}, t)$.

b) Based on the results in part (a), show that $V(S, t)$ can be expressed as a function $f(S_0^*, S_1^*, t)$ and this function is symmetric for S_0^* and S_1^*, i.e., $f(S_0^*, S_1^*, t) = f(S_1^*, S_0^*, t)$. This result indicates that in this option problem, the position of the cash and the position of the value of the stock are symmetric in some sense.

16. As we know, $f = S e^{-D_0(T-t)} - K e^{-r(T-t)}$ is the value of a forward/futures contract. For S we assume $dS = \mu S dt + \sigma S dX$, so for df we have

$$df = \left[(\mu + D_0)\left(f + K e^{-r(T-t)}\right) - rK e^{-r(T-t)}\right] dt$$
$$+ \sigma \left[f + K e^{-r(T-t)}\right] dX$$

according to Itô's lemma.

(a) *Consider an option on a forward/futures and let the price of such an option be $V_1(f, t)$. Derive the PDE for V_1 by using Itô's lemma. (Hint: Set $\Pi = V_1(f, t) - \Delta f$.)

(b) *Let $F = e^{(r-D_0)(T-t)} S$, then for f we have another expression: $f = e^{-r(T-t)}\left(S e^{(r-D_0)(T-t)} - K\right) = e^{-r(T-t)}(F - K)$. Define $V(F, t) = V_1(f(F, t), t) = V_1(e^{-r(T-t)}(F - K), t)$. Derive the PDE for $V(F, t)$ from the PDE obtained in part (a).

(c) Define $V_3(S, t) = V_1(f(S, t), t) = V_1(S e^{-D_0(T-t)} - K e^{-r(T-t)}, t)$. Show that $V_3(S, t)$ satisfies the Black–Scholes equation:

$$\frac{\partial V_3}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_3}{\partial S^2} + (r - D_0)S\frac{\partial V_3}{\partial S} - rV_3 = 0.$$

17. *Describe and derive the generalized Itô's lemma.

18. Suppose that S_1, S_2, $\cdots$, S_n are n lognormal random variables satisfying the following stochastic differential equations:

$$dS_i = \mu_i S_i dt + \sigma_i S_i dX_i, \quad i = 1, 2, \cdots, n,$$

where $\mu_i, \sigma_i, i = 1, 2, \cdots, n$, are constants and $dX_i, i = 1, 2, \cdots, n$, are n Wiener processes, i.e., $dX_i = \phi_i \sqrt{dt}$, ϕ_i being distinct standardized normal random variables, $i = 1, 2, \cdots, n$. ϕ_i and ϕ_j could be correlated and $E[\phi_i \phi_j] = \rho_{ij}$, $i, j = 1, 2, \cdots, n$, where $-1 \le \rho_{ij} \le 1$. Define

$$\xi_{ij} = \frac{S_i}{S_j}, \quad i \ne j.$$

(a) Show that ξ_{ij} satisfies the following stochastic differential equation

$$d\xi_{ij} = (\mu_i - \mu_j + \sigma_j^2 - \rho_{ij}\sigma_i\sigma_j)\xi_{ij}dt + \sigma_{ij}\xi_{ij}dX_{ij},$$

where

$$\sigma_{ij} = \sqrt{\sigma_i^2 - 2\rho_{ij}\sigma_i\sigma_j + \sigma_j^2}$$

and dX_{ij} is a Wiener process defined by

$$dX_{ij} = \frac{\sigma_i dX_i - \sigma_j dX_j}{\sigma_{ij}}.$$

That is, $\xi_{ij} = S_i/S_j$ is also a lognormal variable and its volatility is σ_{ij}.

(b) Let S_0 be a function of t, satisfying

$$dS_0 = \mu_0 S_0 dt.$$

It is clear that if we think S_0 to be a random variable and let its volatility be σ_0, then $\sigma_0 = 0$. Show that if S_i is S_0, then $\sigma_{0j} = \sigma_j$ and $dX_{0j} = -dX_j$; if S_j is S_0, then $\sigma_{i0} = \sigma_i$ and $dX_{i0} = dX_i$.

(c) Define

$$\rho_{ijk} = \frac{\sigma_k^2 - \rho_{ik}\sigma_i\sigma_k - \rho_{jk}\sigma_j\sigma_k + \rho_{ij}\sigma_i\sigma_j}{\sigma_{ik}\sigma_{jk}}.$$

Show

$$\mathrm{E}[dX_{ik}dX_{jk}] = \rho_{ijk}dt,$$

i.e., ρ_{ijk} is the correlation coefficient between the Wiener processes related to ξ_{ik} and ξ_{jk}.

(d) Show that if $S_i = S_0$, then

$$\mathrm{E}[dX_{0k}dX_{jk}] = \rho_{0jk}dt = \frac{\sigma_k - \rho_{jk}\sigma_j}{\sigma_{jk}}dt.$$

19. Suppose that S is the price of a dividend-paying stock and satisfies

$$dS = \mu(S,t)Sdt + \sigma SdX_1, \quad 0 \le S < \infty,$$

where dX_1 is a Wiener process and σ is another random variable. Let the dividend paid during the time period $[t, t+dt]$ be $D(S,t)dt$. Assume that for σ, the stochastic equation

$$d\sigma = p(\sigma,t)dt + q(\sigma,t)dX_2, \quad \sigma_l \le \sigma \le \sigma_u$$

holds. Here, $p(\sigma,t)$ and $q(\sigma,t)$ are differentiable functions. dX_2 is another Wiener process correlated with dX_1, and the correlation coefficient between them is ρdt. For options on such a stock, derive directly the partial differential equation that contains only the unknown market price of risk for the volatility. Here "Directly" means "without using the general PDE for derivatives". (Hint: Take a portfolio in the form $\Pi = \Delta_1 V_1(S, \sigma, t) + \Delta_2 V_2(S, \sigma, t) + S$, where V_1 and V_2 are two different options.)

20. Consider a two-factor convertible bond paying coupons with a rate $k(t)$. For such a convertible bond, derive directly the partial differential equation that contains only the unknown market price of risk for the short-term interest rate. "Directly" means "without using the general PDE for derivatives". (Hint: Take a portfolio in the form $\Pi = \Delta_1 V_1(S, r, t) + \Delta_2 V_2(S, r, t) + S$, where V_1 and V_2 are two different convertible bonds.)

21. *Describe and derive the general equations for derivative securities.

22. (a) Suppose that $V(S, t)$ satisfies

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (a - \lambda b)\frac{\partial V}{\partial S} - rV = 0.$$

Assuming that $V = Se^{-D_0(T-t)}$ is a solution, find $a - \lambda b$.

(b) Let Z_l be a constant and suppose that $V(\xi, t) = Z_l + \xi(1 - Z_l)$ satisfies

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial \xi^2} + (a - \lambda b)\frac{\partial V}{\partial \xi} - rV = 0.$$

Find $a - \lambda b$.

23. Suppose that ξ satisfy the stochastic differential equation:

$$d\xi = a(\xi, t)dt + b(\xi, t)dX,$$

where dX is a Wiener process. Let $S(\xi)$ be the price of a stock which pays dividends $D(S(\xi), t)dt$ during the time period $[t, t + dt]$ and $f(\xi, t)$ represent the value of a derivative security.

(a) Setting a portfolio $\Pi = f(\xi, t) - \Delta S(\xi)$ and using Itô's lemma, derive a PDE for $f(\xi, t)$.

(b) Assume $f(\xi, t) = V(\xi, t)$, $S(\xi) = e^\xi$ and $D(S(\xi), t) = D_0 e^\xi$. Find the PDE for $V(\xi, t)$.

(c) Assume $f(\xi, t) = V(\xi, t)/\xi$, $S(\xi) = 1/\xi$ and $D(S(\xi), t) = D_0/\xi$. Find the PDE for $V(\xi, t)$.

(d) Assume $f(\xi, t) = P_m V(\xi, t)/(1-\xi)$, $S(\xi) = P_m \xi/(1-\xi)$ and $D(S(\xi), t) = D_0 P_m \xi/(1 - \xi)$. Find the PDE for $V(\xi, t)$.

24. As we know, $f = Se^{-D_0(T-t)} - Ke^{-r(T-t)}$ is the value of a forward/futures contract. If we set $F = e^{(r-D_0)(T-t)}S$, then for f we have another expression: $f = e^{-r(T-t)}\left(Se^{(r-D_0)(T-t)} - K\right) = e^{-r(T-t)}(F - K)$. For S we assume $dS = \mu Sdt + \sigma SdX$, so for F we have

$$dF = (\mu - r + D_0)Fdt + \sigma FdX$$

according to Itô's lemma. Consider an option on a forward/futures and let the price of such an option be $V(F, t)$. Derive the PDE for V by using Itô's lemma. (Hint: Set $\Pi = V(F, t) - \Delta f(F, t) = V(F, t) - \Delta e^{-r(T-t)}(F-K)$.)

25. *Suppose that ξ_1 and ξ_2 satisfy the system of stochastic differential equations:
$$d\xi_i = \mu_i(\xi_1, \xi_2, t)dt + \sigma_i(\xi_1, \xi_2, t)dX_i, \quad i = 1, 2,$$
where dX_i are Wiener processes and $\mathrm{E}\,[dX_i dX_j] = \rho_{ij}dt$ with $-1 \le \rho_{ij} \le 1$. Define

$$\begin{cases} Z_1\,(\xi_1) & = Z_{1,l} + \xi_1\,(1 - Z_{1,l})\,, \\[2mm] Z_2\,(\xi_1, \xi_2) = Z_{2,l} + \xi_2\,[Z_1\,(\xi_1) - Z_{2,l}] \\[2mm] \qquad = Z_{2,l} + \xi_2\,[Z_{1,l} + \xi_1\,(1 - Z_{1,l}) - Z_{2,l}]\,. \end{cases}$$

Assume that $Z_1(\xi_1)$ and $Z_2(\xi_1, \xi_2)$ represent prices of two securities. Let $V(\xi_1, \xi_2, t)$ be the value of a derivative security. Setting a portfolio $\Pi = V - \Delta_1 Z_1(\xi_1) - \Delta_2 Z_2(\xi_1, \xi_2)$ and using Itô's lemma, show that $V(\xi_1, \xi_2, t)$ satisfies the following PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_i\sigma_j\rho_{i,j}\frac{\partial^2 V}{\partial\xi_i\partial\xi_j} + \frac{rZ_1}{1 - Z_{1,l}}\frac{\partial V}{\partial\xi_1}$$
$$+ \left[\frac{r\,(Z_2 - Z_1\xi_2)}{Z_1 - Z_{2,l}} - \frac{\sigma_1\sigma_2\rho_{1,2}\,(1 - Z_{1,l})}{Z_1 - Z_{2,l}}\right]\frac{\partial V}{\partial\xi_2} - rV = 0.$$

26. Suppose that ξ_1, ξ_2 and ξ_3 satisfy the system of stochastic differential equations:
$$d\xi_i = \tilde{\mu}_i(\xi_1, \xi_2, \xi_3, t)dt + \tilde{\sigma}_i(\xi_1, \xi_2, \xi_3, t)d\tilde{X}_i, \quad i = 1, 2, 3,$$
where $d\tilde{X}_i$ are the Wiener processes and $\mathrm{E}\left[d\tilde{X}_i d\tilde{X}_j\right] = \tilde{\rho}_{ij}dt$ with $-1 \le \tilde{\rho}_{ij} \le 1$. Define

$$\begin{cases} Z_1\,(\xi_1) & = Z_{1,l} + \xi_1\,(1 - Z_{1,l})\,, \\[2mm] Z_2\,(\xi_1, \xi_2) & = Z_{2,l} + \xi_2\,[Z_1\,(\xi_1) - Z_{2,l}] \\[2mm] & = Z_{2,l} + \xi_2\,[Z_{1,l} + \xi_1\,(1 - Z_{1,l}) - Z_{2,l}]\,, \\[2mm] Z_3\,(\xi_1, \xi_2, \xi_3) = Z_{3,l} + \xi_3\,[Z_2\,(\xi_1, \xi_2) - Z_{3,l}] \\[2mm] \qquad = Z_{3,l} + \xi_3\,\{Z_{2,l} + \xi_2\,[Z_{1,l} + \xi_1\,(1 - Z_{1,l}) - Z_{2,l}] - Z_{3,l}\}\,. \end{cases}$$

Assume that $Z_1(\xi_1)$, $Z_2(\xi_1, \xi_2)$, and $Z_3(\xi_1, \xi_2, \xi_3)$ represent prices of three securities. Let $V(\xi_1, \xi_2, \xi_3, t)$ be the value of a derivative security. Setting a portfolio $\Pi = V - \Delta_1 Z_1(\xi_1) - \Delta_2 Z_2(\xi_1, \xi_2) - \Delta_3 Z_3(\xi_1, \xi_2, \xi_3)$ and using Itô's lemma, derive the PDE that $V(\xi_1, \xi_2, \xi_3, t)$ should satisfy.

27. *Write down the weak-form reversion conditions and the reversion conditions of a stochastic process, describe when the two types of reversion conditions are the same, and give the intuitive meaning of the weak-form reversion conditions.

28. Show the following:
   (a) The Cox–Ingersoll–Ross interest rate model defined on $[0, \infty)$

   $$dr = (\bar{\mu} - \bar{\gamma}r)dt + \sqrt{\alpha r}dX, \quad \bar{\mu}, \bar{\gamma}, \alpha > 0$$

   can be converted into the model

   $$d\xi = \left[\frac{(1-\xi)^2}{P_m}\left(\bar{\mu} - \frac{\bar{\gamma}P_m\xi}{1-\xi}\right) - \frac{\alpha\xi(1-\xi)^2}{P_m}\right]dt + \frac{\sqrt{\alpha}\xi^{1/2}(1-\xi)^{3/2}}{P_m^{1/2}}dX$$

   by introducing a new random variable $\xi = \dfrac{r}{r + P_m}$, where P_m is a positive constant.
   (b) ξ is defined on $[0, 1]$. For the new model, the reversion conditions at $\xi = 0$ hold if and only if $\bar{\mu} - \alpha/2 \geq 0$ and the reversion conditions at $\xi = 1$ always hold.
29. *Consider the following degenerate parabolic problem:

   $$\begin{cases} \dfrac{\partial u}{\partial \tau} = f_1(x, \tau)\dfrac{\partial^2 u}{\partial x^2} + f_2(x, \tau)\dfrac{\partial u}{\partial x} + f_3(x, \tau)u + g(x, \tau), \\[2mm] \qquad\qquad\qquad 0 \leq x \leq 1, \quad 0 \leq \tau \leq T, \\[2mm] u(x, 0) = f(x), \qquad\qquad 0 \leq x \leq 1, \\[2mm] u(0, \tau) \begin{cases} \text{needs not to be given if } f_2(0, \tau) - \dfrac{\partial f_1(0, \tau)}{\partial x} \geq 0, \\[3mm] = f_l(\tau) \text{ if } f_2(0, \tau) - \dfrac{\partial f_1(0, \tau)}{\partial x} < 0, \end{cases} \\[6mm] u(1, \tau) \begin{cases} \text{needs not to be given if } f_2(1, \tau) - \dfrac{\partial f_1(1, \tau)}{\partial x} \leq 0, \\[3mm] = f_u(\tau) \text{ if } f_2(1, \tau) - \dfrac{\partial f_1(1, \tau)}{\partial x} > 0, \end{cases} \end{cases}$$

   where $f_1(0, \tau) = f_1(1, \tau) = 0$ and $f_1(x, \tau) \geq 0$. Suppose that its solution exists and is bounded and that there exist a constant c_1 and two bounded functions $c_2(\tau)$ and $c_3(\tau)$ such that

   $$1 + \max_{0 \leq x \leq 1, \, 0 \leq \tau \leq T}\left(\left|\frac{\partial^2 f_1(x, \tau)}{\partial x^2} - \frac{\partial f_2(x, \tau)}{\partial x} + 2f_3(x, \tau)\right|\right) \leq c_1,$$

   $$-\min\left(0, f_2(0, \tau) - \frac{\partial f_1(0, \tau)}{\partial x}\right) \leq c_2(\tau),$$

   and

   $$\max\left(0, f_2(1, \tau) - \frac{\partial f_1(1, \tau)}{\partial x}\right) \leq c_3(\tau).$$

   Show that in this case, its solution is unique and stable with respect to the initial value $f(x)$, inhomogeneous term $g(x, \tau)$, and the boundary values $f_l(\tau), f_u(\tau)$ if there are any.

30. Suppose $f_1(r, t) \geq 0$ and $f_1(r_l, t) = \dfrac{\partial f_1(r_l, t)}{\partial r} = f_1(r_u, t) = \dfrac{\partial f_1(r_u, t)}{\partial r} = 0$, and $f_2(r_l, t) < 0$, $f_2(r_u, t) > 0$. Explain why problem **A**

$$
\begin{cases}
\dfrac{\partial V}{\partial t} = f_1 \dfrac{\partial^2 V}{\partial r^2} + f_2 \dfrac{\partial V}{\partial r} + f_3 V, & r_l \leq r \leq r_u, \quad 0 \leq t, \\[2mm]
V(r, 0) = V_0(r), & r_l \leq r \leq r_u, \\[2mm]
V(r_l, t) = f_l(t), & 0 \leq t, \\[2mm]
V(r_u, t) = f_u(t), & 0 \leq t
\end{cases}
$$

and problem **B**

$$
\begin{cases}
\dfrac{\partial V}{\partial t} = -f_1 \dfrac{\partial^2 V}{\partial r^2} + f_2 \dfrac{\partial V}{\partial r} + f_3 V, & r_l \leq r \leq r_u, \quad t \leq T, \\[2mm]
V(r, T) = V_T(r), & r_l \leq r \leq r_u
\end{cases}
$$

have unique solutions.

31. (a) Consider a linear hyperbolic partial differential equation

$$
\frac{\partial u}{\partial t} + f(x, t) \frac{\partial u}{\partial x} = 0.
$$

Let $x = x(t)$ be the curve C which is determined by the following ordinary differential equation

$$
\frac{dx(t)}{dt} = f(x, t)
$$

with $x(0) = \xi$. Show that u is a constant along the curve C:

$$
u\left(x(t^*), t^*\right) = u\left(x(t^{**}), t^{**}\right),
$$

where t^* and t^{**} are any two times, and that if

$$
f(x, t) = F(x, t)\delta(t - t_i),
$$

where $\delta(t - t_i)$ is the Dirac delta function, then

$$
u\left(x(t_i^-), t_i^-\right) = u\left(x(t_i^-) + F\left(x(t_i^-), t_i^-\right), t_i^+\right),
$$

where t_i^- and t_i^+ denote the time just before and after t_i, respectively.

(b) Derive the jump condition for options on stocks with discrete dividends and explain its financial meaning.

(c) Find the corresponding jump condition for the following PDE

$$
\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 \eta^2 \frac{\partial^2 W}{\partial \eta^2} + \left[(D_0 - r)\eta + \frac{1}{K} \sum_{i=1}^{K} \delta(t - t_i) \right] \frac{\partial W}{\partial \eta} - D_0 W = 0.
$$

(d) Find the corresponding jump condition for the following PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0) S \frac{\partial V}{\partial S}$$

$$+ \sum_{i=1}^{K} \left[\max\left(S\left(t\right), H\left(t^-\right)\right) - H\left(t^-\right)\right] \delta\left(t - t_i\right) \frac{\partial V}{\partial H} - rV = 0.$$

32. Show that the expression

$$W(\eta, t) = \begin{cases} e^{-r(T-t)}\eta, & t_2 < t \le T, \\[2mm] e^{-r(T-t)}\eta + \frac{1}{2}e^{-r(T-t_2)-D_0(t_2-t)}, & t_1 < t \le t_2, \\[2mm] e^{-r(T-t)}\eta + \frac{1}{2}e^{-r(T-t_1)-D_0(t_1-t)} \\[2mm] \qquad + \frac{1}{2}e^{-r(T-t_2)-D_0(t_2-t)}, & 0 < t \le t_1, \end{cases}$$

is the solution of the problem:

$$\begin{cases} \dfrac{\partial W}{\partial t} + \dfrac{1}{2}\sigma^2\eta^2 \dfrac{\partial^2 W}{\partial \eta^2} + \left[(D_0 - r)\eta + \dfrac{1}{2}\sum_{i=1}^{2}\delta\left(t - t_i\right)\right]\dfrac{\partial W}{\partial \eta} \\[2mm] \quad - D_0 W = 0, & 0 \le \eta, \quad 0 \le t \le T, \\[2mm] W(\eta, T) = \eta, & 0 \le \eta. \end{cases}$$

(This problem is related to discretely sampled average price call options.)

33. Suppose that $V(S,t)$ is the solution of the following PDE:

$$\frac{\partial V}{\partial t} + a(S,t)\frac{\partial^2 V}{\partial S^2} + b(S,t)\frac{\partial V}{\partial S} + c(S,t)V + d(S,t)\delta(t - t_i) = 0.$$

Find the relation between $V(S, t_i^+)$ and $V(S, t_i^-)$, and describe the financial meaning of this relation.

34. Suppose $V(S,t)$ is the solution of the problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r - D_0)S\dfrac{\partial V}{\partial S} - rV = 0, & 0 \le S, \quad t \le T, \\[2mm] V(S,T) = V_T(S), & 0 \le S. \end{cases}$$

Let $x = \dfrac{\sqrt{2}}{\sigma}\left[\ln S + (r - D_0 - \sigma^2/2)(T - t)\right]$, $\tau = T - t$ and $V(S,t) = e^{-r(T-t)}u(x,\tau)$. Show that $u(x,\tau)$ is the solution of the problem

$$\begin{cases} \dfrac{\partial u}{\partial \tau} = \dfrac{\partial^2 u}{\partial x^2}, & -\infty < x < \infty, \quad 0 \le \tau, \\[2mm] u(x,0) = V_T\left(e^{\sigma x/\sqrt{2}}\right), & -\infty < x < \infty. \end{cases}$$

35. *Suppose $V(S, t)$ is the solution of the problem

$$
\begin{cases}
\dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r - D_0)S\dfrac{\partial V}{\partial S} - rV = 0, & 0 \le S, \quad t \le T, \\
V(S, T) = V_T(S), & 0 \le S.
\end{cases}
$$

Let $x = \ln S + (r - D_0 - \sigma^2/2)(T - t)$, $\bar{\tau} = \sigma^2(T - t)/2$ and $V(S, t) = e^{-r(T-t)}u(x, \bar{\tau})$. Show that $u(x, \bar{\tau})$ is the solution of the problem

$$
\begin{cases}
\dfrac{\partial u}{\partial \bar{\tau}} = \dfrac{\partial^2 u}{\partial x^2}, & -\infty < x < \infty, \quad 0 \le \bar{\tau}, \\
u(x, 0) = V_T(e^x), & -\infty < x < \infty.
\end{cases}
$$

36. Consider problem **A**:

$$
\begin{cases}
\dfrac{\partial V}{\partial t} + a(t)S^2 \dfrac{\partial^2 V}{\partial S^2} + b(t)S\dfrac{\partial V}{\partial S} - r(t)V = 0, & 0 \le S, \quad t \le T, \\
V(S, T) = V_T(S), & 0 \le S
\end{cases}
$$

and define

$$
\alpha(t) = \int_t^T a(s)ds, \quad \beta(t) = \int_t^T b(s)ds,
$$

and

$$
\gamma(t) = \int_t^T r(s)ds.
$$

Assume that for this problem the uniqueness of solution is proved. Show that

(a) Let $x = \ln S + \beta(t) - \alpha(t)$, $\bar{\tau} = \alpha(t)$ and $V(S, t) = e^{-\gamma(t)}u(x, \bar{\tau})$, then $u(x, \bar{\tau})$ is the solution of the problem:

$$
\begin{cases}
\dfrac{\partial u}{\partial \bar{\tau}} = \dfrac{\partial^2 u}{\partial x^2}, & -\infty < x < \infty, \quad 0 \le \bar{\tau}, \\
u(x, 0) = V_T(e^x), & -\infty < x < \infty.
\end{cases}
$$

(b) $V(S, t)$ must be in the form

$$
V(S, t) = e^{-\gamma(t)}u(\ln S + \beta(t) - \alpha(t), \alpha(t))
$$

or

$$
V(S, t) = e^{-\gamma(t)}\bar{u}(Se^{\beta(t)}, \alpha(t)).
$$

(c) If

$$
V(S, t) = e^{-r(T-t)}\bar{u}(Se^{b(T-t)}, a(T - t))
$$

is the solution of problem **A** with constant coefficients, then

$$
V(S, t) = e^{-\gamma(t)}\bar{\bar{u}}(Se^{\beta(t)}, \alpha(t))
$$

is the solution of problem **A** with time-dependent coefficients.

37. *Find an integral expression of the solution of the following problem

$$\begin{cases} \dfrac{\partial u}{\partial \bar{\tau}} = \dfrac{\partial^2 u}{\partial x^2}, & -\infty < x < \infty, \quad 0 \le \bar{\tau}, \\ u(x,0) = u_0(x), & -\infty < x < \infty. \end{cases}$$

38. Using the results given in Problems 34 and 37, show that the solution of the following problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r - D_0)S\dfrac{\partial V}{\partial S} - rV = 0, & 0 \le S, \quad t \le T, \\ V(S,T) = V_T(S), & 0 \le S \end{cases}$$

is

$$V(S,t) = e^{-r(T-t)} \int_0^\infty V_T(S')G(S',T;S,t)dS',$$

where

$$G(S',T;S,t)$$
$$= \dfrac{1}{\sigma\sqrt{2\pi(T-t)}S'}e^{-\left[\ln S'-\ln S-(r-D_0-\sigma^2/2)(T-t)\right]^2/2\sigma^2(T-t)}.$$

39. Suppose that S is a random variable which is defined on $[0, \infty)$ and whose probability density function is

$$G(S) = \dfrac{1}{\sqrt{2\pi}bS}e^{-\left[\ln(S/a)+b^2/2\right]^2/2b^2},$$

a and b being positive numbers. Show that
(a)

$$\int_0^c G(S)dS = N\left(\dfrac{\ln(c/a)+b^2/2}{b}\right);$$

(b)

$$\int_0^c SG(S)dS = aN\left(\dfrac{\ln(c/a)-b^2/2}{b}\right);$$

(c) for any real number n

$$\int_0^c S^n G(S)dS = a^n e^{(n^2-n)b^2/2}N\left(\dfrac{\ln(c/a)+b^2/2}{b} - nb\right);$$

(d) for any real number n

$$E[S^n] = a^n e^{(n^2-n)b^2/2};$$

(e) for any real number n

$$\int_c^\infty S^n G(S) dS = a^n e^{(n^2-n)b^2/2} N\left(-\frac{\ln(c/a) + b^2/2}{b} + nb\right);$$

(f)

$$\int_0^c \ln S\, G(S)\, dS$$
$$= \frac{-b}{\sqrt{2\pi}} e^{-[\ln(c/a)+b^2/2]^2/2b^2} + \left(\ln a - b^2/2\right) N\left(\frac{\ln(c/a) + b^2/2}{b}\right);$$

(g)

$$\int_c^\infty \ln S\, G(S)\, dS$$
$$= \frac{b}{\sqrt{2\pi}} e^{-[\ln(c/a)+b^2/2]^2/2b^2} + \left(\ln a - b^2/2\right) N\left(-\frac{\ln(c/a) + b^2/2}{b}\right),$$

where

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-\xi^2/2} d\xi.$$

40. (a) Define $S_0^* = E e^{-r(T-t)}$ and $S_1^* = S e^{-D_0(T-t)}$. Show that there exists a function $f(x_1, x_2, t; \sigma)$ such that the following is true:

$$e^{-r(T-t)} \int_0^E \max(E, S') G(S', T; S, t) dS' = f(S_0^*, S_1^*, t; \sigma)$$

and

$$e^{-r(T-t)} \int_E^\infty \max(E, S') G(S', T; S, t) dS' = f(S_1^*, S_0^*, t; \sigma),$$

where

$$G(S', T; S, t)$$
$$= \frac{1}{\sigma\sqrt{2\pi(T-t)}S'} e^{-\{\ln S' - [\ln S + (r - D_0 - \sigma^2/2)(T-t)]\}^2/2\sigma^2(T-t)}.$$

(b) Let $V(S, t)$ be the solution of the problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r - D_0) S \dfrac{\partial V}{\partial S} - rV = 0, & 0 \le S, \quad t \le T, \\ V(S, T) = \max(E, S), & 0 \le S. \end{cases}$$

Based on the results in part (a), show that in the expression for $V(S, t)$, the positions of S_0^* and S_1^* are symmetric, i.e., exchanging S_0^* and S_1^* in the expression for $V(S, t)$ will generate the same expression.

41. As we know,

$$c(S,t) = e^{-r(T-t)} \int_0^\infty \max(S' - E, 0) G(S', T; S, t) dS'$$

and

$$p(S,t) = e^{-r(T-t)} \int_0^\infty \max(E - S', 0) G(S', T; S, t) dS',$$

where

$$G(S', T; S, t) = \frac{1}{\sigma \sqrt{2\pi(T-t)}S'} e^{-\left[\ln(S'/S) - (r - D_0 - \sigma^2/2)(T-t)\right]^2 / 2\sigma^2(T-t)}.$$

(a) Using the expression above for $c(S,t)$, show that if $D_0 = 0$, then $c(S,t) \geq \max(S - E, 0)$, which means that for this case the value of an American call option is the same as the value of a European call option.

(b) Using the expression above for $p(S,t)$, show that if $r = 0$, then $p(S,t) \geq \max(E - S, 0)$, which means that for this case the value of an American put option is the same as the value of a European put option.

42. Consider the problem

$$\begin{cases} \dfrac{\partial B_c}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 B_c}{\partial S^2} + (r - D_0)S\dfrac{\partial B_c}{\partial S} - rB_c = 0, \\[2mm] \qquad\qquad\qquad\qquad\qquad\qquad 0 \leq S, \quad 0 \leq t \leq T, \\[2mm] B_c(S, T) = \max(Z, nS), \qquad\qquad 0 \leq S, \end{cases}$$

where σ, r, D_0, Z, and n are constants. Show that if $D_0 \leq 0$, then

$$B_c(S,t) \geq \max\left(Ze^{-r(T-t)}, nS\right) \quad \text{for} \quad 0 \leq t \leq T.$$

43. Find the solution in the form of $V(S,t) = V(S)$ for the Black–Scholes equation.

44. Show by substitution that
    (a) $V(S,t) = Se^{-D_0(T-t)}$,
    (b) $V(S,t) = Ee^{-r(T-t)}$
    are solutions of the Black–Scholes equation. What do these solutions represent?

45. *Using the results given in Problems 38 and 39, derive the Black-Scholes formula for a European put option.

46. As we know, the price of a call option on a forward/futures is the solution of the following problem:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 F^2 \dfrac{\partial^2 V}{\partial F^2} - rV = 0, & 0 \le F, \quad t \le T, \\ V(F,T) = \max(F - K, 0), & 0 \le F. \end{cases}$$

Using the general solution of the Black–Scholes equation and the results given in Problem 39, find a closed-form solution for this case.

47. Consider the following problem:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r - D_0)\,S\dfrac{\partial V}{\partial S} - rV + k(t)Z = 0, \\ \hspace{6cm} 0 \le S, \quad 0 \le t \le T, \\ V(S,T) = \max(Z, nS), \hspace{2.5cm} 0 \le S, \end{cases}$$

where σ, r, D_0, Z, n are constants and $k(t)$ is a nonnegative function. Using the general solution of the Black–Scholes equation and the results given in Problem 39, find a closed-form solution for this case. (If $D_0 = 0$, this solution gives the price of a one-factor convertible bond paying coupon.) (Hint: Define $\overline{V}(S,t) = V(S,t) - b_0(t)$, where $b_0(t)$ is the solution of the following problem:

$$\begin{cases} \dfrac{db_0}{dt} - rb_0 + k(t)Z = 0, & 0 \le t \le T, \\ b_0(T) = 0. \end{cases}$$

Find $b_0(t)$ and a closed-form solution of $\overline{V}(S,t)$ first, then putting them together, we have $V(S,t)$.)

48. Consider the following problem

$$\begin{cases} \dfrac{\partial c_b}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 c_b}{\partial S^2} + (r - D_0)S\dfrac{\partial c_b}{\partial S} - rc_b = 0, \\ \hspace{3cm} 0 \le S < \infty, \ \ 0 < t < T, \\ c_b(S,T) = \begin{cases} 0, & \text{if } 0 \le S < S^{**}, \\ f(S), & \text{if } S^{**} \le S < S^*, \\ S - E, & \text{if } S^* \le S < \infty, \end{cases} \end{cases}$$

where

$$f(S) = a_0 + a_1 S + \cdots + a_J S^J.$$

Show that it has a solution in the following closed form:

$$c_b(S,t) = \sum_{n=0}^{J} \left\{ a_n S^n e^{[-r+n(r-D_0)+(n-1)n\sigma^2/2](T-t)} \right.$$

$$\left. \times \left[N\left(d^* - n\sigma\sqrt{T-t} \right) - N\left(d^{**} - n\sigma\sqrt{T-t} \right) \right] \right\}$$

$$+ S e^{-D_0(T-t)} \left[1 - N\left(d^* - \sigma\sqrt{T-t} \right) \right] - E e^{-r(T-t)} [1 - N(d^*)],$$

where

$$d^* = \left[\ln(S^*/S) - \left(r - D_0 - \frac{1}{2}\sigma^2 \right)(T-t) \right] \Big/ \left(\sigma\sqrt{T-t} \right),$$

$$d^{**} = \left[\ln(S^{**}/S) - \left(r - D_0 - \frac{1}{2}\sigma^2 \right)(T-t) \right] \Big/ \left(\sigma\sqrt{T-t} \right).$$

49. Using the Black–Scholes formula for a put option and the result in Problem 36 part (c), find the formula for the price of a put option with time-dependent parameters.

50. Consider a European call option on a non-dividend-paying stock. Use the Black–Scholes formula to find the option price when the stock price is $63, the strike price is $60, the risk-free interest rate is 5 % per annum, the volatility is 35 % per annum, and the time to maturity is six months.

51. Consider a European put option on a dividend-paying stock. Use the Black–Scholes formula to find the option price when the stock price is $55, the strike price is $60, the risk-free interest rate is 5 % per annum, the volatility is 35 % per annum, the dividend yield is 3 % per annum, and the time to maturity is six months.

52. Consider a European call option on a non-dividend-paying stock. The option price is $4.5, the stock price is $86, the exercise price is $92, the risk-free interest rate is 5 % per annum, and the time to maturity is three months. Use the Black–Scholes formula for a call option to find what the corresponding volatility should be. (This volatility is usually referred to as the implied volatility associated with the given option price.)

53. *Show
$$S e^{-D_0(T-t)-d_1^2/2} = E e^{-r(T-t)-d_2^2/2},$$

where

$$d_1 = \left[\ln \frac{S e^{(r-D_0)(T-t)}}{E} + \frac{1}{2}\sigma^2(T-t) \right] \Big/ \left(\sigma\sqrt{T-t} \right),$$

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

54. Verify that the Black–Scholes formula for a put option is the solution of the following problem:

$$\begin{cases} \dfrac{\partial p}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 p}{\partial S^2} + (r - D_0)S\dfrac{\partial p}{\partial S} - rp = 0, & 0 \le S, \quad 0 \le t \le T, \\ p(S,T) = \max(E - S,\, 0), & 0 \le S. \end{cases}$$

(Hint: Show the following identity $Ee^{-r(T-t)-d_2^2/2} = Se^{-D_0(T-t)-d_1^2/2}$ first.)

55. Find the expressions of $\lim_{S\to 0} c(S,t)$ and $\lim_{S\to 0} p(S,t)$.

56. Derive the expressions for derivatives of $c(S,t)$ and $p(S,t)$ with respect to r, D_0, σ, E, and show that $\dfrac{\partial c}{\partial r}, \dfrac{\partial c}{\partial \sigma}, \dfrac{\partial p}{\partial D_0}, \dfrac{\partial p}{\partial \sigma}, \dfrac{\partial p}{\partial E}$ are nonnegative, and others are nonpositive.

57. Let $\bar{c}(\xi,\tau) = c(S,t)/(S + P_m)$ and $\bar{p}(\xi,\tau) = p(S,t)/(S + P_m)$, where $\xi = S/(S + P_m)$ and $\tau = T - t$. Derive the expressions of $\bar{c}(\xi,\tau)$ and $\bar{p}(\xi,\tau)$ and find the limits of $\bar{c}(\xi,\tau)$ and $\bar{p}(\xi,\tau)$ as ξ tends to 0 and 1. Also write down the formulae for the case $P_m = E$.

58. Suppose that S is the price of a stock,

$$dS = \mu S\,dt + \sigma S\,dX,$$

and $V(S,t)$ is the value of an option on the stock. Define $S_0^* = Ee^{-r(T-t)}$,
$$S_1^* = Se^{-D_0(T-t)}, \quad \xi_{10} = \frac{S_1^*}{S_0^*} = \frac{Se^{(r-D_0)(T-t)}}{E}, \quad \xi_{01} = \frac{S_0^*}{S_1^*} = \frac{E}{Se^{(r-D_0)(T-t)}},$$
$V_0(\xi_{10},t) = V(S(\xi_{10},t),t)/S_0^*(t)$, and $V_1(\xi_{01},t) = V(S(\xi_{01},t),t)/S_1^*(\xi_{01},t)$, where E and T are constants, r is the interest rate, and D_0 is the dividend yield of the stock. Assume that we already know that

$$d\xi_{10} = (\mu - r + D_0)\xi_{10}\,dt + \sigma\xi_{10}\,dX.$$

(a) By setting $\Pi = V - \Delta S = S_0^*(t)V_0(\xi_{10},t) - \Delta E e^{-(r-D_0)(T-t)}\xi_{10}$ and using Itô's lemma, show that the PDE for $V_0(\xi_{10},t)$ is

$$\frac{\partial V_0}{\partial t} + \frac{1}{2}\sigma^2\xi_{10}^2\frac{\partial^2 V_0}{\partial\xi_{10}^2} = 0.$$

(b) From the PDE for $V_0(\xi_{10},t)$ obtained in part (a), show that the PDE for $V_1(\xi_{01},t)$ is

$$\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2\xi_{01}^2\frac{\partial^2 V_1}{\partial\xi_{01}^2} = 0.$$

(c) Consider the problem:

$$\begin{cases} \dfrac{\partial W}{\partial t} + \frac{1}{2}\sigma^2\xi^2\dfrac{\partial^2 W}{\partial\xi^2} = 0, & 0 \le \xi, \quad t \le T, \\ W(\xi,T) = \max(\xi,1), & 0 \le \xi. \end{cases}$$

As we know, the solution of this problem is

$$W(\xi, t) = \int_0^\infty \max(\xi', 1) G(\xi', T; \xi, t) d\xi'$$

$$= \int_0^1 G(\xi', T; \xi, t) d\xi' + \int_1^\infty \xi' G(\xi', T; \xi, t) d\xi',$$

where

$$G(\xi', T; \xi, t) = \frac{1}{\sqrt{2\pi} b \xi'} e^{-[\ln(\xi'/\xi) + b^2/2]^2 / 2b^2}, \quad b \text{ being } \sigma\sqrt{T-t}.$$

Let $V(S, t)$ be the price of the option with payoff $\max(S, E)$. In this case $V_0(\xi_{10}, T) = \max(S, E)/E = \max(\xi_{10}, 1)$ and $V_1(\xi_{01}, T) = \max(S, E)/S = \max(\xi_{01}, 1)$. Thus, for $V(S, t)$ we have two expressions:

$$V(S, t) = S_0^* W(\xi_{10}, t)$$

$$= S_0^* \int_0^1 G(\xi'_{10}, T; \xi_{10}, t) d\xi'_{10} + S_0^* \int_1^\infty \xi'_{10} G(\xi'_{10}, T; \xi_{10}, t) d\xi'_{10},$$

and

$$V(S, t) = S_1^* W(\xi_{01}, t)$$

$$= S_1^* \int_0^1 G(\xi'_{01}, T; \xi_{01}, t) d\xi'_{01} + S_1^* \int_1^\infty \xi'_{01} G(\xi'_{01}, T; \xi_{01}, t) d\xi'_{01}.$$

Because at $t = T$ both $\xi'_{10} < 1$ and $\xi'_{01} > 1$ correspond to $S' < E$, both the first term in the first expression and the second term in the second expression represent the contribution which the function $\max(S', E)$ as $S' < E$ makes to the value $V(S, t)$. Consequently, the two terms should be equal. Similarly the second term in the first expression should be equal to the first term in the second expression. Verify this conclusion by direct calculation.

59. *Suppose that $c(S, t)$ and $p(S, t)$ are the prices of European call and put options with the same parameters, respectively. Show the put–call parity

$$c(S, t) - p(S, t) = Se^{-D_0(T-t)} - Ee^{-r(T-t)}$$

without using the Black–Scholes formulae.

60. Consider a European option on a non-dividend-paying stock. The stock price is \$37, the exercise price is \$34, the risk-free interest rate is 5 % per annum, the volatility is 30 % per annum, and the time to maturity is six months. Find the call and put option prices by using the Black–Scholes formulae and verify that the put–call parity holds.

61. By using the put–call parity relation of European options

$$c(S,t) - p(S,t) = Se^{-D_0(T-t)} - Ee^{-r(T-t)},$$

show that the following relations hold:

$$\frac{\partial p}{\partial S} = \frac{\partial c}{\partial S} - e^{-D_0(T-t)}, \qquad \frac{\partial^2 p}{\partial S^2} = \frac{\partial^2 c}{\partial S^2}$$

and

$$\frac{\partial^2 p}{\partial S \partial \sigma} = \frac{\partial^2 c}{\partial S \partial \sigma}, \qquad \frac{\partial p}{\partial \sigma} = \frac{\partial c}{\partial \sigma}, \qquad \frac{\partial^2 p}{\partial \sigma^2} = \frac{\partial^2 c}{\partial \sigma^2}.$$