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Derivative Securities and Difference Methods

Second Edition

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Derivative Securities and Difference Methods

Second Edition

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To our families

Preface to the Second Edition

During the last 9 years, this textbook has been used for financial mathematics courses in M.S. or Ph.D. degree programs. During teaching, some description has been improved, some new material has been added, and many new exercise problems have been provided. Based on these materials accumulated, many changes are made in the second edition. Major changes include:

1. The original chapter of Basic Options has been divided into two chapters: European Style Derivatives and American Style Derivatives. Thus the original Chaps. 3 and 4 become Chaps. 4 and 5. In the chapter of European Style Derivatives, immediately following the derivation of the Black–Scholes equation, the description of deriving partial differential equations for general derivative securities is given, including derivatives depending on random variables which do not represent prices or are not prices but known functions of prices. In that chapter, for two-dimensional case, the solution-uniqueness of final value problem of degenerate parabolic equations is proved when the reversion conditions are satisfied. Thus a clear picture for the formulation of such problems is provided.
2. The original chapter of Basic Numerical Methods is split into two chapters: Basic Numerical Methods and Finite-Difference Methods and the original Chaps. 6–8 becomes Chaps. 8–10. In the chapter of Finite-Difference Methods, strict stability analysis for a popular two-dimensional scheme for derivative securities is added. The proof of solution-uniqueness and the strict stability analysis make this book also suitable to the Ph.D. students who wants to work on numerical methods on partial differential equations for derivative securities as the textbook of main courses.
3. Besides the methods of pricing a variety of derivative securities in the first edition, for two cases, the details of the methods are added into the second edition. Give the details of pricing Asian options in Chap. 8 because of the importance of the Asian options in practice. The material provided can let readers know how to price such options with discrete samplings, for example, daily, weekly, or monthly, and write a code for such a purpose. A very good approximate expression of the cumulative distribution for bivariate standard normal distribution has been added in

the second edition, and in the projects of Chap. 6 readers are asked to write a C++ function on this expression. Using this function, the price of options on two assets with the same exercise prices can be calculated. In Chap. 7, a two-dimensional finite-difference scheme is added, which is easy to perform and can be used to calculate the prices of options on two assets. Thus the tools for pricing options on two assets have been provided.
4. Number of exercise problems increases to more than 250 in this edition from 170 in the first edition. These problems are very helpful for readers to understand the material in this book.
5. This book can be used for financial mathematics courses with different levels. In this edition, for those sections/subsections suitable only to a course with advanced level, we put † in the front of the section/subsection name, and for those sections/subsections suitable only to a course with Ph.D. degree level, we put ‡ in the front of the section/subsection name. At the beginning of most problem sections, we give a table[1] showing which problem is related to which section/subsection. For example, if Problems 1–4 of Chap. 2 are related to Sect. 2.1.1[2], then in the table at the beginning of Problem Section in Chap. 2, 1–4 and Sect. 2.1.1 will appear in a column of Problems and in the closely right-hand column of Subsections, respectively, and they are on the same line. We hope that these might give the user of this book some help.

Besides these major changes, small changes are done throughout the entire book.

As it has been pointed out in Preface to the first edition, this book can be used as a textbook for two courses as a sequence. In the first course, the subject "Partial Differential Equations in Finance" is taught by using the materials in Part I. The second one is a course on "Numerical Methods for Derivative Securities" based on Part II of this book. The following materials are basic and more important:

- Sects. 1.1–1.2;
- Sects. 2.1–2.3, 2.5–2.6;
- Sects. 3.1–3.2, 3.3.1–3.3.3;
- Sects. 4.1, 4.2.1, 4.3.1–4.3.4, 4.4.1–4.4.2;
- Sects. 5.1–5.2, 5.6–5.7;
- Sects. 6.1.1–6.1.2, 6.2.1–6.2.2;
- Sects. 7.1, 7.2.1, 7.3;
- Sects. 8.1.1–8.1.3, 8.1.5, 8.2.1, 8.2.3;
- Sects. 9.1.1, 9.2.1, 9.2.3, 9.3.2;
- Sect. 10.3.

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[1] Here a table is referred to Table 2.1 of Chapter 2, Table 3.1 of Chapter 3, ..., or Table 10.6 of Chapter 10.

[2] In this book, we adopt the following notation: Sect. x.x is the abbreviation of Section x.x and Sect. x.x.x is the abbreviation of Subsection x.x.x.

These materials can be taught in one semester. Thus, if only one course is offered, this book can also be used.

During the procedure of revising the book, we received helps from many persons. Here we would like to express our great thanks to them. Special thanks should go to graduate students Qiang Shi, who provides the expression of standard deviation of the interest model given in this book, and Guanghua Gao, who computes the results of options on two assets. We also would like to express our thanks to Achi Dosanjh, the editor of this book, whose many suggestions have greatly improved the quality of the book.

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Preface to the First Edition

In the past three decades, great progress has been made in the theory and practice of financial derivative securities. Now huge volumes of financial derivative securities are traded on the market every day. This causes a big demand for experts who know how to price financial derivative securities. This book is designed as a textbook for graduate students in a mathematical finance program and as a reference book for the people who already work in this field. We hope that a person who has studied this book and who knows how to write codes for engineering computation can handle the business of providing efficient derivative-pricing codes. In order for this book to be used by various people, the prerequisites to study the majority of this book are multivariable calculus, linear algebra, and basic probability and statistics.

In this book, the determination of the prices of financial derivative securities is reduced to solving partial differential equation problems, i.e., a PDE approach is adopted in order to find the price of a derivative security. This book is divided into two parts. In the first part, we discuss how to establish the corresponding partial differential equations and find the final and necessary boundary conditions for a specific derivative product. If possible, we derive its explicit solution and describe some properties of the solution. In many cases, no explicit solution has been found so far. In these situations, we have to use numerical methods to determine the value of financial derivative securities. Therefore, the second part is devoted to numerical methods for derivative securities. There are two styles of financial derivatives: European and American. The numerical methods for both styles of derivatives are described. The main numerical method discussed is the finite-difference method. The binomial/trinomial method is also introduced as a version of an explicit finite-difference method, and the pseudo-spectral method is discussed as a high-order finite-difference method. In this part, numerical methods for determining the market price of risk are also studied as numerical methods for inverse problems. From the viewpoint of partial differential equations, solving an inverse problem means to determine a function as a variable coefficient in a partial differential equation, according to certain values of some solutions.

During the past few years, a great number of books on financial derivative securities have been published. For example: Duffie [28], Baxter and Rennie [6], Hull [43], James and Webber [47], Jarrow [48], Kwok [54], Lamberton and Lapeyre [55], Lyuu [59], Musiela and Rutkowski [64], Pelsser [66], Tavella and Randall [80], Wilmott, Dewynne, and Howison [84], Wilmott [82], Wilmott [83], and Yan [87] have published books on this subject. However, each book has its own features and gives emphasis to some aspects of this subject. Relatively speaking, this book is similar to the books by Wilmott, Dewynne, and Howison [84], Kwok [54], and Tavella and Randall [80] because all of them deal with the partial differential equation problems in finance and their numerical methods. However, this book pays more attention to numerical methods. At least the following features of this book are unique:

1. The slopes of the payoff functions for many derivative securities are discontinuous, and American-style derivative securities usually have free boundaries. These features downgrade the efficiency of numerical methods. In this book, we will discuss how to make computation more efficient even though the solutions have such types of weak singularities.
2. Many derivative security problems are defined on an infinite domain. When a numerical method is used to solve such a problem, usually a large finite domain is taken, and some artificial boundary conditions are adopted for implicit methods. This book will discuss how to convert such a problem into a problem defined on a finite domain and without requiring any artificial boundary conditions. Also, conditions guaranteeing that a random variable is defined on a finite domain are derived. When these conditions hold, any derivative security problems will be defined on a finite domain and do not need any artificial boundary conditions in order to solve them numerically.
3. A numerical method for an inverse problem in finance, for determination of the market price of risk on the spot interest rate, has been provided. As soon as having the market price of risk on the spot interest rate, we can use partial differential equations for evaluating interest rate derivatives in practice.
4. A three-factor interest rate model has been provided. All the parameters in the model and the final values of derivatives are determined from the market data. Because of this, it can be expected that the model reflects the real market. The evaluation of interest rate derivatives is reduced to solving a final value problem of a three-dimensional partial differential equation on a finite domain. Because the correctness of the formulation of the problem is proven, the numerical method for such a problem can be designed without difficulties.

The first four chapters are related to partial differential equations in finance. Chapter 1 is an introduction, where basic features of several assets and financial derivative securities are briefly described. Chapter 2 discusses basic options. In this chapter, Itô's lemma and the Black–Scholes equation are introduced, along with the derivation of the Black–Scholes formulae. These topics are followed by a discussion on American options as both linear complementarity and free-boundary problems. Also in Chap. 2, the put–call parity relation for European options as well as the put–call symmetry relations for American options are introduced. Finally, the general equations for derivative securities are derived.

In Chap. 3, exotic options such as barrier, Asian, lookback, and multi-asset are introduced. The equations, final conditions, and necessary boundary conditions for these options are provided. In this chapter, we examine a few cases in which a two-dimensional problem may be reduced to a one-dimensional problem. Explicit solutions for some of these options are provided whenever possible. Also, the formulations as free-boundary problems have been given for several American exotic options.

In Chap. 4, one-factor interest rate models, namely, the Vasicek, Cox–Ingersoll–Ross, Ho–Lee, and Hull–White models, are carefully discussed. Then, we describe how the problem of determining the market price of risk from the market data may be formulated as an inverse problem. After that, the formulations of interest rate derivatives such as bond options and swaptions are given. Then, we discuss multi-factor models and give the details of a three-factor model that can reflect the real market and be used in practice readily. The final topics in Chap. 4 are a discussion on two-factor convertible bonds and the derivation of the equivalent free-boundary problem.

Most of basic materials in these four chapters can be found from many books, for example, from the books listed above. Readers who need to know more about these subjects are referred to those books. Some of the materials are the authors' research results. For more details, see those corresponding papers given in the references.

As is well-known, exact solutions to the vanilla American option problems are not known, and the problems need to be solved numerically. For vanilla European options, if σ depends on S or the dividend is paid discretely, then explicit solutions may not exist. Therefore, in order to evaluate their prices, we often rely on numerical methods. For pricing exotic options and interest rate derivatives, we rely on numerical methods even more due to the complexity of these problems.

The next four chapters are devoted to numerical methods for partial differential equations in finance. In Chap. 5, we provide the basic numerical methods that will be used for solving partial differential equation problems and discuss the basic theory on finite-difference methods—stability, convergence and the extrapolation technique of numerical solutions. Most of these concepts can be found in many books. In the next chapter, Initial-Boundary Value and LC (linear complementarity) Problems, we discuss the numerical methods for European-style derivative securities and for American-style derivative securities formulated as an LC problem. In Chap. 7, Free-Boundary Problems, we carefully discuss how to solve one-factor and two-factor American option problems as free-boundary problems by implicit finite-difference methods. We also describe how to solve a two-factor convertible bond problem as a free-boundary problem by the pseudo-spectral method. In this chapter, we provide a comparison among these methods given in this chapter and in Chap. 6 as well. In the last chapter, Interest Rate Modeling, we begin with another formulation of the inverse problem and some numerical examples on the market price of risk. Then, we discuss how to price interest rate derivatives, such as swaptions, using one-factor models with numerical market prices of risk and show some numerical results. Finally, how to use the three-factor model to price interest rate derivatives in practice is discussed. Most of the materials presented in the last three chapters are from research results, especially from the authors' research.

This book can be used as a textbook for two courses as a sequence. In the first course, the subject "Partial Differential Equations in Finance" is taught by using the materials in Part I. The second one is a course on "Numerical Methods for Derivative Securities" based on Part II of this book. In order to help students to understand the materials and check whether or not students have understood them, a number of problems are given at the end of each chapter. Also, at the ends of Chaps. 5–8, some projects are given in order for students to be trained in evaluating derivative securities. This book is considered as a book between a textbook for graduate students and a monograph. If time is not enough, some portions can be omitted and left to students as reference materials. We have used it as a textbook in our mathematical finance program and almost all the materials can be taught in class. The following materials are basic and more important:[3]

- Sects. 1.1–1.2;
- Sects. 2.1–2.4, 2.5.1–2.5.2, 2.6.1–2.6.3, 2.9.1–2.9.4, 2.10.1–2.10.2;
- Sects. 3.1, 3.2.1, 3.3.1–3.3.4, 3.4.1–3.4.2;
- Sects. 4.1–4.2, 4.6–4.7;
- Sects. 5.1.1–5.1.2, 5.2.1–5.2.2, 5.3, 5.4.1, 5.5;
- Sects. 6.1.1–6.1.3, 6.1.5, 6.2.1–6.2.3, 6.3.2–6.3.3, 6.3.6;
- Sects. 7.1, 7.2.1, 7.2.3, 7.2.5–7.2.6, 7.3;
- Sect. 8.3.

These materials can be taught in one semester. Thus, if only one course is offered, this book can also be used.

During the production of this book, we received great help from our colleagues and former and current graduate students. We would like to express our thanks to them, especially, to Bing-mu Chen, Jinliang Li, Yingjun Sun,

[3]The sections and subsections here are referred to the sections and subsections in the first edition of the book.

Xionghua Wu, Chenggao Yang, and Jainqing Zhang, who provided many data and plots for this book. Our gratefulness is also extended to Jeremy Lane for his grammatical editing of the entire text and Doris Huneycutt for her careful and patient word processing. Finally, special thanks go to Achi Dosanjh, the editor of this book, and the reviewers of the book for their many suggestions, which greatly improved the quality of the book.

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Contents

Part I Partial Differential Equations in Finance

Part I

Partial Differential Equations in Finance

1

Introduction

1.1 Assets

We first introduce some basic knowledge on stocks, bonds, foreign currencies, commodities, and indices, all of which are called **assets** in this book.

Huge volumes of stocks are traded on the stock market every day, and the price of a stock changes all the time. Such a price is a typical random variable. As examples, the prices of the stocks issued by IBM and GE during the period 1990–2000 are plotted in Figs. 1.1 and 1.2. Stocks are issued by corporations. A corporation like IBM, for example, is a business unit, which gets its capital through issuing stocks. A holder of a share of stock owns a fixed portion of the corporation. For example, if a corporation issues ten million shares of stock, then the holder of a share of stock owns 10^{-7} portion of the corporation. Stock prices, especially those of high technology stocks, have large volatilities. However, stocks usually have higher returns than bonds, which attracts people to buy them. Many corporations distribute a small amount of cash to its stockholders in proportion to the number of shares of stock held periodically. The amount is not fixed and is determined by the corporation after the stocks have been issued. This payment is commonly known as the dividend. A corporation sometimes splits its stock. When a stock split occurs, the value of the stock changes. If one share splits into two shares, the value of a new share of stock is one half of the value of an old share of stock because the value of the corporation does not change when the stock split occurs.

Bonds and other debt instruments are other types of securities that are traded on the market frequently. Besides issuing stocks, a corporation can also get its capital through issuing bonds. Governments at various levels issue bonds for some special purposes, too. The holder of a bond will get the face value (the par value) at the maturity as long as the issuer has the ability to pay. Therefore, the price of a bond usually goes to the face value as the maturity approaches, which is a feature any price of stock does not have and is called the pull-to-par phenomenon. Periodically, a bondholder will receive a fixed amount of cash, usually a few percent of the face value. This percentage is

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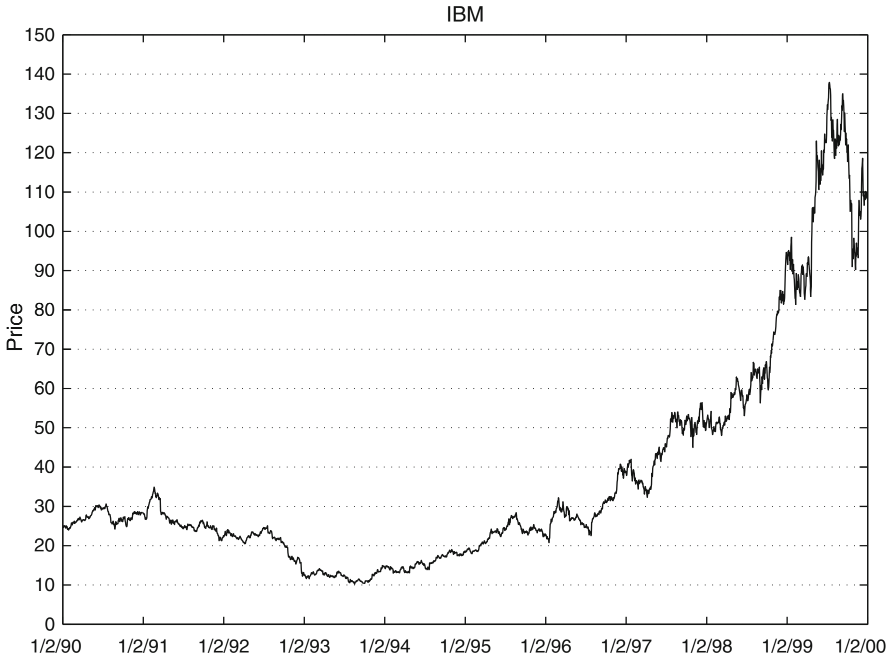

Fig. 1.1. IBM stock

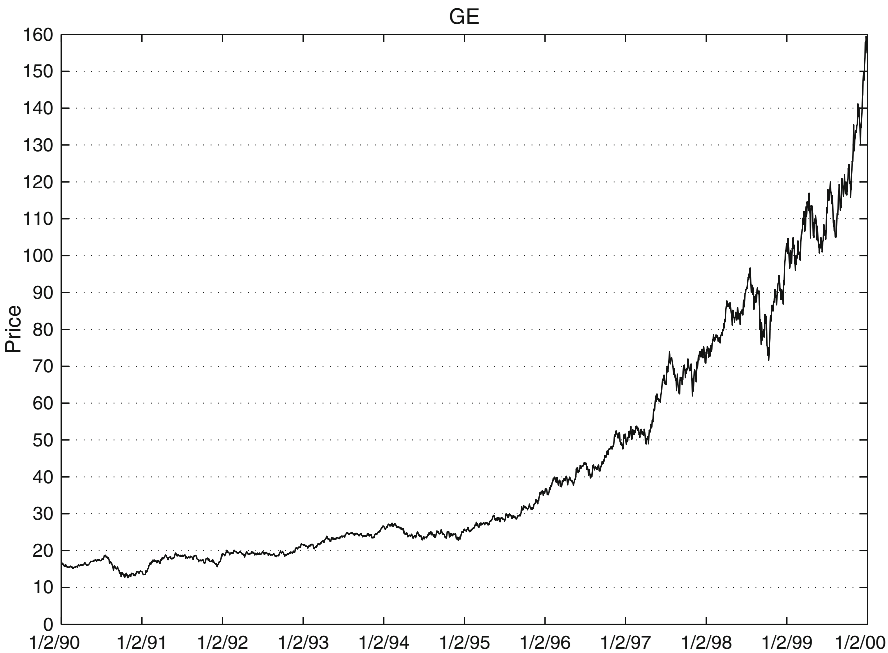

Fig. 1.2. GE stock

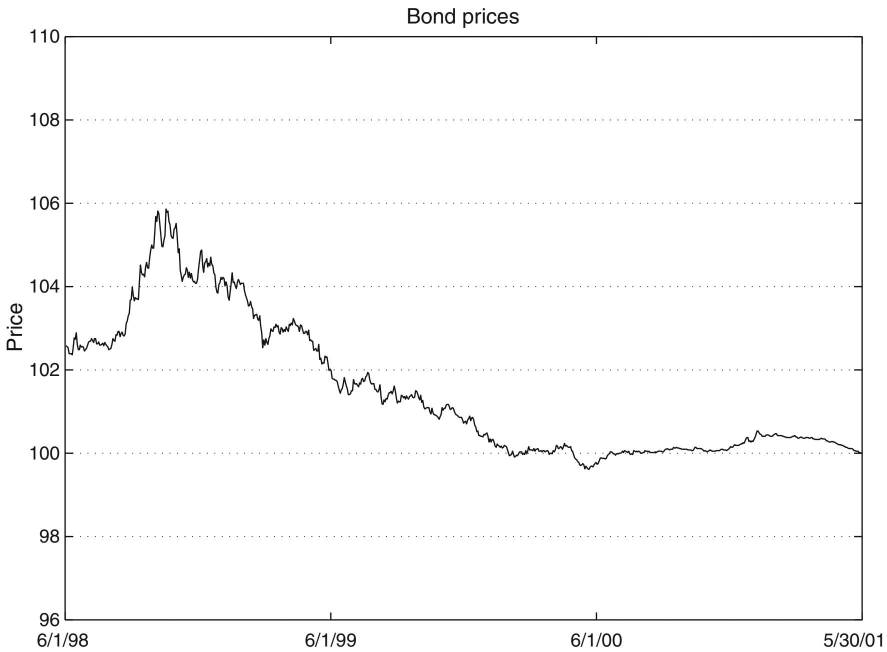

Fig. 1.3. A bond

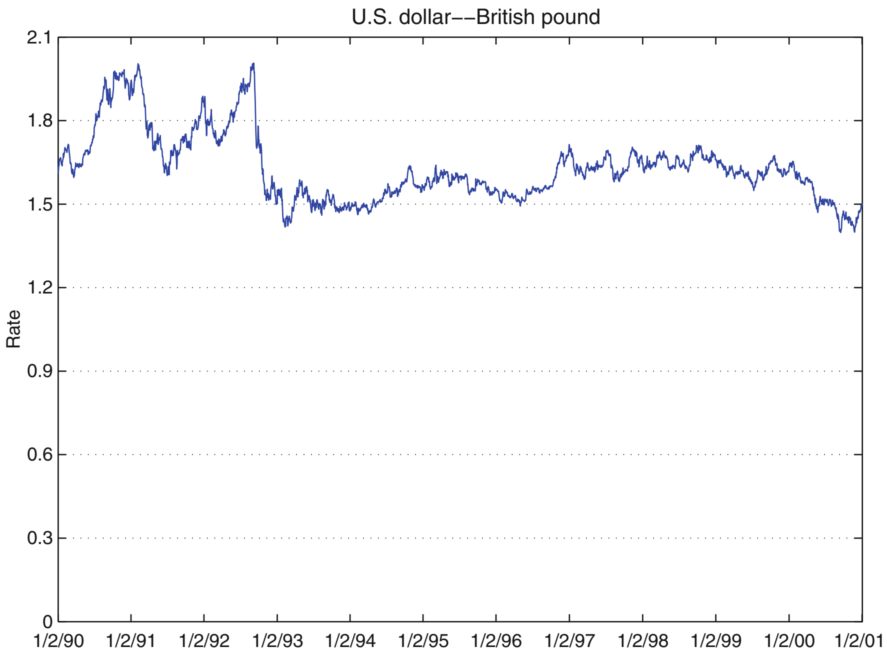

Fig. 1.4. Exchange rate between the British pound and the U.S. dollar (The quote is the number of U.S. dollars per British pound)

Fig. 1.5. Exchange rate between the U.S. dollar and the Japanese yen (The quote is the number of Japanese yen per U.S. dollar)

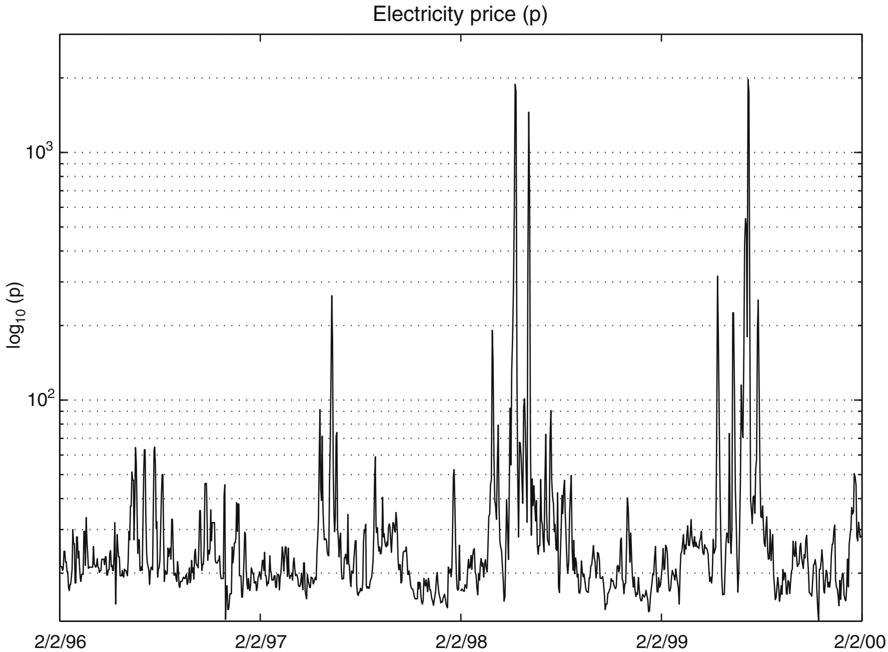

Fig. 1.6. Electricity price

Fig. 1.7. S&P 500

specified when a bond is issued, and this amount of money is called a coupon. In Fig. 1.3, the price of a five-year government bond with a coupon 6.5 % and maturing on May 31, 2001, is plotted, and the pull-to-par phenomenon can be seen clearly. Bonds usually have less risk than stocks. The bonds issued by the central government of a developed country have almost no risk.

Foreign currencies can also be sold or bought on the foreign currencies market. The exchange rate of a foreign currency, similar to the price of a stock, is also changing continuously. Figure 1.4 shows the exchange rate between the British pound and the U.S. dollar during 1990–2000. For the same period, the exchange rate between the Japanese yen and the U.S. dollar is plotted in Fig. 1.5. A person who holds foreign currencies can always deposit them into a bank to earn some interest. The interest is paid every day. Therefore, a foreign currency can be seen as a stock that pays dividends continuously. The interest rate of the foreign currency plays the role of the dividend yield. The price of a foreign currency usually has lower volatility.

Another important financial market is the commodity market. Similar to stocks, bonds, and foreign currencies, commodities are traded on the commodity market. However, a holder of commodities sometimes has to spend money every day in order to store them in a safe place. In this case, commodities pay negative dividend yields. In addition, the prices of some commodities have certain periodicity due to the periodicity in climate. In Fig. 1.6, the electricity price of a company from 1996 to 2000 is given. The unit of the price is dollar/Megawatt. Because the range of the price p is from about \$20 to \$2,000, we use $\log_{10} p$ instead of p as the ordinate in the figure. From this figure, we can see that the price possesses some "periodicity" with a period of one year. In the peak season, late June–late August, the price is higher because the demand is higher. On a few days the price almost reached \$2,000, but usually the price is \$20–\$30. The other markets do not possess such a feature.

We can also sell or buy indices, for example, S&P 500 and S&P 100. An index is a mixture of many stocks in which the percentage for each stock is fixed. Some of the stocks may pay dividends on different days, and the dividend payment of an index can be approximately understood as a continuous payment. Figure 1.7 shows the index level of S&P 500 during 1990–2000.

1.2 Derivative Securities

On markets, not only stocks, bonds, foreign currencies, commodities, and indices can be sold or bought, but also any contracts related to an asset can be traded. A contract is an agreement on something between two parties for a specified period. Those contracts are called **derivative securities** or **contingent claims**. The assets are called the underlying assets because those securities are derived from and their prices are contingent on the assets. Forward contracts, futures contracts, and options are such securities. Moreover, many other types of derivative securities exist. For example, an interest rate derivative is a contract derived from interest rates, rather than an asset. If one party earns a certain amount of money, then the other party loses the same amount of money. When we mention the value of a contract, generally speaking, the party should be specified. For a forward contract, a futures contract, or an option, one party agrees to sell the asset or writes the option and the other party agrees to buy the asset or purchases the option. We often say that the former takes the short position and the latter takes the long position of the contract. The values of the contract for the holders of the long and short positions are the same in magnitude but have the opposite signs. In this book, the value of a forward contract, a futures contract, or an option means the value of the contract for the holder of the long position, i.e., for the buyer. The end of the specified period usually is called the expiry, the expiration date, or the maturity date. The value of the contract at expiry is called the payoff of the contract. If a contract can be exercised at any time during the period, then the derivative is called an American-style derivative; if it can be exercised at a certain time specified in the contract, then we say that it is a European-style derivative. In what follows, we give some details on three types of derivatives: forward and futures contracts, options, and interest rate derivatives.

1.2.1 Forward and Futures Contracts

A forward contract is an agreement between two parties according to which one party will buy an asset from another party at the expiry for a specified

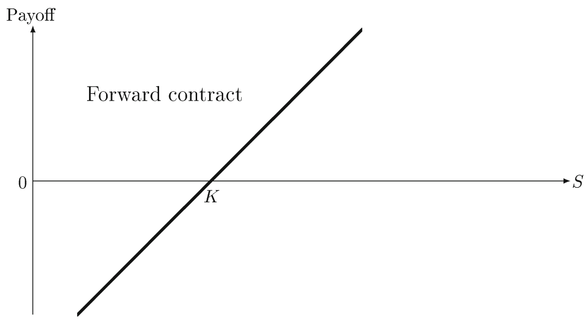

Fig. 1.8. Payoffs from forward contracts

price in the agreement. The specified price, called the delivery or forward price, is chosen so that its value is zero at the initiation of the contract. As time passes, the value of the contract can become positive or negative, depending on movement in the price of the asset. If the price of the asset rises sharply soon after the contract is entered into, the value of the forward contract becomes positive. If the price drops, it becomes negative. Its payoff is positive if the price of the asset is greater than the delivery price. Otherwise, the payoff is less than or equal to zero. Concretely, the payoff of a forward contract is

$$S - K, \tag{1.1}$$

where S is the price of the asset at expiry, and K is the delivery price. The graph of the payoff as a function of S is given in Fig. 1.8.

Corporations facing foreign exchange exposure frequently enter into forward contracts on foreign currencies with financial institutions in order to avoid potential loss in profits caused by the sharp change in foreign currency exchange rates. Such a contract is usually not traded on an exchange, and we say that it is traded on the over-the-counter market.

Like a forward contract, a futures contract is also an agreement between two parties to buy or sell an asset at a certain time in the future for a specified price called the delivery or futures price. However, futures contracts are usually traded on an exchange. In order to guarantee that the contract will be honored, the exchange requests each party to deposit funds in a margin account. At the end of each day, the difference between the closing futures prices on the day and the previous day is added to or subtracted from the margin account of each party, so the net profit or loss is paid over the lifetime of the contract. Another difference between a forward and a futures contract is that an exact delivery date is sometimes not specified in a futures contract. For commodities, the delivery period is often the entire month. These differences make determining how much its holder owns more complicated than evaluating the value of a forward contract for many situations.

1.2.2 Options

Options on stocks were first traded on an organized exchange in 1973. Now, options are traded on a large number of exchanges throughout the world. Huge volumes of options are also traded in the over-the-counter market by financial institutions. Not only can an option be on assets, but it can also be on another derivative, for example, a futures contract.

An option gives the holder a right, not an obligation, to do something. Hence, the holder does not have to exercise this right. There are two basic types of options: call options and put options. A call option gives the holder a right to buy the underlying asset at or by a certain date for a specified price. A put option gives the holder a right to sell the underlying asset at or by a certain date for a specified price. The price in the contract is known as the exercise or strike price. Let E denote this price. If the price of stock is less than E, then a holder of a call option will not exercise the option because there is no point in buying for E a stock that has a market value less than E. That is, the payoff of a call option is

$$\max(S - E, 0), \tag{1.2}$$

where S is the price of the stock at the end of the option's life. Similarly, the payoff of a put option is

$$\max(E - S, 0). \tag{1.3}$$

The graphs of these two functions are given in Figs. 1.9 and 1.10, respectively. If S is greater than E and if the holder of a call option could immediately exercise the option, then the holder would earn some money. In this case, we say that the call option is "in the money." If S is less than E, it is said that the call option is "out of the money" because the holder would lose money if the option were exercised immediately. If $S = E$, we say that the call option is "at the money" because no cash flow would come in or go out if exercising the option. For a put option, the situation is similar.

Unlike a forward or futures contract, where its value is equal to zero at the initiation of the contract, the holder of an option has to pay a certain amount of money to the writer of the option in order to enter into the option contract because the payoff is always nonnegative. This payment is usually called a premium. Because the holder of an option paid a premium when the option was bought, the value of the payoff is not the money earned. The money earned, the profit, is the payoff function minus the value of the premium at the end of the option's life. In Figs. 1.11 and 1.12, the profits are shown for a call option and a put option, respectively.

Many newspapers, such as the *Wall Street Journal* and the *Financial Times*, carry vanilla option quotations, which refer to trading that took place on exchanges on the previous workday. Most of the options that are traded on exchanges are American.

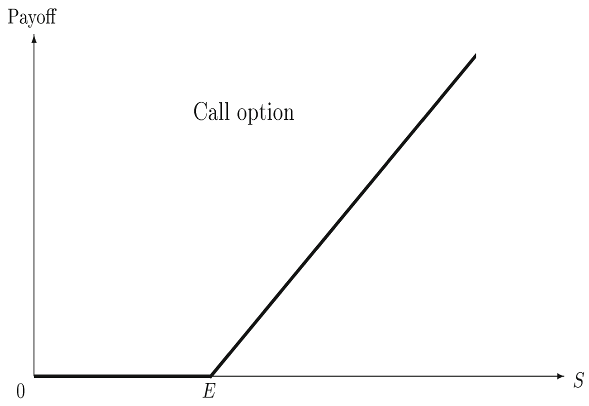

Fig. 1.9. The payoff diagram for a call option

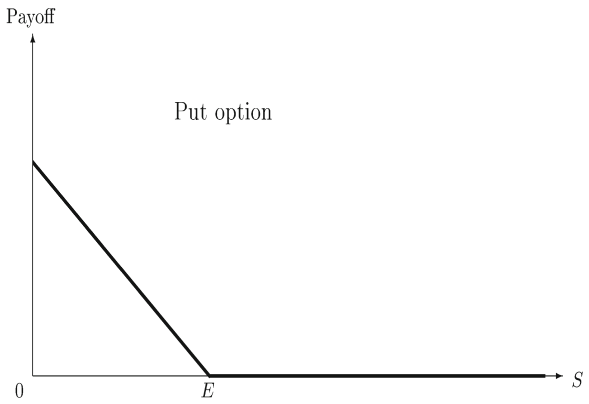

Fig. 1.10. The payoff diagram for a put option

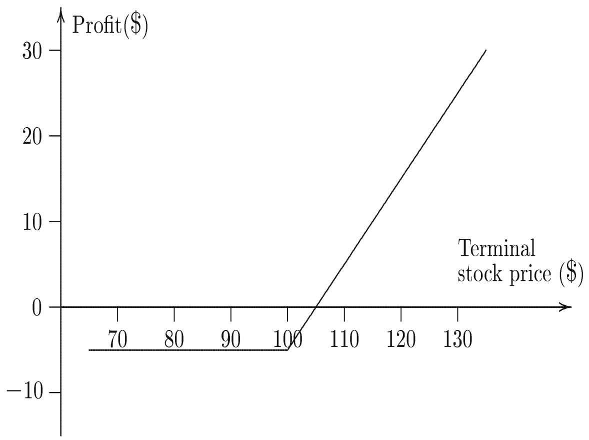

Fig. 1.11. Profit of a call option

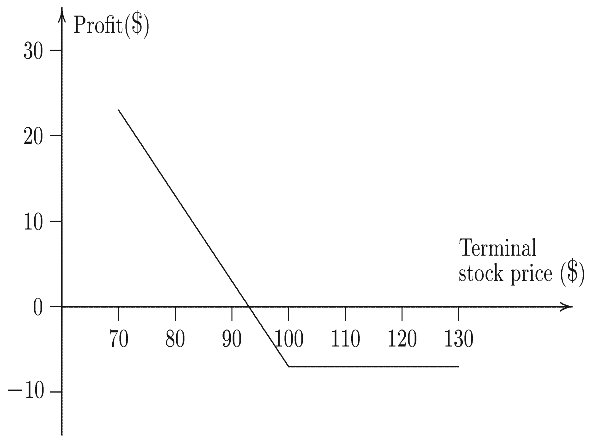

Fig. 1.12. Profit of a put option

1.2.3 Interest Rate Derivatives

On markets, the interest rate of a loan depends on the term of the loan. For example, one-year loans and five-year loans usually have different interest rates. The interest rate for the shortest possible deposit is commonly called the spot rate. The interest rates are constantly changing, at least on a daily basis, therefore in some cases it is necessary to consider interest rates as random variables. A security dependent on interest rates is called an interest rate derivative.

A typical interest rate derivative is a swap. Such a contract is an agreement between two parties A and B, usually between a bank and a corporation. In the contract, they agree that during the next few years A will pay interest on certain capital to B at a fixed interest rate, and B will pay interest on the same capital to A at a floating interest rate. One needs to determine what the fixed interest rate should be according to the current market and the value of the contract with a given fixed interest rate at a specific time. Other than swaps, there are many other types of interest rate derivatives, such as caps, floors and collars and options on swaps, caps, floors, and collars, which will be discussed in Chap. 5.

1.2.4 Factors Affecting Derivative Prices

It is clear that the value of a derivative depends on underlying random variables. For example, the value of an option on a stock depends on the price of the stock, and the manner of dependence is determined by the feature of the option. For a call option, the profit of the holder of the option will increase when the price of the stock rises. Therefore, the price of a call option is an increasing function of the stock price. The price of a put option is a decreasing function because the holder of the option will receive less when the stock price increases. The value of a swap depends on the floating interest rate. For the party who receives the floating interest, the swap is more valuable when the floating interest rate rises.

The price of a derivative also depends on some parameters. For example, the volatility of the underlying random variable, the time to expiration, the strike price of an option, and the fixed interest rate of a swap are such parameters. If the volatility is large, then the chance that the underlying random variable becomes very large or very small increases. For a call option, no matter whether the stock price is \$10 less than the strike price or \$100 less than the strike price at expiry, the value of the option is zero. However, if at expiry the stock price changes from \$10 higher than the strike price to \$100 higher than the strike price, then the value of the call option increases from \$10 to \$100. Therefore, a holder of a call option benefits from a large volatility. Similarly, a holder of a put option also benefits from a large volatility. That is, both calls and puts become more valuable as the volatility increases. If the time to expiration is longer, then the value of an American-style derivative should increase because the holder of the long-life American derivative has more exercise opportunities than the owner of the short-life American derivative. A rise in the strike price makes a call option less valuable because in order to get one share, the owner of the call option needs to pay more. However, a rise in the strike price makes the price of a put option go up because the owner of an option will get more money from one share. Increasing the fixed interest rate of a swap causes the party who receives the floating interest to pay more money, so the value to this party decreases. Derivatives depend on more parameters, such as the short-term interest rate and the dividend.

1.2.5 Functions of Derivative Securities

Generally speaking, derivatives have two primary uses: speculation and hedging.

Speculation. Suppose the price of a particular stock, for example, IBM, is \$120 today. An investor who believes that the stock is going to rise can purchase shares in that company. If he is correct, he makes money; if he is wrong, he loses money. This investor is speculating. If the share price rises from \$120 to \$150, he makes a profit of \$30 per share or

$$\frac{30}{120} = 25\,\%.$$

If it falls to \$90, he takes a loss of \$30 or

$$\frac{30}{120} = 25\,\%.$$

Alternatively, instead of shares, he buys call options with exercise price \$120. Assume that such an option costs \$15. If the share price rises to \$150, the payoff is \$30 with a profit of \$15; if the share price drops to \$90, then the loss is \$15 because the payoff is zero and the premium is \$15. Therefore, the profit or loss is magnified to

$$\frac{15}{15} = 100\,\%!$$

Consequently, options can be a cheap way of exposing a portfolio to a large amount of risk.

On the other hand, if the investor thinks that IBM shares are going to fall, he can do one of two things: sell shares or buy puts. If he speculates by selling shares that he does not own,[1] he will profit from a fall in IBM shares. He can also buy puts and will earn money from a fall of the stock.

Hedging. An owner of an asset will lose money when the price of the asset falls. The value of a put option rises when the asset price falls. What happens to the value of a portfolio containing both the asset and the put when the asset price falls? Clearly, the answer depends on the ratio of assets to options in the portfolio. If the ratio is equal to zero, the value rises, whereas if the ratio is infinity, the value falls. Somewhere between these two extremes is a ratio at which a small movement in the asset does not result in any movement in the value of the portfolio. Such a portfolio is risk-free. The reduction of risk by taking advantage of such correlations between the asset and option price movements is called hedging. This is one example explaining how options are used in hedging. Call options and futures can also be used for the purpose of hedging.

Problems

1. What is the difference between taking a long position in a forward contract and in a call option?
2. Suppose the futures price of gold is currently \$324 per ounce. An investor takes a short position in a futures contract for the delivery of 1,000 ounces. How much does the investor gain or lose if the price of gold at the end of the contract is (a) \$310 per ounce; (b) \$340 per ounce?
3. An investor holds a European call option on a stock with an exercise price of \$88 and the option costs \$3.50. For what value of the stock at maturity will the investor exercise the option, and for what value of the stock at maturity will the investor make a profit?
4. An investor holds a European put option for a stock with an exercise price of \$88 and the option costs \$3.50. Find the gain or loss to the investor if the stock price at maturity is (a) \$93.50; (b) \$81.50.

[1] This is perfectly legal in many markets. This action is called a short selling.

5. A company will receive a certain amount of foreign currency in one year. To reduce the risk of the changes in the exchange rate, what type of contract is appropriate for hedging?
6. Suppose a fund manager holds ten million shares of IBM stock and would like to use options to reduce risk. What action is suitable for reducing the risk of decline of the stock price in the next three months?
7. A stock price is \$67 just before a dividend of \$1.50 is paid. What is the stock price immediately after the payment?

2

European Style Derivatives

2.1 Asset Price Models and Itô's Lemma

2.1.1 Models for Asset Prices

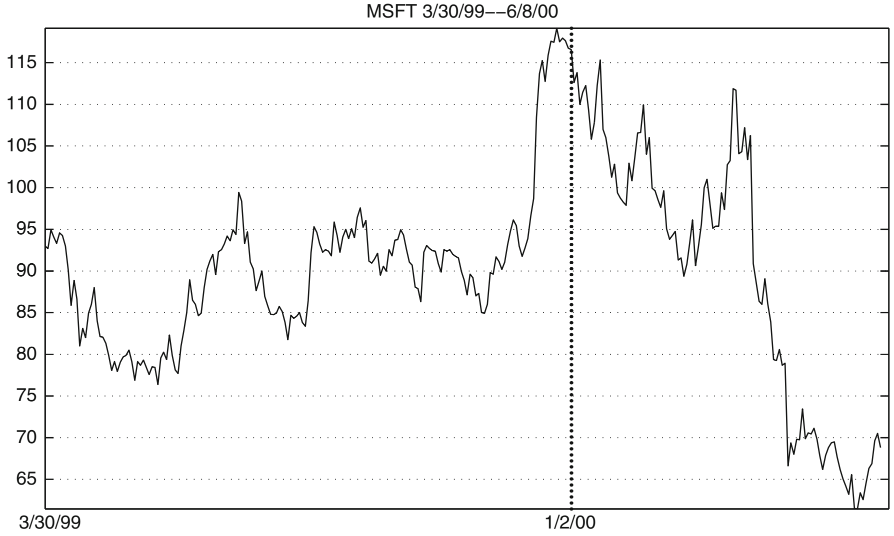

Fig. 2.1. Stock price of Microsoft Inc.

As examples, in Figs. 1.1–1.7 we showed how the prices of assets vary with time t. Figure 2.1 shows the stock price of Microsoft Inc. in the period March 30, 1999, to June 8, 2000. From these figures, we can see the following: the graphs are jagged, and the size of the jags changes all the time. This means that we cannot express S as a smooth function of t, and it is difficult to predict exactly the price at time t from the price before time t. It is natural to think

Y.-l. Zhu et al., *Derivative Securities and Difference Methods*, Springer Finance,
DOI 10.1007/978-1-4614-7306-0_2,

of the price at time t as a random variable. Now let us give a model for such a random variable.

Suppose that at time t the asset price is S. Let us consider a small subsequent time interval dt, during which S changes to $S+dS$. (We use the notation df for the small change in any quantity f over this time interval.) How might we model the corresponding return rate on the asset, dS/S?

Assume that the return rate on the asset can be described by the following stochastic differential equation:

$$\frac{dS}{S} = \mu(S,t)dt + \sigma(S,t)dX, \tag{2.1}$$

where μ and σ are called the **drift** and the **volatility**, respectively, and dX is known as a Wiener process defined by

$$\begin{cases} dX = \phi\sqrt{dt}, \\ \phi \text{ being a standardized normal random variable.} \end{cases}$$

In this model, the first part is an anticipated and deterministic return rate, and the second part is the random return rate of the asset price in response to unexpected events. As we can see, the random increment dS depends solely on today's price. This independence from the past is known as the Markov property. In many situations, it is assumed that μ and σ are constants. Due to its simplicity, this is a popular model for asset prices

For a random variable ψ with a probability density function $f(\psi)$ defined on $(-\infty, \infty)$, the expectation of any function $F(\psi)$, $\mathrm{E}\left[F(\psi)\right]$, is given by

$$\mathrm{E}\left[F(\psi)\right] = \int_{-\infty}^{\infty} F(\psi)f(\psi)d\psi.$$

The variance of $F(\psi)$, $\mathrm{Var}\left[F(\psi)\right]$, is defined by

$$\mathrm{Var}\left[F(\psi)\right] = \mathrm{E}\left[(F(\psi) - \mathrm{E}\left[F(\psi)\right])^2\right].$$

According to these definitions, for any constants a, b, c, and random variable W, we have

$$\begin{aligned} \mathrm{E}\left[aW - b\right] &= a\mathrm{E}\left[W\right] - b, \\ \mathrm{Var}\left[W\right] &= \mathrm{E}\left[(W - \mathrm{E}\left[W\right])^2\right] \\ &= \mathrm{E}\left[W^2\right] - (\mathrm{E}\left[W\right])^2 \end{aligned}$$

and

$$\mathrm{Var}\left[\frac{W}{c}\right] = \frac{1}{c^2}\mathrm{Var}\left[W\right].$$

For a standardized normal random variable ϕ, the probability density function is

$$\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\phi^2/2}, \qquad -\infty < \phi < \infty.$$

As a probability density function, this function satisfies[1]

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\phi^2/2} d\phi = 1.$$

Therefore we have

$$\mathrm{E}\left[\phi\right] = \int_{-\infty}^{\infty} \phi \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\phi^2/2} d\phi = 0$$

and

$$\begin{aligned}
\mathrm{Var}\left[\phi\right] &= \mathrm{E}\left[\phi^2\right] \\
&= \int_{-\infty}^{\infty} \phi^2 \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\phi^2/2} d\phi \\
&= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi d\left(\mathrm{e}^{-\phi^2/2}\right) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-\phi^2/2} d\phi \\
&= 1.
\end{aligned}$$

From these we obtain

$$\mathrm{E}\left[dX\right] = \mathrm{E}\left[\phi\right]\sqrt{dt} = 0$$

and

$$\mathrm{Var}\left[dX\right] = \mathrm{E}\left[dX^2\right] = \mathrm{E}\left[\phi^2\right] dt = dt.$$

Consequently[2]

$$\mathrm{E}\left[dS\right] = \mathrm{E}\left[\sigma S\, dX + \mu S\, dt\right] = \mu S\, dt,$$

and

$$\begin{aligned}
\mathrm{Var}\left[dS\right] &= \mathrm{E}\left[dS^2\right] - \left(\mathrm{E}\left[dS\right]\right)^2 \\
&= \mathrm{E}\left[\sigma^2 S^2 dX^2 + 2\sigma S^2 \mu\, dt\, dX + \mu^2 S^2 dt^2\right] - \mu^2 S^2 dt^2 \\
&= \sigma^2 S^2 dt.
\end{aligned}$$

The square root of the variance is known as the standard deviation. Thus, the deviation of the return on the asset is proportional to σ. This means

[1] Because $\int_0^\infty \mathrm{e}^{-x^2/2}dx \times \int_0^\infty \mathrm{e}^{-y^2/2}dy = \int_0^{\pi/2}\int_0^\infty \mathrm{e}^{-r^2/2} r dr d\theta = \pi/2$, we have $\int_0^\infty \mathrm{e}^{-\phi^2/2}d\phi = \sqrt{\pi/2}$.

[2] Here, dX is a random variable and S is unchanged. In stochastic calculus, it is called conditional expectation (see [51, 6]).

that an asset price with a larger σ would appear more jagged. Typically, for stocks, indices, exchange rates, and bonds, the value of σ is in the range 0.02–0.4. Usually, the volatility of stocks is greater than indices, exchange rates, and bonds, and government bonds have the smallest volatility among these. Among shares, high-tech companies tend to have higher volatility than other companies. For example, assume that the volatility of the price of IBM stock is a constant during 1990–2000, then its value is 0.31. Under the same assumption, for the price of GE stock, $\sigma = 0.23$. For S&P 500, British pound—U.S. dollar exchange rate, Japanese yen—U.S. dollar exchange rate, and a five-year government bond with coupon 6.5 % and maturing on May 31, 2001, $\sigma = 0.10, 0.11, 0.12$, and 0.03, respectively. For the bond, we assume that σ depends on the time to maturity. Clearly, at maturity σ is zero. The value 0.03 means that the maximum value of σ is 0.03. In practice, the volatility is often quoted as a percentage so that $\sigma = 0.2$ would be 20 % volatility.

If $\sigma = 0$, then

$$\frac{dS}{S} = \mu dt \quad \text{and} \quad S(t) = S_0 \mathrm{e}^{\mu(t-t_0)},$$

where S_0 is the value of the asset at $t = t_0$.

In this asset price model, μ and σ are two parameters. In general, these parameters depend on the asset price S and time t, i.e., $\mu = \mu(S,t)$, $\sigma = \sigma(S,t)$. According to the historical data, we can determine these parameters (or parameter functions) for the past by statistical analysis. If we assume that μ and σ depend on S only, then the functions $\mu(S)$ and $\sigma(S)$ determined by the historical data can be used for the future.

A Wiener process is also referred to as a Brownian motion. There are many excellent books on the Brownian motion. Readers interested in this subject can read, for example, [51]. A basic and very important feature of the Wiener process is that the sum of two independent Wiener processes is also a Wiener process, and the variance of the sum is the sum of the two original variances. That is, if $dX_1 = \phi_1\sqrt{dt_1}$ and $dX_2 = \phi_2\sqrt{dt_2}$ are two Wiener processes and they are independent, namely, $\mathrm{E}[\phi_1\phi_2] = 0$, then

$$dX_3 = dX_1 + dX_2 = \phi_1\sqrt{dt_1} + \phi_2\sqrt{dt_2} = \phi_3\sqrt{dt_1 + dt_2}, \qquad (2.2)$$

where ϕ_3 is also a standardized normal random variable. Readers are asked to prove a similar conclusion as a portion of Problem 4.

2.1.2 Itô's Lemma

There is a practical lower bound for the basic time-step dt of the random walk of an asset price. Thus, an asset price is a discrete random variable. However, sometimes the lower bound is so small that we consider an asset price as a continuous random variable. Also, because it is much more efficient to solve the resulting differential equations than to evaluate options by direct simulation of the random walk on a practical time scale, we will assume that an asset price is a continuous random variable even if the basic time-step is not very small.

Before coming to Itô's lemma, we need one result, which we do not prove. This result is, with probability one,

$$dX^2 = \phi^2 dt \to dt \quad \text{as} \quad dt \to 0.$$

This can be explained as follows. Because

$$\mathrm{E}\left[dX^2\right] = \mathrm{E}\left[\phi^2\right] dt = dt$$

and

$$\mathrm{Var}\left[dX^2\right] = \mathrm{E}\left[dX^4\right] - (\mathrm{E}\left[dX^2\right])^2 = O(dt^2),$$

the variance of dX^2 is very small and the smaller dt becomes, the closer dX^2 comes to being equal to dt.

Assume

$$dS = a(S,t)dt + b(S,t)dX$$

and suppose $f(S,t)$ is a smooth function of a random variable S and time t. We need to find a stochastic differential equation for f. If we vary S and t by a small amount dS and dt, then f also varies by a small amount. From the Taylor series expansion we can write

$$df = \frac{\partial f}{\partial S} dS + \frac{\partial f}{\partial t} dt + \frac{1}{2}\left(\frac{\partial^2 f}{\partial S^2} dS^2 + 2\frac{\partial^2 f}{\partial t \partial S} dt\, dS + \frac{\partial^2 f}{\partial t^2} dt^2\right) + \cdots .$$

Because

$$\begin{aligned} dS^2 &= [a(S,t)dt + b(S,t)dX]^2 = \left(adt + b\phi\sqrt{dt}\right)^2 \\ &= a^2(dt)^2 + 2ab\phi(dt)^{3/2} + b^2\phi^2 dt \to b^2 dt \quad \text{as} \quad dt \to 0, \end{aligned}$$

we have[3]

$$df = \frac{\partial f}{\partial S} dS + \left(\frac{\partial f}{\partial t} + \frac{1}{2} b^2 \frac{\partial^2 f}{\partial S^2}\right) dt \quad \text{as} \quad dt \to 0 \tag{2.3}$$

or in the form of a stochastic differential equation

$$df = b\frac{\partial f}{\partial S} dX + \left(\frac{\partial f}{\partial t} + \frac{1}{2} b^2 \frac{\partial^2 f}{\partial S^2} + a\frac{\partial f}{\partial S}\right) dt.$$

This is Itô's lemma. If in the asset price model (2.1), μ and σ are constants, i.e.,

---

[3] As we know, in calculus we have $df(S,t) = \dfrac{\partial f}{\partial S} dS + \dfrac{\partial f}{\partial t} dt$. Thus this relation is the same as the relation in calculus only if $f(S,t)$ is a linear function in S.

$$dS = \mu S dt + \sigma S dX,$$

then Itô's lemma is in the form:

$$\begin{aligned} df &= \frac{\partial f}{\partial S} dS + \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) dt \\ &= \sigma S \frac{\partial f}{\partial S} dX + \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \mu S \frac{\partial f}{\partial S} \right) dt. \end{aligned}$$

2.1.3 Expectation and Variance of Lognormal Random Variables

As a simple example, consider the function $f(S) = \ln S$. Differentiation of this function gives

$$\frac{df}{dS} = \frac{1}{S} \quad \text{and} \quad \frac{d^2 f}{dS^2} = -\frac{1}{S^2}.$$

Suppose that S satisfies Eq. (2.1) with constant μ and σ, i.e., $dS = \mu S dt + \sigma S dX$. Using Itô's lemma, for $\ln S$ we have

$$d \ln S = \sigma dX + \left(\mu - \frac{\sigma^2}{2} \right) dt = m dt + \sigma dX, \tag{2.4}$$

where

$$m = \mu - \frac{\sigma^2}{2}. \tag{2.5}$$

It is clear that

$$\mathrm{E}\left[d \ln S\right] = \mathrm{E}\left[m dt + \sigma dX\right] = m dt$$

and

$$\begin{aligned} \mathrm{Var}\left[d \ln S\right] &= \mathrm{E}\left[(d \ln S)^2\right] - \left(\mathrm{E}\left[d \ln S\right]\right)^2 \\ &= \mathrm{E}\left[\sigma^2 dX^2 + 2\sigma m \, dt \, dX + m^2 dt^2\right] - m^2 dt^2 \\ &= \sigma^2 \mathrm{E}\left[\phi^2 dt\right] = \sigma^2 dt. \end{aligned}$$

From Eq. (2.4), the probability density function for $d \ln S$ is[4]

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[4] • Here $\mathrm{e}^{-(d \ln S - m dt)^2 / 2\sigma^2 dt}$ means $\mathrm{e}^{-(d \ln S - m dt)^2 / (2\sigma^2 dt)}$. That is, in the expression $(d \ln S - m dt)^2 / 2\sigma^2 dt$, the division between $(d \ln S - m dt)^2$ and $2\sigma^2 dt$ should be done after $2 \times \sigma^2 \times dt$ is obtained. Throughout the entire book we use such a notation.

• If x is a normal random variable and its mean and variance are a and b^2, then its probability density function is

$$\frac{1}{b\sqrt{2\pi}} \mathrm{e}^{-(x-a)^2 / 2b^2}.$$

$$\frac{1}{\sigma\sqrt{2\pi dt}}\mathrm{e}^{-(d\ln S - mdt)^2/2\sigma^2 dt}.$$

Let $d\ln S = \ln S' - \ln S$. Then for $\ln S'$, the probability density function is

$$G_1(\ln S') = \frac{1}{\sigma\sqrt{2\pi dt}}\mathrm{e}^{-\left[\ln S' - \ln S - mdt\right]^2/2\sigma^2 dt}.$$

Here, S is the value of the asset at time t and S' is the value of the asset at time $t+dt$ which is a random variable. In Fig. 2.2, the curve of $G_1(\ln S')$ with $\ln S + mdt = 0$ and $\sigma\sqrt{dt} = 0.2$ is shown.

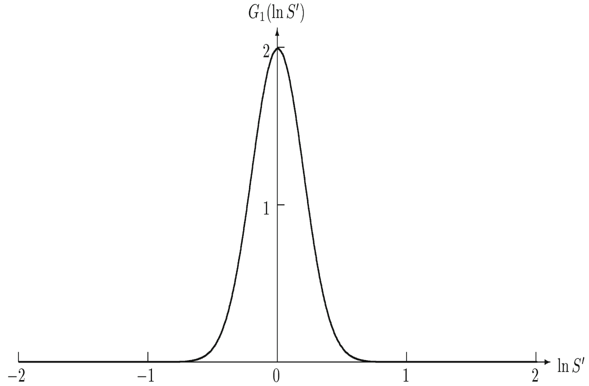

Fig. 2.2. The probability density function for $\ln S'$ with $\ln S + mdt = 0$ and $\sigma\sqrt{dt} = 0.2$

Suppose that for S' the probability density function is $G(S')$. Because[5]

$$G(S')\,dS' = \frac{1}{\sigma\sqrt{2\pi dt}}\mathrm{e}^{-\left(\ln S' - \ln S - mdt\right)^2/2\sigma^2 dt}d\ln S',$$

we have

$$G(S') = \frac{1}{S'\sigma\sqrt{2\pi dt}}\mathrm{e}^{-(\ln S' - \ln S - mdt)^2/2\sigma^2 dt}.$$

[5]If for x the probability density function is $f(x)$, then the probability of $x \in [x, x+dx]$ is $f(x)dx$. If $y = y(x)$ and $y(x)$ is a nondecreasing function, then $x \in [x, x+dx]$ if and only if $y \in [y(x),\, y(x+dx)] \approx \left[y(x),\, y(x) + \frac{dy}{dx}dx\right]$. Thus, the probability of the event $y \in \left[y(x), y(x) + \frac{dy}{dx}dx\right]$ is also $f(x)dx$. If for y the probability density function is $f_1(y)$, then $f_1(y)dy = f(x)dx$, from which we have $f_1(y) = f(x(y))\frac{dx}{dy}$. If $x = \ln S'$ and $y = S'$, then $f_1(S') = f(x(y))\frac{dx}{dy} = f(\ln S')\frac{1}{S'}$.

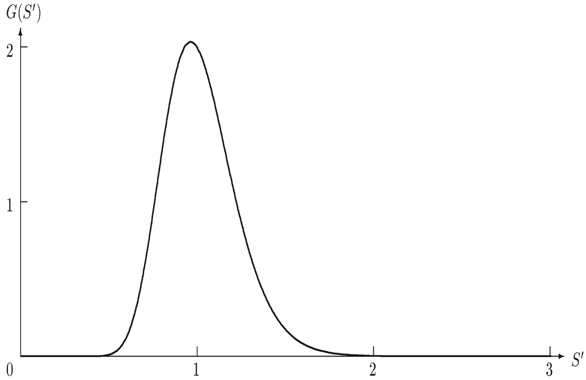

Fig. 2.3. The probability density function for S' with $\ln S + mdt = 0$ and $\sigma\sqrt{dt} = 0.2$

In Fig. 2.3, the corresponding curve of $G(S')$ is given. This is called a lognormal because the corresponding distribution for $\ln S'$ is normal.

Now the question is what are $\mathrm{E}\left[S'\right]$ and $\mathrm{Var}\left[S'\right]$. Because we have the probability density function, let

$$y = \frac{\ln S' - \ln S - mdt}{\sigma\sqrt{dt}}$$

and we have

$$\begin{aligned}
\mathrm{E}\left[S'\right] &= \int_0^\infty G(S')S'dS' \\
&= \frac{1}{\sigma\sqrt{2\pi dt}}\int_0^\infty \mathrm{e}^{-(\ln S' - \ln S - mdt)^2/2\sigma^2 dt}\frac{1}{S'} \times S'dS' \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \mathrm{e}^{-y^2/2}\mathrm{e}^{y\sigma\sqrt{dt}+\ln S + mdt}dy \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \mathrm{e}^{-\left(y-\sigma\sqrt{dt}\right)^2/2} \times \mathrm{e}^{\sigma^2 dt/2+\ln S+mdt}dy \\
&= \mathrm{e}^{\sigma^2 dt/2+\ln S+mdt} = S\mathrm{e}^{\mu dt},
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}\left[S'^2\right] &= \int_0^\infty G(S')S'^2dS' \\
&= \frac{1}{\sigma\sqrt{2\pi dt}}\int_0^\infty \mathrm{e}^{-(\ln S' - \ln S - mdt)^2/2\sigma^2 dt}\frac{1}{S'}S'^2dS' \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \mathrm{e}^{-y^2/2} \times \mathrm{e}^{2(y\sigma\sqrt{dt}+\ln S+mdt)}dy \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \mathrm{e}^{-\left(y-2\sigma\sqrt{dt}\right)^2/2}\mathrm{e}^{2\sigma^2 dt+2(\ln S+mdt)}dy \\
&= \mathrm{e}^{2\sigma^2 dt+\ln S^2+2mdt} = S^2\mathrm{e}^{2\mu dt+\sigma^2 dt}
\end{aligned}$$

and

$$\begin{aligned}\mathrm{Var}\left[S'\right] &= S^2 \mathrm{e}^{2\mu dt+\sigma^2 dt} - S^2 \mathrm{e}^{2\mu dt} \\ &= S^2 \mathrm{e}^{2\mu dt}\left(\mathrm{e}^{\sigma^2 dt} - 1\right),\end{aligned}$$

where we have used the relation (2.5).

If m and σ in the expression (2.4) are constants, then for a large time period $T-t$, we can have

$$\ln S' - \ln S = \int_t^T d\ln S = m\int_t^T dt + \sigma\int_t^T dX(t) = m(T-t) + \sigma\phi\sqrt{T-t},$$

where S' is the stock price at time T, S is the stock price at time t, and ϕ is a standardized normal random variable. Here we used the relation $\int_t^T dX(t) = \phi\sqrt{T-t}$, which can be obtained from the relation (2.2). Therefore, in this case, the probability density function for S' is

$$G(S') = \frac{1}{S'\sigma\sqrt{2\pi(T-t)}}\mathrm{e}^{-[\ln S' - \ln S - m(T-t)]^2/2\sigma^2(T-t)}$$

and

$$\begin{cases} \mathrm{E}\left[S'\right] = S\mathrm{e}^{\mu(T-t)}, \\ \mathrm{Var}\left[S'\right] = S^2\mathrm{e}^{2\mu(T-t)}\left[\mathrm{e}^{\sigma^2(T-t)} - 1\right], \end{cases} \tag{2.6}$$

where μ is given by the relation (2.5):

$$\mu = m + \frac{\sigma^2}{2}.$$

2.2 Derivation of the Black–Scholes Equation

2.2.1 Arbitrage Arguments

In the modern world, financial transactions may be done simultaneously in more than one market. Suppose the price of gold is \$324 per ounce in New York and 37,275 Japanese Yen in Tokyo, while the exchange rate is 1 U.S. dollar for 115 Japanese Yen. An arbitrageur, who is always looking for any arbitrage opportunities to make money, could simultaneously buy 1,000 ounces in New York, sell them in Tokyo to obtain a risk-free profit of

$$37{,}275 \times 1{,}000/115 - 324 \times 1{,}000 = \$130.43$$

if the transaction costs can be ignored. Small investors may not profit from such opportunity due to the transaction costs. However, the transaction costs for large investors might be a small portion of the profit, which makes the arbitrage opportunity very attractive.

Arbitrage opportunities usually cannot last long. As arbitrageurs buy the gold in New York, the price of the gold will rise. Similarly, as they sell the gold in Tokyo, the price of the gold will be driven down. Very quickly, the ratio between the two prices will become closer to the current exchange rate. In practice, due to the existence of arbitrageurs, very few arbitrage opportunities can be observed. Therefore, throughout this book we will assume that there are no arbitrage opportunities for financial transactions.

Let us make the following assumptions: both the borrowing short-term interest rate and the lending short-term interest rate are equal to r, short selling is permitted, the assets and options are divisible, and there is no transaction cost. Then, we can conclude that the absence of arbitrage opportunities is equivalent to all risk-free portfolios having the same return rate r.

Let us show this point. Suppose that Π is the value of a portfolio and that during a time step dt the return of the portfolio $d\Pi$ is risk-free. If

$$d\Pi > r\Pi dt,$$

then an arbitrageur could make a risk-free profit $d\Pi - r\Pi dt$ during the time step dt by borrowing an amount Π from a bank to invest in the portfolio. Conversely, if

$$d\Pi < r\Pi dt,$$

then the arbitrageur would short the portfolio and invest Π in a bank and get a net income $r\Pi dt - d\Pi$ during the time step dt without taking any risk. Only when

$$d\Pi = r\Pi dt$$

holds, is it guaranteed that there are no arbitrage opportunities.

In the next subsection, we will derive the equation the prices of derivative securities should satisfy by using the conclusion that all risk-free portfolios have the same return rate r.

2.2.2 The Black–Scholes Equation

Let V denote the value of an option that depends on the value of the underlying asset S and time t, i.e., $V = V(S, t)$. It is not necessary at this stage to specify whether V is a call or a put; indeed, V can even be the value of a whole portfolio of various options. For simplicity, readers may think of a simple call or put.

Assume that in a time step dt, the underlying asset pays out a dividend SD_0dt, where D_0 is a constant known as the dividend yield.[6] Suppose S satisfies Eq. (2.1):

[6] This dividend structure is a good model for an index. In this case, many discrete dividend payments are paid at many different times, and the dividend payment can be approximated by a continuous yield without serious error. Also, if the asset is a foreign currency, then the interest rate for the foreign currency plays the role of D_0.

$$\frac{dS}{S} = \mu(S,t)dt + \sigma(S,t)dX.$$

According to Itô's lemma (2.3), the random walk followed by V is given by

$$dV = \frac{\partial V}{\partial S}dS + \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt. \tag{2.7}$$

Here we require V to have at least one t derivative and two S derivatives.

Now construct a portfolio consisting of one option and a number $-\Delta$ of the underlying asset. This number is as yet unspecified. The value of this portfolio is

$$\Pi = V - \Delta S. \tag{2.8}$$

Because the portfolio contains one option and a number $-\Delta$ of the underlying asset, and the owner of the portfolio receives SD_0dt for every asset held, the earnings for the owner of the portfolio during the time step dt is

$$d\Pi = dV - \Delta\left(dS + SD_0dt\right).$$

Using the relation (2.7), we find that Π follows the random walk

$$d\Pi = \left(\frac{\partial V}{\partial S} - \Delta\right) dS + \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \Delta S D_0\right) dt.$$

The random component in this random walk can be eliminated by choosing

$$\Delta = \frac{\partial V}{\partial S}. \tag{2.9}$$

This results in a portfolio whose increment is wholly deterministic:

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \Delta S D_0\right) dt. \tag{2.10}$$

Because the return for any risk-free portfolio should be r, we have

$$r\Pi dt = d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \Delta S D_0\right) dt. \tag{2.11}$$

Substituting the relations (2.8) and (2.9) into Eq. (2.11) and dividing by dt, we arrive at

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0)S\frac{\partial V}{\partial S} - rV = 0. \tag{2.12}$$

When we take different Π for different S and t, we can conclude that Eq. (2.12) holds on a domain. In this book, Eq. (2.12) is called the Black–Scholes partial differential equation, or the Black–Scholes equation,[7] even though $D_0 = 0$ in the equation originally given by Black and Scholes (see [11]). With its extensions and variants, it plays the major role in the rest of the book.

About the derivation of this equation and the equation itself, we give more explanation here.

[7] It is also called Black–Scholes–Merton differential equation (see [43]).

- The key idea of deriving this equation is to eliminate the uncertainty or the risk. $d\Pi$ is not a differential in the usual sense. It is the earning of the holder of the portfolio during the time step dt. Therefore, ΔSD_0dt appear. In the derivation, in order to eliminate any small risk, Δ is chosen before an uncertainty appears and does not depend on the coming risk. Therefore, no differential of Δ is needed.
- The linear differential operator given by

$$\frac{\partial}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2}{\partial S^2}+(r-D_0)S\frac{\partial}{\partial S}-r$$

has a financial interpretation as a measure of the difference between the return on a hedged option portfolio

$$\frac{\partial}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2}{\partial S^2}-D_0S\frac{\partial}{\partial S}$$

and the return on a bank deposit

$$r\left(1-S\frac{\partial}{\partial S}\right).$$

Although the difference between the two returns is identically zero for European options, we will later see that the difference between the two returns may be nonzero for American options.
- From the Black–Scholes equation (2.12), we know that the parameter μ in Eq. (2.1) does not affect the option price, i.e., the option price determined by this equation is independent of the average return rate of an asset price per unit time.
- From the derivation procedure of the Black-Scholes equation we know that the Black-Scholes equation still holds if r and D_0 are functions of S and t.
- If dividends are paid only on certain dates, then the money the owner of the portfolio will get during the time period $[t, t+dt]$ is

$$dV-\Delta dS-\Delta D(S,t)dt,$$

where $D(S,t)$ is a sum of several Dirac delta functions. Suppose that a stock pays dividend $D_1(S)$ at time t_1 and $D_2(S)$ at time t_2 for a share, where $D_1(S)\le S$ and $D_2(S)\le S$. Then

$$D(S,t)=D_1(S)\delta(t-t_1)+D_2(S)\delta(t-t_2),$$

where the Dirac delta function[8] $\delta(t)$ is defined as follows:

[8] It is the limit as $\varepsilon\to 0$ of the one-parameter family of functions:

$$\delta_\varepsilon(x)=\begin{cases}\dfrac{1}{2\varepsilon}, & -\varepsilon\le x\le\varepsilon,\\ 0, & |x|>\varepsilon.\end{cases}$$

$$\delta(t) = \begin{cases} 0, & \text{if } t \neq 0, \\ \infty, & \text{if } t = 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t) = 1.$$

In this case, the modified Black–Scholes equation is in the form

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \left[rS - D\left(S,t\right)\right]\frac{\partial V}{\partial S} - rV = 0. \tag{2.13}$$

2.2.3 Final Conditions for the Black–Scholes Equation

From the derivation of the Black–Scholes equation (2.12), we know that this partial differential equation holds for any option (or portfolio of options) whose value depends only on S and t. In order to determine a unique solution of the Black–Scholes equation, the solution at the expiry, $t = T$, needs to be given. This condition is called the final condition for the partial differential equation. Different options satisfy the same partial differential equation, but different final conditions. Therefore, in order to determine the price of an option, we need to know the value of the option at time T. In what follows, we will derive the final conditions for call and put options.

Final Condition for Call Options. Let us examine what a holder of a call option will do just at the moment of expiry. If $S > E$ at expiry, it makes financial sense for the holder to exercise the call option, handing over an amount E for an asset worth S. The money earned by the holder from such a transaction is then $S - E$. On the other hand, if $S < E$ at expiry, the holder should not exercise the option because the holder would lose an amount of $E - S$. In this case, the option expires valueless. Thus, the value of the call option at expiry can be written as

$$C(S,T) = \max(S - E, 0). \tag{2.14}$$

This function giving the value of a call option at expiry is usually called the payoff function of a call option. In Fig. 1.9, we plot $\max(S-E, 0)$ as a function of S, which is usually known as a payoff diagram. A call option with such a payoff is the simplest call option and is known as a vanilla call option.

Final Condition for Put Options. Each option or each portfolio of options has its own payoff at expiry. An argument similar to that given above for the value of a call at expiry leads to the payoff for a put option. At expiry, the put option is worthless if $S > E$ but has the value $E - S$ for $S < E$. Thus, the payoff function of a put option is

$$P(S,T) = \max(E - S, 0). \tag{2.15}$$

The payoff diagram for a put is shown in Fig. 1.10 where the line shows the payoff function $\max(E-S, 0)$. In order to distinguish this put option from other more complicated put options, sometimes it is referred to as the vanilla put option.

2.2.4 Hedging and Greeks

The way to reduce the sensitivity of a portfolio to the movement of something by taking opposite positions in different financial instruments is called hedging. Hedging is a basic concept in finance. When we derived the Black–Scholes equation in Sect. 2.2.2, we chose the delta to be $\dfrac{\partial V}{\partial S}$, so that the portfolio Π became risk-free. This gives an important example on how hedging is applied. Let us see another example of hedging that is similar to what we have used in deriving the Black–Scholes equation.

Consider a call option on a stock. Figure 2.4 shows the relation between the call price and the underlying stock price. When the stock price corresponds to point A, the option price corresponds to point B and the Δ of the call is the slope of the line indicated. As an approximation

$$\Delta = \frac{\delta c}{\delta S},$$

where δS is a small change in the stock price and δc is the corresponding change in the call price.

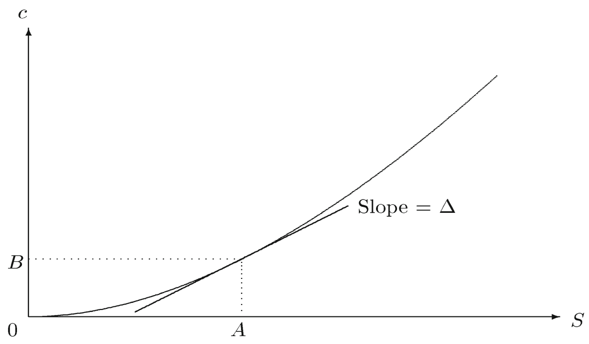

Fig. 2.4. Δ = the slope of a curve

Assume that the delta of the call option is 0.7 and a writer sold 10,000 units of call options. Then, the writer's position could be hedged by buying $0.7 \times 10{,}000 = 7{,}000$ shares of stocks. If the stock price goes up by \$0.50, the writer will earn \$3,500 from the 7,000 shares of stocks held. At the same time, the price of call options will go up approximately $0.7 \times 0.5 = \$0.35$, and he will lose $10{,}000 \times \$0.35 = \$3{,}500$ from 10,000 shares of option he sold. Therefore, the net profit or loss is about zero. If the price falls down by a small amount, the situation is similar. Consequently, the writer's position has been hedged quite well as long as the movement of the price is small. This is called delta hedging.

In the example above, we have considered only a call option. Actually, any portfolio can be hedged in this way. If Π denotes the price of option, then the slope is

$$\Delta = \frac{\partial \Pi}{\partial S}.$$

If the movement of the price is not very small, then it might be necessary to use the value of the second derivative of the portfolio with respect to S in order to eliminate most of the risk. The second derivative is known as gamma

$$\Gamma = \frac{\partial^2 \Pi}{\partial S^2}.$$

When hedging in practice, some other values, for example, $\frac{\partial \Pi}{\partial t}$, $\frac{\partial \Pi}{\partial \sigma}$, $\frac{\partial \Pi}{\partial r}$, $\frac{\partial \Pi}{\partial D_0}$, may need to be known. Usually, $\frac{\partial \Pi}{\partial t}$, $\frac{\partial \Pi}{\partial \sigma}$, and $\frac{\partial \Pi}{\partial r}$ are called theta, vega, and rho, respectively; namely, the following notation is used:

$$\Theta = \frac{\partial \Pi}{\partial t}, \quad \mathcal{V} = \frac{\partial \Pi}{\partial \sigma},$$

and

$$\rho = \frac{\partial \Pi}{\partial r}.$$

In currency options, D_0 is the interest rate in the foreign country. Thus, $\frac{\partial \Pi}{\partial D_0}$ is also known as rho. In order to distinguish $\frac{\partial \Pi}{\partial r}$ and $\frac{\partial \Pi}{\partial D_0}$, here we define

$$\rho_d = \frac{\partial \Pi}{\partial D_0}.$$

These quantities are usually referred to as Greeks.

When σ depends on S, or the coefficient of $\frac{\partial V}{\partial S}$ is more complicated, analytic expressions of option prices may not exist. In this case, we have to use numerical methods. Also sometimes (for example, for a call option), the solution is unbounded. It is not convenient to solve a problem numerically on an infinite domain with an unbounded solution. Therefore in Sect. 2.2.5, we also provide a transformation under which the Black–Scholes equation on $[0, \infty)$ becomes an equation on $[0, 1)$ with a bounded solution.

2.2.5 Transforming the Black–Scholes Equation into an Equation Defined on a Finite Domain

Let us consider the following option problem:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2(S)S^2 \dfrac{\partial^2 V}{\partial S^2} + (r - D_0)S\dfrac{\partial V}{\partial S} - rV = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 \le S < \infty, \quad t \le T, \\ V(S, T) = V_T(S), \qquad\qquad\quad 0 \le S < \infty. \end{cases} \tag{2.16}$$

The transformation to be described in this subsection works even when σ, r, or D_0 depends on S and t. For simplicity, we assume in the derivation that σ depends on S and that r, D_0 are constant. In this case, an analytic expression of the solution $V(S,t)$ may not exist, and numerical methods may be necessary. Also for a call option, the solution $V(S,t)$ is not bounded. Therefore, we introduce new independent variables and dependent variable through the following transformation:

$$\begin{cases} \xi = \dfrac{S}{S+P_m}, \\ \tau = T - t, \\ V(S,t) = (S+P_m)\overline{V}(\xi,\tau). \end{cases} \tag{2.17}$$

From Eq. (2.17) we have

$$S = \frac{P_m \xi}{1-\xi}, \quad S + P_m = \frac{P_m}{1-\xi}$$

and

$$\frac{d\xi}{dS} = \frac{P_m}{(S+P_m)^2} = \frac{(1-\xi)^2}{P_m}.$$

Because

$$\begin{aligned}
\frac{\partial V}{\partial t} &= \frac{\partial}{\partial t}\left[(S+P_m)\overline{V}(\xi,\tau)\right] = -(S+P_m)\frac{\partial \overline{V}}{\partial \tau} = -\frac{P_m}{1-\xi}\frac{\partial \overline{V}}{\partial \tau}, \\
\frac{\partial V}{\partial S} &= \frac{\partial}{\partial S}\left[(S+P_m)\overline{V}(\xi,\tau)\right] = (S+P_m)\frac{\partial \overline{V}}{\partial \xi}\frac{d\xi}{dS} + \overline{V} = (1-\xi)\frac{\partial \overline{V}}{\partial \xi} + \overline{V}, \\
\frac{\partial^2 V}{\partial S^2} &= \frac{\partial}{\partial \xi}\left[(1-\xi)\frac{\partial \overline{V}}{\partial \xi} + \overline{V}\right]\frac{d\xi}{dS} = \frac{(1-\xi)^3}{P_m}\frac{\partial^2 \overline{V}}{\partial \xi^2},
\end{aligned}$$

and let

$$\bar{\sigma}(\xi) = \sigma(S(\xi)) = \sigma\left(\frac{P_m \xi}{1-\xi}\right),$$

the original equation becomes[9]

$$\frac{P_m}{1-\xi}\frac{\partial \overline{V}}{\partial \tau} = \frac{\bar{\sigma}^2(\xi)P_m\xi^2(1-\xi)}{2}\frac{\partial^2 \overline{V}}{\partial \xi^2} + (r-D_0)P_m\xi\frac{\partial \overline{V}}{\partial \xi} + \frac{(r-D_0)\xi - r}{1-\xi}P_m\overline{V}$$

or

$$\begin{aligned}
\frac{\partial \overline{V}}{\partial \tau} = \frac{\bar{\sigma}^2(\xi)\xi^2(1-\xi)^2}{2}\frac{\partial^2 \overline{V}}{\partial \xi^2} + (r-D_0)\xi(1-\xi)\frac{\partial \overline{V}}{\partial \xi} - [r(1-\xi)+D_0\xi]\overline{V}, \\
0 \le \xi < 1, \quad 0 \le \tau.
\end{aligned}$$

[9]Actually, the same equation can be directly obtained by constructing a portfolio and using Itô lemma (see Problem 23).

Assume that $\overline{V}$ is a smooth function of ξ, then the equation also holds at $\xi = 1$. Because $V(S,T) = (S + P_m)\overline{V}(\xi, 0) = \overline{V}(\xi, 0)\dfrac{P_m}{1-\xi}$, the condition $V(S,T) = V_T(S)$ can be rewritten as $\overline{V}(\xi, 0) = V_T\left(\dfrac{P_m\xi}{1-\xi}\right)\dfrac{1-\xi}{P_m}$. Consequently, the problem (2.16) becomes

$$\begin{cases} \dfrac{\partial \overline{V}}{\partial \tau} = \dfrac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1-\xi)^2\dfrac{\partial^2 \overline{V}}{\partial \xi^2} + (r - D_0)\xi(1-\xi)\dfrac{\partial \overline{V}}{\partial \xi} - [r(1-\xi) + D_0\xi]\overline{V}, \\ \qquad\qquad 0 \le \xi \le 1, \quad 0 \le \tau, \\ \overline{V}(\xi, 0) = \dfrac{1-\xi}{P_m}V_T\left(\dfrac{P_m\xi}{1-\xi}\right), \qquad 0 \le \xi \le 1. \end{cases} \tag{2.18}$$

Thus, the transformation (2.17) converts a problem on an infinite domain into a problem on a finite domain. For a parabolic equation defined on a finite domain to have a unique solution, besides an initial condition, boundary conditions are usually needed. However, in this equation the coefficients of $\dfrac{\partial^2 \overline{V}}{\partial \xi^2}$ and $\dfrac{\partial \overline{V}}{\partial \xi}$ at $\xi = 0$ and at $\xi = 1$ are equal to zero, i.e., the equation degenerates to ordinary differential equations at the boundaries. Actually, at $\xi = 0$ the equation becomes

$$\frac{\partial \overline{V}(0,\tau)}{\partial \tau} = -r\overline{V}(0,\tau)$$

and the solution is

$$\overline{V}(0,\tau) = \overline{V}(0,0)\mathrm{e}^{-r\tau}. \tag{2.19}$$

Similarly, at $\xi = 1$ the equation reduces to

$$\frac{\partial \overline{V}(1,\tau)}{\partial \tau} = -D_0\overline{V}(1,\tau),$$

from which we have

$$\overline{V}(1,\tau) = \overline{V}(1,0)\mathrm{e}^{-D_0\tau}. \tag{2.20}$$

Therefore for this equation, the two solutions of the ordinary differential equations provide the values at the boundaries, and no boundary conditions are needed in order for the problem (2.18) to have a unique solution.

Consequently, in order to price an option, we need to solve a problem on a finite domain if this formulation is adopted. From the point view of numerical methods, such a formulation is better. This is its first advantage. Actually, the uniqueness of solution for problem (2.18) can easily be proved (see Sect. 2.4). Indeed, not only the uniqueness can be proved, but the stability of the solution with respect to the initial value can also be shown easily. That is, this formulation makes proof of some theoretical problems easy. This is its other advantage.

For a call option, the payoff is

$$V(S,T) = \max(S-E,0),$$

so the initial condition in the problem (2.18) for a call is

$$\begin{aligned}\overline{V}(\xi,0) &= \max(S-E,0)(1-\xi)/P_m \\ &= \max\left(\frac{P_m\xi}{1-\xi}-E,0\right)(1-\xi)/P_m \\ &= \max\left(\xi-\frac{E}{P_m}(1-\xi),0\right).\end{aligned}$$

For a put option

$$V(S,T) = \max(E-S,0).$$

Therefore

$$\overline{V}(\xi,0) = \max\left(\frac{E}{P_m}(1-\xi)-\xi,0\right).$$

Let $P_m = E$, the two initial conditions become

$$\overline{V}(\xi,0) = \max\left(2\xi-1,0\right) \quad \text{and} \quad \overline{V}(\xi,0) = \max\left(1-2\xi,0\right),$$

respectively. Therefore, a European call option is the solution of the following problem:

$$\begin{cases}\dfrac{\partial\overline{V}}{\partial\tau} = \dfrac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1-\xi)^2\dfrac{\partial^2\overline{V}}{\partial\xi^2}+(r-D_0)\xi(1-\xi)\dfrac{\partial\overline{V}}{\partial\xi}-[r(1-\xi)+D_0\xi]\overline{V}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0\le\xi\le1, \quad 0\le\tau, \\ \overline{V}(\xi,0) = \max(2\xi-1,0), \qquad\qquad\qquad\qquad\qquad 0\le\xi\le1\end{cases} \tag{2.21}$$

and the solution of the problem

$$\begin{cases}\dfrac{\partial\overline{V}}{\partial\tau} = \dfrac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1-\xi)^2\dfrac{\partial^2\overline{V}}{\partial\xi^2}+(r-D_0)\xi(1-\xi)\dfrac{\partial\overline{V}}{\partial\xi}-[r(1-\xi)+D_0\xi]\overline{V}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0\le\xi\le1, \quad 0\le\tau, \\ \overline{V}(\xi,0) = \max(1-2\xi,0), \qquad\qquad\qquad\qquad\qquad 0\le\xi\le1\end{cases} \tag{2.22}$$

gives the price of a European put option. In the problem (2.21) the initial condition is bounded, so $\overline{V}(\xi,\tau)$, as a solution of a linear parabolic equation, is also bounded. Therefore in this case, the solution that needs to be found numerically is bounded.

So far, we assumed that σ depends only on S and that r and D_0 are constant. However, the result will be the same if σ depends on both S and t, and r and D_0 also depend on S and t.

Finally, we would like to point out that from the expression (2.20) we can have an asymptotic expression of the solution of the Black–Scholes equation at infinity. Because at $\xi = 1$ there is an analytic solution (2.20), noticing

$$V(S,t) = (S + P_m)\overline{V}(\xi,\tau),$$

for $S \approx \infty$ we have

$$\begin{aligned} V(S,t) &= (S + P_m)\overline{V}(\xi,\tau) \approx (S + P_m)\overline{V}(1,\tau) \\ &= (S + P_m)\overline{V}(1,0)\mathrm{e}^{-D_0\tau} \\ &\approx V(S,T)\mathrm{e}^{-D_0\tau} = V(S,T)\mathrm{e}^{-D_0(T-t)}. \end{aligned} \tag{2.23}$$

This is an asymptotic expression of the solution of the Black–Scholes equation at infinity.

2.2.6 Derivation of the Equation for Options on Futures

As we know, a futures contract in finance is a standardized contract between two parties to exchange a specified asset of a standardized quantity and quality for a price K (the delivery price) agreed today with delivery occurring at a specified future date, while a forward contract in finance is a nonstandardized contract between two parties to buy or sell an asset at a specified future time at a price K agreed today. There are some differences between a futures contract and a forward contract, but both are a contract in which two parties agree to exchange a specified asset for a specified amount of cash at a specified future date. Here we derive the PDE for options on such a contract.

Suppose that the price of the underlying asset satisfies

$$dS = \mu S dt + \sigma S dX, \tag{2.24}$$

and it pays dividends continuously with a constant dividend yield D_0. We also assume that the interest rate r is a constant. Let T be the expiration date of the contract and t be the time today.

Before deriving the PDE, we point out that the value of a forward/futures contract at time t is

$$f = S\mathrm{e}^{-D_0(T-t)} - K\mathrm{e}^{-r(T-t)}, \tag{2.25}$$

from which we can have

$$S = \mathrm{e}^{D_0(T-t)}\left(f + K\mathrm{e}^{-r(T-t)}\right). \tag{2.26}$$

The reason is as follows. At time t, the seller of this contract, who gets f when the contract is sold, can borrow $K\mathrm{e}^{-r(T-t)}$ from a bank with an interest rate r and buy $\mathrm{e}^{-D_0(T-t)}$ units of the asset by spending $S\mathrm{e}^{-D_0(T-t)}$. At time T, the seller will get K from the holder of the contract, which will be paid to the bank, and give a unit of the asset to the holder. Therefore, there is no risk for seller, and it is a reasonable price for the contract.

Now we consider an option on such a contract. When we consider options on stocks, we let its value be a function of the value of the stock, S, and t. Thus, it is natural to let the value of options on futures be a function of the value of futures contracts, f, and t. That is, let $V_1(f,t)$ denote the price of the option. The PDE for $V_1(f,t)$ can be derived in the following way. Consider a portfolio

$$\Pi = V_1(f,t) - \Delta f.$$

Because we assume that S is a lognormal variable,[10] using Itô's lemma, for f we have

$$\begin{aligned}
df &= \frac{\partial f}{\partial S} dS + \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) dt \\
&= \mathrm{e}^{-D_0(T-t)} dS + \left(D_0 S \mathrm{e}^{-D_0(T-t)} - rK\mathrm{e}^{-r(T-t)} \right) dt \\
&= \mathrm{e}^{-D_0(T-t)} (\mu S dt + \sigma S dX) + \left(D_0 S \mathrm{e}^{-D_0(T-t)} - rK\mathrm{e}^{-r(T-t)} \right) dt \\
&= \left(\mathrm{e}^{-D_0(T-t)} \mu S + D_0 S \mathrm{e}^{-D_0(T-t)} - rK\mathrm{e}^{-r(T-t)} \right) dt + \mathrm{e}^{-D_0(T-t)} \sigma S dX \\
&= \left[(\mu + D_0) \left(f + K\mathrm{e}^{-r(T-t)} \right) - rK\mathrm{e}^{-r(T-t)} \right] dt \\
&\quad + \sigma \left[f + K\mathrm{e}^{-r(T-t)} \right] dX.
\end{aligned}$$

Using this relation and Itô's lemma again, we can further have

$$\begin{aligned}
d\Pi &= \frac{\partial V_1}{\partial f} df + \left[\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2 \left(f + K\mathrm{e}^{-r(T-t)} \right)^2 \frac{\partial^2 V_1}{\partial f^2} \right] dt - \Delta df \\
&= \left(\frac{\partial V_1}{\partial f} - \Delta \right) df + \left[\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2 \left(f + K\mathrm{e}^{-r(T-t)} \right)^2 \frac{\partial^2 V_1}{\partial f^2} \right] dt.
\end{aligned}$$

If we choose $\Delta = \dfrac{\partial V_1}{\partial f}$, then the portfolio $d\Pi$ is risk-free and

$$\begin{aligned}
d\Pi &= \left[\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2 \left(f + K\mathrm{e}^{-r(T-t)} \right)^2 \frac{\partial^2 V_1}{\partial f^2} \right] dt \\
&= r\Pi dt = r \left(V_1(f,t) - \frac{\partial V_1}{\partial f} f \right) dt.
\end{aligned}$$

This relation can be rewritten as

$$\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2 \left(f + K\mathrm{e}^{-r(T-t)} \right)^2 \frac{\partial^2 V_1}{\partial f^2} + rf\frac{\partial V_1}{\partial f} - rV_1 = 0. \tag{2.27}$$

[10] If we assume that f has a lognormal distribution, the PDE will be different.

Actually, if we use another independent variable, the PDE will become simple. This independent variable is the forward price F. What is the forward price? Consider a foreign currency. Let S be the current spot price in dollars of one unit of the foreign currency at time t and F be the forward price in dollars of one unit of the foreign currency in the forward contract issued at time t and expiring at time T. Let D_0 be the interest rate in the foreign country. Then for the forward price F, there is the following expression:

$$F = \mathrm{e}^{(r-D_0)(T-t)}S. \tag{2.28}$$

This is because the seller of the forward contract can borrow $\mathrm{e}^{-D_0(T-t)}S$ to buy $\mathrm{e}^{-D_0(T-t)}$ units of the foreign currency at time t, and at time T he or she can have one unit of the foreign currency and can obtain an amount of $\mathrm{e}^{(r-D_0)(T-t)}S$ from one unit of the foreign currency, which is what he or she needs in order to pay off the borrowing. It is clear that between F and f, there are the following relations:

$$f = \mathrm{e}^{-r(T-t)}\left(S\mathrm{e}^{(r-D_0)(T-t)} - K\right) = \mathrm{e}^{-r(T-t)}\left(F - K\right) \tag{2.29}$$

and

$$F = \mathrm{e}^{r(T-t)}f + K.$$

Let $V(F,t)$ denote the value of that option, and let us find the PDE for the function $V(F,t)$. Set

$$\begin{cases} F = \mathrm{e}^{r(T-t)}f + K, \\ t = t, \\ V_1(f,t) = V(F(f,t),t) = V(\mathrm{e}^{r(T-t)}f + K, t). \end{cases}$$

From these expressions, we have

$$\begin{aligned} \frac{\partial V_1}{\partial t} &= \frac{\partial V}{\partial t} - r\mathrm{e}^{r(T-t)}f\frac{\partial V}{\partial F}, \\ \frac{\partial V_1}{\partial f} &= \mathrm{e}^{r(T-t)}\frac{\partial V}{\partial F}, \end{aligned}$$

and

$$\frac{\partial^2 V_1}{\partial f^2} = \mathrm{e}^{2r(T-t)}\frac{\partial^2 V}{\partial F^2}.$$

Using these relations, we can rewrite the PDE for V_1 as

$$\begin{aligned} &\frac{\partial V}{\partial t} - r\mathrm{e}^{r(T-t)}f\frac{\partial V}{\partial F} + \frac{1}{2}\sigma^2\left(f + K\mathrm{e}^{-r(T-t)}\right)^2\mathrm{e}^{2r(T-t)}\frac{\partial^2 V}{\partial F^2} \\ &\quad + rf\mathrm{e}^{r(T-t)}\frac{\partial V}{\partial F} - rV = 0 \end{aligned}$$

or

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2F^2\frac{\partial^2 V}{\partial F^2}-rV=0. \tag{2.30}$$

Usually, this equation is called the PDE for an option on a futures contract (see [8]). However, the PDE indeed is a variant of the Black–Scholes equation in Sect. 2.2.2. Because F is a function of S and t, we can define a function of S, t as follows: $V_2(S,t)=V(F(S,t),t)$. It is clear that $V_2(S,t)$ also gives the value of the option. The only difference is that it is a function of S, t, not F, t. As we know, any function of S, t, giving the value of a derivative security, should satisfy the Black–Scholes equation; that is, the equation

$$\frac{\partial V_2}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V_2}{\partial S^2}+(r-D_0)S\frac{\partial^2 V_2}{\partial S}-rV_2=0$$

holds. Let us show by direct calculation that $V_2(S,t)$ satisfies the Black–Scholes equation. Because

$$V(F,t)=V_2(S(F,t),t)$$

and

$$S=\mathrm{e}^{-(r-D_0)(T-t)}F,$$

we have

$$\begin{aligned}\frac{\partial V}{\partial t}&=\frac{\partial V_2}{\partial t}+\frac{\partial V_2}{\partial S}\frac{\partial S}{\partial t}=\frac{\partial V_2}{\partial t}+(r-D_0)S\frac{\partial V_2}{\partial S},\\ \frac{\partial V}{\partial F}&=\mathrm{e}^{-(r-D_0)(T-t)}\frac{\partial V_2}{\partial S},\\ \frac{\partial^2 V}{\partial F^2}&=\mathrm{e}^{-2(r-D_0)(T-t)}\frac{\partial^2 V_2}{\partial S^2}.\end{aligned}$$

From Eq. (2.30) we can have

$$\frac{\partial V_2}{\partial t}+(r-D_0)S\frac{\partial V_2}{\partial S}+\frac{1}{2}\sigma^2F^2\mathrm{e}^{-2(r-D_0)(T-t)}\frac{\partial^2 V_2}{\partial S^2}-rV_2=0$$

or

$$\frac{\partial V_2}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V_2}{\partial S^2}+(r-D_0)S\frac{\partial V_2}{\partial S}-rV_2=0.$$

Thus, we have proved that if the value of an option on a futures contract is a function of S and t, then it satisfies the Black–Scholes equation. It can also be proved that if we let $V_3(S,t)=V_1(f(S,t),t)$, then $V_3(S,t)$ also satisfies the Black–Scholes equation. This means that Eq. (2.27) is also a variant of the Black–Scholes equation. The proof is left for readers as a part of Problem 16. When the Black–Scholes equation is derived, the randomness of the value of derivative securities is cancelled by the randomness of the value of the stock, S, and when Eq. (2.27) is derived, the randomness is cancelled by the randomness of the value of the forward/futures contract, f. However, f is a function of S and t given by the expression (2.25). Thus, their randomnesses are related. Consequently, the Black–Scholes equation and the equation for options on futures contracts are the same essentially.

2.3 General Equations for Derivatives

Generally speaking, a financial derivative could depend on several random variables, and a random variable may not represent a price of an asset that can be traded on the market. For example, a derivative could depend on prices of several assets. Also interest rates and volatilities may need to be treated as random variables. As we know, both interest rates and volatilities are not prices of assets. In this section, we will derive the general partial differential equations satisfied by derivatives, where there exist several state variables and a state variable may not be a price of an asset traded on the market or even not be related to a price. The derivation of equations for derivatives with several state variables can be found from other books, for example, the books by Hull [42], and Wilmott, Dewynne, and Howison [84].

2.3.1 Generalization of Itô's Lemma

Suppose a financial derivative depends on time t and n random state variables, namely, $S_1, S_2, \cdots, S_n$. Each of them satisfies a stochastic differential equation

$$dS_i = a_i dt + b_i dX_i, \qquad i = 1, 2, \cdots, n, \tag{2.31}$$

where a_i, b_i are functions of $S_1, S_2, \cdots, S_n$ and t, and $dX_i = \phi_i \sqrt{dt}$ are Wiener processes. In addition, ϕ_1, ϕ_2, $\cdots$, ϕ_n have a joint normal distribution and

$$\mathrm{E}\left[\phi_i \phi_j\right] = \rho_{ij}, \tag{2.32}$$

where

$$-1 \le \rho_{ij} \le 1.$$

If $\rho_{ij} = 0$, then ϕ_i and ϕ_j are not correlated. If $\rho_{ij} = \pm 1$, then ϕ_i and ϕ_j are completely correlated. It is clear that $\rho_{ii} = 1$. In this book ρ_{ij} is referred to as the correlation coefficient between S_i and S_j.

Let $V = V(S_1, S_2, \cdots, S_n, t)$. According to the Taylor expansion, we have

$$\begin{aligned} dV =& V\left(S_1 + dS_1, S_2 + dS_2, \cdots, S_n + dS_n, t + dt\right) - V\left(S_1, S_2, \cdots, S_n, t\right) \\ =& \sum_{i=1}^{n} \frac{\partial V}{\partial S_i} dS_i + \frac{\partial V}{\partial t} dt + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 V}{\partial S_i \partial S_j} dS_i dS_j \\ &+ \sum_{i=1}^{n} \frac{\partial^2 V}{\partial S_i \partial t} dS_i dt + \frac{1}{2} \frac{\partial^2 V}{\partial t^2} (dt)^2 + \cdots . \end{aligned}$$

Because

$$\lim_{dt \to 0} dS_i dS_j / dt = b_i b_j \rho_{ij}$$

and $dS_i dt$ is a quantity of order $(dt)^{3/2}$, the relation above as $dt \to 0$ becomes

$$dV = f dt + \sum_{i=1}^{n} \frac{\partial V}{\partial S_i} dS_i, \qquad (2.33)$$

where

$$f = \frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 V}{\partial S_i \partial S_j} b_i b_j \rho_{ij}.$$

This is called the generalized Itô's lemma.

2.3.2 Derivation of Equations for Financial Derivatives

On the n random variables, we further assume that

$$S_1, S_2, \cdots, \text{ and } S_m, \quad m \leq n,$$

are prices of some assets which can be traded on markets, and that the k-th asset pays a dividend payment $D_k dt$ during the time interval $[t, t + dt]$, D_k being a known function that may depend on $S_1, S_2, \cdots, S_n$ and t. In order to derive the general PDE for financial derivatives, we suppose that there are

$$n - m + 1$$

distinct financial derivatives dependent on S_1, S_2, $\cdots$, S_n and t. Let V_k stand for the value of the k-th derivative, $k = 0, 1, \cdots, n - m$ and assume that the k-th derivative during the time interval $[t, t + dt]$ pays coupon payment $K_k dt$, K_k being a known function that may depend on $S_1, S_2, \cdots, S_n$ and t. They could have different expiries, different exercise prices, or different payoff functions. Even some of the derivatives may depend on only some of the random variables. According to the generalized Itô's lemma, for each derivative, we have

$$dV_k = f_k dt + \sum_{i=1}^{n} \nu_{i,k} dS_i,$$

where

$$f_k = \frac{\partial V_k}{\partial t} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 V_k}{\partial S_i \partial S_j} b_i b_j \rho_{ij}$$

and

$$\nu_{i,k} = \frac{\partial V_k}{\partial S_i}.$$

Consider a portfolio consisting of the $n-m+1$ derivatives and the m assets, whose prices are $S_1, S_2, \cdots, S_m$:

$$\Pi = \sum_{k=0}^{n-m} \Delta_k V_k + \sum_{k=n-m+1}^{n} \Delta_k S_{k-n+m},$$

where Δ_k is the amount of the k-th derivative for $k = 0, 1, \cdots, n-m$ and the amount of the $(k-n+m)$-th asset, for $k = n-m+1, n-m+2, \cdots, n$. During the time interval $[t, t+dt]$, the holder of this portfolio will earn

$$\begin{aligned}
&\sum_{k=0}^{n-m} \Delta_k \left(dV_k + K_k dt\right) + \sum_{k=n-m+1}^{n} \Delta_k \left(dS_{k-n+m} + D_{k-n+m} dt\right) \\
&= \sum_{k=0}^{n-m} \Delta_k \left(f_k dt + \sum_{i=1}^{n} \nu_{i,k} dS_i + K_k dt\right) \\
&\quad + \sum_{k=n-m+1}^{n} \Delta_k \left(dS_{k-n+m} + D_{k-n+m} dt\right) \\
&= \sum_{k=0}^{n-m} \Delta_k \left(f_k + K_k\right) dt + \sum_{i=1}^{n} \left(\sum_{k=0}^{n-m} \Delta_k \nu_{i,k}\right) dS_i \\
&\quad + \sum_{i=1}^{m} \Delta_{i+n-m} dS_i + \sum_{k=n-m+1}^{n} \Delta_k D_{k-n+m} dt \\
&= \sum_{k=0}^{n-m} \Delta_k \left(f_k + K_k\right) dt + \sum_{i=1}^{m} \left(\sum_{k=0}^{n-m} \Delta_k \nu_{i,k} + \Delta_{i+n-m}\right) dS_i \\
&\quad + \sum_{i=m+1}^{n} \left(\sum_{k=0}^{n-m} \Delta_k \nu_{i,k}\right) dS_i + \sum_{k=n-m+1}^{n} \Delta_k D_{k-n+m} dt.
\end{aligned}$$

Let us choose Δ_k so that

$$\sum_{k=0}^{n-m} \Delta_k \nu_{i,k} + \Delta_{i+n-m} = 0, \qquad i = 1, 2, \cdots, m$$

and

$$\sum_{k=0}^{n-m} \Delta_k \nu_{i,k} = 0, \qquad i = m+1, m+2, \cdots, n.$$

In this case the portfolio is risk-free, so its return rate is r, i.e.,

$$\begin{aligned}
&\sum_{k=0}^{n-m} \Delta_k \left(f_k + K_k\right) dt + \sum_{k=n-m+1}^{n} \Delta_k D_{k-n+m} dt \\
&= r \left[\sum_{k=0}^{n-m} \Delta_k V_k + \sum_{k=n-m+1}^{n} \Delta_k S_{k-n+m}\right] dt,
\end{aligned}$$

or

$$\sum_{k=0}^{n-m} \Delta_k \left(f_k + K_k - rV_k\right) + \sum_{k=n-m+1}^{n} \Delta_k \left(D_{k-n+m} - rS_{k-n+m}\right) = 0,$$

or

$$\sum_{k=0}^{n-m} \Delta_k \left(f_k + K_k - rV_k\right) + \sum_{k=1}^{m} \Delta_{n-m+k} \left(D_k - rS_k\right) = 0.$$

This relation and the relations the chosen Δ_k satisfy can be written together in a matrix form:

$$\begin{bmatrix} \nu_{1,0} & \nu_{1,1} & \cdots & \nu_{1,n-m} & 1 & 0 & \cdots & 0 \\ \nu_{2,0} & \nu_{2,1} & \cdots & \nu_{2,n-m} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \nu_{m,0} & \nu_{m,1} & \cdots & \nu_{m,n-m} & 0 & 0 & \cdots & 1 \\ \nu_{m+1,0} & \nu_{m+1,1} & \cdots & \nu_{m+1,n-m} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \nu_{n,0} & \nu_{n,1} & \cdots & \nu_{n,n-m} & 0 & 0 & \cdots & 0 \\ g_0 & g_1 & \cdots & g_{n-m} & h_1 & h_2 & \cdots & h_m \end{bmatrix} \begin{bmatrix} \Delta_0 \\ \Delta 1 \\ \vdots \\ \Delta_{n-m} \\ \Delta n-m+1 \\ \Delta n-m+2 \\ \vdots \\ \Delta_n \end{bmatrix} = 0,$$

where

$$g_k = f_k + K_k - rV_k, \; k = 0, 1, \cdots, n-m$$

and

$$h_k = D_k - rS_k, \; k = 1, 2, \cdots, m.$$

In order for the system to have a non-trivial solution, the determinant of the matrix must be zero, or the $n+1$ row vectors of the matrix must be linearly dependent. Therefore, it is expected that the last row can be expressed as a linear combination of the other rows with coefficients $\tilde{\lambda}_1, \tilde{\lambda}_2, \cdots, \tilde{\lambda}_n$:

$$g_k = \sum_{i=1}^{n} \tilde{\lambda}_i \nu_{i,k}, \quad k = 0, 1, \cdots, n-m$$

and

$$h_k = \tilde{\lambda}_k, \quad k = 1, 2, \cdots, m.$$

Using the last m relations, we can rewrite the first $n-m+1$ relations as

$$g_k - \sum_{i=1}^{m} h_i \nu_{i,k} - \sum_{i=m+1}^{n} \tilde{\lambda}_i \nu_{i,k} = 0, \quad k = 0, 1, \cdots, n-m,$$

which means that any derivative satisfies an equation in the form

$$f + K - rV - \sum_{i=1}^{m} h_i \frac{\partial V}{\partial S_i} - \sum_{i=m+1}^{n} \tilde{\lambda}_i \frac{\partial V}{\partial S_i} = 0,$$

or

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} b_i b_j \rho_{ij} \frac{\partial^2 V}{\partial S_i \partial S_j} + \sum_{i=1}^{m} (rS_i - D_i) \frac{\partial V}{\partial S_i}$$
$$- \sum_{i=m+1}^{n} \tilde{\lambda}_i \frac{\partial V}{\partial S_i} - rV + K = 0,$$

where b_i, ρ_{ij} are given functions in the models of S_i, $\tilde{\lambda}_i$ are unknown functions which are independent of $V_0, V_1, \cdots, V_{n-m}$ and could depend on $S_1, S_2, \cdots, S_n$ and t, and K depends on the individual derivative security. Usually $\tilde{\lambda}_i$ is written in the form:

$$\tilde{\lambda}_i = \lambda_i b_i - a_i$$

and λ_i is called the market price of risk for S_i. Using this notation, we finally arrive at

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} b_i b_j \rho_{ij} \frac{\partial^2 V}{\partial S_i \partial S_j} + \sum_{i=1}^{m} (rS_i - D_i) \frac{\partial V}{\partial S_i}$$
$$+ \sum_{i=m+1}^{n} (a_i - \lambda_i b_i) \frac{\partial V}{\partial S_i} - rV + K = 0. \tag{2.34}$$

It is clear that if $m = n = 1$, $b_1 = \sigma_1 S_1$, $D_1 = D_{01} S_1$, and $K = 0$, then this equation becomes the Black–Scholes equation (2.12) after ignoring the subscript 1.

In the last we give some explanation on why λ_i is called the market price of risk for S_i. For simplicity, assume that none of S_k, $k = 1, 2, \cdots, n$, is a price. In this case the PDE above becomes

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} b_i b_j \rho_{ij} \frac{\partial^2 V}{\partial S_i \partial S_j} + \sum_{i=1}^{n} (a_i - \lambda_i b_i) \frac{\partial V}{\partial S_i} - rV + K = 0.$$

According to Itô's lemma and using this PDE, we have

$$dV = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 V}{\partial S_i \partial S_j} b_i b_j \rho_{ij} \right) dt + \sum_{i=1}^{n} \frac{\partial V}{\partial S_i} dS_i$$
$$= \left[\sum_{i=1}^{n} (\lambda_i b_i - a_i) \frac{\partial V}{\partial S_i} + rV - K \right] dt + \sum_{i=1}^{n} \frac{\partial V}{\partial S_i} (a_i dt + b_i dX_i)$$

or

$$dV + K dt - rV dt = \sum_{i=1}^{n} \frac{\partial V}{\partial S_i} b_i \left(dX_i + \lambda_i dt \right).$$

Here, $dV + K dt$ is the return for the derivative including the coupon payment and $rV dt$ is the return if the investment is risk-free. Therefore, $dV + K dt - rV dt$

is the excess return above the risk-free rate during the time interval $[t, t+dt]$. This equals the right-hand side of the equation. Its expectation is $\sum_{i=1}^{n} \frac{\partial V}{\partial S_i} b_i \lambda_i dt$ because $\mathrm{E}[dX_i] = 0$, $i = 1, 2, \cdots, n$. Therefore, the term $\frac{\partial V}{\partial S_i} b_i \lambda_i dt$ may be interpreted as an excess return above the risk-free return for taking the risk dX_i. Consequently, λ_i is a price of risk for S_i that is associated with dX_i and is often called the market price of risk for S_i.

2.3.3 Three Types of State Variables

When we talk about the market price of risk, we can think that there are three types of state variables.

The first type of state variable is a price of an asset. In this case the coefficient of $\frac{\partial V}{\partial S_i}$ in Eq. (2.34) is $rS_i - D_i$. Thus for such a state variable, there is no market price of risk. However, this fact can also be understood in another way: there still is a market price of risk and the market price of risk for an asset is determined by

$$a_i - \lambda_i b_i = rS_i - D_i(S_1, S_2, \cdots, S_n, t). \tag{2.35}$$

This can be explained as follows. Suppose that the $(m+1)$-th random variable actually is a price of an asset. In this case, let us consider a portfolio consisting of the $n-m$ derivatives and the $m+1$ assets, and derive the PDE. In the new PDE obtained the coefficient of $\frac{\partial V}{\partial S_{m+1}}$ is $rS_{m+1} - D_{m+1}$. The price of any financial derivative dependent on $S_1, S_2, \cdots, S_n, t$ should satisfy the same equation. Thus $a_{m+1} - \lambda_{m+1} b_{m+1}$ should equal $rS_{m+1} - D_{m+1}(S_1, S_2, \cdots, S_n, t)$, which means that the relation (2.35) holds. If $D_i = D_{0i} S_i$, then the following should be true:

$$a_i - \lambda_i b_i = (r - D_{0i}) S_i. \tag{2.36}$$

This can be shown in another way, which is left for readers as Problem 22.

A state variable S_i with $b_i = 0$ in Eq. (2.31) is another type of state variable. From $b_i = 0$, we have

$$a_i - \lambda_i b_i = a_i, \tag{2.37}$$

so λ_i disappears in Eq. (2.34). As we will see from Chap. 4, if S_i' is the price of a stock and S_i is the maximum, minimum, or average price of the stock during a time period, and both of them are state variables, then $dS_i = a_i dt$.

If S_i is the short-term interest rate, then in order to determine λ_i, we have to solve an inverse problem. We will discuss this problem in detail in Chap. 5. This is an example of the third type of state variable. Besides the interest rate, the random volatility also falls into this type of state variable.

2.3.4 Random Variables Not Being But Related to Prices of Assets

In Sect. 2.3.2 we assume that a random variable either is a value of a derivative or is a price of an asset. However, sometimes a random variable is merely related to an asset price. The random variable ξ in Sect. 2.2.5 and the random variable F in Sect. 2.2.6 are such examples. In Sects. 2.2.5 and 2.2.6, the PDEs for $\overline{V}(\xi,\tau)$ and $V(F,t)$ are obtained from the known PDEs by using transformations of independent and dependent variables. However, the two PDEs can also be obtained by setting a portfolio and using Itô's lemma. In Problems 23 and 24, readers are asked to derive the two PDEs and some other PDEs in this way. Here we assume that there are m random variables that do not represent prices of assets, but there exist m known different functions dependent on the m random variables that represent asset prices. In this case, in the procedure of deriving a PDE, determining $\Delta_0,\cdots,\Delta_n$ in the portfolio will involve solving a linear system; the expressions of the coefficients of the first derivatives in the PDE are more complicated. Here we give an example with $m=2$, and readers are asked to do Problem 26 with $m=3$.

Suppose that ξ_1 and ξ_2 satisfy the system of stochastic differential equations

$$d\xi_i = \mu_i(\xi_1,\xi_2,t)dt + \sigma_i(\xi_1,\xi_2,t)dX_i, \quad i=1,2,$$

where dX_i are the Wiener processes and $\mathrm{E}\left[dX_i dX_j\right] = \rho_{ij}dt$ with $-1 \le \rho_{ij} \le 1$. The functions

$$\begin{cases} Z_1\left(\xi_1\right) &= Z_{1,l} + \xi_1\left(1 - Z_{1,l}\right), \\ Z_2\left(\xi_1,\xi_2\right) &= Z_{2,l} + \xi_2\left[Z_1\left(\xi_1\right) - Z_{2,l}\right] \\ &= Z_{2,l} + \xi_2\left[Z_{1,l} + \xi_1\left(1 - Z_{1,l}\right) - Z_{2,l}\right] \end{cases}$$

represent prices of two nondividend-paying assets, where $Z_{1,l}$ and $Z_{2,l}$ are two constants. Let $V(\xi_1,\xi_2,t)$ be the value of a noncoupon-paying derivative security. Because $Z_1(\xi_1)$ and $Z_2(\xi_1,\xi_2)$ are prices of two assets, we can set a portfolio

$$\Pi = V - \Delta_1 Z_1(\xi_1) - \Delta_2 Z_2(\xi_1,\xi_2)$$

when deriving the PDE for $V(\xi_1,\xi_2,t)$. According to Itô's lemma and noticing the form of functions $Z_1\left(\xi_1\right)$ and $Z_2\left(\xi_1,\xi_2\right)$, we have

$$\begin{aligned} dV &= \sum_{i=1}^{2} \frac{\partial V}{\partial \xi_i} d\xi_i + \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} \sigma_i\sigma_j\rho_{i,j}\frac{\partial^2 V}{\partial \xi_i \partial \xi_j} \right) dt, \\ dZ_1 &= \frac{\partial Z_1}{\partial \xi_1} d\xi_1, \\ dZ_2 &= \sum_{i=1}^{2} \frac{\partial Z_2}{\partial \xi_i} d\xi_i + \sigma_1\sigma_2\rho_{1,2}\frac{\partial^2 Z_2}{\partial \xi_1 \partial \xi_2} dt. \end{aligned}$$

Using these expressions, we obtain

$$\begin{aligned}
d\Pi =& \sum_{i=1}^{2} \frac{\partial V}{\partial \xi_i} d\xi_i + \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \sigma_i \sigma_j \rho_{i,j} \frac{\partial^2 V}{\partial \xi_i \partial \xi_j} \right) dt \\
& - \Delta_1 \frac{\partial Z_1}{\partial \xi_1} d\xi_1 - \Delta_2 \left(\sum_{i=1}^{2} \frac{\partial Z_2}{\partial \xi_i} d\xi_i + \sigma_1 \sigma_2 \rho_{1,2} \frac{\partial^2 Z_2}{\partial \xi_1 \partial \xi_2} dt \right) \\
=& \left(\frac{\partial V}{\partial \xi_1} - \Delta_1 \frac{\partial Z_1}{\partial \xi_1} - \Delta_2 \frac{\partial Z_2}{\partial \xi_1} \right) d\xi_1 + \left(\frac{\partial V}{\partial \xi_2} - \Delta_2 \frac{\partial Z_2}{\partial \xi_2} \right) d\xi_2 \\
& + \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \sigma_i \sigma_j \rho_{i,j} \frac{\partial^2 V}{\partial \xi_i \partial \xi_j} \right) dt \\
& - \Delta_2 \sigma_1 \sigma_2 \rho_{1,2} \frac{\partial^2 Z_2}{\partial \xi_1 \partial \xi_2} dt.
\end{aligned}$$

Let us choose

$$\begin{aligned}
\Delta_2 &= \frac{1}{\frac{\partial Z_2}{\partial \xi_2}} \frac{\partial V}{\partial \xi_2}, \\
\Delta_1 &= \frac{1}{\frac{\partial Z_1}{\partial \xi_1}} \left(\frac{\partial V}{\partial \xi_1} - \Delta_2 \frac{\partial Z_2}{\partial \xi_1} \right) = \frac{1}{\frac{\partial Z_1}{\partial \xi_1}} \frac{\partial V}{\partial \xi_1} - \frac{\frac{\partial Z_2}{\partial \xi_1}}{\frac{\partial Z_1}{\partial \xi_1} \frac{\partial Z_2}{\partial \xi_2}} \frac{\partial V}{\partial \xi_2},
\end{aligned}$$

so that

$$\frac{\partial V}{\partial \xi_1} - \Delta_1 \frac{\partial Z_1}{\partial \xi_1} - \Delta_2 \frac{\partial Z_2}{\partial \xi_1} = \frac{\partial V}{\partial \xi_2} - \Delta_2 \frac{\partial Z_2}{\partial \xi_2} = 0.$$

In this case, the portfolio is risk-free and the return rate should be r:

$$\begin{aligned}
& \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \sigma_i \sigma_j \rho_{i,j} \frac{\partial^2 V}{\partial \xi_i \partial \xi_j} \right) dt - \Delta_2 \sigma_1 \sigma_2 \rho_{1,2} \frac{\partial^2 Z_2}{\partial \xi_1 \partial \xi_2} dt \\
& = r \left(V - \Delta_1 Z_1 - \Delta_2 Z_2 \right) dt
\end{aligned}$$

or

$$\begin{aligned}
& \frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \sigma_i \sigma_j \rho_{i,j} \frac{\partial^2 V}{\partial \xi_i \partial \xi_j} + \frac{r Z_1}{\frac{\partial Z_1}{\partial \xi_1}} \frac{\partial V}{\partial \xi_1} \\
& + \left(- \frac{r Z_1 \frac{\partial Z_2}{\partial \xi_1}}{\frac{\partial Z_1}{\partial \xi_1} \frac{\partial Z_2}{\partial \xi_2}} + \frac{r Z_2}{\frac{\partial Z_2}{\partial \xi_2}} - \frac{\sigma_1 \sigma_2 \rho_{1,2} \frac{\partial^2 Z_2}{\partial \xi_1 \partial \xi_2}}{\frac{\partial Z_2}{\partial \xi_2}} \right) \frac{\partial V}{\partial \xi_2} - rV = 0.
\end{aligned}$$

Noticing

$$\frac{\partial Z_1}{\partial \xi_1} = 1 - Z_{1,l},$$
$$\frac{\partial Z_2}{\partial \xi_1} = \xi_2 \left(1 - Z_{1,l}\right), \quad \frac{\partial Z_2}{\partial \xi_2} = Z_1 - Z_{2,l}, \quad \frac{\partial^2 Z_2}{\partial \xi_1 \partial \xi_2} = 1 - Z_{1,l},$$

we can rewrite the PDE as

$$\begin{aligned}&\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_i\sigma_j\rho_{i,j}\frac{\partial^2 V}{\partial \xi_i \partial \xi_j} + \frac{rZ_1}{1 - Z_{1,l}}\frac{\partial V}{\partial \xi_1}\\ &+\left(-\frac{rZ_1\xi_2\left(1 - Z_{1,l}\right)}{\left(1 - Z_{1,l}\right)\left(Z_1 - Z_{2,l}\right)} + \frac{rZ_2}{Z_1 - Z_{2,l}} - \frac{\sigma_1\sigma_2\rho_{1,2}\left(1 - Z_{1,l}\right)}{Z_1 - Z_{2,l}}\right)\frac{\partial V}{\partial \xi_2}\\ &-rV = 0,\end{aligned}$$

which can be simplified to

$$\begin{aligned}&\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_i\sigma_j\rho_{i,j}\frac{\partial^2 V}{\partial \xi_i \partial \xi_j} + \frac{rZ_1}{1 - Z_{1,l}}\frac{\partial V}{\partial \xi_1}\\ &+\left[\frac{r\left(Z_2 - Z_1\xi_2\right) - \sigma_1\sigma_2\rho_{1,2}\left(1 - Z_{1,l}\right)}{Z_1 - Z_{2,l}}\right]\frac{\partial V}{\partial \xi_2} - rV = 0.\end{aligned}$$

From Sect. 5.6 you can see that it could be a PDE for a two-factor interest rate model.

2.4 Uniqueness of Initial-Value Problems for Degenerate Parabolic PDEs

2.4.1 Reversion Conditions for Stochastic Models

In many cases, a stochastic model in finance usually describes a random variable which can take its value on an infinite domain. For such a model, closed-form solutions can be found in many situations. This is an advantage of such a model. However it seems that assuming a random variable (such as interest rates, volatilities) to be defined on a finite domain and designing a model from market data are more realistic. How do we model a random variable with such a property? For simplicity, we consider problems with only one random variable S. Suppose that we want a random variable S to have a lower boundary S_l, i.e., if $S \geq S_l$ at time t, then we want to guarantee that S is still greater than or equal to S_l after time t even though the movement of S possesses some uncertainty. In this case, we need to require that $a\left(S, t\right)$ and $b\left(S, t\right)$ at $S = S_l$ satisfy the conditions

$$\begin{cases} a\left(S_l,t\right) \geq 0, & 0 \leq t \leq T, \\ b\left(S_l,t\right) = 0, & 0 \leq t \leq T. \end{cases} \tag{2.38}$$

This is a necessary condition because if either of the two conditions does not hold, then there is a chance for S to be lower than S_l at time $t + dt$ when $S = S_l$ at time t. In Sect. 2.4.2, we will see that if

$$\begin{cases} a\left(S_l,t\right) - b(S_l,t)\dfrac{\partial}{\partial S}b(S_l,t) \geq 0, & 0 \leq t \leq T, \\ b\left(S_l,t\right) = 0, & 0 \leq t \leq T \end{cases} \tag{2.39}$$

holds, then a unique solution of the corresponding partial differential equation can be determined by a final condition on $[S_l, \infty)$ without any boundary conditions at $S = S_l$. Therefore, what happens at $S = S_l$ will not affect the solution at $t = 0$ for any S. This fact can be interpreted as follows. If the condition (2.39) holds for any $t \in [t_0, T]$, then for any such time t, S will be greater than or equal to S_l if $S > S_l$ at $t = t_0$. That is, S is either reflected into the region $S > S_l$ or is absorbed by the boundary $S = S_l$ in the event S hits the lower bound S_l at some time $t \in [t_0, T]$. This is because if there are paths that pass through a point (S_l, t) and go to the outside of $[S_l, \infty)$, then the solution at the point $(S, 0)$ should depend on the value of the solution at the point (S_l, t). The solution is determined only by the final condition, so there is no path passing the boundary $S = S_l$. Consequently the condition (2.39) is a sufficient condition to guarantee $S \geq S_l$ for any t.

In the popular model

$$dS = \mu S dt + \sigma S dX,$$

we have $a = \mu S$ and $b = \sigma S$. Therefore, the condition (2.39) holds at $S = 0$, and S is always greater than or equal to zero. In the Cox–Ingersoll–Ross interest rate model (see [23])

$$dr = (\bar{\mu} - \bar{\gamma} r)dt + \sqrt{\alpha r}dX, \quad \bar{\mu}, \bar{\gamma}, \alpha > 0,$$

which will be discussed in Chap. 5, $a = \bar{\mu} - \bar{\gamma} r$, $b = \sqrt{\alpha r}$, and the condition (2.39) is reduced to $\bar{\mu} - \alpha/2 \geq 0$ if the lower bound is zero. This means that if $\bar{\mu} - \alpha/2 \geq 0$, then at $r = 0$, no boundary condition is needed. In fact, if $\bar{\mu} - \alpha/2 \geq 0$, the upward drift is sufficiently large to make the origin inaccessible (see [23]). Therefore, no boundary condition at $r = 0$ is related to inaccessibility to the origin.

Actually, S_l may not be zero, and a similar condition

$$\begin{cases} a\left(S_u,t\right) - b(S_u,t)\dfrac{\partial}{\partial S}b(S_u,t) \leq 0, & 0 \leq t \leq T, \\ b\left(S_u,t\right) = 0, & 0 \leq t \leq T \end{cases} \tag{2.40}$$

can also be required at $S = S_u > S_l$ so that S will always be in $[S_l, S_u]$. If $a(S_l, t) \geq 0$ and $a(S_u, t) \leq 0$, then it is usually said that the model has a mean reversion property. However, if $b(S_l, t) \neq 0$ or $b(S_u, t) \neq 0$, then there is still a chance for S to become less than S_l or greater than S_u. If the conditions (2.39) and (2.40) hold, then we say that the model really has a reversion property because S will always be in $[S_l, S_u]$. In this book, the conditions (2.39) and (2.40) will be referred to as the reversion conditions, and (2.38) and the like will be referred to as the weak-form reversion conditions. When $\dfrac{\partial}{\partial S} b(S,t)$ is bounded, the two types of reversion conditions are the same.

The two random variables given above as examples are defined on $[0, \infty)$. In what follows, we will show that they can be converted into new random variables whose domains are $[0, 1)$ and can be naturally extended to $[0, 1]$, and for them the reversion conditions hold at both the lower and upper boundaries.

Let us introduce a new random variable

$$\xi = \frac{S}{S + P_m},$$

where P_m is a positive parameter. From this relation, we can have

$$\begin{aligned} S &= \frac{P_m \xi}{1 - \xi}, \\ S + P_m &= \frac{P_m}{1 - \xi}, \\ \frac{d\xi}{dS} &= \frac{P_m}{(S + P_m)^2} = \frac{(1 - \xi)^2}{P_m}, \end{aligned}$$

and

$$\frac{d^2 \xi}{dS^2} = \frac{-2P_m}{(S + P_m)^3} = \frac{-2(1 - \xi)^3}{P_m^2}.$$

According to Itô's lemma, if S satisfies $dS = \mu S dt + \sigma S dX$, then for ξ the stochastic differential equation is

$$\begin{aligned} d\xi &= \frac{(1 - \xi)^2}{P_m} dS - \frac{(1 - \xi)^3}{P_m^2} \sigma^2 S^2 dt \\ &= \left[\mu \xi (1 - \xi) - \sigma^2 \xi^2 (1 - \xi)\right] dt + \sigma \xi (1 - \xi) dX. \end{aligned}$$

Consequently for ξ, the conditions (2.39) and (2.40) are fulfilled at $\xi = 0$ and $\xi = 1$, respectively.

Similarly for the Cox–Ingersoll–Ross interest rate model, let

$$\xi = \frac{r}{r + P_m},$$

then we get

$$d\xi = \left[\frac{(1-\xi)^2}{P_m} \left(\bar{\mu} - \frac{\bar{\gamma} P_m \xi}{1-\xi} \right) - \frac{\alpha \xi (1-\xi)^2}{P_m} \right] dt + \frac{\sqrt{\alpha} \xi^{1/2} (1-\xi)^{3/2}}{P_m^{1/2}} dX.$$

In this case $\xi_l = 0$ and $\xi_u = 1$ and it is easy to show that both the conditions (2.39) and (2.40) hold if $\bar{\mu} - \alpha/2 \geq 0$. All the proofs here are left for readers as Problem 28. In this book we only talk these models satisfying conditions (2.39) and (2.40) or these models which can become models satisfying conditions (2.39) and (2.40) after introducing new random variables.

Suppose that a model defined on $[S_l, S_u]$ has the property of mean reverting, but it does not satisfy the reversion condition. The model can be modified as follows: the coefficient of dX is multiplied by a function, for example,

$$\Phi(x) = \frac{1 - (1-2x)^2}{1 - 0.975(1-2x)^2},$$

where $x = \frac{(S - S_l)}{(S_u - S_l)}$. Because $\Phi(x)$ are equal to zero at $S = S_l$ and $S = S_u$ and very close to one at $S \in (S_l + \varepsilon, S_u - \varepsilon)$, ε being a very small number, almost all the properties of the original model are kept in the modified model and the reversion conditions will hold after the modification is made.

Now we describe the reversion conditions for the case involving n random variables. Suppose that a financial derivative depends on the time t and n random variables $S_1, S_2, \cdots, S_n$ and that for $i = 1, 2, \cdots, n$, S_i satisfies the equation

$$dS_i = a_i(S_1, S_2, \cdots, S_n, t)dt + b_i(S_1, S_2, \cdots, S_n, t)dX_i \tag{2.41}$$

in a rectangular domain $\Omega : [S_{1l}, S_{1u}] \times [S_{2l}, S_{2u}] \times \cdots \times [S_{nl}, S_{nu}]$. In this case we require that the following conditions hold:

$$\begin{cases} \left[a_i(S_1, \cdots, S_n, t) - b_i(S_1, \cdots, S_n, t) \dfrac{\partial b_i(S_1, \cdots, S_n, t)}{\partial S_i} \right] \Bigg|_{\substack{S_i = S_{il} \\ S_j \in [S_{jl}, S_{ju}] \\ j \neq i}} \geq 0, \\ b_i(S_1, S_2, \cdots, S_n, t)\Big|_{\substack{S_i = S_{il} \\ S_j \in [S_{jl}, S_{ju}] \\ j \neq i}} = 0 \end{cases} \tag{2.42}$$

and

$$\begin{cases} \left[a_i(S_1, \cdots, S_n, t) - b_i(S_1, \cdots, S_n, t) \dfrac{\partial b_i(S_1, \cdots, S_n, t)}{\partial S_i} \right] \Bigg|_{\substack{S_i = S_{iu} \\ S_j \in [S_{jl}, S_{ju}] \\ j \neq i}} \leq 0, \\ b_i(S_1, \cdots, S_n, t)\Big|_{\substack{S_i = S_{iu} \\ S_j \in [S_{jl}, S_{ju}] \\ j \neq i}} = 0. \end{cases} \tag{2.43}$$

These conditions are called the reversion conditions on a rectangular domain Ω. It is clear that if

$$\left.\frac{\partial b_i(S_1,\cdots,S_n,t)}{\partial S_i}\right|_{\substack{S_i=S_{il}\\ S_j\in[S_{jl},S_{ju}]\\ j\neq i}}$$

and

$$\left.\frac{\partial b_i(S_1,\cdots,S_n,t)}{\partial S_i}\right|_{\substack{S_i=S_{iu}\\ S_j\in[S_{jl},S_{ju}]\\ j\neq i}}$$

are bounded, then the two conditions (2.42) and (2.43) can be reduced to

$$\begin{cases} a_i(S_1,\cdots,S_n,t)|_{\substack{S_i=S_{il}\\ S_j\in[S_{jl},S_{ju}]\\ j\neq i}} \geq 0, \\ b_i(S_1,\cdots,S_n,t)|_{\substack{S_i=S_{il}\\ S_j\in[S_{jl},S_{ju}]\\ j\neq i}} = 0 \end{cases} \tag{2.44}$$

and

$$\begin{cases} a_i(S_1,\cdots,S_n,t)|_{\substack{S_i=S_{iu}\\ S_j\in[S_{jl},S_{ju}]\\ j\neq i}} \leq 0, \\ b_i(S_1,\cdots,S_n,t)|_{\substack{S_i=S_{iu}\\ S_j\in[S_{jl},S_{ju}]\\ j\neq i}} = 0. \end{cases} \tag{2.45}$$

If the domain is not rectangular, the form of reversion conditions will be a little different. If all the coefficients in the models are differential, then the form is relatively simple. For example, consider the case of $n = 3$. Let the outer normal vector be $(n_1, n_2, n_3)^T$. Then the reversion conditions are that

$$\begin{cases} n_1a_1 + n_2a_2 + n_3a_3 \geq 0, \\ \mathrm{Var}(n_1b_1dX_1 + n_2b_2dX_2 + n_3b_3dX_3) = 0 \end{cases}$$

hold on the boundary of the domain.

2.4.2 †Uniqueness of Solutions for One-Dimensional Case

Equation (2.34) is a parabolic equation. When S_i is defined on $[S_{il}, S_{iu}]$, $i = 1, 2, \cdots, n$, Eq. (2.34) is defined on the rectangular domain Ω. If $b_i = 0$ at $S_i = S_{i,l}$ and $S_i = S_{i,u}$, $i = 1, 2, \cdots, n$, then we say that the equation is a degenerate parabolic partial differential equation. In this subsection, we are going to discuss when a degenerate equation has a unique solution. The conclusion expected is that if for any i,

$$\left.\left[a_i(S_1,\cdots,S_n,t) - b_i(S_1,\cdots,S_n,t)\frac{\partial b_i(S_1,\cdots,S_n,t)}{\partial S_i}\right]\right|_{\substack{S_i=S_{il}\\ S_j\in[S_{jl},S_{ju}]\\ j\neq i}} \geq 0 \tag{2.46}$$

and

$$\left[a_i(S_1,\cdots,S_n,t)-b_i(S_1,\cdots,S_n,t)\frac{\partial b_i(S_1,\cdots,S_n,t)}{\partial S_i}\right]\Bigg|_{\substack{S_i=S_{iu}\\ S_j\in[S_{jl},S_{ju}]\\ j\neq i}}\leq 0 \tag{2.47}$$

hold, the solution of the degenerate parabolic equation on a rectangular domain with a final condition at $t=T$ is unique.[11] If

$$\left[a_i(S_1,\cdots,S_n,t)-b_i(S_1,\cdots,S_n,t)\frac{\partial b_i(S_1,\cdots,S_n,t)}{\partial S_i}\right]\Bigg|_{\substack{S_i=S_{il}\\ S_j\in[S_{jl},S_{ju}]\\ j\neq i}}< 0 \tag{2.48}$$

or

$$\left[a_i(S_1,\cdots,S_n,t)-b_i(S_1,\cdots,S_n,t)\frac{\partial b_i(S_1,\cdots,S_n,t)}{\partial S_i}\right]\Bigg|_{\substack{S_i=S_{iu}\\ S_j\in[S_{jl},S_{ju}]\\ j\neq i}}> 0, \tag{2.49}$$

then a boundary condition at $S_i = S_{i,l}$ or $S_i = S_{i,u}$ needs to be imposed besides the final condition in order to have a unique solution. In this subsection we now prove this conclusion for the one-dimensional case. In the next subsection we will prove that for a final-value problem the solution is unique if the reversion conditions hold.

In the case $m = 0$ and $n = 1$, after ignoring the subscript 1, Eq. (2.34) becomes

$$\frac{\partial V}{\partial t}+\frac{1}{2}b^2\frac{\partial^2 V}{\partial S^2}+(a-\lambda b)\frac{\partial V}{\partial S}-rV+K=0.$$

Here, the sign of the coefficient of the second derivative is opposite to that of the coefficient of the second derivative in the heat equation. We say that such a parabolic equation has an "anti-directional" time. For a heat equation, an initial condition is given at $t = 0$, and the solution for $t \geq 0$ needs to be determined. Therefore, for the equation with an "anti-directional" time, a final condition should be given at $t = T$, and the solution for $t \leq T$ needs to be determined. Consequently, we consider the following problem:

[11] For a parabolic equation defined on a non-rectangular domain, the conditions for a parabolic partial differential equation to be degenerate and the conditions for the solution of its initial-value problem to be unique, see the paper [91] by Zhu.

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2} b^2 \dfrac{\partial^2 V}{\partial S^2} + (a - \lambda b) \dfrac{\partial V}{\partial S} - rV + K = 0, \\ \qquad\qquad\qquad\qquad 0 \le t \le T, \quad S_l \le S \le S_u, \\ V(S,T) = f(S), \qquad\qquad S_l \le S \le S_u, \\ V(S_l,t) \begin{cases} \text{needs not to be given if the condition (2.46) holds,} \\ = f_l(t) \text{ if the condition (2.46) does not hold,} \end{cases} \\ V(S_u,t) \begin{cases} \text{needs not to be given if the condition (2.47) holds,} \\ = f_u(t) \text{ if the condition (2.47) does not hold.} \end{cases} \end{cases} \tag{2.50}$$

Let $\tau = T - t$ and $x = (S - S_l)/(S_u - S_l)$, then the problem (2.50) is converted into a problem in the form:

$$\begin{cases} \dfrac{\partial u}{\partial \tau} = f_1(x,\tau) \dfrac{\partial^2 u}{\partial x^2} + f_2(x,\tau) \dfrac{\partial u}{\partial x} + f_3(x,\tau) u + g(x,\tau), \\ \qquad\qquad\qquad\qquad 0 \le x \le 1, \quad 0 \le \tau \le T, \\ u(x,0) = f(x), \qquad\qquad 0 \le x \le 1, \\ u(0,\tau) \begin{cases} \text{needs not to be given if } f_2(0,\tau) - \dfrac{\partial f_1(0,\tau)}{\partial x} \ge 0, \\ = f_l(\tau) \text{ if } f_2(0,\tau) - \dfrac{\partial f_1(0,\tau)}{\partial x} < 0, \end{cases} \\ u(1,\tau) \begin{cases} \text{needs not to be given if } f_2(1,\tau) - \dfrac{\partial f_1(1,\tau)}{\partial x} \le 0, \\ = f_u(\tau) \text{ if } f_2(1,\tau) - \dfrac{\partial f_1(1,\tau)}{\partial x} > 0, \end{cases} \end{cases} \tag{2.51}$$

where $f_1(0,\tau) = f_1(1,\tau) = 0$ and $f_1(x,\tau) \ge 0$. Thus, if we can prove the uniqueness of the solution of the problem (2.51), then we have the uniqueness of the solution of the problem (2.50). The third and fourth relations in the problem (2.51) are the boundary conditions for degenerate parabolic equations. For parabolic equations, there is always a boundary condition at any boundary, that is, the number of boundary conditions for parabolic equations is always one. However, for degenerate parabolic equations, sometimes there is a boundary condition and sometimes there is not, depending on the value of $f_2(x,\tau) - \dfrac{\partial f_1(x,\tau)}{\partial x}$ at the boundary. For the problem (2.51), we have the following theorem (see [79]).

Theorem 2.1 *Suppose that the solution of the problem (2.51) exists and is bounded*[12] *and that there exist a constant c_1 and two bounded functions $c_2(\tau)$*

[12] This is proved in the paper [7] by Behboudi.

and $c_3(\tau)$ *such that*

$$1+\max_{0\le x\le 1,\ 0\le\tau\le T}\left(\left|\frac{\partial^2 f_1(x,\tau)}{\partial x^2}-\frac{\partial f_2(x,\tau)}{\partial x}+2f_3(x,\tau)\right|\right)\le c_1,$$

$$-\min\left(0, f_2(0,\tau)-\frac{\partial f_1(0,\tau)}{\partial x}\right)\le c_2(\tau),$$

and

$$\max\left(0, f_2(1,\tau)-\frac{\partial f_1(1,\tau)}{\partial x}\right)\le c_3(\tau).$$

In this case, its solution is unique and stable with respect to the initial value $f(x)$*, inhomogeneous term* $g(x,\tau)$*, and the boundary values* $f_l(\tau), f_u(\tau)$ *if there are any.*

Proof. Because the partial differential equation in the problem (2.51) can be rewritten as

$$\frac{\partial u}{\partial\tau}=\frac{\partial}{\partial x}\left[f_1(x,\tau)\frac{\partial u}{\partial x}\right]+\left[f_2(x,\tau)-\frac{\partial f_1(x,\tau)}{\partial x}\right]\frac{\partial u}{\partial x}+f_3(x,\tau)u+g(x,\tau),$$

multiplying that equation by $2u$, we have

$$\begin{aligned}\frac{\partial(u^2)}{\partial\tau}&=2\frac{\partial}{\partial x}\left(f_1u\frac{\partial u}{\partial x}\right)+\left(f_2-\frac{\partial f_1}{\partial x}\right)\frac{\partial(u^2)}{\partial x}-2f_1\left(\frac{\partial u}{\partial x}\right)^2+2f_3u^2+2gu\\&=2\frac{\partial}{\partial x}\left(f_1u\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial x}\left[\left(f_2-\frac{\partial f_1}{\partial x}\right)u^2\right]-2f_1\left(\frac{\partial u}{\partial x}\right)^2\\&\quad+\left(\frac{\partial^2 f_1}{\partial x^2}-\frac{\partial f_2}{\partial x}+2f_3\right)u^2+2gu.\end{aligned}$$

Integrating this equality with respect to x on the interval $[0,1]$, we obtain the second equality

$$\begin{aligned}&\frac{d}{d\tau}\int_0^1 u^2(x,\tau)dx\\&=2\left(f_1u\frac{\partial u}{\partial x}\right)\bigg|_{x=0}^{1}+\left[\left(f_2-\frac{\partial f_1}{\partial x}\right)u^2\right]\bigg|_{x=0}^{1}-2\int_0^1 f_1\left(\frac{\partial u}{\partial x}\right)^2dx\\&\quad+\int_0^1\left(\frac{\partial^2 f_1}{\partial x^2}-\frac{\partial f_2}{\partial x}+2f_3\right)u^2dx+2\int_0^1 gudx.\end{aligned}$$

Because

$$\begin{aligned}&\left[\left(f_2-\frac{\partial f_1}{\partial x}\right)u^2\right]\bigg|_{x=0}^{1}\\&=\left[f_2(1,\tau)-\frac{\partial f_1(1,\tau)}{\partial x}\right]u^2(1,\tau)-\left[f_2(0,\tau)-\frac{\partial f_1(0,\tau)}{\partial x}\right]u^2(0,\tau)\\&\le\max\left(0,f_2(1,\tau)-\frac{\partial f_1(1,\tau)}{\partial x}\right)f_u^2(\tau)-\min\left(0,f_2(0,\tau)-\frac{\partial f_1(0,\tau)}{\partial x}\right)f_l^2(\tau),\end{aligned}$$

from the equality above and the relations $f_1(0,\tau) = f_1(1,\tau) = 0$ and $f_1(x,\tau) \geq 0$, we have

$$\frac{d}{d\tau}\int_0^1 u^2(x,\tau)dx \leq c_1\int_0^1 u^2(x,\tau)dx + \int_0^1 g^2(x,\tau)dx + c_2(\tau)f_l^2(\tau) + c_3(\tau)f_u^2(\tau).$$

Based on this inequality and by the Gronwall inequality,[13] we arrive at

$$\begin{aligned}&\int_0^1 u^2(x,\tau)dx\\ &\leq \mathrm{e}^{c_1\tau}\left\{\int_0^1 f^2(x)dx + \int_0^\tau \left[\int_0^1 g^2(x,s)dx + c_2(s)f_l^2(s) + c_3(s)f_u^2(s)\right]ds\right\},\\ &\qquad t \in [0,T].\end{aligned}$$

From the last inequality, we know that the solution is stable with respect to $f(x)$ and $g(x,\tau)$. Also if

$$f_2(0,\tau) - \frac{\partial f_1(0,\tau)}{\partial x} \geq 0 \quad \text{and} \quad f_2(1,\tau) - \frac{\partial f_1(1,\tau)}{\partial x} \leq 0$$

hold and

$$f(x) \equiv 0, \quad g(x,\tau) \equiv 0,$$

then the solution of the problem (2.51) must be zero. Hence, the functions $f(x)$ and $g(x,\tau)$ determine the solution uniquely. If

$$f_2(0,\tau) - \frac{\partial f_1(0,\tau)}{\partial x} < 0 \quad \text{and} \quad f_2(1,\tau) - \frac{\partial f_1(1,\tau)}{\partial x} \leq 0$$

hold, then the solution is determined by $f(x)$, $g(x,\tau)$, and $f_l(\tau)$ uniquely. The situation for other cases are similar. Therefore, we may conclude that if the solution of the problem (2.51) exists, then it is unique and stable with respect to the initial value $f(x)$, the inhomogeneous term $g(x,\tau)$, and the boundary values $f_l(\tau), f_u(\tau)$ if there are any. This completes the proof and gives an explanation on when a boundary condition is necessary. □

Here we give some remarks.

- From the probabilistic point of view, a boundary condition on a boundary is needed if and only if there are paths reaching the boundary from a point $x \in (0,1)$ and $t = 0$. Therefore, on whether or not a random variable can reach a boundary from the interior, there are similar conclusions (see [33]).
- This result indicates that a degenerate parabolic equation at boundaries is similar to a hyperbolic equation.[14] Due to this fact, roughly speaking, we might say that the parabolic equation degenerates into a hyperbolic equation at the boundaries. When conditions (2.46) and (2.47) hold,

[13]The inequality $dA(\tau)/d\tau \leq cA(\tau) + B(\tau)$ can be rewritten as $\mathrm{e}^{-c\tau}[dA(\tau)/d\tau - cA(\tau)] \leq \mathrm{e}^{-c\tau}B(\tau)$ or $d(\mathrm{e}^{-c\tau}A(\tau))/d\tau \leq \mathrm{e}^{-c\tau}B(\tau)$, so for positive $\tau, c, B(\tau)$ we have $A(\tau) \leq \mathrm{e}^{c\tau}[A(0) + \int_0^\tau \mathrm{e}^{-c\bar{\tau}}B(\bar{\tau})d\bar{\tau}] \leq \mathrm{e}^{c\tau}[A(0) + \int_0^\tau B(\bar{\tau})d\bar{\tau}]$.

[14]When $f_1(x,t) \equiv 0$, the partial differential equation in Eq. (2.51) is called a hyperbolic equation.

incoming information is not needed at boundaries, that is, the value of V at the boundaries at $t = t^*$ is determined by the value V on the region: $S_l \le S \le S_u$ and $t^* \le t \le T$. Therefore, in this case, in order for a degenerate parabolic equation to have a unique solution, only the final condition is needed.[15]

- When the domain of S is not finite, a final condition is still enough for such an equation to have a unique solution if S can be converted into a random variable for which the reversion conditions hold. The reason is that a final condition can determine a unique solution if the new random variable is used. However, a transformation will not change the nature of the problem. If the problem has a unique solution as a function of a random variable, the problem will also have a unique solution as a function of another random variable associated by a transformation. Applying this theorem to problem (2.18), we know that its solution is unique and stable with respect to the initial value. Problem (2.18) is obtained through a transformation from the European option problem (2.16). Therefore, the European option problem (2.16) also has a unique solution. In Sect. 2.2.5 it is pointed that for problem (2.18) the values at $\xi = 0$ and $\xi = 1$ are given by the expressions (2.19) and (2.20), respectively. This means that when a solution of the problem (2.18) is determined, no boundary condition is needed. The result here points out not only that no boundary condition is needed when a solution of the problem (2.18) is determined, but also that it is impossible for problem (2.18) to have several solutions.

2.4.3 ‡Uniqueness of Solutions for Two-Dimensional Case

On a multidimensional rectangular domain, it can be proved that if the reversion conditions are satisfied, then the final-value problem for degenerate parabolic partial differential equations has a unique solution. In this subsection, we give a detailed proof only for the two-dimensional case; at the end of this subsection, we point out the key part of the proof for the multidimensional case.

Suppose that a financial derivative depends on the time t and two random variables S_1 and S_2, which satisfy Eq. (2.41) and the reversion conditions, and let $V(S_1, S_2, t)$ be the price of the financial derivative. By an arbitrage argument, it can be shown that $V(S_1, S_2, t)$ should satisfy the following equation (see Sect. 2.3.2):

$$\begin{aligned}\frac{\partial V}{\partial t} + \frac{1}{2}b_1^2\frac{\partial^2 V}{\partial S_1^2} + \rho b_1 b_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{1}{2}b_2^2\frac{\partial^2 V}{\partial S_2^2}\\ +(a_1 - \lambda_1 b_1)\frac{\partial V}{\partial S_1} + (a_2 - \lambda_2 b_2)\frac{\partial V}{\partial S_2} - rV + g(S_1, S_2, t) = 0,\end{aligned}$$

[15] Oleĭnik and Radkevič in their book [65] discussed the uniqueness of solutions of this type of partial differential equations under different conditions.

where λ_1 and λ_2 are two bounded functions and called the market prices of risk on S_1 and S_2, respectively, and r is the short-term interest rate.[16] Also, many financial derivatives should be solutions of the final-value problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \frac{1}{2}b_1^2 \dfrac{\partial^2 V}{\partial S_1^2} + \rho b_1 b_2 \dfrac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{1}{2}b_2^2 \dfrac{\partial^2 V}{\partial S_2^2} \\ \quad +(a_1 - \lambda_1 b_1)\dfrac{\partial V}{\partial S_1} + (a_2 - \lambda_2 b_2)\dfrac{\partial V}{\partial S_2} - rV + g(S_1, S_2, t) = 0, \\ \qquad\qquad\qquad\qquad S_1 \in [S_{1l}, S_{1u}], \quad S_2 \in [S_{2l}, S_{2u}], \quad t \in [0, T], \\ V(S_1, S_2, T) = f(S_1, S_2), \quad S_1 \in [S_{1l}, S_{1u}], \quad S_2 \in [S_{2l}, S_{2u}]. \end{cases} \tag{2.52}$$

Now let us discuss when Problem (2.52) has a unique solution. For this question, we have the following theorem:

Theorem 2.2 *If*

(i) the reversion conditions (2.42) and (2.43) hold;
(ii) there exists a constant c_1 such that

$$\max_{\substack{S_{1l}\le S_1\le S_{1u}\\ S_{2l}\le S_2\le S_{2u}}} \left| \frac{\partial}{\partial S_1}\left(a_1 - \lambda_1 b_1 - b_1\frac{\partial b_1}{\partial S_1} - \frac{1}{2}\frac{\partial}{\partial S_2}(\rho b_1 b_2)\right) \right.$$
$$\left. + \frac{\partial}{\partial S_2}\left(a_2 - \lambda_2 b_2 - b_2\frac{\partial b_2}{\partial S_2} - \frac{1}{2}\frac{\partial}{\partial S_1}(\rho b_1 b_2)\right) + 2r \right| + 1 \le c_1;$$

(iii) solutions of Problem (2.52) exist and their first derivatives are bounded,

then the solution of Eq. (2.52) is unique.

Proof. Suppose that $u(S_1, S_2, t)$ is a solution of the problem (2.52). Let $\tau = T - t$ and define

$$W(\tau) = \int_{S_{2l}}^{S_{2u}} \int_{S_{1l}}^{S_{1u}} u^2(S_1, S_2, T-\tau) dS_1 dS_2. \tag{2.53}$$

Since the partial differential equation in the problem (2.52) can be rewritten as

$$\begin{aligned} \frac{\partial u}{\partial \tau} = & \frac{1}{2}\frac{\partial}{\partial S_1}\left(b_1^2\frac{\partial u}{\partial S_1} + \rho b_1 b_2 \frac{\partial u}{\partial S_2}\right) + \frac{1}{2}\frac{\partial}{\partial S_2}\left(\rho b_1 b_2 \frac{\partial u}{\partial S_1} + b_2^2\frac{\partial u}{\partial S_2}\right) \\ & + \left(a_1 - \lambda_1 b_1 - b_1\frac{\partial b_1}{\partial S_1} - \frac{1}{2}\frac{\partial}{\partial S_2}(\rho b_1 b_2)\right)\frac{\partial u}{\partial S_1} \\ & + \left(a_2 - \lambda_2 b_2 - b_2\frac{\partial b_2}{\partial S_2} - \frac{1}{2}\frac{\partial}{\partial S_1}(\rho b_1 b_2)\right)\frac{\partial u}{\partial S_2} - ru + g, \end{aligned}$$

[16] If r is replaced by a bounded function, Theorem 2.2 still holds.

we have

$$\begin{aligned}\frac{1}{2}\frac{dW(\tau)}{d\tau} &= \int_{S_{2l}}^{S_{2u}}\int_{S_{1l}}^{S_{1u}} u\frac{\partial u}{\partial \tau}dS_1dS_2\\ &= \int_{S_{2l}}^{S_{2u}}\int_{S_{1l}}^{S_{1u}} \frac{u}{2}\frac{\partial}{\partial S_1}\left(b_1^2\frac{\partial u}{\partial S_1}+\rho b_1b_2\frac{\partial u}{\partial S_2}\right)dS_1dS_2\\ &\quad+\int_{S_{2l}}^{S_{2u}}\int_{S_{1l}}^{S_{1u}} \frac{u}{2}\frac{\partial}{\partial S_2}\left(\rho b_1b_2\frac{\partial u}{\partial S_1}+b_2^2\frac{\partial u}{\partial S_2}\right)dS_1dS_2\\ &\quad+\int_{S_{2l}}^{S_{2u}}\int_{S_{1l}}^{S_{1u}} u\left(a_1-\lambda_1b_1-b_1\frac{\partial b_1}{\partial S_1}-\frac{1}{2}\frac{\partial}{\partial S_2}(\rho b_1b_2)\right)\frac{\partial u}{\partial S_1}dS_1dS_2\\ &\quad+\int_{S_{2l}}^{S_{2u}}\int_{S_{1l}}^{S_{1u}} u\left(a_2-\lambda_2b_2-b_2\frac{\partial b_2}{\partial S_2}-\frac{1}{2}\frac{\partial}{\partial S_1}(\rho b_1b_2)\right)\frac{\partial u}{\partial S_2}dS_1dS_2\\ &\quad-\int_{S_{2l}}^{S_{2u}}\int_{S_{1l}}^{S_{1u}} ru^2dS_1dS_2+\int_{S_{2l}}^{S_{2u}}\int_{S_{1l}}^{S_{1u}} gudS_1dS_2. \end{aligned}\tag{2.54}$$

Now let us look at the first four terms in the right-hand side of the relation (2.54). Using integration by parts and the equality conditions in the conditions (2.42) and (2.43), we can rewrite the first and second terms as follows:

$$\begin{aligned}&\int_{S_{2l}}^{S_{2u}}\int_{S_{1l}}^{S_{1u}} \frac{u}{2}\frac{\partial}{\partial S_1}\left(b_1^2\frac{\partial u}{\partial S_1}+\rho b_1b_2\frac{\partial u}{\partial S_2}\right)dS_1dS_2\\ &\quad=\frac{1}{2}\int_{S_{2l}}^{S_{2u}}\left\{\left[u\left(b_1^2\frac{\partial u}{\partial S_1}+\rho b_1b_2\frac{\partial u}{\partial S_2}\right)\right]\Bigg|_{S_{1l}}^{S_{1u}}\right.\\ &\qquad\left.-\int_{S_{1l}}^{S_{1u}}\left(b_1^2\frac{\partial u}{\partial S_1}+\rho b_1b_2\frac{\partial u}{\partial S_2}\right)\frac{\partial u}{\partial S_1}dS_1\right\}dS_2\\ &\quad=-\frac{1}{2}\int_{S_{2l}}^{S_{2u}}\int_{S_{1l}}^{S_{1u}}\left(b_1^2\frac{\partial u}{\partial S_1}+\rho b_1b_2\frac{\partial u}{\partial S_2}\right)\frac{\partial u}{\partial S_1}dS_1dS_2\end{aligned}\tag{2.55}$$

and

$$\begin{aligned}&\int_{S_{2l}}^{S_{2u}}\int_{S_{1l}}^{S_{1u}} \frac{u}{2}\frac{\partial}{\partial S_2}\left(\rho b_1b_2\frac{\partial u}{\partial S_1}+b_2^2\frac{\partial u}{\partial S_2}\right)dS_1dS_2\\ &\quad=\frac{1}{2}\int_{S_{1l}}^{S_{1u}}\left\{\left[u\left(\rho b_1b_2\frac{\partial u}{\partial S_1}+b_2^2\frac{\partial u}{\partial S_2}\right)\right]\Bigg|_{S_{2l}}^{S_{2u}}\right.\\ &\qquad\left.-\int_{S_{2l}}^{S_{2u}}\left(\rho b_1b_2\frac{\partial u}{\partial S_1}+b_2^2\frac{\partial u}{\partial S_2}\right)\frac{\partial u}{\partial S_2}dS_2\right\}dS_1\\ &\quad=-\frac{1}{2}\int_{S_{2l}}^{S_{2u}}\int_{S_{1l}}^{S_{1u}}\left(\rho b_1b_2\frac{\partial u}{\partial S_1}+b_2^2\frac{\partial u}{\partial S_2}\right)\frac{\partial u}{\partial S_2}dS_1dS_2.\end{aligned}\tag{2.56}$$

Also, according to the equality condition in the condition (2.42), $b_1(S_{1l}, S_2, t) = 0$ holds for any S_2, so

$$\left.\frac{\partial}{\partial S_2}(\rho b_1 b_2)\right|_{S_1=S_{1l}} = 0.$$

Similarly, we have

$$\left.\frac{\partial}{\partial S_2}(\rho b_1 b_2)\right|_{S_1=S_{1u}} = \left.\frac{\partial}{\partial S_1}(\rho b_1 b_2)\right|_{S_2=S_{2l}} = \left.\frac{\partial}{\partial S_1}(\rho b_1 b_2)\right|_{S_2=S_{2u}} = 0.$$

Noticing these facts and the inequality conditions in the conditions (2.42) and (2.43), for the third and fourth integrals in the right-hand side of the relation (2.54), we have

$$\begin{aligned}
&\int_{S_{2l}}^{S_{2u}}\int_{S_{1l}}^{S_{1u}} u\left(a_1 - \lambda_1 b_1 - b_1\frac{\partial b_1}{\partial S_1} - \frac{1}{2}\frac{\partial}{\partial S_2}(\rho b_1 b_2)\right)\frac{\partial u}{\partial S_1}dS_1 dS_2\\
&= \frac{1}{2}\int_{S_{2l}}^{S_{2u}}\left\{\left[u^2\left(a_1 - \lambda_1 b_1 - b_1\frac{\partial b_1}{\partial S_1} - \frac{1}{2}\frac{\partial}{\partial S_2}(\rho b_1 b_2)\right)\right]\right|_{S_{1l}}^{S_{1u}}\\
&\quad -\left.\int_{S_{1l}}^{S_{1u}} u^2\frac{\partial}{\partial S_1}\left(a_1 - \lambda_1 b_1 - b_1\frac{\partial b_1}{\partial S_1} - \frac{1}{2}\frac{\partial}{\partial S_2}(\rho b_1 b_2)\right)dS_1\right\}dS_2\\
&\le -\frac{1}{2}\int_{S_{2l}}^{S_{2u}}\int_{S_{1l}}^{S_{1u}} u^2\frac{\partial}{\partial S_1}\left(a_1 - \lambda_1 b_1 - b_1\frac{\partial b_1}{\partial S_1} - \frac{1}{2}\frac{\partial}{\partial S_2}(\rho b_1 b_2)\right)dS_1 dS_2
\end{aligned} \tag{2.57}$$

and

$$\begin{aligned}
&\int_{S_{1l}}^{S_{1u}}\int_{S_{2l}}^{S_{2u}} u\left(a_2 - \lambda_2 b_2 - b_2\frac{\partial b_2}{\partial S_2} - \frac{1}{2}\frac{\partial}{\partial S_1}(\rho b_1 b_2)\right)\frac{\partial u}{\partial S_2}dS_2 dS_1\\
&= \frac{1}{2}\int_{S_{1l}}^{S_{1u}}\left\{\left[u^2\left(a_2 - \lambda_2 b_2 - b_2\frac{\partial b_2}{\partial S_2} - \frac{1}{2}\frac{\partial}{\partial S_1}(\rho b_1 b_2)\right)\right]\right|_{S_{2l}}^{S_{2u}}\\
&\quad -\left.\int_{S_{2l}}^{S_{2u}} u^2\frac{\partial}{\partial S_2}\left(a_2 - \lambda_2 b_2 - b_2\frac{\partial b_2}{\partial S_2} - \frac{1}{2}\frac{\partial}{\partial S_1}(\rho b_1 b_2)\right)dS_2\right\}dS_1\\
&\le -\frac{1}{2}\int_{S_{2l}}^{S_{2u}}\int_{S_{1l}}^{S_{1u}} u^2\frac{\partial}{\partial S_2}\left(a_2 - \lambda_2 b_2 - b_2\frac{\partial b_2}{\partial S_2} - \frac{1}{2}\frac{\partial}{\partial S_1}(\rho b_1 b_2)\right)dS_1 dS_2.
\end{aligned} \tag{2.58}$$

Adding the relations (2.55) and (2.56) together, due to $|\rho| \le 1$, we have

$$\begin{aligned}
&\int_{S_{2l}}^{S_{2u}} \int_{S_{1l}}^{S_{1u}} \frac{u}{2}\frac{\partial}{\partial S_1}\left(b_1^2\frac{\partial u}{\partial S_1}+\rho b_1 b_2\frac{\partial u}{\partial S_2}\right) dS_1 dS_2 \\
&\quad+\int_{S_{2l}}^{S_{2u}} \int_{S_{1l}}^{S_{1u}} \frac{u}{2}\frac{\partial}{\partial S_2}\left(\rho b_1 b_2\frac{\partial u}{\partial S_1}+b_2^2\frac{\partial u}{\partial S_2}\right) dS_1 dS_2 \\
&\quad=-\frac{1}{2}\int_{S_{2l}}^{S_{2u}} \int_{S_{1l}}^{S_{1u}} \left[\left(b_1\frac{\partial u}{\partial S_1}\right)^2+2\rho b_1 b_2\frac{\partial u}{\partial S_1}\frac{\partial u}{\partial S_2}+\left(b_2\frac{\partial u}{\partial S_2}\right)^2\right] dS_1 dS_2 \le 0.
\end{aligned} \tag{2.59}$$

Substituting the relations (2.55)–(2.56) and the inequalities (2.57)–(2.58) into the relation (2.54) and applying the inequality (2.59) and condition (ii), we have

$$\begin{aligned}
\frac{1}{2}\frac{dW(\tau)}{d\tau} \le& -\frac{1}{2}\int_{S_{2l}}^{S_{2u}} \int_{S_{1l}}^{S_{1u}} u^2\left\{\frac{\partial}{\partial S_1}\left(a_1-\lambda_1 b_1-b_1\frac{\partial b_1}{\partial S_1}-\frac{1}{2}\frac{\partial}{\partial S_2}(\rho b_1 b_2)\right)\right. \\
&\left.+\frac{\partial}{\partial S_2}\left(a_2-\lambda_2 b_2-b_2\frac{\partial b_2}{\partial S_2}-\frac{1}{2}\frac{\partial}{\partial S_1}(\rho b_1 b_2)\right)+2r\right\} dS_1 dS_2 \\
&+\int_{S_{2l}}^{S_{2u}} \int_{S_{1l}}^{S_{1u}} g(S_1,S_2,T-\tau)u\, dS_1 dS_2 \\
\le& \frac{1}{2}c_1 W(\tau)+\frac{1}{2}\int_{S_{2l}}^{S_{2u}} \int_{S_{1l}}^{S_{1u}} g^2(S_1,S_2,T-\tau) dS_1 dS_2.
\end{aligned}$$

Therefore, according to the Gronwall inequality, we arrive at

$$0 \le W(\tau) \le \mathrm{e}^{c_1\tau}\left[W(0)+\int_0^\tau \int_{S_{2l}}^{S_{2u}} \int_{S_{1l}}^{S_{1u}} g^2(S_1,S_2,T-\tau) dS_1 dS_2 d\tau\right].$$

Suppose that u_1 and u_2 are two solutions of the problem (2.52) and let $u = u_1 - u_2$. It is clear that u is the solution of the problem (2.52) with $V(S_1,S_2,T) = f(S_1,S_2) \equiv 0$ and $g(S_1,S_2,t) \equiv 0$. In this case, we get $W(\tau) \equiv 0$. Then, $u \equiv 0$, or $u_1 \equiv u_2$; that is, the solution of the problem (2.52) is unique. □

Here we would like to make some remarks. The first one is about the conditions given in the theorem. If a_1, a_2, b_1, b_2, λ_1, λ_2, r, the first derivatives of a_1, a_2, λ_1, and λ_2, and the first and second derivatives of ρ, b_1, and b_2 are bounded, then conditions (2.42), (2.43) are reduced to the conditions (2.44), (2.45) respectively, and condition (ii) is always satisfied. The partial differential equation in the problem (2.52) is called a degenerate parabolic partial differential equation because of the equality conditions in the conditions (2.42) and (2.43). It is clear that the result can be applied to any degenerate parabolic problems from various fields.

When there are K random variables, $K \geq 3$, governed by

$$dS_i = a_i(S_1, S_2, \cdots, S_K, t)dt + b_i(S_1, S_2, \cdots, S_K, t)dX_i, \quad i = 1, 2, \cdots, K,$$

similar results can still be proved. For the proof above, a key fact we used is $|\rho| \leq 1$, which means that the correlation matrix is semi-positive. For multidimensional cases, the key fact we need is that the correlation matrix

$$\begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1K} \\ \rho_{21} & 1 & \cdots & \rho_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{K1} & \rho_{K2} & \cdots & 1 \end{pmatrix}$$

is semi-positive definite. Here $\rho_{i,j} = \mathrm{E}[dX_i dX_j]/dt$.

The meaning of the final-value problem (2.52) having a unique solution is that the solution of the problem (2.52) is completely determined by the PDE and the final condition. This also means that the random variables will never reach the boundaries if they are inside the domain at the beginning [33]. This is because if the random variables reach the boundaries, then the solution must also be affected by what happens at the boundaries. Therefore, if stochastic models satisfy the reversion conditions, then those random variables should be guaranteed on the finite domain $[S_{1l}, S_{1u}] \times [S_{2l}, S_{2u}]$. When $\left.\dfrac{\partial b_i(S_1, S_2, t)}{\partial S_i}\right|_{S_i = S_{il}}$ and $\left.\dfrac{\partial b_i(S_1, S_2, t)}{\partial S_i}\right|_{S_i = S_{iu}}$ are bounded, then conditions (2.42) and (2.43) are reduced to the conditions (2.44) and (2.45). Under conditions (2.44) and (2.45), the fact that the random variable will never reach the boundaries has been proved for the one-dimensional case in [33]. It can be expected that the same result is still true for multidimensional cases and when conditions (2.42) and (2.43) cannot be reduced to the conditions (2.44) and (2.45).

2.4.4 ‡Uniqueness of Solutions for European Options on Assets with Stochastic Volatilities

In this subsection, we consider a special two-factor financial derivative: options on assets with stochastic volatilities. We assume that the asset price S follows the following stochastic process:

$$dS = \mu S dt + \sigma S dX_1, \quad 0 \leq S \tag{2.60}$$

and that the volatility σ is also a random variable and its evolution is governed by

$$d\sigma = p(\sigma, t)dt + q(\sigma, t)dX_2, \quad \sigma_l \leq \sigma \leq \sigma_u, \tag{2.61}$$

where the two random increments dX_1 and dX_2 are two Wiener processes. dX_1 and dX_2 are correlated and $\mathrm{E}[dX_1 dX_2] = \rho dt$. Furthermore, we assume that the stochastic model for σ satisfies reversion conditions; that is, the following relations hold:

$$\begin{cases} p(\sigma_l, t) - q(\sigma_l, t)\dfrac{\partial q}{\partial \sigma}(\sigma_l, t) \geq 0, \\ q(\sigma_l, t) = 0 \end{cases} \tag{2.62}$$

and

$$\begin{cases} p(\sigma_u, t) - q(\sigma_u, t)\dfrac{\partial q}{\partial \sigma}(\sigma_u, t) \leq 0, \\ q(\sigma_u, t) = 0, \end{cases} \tag{2.63}$$

or when $\dfrac{\partial q}{\partial \sigma}(\sigma_l, t)$ and $\dfrac{\partial q}{\partial \sigma}(\sigma_u, t)$ are bounded,

$$\begin{cases} p(\sigma_l, t) \geq 0, \\ q(\sigma_l, t) = 0 \end{cases} \tag{2.64}$$

and

$$\begin{cases} p(\sigma_u, t) \leq 0, \\ q(\sigma_u, t) = 0 \end{cases} \tag{2.65}$$

hold. Suppose that $V(S, \sigma, t)$ is the value of such an option. $V(S, \sigma, t)$ satisfies

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma q S \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial \sigma^2} + (r - D_0) S \frac{\partial V}{\partial S} + (p - \lambda q) \frac{\partial V}{\partial \sigma} - rV = 0. \tag{2.66}$$

This equation holds for $S \in [0, \infty)$. In order to convert the problem on an infinite domain into one on a finite domain, we introduce the following transformation:

$$\begin{cases} \xi = \dfrac{S}{S + P_m}, \\ \sigma = \sigma, \\ t = t, \\ \overline{V} = \dfrac{V}{S + P_m}, \end{cases} \tag{2.67}$$

where P_m is a positive constant. Since the following expressions exist:

$$\begin{aligned}
&S = \frac{\xi P_m}{1-\xi}, && S + P_m = \frac{P_m}{1-\xi},\\
&\frac{d\xi}{dS} = \frac{(1-\xi)^2}{P_m}, && \frac{\partial V}{\partial t} = \frac{P_m}{1-\xi}\frac{\partial \overline{V}}{\partial t},\\
&\frac{\partial V}{\partial S} = \overline{V} + (1-\xi)\frac{\partial \overline{V}}{\partial \xi}, && \frac{\partial V}{\partial \sigma} = \frac{P_m}{1-\xi}\frac{\partial \overline{V}}{\partial \sigma},\\
&\frac{\partial^2 V}{\partial S^2} = \frac{(1-\xi)^3}{P_m}\frac{\partial^2 \overline{V}}{\partial \xi^2}, && \frac{\partial^2 V}{\partial S \partial \sigma} = \frac{\partial \overline{V}}{\partial \sigma} + (1-\xi)\frac{\partial^2 \overline{V}}{\partial \xi \partial \sigma},\\
&\frac{\partial^2 V}{\partial \sigma^2} = \frac{P_m}{1-\xi}\frac{\partial^2 \overline{V}}{\partial \sigma^2},
\end{aligned}$$

we can rewrite Eq. (2.66) as

$$\begin{aligned}
&\frac{\partial \overline{V}}{\partial t} + \frac{1}{2}\sigma^2\xi^2(1-\xi)^2\frac{\partial^2 \overline{V}}{\partial \xi^2} + \rho\sigma\xi(1-\xi)q\frac{\partial^2 \overline{V}}{\partial \xi \partial \sigma} + \frac{1}{2}q^2\frac{\partial^2 \overline{V}}{\partial \sigma^2}\\
&\quad +(r-D_0)\xi(1-\xi)\frac{\partial \overline{V}}{\partial \xi} + \left[p - (\lambda - \rho\sigma\xi)q\right]\frac{\partial \overline{V}}{\partial \sigma} - \left[r - (r-D_0)\xi\right]\overline{V} = 0.
\end{aligned}$$

Since the transformation above converts a value of $S \in [0,\infty)$ into a value of $\xi \in [0,1)$, $\overline{V}(\xi,\sigma,t)$ is defined on the domain $[0,1]\times[\sigma_l,\sigma_u]\times[0,T]$. Therefore, the determination of European option prices in this case reduces to finding the solution of the following final-value problem:

$$\begin{cases}
\dfrac{\partial \overline{V}}{\partial t} + \frac{1}{2}\sigma^2\xi^2(1-\xi)^2\dfrac{\partial^2 \overline{V}}{\partial \xi^2} + \rho\sigma\xi(1-\xi)q\dfrac{\partial^2 \overline{V}}{\partial \xi \partial \sigma} + \frac{1}{2}q^2\dfrac{\partial^2 \overline{V}}{\partial \sigma^2}\\
\quad +(r-D_0)\xi(1-\xi)\dfrac{\partial \overline{V}}{\partial \xi} + \left[p - (\lambda - \rho\sigma\xi)q\right]\dfrac{\partial \overline{V}}{\partial \sigma} - \left[r - (r-D_0)\xi\right]\overline{V} = 0,\\
\qquad\qquad \xi \in [0,1], \quad \sigma \in [\sigma_l,\sigma_u], \quad t \in [0,T],\\
\overline{V}(\xi,\sigma,T) = f(\xi,\sigma), \quad \xi \in [0,1], \quad \sigma \in [\sigma_l,\sigma_u].
\end{cases} \tag{2.68}$$

It is easy to see $d\xi = a_1(\xi)dt + b_1(\xi)dX_1$, where $a_1(\xi) = (\mu - \sigma\xi)\xi(1-\xi)$ and $b_1(\xi) = \sigma\xi(1-\xi)$. Thus, this problem is in the form of the problem (2.52) with

$$\begin{aligned}
&\lambda_1 = \frac{\mu - \sigma\xi - r + D_0}{\sigma}, && a_2 = p(\sigma,t),\\
&b_2 = q(\sigma,t), && \lambda_2 = \lambda - \rho\sigma\xi,
\end{aligned}$$

and the coefficient of $\overline{V}$ here is $-[r-(r-D_0)\xi]$ instead of $-r$. In order to have a unique solution, the key is that a_1, b_1, a_2, and b_2 should satisfy the reversion conditions (2.42) and (2.43). In this case, a_1 and b_1 always satisfy the conditions (2.42) and (2.43). That a_2 and b_2 satisfy the reversion conditions is equivalent to fulfillment of the conditions (2.62) and (2.63). Therefore, if the conditions (2.62), (2.63), conditions (ii) and (iii) of Theorem 2.2 are satisfied, then the problem (2.68) has a unique solution.

Suppose that a problem is defined on an infinite domain and its closed-form solution cannot be found. In order to get its solution, we need to solve the problem numerically on a finite domain. In this case, an artificial boundary condition will be needed, which causes some error and problems. The problem here is defined on a finite domain, so its numerical solution can be obtained without using any artificial boundary conditions; if the singularity-separating method and extrapolation techniques are used, then numerical solutions are very good even on quite coarse meshes.

2.5 Jump Conditions

2.5.1 Hyperbolic Equations with a Dirac Delta Function

Consider the following linear hyperbolic partial differential equation

$$\frac{\partial u}{\partial t} + f_1(x_1, x_2, \cdots, x_{\scriptscriptstyle K}, t)\frac{\partial u}{\partial x_1} + \cdots + f_{\scriptscriptstyle K}(x_1, x_2, \cdots, x_{\scriptscriptstyle K}, t)\frac{\partial u}{\partial x_{\scriptscriptstyle K}} = 0.$$

Let C be a curve defined by the system of ordinary differential equations

$$\begin{aligned}\frac{dx_1(t)}{dt} &= f_1(x_1, x_2, \cdots, x_{\scriptscriptstyle K}, t),\\ &\vdots\\ \frac{dx_{\scriptscriptstyle K}(t)}{dt} &= f_{\scriptscriptstyle K}(x_1, x_2, \cdots, x_{\scriptscriptstyle K}, t)\end{aligned}$$

with initial conditions

$$x_1(0) = \xi_1,\ x_2(0) = \xi_2,\ \cdots,\ x_{\scriptscriptstyle K}(0) = \xi_{\scriptscriptstyle K}.$$

Along the curve C we have

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x_1}\frac{dx_1}{dt} + \cdots + \frac{\partial u}{\partial x_{\scriptscriptstyle K}}\frac{dx_{\scriptscriptstyle K}}{dt} = 0.$$

Therefore, u is a constant along the curve:

$$u\left(x_1(t^*), x_2(t^*), \cdots, x_{\scriptscriptstyle K}(t^*), t^*\right) = u\left(x_1(t^{**}), x_2(t^{**}), \cdots, x_{\scriptscriptstyle K}(t^{**}), t^{**}\right),$$

where t^* and t^{**} are any two times. If

$$f_k(x_1, x_2, \cdots, x_K, t) = F_k(x_1, x_2, \cdots, x_K, t)\delta(t - t_i),$$

where $\delta(t - t_i)$ is the Dirac delta function, then [17]

$$\begin{aligned} x_k(t_i^+) - x_k(t_i^-) &= \int_{t_i^-}^{t_i^+} F_k(x_1(t), x_2(t), \cdots, x_K(t), t)\delta(t - t_i)dt \\ &= F_k\left(x_1(t_i^-), x_2(t_i^-), \cdots, x_K(t_i^-), t_i^-\right) \end{aligned}$$

and

$$\begin{aligned} &u\left(x_1(t_i^-), x_2(t_i^-), \cdots, x_K(t_i^-), t_i^-\right) \\ &= u\left(x_1(t_i^+), x_2(t_i^+), \cdots, x_K(t_i^+), t_i^+\right) \\ &= u\left(x_1(t_i^-) + F_{1i}^-,\ x_2(t_i^-) + F_{2i}^-,\ \cdots,\ x_K(t_i^-) + F_{Ki}^-,\ t_i^+\right), \end{aligned} \quad (2.69)$$

where t_i^- and t_i^+ denote the time just before and after t_i, respectively, and

$$F_{ki}^- \equiv F_k\left(x_1(t_i^-), x_2(t_i^-), \cdots, x_K(t_i^-), t_i^-\right).$$

For such a jump condition, a similar derivation is given in the book [84] by Wilmott, Dewynne, and Howison.

2.5.2 Jump Conditions for Options on Stocks with Discrete Dividends and Discrete Sampling

From the relation (2.69), jump conditions of various options can be derived. Here, we give three examples. Two are simple and the other is quite complicated. Jump conditions for other options will be given when they are discussed.

Suppose $V(S, t)$ is the value of an option on a stock, which pays a dividend D_i at time t_i, $i = 1, 2, \cdots, I$. Here, we assume that $t_i \leq T$, T being expiry. From Sect. 2.2, we know that $V(S, t)$ satisfies Eq. (2.13):

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + [rS - D(S, t)]\frac{\partial V}{\partial S} - rV = 0,$$

[17] Here an integral is defined in the following way. Suppose that on $[0, T]$ we have a partition with $N+1$ points: $0 = t_0 < t_1 < \cdots < T_N = T$. The definition of an integral is

$$\int_0^T f(t)dt = \lim_{dt \to 0} \sum_{n=0}^{n=N-1} f(t_n)(t_{n+1} - t_n),$$

where $dt = \max\limits_{0 \leq n \leq N-1}(t_{n+1} - t_n)$. Let us call it an Itô integral. Such a definition is usually used in financial calculus.

where

$$D(S,t)=\sum_{i=1}^{I}D_i(S)\delta(t-t_i),\quad \text{with}\quad D_i(S)\le S.$$

This means that at $t\neq t_i, i=1,2,\cdots,I$, V satisfies

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0$$

and at $t=t_i, i=1,2,\cdots,$ or I, the equation

$$\frac{\partial V}{\partial t}-D_i(S)\delta(t-t_i)\frac{\partial V}{\partial S}=0$$

holds. According to Eq. (2.69), at $t=t_i$ we have

$$V(S,t_i^-)=V(S-D_i(S),t_i^+). \tag{2.70}$$

This is the jump condition for options on stocks with discrete dividends. We now explain the financial meaning of this relation. At $t=t_i$, the stock pays a dividend D_i, so the stock price will drop by D_i. If the price is S at t_i^-, then the price is $S-D_i$ at t_i^+. However, the price of the option is unchanged at time t_i because the holder of the option does not receive any money at time t_i.

The second example is similar to the first one. Suppose that $W(\eta,t)$ satisfies

$$\frac{\partial W}{\partial t}+\frac{1}{2}\sigma^2\eta^2\frac{\partial^2 W}{\partial \eta^2}+\left[(D_0-r)\,\eta+\frac{1}{K}\sum_{i=1}^{K}\delta\,(t-t_i)\right]\frac{\partial W}{\partial \eta}-D_0W=0,$$

Then at $t=t_i, i=1,2,\cdots,$ or K, W satisfies

$$\frac{\partial W}{\partial t}+\frac{1}{K}\delta\,(t-t_i)\,\frac{\partial W}{\partial \eta}=0$$

Thus according to the relation (2.69), at $t=t_i$ we have

$$W\left(\eta,t_i^-\right)=W\left(\eta+\frac{1}{K},t_i^+\right). \tag{2.71}$$

We will see in Chap. 4 that this jump condition will be often used when pricing Asian options because usually the average is measured discretely.

The third example involves several independent variables. Suppose the stock price is measured discretely and let $S_1, S_2, \cdots, S_N$ be the first N largest sampled stock prices until time t and $S_1\ge S_2\ge\cdots\ge S_N$. Assume that the value of option V depends on S, S_1, $\cdots$, S_N, t. From Sect. 4.4.6, we will see that if sampling occurs at $t=t_i$, then

$$\frac{dS_n}{dt} = \begin{cases} \left[\max(S, S_1(t_i^-)) - S_1(t_i^-)\right]\delta(t-t_i), & \text{if } n=1, \\ \left[\max(\min(S, S_{n-1}(t_i^-)),\ S_n(t_i^-)) - S_n(t_i^-)\right]\delta(t-t_i), & \\ & \text{if } n = 2,3,\cdots,N; \end{cases}$$

otherwise

$$\frac{dS_n}{dt} = 0.$$

According to Sect. 2.3, in this case, the option price is the solution of

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0)S\frac{\partial V}{\partial S} \\ &+ \frac{\partial V}{\partial S_1}\frac{dS_1}{dt} + \frac{\partial V}{\partial S_2}\frac{dS_2}{dt} + \cdots + \frac{\partial V}{\partial S_N}\frac{dS_N}{dt} - rV = 0. \end{aligned}$$

Consequently, at $t = t_i$, V satisfies

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S_1}\frac{dS_1}{dt} + \frac{\partial V}{\partial S_2}\frac{dS_2}{dt} + \cdots + \frac{\partial V}{\partial S_N}\frac{dS_N}{dt} = 0.$$

From the relation (2.69) we know when $t = t_i$, the jump condition

$$\begin{aligned} V(S, S_1^-, S_2^-, \cdots, S_N^-, t_i^-) = V(S,\ &\max(S, S_1^-),\ \max(\min(S, S_1^-), S_2^-), \\ &\cdots, \max(\min(S, S_{N-1}^-), S_N^-), t_i^+) \end{aligned} \quad (2.72)$$

holds, where S_n^- denotes $S_n(t_i^-)$ for brevity.

It is clear how to use such a jump condition when a European-style derivative is evaluated. When the price of an American-style derivative needs to be calculated, such a condition should be used on the solution obtained by the PDE. After that, taking the maximum between the new solution and the constraint yields the solution for the American derivative.

2.6 Solutions of European Options

A linear partial differential equation

$$A\frac{\partial^2 u}{\partial t^2} + 2B\frac{\partial^2 u}{\partial t \partial x} + C\frac{\partial^2 u}{\partial x^2} = F\left(x, t, u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}\right)$$

is called a parabolic partial differential equation if $AC - B^2 = 0$, where A, B, and C are not all equal to zero. The diffusion equation is the simplest parabolic equation. The Black–Scholes equation is another parabolic equation. In this section we mainly do two things. We reduce the Black–Scholes equation to a diffusion equation, and find out the analytic expression of the solution of the Black–Scholes equation and the Black–Scholes formulae for European options based on the analytic solution of the diffusion equation.

2.6.1 Converting the Black–Scholes Equation into a Heat Equation

In this subsection, we introduce one transformation that reduces the Black–Scholes equation to the heat equation. Because Green's function[18] of the heat equation has an analytic expression, we can obtain an analytic expression of Green's function for the Black–Scholes equation using the inverse transformation. Based on this, analytic expressions of European call and put option prices can be derived. These are the famous Black–Scholes formulae.

The price of a European option is a solution of the following problem:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r - D_0)S\dfrac{\partial V}{\partial S} - rV = 0, \quad 0 \le S, \quad t \le T, \\ V(S,T) = V_T(S), \quad 0 \le S. \end{cases} \tag{2.73}$$

The payoff function $V_T(S)$ is determined by the feature of the option. For example, the payoffs of European calls and puts are given by

$$V_T(S) = \max(\pm(S-1),\ 0), \quad 0 \le S,$$

where $+$ and $-$ in $\pm$ correspond to call and put options, respectively. Here, the exercise price is 1 because we assume that both the price of the stock and the price of option have been divided by the exercise price. We call a problem with such a payoff a standard put/call problem. Let us set

$$\begin{cases} y = \ln S, \\ \tau = T - t, \\ V(S,t) = \mathrm{e}^{-r(T-t)} v(y,\tau). \end{cases} \tag{2.74}$$

Because

$$\begin{aligned} \frac{\partial V}{\partial t} &= \mathrm{e}^{-r(T-t)}\left(rv - \frac{\partial v}{\partial \tau}\right), \\ \frac{\partial V}{\partial S} &= \mathrm{e}^{-r(T-t)}\frac{\partial v}{\partial y}\frac{dy}{dS} = \mathrm{e}^{-r(T-t)}\frac{1}{S}\frac{\partial v}{\partial y}, \\ \frac{\partial^2 V}{\partial S^2} &= \frac{\partial}{\partial S}\left(\frac{\partial V}{\partial S}\right) = \frac{\partial}{\partial S}\left(\mathrm{e}^{-r(T-t)}\frac{1}{S}\frac{\partial v}{\partial y}\right) \\ &= \mathrm{e}^{-r(T-t)}\left(-\frac{1}{S^2}\frac{\partial v}{\partial y} + \frac{1}{S^2}\frac{\partial^2 v}{\partial y^2}\right), \end{aligned}$$

the Black–Scholes equation is converted into

$$-\frac{\partial v}{\partial \tau} + \frac{1}{2}\sigma^2\left(\frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial y}\right) + (r - D_0)\frac{\partial v}{\partial y} = 0,$$

[18] The definitions of Green's functions of the heat equation and the Black–Scholes equation are given in Sect. 2.6.

and the problem above becomes

$$\begin{cases} \dfrac{\partial v}{\partial \tau} = \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 v}{\partial y^2} + \left(r - D_0 - \dfrac{1}{2}\sigma^2\right)\dfrac{\partial v}{\partial y}, & -\infty < y < \infty, \quad 0 \le \tau, \\ v(y,0) = V_T(\mathrm{e}^y), \quad -\infty < y < \infty. \end{cases} \tag{2.75}$$

Furthermore, we let

$$\begin{cases} x = y + \left(r - D_0 - \dfrac{1}{2}\sigma^2\right)\tau, \\ \bar{\tau} = \dfrac{1}{2}\sigma^2 \tau, \\ v(y,\tau) = u(x,\bar{\tau}). \end{cases} \tag{2.76}$$

Noticing the relations

$$\begin{aligned} \frac{\partial v}{\partial \tau} &= \frac{1}{2}\sigma^2 \frac{\partial u}{\partial \bar{\tau}} + \left(r - D_0 - \frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial x}, \\ \frac{\partial v}{\partial y} &= \frac{\partial u}{\partial x}, \\ \frac{\partial^2 v}{\partial y^2} &= \frac{\partial^2 u}{\partial x^2}, \end{aligned}$$

we finally arrive at

$$\begin{cases} \dfrac{\partial u}{\partial \bar{\tau}} = \dfrac{\partial^2 u}{\partial x^2}, & -\infty < x < \infty, \quad 0 \le \bar{\tau}, \\ u(x,0) = V_T(\mathrm{e}^x), \quad -\infty < x < \infty, \end{cases} \tag{2.77}$$

where $V_T(\mathrm{e}^x) = \max(\pm(\mathrm{e}^x - 1), 0)$ for the European call and put options. The partial differential equation in this problem is usually called the heat or diffusion equation.

Before we go to the next subsection, we point out the following:

1. From the relations (2.74) and (2.76), we know

$$\begin{aligned} V(S,t) &= \mathrm{e}^{-r(T-t)} u(\ln S + (r - D_0 - \sigma^2/2)(T-t), \ \ \sigma^2(T-t)/2) \\ &= \mathrm{e}^{-r(T-t)} u\left(\ln \frac{S\mathrm{e}^{-D_0(T-t)}}{\mathrm{e}^{-r(T-t)}} - \sigma^2(T-t)/2, \ \ \sigma^2(T-t)/2\right). \end{aligned}$$

   Therefore, besides those parameters in the payoff function $V_T(S)$, $V(S,t)$ depends on only three parameters: $S\mathrm{e}^{-D_0(T-t)}$, $\mathrm{e}^{-r(T-t)}$, and $\sigma^2(T-t)/2$.
2. Actually, the transformations (2.74) and (2.76) can be combined into one transformation[19]

[19]The transform converting the Black-Scholes equation into a heat equation is not unique. For example, let $\bar{x} = \sqrt{2}\left[\ln S + \left(r - D_0 - \frac{1}{2}\sigma^2\right)(T-t)\right]/\sigma, \tau = T - t$, and

$$\begin{cases} x = \ln S + \left(r - D_0 - \dfrac{1}{2}\sigma^2\right)(T-t), \\ \bar{\tau} = \dfrac{1}{2}\sigma^2(T-t), \\ V(S,t) = \mathrm{e}^{-r(T-t)}u(x,\bar{\tau}). \end{cases} \tag{2.78}$$

That is, through the transformation (2.78), the Black–Scholes equation can be directly converted into the heat equation. The reason we complete the transformation through two steps is to see the function of each single transformation. In fact, from the derivation we know the following:

- Through setting $\tau = T - t$, we change a problem with a final condition to a problem with an initial condition and let the initial time be zero.
- The transformation $y = \ln S$ is to reduce an equation with variable coefficients to one with constant coefficients. This is the transformation by which the Euler equation in ordinary differential equations becomes a differential equation with constant coefficients.
- Letting $V = \mathrm{e}^{-r(T-t)}v(y,\tau)$, we eliminate the term rV in the equation. This is similar to the fact that an equation $\dfrac{dV}{d\tau} - rV = f$ can be written as $\dfrac{d(\mathrm{e}^{-r\tau}V)}{d\tau} = \mathrm{e}^{-r\tau}f$ after the equation is multiplied by $\mathrm{e}^{-r\tau}$. The factor $\mathrm{e}^{-r\tau}$ is called an integrating factor for the ordinary differential equation. If r depends on t, then the integrating factor is $\mathrm{e}^{-\int_0^\tau r(T-s)ds}$ $= \mathrm{e}^{-\int_t^T r(s)ds}$ and the term rV can be eliminated in the same way.
- The transformation $x = y + (r - D_0 - \sigma^2/2)\tau$ is to eliminate the term $(r - D_0 - \sigma^2/2)\dfrac{\partial v}{\partial y}$. This is similar to reducing the simplest hyperbolic partial differential equation $\dfrac{\partial v}{\partial \tau} - a\dfrac{\partial v}{\partial y} = 0$ to an ordinary differential equation. For this case, the characteristic equation is $\dfrac{dy}{d\tau} = -a$ and its solution is $y = -a\tau + c$ or $y + a\tau = c$. Let $x = y + a\tau$ and $v(y,\tau) = u(x,\tau)$, then the hyperbolic partial differential equation becomes $\dfrac{\partial u(x,\tau)}{\partial \tau} = 0$. If a depends on t, then the solution of the characteristic equation is $y = -\int_0^\tau a(T-s)ds + c = -\int_t^T a(s)ds + c$. Letting $x = y + \int_t^T a(s)ds$ and $v(y,\tau) = u(x,\tau)$, we still have $\dfrac{\partial u(x,\tau)}{\partial \tau} = 0$.

$V(S,t) = \mathrm{e}^{-r(T-t)}u(\bar{x},\tau)$, then $u(\bar{x},\tau)$ is a solution of the problem

$$\begin{cases} \dfrac{\partial u}{\partial \tau} = \dfrac{\partial^2 u}{\partial \bar{x}^2}, & -\infty < \bar{x} < \infty, \quad 0 \le \tau, \\ u(x,0) = V_T(\mathrm{e}^{\sigma\bar{x}/\sqrt{2}}), & -\infty < \bar{x} < \infty. \end{cases}$$

- In order for the coefficient of $\dfrac{\partial^2 u}{\partial x^2}$ to be one, we let $\bar{\tau} = \sigma^2\tau/2$. If σ depends on t, then letting $\bar{\tau} = \frac{1}{2}\int_0^{\tau} \sigma^2(T-s)ds = \frac{1}{2}\int_t^T \sigma^2(s)ds$ can still make the coefficient of $\dfrac{\partial^2 u}{\partial x^2}$ be one.

3. From the explanation on the function of each single transformation given above, we can see that if r, D_0, and σ are not constant, but depend on t only, then the Black–Scholes equation can still be converted into a heat equation by the following transformation

$$\begin{cases} x = \ln S + \int_t^T \left[r(s) - D_0(s) - \sigma^2(s)/2\right] ds, \\ \bar{\tau} = \dfrac{1}{2}\int_t^T \sigma^2(s)ds, \\ V(S,t) = \mathrm{e}^{-\int_t^T r(s)ds} u(x,\bar{\tau}) \end{cases} \tag{2.79}$$

and the solution $V(S,t)$ possesses the following form:

$$\mathrm{e}^{-\int_t^T r(s)ds} u\left(\ln \frac{S\mathrm{e}^{-\int_t^T D_0(s)ds}}{\mathrm{e}^{-\int_t^T r(s)ds}} - \frac{1}{2}\int_t^T \sigma^2(s)ds,\ \frac{1}{2}\int_t^T \sigma^2(s)ds\right), \tag{2.80}$$

where $u(x,\bar{\tau})$ is a solution of the heat equation (see [84]). This is left for readers as an exercise (Problem 36). There, in order to see the function of each part of the transformation, readers are asked to reduce the Black–Scholes equation with time-dependent parameters to a heat equation through two steps.

4. The transformation to convert the Black–Scholes equation into a heat equation is not unique. In fact, we can let $x = \ln S$, $\bar{\tau} = \dfrac{1}{2}\sigma^2(T-t)$, $V(S,t) = \mathrm{e}^{\alpha x + \beta\bar{\tau}} u(x,\bar{\tau})$, and choose constants α and β such that $u(x,\bar{\tau})$ satisfies the heat equation (see [84]).

2.6.2 The Solutions of Parabolic Equations

In order for a parabolic differential equation to have a unique solution, one has to specify some conditions. For example, the initial value problem for a heat equation

$$\frac{\partial u}{\partial \bar{\tau}} = \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad \bar{\tau} \geq 0 \tag{2.81}$$

with

$$u(x,0) = u_0(x) \tag{2.82}$$

has a unique solution under certain conditions that usually hold for cases considered in this book.

Let us find the solution of Eq. (2.81) with initial condition (2.82). The way to find the solution is not unique. Here, we use the following method (see [52]). We first try to find a special solution of Eq. (2.81) in the form

$$u(x,\bar{\tau}) = \bar{\tau}^{-1/2}U(\eta),$$

where

$$\eta = \frac{x-\xi}{\sqrt{\bar{\tau}}}, \qquad \xi \text{ being a parameter.}$$

Because

$$\begin{aligned}
\frac{\partial u}{\partial \bar{\tau}} &= -\frac{\bar{\tau}^{-3/2}}{2}\left(U + \eta\frac{dU}{d\eta}\right) = -\frac{\bar{\tau}^{-3/2}}{2}\frac{d}{d\eta}[\eta U(\eta)],\\
\frac{\partial u}{\partial x} &= \bar{\tau}^{-1/2}\frac{dU}{d\eta}\frac{1}{\sqrt{\bar{\tau}}} = \bar{\tau}^{-1}\frac{dU}{d\eta},\\
\frac{\partial^2 u}{\partial x^2} &= \bar{\tau}^{-3/2}\frac{d^2U}{d\eta^2},
\end{aligned}$$

from Eq. (2.81) we have

$$-\frac{\bar{\tau}^{-3/2}}{2}\frac{d}{d\eta}(\eta U) = \bar{\tau}^{-3/2}\frac{d^2U}{d\eta^2},$$

that is,

$$\frac{d^2U}{d\eta^2} + \frac{1}{2}\frac{d}{d\eta}(\eta U) = 0.$$

Integrating this equation, we have

$$\frac{dU}{d\eta} + \frac{\eta}{2}U = c_1,$$

where c_1 is a constant. Let us choose $c_1 = 0$, so now we have a linear homogeneous equation. The solution of this equation is

$$U(\eta) = c\mathrm{e}^{-\eta^2/4},$$

where c is a constant. Thus, for the diffusion equation we have a special solution in the form

$$c\bar{\tau}^{-1/2}\mathrm{e}^{-(x-\xi)^2/4\bar{\tau}}.$$

If we further require

$$\int_{-\infty}^{\infty} c\bar{\tau}^{-1/2}\mathrm{e}^{-(x-\xi)^2/4\bar{\tau}}d\xi = 1,$$

then

$$c = \frac{1}{\displaystyle\int_{-\infty}^{\infty} \bar{\tau}^{-1/2} \mathrm{e}^{-(x-\xi)^2/4\bar{\tau}} d\xi} = \frac{1}{\sqrt{2}\displaystyle\int_{-\infty}^{\infty} \mathrm{e}^{-\eta^2/2} d\eta} = \frac{1}{2\sqrt{\pi}}$$

and the special solution is

$$\frac{1}{2\sqrt{\pi\bar{\tau}}} \mathrm{e}^{-(x-\xi)^2/4\bar{\tau}}.$$

This solution is called the fundamental solution, or Green's function, for the heat equation (2.81). Let $g(\xi; x, \bar{\tau})$ represent this class of functions with ξ as parameters. It is clear that the relation

$$\frac{\partial g(\xi; x, \bar{\tau})}{\partial \bar{\tau}} = \frac{\partial^2 g(\xi; x, \bar{\tau})}{\partial x^2}$$

holds for any ξ. Thus, for any $u_0(\xi)$ we have

$$\int_{-\infty}^{\infty} u_0(\xi) \frac{\partial g(\xi; x, \bar{\tau})}{\partial \bar{\tau}} d\xi = \int_{-\infty}^{\infty} u_0(\xi) \frac{\partial^2 g(\xi; x, \bar{\tau})}{\partial x^2} d\xi,$$

that is,

$$\frac{\partial \left[\displaystyle\int_{-\infty}^{\infty} u_0(\xi) g(\xi; x, \bar{\tau}) d\xi\right]}{\partial \bar{\tau}} = \frac{\partial^2 \left[\displaystyle\int_{-\infty}^{\infty} u_0(\xi) g(\xi; x, \bar{\tau}) d\xi\right]}{\partial x^2}.$$

Consequently,

$$u(x, \bar{\tau}) = \int_{-\infty}^{\infty} u_0(\xi) \times \frac{1}{2\sqrt{\pi\bar{\tau}}} \mathrm{e}^{-(x-\xi)^2/4\bar{\tau}} d\xi \tag{2.83}$$

is also a solution of Eq. (2.81). Because

$$\lim_{\bar{\tau}\to 0} \frac{1}{2\sqrt{\pi\bar{\tau}}} \mathrm{e}^{-(x-\xi)^2/4\bar{\tau}} = \begin{cases} 0, & x - \xi \neq 0, \\ \infty, & x - \xi = 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi\bar{\tau}}} \mathrm{e}^{-(x-\xi)^2/4\bar{\tau}} d\xi = 1$$

is true for any $\bar{\tau}$, we have

$$\lim_{\bar{\tau}\to 0} \frac{1}{2\sqrt{\pi\bar{\tau}}} \mathrm{e}^{-(x-\xi)^2/4\bar{\tau}} = \delta(x - \xi)$$

and

$$\lim_{\bar{\tau}\to 0} \int_{-\infty}^{\infty} u_0(\xi) \times \frac{1}{2\sqrt{\pi\bar{\tau}}} \mathrm{e}^{-(x-\xi)^2/4\bar{\tau}} d\xi = u_0(x).$$

Consequently, Eq. (2.83) is the solution of the initial-value problem

$$\begin{cases} \dfrac{\partial u}{\partial \bar{\tau}} = \dfrac{\partial^2 u}{\partial x^2}, & -\infty < x < \infty, \quad \bar{\tau} > 0, \\ u(x, 0) = u_0(x), -\infty < x < \infty. \end{cases}$$

2.6.3 Solutions of the Black–Scholes Equation

Because the solution of the problem (2.77) is the expression (2.83), from the relation (2.78) we know that the solution of the final value problem

$$\begin{cases} \dfrac{\partial V}{\partial t}+\dfrac{1}{2}\sigma^2 S^2\dfrac{\partial^2 V}{\partial S^2}+(r-D_0)S\dfrac{\partial V}{\partial S}-rV=0, & 0\le S, \quad 0\le t\le T, \\ V(S,T)=V_T(S), & 0\le S \end{cases}$$

is

$$\begin{aligned} V(S,t) &= \mathrm{e}^{-r(T-t)}\int_{-\infty}^{\infty} u_0(\xi)\frac{1}{2\sqrt{\pi\bar{\tau}}}\mathrm{e}^{-(x-\xi)^2/4\bar{\tau}}d\xi \\ &= \mathrm{e}^{-r(T-t)}\int_{-\infty}^{\infty} V_T\left(\mathrm{e}^{\xi}\right)\frac{1}{2\sqrt{\pi\bar{\tau}}}\mathrm{e}^{-(\xi-x)^2/4\bar{\tau}}d\xi \\ &= \mathrm{e}^{-r(T-t)}\frac{1}{\sigma\sqrt{2\pi(T-t)}} \\ &\quad \times\int_0^{\infty} V_T(S')\mathrm{e}^{-\left\{\ln S'-\left[\ln S+(r-D_0-\sigma^2/2)(T-t)\right]\right\}^2/2\sigma^2(T-t)}\frac{dS'}{S'}. \end{aligned}$$

This result can be written as

$$V(S,t)=\mathrm{e}^{-r(T-t)}\int_0^{\infty} V_T(S')G(S',T;S,t)dS', \tag{2.84}$$

where

$$\begin{aligned} &G(S',T;S,t) \\ &= \frac{1}{\sigma\sqrt{2\pi(T-t)}S'}\mathrm{e}^{-\left\{\ln S'-\left[\ln S+(r-D_0-\sigma^2/2)(T-t)\right]\right\}^2/2\sigma^2(T-t)}. \end{aligned} \tag{2.85}$$

Equations (2.84) and (2.85) are usually referred to as the general solution and Green's function of the Black–Scholes equation, respectively. From Sect. 2.1.3, we know that this function is also the probability density function for a lognormal distribution, that is, we can say that S' is a lognormal random variable and according to the result (2.6) its expectation is

$$\mathrm{E}\left[S'\right]=S\mathrm{e}^{(r-D_0)(T-t)}. \tag{2.86}$$

In order to make the expression of this function short, we rewrite it as

$$G(S',T;S,t)=\frac{1}{\sqrt{2\pi}bS'}\mathrm{e}^{-\left[\ln(S'/a)+b^2/2\right]^2/2b^2},$$

where

$$a=S\mathrm{e}^{(r-D_0)(T-t)} \quad \text{and} \quad b=\sigma\sqrt{T-t}.$$

For this function, there are the following useful formulae:

$$\int_c^{\infty} G(S', T; S, t)dS' = N\left(\frac{\ln(a/c) - b^2/2}{b}\right) \tag{2.87}$$

and

$$\int_c^{\infty} S'G(S', T; S, t)dS' = aN\left(\frac{\ln(a/c) + b^2/2}{b}\right), \tag{2.88}$$

where $N(z)$ is the cumulative distribution function for the standard normal distribution defined by[20]

$$N(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} \mathrm{e}^{-\xi^2/2}d\xi. \tag{2.89}$$

The proof of the two formulae is straightforward. Let

$$\eta(S') = \frac{\ln(S'/a) + b^2/2}{b},$$

that is,

$$S' = a\mathrm{e}^{b\eta - b^2/2}.$$

Thus

$$dS' = a\mathrm{e}^{b\eta - b^2/2}bd\eta = S'bd\eta.$$

Consequently, we have

$$\begin{aligned}
&\int_c^{\infty} \frac{1}{\sqrt{2\pi}bS'}\mathrm{e}^{-\left[\ln(S'/a)+b^2/2\right]^2/2b^2}dS' \\
&= \int_{\eta(c)}^{\infty} \frac{1}{\sqrt{2\pi}bS'}\mathrm{e}^{-\eta^2/2}S'bd\eta \\
&= N(-\eta(c)) \\
&= N\left(-\frac{\ln(c/a) + b^2/2}{b}\right) \\
&= N\left(\frac{\ln(a/c) - b^2/2}{b}\right)
\end{aligned}$$

---

[20]The value of this function has to be obtained by numerical methods. If $z \le 0$, this function can be approximated by

$$\begin{aligned}
N(z) = 0.5t\exp(&-x^2 - 1.26551223 + t(1.00002368 + t(0.37409196 + t(0.09678418 \\
&+t(-0.18628806 + t(0.27886807 + t(-1.13520398 + t(1.48851587 \\
&+t(-0.82215223 + t \times 0.17087277))))))))),
\end{aligned}$$

where $x = -z \times 0.707106781186550$ and $t = 1.0/(1.0 + 0.5x)$. If $z > 0$, then $N(z) = 1 - N(-z)$. The fractional error is less than 0.6×10^{-7} everywhere. See NUMERICAL RECIPES IN C: The Art of Scientific Computing. Cambridge University Press, Cambridge (1988–1992).

and

$$\begin{aligned}
&\int_c^\infty S' \frac{1}{\sqrt{2\pi}bS'} \mathrm{e}^{-\left[\ln(S'/a)+b^2/2\right]^2/2b^2} dS' \\
&= \int_{\eta(c)}^\infty \frac{1}{\sqrt{2\pi}b} \mathrm{e}^{-\eta^2/2} a \mathrm{e}^{b\eta-b^2/2} b d\eta \\
&= \frac{a}{\sqrt{2\pi}} \int_{\eta(c)}^\infty \mathrm{e}^{-(\eta-b)^2/2} d\eta \\
&= \frac{a}{\sqrt{2\pi}} \int_{\eta(c)-b}^\infty \mathrm{e}^{-\xi^2/2} d\xi \\
&= aN\left(-\frac{\ln(c/a)+b^2/2}{b} + b\right) \\
&= aN\left(\frac{\ln(a/c)+b^2/2}{b}\right).
\end{aligned}$$

2.6.4 Prices of Forward Contracts and Delivery Prices

From Sect. 1.2.1, we know that the payoff function for a forward contract is

$$V(S,T) = S - K.$$

Therefore, according to the formula (2.84) and using the result (2.86), we see that its price is

$$\begin{aligned}
V(S,t) &= \mathrm{e}^{-r(T-t)} \int_0^\infty (S'-K)G(S',T;S,t)dS' \\
&= \mathrm{e}^{-r(T-t)}\left(S\mathrm{e}^{(r-D_0)(T-t)} - K\right) \\
&= S\mathrm{e}^{-D_0(T-t)} - K\mathrm{e}^{-r(T-t)}.
\end{aligned}$$

Because for a forward contract the buyer does not need to pay any premium at $t = 0$, we have

$$V(S,0) = S\mathrm{e}^{-D_0 T} - K\mathrm{e}^{-rT} = 0.$$

Consequently, the delivery price should be

$$K = \mathrm{e}^{(r-D_0)T} S_0,$$

where in order to make it clear, we use S_0, instead of S, to denote the price of the underlying asset at the initiation of the contract.

2.6.5 Derivation of the Black–Scholes Formulae

At $t = T$, the value of a call option is

$$c(S,T) = \max(S - E, 0).$$

According to the formulae (2.84), (2.87), and (2.88), the value of a European call is

$$\begin{aligned}
c(S,t) &= \mathrm{e}^{-r(T-t)} \int_0^\infty \max(S'-E,0)G(S',T;S,t)dS' \\
&= \mathrm{e}^{-r(T-t)} \int_E^\infty (S'-E)G(S',T;S,t)dS' \\
&= \mathrm{e}^{-r(T-t)} \left[\int_E^\infty S'G(S',T;S,t)dS' - \int_E^\infty EG(S',T;S,t)dS'\right] \\
&= \mathrm{e}^{-r(T-t)} \left[S\mathrm{e}^{(r-D_0)(T-t)}N(d_1) - EN(d_2)\right] \\
&= S\mathrm{e}^{-D_0(T-t)}N(d_1) - E\mathrm{e}^{-r(T-t)}N(d_2),
\end{aligned} \tag{2.90}$$

where

$$\begin{aligned}
d_1 &= \left[\ln\frac{S\mathrm{e}^{(r-D_0)(T-t)}}{E} + \frac{1}{2}\sigma^2(T-t)\right] \Big/ \left(\sigma\sqrt{T-t}\right) \\
&= \left[\ln\frac{S\mathrm{e}^{-D_0(T-t)}}{E\mathrm{e}^{-r(T-t)}} + \frac{1}{2}\sigma^2(T-t)\right] \Big/ \left(\sigma\sqrt{T-t}\right), \\
d_2 &= \left[\ln\frac{S\mathrm{e}^{(r-D_0)(T-t)}}{E} - \frac{1}{2}\sigma^2(T-t)\right] \Big/ \left(\sigma\sqrt{T-t}\right) \\
&= \left[\ln\frac{S\mathrm{e}^{-D_0(T-t)}}{E\mathrm{e}^{-r(T-t)}} - \frac{1}{2}\sigma^2(T-t)\right] \Big/ \left(\sigma\sqrt{T-t}\right) \\
&= d_1 - \sigma\sqrt{T-t}.
\end{aligned}$$

For a put, the final value is

$$p(S,T) = \max(E-S,0).$$

Thus, the value of a European put is

$$\begin{aligned}
p(S,t) &= \mathrm{e}^{-r(T-t)} \int_0^\infty \max(E-S',0)G(S',T;S,t)dS' \\
&= \mathrm{e}^{-r(T-t)} \int_0^E (E-S')G(S',T;S,t)dS' \\
&= \mathrm{e}^{-r(T-t)} \left[E\int_0^E G(S',T;S,t)dS' - \int_0^E S'G(S',T;S,t)dS'\right] \\
&= \mathrm{e}^{-r(T-t)} \left\{E[1-N(d_2)] - S\mathrm{e}^{(r-D_0)(T-t)}[1-N(d_1)]\right\} \\
&= E\mathrm{e}^{-r(T-t)}N(-d_2) - S\mathrm{e}^{-D_0(T-t)}N(-d_1).
\end{aligned} \tag{2.91}$$

It is interesting that the values of European call and put options can be expressed in terms of the cumulative distribution function for the standardized normal random variable, $N(z)$. Expressions (2.90) and (2.91) give closed-form solutions for European vanilla options and are usually referred to as the Black–Scholes formulae.

When hedging is involved, we not only seek the value of options, but also the value of the first and second derivatives with respect to S, Δ, and Γ. For European call, $\Delta = \dfrac{\partial c}{\partial S}$ is

$$\begin{aligned}\frac{\partial c}{\partial S} &= \mathrm{e}^{-D_0(T-t)}N(d_1) + S\mathrm{e}^{-D_0(T-t)}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-d_1^2/2}\frac{\partial d_1}{\partial S}\\ &\quad -E\mathrm{e}^{-r(T-t)}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-d_2^2/2}\frac{\partial d_2}{\partial S}\\ &= \mathrm{e}^{-D_0(T-t)}N(d_1) + \frac{1}{\sqrt{2\pi}}\frac{\partial d_1}{\partial S}\left(S\mathrm{e}^{-D_0(T-t)-d_1^2/2} - E\mathrm{e}^{-r(T-t)-d_2^2/2}\right).\end{aligned}$$

Noticing

$$\begin{aligned}&-r(T-t) - d_2^2/2\\ =&-r(T-t) - \left[d_1^2 - 2d_1\sigma\sqrt{T-t} + \sigma^2(T-t)\right]\Big/2\\ =&-r(T-t) - \left[d_1^2 - 2\ln(S/E) - 2\left(r - D_0 + \sigma^2/2\right)(T-t) + \sigma^2(T-t)\right]\Big/2\\ =&-d_1^2/2 - D_0(T-t) + \ln(S/E),\end{aligned}$$

that is,

$$S\mathrm{e}^{-D_0(T-t)-d_1^2/2} = E\mathrm{e}^{-r(T-t)-d_2^2/2},$$

we have

$$\frac{\partial c}{\partial S} = \mathrm{e}^{-D_0(T-t)}N(d_1).$$

Taking the derivative with respect to S again yields

$$\frac{\partial^2 c}{\partial S^2} = \frac{1}{S\sigma\sqrt{2\pi(T-t)}}\mathrm{e}^{-D_0(T-t)-d_1^2/2}.$$

Similarly, for put options

$$\frac{\partial p}{\partial S} = -\mathrm{e}^{-D_0(T-t)}N(-d_1) \quad \text{and} \quad \frac{\partial^2 p}{\partial S^2} = \frac{\partial^2 c}{\partial S^2}.$$

We need to point out that if the value of an option and the price of the underlying asset are divided by E, then the dimensionless option value V/E and the derivatives of V/E can still be obtained by the same formulae. The only change is to let $E = 1$ and S should have dimensionless value.

What the values of $c(S,t)$ and $p(S,t)$ are? What do the functions $c(S,t)$ and $p(S,t)$ look like? For the case $S = E$, $r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$ and $T - t = 1$,

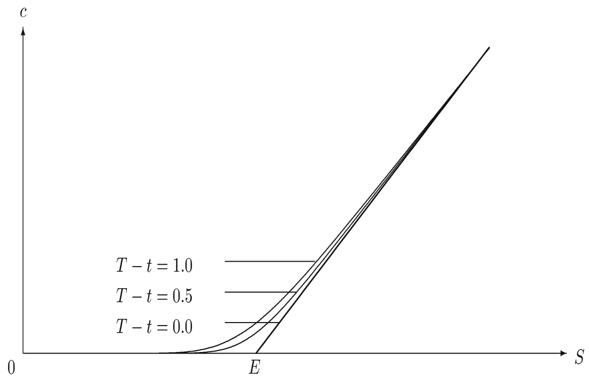

Fig. 2.5. The European call value $c(S, t)$ as a function of S ($r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$, and $T - t = 0$, 0.5, and 1.0)

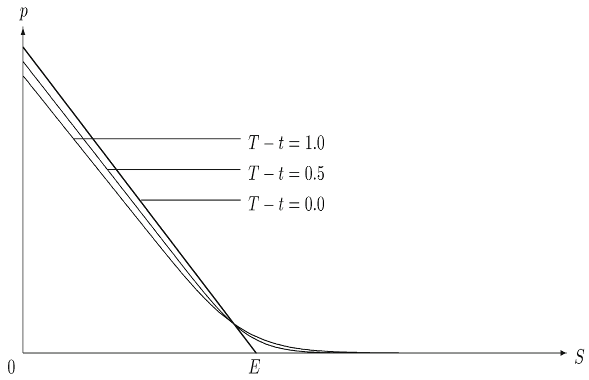

Fig. 2.6. The European put value $p(S, t)$ as a function of S ($r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$, and $T - t = 0$, 0.5, and 1.0)

$$c(S, t)/E = 0.0994 \quad \text{and} \quad p(S, t)/E = 0.0530$$

and for the case $S = E$, $r = 0.02$, $D_0 = 0.01$, $\sigma = 0.2$ and $T - t = 1$,

$$c(S, t)/E = 0.0835 \quad \text{and} \quad p(S, t)/E = 0.0736.$$

The functions of the European call and put options for the case $r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$, $T - t = 0, 0.5, 1$ are shown in Figs. 2.6 and 2.5. Clearly, the curves should approach the payoff functions as $t \to T$, which can be seen from the two figures. From Fig. 2.6, we can also see that when S is close to zero, the curves approach the payoff from the bottom and when S is large, the curves tend to the payoff from the top. That is, $p(S, t)$ is less than the payoff for small S and greater than the payoff for large S. In Sect. 3.1, we will see that for American options, the price should always be at least the payoff. Because of this, the Black–Scholes equation cannot be used to determine the price of American options in some situations.

When σ, r, and D_0 depend on t, closed-form solutions can still be obtained (see [63, 84]). Actually, through the transformation (2.79), the Black–Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2(t)S^2\frac{\partial^2 V}{\partial S^2} + [r(t) - D_0(t)]S\frac{\partial V}{\partial S} - r(t)V = 0$$

can still be reduced to a diffusion equation. Let

$$\begin{cases} \alpha(t) = \frac{1}{2}\int_t^T \sigma^2(s)ds, \\ \delta(t) = \int_t^T D_0(s)ds, \\ \gamma(t) = \int_t^T r(s)ds, \end{cases}$$

then the solution of the Black–Scholes equation in this case is

$$V(S,t) = \mathrm{e}^{-\gamma(t)}\int_{-\infty}^{\infty} V_T\left(\mathrm{e}^{\xi}\right)\frac{1}{2\sqrt{\pi\bar{\tau}}}\mathrm{e}^{-(\xi-x)^2/4\bar{\tau}}d\xi,$$

where $x = \ln S + \gamma(t) - \delta(t) - \alpha(t)$ and $\bar{\tau} = \alpha(t)$. Therefore, for a call with coefficients $r(t)$, $D_0(t)$, and $\sigma(t)$, the solution should be

$$c(S,t) = S\mathrm{e}^{-\delta(t)}N(\overline{d}_1) - E\mathrm{e}^{-\gamma(t)}N(\overline{d}_2),$$

where

$$\begin{aligned} \overline{d}_1 &= \left[\ln\frac{S\mathrm{e}^{-\delta(t)}}{E\mathrm{e}^{-\gamma(t)}} + \alpha(t)\right] \Big/ [2\alpha(t)]^{1/2}, \\ \overline{d}_2 &= \left[\ln\frac{S\mathrm{e}^{-\delta(t)}}{E\mathrm{e}^{-\gamma(t)}} - \alpha(t)\right] \Big/ [2\alpha(t)]^{1/2}. \end{aligned}$$

2.6.6 Put–Call Parity Relation

Although call and put options are superficially different, they can be combined in such a way that they are perfectly correlated. In fact, there is the following relation:

$$c(S,t) - p(S,t) = S\mathrm{e}^{-D_0(T-t)} - E\mathrm{e}^{-r(T-t)}, \tag{2.92}$$

which is usually called the put–call parity relation. It can be obtained in different ways. From the Black–Scholes formulae (2.90) and (2.91), we can have

$$\begin{aligned} c(S,t) - p(S,t) &= S\mathrm{e}^{-D_0(T-t)}N(d_1) - E\mathrm{e}^{-r(T-t)}N(d_2) \\ &\quad - E\mathrm{e}^{-r(T-t)}N(-d_2) + S\mathrm{e}^{-D_0(T-t)}N(-d_1) \\ &= S\mathrm{e}^{-D_0(T-t)} - E\mathrm{e}^{-r(T-t)}. \end{aligned}$$

This is one way to get it.

We can also find this relation without finding the concrete expressions of $c(S,t)$ and $p(S,t)$. Let us look at a portfolio whose payoff is

$$\Pi(S,T) = S + \max(E-S,0) - \max(S-E,0) - E = 0.$$

According to the formula (2.84), $\Pi(S,t) = 0$ and we also have

$$\begin{aligned}&\Pi(S,t)\\ =&\,\mathrm{e}^{-r(T-t)}\int_0^\infty [S' + \max(E-S',0) - \max(S'-E,0) - E]G(S',T;S,t)dS'\\ =&\,S\mathrm{e}^{-D_0(T-t)} + p(S,t) - c(S,t) - E\mathrm{e}^{-r(T-t)}.\end{aligned}$$

Here, we are actually using the superposition principle of homogeneous linear partial differential equations. From these relations, we immediately have the put–call parity. In Sect. 3.4, we will derive this relation again without using a partial differential equation. Here, we need to point out that the put–call parity relation is true only for European options. For American options, the equality becomes an inequality, which will be discussed in Sect. 3.4.

2.6.7 An Explanation in Terms of Probability

The function $G(S',T;S,t)$ given by the expression (2.85) represents a probability density function of a random variable S', and S' can be interpreted as the random price of a stock at time T. Then, we can understand S as the price of the stock at time t because $G(S',T;S,t)$ goes to a Dirac delta function $\delta(S'-S)$ as $T \to t$. $V_T(S')$ is the value of an option at time T if the price is S'. Therefore

$$\int_0^\infty V_T(S')G(S',T;S,t)dS'$$

is the expectation of the value of the option at time T if the price is S at time t, and

$$\mathrm{e}^{-r(T-t)}\int_0^\infty V_T(S')G(S',T;S,t)dS'$$

is the present (or discounted) value of the expectation at time T. That is, the price of an option at time t given by the formula (2.84) is the present value of the expectation of the option value at time T. This is the explanation of the solution given by the formula (2.84) in terms of probability.

Suppose that S and S' are the prices of a stock at time $T-\Delta t$ and time T, respectively, and that S' has the probability density function $G(S',T;S,T-\Delta t)$. According to the result (2.6) we have

$$\mathrm{E}\left[S'\right] = S\mathrm{e}^{(r-D_0)\Delta t}$$

and

$$\mathrm{Var}\left[S'\right] = S^2 \mathrm{e}^{2(r-D_0)\Delta t}\left(\mathrm{e}^{\sigma^2 \Delta t} - 1\right) \approx S^2\sigma^2\Delta t.$$

Therefore[21]

$$\mathrm{E}\left[\frac{S'-S}{S}\right] = \frac{S\mathrm{e}^{(r-D_0)\Delta t} - S}{S} \approx (r-D_0)\Delta t$$

and

$$\mathrm{Var}\left[\frac{S'-S}{S}\right] \approx \sigma^2\Delta t.$$

Consequently

$$\frac{dS}{S} = (r-D_0)dt + \sigma dX.$$

However, in the real world

$$\frac{dS}{S} = \mu dt + \sigma dX.$$

Therefore, the random variable in the expression of the solution is a different random variable from that in the real world. Usually, we say that the random variable in the expression of the solution is in a "risk-neutral" world. In this case, the expected return rate per unit time on any asset is the difference between the riskless interest rate r and the dividend yield D_0.

It is clear that if we let

$$\overline{V}(S,t) = \mathrm{e}^{r(T-t)}V(S,t),$$

then $\overline{V}$ is the solution of the problem

$$\begin{cases} \dfrac{\partial \overline{V}}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 \overline{V}}{\partial S^2} + (r-D_0)S\dfrac{\partial \overline{V}}{\partial S} = 0, \ 0 \le S, & t \le T, \\ \overline{V}(S,T) = \overline{V}_T(S), & 0 \le S \end{cases}$$

and

$$\overline{V}(S,t) = \int_0^\infty \overline{V}_T(S')G(S',T;S,t)dS' = \mathrm{E}\left[\overline{V}_T(S')\right].$$

In probability theory, when this relation holds, it is said that $\overline{V}(S,t)$ is a martingale under the probability density function $G(S',T;S,t)$.

[21] Here we take a conditional expectation, i.e., S' is a random variable and S is fixed.

Problems

Table 2.1. Problems and subsections

Problems	Subsections	Problems	Subsections	Problems	Subsections
1–4	2.1.1	5–10	2.1.2	11(a)	2.2.1
11(b)	2.2.2	12–15	2.2.5	16	2.2.6
17–18	2.3.1	19–21	2.3.2	22	2.3.3
23–26	2.3.4	27–28	2.4.1	29–30	2.4.2
31(a)	2.5.1	31(b–d)–33	2.5.2	34–36	2.6.1
37	2.6.2	38–42	2.6.3	43–58	2.6.5
59–61	2.6.6				

1. (a) Show

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2} dx = 1.$$

   (b) Show that

$$\int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} \mathrm{e}^{-(x-a)^2/2b^2} dx = 1$$

   holds for any a and b. (Because this is true and the integrand is always positive, it can be a probability density function.)

   (c) If the probability density function of a random variable x is

$$\frac{1}{b\sqrt{2\pi}} \mathrm{e}^{-(x-a)^2/2b^2},$$

   then it is called a normal random variable. Show $\mathrm{E}[x] = a$ and $\mathrm{Var}[x] = b^2$.

2. Define $dX = \phi\sqrt{dt}$, where ϕ is a standardized normal random variable and its probability density function is

$$\frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\phi^2/2}, \qquad -\infty < \phi < \infty.$$

   Find $\mathrm{E}\left[dX\right]$, $\mathrm{Var}\left[dX\right]$, $\mathrm{E}\left[(dX)^2\right]$, and $\mathrm{Var}\left[(dX)^2\right]$.

3. Suppose that S has the probability density function

$$G(S) = \frac{1}{\sqrt{2\pi}bS} \mathrm{e}^{-\left[\ln(S/a)+b^2/2\right]^2/2b^2}.$$

   Find the probability density function for $\xi = \dfrac{1}{S}$, $\mathrm{E}\left[\xi\right]$ and $\mathrm{Var}\left[\xi\right]$.

4. (a) Suppose that S_1 and S_2 are two independent normal random variables. The mean and variance of S_1 are μ_1 and σ_1^2 and for S_2 they are μ_2 and σ_2^2. Find the probability density function of the random variable S_1+S_2 and using this function, show that S_1+S_2 is a normal random variable with mean $\mu_1+\mu_2$ and variance $\sigma_1^2+\sigma_2^2$.[22]
   (b) Suppose that $\varDelta t = t/n$ and ϕ_i, $i=1,2,\cdots,n$, are independent standardized normal random variables. Show that
$$X(t) = \lim_{n\to\infty}\left(\phi_1\sqrt{\varDelta t}+\phi_2\sqrt{\varDelta t}+\cdots+\phi_n\sqrt{\varDelta t}\right)$$
   is a normal random variable with mean zero and variance t.
   (c) Define $dX = X(t+dt)-X(t)$. Show that it is a normal random variable with mean zero and variance dt.
   (d) Suppose $S(t) = \mathrm{e}^{\mu t+\sigma X(t)}$. Show that $d\ln S(t) \equiv \ln S(t+dt) - \ln S(t) = \mu dt+\sigma dX$ without using Itô's lemma. (This result shows that $S(t)=\mathrm{e}^{\mu t+\sigma X(t)}$ is a solution of the equation $d\ln S(t) = \mu dt+\sigma dX$.)
5. *[23]Suppose
$$dS = a(S,t)dt + b(S,t)dX,$$
   where dX is a Wiener process. Let f be a function of S and t. Show that
$$\begin{aligned} df &= \frac{\partial f}{\partial S}dS + \left(\frac{\partial f}{\partial t}+\frac{1}{2}b^2\frac{\partial^2 f}{\partial S^2}\right)dt \\ &= b\frac{\partial f}{\partial S}dX + \left(\frac{\partial f}{\partial t}+\frac{1}{2}b^2\frac{\partial^2 f}{\partial S^2}+a\frac{\partial f}{\partial S}\right)dt. \end{aligned}$$
   This result is usually referred to as Itô's lemma.
6. Suppose that a random variable satisfies
$$dS = \mu S dt + \sigma S dX,$$
   where dX is a Wiener process. Find the stochastic equation for $\xi = \dfrac{1}{S}$ by using Itô's lemma and determine the mean and variance of $\dfrac{d\xi}{\xi}$.
7. Suppose that S satisfies
$$dS = \mu S dt + \sigma S dX.$$

[22]You have to show directly the relation
$$\begin{aligned} &\frac{1}{\sqrt{2\pi}\sigma_1}\int_{-\infty}^{\infty}\mathrm{e}^{tS_1}\mathrm{e}^{-(S_1-\mu_1)^2/2\sigma_1^2}dS_1\cdot\frac{1}{\sqrt{2\pi}\sigma_2}\int_{-\infty}^{\infty}\mathrm{e}^{tS_2}\mathrm{e}^{-(S_2-\mu_2)^2/2\sigma_2^2}dS_2 \\ &= \frac{1}{\sqrt{2\pi}\sqrt{\sigma_1^2+\sigma_1^2}}\int_{-\infty}^{\infty}\mathrm{e}^{tS}\mathrm{e}^{-(S-\mu_1-\mu_2)^2/2\left(\sigma_1^2+\sigma_2^2\right)}dS \end{aligned}$$
if such a conclusion is used.

[23]A problem with * in this book means that you can find the answer in this book. It is suggested that a student should first read and understand the corresponding material and then do the problem without looking at the book.

(a) Let $F = \mathrm{e}^{(r-D_0)(T-t)}S$, which is called the forward/futures price, and $f = S\mathrm{e}^{-D_0(T-t)} - K\mathrm{e}^{-r(T-t)}$, which is the value of a forward/futures contract. Here K is a constant and we assume that r and D_0 are constant. By Itô's lemma, show that F and f satisfy

$$dF = (\mu - r + D_0)F dt + \sigma F dX$$

and

$$\begin{aligned} df = &\left[(\mu + D_0)\left(f + K\mathrm{e}^{-r(T-t)}\right) - rK\mathrm{e}^{-r(T-t)}\right] dt \\ &+\sigma\left[f + K\mathrm{e}^{-r(T-t)}\right] dX, \end{aligned}$$

respectively.

(b) Define $\xi_{10} = \dfrac{S\mathrm{e}^{-D_0(T-t)}}{E\mathrm{e}^{-r(T-t)}} = \dfrac{S\mathrm{e}^{(r-D_0)(T-t)}}{E}$ and $\xi_{01} = \dfrac{E\mathrm{e}^{-r(T-t)}}{S\mathrm{e}^{-D_0(T-t)}} = \dfrac{E}{S\mathrm{e}^{(r-D_0)(T-t)}}$, where E is a constant. Show

$$d\xi_{10} = (\mu - r + D_0)\xi_{10} dt + \sigma\xi_{10} dX$$

and

$$d\xi_{01} = (-\mu + r - D_0 + \sigma^2)\xi_{01} dt - \sigma\xi_{01} dX.$$

8. Suppose that S satisfies

$$dS = a(S,t)dt + b(S,t)dX.$$

Show that for any functions $f_1(S,t)$ and $f_2(S,t)$, the following is true:

$$d\left(f_1 f_2\right) = f_1 df_2 + f_2 df_1 + b^2\frac{\partial f_1}{\partial S}\frac{\partial f_2}{\partial S}dt.$$

9. Suppose that S satisfies

$$dS = \mu S dt + \sigma S dX, \qquad 0 \le S < \infty,$$

where μ, σ are positive constants and dX is a Wiener process. Let

$$\xi = \frac{S}{S + P_m},$$

where P_m is a positive constant. The range of ξ is $[0,1)$ and the stochastic differential equation for ξ is in the form:

$$d\xi = a(\xi)dt + b(\xi)dX.$$

Find the concrete expressions for $a(\xi)$ and $b(\xi)$ by Itô's lemma and show

$$\begin{cases} a(0) = 0, \\ b(0) = 0, \end{cases} \quad \text{and} \quad \begin{cases} a(1) = 0, \\ b(1) = 0. \end{cases}$$

10. Consider a random variable r satisfying the stochastic differential equation

$$dr = (\mu - \gamma r)dt + wdX, \quad -\infty < r < \infty,$$

where μ, γ, w are positive constants and dX is a Wiener process. Define

$$\xi = \frac{r}{|r| + P_m}, \qquad P_m > 0,$$

which transforms the domain $(-\infty, \infty)$ for r into $(-1, 1)$ for ξ. Suppose the stochastic equation for the new random variable ξ is

$$d\xi = a(\xi)dt + b(\xi)dX.$$

Find the concrete expressions of $a(\xi)$ and $b(\xi)$ and show that $a(\xi)$ and $b(\xi)$ fulfill the conditions

$$\begin{cases} a(-1) = 0, \\ b(-1) = \dfrac{db(-1)}{d\xi} = 0, \end{cases} \quad \text{and} \quad \begin{cases} a(1) = 0, \\ b(1) = \dfrac{db(1)}{d\xi} = 0. \end{cases}$$

11. (a) *Show that if an investment is risk-free, then theoretically its return rate must be the short-term interest rate.
(b) *Using this fact and Itô's lemma, derive the Black–Scholes equation.
12. *Suppose $V(S,t)$ is the solution of the problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2(S)S^2\dfrac{\partial^2 V}{\partial S^2} + (r - D_0)S\dfrac{\partial V}{\partial S} - rV = 0\ ,\ 0 \le S, \ \ t \le T, \\ V(S,T) = V_T(S), \ \ 0 \le S. \end{cases}$$

Let $\xi = \dfrac{S}{S + P_m}$, $\tau = T - t$, and $V(S,t) = (S + P_m)\overline{V}(\xi, \tau)$, where P_m is a positive constant.
(a) Show that $\overline{V}(\xi,\tau)$ is the solution of the problem

$$\begin{cases} \dfrac{\partial \overline{V}}{\partial \tau} = \dfrac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1-\xi)^2\dfrac{\partial^2 \overline{V}}{\partial \xi^2} + (r - D_0)\xi(1-\xi)\dfrac{\partial \overline{V}}{\partial \xi} \\ \qquad -[r(1-\xi) + D_0\xi]\overline{V}, \qquad 0 \le \xi \le 1, \quad 0 \le \tau, \\ \overline{V}(\xi, 0) = \dfrac{1-\xi}{P_m}V_T\left(\dfrac{P_m\xi}{1-\xi}\right), \qquad 0 \le \xi \le 1, \end{cases}$$

where $\bar{\sigma}(\xi) = \sigma\left(\dfrac{P_m\xi}{1-\xi}\right)$.
(b) What are the advantages of reformulating the problem on a finite domain?

13. Consider the problem

$$\begin{cases} \dfrac{\partial W}{\partial t} + \dfrac{1}{2}\sigma^2\eta^2\dfrac{\partial^2 W}{\partial \eta^2} + \left[(D_0 - r)\,\eta + \dfrac{1}{T}\right]\dfrac{\partial W}{\partial \eta} - D_0 W = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad -\infty < \eta < \infty, \quad t \le T, \\ W(\eta, T) = W_T(\eta), \qquad\qquad -\infty < \eta < \infty, \end{cases}$$

which is related to the European average price options. Let us introduce the following transformation:

$$\begin{cases} \xi = \dfrac{\eta}{|\eta| + P_m}, \\ \tau = T - t, \\ W(\eta, t) = (|\eta| + P_m)\overline{u}(\xi, \tau), \end{cases}$$

where $P_m > 0$. Find the PDE and the initial condition $\overline{u}(\xi, \tau)$ should satisfy.

14. As we know, the prices of European call and put options are solutions of the problem

$$\begin{cases} \dfrac{\partial c}{\partial t} + \dfrac{1}{2}\sigma^2 S^2\dfrac{\partial^2 c}{\partial S^2} + (r - D_0)S\dfrac{\partial c}{\partial S} - rc = 0, \quad 0 \le S, \quad t \le T, \\ c(S, T) = \max(S - E, 0), \qquad 0 \le S, \end{cases}$$

and the problem

$$\begin{cases} \dfrac{\partial p}{\partial t} + \dfrac{1}{2}\sigma^2 S^2\dfrac{\partial^2 p}{\partial S^2} + (r - D_0)S\dfrac{\partial p}{\partial S} - rp = 0, \quad 0 \le S, \quad t \le T, \\ p(S, T) = \max(E - S, 0), \qquad 0 \le S, \end{cases}$$

respectively.

(a) Let $S_0^* = Ee^{-r(T-t)}$, $S_1^* = Se^{-D_0(T-t)}$, $\xi_{10} = S_1^*/S_0^*$, and $\xi_{01} = S_0^*/S_1^*$. Define $V_0(\xi_{10}, t) = c(S, t)/S_0^*$ and $V_1(\xi_{01}, t) = p(S, t)/S_1^*$. Find the PDEs and final conditions for $V_0(\xi_{10}, t)$ and $V_1(\xi_{01}, t)$.

(b) Based on the results in part (a), show that when S_1^* is replaced by S_0^* and S_0^* by S_1^* at the same time, the expression for $c(S, t) = S_0^* V_0(S_1^*/S_0^*, t)$ becomes the expression for $p(S, t) = S_1^* V_1(S_0^*/S_1^*, t)$.

15. Consider the following option problem:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2\dfrac{\partial^2 V}{\partial S^2} + (r - D_0)S\dfrac{\partial V}{\partial S} - rV = 0, \quad 0 \le S, \quad t \le T, \\ V(S, T) = \max(E, S), \qquad 0 \le S. \end{cases}$$

Suppose that the uniqueness of the solution has been proved.

a) Let $S_0^* = E\mathrm{e}^{-r(T-t)}$, $S_1^* = S\mathrm{e}^{-D_0(T-t)}$, $\xi_{10} = S_1^*/S_0^*$, and $\xi_{01} = S_0^*/S_1^*$. Define $V_0(\xi_{10}, t) = V(S,t)/S_0^*$ and $V_1(\xi_{01}, t) = V(S,t)/S_1^*$. Based on these relations, find the PDEs and final conditions for $V_0(\xi_{10}, t)$ and $V_1(\xi_{01}, t)$.

b) Based on the results in part (a), show that $V(S,t)$ can be expressed as a function $f(S_0^*, S_1^*, t)$ and this function is symmetric for S_0^* and S_1^*, i.e., $f(S_0^*, S_1^*, t) = f(S_1^*, S_0^*, t)$. This result indicates that in this option problem, the position of the cash and the position of the value of the stock are symmetric in some sense.

16. As we know, $f = S\mathrm{e}^{-D_0(T-t)} - K\mathrm{e}^{-r(T-t)}$ is the value of a forward/futures contract. For S we assume $dS = \mu S dt + \sigma S dX$, so for df we have

$$df = \left[(\mu + D_0)\left(f + K\mathrm{e}^{-r(T-t)}\right) - rK\mathrm{e}^{-r(T-t)}\right] dt$$
$$+\sigma\left[f + K\mathrm{e}^{-r(T-t)}\right] dX$$

according to Itô's lemma.

(a) *Consider an option on a forward/futures and let the price of such an option be $V_1(f,t)$. Derive the PDE for V_1 by using Itô's lemma. (Hint: Set $\Pi = V_1(f,t) - \Delta f$.)

(b) *Let $F = \mathrm{e}^{(r-D_0)(T-t)}S$, then for f we have another expression: $f = \mathrm{e}^{-r(T-t)}\left(S\mathrm{e}^{(r-D_0)(T-t)} - K\right) = \mathrm{e}^{-r(T-t)}(F-K)$. Define $V(F,t) = V_1(f(F,t),t) = V_1(\mathrm{e}^{-r(T-t)}(F-K), t)$. Derive the PDE for $V(F,t)$ from the PDE obtained in part (a).

(c) Define $V_3(S,t) = V_1(f(S,t),t) = V_1(S\mathrm{e}^{-D_0(T-t)} - K\mathrm{e}^{-r(T-t)}, t)$. Show that $V_3(S,t)$ satisfies the Black–Scholes equation:

$$\frac{\partial V_3}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_3}{\partial S^2} + (r - D_0)S\frac{\partial V_3}{\partial S} - rV_3 = 0.$$

17. *Describe and derive the generalized Itô's lemma.

18. Suppose that S_1, S_2, $\cdots$, S_n are n lognormal random variables satisfying the following stochastic differential equations:

$$dS_i = \mu_i S_i dt + \sigma_i S_i dX_i, \qquad i = 1, 2, \cdots, n,$$

where $\mu_i, \sigma_i, i = 1, 2, \cdots, n$, are constants and $dX_i, i = 1, 2, \cdots, n$, are n Wiener processes, i.e., $dX_i = \phi_i\sqrt{dt}$, ϕ_i being distinct standardized normal random variables, $i = 1, 2, \cdots, n$. ϕ_i and ϕ_j could be correlated and $\mathrm{E}[\phi_i\phi_j] = \rho_{ij},\ i, j = 1, 2, \cdots, n$, where $-1 \le \rho_{ij} \le 1$. Define

$$\xi_{ij} = \frac{S_i}{S_j}, \quad i \ne j.$$

(a) Show that ξ_{ij} satisfies the following stochastic differential equation

$$d\xi_{ij} = (\mu_i - \mu_j + \sigma_j^2 - \rho_{ij}\sigma_i\sigma_j)\xi_{ij} dt + \sigma_{ij}\xi_{ij} dX_{ij},$$

where

$$\sigma_{ij} = \sqrt{\sigma_i^2 - 2\rho_{ij}\sigma_i\sigma_j + \sigma_j^2}$$

and dX_{ij} is a Wiener process defined by

$$dX_{ij} = \frac{\sigma_i dX_i - \sigma_j dX_j}{\sigma_{ij}}.$$

That is, $\xi_{ij} = S_i/S_j$ is also a lognormal variable and its volatility is σ_{ij}.

(b) Let S_0 be a function of t, satisfying

$$dS_0 = \mu_0 S_0 dt.$$

It is clear that if we think S_0 to be a random variable and let its volatility be σ_0, then $\sigma_0 = 0$. Show that if S_i is S_0, then $\sigma_{0j} = \sigma_j$ and $dX_{0j} = -dX_j$; if S_j is S_0, then $\sigma_{i0} = \sigma_i$ and $dX_{i0} = dX_i$.

(c) Define

$$\rho_{ijk} = \frac{\sigma_k^2 - \rho_{ik}\sigma_i\sigma_k - \rho_{jk}\sigma_j\sigma_k + \rho_{ij}\sigma_i\sigma_j}{\sigma_{ik}\sigma_{jk}}.$$

Show

$$\mathrm{E}[dX_{ik}dX_{jk}] = \rho_{ijk}dt,$$

i.e., ρ_{ijk} is the correlation coefficient between the Wiener processes related to ξ_{ik} and ξ_{jk}.

(d) Show that if $S_i = S_0$, then

$$\mathrm{E}[dX_{0k}dX_{jk}] = \rho_{0jk}dt = \frac{\sigma_k - \rho_{jk}\sigma_j}{\sigma_{jk}}dt.$$

19. Suppose that S is the price of a dividend-paying stock and satisfies

$$dS = \mu(S,t)Sdt + \sigma S dX_1, \qquad 0 \le S < \infty,$$

where dX_1 is a Wiener process and σ is another random variable. Let the dividend paid during the time period $[t, t+dt]$ be $D(S,t)dt$. Assume that for σ, the stochastic equation

$$d\sigma = p(\sigma,t)dt + q(\sigma,t)dX_2, \qquad \sigma_l \le \sigma \le \sigma_u$$

holds. Here, $p(\sigma,t)$ and $q(\sigma,t)$ are differentiable functions. dX_2 is another Wiener process correlated with dX_1, and the correlation coefficient between them is ρdt. For options on such a stock, derive directly the partial differential equation that contains only the unknown market price of risk for the volatility. Here "Directly" means "without using the general PDE for derivatives". (Hint: Take a portfolio in the form $\Pi = \Delta_1 V_1(S,\sigma,t) + \Delta_2 V_2(S,\sigma,t) + S$, where V_1 and V_2 are two different options.)

20. Consider a two-factor convertible bond paying coupons with a rate $k(t)$. For such a convertible bond, derive directly the partial differential equation that contains only the unknown market price of risk for the short-term interest rate. "Directly" means "without using the general PDE for derivatives". (Hint: Take a portfolio in the form $\Pi = \Delta_1 V_1(S,r,t) + \Delta_2 V_2(S,r,t) + S$, where V_1 and V_2 are two different convertible bonds.)
21. *Describe and derive the general equations for derivative securities.
22. (a) Suppose that $V(S,t)$ satisfies

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (a - \lambda b)\frac{\partial V}{\partial S} - rV = 0.$$

Assuming that $V = S\mathrm{e}^{-D_0(T-t)}$ is a solution, find $a - \lambda b$.

(b) Let Z_l be a constant and suppose that $V(\xi,t) = Z_l + \xi(1 - Z_l)$ satisfies

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial \xi^2} + (a - \lambda b)\frac{\partial V}{\partial \xi} - rV = 0.$$

Find $a - \lambda b$.

23. Suppose that ξ satisfy the stochastic differential equation:

$$d\xi = a(\xi,t)dt + b(\xi,t)dX,$$

where dX is a Wiener process. Let $S(\xi)$ be the price of a stock which pays dividends $D(S(\xi),t)dt$ during the time period $[t, t+dt]$ and $f(\xi,t)$ represent the value of a derivative security.

(a) Setting a portfolio $\Pi = f(\xi,t) - \Delta S(\xi)$ and using Itô's lemma, derive a PDE for $f(\xi,t)$.

(b) Assume $f(\xi,t) = V(\xi,t)$, $S(\xi) = \mathrm{e}^{\xi}$ and $D(S(\xi),t) = D_0\mathrm{e}^{\xi}$. Find the PDE for $V(\xi,t)$.

(c) Assume $f(\xi,t) = V(\xi,t)/\xi$, $S(\xi) = 1/\xi$ and $D(S(\xi),t) = D_0/\xi$. Find the PDE for $V(\xi,t)$.

(d) Assume $f(\xi,t) = P_m V(\xi,t)/(1-\xi)$, $S(\xi) = P_m\xi/(1-\xi)$ and $D(S(\xi),t) = D_0 P_m \xi/(1-\xi)$. Find the PDE for $V(\xi,t)$.

24. As we know, $f = S\mathrm{e}^{-D_0(T-t)} - K\mathrm{e}^{-r(T-t)}$ is the value of a forward/futures contract. If we set $F = \mathrm{e}^{(r-D_0)(T-t)}S$, then for f we have another expression: $f = \mathrm{e}^{-r(T-t)}\left(S\mathrm{e}^{(r-D_0)(T-t)} - K\right) = \mathrm{e}^{-r(T-t)}(F-K)$. For S we assume $dS = \mu S dt + \sigma S dX$, so for F we have

$$dF = (\mu - r + D_0)F dt + \sigma F dX$$

according to Itô's lemma. Consider an option on a forward/futures and let the price of such an option be $V(F,t)$. Derive the PDE for V by using Itô's lemma. (Hint: Set $\Pi = V(F,t) - \Delta f(F,t) = V(F,t) - \Delta\mathrm{e}^{-r(T-t)}(F-K)$).

25. *Suppose that ξ_1 and ξ_2 satisfy the system of stochastic differential equations:

$$d\xi_i = \mu_i(\xi_1, \xi_2, t)dt + \sigma_i(\xi_1, \xi_2, t)dX_i, \quad i = 1, 2,$$

where dX_i are Wiener processes and $\mathrm{E}\left[dX_i dX_j\right] = \rho_{ij}dt$ with $-1 \le \rho_{ij} \le 1$. Define

$$\begin{cases} Z_1(\xi_1) &= Z_{1,l} + \xi_1(1 - Z_{1,l}), \\ Z_2(\xi_1, \xi_2) &= Z_{2,l} + \xi_2\left[Z_1(\xi_1) - Z_{2,l}\right] \\ &= Z_{2,l} + \xi_2\left[Z_{1,l} + \xi_1(1 - Z_{1,l}) - Z_{2,l}\right]. \end{cases}$$

Assume that $Z_1(\xi_1)$ and $Z_2(\xi_1, \xi_2)$ represent prices of two securities. Let $V(\xi_1, \xi_2, t)$ be the value of a derivative security. Setting a portfolio $\Pi = V - \Delta_1 Z_1(\xi_1) - \Delta_2 Z_2(\xi_1, \xi_2)$ and using Itô's lemma, show that $V(\xi_1, \xi_2, t)$ satisfies the following PDE:

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_i\sigma_j\rho_{i,j}\frac{\partial^2 V}{\partial \xi_i \partial \xi_j} + \frac{rZ_1}{1 - Z_{1,l}}\frac{\partial V}{\partial \xi_1} \\ &+ \left[\frac{r(Z_2 - Z_1\xi_2)}{Z_1 - Z_{2,l}} - \frac{\sigma_1\sigma_2\rho_{1,2}(1 - Z_{1,l})}{Z_1 - Z_{2,l}}\right]\frac{\partial V}{\partial \xi_2} - rV = 0. \end{aligned}$$

26. Suppose that ξ_1, ξ_2 and ξ_3 satisfy the system of stochastic differential equations:

$$d\xi_i = \tilde{\mu}_i(\xi_1, \xi_2, \xi_3, t)dt + \tilde{\sigma}_i(\xi_1, \xi_2, \xi_3, t)d\tilde{X}_i, \quad i = 1, 2, 3,$$

where $d\tilde{X}_i$ are the Wiener processes and $\mathrm{E}\left[d\tilde{X}_i d\tilde{X}_j\right] = \tilde{\rho}_{ij}dt$ with $-1 \le \tilde{\rho}_{ij} \le 1$. Define

$$\begin{cases} Z_1(\xi_1) &= Z_{1,l} + \xi_1(1 - Z_{1,l}), \\ Z_2(\xi_1, \xi_2) &= Z_{2,l} + \xi_2\left[Z_1(\xi_1) - Z_{2,l}\right] \\ &= Z_{2,l} + \xi_2\left[Z_{1,l} + \xi_1(1 - Z_{1,l}) - Z_{2,l}\right], \\ Z_3(\xi_1, \xi_2, \xi_3) &= Z_{3,l} + \xi_3\left[Z_2(\xi_1, \xi_2) - Z_{3,l}\right] \\ &= Z_{3,l} + \xi_3\left\{Z_{2,l} + \xi_2\left[Z_{1,l} + \xi_1(1 - Z_{1,l}) - Z_{2,l}\right] - Z_{3,l}\right\}. \end{cases}$$

Assume that $Z_1(\xi_1)$, $Z_2(\xi_1, \xi_2)$, and $Z_3(\xi_1, \xi_2, \xi_3)$ represent prices of three securities. Let $V(\xi_1, \xi_2, \xi_3, t)$ be the value of a derivative security. Setting a portfolio $\Pi = V - \Delta_1 Z_1(\xi_1) - \Delta_2 Z_2(\xi_1, \xi_2) - \Delta_3 Z_3(\xi_1, \xi_2, \xi_3)$ and using Itô's lemma, derive the PDE that $V(\xi_1, \xi_2, \xi_3, t)$ should satisfy.

27. *Write down the weak-form reversion conditions and the reversion conditions of a stochastic process, describe when the two types of reversion conditions are the same, and give the intuitive meaning of the weak-form reversion conditions.

28. Show the following:
    (a) The Cox–Ingersoll–Ross interest rate model defined on $[0, \infty)$

$$dr = (\bar{\mu} - \bar{\gamma} r)dt + \sqrt{\alpha r} dX, \quad \bar{\mu}, \bar{\gamma}, \alpha > 0$$

can be converted into the model

$$d\xi = \left[\frac{(1-\xi)^2}{P_m} \left(\bar{\mu} - \frac{\bar{\gamma} P_m \xi}{1-\xi} \right) - \frac{\alpha \xi (1-\xi)^2}{P_m} \right] dt + \frac{\sqrt{\alpha} \xi^{1/2} (1-\xi)^{3/2}}{P_m^{1/2}} dX$$

by introducing a new random variable $\xi = \dfrac{r}{r + P_m}$, where P_m is a positive constant.

    (b) ξ is defined on $[0, 1]$. For the new model, the reversion conditions at $\xi = 0$ hold if and only if $\bar{\mu} - \alpha/2 \geq 0$ and the reversion conditions at $\xi = 1$ always hold.

29. *Consider the following degenerate parabolic problem:

$$\begin{cases} \dfrac{\partial u}{\partial \tau} = f_1(x, \tau) \dfrac{\partial^2 u}{\partial x^2} + f_2(x, \tau) \dfrac{\partial u}{\partial x} + f_3(x, \tau) u + g(x, \tau), \\ \qquad\qquad\qquad\qquad 0 \leq x \leq 1, \quad 0 \leq \tau \leq T, \\ u(x, 0) = f(x), \qquad\qquad 0 \leq x \leq 1, \\ u(0, \tau) \begin{cases} \text{needs not to be given if } f_2(0, \tau) - \dfrac{\partial f_1(0, \tau)}{\partial x} \geq 0, \\ = f_l(\tau) \text{ if } f_2(0, \tau) - \dfrac{\partial f_1(0, \tau)}{\partial x} < 0, \end{cases} \\ u(1, \tau) \begin{cases} \text{needs not to be given if } f_2(1, \tau) - \dfrac{\partial f_1(1, \tau)}{\partial x} \leq 0, \\ = f_u(\tau) \text{ if } f_2(1, \tau) - \dfrac{\partial f_1(1, \tau)}{\partial x} > 0, \end{cases} \end{cases}$$

where $f_1(0, \tau) = f_1(1, \tau) = 0$ and $f_1(x, \tau) \geq 0$. Suppose that its solution exists and is bounded and that there exist a constant c_1 and two bounded functions $c_2(\tau)$ and $c_3(\tau)$ such that

$$1 + \max_{0 \leq x \leq 1,\, 0 \leq \tau \leq T} \left(\left| \frac{\partial^2 f_1(x, \tau)}{\partial x^2} - \frac{\partial f_2(x, \tau)}{\partial x} + 2 f_3(x, \tau) \right| \right) \leq c_1,$$

$$-\min \left(0, f_2(0, \tau) - \frac{\partial f_1(0, \tau)}{\partial x} \right) \leq c_2(\tau),$$

and

$$\max \left(0, f_2(1, \tau) - \frac{\partial f_1(1, \tau)}{\partial x} \right) \leq c_3(\tau).$$

Show that in this case, its solution is unique and stable with respect to the initial value $f(x)$, inhomogeneous term $g(x, \tau)$, and the boundary values $f_l(\tau), f_u(\tau)$ if there are any.

30. Suppose $f_1(r,t) \geq 0$ and $f_1(r_l,t) = \dfrac{\partial f_1(r_l,t)}{\partial r} = f_1(r_u,t) = \dfrac{\partial f_1(r_u,t)}{\partial r} = 0$, and $f_2(r_l,t) < 0$, $f_2(r_u,t) > 0$. Explain why problem **A**

$$\begin{cases} \dfrac{\partial V}{\partial t} = f_1 \dfrac{\partial^2 V}{\partial r^2} + f_2 \dfrac{\partial V}{\partial r} + f_3 V, & r_l \leq r \leq r_u, \quad 0 \leq t, \\ V(r,0) = V_0(r), & r_l \leq r \leq r_u, \\ V(r_l,t) = f_l(t), & 0 \leq t, \\ V(r_u,t) = f_u(t), & 0 \leq t \end{cases}$$

and problem **B**

$$\begin{cases} \dfrac{\partial V}{\partial t} = -f_1 \dfrac{\partial^2 V}{\partial r^2} + f_2 \dfrac{\partial V}{\partial r} + f_3 V, & r_l \leq r \leq r_u, \quad t \leq T, \\ V(r,T) = V_T(r), & r_l \leq r \leq r_u \end{cases}$$

have unique solutions.

31. (a) Consider a linear hyperbolic partial differential equation

$$\frac{\partial u}{\partial t} + f(x,t)\frac{\partial u}{\partial x} = 0.$$

Let $x = x(t)$ be the curve C which is determined by the following ordinary differential equation

$$\frac{dx(t)}{dt} = f(x,t)$$

with $x(0) = \xi$. Show that u is a constant along the curve C:

$$u\left(x(t^*), t^*\right) = u\left(x(t^{**}), t^{**}\right),$$

where t^* and t^{**} are any two times, and that if

$$f(x,t) = F(x,t)\delta(t - t_i),$$

where $\delta(t - t_i)$ is the Dirac delta function, then

$$u\left(x(t_i^-), t_i^-\right) = u\left(x(t_i^-) + F\left(x(t_i^-), t_i^-\right),\, t_i^+\right),$$

where t_i^- and t_i^+ denote the time just before and after t_i, respectively.

(b) Derive the jump condition for options on stocks with discrete dividends and explain its financial meaning.

(c) Find the corresponding jump condition for the following PDE

$$\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2\eta^2\frac{\partial^2 W}{\partial \eta^2} + \left[(D_0 - r)\,\eta + \frac{1}{K}\sum_{i=1}^{K}\delta\left(t - t_i\right)\right]\frac{\partial W}{\partial \eta} - D_0 W = 0.$$

(d) Find the corresponding jump condition for the following PDE

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+(r-D_0)\,S\frac{\partial V}{\partial S}$$
$$+\sum_{i=1}^{K}\left[\max\left(S\left(t\right),H\left(t^-\right)\right)-H\left(t^-\right)\right]\delta\left(t-t_i\right)\frac{\partial V}{\partial H}-\ rV=0.$$

32. Show that the expression

$$W(\eta,t)=\begin{cases} \mathrm{e}^{-r(T-t)}\eta, & t_2<t\le T,\\ \mathrm{e}^{-r(T-t)}\eta+\frac{1}{2}\mathrm{e}^{-r(T-t_2)-D_0(t_2-t)}, & t_1<t\le t_2,\\ \mathrm{e}^{-r(T-t)}\eta+\frac{1}{2}\mathrm{e}^{-r(T-t_1)-D_0(t_1-t)} & \\ \qquad +\frac{1}{2}\mathrm{e}^{-r(T-t_2)-D_0(t_2-t)}, & 0<t\le t_1,\end{cases}$$

is the solution of the problem:

$$\begin{cases} \dfrac{\partial W}{\partial t}+\dfrac{1}{2}\sigma^2\eta^2\dfrac{\partial^2 W}{\partial \eta^2}+\left[(D_0-r)\,\eta+\dfrac{1}{2}\sum\limits_{i=1}^{2}\delta\left(t-t_i\right)\right]\dfrac{\partial W}{\partial \eta} & \\ -D_0W=0, & 0\le\eta,\quad 0\le t\le T,\\ W(\eta,T)=\eta, & 0\le\eta.\end{cases}$$

(This problem is related to discretely sampled average price call options.)

33. Suppose that $V(S,t)$ is the solution of the following PDE:

$$\frac{\partial V}{\partial t}+a(S,t)\frac{\partial^2 V}{\partial S^2}+b(S,t)\frac{\partial V}{\partial S}+c(S,t)V+d(S,t)\delta(t-t_i)=0.$$

Find the relation between $V(S,t_i^+)$ and $V(S,t_i^-)$, and describe the financial meaning of this relation.

34. Suppose $V(S,t)$ is the solution of the problem

$$\begin{cases} \dfrac{\partial V}{\partial t}+\dfrac{1}{2}\sigma^2S^2\dfrac{\partial^2 V}{\partial S^2}+(r-D_0)S\dfrac{\partial V}{\partial S}-rV=0, & 0\le S,\quad t\le T,\\ V(S,T)=V_T(S), & 0\le S.\end{cases}$$

Let $x=\dfrac{\sqrt{2}}{\sigma}\left[\ln S+(r-D_0-\sigma^2/2)(T-t)\right]$, $\tau=T-t$ and $V(S,t)=\mathrm{e}^{-r(T-t)}u(x,\tau)$. Show that $u(x,\tau)$ is the solution of the problem

$$\begin{cases} \dfrac{\partial u}{\partial \tau}=\dfrac{\partial^2 u}{\partial x^2}, & -\infty<x<\infty,\quad 0\le\tau,\\ u(x,0)=V_T\left(\mathrm{e}^{\sigma x/\sqrt{2}}\right), & -\infty<x<\infty.\end{cases}$$

35. *Suppose $V(S,t)$ is the solution of the problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r - D_0)S\dfrac{\partial V}{\partial S} - rV = 0, & 0 \le S, \quad t \le T, \\ V(S,T) = V_T(S), & 0 \le S. \end{cases}$$

Let $x = \ln S + (r - D_0 - \sigma^2/2)(T - t)$, $\bar{\tau} = \sigma^2(T - t)/2$ and $V(S,t) = \mathrm{e}^{-r(T-t)}u(x,\bar{\tau})$. Show that $u(x,\bar{\tau})$ is the solution of the problem

$$\begin{cases} \dfrac{\partial u}{\partial \bar{\tau}} = \dfrac{\partial^2 u}{\partial x^2}, & -\infty < x < \infty, \quad 0 \le \bar{\tau}, \\ u(x,0) = V_T(\mathrm{e}^x), & -\infty < x < \infty. \end{cases}$$

36. Consider problem **A**:

$$\begin{cases} \dfrac{\partial V}{\partial t} + a(t)S^2 \dfrac{\partial^2 V}{\partial S^2} + b(t)S\dfrac{\partial V}{\partial S} - r(t)V = 0, & 0 \le S, \quad t \le T, \\ V(S,T) = V_T(S), & 0 \le S \end{cases}$$

and define

$$\alpha(t) = \int_t^T a(s)ds, \quad \beta(t) = \int_t^T b(s)ds,$$

and

$$\gamma(t) = \int_t^T r(s)ds.$$

Assume that for this problem the uniqueness of solution is proved. Show that

(a) Let $x = \ln S + \beta(t) - \alpha(t)$, $\bar{\tau} = \alpha(t)$ and $V(S,t) = \mathrm{e}^{-\gamma(t)}u(x,\bar{\tau})$, then $u(x,\bar{\tau})$ is the solution of the problem:

$$\begin{cases} \dfrac{\partial u}{\partial \bar{\tau}} = \dfrac{\partial^2 u}{\partial x^2}, & -\infty < x < \infty, \quad 0 \le \bar{\tau}, \\ u(x,0) = V_T(\mathrm{e}^x), & -\infty < x < \infty. \end{cases}$$

(b) $V(S,t)$ must be in the form

$$V(S,t) = \mathrm{e}^{-\gamma(t)}u(\ln S + \beta(t) - \alpha(t), \alpha(t))$$

or

$$V(S,t) = \mathrm{e}^{-\gamma(t)}\bar{u}(S\mathrm{e}^{\beta(t)}, \alpha(t)).$$

(c) If

$$V(S,t) = \mathrm{e}^{-r(T-t)}\bar{\bar{u}}(S\mathrm{e}^{b(T-t)}, a(T-t))$$

is the solution of problem **A** with constant coefficients, then

$$V(S,t) = \mathrm{e}^{-\gamma(t)}\bar{\bar{u}}(S\mathrm{e}^{\beta(t)}, \alpha(t))$$

is the solution of problem **A** with time-dependent coefficients.

37. *Find an integral expression of the solution of the following problem

$$\begin{cases} \dfrac{\partial u}{\partial \bar{\tau}} = \dfrac{\partial^2 u}{\partial x^2}, & -\infty < x < \infty, \quad 0 \le \bar{\tau}, \\ u(x,0) = u_0(x), & -\infty < x < \infty. \end{cases}$$

38. Using the results given in Problems 34 and 37, show that the solution of the following problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r - D_0) S \dfrac{\partial V}{\partial S} - rV = 0, & 0 \le S, \quad t \le T, \\ V(S,T) = V_T(S), & 0 \le S \end{cases}$$

is

$$V(S,t) = \mathrm{e}^{-r(T-t)} \int_0^\infty V_T(S') G(S', T; S, t) dS',$$

where

$$G(S', T; S, t) = \frac{1}{\sigma \sqrt{2\pi(T-t)} S'} \mathrm{e}^{-\left[\ln S' - \ln S - (r - D_0 - \sigma^2/2)(T-t)\right]^2 / 2\sigma^2 (T-t)}.$$

39. Suppose that S is a random variable which is defined on $[0, \infty)$ and whose probability density function is

$$G(S) = \frac{1}{\sqrt{2\pi} b S} \mathrm{e}^{-\left[\ln(S/a) + b^2/2\right]^2 / 2b^2},$$

a and b being positive numbers. Show that

(a)
$$\int_0^c G(S) dS = N\left(\frac{\ln(c/a) + b^2/2}{b}\right);$$

(b)
$$\int_0^c S G(S) dS = aN\left(\frac{\ln(c/a) - b^2/2}{b}\right);$$

(c) for any real number n
$$\int_0^c S^n G(S) dS = a^n \mathrm{e}^{(n^2 - n) b^2 / 2} N\left(\frac{\ln(c/a) + b^2/2}{b} - nb\right);$$

(d) for any real number n
$$\mathrm{E}\left[S^n\right] = a^n \mathrm{e}^{(n^2 - n) b^2 / 2};$$

(e) for any real number n

$$\int_c^{\infty} S^n G(S)dS = a^n \mathrm{e}^{(n^2-n)b^2/2} N\left(-\frac{\ln(c/a)+b^2/2}{b}+nb\right);$$

(f)

$$\int_0^c \ln S\, G(S)\, dS$$
$$= \frac{-b}{\sqrt{2\pi}}\mathrm{e}^{-\left[\ln(c/a)+b^2/2\right]^2/2b^2} + \left(\ln a - b^2/2\right) N\left(\frac{\ln(c/a)+b^2/2}{b}\right);$$

(g)

$$\int_c^{\infty} \ln S\, G(S)\, dS$$
$$= \frac{b}{\sqrt{2\pi}}\mathrm{e}^{-\left[\ln(c/a)+b^2/2\right]^2/2b^2} + \left(\ln a - b^2/2\right) N\left(-\frac{\ln(c/a)+b^2/2}{b}\right),$$

where

$$N(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} \mathrm{e}^{-\xi^2/2} d\xi.$$

40. (a) Define $S_0^* = E\mathrm{e}^{-r(T-t)}$ and $S_1^* = S\mathrm{e}^{-D_0(T-t)}$. Show that there exists a function $f(x_1, x_2, t; \sigma)$ such that the following is true:

$$\mathrm{e}^{-r(T-t)}\int_0^E \max(E, S')G(S', T; S, t)dS' = f(S_0^*, S_1^*, t; \sigma)$$

and

$$\mathrm{e}^{-r(T-t)}\int_E^{\infty} \max(E, S')G(S', T; S, t)dS' = f(S_1^*, S_0^*, t; \sigma),$$

where

$$G(S', T; S, t)$$
$$= \frac{1}{\sigma\sqrt{2\pi(T-t)}S'}\mathrm{e}^{-\left\{\ln S' - \left[\ln S + \left(r - D_0 - \sigma^2/2\right)(T-t)\right]\right\}^2/2\sigma^2(T-t)}.$$

(b) Let $V(S, t)$ be the solution of the problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r - D_0)S\dfrac{\partial V}{\partial S} - rV = 0, & 0 \le S, \quad t \le T, \\ V(S, T) = \max(E, S), & 0 \le S. \end{cases}$$

Based on the results in part (a), show that in the expression for $V(S, t)$, the positions of S_0^* and S_1^* are symmetric, i.e., exchanging S_0^* and S_1^* in the expression for $V(S, t)$ will generate the same expression.

41. As we know,

$$c(S,t) = \mathrm{e}^{-r(T-t)} \int_0^\infty \max(S' - E, 0) G(S', T; S, t) dS'$$

and

$$p(S,t) = \mathrm{e}^{-r(T-t)} \int_0^\infty \max(E - S', 0) G(S', T; S, t) dS',$$

where

$$G(S', T; S, t) = \frac{1}{\sigma\sqrt{2\pi(T-t)}S'} \mathrm{e}^{-\left[\ln(S'/S) - (r - D_0 - \sigma^2/2)(T-t)\right]^2 / 2\sigma^2(T-t)}.$$

(a) Using the expression above for $c(S,t)$, show that if $D_0 = 0$, then $c(S,t) \geq \max(S - E, 0)$, which means that for this case the value of an American call option is the same as the value of a European call option.

(b) Using the expression above for $p(S,t)$, show that if $r = 0$, then $p(S,t) \geq \max(E - S, 0)$, which means that for this case the value of an American put option is the same as the value of a European put option.

42. Consider the problem

$$\begin{cases} \dfrac{\partial B_c}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 B_c}{\partial S^2} + (r - D_0) S \dfrac{\partial B_c}{\partial S} - r B_c = 0, \\ \qquad\qquad\qquad\qquad 0 \leq S, \quad 0 \leq t \leq T, \\ B_c(S,T) = \max(Z, nS), \qquad 0 \leq S, \end{cases}$$

where σ, r, D_0, Z, and n are constants. Show that if $D_0 \leq 0$, then

$$B_c(S,t) \geq \max\left(Z\mathrm{e}^{-r(T-t)}, nS\right) \quad \text{for} \quad 0 \leq t \leq T.$$

43. Find the solution in the form of $V(S,t) = V(S)$ for the Black–Scholes equation.

44. Show by substitution that

(a) $V(S,t) = S\mathrm{e}^{-D_0(T-t)}$,

(b) $V(S,t) = E\mathrm{e}^{-r(T-t)}$

are solutions of the Black–Scholes equation. What do these solutions represent?

45. *Using the results given in Problems 38 and 39, derive the Black-Scholes formula for a European put option.

46. As we know, the price of a call option on a forward/futures is the solution of the following problem:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 F^2 \dfrac{\partial^2 V}{\partial F^2} - rV = 0, & 0 \le F, \quad t \le T, \\ V(F,T) = \max(F-K,0), & 0 \le F. \end{cases}$$

Using the general solution of the Black–Scholes equation and the results given in Problem 39, find a closed-form solution for this case.

47. Consider the following problem:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r - D_0)\, S \dfrac{\partial V}{\partial S} - rV + k(t)Z = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le S, \quad 0 \le t \le T, \\ V(S,T) = \max(Z, nS), \qquad\qquad\qquad\qquad 0 \le S, \end{cases}$$

where σ, r, D_0, Z, n are constants and $k(t)$ is a nonnegative function. Using the general solution of the Black–Scholes equation and the results given in Problem 39, find a closed-form solution for this case. (If $D_0 = 0$, this solution gives the price of a one-factor convertible bond paying coupon.) (Hint: Define $\overline{V}(S,t) = V(S,t) - b_0(t)$, where $b_0(t)$ is the solution of the following problem:

$$\begin{cases} \dfrac{db_0}{dt} - rb_0 + k(t)Z = 0, & 0 \le t \le T, \\ b_0(T) = 0. \end{cases}$$

Find $b_0(t)$ and a closed-form solution of $\overline{V}(S,t)$ first, then putting them together, we have $V(S,t)$.)

48. Consider the following problem

$$\begin{cases} \dfrac{\partial c_b}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 c_b}{\partial S^2} + (r - D_0) S \dfrac{\partial c_b}{\partial S} - rc_b = 0, \\ \qquad\qquad 0 \le S < \infty, \;\; 0 < t < T, \\ c_b(S,T) = \begin{cases} 0, & \text{if } 0 \le S < S^{**}, \\ f(S), & \text{if } S^{**} \le S < S^{*}, \\ S - E, & \text{if } S^{*} \le S < \infty, \end{cases} \end{cases}$$

where

$$f(S) = a_0 + a_1 S + \cdots + a_J S^J.$$

Show that it has a solution in the following closed form:

$$\begin{aligned}c_b(S,t) = \sum_{n=0}^{J} \Big\{ & a_n S^n \mathrm{e}^{[-r+n(r-D_0)+(n-1)n\sigma^2/2](T-t)} \\ & \times \Big[N\left(d^* - n\sigma\sqrt{T-t}\right) - N\left(d^{**} - n\sigma\sqrt{T-t}\right)\Big]\Big\} \\ + S\mathrm{e}^{-D_0(T-t)} & \left[1 - N\left(d^* - \sigma\sqrt{T-t}\right)\right] - E\mathrm{e}^{-r(T-t)}[1 - N(d^*)],\end{aligned}$$

where

$$d^* = \left[\ln(S^*/S) - \left(r - D_0 - \frac{1}{2}\sigma^2\right)(T-t)\right] \Big/ \left(\sigma\sqrt{T-t}\right),$$

$$d^{**} = \left[\ln(S^{**}/S) - \left(r - D_0 - \frac{1}{2}\sigma^2\right)(T-t)\right] \Big/ \left(\sigma\sqrt{T-t}\right).$$

49. Using the Black–Scholes formula for a put option and the result in Problem 36 part (c), find the formula for the price of a put option with time-dependent parameters.
50. Consider a European call option on a non-dividend-paying stock. Use the Black–Scholes formula to find the option price when the stock price is \$63, the strike price is \$60, the risk-free interest rate is 5 % per annum, the volatility is 35 % per annum, and the time to maturity is six months.
51. Consider a European put option on a dividend-paying stock. Use the Black–Scholes formula to find the option price when the stock price is \$55, the strike price is \$60, the risk-free interest rate is 5 % per annum, the volatility is 35 % per annum, the dividend yield is 3 % per annum, and the time to maturity is six months.
52. Consider a European call option on a non-dividend-paying stock. The option price is \$4.5, the stock price is \$86, the exercise price is \$92, the risk-free interest rate is 5 % per annum, and the time to maturity is three months. Use the Black–Scholes formula for a call option to find what the corresponding volatility should be. (This volatility is usually referred to as the implied volatility associated with the given option price.)
53. *Show

$$S\mathrm{e}^{-D_0(T-t)-d_1^2/2} = E\mathrm{e}^{-r(T-t)-d_2^2/2},$$

where

$$d_1 = \left[\ln\frac{S\mathrm{e}^{(r-D_0)(T-t)}}{E} + \frac{1}{2}\sigma^2(T-t)\right] \Big/ \left(\sigma\sqrt{T-t}\right),$$

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

54. Verify that the Black–Scholes formula for a put option is the solution of the following problem:

$$\begin{cases} \dfrac{\partial p}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 p}{\partial S^2} + (r - D_0)S\dfrac{\partial p}{\partial S} - rp = 0, & 0 \le S, \quad 0 \le t \le T, \\ p(S,T) = \max(E - S,\ 0), & 0 \le S. \end{cases}$$

(Hint: Show the following identity $E\mathrm{e}^{-r(T-t)-d_2^2/2} = S\mathrm{e}^{-D_0(T-t)-d_1^2/2}$ first.)

55. Find the expressions of $\lim_{S\to 0} c(S,t)$ and $\lim_{S\to 0} p(S,t)$.

56. Derive the expressions for derivatives of $c(S,t)$ and $p(S,t)$ with respect to r, D_0, σ, E, and show that $\dfrac{\partial c}{\partial r}, \dfrac{\partial c}{\partial \sigma}, \dfrac{\partial p}{\partial D_0}, \dfrac{\partial p}{\partial \sigma}, \dfrac{\partial p}{\partial E}$ are nonnegative, and others are nonpositive.

57. Let $\bar{c}(\xi,\tau) = c(S,t)/(S + P_m)$ and $\bar{p}(\xi,\tau) = p(S,t)/(S + P_m)$, where $\xi = S/(S + P_m)$ and $\tau = T - t$. Derive the expressions of $\bar{c}(\xi,\tau)$ and $\bar{p}(\xi,\tau)$ and find the limits of $\bar{c}(\xi,\tau)$ and $\bar{p}(\xi,\tau)$ as ξ tends to 0 and 1. Also write down the formulae for the case $P_m = E$.

58. Suppose that S is the price of a stock,

$$dS = \mu S dt + \sigma S dX,$$

and $V(S,t)$ is the value of an option on the stock. Define $S_0^* = E\mathrm{e}^{-r(T-t)}$, $S_1^* = S\mathrm{e}^{-D_0(T-t)}$, $\xi_{10} = \dfrac{S_1^*}{S_0^*} = \dfrac{S\mathrm{e}^{(r-D_0)(T-t)}}{E}$, $\xi_{01} = \dfrac{S_0^*}{S_1^*} = \dfrac{E}{S\mathrm{e}^{(r-D_0)(T-t)}}$, $V_0(\xi_{10},t) = V(S(\xi_{10},t),t)/S_0^*(t)$, and $V_1(\xi_{01},t) = V(S(\xi_{01},t),t)/S_1^*(\xi_{01},t)$, where E and T are constants, r is the interest rate, and D_0 is the dividend yield of the stock. Assume that we already know that

$$d\xi_{10} = (\mu - r + D_0)\xi_{10}dt + \sigma\xi_{10}dX.$$

(a) By setting $\Pi = V - \Delta S = S_0^*(t)V_0(\xi_{10},t) - \Delta E\mathrm{e}^{-(r-D_0)(T-t)}\xi_{10}$ and using Itô's lemma, show that the PDE for $V_0(\xi_{10},t)$ is

$$\frac{\partial V_0}{\partial t} + \frac{1}{2}\sigma^2\xi_{10}^2\frac{\partial^2 V_0}{\partial \xi_{10}^2} = 0.$$

(b) From the PDE for $V_0(\xi_{10},t)$ obtained in part (a), show that the PDE for $V_1(\xi_{01},t)$ is

$$\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2\xi_{01}^2\frac{\partial^2 V_1}{\partial \xi_{01}^2} = 0.$$

(c) Consider the problem:

$$\begin{cases} \dfrac{\partial W}{\partial t} + \frac{1}{2}\sigma^2\xi^2\dfrac{\partial^2 W}{\partial \xi^2} = 0, & 0 \le \xi, \qquad t \le T, \\ W(\xi,T) = \max(\xi,1), & 0 \le \xi. \end{cases}$$

As we know, the solution of this problem is

$$\begin{aligned} W(\xi,t) &= \int_0^\infty \max(\xi',1)G(\xi',T;\xi,t)d\xi' \\ &= \int_0^1 G(\xi',T;\xi,t)d\xi' + \int_1^\infty \xi' G(\xi',T;\xi,t)d\xi', \end{aligned}$$

where

$$G(\xi',T;\xi,t) = \frac{1}{\sqrt{2\pi}b\xi'}\mathrm{e}^{-\left[\ln(\xi'/\xi)+b^2/2\right]^2/2b^2}, \quad b \text{ being } \sigma\sqrt{T-t}.$$

Let $V(S,t)$ be the price of the option with payoff $\max(S,E)$. In this case $V_0(\xi_{10},T) = \max(S,E)/E = \max(\xi_{10},1)$ and $V_1(\xi_{01},T) = \max(S,E)/S = \max(\xi_{01},1)$. Thus, for $V(S,t)$ we have two expressions:

$$\begin{aligned} V(S,t) &= S_0^* W(\xi_{10},t) \\ &= S_0^* \int_0^1 G(\xi_{10}',T;\xi_{10},t)d\xi_{10}' + S_0^* \int_1^\infty \xi_{10}' G(\xi_{10}',T;\xi_{10},t)d\xi_{10}', \end{aligned}$$

and

$$\begin{aligned} V(S,t) &= S_1^* W(\xi_{01},t) \\ &= S_1^* \int_0^1 G(\xi_{01}',T;\xi_{01},t)d\xi_{01}' + S_1^* \int_1^\infty \xi_{01}' G(\xi_{01}',T;\xi_{01},t)d\xi_{01}'. \end{aligned}$$

Because at $t = T$ both $\xi_{10}' < 1$ and $\xi_{01}' > 1$ correspond to $S' < E$, both the first term in the first expression and the second term in the second expression represent the contribution which the function $\max(S',E)$ as $S' < E$ makes to the value $V(S,t)$. Consequently, the two terms should be equal. Similarly the second term in the first expression should be equal to the first term in the second expression. Verify this conclusion by direct calculation.

59. *Suppose that $c(S,t)$ and $p(S,t)$ are the prices of European call and put options with the same parameters, respectively. Show the put–call parity

$$c(S,t) - p(S,t) = S\mathrm{e}^{-D_0(T-t)} - E\mathrm{e}^{-r(T-t)}$$

without using the Black–Scholes formulae.

60. Consider a European option on a non-dividend-paying stock. The stock price is \$37, the exercise price is \$34, the risk-free interest rate is 5 % per annum, the volatility is 30 % per annum, and the time to maturity is six months. Find the call and put option prices by using the Black–Scholes formulae and verify that the put–call parity holds.

61. By using the put–call parity relation of European options

$$c(S,t) - p(S,t) = S\mathrm{e}^{-D_0(T-t)} - E\mathrm{e}^{-r(T-t)},$$

show that the following relations hold:

$$\frac{\partial p}{\partial S} = \frac{\partial c}{\partial S} - \mathrm{e}^{-D_0(T-t)}, \quad \frac{\partial^2 p}{\partial S^2} = \frac{\partial^2 c}{\partial S^2}$$

and

$$\frac{\partial^2 p}{\partial S \partial \sigma} = \frac{\partial^2 c}{\partial S \partial \sigma}, \quad \frac{\partial p}{\partial \sigma} = \frac{\partial c}{\partial \sigma}, \quad \frac{\partial^2 p}{\partial \sigma^2} = \frac{\partial^2 c}{\partial \sigma^2}.$$

3

American Style Derivatives

In this Chapter, we will discuss that in order to find the value of an American style derivative, what kind of mathematical problems needs to be solved. When we have such discussions, we mainly take American options as examples. However the methods can be used for other American style derivatives. In the first section, we will derive the additional constraints on American style derivatives and discuss how the constraints affect the way the price is determined. In Sect. 3.2, we formulate the American call and put problems as linear complementarity (LC) problems and point out how to get the formulation for an American style derivative from the formulation for the corresponding European style derivative. In Sect. 3.3, we will discuss how to formulate an American option problem as a free-boundary problem (FBP) from a linear complementarity problem. For other American style derivatives, the method is similar. Finally we discuss some properties of options, including the relations between European and American options, by the arbitrage theory in the last section.

3.1 Constraints on American Style Derivatives

3.1.1 Constraints on American Options

Let $C(S,t)$ and $P(S,t)$ denote the prices of American call and put options, respectively. As we know from Sect. 1.2, an American option has the additional feature that it may be exercised at any time during the life of the option. What does this additional feature mean in mathematics? It means that the value of an American call option must satisfy the condition

$$C(S,t) \geq \max(S-E,0), \tag{3.1}$$

and that the value of an American put option must fulfill the inequality

$$P(S,t) \geq \max(E-S,0). \tag{3.2}$$

Y.-l. Zhu et al., *Derivative Securities and Difference Methods*, Springer Finance, DOI 10.1007/978-1-4614-7306-0_3,

Usually, $\max(S-E,0)$ and $\max(E-S,0)$ are called the intrinsic values of call and put options, respectively. Thus, satisfying the two inequalities above means that the value of an option must be at least equal to its intrinsic value. Conditions (3.1) and (3.2) are usually referred to as the constraints on American vanilla options. These conclusions can be proved by arbitrage arguments as follows.

First, let us consider an American call option. For $S \le E$, the condition (3.1) means $C(S,t) \ge 0$. This is always true because a solution of the Black–Scholes equation with a nonnegative payoff function as a final condition is always nonnegative. From the financial point of view, it is also clear that a holder of an option has only rights, no obligation, so he/she needs to pay something in order to get it, i.e., the option price should not be negative. Thus, the condition (3.1) always holds for any $S \in [0,E]$. Suppose that for a price $S > E$, the condition (3.1) is not fulfilled, i.e., $C(S,t) < S-E$. Then, an obvious arbitrage opportunity arises: by short selling the asset on the market for S, purchasing the option for C, and exercising the call option, a risk-free profit of $S-C-E$ is made. Of course, such an opportunity would not last long before the value of the option was pushed up by the demand of arbitrageurs. We conclude that on a value of an American call, we must impose the constraint (3.1). For an American put option the situation is similar. For any $S \ge E$, the condition (3.2) holds naturally. Suppose the option price satisfies $P(S,t) < E-S$ for a price $S < E$. Then, by purchasing the option for P, purchasing the asset from the market for S, and exercising the put option, an immediate risk-free profit of $E-P-S$ is made, and the demand will push the option price up so that condition (3.2) holds.

Bermudan options are similar to American options but can be exercised only at several predetermined dates, instead of the entire period $[0,T]$. This means that for a Bermudan option, condition (3.1) or condition (3.2) should be required at several predetermined dates but not on the entire period $[0,T]$, which is the only difference between American and Bermudan options.

How does a constraint affect the way of determining the price of an option? Let us take an American put option as an example. As we easily see, at $S=0$ the Black–Scholes equation degenerates to an ordinary differential equation

$$\frac{\partial V(0,t)}{\partial t} - rV(0,t) = 0$$

and its solution is

$$V(0,t) = V(0,T)\mathrm{e}^{-r(T-t)}.$$

For a put, $V(0,T) = E$. Therefore, the price of a European put option at $S=0$ is

$$p(0,t) = E\mathrm{e}^{-r(T-t)} < E$$

for any $t < T$ if $r > 0$. Consequently, the price of a European put option will not satisfy the constraint (3.2). Thus, in order to price an American put, we must modify the method for determining the price of an option if $r > 0$.

Roughly speaking, the way of determining the price of an American style derivative is as follows. Let $V(S,t)$ be the price of an American style derivative and $G_v(S,t)$ be the constraint. Suppose that for a time t, $V(S,t)$ is known for any S. Based on $V(S,t)$ and using the Black–Scholes equation, we can obtain the price of a derivative security at time $t-\Delta t$ for a small positive Δt. If the value satisfies the constraint condition $V(S,t-\Delta t) \geq G_v(S,t-\Delta t)$, it gives the price of the American style derivative; if not, the constraint is the value of the American style derivative, i.e., the Black–Scholes equation cannot be used for determining the price of the American style derivative in this case.

Let us explain why the price of the American style derivative is determined in this way. If $V(S,t) > G_v(S,t)$ in a neighborhood of a point $S = S^*$ at time t, then the solution $V(S,t-\Delta t)$ obtained by using the Black–Scholes equation must still satisfy the condition $V(S,t-\Delta t) > G_v(S,t-\Delta t)$ at that point if Δt is small enough. Therefore the event "the Black–Scholes equation cannot be used" only occurs at a point $S = S^*$ where $V(S,t) = G_v(S,t)$. Thus we need to discuss when the Black–Scholes equation can be used and when the Black–Scholes equation cannot be used only if $V(S,t) = G_v(S,t)$. On this question, we have the following theorem. In the future we will also consider other problems besides option problems, thus in the theorem, we consider a general partial differential equation (PDE) similar to the Black–Scholes equation. The theorem is described as follows.

Theorem 3.1 *Let* $\mathbf{L_{S,t}}$ *be an operator in a derivative security problem in the form:*

$$\mathbf{L_{S,t}} = a(S,t)\frac{\partial^2}{\partial S^2} + b(S,t)\frac{\partial}{\partial S} + c(S,t)$$

and $G_v(S,t)$ *be the constraint function for an American style derivative. Furthermore, we assume that* $\dfrac{\partial G_v}{\partial t} + \mathbf{L_{S,t}} G_v$ *exists. Suppose* $V(S,t^*) = G_v(S,t^*)$ *on an open interval* (A,B) *on the* S*-axis. Let* $t = t^* - \Delta t$*, where* Δt *is a sufficiently small positive number. For this case we have the following conclusions: If for any* $S \in (A,B)$,

$$\frac{\partial G_v}{\partial t}(S,t^*) + \mathbf{L_{S,t^*}} G_v(S,t^*) \geq 0,$$

then the value $V(S,t)$ *determined by the equation*

$$\frac{\partial V}{\partial t}(S,t) + \mathbf{L_{S,t}} V(S,t) = 0$$

satisfies the condition $V(S,t) - G_v(S,t) \geq 0$ *on* (A,B)*, which means the PDE can be used for determining the price of the American style derivative; and if for any* $S \in (A,B)$,

$$\frac{\partial G_v}{\partial t}(S,t^*) + \mathbf{L_{S,t^*}} G_v(S,t^*) < 0,$$

then the equation

$$\frac{\partial V}{\partial t}(S,t) + \mathbf{L_{S,t}} V(S,t) = 0$$

cannot give a solution satisfying the condition $V(S,t) - G_v(S,t) \geq 0$ *for any* $S \in (A,B)$*, which means the PDE cannot be used for determining the price of the American style derivative.*

Proof. Because $V(S,t^*) = G_v(S,t^*)$, the fact that $V(S,t) - G_v(S,t) > 0$ holds for any $t = t^* - \Delta t$, Δt being a sufficiently small positive number, is equivalent to that at time t^*, $V(S,t) - G_v(S,t)$ is a decreasing function with respect to t, that is,

$$\frac{\partial V}{\partial t}(S,t^*) - \frac{\partial G_v}{\partial t}(S,t^*) < 0.$$

If

$$\frac{\partial G_v}{\partial t}(S,t^*) + \mathbf{L_{S,t^*}} G_v(S,t^*) > 0$$

and

$$\frac{\partial V}{\partial t}(S,t^*) + \mathbf{L_{S,t^*}} V(S,t^*) = \frac{\partial V}{\partial t}(S,t^*) + \mathbf{L_{S,t^*}} G_v(S,t^*) = 0,$$

then

$$\frac{\partial G_v}{\partial t}(S,t^*) > -\mathbf{L_{S,t^*}} G_v(S,t^*) = \frac{\partial V}{\partial t}(S,t^*)$$

or

$$\frac{\partial V}{\partial t}(S,t^*) - \frac{\partial G_v}{\partial t}(S,t^*) < 0.$$

Therefore in this case we can use the equation

$$\frac{\partial V}{\partial t}(S,t) + \mathbf{L_{S,t}} V(S,t) = 0$$

to get a solution satisfying the condition $V(S,t) - G_v(S,t) > 0$, which means the PDE can be used for determining the price of the American style derivative.

If on a point (S,t^*) with $S \in (A,B)$

$$\frac{\partial G_v}{\partial t}(S,t^*) + \mathbf{L_{S,t^*}} G_v(S,t^*) = 0,$$

then $G_v(S,t)$ is the solution $V(S,t)$ in a sufficiently small neighborhood of the point (S,t^*). Putting this result and the result above together, we know that if

$$\frac{\partial G_v}{\partial t}(S,t^*) + \mathbf{L_{S,t^*}} G_v(S,t^*) \geq 0$$

then the PDE can be used for determining the price of the American style derivative.

If

$$\frac{\partial G_v}{\partial t}(S, t^*) + \mathbf{L}_{\mathbf{S},\mathbf{t}^*} G_v(S, t^*) < 0$$

and

$$\frac{\partial V}{\partial t}(S, t^*) + \mathbf{L}_{\mathbf{S},\mathbf{t}^*} V(S, t^*) = \frac{\partial V}{\partial t}(S, t^*) + \mathbf{L}_{\mathbf{S},\mathbf{t}^*} G_v(S, t^*) = 0,$$

then

$$\frac{\partial G_v}{\partial t}(S, t^*) < -\mathbf{L}_{\mathbf{S},\mathbf{t}^*} G_v(S, t^*) = \frac{\partial V}{\partial t}(S, t^*)$$

or

$$\frac{\partial V}{\partial t}(S, t^*) - \frac{\partial G_v}{\partial t}(S, t^*) > 0,$$

which will cause $V(S, t) - G_v(S, t) < 0$ for any $t = t^* - \Delta t$. Therefore, we cannot get the solution by using the equation

$$\frac{\partial V}{\partial t}(S, t) + \mathbf{L}_{\mathbf{S},\mathbf{t}} V(S, t) = 0,$$

which means the PDE cannot be used for determining the price of the American style derivative. □

About this theorem, we would like to make the following remark.

- Let us adopt $\tau = T - t$ instead of t and we want to have the solution at $\tau = \tau^* + \Delta\tau$ from the solution at $\tau = \tau^*$, where $\Delta\tau > 0$. Then the theorem is still true if the condition

$$\frac{\partial G_v}{\partial t}(S, t) + \mathbf{L}_{\mathbf{S},\mathbf{t}} G_v(S, t) \geq 0$$

is changed into

$$\frac{\partial \overline{G}_v}{\partial \tau}(S, \tau) - \mathbf{L}_{\mathbf{S},\tau} \overline{G}_v(S, \tau) \leq 0$$

and the condition

$$\frac{\partial G_v}{\partial t}(S, t) + \mathbf{L}_{\mathbf{S},\mathbf{t}} G_v(S, t) < 0$$

is changed into

$$\frac{\partial \overline{G}_v}{\partial \tau}(S, \tau) - \mathbf{L}_{\mathbf{S},\tau} \overline{G}_v(S, \tau) > 0,$$

where $\overline{G}_v(S, \tau) = G_v(S, t)$ and $\mathbf{L}_{\mathbf{S},\tau} \overline{\mathbf{G}}_{\mathbf{v}}(\mathbf{S}, \tau) = \mathbf{L}_{\mathbf{S},\mathbf{t}} \mathbf{G}_{\mathbf{v}}(\mathbf{S}, \mathbf{t})$.

From Theorem 3.1, we know that when $V(S, t) = G_v(S, t)$ and

$$\left[\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0) S \frac{\partial}{\partial S} - r\right] G_v < 0,$$

we cannot use the Black–Scholes equation to determine $V(S, t - \Delta t)$. What $V(S, t - \Delta t)$ should be in this case? In the above, we have pointed that in this case, $V(S,t-\Delta t) = G_v(S,t-\Delta t)$. Here let us explain why it should be. It is clear that a buyer of a derivative security wants the price to be as low as possible and that the price of an American style derivative cannot be less than the constraint as we discussed above. Thus for $V(S,t-\Delta t)$ the constraint $G_v(S,t-\Delta t)$ is the lowest price the buyer can expect. A seller wants the price to be as high as possible. Can the seller accept that the constraint is the price in this case? The answer is "yes", so the constraint is the price both the buyer and the seller accept. Let us explain why the seller accepts this price. Suppose that the seller sells the derivative security for $V(S,t-\Delta t) = G_v(S,t-\Delta t)$. After the derivative security is sold, using the money obtained, the seller buys $\dfrac{\partial V}{\partial S}$ shares and deposits the remains $V - \dfrac{\partial V}{\partial S}S$ into a money market account.[1] Because $V(S,t-\Delta t) = G_v(S,t-\Delta t)$, we will have

$$\begin{aligned}&\left[\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0)S\frac{\partial}{\partial S} - r\right] V \\ =& \left[\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0)S\frac{\partial}{\partial S} - r\right] G_v < 0,\end{aligned}$$

i.e.,

$$\begin{aligned}\frac{\partial V}{\partial S}dS + \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt &< \frac{\partial V}{\partial S}dS - \left[(r - D_0)S\frac{\partial V}{\partial S} - rV\right] dt \\ &= r\left(V - \frac{\partial V}{\partial S}S\right) dt + \frac{\partial V}{\partial S}(dS + D_0 S dt).\end{aligned}$$

This means that the return from the derivative security during a time step dt,

$$\frac{\partial V}{\partial S}dS + \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt,$$

will be less than the return from the money market account with a value of $V - \dfrac{\partial V}{\partial S}S$ and $\dfrac{\partial V}{\partial S}$ shares,

$$r\left(V - \frac{\partial V}{\partial S}S\right) dt + \frac{\partial V}{\partial S}(dS + D_0 S dt).$$

Thus the amount of money the seller obtains from the money market account and shares is more than the change of the derivative value, which means the seller will earn money. Hence the seller can accept this price.

[1] If $V - \dfrac{\partial V}{\partial S}S < 0$, the seller indeed borrows $-\left(V - \dfrac{\partial V}{\partial S}S\right)$, the money needed to buy $\dfrac{\partial V}{\partial S}$ shares, from somewhere.

3.1.2 Some Properties of American Style Derivatives

Consider a European style derivative and an American style derivative with identical payoffs and identical operators. Let $V(S,t)$, $v(S,t)$ denote the prices of the American and European style derivatives, respectively, let $G_v(S,t)$ be the constraint for the American style derivative, and the operator for the two derivatives is

$$\frac{\partial}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2}{\partial S^2}+(r-D_0)S\frac{\partial}{\partial S}-r$$

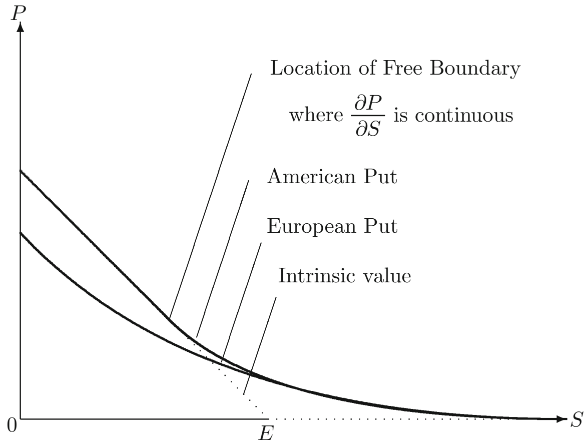

Fig. 3.1. The price of an American put option before expiry

with constant σ, r, and D_0. Using the results we have obtained, we can prove that the price of the European style derivative is never higher than the price of the American style derivative, i.e., $V(S,t) \geq v(S,t)$ holds. Let us prove this conclusion. Suppose that

$$V(S,T)=v(S,T)=G_v(S,T).$$

Set $\Delta t = T/N$, N being a positive integer and define $t_n = n\Delta t$, $n = N, N-1, \cdots, 0$. For the European style derivative, from the formula (2.84) we have the relation between $v(S,t_n)$ and $v(S,t_{n+1})$

$$v(S,t_n)=\mathrm{e}^{-r\Delta t}\int_0^{\infty} v(S',t_{n+1})G(S',t_{n+1};S,t_n)dS'$$

for $n = N-1, N-2, \cdots, 0$. Let

$$\tilde{V}(S,t_{\scriptscriptstyle N})=G_v(S,T)$$

and for $n = N-1, N-2, \cdots, 0$, define

$$\tilde{V}(S, t_n) = \max\left(\mathrm{e}^{-r\Delta t}\int_0^\infty \tilde{V}(S', t_{n+1})G(S', t_{n+1}; S, t_n)dS', G_v(S, t_n)\right), \tag{3.3}$$

where $G(S', t_{n+1}; S, t_n)$ is given by the formula (2.85). Suppose $\tilde{V}(S, t_{n+1}) \geq v(S, t_{n+1})$, then we know

$$\begin{aligned} v(S, t_n) &= \mathrm{e}^{-r\Delta t}\int_0^\infty v(S', t_{n+1})G(S', t_{n+1}; S, t_n)dS' \\ &\leq \mathrm{e}^{-r\Delta t}\int_0^\infty \tilde{V}(S', t_{n+1})G(S', t_{n+1}; S, t_n)dS' \\ &\leq \tilde{V}(S, t_n). \end{aligned}$$

At $t = t_N = T$, the condition

$$\tilde{V}(S, t_{\scriptscriptstyle N}) = G_v(S, T) \geq v(S, T) = G_v(S, T)$$

holds. Therefore, using the induction method, we can prove $\tilde{V}(S, t_n) \geq v(S, t_n)$ for $n = N-1, N-2, \cdots, 0$ successively. Letting $N \to \infty$ and noticing that $\tilde{V}\left(S, \text{ int}\left(\frac{tN}{T}\right) \cdot \frac{T}{N}\right)$ generates $V(S, t)$ as $N \to \infty$, where $\text{int}\left(\frac{tN}{T}\right)$ is the integer part of $\frac{tN}{T}$, we can have the conclusion:

$$V(S, t) \geq v(S, t) \quad \text{ for any } S \text{ and } t.$$

The put and call options are such type of derivatives. Thus $C(S, t) \geq c(S, t)$ and $P(S, t) \geq p(S, t)$. This result has the following financial meaning. Because an American option can be exercised at any time by expiry, a holder of an American option has more rights than does a holder of a European option. Thus, the holder of an American option needs to pay at least as much premium as does the holder of a European option with the same parameters. Figure 3.1 shows this fact and other related facts for put options. From the figure, we can see that the price of America put option is always greater than the price of European put option and the intrinsic value, but the price of the European put option is greater than the intrinsic value for some S and less than the intrinsic value for other S. It can also be proved that the price of a Bermudan option should be between these of European and American options and the financial meaning can be expressed as follows. The Bermudan option can be exercised at several predetermined dates including the expiration date, its holder has less rights than does the holder of an American option and more rights than does the holder of a European option. Thus, its premium should be between the premiums of the American and European options with the same parameters.

The price of an American style derivative has another property: $V(S,t^*) \geq V(S,t^{**})$ for $t^* \leq t^{**}$ if $G_v(S,t) = G_v(S)$ or, more generally, the condition $G_v(S,t^*) \geq G_v(S,t^{**})$ for $t^* \leq t^{**}$ holds. Let us explain this fact by using mathematical tools. Suppose $\tilde{V}(S,t_n) \geq \tilde{V}(S,t_{n+1})$. According to the definition of $\tilde{V}(S,t_n)$, we have

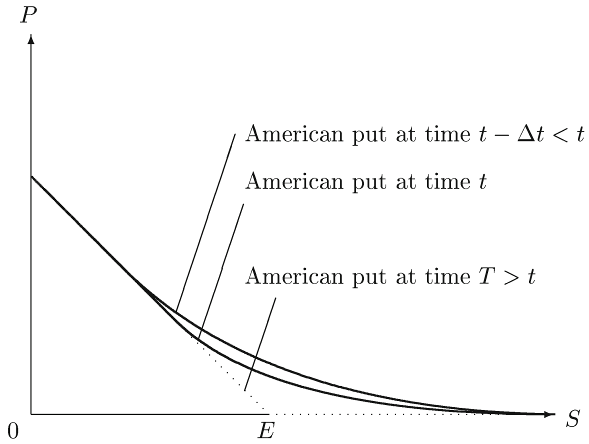

Fig. 3.2. $P(S, t-\Delta t) \geq P(S,t)$ for any positive Δt

$$\begin{aligned}\tilde{V}(S,t_n) &= \max\left(\mathrm{e}^{-r\Delta t}\int_0^\infty \tilde{V}(S',t_{n+1})G(S',t_{n+1};S,t_n)dS', G_v(S,t_n)\right)\\ &\leq \max\left(\mathrm{e}^{-r\Delta t}\int_0^\infty \tilde{V}(S',t_n)G(S',t_n;S,t_{n-1})dS', G_v(S,t_{n-1})\right)\\ &= \tilde{V}(S,t_{n-1}).\end{aligned}$$

Here we have used the facts

$$G(S',t_{n+1};S,t_n) = G(S',t_n;S,t_{n-1})$$

and

$$G_v(S,t_n) \leq G_v(S,t_{n-1}).$$

Because

$$\begin{aligned}\tilde{V}(S,t_{N-1}) &= \max\left(\mathrm{e}^{-r\Delta t}\int_0^\infty \tilde{V}(S',t_N)G(S',t_N;S,t_{N-1})dS', G_v(S,t_{N-1})\right)\\ &\geq G_v(S,t_{N-1}) \geq G_v(S,t_N) = \tilde{V}(S,t_N),\end{aligned}$$

we can prove

$$\tilde{V}(S,t_n) \geq \tilde{V}(S,t_{n+1}) \quad \text{for} \quad n = N-2, N-3, \cdots, 0$$

successively. This means

$$\tilde{V}(S,t_n) \geq \tilde{V}(S,t_m) \quad \text{for} \quad n \leq m \leq N.$$

Letting $N \to \infty$ and noticing that $\tilde{V}(S,t)$ generates $V(S,t)$ as $N \to \infty$, we arrive at the conclusion

$$V(S,t^*) \geq V(S,t^{**}) \quad \text{if} \quad t^* \leq t^{**}.$$

For the American call/put option, $G_v(S,t) = G_v(S)$, so we have $C(S,t^*) \geq C(S,t^{**})$ and $P(S,t^*) \geq P(S,t^{**})$ if $t^* \leq t^{**}$. Figure 3.2 shows this fact graphically for an American put option. From the point of financial view, when $t^* < t^{**}$, a holder of an American call/put option at time t^* has more rights than does a holder at time t^{**}, so the premium of the option at time t^* should be higher than the premium of the option at time t^{**}.

As we have pointed out, $C(S,t) \geq \max(S-E,0)$ and $P(S,t) \geq \max(E-S,0)$, which means that $C(S,t) - \max(S-E,0)$ and $P(S,t) - \max(E-S,0)$ must be nonnegative. Because these two functions are usually called the time values of the American call and put options, respectively, this fact can be expressed as that the time values must be nonnegative. Using the result here $V(S,t^*) \geq V(S,t^{**})$ for $t^* \leq t^{**}$, we can have another conclusion: the time values $C(S,t) - \max(S-E,0)$ and $P(S,t) - \max(E-S,0)$ are non-increasing functions in time because

$$V(S,t^*) \geq V(S,t^{**}) \quad \text{for} \quad t^* \leq t^{**}$$

is equivalent to

$$V(S,t^*) - G_v(S) \geq V(S,t^{**}) - G_v(S) \quad \text{for} \quad t^* \leq t^{**}.$$

However not all American style derivatives have such a property. Here we give an example. Consider the following derivative security. It is a bond with a face value Z and it can be converted into n shares at any time. We assume that the price of the stock is a random variable, the interest rate is a constant and the bond pays no coupon. This problem is referred to as the problem of one-factor convertible bond paying no coupon. Let $B_c(S,t)$ stand for its value. It is clear that $B_c(S,T) = \max(Z, nS)$, $B_c(S,t) \geq nS$ for $t < T$, and the basic PDE for this problem is the Black-Scholes equation. Thus this derivative security problem is close to the American option problem and its some properties can be studied by using a similar way given in this subsection. For example, using the method given here, it can be shown that

$$B_c(S,t^*) - Z\mathrm{e}^{-r(T-t^*)} \geq B_c(S,t^{**}) - Z\mathrm{e}^{-r(T-t^{**})} \quad \text{if} \quad t^* \leq t^{**}$$

holds and

$$B_c(S,t^*) \geq B_c(S,t^{**}) \quad \text{if} \quad t^* \leq t^{**}$$

does not hold at least for $S = 0$. These results are left for readers to prove as Problem 6. Here we give an explanation for such results. The final condition

$B_c(S,T) = \max(Z, nS)$ can be rewritten as $B_c(S,T) = Z + \max(nS - Z, 0)$, so it consists of two problems, one is a bond problem with a solution of $Z\mathrm{e}^{-r(T-t)}$ and the other is a special American call problem with a payoff of $\max(nS - Z, 0)$ and a constraint $nS - Z\mathrm{e}^{-r(T-t)}$. For this special American call option, the price is a non-increasing function like the American option. However for the bond problem, the price is an increasing function. Thus the total is not a non-increasing function. Consequently, even the holder of this American derivative at t^* has "more rights" than does the holder at t^{**} if $t^* \leq t^{**}$, but the price at t^* is not always greater than or equal to the price at t^{**}.

3.2 American Options Problems as Linear Complementarity Problems

3.2.1 Formulation of the Linear Complementarity Problem in (S,t)-Plane

From Sect. 3.1.1, we know that the price of an American option usually is not a solution of the problem (2.73) anymore because usually in some regions the solution satisfies the PDE and in other regions it is not determined by the PDE. For American option problems, the price is given by a solution of a so-called linear complementarity (LC) problem.

Now let us formulate the LC problem the price of an American option should satisfy. Let us take an American put option as an example. Assume that at time t we have obtained $P(S,t)$ satisfying (3.2) and we need to determine $P(S, t - \Delta t)$ satisfying (3.2), where Δt is a sufficiently small positive number. Define $G_p(S,t) = \max(E - S, 0)$. For simplicity, we assume that the entire interval consists of three open intervals plus their boundaries. On the first open interval, $P(S,t) > G_p(S,t)$. For any point in this interval, we can use the Black–Scholes equation to determine $P(S, t - \Delta t)$ and $P(S, t - \Delta t)$ must be still greater than $G_p(S, t - \Delta t)$ if Δt is small enough. Therefore, at any point in this open interval

$$\begin{cases} \dfrac{\partial P}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 P}{\partial S^2} + (r - D_0) S \dfrac{\partial P}{\partial S} - rP = 0, \\ P(S,t) > G_p(S,t). \end{cases}$$

On the second open interval $P(S,t) = G_p(S,t)$ and

$$\left[\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0) S \frac{\partial}{\partial S} - r\right] G_p(S,t) \geq 0$$

and on the third open interval $P(S,t) = G_p(S,t)$ and

$$\left[\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0) S \frac{\partial}{\partial S} - r\right] G_p(S,t) < 0.$$

According to Theorem 3.1, for a point (S,t) in the second open interval the Black–Scholes equation can be used to determine $P(S,t-\Delta t)$ and the following is true:

$$\begin{cases} \dfrac{\partial P}{\partial t}+\dfrac{1}{2}\sigma^2 S^2\dfrac{\partial^2 P}{\partial S^2}+(r-D_0)S\dfrac{\partial P}{\partial S}-rP=0,\\ P(S,t)=G_p(S,t). \end{cases}$$

On the third interval, the Black–Scholes equation cannot be used to determine $P(S,t-\Delta t)$. Instead, $P(S,t-\Delta t)$ should equal $G_p(S,t-\Delta t)$. In this situation

$$\frac{P(S,t)-P(S,t-\Delta t)}{\Delta t}=\frac{G_p(S,t)-G_p(S,t-\Delta t)}{\Delta t}\rightarrow\frac{\partial G_p(S,t)}{\partial t}$$

as $\Delta t\rightarrow 0$ and we have

$$\begin{cases} \left[\dfrac{\partial}{\partial t}+\dfrac{1}{2}\sigma^2 S^2\dfrac{\partial^2}{\partial S^2}+(r-D_0)S\dfrac{\partial}{\partial S}-r\right]P\\ =\left[\dfrac{\partial}{\partial t}+\dfrac{1}{2}\sigma^2 S^2\dfrac{\partial^2}{\partial S^2}+(r-D_0)S\dfrac{\partial}{\partial S}-r\right]G_p(S,t)<0,\\ P(S,t)=G_p(S,t). \end{cases}$$

Because $P(S,T)=G_p(S,T)$, we can use this argument from T to 0. Putting all the cases together, for $S\in[0,\infty)$ and $t\le T$ we have

$$\begin{cases} \left[\dfrac{\partial P}{\partial t}+\dfrac{1}{2}\sigma^2 S^2\dfrac{\partial^2 P}{\partial S^2}+(r-D_0)S\dfrac{\partial P}{\partial S}-rP\right](P-G_p)=0,\\ \dfrac{\partial P}{\partial t}+\dfrac{1}{2}\sigma^2 S^2\dfrac{\partial^2 P}{\partial S^2}+(r-D_0)S\dfrac{\partial P}{\partial S}-rP\le 0,\\ P(S,t)-G_p(S,t)\ge 0,\\ P(S,T)=G_p(S,T), \end{cases}$$

where $G_p(S,t)=\max(E-S,0)$. Here, we use the fact that these relations in the formulation are also true in some sense at the boundary points of these open intervals because these relations are true on the two sides of a boundary point. It is clear that the formulation above can also be written in the following short form:

$$\begin{cases} \min\left(-\dfrac{\partial P}{\partial t}-\mathbf{L}_S P,\ P(S,t)-G_p(S,t)\right)=0, & 0\le S,\ t\le T,\\ P(S,T)=G_p(S,T), & 0\le S, \end{cases} \tag{3.4}$$

where

$$\mathbf{L}_S=\frac{1}{2}\sigma^2 S^2\frac{\partial^2}{\partial S^2}+(r-D_0)S\frac{\partial}{\partial S}-r$$

and

$$G_p(S,t) = \max(E - S, 0).$$

This problem is called the linear complementarity problem for an American put option. In order to determine the price of an American put option, we need to solve this problem.

Similarly, for an American call option, the corresponding linear complementarity problem is

$$\begin{cases} \min\left(-\dfrac{\partial C}{\partial t} - \mathbf{L}_S C,\ C(S,t) - G_c(S,t)\right) = 0, & 0 \le S,\ t \le T, \\ C(S,T) = G_c(S,T), & 0 \le S, \end{cases} \tag{3.5}$$

where $G_c(S,t) = \max(S - E, 0)$. From the derivation of the problem (3.4), we can see that the formulations are still correct when σ, r, D_0 depend on S and t.

3.2.2 Formulation of the Linear Complementarity Problem in $(x, \bar{\tau})$-Plane

As we know from Sect. 2.6.1, if we set

$$\begin{cases} x = \ln S + \left(r - D_0 - \dfrac{1}{2}\sigma^2\right)(T-t), \\ \bar{\tau} = \dfrac{1}{2}\sigma^2 (T-t), \\ V(S,t) = \mathrm{e}^{-r(T-t)} u(x, \bar{\tau}), \end{cases}$$

then

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0) S \frac{\partial V}{\partial S} - rV$$

becomes

$$-\frac{1}{2}\sigma^2 \mathrm{e}^{-r(T-t)} \left(\frac{\partial u}{\partial \bar{\tau}} - \frac{\partial^2 u}{\partial x^2}\right).$$

Thus,

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial P}{\partial S^2} + (r - D_0) S \frac{\partial P}{\partial S} - rP < 0$$

is equivalent to

$$\frac{\partial u}{\partial \bar{\tau}} - \frac{\partial^2 u}{\partial x^2} > 0$$

and the Black–Scholes equation holds if and only if

$$\frac{\partial u}{\partial \bar{\tau}} - \frac{\partial^2 u}{\partial x^2} = 0.$$

Let us define

$$g_p(x,\bar\tau) = \max\left(\mathrm{e}^{2r\bar\tau/\sigma^2} - \mathrm{e}^{x+(2D_0/\sigma^2+1)\bar\tau}, 0\right),$$

then

$$\begin{aligned}
P - G_p &= P(S,t) - \max(1-S,0)\\
&= \mathrm{e}^{-r(T-t)}u(x,\bar\tau) - \max\left(1-\mathrm{e}^{x-(r-D_0-\sigma^2/2)(T-t)}, 0\right)\\
&= \mathrm{e}^{-r(T-t)}\left[u(x,\bar\tau) - \max\left(\mathrm{e}^{r(T-t)} - \mathrm{e}^{x+(D_0+\sigma^2/2)(T-t)}, 0\right)\right]\\
&= \mathrm{e}^{-r(T-t)}[u(x,\bar\tau) - g_p(x,\bar\tau)],
\end{aligned}$$

where we suppose $E = 1$ for simplicity. Thus, $P - G_p > 0$ is equivalent to

$$u(x,\bar\tau) - g_p(x,\bar\tau) > 0$$

and $P - G_p = 0$ if and only if

$$u(x,\bar\tau) - g_p(x,\bar\tau) = 0.$$

Therefore, the American put option is the solution of the following problem:

$$\begin{cases}
\min\left(\dfrac{\partial u}{\partial \bar\tau} - \dfrac{\partial^2 u}{\partial x^2},\ u(x,\bar\tau) - g_p(x,\bar\tau)\right) = 0, & -\infty < x < \infty,\ 0 \le \bar\tau,\\
u(x,0) = g_p(x,0), & -\infty < x < \infty.
\end{cases} \tag{3.6}$$

Similarly, for American call options we have

$$\begin{cases}
\min\left(\dfrac{\partial u}{\partial \bar\tau} - \dfrac{\partial^2 u}{\partial x^2},\ u(x,\bar\tau) - g_c(x,\bar\tau)\right) = 0, & -\infty < x < \infty,\ 0 \le \bar\tau,\\
u(x,0) = g_c(x,0), & -\infty < x < \infty,
\end{cases} \tag{3.7}$$

where

$$g_c(x,\bar\tau) = \max\left(\mathrm{e}^{x+(2D_0/\sigma^2+1)\bar\tau} - \mathrm{e}^{2r\bar\tau/\sigma^2}, 0\right).$$

The derivation of the problem (3.7) is almost identical to the American put. The only difference is that instead of using $P - G_p = \mathrm{e}^{-r(T-t)}\left[u(x,\bar\tau) - g_p(x,\bar\tau)\right]$, we need to use the relation

$$\begin{aligned}
C - G_c &= C(S,t) - \max(S-1,0)\\
&= \mathrm{e}^{-r(T-t)}u(x,\bar\tau) - \max\left(\mathrm{e}^{x-(r-D_0-\sigma^2/2)(T-t)} - 1, 0\right)\\
&= \mathrm{e}^{-r(T-t)}\left[u(x,\bar\tau) - \max\left(\mathrm{e}^{x+(D_0+\sigma^2/2)(T-t)} - \mathrm{e}^{r(T-t)}, 0\right)\right]\\
&= \mathrm{e}^{-r(T-t)}\left[u(x,\bar\tau) - g_c(x,\bar\tau)\right],
\end{aligned}$$

where we also assume $E = 1$.

It is clear that if r, D_0, and σ depend on t, then similar results hold. However, if σ depends on S, then we may not be able to convert the problems(3.4) and (3.5) into (3.6) and (3.7) by a simple transformation.

3.2.3 Formulation of the Linear Complementarity Problem on a Finite Domain

Generally speaking, r, D_0, and σ are not constants. For simplicity, we assume that σ depends on S in this subsection even though the derivation is almost the same when r, D_0, and σ all depend on S and t.

From Sect. (2.2.5), we know that through the transformation

$$\begin{cases} \xi = \dfrac{S}{S+E}, \\ \tau = T - t, \\ V(S,t) = (S+E)\overline{V}(\xi,\tau) = \dfrac{E}{1-\xi}\overline{V}(\xi,\tau), \end{cases}$$

the operator

$$\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2(S)S^2\frac{\partial^2}{\partial S^2} + (r - D_0)S\frac{\partial}{\partial S} - r$$

is converted into

$$\frac{-E}{1-\xi}\left\{\frac{\partial}{\partial \tau} - \frac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1-\xi)^2\frac{\partial^2}{\partial \xi^2} - (r - D_0)\xi(1-\xi)\frac{\partial}{\partial \xi} + [r(1-\xi) + D_0\xi]\right\},$$

where $\bar{\sigma}(\xi) = \sigma(E\xi/(1-\xi))$, and the function $\max(\pm(S-E), 0)$ becomes

$$\frac{E}{1-\xi}\max(\pm(2\xi - 1), 0).$$

Therefore, problem (3.4) can be rewritten as

$$\begin{cases} \min\left(\dfrac{\partial \overline{V}}{\partial \tau} - \mathbf{L}_\xi \overline{V},\ \overline{V}(\xi,\tau) - \max(1-2\xi, 0)\right) = 0, & 0 \le \xi \le 1, \\ & 0 \le \tau, \\ \overline{V}(\xi, 0) = \max(1 - 2\xi, 0), & 0 \le \xi \le 1, \end{cases} \tag{3.8}$$

where

$$\mathbf{L}_\xi = \frac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1-\xi)^2\frac{\partial^2}{\partial \xi^2} + (r - D_0)\xi(1-\xi)\frac{\partial}{\partial \xi} - [r(1-\xi) + D_0\xi].$$

This is the American put option problem reformulated as a linear complementarity problem on a finite domain. Similarly, from the problem (3.5) we know that the American call option problem can be reformulated as the following linear complementarity problem:

$$\begin{cases} \min\left(\dfrac{\partial \overline{V}}{\partial \tau} - \mathbf{L}_\xi \overline{V},\ \overline{V}(\xi,\tau) - \max(2\xi - 1, 0)\right) = 0, & 0 \le \xi \le 1, \\ & 0 \le \tau, \\ \overline{V}(\xi, 0) = \max(2\xi - 1, 0), & 0 \le \xi \le 1. \end{cases} \tag{3.9}$$

In this section, an American option is reduced to a linear complementarity problem. Such a problem usually needs to be solved numerically. Here, we need to point out that the version given in Sect. 3.2.2 can be applied only if σ does not depend on S and that the other two versions can be applied for any case. However, the version given in Sect. 3.2.2 has the simplest equation. Also, if an implicit scheme is used, then for the versions given in Sects. 3.2.1 and 3.2.2, artificial boundary conditions are needed at the boundaries because numerical methods have to be performed on a finite domain. However, the version given in this subsection does not have such a problem.

3.2.4 More General Form of the Linear Complementarity Problems

From the three previous subsections, we see that a linear complementarity problem could be in the form:

$$\begin{cases} \min\left(-\dfrac{\partial V(S,t)}{\partial t} - \mathbf{L_{S,t}}V(S,t),\ V(S,t) - G_v(S,t)\right) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad S_l \le S \le S_u,\ t \le T, \\ V(S,T) = G_v(S,T), \qquad\qquad S_l \le S \le S_u, \end{cases}$$

where[2]

$$\mathbf{L_{S,t}} = a(S,t)\frac{\partial^2}{\partial S^2} + b(S,t)\frac{\partial}{\partial S} + c(S,t).$$

However, a linear complementarity problem could have a more general form such as

$$\begin{cases} \min\left(-\dfrac{\partial V(S,t)}{\partial t} - \mathbf{L_{S,t}}V(S,t) - d(S,t),\ V(S,t) - G_v(S,t)\right) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad S_l \le S \le S_u,\ t \le T, \\ V(S,T) = G_1(S) \ge G_v(S,T), \qquad\qquad S_l \le S \le S_u. \end{cases} \tag{3.10}$$

In this problem there are two new features. There is a new function $d(S,t)$ called the nonhomogeneous term of the problem and the payoff $G_1(S)$ is not equal to $G_v(S,T)$. The linear complementarity problem for one-factor convertible bonds has such a form. For two-factor convertible bonds, the form of the linear complementarity problem is similar, but the operator $\mathbf{L_{S,t}}$ is replaced by a two-dimensional one (see Chap. 5).

From what we have done in this section, we know the following. Consider a European style derivative and an American style derivative with identical payoffs $G_1(S)$, identical operators, and identical nonhomogeneous terms.

[2] If $S_l = -\infty$, then the first "$\le$" needs to be changed into "$<$," and if $S_u = \infty$, then the second "$\le$" needs to be changed into "$<$." In what follows, the same notation is used.

Suppose that the American style derivative has a constraint $G_v(S,t)$ satisfying $G_v(S,T) \le G_1(S)$. If the price of the European style derivative is the solution of the PDE problem

$$\begin{cases} \dfrac{\partial v(S,t)}{\partial t} + \mathbf{L_{s,t}} v(S,t) + d(S,t) = 0, & S_l \le S \le S_u,\ t \le T, \\ v(S,T) = G_1(S) & S_l \le S \le S_u, \end{cases}$$

then the price of the American style derivative with a constraint $G_v(S,t)$ satisfying $G_v(S,T) \le G_1(S)$ is the solution of LC problem (3.10).

3.3 American Option Problems as Free-Boundary Problems

3.3.1 Free Boundaries

From the past two sections, we discovered that there are some regions where the Black–Scholes equation cannot be used. Therefore, there exist two different types of regions: one where the Black–Scholes equation is valid, and the other where the Black–Scholes equation cannot be used and the solution is equal to the constraint. Because we do not know *a priori* the location of the boundaries between the two types of different regions, these boundaries are called free boundaries. Because in some regions the solution is known, we only need to determine the price in other regions and the locations of these free boundaries. In order to do that, we reformulate the American option problems as so-called free-boundary problems (FBPs).

Let us first discuss how to find the locations of the free boundaries at time T. Using Theorem 3.1, we can easily determine the locations of free boundaries at time T, namely, the starting points of free boundaries. We will show that for an American put option with $r > 0$, there is a free boundary starting from the point $(\min(E, rE/D_0), T)$ on the (S,t)-plane. If $r = 0$, then there is no free boundary. This implies that the Black–Scholes equation is valid everywhere and that the prices of the American and European put options are the same if $r = 0$. For an American call option, the situation is similar. If $D_0 > 0$, then there is a free boundary starting from the point $(\max(E, rE/D_0), T)$ on the (S,t)-plane. If $D_0 = 0$, then there is no free boundary, implying that an American call option is the same as a European call option.

First, let us consider an American put option and let $P(S,t)$ denote its value as we did in Sect. 3.1.1. In this case

$$G_p(S,t) = \max(E - S, 0) = \begin{cases} E - S, & \text{for } S < E, \\ 0, & \text{for } S \ge E \end{cases}$$

and the operator $\mathbf{L_{S,t}}$ in this case does not depend on t and is equal to

$$\mathbf{L_S} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0)S\frac{\partial}{\partial S} - r.$$

For $S \in (E, \infty)$, we have $G_p(S,t) = 0$ and

$$\frac{\partial G_p}{\partial t}(S,T) + \mathbf{L_S} G_p(S,T) = 0,$$

which means that the PDE can be used on (E, ∞). For $S \in (0, E)$, we have $G_p(S,t) = E - S$ and

$$\begin{aligned}
&\frac{\partial G_p}{\partial t}(S,T) + \mathbf{L_S} G_p(S,T) \\
&= \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 G_p}{\partial S^2} + (r - D_0)S\frac{\partial G_p}{\partial S} - rG_p \\
&= \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2}(E - S) + (r - D_0)S\frac{\partial}{\partial S}(E - S) - r(E - S) \\
&= D_0 S - rE.
\end{aligned}$$

The root of the equation $D_0 S - rE = 0$ is $S^* = rE/D_0$. If $E > rE/D_0$, then there are two situations: $S \in (0, rE/D_0)$ and $S \in (rE/D_0, E)$. On $(0, rE/D_0)$

$$\frac{\partial G_p}{\partial t}(S,T) + \mathbf{L_S} G_p(S,T) = D_0 S - rE < 0$$

and on $(rE/D_0, E)$

$$\frac{\partial G_p}{\partial t}(S,T) + \mathbf{L_S} G_p(S,T) = D_0 S - rE > 0.$$

Thus in this case, the entire S-axis is divided into two parts: $(0, rE/D_0)$ where the Black–Scholes equation cannot be used and $(rE/D_0, \infty)$ where the Black–Scholes equation gives the price of the American put option. Consequently, if $E > rE/D_0$, there is only one free boundary at time T when $r > 0$ and the location of the free boundary is $S = rE/D_0$. If $E < rE/D_0$, then on the entire interval $(0, E)$

$$\frac{\partial G_p}{\partial t}(S,T) + \mathbf{L_S} G_p(S,T) = D_0 S - rE < 0.$$

Thus in this case, the entire S-axis is divided into two parts: $(0, E)$ where the Black–Scholes equation cannot be used and (E, ∞) where the Black–Scholes equation gives the price of the American put option. Consequently, if $E < rE/D_0$, then there is also only one free boundary at time T when $r > 0$ and the location of the free boundary is $S = E$. Put them together, we have that there is only one free boundary at time T when $r > 0$ and the location of the free boundary is $S = min(E, rE/D_0)$. Let $S_f(t)$ denote this free boundary. Because it starts from the point $(\min(E, rE/D_0), T)$, we have

$$S_f(T) = \min\left(E, \frac{rE}{D_0}\right). \tag{3.11}$$

If $r = 0$, then $\min(E, rE/D_0) = 0$, so in the entire interval $(0, \infty)$, the Black–Scholes equation can be used, and there is no free boundary.

Now let us explain that in the case $r > 0$, no new free boundary can appear at any time $t < T$, so $S_f(t)$ is the only free boundary in this problem, and that $S_f(t)$ is not a constant, but an increasing function in t (see Fig. 3.3). First let us explain this when t is discrete. Similarly to what we did in Sect. 3.1.2, set $\Delta t = T/N$ and $t_n = n\Delta t, n = 0, 1, \cdots, N, N$ being a large integer, let $\tilde{P}(S, t_N) = G_p(S)$ and $\tilde{S}_f(t_N) = S_f(T)$, and for $n = N-1, N-2, \cdots, 0$, successively, define $\tilde{P}(S, t_n)$ by

$$\tilde{P}(S, t_n) = \max\left(\tilde{p}(S, t_n), G_p(S)\right),$$

where

$$\tilde{p}(S, t_n) = \mathrm{e}^{-r\Delta t}\int_0^\infty \tilde{P}(S', t_{n+1})G(S', t_{n+1}; S, t_n)dS'.$$

At $S = 0$, $G_p(0) = E > \tilde{p}(0, t_n) = \mathrm{e}^{-r\Delta t}E$ and at $S = S^* \approx \infty$, $G_p(S^*) = 0$ and $\tilde{p}(S^*, t_n) > 0$, the two continuous curves $\tilde{p}(S, t_n)$ and $G_p(S)$ must have at least one intersection point and let us denote the location of the intersection point with the largest S value by $\tilde{S}(t_n)$. Thus for any $S \in (\tilde{S}(t_n), \infty)$, $\tilde{P}(S, t_n) = \tilde{p}(S, t_n)$. If $S \in (E, \infty)$, for $\tilde{P}(S, t_{N-1})$ we have

$$\begin{aligned}
\tilde{P}(S, t_{N-1}) &= \max\left(\tilde{p}(S, t_{N-1}), G_p(S)\right) \\
&= \max\left(\tilde{p}(S, t_{N-1}), \max(E - S, 0)\right) \\
&= \tilde{p}(S, t_{N-1}) \\
&= \mathrm{e}^{-r\Delta t}\int_0^\infty \max(E - S', 0)G(S', t_N; S, t_{N-1})dS' > 0.
\end{aligned}$$

Thus for the case $S_f(t_N) = E$, then $\tilde{P}(S, t_{N-1}) > G_p(S) = \max(E - S, 0) = 0$ for $S \in (E, \infty)$; for the case $S_f(t_N) = rE/D_0$, for any point in $(rE/D_0, E)$,

$$\frac{\partial G_p(S)}{\partial t} + \mathbf{L}_{\mathbf{S}} G_p(S) = D_0 S - rE > 0,$$

so $\tilde{P}(S, t_{N-1}) > G_p(S)$ also holds for $S \in (rE/D_0, E)$. Consequently, put them together, we have that for any $S \in (S_f(t_N), \infty)$, $\tilde{P}(S, t_{N-1}) > G_p(S)$, from which we know $\tilde{P}(S_f(t_N), t_{N-1}) > G_p(S_f(t_N))$ holds also. Thus we have $\tilde{S}_f(t_{N-1}) < \tilde{S}_f(t_N)$ and $\tilde{P}(S, t_{N-1}) > G_p(S)$ on $(\tilde{S}_f(t_{N-1}), \tilde{S}_f(t_N))$.

Now let us assume that for certain n we have $\tilde{P}(S, t_{n+1}) > \tilde{P}(S, t_{n+2})$ for $S \in (\tilde{S}_f(t_{n+1}), \infty)$, and show $\tilde{P}(S, t_n) > \tilde{P}(S, t_{n+1})$ on $(\tilde{S}_f(t_n), \infty)$ and $\tilde{S}_f(t_n) < \tilde{S}_f(t_{n+1})$. In order to show this result, we only need to show $\tilde{p}(S, t_n) > \tilde{p}(S, t_{n+1})$ for $S \in (0, \infty)$. This is easy to see: for $S \in (0, \infty)$

$$\begin{aligned}\tilde{p}(S, t_n) &= \mathrm{e}^{-r\Delta t} \int_0^\infty \tilde{P}(S', t_{n+1}) G(S', t_{n+1}; S, t_n) dS' \\ &> \mathrm{e}^{-r\Delta t} \int_0^\infty \tilde{P}(S', t_{n+2}) G(S', t_{n+2}; S, t_{n+1}) dS' \\ &= \tilde{p}(S, t_{n+1}) \end{aligned} \tag{3.12}$$

because from Sect. 3.1.2 we have $\tilde{P}(S, t_{n+1}) \geq \tilde{P}(S, t_{n+2})$ for any $S \in (0, \infty)$ and it is given that $\tilde{P}(S, t_{n+1}) > \tilde{P}(S, t_{n+2})$ on $(\tilde{S}_f(t_{n+1}), \infty)$. Here we also have used the fact that $G(S', t_{n+1}; S, t_n) = G(S', t_{n+2}; S, t_{n+1}) > 0$ for $S \in (0, \infty)$ and $S' \in (0, \infty)$. From the relation (3.12) we know $\tilde{P}(S, t_n) > \tilde{P}(S, t_{n+1})$ on $(\tilde{S}_f(t_{n+1}), \infty)$ because on this interval $\tilde{P}(S, t_n) = \tilde{p}(S, t_n)$ and $\tilde{P}(S, t_{n+1}) = \tilde{p}(S, t_{n+1})$, which means that we can have $\tilde{S}_f(t_n) < \tilde{S}_f(t_{n+1})$. From the definition of $\tilde{S}_f(t_n)$, we further know $\tilde{P}(S, t_n) > \tilde{P}(S, t_{n+1})$ on $(\tilde{S}_f(t_n), \tilde{S}_f(t_{n+1}))$. For $n = N - 1$, we already have $\tilde{P}(S, t_{N-1}) > G_p(S) = \tilde{P}(S, t_N)$ for $S \in (S_f(t_N), \infty)$. Thus this procedure can be done for $n = N-2, N-3, \cdots, 0$, successively.

On $(0, S_f(t_N))$, $\tilde{P}(S, t_N) = G_p(S)$ and the following inequality

$$\frac{\partial G_p}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 G_p}{\partial S^2} + (r - D_0) S \frac{\partial G_p}{\partial S} - r G_p < 0$$

holds, which means that $\tilde{p}(S, t_{N-1}) < G_p(S)$ on that interval if Δt is small enough. Therefore the inequality $\tilde{p}(S, t_{N-1}) < G_p(S)$ must hold on $(0, \tilde{S}_f(t_{N-1}))$ at least for a very small Δt. Consequently, no more intersection points exist. This procedure can also be done for $n = N-2, N-3, \cdots, 0$, successively. Consequently no new free boundary will appear during the entire procedure if Δt is small enough. Let $N \to \infty$, we will have the conclusion we need to explain.

Consequently, if $r > 0$, then there is a unique free boundary, and the entire domain is divided into two regions by the free boundary (see Fig. 3.3): one region is $[0, S_f(t)) \times [0, T]$, where

$$\begin{cases} P = \max(E - S, 0) = E - S, \\ \dfrac{\partial P}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 P}{\partial S^2} + (r - D_0) S \dfrac{\partial P}{\partial S} - rP < 0 \end{cases}$$

and the other is $(S_f(t),\ \infty) \times [0, T]$, where

$$\begin{cases} P > \max(E - S, 0), \\ \dfrac{\partial P}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 P}{\partial S^2} + (r - D_0) S \dfrac{\partial P}{\partial S} - rP = 0 \end{cases}$$

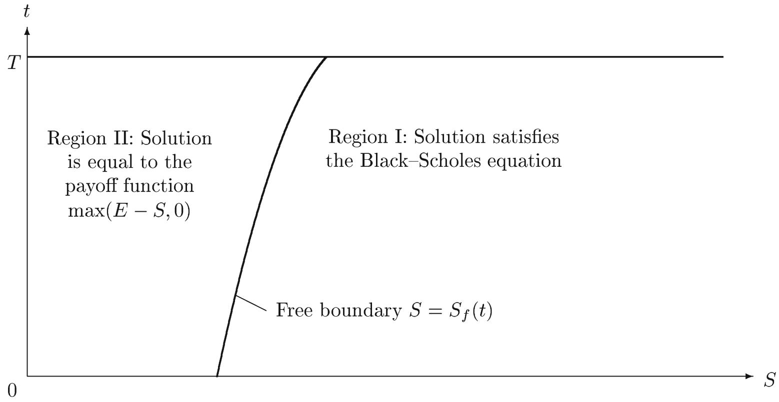

Fig. 3.3. Structure of solution to American put options ($r > 0$)

if $t < T$. Also if at a point (S, t), $P(S, t) > \max(E - S, 0)$, then $P(S, t - \Delta t) > P(S, t)$ for any positive Δt, and the location of the free boundary has the following property (see Fig. 3.2):

$$S_f(t) > S_f(t - \Delta t), \quad \Delta t > 0,$$

implying that $S_f(t)$ is an increasing function of t (see Fig. 3.3).

Before going further, we would like to give some remarks.

- What is the meaning of the inequality

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r - D_0)S\frac{\partial P}{\partial S} - rP < 0?$$

  As pointed out in Sect. 3.1.1, this means that holding the money market account with a value of $P - \frac{\partial P}{\partial S}S$ and $\frac{\partial P}{\partial S}$ shares will be better than holding the option. In this case exercising the option and holding a money market account with a value of $P - \frac{\partial P}{\partial S}S$ and $\frac{\partial P}{\partial S}$ shares will have better return than holding the option. Therefore the option should be exercised. If $P(S, t) > \max(E - S, 0)$, one should hold the option, as one should not give up a higher value (the option) for a lower value (the intrinsic value). Therefore, the free boundary is the optimal exercise price that divides the exercise region and the non-exercise region.
- Let $\mathbb{D}_{ge}$ denote the open domain where $\frac{\partial G_v}{\partial t}(S, t) + \mathbf{L}_{\mathbf{S},\mathbf{t}} G_v(S, t) \geq 0$ and $\mathbb{D}_l$ the open domain where $\frac{\partial G_v}{\partial t}(S, t) + \mathbf{L}_{\mathbf{S},\mathbf{t}} G_v(S, t) < 0$. For the put option case, $G_v(S, t) = \max(E - S, 0)$ and the open domain $\mathbb{D}_{ge}$ is $(\min(E, rE/D_0), \infty) \times [0, T]$ and $\mathbb{D}_l$ is $(0, \min(E, rE/D_0)) \times [0, T]$. In a neighborhood of a point in the open domain $\mathbb{D}_{ge}$, if $V(S, t) > G_v(S, t)$, then

the PDE can be used because we can let a positive Δt be small enough to guarantee $V(S,t-\Delta t) > G_v(S,t-\Delta t)$, and if $V(S,T) = G_v(S,T)$, then the PDE can also be used because $\dfrac{\partial G_v}{\partial t}(S,t) + \mathbf{L_{S,t}} G_v(S,t) \geq 0$. Thus a point on a free boundary cannot appear in the open domain $\mathbb{D}_{ge}$. In a neighborhood of a point in the open domain $\mathbb{D}_l$, if $V(S,t) > G_v(S,t)$, then the PDE can be used, and if $V(S,T) = G_v(S,T)$, then the PDE cannot be used because $\dfrac{\partial G_v}{\partial t}(S,t) + \mathbf{L_{S,t}} G_v(S,t) < 0$. Thus a point on a free boundary may appear in the open domain $\mathbb{D}_l$.

- From theorem 3.1, we can find that there are two types of points on free boundaries. The first type of points is: in a neighborhood of the point, $V(S,t) = G_v(S,t)$ and some portion of the neighborhood belongs to $\mathbb{D}_{ge}$ and another portion of the neighborhood belongs to $\mathbb{D}_l$. The second type of points is: in some portion of a neighborhood of the point, $V(S,t) > G_v(S,t)$ and in another portion of the neighborhood, $V(S,t) = G_v(S,t)$ and this portion belongs to $\mathbb{D}_l$. Thus a free boundary will appear only in the open domain $\mathbb{D}_l$ and on the boundary between the open domains $\mathbb{D}_{ge}$ and $\mathbb{D}_l$. If $V(S,T) = G_v(S,T)$, then a free boundary will start at a point between the open domains $\mathbb{D}_{ge}$ and $\mathbb{D}_l$. For example, the free boundary of an American put option starts at such a point. If $V(S,T) > G_v(S,T)$ on some portion of the entire domain and $V(S,T) = G_v(S,T)$ on another portion, then a free boundary might also start at a boundary between an open interval belonging to $\mathbb{D}_l$ and an open interval where $V(S,T) > G_v(S,T)$. As we will see in Sect. 5.7, the free boundary of a one-factor convertible bond can start from a point of the first type of points or a point of the second type of points. Later, a free boundary may move but never move into the open domain $\mathbb{D}_{ge}$.

Now let us consider an American call option. From Sect. 2.2.5 we know, at very large S, the solution of the Black–Scholes equation with final condition $V(S,t) = \max(S-E,0)$ has the following asymptotic expression

$$V(S,t) \approx V(S,T)\mathrm{e}^{-D_0(T-t)} = \max(S-E,0)\mathrm{e}^{-D_0(T-t)},$$

so if $D_0 > 0$, then $V(S,t) < \max(S-E,0)$ for any $t < T$. Therefore, if $D_0 > 0$, the American call problem is a free-boundary problem. Now let us show that the free-boundary problem has only one free boundary, which is also denoted by $S_f(t)$ in what follows, and determine the location of the free boundary at $t = T$ from the constraint condition $C(S,t) \geq G_c(S,t)$.

In the case of an American call option,

$$G_c(S,t) = \max(S-E,0) = \begin{cases} S-E, & S > E, \\ 0, & S \leq E. \end{cases}$$

Let $S > \max\left(E, \frac{rE}{D_0}\right)$. In this case

$$G_c(S,t) = S - E$$

and

$$\begin{aligned}
&\frac{\partial G_c}{\partial t}(S,T) + \mathbf{L_S} G_c(S,T)\\
&= \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 G_c}{\partial S^2} + (r - D_0)S\frac{\partial G_c}{\partial S} - rG_c\\
&= \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 (S-E)}{\partial S^2} + (r - D_0)S\frac{\partial (S-E)}{\partial S} - r(S-E)\\
&= rS - D_0 S - rS + rE = -D_0 S + rE < 0
\end{aligned}$$

because $S > \frac{rE}{D_0}$. Therefore, the Black–Scholes equation cannot hold in this case, and $C(S, T - \Delta t)$ should be equal to $S - E$ for $S > \max\left(E, \frac{rE}{D_0}\right)$. Just like the case of the American put option, we can know that for $S < \max\left(E, \frac{rE}{D_0}\right)$, the Black–Scholes equation can hold. Thus, a free boundary starts at $S = \max\left(E, \frac{rE}{D_0}\right)$, i.e.,

$$S_f(T) = \max\left(E, \frac{rE}{D_0}\right). \tag{3.13}$$

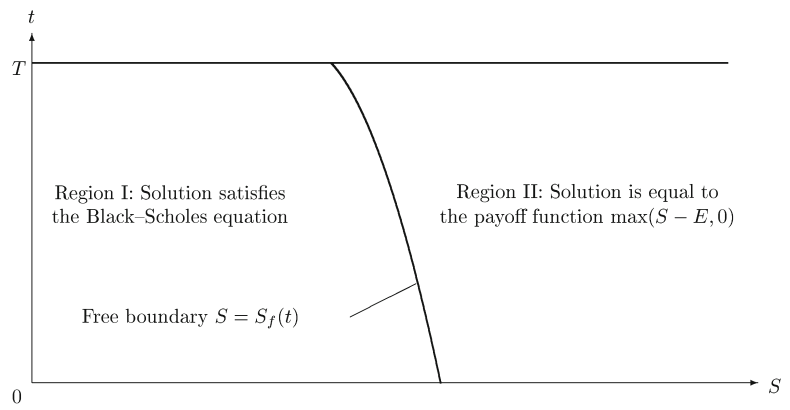

Fig. 3.4. Structure of solution to American call options ($D_0 > 0$)

Using the same argument we have used for an American put option, we can show that the free boundary starting from the point $(\max(E, rE/D_0), T)$ is the only free boundary because no new free boundary can appear at time $t < T$. Just like the put case, the entire domain is divided into two parts by the free boundary. However, the situation is a little different from the American put. Here in the region $[0, S_f(t)) \times [0, T]$, the Black–Scholes equation holds, whereas in the region $(S_f(t), \infty) \times [0, T]$, the Black–Scholes equation cannot be used. In other words, for $S \in [0, S_f(t))$ and $t < T$,

$$\begin{cases} C(S,t) \geq \max(S-E, 0), \\ \dfrac{\partial C}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 C}{\partial S^2} + (r - D_0) S \dfrac{\partial C}{\partial S} - rC = 0, \end{cases}$$

where the equal sign in $C(S,t) \geq \max(S-E, 0)$ holds only at $S = 0$; whereas for $S \in (S_f(t), \infty)$,

$$\begin{cases} C(S,t) = \max(S-E, 0) = S - E, \\ \dfrac{\partial C}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 C}{\partial S^2} + (r - D_0) S \dfrac{\partial C}{\partial S} - rC < 0 \end{cases}$$

and the option should be exercised. It can also be shown that for an American call option, the free boundary $S_f(t)$ is a decreasing function of t, as graphed in Fig. 3.4, and that the price of an American call option is the same as a European call if $D_0 = 0$.

3.3.2 Free-Boundary Problems

In this subsection, we will describe the formulation of American option problems as free-boundary problems. In order to give a complete formulation, we need to give the conditions on the free boundary. For an initial-boundary value problem of a parabolic equation on a finite interval, if the locations of the boundaries are given and if the coefficient of the second derivative at the boundaries is not equal to zero, one boundary condition at each boundary is needed in order for the problem to have a unique solution. However, the location of the free boundary is unknown, so two conditions are needed at the free boundary in order for the problem to have a unique solution. One boundary condition determines the option value on the free boundary and the other boundary condition determines the location of the free boundary. Now the question is what the two conditions should be. For some other linear complementarity problems, it has been proved that on the free boundary the value and the first derivative are continuous (see [31]). For this problem, from the proof given by Badea and Wang (see [4] and [5]), the situation is still the same. Therefore, the two conditions on the free boundary are: both the value and the derivative with respect to S are continuous.

For an American put option, in the region $[0,\ S_f(t))$,

$$P(S,t) = E - S$$

and

$$\frac{\partial P}{\partial S} = -1.$$

Therefore, the boundary conditions on the free boundary $S_f(t)$ are

$$P(S_f(t), t) = E - S_f(t) \tag{3.14}$$

and

$$\frac{\partial P}{\partial S}(S_f(t), t) = -1. \tag{3.15}$$

It is clear that when the boundary condition (3.15) holds, the gradient $\dfrac{\partial P}{\partial S}$ must be continuous at $S = S_f$, which is shown in Fig. 3.1.

Now we can formulate the American put option problem. On the domain $[0, S_f(t)) \times [0, T]$,

$$P(S, t) = E - S,$$

while on the domain $[S_f(t), \infty) \times [0, T]$, $P(S, t)$ is the solution of the free-boundary problem[3] for American put options

$$\begin{cases} \dfrac{\partial P}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 P}{\partial S^2} + (r - D_0) S \dfrac{\partial P}{\partial S} - rP = 0, & \\ \qquad\qquad\qquad\qquad\qquad\qquad S_f(t) \le S, \quad 0 \le t \le T, & \\ P(S, T) = \max(E - S, 0), & S_f(T) \le S, \\ P(S_f(t), t) = E - S_f(t), & t \le T, \\ \dfrac{\partial P(S_f(t), t)}{\partial S} = -1, & t \le T, \\ S_f(T) = \min\left(E, \dfrac{rE}{D_0}\right). & \end{cases} \tag{3.16}$$

Similarly, for call options we need two boundary conditions on the free boundary. One is

$$C(S_f(t), t) = S_f(t) - E \tag{3.17}$$

and the other still can be obtained by requiring the continuity of the slope of the solution at $S = S_f(t)$. In this case, the condition is

$$\frac{\partial C(S_f(t), t)}{\partial S} = 1. \tag{3.18}$$

[3]In this book we call this problem and the like a free-boundary problem. An LC problem usually involves free boundaries. Thus it is not strange that some people call an LC problem a free-boundary problem.

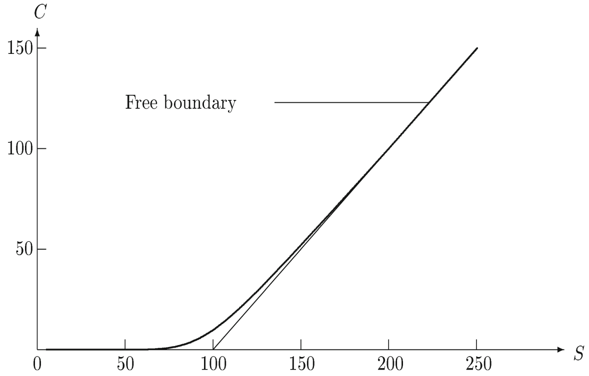

Fig. 3.5. Numerically calculated solution of an American call problem with $E = 100$, $r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$, and $T = 1$ year

Therefore for the American call option, the formulation is as follows. On the domain $[0, S_f(t)] \times [0, T]$, $C(S,t)$ is the solution of the free-boundary problem for American call options

$$\begin{cases} \dfrac{\partial C}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 C}{\partial S^2} + (r - D_0)S\dfrac{\partial C}{\partial S} - rC = 0, \\ \qquad\qquad\qquad\qquad 0 \leq S \leq S_f(t), \quad 0 \leq t \leq T, \\ C(S,T) = \max(S - E, 0), \quad 0 \leq S \leq S_f(T), \\ C(S_f(t),t) = S_f(t) - E, \quad 0 \leq t \leq T, \\ \dfrac{\partial C}{\partial S}(S_f(t),t) = 1, \quad 0 \leq t \leq T, \\ S_f(T) = \max\left(E, \dfrac{rE}{D_0}\right); \end{cases} \tag{3.19}$$

whereas on the domain $(S_f(t), \infty) \times [0, T]$, $C(S,t) = S - E$. In Fig. 3.5, the value of an American call option is plotted, from which we can see that the two parts of solution are connected smoothly. The parameters of the problem are $E = 100$, $r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$, and $T = 1$ year.

Here we need to point out that $S_f(T)$ is determined by the partial differential operator and the final condition. Therefore, in a free-boundary problem, the starting location of the free boundary is not arbitrary and should be consistent with the partial differential operator and the final condition.

As has been pointed, there are two formulations for American option problems. It is clear that the solutions obtained from the two formulations should be the same. In this book, we will not carefully study this problem. However in Sect. 3.3.5 for the perpetual American call option, we will prove that the solution obtained by solving the free-boundary problem is the solution of the

LC problem. Here we just show the following. For the American put problem, if the solution of the problem (3.16) satisfies the conditions $P(S,t) \geq 0$ and $\dfrac{\partial P^2(S,t)}{\partial S^2} \geq 0$ for $S_f(t) < S$, then the solution, including the part on the domain $[0, S_f(t)) \times [0,T]$ and the part on the domain $[S_f(t), \infty) \times [0,T]$, satisfies the LC relation:

$$\min\left(-\frac{\partial P}{\partial t} - \mathbf{L_s}P,\ P(S,t) - \max(E-S,0)\right) = 0, \quad 0 < S, \quad t \leq T,$$

where

$$\mathbf{L_s} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0)S\frac{\partial}{\partial S} - r.$$

The proof is as follows. On the interval $(0, S_f(t))$, $P(S,t) = G_p(S)$ and this interval belongs to the domain $\mathbb{D}_l$, which means

$$-\frac{\partial G_p(S)}{\partial t} - \mathbf{L}_S G_p(S) > 0.$$

Consequently, the LC relation holds for this case. For the case $S_f(t) < S \leq E$, we have $-\dfrac{\partial P}{\partial t} - \mathbf{L}_S P = 0$, and we need to show $P(S,t) - (E-S) \geq 0$ in order to prove our conclusion. Define $f(S,t) = P(S,t) - (E-S)$. We know that $f(S_f(t),t) = P(S_f(t),t) - (E - S_f(t)) = 0$ and $\dfrac{\partial f(S_f(t),t)}{\partial S} = \dfrac{\partial P(S_f(t),t)}{\partial S} + 1 = 0$. Thus for a fixed t, we have

$$\begin{aligned} f(S,t) &= f(S_f(t),t) + \frac{\partial f(S_f(t),t)}{\partial S}\left[S - S_f(t)\right] + \frac{1}{2}\frac{\partial^2 f(S^*,t)}{\partial S^2}\left[S - S_f(t)\right]^2 \\ &= \frac{1}{2}\frac{\partial^2 P(S^*,t)}{\partial S^2}\left[S - S_f(t)\right]^2 \geq 0, \end{aligned}$$

where $S^* \in (S_f(t), S)$ and we have used the condition $\dfrac{\partial P^2(S,t)}{\partial S^2} \geq 0$ for $S_f(t) < S$. For the case $E < S$, we have $-\dfrac{\partial P}{\partial t} - \mathbf{L}_S P = 0$ and $P(S,t) - \max(E-S,0) = P(S,t) \geq 0$, and thus the LC relation holds on $(S_f(t), \infty)$. Because the LC relation holds on $(0, S_f(t))$, $(S_f(t), \infty)$, the LC relation at the points 0 and $S_f(t)$ also holds, which can be shown by letting S go to these points. Consequently the LC relation holds for all the cases and the proof is completed.

As is proved in Problem 41 of Chap. 2, if $D_0 = 0$, then the value of an American call option is equal to the value of a European call option. Thus in this case there is no free boundary, that is, there is no optimal exercise price. A new question is: does the optimal exercise price exist when the dividends are paid discretely? The answer is that when there are discrete dividends, the American call option can only be optimal to exercise at a time immediately before the stock goes ex-dividend and that an optimal exercise price does not always exist even at those moments. Readers are asked to prove these conclusions as Problem 15.

3.3.3 Put–Call Symmetry Relations

As we know, the price of an American put option is the solution of the following LC problem:

$$\begin{cases} \min\left(-\dfrac{\partial P}{\partial t} - \mathbf{L}_S P,\ P(S,t) - \max(E-S,0)\right) = 0, & 0 \le S,\ t \le T, \\ P(S,T) = \max(E-S,0), & 0 \le S, \end{cases}$$

where

$$\mathbf{L}_S = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0) S \frac{\partial}{\partial S} - r.$$

Let

$$\begin{cases} \zeta = \dfrac{E^2}{S}, \\ C(\zeta, t) = \dfrac{EP(S,t)}{S}. \end{cases} \tag{3.20}$$

Because

$$\frac{E}{S}\max(E-S,0) = \max(\zeta - E, 0),$$

for $C(\zeta,t)$ the payoff and constraint are $\max(\zeta - E, 0)$. Noticing

$$\begin{aligned} \frac{\partial P}{\partial t} &= \frac{S}{E}\frac{\partial C}{\partial t}, \\ \frac{\partial P}{\partial S} &= \frac{1}{E}\left[C + S\frac{\partial C}{\partial \zeta}\left(-\frac{E^2}{S^2}\right)\right] = \frac{1}{E}\left(C - \zeta\frac{\partial C}{\partial \zeta}\right), \\ \frac{\partial^2 P}{\partial S^2} &= \frac{\zeta^3}{E^3}\frac{\partial^2 C}{\partial \zeta^2}, \end{aligned}$$

we have

$$\begin{aligned} &\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r - D_0) S \frac{\partial P}{\partial S} - rP \\ &= \frac{S}{E}\left\{\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 \zeta^2 \frac{\partial^2 C}{\partial \zeta^2} + (D_0 - r)\zeta\frac{\partial C}{\partial \zeta} - D_0 C\right\}. \end{aligned}$$

Therefore the function $C(\zeta,t)$ is the solution of the following American call option problem:

$$\begin{cases} \min\left(-\dfrac{\partial C}{\partial t} - \mathbf{L}_\zeta C,\ C(\zeta,t) - \max(\zeta - E, 0)\right) = 0, & 0 \le \zeta,\ t \le T, \\ C(\zeta, T) = \max(\zeta - E, 0), & 0 \le \zeta, \end{cases} \tag{3.21}$$

where

$$\mathbf{L}_\zeta = \frac{1}{2}\sigma^2\zeta^2\frac{\partial^2}{\partial\zeta^2} + (D_0 - r)\zeta\frac{\partial}{\partial\zeta} - D_0.$$

Consequently, an American put problem can be converted into an American call problem. However in the two problems, the state variable and the parameters are different. From the definitions of $\mathbf{L}_S$ and $\mathbf{L}_\zeta$, we know that the volatilities of the put and call problems are the same, but the interest rate and the dividend yield of the call problem are equal to the dividend yield and the interest rate of the put problem, respectively. In order to explain these facts, we express the dependency of the options on interest rate and dividend yield explicitly. Let $P(S,t;b,a)$ denote the price of the put option and $C(\zeta,t;a,b)$ the price of the call option, where the first and second parameters after the semicolon are the interest rate and the dividend yield, respectively. From the definition of ζ and $C(\zeta,t;a,b)$, we know

$$P(S,t;b,a) = C\left(\zeta,t;a,b\right) S/E,$$

where $\zeta = \dfrac{E^2}{S}$. This can also be rewritten as

$$P(S,t;b,a) = C\left(E^2/S,t;a,b\right) S/E,$$

or

$$C\left(\zeta,t;a,b\right) = P\left(E^2/\zeta,t;b,a\right) \zeta/E,$$

where we used the relation $E/S = \zeta/E$. In the last relation, we can use S, instead of ζ, as the state variable. That is, we can write this relation as

$$C\left(S,t;a,b\right) = P\left(E^2/S,t;b,a\right) S/E.$$

Finally, putting them together, we have

$$\begin{cases} C(S,t;a,b) = P\left(E^2/S,t;b,a\right) S/E, & \text{or} \\ P(S,t;b,a) = C\left(E^2/S,t;a,b\right) S/E. \end{cases} \tag{3.22}$$

For the special case $S = E$, it becomes

$$P(E,t;b,a) = C(E,t;a,b).$$

Also, the location of free boundary in the latter problem, $\zeta_{cf}(t;a,b)$, must be equal to E^2 divided by the location of free boundary of the former problem, $E^2/S_{pf}(t;b,a)$, because $\zeta = E^2/S$, i.e.,

$$\zeta_{cf}(t;a,b) = E^2/S_{pf}(t;b,a)$$

or

$$S_{cf}(t;a,b) \times S_{pf}(t;b,a) = E^2, \tag{3.23}$$

where in the last relation, instead of ζ_{cf}, we use S_{cf} as the name of the function representing the location of the free boundary. From the derivation above we know that for European options, the following relations also hold:

$$\begin{cases} c(S,t;a,b) = p\left(E^2/S,t;b,a\right)S/E, \quad \text{or} \\ p(S,t;b,a) = c\left(E^2/S,t;a,b\right)S/E. \end{cases} \tag{3.24}$$

The relations (3.22)–(3.24) are called the put–call symmetry relations.

Now let us discuss the financial meaning of the put–call symmetry relations. Suppose that one British pound is worth S U.S. dollars and that E^2 U.S. dollars are worth ζ British pounds. It is clear that $\zeta = E^2/S$. Let P be the value of a put option whose holder can always sell one pound for E dollars if the holder wants. This means that the payoff and constraint of the put option is $\max(E-S,0)$ in dollars. Let C be the value of a call option whose holder can buy E^2 dollars by paying E pounds if the holder wants. This means that the payoff and constraint of the call option are $\max(E^2/S-E,0) = \max(\zeta-E,0)$ in pounds. The holder of the put option has the right to sell one pound for E U.S. dollars even if $S \le E$. The holder of $1/E$ units of the call option has the right to buy E dollars by paying one British pound even if $\zeta \ge E$. The condition $S \le E$ is equivalent to $E^2/S = \zeta \ge E$. Thus, both the holder of one unit of the put option and the holder of $1/E$ units of the call option have the right to exchange one pound for E dollars even if $S < E$. The two holders have the same rights, so the value of one unit of the put option and the value of $1/E$ units of the call option in U.S. dollars, which is equal to $S \cdot C/E$, should be equal, i.e.,

$$P = S \cdot C/E.$$

Here, we need to notice that P and C have different but related volatilities, interest rates, and dividend yields. According to Itô's lemma, if

$$dS = \mu S dt + \sigma S dX,$$

then

$$d\zeta = (-\mu + \sigma^2)\zeta dt - \sigma\zeta dX.$$

Hence, the volatilities of S and $\zeta = E^2/S$ are the same if the volatilities are constants. Suppose that σ, r, and D_0 are constant and that the interest rates of the British pound and the U.S. dollar are a and b, respectively. Then $r = a$ and $D_0 = b$ for the call and $r = b$ and $D_0 = a$ for the put, and the volatilities are the same. In this case, the relation above can be written as

$$P(S,t;b,a) = C\left(E^2/S,t;a,b\right)S/E.$$

The first relation in the set of relations (3.22) (or (3.24)) actually is another form of the second relation in the set of relations (3.22) (or (3.24)). Thus from the financial reasoning here, we know that all the relations in the sets of relations (3.22) and (3.24) hold. Because the state variable ζ for the call with $r = a$ and $D_0 = b$ and the state variable S for the put with $r = b$ and $D_0 = a$ have the relation $\zeta = E^2/S$, the argument above to obtain the relation (3.23) can still be used here. Hence from the financial reasoning above, we can also have the relation (3.23).

Actually such relations exist for more complicated cases. If σ depends upon S, then the following relations hold:

$$\begin{cases} C(S, t; a, b, \sigma(S)) = P\left(\dfrac{E^2}{S}, t; b, a, \sigma(S)\right) S/E, \quad \text{or} \\ P(S, t; b, a, \sigma(S)) = C\left(\dfrac{E^2}{S}, t; a, b, \sigma(S)\right) S/E \end{cases}$$

and

$$S_{cf}(t; a, b, \sigma(S)) \times S_{pf}(t; b, a, \sigma(E^2/S)) = E^2.$$

Here, the third argument after the semicolon is the function for the volatility. The proof is left for readers as an exercise (Problem 17).

The symmetry relations can be used when we write codes for pricing American options or calculate prices of options. Suppose that we need codes for pricing American call and put options and that we already have a code for pricing American call options. If it is very easy for the code to be modified to a code for pricing American put options, then we can have another code for put options by modifying the code we already have. If the code for put options will be quite a different from the code for call options, then we can use the code for call options to find $C(E^2/S, t; a, b)$ first and then obtain $P(S, t; b, a)$ by using the relation $P(S, t; b, a) = C(E^2/S, t; a, b) \cdot S/E$. If one already has a code that can deal with both American call and put options, then the symmetry relations can be used for checking the accuracy of the numerical results. Because the numerical results have errors, they will not exactly satisfy the symmetry relation and can be used as indicators to show how accurate the numerical results are if the values of a call and the corresponding put option have been obtained. For details, see the paper [98] by Zhu, Ren, and Xu. For more about symmetry relations and similar results, see [53, 54, 62] and [24].

3.3.4 Equations for Some Greeks

Here, for American options we would like to derive the equations and boundary conditions that $\mathcal{V} = \dfrac{\partial \Pi}{\partial \sigma}$, $\rho = \dfrac{\partial \Pi}{\partial r}$, and $\rho_d = \dfrac{\partial \Pi}{\partial D_0}$ should satisfy. Let us first consider American call options and write the dependence of C and S_f on r, D_0, and σ explicitly, that is, instead of $C(S, t)$ and $S_f(t)$, we use $C(S, t; r, D_0, \sigma)$ and $S_f(t; r, D_0, \sigma)$ to denote the price of American call options and the free boundary in what follows. Differentiating the partial differential equation in the problem (3.19) with respect to r, D_0, or σ yields the equations for $\dfrac{\partial C}{\partial r}$, $\dfrac{\partial C}{\partial D_0}$ or $\dfrac{\partial C}{\partial \sigma}$. For example, for $\dfrac{\partial C}{\partial \sigma}$ we have

$$\frac{\partial C_\sigma}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_\sigma}{\partial S^2} + (r - D_0)S\frac{\partial C_\sigma}{\partial S} - rC_\sigma + \sigma S^2 \frac{\partial^2 C}{\partial S^2} = 0,$$

where C_σ denotes the partial derivative of the call option with respect to σ. The final condition for the price of American call options is

$$C(S, T; r, D_0, \sigma) = \max(S - E, 0).$$

Therefore $\dfrac{\partial C}{\partial \sigma} = 0$ at $t = T$. The boundary conditions on the free boundary are

$$C(S_f(t; r, D_0, \sigma), t; r, D_0, \sigma) = S_f(t; r, D_0, \sigma) - E \tag{3.25}$$

and

$$\frac{\partial C(S_f(t; r, D_0, \sigma), t; r, D_0, \sigma)}{\partial S} = 1. \tag{3.26}$$

From the relation (3.25) we have

$$\frac{\partial C}{\partial S}\frac{\partial S_f}{\partial \sigma} + \frac{\partial C}{\partial \sigma} = \frac{\partial S_f}{\partial \sigma}$$

on the free boundary. Noticing (3.26), we have $\dfrac{\partial C}{\partial \sigma} = 0$ at the free boundary. Consequently, $\dfrac{\partial C}{\partial \sigma}$ is the solution of the following final-boundary value problem

$$\begin{cases} \dfrac{\partial C_\sigma}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 C_\sigma}{\partial S^2} + (r - D_0)S\dfrac{\partial C_\sigma}{\partial S} - rC_\sigma + \sigma S^2 \dfrac{\partial^2 C}{\partial S^2} = 0, \\ \qquad\qquad\qquad\qquad\qquad 0 \le S \le S_f(t), \quad 0 \le t \le T, \\ C_\sigma(S, T) = 0, \qquad\qquad 0 \le S \le S_f(T), \\ C_\sigma(S_f(t), t) = 0, \qquad\quad 0 \le t \le T, \end{cases} \tag{3.27}$$

where $\dfrac{\partial^2 C}{\partial S^2}$ and $S_f(t)$ are known functions obtained from the solution of problem (3.19).

For $\dfrac{\partial C}{\partial r}$ and $\dfrac{\partial C}{\partial D_0}$, we can derive the same final and boundary conditions as $\dfrac{\partial C}{\partial \sigma}$, namely,

$$\frac{\partial C}{\partial r} = \frac{\partial C}{\partial D_0} = 0 \tag{3.28}$$

at $t = T$ and

$$\frac{\partial C}{\partial r} = \frac{\partial C}{\partial D_0} = 0 \tag{3.29}$$

at the free boundary. The only difference is the equation. Differentiating the partial differential equation in the problem (3.19) with respect to r and D_0 yields

$$\frac{\partial C_r}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_r}{\partial S^2} + (r - D_0) S \frac{\partial C_r}{\partial S} - rC_r + S\frac{\partial C}{\partial S} - C = 0 \tag{3.30}$$

and

$$\frac{\partial C_{D_0}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_{D_0}}{\partial S^2} + (r - D_0) S \frac{\partial C_{D_0}}{\partial S} - rC_{D_0} - S\frac{\partial C}{\partial S} = 0 \tag{3.31}$$

respectively, where C_r stands for $\dfrac{\partial C}{\partial r}$ and C_{D_0} for $\dfrac{\partial C}{\partial D_0}$.

For American put options, the Greeks are solutions of similar problems. This is left for readers to show as Problem 19 of this chapter.

3.3.5 Solutions for Perpetual American Call Options

If an option does not have an expiry date but rather an infinite time zone, then the option is called a perpetual option. Let $C(S, 0; T)$ be the today's price of an American call option with expiry T, and let $C_\infty(S)$ be the price of the corresponding perpetual American call option. Between them, there is the following relation:

$$C_\infty(S) = \lim_{T\to\infty} C(S, 0; T).$$

Since $\left.\dfrac{\partial C(S, t; T)}{\partial t}\right|_{t=0} = 0$ as $T \to \infty$, we know from the problem (3.19) that for $S \in [0, S_f]$, S_f standing for the location of the corresponding free boundary, $C_\infty(S)$ is the solution of the following problem

$$\begin{cases} \dfrac{1}{2}\sigma^2 S^2 \dfrac{d^2 C_\infty}{dS^2} + (r - D_0) S \dfrac{dC_\infty}{dS} - rC_\infty = 0, \quad 0 \le S \le S_f, \\ C_\infty(S_f) = S_f - E, \\ \dfrac{dC_\infty(S_f)}{dS} = 1. \end{cases} \tag{3.32}$$

Let

$$C_\infty(S) = S^\alpha,$$

then

$$\frac{dC_\infty}{dS} = \alpha S^{\alpha-1}$$

and

$$\frac{d^2C_\infty}{dS^2} = \alpha(\alpha-1)S^{\alpha-2}.$$

Substituting these into the ordinary differential equation in the problem (3.32), we get

$$\frac{1}{2}\sigma^2\alpha^2 + \left(r - D_0 - \frac{1}{2}\sigma^2\right)\alpha - r = 0.$$

The two roots of this equation are

$$\alpha_\pm = \frac{1}{\sigma^2}\left[-\left(r - D_0 - \frac{1}{2}\sigma^2\right) \pm \sqrt{\left(r - D_0 - \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 r}\right].$$

Thus

$$C_\infty(S) = C_+(S/S_f)^{\alpha_+} + C_-(S/S_f)^{\alpha_-}.$$

It is clear that $\alpha_+ > 0$ and $\alpha_- < 0$. In order to guarantee the solution to be bounded at $S = 0$, C_- should equal zero. Consequently, we arrive at

$$C_\infty(S) = C_+(S/S_f)^{\alpha_+}.$$

From the free-boundary conditions in the problem (3.32) we obtain

$$C_+ = S_f - E,$$
$$C_+\alpha_+S_f^{-1} = 1.$$

Solving these two equations we get

$$S_f = \frac{E}{1 - 1/\alpha_+} \quad \text{and} \quad C_+ = \frac{1}{\alpha_+S_f^{-1}}.$$

Thus, the solution of problem (3.32) is

$$C_\infty(S) = \frac{S_f}{\alpha_+}\left(\frac{S}{S_f}\right)^{\alpha_+}. \tag{3.33}$$

On $[0, \infty)$, the solution of the perpetual American call option is

$$C_\infty(S) = \begin{cases} \text{the solution of the free-boundary problem}, & 0 \le S \le S_f, \\ S - E, & S_f < S. \end{cases}$$

$C_\infty(S)$ should satisfy the following LC relation of the perpetual American call option for any S:

$$\min\left(-\left[\frac{1}{2}\sigma^2 S^2 \frac{d^2 C_\infty}{dS^2} + (r - D_0)S\frac{dC_\infty}{dS} - rC_\infty\right],\right.$$
$$\left. C_\infty - \max(S - E, 0)\right) = 0.$$

Here let us verify this conclusion by direct computation. Before doing that, we point out that the following is true: $S_f = E/(1-1/\alpha_+) \geq E\max(1, r/D_0)$. As we know, for a vanilla call option, $S_f(T) = E\max(1, r/D_0)$ and $S_f(0) \geq S_f(T) = E\max(1, r/D_0)$. This still holds as $T \to \infty$. Thus[4]

$$S_f \geq E\max(1, r/D_0).$$

For $S \in (0, E)$, C_∞ satisfies the ODE and is greater than 0, and $\max(S - E, 0) = 0$. Thus the LC relation

$$\min\left(-\left[\frac{1}{2}\sigma^2 S^2 \frac{d^2 C_\infty}{dS^2} + (r - D_0)S\frac{dC_\infty}{dS} - rC_\infty\right],\right.$$
$$\left. C_\infty - \max(E - S, 0)\right) = 0$$

holds. Now let us check if the LC relation holds for $S \in (E, S_f)$. Suppose that $f(x)$, $f'(x)$, and $f''(x)$ are continuous functions on $[a, b]$. As we know, if $f(b) = 0$ and $f'(b) = 0$, then the following relation is true: $f(x) = \frac{1}{2}f''(\xi)(x - b)^2$, where $x \in [a, b]$ and $\xi \in [x, b]$. Using this fact, we know that because $C_\infty(S_f) - (S_f - E) = 0$ and $\dfrac{dC_\infty(S_f)}{dS} - 1 = 0$, $C_\infty(S) - (S - E) \geq 0$ on (E, S_f) if $\dfrac{d^2 C_\infty(S)}{dS^2} \geq 0$ on (E, S_f). From the expression of C_∞, we have

$$\frac{d^2 C_\infty}{dS^2} = \frac{\alpha_+ - 1}{S_f}\left(\frac{S}{S_f}\right)^{\alpha_+ - 2}.$$

Because $\dfrac{\alpha_+ - 1}{S_f} = \dfrac{(\alpha_+ - 1)^2}{E\alpha_+} > 0$, we know $\dfrac{d^2 C_\infty}{dS^2} \geq 0$ and the LC relation holds on (E, S_f). For $S \in (S_f, \infty)$, because $S_f \geq E\max(1, r/D_0)$, we have $C_\infty(S) = S - E = \max(S - E, 0)$, which means $C_\infty(S) - \max(S - E, 0) = 0$, and

$$\begin{aligned}&-\frac{\sigma^2 S^2}{2}\frac{d^2 C_\infty}{dS^2} - (r - D_0)\,S\frac{dC_\infty}{dS} + rC_\infty \\ &= D_0 S - rE = D_0(S - rE/D_0) \geq 0.\end{aligned}$$

[4]This result can also be obtained from direct calculation, which is left for readers as Problem 20.

Thus the LC relation also holds for $S \in (S_f, \infty)$. Consequently, we have proved our conclusion for all the cases.

For an American put option, as $T \to \infty$, the free-boundary problem (3.16) becomes

$$\begin{cases} \dfrac{1}{2}\sigma^2 S^2 \dfrac{d^2 P_\infty}{dS^2} + (r - D_0) S \dfrac{dP_\infty}{dS} - rP_\infty = 0, \quad S_f \le S, \\ P_\infty(S_f) = E - S_f, \\ \dfrac{dP_\infty(S_f)}{dS} = -1. \end{cases}$$

Similar to the call option, for $S \ge S_f$ the price of a perpetual American put option is

$$P_\infty(S) = \frac{-S_f}{\alpha_-}\left(\frac{S}{S_f}\right)^{\alpha_-}, \tag{3.34}$$

where

$$S_f = \frac{E}{1 - 1/\alpha_-}.$$

3.4 Some Conclusion from Arbitrage Theory

In Sect. 2.2, we derived the Black–Scholes equation by using arbitrage arguments. Here, we will further use arbitrage arguments to obtain some properties of option prices. Similar materials can be found in the book [43] by Hull.

3.4.1 Three Conclusions and Some Portfolios

Consider two portfolios $\mathbf{X}$ and $\mathbf{Y}$, whose values depend on a stock price S and time t. Let $\mathbf{X}(S,t)$ and $\mathbf{Y}(S,t)$ denote the values of portfolios $\mathbf{X}$ and $\mathbf{Y}$, respectively. $\mathbf{X}$ and $\mathbf{Y}$ could involve options, and all their expiries are T. By using arbitrage arguments, we can have three conclusions, which are written in the form of theorems.

Theorem 3.2 *If only European options are involved and* $\mathbf{X}(S,T) \ge \mathbf{Y}(S,T)$ *for any* S*, then for any* $t \le T$*,* $\mathbf{X}(S,t)$ *must be greater than or equal to* $\mathbf{Y}(S,t)$*.*

Proof. Suppose that at time $\bar{t}$ the value of portfolio $\mathbf{X}$ is less than the value of portfolio $\mathbf{Y}$ and that the latter is higher than the former by an amount of $Z(\bar{t})$. In this case, an arbitrageur can earn at least $Z(\bar{t})\mathrm{e}^{r(T-\bar{t})}$ at time T by doing the following: sell $\mathbf{Y}$, buy $\mathbf{X}$, and invest $Z(\bar{t})$ into a bank at an interest rate r at time $\bar{t}$, and get $\mathbf{X}(S,T)$ from portfolio $\mathbf{X}$, pay $\mathbf{Y}(S,T)$ for portfolio $\mathbf{Y}$, and obtain $Z(\bar{t})\mathrm{e}^{r(T-\bar{t})}$ from the risk-free investment at time T. Because $\mathbf{X}(S,T) \ge \mathbf{Y}(S,T)$ for any S, the arbitrageur will always earn at least

$Z(\bar{t})e^{r(T-\bar{t})}$ at the time T, which means that the earning is risk-free. Thus, everyone will do such a thing. Because so many people sell $\mathbf{Y}$ and buy $\mathbf{X}$, the price of $\mathbf{Y}$ will drop and the price of $\mathbf{X}$ will rise and will be immediately equal to or greater than the price of $\mathbf{Y}$. Therefore, Theorem 3.2 holds. □

From this result, assuming $\mathbf{X}(S,T) \le \mathbf{Y}(S,T)$, we can immediately get that for any time $t \le T$, $\mathbf{X}(S,t) \le \mathbf{Y}(S,t)$ and furthermore we can have

Theorem 3.3 *If $\mathbf{X}(S,T) = \mathbf{Y}(S,T)$ for any S, then for any $t \le T$, $\mathbf{X}(S,t)$ must be equal to $\mathbf{Y}(S,t)$ for any S.*

Proof. Because $\mathbf{X}(S,T) = \mathbf{Y}(S,T)$ means $\mathbf{X}(S,T) \ge \mathbf{Y}(S,T)$ and $\mathbf{X}(S,T) \le \mathbf{Y}(S,T)$, from the conclusion above we have for any t

$$\mathbf{X}(S,t) \ge \mathbf{Y}(S,t) \quad \text{and} \quad \mathbf{X}(S,t) \le \mathbf{Y}(S,t),$$

which means

$$\mathbf{X}(S,t) = \mathbf{Y}(S,t).$$

Thus we have Theorem 3.3. □

We can also have the following conclusion.

Theorem 3.4 *Suppose that portfolio $\mathbf{Y}$ involves only one American option and no European option and that portfolio $\mathbf{X}$ involves only European options. If $\mathbf{X}(S,T) \ge \mathbf{Y}(S,T)$ at time T and if the amount of cash and the number of stocks in $\mathbf{X}$ is greater than or equal to the amount of cash and the number of stocks the holder of $\mathbf{Y}$ has when the American option is exercised at time $\bar{t} < T$, then $\mathbf{X}(S,t) \ge \mathbf{Y}(S,t)$ for any time t.*

Proof. The argument is similar to the argument for proving Theorem 3.2. Suppose $\mathbf{X}(S,t) < \mathbf{Y}(S,t)$ at time $t < T$. Then, an arbitrageur can purchase $\mathbf{X}$, sell $\mathbf{Y}$, and earn some money. Later, when the American option is exercised early at time $\bar{t} < T$, the arbitrageur will never lose money because the amount of cash and the number of stocks in $\mathbf{X}$ are greater than or equal to the amount of cash and the number of stocks the holder of $\mathbf{Y}$ has. When the American option is not exercised before time T, the arbitrageur will also never lose any money because the value of $\mathbf{X}$ is greater than or equal to the value of $\mathbf{Y}$ at time T. Therefore, the earning is risk-free, which means $\mathbf{X}(S,t)$ should not be less than $\mathbf{Y}(S,t)$ at any time. □

Before applying these conclusions, we define some portfolios and find their values at time T along with what their holders will have if American options are exercised at time $\bar{t} < T$.

Portfolio A: An amount of cash equal to $Ee^{-r(T-t)}$ invested at an interest rate r. It is clear that its value at time T is E.

Portfolio B: $e^{-D_0(T-t)}$ shares of a stock with dividends being reinvested in the stock if the stock pays the dividend continuously or one share of a stock plus a loan $D_p(S,t)$[5] if the stock pays cash dividends discretely. Here, $D_p(S,t)$ is equal to the present value of these dividends to be paid from time t to time T, and the money will be returned to the loaner as soon as the stock pays a dividend. Obviously, its value at time T is the price of the stock S.

Portfolio C: One European call option plus portfolio **A**. The value of this portfolio at time T is $\max(S-E,0)+E=\max(S,E)$.

Portfolio D: One European put option plus portfolio **B**. Its value at time T is $\max(E-S,0)+S=\max(S,E)$.

Portfolio E: One American call option plus portfolio **A**. If the American call option is not exercised before time T, its value at time T is $\max(S-E,0)+E=\max(S,E)$. If at some time $\bar{t}<T$, the stock price S is greater than E and the American option is exercised, then the holder of the portfolio has one share plus a loan of $(1-\mathrm{e}^{-r(T-\bar{t})})E$.

Portfolio F: One American put option plus portfolio **B**. $\max(S,E)$ is its value at time T if the put option is not exercised before time T; while its holder has an amount of cash E minus $(1-\mathrm{e}^{-D_0(T-\bar{t})})$ shares or an amount of cash $E-D_p(S,\bar{t})$ if the stock price S is less than E and the put option is exercised at some time $\bar{t}<T$.

Portfolio G: One European call option plus E. Its value at time T is equal to $\max(S,E)$.

Portfolio H: One European put option plus one share. Its value is equal to $\max(S,E)$ at expiry.

3.4.2 Bounds of Option Prices

Consider a European call option and portfolio **B**. At time T, $c(S,T)=\max(S-E,0)\le \mathbf{B}(S,T)=S$. From Theorem 3.2, we have

$$c(S,t)\le S\mathrm{e}^{-D_0(T-t)}$$

or

$$c(S,t)\le S-D_p(S,t).$$

Now let us compare portfolio **C** with portfolio **B**. Because at time T

$$\mathbf{C}(S,T)=\max(S,E)\ge \mathbf{B}(S,T)=S,$$

we have

$$c(S,t)+E\mathrm{e}^{-r(T-t)}\ge S\mathrm{e}^{-D_0(T-t)}$$

or

$$c(S,t)+E\mathrm{e}^{-r(T-t)}\ge S-D_p(S,t).$$

[5]Here we assume that the value of the dividends depends on S, just like what we did in Sect. 2.2.2.

Clearly, $c(S,t) \geq 0$ for any case. Therefore, for a European call option we have

$$\max\left(S\mathrm{e}^{-D_0(T-t)} - E\mathrm{e}^{-r(T-t)}, 0\right) \leq c(S,t) \leq S\mathrm{e}^{-D_0(T-t)} \tag{3.35}$$

or

$$\max\left(S - D_p(S,t) - E\mathrm{e}^{-r(T-t)}, 0\right) \leq c(S,t) \leq S - D_p(S,t). \tag{3.36}$$

Consequently, the lower bound of $c(S,t)$ is $\max\left(S\mathrm{e}^{-D_0(T-t)} - E\mathrm{e}^{-r(T-t)}, 0\right)$ or $\max\left(S - D_p(S,t) - E\mathrm{e}^{-r(T-t)}, 0\right)$ and the upper bound is $S\mathrm{e}^{-D_0(T-t)}$ or $S - D_p(S,t)$. Here, we assume that $S - D_p(S,t)$ is always greater than zero. If $S < D_p(S,t)$ at time t, then any person will buy one share of the stock by finding a loan of amount S at time t and returning the loan as soon as the stock pays a dividend. In this way, the person gets one share and some cash free at time T. Therefore, the price must rise until $S \geq D_p(S,t)$.

Because $C(S,t) \geq c(S,t)$, we require that $C(S,t)$ is greater than or equal to the lower bound of $c(S,t)$. Also, $C(S,t)$ needs to be greater than or equal to the constraint $\max(S-E,0)$. Thus

$$\max\left(S\mathrm{e}^{-D_0(T-t)} - E\mathrm{e}^{-r(T-t)}, S-E, 0\right)$$

or

$$\max\left(S - D_p(S,t) - E\mathrm{e}^{-r(T-t)}, S-E, 0\right)$$

is a lower bound. Clearly, S is an upper bound for an American call option. Consequently, for the price of an American call option, we have

$$\max\left(S\mathrm{e}^{-D_0(T-t)} - E\mathrm{e}^{-r(T-t)}, S-E, 0\right) \leq C(S,t) \leq S \tag{3.37}$$

or

$$\max\left(S - D_p(S,t) - E\mathrm{e}^{-r(T-t)}, S-E, 0\right) \leq C(S,t) \leq S. \tag{3.38}$$

Now let us compare a European put option with portfolio **A**. At time T,

$$p(S,T) = \max(E-S,0) \leq \mathbf{A}(S,T) = E.$$

Thus

$$p(S,t) \leq E\mathrm{e}^{-r(T-t)}.$$

In order to get a lower bound of $p(S,t)$, let us look at portfolios **D** and **A**. Because at time T,

$$\mathbf{D}(S,T) = \max(S,E) \geq \mathbf{A}(S,T) = E,$$

we arrive at

$$p(S,t) + S\mathrm{e}^{-D_0(T-t)} \geq E\mathrm{e}^{-r(T-t)}$$

or

$$p(S,t) + S - D_p(S,t) \geq E\mathrm{e}^{-r(T-t)}.$$

Also, $p(S,t)$ must be nonnegative. Therefore, we have

$$\max\left(E\mathrm{e}^{-r(T-t)} - S\mathrm{e}^{-D_0(T-t)}, 0\right) \leq p(S,t) \leq E\mathrm{e}^{-r(T-t)} \tag{3.39}$$

or

$$\max\left(E\mathrm{e}^{-r(T-t)} - S + D_p(S,t), 0\right) \leq p(S,t) \leq E\mathrm{e}^{-r(T-t)}. \tag{3.40}$$

These give the lower and upper bounds of European put options.

For an American put option, we can also get the lower and upper bounds. Because $P(S,t) \geq p(S,t)$, we have

$$P(S,t) \geq \max\left(E\mathrm{e}^{-r(T-t)} - S\mathrm{e}^{-D_0(T-t)}, 0\right)$$

or

$$P(S,t) \geq \max\left(E\mathrm{e}^{-r(T-t)} - S + D_p(S,t), 0\right).$$

Also, $P(S,t)$ must be greater than or equal to $\max(E - S, 0)$. Therefore, we further obtain

$$P(S,t) \geq \max\left(E\mathrm{e}^{-r(T-t)} - S\mathrm{e}^{-D_0(T-t)}, E - S, 0\right)$$

or

$$P(S,t) \geq \max\left(E\mathrm{e}^{-r(T-t)} - S + D_p(S,t), E - S, 0\right).$$

E is a upper bound of $P(S,t)$, consequently we have

$$\max\left(E\mathrm{e}^{-r(T-t)} - S\mathrm{e}^{-D_0(T-t)}, E - S, 0\right) \leq P(S,t) \leq E \tag{3.41}$$

or

$$\max\left(E\mathrm{e}^{-r(T-t)} - S + D_p(S,t), E - S, 0\right) \leq P(S,t) \leq E. \tag{3.42}$$

From the proofs we know that if one of these relations is not true, then we can find an arbitrage opportunity to earn some money. This means that the lower bound is the greatest lower bound and that the upper bound is the least upper bound. From Sect. 1.2.4, we know that the price of an option is an increasing function of the volatility. Therefore, if the lower bound is the greatest lower bound, then as the volatility approaches zero, the limit of option should be the lower bound. Similarly, if the upper bound is the least upper bound, then as the volatility approaches infinity, the limit of the option should be the upper bound. When r, D_0, and σ are constant, the European option price is given by the Black–Scholes formulae in Sect. 2.6.5:

$$c(S,t) = S\mathrm{e}^{-D_0(T-t)}N(d_1) - E\mathrm{e}^{-r(T-t)}N(d_2)$$

and

$$p(S,t) = E\mathrm{e}^{-r(T-t)}N(-d_2) - S\mathrm{e}^{-D_0(T-t)}N(-d_1),$$

where

$$d_1 = \left[\ln \frac{S\mathrm{e}^{-D_0(T-t)}}{E\mathrm{e}^{-r(T-t)}} + \frac{1}{2}\sigma^2(T-t)\right] \Big/ \left(\sigma\sqrt{T-t}\right)$$

and

$$d_2 = \left[\ln \frac{S\mathrm{e}^{-D_0(T-t)}}{E\mathrm{e}^{-r(T-t)}} - \frac{1}{2}\sigma^2(T-t)\right] \Big/ \left(\sigma\sqrt{T-t}\right).$$

Therefore we have

$$\begin{cases} \lim\limits_{\sigma\to 0} c(S,t) &= \begin{cases} 0, & \text{if } S\mathrm{e}^{-D_0(T-t)} < E\mathrm{e}^{-r(T-t)}, \\ S\mathrm{e}^{-D_0(T-t)} - E\mathrm{e}^{-r(T-t)}, & \text{if } S\mathrm{e}^{-D_0(T-t)} > E\mathrm{e}^{-r(T-t)}, \end{cases} \\ \lim\limits_{\sigma\to\infty} c(S,t) &= S\mathrm{e}^{-D_0(T-t)}, \\ \lim\limits_{\sigma\to 0} p(S,t) &= \begin{cases} 0, & \text{if } E\mathrm{e}^{-r(T-t)} < S\mathrm{e}^{-D_0(T-t)}, \\ E\mathrm{e}^{-r(T-t)} - S\mathrm{e}^{-D_0(T-t)}, & \text{if } E\mathrm{e}^{-r(T-t)} > S\mathrm{e}^{-D_0(T-t)}, \end{cases} \\ \lim\limits_{\sigma\to\infty} p(S,t) &= E\mathrm{e}^{-r(T-t)}. \end{cases}$$

That is, the inequalities (3.35) and (3.39) truly provide the least upper and greatest lower bounds of European options, respectively.

Here, we give an example to show that if the price of an option does not satisfy a related condition, then there exists an arbitrage opportunity. More examples are given as problems for readers to study.

Example 1. Consider a European call option on a dividend-paying stock. Suppose the following: $S = \$102, E = \$100, c = \$8.50, r = 0.1$, the time to maturity is 9 months, and the present value of the dividend $D_p(102,t)$ is \$0.50. Is there any arbitrage opportunity?

Solution: As we know, the price of a call option has to satisfy the condition (3.36):

$$\max\left(S - D_p(102,t) - E\mathrm{e}^{-r(T-t)}, 0\right) \le c(S,t) \le S - D_p(102,t).$$

In this case

$$\begin{aligned} \max\left(S - D_p(102,t) - E\mathrm{e}^{-r(T-t)}, 0\right) &= \max\left(102 - 0.5 - 100\mathrm{e}^{-0.9/12}, 0\right) \\ &= 8.73. \end{aligned}$$

Therefore, the price of the call option is less than the lower bound. In this case, if we own one share of the stock or if you can borrow one share of the stock for the period $[t, T]$, then we should take a long position in a portfolio **C** and a short position in a portfolio **B**. In other words, buy one call option, sell one share, and deposit $Ee^{-r(T-t)} + D_p(102, t)$ in a bank at time t. In this case we will get $-8.5 + 102 - 100e^{-0.9/12} - 0.5 = \$0.23$ at time t, and this is a risk-free earning. This is because we can get the money from the bank to pay the dividends on the stock during the time interval $[t, T]$ and get E from the bank at time T. If $S \geq E$ at time T, we can exercise the call option and get one share. If $S < E$, we can have one share of the stock that is bought from the market and an amount of cash $E - S$. In any case, we have one share plus at least \$0.23. That is, we can get one share back or return one share to the borrower and earn at least \$0.23 free at time T.

3.4.3 Relations Between Call and Put Prices

Let us look at portfolios **C** and **D**. Because $\mathbf{C}(S, T) = \mathbf{D}(S, T)$, we have

$$c(S, t) + Ee^{-r(T-t)} = p(S, t) + Se^{-D_0(T-t)} \tag{3.43}$$

or

$$c(S, t) + Ee^{-r(T-t)} = p(S, t) + S - D_p(S, t) \tag{3.44}$$

according to Theorem 3.3. These are called put–call parities of European options. For stocks with continuous dividends, we obtained such a relation through a very long procedure in Sect. 2.6. However, the derivation here is so simple. This shows that arbitrage theory is a very powerful tool.

The put–call parity relations hold only for European options. For American options they are not true, but the following inequalities on the difference between the American call and put option prices are fulfilled

$$Se^{-D_0(T-t)} - E \leq C(S, t) - P(S, t) \leq S - Ee^{-r(T-t)} \tag{3.45}$$

or

$$S - D_p(S, t) - E \leq C(S, t) - P(S, t) \leq S - Ee^{-r(T-t)}. \tag{3.46}$$

The two inequalities can also be written as

$$\begin{cases} Se^{-D_0(T-t)} - E + P(S, t) \leq C(S, t) \leq S - Ee^{-r(T-t)} + P(S, t), \\ C(S, t) - S + Ee^{-r(T-t)} \leq P(S, t) \leq C(S, t) - Se^{-D_0(T-t)} + E \end{cases}$$

or

$$\begin{cases} S - D_p(S, t) - E + P(S, t) \leq C(S, t) \leq S - Ee^{-r(T-t)} + P(S, t), \\ C(S, t) - S + Ee^{-r(T-t)} \leq P(S, t) \leq C(S, t) - S + D_p(S, t) + E, \end{cases}$$

which gives the lower and upper bounds of an American call (put) option if the price of the corresponding American put (call) option is known.

First, let us prove the left portions of the inequalities (3.45) and (3.46). Consider portfolios **G** and **F**. Because **G** contains European options only and **F** contains only one American option, it is possible to use Theorem 3.4. According to Theorem 3.4, the value of **G** is always greater than or equal to the value of **F** if we can prove two things:

1. The value of **G** is greater than or equal to the value of **F** at time T;
2. The amount of cash and the number of stocks in **G** is greater than or equal to the amount of cash and the number of stocks in **F** when the American option is exercised at time $\bar{t} < T$.

At time T, the value of **G** is equal to the value of **F**. At any time $\bar{t} < T$, there is an amount of cash E and no stock in **G**. If the American put option in **F** is exercised before time T, **F** contains an amount of cash E and $-(1-\mathrm{e}^{-D_0(T-\bar{t})})$ shares or an amount of cash $E - D_p(S,t)$. Therefore, both the amount of cash and the number of stocks in **G** is greater than or equal to those in **F** if the American option in **F** is exercised at some time $\bar{t} < T$. Consequently, according to Theorem 3.4, the value of **G** is greater than or equal to the value of **F** for any case, that is,

$$P(S,t) + S\mathrm{e}^{-D_0(T-t)} \le c(S,t) + E$$

or

$$P(S,t) + S - D_p(S,t) \le c(S,t) + E.$$

Because $C(S,t) \ge c(S,t)$, we further have

$$S\mathrm{e}^{-D_0(T-t)} - E \le C(S,t) - P(S,t)$$

or

$$S - D_p(S,t) - E \le C(S,t) - P(S,t).$$

In order to prove the right portions of the relations, we need to look at portfolios **H** and **E**. In **H** there is only one European option and in **E** the American option is the only option, so we can use Theorem 3.4 again. When the American call option in **E** is exercised before time T, the amount of cash and the number of stocks in **H** is greater than or equal to those in **E**. When it is not exercised before expiry, the value of **H** is equal to the value of **E** at time T. Therefore

$$C(S,t) + E\mathrm{e}^{-r(T-t)} \le p(S,t) + S.$$

Noticing $P(S,t) \ge p(S,t)$, we have

$$C(S,t) - P(S,t) \le S - E\mathrm{e}^{-r(T-t)}.$$

This completes our proof.

Example 2. Suppose there are an American call option and an American put option on the same stock. The stock pays dividends continuously, and $D_0 = 0.05$. For both options, $E = \$100$ and $T = 1$ month. At present, $r = 0.1, S = \$103$, and $C = \$5.50$. Find the upper and lower bounds for the price of the American put option by using the relation (3.45). How do we take the arbitrage opportunity if the price of the American put option is greater than the calculated upper bound?

Solution: According to the relation (3.45), the lower bound of $P(S,t)$ is

$$C(S,t) - S + Ee^{-r(T-t)} = 5.5 - 103 + 100e^{-0.1/12} = 1.67$$

and the upper bound is

$$C(S,t) - Se^{-D_0(T-t)} + E = 5.5 - 103e^{-0.05/12} + 100 = 2.93.$$

Suppose that on the market $P(103,t) = \$3.50$. Now we describe how to take advantage of the arbitrage opportunity. At time t, we can sell the American put option and short-sell $e^{-0.05/12}$ shares, purchase one European call option that is less than or equal to \$5.50, and hold at least an amount of cash $3.5 + 103e^{-0.05/12} - 5.5 = \$100.57$. If we want, it can be deposited into a bank. At any time $\bar{t} \in [t,T)$, the holder of the American put option wants to exercise the option, we pay \$100 and get one share. In this case, we have at least one share of stock and at least an amount of cash equal to \$0.57 at time T. If the holder of the American put option does not exercise the option before time T, we also will always have at least \$0.57 in cash plus one share of stock at time T. The reason is that we can exercise the European call option and get one share if $S > E$, whereas we can purchase one share from the market if $S \le E$. Because we need to return only one share to the borrower at time T, we always have enough shares of stocks. Therefore, the risk-free earning in this case is at least \$0.57.

Problems

Table 3.1. Problems and subsections

Problems	Subsections	Problems	Subsections	Problems	Subsections
1–2	3.1.1	3–7	3.1.2	8	3.2.1
9–15	3.3.2	16–18	3.3.3	19	3.3.4
20–23	3.3.5	24–25	3.4.1	26–27	3.4.2
28–30	3.4.3				

1. Let $\mathbf{L_{S,t}}$ be an operator in an option problem in the form:

$$\mathbf{L_{S,t}} = a(S,t)\frac{\partial^2}{\partial S^2} + b(S,t)\frac{\partial}{\partial S} + c(S,t)$$

and $G_v(S,t)$ be the constraint function for an American option. Furthermore we assume that $\dfrac{\partial G_v}{\partial t} + \mathbf{L_{S,t}}G_v$ exists. Suppose $V(S,t^*) = G_v(S,t^*)$ on an open interval (A,B) on the S-axis. Let $t = t^* - \Delta t$, where Δt is a sufficiently small positive number. Show the following conclusions: If for any $S \in (A,B)$,

$$\frac{\partial G_v}{\partial t}(S,t^*) + \mathbf{L_{S,t^*}}G_v(S,t^*) + d(S,t^*) \geq 0,$$

then the value $V(S,t)$ determined by the equation

$$\frac{\partial V}{\partial t}(S,t) + \mathbf{L_{S,t}}V(S,t) + d(S,t) = 0$$

satisfies the condition $V(S,t) - G_v(S,t) \geq 0$ on (A,B); and if for any $S \in (A,B)$,

$$\frac{\partial G_v}{\partial t}(S,t^*) + \mathbf{L_{S,t^*}}G_v(S,t^*) + d(S,t^*) < 0,$$

then the equation

$$\frac{\partial V}{\partial t}(S,t) + \mathbf{L_{S,t}}V(S,t) + d(S,t) = 0$$

cannot give a solution satisfying the condition $V(S,t) - G_v(S,t) \geq 0$ for any $S \in (A,B)$.
2. *Suppose that for an American option, the constraint is $G_v(S,t)$, its value at time t is $V(S,t)$, and $V(S,t) = G_v(S,t)$ on (A,B). Assume that when $V(S,t)$ were given as the value of a European option at t, the value of the European option at $t - \Delta t$ for a positive and very small Δt is $v(S, t-\Delta t)$. Explain that if in an open interval containing $S^* \in (A,B)$, $v(S,t-\Delta t) < G_v(S,t-\Delta t)$, then for the American option a fair value at the point $(S^*, t-\Delta t)$ should be $G_v(S^*, t-\Delta t)$.
3. *Show that an American option is always worth at least as much as a European option on the same asset with the same strike price and exercise date if r, D_0, σ are constant, and give a financial explanation.
4. Show that a Bermudan option is always worth at least as much as a European option on the same asset with the same strike price and exercise date if r, D_0, σ are constant, and give a financial explanation of this fact. (Hint: For a Bermudan option, the approximate relation between the price at t_n and the price at t_{n+1} is the same as for a European option if at $t = t_n$ the option cannot be exercised, and the same as for an American option if at $t = t_n$ the option can be exercised.)

5. (a) *Explain why an American option is always worth at least as much as its intrinsic value. What is the definition of the time value of an American option?
   (b) *Let $V(S,t)$ be the price of a vanilla American option. Show that $V(S,t^*) \geq V(S,t^{**})$ is always true, where $t^* \leq t^{**}$. This means that the time value of a vanilla American option for a fixed S is decreasing as $t \to T$, and give a financial explanation of this fact.
6. (a) The price of a one-factor convertible bond paying no coupon is the solution of the following linear complementarity problem

$$\begin{cases} \min\left(-\dfrac{\partial V}{\partial t} - \mathbf{L}_S V,\ V(S,t) - nS\right) = 0, & 0 \leq S,\ 0 \leq t \leq T, \\ V(S,T) = \max(Z, nS) \geq nS, & 0 \leq S, \end{cases}$$

   where

$$\mathbf{L}_S = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0)\, S \frac{\partial}{\partial S} - r$$

   and n, Z, σ, r, and D_0 are positive constants. Show

$$V(S,t^*) - Z\mathrm{e}^{-r(T-t^*)} \geq V(S,t^{**}) - Z\mathrm{e}^{-r(T-t^{**})} \quad \text{if} \quad t^* \leq t^{**}.$$

   (Hint: Define $\overline{V}(S,t) = V(S,t) - Z\mathrm{e}^{-r(T-t)}$ and show $\overline{V}(S,t^*) \geq \overline{V}(S,t^{**})$ if $t^* \leq t^{**}$.)
   (b) Can you prove that $V(S,t^*) \geq V(S,t^{**})$ for $t^* \leq t^{**}$ by using the method used in part (a)? If your answer is “Yes”, give a proof; otherwise explain why you cannot.
   (c) “A holder of a convertible bond at time t^* has “more rights” than a holder of a convertible bond at time t^{**} does if $t^* \leq t^{**}$, so the premium at t^* should be higher than the premium at t^{**}, i.e., the inequality $V(S,t^*) \geq V(S,t^{**})$ should hold for any $t^* \leq t^{**}$.” Do you think that this statement is true and why?
7. The price of a one-factor convertible bond paying constant coupon is the solution of the following linear complementarity problem

$$\begin{cases} \min\left(-\dfrac{\partial V}{\partial t} - \mathbf{L}_S V - kZ,\ V(S,t) - nS\right) = 0, & 0 \leq S,\ 0 \leq t \leq T, \\ V(S,T) = \max(Z, nS) \geq nS, & 0 \leq S, \end{cases}$$

   where

$$\mathbf{L}_S = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0)\, S \frac{\partial}{\partial S} - r$$

   and k, Z, n, σ, r, and D_0 are positive constants. Study whether or not $V(S,t^*) \geq V(S,t^{**})$ for $t^* \leq t^{**}$ holds in the cases $r > k$ and $r = k$, and if not, try to find a relation between $V(S,t^*)$ and $V(S,t^{**})$.

8. A European option is the solution of the problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \mathbf{L}_S V = 0, & 0 \le S, \quad t \le T, \\ V(S,T) = V_T(S), & 0 \le S, \end{cases}$$

where

$$\mathbf{L}_S = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0)S\frac{\partial}{\partial S} - r.$$

For an American option, the constraint is that the inequality

$$V(S,t) \ge G(S,t)$$

holds for any S and t, where $G(S,T) = V_T(S)$. Derive the linear complementarity problem for the American option.

9. The American call option is the solution of the following linear complementarity problem on a finite domain:

$$\begin{cases} \min\left(\dfrac{\partial \overline{V}}{\partial \tau} - \mathbf{L}_\xi \overline{V},\ \overline{V}(\xi,\tau) - \max(2\xi - 1, 0)\right) = 0, & 0 \le \xi \le 1,\ 0 \le \tau, \\ \overline{V}(\xi, 0) = \max(2\xi - 1, 0), & 0 \le \xi \le 1, \end{cases}$$

where

$$\mathbf{L}_\xi = \frac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1-\xi)^2\frac{\partial^2}{\partial \xi^2} + (r - D_0)\xi(1-\xi)\frac{\partial}{\partial \xi} - [r(1-\xi) + D_0\xi].$$

Reformulate this problem as a free-boundary problem if $D_0 > 0$.

10. The American put option is the solution of the following linear complementarity problem:

$$\begin{cases} \min\left(\dfrac{\partial u}{\partial \bar{\tau}} - \dfrac{\partial^2 u}{\partial x^2},\ u(x,\bar{\tau}) - g_p(x,\bar{\tau})\right) = 0, & -\infty < x < \infty,\ 0 \le \bar{\tau}, \\ u(x,0) = g_p(x,0), & -\infty < x < \infty, \end{cases}$$

where

$$g_p(x,\bar{\tau}) = \max\left(\mathrm{e}^{2r\bar{\tau}/\sigma^2} - \mathrm{e}^{x + (2D_0/\sigma^2 + 1)\bar{\tau}}, 0\right).$$

Find the domain where a free boundary may appear and the domain where it is impossible for a free boundary to appear, show that there is only one free boundary at $\bar{\tau} = 0$, and give the starting location of this free boundary.

11. The price of a one-factor convertible bond is the solution of the linear complementarity problem

$$\begin{cases} \min\left(-\dfrac{\partial V}{\partial t} - \mathbf{L}_S V - kZ,\ V(S,t) - nS\right) = 0, & 0 \le S,\ 0 \le t \le T, \\ V(S,T) = \max(Z, nS) \ge nS, & 0 \le S, \end{cases}$$

where

$$\mathbf{L}_S = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0) S \frac{\partial}{\partial S} - r,$$

and k, Z, n, σ, r and D_0 are constants. Show that if $D_0 > 0$, then the solution of a one-factor convertible bond must involve a free boundary and its location at $t = T$ is $S = \max\left(\frac{Z}{n}, \frac{kZ}{D_0 n}\right)$. Also, derive the corresponding free-boundary problem if this problem has only one free boundary.

12. Consider the following LC problem:

$$\begin{cases} \min\left(-\dfrac{\partial W}{\partial t} - \mathbf{L}_{\alpha,t} W,\ W(\eta,t) - \max(\alpha - \eta, 0)\right) = 0, & 0 \le \eta,\ \ t \le T, \\ W(\eta, T) = \max(\alpha - \eta, 0), & 0 \le \eta, \end{cases}$$

where the operator $\mathbf{L}_{a,t}$ is defined by

$$\mathbf{L}_{a,t} = \frac{1}{2}\sigma^2 \eta^2 \frac{\partial^2}{\partial \eta^2} + \left[(D_0 - r)\eta + \frac{1-\eta}{t}\right] \frac{\partial}{\partial \eta} - D_0.$$

Suppose that there is only one free-boundary for this problem, reformulate this problem as a free-boundary problem.

13. Consider the following LC problem:

$$\begin{cases} \min\left(-\dfrac{\partial W}{\partial t} - \mathbf{L}_{\eta} W,\ W(\eta,t) - G_{lsp}(\eta, t)\right) = 0, & 1 \le \eta,\ \ t \le T, \\ W(\eta, T) = G_{lsp}(\eta, T), & 1 \le \eta, \\ \dfrac{\partial W}{\partial \eta}(1, t) = 0, & t \le T, \end{cases}$$

where $G_{lsp}(\eta, t) = \max(\eta - \beta, 0)$ with $\beta \ge 1$ and $\mathbf{L}_\eta = \frac{1}{2}\sigma^2\eta^2 \frac{\partial^2}{\partial \eta^2} + (D_0 - r)$ $\eta \frac{\partial}{\partial \eta} - D_0$. Find the domain where it is impossible for a free boundary to appear and the domain where a free boundary may appear.

14. As we know, when the LC problem of an American call option is formulated as a free-boundary problem, on the free boundary $S = S_f(t) \ge \max(E, rE/D_0)$, we need to require $C(S_f(t), t) = \max(S_f(t) - E, 0) = S_f(t) - E$ and $\frac{\partial C(S_f(t), t)}{\partial S} = 1$, where $C(S, t)$ and $\max(S - E, 0)$ are the solution of the free-boundary problem and the constraint. Show that if $C(S, t) \ge 0$ and $\frac{\partial C^2(S, t)}{\partial S^2} \ge 0$ for $S < S_f(t)$, then the solution of the free-boundary problem satisfies the LC condition

$$\min\left(-\frac{\partial C}{\partial t} - \mathbf{L}_S C,\ C - \max(S - E, 0)\right) = 0,$$

where $\mathbf{L}_S = \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2}{\partial S^2} + (r - D_0)S\dfrac{\partial}{\partial S} - r$, that is, $C(S,t)$ truly is the solution of the LC problem for $S \in [0, S_f(t))$.

15. Consider an American call option on a stock paying discrete dividends.
    (a) Show that in this case, the optimal exercise price cannot appear for t between two successive ex-dividend dates.
    (b) Suppose that t_n, t_{n+1} are two successive ex-dividend dates with $t_n < t_{n+1}$. Assume $D_n(S)$ be the dividend payment at time t_n. Show that if $D_n(S) \le E\left(1 - \mathrm{e}^{-r(t_{n+1}-t_n)}\right)$, then there is no chance for an optimal exercise price to appear at time t_n^-; if $D_n(S) > E\left(1 - \mathrm{e}^{-r(t_{n+1}-t_n)}\right)$, it is possible for an optimal exercise price to appear at time t_n^-.
16. *Suppose r, D_0, and σ are constant.
    (a) Derive the put–call symmetry relations.
    (b) Explain the financial meaning of the symmetry relation.
    (c) Explain how to use these relations when we write codes if a code for put options is quite a different from a code for call options.
17. (a) Suppose $\sigma = \sigma(S,t)$, $r = r(t)$, and $D_0 = D_0(S,t)$. Show that the problem of pricing a put option can always be converted into a problem of pricing a call option. Also explain how to use this conclusion when we write codes if a code for put options is quite a different from a code for call options.
    (b) Let the exercise price be E. Suppose that r, D_0 are constants and $\sigma = \sigma(S)$. Show

$$\begin{aligned} P(S,t;b,\ a,\ \sigma(S)) &= C\left(E^2/S,t;a,\ b,\ \sigma(S)\right)S/E, \\ C\left(S,t;a,\ b,\ \sigma(S)\right) &= P\left(E^2/S,t;b,\ a,\ \sigma(S)\right)S/E \end{aligned}$$

    and

$$S_{cf}(t;a,\ b,\ \sigma(S)) \times S_{pf}(t;b,\ a,\ \sigma(E^2/S)) = E^2.$$

    Here, the first, second, and third parameters after the semicolon in P, C, S_{pf}, and S_{cf} are the interest rate, the dividend yield and the volatility function, respectively.
    (c) Show that for Bermudan options the symmetry relation is still true.
18. Suppose that σ, r, D_0 are constants. In this case we have the following symmetry relation for European options

$$p(S,t;b,\ a) = c\left(\frac{E^2}{S},t;a,\ b\right)S/E,$$

where the first and second arguments after the semicolon in p and c are the values of the interest rate and the dividend yield, respectively. For a European call option, the price is

$$c(S,t) = S\mathrm{e}^{-D_0(T-t)}N(d_1) - E\mathrm{e}^{-r(T-t)}N(d_2),$$

where

$$d_1 = \frac{\ln \dfrac{Se^{-D_0(T-t)}}{Ee^{-r(T-t)}} + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\ln \dfrac{Se^{-D_0(T-t)}}{Ee^{-r(T-t)}} - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}.$$

Find the price of a European put option by using the symmetry relation.

19. Derive the formulation of the problem for $\dfrac{\partial P}{\partial r}$ and write down the formulation of the problems for $\dfrac{\partial P}{\partial \sigma}$ and $\dfrac{\partial P}{\partial D_0}$, where P is the price of an American put option.
20. Define

$$\alpha_\pm = \frac{1}{\sigma^2}\left[-\left(r - D_0 - \frac{1}{2}\sigma^2\right) \pm \sqrt{\left(r - D_0 - \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 r}\right],$$

where $r \geq 0$ and $D_0 \geq 0$.

(a) Show that $\alpha_+ \geq 1$, $\alpha_- \leq 0$, and $-(r - D_0)\alpha_\pm + r \geq 0$.

(b) Based on the results in part (a), show that $1/(1-1/\alpha_+) \geq \max(1, r/D_0)$ and $1/(1 - 1/\alpha_-) \leq \min(1, r/D_0)$.

21. (a) Find the solution of the following free-boundary problem:

$$\begin{cases} \dfrac{1}{2}\sigma^2 S^2 \dfrac{d^2 P_\infty}{dS^2} + (r - D_0)S\dfrac{dP_\infty}{dS} - rP_\infty = 0, & S_f \leq S, \\ P_\infty(S_f) = E - S_f, \\ \dfrac{dP_\infty(S_f)}{dS} = -1. \end{cases}$$

(b) Define

$$P_\infty(S) = \begin{cases} E - S, & 0 \leq S < S_f, \\ \text{the solution of the free-boundary problem}, & S_f \leq S. \end{cases}$$

Show that $P_\infty(S)$ satisfies

$$\min\left(-\left[\frac{1}{2}\sigma^2 S^2 \frac{d^2 P_\infty}{dS^2} + (r - D_0)S\frac{dP_\infty}{dS} - rP_\infty\right], \right.$$
$$\left. P_\infty - \max(E - S, 0)\right) = 0,$$

that is, $P_\infty(S)$ is a solution of the perpetual American put option.

22. (a) Find the solution of the following free-boundary problem:

$$\begin{cases} \dfrac{1}{2}\sigma^2\eta^2\dfrac{d^2W_\infty}{d\eta^2} + (D_0 - r)\,\eta\dfrac{dW_\infty}{d\eta} - D_0W_\infty = 0, \quad 1 \le \eta \le \eta_f, \\ \dfrac{dW_\infty(1)}{d\eta} = 0, \\ W_\infty\left(\eta_f\right) = \eta_f, \\ \dfrac{dW_\infty\left(\eta_f\right)}{d\eta} = 1, \end{cases}$$

where η_f is a number representing the location of this free boundary.

(b) Define

$$W_\infty(\eta) = \begin{cases} \text{the solution of the free-boundary problem}, & 1 \le \eta \le \eta_f, \\ \eta, & \eta_f < \eta. \end{cases}$$

Show that $W_\infty(\eta)$ is a solution of the following LC problem

$$\begin{cases} \min\left(-\dfrac{\sigma^2\eta^2}{2}\dfrac{d^2W_\infty}{d\eta^2} - (D_0 - r)\,\eta\dfrac{dW_\infty}{d\eta} + D_0W_\infty,\ W_\infty - \eta\right) = 0, \\ \qquad\qquad 1 \le \eta, \\ \dfrac{dW_\infty(1)}{d\eta} = 0. \end{cases}$$

(This problem is related to the Russian option.)

23. Find the solution of the problem:

$$\begin{cases} \dfrac{1}{2}\sigma^2\xi^2\dfrac{d^2W_\infty}{d\xi^2} + (D_{02} - D_{01})\,\xi\dfrac{dW_\infty}{d\xi} - D_{02}W_\infty = 0, \quad \xi_{f_1} \le \xi \le \xi_{f_2}, \\ W_\infty\left(\xi_{f_1}\right) = 1, \\ \dfrac{dW_\infty}{d\xi}\left(\xi_{f_1}\right) = 0, \\ W_\infty\left(\xi_{f_2}\right) = \xi_{f_2}, \\ \dfrac{dW_\infty}{d\xi}\left(\xi_{f_2}\right) = 1, \end{cases}$$

where $\xi_{f_1} < \xi_{f_2}$. (This problem is related to the perpetual American better-of option.)

24. Suppose that $c_1(S, t)$ and $c_2(S, t)$ are the prices of European call options with strikes E_1 and E_2, respectively, where $E_1 < E_2$. Also assume that the two options have the same maturity T and that the interest rate r is a constant. Show

$$0 \le c_1(S,t) - c_2(S,t) \le (E_2 - E_1)\mathrm{e}^{-r(T-t)}.$$

25. Suppose that p_1, p_2, and p_3 are the prices of European put options with strike prices E_1, E_2, and E_3, respectively, where $E_2 = \dfrac{1}{2}(E_1 + E_3)$. All the options have the same maturity. Show

$$p_2 \le \frac{1}{2}(p_1 + p_3).$$

26. Consider a European call option with $T = 6$ months and $E = \$80$ on a dividend-paying stock. The dividend is paid continuously with a dividend yield $D_0 = 0.05$. Today, $t = 0$, $r = 0.1$ and $S = \$82$.
   (a) Find the lower bound of the call option.
   (b) What are the least profits we could make at time T by arbitrage if the call option price today is \$0.10 less than the lower bound and why?
27. Consider a European put option with $T = 3$ months and $E = \$60$ on a dividend-paying stock. Today $t = 0$, $r = 0.05$, and $S = \$55$. The dividends are paid discretely, and the total present value of them is $D_p(55, 0) = \$0.30$.
   (a) Find the lower bound of the put option.
   (b) What are the least profits we could make at time T by arbitrage if the put option price today is \$0.20 less than the lower bound and why?
28. *Use arbitrage arguments to show the put–call parity of European options for the following two cases.
   (a) When the dividend is paid continuously, the put–call parity is

$$c(S,t) - p(S,t) = S\mathrm{e}^{-D_0(T-t)} - E\mathrm{e}^{-r(T-t)};$$

   (b) when the dividend is paid discretely, the put–call parity is

$$c(S,t) - p(S,t) = S - D_p(S,t) - E\mathrm{e}^{-r(T-t)},$$

   where $D_p(S,t)$ is the value of "will-be-paid" dividends at time t.
29. *Use arbitrage arguments to show the inequalities of American options for the following two cases.
   (a) When the dividend is paid continuously, there is the inequality

$$S\mathrm{e}^{-D_0(T-t)} - E \le C(S,t) - P(S,t) \le S - E\mathrm{e}^{-r(T-t)}$$

   between American put option $P(S,t)$ and American call option $C(S,t)$ with the same parameters.
   (b) When the dividend is paid discretely, there is the inequality

$$S - D_p(S,t) - E \le C(S,t) - P(S,t) \le S - E\mathrm{e}^{-r(T-t)}$$

   between American put option $P(S,t)$ and American call option $C(S,t)$ with the same parameters.

30. Suppose that there are an American call option and an American put option on the same stock that pays dividends discretely. For both of them, $E = \$90$ and $T = 3\,\text{months}$. At time $t = 0$, the stock price is \$93 and the present value of dividend payments during the period $[0, T]$ is $D_p(93, 0) = \$0.50$. Assume that $r = 0.10$ and $P(93, 0) = \$2.50$.
    (a) Find the upper and lower bounds of the price of the American call option.
    (b) What are the risk-free profits we could make today by arbitrage if the price of the call option today is \$0.10 greater than the calculated upper bound and why?

4

Exotic Options

4.1 Introduction

In order to meet a variety of demands, modern financial institutions issue many exotic options besides the vanilla options we have introduced in Chaps. 2 and 3. An exotic option is an option that is not a vanilla put or call. It usually is traded between companies and banks and not quoted on an exchange. In this case, we usually say that it is traded in the over-the-counter market. Most exotic options are quite complicated, and their final values depend not only on the asset price at expiry but also on the asset price at previous times. They are determined by a part or the whole of the path of the asset price during the life of option. These options are called path-dependent exotic options. Barrier options, Asian options, and lookback options are important examples of path-dependent exotic options.

A barrier option is a derivative product that either becomes worthless, must be exercised, or comes into existence if the underlying asset price reaches a certain level during a certain period of time. For example, a down-and-out call has similar features to a vanilla call option, except that it becomes nullified when the asset price falls below a knock-out level. Because the holder of the option loses some of the rights, the price of such an option is lower than a vanilla call option. However, if the asset price is always higher than the knock-out level (which is expected by any holder of a call option), then the two options are actually the same. Therefore, such a call option is more attractive than a vanilla call option for people who expect the price to rise.

An Asian option is an option whose payoff depends on some form of the average of the underlying asset price over a part or the whole of the life of the option. Consider a call option and let the price of underlying asset in its payoff be replaced by the average of the asset price over a period. Suppose that a manufacturer expects to make a series of crude oil purchases for his factory during some fixed time period. If the average price of crude oil drops, then he will be happy because the cost of his product declines; if the average price of crude oil rises, then he might lose money because the cost rises. In this case,

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such an option can be a hedging instrument for him. He can avoid the risk caused by the rise in average price and keep the advantage due to the drop in average price by holding an average call option on crude oil for that period. Asian options are more appropriate than vanilla options for such a case.

A lookback option is a contract whose payoff depends on the maximum or minimum stock price reached during the life of the option. For example, a lookback put option has a payoff that is the difference between the maximum realized price and the price at expiry. Therefore, the holder of such an option can sell the asset at the highest price.

We have described three examples through which we explain how those exotic options are designed by financial institutions to meet the requirements of their clients. Besides the examples mentioned above, there are many other types of barrier, Asian, lookback options, and other exotic options. Multi-asset options, binary options, forward start options, compound options, and chooser options are all examples of exotic options. In the following sections, we will give some details on these options.

4.2 Barrier Options

4.2.1 Knock-Out and Knock-In Options

As pointed out in Sect. 4.1, a barrier option is a derivative product that either becomes worthless, must be exercised, or comes into existence if the underlying asset price reaches a certain level during a certain period of time.

A knock-out option is an option that either becomes worthless or must be exercised if the underlying asset value reaches the knock-out level, which is called a barrier. The simplest knock-out options are the down-and-out call and the up-and-out put. An option is called a down-and-out call if it is actually a call when S is always greater than the barrier during the life of the option, and it becomes worthless when S reaches the barrier from above at some time before or at expiry. We call such a barrier a lower barrier B_l, and in this section we mainly consider the case that such a barrier is below the exercise price E. A down-and-out call could be a European-style or an American-style option just like a vanilla option. An up-and-out put is similar to a down-and-out call. However, instead of a lower barrier, it has an upper barrier B_u, which we usually assume is greater than E. It is a put if S is never above B_u and becomes worthless when S crosses the barrier B_u from below at some time prior to expiry. More complicated knock-out options have two barriers B_l and B_u that might be given as functions of time, and such an option becomes worthless if S enters $[0, B_l]$ or $[B_u, \infty)$ from (B_l, B_u) at some time during the life of the option. Sometimes the holder of the option receives a specified amount of money as a rebate if a barrier is reached. For example, an option is called a knock-out call with a rebate if it has the following three properties. It is actually a call when S is always in (B_l, B_u) during the life of the option, it becomes worthless when S enters $[0, B_l]$ at any time, and it must be exercised for a rebate $B_u - E$ when S crosses the upper barrier B_u from below at any time.

A knock-in option is a contract that comes into existence if the asset price crosses a barrier. For example, a down-and-in call with a lower barrier B_l expires worthless unless the asset price reaches the lower barrier from above prior to or at expiry. If it crosses the lower barrier from above at some time before expiry, then the option becomes a vanilla option. An up-and-in put is similar to a down-and-in call, but the barrier is an upper one and the put option is activated when S crosses the upper barrier from below.

4.2.2 Closed-Form Solutions of Some European Barrier Options

For some European barrier options, closed-form solutions can be obtained. As examples, we first derive such a solution for a European down-and-out call by the method of images and then obtain the solution for a European down-and-in call from the solution for the European down-and-out call.

Now let us look at a European down-and-out call option. Let $c_o(S, t)$ denote the value of this option. As an option, $c_o(S, t)$ satisfies the Black–Scholes equation for $S > B_l$. If S is always greater than B_l, then it is a call option. Therefore the final condition should be

$$c_o(S, T) = \max(S - E, 0), \text{ for } S > B_l.$$

The option becomes worthless if S ever reaches B_l, which means that the boundary condition at $S = B_l$ should be

$$c_o(B_l, t) = 0.$$

Therefore, the fair value of such an option should be the solution of the problem

$$\begin{cases} \dfrac{\partial c_o}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial c_o}{\partial S^2} + (r - D_0) S \dfrac{\partial c_o}{\partial S} - r c_o = 0, & S \geq B_l, \quad t \leq T, \\ c_o(S, T) = \max(S - E, 0), & S \geq B_l, \\ c_o(B_l, t) = 0, & t \leq T. \end{cases}$$

In order to find analytic solutions of European Barrier options, let us consider the following problem:

$$\begin{cases} \dfrac{\partial \overline{V}}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 \overline{V}}{\partial S^2} + (r - D_0) S \dfrac{\partial \overline{V}}{\partial S} - r\overline{V} = 0, \quad 0 \leq S, \quad t \leq T, \\ \overline{V}(S, T) = \begin{cases} \varphi_1(S), \ 0 \leq S \leq B, \\ \varphi_2(S), \ B < S, \end{cases} \end{cases} \tag{4.1}$$

where $\varphi_1(S)$ and $\varphi_2(S)$ are continuous functions and $\varphi_1(B) = \varphi_2(B)$ might not hold. Let us show that if the relation

$$\varphi_1(S) = -\left(\frac{B}{S}\right)^{2\left(r-D_0-\sigma^2/2\right)/\sigma^2} \varphi_2\left(\frac{B^2}{S}\right)$$

or

$$\varphi_2(S) = -\left(\frac{B}{S}\right)^{2\left(r-D_0-\sigma^2/2\right)/\sigma^2} \varphi_1\left(\frac{B^2}{S}\right)$$

holds, then the solution $\overline{V}(S,t)$ must satisfy the condition $\overline{V}(B,t) = 0$.

As we know, the solution of the above problem can be expressed by

$$\overline{V}(S,t) = \mathrm{e}^{-r(T-t)} \int_0^\infty \overline{V}(S',T)\, G(S',T;S,t)\, dS',$$

where

$$\begin{aligned} &G(S',T;S,t) \\ &= \frac{1}{S'\sigma\sqrt{2\pi(T-t)}} \mathrm{e}^{-\left[\ln\left(S'/S\right)-\left(r-D_0-\sigma^2/2\right)(T-t)\right]^2/2\sigma^2(T-t)}. \end{aligned}$$

Therefore, $\overline{V}(B,t) = 0$ is equivalent to

$$\begin{aligned} &\int_0^\infty \overline{V}(S',T)\, G(S',T;B,t)\, dS' \\ &= \int_0^B \varphi_1(S')\, G(S',T;B,t)\, dS' + \int_B^\infty \varphi_2(S')\, G(S',T;B,t)\, dS' \\ &= \int_\infty^B \varphi_1\left(\frac{B^2}{S''}\right) G\left(B^2/S'',T;B,t\right)\left(-\frac{B^2}{S''^2}\right) dS'' \\ &\quad + \int_B^\infty \varphi_2(S')\, G(S',T;B,t)\, dS' \\ &= \int_B^\infty \left[\varphi_1\left(\frac{B^2}{S'}\right)\frac{B^2}{S'^2} G\left(B^2/S',T;B,t\right) + \varphi_2(S')\, G(S',T;B,t)\right] dS' \\ &= 0. \end{aligned}$$

Thus if

$$\varphi_1\left(\frac{B^2}{S'}\right)\frac{B^2}{S'^2} G\left(B^2/S',T;B,t\right) + \varphi_2(S')\, G(S',T;B,t) = 0$$

holds for $S' \in (B,\infty)$ and $t \in [0,T]$, then $\overline{V}(B,t) = 0$ for $t \in [0,T]$. Because

$$\begin{aligned} \frac{G\left(B^2/S',T;B,t\right)}{G(S',T;B,t)} &= \frac{\frac{S'}{B^2}\mathrm{e}^{-\left[\ln\left(B/S'\right)-\left(r-D_0-\sigma^2/2\right)(T-t)\right]^2/2\sigma^2(T-t)}}{\frac{1}{S'}\mathrm{e}^{-[\ln(S'/B)-(r-D_0-\sigma^2/2)(T-t)]^2/2\sigma^2(T-t)}} \\ &= \frac{S'^2}{B^2}\mathrm{e}^{4\ln\left(B/S'\right)\left(r-D_0-\sigma^2/2\right)(T-t)/2\sigma^2(T-t)} \\ &= \left(\frac{B}{S'}\right)^{2\left(r-D_0-\sigma^2/2\right)/\sigma^2-2} \end{aligned}$$

is a function of S', when φ_1 and φ_2 satisfy the relation

$$\varphi_2\left(S'\right)=-\left(\frac{B}{S'}\right)^{2\left(r-D_0-\sigma^2/2\right)/\sigma^2}\varphi_1\left(\frac{B^2}{S'}\right),$$

the condition $\overline{V}\left(B,t\right)=0$ holds. Let $S'=\dfrac{B^2}{S''}$, then this relation can also be rewritten as

$$\varphi_2\left(\frac{B^2}{S''}\right)=-\left(\frac{S''}{B}\right)^{2\left(r-D_0-\sigma^2/2\right)/\sigma^2}\varphi_1\left(S''\right),$$

or

$$\varphi_1\left(S'\right)=-\left(\frac{B}{S'}\right)^{2\left(r-D_0-\sigma^2/2\right)/\sigma^2}\varphi_2\left(\frac{B^2}{S'}\right).$$

Therefore we obtain our conclusion.

Now let us show that for the problem

$$\begin{cases}\dfrac{\partial V}{\partial t}+\dfrac{1}{2}\sigma^2S^2\dfrac{\partial^2V}{\partial S^2}+\left(r-D_0\right)S\dfrac{\partial V}{\partial S}-rV=0, & B_l\le S,\ t\le T,\\ V\left(S,T\right)=V_T\left(S\right), & B_l\le S,\\ V\left(B_l,t\right)=0, & t\le T,\end{cases}\tag{4.2}$$

the solution is

$$V\left(S,t\right)=\mathrm{e}^{-r(T-t)}\int_{B_l}^{\infty}V_T\left(S'\right)G_1\left(S',T;S,t,B_l\right)dS',$$

where

$$\begin{aligned}G_1\left(S',T;S,t,B_l\right)&=G\left(S',T;S,t\right)\\&\quad-\left(B_l/S\right)^{2(r-D_0-\sigma^2/2)/\sigma^2}G\left(S',T;B_l^2/S,t\right).\end{aligned}$$

Usually G_1 is called Green's function of down-and-out option problems.[1] Let us set $B=B_l$ and $\varphi_2(S)=V_T(S)$ in the problem (4.1). From the result above we know if

[1]Actually, $G_1\left(S',T;S,t,B_l\right)dS'$ is the probability of the price at time T being in $[S',S'+dS']$ with the lowest price during the time period $[t,T]$ being greater than B_l. Let us explain this fact. Consider all the paths of the price during the time period $[t,T]$ that start from S at time t. For any path that hits the lower barrier, the contribution to the option value is 0 because the option dies. Only those paths that never hit the lower barrier have contribution to the option value. A path that never hits the lower barrier $S=B_l$ during the time period $[t,T]$ is a path whose lowest price during the time period $[t,T]$ is greater than B_l. From the expression for $V(S,t)$, we see that the value of a down-and-out option is equal to the discounting factor times an integral of the product of the payoff function and $G_1\left(S',T;S,t,B_l\right)$ on $[B_l,\infty)$. Consequently, $G_1\left(S',T;S,t,B_l\right)dS'$ actually is the probability of the price at time T being in $[S',S'+dS']$ with the lowest price during the time period $[t,T]$ being greater than B_l. This fact will be used when we derive closed-form solutions for lookback options in Sect. 4.4.3.

$$\varphi_1(S) = -(B_l/S)^{2\left(r-D_0-\sigma^2/2\right)/\sigma^2} V_T\left(B_l^2/S\right),$$

then the solution of the problem on $[B_l, \infty)$ is the solution of the problem (4.2). Thus

$$\begin{aligned}
V(S,t) &= \mathrm{e}^{-r(T-t)}\left[-\int_0^{B_l} V_T\left(\frac{B_l^2}{S'}\right)\left(\frac{B_l}{S'}\right)^{2(r-D_0-\sigma^2/2)/\sigma^2}\right. \\
&\qquad\times G(S',T;S,t)dS' \\
&\qquad\left.+\int_{B_l}^{\infty} V_T(S')G(S',T;S,t)dS'\right] \\
&= \mathrm{e}^{-r(T-t)}\left[\int_{\infty}^{B_l} V_T(S'')\left(\frac{S''}{B_l}\right)^{2(r-D_0-\sigma^2/2)/\sigma^2}\right. \\
&\qquad\times G(B_l^2/S'',T;S,t)\frac{B_l^2}{S''^2}dS'' \\
&\qquad\left.+\int_{B_l}^{\infty} V_T(S')G(S',T;S,t)dS'\right] \\
&= \mathrm{e}^{-r(T-t)}\int_{B_l}^{\infty} V_T(S')\left[-\left(\frac{S'}{B_l}\right)^{2\left(r-D_0-\sigma^2/2\right)/\sigma^2-2}\right. \\
&\qquad\times\frac{G\left(B_l^2/S',T;S,t\right)}{G\left(S',T;B_l^2/S,t\right)}G\left(S',T;B_l^2/S,t\right) \\
&\qquad\left.+G(S',T;S,t)\right]dS' \\
&= \mathrm{e}^{-r(T-t)}\int_{B_l}^{\infty} V_T(S')\left[G(S',T;S,t)\right. \\
&\qquad\left.-\left(\frac{B_l}{S}\right)^{2(r-D_0-\sigma^2/2)/\sigma^2} G\left(S',T;B_l^2/S,t\right)\right]dS' \\
&= \mathrm{e}^{-r(T-t)}\int_{B_l}^{\infty} V_T(S')\,G_1(S',T;S,t,B_l)\,dS'.
\end{aligned}$$

Here we used the relation:

$$\begin{aligned}
\frac{G\left(B_l^2/S',T;S,t\right)}{G\left(S',T;B_l^2/S,t\right)} &= \frac{\frac{S'}{B_l^2}\mathrm{e}^{-\left[\ln\left(B_l^2/S'S\right)-\left(r-D_0-\sigma^2/2\right)(T-t)\right]^2/2\sigma^2(T-t)}}{\frac{1}{S'}\mathrm{e}^{-\left[\ln\left(S'S/B_l^2\right)-(r-D_0-\sigma^2/2)(T-t)\right]^2/2\sigma^2(T-t)}} \\
&= \frac{S'^2}{B_l^2}\mathrm{e}^{4\ln\left(B_l^2/S'S\right)\left(r-D_0-\sigma^2/2\right)(T-t)/2\sigma^2(T-t)} \\
&= \frac{S'^2}{B_l^2}\left(\frac{B_l^2}{S'S}\right)^{2\left(r-D_0-\sigma^2/2\right)/\sigma^2}.
\end{aligned}$$

Based on this result we know that if $B_l \le E$, then for $S \ge B_l$,

$$\begin{aligned}
c_o(S,t) &= \mathrm{e}^{-r(T-t)} \int_{B_l}^{\infty} \max(S'-E,0)\, G_1(S',T;S,t,B_l)\, dS' \\
&= \mathrm{e}^{-r(T-t)} \int_{0}^{\infty} \max(S'-E,0)\, G_1(S',T;S,t,B_l)\, dS' \\
&= \mathrm{e}^{-r(T-t)} \int_{0}^{\infty} \max(S'-E,0)\, G(S',T;S,t)\, dS' \\
&\quad - \left(\frac{B_l}{S}\right)^{2(r-D_0-\sigma^2/2)/\sigma^2} \mathrm{e}^{-r(T-t)} \\
&\quad \times \int_{0}^{\infty} \max(S'-E,0)\, G\left(S',T;B_l^2/S,t\right) dS' \\
&= c(S,t) - \left(\frac{B_l}{S}\right)^{2(r-D_0-\sigma^2/2)/\sigma^2} c\left(\frac{B_l^2}{S},t\right). \qquad (4.3)
\end{aligned}$$

The formula (4.3) is a closed-form solution for a down-and-out call option if $B_l \le E$. From this formula, we know that the price of a down-and-out call option is cheaper than the price of a vanilla call option. From the financial point of view, it is clear that a holder of a down-and-out call option has less rights than a holder of a vanilla call option and should pay less premium. However, if the price is always greater than B_l (which is what a holder of a call expects), then it is the same as a call. This is why a down-and-out call option is so attractive for many people. The method used to derive this formula is called the method of images because an "artificial" final condition for $S \in [0, B_l)$ is used, which is generated from the condition for $S > B_l$ and can be called an "image" of the condition for $S > B_l$.

Let us now consider a down-and-in European call option and let $c_i(S,t)$ stand for its value. The option value $c_i(S,t)$ satisfies the Black–Scholes equation for $S > B_l$, and all we need to do is to determine the correct final and boundary conditions. The down-and-in option expires worthless unless the asset price reaches the lower barrier B_l by expiry, i.e., if S has been greater than B_l right up to time T, then the option is not activated and expires worthless. Thus for $S > B_l$, the final condition is

$$c_i(S,T) = 0.$$

If the asset price S reaches B_l by expiry, then the option immediately turns into a vanilla call and must have the identical value as the vanilla call. The boundary condition is

$$c_i(B_l,t) = c(B_l,t).$$

Therefore, the fair value of a down-and-in option is the solution of the following final-boundary value problem:

$$\begin{cases} \dfrac{\partial c_i}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 c_i}{\partial S^2} + (r - D_0)\, S \dfrac{\partial c_i}{\partial S} - r c_i = 0, & S \geq B_l, \quad t \leq T, \\ c_i\,(S, T) = 0, & S \geq B_l, \\ c_i\,(B_l, t) = c\,(B_l, t)\,, & t \leq T. \end{cases}$$

Let

$$\bar{c}\,(S, t) = c\,(S, t) - c_i\,(S, t)\,.$$

Because both $c\,(S, t)$ and $c_i\,(S, t)$ satisfy the Black–Scholes equation, $\bar{c}\,(S, t)$ also satisfies the same equation. The final and boundary conditions for $\bar{c}\,(S, t)$ is

$$\bar{c}\,(S, T) = c\,(S, T) - c_i\,(S, T) = c\,(S, T) = \max\,(S - E, 0)\,,$$

and

$$\bar{c}\,(B_l, t) = c\,(B_l, t) - c_i\,(B_l, t) = 0.$$

Therefore, $\bar{c}\,(S, t)$ actually is $c_o(S, t)$. In other words, we have the identity:

$$c\,(S, t) = c_o\,(S, t) + c_i\,(S, t) \;\text{ for } S \geq B_l.$$

According to this identity and using the expression for the fair value of a down-and-out call, we have the fair value of a down-and-in option as follows:

$$c_i(S, t) = \left(\frac{B_l}{S}\right)^{2(r - D_0 - \sigma^2/2)/\sigma^2} c\left(\frac{B_l^2}{S}, t\right) \text{ for } S \geq B_l \text{ if } B_l \leq E. \tag{4.4}$$

Obviously, $c_o(S, t) = 0$ and $c_i(S, t) = c(S, t)$ for $S < B_l$. Therefore the identity

$$c\,(S, t) = c_o\,(S, t) + c_i\,(S, t)$$

still holds for $S < B_l$.

For a European up-and-out put option with $B_u > E$, the solution is similar to the formula (4.3). It can also be shown that the sum of a European up-and-out put option and a European up-and-in put option equals a European vanilla put option. These problems are left for readers as a part of Problem 1 and Problem 3.

Solutions in closed form can still be obtained for more complicated cases. For example, if $B_l = bE\mathrm{e}^{-\alpha(T-t)}$, where b and α are constants and $b \in [0, 1]$ and $\alpha \geq 0$, then such a solution for a down-and-out call with $D_0 = 0$ is given in the paper [63] by Merton and in the book [54] by Kwok. The solution for $D_0 \neq 0$ can still be obtained (see Problem 4).

If $B_l \geq E$, the closed-form solution for down-and-out call option can still be obtained and it is

$$\begin{aligned}c_o = {} & S\mathrm{e}^{-D_0(T-t)}N\left(\tilde{d}_1(B_l)\right) - E\mathrm{e}^{-r(T-t)}N\left(\tilde{d}_1(B_l) - \sigma\sqrt{T-t}\right) \\ & - (B_l/S)^{2(r-D_0-\sigma^2/2)/\sigma^2} \\ & \times \left[\frac{B_l^2}{S}\mathrm{e}^{-D_0(T-t)}N\left(\bar{d}_1(B_l)\right) - E\mathrm{e}^{-r(T-t)}N\left(\bar{d}_1(B_l) - \sigma\sqrt{T-t}\right)\right],\end{aligned}$$

where

$$\tilde{d}_1(B_l) = \left[\ln\frac{S\mathrm{e}^{(r-D_0)(T-t)}}{B_l} + \frac{1}{2}\sigma^2(T-t)\right] \bigg/ \left(\sigma\sqrt{T-t}\right),$$

$$\bar{d}_1(B_l) = \left[\ln\frac{B_l\mathrm{e}^{(r-D_0)(T-t)}}{S} + \frac{1}{2}\sigma^2(T-t)\right] \bigg/ \left(\sigma\sqrt{T-t}\right).$$

For an up-and-out put option with $B_u \le E$, the closed-form solution can also be obtained. Readers are asked to derive the closed-form formulae for both cases as an exercise. If $B_l \ge E$ or $B_u \le E$, the total price of the knock-out and knock-in options is still equal to the price of a vanilla option. Therefore, from these two formulae, we can easily obtain the closed-form solutions for a down-and-in call option with $B_l \ge E$ and for a up-and-in put option with $B_u \le E$.

4.2.3 Formulation of American Barrier Options

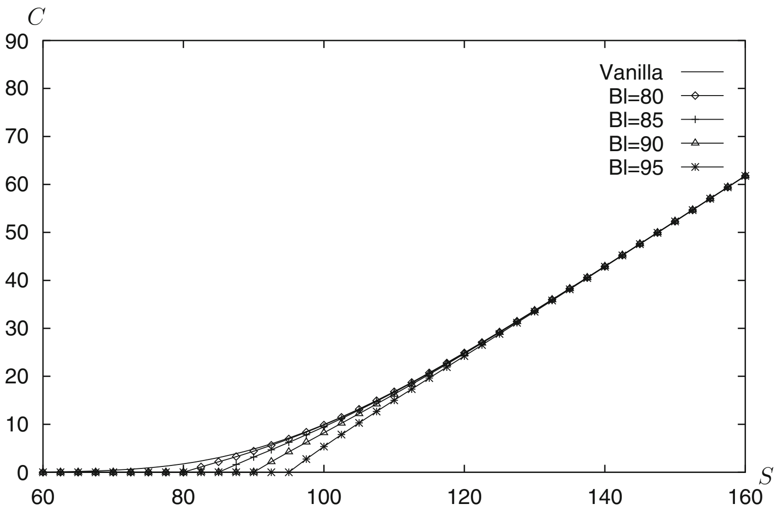

Fig. 4.1. Values of American down-and-out call options with $B_l = 0.8, 0.85, 0.9, 0.95$, and an American vanilla call option ($r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$, $T = 1$ year, and $E = 100$)

A barrier option could be an American one. As an example, here we give the formulation of an American down-and-out call option. Let $C_o(S,t)$ be its price. If $D_0 \neq 0$, then the American down-and-out call option problem with $B_l < E$ has a free boundary $S_f(t)$. The solution between the free boundary $S = S_f(t)$ and $S = B_l$ is determined by the following problem:

$$\begin{cases} \dfrac{\partial C_o}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 C_o}{\partial S^2} + (r - D_0)S\dfrac{\partial C_o}{\partial S} - rC_o = 0, & \\ & B_l \leq S \leq S_f(t), \quad t \leq T, \\ C_o(S,T) = \max(S - E, 0), & B_l \leq S \leq S_f(T), \\ C_o\left(B_l, t\right) = 0, & t \leq T, \\ C_o(S_f(t), t) = S_f(t) - E, & t \leq T, \\ \dfrac{\partial C_o}{\partial S}(S_f(t), t) = 1, & t \leq T, \\ S_f(T) = \max\left(E, \dfrac{rE}{D_0}\right). & \end{cases} \tag{4.5}$$

The only difference between the formulations of a vanilla call and a down-and-out call is that a boundary condition at $S = B_l$ is imposed on the solution of the down-and-out call option. The problem (4.5) will be referred to as the free-boundary problem for American down-and-out call options.

For American and some other complicated European options, numerical methods might become necessary. Some details on numerical methods for these cases can be found in Chaps. 8 and 9. In Fig. 4.1, the prices of American down-and-out call options with $B_l = 0.8, 0.85, 0.9$, and 0.95 and the American vanilla call option obtained by numerical methods are given. As we have pointed out in the case of European barrier call options, the price of a barrier option is cheaper than a vanilla option because the holder of a barrier option has less rights than a holder of a vanilla option. Clearly, this should still be true for the American case, and the greater B_l is, the lower the option price. Obviously, the vanilla call option actually is a down-and-out call option with the smallest $B_l(= 0)$ and it has the highest value. This fact has been confirmed by the figure. Indeed all these properties can be proved by using mathematical tools just like the case of vanilla call options. These proofs are left for readers to do as a part of Problem 2.

4.2.4 Parisian Options

A Parisian option is a barrier option with the feature that a knock-in or knock-out event only occurs when the price of the underlying asset has been above or below the barrier price for a prescribed continuous length of time if sampling is done continuously or for a prescribed number of contiguous samples if sampling is done discretely (see [20, 37, 80]). As we know, if the knock-in or knock-out event can be activated by one touch of the barrier, such an event can be triggered by manipulating the price of the underlying asset for a short time. Such a thing does not happen with a Parisian option.

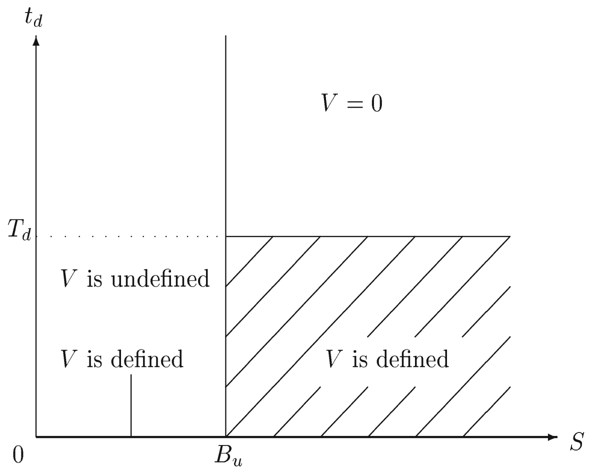

Fig. 4.2. The area of computation for Parisian up-and-out options

Let us consider European Parisian up-and-out options with an upper barrier $B_u > E$. Sampling is done continuously, and the prescribed length of time is T_d. Let t_d be the length of the time period of the stock price being continuously greater than or equal to B_u. It is clear that the option price depends on S, t_d, and t. Let $V(S, t_d, t)$ be the option price, and let time $t = 0$ represent today. We need to find the option price for $t \in [0, T]$. If $S < B_u$ at time t, then t_d must be 0 for that time. Therefore, $V(S, t_d, t)$ should have a value only for $t_d = 0$. Suppose that we have the stock price for $t \in [-T_d, 0]$, so t_d can be defined on $[0, T_d]$ for any $t \in [0, T]$. For $t \in [0, T]$ if $S \geq B_u$ and $t_d \geq T_d$, then $V(S, t_d, t)$ must be 0. Therefore, for any $t \in [0, T]$, we only need to find $V(S, t_d, t)$ on the interval $[0, B_u) \times [0, 0]$ and on the domain $[B_u, \infty) \times [0, T_d]$ (see Fig. 4.2). For any $S \in [0, B_u)$ and $t_d = 0$, $V(S, 0, t)$ satisfies the Black–Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0)\, S \frac{\partial V}{\partial S} - rV = 0, \quad 0 \leq S < B_u, \quad 0 \leq t \leq T.$$

If $S \in [B_u, \infty)$, then

$$t_d(t + dt) = t_d(t) + dt,$$

and

$$\frac{dt_d}{dt} = 1.$$

Therefore, according to the results in Sect. 2.3.2, $V(S, t_d, t)$ should satisfy

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0)\, S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t_d} - rV = 0,$$
$$B_u \leq S, \quad 0 \leq t_d \leq T_d, \quad 0 \leq t \leq T.$$

Putting these two cases together, we have for $t \in [0, T]$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0)\, S \frac{\partial V}{\partial S} + H(S - B_u) \frac{\partial V}{\partial t_d} - rV = 0,$$
$$0 \leq S, \quad t_d = 0 \quad \text{and} \quad B_u \leq S, \quad 0 < t_d \leq T_d,$$

where

$$H(x) = \begin{cases} 0, \; x < 0, \\ 1, \; 0 \leq x. \end{cases}$$

Now we consider the case of discrete sampling. Let sampling be done at $T_k = kT/K, k = 1, 2, \cdots, K$, and let the prescribed number of contiguous samples be $N < K$. In this case, for $S \geq B_u$

$$\frac{dt_d}{dt} = \frac{T}{K} \sum_{k=1}^{K} \delta(t - T_k).$$

Therefore, $V(S, t_d, t)$ should satisfy

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0)\, S \frac{\partial V}{\partial S} + H(S - B_u) \frac{T}{K} \sum_{k=1}^{K} \delta(t - T_k) \frac{\partial V}{\partial t_d} - rV = 0,$$
$$0 \leq S, \quad t_d = 0 \quad \text{and} \quad B_u \leq S, \quad 0 < t_d \leq T_d, \quad 0 \leq t \leq T,$$

where $T_d = NT/K$. This partial differential equation can also be rewritten as follows. For any $t \neq T_k, k = 1, 2, \cdots, K$,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0)\, S \frac{\partial V}{\partial S} - rV = 0,$$
$$0 \leq S, \quad t_d = 0 \quad \text{and} \quad B_u \leq S, \quad 0 < t_d \leq T_d$$

and at $t = T_k, k = 1, 2, \cdots$ or K,

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r - D_0)\, S \dfrac{\partial V}{\partial S} - rV = 0, \; 0 \leq S < B_u, \quad t_d = 0, \\ V(S, t_d, T_k^-) = V(S, t_d + T/K, T_k^+), \qquad B_u \leq S, \quad 0 \leq t_d \leq T_d. \end{cases}$$

Here, we have used the jump condition given in Sect. 2.5, and in order to have the value $V(S, T_d + T/k, T_k^+)$ with $S \geq B_u$, we need to use the fact $V(S, t_d, t) = 0$ for $S \geq B_u$ and $t_d \geq T_d$.

Suppose that at time t the price is B_u and t_d is not equal to 0 and could be very close to T_d. If at time $t + dt > t$, S becomes less than B_u, then t_d becomes 0. Thus, the situation $S = B_u$ and $t_d \in [0, T_d]$ can easily become the situation $S = B_u^-$ and $t_d = 0$. Consequently, we require

$$V(B_u, t_d, t) = V(B_u, 0, t), \quad t_d \in (0, T_d).$$

In order to determine the value of an option, we also need to give the value of the option at time T. This is related to the type of the option. As an example, we consider a European Parisian up-and-out call option. Let $c_p(S, t_d, t)$ be its price. For this option, the payoff function is

$$c_p(S, t_d, T) = \begin{cases} \max(S - E, 0), & 0 \leq S < B_u, \quad t_d = 0, \\ \max(S - E, 0) = S - E, & B_u \leq S, \quad 0 \leq t_d < T_d, \\ 0, & B_u \leq S, \quad t_d = T_d. \end{cases}$$

Consequently, when sampling is done continuously, $c_p(S, t_d, t)$ is the solution of the problem

$$\begin{cases} \dfrac{\partial c_p}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 c_p}{\partial S^2} + (r - D_0) S \dfrac{\partial c_p}{\partial S} + H(S - B_u)\dfrac{\partial c_p}{\partial t_d} - r c_p = 0, \\ \qquad 0 \leq S, \quad t_d = 0 \quad \text{and} \quad B_u \leq S, \quad 0 < t_d \leq T_d, \quad 0 \leq t \leq T, \\ c_p(S, t_d, T) = \begin{cases} \max(S - E, 0), & 0 \leq S < B_u, \quad t_d = 0, \\ S - E, & B_u \leq S, \quad 0 \leq t_d < T_d, \\ 0, & B_u \leq S, \quad t_d = T_d, \end{cases} \\ c_p(B_u, t_d, t) = c_p(B_u, 0, t), \quad t_d \in (0, T_d), \quad 0 \leq t \leq T, \\ c_p(S, T_d, t) = 0, \qquad B_u \leq S, \quad 0 \leq t \leq T; \end{cases} \tag{4.6}$$

and when sampling is done discretely, $c_p(S, t_d, t)$ satisfies

$$
\left\{
\begin{array}{l}
\dfrac{\partial c_p}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 c_p}{\partial S^2} + (r - D_0)\, S \dfrac{\partial c_p}{\partial S} \\
\qquad\qquad\qquad + H(S - B_u)\dfrac{T}{K}\sum\limits_{k=1}^{K} \delta(t - T_k)\dfrac{\partial c_p}{\partial t_d} - rc_p = 0, \\
\quad 0 \le S, \quad t_d = 0 \quad \text{and} \quad B_u \le S, \quad 0 < t_d \le T_d, \quad 0 \le t \le T, \\
c_p(S, t_d, T) = \begin{cases} \max(S - E, 0), & 0 \le S < B_u, \quad t_d = 0, \\ S - E, & B_u \le S, \quad 0 \le t_d < T_d, \\ 0, & B_u \le S, \quad t_d = T_d, \end{cases} \\
c_p(B_u, t_d, t) = c_p(B_u, 0, t), \quad t_d \in (0, T_d), \quad 0 \le t \le T, \\
c_p(S, T_d, t) = 0, \qquad\qquad B_u \le S, \quad 0 \le t \le T.
\end{array}
\right.
\tag{4.7}
$$

Table 4.1. Parisian up-and-out call option prices

($r = 0.1$, $D_0 = 0.05$, $\sigma = 0.25$, $E = 100$, $T = 0.5$, and $B_u = 150$)

$T_d \setminus S$	100	120	150
0	6.8669	12.1036	0
0.004	7.1806	13.9689	2.5134
0.02	7.3930	15.1938	4.8226
0.04	7.5759	16.6243	7.3770
0.08	7.7836	18.3608	11.3442

Solving the problems (4.6) or (4.7), we can have the price of Parisian up-and-out call options. In Table 4.1 (see [58]), the prices of some Parisian up-and-out call options are given. From there, we see that the larger the parameter T_d, the higher the Parisian up-and-out call option price. The financial reason is clear. The larger the parameter T_d, the less the chance of the event of "up-and-out," so the higher the option price.

4.3 Asian Options

4.3.1 Average Price, Average Strike, and Double Average Options

The Asian options are another type of popular path-dependent options. One of Asian call options can be used by a company to reduce its risk in frequently purchasing raw materials as pointed out in Sect. 4.1. It can also reduce the risk in frequently selling foreign currency through buying a put whose payoff depends on the difference between the exercise price and the average exchange rate. If the average exchange rate drops, the company can get some compensation from the option for the loss in frequently selling foreign currency. In practice, the asset price might be manipulated by some groups so that the asset price can be at a certain level desired by the groups for a short period. An Asian option may also protect option holders from the kind of asset price manipulation that occurs, especially near the end of the option's life.

Asian options can be divided into three types: average price, average strike, and double average options, where average price options are also called average rate options if the underlying asset is an exchange rate. Let A be some type of average price, which depends on the path of the price. The payoff of an average price option is a function of $A-E$, i.e., A is in the position of S. Sometimes, it is also called an average value option. Here, the strike price is fixed. Therefore, it is called a fixed strike Asian option as well. Let $c_{pr}(S,A,t)$ and $p_{pr}(S,A,t)$ denote the prices of average price call and put options, respectively. For an average price call option, its payoff is

$$c_{pr}(S,A,T) = \max(A-E,0), \tag{4.8}$$

whereas for an average price put option, the payoff is

$$p_{pr}(S,A,T) = \max(E-A,0). \tag{4.9}$$

In an average strike option, a payoff function depends on $\alpha S - A$ instead of $S-E$, where $\alpha \approx 1$. Thus, it is also called a floating strike option. Let $c_{st}(S,A,t)$ and $p_{st}(S,A,t)$ denote the prices of average strike call and put options, respectively. It is clear that the payoff of an average strike call option is

$$c_{st}(S,A,T) = \max(\alpha S - A,0) \tag{4.10}$$

and the payoff of an average strike put option is

$$p_{st}(S,A,T) = \max(A-\alpha S,0). \tag{4.11}$$

A double average option has a payoff function of $A-A_1$, where A is an average over one period $[T_s,T_e]$ and A_1 is an average over another period $[T_{s_1},T_{e_1}]$. Here, we assume $T_e \le T_{s_1}$. In what follows, a double average call option is referred to as an option with a payoff

$$\max(A_1-A,0) \tag{4.12}$$

and a double average put option is an option with a payoff

$$\max(A-A_1,0). \tag{4.13}$$

4.3.2 Continuously and Discretely Sampled Arithmetic Averages

Sampling for an arithmetic average may be done either continuously or discretely. Suppose $S(t)$ is the asset price at time t. A continuously sampled average over $[T_s,t] \subset [T_s,T_e]$ is given by

$$A = \frac{1}{t - T_s} \int_{T_s}^{t} S(\tau)\, d\tau.$$

A discretely sampled average over $t_1, t_2, \cdots, t_k$ is

$$A = \frac{1}{k} \sum_{i=1}^{k} S(t_i).$$

In the following, we assume that over $[T_s, T_e]$ the price is sampled K times at $t_1, t_2, \cdots, t_K$, where $T_s = t_1 < t_2 < \cdots < t_K = T_e$. Assume $t_k \le t < t_{k+1}$. Then, the discretely sampled average over $t_1, t_2, \cdots, t_k$ can be rewritten as an average over $[T_s, t] \subset [T_s, T_e]$ with a weight function

$$\begin{aligned} A &= \frac{1}{\displaystyle\int_{T_s}^{t} \sum_{i=1}^{k} \delta(\tau - t_i)\, d\tau} \int_{T_s}^{t} S(\tau) \sum_{i=1}^{k} \delta(\tau - t_i)\, d\tau, \\ &= \frac{1}{\displaystyle\int_{T_s}^{t} \sum_{i=1}^{K} \delta(\tau - t_i)\, d\tau} \int_{T_s}^{t} S(\tau) \sum_{i=1}^{K} \delta(\tau - t_i)\, d\tau, \end{aligned}$$

where $\delta(t)$ is the Dirac delta function defined in Sect. 2.2.2. Therefore, an average price over $[T_s, t] \subset [T_s, T_e]$ can be written as

$$A = \frac{1}{\displaystyle\int_{T_s}^{t} f(\tau)\, d\tau} \int_{T_s}^{t} S(\tau) f(\tau)\, d\tau,$$

where

$$f(\tau) = \begin{cases} 1, & \text{if sampled continuously} \\ \displaystyle\sum_{i=1}^{K} \delta(\tau - t_i), & \text{if sampled at } t_1, t_2, \cdots, t_K. \end{cases}$$

Here, A and f are defined on $[T_s, T_e]$. We can extend the domain of A from $[T_s, T_e]$ to $[0, T] \supset [T_s, T_e]$ by defining

$$A = \frac{1}{\displaystyle\int_{0}^{t} f(\tau)\, d\tau} \int_{0}^{t} S(\tau) f(\tau)\, d\tau, \tag{4.14}$$

where

$$f(\tau) = \begin{cases} 0, & \text{if } \tau \notin [T_s, T_e], \\ 1, & \text{if } \tau \in [T_s, T_e] \text{ and sampled continuously,} \\ \displaystyle\sum_{i=1}^{K} \delta(\tau - t_i), & \text{if sampled at } t_1, t_2, \cdots, t_K. \end{cases}$$

If we give another interval $[T_{s_1}, T_{e_1}]$ or K_1 specified times satisfying $T_{s_1} = t_{11} < t_{12} < \cdots < t_{1K_1} = T_{e_1}$, then we can define another function whose domain is $[0, T]$:

$$A_1 = \frac{1}{\displaystyle\int_0^t f_1(\tau)\, d\tau} \int_0^t S(\tau) f_1(\tau)\, d\tau, \tag{4.15}$$

where

$$f_1(\tau) = \begin{cases} 0, & \text{if } \tau \notin [T_{s_1}, T_{e_1}], \\ 1, & \text{if } \tau \in [T_{s_1}, T_{e_1}] \text{ and sampled continuously,} \\ \sum_{i=1}^{K_1} \delta(\tau - t_{1i}), & \text{if sampled at } t_{11}, t_{12}, \cdots, t_{1K_1}. \end{cases}$$

Let us define I and I_1 as follows:

$$I = \frac{1}{\displaystyle\int_0^T f(\tau)\, d\tau} \int_0^t S(\tau) f(\tau)\, d\tau \tag{4.16}$$

and

$$I_1 = \frac{1}{\displaystyle\int_0^T f_1(\tau)\, d\tau} \int_0^t S(\tau) f_1(\tau)\, d\tau. \tag{4.17}$$

Because

$$A = I \cdot \frac{\displaystyle\int_0^T f(\tau)\, d\tau}{\displaystyle\int_0^t f(\tau)\, d\tau},$$

$$A_1 = I_1 \cdot \frac{\displaystyle\int_0^T f_1(\tau)\, d\tau}{\displaystyle\int_0^t f_1(\tau)\, d\tau}$$

and

$$\frac{\displaystyle\int_0^T f(\tau)\, d\tau}{\displaystyle\int_0^t f(\tau)\, d\tau} \quad \text{and} \quad \frac{\displaystyle\int_0^T f_1(\tau)\, d\tau}{\displaystyle\int_0^t f_1(\tau)\, d\tau}$$

are given functions of t, we can replace A by I or A_1 by I_1 as an independent variable. Furthermore, we will discover in the next two subsections that for some cases, it is more convenient to use I or $D_I = I - I_1$ as an independent variable.

4.3.3 Partial Differential Equations for Asian Options

Consider an option whose value V depends on three independent variables, say, S, Y, t. Suppose S satisfies

$$dS = \mu S dt + \sigma S dX \tag{4.18}$$

and for Y we have

$$dY = g\left(S, Y, t\right) dt. \tag{4.19}$$

From Sect. 2.3.2, we know that $V(S, Y, t)$ satisfies

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \left(r - D_0\right) S \frac{\partial V}{\partial S} + g \frac{\partial V}{\partial Y} - rV = 0. \tag{4.20}$$

If we take

$$Y = A, \quad I \quad \text{or} \quad D_I = I - I_1,$$

from the definitions (4.14), (4.16), and (4.17) we have

$$g = \begin{cases} \dfrac{S - A}{\int_0^t f\left(\tau\right) d\tau} f\left(t\right), & \text{if} \quad Y = A, \\ \dfrac{1}{\int_0^T f\left(\tau\right) d\tau} S f\left(t\right), & \text{if} \quad Y = I, \\ \dfrac{1}{\int_0^T f\left(\tau\right) d\tau} S f\left(t\right) - \dfrac{1}{\int_0^T f_1\left(\tau\right) d\tau} S f_1\left(t\right), & \text{if} \quad Y = D_I. \end{cases}$$

Therefore for these cases, we can have a partial differential equation that involves A, I or D_I besides S and t. For example, if the independent variables are S, A, t, and the average is measured continuously, then $V\left(S, A, t\right)$ satisfies

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + \left(r - D_0\right) S \dfrac{\partial V}{\partial S} + \dfrac{S - A}{t - T_s} \dfrac{\partial V}{\partial A} - rV = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad t \in \left[T_s, T_e\right], \\ \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + \left(r - D_0\right) S \dfrac{\partial V}{\partial S} - rV = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad t \notin \left[T_s, T_e\right]; \end{cases} \tag{4.21}$$

If the independent variables are S, I, t, and the average is measured discretely, then $V\left(S, I, t\right)$ satisfies

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \left(r - D_0\right) S \frac{\partial V}{\partial S} + \frac{S}{K} \sum_{i=1}^{K} \delta\left(t - t_i\right) \frac{\partial V}{\partial I} - rV = 0; \tag{4.22}$$

and if the independent variables are S, D_I, t, and the average is measured discretely, then $V(S, D_I, t)$ satisfies

$$\begin{aligned}&\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+(r-D_0)\,S\frac{\partial V}{\partial S}\\+S&\left[\frac{1}{K}\sum_{i=1}^{K}\delta(t-t_i)-\frac{1}{K_1}\sum_{i=1}^{K_1}\delta(t-t_{1i})\right]\frac{\partial V}{\partial D_I}-rV=0.\end{aligned}\qquad(4.23)$$

For average price and average strike options, payoff functions at time T are in the form of $f(S, A)$ or $f(S, I)$, so for a European option we can use Eq. (4.21) or Eq. (4.22) and the payoff function to determine the value of the option. For double average options, payoff functions at time T are in the form of $f(A-A_1) = f(I-I_1) = f(D_I)$, so for a European option we can use Eq. (4.23) and the payoff to compute the option price.

4.3.4 Reducing to One-Dimensional Problems

In the problems mentioned above, there are three independent variables. Usually they are called two-dimensional problems as besides t, the values depend on two independent variables. In many cases, such an Asian option problem can be reduced to a one-dimensional problem, i.e., for a fixed t, the solution depends only on one independent variable. If a problem can be reduced to a one-dimensional problem, the amount of computation needed to obtain numerical solutions will be greatly decreased.

Let us consider an average strike call option. For this case, the payoff function is $\max(\alpha S - A, 0)$. Let

$$\eta = \frac{A}{S} \quad \text{and} \quad W = \frac{V}{S},$$

where the function W actually is the option value in units of the stock price, and consider the case of continuous sampling. Suppose W is a function of η, t. Because

$$\begin{cases}\dfrac{\partial V}{\partial t}=S\dfrac{\partial W}{\partial t},\\[2mm] \dfrac{\partial V}{\partial S}=S\dfrac{\partial W}{\partial \eta}\dfrac{\partial \eta}{\partial S}+W=W-\eta\dfrac{\partial W}{\partial \eta},\\[2mm] \dfrac{\partial^2 V}{\partial S^2}=\dfrac{\eta^2}{S}\dfrac{\partial^2 W}{\partial \eta^2},\\[2mm] \dfrac{\partial V}{\partial A}=\dfrac{\partial W}{\partial \eta},\end{cases}\qquad(4.24)$$

from Eq. (4.21) we know that W satisfies

$$\begin{cases} \dfrac{\partial W}{\partial t}+\dfrac{1}{2}\sigma^2\eta^2\dfrac{\partial^2 W}{\partial \eta^2}+\left[\left(D_0-r\right)\eta+\dfrac{1-\eta}{t-T_s}\right]\dfrac{\partial W}{\partial \eta}-D_0W=0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } t\in\left[T_s,T_e\right], \\ \dfrac{\partial W}{\partial t}+\dfrac{1}{2}\sigma^2\eta^2\dfrac{\partial^2 W}{\partial \eta^2}+\left(D_0-r\right)\eta\dfrac{\partial W}{\partial \eta}-D_0W=0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } t\notin\left[T_s,T_e\right] \end{cases} \tag{4.25}$$

and from the payoff we have

$$W\left(\eta,T\right)=\max\left(\alpha-\eta,0\right). \tag{4.26}$$

Equation (4.25) with the final condition (4.26) has a solution, so for a European average strike call option, we only need to solve a one-dimensional problem. For a European average strike put option, the only difference is the payoff.

In the case of an average being measured discretely and $T_s=0$, the procedure above can still be used to reduce an original average strike option to a one-dimensional problem (see Problem 6).

For an average price call option, the payoff is

$$\max\left(A-E,0\right).$$

We cannot use the transformation mentioned above to reduce the problem into a one-dimensional problem. However, if we consider the fact that the value of the option is a function of S, I, t, and use the transformations

$$\eta=\frac{I-E}{S},\quad W=\frac{V}{S},$$

then for such a European option, we still only need to solve a one-dimensional problem in order to get its price. Here, taking a discretely sampled average price call option as an example, we explain the situation. Let $V=V(S,I,t)$, then V is the solution of the problem

$$\begin{cases} \dfrac{\partial V}{\partial t}+\dfrac{1}{2}\sigma^2S^2\dfrac{\partial^2 V}{\partial S^2}+\left(r-D_0\right)S\dfrac{\partial V}{\partial S}+\dfrac{S}{K}\sum\limits_{i=1}^{K}\delta\left(t-t_i\right)\dfrac{\partial V}{\partial I}-rV=0, \\ \qquad\qquad 0\le S<\infty,\quad 0\le I<\infty,\quad t\le T, \\ V\left(S,I,T\right)=\max\left(A-E,0\right)=\max\left(I-E,0\right), \\ \qquad\qquad 0\le S<\infty,\quad 0\le I<\infty. \end{cases}$$

Let $\eta = \dfrac{I-E}{S}$, $W = \dfrac{V}{S}$. In this case, the first three relations in the set of relations (4.24) are still true and

$$\frac{\partial V}{\partial I} = \frac{\partial W}{\partial \eta}.$$

Furthermore, we have

$$W(\eta, T) = \max(\eta, 0).$$

Therefore, $W(\eta, t)$ satisfies

$$\begin{cases} \dfrac{\partial W}{\partial t} + \dfrac{1}{2}\sigma^2\eta^2\dfrac{\partial^2 W}{\partial \eta^2} + \left[(D_0 - r)\eta + \dfrac{1}{K}\sum\limits_{i=1}^{K}\delta(t-t_i)\right]\dfrac{\partial W}{\partial \eta} \\ -D_0 W = 0, \qquad -\infty < \eta < \infty, \quad t \le T, \\ W(\eta, T) = \max(\eta, 0), \quad -\infty < \eta < \infty. \end{cases} \tag{4.27}$$

That is, this problem can be reduced to a one-dimensional problem. For such a put option or for such a continuously sampled option, the situation is similar.

For a European double average option, we can assume that the value of such an option is a function of S, D_I, t. In this case, we use the transformations

$$\eta = \frac{D_I}{S} \quad \text{and} \quad W = \frac{V}{S},$$

then determining the option price can be reduced to solving a one-dimensional problem, and W satisfies

$$\begin{cases} \dfrac{\partial W}{\partial t} + \dfrac{1}{2}\sigma^2\eta^2\dfrac{\partial^2 W}{\partial \eta^2} + \left[(D_0 - r)\eta + \dfrac{1}{K}\sum\limits_{i=1}^{K}\delta(t-t_i)\right. \\ \qquad \left. -\dfrac{1}{K_1}\sum\limits_{i=1}^{K_1}\delta(t-t_{1i})\right]\dfrac{\partial W}{\partial \eta} - D_0 W = 0, \\ \qquad -\infty < \eta < \infty, \quad t \le T, \\ W(\eta, T) = \max(\eta, 0) \ \text{or} \ \max(-\eta, 0), \quad -\infty < \eta < \infty. \end{cases} \tag{4.28}$$

Reducing an Asian option problem to a one-dimensional problem can be done in other ways. For example, see [46, 84, 68, 54, 3, 80].

4.3.5 Jump Conditions

As we can see from Sect. 4.3.4, when the average is measured discretely, at the time a sample is taken, i.e., at $t = t_i \in \mathcal{T} \equiv \{t_1, t_2, \cdots, t_K\}$ or $t = t_{1i} \in \mathcal{T}_1 \equiv \{t_{11}, t_{12}, \cdots, t_{1K_1}\}$, W satisfies

$$\frac{\partial W}{\partial t} + \frac{1}{K}\delta\left(t - t_i\right)\frac{\partial W}{\partial \eta} = 0$$

or

$$\frac{\partial W}{\partial t} - \frac{1}{K_1}\delta\left(t - t_{i1}\right)\frac{\partial W}{\partial \eta} = 0$$

respectively, as the other terms in the equations can be neglected in this case. From Sect. 2.5.2, we know that W fulfills the relation

$$W\left(\eta, t_i^-\right) = W\left(\eta + \frac{1}{K}, t_i^+\right) \tag{4.29}$$

for $t_i \in \mathcal{T}$ and

$$W\left(\eta, t_{1i}^-\right) = W\left(\eta - \frac{1}{K_1}, t_{1i}^+\right) \tag{4.30}$$

for $t_{1i} \in \mathcal{T}_1$. These relations will be referred to as the jump conditions for Asian options with a discrete average and be used when W is determined if a discretely sampled arithmetic average is adopted.

4.3.6 American Asian Options

Some Asian options could be American style. Let us consider an American average strike call option with continuous sampling. Suppose $[T_s, T_e] = [0, T]$. In this case, $V(S, A, t)$ needs to satisfy the following constraint on American average strike call options:

$$V\left(S, A, t\right) \geq \max\left(\alpha S - A, 0\right) \quad \text{for} \quad t \in [0, T],$$

which is equivalent to

$$W\left(\eta, t\right) \geq \max\left(\alpha - \eta, 0\right) \quad \text{for} \quad t \in [0, T].$$

Thus, for an American average strike call option, we still only need to solve a one-dimensional problem. From Eq. (4.25), when we define

$$\mathbf{L}_{a,t} = \frac{1}{2}\sigma^2\eta^2\frac{\partial^2}{\partial\eta^2} + \left[(D_0 - r)\eta + \frac{1-\eta}{t}\right]\frac{\partial}{\partial\eta} - D_0,$$

this American option should be the solution of the following linear complementarity problem:

$$\begin{cases} \min\left(-\dfrac{\partial W}{\partial t} - \mathbf{L}_{a,t}W,\ W\left(\eta, t\right) - \max\left(\alpha - \eta, 0\right)\right) = 0, & 0 \leq \eta,\ t \leq T, \\ W\left(\eta, T\right) = \max\left(\alpha - \eta, 0\right) & 0 \leq \eta. \end{cases}$$

Now let us reformulate this problem as a free-boundary problem. In order to do this, we need to find out how many free boundaries it has and where they are located at time T. Theorem 3.1 in Sect. 3.1 tells us that these locations are the boundaries between the regions where

$$\left(\frac{\partial}{\partial t} + \mathbf{L}_{a,T}\right) \max(\alpha - \eta, 0) \geq 0$$

and the regions where

$$\left(\frac{\partial}{\partial t} + \mathbf{L}_{a,T}\right) \max(\alpha - \eta, 0) < 0.$$

First, we consider the case $\eta > \alpha$. Because $\max(\alpha - \eta, 0) = 0$, we have

$$\left(\frac{\partial}{\partial t} + \mathbf{L}_{a,T}\right) \max(\alpha - \eta, 0) = 0.$$

Now let us look at the case $\eta < \alpha$. In this case,

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \mathbf{L}_{a,T}\right) \max(\alpha - \eta, 0) &= -\left[(D_0 - r)\,\eta + \frac{1-\eta}{T}\right] - D_0(\alpha - \eta)\\ &= -\alpha D_0 + r\eta - \frac{1-\eta}{T}.\end{aligned}$$

The inequality

$$-\alpha D_0 + r\eta - \frac{1-\eta}{T} > 0$$

is equivalent to

$$\eta > \frac{1 + \alpha D_0 T}{1 + rT}.$$

Thus, when $\frac{1 + \alpha D_0 T}{1 + rT} < \alpha$, there exists an interval $\left(\frac{1 + \alpha D_0 T}{1 + rT}, \alpha\right)$ where $\left(\frac{\partial}{\partial t} + \mathbf{L}_{a,T}\right) \max(\alpha - \eta, 0) > 0$. Therefore, if $\eta > \min\left(\alpha, \frac{1 + \alpha D_0 T}{1 + rT}\right)$, then

$$\left(\frac{\partial}{\partial t} + \mathbf{L}_{a,T}\right) \max(\alpha - \eta, 0) \geq 0$$

and we can determine the solution by the partial differential equation; and if $\eta < \min\left(\alpha, \frac{1 + \alpha D_0 T}{1 + rT}\right)$, then

$$\left(\frac{\partial}{\partial t} + \mathbf{L}_{a,T}\right) \max(\alpha - \eta, 0) < 0$$

and the solution should be equal to $\alpha-\eta$. Consequently, there is only one free boundary at time T, the location of the free boundary is $\min\left(\alpha,\ \dfrac{1+\alpha D_0 T}{1+rT}\right)$, and in the region $\eta \geq \min\left(\alpha,\ \dfrac{1+\alpha D_0 T}{1+rT}\right)$, we need to solve the partial differential equation. Let $\eta_f(t)$ be the location function of the free boundary. Then

$$\eta_f(T)=\min\left(\alpha,\ \frac{1+\alpha D_0 T}{1+rT}\right).$$

At the free boundary, the solution of the option and its derivative should be continuous, i.e.,

$$\begin{cases} W(\eta_f,t)=\alpha-\eta_f, \\ \dfrac{\partial}{\partial\eta}W(\eta_f,t)=-1 \end{cases}$$

should hold. Therefore, if

$$\eta \geq \eta_f(t),$$

then $W(\eta,t)$ should be the solution of the free-boundary problem

$$\begin{cases} \dfrac{\partial W}{\partial t}+\dfrac{1}{2}\sigma^2\eta^2\dfrac{\partial^2 W}{\partial \eta^2}+\left[(D_0-r)\eta+\dfrac{1-\eta}{t}\right]\dfrac{\partial W}{\partial\eta}-D_0 W=0, \\ \qquad\qquad\qquad\qquad \eta_f(t)\leq\eta, \quad t\leq T, \\ W(\eta,T)=\max(\alpha-\eta,0), \qquad \eta_f(T)\leq\eta, \\ W(\eta_f,t)=\alpha-\eta_f, \qquad t\leq T, \\ \dfrac{\partial W}{\partial\eta}(\eta_f,t)=-1, \qquad t\leq T, \\ \eta_f(T)=\min\left(\alpha,\ \dfrac{1+\alpha D_0 T}{1+rT}\right); \end{cases} \tag{4.31}$$

while if

$$0\leq\eta<\eta_f(t),$$

then

$$W(\eta,t)=\alpha-\eta.$$

This problem will be referred to as the free-boundary problem for American average strike call options. For an American average strike put option, the two-dimensional problem can also be reduced into a one-dimensional problem. Furthermore the one-dimensional linear complementarity problem can be converted into a free-boundary problem. We leave this as Problem 7 for readers to derive.

For an American average price call option, $V(S, I, t)$ satisfies

$$V(S, I, t) \geq \max(A - E, 0)$$

or

$$\frac{V(S, I, t)}{S} \geq \frac{\max(A - E, 0)}{S}.$$

The right-hand side cannot be a function of $\eta = (I - E)/S$ and t. Therefore, for an American average price call option, we cannot use the method described in Sect. 4.3.4 for reducing the problem to a one-dimensional problem. For an American average rate put option, the situation is the same. This means that it is necessary to solve a two-dimensional problem in these cases.

4.3.7 Some Examples

According to the equations, final conditions, boundary conditions, and jump conditions given in Sect. 4.3.4–4.3.6, and using the numerical methods described in Chaps. 8 and 9, we can obtain $W(\eta, t)$ numerically and furthermore find the option price by $V = SW$. In what follows, we give some results in the forms of tables and figures.

Table 4.2. Average strike put option prices for various α

($r = 0.05$, $D_0 = 0$, $\sigma = 0.2$, $S = 100$, $T = 1$, $T_s = 0.1$, $T_e = 1.0$, $K = 10$, and the payoff $= \max(A - \alpha S, 0)$)

Parameter α	"Exact" solution
0.900	8.981655
0.925	7.175189
0.950	5.599918
0.975	4.267895
1.000	3.176202
1.025	2.308797
1.050	1.640145
1.075	1.139517
1.100	0.774976

In Table 4.2, we list the prices of European average strike put options with various α. The other parameters are $r = 0.05$, $D_0 = 0$, $\sigma = 0.2$, $S = 100$, and $T = 1$. The number of discrete samplings K is 10 and $t_k = k/K$, $k = 1, 2, \cdots, K$. From Table 4.2, we see that the price decreases as α increases.

Table 4.3. Average price call option prices for various strike prices

($r = 0.05$, $D_0 = 0$, $\sigma = 0.2$, $S = 100$, $T = 1$, $T_s = 0.1$, $T_e = 1.0$, $K = 10$, and the payoff $= \max(A - E, 0)$)

Strike price E	"Exact" solution
90.0	12.985323
92.5	11.050426
95.0	9.269009
97.5	7.659745
100.0	6.234515
102.5	4.997539
105.0	3.945496
107.5	3.068492
110.0	2.351591

The reason for this fact is clear: when α increases, the money the holder of the option gets at expiry, the payoff $\max(A - \alpha S)$, decreases or does not change.

Table 4.3 shows the results of European average price call options with various strike prices. There, the parameters are the same as those in Table 4.2. From Table 4.3, we see that the option price is a decreasing function of the strike price E. The reason is as follows. When E increases, the money its holder gets at expiry, the payoff $\max(A - E, 0)$, decreases or does not change.

European double average call option prices for various sampling intervals Δt are listed in Table 4.4. For a given $\Delta t = 0.1/2^{n-1}$, n being a positive integer, $t_k = k\Delta t$, $k = 1, 2, \cdots, 10 \times 2^{n-2}$, and $t_{1k_1} = 0.5 + k_1 \Delta t$, $k_1 = 1, 2, \cdots, 10 \times 2^{n-2}$. The data show that when the interval goes to 0, the price tends to $5.813\cdots$. As we know, when the interval goes to 0, discrete sampling becomes continuous sampling. Therefore, the limit should be the price of the option with continuous sampling. From the data, we also see that if the length of interval is less than 0.025 (6.25 business days), then the difference of the prices between an option with discrete sampling and the option with continuous sampling is 10^{-2}. Therefore, if such an error is acceptable, we can use a continuous model to replace a discrete sampling with an interval less than 0.025. Usually, the CPU time needed for continuous sampling is less than that needed for a discrete model with a small Δt, so such a replacement can save CPU time needed. Finally, we would like to point out that in these tables, "exact" solution means that the error of these values is about 10^{-6}.

In Fig. 4.3, for an American average strike put option, the value of $W(\eta, t)$ as a function of η is given. The price of the option is $V(S, A, t) = SW(A/S, t)$. Thus if $t > 0$, then S and A need to be given, whereas if $t = 0$, then $A = S$ and only S needs to be given in order to find the price from the figure.

Table 4.4. Double average call option prices with various sampling intervals Δt

($r = 0.05$, $D_0 = 0$, $\sigma = 0.2$, $S = 100$, $T = 1$,
$T_s = \Delta t$, $T_e = 0.5$, $T_{s_1} = 0.5 + \Delta t$, $T_{e_1} = 1$,
and the payoff $= \max(A_1 - A, 0)$)

Sampling interval Δt (in years)	"Exact" solution
0.1	5.872133
0.05	5.831998
0.025	5.820122
0.0125	5.816244
0.00625	5.814820
0.003125	5.814237
0.0015625	5.813978

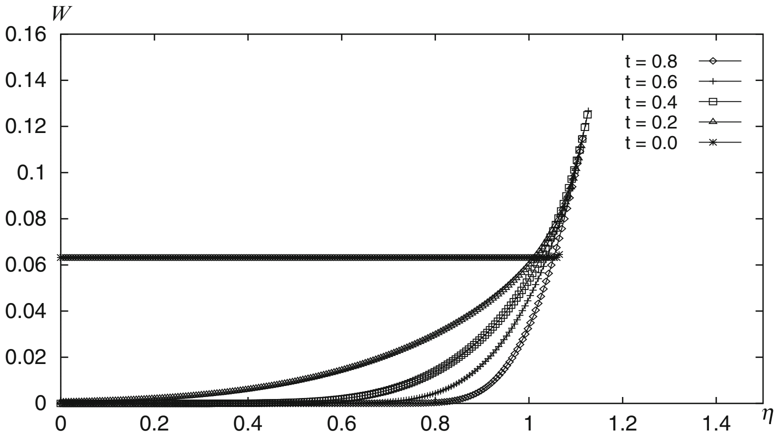

Fig. 4.3. $W(\eta, t)$ of an American average strike put option ($r = 0.1$, $D_0 = 0.1$, $\sigma = 0.2$, and $\alpha = 1$)

4.4 Lookback Options

4.4.1 Equations for Lookback Options

Sometimes the payoff of a derivative product depends on the maximum or minimum realized asset price over the life of the option. Such an option is called a lookback option. If the strike price in the payoff depends on the maximum or minimum, then the lookback option is called a lookback strike option or a lookback option with a floating strike; whereas if the stock price in the payoff is replaced by the maximum or minimum, then it is called a lookback price option or a lookback option with fixed strike. When the underlying asset is an exchange rate, a lookback price option is also called a lookback rate option. If the payoff is

$$\max(\alpha S - L, 0),$$

where L is the minimum realized price and α is a constant satisfying

$$0 < \alpha \leq 1,$$

then the option is called a lookback strike call option; whereas if the payoff is

$$\max(H - \beta S, 0),$$

where H is the maximum realized price and β is a constant satisfying

$$1 \leq \beta,$$

then the option is called a lookback strike put option. Similarly, lookback price call and put options have the payoffs

$$\max(H - E, 0)$$

and

$$\max(E - L, 0)$$

respectively. A person who holds a European lookback strike call option with $\alpha = 1$ can buy the underlying asset at the lowest realized price at expiration time, and an investor who holds a European lookback strike put option with $\beta = 1$ can sell the asset at the highest realized price. These attractive features are the reason why there exist such options on the market.

As for Asian options, the maximum or minimum realized asset price may be measured continuously, or more commonly, discretely. If it is measured continuously, then

$$L(t) = \min_{0 \leq \tau \leq t} S(\tau)$$

and

$$H(t) = \max_{0 \leq \tau \leq t} S(\tau).$$

If the sampling is done discretely and the sampling times are $t_1, t_2, \cdots, t_K$, where

$$0 \leq t_1 < t_2 < \cdots < t_K \leq T,$$

then

$$L(t) = \min\left(S(t_1), \cdots, S\left(t_{i^*(t)}\right)\right)$$

and

$$H(t) = \max\left(S(t_1), \cdots, S\left(t_{i^*(t)}\right)\right),$$

where $i^*(t)$ is the number of samplings before time t.

We see that the value V of such an option depends on not only S and t but also L or H, i.e., $V = V(S, L, t)$ or $V(S, H, t)$. In what follows, let us find out the concrete equation for the price of a lookback strike put option or a lookback price call option. For such an option, $V = V(S, H, t)$. First, suppose discrete sampling is taken. In this case

$$dH(t) = \begin{cases} 0, & \text{if } t \neq t_i,\ i = 1, 2, \cdots, K, \\ \max\left(S(t_i), H\left(t_i^-\right)\right) - H\left(t_i^-\right), & \text{if } t = t_i,\ i = 1, 2, \cdots, \text{ or } K \end{cases}$$

and

$$\frac{dH(t)}{dt} = \sum_{i=1}^{K} \left[\max\left(S(t), H\left(t^-\right)\right) - H\left(t^-\right)\right] \delta(t - t_i),$$

where

$$H\left(t^-\right) = \lim_{\varepsilon \to 0} H(t - \varepsilon)$$

with $\varepsilon > 0$.

Suppose dS satisfies Eq. (4.18). According to the results given in Sect. 2.3.2, $V(S, H, t)$ satisfies Eq. (4.20) with $g = \frac{dH}{dt}$, i.e.,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0) S \frac{\partial V}{\partial S}$$

$$+ \sum_{i=1}^{K} \left[\max\left(S(t), H\left(t^-\right)\right) - H\left(t^-\right)\right] \delta(t - t_i) \frac{\partial V}{\partial H} - rV = 0, \quad 0 \leq S, \quad 0 \leq H.$$

This means that at $t \neq t_i$, $i = 1, 2, \cdots, K$, V fulfills

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0) S \frac{\partial V}{\partial S} - rV = 0,\ 0 \leq S,\ 0 \leq H \tag{4.32}$$

and at $t = t_i$, $i = 1, 2, \cdots$, or K, the equation

$$\frac{\partial V}{\partial t} + \sum_{i=1}^{K} \left[\max\left(S(t), H\left(t^-\right)\right) - H\left(t^-\right)\right] \delta(t - t_i) \frac{\partial V}{\partial H} = 0, \quad 0 \leq S, \quad 0 \leq H$$

holds. It is a hyperbolic equation, and the characteristic relation is

$$\frac{dH}{dt} = \sum_{i=1}^{K} \left[\max\left(S(t), H\left(t^-\right)\right) - H\left(t^-\right)\right] \delta(t - t_i).$$

According to the results given in Sect. 2.5, the solution of the characteristic equation is

$$H\left(t_i^+\right) - H\left(t_i^-\right) = \max\left(S(t_i), H\left(t_i^-\right)\right) - H\left(t_i^-\right)$$

or

$$H\left(t_i^+\right) = \max\left(S(t_i), H\left(t_i^-\right)\right).$$

From this relation and the results given in Sect. 2.5, the solution of the hyperbolic equation above is

$$V\left(S\left(t_i\right), H\left(t_i^-\right), t_i^-\right) = V\left(S\left(t_i\right), H\left(t_i^+\right), t_i^+\right)$$
$$= V\left(S\left(t_i\right), \max\left(S\left(t_i\right), H\left(t_i^-\right)\right), t_i^+\right)$$

or

$$V\left(S, H, t_i^-\right) = V\left(S, \max\left(S, H\right), t_i^+\right), \quad 0 \le S, \quad 0 \le H. \tag{4.33}$$

For simplicity, H stands for $H\left(t_i^-\right)$ here. This is a jump condition for lookback options with a discrete maximum.

Therefore, if the maximum realized asset price is measured discretely, the price of a lookback strike put option or a lookback price call option satisfies Eqs. (4.32) and (4.33). Consequently, the price of a European lookback strike put option is the solution of the following problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + \left(r - D_0\right) S \dfrac{\partial V}{\partial S} - rV = 0, \\ \qquad\qquad\qquad\qquad \text{if } T \ge t \ne t_i,\ i = 1, 2, \cdots, K, \\ V\left(S, H, t_i^-\right) = V\left(S, \max\left(S, H\right), t_i^+\right), \\ \qquad\qquad\qquad\qquad \text{if } t = t_i,\ \ i = 1, 2, \cdots,\ \text{ or } K, \\ V\left(S, H, T\right) = \max\left(H - \beta S, 0\right), \end{cases} \tag{4.34}$$

where

$$0 \le H, \quad 0 \le S, \quad \text{and} \quad \beta \ge 1.$$

For an American one, the solution must fulfill the constraint on American lookback strike put options:

$$V\left(S, H, t\right) \ge \max\left(H - \beta S, 0\right). \tag{4.35}$$

Therefore, if the first equation in the problem (4.34) gives a value of V that is less than the constraint $\max\left(H - \beta S, 0\right)$ at some point, the value should be replaced by $\max\left(H - \beta S, 0\right)$. In this case, the formulation of problem can be written as a linear complementarity problem.

For a lookback price call option, the final condition and the constraint are

$$V\left(S, H, T\right) = \max\left(H - E, 0\right) \tag{4.36}$$

and

$$V\left(S, H, t\right) \ge \max\left(H - E, 0\right) \tag{4.37}$$

respectively, and the value of $V(S, H, t)$ is determined by the equation (4.32) or the relation (4.33) before using the constraint.

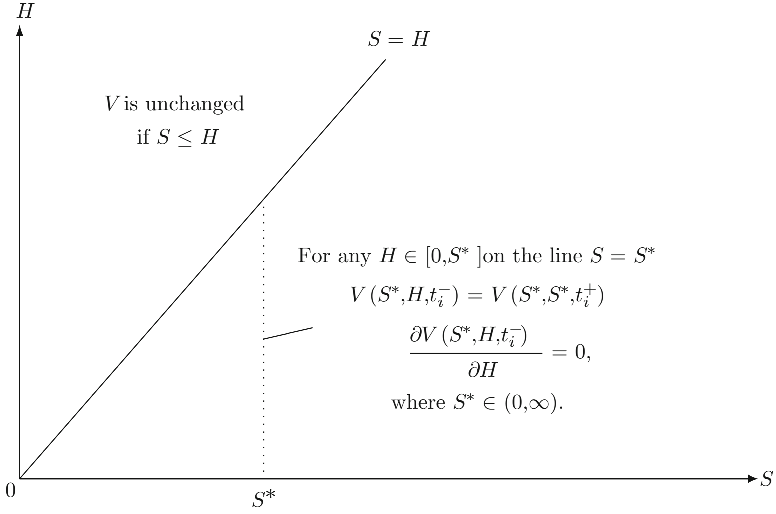

Fig. 4.4. When $t = t_i$ a jump condition is used

Now let us derive the equation if continuous sampling is done. Equation (4.33) can be rewritten as

$$V\left(S, H, t_i^-\right) = \begin{cases} V\left(S, S, t_i^+\right), & \text{if } 0 \le S,\ 0 \le H < S, \\ V\left(S, H, t_i^+\right), & \text{if } 0 \le S,\ S \le H. \end{cases}$$

Therefore, from t_i^+ to t_i^-,

$$V\left(S, H, t_i^-\right) = V\left(S, H, t_i^+\right) \quad \text{if} \quad S \le H$$

and $V\left(S, H, t_i^-\right)$ does not depend on H or

$$\frac{\partial V}{\partial H}\left(S, H, t_i^-\right) = 0 \quad \text{if} \quad 0 \le H < S$$

(see Fig. 4.4). Consequently, if $S < H$, then $V(S, H, t)$ always satisfies the Eq. (4.32) and if $0 \le H < S$ and $\max\limits_{0 \le i \le K}(t_{i+1} - t_i)$ is very small, where we let $t_0 = 0$ and $t_{K+1} = T$, then $\frac{\partial V}{\partial H}$ should be very close to 0 because the condition $V\left(S, H, t_i^-\right) = V\left(S, S, t_i^+\right)$ for any $H < S$ is used very frequently. The solution for $t \ne t_i$ should be smooth, thus $\frac{\partial V}{\partial H}(S, S, t)$ should be close to 0 and becomes closer to 0 when $\max\limits_{0 \le i \le K}(t_{i+1} - t_i)$ goes to 0, i.e., when the measure becomes continuous. Therefore, if sampling is done continuously, then the price of a European lookback strike put option is determined by the problem

$$\begin{cases} \dfrac{\partial V}{\partial t}+\dfrac{1}{2}\sigma^2 S^2\dfrac{\partial^2 V}{\partial S^2}+(r-D_0)\,S\dfrac{\partial V}{\partial S}-rV=0, & \\ & 0\le S\le H,\ t\le T, \\ V(S,H,T)=\max(H-\beta S,0), & 0\le S\le H, \\ \dfrac{\partial V}{\partial H}(S,S,t)=0, & 0\le S,\ t\le T. \end{cases} \tag{4.38}$$

For a European lookback price call option, the equation and the boundary condition are the same. The final condition in the problem (4.38) should be replaced by the condition (4.36). The constraint for an American option will be unchanged if the measure is changed from discrete to continuous. Thus the constraints for an American lookback strike put and price call options still are the inequalities (4.35) and (4.37) respectively.

Similarly, if the minimum realized asset price is measured discretely, then the price of a European lookback strike call option satisfies

$$\begin{cases} \dfrac{\partial V}{\partial t}+\dfrac{1}{2}\sigma^2 S^2\dfrac{\partial^2 V}{\partial S^2}+(r-D_0)\,S\dfrac{\partial V}{\partial S}-rV=0, & \\ & \text{if } T\ge t\ne t_i, i=1,2,\cdots,K, \\ V\left(S,L,t_i^-\right)=V\left(S,\min(S,L),t_i^+\right), & \text{if } t=t_i,\ i=1,2,\cdots,\text{ or } K, \\ V(S,L,T)=\max(\alpha S-L,0), & \end{cases}$$

where

$$0\le L,\quad 0\le S,\quad \text{and}\quad 0<\alpha\le 1.$$

If the asset price is measured continuously, then the price is the solution of the problem

$$\begin{cases} \dfrac{\partial V}{\partial t}+\dfrac{1}{2}\sigma^2 S^2\dfrac{\partial^2 V}{\partial S^2}+(r-D_0)\,S\dfrac{\partial V}{\partial S}-rV=0, & 0\le L\le S,\quad t\ge T, \\ V(S,L,T)=\max(\alpha S-L,0), & 0\le L\le S, \\ \dfrac{\partial V}{\partial L}(S,S,t)=0, & 0\le S,\quad t\le T. \end{cases}$$

For a European lookback price put option, the only change is that the final condition should be replaced by

$$V(S,L,T)=\max(E-L,0).$$

Readers are asked to prove these results as exercises. Clearly, the constraints for such American options are

$$V(S,L,t)\ge\max(\alpha S-L,0)\quad\text{or}\quad V(S,L,t)\ge\max(E-L,0).$$

4.4.2 Reducing to One-Dimensional Problems

As with Asian options, some lookback option problems can be reduced to one-dimensional problems.[2] For example, a lookback strike put option can be reduced to a one-dimensional problem. We show this here.

Let

$$\eta = \frac{H}{S}$$

and

$$W = \frac{V}{S}.$$

Suppose W depends only on η and t. Using the first three relations in the set of relations (4.24) and

$$\frac{\partial V}{\partial H} = \frac{\partial W}{\partial \eta},$$

the first equation in the problem (4.34) can be rewritten as

$$S\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\eta^2}{S}\frac{\partial^2 W}{\partial \eta^2} + (r - D_0)\, S\left(W - \eta\frac{\partial W}{\partial \eta}\right) - rSW = 0$$

or

$$\frac{\partial W}{\partial t} + \mathbf{L}_\eta W = 0,$$

where

$$\mathbf{L}_\eta = \frac{1}{2}\sigma^2\eta^2\frac{\partial^2}{\partial \eta^2} + (D_0 - r)\,\eta\frac{\partial}{\partial \eta} - D_0,$$

and from the second and third relations in Eq. (4.34), we have

$$\begin{aligned} W\left(\eta, t_i^-\right) &= \frac{V\left(S, H, t_i^-\right)}{S} \\ &= \frac{V\left(S, \max\left(S, H\right), t_i^+\right)}{S} \\ &= W\left(\max\left(1, \eta\right), t_i^+\right) \end{aligned}$$

and

$$\begin{aligned} W\left(\eta, T\right) &= \frac{V\left(S, H, T\right)}{S} \\ &= \frac{\max\left(H - \beta S, 0\right)}{S} \\ &= \max\left(\eta - \beta, 0\right). \end{aligned}$$

Therefore, it is true that the problem (4.34) can be reduced to the following one-dimensional problem

[2]The way to reduce some lookback option problems to one-dimensional problems is not unique (see [84, 54, 3, 80]).

$$\begin{cases} \dfrac{\partial W}{\partial t} + \mathbf{L}_\eta W = 0, & \text{if } T \geq t \neq t_i,\ i = 1, 2, \cdots, K, \\ W\left(\eta, t_i^-\right) = W\left(\max\left(1, \eta\right), t_i^+\right), & \text{if } t = t_i,\ \ i = 1, 2, \cdots,\ \text{or } K, \\ W\left(\eta, T\right) = \max\left(\eta - \beta, 0\right), \end{cases} \tag{4.39}$$

where $0 \leq \eta$. This is a formulation of a European lookback strike put option as a one-dimensional problem if sampling is done discretely. Similarly, the problem (4.38) can be written as

$$\begin{cases} \dfrac{\partial W}{\partial t} + \mathbf{L}_\eta W = 0, & 1 \leq \eta,\ t \leq T, \\ W\left(\eta, T\right) = \max\left(\eta - \beta, 0\right), & 1 \leq \eta, \\ \dfrac{\partial W}{\partial \eta}\left(1, t\right) = 0, & t \leq T. \end{cases} \tag{4.40}$$

This is a formulation of a European lookback strike put option as a one-dimensional problem if sampling is done continuously. The constraint for such an American option, the constraint (4.35), can also be rewritten in a form involving η only:

$$W\left(\eta, t\right) \geq \max\left(\eta - \beta, 0\right). \tag{4.41}$$

For a European lookback strike call option with continuous sampling, the corresponding one-dimensional problem is

$$\begin{cases} \dfrac{\partial W}{\partial t} + \mathbf{L}_\eta W = 0, & 0 \leq \eta \leq 1,\ t \leq T, \\ W\left(\eta, T\right) = \max\left(\alpha - \eta, 0\right), & 0 \leq \eta \leq 1, \\ \dfrac{\partial W}{\partial \eta}\left(1, t\right) = 0, & t \leq T, \end{cases}$$

where

$$\eta = L/S \quad \text{and} \quad W = V/S$$

and for a corresponding American option, the constraint is

$$W\left(\eta, t\right) \geq \max\left(\alpha - \eta, 0\right).$$

These are left for the reader to derive as exercises.

It is useful to write an American option problem as a linear complementarity problem. Let us write the American lookback strike put problem as a linear complementarity problem for the case of continuous sampling. Just like other American options we have met, the price of the American lookback strike put option is the solution of the following linear complementarity problem

$$\begin{cases} \min\left(-\dfrac{\partial W}{\partial t} - \mathbf{L}_\eta W,\ W - \max\left(\eta - \beta, 0\right)\right) = 0, & 1 \le \eta, \quad t \le T, \\ W\left(\eta, T\right) = \max\left(\eta - \beta, 0\right), & 1 \le \eta, \\ \dfrac{\partial W}{\partial \eta}\left(1, t\right) = 0, & t \le T. \end{cases}$$

For the lookback strike call option, similar results can be obtained (Problem 17).

4.4.3 Closed-Form Solutions for European Lookback Options

For some European lookback options with continuous sampling, the closed-form solutions have been found. For example, such solutions for lookback strike options with $\alpha = 1$ or $\beta = 1$ were found by Goldman, Sosin, and Gatto (see [34]). Later, Conze and Viswanathan (see [21]) derived explicit solutions for the lookback strike options with $\alpha < 1$ or $\beta > 1$ and the lookback price options. In Kwok's book [54] and Jiang's book [49], some details of those derivations are given.

In the following, we will derive the closed-form solution for a lookback strike call option with $\alpha \le 1$ in a way close to the way given in Jiang's book [49].

In order to find closed-form solutions of European lookback options, let us consider the following problem:

$$\begin{cases} \dfrac{\partial \overline{W}}{\partial t} + \mathbf{L}_\eta \overline{W} = 0, \quad 0 \le \eta, \ t \le T, \\ \overline{W}\left(\eta, T\right) = \begin{cases} \varphi_1\left(\eta\right), \ 0 \le \eta \le 1, \\ \varphi_2\left(\eta\right), \ 1 < \eta, \end{cases} \end{cases} \tag{4.42}$$

where $\varphi_1\left(\eta\right)$ and $\varphi_2\left(\eta\right)$ are continuous functions, $\varphi_1\left(1\right) = \varphi_2\left(1\right)$ might not hold, and

$$\mathbf{L}_\eta = \frac{1}{2}\sigma^2\eta^2\frac{\partial^2}{\partial \eta^2} + \left(D_0 - r\right)\eta\frac{\partial}{\partial \eta} - D_0.$$

Let us show that if

$$\begin{cases} \varphi_1\left(1\right) = \varphi_2\left(1\right), \\ \dfrac{d\varphi_1\left(\eta\right)}{d\eta} = \eta^{2\left(r - D_0 + \sigma^2/2\right)/\sigma^2}\dfrac{d\varphi_2\left(1/\eta\right)}{d\eta}, \end{cases}$$

or

$$\begin{cases} \varphi_2(1) = \varphi_1(1), \\ \dfrac{d\varphi_2(\eta)}{d\eta} = \eta^{2\left(r-D_0+\sigma^2/2\right)/\sigma^2} \dfrac{d\varphi_1(1/\eta)}{d\eta}, \end{cases}$$

then $\dfrac{\partial \overline{W}(1,t)}{\partial \eta} = 0.$

As we know, the solution of the above problem can be expressed by

$$\overline{W}(\eta, t) = \mathrm{e}^{-D_0(T-t)} \int_0^\infty \overline{W}(\eta', T)\, \tilde{G}(\eta', T; \eta, t)\, d\eta',$$

where

$$\tilde{G}(\eta', T; \eta, t) = \frac{1}{\sigma\sqrt{2\pi(T-t)}\eta'} \mathrm{e}^{-\left[\ln\left(\eta'/\eta\right)-\left(D_0-r-\sigma^2/2\right)(T-t)\right]^2/2\sigma^2(T-t)}.$$

Let

$$\overline{G}(\eta', T; \eta, t) = \mathrm{e}^{-\left[\ln\left(\eta'/\eta\right)-\left(D_0-r-\sigma^2/2\right)(T-t)\right]^2/2\sigma^2(T-t)}.$$

For $\overline{G}(\eta', T; \eta, t)$ the following is true:

$$-\eta \frac{\partial \overline{G}}{\partial \eta} = \eta' \frac{\partial \overline{G}}{\partial \eta'}.$$

Then the expression for $\overline{W}(\eta, t)$ can be rewritten as

$$\overline{W}(\eta, t) = \frac{\mathrm{e}^{-D_0(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_0^\infty \frac{\overline{W}(\eta', T)}{\eta'} \overline{G}(\eta', T; \eta, t)\, d\eta'$$

and we can have

$$\begin{aligned} \frac{\partial \overline{W}(\eta, t)}{\partial \eta} &= \frac{\mathrm{e}^{-D_0(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_0^\infty \frac{\overline{W}(\eta', T)}{\eta'} \frac{\partial \overline{G}(\eta', T; \eta, t)}{\partial \eta} d\eta' \\ &= \frac{-\mathrm{e}^{-D_0(T-t)}}{\eta\sigma\sqrt{2\pi(T-t)}} \int_0^\infty \overline{W}(\eta', T) \frac{\partial \overline{G}(\eta', T; \eta, t)}{\partial \eta'} d\eta' \\ &= \frac{-\mathrm{e}^{-D_0(T-t)}}{\eta\sigma\sqrt{2\pi(T-t)}} \left[\overline{W}(\eta', T)\, \overline{G}(\eta', T; \eta, t) \Big|_0^\infty \right. \\ &\quad \left. - \int_0^\infty \overline{G}(\eta', T; \eta, t) \frac{\partial \overline{W}(\eta', T)}{\partial \eta'} d\eta' \right] \\ &= \frac{\mathrm{e}^{-D_0(T-t)}}{\eta\sigma\sqrt{2\pi(T-t)}} \int_0^\infty \overline{G}(\eta', T; \eta, t) \frac{\partial \overline{W}(\eta', T)}{\partial \eta'} d\eta'. \end{aligned}$$

Thus, $\dfrac{\partial \overline{W}(1,t)}{\partial \eta} = 0$ is equivalent to that the last integral above at $\eta = 1$ is equal to 0, i.e.,

$$\begin{aligned}
&\int_0^\infty \overline{G}(\eta', T; 1, t) \frac{\partial \overline{W}(\eta', T)}{\partial \eta'} d\eta' \\
&\quad = \int_0^{1^-} \overline{G}(\eta', T; 1, t) \frac{\partial \overline{W}(\eta', T)}{\partial \eta'} d\eta' + \int_{1^-}^{1^+} \overline{G}(\eta', T; 1, t) \frac{\partial \overline{W}(\eta', T)}{\partial \eta'} d\eta' \\
&\qquad + \int_{1^+}^\infty \overline{G}(\eta', T; 1, t) \frac{\partial \overline{W}(\eta', T)}{\partial \eta'} d\eta' \\
&\quad = \int_0^1 \overline{G}(\eta', T; 1, t) \frac{d\varphi_1(\eta')}{d\eta'} d\eta' + [\varphi_2(1) - \varphi_1(1)] \overline{G}(1, T; 1, t) \\
&\qquad + \int_1^\infty \overline{G}(\eta', T; 1, t) \frac{d\varphi_2(\eta')}{d\eta'} d\eta' \\
&\quad = [\varphi_2(1) - \varphi_1(1)] \overline{G}(1, T; 1, t) + \int_0^1 \overline{G}(\eta', T; 1, t) \frac{d\varphi_1(\eta')}{d\eta'} d\eta' \\
&\qquad + \int_1^0 \overline{G}(1/\eta', T; 1, t) \frac{d\varphi_2(1/\eta')}{d(1/\eta')} d(1/\eta') \\
&\quad = [\varphi_2(1) - \varphi_1(1)] \overline{G}(1, T; 1, t) \\
&\qquad + \int_0^1 \left[\overline{G}(\eta', T; 1, t) \frac{d\varphi_1(\eta')}{d\eta'} - \overline{G}(1/\eta', T; 1, t) \frac{d\varphi_2(1/\eta')}{d\eta'} \right] d\eta' \\
&\quad = 0.
\end{aligned}$$

Consequently, if the two conditions

$$\varphi_1(1) = \varphi_2(1)$$

and

$$\frac{d\varphi_1(\eta')}{d\eta'} = \frac{\overline{G}(1/\eta', T; 1, t)}{\overline{G}(\eta', T; 1, t)} \frac{d\varphi_2(1/\eta')}{d\eta'}$$

hold, then

$$\frac{\partial \overline{W}(1, t)}{\partial \eta} = 0.$$

Because

$$\begin{aligned}
\frac{\overline{G}(1/\eta', T; 1, t)}{\overline{G}(\eta', T; 1, t)} &= \frac{\mathrm{e}^{-\left[\ln \eta' + \left(D_0 - r - \sigma^2/2\right)(T-t)\right]^2 / 2\sigma^2 (T-t)}}{\mathrm{e}^{-\left[\ln \eta' - \left(D_0 - r - \sigma^2/2\right)(T-t)\right]^2 / 2\sigma^2 (T-t)}} \\
&= \mathrm{e}^{-4 \ln \eta' \left(D_0 - r - \sigma^2/2\right)(T-t)/2\sigma^2 (T-t)} \\
&= (\eta')^{2\left(r - D_0 + \sigma^2/2\right)/\sigma^2},
\end{aligned}$$

the second condition above can be rewritten as

$$\frac{d\varphi_1(\eta')}{d\eta'} = (\eta')^{2\left(r-D_0+\sigma^2/2\right)/\sigma^2} \frac{d\varphi_2(1/\eta')}{d\eta'}.$$

Thus when

$$\begin{cases} \varphi_1(1) = \varphi_2(1), \\ \dfrac{d\varphi_1(\eta)}{d\eta} = \eta^{2\left(r-D_0+\sigma^2/2\right)/\sigma^2} \dfrac{d\varphi_2(1/\eta)}{d\eta}, \end{cases}$$

$\dfrac{\partial\overline{W}(1,t)}{\partial\eta} = 0$ holds. Let $\xi = \dfrac{1}{\eta}$. Then from

$$d\xi = -\frac{1}{\eta^2}d\eta = -\xi^2 d\eta$$

and the relation $\dfrac{d\varphi_2(\eta)}{d\eta} = \eta^{2\left(r-D_0+\sigma^2/2\right)/\sigma^2} \dfrac{d\varphi_1(1/\eta)}{d\eta}$, we have

$$\frac{d\varphi_2(\eta)}{d\eta} = \frac{d\varphi_2(1/\xi)}{-\xi^{-2}d\xi} = \xi^{-2\left(r-D_0+\sigma^2/2\right)/\sigma^2} \cdot \frac{d\varphi_1(\xi)}{-\xi^{-2}d\xi}.$$

Thus

$$\frac{d\varphi_1(\xi)}{d\xi} = \xi^{2\left(r-D_0+\sigma^2/2\right)/\sigma^2} \frac{d\varphi_2(1/\xi)}{d\xi},$$

or

$$\frac{d\varphi_1(\eta)}{d\eta} = \eta^{2\left(r-D_0+\sigma^2/2\right)/\sigma^2} \frac{d\varphi_2(1/\eta)}{d\eta}.$$

Therefore, from the result we have obtained we can further have that if

$$\begin{cases} \varphi_2(1) = \varphi_1(1), \\ \dfrac{d\varphi_2(\eta)}{d\eta} = \eta^{2\left(r-D_0+\sigma^2/2\right)/\sigma^2} \dfrac{d\varphi_1(1/\eta)}{d\eta}, \end{cases}$$

then $\dfrac{\partial\overline{W}(1,t)}{\partial\eta} = 0$ also holds.

Remark. The condition $\dfrac{\partial\overline{W}(1,t)}{\partial\eta} = 0$ is also equivalent to

$$\begin{aligned} &\int_0^\infty \overline{W}(\eta',T) \frac{\partial\overline{G}(\eta',T;1,t)}{\partial\eta'} d\eta' \\ &= \int_0^1 \left[\varphi_1(\eta') \frac{\partial\overline{G}(\eta',T;1,t)}{\partial\eta'} - \varphi_2(1/\eta') \frac{\partial\overline{G}(1/\eta',T;1,t)}{\partial\eta'}\right] d\eta' \\ &= 0. \end{aligned}$$

However, we cannot find the relation between $\varphi_1(\eta')$ and $\varphi_2(\eta')$ by assuming

$$\varphi_1(\eta')\frac{\partial\overline{G}(\eta',T;1,t)}{\partial\eta'}-\varphi_2(1/\eta')\frac{\partial\overline{G}(1/\eta',T;1,t)}{\partial\eta'}=0$$

because $\dfrac{\partial\overline{G}(\eta',T;1,t)}{\partial\eta'}\Big/\dfrac{\partial\overline{G}(1/\eta',T;1,t)}{\partial\eta'}$ depends on not only η' but also $T-t$.

Let $c_{ls}(S,L,t)$ denote the price of a European lookback strike call option. As we know, set $W=\dfrac{c_{ls}(S,L,t)}{S}$ and $\eta=\dfrac{L}{S}$, then $W(\eta,t)$ satisfies

$$\begin{cases}\dfrac{\partial W}{\partial t}+\mathbf{L}_\eta W=0, & 0\le\eta\le1,\quad t\le T,\\ W(\eta,T)=\max(\alpha-\eta,0), & 0\le\eta\le1,\\ \dfrac{\partial W}{\partial\eta}(1,t)=0, & t\le T,\end{cases}\tag{4.43}$$

where $0<\alpha\le1$.

From the result we just obtain, we know that if $\varphi_1(\eta)=\max(\alpha-\eta,0)$ and $\varphi_2(\eta)$ satisfies $\dfrac{d\varphi_2(\eta)}{d\eta}=\eta^{2\left(r-D_0+\sigma^2/2\right)/\sigma^2}\dfrac{d\varphi_1(1/\eta)}{d\eta}$ with $\varphi_2(1)=\varphi_1(1)$, then the solution of the problem (4.42) for $\eta\in[0,1]$ is the solution of the problem (4.43). In order to find such a solution, first let us find $\varphi_2(\eta)$. In this case $\varphi_1(\eta)=\alpha-\eta$ if $\eta\in[0,\alpha]$ and $\varphi_1(\eta)=0$ for $\eta\in(\alpha,1]$, so $\dfrac{d\varphi_1(1/\eta)}{d\eta}=0$ for $\eta\in(1,1/\alpha)$ and $\dfrac{d\varphi_1(1/\eta)}{d\eta}=1/\eta^2$ for $\eta\in(1/\alpha,\infty)$. Thus from $\varphi_1(1)=0$, we have $\varphi_2(\eta)=0$ for $\eta\in(1,1/\alpha)$ and for $\eta\in(1/\alpha,\infty)$,

$$\begin{aligned}\varphi_2(\eta)&=\varphi_2(1/\alpha)+\int_{1/\alpha}^{\eta}\frac{d\varphi_2(\eta)}{d\eta}d\eta=\int_{1/\alpha}^{\eta}\eta^{2\left(r-D_0+\sigma^2/2\right)/\sigma^2-2}d\eta\\&=\frac{(\eta)^{2(r-D_0)/\sigma^2}-\alpha^{-2(r-D_0)/\sigma^2}}{2(r-D_0)/\sigma^2}.\end{aligned}$$

Here we assume $r-D_0\neq0$. Therefore if set $\tau=T-t$, then for $W(\eta,t)$ we have

$$
\begin{aligned}
& W(\eta,t) \\
&= \mathrm{e}^{-D_0\tau}\left[\int_0^1 \varphi_1(\eta')\,\tilde{G}(\eta',T;\eta,t)\,d\eta' + \int_1^\infty \varphi_2(\eta')\,\tilde{G}(\eta',T;\eta,t)\,d\eta'\right] \\
&= \mathrm{e}^{-D_0\tau}\left[\int_0^\alpha (\alpha-\eta')\,\tilde{G}(\eta',T;\eta,t)\,d\eta' + \int_{1/\alpha}^\infty \varphi_2(\eta')\,\tilde{G}(\eta',T;\eta,t)\,d\eta'\right] \\
&= \mathrm{e}^{-D_0\tau}\left[\alpha N\left(\frac{\ln\dfrac{\alpha}{\eta \mathrm{e}^{(D_0-r)\tau}}+\sigma^2\tau/2}{\sigma\sqrt{\tau}}\right)\right. \\
&\qquad -\eta \mathrm{e}^{(D_0-r)\tau} N\left(\frac{\ln\dfrac{\alpha}{\eta \mathrm{e}^{(D_0-r)\tau}}-\sigma^2\tau/2}{\sigma\sqrt{\tau}}\right) \\
&\qquad \left.+\frac{\sigma^2}{2(r-D_0)}\int_{1/\alpha}^\infty \left[(\eta')^{2(r-D_0)/\sigma^2}-\alpha^{-2(r-D_0)/\sigma^2}\right]\tilde{G}(\eta',T;\eta,t)\,d\eta'\right] \\
&= \mathrm{e}^{-D_0\tau}\left[\alpha N\left(\frac{\ln(\alpha/\eta)+\left(r-D_0+\sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}\right)\right. \\
&\qquad -\eta \mathrm{e}^{(D_0-r)\tau} N\left(\frac{\ln(\alpha/\eta)+\left(r-D_0-\sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}\right) \\
&\qquad +\frac{\sigma^2}{2(r-D_0)}\left(\eta \mathrm{e}^{(D_0-r)\tau}\right)^{2(r-D_0)/\sigma^2}\cdot \mathrm{e}^{\left[4(r-D_0)^2/\sigma^4-2(r-D_0)/\sigma^2\right]\sigma^2\tau/2} \\
&\qquad \times N\left(\frac{\ln\left(\alpha\eta \mathrm{e}^{(D_0-r)\tau}\right)-\sigma^2\tau/2+2(r-D_0)\tau}{\sigma\sqrt{\tau}}\right) \\
&\qquad \left.-\frac{\sigma^2}{2(r-D_0)}\alpha^{-2(r-D_0)/\sigma^2}N\left(\frac{\ln\left(\alpha\eta \mathrm{e}^{(D_0-r)\tau}\right)-\sigma^2\tau/2}{\sigma\sqrt{\tau}}\right)\right] \\
&= \mathrm{e}^{-D_0\tau}\left[\alpha N\left(\frac{\ln(\alpha/\eta)+\left(r-D_0+\sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}\right)\right. \\
&\qquad -\eta \mathrm{e}^{(D_0-r)\tau} N\left(\frac{\ln(\alpha/\eta)+\left(r-D_0-\sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}\right) \\
&\qquad +\frac{\sigma^2}{2(r-D_0)}\eta^{2(r-D_0)/\sigma^2}\mathrm{e}^{-(r-D_0)\tau}N\left(\frac{\ln(\alpha\eta)+\left(r-D_0-\sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}\right) \\
&\qquad \left.-\frac{\sigma^2}{2(r-D_0)}\alpha^{-2(r-D_0)/\sigma^2}N\left(\frac{\ln(\alpha\eta)-\left(r-D_0+\sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}\right)\right],
\end{aligned}
$$

and if set

$$\mu = r - D_0 - \sigma^2/2,$$

then for $c_{ls}(S,L,t)$, we have

$$\begin{aligned}
&c_{ls}(S,L,t) = SW(\eta,t) = \mathrm{e}^{-r(T-t)}S\mathrm{e}^{r(T-t)}W(\eta,t)\\
&= \mathrm{e}^{-r(T-t)}S\left[\frac{\sigma^2}{2(r-D_0)}\left(\frac{L}{S}\right)^{2(r-D_0)/\sigma^2} N\left(\frac{\ln(\alpha L/S)+\mu(T-t)}{\sigma\sqrt{T-t}}\right)\right.\\
&-\frac{\sigma^2}{2(r-D_0)}\alpha^{-2(r-D_0)/\sigma^2}\mathrm{e}^{(r-D_0)(T-t)}N\left(\frac{\ln(\alpha L/S)-\left(\mu+\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right)\\
&+\alpha\mathrm{e}^{(r-D_0)(T-t)}N\left(\frac{\ln(\alpha S/L)+\left(\mu+\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right)\\
&\left.-\frac{L}{S}N\left(\frac{\ln(\alpha S/L)+\mu(T-t)}{\sigma\sqrt{T-t}}\right)\right].
\end{aligned}$$

When $r-D_0=0$, this expression cannot be used because $r-D_0$ appears on denominators. In this case for $\eta\in(1/\alpha,\infty)$,

$$\frac{d\varphi_2(\eta)}{d\eta}=\eta^{-1} \quad \text{and} \quad \varphi_2(\eta)=\ln\eta+\ln\alpha.$$

Thus we have

$$\begin{aligned}
&\int_{1/\alpha}^{\infty}\varphi_2(\eta')\tilde{G}(\eta',T;\eta,t)\,d\eta'\\
&=\int_{1/\alpha}^{\infty}(\ln\eta'+\ln\alpha)\tilde{G}(\eta',T;\eta,t)\,d\eta'\\
&=\frac{\sigma\sqrt{\tau}}{\sqrt{2\pi}}\mathrm{e}^{-\left[\ln(\alpha\eta)-\sigma^2\tau/2\right]^2/2\sigma^2\tau}+\left(\ln\eta-\sigma^2\tau/2\right)N\left(\frac{\ln(\alpha\eta)-\sigma^2\tau/2}{\sigma\sqrt{\tau}}\right)\\
&\quad+\ln\alpha N\left(\frac{\ln(\alpha\eta)-\sigma^2\tau/2}{\sigma\sqrt{\tau}}\right)\\
&=\frac{\sigma\sqrt{\tau}}{\sqrt{2\pi}}\mathrm{e}^{-\left[\ln(\alpha\eta)-\sigma^2\tau/2\right]^2/2\sigma^2\tau}+\left[\ln(\alpha\eta)-\sigma^2\tau/2\right]N\left(\frac{\ln(\alpha\eta)-\sigma^2\tau/2}{\sigma\sqrt{\tau}}\right)
\end{aligned}$$

and the expression usable in practice for $c_{ls}(S,L,t)$ is

$$\begin{aligned}
&c_{ls}(S,L,t)\\
&=\mathrm{e}^{-r(T-t)}S\left\{\left[\ln(\alpha L/S)-\sigma^2(T-t)/2\right]N\left(\frac{\ln(\alpha L/S)-\sigma^2(T-t)/2}{\sigma\sqrt{T-t}}\right)\right.\\
&\quad+\frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}}\mathrm{e}^{-\left[\ln(\alpha L/S)-\sigma^2(T-t)/2\right]^2/2\sigma^2(T-t)}
\end{aligned}$$

$$+\alpha N\left(\frac{\ln(\alpha S/L)+\left(\mu+\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right)$$
$$-\frac{L}{S}N\left(\frac{\ln(\alpha S/L)+\mu(T-t)}{\sigma\sqrt{T-t}}\right)\Bigg\}.$$

This expression can also be obtained by finding the following limit:

$$\lim_{r-D_0\to 0}\frac{\sigma^2}{2(r-D_0)}\left[\left(\frac{L}{S}\right)^{2(r-D_0)/\sigma^2}N\left(\frac{\ln(\alpha L/S)+\mu(T-t)}{\sigma\sqrt{T-t}}\right)\right.$$
$$\left.-\alpha^{-2(r-D_0)/\sigma^2}\mathrm{e}^{(r-D_0)(T-t)}N\left(\frac{\ln(\alpha L/S)-\left(\mu+\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right)\right].$$

In the above we have found the closed-form solution of the lookback strike call option. In a similar way, we can also find a closed-form solution of the lookback strike put option. Because a lookback price option problem cannot be reduced to a one-dimensional problem by the way given in Sect. 4.4.2, we cannot find a closed-form solution of a lookback price option in the way above. However their closed-form solutions can be obtained in another way.

Suppose that $t\le t'$, and let S and S' stand for the price at time t and t', respectively. Also let $L_t^{t'}$ be the lowest price during the time period $[t,t']$. In this notation the lowest price during the time period $[0,t]$ is L_0^t. However sometimes we also write it as L, i.e., we define $L\equiv L_0^t$. Among $L_0^t, L_0^{t'}$, and $L_t^{t'}$, there is the relation $L_0^{t'}=\min\left(L_0^t,L_t^{t'}\right)$. Let $V\left(S,L_0^t,t\right)$ be the value of a European option depending on S, L_0^t, t. Suppose that we know

$$V\left(S',L_0^{t'},t'\right)=V\left(S',\min\left(L_0^t,L_t^{t'}\right),t'\right),$$

and we want to find $V\left(S,L_0^t,t\right)$. In this case, S' and $L_t^{t'}$ are two random variables, and the value of the European option at time t, $V\left(S,L_0^t,t\right)$, can be expressed as the expectation of $V\left(S',L_0^{t'},t'\right)$ times a discounting factor:

$$V\left(S,L_0^t,t\right)=\mathrm{e}^{-r\left(t'-t\right)}$$
$$\times\int_0^S\int_{L_t^{t'}}^{\infty}V\left(S',\min\left(L_0^t,L_t^{t'}\right),t'\right)g\left(S',L_t^{t'};S,t'-t\right)dS'dL_t^{t'},$$

where $g\left(S',L_t^{t'};S,t'-t\right)$ is a two-dimensional probability density function in the "risk-neutral" world, and we have used the fact that the two random variables S' and $L_t^{t'}$ appear only in the domain (the shaded area in Fig. 4.5): $0\le L_t^{t'}\le S$ and $L_t^{t'}\le S'$ because of $L_t^{t'}\le S$ and $L_t^{t'}\le S'$.

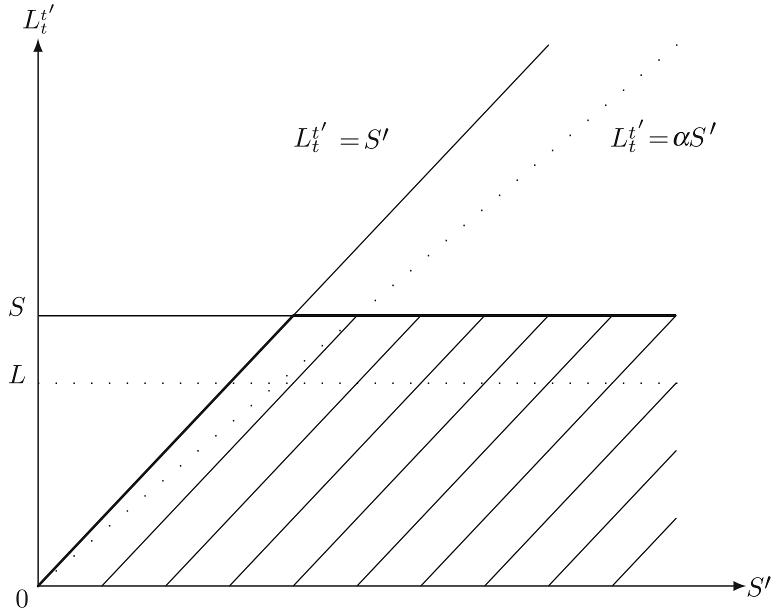

Fig. 4.5. The area of integration for a lookback strike call option

From Sect. 4.2.2, the probability of the price at time t' being in $[S', S'+dS')$ and the price during the time period $[t, t']$ never being lower than $L_t^{t'}$ is

$$G_1\left(S', t'; S, t, L_t^{t'}\right) dS',$$

where

$$\begin{aligned} G_1\left(S', t'; S, t, L_t^{t'}\right) &= \frac{1}{S'\sigma\sqrt{2\pi\tau'}} \\ &\times\left[\mathrm{e}^{-\left[\ln\left(S'/S\right)-\mu\tau'\right]^2/2\sigma^2\tau'} - \left(L_t^{t'}/S\right)^{2\mu/\sigma^2} \mathrm{e}^{-\left[\ln\left(S'S/L_t^{t'2}\right)-\mu\tau'\right]^2\Big/2\sigma^2\tau'}\right]. \end{aligned}$$

In this expression, $\mu = r - D_0 - \sigma^2/2$ and $\tau' = t' - t$. Thus, the probability of the event $S' \in [S', S' + dS')$ and $L_t^{t'} \in \left[L_t^{t'}, L_t^{t'} + dL_t^{t'}\right)$ is

$$\begin{aligned} &g\left(S', L_t^{t'}; S, t'-t\right) dS' dL_t^{t'} \\ &= G_1\left(S', t'; S, t, L_t^{t'}\right) dS' - G_1\left(S', t'; S, t, L_t^{t'} + dL_t^{t'}\right) dS' = -\frac{\partial G_1}{\partial L_t^{t'}} dS' dL_t^{t'}, \end{aligned}$$

that is,

$$g\left(S', L_t^{t'}; S, t'-t\right) = -\frac{\partial G_1\left(S', t'; S, t, L_t^{t'}\right)}{\partial L_t^{t'}}.$$

Here we use the fact that indeed the function $G_1\left(S', t'; S, t, L_t^{t'}\right)$ depends on $S', S, t'-t$ and $L_t^{t'}$. Because the first term in $G_1\left(S', t'; S, t, L_t^{t'}\right)$ does not depend on $L_t^{t'}$, $g\left(S', L_t^{t'}; S, t'-t\right)$ actually is equal to the partial derivative of the second term with respect to $L_t^{t'}$, i.e.,

$$g\left(S', L_t^{t'}; S, t'-t\right) = \frac{\partial f_1\left(S', L_t^{t'}; S, t'-t\right)}{\partial L_t^{t'}},$$

where

$$f_1\left(S', L_t^{t'}; S, t'-t\right) = \frac{1}{S'\sigma\sqrt{2\pi\tau'}}\left(\frac{L_t^{t'}}{S}\right)^{2\mu/\sigma^2} \mathrm{e}^{-\left[\ln\left(S'S/L_t^{t'2}\right)-\mu\tau'\right]^2/2\sigma^2\tau'}.$$

It can be proved that $g \geq 0$ if $0 \leq L_t^{t'} \leq S$ and $L_t^{t'} \leq S'$. The function $g\left(S', L_t^{t'}; S, t'-t\right)$ is usually called Green's function for lookback options depending on S, L, t. The proof of this result is left to readers as a part of Problem 16.

In almost the same way we can find out Green's function for lookback options depending on S, H, t. Based on these Green's functions, we can prove that the value of an American option is not less than the value of the corresponding European option and the value of such an American option at time t^* is not less than the value of this option at time t^{**} if $t^* < t^{**}$ by using the method given in Sect. 3.1. Also when we have Green's functions for such lookback option problems, the closed-form solution for European lookback strike/price options can be obtained in the following way which is close to the way given in Kwok's book [54].

Now let us find out the closed-form solution to the European lookback strike call option by using $g\left(S', L_t^{t'}; S, t'-t\right)$. For this option, the payoff is

$$c_{ls}(S, L, T) = \max(\alpha S - L, 0) = \max\left(\alpha S - \min\left(L_0^t, L_t^T\right), 0\right).$$

We need to find $c_{ls}(S, L, t)$ from $c_{ls}(S, L, T)$, so $\tau' = t'-t = T-t = \tau$. Hence we have

$$\begin{aligned} &c_{ls}(S, L, t) \\ &= \mathrm{e}^{-r(T-t)} \int_0^S \int_{L_t^{t'}}^\infty \max\left(\alpha S' - \min\left(L, L_t^{t'}\right), 0\right) g\left(S', L_t^{t'}; S, T-t\right) dS' dL_t^{t'}. \end{aligned}$$

The integral here can be expressed in terms of the cumulative distribution function for the standardized normal variable. In the procedure of finding such an expression, in order to make writing short, instead of $f_1(S', x; S, \tau)$, we just write it as f_1 or $f_1(S', x)$, x being $L_t^{t'}, S', L, S$, or $\alpha S'$. This procedure is as follows. Because $L \leq S$, we have

$$\int_0^S \int_{L_t^{t'}}^\infty \max\left(\alpha S' - \min\left(L, L_t^{t'}\right), 0\right) \frac{\partial f_1}{\partial L_t^{t'}} dS' dL_t^{t'}$$

$$= \int_0^L \int_{L_t^{t'}}^\infty \max\left(\alpha S' - L_t^{t'}, 0\right) \frac{\partial f_1}{\partial L_t^{t'}} dS' dL_t^{t'}$$

$$+ \int_L^S \int_{L_t^{t'}}^\infty \max\left(\alpha S' - L, 0\right) \frac{\partial f_1}{\partial L_t^{t'}} dS' dL_t^{t'}$$

$$= \int_0^L \int_{L_t^{t'}/\alpha}^\infty \left(\alpha S' - L_t^{t'}\right) \frac{\partial f_1}{\partial L_t^{t'}} dS' dL_t^{t'} + \int_L^S \int_L^{S'} \max\left(\alpha S' - L, 0\right) \frac{\partial f_1}{\partial L_t^{t'}} dL_t^{t'} dS'$$

$$+ \int_S^\infty \int_L^S \max\left(\alpha S' - L, 0\right) \frac{\partial f_1}{\partial L_t^{t'}} dL_t^{t'} dS'$$

$$= \int_0^{L/\alpha} \int_0^{\alpha S'} \left(\alpha S' - L_t^{t'}\right) \frac{\partial f_1}{\partial L_t^{t'}} dL_t^{t'} dS' + \int_{L/\alpha}^\infty \int_0^L \left(\alpha S' - L_t^{t'}\right) \frac{\partial f_1}{\partial L_t^{t'}} dL_t^{t'} dS'$$

$$+ \int_L^S \max\left(\alpha S' - L, 0\right) \left[f_1\left(S', S'\right) - f_1\left(S', L\right)\right] dS'$$

$$+ \int_S^\infty \max\left(\alpha S' - L, 0\right) \left[f_1\left(S', S\right) - f_1\left(S', L\right)\right] dS'$$

$$= \int_0^{L/\alpha} \alpha S' f_1\left(S', \alpha S'\right) dS' - \int_0^{L/\alpha} \int_0^{\alpha S'} L_t^{t'} \frac{\partial f_1}{\partial L_t^{t'}} dL_t^{t'} dS'$$

$$+ \int_{L/\alpha}^\infty \alpha S' f_1\left(S', L\right) dS' - \int_{L/\alpha}^\infty \int_0^L L_t^{t'} \frac{\partial f_1}{\partial L_t^{t'}} dL_t^{t'} dS'$$

$$+ \int_L^S \max\left(\alpha S' - L, 0\right) f_1\left(S', S'\right) dS'$$

$$+ \int_S^\infty \max\left(\alpha S' - L, 0\right) f_1\left(S', S\right) dS' - \int_{L/\alpha}^\infty \left(\alpha S' - L\right) f_1\left(S', L\right) dS'$$

$$= \int_0^{L/\alpha} \alpha S' f_1(S', \alpha S')\, dS' - \int_0^{L/\alpha} \left[L_t^{t'} f_1 |_0^{\alpha S'} - \int_0^{\alpha S'} f_1\left(S', L_t^{t'}\right) dL_t^{t'} \right] dS'$$

$$- \int_{L/\alpha}^{\infty} \left[L_t^{t'} f_1 \Big|_0^L - \int_0^L f_1\left(S', L_t^{t'}\right) dL_t^{t'} \right] dS'$$

$$+ \int_L^S \max(\alpha S' - L, 0)\, f_1(S', S')\, dS'$$

$$+ \int_S^{\infty} \max(\alpha S' - L, 0)\, f_1(S', S)\, dS' + \int_{L/\alpha}^{\infty} L f_1(S', L)\, dS'$$

$$= \int_0^{L/\alpha} \int_0^{\alpha S'} f_1\left(S', L_t^{t'}\right) dL_t^{t'} dS' + \int_{L/\alpha}^{\infty} \int_0^L f_1\left(S', L_t^{t'}\right) dL_t^{t'} dS'$$

$$+ \int_L^S \max(\alpha S' - L, 0)\, f_1(S', S')\, dS'$$

$$+ \int_S^{\infty} \max(\alpha S' - L, 0)\, f_1(S', S)\, dS'$$

$$= \int_0^L \int_{L_t^{t'}/\alpha}^{\infty} f_1\left(S', L_t^{t'}\right) dS' dL_t^{t'} + \int_L^S \max(\alpha S' - L, 0)\, f_1(S', S')\, dS'$$

$$+ \int_S^{\infty} \max(\alpha S' - L, 0)\, f_1(S', S)\, dS'.$$

Now let us find the result for each integral. The first integral is equal to

$$\int_0^L \int_{L_t^{t'}/\alpha}^{\infty} f_1\left(S', L_t^{t'}\right) dS' dL_t^{t'}$$

$$= \int_0^L \int_{L_t^{t'}/\alpha}^{\infty} \frac{1}{S'\sigma\sqrt{2\pi\tau}} \left(\frac{L_t^{t'}}{S}\right)^{2\mu/\sigma^2} \mathrm{e}^{-\left[\ln\left(S'S/L_t^{t'\,2}\right) - \mu\tau\right]^2 / 2\sigma^2\tau} dS' dL_t^{t'}$$

$$= \int_0^L \left(\frac{L_t^{t'}}{S}\right)^{2\mu/\sigma^2} N\left(\frac{-\ln\frac{S}{\alpha L_t^{t'}} + \mu\tau}{\sigma\sqrt{\tau}}\right) dL_t^{t'}$$

$$= \frac{S^{-2\mu/\sigma^2}}{2\mu/\sigma^2+1} \int_0^L N\left(\frac{\ln \frac{\alpha L_t^{t'}}{S} + \mu\tau}{\sigma\sqrt{\tau}}\right) d\left(L_t^{t'}\right)^{2\mu/\sigma^2+1}$$

$$= \frac{S^{-2\mu/\sigma^2}}{2\mu/\sigma^2+1} \left[\left(L_t^{t'}\right)^{2\mu/\sigma^2+1} N\left(\frac{\ln \frac{\alpha L_t^{t'}}{S} + \mu\tau}{\sigma\sqrt{\tau}}\right)\Bigg|_0^L \right.$$

$$\left. - \int_0^L \left(L_t^{t'}\right)^{2\mu/\sigma^2+1} dN\left(\frac{\ln \frac{\alpha L_t^{t'}}{S} + \mu\tau}{\sigma\sqrt{\tau}}\right)\right]$$

$$= \frac{\sigma^2 S^{-2\mu/\sigma^2}}{2(r-D_0)} L^{2\mu/\sigma^2+1} N\left(\frac{\ln \frac{\alpha L}{S} + \mu\tau}{\sigma\sqrt{\tau}}\right)$$

$$- \frac{\sigma^2 S^{-2\mu/\sigma^2}}{2(r-D_0)} \int_0^L \left(L_t^{t'}\right)^{2\mu/\sigma^2+1} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\left[\ln\left(\alpha L_t^{t'}/S\right)+\mu\tau\right]^2 \big/ 2\sigma^2\tau} \frac{dL_t^{t'}}{\sigma\sqrt{\tau} L_t^{t'}}$$

$$= \frac{\sigma^2 S}{2(r-D_0)} \left(\frac{L}{S}\right)^{2(r-D_0)/\sigma^2} N\left(\frac{\ln \frac{\alpha L}{S} + \mu\tau}{\sigma\sqrt{\tau}}\right) - \frac{\sigma^2 S}{2(r-D_0)} \alpha^{-2\mu/\sigma^2-1}$$

$$\times \mathrm{e}^{\left[-\mu^2+\left(\mu+\sigma^2\right)^2\right]\tau/2\sigma^2} \int_0^L \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\left[\ln\left(\alpha L_t^{t'}/S\right)-\left(\mu+\sigma^2\right)\tau\right]^2/2\sigma^2\tau} \frac{dL_t^{t'}}{\sigma\sqrt{\tau} L_t^{t'}}$$

$$= \frac{\sigma^2 S}{2(r-D_0)} \left(\frac{L}{S}\right)^{2(r-D_0)/\sigma^2} N\left(\frac{\ln \frac{\alpha L}{S} + \mu\tau}{\sigma\sqrt{\tau}}\right)$$

$$- \frac{\sigma^2 S}{2(r-D_0)} \alpha^{-2(r-D_0)/\sigma^2} \mathrm{e}^{(r-D_0)\tau} N\left(\frac{\ln \frac{\alpha L}{S} - \left(\mu+\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right).$$

If $S > L/\alpha$, then the second and the third integrals can be written as

$$\int_L^S \max(\alpha S' - L, 0) f_1(S', S') dS' + \int_S^\infty \max(\alpha S' - L, 0) f_1(S', S) dS'$$

$$= \int_{L/\alpha}^S (\alpha S' - L) f_1(S', S') dS' + \int_S^\infty (\alpha S' - L) f_1(S', S) dS'$$

$$= \alpha \int_{L/\alpha}^S S' \frac{1}{S'\sigma\sqrt{2\pi\tau}} \left(\frac{S'}{S}\right)^{2\mu/\sigma^2} \mathrm{e}^{-\left[\ln\left(S/S'\right)-\mu\tau\right]^2/2\sigma^2\tau} dS'$$

$$-L \int_{L/\alpha}^S \frac{1}{S'\sigma\sqrt{2\pi\tau}} \left(\frac{S'}{S}\right)^{2\mu/\sigma^2} \mathrm{e}^{-\left[\ln\left(S/S'\right)-\mu\tau\right]^2/2\sigma^2\tau} dS'$$

$$+\alpha \int_S^\infty S' \frac{1}{S'\sigma\sqrt{2\pi\tau}} \mathrm{e}^{-\left[\ln\left(S'/S\right)-\mu\tau\right]^2/2\sigma^2\tau} dS'$$

$$-L \int_S^\infty \frac{1}{S'\sigma\sqrt{2\pi\tau}} \mathrm{e}^{-\left[\ln\left(S'/S\right)-\mu\tau\right]^2/2\sigma^2\tau} dS'$$

$$= \alpha S \mathrm{e}^{\left[-\mu^2+\left(\mu+\sigma^2\right)^2\right]\tau/2\sigma^2} \int_{L/\alpha}^S \frac{1}{S'\sigma\sqrt{2\pi\tau}} \mathrm{e}^{-\left[\ln\left(S'/S\right)-\left(\mu+\sigma^2\right)\tau\right]^2/2\sigma^2\tau} dS'$$

$$-L \mathrm{e}^{\left[-\mu^2+\mu^2\right]\tau/2\sigma^2} \int_{L/\alpha}^S \frac{1}{S'\sigma\sqrt{2\pi\tau}} \mathrm{e}^{-\left[\ln\left(S'/S\right)-\mu\tau\right]^2/2\sigma^2\tau} dS'$$

$$+\alpha S \mathrm{e}^{\left[-\mu^2+\left(\mu+\sigma^2\right)^2\right]\tau/2\sigma^2} \int_S^\infty \frac{1}{S'\sigma\sqrt{2\pi\tau}} \mathrm{e}^{-\left[\ln\left(S'/S\right)-\left(\mu+\sigma^2\right)\tau\right]^2/2\sigma^2\tau} dS'$$

$$-L \int_S^\infty \frac{1}{S'\sigma\sqrt{2\pi\tau}} \mathrm{e}^{-\left[\ln\left(S'/S\right)-\mu\tau\right]^2/2\sigma^2\tau} dS'$$

$$= \alpha S \mathrm{e}^{(r-D_0)\tau} \int_{L/\alpha}^\infty \frac{1}{S'\sigma\sqrt{2\pi\tau}} \mathrm{e}^{-\left[\ln\left(S'/S\right)-\left(\mu+\sigma^2\right)\tau\right]^2/2\sigma^2\tau} dS'$$

$$-L \int_{L/\alpha}^\infty \frac{1}{S'\sigma\sqrt{2\pi\tau}} \mathrm{e}^{-\left[\ln\left(S'/S\right)-\mu\tau\right]^2/2\sigma^2\tau} dS'$$

$$= \alpha S \mathrm{e}^{(r-D_0)\tau} N\left(\frac{-\ln\frac{L}{\alpha S}+\left(\mu+\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right) - LN\left(\frac{-\ln\frac{L}{\alpha S}+\mu\tau}{\sigma\sqrt{\tau}}\right).$$

If $S \le L/\alpha$, then for the second and the third integrals, we have

$$\int_L^S \max\left(\alpha S'-L,0\right) f_1\left(S',S'\right) dS' + \int_S^\infty \max\left(\alpha S'-L,0\right) f_1\left(S',S\right) dS'$$

$$= \int_{L/\alpha}^\infty \left(\alpha S'-L\right) f_1\left(S',S\right) dS'$$

$$= \alpha \int_{L/\alpha}^\infty S' \frac{1}{S'\sigma\sqrt{2\pi\tau}} \mathrm{e}^{-\left[\ln\left(S'/S\right)-\mu\tau\right]^2/2\sigma^2\tau} dS'$$

$$-L \int_{L/\alpha}^\infty \frac{1}{S'\sigma\sqrt{2\pi\tau}} \mathrm{e}^{-\left[\ln\left(S'/S\right)-\mu\tau\right]^2/2\sigma^2\tau} dS'$$

$$= \alpha S \mathrm{e}^{(r-D_0)\tau} \int_{L/\alpha}^{\infty} \frac{1}{S'\sigma\sqrt{2\pi\tau}} \mathrm{e}^{-\left[\ln\left(S'/S\right)-\left(\mu+\sigma^2\right)\tau\right]^2/2\sigma^2\tau} dS'$$

$$-L \int_{L/\alpha}^{\infty} \frac{1}{S'\sigma\sqrt{2\pi\tau}} \mathrm{e}^{-\left[\ln\left(S'/S\right)-\mu\tau\right]^2/2\sigma^2\tau} dS'$$

$$= \alpha S \mathrm{e}^{(r-D_0)\tau} N\left(\frac{-\ln\frac{L}{\alpha S}+\left(\mu+\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right) - LN\left(\frac{-\ln\frac{L}{\alpha S}+\mu\tau}{\sigma\sqrt{\tau}}\right).$$

Thus, the results are the same for $S \le L/\alpha$ and $S > L/\alpha$. Consequently, the price of the European lookback strike call option is

$$\begin{aligned}
&c_{ls}(S,L,t) \\
&= \mathrm{e}^{-r\tau} S \left[\frac{\sigma^2}{2(r-D_0)}\left(\frac{L}{S}\right)^{2(r-D_0)/\sigma^2} N\left(\frac{\ln\frac{\alpha L}{S}+\left(r-D_0-\sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}\right)\right. \\
&-\frac{\sigma^2}{2(r-D_0)}\alpha^{-2(r-D_0)/\sigma^2}\mathrm{e}^{(r-D_0)\tau} N\left(\frac{\ln\frac{\alpha L}{S}-\left(r-D_0+\sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}\right) \\
&+\alpha \mathrm{e}^{(r-D_0)\tau} N\left(\frac{\ln\frac{\alpha S}{L}+\left(r-D_0+\sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}\right) \\
&\left.-\frac{L}{S} N\left(\frac{\ln\frac{\alpha S}{L}+\left(r-D_0-\sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}\right)\right].
\end{aligned}$$

This is a closed-form solution for a lookback strike call option with $\alpha \le 1$. From this formula, we know that c_{ls}/S is a function of L/S and t, so c_{ls}/S is a solution of a one-dimensional problem.

Using a similar procedure, we can have explicit formulae for lookback strike put options and lookback price options. We leave these for readers as Problems 12–14.

4.4.4 American Options Formulated as Free-Boundary Problems

First, consider the American lookback strike put option with continuous sampling. In Sect. 4.4.2, this problem was formulated as a LC problem. Here let us formulate this problem as a free-boundary problem. According to Theorem 3.1 in Sect. 3.1, we need to check

$$\left(\frac{\partial}{\partial t}+\mathbf{L}_\eta\right)\max\left(\eta-\beta,0\right).$$

When $\eta < \beta$, we have

$$\left(\frac{\partial}{\partial t}+\mathbf{L}_\eta\right)\max\left(\eta-\beta,0\right)=0;$$

and when $\beta < \eta$, we obtain

$$\left(\frac{\partial}{\partial t} + \mathbf{L}_\eta\right) \max(\eta - \beta, 0) = (D_0 - r)\eta - D_0(\eta - \beta) = -r\eta + \beta D_0.$$

The inequality $-r\eta + \beta D_0 > 0$ is equivalent to $\eta < \beta D_0/r$. Therefore, when $\beta D_0/r > \beta$, there exists an interval $[\beta, \beta D_0/r]$ where

$$\left(\frac{\partial}{\partial t} + \mathbf{L}_\eta\right) \max(\eta - \beta, 0) > 0;$$

when $\beta D_0/r < \beta$, no such an interval exists. Thus when

$$\eta < \beta \max(1, D_0/r),$$

we have

$$\left(\frac{\partial}{\partial t} + \mathbf{L}_\eta\right) \max(\eta - \beta, 0) \geq 0.$$

Otherwise, i.e., when

$$\beta \max(1, D_0/r) < \eta,$$

we have

$$\left(\frac{\partial}{\partial t} + \mathbf{L}_\eta\right) \max(\eta - \beta, 0) < 0.$$

Consequently, at time T there is only one free boundary, and its location is $\beta \max(1, D_0/r)$. Let $V(S, H, t)$ be the price of an American lookback strike put option. In Sect. 3.1.2, for American vanilla options we proved that $V(S, t^*) \geq V(S, t^{**})$ if $t^* < t^{**}$. For American lookback options, the situation is similar. By the same method it can be proved[3] that

$$V(S, H, t^*) \geq V(S, H, t^{**}) \quad \text{if} \quad t^* < t^{**}.$$

From this inequality, we can further have

$$W(\eta, t^*) \geq W(\eta, t^{**}) \quad \text{if} \quad t^* < t^{**}.$$

Therefore, no new free boundary can appear at $t < T$. Consequently, the rectangular domain $[1, \infty) \times [0, T]$ can be divided into two parts:

$$[1, \eta_f(t)] \times [0, T]$$

and

$$(\eta_f(t), \infty) \times [0, T],$$

where the curve $\eta = \eta_f(t)$ is the free boundary with $\eta_f(T) = \beta \max(1, D_0/r)$. In the left region, the price of the American option, W, satisfies the partial differential equation in the linear complementarity problem, and in the right

[3] A similar problem is given as a part of Problem 16 for readers to prove.

region, W is equal to $\max(\eta-\beta,0)$. That is, in the left region, the solution of the following free-boundary problem

$$\begin{cases} \dfrac{\partial W}{\partial t}+\dfrac{1}{2}\sigma^2\eta^2\dfrac{\partial^2 W}{\partial\eta^2}+(D_0-r)\,\eta\dfrac{\partial W}{\partial\eta}-D_0W=0, & \\ \qquad\qquad\qquad\qquad 1\le\eta\le\eta_f(t),\ t\le T, & \\ W(\eta,T)=\max(\eta-\beta,0), & 1\le\eta\le\eta_f(T), \\ \dfrac{\partial W}{\partial\eta}(1,t)=0, & t\le T, \\ W(\eta_f,t)=\eta_f-\beta, & t\le T, \\ \dfrac{\partial W}{\partial\eta}(\eta_f,t)=1, & t\le T, \\ \eta_f(T)=\beta\max(1,D_0/r) & \end{cases} \tag{4.44}$$

provides the price of the American option, and in the domain $(\eta_f(t),\infty)\times[0,T]$, the value of W is $\max(\eta-\beta,0)$. We will call the problem (4.44) the free-boundary problem for American lookback strike put options. This problem has boundary conditions on the both sides, which is similar to the American barrier options but different from American vanilla options and American Asian options.

Using the same procedure, we can find that in the domain $[\eta_f(t),1]\times[0,T]$ the price of an American lookback strike call option with continuous sampling is the solution of the free-boundary problem

$$\begin{cases} \dfrac{\partial W}{\partial t}+\dfrac{1}{2}\sigma^2\eta^2\dfrac{\partial^2 W}{\partial\eta^2}+(D_0-r)\,\eta\dfrac{\partial W}{\partial\eta}-D_0W=0, & \\ \qquad\qquad\qquad\qquad \eta_f(t)\le\eta\le 1,\ t\le T, & \\ W(\eta,T)=\max(\alpha-\eta,0), & \eta_f(T)\le\eta\le 1, \\ W(\eta_f,t)=\alpha-\eta_f, & t\le T, \\ \dfrac{\partial W}{\partial\eta}(\eta_f,t)=-1, & t\le T, \\ \dfrac{\partial W}{\partial\eta}(1,t)=0, & t\le T, \\ \eta_f(T)=\alpha\min(1,D_0/r); & \end{cases} \tag{4.45}$$

whereas in the domain $[0,\eta_f(t))\times[0,T]$, the value of W is $\max(\alpha-\eta,0)$. This is left for readers as a part of Problem 17.

4.4.5 A Closed-Form Solution for a Perpetual American Lookback Option

In Sect. 3.3.5, closed-form solutions for perpetual American vanilla options were derived. Here, a closed-form solution for a perpetual American lookback option will be derived. As explained in Sect. 3.3.5, a perpetual option means an option with $T = \infty$. Because $T = \infty$, the solution does not depend on t and the final condition. Let us look at a perpetual American lookback option that depends on S and H and satisfies the constraint $V \ge H$. This option is called the Russian option (see [29]). We again let $\eta = H/S$ and $W = V/S$. The constraint now should be $W \ge \eta$. Therefore, from the procedure to derive the problem (4.44), we can see that the free-boundary problem for this case is

$$\begin{cases} \dfrac{1}{2}\sigma^2\eta^2\dfrac{d^2W}{d\eta^2} + (D_0 - r)\,\eta\dfrac{dW}{d\eta} - D_0W = 0, \qquad 1 \le \eta \le \eta_f, \\[2mm] \dfrac{dW(1)}{d\eta} = 0, \\[2mm] W(\eta_f) = \eta_f, \\[2mm] \dfrac{dW(\eta_f)}{d\eta} = 1, \end{cases}$$

where η_f is a number representing the location of the free boundary. Let us look for a solution of the ordinary differential equation in the form η^α. After substituting this function into the equation, we know that α is a root of the quadratic equation

$$\frac{1}{2}\sigma^2\alpha^2 + \left(D_0 - r - \frac{1}{2}\sigma^2\right)\alpha - D_0 = 0.$$

Therefore, the solution of the problem is in the form

$$W = C_+\left(\frac{\eta}{\eta_f}\right)^{\alpha_+} + C_-\left(\frac{\eta}{\eta_f}\right)^{\alpha_-},$$

where

$$\alpha_+ = \frac{-D_0 + r + \dfrac{1}{2}\sigma^2 + \sqrt{\left(D_0 - r - \dfrac{1}{2}\sigma^2\right)^2 + 2\sigma^2 D_0}}{\sigma^2}$$

and

$$\alpha_- = \frac{-D_0 + r + \dfrac{1}{2}\sigma^2 - \sqrt{\left(D_0 - r - \dfrac{1}{2}\sigma^2\right)^2 + 2\sigma^2 D_0}}{\sigma^2}.$$

Consequently, these boundary conditions become

$$\begin{cases} \dfrac{dW(1)}{d\eta} = C_+\alpha_+\eta_f^{-\alpha_+} + C_-\alpha_-\eta_f^{-\alpha_-} = 0, \\ W(\eta_f) = C_+ + C_- = \eta_f, \\ \dfrac{dW(\eta_f)}{d\eta} = C_+\alpha_+\eta_f^{-1} + C_-\alpha_-\eta_f^{-1} = 1. \end{cases}$$

From the last two equations, we have

$$C_+(1-\alpha_+) + C_-(1-\alpha_-) = 0.$$

Comparing this equation with $C_+\alpha_+\eta_f^{-\alpha_+} + C_-\alpha_-\eta_f^{-\alpha_-} = 0$, we know

$$\frac{1-\alpha_+}{1-\alpha_-} = \frac{\alpha_+\eta_f^{-\alpha_+}}{\alpha_-\eta_f^{-\alpha_-}},$$

which gives

$$\eta_f = \left[\frac{\alpha_+(1-\alpha_-)}{\alpha_-(1-\alpha_+)}\right]^{\dfrac{1}{\alpha_+-\alpha_-}}.$$

From $C_+ + C_- = \eta_f$ and $C_+(1-\alpha_+) + C_-(1-\alpha_-) = 0$, we have

$$C_+ = \frac{\eta_f(1-\alpha_-)}{\alpha_+-\alpha_-},$$
$$C_- = \frac{-\eta_f(1-\alpha_+)}{\alpha_+-\alpha_-}.$$

Therefore, the solution of the free-boundary problem is

$$W(\eta) = \frac{\eta_f}{\alpha_+-\alpha_-}\left[(1-\alpha_-)\left(\frac{\eta}{\eta_f}\right)^{\alpha_+} - (1-\alpha_+)\left(\frac{\eta}{\eta_f}\right)^{\alpha_-}\right]$$

and the price of a Russian option is

$$V(S,H) = \begin{cases} \dfrac{\eta_f S}{\alpha_+-\alpha_-}\left[(1-\alpha_-)\left(\dfrac{H}{\eta_f S}\right)^{\alpha_+} - (1-\alpha_+)\left(\dfrac{H}{\eta_f S}\right)^{\alpha_-}\right], \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } 1 \le \dfrac{H}{S} < \eta_f, \\ H, \qquad\qquad\qquad\qquad\qquad\quad \text{if } \eta_f \le \dfrac{H}{S}. \end{cases}$$

Using the method given in Sect. 3.3.5 for perpetual American call option, it is not difficult to show that this function is a solution of the LC problem:

$$\begin{cases} \min\left(-\left[\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0)\, S \frac{\partial V}{\partial S} - rV\right],\ V(S,H) - H\right) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le S \le H, \\ \frac{\partial V}{\partial H}(S,S) = 0, \qquad\qquad\qquad\qquad\qquad 0 \le S. \end{cases}$$

This is left for readers to prove as a part of Problem 19. If $D_0 = 0$, then $\alpha_- = 0$ and $\eta_f = \infty$, implying that there is no free boundary in this case.

4.4.6 Lookback-Asian Options

Suppose the sampling is done discretely. An option could depend on the average of N largest or smallest sampled prices. For example, an European option could have such a payoff function

$$\max\left(\frac{1}{N}\sum_{n=1}^{N} S_n - S,\ 0\right), \tag{4.46}$$

where S_n is the nth largest sampled price during a time period $[0, T]$, T being the expiry. Let us call it a European lookback-Asian option. Consider the following scenario. During the peak season of July–September, the price of electricity might be very high for only a few days, and it is uncertain about how high the price will be. Therefore, there is a risk for consumers of electricity. In October, the price of electricity is normal, so if the expiry of the option is in October, a holder of such an option will receive certain amount of money that is proportional to the difference between the average of uncertain high prices and the normal price. Therefore, a consumer of electricity can be protected from the risk if the consumer holds such an option. Consequently, in May or so, electricity companies sell options with a payoff function (4.46) and an expiry in October, which are interesting for many companies who consume a large amount of electricity.

Let us assume that S satisfies

$$dS = \mu S dt + \sigma(t) S dX,$$

where dX is a Wiener process. In this model, we require that $\sigma(t)$ is a function of t so that it is possible to have a big volatility in one period, for example, in July–September, and to have a normal volatility in other periods, for example, in May and in October.

Now let us determine $\frac{dS_n}{dt}$. Suppose that sampling is performed at $t = t_i$, $i = 1, 2, \cdots, K$ with $K \ge N$, and $0 < t_1 < t_2 < \cdots < t_K < T$. Let $S_n(t_i^-)$ and

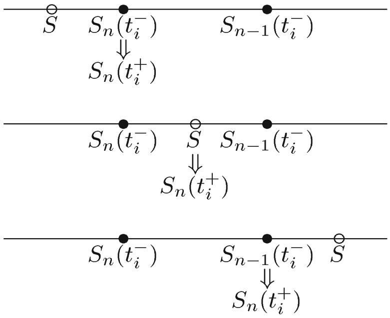

Fig. 4.6. Different $S_n(t_i^+)$ for three cases

$S_n(t_i^+)$ denote the nth largest price before and after the sampling procedure at $t = t_i$, respectively, and let S be the price at $t = t_i$. In this case, $S_1(t_i^+)$ is determined by S and $S_1(t_i^-)$ according to the following relation:

$$S_1(t_i^+) = \begin{cases} S_1(t_i^-), \text{ if } & S \le S_1(t_i^-), \\ S, \quad \text{if} & S_1(t_i^-) < S \end{cases}$$

or

$$S_1(t_i^+) = \max(S, S_1(t_i^-)).$$

For $1 < n \le N$, $S_n(t_i^+)$ can be expressed by S, $S_{n-1}(t_i^-)$, and $S_n(t_i^-)$ as follows (see Fig. 4.6):

$$S_n(t_i^+) = \begin{cases} S_n(t_i^-), & \text{if} \quad S \le S_n(t_i^-), \\ S, & \text{if} \quad S_n(t_i^-) < S \le S_{n-1}(t_i^-), \\ S_{n-1}(t_i^-), & \text{if} \quad S_{n-1}(t_i^-) < S \end{cases}$$

or[4]

$$S_n(t_i^+) = \max(\min(S, S_{n-1}(t_i^-)),\ S_n(t_i^-)).$$

Therefore at $t = t_i$

$$\frac{dS_n}{dt} = \begin{cases} \left[\max(S, S_1(t_i^-)) - S_1(t_i^-)\right] \delta(t - t_i), & \text{if } n = 1, \\ \left[\max(\min(S, S_{n-1}(t_i^-)),\ S_n(t_i^-)) - S_n(t_i^-)\right] \delta(t - t_i), & \\ & \text{if } n = 2, 3, \cdots, N. \end{cases}$$

If $t \ne t_i$, $i = 1, 2, \cdots, K$, then $\dfrac{dS_n}{dt} = 0$. Let $V(S, S_1, S_2, \cdots, S_N, t)$ denote the price of the option. From Sect. 2.3, we know that V satisfies

[4] $S_n(t_i^+)$ can also be expressed as $S_n(t_i^+) = \min(S_{n-1}(t_i^-), \max(S, S_n(t_i^-)))$.

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}+(r-D_0)S\frac{\partial V}{\partial S}+\sum_{n=1}^{N}\frac{\partial V}{\partial S_n}\frac{dS_n}{dt}-rV=0. \tag{4.47}$$

At $t\neq t_i$, $i=1,2,\cdots,K$, because $\dfrac{dS_n}{dt}=0$, this equation reduces to

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}+(r-D_0)S\frac{\partial V}{\partial S}-rV=0; \tag{4.48}$$

whereas at $t=t_i$, $i=1,2,\cdots$, or K, according to Sect. 2.5, V satisfies the following jump condition:

$$\begin{aligned}V(S,S_1^-,S_2^-,\cdots,S_N^-,t_i^-)=&\,V(S,\max(S,S_1^-),\max(\min(S,S_1^-),S_2^-),\\&\cdots,\max(\min(S,S_{N-1}^-),S_N^-),t_i^+),\end{aligned} \tag{4.49}$$

where S_n^- stands for $S_n(t_i^-)$. This is the jump condition for lookback-Asian options.

In order to get the solution of this problem, it might be necessary to use numerical methods. This problem actually is an $(N+1)$-dimensional problem, so solving such a problem is very time-consuming. In order to reduce the time needed to get a numerical solution, we can reduce it to an N-dimensional problem by letting

$$\xi=\frac{S}{S_N},\ \xi_1=\frac{S_1}{S_N},\ \cdots,\ \xi_{N-1}=\frac{S_{N-1}}{S_N},$$

and

$$U(\xi,\xi_1,\cdots,\xi_{N-1},t)=\frac{V(S,S_1,\cdots,S_{N-1},S_N,t)}{S_N}.$$

Moreover, in this case we need to solve a problem defined on an infinite domain. As pointed out in Sect. 2.2.5, it is not convenient to solve a problem on an infinite domain numerically. In order to avoid such a problem, we can transform the problem on an infinite domain to a problem on a finite domain by using a way similar to the way given in Sect. 2.2.5.

4.4.7 Some Examples

In this subsection, we give some results for some of those problems formulated in the previous subsections. These results are obtained by using the numerical methods in Part II.

In Fig. 4.7, the function $W(\eta,t)$ of an American lookback strike call option with continuous sampling for $t=0$, 0.2, 0.4, 0.6, 0.8 is shown. From the figure, we know that $W(\eta,t)=V(S,L,t)/S$ is a decreasing function in $\eta=L/S$, i.e., if S is fixed, then $V(S,L,t)$ is a decreasing function in L. This is because the value of the payoff $\max(S-L,0)$ decreases or does not change as L increases. The lowest price up to time t is of course less than or equal to the price at

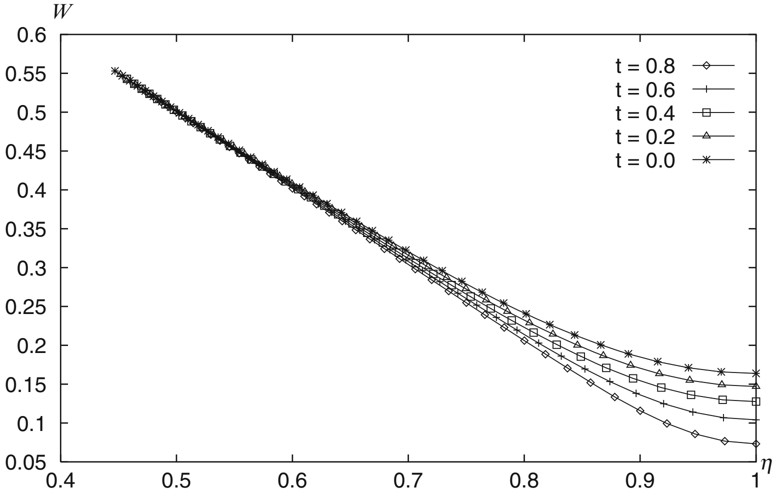

Fig. 4.7. $W(\eta, t)$ of an American lookback strike call option ($r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$, and $\alpha = 1$)

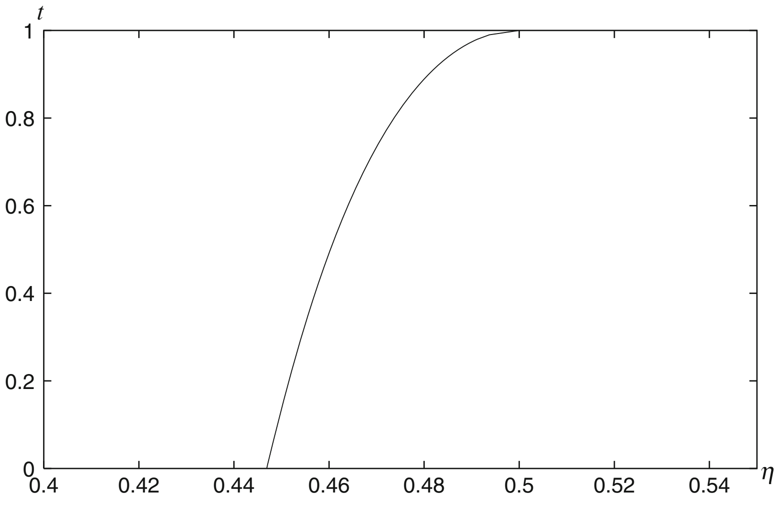

Fig. 4.8. The free boundary of an American lookback strike call option ($r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$, and $\alpha = 1$)

time t. Thus, $\eta = L/S$ must be less than or equal to 1. Therefore, for a fixed t, the price has a minimum at $\eta = L/S = 1$. The minimum price at $t = 0$ is 16.37 % of S, which is much higher than 9.94 %—the value for the vanilla case with $S = E$. In Fig. 4.8, the location of the free boundary on (η, t)-plane is also given, which is a monotone function in t.

In Table 4.5, today's prices of some lookback strike put options with discrete sampling are given. The parameters are given there. The sampling times are $t_k = t_0 + (2k-1)T/2K$, $k = 1, 2, \cdots, K$, where t_0 is the time today. Both European and American option prices are given. From the table, we see that the American option prices are about 9 % higher than the European option prices for these cases. The table also shows that the larger the number K, the higher the price. This is because H and the payoff $\max(H-S, 0)$ increase or do not change as K increases. For comparison, the values of the vanilla options are also given. From those values, we can see that the prices of lookback options are much higher than those of the corresponding vanilla options.

Table 4.5. Lookback strike put option prices

($r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$, $S = H = 100$, $T = 1$, and $\beta = 1$)

	$K = 48$	$K = 24$	$K = 12$	$K = 6$	$K = 3$	Vanilla
European option	11.865	11.283	10.499	9.535	8.456	5.302
American option	12.893	12.280	11.452	10.427	9.277	5.928

4.5 Multi-Asset Options

Sometimes an option involves several assets. Such an option is called a multi-asset option. For example, a U.S. company buys raw materials from foreign country A and sells its product in foreign country B. Its income depends on the exchange rates of the currencies of the foreign countries A and B. The higher the exchange rate of the currency of country A, the higher the cost, and the higher the exchange rate of the currency of country B, the higher the revenue. Here, an exchange rate of a foreign currency is referred to as the price of one unit of the foreign currency in U.S. dollars. Therefore, the company is interested in an option of exchanging the currency of country B into the currency of country A with a fixed rate because such an option protects the company from the exchange rate risk. Such an option involves two assets and is called an exchange option or a cross-currency option. Besides this, there are many options involving more than one asset. For example, options on the maximum or the minimum of several assets, multi-asset call/put options, and basket options are such options. This section is devoted to such options.

In this section, first we give the equation and Green's formula of solution of European multi-asset options. Then, we study the exchange options and options on the extremum of several assets. If these options are European, it is possible to express their solution in terms of multivariate cumulative normal distribution functions. Derivation of such expressions are given when we study them. Finally, the formulation of multi-asset option problems on a finite domain is given, which is useful when such a option problem has to be solved numerically, and such American option problems are briefly mentioned. Kwok in his book [54] gave an excellent summary on multi-asset options. For more details on this subject, readers are referred to that book.

4.5.1 †Equations for Multi-Asset Options and Green's Formula

Consider an option dependent on n assets. Let S_i be the price of the ith underlying asset and $V(S_1, S_2, \cdots, S_n, t)$ represent the price of the option. For simplicity, $V(S_1, S_2, \cdots, S_n, t)$ is sometimes written as $V(\mathbf{S}, t)$, $\mathbf{S}$ being $(S_1, S_2, \cdots, S_n)^T$. Suppose that S_i satisfies

$$dS_i = \mu_i S_i dt + \sigma_i S_i dX_i, \quad 0 \le S_i, \tag{4.50}$$

and the ith asset pays a dividend $D_{0i} S_i dt$ in a time step dt. $dX_i = \phi_i \sqrt{dt}$, $i = 1, 2, \cdots, n$, are Wiener processes. ϕ_1, ϕ_2, $\cdots$, ϕ_n have a joint normal distribution and

$$\mathrm{E}\left[\phi_i \phi_j\right] = \rho_{ij}, \quad i, j = 1, 2, \cdots, n. \tag{4.51}$$

It is clear that $\rho_{ij} = \rho_{ji}$ and $\rho_{ii} = 1, i = 1, 2, \cdots, n$. Let us call ρ_{ij} the correlation coefficient between the two standardized normal random variables associated with S_i and S_j, or simply, the correlation coefficient between S_i and S_j. According to Sect. 2.3, $V(\mathbf{S}, t)$ satisfies

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2} \displaystyle\sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} \sigma_i \sigma_j S_i S_j \dfrac{\partial^2 V}{\partial S_i \partial S_j} + \sum_{i=1}^{n} (r - D_{0i}) S_i \dfrac{\partial V}{\partial S_i} - rV = 0, \\ \qquad\qquad\qquad\qquad 0 \le \mathbf{S}, \quad 0 \le t \le T, \end{cases} \tag{4.52}$$

where the inequality $0 \le \mathbf{S}$ means $0 \le S_i$, $i = 1, 2, \cdots, n$, and similar notation will be used later on. Suppose the payoff function of an option is

$$V(\mathbf{S}, T) = V_T(\mathbf{S}), \ 0 \le \mathbf{S}, \tag{4.53}$$

we need to evaluate the price of the option dependent on n assets and with the payoff above.

In what follows, we assume that μ_i, σ_i and D_{0i} are constants and that the option is European, and we want to find the solution in an integral form. Let $V(\mathbf{S}, t) = \mathrm{e}^{-r(T-t)} \overline{V}(\mathbf{S}, t)$. Because

$$\begin{aligned} \frac{\partial V}{\partial t} &= rV + \mathrm{e}^{-r(T-t)} \frac{\partial \overline{V}}{\partial t}, \\ \frac{\partial V}{\partial S_i} &= \mathrm{e}^{-r(T-t)} \frac{\partial \overline{V}}{\partial S_i}, \\ \frac{\partial^2 V}{\partial S_i \partial S_j} &= \mathrm{e}^{-r(T-t)} \frac{\partial^2 \overline{V}}{\partial S_i \partial S_j}, \end{aligned}$$

we know that $\overline{V}(\mathbf{S},t)$ is the solution of the problem

$$\begin{cases} \dfrac{\partial \overline{V}}{\partial t} + \dfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}\sigma_i\sigma_j S_i S_j \dfrac{\partial^2 \overline{V}}{\partial S_i \partial S_j} + \sum_{i=1}^{n}(r-D_{0i})S_i\dfrac{\partial \overline{V}}{\partial S_i} = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 \le \mathbf{S}, \quad 0 \le t \le T, \\ \overline{V}(\mathbf{S},T) = V_T(\mathbf{S}), \qquad\qquad 0 \le \mathbf{S}. \end{cases} \tag{4.54}$$

Furthermore, we introduce the following transformation

$$\begin{cases} y_i = \dfrac{\sqrt{2}}{\sigma_i}[\ln S_i + (r - D_{0i} - \sigma_i^2/2)(T-t)], \qquad i = 1,2,\cdots,n, \\ \tau = T - t, \end{cases}$$

and let $\overline{V}_1(\mathbf{y},\tau) = \overline{V}(\mathbf{S},t)$, where $\mathbf{y}$ stands for $(y_1, y_2, \cdots, y_n)^T$. Noticing

$$\begin{aligned} \frac{\partial \overline{V}}{\partial t} &= -\frac{\partial \overline{V}_1}{\partial \tau} - \sum_{i=1}^{n}\frac{\sqrt{2}}{\sigma_i}\left(r - D_{0i} - \frac{\sigma_i^2}{2}\right)\frac{\partial \overline{V}_1}{\partial y_i}, \\ \frac{\partial \overline{V}}{\partial S_i} &= \frac{\sqrt{2}}{\sigma_i S_i}\frac{\partial \overline{V}_1}{\partial y_i}, \\ \frac{\partial^2 \overline{V}}{\partial S_i^2} &= -\frac{\sqrt{2}}{\sigma_i S_i^2}\frac{\partial \overline{V}_1}{\partial y_i} + \frac{2}{\sigma_i^2 S_i^2}\frac{\partial^2 \overline{V}_1}{\partial y_i^2}, \\ \frac{\partial^2 \overline{V}}{\partial S_i \partial S_j} &= \frac{2}{\sigma_i\sigma_j S_i S_j}\frac{\partial^2 \overline{V}_1}{\partial y_i \partial y_j}, \qquad i \ne j, \end{aligned}$$

we can rewrite the problem above as follows:

$$\begin{cases} \dfrac{\partial \overline{V}_1}{\partial \tau} = \sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}\dfrac{\partial^2 \overline{V}_1}{\partial y_i \partial y_j}, \qquad -\infty < \mathbf{y} < \infty, \quad 0 \le \tau \le T, \\ \overline{V}_1(\mathbf{y},0) = V_{1T}(\mathbf{y}), \qquad -\infty < \mathbf{y} < \infty, \end{cases} \tag{4.55}$$

where

$$V_{1T}(\mathbf{y}) = V_T\left(\mathrm{e}^{\sigma_1 y_1/\sqrt{2}}, \mathrm{e}^{\sigma_2 y_2/\sqrt{2}}, \cdots, \mathrm{e}^{\sigma_n y_n/\sqrt{2}}\right).$$

Define

$$\mathbf{P} = \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & 1 & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \cdots & 1 \end{bmatrix}.$$

Because $\mathbf{P}$ should be a symmetric positive definite matrix, there exists an orthogonal matrix[5] $\mathbf{Q}$ and a diagonal positive definite matrix $\mathbf{\Lambda}$ such that

[5]If $\mathbf{Q}^{-1} = \mathbf{Q}^T$, which is equivalent to $\mathbf{Q}\mathbf{Q}^T = \mathbf{I}$ or $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$, then $\mathbf{Q}$ is called an orthogonal matrix.

$$\mathbf{P} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^T = \mathbf{Q}\boldsymbol{\Lambda}^{1/2}\boldsymbol{\Lambda}^{1/2}\mathbf{Q}^T, \quad \mathbf{P}^{-1} = \mathbf{Q}\boldsymbol{\Lambda}^{-1/2}\boldsymbol{\Lambda}^{-1/2}\mathbf{Q}^T$$

and

$$\boldsymbol{\Lambda}^{-1/2}\mathbf{Q}^T\mathbf{P}\mathbf{Q}\boldsymbol{\Lambda}^{-1/2} = \mathbf{I}.$$

Let

$$\mathbf{R} = \boldsymbol{\Lambda}^{-1/2}\mathbf{Q}^T \equiv \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{bmatrix},$$

then the relations above can further be rewritten as

$$\mathbf{P}^{-1} = \mathbf{R}^T\mathbf{R}$$

and

$$\mathbf{R}\mathbf{P}\mathbf{R}^T = \mathbf{I}$$

or in component form

$$\sum_{i=1}^{n}\sum_{j=1}^{n} r_{li}\rho_{ij}r_{kj} = \delta_{lk}, \;\; l,k = 1,2,\cdots,n.$$

Now we define new variables as follows:

$$\mathbf{x} \equiv \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

and further rewrite the partial differential equation in the problem (4.55) as

$$\begin{aligned} \frac{\partial \overline{V}_1}{\partial \tau} &= \sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{ij}\frac{\partial^2 \overline{V}_1}{\partial y_i \partial y_j} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \left[\rho_{ij}\sum_{l=1}^{n}\sum_{k=1}^{n} r_{li}r_{kj}\frac{\partial^2 \overline{V}_2}{\partial x_l \partial x_k}\right] \\ &= \sum_{l=1}^{n}\sum_{k=1}^{n} \delta_{lk}\frac{\partial^2 \overline{V}_2}{\partial x_l \partial x_k} \\ &= \sum_{l=1}^{n} \frac{\partial^2 \overline{V}_2}{\partial x_l^2}, \end{aligned}$$

where $\overline{V}_2(\mathbf{x},\tau) = \overline{V}_1(\mathbf{y},\tau)$. Consequently, the problem is now reduced to

$$\begin{cases} \dfrac{\partial \overline{V}_2(\mathbf{x},\tau)}{\partial \tau} = \displaystyle\sum_{l=1}^{n} \frac{\partial^2 \overline{V}_2(\mathbf{x},\tau)}{\partial x_l^2}, & -\infty < \mathbf{x} < \infty, \quad 0 \le \tau \le T, \\ \overline{V}_2(\mathbf{x},0) = V_{2T}(\mathbf{x}), & -\infty < \mathbf{x} < \infty, \end{cases} \tag{4.56}$$

where $V_{2T}(\mathbf{x}) = V_{1T}(\mathbf{R}^{-1}\mathbf{x})$. Similar to the one-dimensional case, the function[6]

$$\phi(\mathbf{x}_0; \mathbf{x}, \tau) = \frac{1}{(4\pi\tau)^{n/2}} \mathrm{e}^{-\sum_{i=1}^{n}(x_i - x_{i0})^2/(4\tau)}$$

is a solution to the equation above, where x_{i0}, $i = 1, 2, \cdots, n$ are constants, and $\mathbf{x}_0$ stands for $(x_{10}, x_{20}, \cdots, x_{n0})^T$. This can be verified by finding the derivatives $\dfrac{\partial \phi}{\partial \tau}, \dfrac{\partial^2 \phi}{\partial x_l^2}, l = 1, 2, \cdots, n$ and substituting them in the equation above. It can also be shown straightforwardly that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \phi(\mathbf{x}_0; \mathbf{x}, \tau) dx_{10} dx_{20} \cdots dx_{n0} = 1$$

and

$$\lim_{\tau \to 0} \phi(\mathbf{x}_0; \mathbf{x}, \tau) = \begin{cases} \infty, & \text{at } \mathbf{x} = \mathbf{x}_0, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, the expression

$$\overline{V}_2(\mathbf{x}, \tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} V_{2T}(\mathbf{x}_0) \phi(\mathbf{x}_0; \mathbf{x}, \tau) dx_{10} dx_{20} \cdots dx_{n0}$$

is a solution to the problem (4.56), which shows how $\overline{V}(\mathbf{x}, \tau)$ depends on the solution at $\tau = 0$, $V_{2T}(\mathbf{x}_0)$. This is left as an exercise to readers. Consequently, the solution of Eq. (4.52) with condition (4.53) is

$$\begin{aligned} & V(\mathbf{S}, t) \\ &= \mathrm{e}^{-r(T-t)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} V_{2T}(\mathbf{x}_0) \phi(\mathbf{x}_0; \mathbf{x}, \tau) dx_{10} dx_{20} \cdots dx_{n0} \\ &= \mathrm{e}^{-r(T-t)} \int_0^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} V_T(\mathbf{S}') \phi J_{\mathbf{x}_0 \mathbf{y}_0} J_{\mathbf{y}_0 \mathbf{S}'} dS_1' dS_2' \cdots dS_n' \\ &= \mathrm{e}^{-r(T-t)} \int_0^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} V_T(\mathbf{S}') \psi(\mathbf{S}'; \mathbf{S}, t) dS_1' dS_2' \cdots dS_n'. \end{aligned} \tag{4.57}$$

Here, $\mathbf{y}_0 = (y_{10}, y_{20}, \cdots, y_{n0})^T = \mathbf{R}^{-1}\mathbf{x}_0$. $J_{\mathbf{x}_0 \mathbf{y}_0}$ and $J_{\mathbf{y}_0 \mathbf{S}'}$ are the Jacobians of the transformation from $\mathbf{x}_0$ to $\mathbf{y}_0$ and from $\mathbf{y}_0$ to $\mathbf{S}'$:

$$J_{\mathbf{x}_0 \mathbf{y}_0} = \frac{\partial(x_{10}, x_{20}, \cdots, x_{n0})}{\partial(y_{10}, y_{20}, \cdots, y_{n0})} = \det \mathbf{R} = \frac{1}{\sqrt{\det \mathbf{P}}},$$

$$J_{\mathbf{y}_0 \mathbf{S}'} = \frac{\partial(y_{10}, y_{20}, \cdots, y_{n0})}{\partial(S_1', S_2', \cdots, S_n')} = \prod_{i=1}^{n} \frac{\sqrt{2}}{\sigma_i S_i'}$$

[6]This function is referred to as Green's function for the n-dimensional heat equation.

and

$$\begin{aligned}
\psi(\mathbf{S}';\mathbf{S},t) &= \phi J_{\mathbf{x}_0\mathbf{y}_0} J_{\mathbf{y}_0\mathbf{S}'} \\
&= \frac{1}{(4\pi\tau)^{n/2}} \mathrm{e}^{-(\mathbf{x}_0-\mathbf{x})^T(\mathbf{x}_0-\mathbf{x})/(4\tau)} \cdot \frac{1}{\sqrt{\det \mathbf{P}}} \cdot \prod_{i=1}^{n} \frac{\sqrt{2}}{\sigma_i S_i'} \\
&= \frac{1}{(2\pi\tau)^{n/2}\sqrt{\det \mathbf{P}} \prod\limits_{i=1}^{n} (\sigma_i S_i')} \mathrm{e}^{-\eta^T \mathbf{R}^T \mathbf{R} \eta/2} \\
&= \frac{1}{(2\pi\tau)^{n/2}\sqrt{\det \mathbf{P}} \prod\limits_{i=1}^{n} (\sigma_i S_i')} \mathrm{e}^{-\eta^T \mathbf{P}^{-1} \eta/2},
\end{aligned} \tag{4.58}$$

where $\eta = (\mathbf{y_0} - \mathbf{y})/\sqrt{2\tau}$, $\mathbf{y_0}$ standing for $\mathbf{R}^{-1}\mathbf{x_0}$, and its ith component is

$$\eta_i\left(S_i'\right) = \frac{1}{\sqrt{2\tau}}\left(\frac{\sqrt{2}}{\sigma_i}\ln S_i' - y_i\right) = \frac{\ln S_i' - \left[\ln S_i + (r - D_{0i} - \sigma_i^2/2)\tau\right]}{\sigma_i\sqrt{\tau}}.$$

The expression (4.57) is called Green's formula. For some payoff function $V_T(\mathbf{S})$, it can be written in terms of multivariate cumulative distribution functions for standardized normal variables, which will be called multivariate cumulative distribution functions for brevity in what follows, and we can have closed-form solutions. In Sect. 4.5.3, some examples will be given. If σ_i depends on S_i, it might be necessary to use numerical methods to price those options.

4.5.2 Exchange Options

Sometimes, a two-asset option problem can be reduced to a one-asset option, and its closed-form solution can be found for such a European option. Exchange options on two assets is such an example.

An exchange option is a contract that gives its holder a right to exchange certain assets for some other assets. In this subsection, we consider an exchange option to exchange n_B shares of asset B for n_A shares of asset A. Suppose S_A and S_B are the prices of assets A and B, respectively. Let

$$\begin{cases} S_1 = n_A S_A, \\ S_2 = n_B S_B. \end{cases}$$

Then, the payoff function of the exchange option is

$$\max(n_A S_A - n_B S_B, 0) = \max(S_1 - S_2, 0).$$

Therefore, if we consider n_A shares of asset A as an asset $\bar{A}$, and n_B shares of asset B as an asset $\bar{B}$, then the exchange option is an option to exchange an asset $\bar{B}$ for another asset $\bar{A}$. Suppose that S_A and S_B satisfy

$$\begin{cases} dS_A = \mu_1 S_A dt + \sigma_1 S_A dX_1, \\ dS_B = \mu_2 S_B dt + \sigma_2 S_B dX_2, \end{cases} \tag{4.59}$$

where dX_1 and dX_2 satisfy

$$E\left[dX_1 dX_2\right] = \rho_{12} dt.$$

Then, using Itô's lemma, we have

$$\begin{cases} dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 dX_1, \\ dS_2 = \mu_2 S_2 dt + \sigma_2 S_2 dX_2. \end{cases}$$

Thus, S_1 and S_2 have the same volatilities as S_A and S_B, respectively. The dividend yields related to the assets A and $\bar{A}$ are the same, and for the assets B and $\bar{B}$, this is also true. In what follows, D_{01} and D_{02} denote the dividend yields related to the assets A and B, respectively. Let $V(S_1, S_2, t)$ be the value of a European exchange option. According to Sect. 4.5.1, V should be the solution of the problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma_1^2 S_1^2 \dfrac{\partial^2 V}{\partial S_1^2} + \rho_{12}\sigma_1\sigma_2 S_1 S_2 \dfrac{\partial^2 V}{\partial S_1 \partial S_2} + \dfrac{1}{2}\sigma_2^2 S_2^2 \dfrac{\partial^2 V}{\partial S_2^2} \\ +(r - D_{01}) S_1 \dfrac{\partial V}{\partial S_1} + (r - D_{02}) S_2 \dfrac{\partial V}{\partial S_2} - rV = 0, \\ \qquad\qquad\qquad\qquad 0 \le S_1,\ 0 \le S_2,\ 0 \le t \le T, \\ V(S_1, S_2, T) = \max(S_1 - S_2, 0), \qquad 0 \le S_1,\ 0 \le S_2. \end{cases} \tag{4.60}$$

This is a two-dimensional problem, but it can be reduced to a one-dimensional problem. In fact, let

$$W = \frac{V}{S_2} \quad \text{and} \quad \xi = \frac{S_1}{S_2},$$

then we have

$$\frac{\partial V}{\partial t} = S_2 \frac{\partial W}{\partial t}, \qquad \frac{\partial V}{\partial S_1} = \frac{\partial W}{\partial \xi}, \qquad \frac{\partial^2 V}{\partial S_1^2} = \frac{\partial^2 W}{\partial \xi^2}\frac{1}{S_2},$$

$$\frac{\partial^2 V}{\partial S_1 \partial S_2} = \frac{-\xi}{S_2}\frac{\partial^2 W}{\partial \xi^2}, \quad \frac{\partial V}{\partial S_2} = W - \xi\frac{\partial W}{\partial \xi}, \quad \frac{\partial^2 V}{\partial S_2^2} = \frac{\xi^2}{S_2}\frac{\partial^2 W}{\partial \xi^2},$$

and the problem (4.60) can be rewritten as

$$\begin{cases} \dfrac{\partial W}{\partial t} + \dfrac{1}{2}(\sigma_1^2 - 2\rho_{12}\sigma_1\sigma_2 + \sigma_2^2)\xi^2 \dfrac{\partial^2 W}{\partial \xi^2} + (D_{02} - D_{01})\xi \dfrac{\partial W}{\partial \xi} \\ -D_{02}\, W = 0, \qquad\qquad 0 \le \xi, \quad 0 \le t \le T, \\ W(\xi, T) = \max(\xi - 1, 0), \qquad 0 \le \xi. \end{cases} \tag{4.61}$$

This is a European call option problem with $r = D_{02}$, $D_0 = D_{01}$, $E = 1$, and

$$\sigma = \sigma_{12} \equiv \sqrt{\sigma_1^2 - 2\rho_{12}\sigma_1\sigma_2 + \sigma_2^2}.$$

Hence

$$W(\xi, t) = \xi e^{-D_{01}(T-t)} N(d_{10}) - e^{-D_{02}(T-t)} N(d_{20}),$$

where

$$d_{10} = \frac{\ln \dfrac{\xi e^{-D_{01}(T-t)}}{e^{-D_{02}(T-t)}} + \dfrac{\sigma_{12}^2}{2}(T-t)}{\sigma_{12}\sqrt{T-t}}$$

and

$$d_{20} = d_{10} - \sigma_{12}\sqrt{T-t}.$$

Therefore

$$\begin{aligned} V(S_1, S_2, t) &= S_1 e^{-D_{01}(T-t)} N(d_{10}) - S_2 e^{-D_{02}(T-t)} N(d_{20}) \\ &= n_A S_A e^{-D_{01}(T-t)} N(d_{10}) - n_B S_B e^{-D_{02}(T-t)} N(d_{20}), \end{aligned} \quad (4.62)$$

where

$$d_{10} = \frac{\ln \dfrac{n_A S_A e^{-D_{01}(T-t)}}{n_B S_B e^{-D_{02}(T-t)}} + \dfrac{\sigma_{12}^2}{2}(T-t)}{\sigma_{12}\sqrt{T-t}}$$

and

$$d_{20} = d_{10} - \sigma_{12}\sqrt{T-t}.$$

Maryrabe [60] derived this closed-form solution with $D_{01} = D_{02} = 0$, and Rumsey [70] and Brooks [15] gave this closed-form solution and called this exchange option a cross-currency option because the assets there were foreign currencies.

An exchange option could be an American option. We can also introduce the transformation $W = \dfrac{V}{S_2}$ and $\xi = \dfrac{S_1}{S_2}$ and reduce the two-dimensional problem to a one-dimensional problem. W is the solution of an American call option with

$$\sigma = \sqrt{\sigma_1^2 - 2\rho_{12}\sigma_1\sigma_2 + \sigma_2^2}, \; r = D_{02}, \; D_0 = D_{01} \text{ and } E = 1.$$

After finding $W(\xi, t)$, the function V is given by

$$V = S_2 W\left(\frac{S_1}{S_2}, t\right) = n_B S_B W\left(\frac{n_A S_A}{n_B S_B}, t\right).$$

4.5.3 †Options on the Extremum of Several Assets

The price of certain European multi-asset options can be expressed in terms of multivariate cumulative distribution functions. Options on the extremum of several assets are such options. In this subsection, we first explain how these options appear in practice. Then, the price expression of such European options in terms of multivariate cumulative distribution functions are derived out if the volatilities, the interest rate, and the dividend yields are constants.

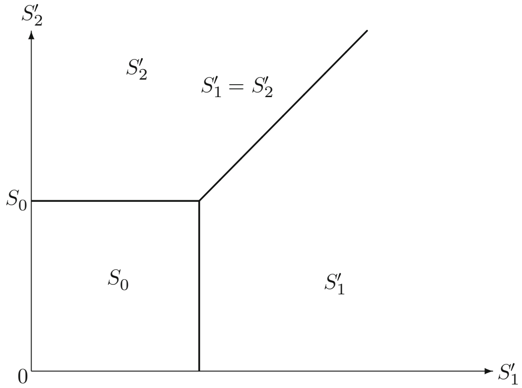

Fig. 4.9. The maximum values on three domains for options with a payoff $\max(S_0, S_1, S_2)$

Let us consider a zero-coupon bond that at maturity date T pays, at the choice of holder, either Z_0 units of domestic currency, Z_A units of currency of country A, or Z_B units of currency of country B. Let S_A denote the domestic price of currency of country A and S_B that of country B. Then, the payment at maturity (payoff) is $\max(Z_0, S_A Z_A, S_B Z_B)$. Let $S_0 = Z_0$, $S_1 = S_A Z_A$, and $S_2 = S_B Z_B$, then the payoff function of the option on the maximum among two assets and cash becomes

$$\max(S_0, S_1, S_2).$$

The question is what is the present value of the option. The value depends on S_1, S_2, and t. Let r be the interest rate of domestic currency. Suppose that S_A and S_B are governed by the system of equations (4.59) and that the interest rates of currencies of the countries A and B are D_{01} and D_{02}, respectively. As shown previously, S_1 and S_2 satisfy the same stochastic differential equations as S_A and S_B, and the dividend yields for the assets S_1 and S_2 also are D_{01} and D_{02}. Therefore, the value of the bond, $V(S_1, S_2, t)$, is the solution of the problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma_1^2 S_1^2 \dfrac{\partial^2 V}{\partial S_1^2} + \rho_{12}\sigma_1\sigma_2 S_1 S_2 \dfrac{\partial^2 V}{\partial S_1 \partial S_2} + \dfrac{1}{2}\sigma_2^2 S_2^2 \dfrac{\partial^2 V}{\partial S_2^2} \\ +(r - D_{01})S_1 \dfrac{\partial V}{\partial S_1} + (r - D_{02})S_2 \dfrac{\partial V}{\partial S_2} - rV = 0, \\ \qquad\qquad\qquad\qquad\qquad 0 \le S_1,\ 0 \le S_2,\ 0 \le t \le T, \\ V(S_1, S_2, T) = \max(S_0, S_1, S_2), \qquad 0 \le S_1,\ 0 \le S_2. \end{cases} \tag{4.63}$$

This problem is usually called a European option on the maximum of assets S_0, S_1, and S_2. From the formula (4.57), the solution of the problem (4.63) is

$$\begin{aligned} V(S_1, S_2, t) &= \mathrm{e}^{-r(T-t)} \int_0^\infty \int_0^\infty \max(S_0, S_1', S_2')\psi(\mathbf{S}'; \mathbf{S}, t) dS_1' dS_2' \\ &= \mathrm{e}^{-r(T-t)} \int_0^{S_0} \int_0^{S_0} S_0 \psi(\mathbf{S}'; \mathbf{S}, t) dS_1' dS_2' \\ &\quad + \mathrm{e}^{-r(T-t)} \int_{S_0}^\infty \int_0^{S_1'} S_1' \psi(\mathbf{S}'; \mathbf{S}, t) dS_2' dS_1' \\ &\quad + \mathrm{e}^{-r(T-t)} \int_{S_0}^\infty \int_0^{S_2'} S_2' \psi(\mathbf{S}'; \mathbf{S}, t) dS_1' dS_2'. \end{aligned} \tag{4.64}$$

In the expression, there are three terms that represent the contributions to the solution from the three domains (see Fig. 4.9):

$$0 \le S_1' \le S_0,\ 0 \le S_2' \le S_0; \quad S_0 \le S_1',\ 0 \le S_2' \le S_1';$$

and

$$S_0 \le S_2',\ 0 \le S_1' \le S_2'.$$

Indeed, every term in Eq. (4.64) can be expressed in terms of the cumulative distribution function for the bivariate standard normal distribution[7]

$$N_2(x_1, x_2; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{x_2} \int_{-\infty}^{x_1} \mathrm{e}^{-\frac{1}{2}(\eta_1^2 - 2\rho\eta_1\eta_2 + \eta_2^2)/(1-\rho^2)} d\eta_1 d\eta_2, \tag{4.65}$$

[7]The value of this function has to be obtained by numerical methods. Here we write down the approximate expression derived in [26] by Drezner and Wesolovsky with the coefficients based on the abscissas and weight factors for Gaussian integration of $\int_{-1}^{1} f(x)dx$ on p. 916 of the handbook [1] by Abramowity and Stegum (editors). When $|\rho| < 0.7$, it is approximated by

$$\rho \sum_{i=1}^{5} \left\{ W_i \mathrm{e}^{\left[X_i \rho x_1 x_2 - \frac{1}{2}(x_1^2 + x_2^2)\right] \Big/ \left[1 - (X_i \rho)^2\right]} \Big/ \sqrt{1 - (X_i \rho)^2} \right\} + N(x_1)N(x_2);$$

where ρ is a parameter. Stulz gave such a result in [75]. Later, Johnson [50] improved the method for deriving this result. Here, we show how every term in the expression (4.64) can be expressed in terms of the function (4.65). Let us begin with looking at the first term. Noticing the concrete expression (4.58) for ψ, we have

$$\begin{aligned}
&\mathrm{e}^{-r(T-t)}\int_0^{S_0}\int_0^{S_0} S_0\psi(\mathbf{S}';\mathbf{S},t)dS_1'dS_2' \\
&= \mathrm{e}^{-r\tau}S_0\int_0^{S_0}\int_0^{S_0}\frac{1}{2\pi\tau\sqrt{\det\mathbf{P}}\sigma_1\sigma_2 S_1'S_2'}\mathrm{e}^{-\eta^T\mathbf{P}^{-1}\eta/2}dS_1'dS_2' \\
&= \mathrm{e}^{-r\tau}S_0\int_{-\infty}^{\eta_2(S_0)}\int_{-\infty}^{\eta_1(S_0)}\frac{1}{2\pi\sqrt{\det\mathbf{P}}}\mathrm{e}^{-\eta^T\mathbf{P}^{-1}\eta/2}d\eta_1 d\eta_2
\end{aligned}$$

---

when $|\rho| \geq 0.7$, it is approximated by

$$\begin{aligned}
&N_2(x_1,x_2;\ \mathrm{sgn}\ (\rho)) -\ \mathrm{sgn}\ (\rho)a\mathrm{e}^{-x_1x_2'/2}\left\{\frac{1}{6\pi}\left(3-cb^2+ca^2\right)\mathrm{e}^{-b^2/(2a^2)}\right. \\
&\left.-\frac{1}{3\sqrt{2\pi}}\frac{b}{a}\left(3-cb^2\right)N(-b/a)+\sum_{i=1}^5 W_i\mathrm{e}^{-\frac{b^2}{2y_i^2}}\left[\frac{\mathrm{e}^{-x_1x_2'/(1+\sqrt{1-y_i^2})}}{\mathrm{e}^{-x_1x_2'/2}\sqrt{1-y_i^2}}-1-cy_i^2\right]\right\},
\end{aligned}$$

where

$$\begin{aligned}
&W_1 = 0.0188540425, \quad W_2 = 0.0380880594, \quad W_3 = 0.0452707394, \\
&W_4 = 0.0380880594, \quad W_5 = 0.0188540425, \\
&X_1 = 0.0469100770, \quad X_2 = 0.2307653449, \quad X_3 = 0.5000000000, \\
&X_4 = 0.7692346551, \quad X_5 = 0.9530899230, \\
&\mathrm{sgn}(\rho) = \begin{cases} 1, & \text{if} \quad \rho \geq 0, \\ -1, & \text{if} \quad \rho < 0, \end{cases} \\
&N_2(x_1,x_2;1) = N(\min(x_1,x_2)), \quad N_2(x_1,x_2;-1) = \max(0, N(x_1)-N(-x_2)), \\
&x_2' = \mathrm{sgn}(\rho)x_2, \quad a = \sqrt{1-\rho^2}, \quad b = |x_1 - x_2'|, \quad c = (4-x_1x_2')/8, \\
&y_i = aX_i, \quad i = 1,2,3,4,5.
\end{aligned}$$

The authors claim that for any x_1, x_2, $-1 \leq \rho \leq 1$, its maximum error is 2×10^{-7}. However in the FORTRAN program given in that paper there are two typos. In the book [43] by Hull, another approximate expression given in [25] by Drezner is also shown. (In order to know how to get its coefficients of expression, see the paper [74] by Steen, Byrne, and Gelbard.) Its accuracy is four decimal places and, averagely speaking, it needs more computational time than the expression shown here.

$$= \mathrm{e}^{-r\tau} S_0 N_2 \left(\frac{\ln \dfrac{S_0}{S_1} - \left(r - D_{01} - \dfrac{\sigma_1^2}{2}\right)\tau}{\sigma_1\sqrt{\tau}}, \frac{\ln \dfrac{S_0}{S_2} - \left(r - D_{02} - \dfrac{\sigma_2^2}{2}\right)\tau}{\sigma_2\sqrt{\tau}}; \rho_{12} \right)$$

$$= S_0^* N_2 \left(\frac{\ln \dfrac{S_0^*}{S_1^*} + \dfrac{\sigma_1^2}{2}\tau}{\sigma_1\sqrt{\tau}}, \frac{\ln \dfrac{S_0^*}{S_2^*} + \dfrac{\sigma_2^2}{2}\tau}{\sigma_2\sqrt{\tau}}; \rho_{12} \right), \tag{4.66}$$

where

$$S_0^* = S_0 \mathrm{e}^{-r\tau}, \quad S_1^* = S_1 \mathrm{e}^{-D_{01}\tau}, \quad S_2^* = S_2 \mathrm{e}^{-D_{02}\tau}.$$

Here, we have used the following formulae

$$\det \mathbf{P} = \begin{vmatrix} 1 & \rho_{12} \\ \rho_{12} & 1 \end{vmatrix} = 1 - \rho_{12}^2,$$

$$\mathbf{P}^{-1} = \begin{bmatrix} 1 & \rho_{12} \\ \rho_{12} & 1 \end{bmatrix}^{-1} = \frac{1}{1 - \rho_{12}^2} \begin{bmatrix} 1 & -\rho_{12} \\ -\rho_{12} & 1 \end{bmatrix},$$

and

$$\eta^T \mathbf{P}^{-1} \eta = \frac{\eta_1^2 - 2\rho_{12}\eta_1\eta_2 + \eta_2^2}{1 - \rho_{12}^2}.$$

The value of the bivariate cumulative distribution function in the expression (4.66) can be interpreted as the probability of the event $\max(S_1', S_2') \le S_0$ in the "risk-neutral" world.

Now let us work on the second and third terms in the expression (4.64). Actually, we can find their expressions from the first term.

From Itô's lemma, we know that the random variable S_i^* has the same volatility as S_i has, which is σ_i, $i = 1, 2$. We can understand S_0^* as a random variable with $\sigma_0 = 0$. The correlation coefficient between S_i^* and S_j^* is the same as that between S_i and S_j, which is denoted by $\rho_{ij}, 0 \le i \le 2$ and $0 \le j \le 2$. Because S_0^* is not a random variable, we have $\rho_{i0} = \rho_{0i} = 0$ for any $i \ne 0$. Define

$$\xi_{ij} = S_i^*/S_j^* \quad \text{for any } i \ne j.$$

Let σ_{ij}^2 be the variance of the random variable ξ_{ij} and ρ_{ijk} be the correlation coefficient between ξ_{ik} and ξ_{jk}. Using Itô's lemma, we have

$$\sigma_{ij} = \sqrt{\sigma_i^2 - 2\rho_{ij}\sigma_i\sigma_j + \sigma_j^2} \tag{4.67}$$

and

$$\rho_{ijk} = \frac{\sigma_k^2 - \rho_{ik}\sigma_i\sigma_k - \rho_{jk}\sigma_j\sigma_k + \rho_{ij}\sigma_i\sigma_j}{\sigma_{ik}\sigma_{jk}}. \tag{4.68}$$

Let us take

$$\xi_{10} = S_1^*/S_0^*$$

and

$$\xi_{20} = S_2^*/S_0^*$$

as independent variables and $V_0 = V/S_0^*$ as the unknown function. Because

$$\begin{aligned}
\frac{\partial V}{\partial t} &= S_0 \mathrm{e}^{-r\tau}\left[rV_0 + \frac{\partial V_0}{\partial t} + (D_{01}-r)\xi_{10}\frac{\partial V_0}{\partial \xi_{10}} + (D_{02}-r)\xi_{20}\frac{\partial V_0}{\partial \xi_{20}}\right],\\
\frac{\partial V}{\partial S_1} &= S_0 \mathrm{e}^{-r\tau}\frac{\partial V_0}{\partial \xi_{10}}\frac{\mathrm{e}^{-D_{01}\tau}}{S_0\mathrm{e}^{-r\tau}} = \mathrm{e}^{-D_{01}\tau}\frac{\partial V_0}{\partial \xi_{10}},\\
\frac{\partial^2 V}{\partial S_1^2} &= \frac{\mathrm{e}^{-2D_{01}\tau}}{S_0\mathrm{e}^{-r\tau}}\frac{\partial^2 V_0}{\partial \xi_{10}^2},\\
\frac{\partial V}{\partial S_2} &= \mathrm{e}^{-D_{02}\tau}\frac{\partial V_0}{\partial \xi_{20}},\\
\frac{\partial^2 V}{\partial S_2^2} &= \frac{\mathrm{e}^{-2D_{02}\tau}}{S_0\mathrm{e}^{-r\tau}}\frac{\partial^2 V_0}{\partial \xi_{20}^2},\\
\frac{\partial^2 V}{\partial S_1 \partial S_2} &= -\frac{\mathrm{e}^{-D_{01}\tau}\mathrm{e}^{-D_{02}\tau}}{S_0\mathrm{e}^{-r\tau}}\frac{\partial^2 V_0}{\partial \xi_{10}\partial \xi_{20}},
\end{aligned}$$

problem (4.63) becomes

$$\begin{cases}
\dfrac{\partial V_0}{\partial t} + \dfrac{1}{2}\sigma_1^2\xi_{10}^2\dfrac{\partial^2 V_0}{\partial \xi_{10}^2} + \rho_{12}\sigma_1\sigma_2\xi_{10}\xi_{20}\dfrac{\partial^2 V_0}{\partial \xi_{10}\partial \xi_{20}} + \dfrac{1}{2}\sigma_2^2\xi_{20}^2\dfrac{\partial^2 V_0}{\partial \xi_{20}^2} = 0,\\
\qquad\qquad 0 \le \xi_{10}, \quad 0 \le \xi_{20}, \quad 0 \le t \le T,\\
V_0(\xi_{10}, \xi_{20}, T) = \max(1, \xi_{10}, \xi_{20}), \qquad 0 \le \xi_{10}, \quad 0 \le \xi_{20}.
\end{cases}$$

Because $\sigma_{10} = \sigma_1$, $\sigma_{20} = \sigma_2$, and $\rho_{120} = \rho_{12}$, the problem above can be rewritten as

$$\begin{cases}
\dfrac{\partial V_0}{\partial t} + \dfrac{1}{2}\sigma_{10}^2\xi_{10}^2\dfrac{\partial^2 V_0}{\partial \xi_{10}^2} + \rho_{120}\sigma_{10}\sigma_{20}\xi_{10}\xi_{20}\dfrac{\partial^2 V_0}{\partial \xi_{10}\partial \xi_{20}} + \dfrac{1}{2}\sigma_{20}^2\xi_{20}^2\dfrac{\partial^2 V_0}{\partial \xi_{20}^2} = 0,\\
\qquad\qquad 0 \le \xi_{10}, \ 0 \le \xi_{20}, \ 0 \le t \le T,\\
V_0(\xi_{10}, \xi_{20}, T) = \max(1, \xi_{10}, \xi_{20}), \quad 0 \le \xi_{10}, \ 0 \le \xi_{20}.
\end{cases} \tag{4.69}$$

By using the integral expression of the solution of the problem (4.69), $V(S_1, S_2, t)$ can be expressed as

$$\begin{aligned}
V(S_1, S_2, t) &= S_0^* V_0\left(\xi_{10}, \xi_{20}, t\right)\\
&= S_0^* \int_0^\infty \int_0^\infty \max\left(1, \xi_{10}', \xi_{20}'\right)\psi d\xi_{10}' d\xi_{20}'
\end{aligned}$$

$$= S_0^* \int_0^1 \int_0^1 \psi d\xi'_{10} d\xi'_{20} + S_0^* \int_1^\infty \int_0^{\xi'_{10}} \xi'_{10} \psi d\xi'_{20} d\xi'_{10}$$
$$+ S_0^* \int_1^\infty \int_0^{\xi'_{20}} \xi'_{20} \psi d\xi'_{10} d\xi'_{20}. \tag{4.70}$$

Clearly, the first term in the expression (4.70) is the same as the first term in the expression (4.64). Thus, the first term in the expression (4.70) is equal to

$$S_0^* N_2 \left(\frac{\ln \dfrac{S_0^*}{S_1^*} + \dfrac{\sigma_1^2}{2}\tau}{\sigma_1\sqrt{\tau}}, \frac{\ln \dfrac{S_0^*}{S_2^*} + \dfrac{\sigma_2^2}{2}\tau}{\sigma_2\sqrt{\tau}}; \rho_{12} \right)$$
$$= S_0^* N_2 \left(\frac{-\ln \xi_{10} + \dfrac{\sigma_{10}^2}{2}\tau}{\sigma_{10}\sqrt{\tau}}, \frac{-\ln \xi_{20} + \dfrac{\sigma_{20}^2}{2}\tau}{\sigma_{20}\sqrt{\tau}}; \rho_{120} \right). \tag{4.71}$$

Now we take

$$\xi_{21} = S_2^*/S_1^*$$

and

$$\xi_{01} = S_0^*/S_1^*$$

as independent variables and $V_1 = V/S_1^*$ as the unknown function. Because

$$\begin{aligned}
\frac{\partial V}{\partial t} &= S_1 e^{-D_{01}\tau} \left[D_{01} V_1 + \frac{\partial V_1}{\partial t} + (D_{02} - D_{01})\xi_{21} \frac{\partial V_1}{\partial \xi_{21}} \right. \\
&\qquad \left. + (r - D_{01})\xi_{01} \frac{\partial V_1}{\partial \xi_{01}} \right], \\
\frac{\partial V}{\partial S_1} &= e^{-D_{01}\tau} \left(V_1 - \xi_{21} \frac{\partial V_1}{\partial \xi_{21}} - \xi_{01} \frac{\partial V_1}{\partial \xi_{01}} \right), \\
\frac{\partial^2 V}{\partial S_1^2} &= \frac{e^{-D_{01}\tau}}{S_1} \left(\xi_{21}^2 \frac{\partial^2 V_1}{\partial \xi_{21}^2} + 2\xi_{21}\xi_{01} \frac{\partial^2 V_1}{\partial \xi_{21} \partial \xi_{01}} + \xi_{01}^2 \frac{\partial^2 V_1}{\partial \xi_{01}^2} \right), \\
\frac{\partial V}{\partial S_2} &= e^{-D_{02}\tau} \frac{\partial V_1}{\partial \xi_{21}}, \\
\frac{\partial^2 V}{\partial S_2^2} &= \frac{e^{-2D_{02}\tau}}{S_1 e^{-D_{01}\tau}} \frac{\partial^2 V_1}{\partial \xi_{21}^2}, \\
\frac{\partial^2 V}{\partial S_1 \partial S_2} &= -\frac{e^{-D_{02}\tau}}{S_1} \left(\xi_{21} \frac{\partial^2 V_1}{\partial \xi_{21}^2} + \xi_{01} \frac{\partial^2 V_1}{\partial \xi_{21} \partial \xi_{01}} \right),
\end{aligned}$$

we have

$$\begin{cases} \dfrac{\partial V_1}{\partial t} + \dfrac{1}{2}\sigma_{21}^2\xi_{21}^2\dfrac{\partial^2 V_1}{\partial \xi_{21}^2} + \rho_{201}\sigma_{21}\sigma_1\xi_{21}\xi_{01}\dfrac{\partial^2 V_1}{\partial \xi_{21}\partial \xi_{01}} + \dfrac{1}{2}\sigma_1^2\xi_{01}^2\dfrac{\partial^2 V_1}{\partial \xi_{01}^2} = 0, \\ \qquad\qquad 0 \le \xi_{21}, \quad 0 \le \xi_{01}, \quad 0 \le t \le T, \\ V_1(\xi_{21}, \xi_{01}, T) = \max(1, \xi_{21}, \xi_{01}), \qquad 0 \le \xi_{21}, \quad 0 \le \xi_{01}, \end{cases}$$

where σ_{21} and ρ_{201} are defined by the formulae (4.67) and (4.68), namely,

$$\sigma_{21} = \sqrt{\sigma_1^2 - 2\rho_{12}\sigma_1\sigma_2 + \sigma_2^2}$$

and

$$\rho_{201} = \frac{\sigma_1^2 - \rho_{12}\sigma_1\sigma_2 - \rho_{01}\sigma_0\sigma_1 + \rho_{20}\sigma_2\sigma_0}{\sigma_{21}\sigma_{01}} = \frac{\sigma_1 - \rho_{12}\sigma_2}{\sigma_{21}}.$$

Because $\sigma_{01} = \sigma_1$, the problem above can be rewritten as

$$\begin{cases} \dfrac{\partial V_1}{\partial t} + \dfrac{1}{2}\sigma_{21}^2\xi_{21}^2\dfrac{\partial^2 V_1}{\partial \xi_{21}^2} + \rho_{201}\sigma_{21}\sigma_{01}\xi_{21}\xi_{01}\dfrac{\partial^2 V_1}{\partial \xi_{21}\partial \xi_{01}} + \dfrac{1}{2}\sigma_{01}^2\xi_{01}^2\dfrac{\partial^2 V_1}{\partial \xi_{01}^2} = 0, \\ \qquad\qquad 0 \le \xi_{21}, \ \ 0 \le \xi_{01}, \ \ 0 \le t \le T, \\ V_1(\xi_{21}, \xi_{01}, T) = \max(1, \xi_{21}, \xi_{01}), \quad 0 \le \xi_{21}, \ \ 0 \le \xi_{01}. \end{cases} \tag{4.72}$$

Therefore, we have

$$\begin{aligned} V(S_1, S_2, t) &= S_1^* V_1\left(\xi_{21}, \xi_{01}, t\right) \\ &= S_1^* \int_0^\infty \int_0^\infty \max\left(1, \xi_{21}', \xi_{01}'\right) \psi d\xi_{21}' d\xi_{01}' \\ &= S_1^* \int_0^1 \int_0^1 \psi d\xi_{21}' d\xi_{01}' + S_1^* \int_1^\infty \int_0^{\xi_{21}'} \xi_{21}' \psi d\xi_{01}' d\xi_{21}' \\ &\quad + S_1^* \int_1^\infty \int_0^{\xi_{01}'} \xi_{01}' \psi d\xi_{21}' \xi_{01}'. \end{aligned} \tag{4.73}$$

Because we can have the problem (4.72) from the problem (4.69) by the rule of substitution of subscripts:

$$0 \to 1, \quad 1 \to 2, \quad 2 \to 0,$$

we can obtain the result of the first term in the expression (4.73) from the result of the first term in the expression (4.70) by the same rule. The result of the first term in the expression (4.70) is the expression (4.71), so for the first term in the expression (4.73) we have

$$
\begin{aligned}
&S_1^*\int_0^1\int_0^1 \psi d\xi_{21}' d\xi_{01}'\\
&= S_1^* N_2\left(\frac{-\ln\xi_{21}+\dfrac{\sigma_{21}^2}{2}\tau}{\sigma_{21}\sqrt{\tau}}, \frac{-\ln\xi_{01}+\dfrac{\sigma_{01}^2}{2}\tau}{\sigma_{01}\sqrt{\tau}}; \rho_{201}\right)\\
&= S_1^* N_2\left(\frac{\ln\dfrac{S_1^*}{S_2^*}+\dfrac{\sigma_{12}^2}{2}\tau}{\sigma_{12}\sqrt{\tau}}, \frac{\ln\dfrac{S_1^*}{S_0^*}+\dfrac{\sigma_1^2}{2}\tau}{\sigma_1\sqrt{\tau}}; \frac{\sigma_1-\rho_{12}\sigma_2}{\sigma_{12}}\right). \qquad (4.74)
\end{aligned}
$$

Because

$$
\max(\xi_{21}', \xi_{01}') = \max\left(\frac{S_2'}{S_1'}, \frac{S_0}{S_1'}\right) \le 1,
$$

which is equivalent to

$$
S_1' \ge \max(S_0, S_2'),
$$

both the first term in the expression (4.73) and the second term in the expression (4.64) represent the contribution to the solution from the same domain where $S_1' \ge \max(S_0, S_2')$. This domain is

$$
S_0 \le S_1',\ 0 \le S_2' \le S_1'.
$$

Thus, the second term in the expression (4.64) and the first term in the expression (4.73) should have the same result (4.74).

Similarly, we can prove that the result of the third term in the expression (4.64) can be obtained from the second term in the expression (4.64) by the same rule of substitution of subscripts, namely, it is equal to

$$
\begin{aligned}
&S_2^* N_2\left(\frac{-\ln\xi_{02}+\dfrac{\sigma_{02}^2}{2}\tau}{\sigma_{02}\sqrt{\tau}}, \frac{-\ln\xi_{12}+\dfrac{\sigma_{12}^2}{2}\tau}{\sigma_{12}\sqrt{\tau}}; \rho_{012}\right)\\
&= S_2^* N_2\left(\frac{\ln\dfrac{S_2^*}{S_0^*}+\dfrac{\sigma_2^2}{2}\tau}{\sigma_2\sqrt{\tau}}, \frac{\ln\dfrac{S_2^*}{S_1^*}+\dfrac{\sigma_{12}^2}{2}\tau}{\sigma_{12}\sqrt{\tau}}; \frac{\sigma_2-\rho_{12}\sigma_1}{\sigma_{12}}\right).
\end{aligned}
$$

Therefore, we finally arrive at

$$
\begin{aligned}
&V(S_1, S_2, t)\\
&= S_0^* N_2\left(\frac{\ln\dfrac{S_0^*}{S_1^*}+\dfrac{\sigma_1^2}{2}\tau}{\sigma_1\sqrt{\tau}}, \frac{\ln\dfrac{S_0^*}{S_2^*}+\dfrac{\sigma_2^2}{2}\tau}{\sigma_2\sqrt{\tau}}; \rho_{12}\right)
\end{aligned}
$$

$$+ S_1^* N_2 \left(\frac{\ln \frac{S_1^*}{S_2^*} + \frac{\sigma_{12}^2}{2}\tau}{\sigma_{12}\sqrt{\tau}}, \frac{\ln \frac{S_1^*}{S_0^*} + \frac{\sigma_1^2}{2}\tau}{\sigma_1\sqrt{\tau}}; \frac{\sigma_1 - \rho_{12}\sigma_2}{\sigma_{12}} \right)$$

$$+ S_2^* N_2 \left(\frac{\ln \frac{S_2^*}{S_0^*} + \frac{\sigma_2^2}{2}\tau}{\sigma_2\sqrt{\tau}}, \frac{\ln \frac{S_2^*}{S_1^*} + \frac{\sigma_{12}^2}{2}\tau}{\sigma_{12}\sqrt{\tau}}; \frac{\sigma_2 - \rho_{12}\sigma_1}{\sigma_{12}} \right). \quad (4.75)$$

This expression will also be referred to as $V_{\max}(S_1, S_2, t)$ later because the payoff of the option is $\max(S_0, S_1, S_2)$. In the expression (4.75), it seems that the rule of substitution of subscripts does not work. Actually, the rule should be used in the following way. First, the rule is applied to S_i^*. Then, you should determine the volatilities and correlation coefficient in a function as follows. If $\frac{S_i^*}{S_j^*}$ appears in an argument expression, then the volatility in the expression is the volatility of $\frac{S_i}{S_j}$. If $\frac{S_i^*}{S_k^*}$ and $\frac{S_j^*}{S_k^*}$ (or equivalently, $\frac{S_k^*}{S_i^*}$ and $\frac{S_k^*}{S_j^*}$) appear in the function, then the third argument is the correlation coefficient between $\frac{S_i}{S_k}$ and $\frac{S_j}{S_k}$.

Here, we would like to make the following four remarks:

1. Through a procedure similar to what we used to derive the expression (4.75), it can be shown that for a European option on the minimum among two assets S_1, S_2 and cash S_0, whose payoff function is $\min(S_0, S_1, S_2)$, the price is

$$\begin{aligned} &V_{\min}(S_1, S_2, t) \\ &= S_0^* N_2 \left(\frac{\ln \frac{S_1^*}{S_0^*} - \frac{\sigma_1^2}{2}\tau}{\sigma_1\sqrt{\tau}}, \frac{\ln \frac{S_2^*}{S_0^*} - \frac{\sigma_2^2}{2}\tau}{\sigma_2\sqrt{\tau}}; \rho_{12} \right) \\ &\quad + S_1^* N_2 \left(\frac{\ln \frac{S_2^*}{S_1^*} - \frac{\sigma_{12}^2}{2}\tau}{\sigma_{12}\sqrt{\tau}}, \frac{\ln \frac{S_0^*}{S_1^*} - \frac{\sigma_1^2}{2}\tau}{\sigma_1\sqrt{\tau}}; \frac{\sigma_1 - \rho_{12}\sigma_2}{\sigma_{12}} \right) \\ &\quad + S_2^* N_2 \left(\frac{\ln \frac{S_0^*}{S_2^*} - \frac{\sigma_2^2}{2}\tau}{\sigma_2\sqrt{\tau}}, \frac{\ln \frac{S_1^*}{S_2^*} - \frac{\sigma_{12}^2}{2}\tau}{\sigma_{12}\sqrt{\tau}}; \frac{\sigma_2 - \rho_{12}\sigma_1}{\sigma_{12}} \right). \end{aligned}$$

   This is left for readers as Problem 27.
2. Because

$$\max(E, S) = \max(S - E, 0) + E,$$

for a European option with a payoff $\max(S, E)$, the solution is

$$\begin{aligned} c(S,t)+E\mathrm{e}^{-r(T-t)} &= S\mathrm{e}^{-D_0(T-t)}N(d_1)-E\mathrm{e}^{-r(T-t)}N(d_2)+E\mathrm{e}^{-r(T-t)} \\ &= E\mathrm{e}^{-r(T-t)}N(-d_2)+S\mathrm{e}^{-D_0(T-t)}N(d_1) \\ &= S_0^* N\left(\frac{\ln\dfrac{S_0^*}{S_1^*}+\dfrac{\sigma^2}{2}\tau}{\sigma\sqrt{\tau}}\right)+S_1^* N\left(\frac{\ln\dfrac{S_1^*}{S_0^*}+\dfrac{\sigma^2}{2}\tau}{\sigma\sqrt{\tau}}\right), \end{aligned}$$

where

$$S_0^* = E\mathrm{e}^{-r(T-t)}, \quad S_1^* = S\mathrm{e}^{-D_0(T-t)}.$$

For this case, obtaining the second term from the first term follows the rule:

$$0 \to 1, \quad 1 \to 0.$$

Therefore, the expression (4.75) can be understood as a generalization of this expression. For an option with a payoff $\max(S_0, S_1, S_2, \cdots, S_n)$, where S_0 is a constant and $S_1, S_2, \cdots, S_n$ are random variables, a term can be obtained from another term by the following rule

$$0 \to 1, \quad 1 \to 2, \quad \cdots, \quad n-1 \to n, \quad n \to 0$$

if the solution is written in a similar form to the expression (4.75). The solution of the option with a payoff $\max(S_0, S_1, S_2, S_3)$ can be found in Problem 31, and readers are asked to show this result. In Problem 32, readers are asked to guess the solution for an option with a payoff $\max(S_0, S_1, S_2, \cdots, S_n)$. For options with a payoff $\min(S_0, S_1, S_2, \cdots, S_n)$, the situation is similar, and Problems 31 and 32 also involve these options.

3. We have pointed out that the value of the bivariate cumulative distribution function appearing in the first term in the expression (4.75) denotes the probability of the event $\max(S_1', S_2') \le S_0$ in the so-called "risk-neutral" world. Now we look at

$$N_2\left(\frac{\ln\dfrac{S_1^*}{S_2^*}+\dfrac{\sigma_{12}^2}{2}\tau}{\sigma_{12}\sqrt{\tau}}, \frac{\ln\dfrac{S_1^*}{S_0^*}+\dfrac{\sigma_1^2}{2}\tau}{\sigma_1\sqrt{\tau}}; \frac{\sigma_1-\rho_{12}\sigma_{12}}{\sigma_{12}}\right).$$

As we saw, it is the value of the integral

$$\int_0^1\int_0^1 \psi\left(\xi_{21}', \xi_{01}'; \xi_{21}, \xi_{01}, t\right) d\xi_{21}' d\xi_{01}'.$$

Therefore, the value of the bivariate cumulative distribution function appearing in the second term of $V(S_1, S_2, t)$ can be interpreted as the probability of the event $\max(\xi_{21}', \xi_{01}') = \max\left(\dfrac{S_2'}{S_1'}, \dfrac{S_0}{S_1'}\right) \le 1$, which is equivalent to $S_1' \geq \max(S_0, S_2')$, in another "risk-neutral" world with the probability density function

$$\psi(\xi_{21}', \xi_{01}'; \xi_{21}, \xi_{01}, t) = \frac{1}{2\pi\tau\sqrt{\det \overline{\mathbf{P}}}\sigma_1\sigma_{12}\xi_{21}'\xi_{01}'} \mathrm{e}^{-\zeta^T\overline{\mathbf{P}}^{-1}\zeta/2},$$

where

$$\overline{\mathbf{P}} = \begin{bmatrix} 1 & \rho_{201} \\ \rho_{201} & 1 \end{bmatrix}$$

and

$$\zeta = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \begin{bmatrix} \dfrac{\ln \xi_{21}' - \left(\ln \xi_{21} - \sigma_{12}^2\tau/2\right)}{\sigma_{12}\sqrt{\tau}} \\ \dfrac{\ln \xi_{01}' - \left(\ln \xi_{01} - \sigma_1^2\tau/2\right)}{\sigma_1\sqrt{\tau}} \end{bmatrix}.$$

Similarly, the value of the bivariate cumulative distribution function appearing in the third term of $V(S_1, S_2, t)$ represents the probability of the event $\max\left(\dfrac{S_0}{S_2'}, \dfrac{S_1'}{S_2'}\right) \leq 1$ or $S_2' \geq \max(S_0, S_1')$ in a third "risk-neutral" world.

4. As the option with a payoff $\max(S, E)$ is related to the European vanilla call option, the option with a payoff $\max(E, S_1, S_2)$ is related to the European call option on the maximum of two assets, whose payoff is $\max(\max(S_1, S_2) - E, 0)$. Let the price of this option be $c(S_1, S_2, t)$. Because

$$\max(\max(S_1, S_2) - E, 0) = \max(S_1 - E, S_2 - E, 0) = \max(E, S_1, S_2) - E,$$

the price of this option is

$$c(S_1, S_2, t) = V_{\max}(S_1, S_2, t) - E\mathrm{e}^{-r(T-t)}, \tag{4.76}$$

where $V_{\max}(S_1, S_2, t)$ is the expression given by the expression (4.75). This is the closed-form solution for the European call option on the maximum of two assets. Similarly, we can show that the price of a European put option on the minimum of two assets is

$$p(S_1, S_2, t) = E\mathrm{e}^{-r(T-t)} - V_{\min}(S_1, S_2, t) \tag{4.77}$$

because its payoff is

$$\max(E - \min(S_1, S_2), 0) = \max(E - S_1, E - S_2, 0) = E - \min(E, S_1, S_2).$$

This is the closed-form solution for the European put option on the minimum of two assets. Using a similar procedure, we can prove some other relations, which are left for readers to show as Problems 33 and 34.

Finally, we give some values of the European call option on the maximum of two assets and the European put option on the minimum of two assets obtained by the formulae (4.76) and (4.77). In Tables 4.6 and 4.7 the prices of some call and put options are given, respectively. Table 4.8 shows how the price varies when ρ changes.

Table 4.6. Prices of European call option on the maximum of two assets

($r = 0.02$, $D_{01} = 0.01$, $D_{02} = 0.01$, $\rho = 0.8$, $E = 100$ and $T = 1$)

Call option price	$\sigma_1 = 0.20$ $\sigma_2 = 0.15$	$\sigma_1 = 0.15$ $\sigma_2 = 0.20$	$\sigma_1 = 0.20$ $\sigma_2 = 0.20$
$S_1 = 100$, $S_2 = 100$	9.99	9.99	11.14
$S_1 = 95$, $S_2 = 105$	10.58	11.69	12.20
$S_1 = 105$, $S_2 = 95$	11.69	10.58	12.20

Table 4.7. Prices of European put option on the minimum of two assets

($r = 0.02$, $D_{01} = 0.01$, $D_{02} = 0.01$, $\rho = 0.8$, $E = 100$ and $T = 1$)

Put option price	$\sigma_1 = 0.20$ $\sigma_2 = 0.15$	$\sigma_1 = 0.15$ $\sigma_2 = 0.20$	$\sigma_1 = 0.20$ $\sigma_2 = 0.20$
$S_1 = 100$, $S_2 = 100$	8.52	8.52	9.57
$S_1 = 95$, $S_2 = 105$	10.02	8.95	10.47
$S_1 = 105$, $S_2 = 95$	8.95	10.02	10.47

Table 4.8. Prices of European options on two assets

($r = 0.02$, $D_{01} = 0.01$, $D_{02} = 0.01$, $\sigma_1 = 0.2$,
$\sigma_2 = 0.2$, $S_1 = 100$, $S_2 = 100$, $E = 100$ and $T = 1$)

ρ	Call price	Put price
0.80	11.14	9.57
0.85	10.76	9.27
0.90	10.32	8.92
0.95	9.75	8.47
1.00	8.35	7.36

4.5.4 †Formulation of Multi-Asset Option Problems on a Finite Domain

In the last two subsections, we studied some options on multi-assets whose solutions can be expressed in terms of multivariate cumulative distribution functions if every σ_i is constant and pricing of which is reduced to find the value of these functions. If σ_i depends on S_i, it may not be possible to express their solutions in terms of such functions. Moreover, for some other options, it might be hard to express their solutions in such a form even though σ_i are constants. Here we give some examples:

1. Multi-asset call options. For such an option, the payoff function is

$$\max(S_1 - E_1, S_2 - E_2, \cdots, S_n - E_n, 0).$$

2. Multi-asset put options. The payoff function is in the form

$$\max(E_1 - S_1, E_2 - S_2, \cdots, E_n - S_n, 0).$$

3. Basket options and index options. The payoff function of a basket option (see [45]) is

$$V(S_1, S_2, \cdots, S_n, T) = \max\left(\sum_{i=1}^{n} \lambda_i S_i - E, 0\right),$$

where

$$\lambda_i \geq 0 \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i = 1.$$

The payoff function of an index option (see [16]) is

$$V(S_1, S_2, \cdots, S_n, T) = \max\left(\sum_{i=1}^{n} n_i S_i - E, 0\right),$$

where n_i is the number of shares of asset i held in the index. An index option is equivalent to a basket option because

$$\max\left(\sum_{i=1}^{n} n_i S_i - E, 0\right) = \left(\sum_{i=1}^{n} n_i\right) \max\left(\sum_{i=1}^{n} \lambda_i S_i - \overline{E}, 0\right),$$

where

$$\lambda_i = n_i \Big/ \sum_{i=1}^{n} n_i\ , \quad \overline{E} = E \Big/ \sum_{i=1}^{n} n_i,$$

and λ_i satisfies the relation $\sum\limits_{i=1}^{n} \lambda_i = 1$.

From Smithson's paper [73], readers can find these and some other options on multi-assets whose solutions might not be expressed in terms of multivariate cumulative distribution functions. For these cases, it might be necessary to use numerical methods. It will become much easier to get numerical solutions if the problem can be reformulated on a finite domain. In this section, we reformulate the two-asset option problems on a finite domain, even though the method can be generalized to multi-asset option problems.

Let us first introduce a new coordinate system:

$$\begin{cases} S_1 = P_1 S \cos\theta, \\ S_2 = P_2 S \sin\theta, \end{cases}$$

that is,

$$\begin{cases} S = \sqrt{\left(\dfrac{S_1}{P_1}\right)^2 + \left(\dfrac{S_2}{P_2}\right)^2}, \\ \theta = \tan^{-1} \dfrac{P_1 S_2}{P_2 S_1}. \end{cases}$$

Under the transformation, the domain $[0, \infty) \times [0, \infty)$ on the (S_1, S_2)-plane becomes the domain $[0, \infty) \times \left[0, \dfrac{\pi}{2}\right]$ on the (S, θ)-plane. Noticing

$$\begin{aligned} \frac{\partial S}{\partial S_1} &= \frac{\cos\theta}{P_1}, & \frac{\partial S}{\partial S_2} &= \frac{\sin\theta}{P_2}, \\ \frac{\partial \theta}{\partial S_1} &= \frac{-\sin\theta}{P_1 S}, & \frac{\partial \theta}{\partial S_2} &= \frac{\cos\theta}{P_2 S}, \end{aligned}$$

we have the following relations

$$\begin{aligned} \frac{\partial V}{\partial S_1} &= \frac{1}{P_1}\left(\frac{\partial V}{\partial S}\cos\theta - \frac{1}{S}\frac{\partial V}{\partial \theta}\sin\theta\right), \\ \frac{\partial^2 V}{\partial S_1^2} &= \frac{1}{P_1^2}\left(\frac{\partial^2 V}{\partial S^2}\cos^2\theta - \frac{2}{S}\frac{\partial^2 V}{\partial S \partial \theta}\sin\theta\cos\theta + \frac{1}{S^2}\frac{\partial^2 V}{\partial \theta^2}\sin^2\theta \right. \\ &\qquad \left. + \frac{1}{S}\frac{\partial V}{\partial S}\sin^2\theta + \frac{2}{S^2}\frac{\partial V}{\partial \theta}\sin\theta\cos\theta\right), \\ \frac{\partial V}{\partial S_2} &= \frac{1}{P_2}\left(\frac{\partial V}{\partial S}\sin\theta + \frac{1}{S}\frac{\partial V}{\partial \theta}\cos\theta\right), \\ \frac{\partial^2 V}{\partial S_2^2} &= \frac{1}{P_2^2}\left(\frac{\partial^2 V}{\partial S^2}\sin^2\theta + \frac{2}{S}\frac{\partial^2 V}{\partial S \partial \theta}\sin\theta\cos\theta + \frac{1}{S^2}\frac{\partial^2 V}{\partial \theta^2}\cos^2\theta \right. \\ &\qquad \left. + \frac{1}{S}\frac{\partial V}{\partial S}\cos^2\theta - \frac{2}{S^2}\frac{\partial V}{\partial \theta}\sin\theta\cos\theta\right), \\ \frac{\partial^2 V}{\partial S_1 \partial S_2} &= \frac{1}{P_1 P_2}\left[\frac{\partial^2 V}{\partial S^2}\sin\theta\cos\theta + \frac{1}{S}\frac{\partial^2 V}{\partial S \partial \theta}(\cos^2\theta - \sin^2\theta) \right. \\ &\qquad \left. - \frac{1}{S^2}\frac{\partial^2 V}{\partial \theta^2}\sin\theta\cos\theta - \frac{1}{S}\frac{\partial V}{\partial S}\sin\theta\cos\theta - \frac{1}{S^2}\frac{\partial V}{\partial \theta}\left(\cos^2\theta - \sin^2\theta\right)\right]. \end{aligned}$$

From these, we can rewrite Eq. (4.52) with $n = 2$ as

$$\begin{cases} \dfrac{\partial V}{\partial t} + a_1 S^2 \dfrac{\partial^2 V}{\partial S^2} + a_{12} S \dfrac{\partial^2 V}{\partial S \partial \theta} + a_2 \dfrac{\partial^2 V}{\partial \theta^2} + b_1 S \dfrac{\partial V}{\partial S} + b_2 \dfrac{\partial V}{\partial \theta} - rV = 0, \\ \qquad 0 \le S, \quad 0 \le \theta \le \dfrac{\pi}{2}, \quad 0 \le t \le T, \end{cases} \tag{4.78}$$

where

$$a_1 = \frac{1}{2}\left(\sigma_1^2 \cos^4\theta + 2\rho_{12}\sigma_1\sigma_2 \sin^2\theta\cos^2\theta + \sigma_2^2\sin^4\theta\right),$$

$$\begin{aligned} a_{12} &= -\sigma_1^2 \sin\theta\cos^3\theta + \rho_{12}\sigma_1\sigma_2\cos\theta\sin\theta(\cos^2\theta - \sin^2\theta) + \sigma_2^2\sin^3\theta\cos\theta \\ &= \left[-\sigma_1(\sigma_1 - \rho_{12}\sigma_2)\cos^2\theta + \sigma_2\left(\sigma_2 - \rho_{12}\sigma_1\right)\sin^2\theta\right]\sin\theta\cos\theta, \end{aligned}$$

$$a_2 = \frac{1}{2}(\sigma_1^2 - 2\rho_{12}\sigma_1\sigma_2 + \sigma_2^2)\cos^2\theta\sin^2\theta,$$

$$\begin{aligned} b_1 &= (r - D_{01})\cos^2\theta + (r - D_{02})\sin^2\theta + \frac{1}{2}(\sigma_1^2 - 2\rho_{12}\sigma_1\sigma_2 + \sigma_2^2)\sin^2\theta\cos^2\theta \\ &= r + \left[-D_{01}\cos^2\theta - D_{02}\sin^2\theta + \frac{1}{2}(\sigma_1^2 - 2\rho_{12}\sigma_1\sigma_2 + \sigma_2^2)\sin^2\theta\cos^2\theta\right], \end{aligned}$$

$$\begin{aligned} b_2 &= [-(r - D_{01}) + (r - D_{02})]\sin\theta\cos\theta + \sigma_1^2\sin\theta\cos^3\theta \\ &\quad -\rho_{12}\sigma_1\sigma_2\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta) - \sigma_2^2\sin^3\theta\cos\theta \\ &= \left[D_{01} - D_{02} + \sigma_1^2\cos^2\theta - \rho_{12}\sigma_1\sigma_2(\cos^2\theta - \sin^2\theta) - \sigma_2^2\sin^2\theta\right]\sin\theta\cos\theta. \end{aligned}$$

Now let us introduce another transformation

$$\begin{cases} \xi = \dfrac{S}{S + P_m}, \\ \theta = \theta, \\ w = \dfrac{V}{S + P_m}. \end{cases}$$

Under this transformation, the domain $[0, \infty)$ on the S-axis is transformed into the domain $[0, 1)$ on the ξ-axis. Because

$$\begin{aligned} S &= \frac{\xi P_m}{1 - \xi}, \\ S + P_m &= \frac{P_m}{1 - \xi}, \\ \frac{d\xi}{dS} &= \frac{P_m}{(S + P_m)^2} = \frac{(1 - \xi)^2}{P_m}, \end{aligned}$$

we get

$$
\begin{aligned}
\frac{\partial V}{\partial t} &= (S+P_m)\frac{\partial w}{\partial t} = \frac{P_m}{1-\xi}\frac{\partial w}{\partial t},\\
\frac{\partial V}{\partial S} &= w + (S+P_m)\frac{\partial w}{\partial \xi}\frac{d\xi}{dS} = w + (1-\xi)\frac{\partial w}{\partial \xi},\\
\frac{\partial V}{\partial \theta} &= \frac{P_m}{1-\xi}\frac{\partial w}{\partial \theta},\\
\frac{\partial^2 V}{\partial S^2} &= \frac{(1-\xi)^3}{P_m}\frac{\partial^2 w}{\partial \xi^2},\\
\frac{\partial^2 V}{\partial \theta \partial S} &= \frac{\partial w}{\partial \theta} + (1-\xi)\frac{\partial^2 w}{\partial \theta \partial \xi},\\
\frac{\partial^2 V}{\partial \theta^2} &= \frac{P_m}{1-\xi}\frac{\partial^2 w}{\partial \theta^2}.
\end{aligned}
$$

Consequently, we arrive at the final equation

$$
\begin{cases}
\dfrac{\partial w}{\partial t} + a_1\xi^2(1-\xi)^2\dfrac{\partial^2 w}{\partial \xi^2} + a_{12}\xi(1-\xi)\dfrac{\partial^2 w}{\partial \theta \partial \xi} + a_2\dfrac{\partial^2 w}{\partial \theta^2}\\
\quad + b_1\xi(1-\xi)\dfrac{\partial w}{\partial \xi} + (b_2 + a_{12}\xi)\dfrac{\partial w}{\partial \theta} - (r - b_1\xi)w = 0,\\
\qquad 0 \le \xi \le 1, \quad 0 \le \theta \le \dfrac{\pi}{2}, \quad 0 \le t \le T.
\end{cases}
\tag{4.79}
$$

The combination of the above two transformations is

$$
\begin{cases}
\xi = \dfrac{\sqrt{\left(\dfrac{S_1}{P_1}\right)^2 + \left(\dfrac{S_2}{P_2}\right)^2}}{\sqrt{\left(\dfrac{S_1}{P_1}\right)^2 + \left(\dfrac{S_2}{P_2}\right)^2} + P_m},\\
\theta = \tan^{-1}\dfrac{P_1 S_2}{P_2 S_1},\\
w = \dfrac{V}{\sqrt{\left(\dfrac{S_1}{P_1}\right)^2 + \left(\dfrac{S_2}{P_2}\right)^2} + P_m}
\end{cases}
$$

and we can derive Eq. (4.79) directly from Eq. (4.52) with $n = 2$ by using this transformation. Here, we do it through two steps in order to make the idea clear.

It can be proved that in order to determine a unique solution, only the final condition

$$
w(\xi, \theta, T) = f(\xi, \theta), \qquad 0 \le \xi \le 1, \quad 0 \le \theta \le \frac{\pi}{2} \tag{4.80}
$$

is needed. The proof is similar to that given in the paper by Zhu and Li [94]. Here, we do not give the proof but an explanation on this issue. At $\theta = 0$ and $\theta = \dfrac{\pi}{2}$, Eq. (4.79) becomes

$$\frac{\partial w}{\partial t} + \frac{1}{2}\sigma_1^2\xi^2(1-\xi)^2\frac{\partial^2 w}{\partial \xi^2} + (r - D_{01})\xi(1-\xi)\frac{\partial w}{\partial \xi} - \left[r(1-\xi) + D_{01}\xi\right] w = 0,$$

$$0 \le \xi \le 1, \quad 0 \le t \le T$$

and

$$\frac{\partial w}{\partial t} + \frac{1}{2}\sigma_2^2\xi^2(1-\xi)^2\frac{\partial^2 w}{\partial \xi^2} + (r - D_{02})\xi(1-\xi)\frac{\partial w}{\partial \xi} - \left[r(1-\xi) + D_{02}\xi\right] w = 0,$$

$$0 \le \xi \le 1, \quad 0 \le t \le T$$

respectively. These are one-dimensional parabolic equations that degenerate into ordinary differential equations at $\xi = 0$ and $\xi = 1$. Therefore, if a final condition is given for each equation, the solution for each equation is unique. These two solutions give the value of the solution to Eq. (4.79) at the boundaries $\theta = 0$ and $\theta = \pi/2$. At $\xi = 0$ and $\xi = 1$, Eq. (4.79) becomes

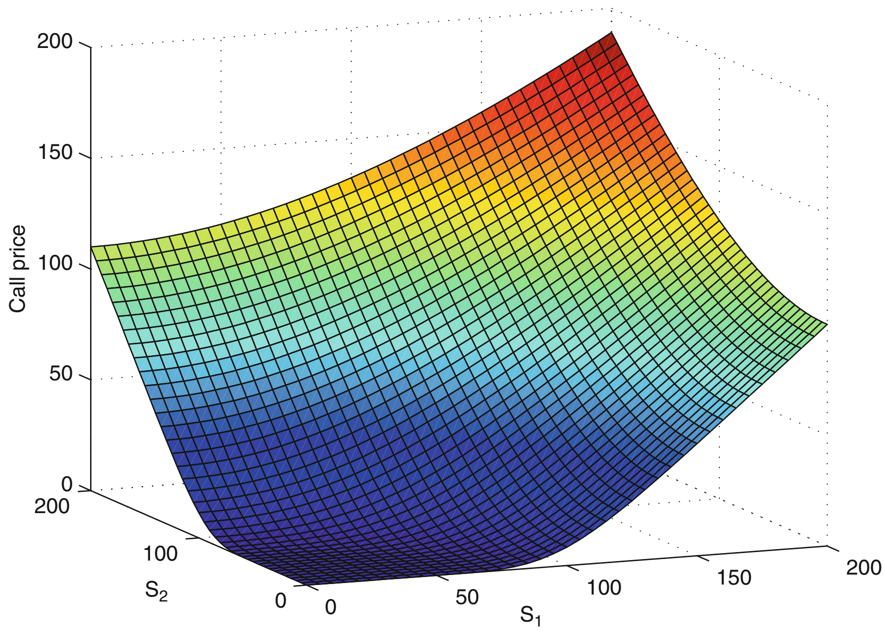

Fig. 4.10. The values of a European two-asset call option ($r = 0.02$, $D_{01} = 0.01$, $D_{02} = 0.01$, $\sigma_1 = 0.2$, $\sigma_2 = 0.15$, $\rho = 0.8$, $E_1 = 100$, $E_2 = 95$, and $T = 1$)

$$\frac{\partial w}{\partial t} + a_2 \frac{\partial^2 w}{\partial \theta^2} + b_2 \frac{\partial w}{\partial \theta} - rw = 0, \quad 0 \le \theta \le \frac{\pi}{2}, \quad 0 \le t \le T$$

and

$$\frac{\partial w}{\partial t} + a_2 \frac{\partial^2 w}{\partial \theta^2} + (b_2 + a_{12}) \frac{\partial w}{\partial \theta} - (r - b_1)w = 0, \quad 0 \le \theta \le \frac{\pi}{2}, \quad 0 \le t \le T$$

respectively. These two parabolic partial differential equations also degenerate to ordinary differential equations at $\theta = 0$ and $\theta = \pi/2$, so in order for them to have unique solutions, only final conditions are enough. Just like the situation on the boundaries $\theta = 0$ and $\theta = \pi/2$, the solutions of these two equations provide the value of the solution to Eq. (4.79) at $\xi = 0$ and $\xi = 1$. Consequently, the final condition (4.80) determines a unique solution to Eq. (4.79).

Instead of t, using a new variable $\tau = T - t$, Eq. (4.79) will change slightly. Let us call this equation the modified Eq. (4.79). It can be discretized by the implicit scheme (7.46). Using the final condition (4.80) and the scheme (7.46) obtained by discretizing the modified Eq. (4.79), a numerical solution of European option problems involving two assets, for example, a European two-asset call option, can be obtained. In Fig. 4.10 the price of the European two-asset call option with $r = 0.02$, $D_{01} = 0.01$, $D_{02} = 0.01$, $\sigma_1 = 0.2$, $\sigma_2 = 0.15$, $\rho = 0.8$, $E_1 = 100$, $E_2 = 95$, and $T = 1$ for $(S_1, S_2) \in [0, 200] \times [0, 200]$ is shown.

For an American multi-asset option, when the problem is formulated as a linear complementarity problem and this transformation given in this subsection is adopted, the problem is defined on a finite rectangular domain. Such a linear complementarity problem is not difficult to solve numerically.

4.6 Some Other Exotic Options

In this section, we introduce some other exotic options, namely, binary options, forward start options, compound options, and chooser options. In what follows, we discuss each case in each subsection.

4.6.1 Binary Options

Binary options are options with discontinuous payoffs. A simple example is a cash-or-nothing call. For this case, the payoff is

$$V(S, T) = B \times H(S - E),$$

where B is a constant and H is the Heaviside function:

$$H(S - E) = \begin{cases} 0, \ S < E, \\ 1, \ S > E. \end{cases}$$

From the formula (2.84) in Sect. 2.6.3, we know that $V(S,t)$ is given by

$$\begin{aligned}V(S,t) &= \mathrm{e}^{-r(T-t)}\int_0^\infty B\times H(S'-E)\,G(S',T;S,t)\,dS'\\ &= B\mathrm{e}^{-r(T-t)}\int_E^\infty G(S',T;S,t)\,dS'\\ &= B\mathrm{e}^{-r(T-t)}N(d_2),\end{aligned}$$

where $N(z)$ is defined by Eq. (2.89) in Sect. 2.6.3 and

$$d_2 = \frac{\ln(S/E) + \left(r - D_0 - \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}.$$

Besides cash-or-nothing calls, we can also have cash-or-nothing puts, asset-or-nothing calls, and asset-or-nothing puts. Their payoffs are

$$B\times H(E-S), \quad S\times H(S-E), \quad \text{and} \quad S\times H(E-S),$$

respectively. Their closed-form solutions are left for the reader to derive as an exercise.

4.6.2 Forward Start Options (Delayed Strike Options)

Forward start (delayed strike) options are options that are paid for now but will start at some time T_1 in the future, and the exercise price of which depends on the stock price at time T_1. Here, we assume that the exercise price $E = \alpha S_{T_1}$, where S_{T_1} is the stock price at time T_1 and α is a positive number.

Suppose that the current time is t_0 and consider a forward start American call option that will start at time $T_1 > t_0$ and mature at time $T_2 > T_1$. In order to have the value of the forward start option, we first need to find the solution of the American call problem with a payoff

$$C(S,T_2) = \max(S - \alpha S_{T_1}, 0)$$

at time T_2. Actually, the solution of this problem $C(S,t)$ is equal to

$$\alpha S_{T_1} C^*\left(\frac{S}{\alpha S_{T_1}},\, t\right),$$

where $C^*\left(\bar{S},t\right)$ is the solution of the American option with the following standard payoff

$$C^*\left(\bar{S},T_2\right)=\max\left(\bar{S}-1,0\right)$$

at time T_2. Therefore,

$$C\left(S_{T_1},T_1\right)=\alpha S_{T_1}C^*\left(\frac{1}{\alpha},T_1\right).$$

At time t_0, we do not know the stock price at time T_1, implying that S_{T_1} is a random variable. Therefore, the value of the forward start option at time t_0 is

$$\begin{aligned}&\mathrm{e}^{-r(T_1-t_0)}\mathrm{E}\left[C(S_{T_1},T_1)\right]=\alpha\mathrm{e}^{-r(T_1-t_0)}C^*\left(\frac{1}{\alpha},T_1\right)\mathrm{E}\left[S_{T_1}\right]\\&=\alpha\mathrm{e}^{-r(T_1-t_0)}C^*\left(\frac{1}{\alpha},T_1\right)S\mathrm{e}^{(r-D_0)(T_1-t_0)}=\alpha S\mathrm{e}^{-D_0(T_1-t_0)}C^*\left(\frac{1}{\alpha},T_1\right).\end{aligned}$$

Here, we assume that we are in a "risk-neutral" world. If the option is European, we can have a similar formula, and $C^*\left(\frac{1}{\alpha},T_1\right)$ should be replaced by $c^*\left(\frac{1}{\alpha},T_1\right)$, which is the solution of the standard European call option and has an analytic expression.

4.6.3 Compound Options

Compound options are options on options. There are four main types of compound options: a call on a call, a put on a call, a call on a put, and a put on a put. Compound options have two strike prices and two exercise dates. For example, the holder of a call on call option is entitled to pay the first strike price E_1 and receive a call option on the first date T_1, and is given the right to buy the underlying asset for the second strike price E_2 on the second exercise date $T_2>T_1$ if the first option is exercised. Here we assume that both options are European. Let c_1 denote the value of the option when the first option is a European call, and let p_1 denote the value of the option when the first one is a put. Furthermore, let $c_1(S,t;C_2)$ and $c_1(S,t;c_2)$ denote the price of a European call option on an American call option and a European call option, respectively. For the other cases, we adopt similar notation. When the first option is a European call, the compound option will only be exercised on the first exercise date if the value of the option on that date is greater than the first strike price. Assume that the second call option is American. Then, the price of the compound option is

$$c_1\left(S,t;C_2\right)=\mathrm{e}^{-r(T_1-t)}\int_0^\infty c_1\left(S',T_1;C_2\right)G\left(S',T_1;S,t\right)dS',$$

where $c_1(S', T_1; C_2) = \max(C_2(S', T_1) - E_1, 0)$ and $G(S', T_1; S, t)$ is given by the expression (2.85) in Sect. 2.6.3.

Clearly, in this case, the price can be written as

$$c_1(S, t; C_2) = \mathrm{e}^{-r(T_1-t)} \int_{S^*}^{\infty} [C_2(S', T_1) - E_1] \, G(S', T_1; S, t) \, dS',$$

where S^* satisfies the condition $C_2(S^*, T_1) - E_1 = 0$. In order to get $c_1(S, t; C_2)$, we can find the value of the integral numerically or we can solve the following final value problem by a numerical method:

$$\begin{cases} \dfrac{\partial c_1}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 c_1}{\partial S^2} + (r - D_0) S \dfrac{\partial c_1}{\partial S} - rc_1 = 0, & 0 \le S, \quad 0 \le t \le T_1, \\ c_1(S, T_1; C_2) = \max(C_2(S, T_1) - E_1, 0), & 0 \le S. \end{cases}$$

Before that, we need to solve an American call option with a payoff function

$$C_2(S, T_2) = \max(S - E_2, 0)$$

and satisfying the constraint

$$C_2(S, t) \ge \max(S - E_2, 0)$$

from T_2 to T_1, so that we can have the value of the function $C_2(S, T_1)$ for any S. Because there is no analytic expression for $C_2(S, T_1)$ if $D_0 \ne 0$, we also have to get $C_2(S, T_1)$ numerically. In Fig. 4.11, the price of a European call on an American call option at time $t = 0$ is given. The result is obtained by solving the partial differential equation problem, and the parameters of this problem are given in the figure. For a European call on an American put, the price can be determined similarly.

The price of a European put on an American put is the solution of the problem

$$\begin{cases} \dfrac{\partial p_1}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 p_1}{\partial S^2} + (r - D_0) S \dfrac{\partial p_1}{\partial S} - rp_1 = 0, & 0 \le S, \quad 0 \le t \le T_1, \\ p_1(S, T_1; P_2) = \max(E_1 - P_2(S, T_1), 0), & 0 \le S, \end{cases}$$

where P_2 is the price of an American put option with a payoff $P_2(S, T_2) = \max(E_2 - S, 0)$ and satisfying the constraint $P_2(S, t) \ge \max(E_2 - S, 0)$. The price can be obtained numerically. In Fig. 4.12, the price of a European put on an American put at time $t = 0$ is shown. The price of a European put on an American call option can be obtained in a similar way.

For a European option on a European option, the compound option can be valued analytically in terms of integrals of the bivariate standardized normal distribution (see [32, 43], or [54]). These closed-form solutions are

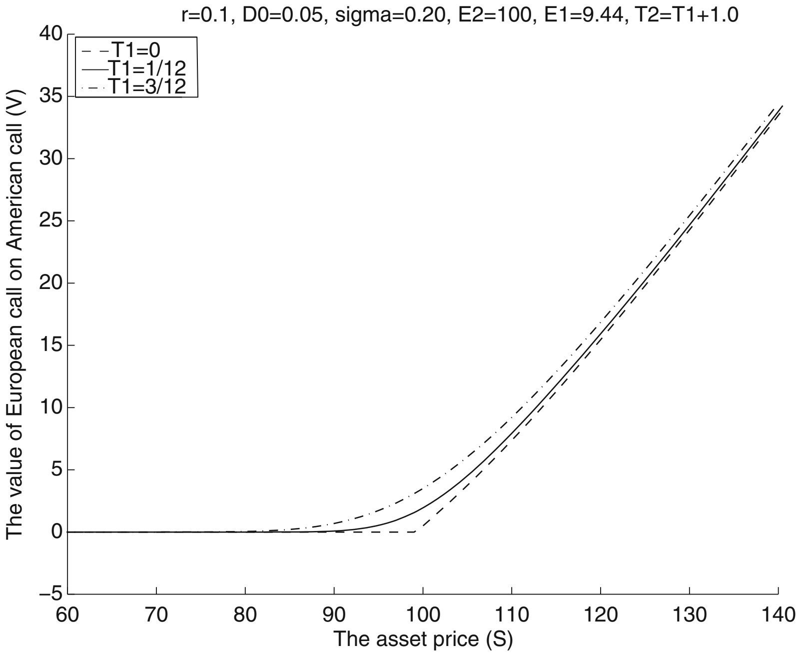

Fig. 4.11. The price of a European call on an American call

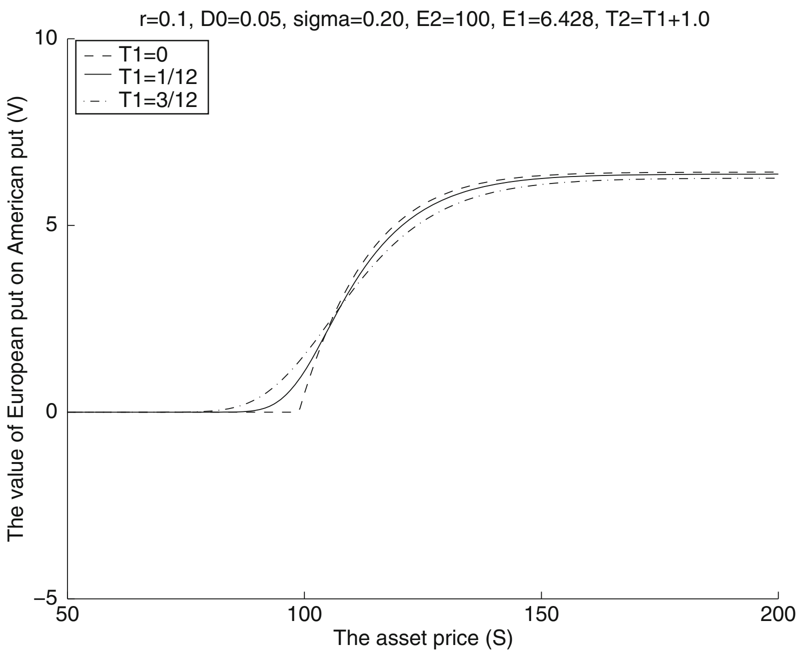

Fig. 4.12. The price of a European put on an American put

$$\begin{aligned}
c_1(S,t;c_2) &= S\mathrm{e}^{-D_0(T_2-t)}N_2(d_{11},d_{12};\rho) - E_2\mathrm{e}^{-r(T_2-t)}N_2(d_{21},d_{22};\rho) \\
&\quad -E_1\mathrm{e}^{-r(T_1-t)}N_2(d_{21}), \\
c_1(S,t;p_2) &= E_2\mathrm{e}^{-r(T_2-t)}N_2(-d_{23},-d_{22};\rho) - S\mathrm{e}^{-D_0(T_2-t)}N_2(-d_{13},-d_{12};\rho) \\
&\quad -E_1\mathrm{e}^{-r(T_1-t)}N(-d_{23}), \\
p_1(S,t;c_2) &= E_1\mathrm{e}^{-r(T_1-t)}N(-d_{21}) - S\mathrm{e}^{-D_0(T_2-t)}N_2(-d_{11},d_{12};-\rho) \\
&\quad +E_2\mathrm{e}^{-r(T_2-t)}N_2(-d_{21},d_{22};-\rho), \\
p_1(S,t;p_2) &= E_1\mathrm{e}^{-r(T_1-t)}N(d_{23}) - E_2\mathrm{e}^{-r(T_2-t)}N_2(d_{23},-d_{22};-\rho) \\
&\quad +S\mathrm{e}^{-D_0(T_2-t)}N_2(d_{13},-d_{12};-\rho),
\end{aligned}$$

where

$$\begin{aligned}
d_{11} &= \frac{\ln(S/S^*) + \left(r - D_0 + \sigma^2/2\right)(T_1 - t)}{\sigma\sqrt{T_1 - t}}, \\
d_{21} &= \frac{\ln(S/S^*) + \left(r - D_0 - \sigma^2/2\right)(T_1 - t)}{\sigma\sqrt{T_1 - t}}, \\
d_{12} &= \frac{\ln(S/E_2) + \left(r - D_0 + \sigma^2/2\right)(T_2 - t)}{\sigma\sqrt{T_2 - t}}, \\
d_{22} &= \frac{\ln(S/E_2) + \left(r - D_0 - \sigma^2/2\right)(T_2 - t)}{\sigma\sqrt{T_2 - t}}, \\
d_{13} &= \frac{\ln(S/S^{**}) + \left(r - D_0 + \sigma^2/2\right)(T_1 - t)}{\sigma\sqrt{T_1 - t}}, \\
d_{23} &= \frac{\ln(S/S^{**}) + \left(r - D_0 - \sigma^2/2\right)(T_1 - t)}{\sigma\sqrt{T_1 - t}}, \\
\rho &= \sqrt{\frac{T_1 - t}{T_2 - t}}.
\end{aligned}$$

Here, S^* and S^{**} are the solutions of the following equations:

$$c_2(S^*, T_1) = E_1$$

and

$$p_2(S^{**}, T_1) = E_1.$$

Let us derive the formula for the price of a put on a call, and other formulae are left for readers to prove. For a put on a call, we have

$$\begin{aligned}
p_1(S,t;c_2) &= \mathrm{e}^{-r(T_1-t)}\int_0^\infty \max(E_1 - c_2(S',T_1), 0)G(S',T_1;S,t)\,dS' \\
&= \mathrm{e}^{-r(T_1-t)}\int_0^{S^*} [E_1 - c_2(S',T_1)]\,G(S',T_1;S,t)\,dS'
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{e}^{-r(T_1-t)} E_1 \int_0^{S^*} G(S', T_1; S, t)\, dS' \\
&\quad - \mathrm{e}^{-r(T_1-t)} \int_0^{S^*} \mathrm{e}^{-r(T_2-T_1)} \int_{E_2}^{\infty} (S''-E_2) G(S'', T_2; S', T_1) \\
&\qquad\qquad \times G(S', T_1; S, t)\, dS'' dS' \\
&= \mathrm{e}^{-r(T_1-t)} E_1 \int_0^{S^*} G(S', T_1; S, t)\, dS' \\
&\quad - \mathrm{e}^{-r(T_2-t)} \int_0^{S^*} \int_{E_2}^{\infty} S'' G(S'', T_2; S', T_1)\, G(S', T_1; S, t)\, dS'' dS' \\
&\quad + E_2 \mathrm{e}^{-r(T_2-t)} \int_0^{S^*} \int_{E_2}^{\infty} G(S'', T_2; S', T_1)\, G(S', T_1; S, t)\, dS'' dS',
\end{aligned}
$$

where

$$
G(S', T_1; S, t) = \frac{1}{\sigma\sqrt{2\pi(T_1-t)}S'} \mathrm{e}^{-\left[\ln S' - \ln S - \left(r - D_0 - \sigma^2/2\right)(T_1-t)\right]^2 / 2\sigma^2 (T_1-t)}
$$

and $G(S'', T_2; S', T_1)$ is defined in the same way.

Now let us express the three integrals above by cumulative distribution functions. Noticing

$$
\int_0^{S^*} G(S', T_1; S, t)\, dS' = N\left(\frac{\ln(S^*/a) + b^2/2}{b}\right),
$$

where

$$
a = S\mathrm{e}^{(r-D_0)(T_1-t)}
$$

and

$$
b = \sigma\sqrt{T_1 - t},
$$

we know that the first term is equal to $E_1 \mathrm{e}^{-r(T_1-t)} N(-d_{21})$.

Now we find out the result of the integral in the third term. Let

$$
\begin{aligned}
x &= \frac{\ln S' - \ln S - \left(r - D_0 - \sigma^2/2\right)(T_1 - t)}{\sigma\sqrt{T_1 - t}}, \\
y &= \frac{\ln S'' - \ln S - \left(r - D_0 - \sigma^2/2\right)(T_2 - t)}{\sigma\sqrt{T_2 - t}}, \\
\rho &= \sqrt{\frac{T_1 - t}{T_2 - t}}.
\end{aligned}
$$

Then

$$\sqrt{1-\rho^2} = \sqrt{\frac{T_2 - T_1}{T_2 - t}},$$
$$\frac{y - \rho x}{\sqrt{1-\rho^2}} = \frac{\ln S'' - \ln S' - \left(r - D_0 - \sigma^2/2\right)(T_2 - T_1)}{\sigma\sqrt{T_2 - T_1}}.$$

Therefore

$$\int_0^{S^*}\int_{E_2}^{\infty} G\left(S'', T_2; S', T_1\right) G\left(S', T_1; S, t\right) dS'' dS'$$
$$= \int_0^{S^*}\int_{E_2}^{\infty} \frac{1}{\sigma\sqrt{2\pi\left(T_2 - T_1\right)}S''} \mathrm{e}^{-\left[\ln S'' - \ln S' - \left(r - D_0 - \sigma^2/2\right)(T_2 - T_1)\right]^2/2\sigma^2(T_2 - T_1)}$$
$$\times \frac{1}{\sigma\sqrt{2\pi\left(T_1 - t\right)}S'} \mathrm{e}^{-\left[\ln S' - \ln S - \left(r - D_0 - \sigma^2/2\right)(T_1 - t)\right]^2/2\sigma^2(T_1 - t)} dS'' dS'$$

$$= \int_{-\infty}^{-d_{21}}\int_{-d_{22}}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} \mathrm{e}^{-\frac{1}{2}(y-\rho x)^2/(1-\rho^2) - \frac{1}{2}x^2} dy dx$$
$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{-d_{21}}\int_{-d_{22}}^{\infty} \mathrm{e}^{-(y^2 - 2\rho xy + x^2)/2(1-\rho^2)} dy dx$$
$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{-d_{21}}\int_{-\infty}^{d_{22}} \mathrm{e}^{-(y^2 + 2\rho xy + x^2)/2(1-\rho^2)} dy dx$$
$$= N_2\left(-d_{21}, d_{22}; -\rho\right).$$

Now let us calculate the integral in the second term. Because

$$S'' \mathrm{e}^{-\left[\ln S'' - \ln S' - \left(r - D_0 - \sigma^2/2\right)(T_2 - T_1)\right]^2/2\sigma^2(T_2 - T_1)}$$
$$= S' \mathrm{e}^{(r - D_0)(T_2 - T_1)} \mathrm{e}^{-\left[\ln S'' - \ln S' - \left(r - D_0 + \sigma^2/2\right)(T_2 - T_1)\right]^2/2\sigma^2(T_2 - T_1)}$$

and

$$S' \mathrm{e}^{-\left[\ln S' - \ln S - \left(r - D_0 - \sigma^2/2\right)(T_1 - t)\right]^2/2\sigma^2(T_1 - t)}$$
$$= S \mathrm{e}^{(r - D_0)(T_1 - t)} \mathrm{e}^{-\left[\ln S' - \ln S - \left(r - D_0 + \sigma^2/2\right)(T_1 - t)\right]^2/2\sigma^2(T_1 - t)},$$

we have

$$\int_0^{S^*}\int_{E_2}^{\infty} S''G\left(S'',T_2;S',T_1\right)G\left(S',T_1;S,t\right)dS''dS'$$

$$=\int_0^{S^*}\int_{E_2}^{\infty}\frac{S''}{\sigma\sqrt{2\pi\left(T_2-T_1\right)}S''}\mathrm{e}^{-\left[\ln S''-\ln S'-\left(r-D_0-\sigma^2/2\right)\left(T_2-T_1\right)\right]^2/2\sigma^2\left(T_2-T_1\right)}$$

$$\times\frac{1}{\sigma\sqrt{2\pi\left(T_1-t\right)}S'}\mathrm{e}^{-\left[\ln S'-\ln S-\left(r-D_0-\sigma^2/2\right)\left(T_1-t\right)\right]^2/2\sigma^2\left(T_1-t\right)}dS''dS'$$

$$=S\mathrm{e}^{\left(r-D_0\right)\left(T_1-t\right)}\mathrm{e}^{\left(r-D_0\right)\left(T_2-T_1\right)}$$

$$\times\int_0^{S^*}\int_{E_2}^{\infty}\frac{1}{\sigma\sqrt{2\pi\left(T_2-T_1\right)}S''}\mathrm{e}^{-\left[\ln S''-\ln S'-\left(r-D_0+\sigma^2/2\right)\left(T_2-T_1\right)\right]^2/2\sigma^2\left(T_2-T_1\right)}$$

$$\times\frac{1}{\sigma\sqrt{2\pi\left(T_1-t\right)}S'}\mathrm{e}^{-\left[\ln S'-\ln S-\left(r-D_0+\sigma^2/2\right)\left(T_1-t\right)\right]^2/2\sigma^2\left(T_1-t\right)}dS''dS'$$

$$=S\mathrm{e}^{\left(r-D_0\right)\left(T_2-t\right)}N_2\left(-d_{11},d_{12};-\rho\right).$$

Here, we use the fact that the only difference between the last integral above and the integral related to the third term is that $\left(r-D_0-\sigma^2/2\right)$ is replaced by $\left(r-D_0+\sigma^2/2\right)$, so replacing d_{21} and d_{22} by d_{11} and d_{12} yields the result here. Consequently, we arrive at

$$p_1\left(S,t;c_2\right)=E_1\mathrm{e}^{-r\left(T_1-t\right)}N\left(-d_{21}\right)-S\mathrm{e}^{-D_0\left(T_2-t\right)}N_2\left(-d_{11},d_{12};-\rho\right)$$
$$+E_2\mathrm{e}^{-r\left(T_2-t\right)}N_2\left(-d_{21},d_{22};-\rho\right).$$

4.6.4 Chooser Options

Chooser (as-you-like-it) options are only slightly more complicated than compound options (see [82]). A chooser option gives its owner the right to purchase either a call for an amount E_{1c} or a put for an amount E_{1p} at time T_1. We suppose that both the call and put options are expired at time T_2 and with an exercise price E_2. We still assume that the first option is European and the second options are American. Therefore, in order to find the price of such a chooser option, we need to do the following. First, find the price functions at time T_1 of the American call option and the American put option with exercise price E_2 and expiry T_2. Then, calculate the price of the chooser option by

$$c_1\left(S,t\right)=\mathrm{e}^{-r\left(T_1-t\right)}\int_0^{\infty}\max\left(C_2\left(S',T_1\right)-E_{1c},P_2\left(S',T_1\right)-E_{1p},0\right)$$
$$\times G\left(S',T_1;S,t\right)dS',$$

where $C_2(S', T_1)$ and $P_2(S', T_1)$ are the values of the second American call and put options, respectively. Just like the case of compound options, $c_1(S, t)$ can be obtained by numerical integration or by solving a final value problem.

A chooser option could be more complicated (see [43, 83]). It could also be less complicated. If $E_{1c} = E_{1p} = 0$, then the chooser option is called a standard chooser option (see [69, 54], or [43]). In this case, if both the underlying options are European, then the chooser option price is given by

$$\begin{aligned} V(S,t) = &S\mathrm{e}^{-D_0(T_2-t)}N(d_1) - E_2\mathrm{e}^{-r(T_2-t)}N(d_2) \\ &+E_2\mathrm{e}^{-r(T_2-t)}N(-d_4) - S\mathrm{e}^{-D_0(T_2-t)}N(-d_3) \end{aligned}$$

where

$$\begin{aligned} d_1 &= \left[\ln(S/E_2) + \left(r - D_0 + \frac{1}{2}\sigma^2\right)(T_2 - t)\right] \Big/ \sigma\sqrt{T_2 - t}, \\ d_2 &= d_1 - \sigma\sqrt{T_2 - t}, \\ d_3 &= \left[\ln(S/E_2) + (r - D_0)(T_2 - t) + \frac{1}{2}\sigma^2(T_1 - t)\right] \Big/ \sigma\sqrt{T_1 - t}, \\ d_4 &= d_3 - \sigma\sqrt{T_1 - t}. \end{aligned}$$

This is left for readers to prove as an exercise.

Problems

Table 4.9. Problems and sections

Problems	Sections	Problems	Sections	Problems	Sections
1–5	4.2	6–8	4.3	9–20	4.4
21–34	4.5	35–40	4.6		

1. Consider the following problem:

$$\begin{cases} \dfrac{\partial \overline{V}}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 \overline{V}}{\partial S^2} + (r - D_0) S \dfrac{\partial \overline{V}}{\partial S} - r\overline{V} = 0, \quad 0 \le S, \quad t \le T, \\ \overline{V}(S,T) = \begin{cases} \varphi_1(S), \ 0 \le S \le B, \\ \varphi_2(S), \ B < S, \end{cases} \end{cases}$$

where $\varphi_1(S)$ and $\varphi_2(S)$ are continuous functions and

$$\varphi_1(B) = \varphi_2(B)$$

may not hold.

(a) *Try to find such a relation between $\varphi_1(S)$ and $\varphi_2(S)$ that $\overline{V}(B,t) = 0$.

(b) *Based on the result in part (a), show that for the problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r - D_0)\, S\dfrac{\partial V}{\partial S} - rV = 0, & B_l \le S, \quad t \le T, \\ V(S,T) = V_T(S), & B_l \le S, \\ V(B_l, t) = 0, & t \le T, \end{cases}$$

the solution is

$$V(S,t) = \mathrm{e}^{-r(T-t)} \int_{B_l}^{\infty} V_T(S')\, G_1(S', T; S, t, B_l)\, dS',$$

where

$$G_1(S', T; S, t, B_l) = G(S', T; S, t) - (B_l/S)^{2(r-D_0-\sigma^2/2)/\sigma^2}\, G\left(S', T; B_l^2/S, t\right).$$

Here

$$G(S', T; S, t) = \frac{1}{S'\sigma\sqrt{2\pi(T-t)}} \mathrm{e}^{-\left[\ln(S'/S) - \left(r - D_0 - \sigma^2/2\right)(T-t)\right]^2 / 2\sigma^2 (T-t)}.$$

(c) The value of a European down-and-out call option is the solution of the problem:

$$\begin{cases} \dfrac{\partial c_o}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial c_o}{\partial S^2} + (r - D_0)\, S\dfrac{\partial c_o}{\partial S} - rc_o = 0, & B_l \le S, \quad t \le T, \\ c_o(S,T) = \max(S - E, 0), & B_l \le S, \\ c_o(B_l, t) = 0, & t \le T. \end{cases}$$

Based on the result in part (b), show that for the case $B_l \le E$, the expression of c_o is

$$c_o(S,t) = c(S,t) - \left(\frac{B_l}{S}\right)^{2(r-D_0-\sigma^2/2)/\sigma^2} c\left(\frac{B_l^2}{S}, t\right);$$

and for the case $B_l \ge E$, its expression is

$$\begin{aligned} c_o(S,t) = {} & S\mathrm{e}^{-D_0(T-t)} N\left(\tilde{d}_1(B_l)\right) - E\mathrm{e}^{-r(T-t)} N\left(\tilde{d}_1(B_l) - \sigma\sqrt{T-t}\right) \\ & - (B_l/S)^{2(r-D_0-\sigma^2/2)/\sigma^2} \left[\frac{B_l^2}{S}\mathrm{e}^{-D_0(T-t)} N\left(\bar{d}_1(B_l)\right) \right. \\ & \left. - E\mathrm{e}^{-r(T-t)} N\left(\bar{d}_1(B_l) - \sigma\sqrt{T-t}\right)\right], \end{aligned}$$

where

$$\tilde{d}_1(B_l) = \left[\ln \frac{S\mathrm{e}^{(r-D_0)(T-t)}}{B_l} + \frac{1}{2}\sigma^2(T-t)\right] \Big/ \left(\sigma\sqrt{T-t}\right),$$

$$\bar{d}_1(B_l) = \left[\ln \frac{B_l\mathrm{e}^{(r-D_0)(T-t)}}{S} + \frac{1}{2}\sigma^2(T-t)\right] \Big/ \left(\sigma\sqrt{T-t}\right).$$

(d) Based on the result in part (a), show that for the problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r-D_0)\, S\dfrac{\partial V}{\partial S} - rV = 0, & 0 \le S \le B_u,\ t \le T, \\ V(S,T) = V_T(S), & 0 \le S \le B_u, \\ V(B_u, t) = 0, & t \le T, \end{cases}$$

the solution is

$$V(S,t) = \mathrm{e}^{-r(T-t)} \int_0^{B_u} V_T(S')\, G_1(S', T; S, t, B_u)\, dS',$$

where

$$\begin{aligned} G_1(S', T; S, t, B_u) &= G(S', T; S, t) \\ &\quad - (B_u/S)^{2(r-D_0-\sigma^2/2)/\sigma^2}\, G\left(S', T; B_u^2/S, t\right). \end{aligned}$$

Here

$$\begin{aligned} &G(S', T; S, t) \\ &= \frac{1}{S'\sigma\sqrt{2\pi(T-t)}} \mathrm{e}^{-\left[\ln(S'/S) - \left(r-D_0-\sigma^2/2\right)(T-t)\right]^2/2\sigma^2(T-t)}. \end{aligned}$$

(e) Based on the result in part (d), find the closed-form solution of a European up-and-out put option for both the case $B_u \ge E$ and the case $0 < B_u \le E$.

2. Show the following results which are related to the down-and-out call options:

(a) If $S \ge B_l$ and $S' \ge B_l$, then

$$\begin{aligned} &G_1(S', T; S, t, B_l) \\ &= G(S', T; S, t) - (B_l/S)^{2(r-D_0-\sigma^2/2)/\sigma^2}\, G\left(S', T; B_l^2/S, t\right) \ge 0, \end{aligned}$$

where

$$\begin{aligned} &G(S', T; S, t) \\ &= \frac{1}{S'\sigma\sqrt{2\pi(T-t)}} \mathrm{e}^{-\left[\ln(S'/S) - \left(r-D_0-\sigma^2/2\right)(T-t)\right]^2/2\sigma^2(T-t)}. \end{aligned}$$

(Hint: First it should be shown that this inequality is equivalent to the following inequalities:

$$\ln G\left(S', T; S, t\right) \geq \ln\left[\left(B_l/S\right)^{2(r-D_0-\sigma^2/2)/\sigma^2} G\left(S', T; B_l^2/S, t\right)\right]$$

and

$$\left(\ln\frac{S'}{B_l} + \ln\frac{S}{B_l}\right)^2 \geq \left(\ln\frac{S'}{B_l} - \ln\frac{S}{B_l}\right)^2.$$

)

(b) Let $c_o(S, t; B_l)$ be the price of the European down-and-out call option, where B_l is a parameter. For $S \geq B_l$,

$$\frac{\partial c_o(S, t; B_l)}{\partial B_l} \leq 0.$$

(Hint: Show $\dfrac{\partial G_1}{\partial B_l} \leq 0$ first.)

(c) Let $c_o(S, t)$ and $C_o(S, t)$ be the prices of the European and American down-and-out call options, respectively. Between them the following is true:

$$C_o(S, t) \geq c_o(S, t) \quad \text{for any } t.$$

(d) For $C_o(S, t)$ the following is true:

$$C_o(S, t^*) \geq C_o(S, t^{**}) \quad \text{if} \quad t^* \leq t^{**}.$$

(e) Let $C_o(S, t; B_l)$ be the price of the American down-and-out call option, where B_l is a parameter. For $C_o(S, t; B_l)$ the following is true:

$$C_o(S, t; B_l^*) \geq C_o(S, t; B_l^{**}) \quad \text{if} \quad 0 \leq B_l^* \leq B_l^{**}.$$

3. Show that a European up-and-out put option with $B_u > E$ plus a European up-and-in put option with the same parameters is equal to a vanilla European put option.
4. Find the solution of the European down-and-out call option

$$\begin{cases} \dfrac{\partial c_o}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 c_o}{\partial S^2} + (r - D_0)\, S \dfrac{\partial c_o}{\partial S} - r c_o = 0, & B_l(t) \leq S, \quad t \leq T, \\ c_o\left(S, T\right) = \max(S - E, 0), & B_l(t) \leq S, \\ c_o\left(B_l(t), t\right) = 0, & t \leq T, \end{cases}$$

where $B_l(t) = bE\mathrm{e}^{-\alpha(T-t)}$ with $b \in [0, 1]$ and $\alpha \geq 0$. (Hint: Let $\eta = S\mathrm{e}^{\alpha(T-t)}$, the moving barrier becomes a fixed barrier in the (η, t)-plane. Then, solve a barrier option problem with a fixed barrier.)

5. Let $P_o(S,t)$ denote the price of an American up-and-out put option. Show that under the following transformation

$$\begin{cases} \zeta = \dfrac{E^2}{S}, \\ C_o(\zeta,t) = \dfrac{EP_o(S,t)}{S}, \end{cases}$$

the new function $C_o(\zeta,t)$ represents the price of an American down-and-out call option. Based on this result, derive the symmetry relations between American down-and-out call and up-and-out put options.

6. Consider an average strike option with discrete arithmetic averaging. Assume that the stock pays dividends and that during the time step $[t, t+dt]$, the dividend payment is $D(S,t)dt$. Take S and

$$I = \frac{1}{K}\int_0^t S(\tau)f(\tau)d\tau$$

as state variables, where

$$f(t) = \sum_{i=1}^{K} \delta(t - t_i).$$

   (a) Derive the equation for such an option directly by using a portfolio $\Pi = V - \Delta S$.
   (b) Find the jump condition at $t = t_i$, $i = 1, 2, \cdots, K$ if at $t = t_i$ no discrete dividend is paid.
   (c) Finally under the assumption $D(S,t) = D_0 S$ and $V(S,I,T) = \max(\pm(\alpha S - I), 0)$, reduce an average strike option problem to a problem with only two independent variables and the payoff to a function with only one independent variable.

7. Let $V(S,A,t)$ be the price of a European Asian option with continuous arithmetic averaging, where A is the average of the price during the time period $[0,t]$. As we know, the equation for European Asian option with continuous arithmetic averaging is

$$\frac{\partial W(\eta,t)}{\partial t} + \mathbf{L}_{a,t}W(\eta,t) = 0,$$

where $W = V(S,A,t)/S$, $\eta = A/S$ and $\mathbf{L}_{a,t}$ is the time-dependent operator related to Asian options and given by

$$\mathbf{L}_{a,t} = \frac{1}{2}\sigma^2\eta^2\frac{\partial^2}{\partial\eta^2} + \left[(D_0 - r)\eta + \frac{1-\eta}{t}\right]\frac{\partial}{\partial\eta} - D_0.$$

   (a) Write down the LC problem for an American Asian put option with a continuous arithmetic average strike price.

(b) Determine where the PDE can always be used and a free boundary cannot appear and where a free boundary may appear.
(c) Derive the free-boundary problem for this case. (Assume that there exists at most one free boundary.)

8. Define

$$\mathbf{L}_{SAt} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0)S\frac{\partial}{\partial S} + \frac{S - A}{t}\frac{\partial}{\partial A} - r.$$

(a) For the LC problem:

$$\begin{cases} \min\left(-\dfrac{\partial V}{\partial t} - \mathbf{L}_{SAt}V,\ V - \max(\alpha S - A, 0)\right) = 0, & 0 \le S,\ 0 \le A, \\ & t \le T, \\ V(S, A, T) = \max(\alpha S - A, 0), & 0 \le S,\ 0 \le A, \end{cases}$$

find the function of location of the free boundary at $t = T$, $S = S_f(A, T)$.

(b) For the LC problem:

$$\begin{cases} \min\left(-\dfrac{\partial V}{\partial t} - \mathbf{L}_{SAt}V,\ V - \max(A - E, 0)\right) = 0, & 0 \le S,\ 0 \le A, \\ & t \le T, \\ V(S, A, T) = \max(A - E, 0), & 0 \le S,\ 0 \le A, \end{cases}$$

find the function of location of the free boundary at $t = T$, $A = A_f(S, T)$.

9. Suppose that sampling is done discretely at $t = t_1,\ t_2,\ \cdots,\ t_K$, where $0 \le t_1 < t_2 < \cdots < t_K \le T$. Let $H(t) = \max\left(S(t_1), \cdots, S\left(t_{i^*(t)}\right)\right)$, where $i^*(t)$ is the number of samplings before time t. Assume $dS = \mu S dt + \sigma S dX$ and the dividends are paid continuously with dividend yield D_0. Let $V(S, H, t)$ be the value of a lookback option with discrete sampling. Derive the PDE and the jump condition for such a lookback option by using a portfolio $\Pi = V(S, H, t) - \Delta S$ (without using the general PDE for derivative securities).

10. *Consider a lookback option $V(S, H, t)$ with continuous sampling. Describe how to get the PDE and the boundary condition for such an option from the PDE and the jump condition for an identical lookback option with discrete sampling and reduce the PDE to a PDE involving only two independent variables and the boundary condition to a boundary condition involving only one independent variable.

11. *Consider the following problem:

$$\begin{cases} \dfrac{\partial \overline{W}}{\partial t} + \dfrac{1}{2}\sigma^2\eta^2 \dfrac{\partial^2 \overline{W}}{\partial \eta^2} + (D_0 - r)\,\eta \dfrac{\partial \overline{W}}{\partial \eta} - D_0 \overline{W} = 0, \quad 0 \le \eta, \quad t \le T, \\ \overline{W}(\eta, T) = \begin{cases} \varphi_1(\eta), \, 0 \le \eta \le 1, \\ \varphi_2(\eta), \, 1 \le \eta, \end{cases} \end{cases}$$

where $\varphi_1(\eta)$ and $\varphi_2(\eta)$ are continuous functions, and

$$\varphi_1(1) = \varphi_2(1)$$

may not hold. Show that if

$$\begin{cases} \varphi_1(1) = \varphi_2(1), \\ \dfrac{d\varphi_1(\eta)}{d\eta} = \eta^{2\left(r - D_0 + \sigma^2/2\right)/\sigma^2} \dfrac{d\varphi_2(1/\eta)}{d\eta}, \end{cases}$$

or

$$\begin{cases} \varphi_2(1) = \varphi_1(1), \\ \dfrac{d\varphi_2(\eta)}{d\eta} = \eta^{2\left(r - D_0 + \sigma^2/2\right)/\sigma^2} \dfrac{d\varphi_1(1/\eta)}{d\eta}, \end{cases}$$

then $\dfrac{\partial \overline{W}(1,t)}{\partial \eta} = 0$.

12. Suppose that the payoff of a lookback strike put option is

$$\max(H - \beta S, 0),$$

where $\beta \ge 1$. Show that if $r \ne D_0$, its solution is

$$\begin{aligned} & p_{ls}(S, H, t) \\ = {} & e^{-r(T-t)} S \left[\frac{H}{S} N\left(\frac{\ln \frac{H}{\beta S} - \mu(T-t)}{\sigma\sqrt{T-t}} \right) \right. \\ & - \beta e^{(r-D_0)(T-t)} N\left(\frac{\ln \frac{H}{\beta S} - \left(\mu + \sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \right) \\ & - \frac{\sigma^2}{2(r-D_0)} \left(\frac{H}{S}\right)^{2(r-D_0)/\sigma^2} N\left(\frac{\ln \frac{S}{\beta H} - \mu(T-t)}{\sigma\sqrt{T-t}} \right) \\ & \left. + \frac{\sigma^2 \beta^{-2(r-D_0)/\sigma^2} e^{(r-D_0)(T-t)}}{2(r-D_0)} N\left(\frac{\ln \frac{S}{\beta H} + \left(\mu + \sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \right) \right], \end{aligned}$$

where

$$\mu = r - D_0 - \sigma^2/2,$$

and if $r = D_0$, the solution is

$$\begin{aligned} p_{ls}(S,H,t) = \mathrm{e}^{-r(T-t)} S \Bigg[& \frac{H}{S} N\left(\frac{\ln\frac{H}{\beta S} - \mu(T-t)}{\sigma\sqrt{T-t}}\right) \\ & -\beta N\left(\frac{\ln\frac{H}{\beta S} - \left(\mu+\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\ & + \left[\ln(S/\beta H) + \sigma^2(T-t)/2\right] \\ & \quad \times N\left(\frac{\ln(S/\beta H) + \sigma^2(T-t)/2}{\sigma\sqrt{T-t}}\right) \\ & + \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}} \mathrm{e}^{-\left[\ln(S/\beta H)+\sigma^2(T-t)/2\right]^2/2\sigma^2(T-t)} \Bigg]. \end{aligned}$$

13. Suppose the payoff of a lookback price call option is

$$\max(H - E, 0).$$

Show that if $H > E$, the price is

$$\begin{aligned} & c_{lp}(S,H,t) \\ = {} & \mathrm{e}^{-r(T-t)} \left\{ HN\left(\frac{\ln\frac{H}{S} - \mu(T-t)}{\sigma\sqrt{T-t}}\right) - E \right. \\ & + \left[1 + \frac{\sigma^2}{2(r-D_0)}\right] S\mathrm{e}^{(r-D_0)(T-t)} N\left(\frac{\ln\frac{S}{H} + \left(\mu+\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\ & \left. -\frac{\sigma^2 S}{2(r-D_0)} \left(\frac{H}{S}\right)^{2(r-D_0)/\sigma^2} N\left(\frac{\ln\frac{S}{H} - \mu(T-t)}{\sigma\sqrt{T-t}}\right) \right\}; \end{aligned}$$

and that if $H \le E$, the price is

$$\begin{aligned} & c_{lp}(S,H,t) \\ = {} & \mathrm{e}^{-r(T-t)} \left\{ -EN\left(\frac{\ln\frac{S}{E} + \mu(T-t)}{\sigma\sqrt{T-t}}\right) \right. \\ & + \left[1 + \frac{\sigma^2}{2(r-D_0)}\right] S\mathrm{e}^{(r-D_0)(T-t)} N\left(\frac{\ln\frac{S}{E} + \left(\mu+\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\ & \left. -\frac{\sigma^2 S}{2(r-D_0)} \left(\frac{E}{S}\right)^{2(r-D_0)/\sigma^2} N\left(\frac{\ln\frac{S}{E} - \mu(T-t)}{\sigma\sqrt{T-t}}\right) \right\}, \end{aligned}$$

where

$$\mu = r - D_0 - \sigma^2/2.$$

14. Suppose the payoff of a lookback price put option is

$$\max(E - L, 0).$$

Show that for the case $E > L$, the price is

$$\begin{aligned}
&p_{lp}(S, L, t) \\
&= e^{-r(T-t)} \left\{ E - LN\left(\frac{\ln \frac{S}{L} + \mu (T-t)}{\sigma\sqrt{T-t}} \right) \right. \\
&\quad - \left[1 + \frac{\sigma^2}{2(r - D_0)} \right] S e^{(r-D_0)(T-t)} N\left(\frac{\ln \frac{L}{S} - (\mu + \sigma^2)(T-t)}{\sigma\sqrt{T-t}} \right) \\
&\quad \left. + \frac{\sigma^2 S}{2(r - D_0)} \left(\frac{L}{S} \right)^{2(r-D_0)/\sigma^2} N\left(\frac{\ln \frac{L}{S} + \mu (T-t)}{\sigma\sqrt{T-t}} \right) \right\};
\end{aligned}$$

and that for $E \le L$, the price is

$$\begin{aligned}
&p_{lp}(S, L, t) \\
&= e^{-r(T-t)} \left\{ EN\left(\frac{\ln \frac{E}{S} - \mu (T-t)}{\sigma\sqrt{T-t}} \right) \right. \\
&\quad - \left[1 + \frac{\sigma^2}{2(r - D_0)} \right] S e^{(r-D_0)(T-t)} N\left(\frac{\ln \frac{E}{S} - (\mu + \sigma^2)(T-t)}{\sigma\sqrt{T-t}} \right) \\
&\quad \left. + \frac{\sigma^2 S}{2(r - D_0)} \left(\frac{E}{S} \right)^{2(r-D_0)/\sigma^2} N\left(\frac{\ln \frac{E}{S} + \mu (T-t)}{\sigma\sqrt{T-t}} \right) \right\},
\end{aligned}$$

where

$$\mu = r - D_0 - \sigma^2/2.$$

15. *Show that for lookback options depending on S, L, t, the Green's function is

$$\begin{aligned}
&g\left(S', L_t^{t'}; S, t' - t\right) \\
&= \frac{\partial}{\partial L_t^{t'}} \left[\frac{1}{S'\sigma\sqrt{2\pi\tau'}} \left(\frac{L_t^{t'}}{S} \right)^{2\mu/\sigma^2} e^{-\left[\ln\left(S'S/(L_t^{t'})^2\right) - \mu\tau'\right]^2 / 2\sigma^2\tau'} \right],
\end{aligned}$$

where $\tau' = t' - t$ and $\mu = r - D_0 - \sigma^2/2$.

16. Let $g\left(S', L_t^{t'}; S, t' - t\right)$ be Green's function for lookback options depending on S, L, t, and let $c_{ls}(S, L, t)$ and $C_{ls}(S, L, t)$ be the prices of the European and American lookback strike call options with continuous sampling, respectively.

(a) As we know,

$$g\left(S', L_t^{t'}; S, t'-t\right)$$
$$= \frac{\partial}{\partial L_t^{t'}}\left[\frac{1}{S'\sigma\sqrt{2\pi\tau'}}\left(\frac{L_t^{t'}}{S}\right)^{2\mu/\sigma^2} e^{-\left[\ln\left(S'S/(L_t^{t'})^2\right)-\mu\tau'\right]^2/2\sigma^2\tau'}\right],$$

where $\tau' = t'-t$ and $\mu = r - D_0 - \sigma^2/2$. Show

$$g\left(S', L_t^{t'}; S, t'-t\right) \ge 0$$

for any $L_t^{t'} \le \min(S, S')$.

(b) Show

$$C_{ls}(S, L, t) \ge c_{ls}(S, L, t)$$

always holds.

(c) Show $C_{ls}(S, L, t^*) \ge C_{ls}(S, L, t^{**})$ if $t^* < t^{**}$.

17. As we know, for a European lookback strike call option with continuous sampling, the corresponding one-dimensional problem is

$$\begin{cases} \dfrac{\partial W}{\partial t} + \mathbf{L}_\eta W = 0, & 0 \le \eta \le 1, \quad t \le T, \\ W(\eta, T) = \max(\alpha - \eta, 0), & 0 \le \eta \le 1, \\ \dfrac{\partial W}{\partial \eta}(1, t) = 0, & t \le T, \end{cases}$$

where

$$\eta = L/S, \quad W(\eta, t) = v(S, L, t)/S, \quad 0 < \alpha \le 1,$$
$$\mathbf{L}_\eta = \frac{1}{2}\sigma^2\eta^2\frac{\partial^2}{\partial\eta^2} + (D_0 - r)\eta\frac{\partial}{\partial\eta} - D_0,$$

and $v(S, L, t)$ is the price of the European lookback strike call option with continuous sampling.

(a) Let $V(S, L, t)$ denote the price of the American lookback strike call option with continuous sampling and define $W = V/S$. Derive the linear complementarity problem for W.

(b) Assume that we have proved $V(S, L, t^*) \ge V(S, L, t^{**})$ for any $t^* \le t^{**}$. Derive the free-boundary problem for W.

18. (a) The price of a European better-of option is the solution of the following problem:

$$\begin{cases} \dfrac{\partial V}{\partial t} + LV = 0, & 0 \le S_1,\, 0 \le S_2,\, 0 \le t \le T, \\ V(S_1, S_2, T) = \max(S_1, S_2), & 0 \le S_1,\, 0 \le S_2, \end{cases}$$

where

$$L = \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2}{\partial S_1^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2}{\partial S_1 \partial S_2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2}{\partial S_2^2} + (r - D_{01}) S_1 \frac{\partial}{\partial S_1} + (r - D_{02}) S_2 \frac{\partial}{\partial S_2} - r.$$

Let

$$\xi = \frac{S_1}{S_2}, \quad W = \frac{V}{S_2},$$

and

$$\tau = T - t.$$

Show that W is the solution of the problem:

$$\begin{cases} \dfrac{\partial W}{\partial \tau} = L_\xi W, & 0 \le \xi, 0 \le \tau, \\ W(\xi, 0) = \max(\xi, 1), & 0 \le \xi, \end{cases}$$

where

$$L_\xi = \frac{1}{2}\left[\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2\right]\xi^2 \frac{\partial^2}{\partial \xi^2} + (D_{02} - D_{01})\xi \frac{\partial}{\partial \xi} - D_{02}.$$

(b) For W, the corresponding American-style problem is

$$\begin{cases} \min\left(\dfrac{\partial W}{\partial \tau} - L_\xi W, \quad W(\xi,\tau) - \max(\xi, 1)\right) = 0, & 0 \le \xi, \quad 0 \le \tau, \\ W(\xi, 0) = \max(\xi, 1) & 0 \le \xi. \end{cases}$$

Based on this formulation and assuming that there exist two free boundaries, show that the corresponding free-boundary problem can be written as

$$\begin{cases} \dfrac{\partial W}{\partial \tau} = L_\xi W, & \xi_{f_1}(\tau) \le \xi \le \xi_{f_2}(\tau), \quad \tau \ge 0, \\ W(\xi, 0) = \max(\xi, 1), & \xi_{f_1}(0) \le \xi \le \xi_{f_2}(0), \\ W(\xi_{f_1}(\tau), \tau) = 1, & \tau \ge 0, \\ \dfrac{\partial W}{\partial \xi}(\xi_{f_1}(\tau), \tau) = 0, & \tau \ge 0, \\ W(\xi_{f_2}(\tau), \tau) = \xi_{f_2}(\tau), & \tau \ge 0, \\ \dfrac{\partial W}{\partial \xi}(\xi_{f_2}(\tau), \tau) = 1, & \tau \ge 0, \\ \xi_{f_1}(0) = 1, \\ \xi_{f_2}(0) = 1. \end{cases}$$

19. Suppose that the value V of an option depends on S, H, and t, i.e., $V = V(S,H,t)$. As we know, for such any European option, V satisfies the equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0) S \frac{\partial V}{\partial S} - rV = 0, \quad 0 \le S \le H,\ t \le T,$$

and the condition

$$\frac{\partial V}{\partial H}(S,S,t) = 0, \quad 0 \le S.$$

For such a perpetual American option with the constraint $V \ge H$, which is called the Russian option, $V = V(S,H)$ and V is the solution of the following LC problem

$$\begin{cases} \min\left(-\left[\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0) S \frac{\partial V}{\partial S} - rV\right],\ V(S,H) - H\right) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le S \le H, \\ \frac{\partial V}{\partial H}(S,S) = 0, \qquad\qquad\qquad\qquad\qquad\quad 0 \le S. \end{cases}$$

We can find a solution of this problem in the following way:

(a) Find the solution of the following free-boundary problem:

$$\begin{cases} \frac{1}{2}\sigma^2 \eta^2 \frac{d^2 W_\infty}{d\eta^2} + (D_0 - r)\eta \frac{dW_\infty}{d\eta} - D_0 W_\infty = 0, \quad 1 \le \eta \le \eta_f, \\ \frac{dW_\infty(1)}{d\eta} = 0, \\ W_\infty(\eta_f) = \eta_f, \\ \frac{dW_\infty(\eta_f)}{d\eta} = 1, \end{cases}$$

where η_f is a number representing the location of this free boundary.

(b) Define

$$W_\infty(\eta) = \begin{cases} \text{the solution of the free-boundary problem}, & 1 \le \eta \le \eta_f, \\ \eta, & \eta_f < \eta. \end{cases}$$

Show that $W_\infty(\eta)$ is a solution of the following LC problem

$$\begin{cases} \min\left(-\frac{\sigma^2\eta^2}{2}\frac{d^2 W_\infty}{d\eta^2} - (D_0 - r)\eta \frac{dW_\infty}{d\eta} + D_0 W_\infty,\ W_\infty - \eta\right) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1 \le \eta, \\ \frac{dW_\infty}{d\eta}(1) = 0. \end{cases}$$

(c) Show that the function $SW_\infty(H/S)$ is a solution of the LC problem given at the beginning.

20. Find the solution of the problem:

$$\begin{cases} \dfrac{1}{2}\sigma^2\xi^2\dfrac{d^2W_\infty}{d\xi^2} + (D_{02} - D_{01})\,\xi\dfrac{dW_\infty}{d\xi} - D_{02}W_\infty = 0, \quad \xi_{f_1} \le \xi \le \xi_{f_2}, \\ W_\infty\left(\xi_{f_1}\right) = 1, \\ \dfrac{dW_\infty}{d\xi}\left(\xi_{f_1}\right) = 0, \\ W_\infty\left(\xi_{f_2}\right) = \xi_{f_2}, \\ \dfrac{dW_\infty}{d\xi}\left(\xi_{f_2}\right) = 1, \end{cases}$$

where $\xi_{f_1} < \xi_{f_2}$. (This problem is related to the perpetual American better-of option.)

21. Consider the following problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n}\rho_{ij}\sigma_i\sigma_j S_i S_j\dfrac{\partial^2 V}{\partial S_i\partial S_j} + \sum\limits_{i=1}^{n}(r - D_{0i})S_i\dfrac{\partial V}{\partial S_i} - rV = 0, \\ \qquad\qquad\qquad 0 \le \mathbf{S}, \quad 0 \le t \le T, \\ V(\mathbf{S}, T) = V_T(S_1, S_2, \cdots, S_n), \quad 0 \le \mathbf{S}. \end{cases}$$

(a) *Let $V(\mathbf{S}, t) = \mathrm{e}^{-r(T-t)}\overline{V}(\mathbf{S}, t)$. Show that $\overline{V}(\mathbf{S}, t)$ is the solution of the problem

$$\begin{cases} \dfrac{\partial \overline{V}}{\partial t} + \dfrac{1}{2}\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n}\rho_{ij}\sigma_i\sigma_j S_i S_j\dfrac{\partial^2 \overline{V}}{\partial S_i\partial S_j} + \sum\limits_{i=1}^{n}(r - D_{0i})S_i\dfrac{\partial \overline{V}}{\partial S_i} = 0, \\ \qquad\qquad\qquad 0 \le \mathbf{S}, \quad 0 \le t \le T, \\ \overline{V}(\mathbf{S}, T) = V_T(S_1, S_2, \cdots, S_n), \qquad 0 \le \mathbf{S}. \end{cases}$$

(b) Let

$$\begin{cases} y_i = a_i\left[\ln S_i + b_i(T - t)\right], \qquad i = 1, 2, \cdots, n, \\ \tau = T - t, \end{cases}$$

and $\overline{V}_1(\mathbf{y}, \tau) = \overline{V}(\mathbf{S}, t)$, $\mathbf{y}$ standing for $(y_1, y_2, \cdots, y_n)^T$. Find a_i and b_i such that $\overline{V}_1(\mathbf{y}, \tau)$ satisfies

$$\begin{cases} \dfrac{\partial \overline{V}_1(\mathbf{y}, \tau)}{\partial \tau} = \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n}\rho_{ij}\dfrac{\partial^2 \overline{V}_1(\mathbf{y}, \tau)}{\partial y_i\partial y_j}, \quad -\infty < \mathbf{y} < \infty, \ 0 \le \tau \le T, \\ \overline{V}(\mathbf{y}, 0) = V_{1T}(\mathbf{y}), \qquad -\infty < \mathbf{y} < \infty, \end{cases}$$

where

$$V_{1T}(\mathbf{y}) \equiv V_T(\mathrm{e}^{\sigma_1 y_1/\sqrt{2}}, \mathrm{e}^{\sigma_2 y_2/\sqrt{2}}, \cdots, \mathrm{e}^{\sigma_n y_n/\sqrt{2}}).$$

(c) Let r_{ij} denote the element on the ith row and the jth column of a matrix $\mathbf{R}$, where $\mathbf{R}$ represents any letter. $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ are $M \times M$ matrices. Define $\mathbf{D} = \mathbf{AB}$ and $\mathbf{E} = \mathbf{ABC}$. According to the definition of multiplication of two matrices, we have

$$d_{ij} = \sum_{k=1}^{M} a_{ik} b_{kj}.$$

Show

$$e_{ij} = \sum_{l=1}^{M} \sum_{k=1}^{M} a_{ik} b_{kl} c_{lj}.$$

(d) Let

$$\mathbf{x} = \mathbf{R}\mathbf{y}$$

and

$$\overline{V}_2(\mathbf{x}, \tau) = \overline{V}_1(\mathbf{y}, \tau),$$

where $\mathbf{R}$ is a constant matrix:

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{bmatrix}.$$

Find the equation and initial condition for $\overline{V}_2(\mathbf{x}, \tau)$.

(e) Find $\mathbf{R}$ such that $\overline{V}_2(\mathbf{x}, \tau)$ satisfies

$$\begin{cases} \dfrac{\partial \overline{V}_2(\mathbf{x}, \tau)}{\partial \tau} = \displaystyle\sum_{l=1}^{n} \dfrac{\partial^2 \overline{V}_2(\mathbf{x}, \tau)}{\partial x_l^2}, & -\infty < \mathbf{x} < \infty, \quad 0 \le \tau \le T, \\ \overline{V}_2(\mathbf{x}, 0) = V_{2T}(\mathbf{x}), & -\infty < \mathbf{x} < \infty, \end{cases}$$

where $V_{2T}(\mathbf{x}) \equiv V_{1T}(\mathbf{R}^{-1}\mathbf{x})$.

22. (a) Show that

$$\phi(\mathbf{x}_0; \mathbf{x}, \tau) = \frac{1}{(4\pi\tau)^{n/2}} \mathrm{e}^{-\sum_{i=1}^{n} (x_i - x_{i0})^2/(4\tau)}$$

is a solution to

$$\frac{\partial \phi}{\partial \tau} = \sum_{i=1}^{n} \frac{\partial^2 \phi}{\partial x_i^2}, \qquad -\infty < \mathbf{x} < \infty, \quad 0 \le \tau,$$

where $\mathbf{x}$ and $\mathbf{x}_0$ are two n-dimensional vectors with components x_i and x_{i0}, $i = 1, 2, \cdots, n$, respectively and $-\infty < \mathbf{x} < \infty$ means

$$-\infty < x_i < \infty, \qquad i = 1, 2, \cdots, n.$$

(b) Show that the function $\phi(\mathbf{x}_0;\mathbf{x},\tau)$ satisfies the conditions

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\phi(\mathbf{x}_0;\mathbf{x},\tau)dx_{10}dx_{20}\cdots dx_{n0}=1$$

and

$$\lim_{\tau\to 0}\phi(\mathbf{x}_0;\mathbf{x},\tau)=\begin{cases}\infty, \ \text{at } \mathbf{x}=\mathbf{x}_0,\\ \\ 0, \ \text{otherwise,}\end{cases}$$

that is,

$$\lim_{\tau\to 0}\phi(\mathbf{x}_0;\mathbf{x},\tau)=\delta(\mathbf{x}-\mathbf{x}_0).$$

(c) Show that

$$V(\mathbf{x},\tau)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}V_0(\mathbf{x}_0)\phi(\mathbf{x}_0;\mathbf{x},\tau)dx_{10}dx_{20}\cdots dx_{n0}$$

is the solution of the problem

$$\begin{cases}\dfrac{\partial V}{\partial \tau}=\sum\limits_{i=1}^{n}\dfrac{\partial^2 V}{\partial x_i^2}, \quad -\infty<\mathbf{x}<\infty, \quad 0\le\tau,\\ V(\mathbf{x},0)=V_0(\mathbf{x}), \ -\infty<\mathbf{x}<\infty.\end{cases}$$

23. Let $\mathbf{P}$ be a positive definite matrix. As we know, in this case there exist a matrix $\mathbf{Q}$ and a diagonal matrix $\mathbf{\Lambda}$ such that $\mathbf{P}=\mathbf{Q\Lambda Q}^T$, where all the components of $\mathbf{\Lambda}$ are positive and $\mathbf{Q}$ satisfies the conditions $\mathbf{Q}^T\mathbf{Q}=\mathbf{I}$ and $\det\mathbf{Q}=1$. Let $\mathbf{y}$ and $\mathbf{y}_0$ be two vectors and define $\mathbf{R}=\mathbf{\Lambda}^{-1/2}\mathbf{Q}^T$, $\mathbf{x}=\mathbf{Ry}, \ \mathbf{x}_0=\mathbf{Ry}_0$, and $\eta=\dfrac{\mathbf{y}_0-\mathbf{y}}{\sqrt{2\tau}}$. Show
(a) $\det\mathbf{R}=\dfrac{1}{\sqrt{\det\mathbf{P}}}$.
(b) $\dfrac{(\mathbf{x}_0-\mathbf{x})^T(\mathbf{x}_0-\mathbf{x})}{4\tau}=\dfrac{\eta^T\mathbf{P}^{-1}\eta}{2}$.

24. *Reduce the problem of the European exchange option

$$\begin{cases}\dfrac{\partial V}{\partial t}+\dfrac{1}{2}\sigma_1^2S_1^2\dfrac{\partial^2 V}{\partial S_1^2}+\rho_{12}\sigma_1\sigma_2S_1S_2\dfrac{\partial^2 V}{\partial S_1\partial S_2}+\dfrac{1}{2}\sigma_2^2S_2^2\dfrac{\partial^2 V}{\partial S_2^2}\\ +(r-D_{01})S_1\dfrac{\partial V}{\partial S_1}+(r-D_{02})S_2\dfrac{\partial V}{\partial S_2}-rV=0,\\ \qquad\qquad\qquad\qquad\qquad\qquad 0\le S_1, \ 0\le S_2, \ 0\le t\le T,\\ V(S_1,S_2,T)=\max(S_1-S_2,0), \qquad 0\le S_1, \ 0\le S_2\end{cases}$$

into a one-dimensional problem and find its closed-form solution.

25. (a) Suppose that $V(S_1, S_2, t)$ satisfies

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma_1^2 S_1^2 \dfrac{\partial^2 V}{\partial S_1^2} + \rho_{12}\sigma_1\sigma_2 S_1 S_2 \dfrac{\partial^2 V}{\partial S_1 \partial S_2} + \dfrac{1}{2}\sigma_2^2 S_2^2 \dfrac{\partial^2 V}{\partial S_2^2} \\ +(r - D_{01})S_1 \dfrac{\partial V}{\partial S_1} + (r - D_{02})S_2 \dfrac{\partial V}{\partial S_2} - rV = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 \le S_1,\ 0 \le S_2,\ 0 \le t \le T, \\ V(S_1, S_2, T) = \max(S_0, S_1, S_2), \qquad 0 \le S_1,\ 0 \le S_2. \end{cases}$$

Define $S_0^* = S_0 e^{-r(T-t)}$, $S_i^* = S_i e^{-D_{0i}(T-t)}$, $i = 1, 2$, $\xi_{02} = S_0^*/S_2^* = S_0 e^{-(r-D_{02})(T-t)}/S_2$, $\xi_{12} = S_1^*/S_2^* = S_1 e^{-(D_{01}-D_{02})(T-t)}/S_2$, and $V_2(\xi_{02}, \xi_{12}, t) = V(S_1, S_2, t)/S_2^* = V(S_1, S_2, t)/(S_2 e^{-D_{02}(T-t)})$. Derive the final-value problem for $V_2(\xi_{02}, \xi_{12}, t)$.

(b) As we know, if let $\xi_{10} = S_1^*/S_0^*$ and $\xi_{20} = S_2^*/S_0^*$ as independent variables and $V_0(\xi_{10}, \xi_{20}, t) = V(S_1, S_2, t)/S_0^*$, then $V_0(\xi_{10}, \xi_{20}, t)$ is the solution of the following problem:

$$\begin{cases} \dfrac{\partial V_0}{\partial t} + \dfrac{1}{2}\sigma_{10}^2 \xi_{10}^2 \dfrac{\partial^2 V_0}{\partial \xi_{10}^2} + \rho_{120}\sigma_{10}\sigma_{20}\xi_{10}\xi_{20} \dfrac{\partial^2 V_0}{\partial \xi_{10}\partial \xi_{20}} \\ \qquad + \dfrac{1}{2}\sigma_{20}^2 \xi_{20}^2 \dfrac{\partial^2 V_0}{\partial \xi_{20}^2} = 0, \qquad 0 \le \xi_{10},\ \ 0 \le \xi_{20},\ \ 0 \le t \le T, \\ V_0(\xi_{10}, \xi_{20}, T) = \max(1, \xi_{10}, \xi_{20}), \qquad 0 \le \xi_{10},\ \ 0 \le \xi_{20}. \end{cases}$$

Suppose that we know

$$\begin{aligned} V(S_1, S_2, t) = S_0^* & \iint_{\max(\xi'_{10}, \xi'_{20}) \le 1} \psi d\xi'_{10} d\xi'_{20} \\ & + S_0^* \iint_{\max(1, \xi'_{20}) \le \xi'_{10}} \xi'_{10}\psi d\xi'_{10} d\xi'_{20} \\ & + S_0^* \iint_{\max(1, \xi'_{10}) \le \xi'_{20}} \xi'_{20}\psi d\xi'_{10} d\xi'_{20}, \end{aligned}$$

where $\psi = \psi(\xi'_{10}, \xi'_{20}; \xi_{10}, \xi_{20}, t, \sigma_{10}, \sigma_{20}, \rho_{120})$ and ξ'_{ij} stands for ξ_{ij} at time T, and the first term in the expression is equal to

$$S_0^* N_2 \left(\frac{\ln \dfrac{S_0^*}{S_1^*} + \dfrac{\sigma_{10}^2}{2}\tau}{\sigma_{10}\sqrt{\tau}}, \frac{\ln \dfrac{S_0^*}{S_2^*} + \dfrac{\sigma_{20}^2}{2}\tau}{\sigma_{20}\sqrt{\tau}}; \rho_{120} \right),$$

where $N_2(x_1, x_2; \rho)$ is a function of x_1, x_2, and ρ. Using the result in part (a), show that the third term should be equal to

$$S_2^* N_2\left(\frac{\ln\dfrac{S_2^*}{S_0^*}+\dfrac{\sigma_{02}^2}{2}\tau}{\sigma_{02}\sqrt{\tau}},\frac{\ln\dfrac{S_2^*}{S_1^*}+\dfrac{\sigma_{12}^2}{2}\tau}{\sigma_{12}\sqrt{\tau}};\rho_{012}\right).$$

26. Show

$$\mathrm{e}^{-r\tau}\int_{S_0}^{\infty}\int_0^{S_1'} S_1'\psi(S_1',S_2';S_1,S_2,t)dS_2'dS_1'$$

$$= S_1^* N_2\left(\frac{\ln\dfrac{S_1^*}{S_2^*}+\dfrac{\sigma_{12}^2}{2}\tau}{\sigma_{12}\sqrt{\tau}},\frac{\ln\dfrac{S_1^*}{S_0^*}+\dfrac{\sigma_1^2}{2}\tau}{\sigma_1\sqrt{\tau}};\frac{\sigma_1-\rho_{12}\sigma_2}{\sigma_{12}}\right)$$

by direct calculation, i.e., without using solutions of PDEs. Here

$$\begin{aligned}
&\tau = T-t,\\
&\psi(S_1',S_2';S_1,S_2,t)=\frac{1}{2\pi\tau\sqrt{\det\mathbf{P}}\prod_{i=1}^{2}(\sigma_i S_i')}\mathrm{e}^{-\eta^T\mathbf{P}^{-1}\eta/2},\\
&S_0^*=S_0\mathrm{e}^{-r\tau},\quad S_1^*=S_1\mathrm{e}^{-D_{01}\tau},\quad S_2^*=S_2\mathrm{e}^{-D_{02}\tau},\\
&\sigma_{12}=\sqrt{\sigma_1^2-2\rho_{12}\sigma_1\sigma_2+\sigma_2^2},\\
&N_2(x_1,x_2;\rho)=\frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{x_2}\int_{-\infty}^{x_1}\mathrm{e}^{-\frac{1}{2}(\eta_1^2-2\rho\eta_1\eta_2+\eta_2^2)/(1-\rho^2)}d\eta_1\,d\eta_2,
\end{aligned}$$

where

$$\mathbf{P}=\begin{bmatrix}1 & \rho_{12}\\ \rho_{12} & 1\end{bmatrix},$$

the ith component of η in $\psi(S_1',S_2';S_1,S_2,t)$ is given by

$$\eta_i\left(S_i'\right)=\frac{\ln S_i'-\left[\ln S_i+(r-D_{0i}-\sigma_i^2/2)\tau\right]}{\sigma_i\sqrt{\tau}},\quad i=1,2,$$

and $r, D_{01}, D_{02}, \sigma_1, \sigma_2, \rho_{12}, T, S_0, S_1, S_2, t$ are parameters.

27. Suppose S_1 and S_2 are the prices of two assets A and B, respectively. The random variables S_1 and S_2 satisfy

$$dS_1=\mu_1 S_1 dt+\sigma_1 S_1 dX_1,$$

$$dS_2=\mu_2 S_2 dt+\sigma_2 S_2 dX_2,$$

where μ_1,μ_2,σ_1, and σ_2 are constants, and dX_1 and dX_2 are two Wiener processes with

$$\mathrm{E}\left[dX_1 dX_2\right] = \rho_{12} dt.$$

Also, suppose that the two assets pay dividends continuously and that the dividend yields of the assets A and B are D_{01} and D_{02}, respectively. Consider a European option on the minimum of S_1, S_2, and S_0, i.e., its payoff function is

$$\min(S_0, S_1, S_2),$$

where S_0 is a constant. Let $V_{\min}(S_1, S_2, t)$ be the price of the option. Show that

$$\begin{aligned}
&V_{\min}(S_1, S_2, t)\\
&= S_0^* N_2\left(\frac{\ln\dfrac{S_1^*}{S_0^*} - \dfrac{\sigma_1^2}{2}\tau}{\sigma_1\sqrt{\tau}}, \frac{\ln\dfrac{S_2^*}{S_0^*} - \dfrac{\sigma_2^2}{2}\tau}{\sigma_2\sqrt{\tau}}; \rho_{12}\right)\\
&\quad + S_1^* N_2\left(\frac{\ln\dfrac{S_2^*}{S_1^*} - \dfrac{\sigma_{12}^2}{2}\tau}{\sigma_{12}\sqrt{\tau}}, \frac{\ln\dfrac{S_0^*}{S_1^*} - \dfrac{\sigma_1^2}{2}\tau}{\sigma_1\sqrt{\tau}}; \frac{\sigma_1 - \rho_{12}\sigma_2}{\sigma_{12}}\right)\\
&\quad + S_2^* N_2\left(\frac{\ln\dfrac{S_0^*}{S_2^*} - \dfrac{\sigma_2^2}{2}\tau}{\sigma_2\sqrt{\tau}}, \frac{\ln\dfrac{S_1^*}{S_2^*} - \dfrac{\sigma_{12}^2}{2}\tau}{\sigma_{12}\sqrt{\tau}}; \frac{\sigma_2 - \rho_{12}\sigma_1}{\sigma_{12}}\right),
\end{aligned}$$

where $S_0^* = S_0\mathrm{e}^{-r\tau}$, $S_1^* = S_1\mathrm{e}^{-D_{01}\tau}$ and $S_2^* = S_2\mathrm{e}^{-D_{02}\tau}$, τ denoting $T - t$.

28. As we know, the cumulative distribution function for the bivariate standard normal distribution is

$$N_2(x_1, x_2; \rho) = \int_{-\infty}^{x_2}\int_{-\infty}^{x_1} p(\eta_1, \eta_2; \rho)d\eta_1 d\eta_2,$$

where $p(\eta_1, \eta_2; \rho)$ is the probability density function for the bivariate standard normal distribution, which has the following expression:

$$p(\eta_1, \eta_2; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}}\mathrm{e}^{-\frac{1}{2}(\eta_1^2 - 2\rho\eta_1\eta_2 + \eta_2^2)/(1-\rho^2)}.$$

Here $\rho \in [-1, 1]$ is a parameter. Show
(a)

$$N_2(x_1, \infty; \rho) = N(x_1)$$

and

$$N_2(\infty, x_2; \rho) = N(x_2).$$

Here $N(z)$ is the cumulative distribution function for the standard normal distribution, i.e.,

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \mathrm{e}^{-\xi^2/2} d\xi.$$

(b) For $N_2(x_1, x_2; \rho)$ there is another expression:

$$N_2(x_1, x_2; \rho) = \int_0^{\rho} p(x_1, x_2; \rho) d\rho + N(x_1)N(x_2),$$

from which we can have

$$N_2(0, 0; \rho) = \frac{\sin^{-1} \rho}{2\pi} + \frac{1}{4}.$$

(Hint: Show $\dfrac{\partial^2 p(\eta_1, \eta_2; \rho)}{\partial \eta_1 \partial \eta_2} = \dfrac{\partial p(\eta_1, \eta_2; \rho)}{\partial \rho}$ first.)

(c) Express the values of $N_2(x_1, x_2; 1)$ and $N_2(x_1, x_2; -1)$ in terms of $N(z)$.

29. The payoffs of two-asset European call and put options with the identical exercise price E for the two assets are

$$c(S_1, S_2, T) = \max(S_1 - E, S_2 - E, 0)$$

and

$$p(S_1, S_2, T) = \max(E - S_1, E - S_2, 0).$$

(a) Drive the closed-form solutions of these two options. (The closed-form solutions of the options with payoffs $\max(S_1, S_2, E)$ and $\min(S_1, S_2, E)$ can be used as given results.)

(b) Show that if $S_1 = S_2$, $D_{01} = D_{02}$, and $\sigma_1 = \sigma_2$, the limits of the two option prices when ρ_{12} goes to 1 are the functions given by the Black–Scholes formulae.

(c) Derive the limits of the two option prices when ρ_{12} goes to 1 and $\sigma_1 = \sigma_2$.

30. $\mathbf{S}$, $\mathbf{S}'$, ξ, and η are n-dimensional vectors. The ith components of $\mathbf{S}$ and $\mathbf{S}'$ are S_i and S_i' respectively. $\tau, S_0, r, D_{0i}, \sigma_i, i = 1, 2, \cdots, n$, are numbers. We further define

$$S_0^* = S_0 \mathrm{e}^{-r\tau}, \quad S_i^* = S_i \mathrm{e}^{-D_{0i}\tau}, \quad i = 1, 2, \cdots, n.$$

The ith components of ξ and η are given by

$$\xi_i\left(S_i'\right) = \frac{\ln S_i' - \ln S_0}{\sigma_i \sqrt{\tau}},$$

$$\eta_i\left(S_i'\right) = \frac{\ln S_i' - \left[\ln S_i + (r - D_{0i} - \sigma_i^2/2)\tau\right]}{\sigma_i \sqrt{\tau}}$$

$$= \frac{\ln S_i' - \ln S_0 - [\ln S_i^* - \ln S_0^*] + \sigma_i^2 \tau/2}{\sigma_i \sqrt{\tau}}$$

$$= \xi_i + \frac{\ln S_0^* - \ln S_i^*}{\sigma_i \sqrt{\tau}} + \sigma_i \sqrt{\tau}/2.$$

Define

$$\mathbf{P} = \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & 1 & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \cdots & 1 \end{bmatrix} \quad \text{with} \quad \rho_{ij} = \rho_{ji},$$

$$\psi(\mathbf{S}'; \mathbf{S}, t) = \frac{1}{(2\pi\tau)^{n/2} \sqrt{\det \mathbf{P}} \prod_{i=1}^{n} (\sigma_i S_i')} \mathrm{e}^{-\frac{1}{2}\eta^T \mathbf{P}^{-1} \eta},$$

and

$$\sigma_{ij} = \sqrt{\sigma_i^2 - 2\rho_{ij}\sigma_i\sigma_j + \sigma_j^2}, \quad \text{for any } i \text{ and } j.$$

(a) Show

$$\mathrm{e}^{-r\tau} S_i' \mathrm{e}^{-\frac{1}{2}\eta^T \mathbf{P}^{-1}\eta} = S_i^* \mathrm{e}^{-\frac{1}{2}\left(\eta - \sigma_i\sqrt{\tau}\mathbf{P}\mathbf{e}_i\right)^T \mathbf{P}^{-1}\left(\eta - \sigma_i\sqrt{\tau}\mathbf{P}\mathbf{e}_i\right)},$$

where $\mathbf{e}_i$ is the vector, whose ith component is one and whose other components are zero.

(b) $\mathbf{R}$ is a given matrix. Define

$$\mathbf{z} = \mathbf{R}\xi, \quad \mathbf{b} = \mathbf{R}\mathbf{a}, \quad \zeta = \mathbf{z} + \mathbf{b}, \quad \text{and} \quad \mathbf{Q} = \mathbf{R}\mathbf{P}\mathbf{R}^T.$$

Show

$$(\xi + \mathbf{a})^T \mathbf{P}^{-1} (\xi + \mathbf{a}) = (\mathbf{z} + \mathbf{b})^T \mathbf{Q}^{-1} (\mathbf{z} + \mathbf{b}) = \zeta^T \mathbf{Q}^{-1} \zeta.$$

(c) Show

$$\frac{1}{\tau^{n/2} \prod_{i=1}^{n} (\sigma_i S_i') \sqrt{\det \mathbf{P}}} dS_1' dS_2' \cdots dS_n'$$

$$= \frac{1}{\sqrt{\det \mathbf{Q}}} dz_1 dz_2 \cdots dz_n.$$

(d) Suppose that the domain Ω in the $(S_1', S_2', \cdots, S_n')$-space is equivalent to the domain $\Omega^* : -\infty < z_i \le 0, \quad i = 1, 2, \cdots, n,$ in the $(z_1, z_2, \cdots, z_n)$-space. Show

$$\mathrm{e}^{-r\tau} \iint \cdots \int_{\Omega} S_i' \psi(\mathbf{S}'; \mathbf{S}, t) dS_1' dS_2' \cdots dS_n' = S_i^* N_n(\mathbf{b}; \mathbf{Q}),$$

where

$$N_n(\mathbf{b};\mathbf{Q}) = \frac{1}{(2\pi)^{n/2}\sqrt{\det \mathbf{Q}}} \cdot \int_{-\infty}^{b_n}\int_{-\infty}^{b_{n-}} \cdots \int_{-\infty}^{b_1} \mathrm{e}^{-\frac{1}{2}\zeta^T \mathbf{Q}^{-1}\zeta} d\zeta_1 d\zeta_2 \cdots d\zeta_n,$$

b_i being the ith component of $\mathbf{b}$

and

$$\mathbf{b} = \mathbf{R}\left(\eta - \xi - \sigma_i\sqrt{\tau}\mathbf{P}\mathbf{e}_i\right).$$

(e) In the space $(S_1', S_2') \in [0,\infty) \times [0,\infty)$, the domain Ω is defined by

$$\max(S_0, S_2') \le S_1'.$$

Show

$$\mathrm{e}^{-r\tau}\iint_\Omega S_1'\psi(S_1', S_2'; S_1, S_2, t)dS_2'dS_1' = S_1^* N_2\left(\frac{\ln\frac{S_1^*}{S_2^*} + \frac{\sigma_{12}^2}{2}\tau}{\sigma_{12}\sqrt{\tau}}, \frac{\ln\frac{S_1^*}{S_0^*} + \frac{\sigma_1^2}{2}\tau}{\sigma_1\sqrt{\tau}}; \frac{\sigma_1 - \rho_{12}\sigma_2}{\sigma_{12}}\right),$$

where

$$N_2(x_1, x_2; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{x_2}\int_{-\infty}^{x_1} \mathrm{e}^{-\frac{1}{2}(\eta_1^2 - 2\rho\eta_1\eta_2 + \eta_2^2)/(1-\rho^2)} d\eta_1 d\eta_2.$$

(f) In the space $(S_1', S_2', S_3') \in [0,\infty) \times [0,\infty) \times [0,\infty)$, the domain Ω is defined by

$$\max(S_0, S_1', S_2') \le S_3'.$$

Show

$$\mathrm{e}^{-r\tau}\iiint_\Omega S_3'\psi(S_1', S_2', S_3'; S_1, S_2, S_3, t)dS_1'dS_2'dS_3'$$
$$= S_3^* N_3\left(\frac{\ln\frac{S_3^*}{S_0^*} + \frac{\sigma_3^2}{2}\tau}{\sigma_3\sqrt{\tau}}, \frac{\ln\frac{S_3^*}{S_1^*} + \frac{\sigma_{13}^2}{2}\tau}{\sigma_{13}\sqrt{\tau}}, \frac{\ln\frac{S_3^*}{S_2^*} + \frac{\sigma_{23}^2}{2}\tau}{\sigma_{23}\sqrt{\tau}}; \rho_{013}, \rho_{023}, \rho_{123}\right),$$

where

$$N_3(x_1, x_2, x_3; \rho_{12}, \rho_{13}, \rho_{23})$$
$$= \frac{1}{(2\pi)^{3/2}\sqrt{\det \mathbf{P}}} \int_{-\infty}^{x_3}\int_{-\infty}^{x_2}\int_{-\infty}^{x_1} \mathrm{e}^{-\frac{1}{2}\eta^T\mathbf{P}^{-1}\eta} d\eta_1 d\eta_2 d\eta_3,$$
$$\mathbf{P} \text{ being } \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{bmatrix} \text{ and } \eta \text{ being } \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}.$$

(g) In the space $(S_1', S_2', \cdots, S_n') \in [0, \infty) \times [0, \infty) \cdots \times [0, \infty)$, the domain Ω is defined by

$$\max(S_0, S_1', \cdots, S_{i-1}', S_{i+1}', \cdots, S_n') \le S_i'.$$

Show that in this case if let

$$\mathbf{R} = \begin{bmatrix} 0 & \cdots & 0 & \dfrac{-\sigma_i}{\sigma_{i+1,i}} & \dfrac{\sigma_{i+1}}{\sigma_{i+1,i}} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \dfrac{-\sigma_i}{\sigma_{ni}} & 0 & \cdots & \dfrac{\sigma_n}{\sigma_{ni}} \\ 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ \dfrac{\sigma_1}{\sigma_{1i}} & \cdots & 0 & \dfrac{-\sigma_i}{\sigma_{1i}} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \dfrac{\sigma_{i-1}}{\sigma_{i-1,i}} & \dfrac{-\sigma_i}{\sigma_{i-1,i}} & 0 & \cdots & 0 \end{bmatrix},$$

then the domain Ω is equivalent to the domain Ω^* in part (d) and for $\mathbf{b}$ we have

$$\mathbf{b} = \begin{bmatrix} \dfrac{\ln(S_i^*/S_{i+1}^*)}{\sigma_{i+1,i}\sqrt{\tau}} + \dfrac{\sigma_{i+1,i}\sqrt{\tau}}{2} \\ \cdots \\ \dfrac{\ln(S_i^*/S_n^*)}{\sigma_{ni}\sqrt{\tau}} + \dfrac{\sigma_{ni}\sqrt{\tau}}{2} \\ \dfrac{\ln(S_i^*/S_0^*)}{\sigma_i\sqrt{\tau}} + \dfrac{\sigma_i\sqrt{\tau}}{2} \\ \dfrac{\ln(S_i^*/S_1^*)}{\sigma_{1i}\sqrt{\tau}} + \dfrac{\sigma_{1i}\sqrt{\tau}}{2} \\ \cdots \\ \dfrac{\ln(S_i^*/S_{i-1}^*)}{\sigma_{i-1,i}\sqrt{\tau}} + \dfrac{\sigma_{i-1,i}\sqrt{\tau}}{2} \end{bmatrix}.$$

Here the diagonal dots in the matrix $\mathbf{R}$ from $\dfrac{\sigma_{i+1}}{\sigma_{i+1,i}}$ to $\dfrac{\sigma_n}{\sigma_{ni}}$ and from $\dfrac{\sigma_1}{\sigma_{1i}}$ to $\dfrac{\sigma_{i-1}}{\sigma_{i-1,i}}$ represent $\dfrac{\sigma_{i+2}}{\sigma_{i+2,i}}, \cdots, \dfrac{\sigma_{n-1}}{\sigma_{n-1,i}}$ and $\dfrac{\sigma_2}{\sigma_{2i}}, \cdots, \dfrac{\sigma_{i-2}}{\sigma_{i-2,i}}$, respectively, and the other dots represent 0.

31. Suppose that S_1, S_2, and S_3 are the prices of three assets satisfying

$$dS_i = \mu_i S_i dt + \sigma_i S_i dX_i, \qquad i = 1,2,3,$$

where $\mu_i, \sigma_i,\ i = 1,2,3$ are constants and $dX_i,\ i = 1,2,3$ are the Wiener processes with

$$\mathrm{E}\left[dX_i dX_j\right] = \rho_{ij} dt, \qquad i,j = 1,2,3.$$

Also, suppose that the three assets pay dividends continuously and that the dividend yields of the three assets are $D_{0i},\ i = 1,2,3$.

(a) Consider a European option on the maximum of S_1, S_2, S_3, and S_0, i.e., its payoff is

$$\max(S_0, S_1, S_2, S_3),$$

where S_0 is a certain amount of cash. If we understand S_0 as a random variable, then its volatility, σ_0, is equal to 0. Let $V_{\max}(S_1, S_2, S_3, t)$ be the price of such a T-year option. Show that

$$\begin{aligned} V_{\max}(S_1, S_2, S_3, t) = {} & S_0^* N_3\left(A_{10}, A_{20}, A_{30}; \rho_{120}, \rho_{130}, \rho_{230}\right) \\ & + S_1^* N_3\left(A_{21}, A_{31}, A_{01}; \rho_{231}, \rho_{201}, \rho_{301}\right) \\ & + S_2^* N_3\left(A_{32}, A_{02}, A_{12}; \rho_{302}, \rho_{312}, \rho_{012}\right) \\ & + S_3^* N_3\left(A_{03}, A_{13}, A_{23}; \rho_{013}, \rho_{023}, \rho_{123}\right), \end{aligned}$$

where

$$\begin{aligned} S_0^* &= S_0 \mathrm{e}^{-r(T-t)}, \\ S_i^* &= S_i \mathrm{e}^{-D_{0i}(T-t)}, \qquad i = 1,2,3, \\ A_{ij} &= \frac{\ln \dfrac{S_j^*}{S_i^*} + \dfrac{\sigma_{ij}^2 (T-t)}{2}}{\sigma_{ij}\sqrt{T-t}}, \qquad i,j = 0,1,2,3 \text{ but } i \neq j, \\ & \quad \sigma_{ij} \text{ being } \sqrt{\sigma_i^2 - 2\rho_{ij}\sigma_i\sigma_j + \sigma_j^2}, \\ \rho_{ijk} &= \frac{\sigma_k^2 - \rho_{ik}\sigma_i\sigma_k - \rho_{jk}\sigma_j\sigma_k + \rho_{ij}\sigma_i\sigma_j}{\sigma_{ik}\sigma_{jk}}, \\ & \quad i,j,k = 0,1,2,3 \text{ but } i,j,k \text{ being distinct}, \end{aligned}$$

and $N_3(x_1, x_2, x_3; \rho_{12}, \rho_{13}, \rho_{23})$ is the trivariate cumulative distribution function:

$$\begin{aligned} & N_3(x_1, x_2, x_3; \rho_{12}, \rho_{13}, \rho_{23}) \\ = {} & \frac{1}{(2\pi)^{3/2}\sqrt{\det \mathbf{P}}} \int_{-\infty}^{x_3}\int_{-\infty}^{x_2}\int_{-\infty}^{x_1} \mathrm{e}^{-\frac{1}{2}\eta^T \mathbf{P}^{-1}\eta} d\eta_1 d\eta_2 d\eta_3, \\ & \mathbf{P} \text{ being } \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{bmatrix} \text{ and } \eta \text{ being } \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}. \end{aligned}$$

(b) Consider a European option on the minimum of S_1, S_2, S_3, and S_0, i.e., its payoff is

$$\min(S_0, S_1, S_2, S_3),$$

where S_0 is a certain amount of cash. Find the expression of the price for this option in terms of the trivariate cumulative distribution function.

32. Suppose that $S_1, S_2, \cdots, S_n$ are the prices of n assets and that each asset pays a dividend continuously, the dividend yield for S_i being D_{0i}, $i = 1, 2, \cdots, n$. Each price S_i satisfies the stochastic equation

$$dS_i = \mu_i S_i dt + \sigma_i S_i dX_i,$$

where μ_i, σ_i are constants and dX_i is a Wiener process, and

$$\mathrm{E}\left[dX_i dX_j\right] = \rho_{ij} dt, \qquad i, j = 1, 2, \cdots, n.$$

(a) Guess the expression of price of the European option on the maximum of $S_1, S_2, \cdots, S_n$ and S_0 according to the result given in part (a) of Problem 31, where S_0 is a certain amount of cash.

(b) Guess the expression of price of the European option on the minimum of $S_1, S_2, \cdots, S_n$ and S_0 according to the result given in part (b) of Problem 31.

33. Suppose that $c_{\max}(S_1, S_2, t)$, $c_{\min}(S_1, S_2, t)$, $c(S_1, t)$ and $c(S_2, t)$ are the prices of four call options with payoff functions

$$\max(\max(S_1, S_2) - E, 0), \quad \max(\min(S_1, S_2) - E, 0), \quad \max(S_1 - E, 0),$$

and

$$\max(S_2 - E, 0),$$

respectively. Show

$$c_{\max}(S_1, S_2, t) + c_{\min}(S_1, S_2, t) = c(S_1, t) + c(S_2, t).$$

(Hint: Show that the total payoff of the two options on the left-hand side is equal to the total payoff of the two options on the right-hand side.)

34. Let $p_{\max}(S_1, S_2, t)$ and $p_{\min}(S_1, S_2, t)$ be the prices of two European put options with payoff functions

$$\max(E - \max(S_1, S_2), 0) \quad \text{and} \quad \max(E - \min(S_1, S_2), 0),$$

respectively. Suppose that $c_{\max}(S_1, S_2, t)$ and $c_{\min}(S_1, S_2, t)$ are the prices of two European call options with payoff functions

$$\max(\max(S_1, S_2) - E, 0) \quad \text{and} \quad \max(\min(S_1, S_2) - E, 0),$$

respectively. $\bar{c}_{\max}(S_1, S_2, t)$ and $\bar{c}_{\min}(S_1, S_2, t)$ denote the prices of European options with payoff functions

$$\max(S_1, S_2) \quad \text{and} \quad \min(S_1, S_2).$$

Show
(a) $p_{\max}(S_1, S_2, t) = E\mathrm{e}^{-r(T-t)} - \bar{c}_{\max}(S_1, S_2, t) + c_{\max}(S_1, S_2, t)$.
(b) $p_{\min}(S_1, S_2, t) = E\mathrm{e}^{-r(T-t)} - \bar{c}_{\min}(S_1, S_2, t) + c_{\min}(S_1, S_2, t)$.
(Hint: Show that the payoff of the option on the left-hand side is equal to the total value of the three terms at $t = T$ on the right-hand side.)

35. Show that the closed-form solutions of cash-or-nothing puts, asset-or-nothing calls, and asset-or-nothing puts are

$$B\mathrm{e}^{-r(T-t)}N(-d_2), \quad S\mathrm{e}^{-D_0(T-t)}N(d_1), \quad \text{and} \quad S\mathrm{e}^{-D_0(T-t)}N(-d_1),$$

respectively. Here

$$d_1 = \frac{\ln\left(S\mathrm{e}^{(r-D_0)(T-t)}/E\right) + \sigma^2(T-t)/2}{\sigma\sqrt{T-t}},$$
$$d_2 = \frac{\ln\left(S\mathrm{e}^{(r-D_0)(T-t)}/E\right) - \sigma^2(T-t)/2}{\sigma\sqrt{T-t}}.$$

36. Show that the value of a forward start American put option with exercise price $E = \alpha S_{T_1}$ at time $t_0 < T_1$ is

$$\alpha S\mathrm{e}^{-D_0(T_1-t_0)}P^*\left(\frac{1}{\alpha}, T_1\right),$$

where $P^*\left(\frac{1}{\alpha}, T_1\right)$ is the value of a standard American put option.

37. Consider compound options and assume that both options are European. Let $c_1(S,t;c_2)$, $c_1(S,t;p_2)$, and $p_1(S,t;p_2)$ denote the prices of a call on a call, a call on a put, and a put on a put, respectively. Show that their closed-form solutions are

$$\begin{aligned}
c_1(S,t;c_2) &= S\mathrm{e}^{-D_0(T_2-t)}N_2(d_{11}, d_{12}; \rho) - E_2\mathrm{e}^{-r(T_2-t)}N_2(d_{21}, d_{22}; \rho) \\
&\quad -E_1\mathrm{e}^{-r(T_1-t)}N(d_{21}), \\
c_1(S,t;p_2) &= E_2\mathrm{e}^{-r(T_2-t)}N_2(-d_{23}, -d_{22}; \rho) \\
&\quad -S\mathrm{e}^{-D_0(T_2-t)}N_2(-d_{13}, -d_{12}; \rho) \\
&\quad -E_1\mathrm{e}^{-r(T_1-t)}N(-d_{23}), \\
p_1(S,t;p_2) &= E_1\mathrm{e}^{-r(T_1-t)}N(d_{23}) - E_2\mathrm{e}^{-r(T_2-t)}N_2(d_{23}, -d_{22}; -\rho) \\
&\quad +S\mathrm{e}^{-D_0(T_2-t)}N_2(d_{13}, -d_{12}; -\rho),
\end{aligned}$$

where

$$d_{11} = \frac{\ln(S/S^*) + \left(r - D_0 + \sigma^2/2\right)(T_1 - t)}{\sigma\sqrt{T_1 - t}},$$
$$d_{21} = \frac{\ln(S/S^*) + \left(r - D_0 - \sigma^2/2\right)(T_1 - t)}{\sigma\sqrt{T_1 - t}},$$
$$d_{12} = \frac{\ln(S/E_2) + \left(r - D_0 + \sigma^2/2\right)(T_2 - t)}{\sigma\sqrt{T_2 - t}},$$

$$d_{22} = \frac{\ln(S/E_2) + \left(r - D_0 - \sigma^2/2\right)(T_2 - t)}{\sigma\sqrt{T_2 - t}},$$

$$d_{13} = \frac{\ln(S/S^{**}) + \left(r - D_0 + \sigma^2/2\right)(T_1 - t)}{\sigma\sqrt{T_1 - t}},$$

$$d_{23} = \frac{\ln(S/S^{**}) + \left(r - D_0 - \sigma^2/2\right)(T_1 - t)}{\sigma\sqrt{T_1 - t}},$$

$$\rho = \sqrt{\frac{T_1 - t}{T_2 - t}}.$$

Here, S^* and S^{**} are the solutions of the following equations:

$$c_2\left(S^*, T_1\right) = E_1$$

and

$$p_2\left(S^{**}, T_1\right) = E_1.$$

38. How do we determine the price of a European put option on an American put option?
39. Show

$$\int_0^\infty G\left(S'', T_2; S', T_1\right) G\left(S', T_1; S, t\right) dS' = G\left(S'', T_2; S, t\right),$$

where

$$G\left(S', T_1; S, t\right) = \frac{1}{\sigma\sqrt{2\pi\left(T_1 - t\right)}S'} \mathrm{e}^{-\left[\ln S' - \ln S - \left(r - D_0 - \sigma^2/2\right)(T_1 - t)\right]^2/2\sigma^2(T_1 - t)}$$

and $G\left(S'', T_2; S', T_1\right)$ and $G\left(S'', T_2; S, t\right)$ are defined in the same way.
40. The payoff of a standard chooser option is

$$V\left(S, T_1\right) = \max\left(c\left(S, T_1\right),\ p\left(S, T_1\right)\right),$$

where $c\left(S, T_1\right)$ and $p\left(S, T_1\right)$ are the prices of European call and put options with the same exercise price E_2 and the same expiration date T_2. Find its closed-form solution. (Hint: Use the put–call parity relation and the result of Problem 39.)

5

Interest Rate Derivative Securities

5.1 Introduction

This chapter is devoted to interest rate derivatives. Interest rate derivatives are financial products derived from interest rates. There are various interest rates that will be mentioned in this chapter. Here we first give the meaning of each interest rate and derive some relations among them.

An N-year zero-coupon yield or an N-year spot interest rate is the interest rate on an investment starting at time t and lasting for N years. The investment is a "pure" N-year investment with no intermediate payments. Assume that the interest is compounded continuously. In this case, suppose that at time t the N-year zero-coupon yield is $Y(t, t+N)$, then the investor will get

$$\mathrm{e}^{Y(t,t+N)N}$$

at the end of year N for each dollar invested. A zero-coupon yield curve is a curve showing the relation between $Y(t, t+N)$ and N.

A zero-coupon bond with a face value or a par value of one dollar is a contract whose holder will get one dollar at the maturity of the contract from its issuer. Let $Z(t;T)$ denote the money a person needs to pay in order to have the contract with maturity date T at time t. Then, between $Y(t,T)$ and $Z(t;T)$, there is the following relation

$$Z(t;T) = \mathrm{e}^{-Y(t,T)(T-t)}, \tag{5.1}$$

or

$$Y(t,T) = \frac{-\ln Z(t;T)}{T-t}.$$

Suppose $t \le T_1 \le T_2$. An interest rate determined at time t for a period $[T_1, T_2]$ and paid at time T_2 is called a forward interest rate. Let us denote this rate by $f(t, T_1, T_2)$ and again assume that the interest is compounded continuously. Among $f(t, T_1, T_2)$, $Z(t; T_1)$, and $Z(t; T_2)$, there is the following relation:

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$$Z(t;T_1) = Z(t;T_2)\mathrm{e}^{f(t,T_1,T_2)(T_2-T_1)},$$

or

$$f(t,T_1,T_2) = \frac{1}{T_2-T_1}\ln\frac{Z(t;T_1)}{Z(t;T_2)}. \tag{5.2}$$

The reason is the following. If we borrow one dollar at time T_1, then we need to return $\mathrm{e}^{f(t,T_1,T_2)(T_2-T_1)}$ dollars at time T_2 according to the forward interest rate at time t. At time t, the values of one dollar at time T_1 and $\mathrm{e}^{f(t,T_1,T_2)(T_2-T_1)}$ dollars at time T_2 should be the same, otherwise there is an arbitrage opportunity.

An instantaneous forward interest rate $F(t,T_1)$ is the limit of $f(t,T_1,T_2)$ as $T_2 \to T_1$, written as

$$\begin{aligned} F(t,T_1) &= \lim_{T_2\to T_1} f(t,T_1,T_2) = \lim_{T_2\to T_1} \frac{-[\ln Z(t;T_2) - \ln Z(t;T_1)]}{T_2-T_1} \\ &= \frac{-1}{Z(t;T_1)}\frac{\partial Z(t;T_1)}{\partial T_1}. \end{aligned} \tag{5.3}$$

This gives

$$Z(t;T) = Z(t;t)\mathrm{e}^{-\int_t^T F(t,u)du} = \mathrm{e}^{-\int_t^T F(t,u)du}.$$

Furthermore, combining this expression for $Z(t,T)$ with the relation (5.1) yields

$$Y(t,T) = \frac{1}{T-t}\int_t^T F(t,u)du. \tag{5.4}$$

The limit of $Y(t,T)$ as $T \to t$ is called the instantaneous short rate (see [43]), the short-term interest rate, the short rate, or the spot rate (see [84]), denoted by $r(t)$, so

$$r(t) = \lim_{T\to t} Y(t,T) = Y(t,t).$$

Because from Eq. (5.4) we also have

$$\lim_{T\to t} Y(t,T) = \lim_{T\to t}\frac{1}{T-t}\int_t^T F(t,u)du = F(t,t),$$

we get

$$r(t) = Y(t,t) = F(t,t). \tag{5.5}$$

Clearly, if $Y(t,T)$ is equal to a constant r, then

$$Z(t;T) = \mathrm{e}^{-r(T-t)},$$

and

$$f(t,T_1,T_2) = F(t,T_1) = F(t,t) = Y(t,t) = r(t) = r.$$

In practice, the interest is often compounded discretely. If a loan of one dollar is required to pay at an interest rate $\bar{r}$ compounded m times per year, then the amount of each payment is

$$\frac{\bar{r}}{m}.$$

For an investment with an interest rate r compounded continuously, the interest payment for a period $\dfrac{1}{m}$ years is

$$\mathrm{e}^{r/m} - 1.$$

If

$$\mathrm{e}^{r/m} - 1 = \frac{\bar{r}}{m},$$

that is,

$$r = m \ln(1 + \bar{r}/m),$$

then the two investments are equivalent. Suppose that a forward interest rate at time t for the period $[T_1, T_1 + 1/m]$ is an interest rate compounded m times per year and we use $\bar{f}(t, T_1, T_1 + 1/m)$ to denote this forward interest rate. Let $f(t, T_1, T_1 + 1/m)$ be equivalent to the interest rate $\bar{f}(t, T_1, T_1 + 1/m)$. Then we have

$$f(t, T_1, T_1 + 1/m) = m \ln\left(1 + \frac{\bar{f}(t, T_1, T_1 + 1/m)}{m}\right)$$

and the relation (5.2) can be rewritten as

$$m \ln\left(1 + \frac{\bar{f}(t, T_1, T_1 + 1/m)}{m}\right) = m \ln\left(\frac{Z(t; T_1)}{Z(t; T_1 + 1/m)}\right)$$

or

$$\bar{f}(t, T_1, T_1 + 1/m) = m\left[\frac{Z(t; T_1)}{Z(t; T_1 + 1/m)} - 1\right]. \tag{5.6}$$

This is the counterpart of the relation (5.2) for an interest rate compounded m times per year. Actually, this relation can also be derived directly. For the formulae (5.1) and (5.3)–(5.5), we can also have their counterparts for interest rates compounded discretely.

As we know, the value of a bond is related to interest rates. There are many other financial contracts related to interest rates, which are signed between two parties, for example, a bank and a company. These are called interest rate derivatives. For an equity option, a typical life span is 9 months or less. In this case, the assumption of a short rate being a deterministic function of t, even a constant, is acceptable. If this is not the case, it may be necessary to consider a short rate as a random variable. For example, a life span of a bond may be 5 years, 10 years, even 30 years. Therefore, it is more realistic to deal with a short-term interest rate as a random variable. An interest rate cannot be traded on the market. In Chap. 2, we pointed out that there is a unknown function called the market price of risk for a short rate in the governing partial differential equation (PDE) for interest rate derivatives. Before using such an equation to price a derivative security, one has to find this function. From the mathematical point of view, to find a unknown function in the partial differential equation is to solve an inverse problem. This function in the PDE is determined by some data associated with solutions of the equation. The values of zero-coupon bonds with various maturity dates on the market or some other data can be taken as the data needed. Moreover, reducing the randomness of a zero-coupon bond curve to the randomness of the short rate is not a good approximation in many cases. Thus, describing the randomness of a zero-coupon bond curve by the randomness of several interest rates, namely, considering multi-factor models, is necessary.

Therefore, the rest of this chapter is organized as follows. In Sects. 5.2 and 5.3, the problem for a bond is formulated and for four special models, explicit solutions are derived. In Sect. 5.4, we discuss the inverse problem of determining the market price of risk and give a formulation of the inverse problem so that the determination of the unknown function can be reduced to solving such a problem. Then, we discuss bond options, swaps, swaptions, and so forth in Sect. 5.5. Section 5.6 is devoted to multi-factor interest rate models, especially, a three-factor model that can used in practice easily. Finally, two-factor convertible bonds are discussed in Sect. 5.7.

5.2 Bonds

A bond is a long-term contract under which the issuer promises to pay the holder a specified amount of money on a specified date. The specified amount is called the face value of the bond, which is denoted by Z in this chapter, and the specified date is named the maturity date T. Usually, the holder is also paid a specified amount at fixed times during the life of the contract. Such a specified amount is called a coupon. If there is no coupon payment, the bond is known as a zero-coupon bond. Clearly, the bondholder must pay a certain amount of money to the issuer when the bond is purchased. This amount is called the upfront premium. In this section, we will mainly derive the equations by which one can determine a fair value of the bond for any time t, including the upfront premium.

5.2.1 Bond Values for Deterministic Short Rates

Let r be the interest rate for the shortest possible deposit, which is called the short-term interest rate or, for short, the short rate in this book. For a short period, r may be assumed to be a constant. For a long period, for example, a few years, it is unreasonable to consider r as a constant. As a starting point, we assume that the short rate is a known function of t, i.e., $r = r(t)$. Let $V(t)$ stand for the value of a bond with coupon rate $k(t)$ at time t. Assume that the return rate of a bond during the time interval $[t, t+dt]$ be the risk-free short rate, so we have

$$dV + Zk(t)dt = r(t)Vdt,$$

where $Zk(t)dt$ is the coupon payment the bondholder receives during the time interval. If the coupon is paid continuously, $k(t)$ is a continuous function of t. If it is paid at fixed times, $k(t)$ is a linear combination of Dirac delta functions, i.e., $k(t) = \sum\limits_i k_i \delta(t - t_i)$, $t_i \le T$. The relation above can also be written as

$$dV - r(t)Vdt = -Zk(t)dt.$$

Multiplying both sides of the equation by $\mathrm{e}^{\int_t^T r(\tau)d\tau}$, which is usually referred to as the integrating factor, yields

$$\mathrm{e}^{\int_t^T r(\tau)d\tau}\left[dV - r(t)Vdt\right] = -Zk(t)\mathrm{e}^{\int_t^T r(\tau)d\tau}dt.$$

The left-hand side actually is $d\left(\mathrm{e}^{\int_t^T r(\tau)d\tau}V\right)$. Therefore, we have

$$\int_t^T d\left(\mathrm{e}^{\int_t^T r(\tau)d\tau}V\right) = V(T) - \mathrm{e}^{\int_t^T r(\tau)d\tau}V(t) = -Z\int_t^T k(\bar{\tau})\mathrm{e}^{\int_{\bar{\tau}}^T r(\tau)d\tau}d\bar{\tau}$$

and

$$\begin{aligned} V(t) &= \mathrm{e}^{-\int_t^T r(\tau)d\tau}\left[V(T) + Z\int_t^T k(\bar{\tau})\mathrm{e}^{\int_{\bar{\tau}}^T r(\tau)d\tau}d\bar{\tau}\right] \\ &= V(T)\mathrm{e}^{-\int_t^T r(\tau)d\tau}\left[1 + \int_t^T k(\bar{\tau})\mathrm{e}^{\int_{\bar{\tau}}^T r(\tau)d\tau}d\bar{\tau}\right], \end{aligned} \tag{5.7}$$

where we have used the condition $Z = V(T)$. For a zero-coupon bond, $k(t) = 0$ and

$$V(t) = V(T)\mathrm{e}^{-\int_t^T r(\tau)d\tau} = Z\mathrm{e}^{-\int_t^T r(\tau)d\tau}.$$

From the right-hand side, we see that the value of $V(t)$ depends on T. However, this dependence is suppressed in this expression. In order to express this dependence explicitly, the relation above can be rewritten as

$$V(t;T) = V(T;T)\mathrm{e}^{-\int_t^T r(\tau)d\tau}, \tag{5.8}$$

where $V(T;T) = Z$.

At time t, the values of zero-coupon bonds with various maturities can be obtained from the market, i.e., $V(t;T)$ with a fixed t and various T is observable. Suppose we have such a function. Differentiating the formula (5.8) with respect to T yields

$$\frac{\partial V(t;T)}{\partial T} = -V(T;T)\mathrm{e}^{-\int_t^T r(\tau)d\tau} r(T) = -V(t;T)r(T)$$

and

$$r(T) = \frac{-1}{V(t;T)} \frac{\partial V(t;T)}{\partial T}.$$

This means that the short rate at time T can be determined by the value and the slope of the function $V(t;T)$. It is clear that $r(T)$ does not depend on Z. Let $Z = 1$, then comparing the expression for $r(T)$ and the formula (5.3) yields

$$F(t,T) = r(T)$$

and

$$Z(t;T) = \mathrm{e}^{-\int_t^T r(u)du} \tag{5.9}$$

if $Z = 1$. Also for a zero-coupon bond,

$$\frac{V(t;T)}{V(T;T)} = Z(t;T).$$

Thus, from the relation (5.1) we have

$$Y(t,T) = \frac{-\ln Z(t;T)}{T-t} = \frac{-\ln\left(V(t;T)/V(T;T)\right)}{T-t}, \tag{5.10}$$

which is usually called the yield of a bond during the time interval $[t,T]$. A plot of Y against the time to maturity, $T-t$, is called the yield curve. The dependence of the yield on $T-t$ is called the term structure of interest rates. The historical data on bonds are usually given in the form of yields for various $T-t$.

5.2.2 Bond Equations for Random Short Rates

It will be more realistic to consider the short rate r as a random variable. Suppose

$$dr = u(r,t)dt + w(r,t)dX. \tag{5.11}$$

From Sect. 2.3, we know that the value of a bond as a short rate derivative, $V(r,t)$, satisfies Eq. (2.34) with only one random variable r:

$$\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV + kZ = 0, \tag{5.12}$$

where kZ is the coupon payment and $\lambda = \lambda(r,t)$ is the market price of risk for r. For a bond the value at maturity date T is a constant Z, i.e.,

$$V(r,T) = Z. \tag{5.13}$$

If the short rate model satisfies the conditions (2.39) and (2.40), then no boundary condition is needed, i.e., Eq. (5.12) with the final condition (5.13) has a unique solution.

5.3 Some Explicit Solutions of Bond Equations

There exist many short rate models. Here, we discuss the following model (see [84]):

$$dr = [\bar{\mu}(t) - \bar{\gamma}(t)r]\,dt + \sqrt{\alpha(t)r - \beta(t)}dX, \tag{5.14}$$

where $\alpha(t)$, $\beta(t)$, $\bar{\gamma}(t)$, and $\bar{\mu}(t)$ are given functions of t. Several important models, for example, the Vasicek model (see [81]), the Cox–Ingersoll–Ross model (see [23]), the Ho–Lee model (see [41]), and the Hull–White model (see [44]) possess this form. For the models in the form (5.14), the determination of the value of a zero-coupon bond can be reduced to solving two ordinary differential equations. Sometimes we can find analytic solutions or the solution can be expressed in terms of integrals with known integrands. Such a solution is referred to as an explicit solution here.

If a short rate model is in the form (5.14) and we take

$$\lambda\,(r,t) = \bar{\lambda}(t)\sqrt{\alpha(t)r - \beta(t)}, \tag{5.15}$$

then Eq. (5.12) can be written as

$$\frac{\partial V}{\partial t} + \frac{1}{2}\left[\alpha(t)r - \beta(t)\right]\frac{\partial^2 V}{\partial r^2} + \left[\mu(t) - \gamma(t)r\right]\frac{\partial V}{\partial r} - rV = 0, \tag{5.16}$$

where

$$\mu(t) = \bar{\mu}(t) + \bar{\lambda}(t)\beta(t) \tag{5.17}$$

and

$$\gamma(t) = \bar{\gamma}(t) + \bar{\lambda}(t)\alpha(t). \tag{5.18}$$

Here, we let $k = 0$ because we are going to determine the value of a zero-coupon bond. Because the coefficients of $\dfrac{\partial^2 V}{\partial r^2}$ and $\dfrac{\partial V}{\partial r}$ are linear functions in r, the solution of Eq. (5.16) with the condition (5.13) has the following form

$$V(r,t) = Z\mathrm{e}^{A(t,T) - rB(t,T)} \tag{5.19}$$

with

$$A(T,T) = 0 \tag{5.20}$$

and

$$B(T,T) = 0. \tag{5.21}$$

In fact, because the conditions (5.20) and (5.21) hold, we have

$$V(r,T) = Z.$$

Substituting the expression (5.19) into Eq. (5.16) yields

$$\frac{dA}{dt} - r\frac{dB}{dt} + \frac{1}{2}\left[\alpha(t)r - \beta(t)\right]B^2 - \left[\mu(t) - \gamma(t)r\right]B - r = 0.$$

If the sum of the terms independent of r is equal to zero, i.e.,

$$\frac{dA}{dt} - \frac{1}{2}\beta(t)B^2 - \mu(t)B = 0$$

and the sum of all coefficients of r is equal to zero, i.e.,

$$-\frac{dB}{dt} + \frac{1}{2}\alpha B^2 + \gamma(t)B - 1 = 0,$$

then the expression (5.19) is a solution to a zero-coupon bond. These two equations above, which can be rewritten as

$$\frac{dA}{dt} = \frac{1}{2}\beta(t)B^2 + \mu(t)B \tag{5.22}$$

and

$$\frac{dB}{dt} = \frac{1}{2}\alpha(t)B^2 + \gamma(t)B - 1, \tag{5.23}$$

have unique solutions satisfying the conditions (5.20) and (5.21). Thus, it is true that Eq. (5.16) with the condition (5.13) has a solution in the form (5.19) satisfying the conditions (5.20) and (5.21), and the solution of the problem can be reduced to solving the two ordinary differential equations (5.22) and (5.23) with the conditions (5.20) and (5.21).

5.3.1 Analytic Solutions for the Vasicek and Cox–Ingersoll–Ross Models

If $\alpha, \beta, \gamma, \mu$ in Eqs. (5.22) and (5.23) are constant, then we can find analytic expressions for A and B. When A and B have such expressions, the expression (5.19) gives an analytic solution for a zero-coupon bond. In this case, Eq. (5.23) can be rewritten as

$$\frac{dB}{B^2 + \dfrac{2\gamma}{\alpha}B - \dfrac{2}{\alpha}} = \frac{\alpha}{2}dt. \tag{5.24}$$

Since

$$B^2 + \frac{2\gamma}{\alpha}B - \frac{2}{\alpha} = \left(B + \frac{\gamma - \psi}{\alpha}\right)\left(B + \frac{\gamma + \psi}{\alpha}\right),$$

where

$$\psi = \sqrt{\gamma^2 + 2\alpha}, \tag{5.25}$$

using the method of partial fraction decomposition, we can have

$$\frac{1}{B^2 + \frac{2\gamma}{\alpha}B - \frac{2}{\alpha}} = \frac{\frac{\alpha}{2\psi}}{B + \frac{\gamma - \psi}{\alpha}} - \frac{\frac{\alpha}{2\psi}}{B + \frac{\gamma + \psi}{\alpha}}.$$

Noticing this relation, we can easily find the solution to Eq. (5.24) by integrating both sides of the equation:

$$\begin{aligned}
&\int_{B(t,T)}^{B(T,T)} \frac{dB}{B^2 + \frac{2\gamma}{\alpha}B - \frac{2}{\alpha}} \\
&= \frac{\alpha}{2\psi}\left[\int_{B(t,T)}^{0} \frac{dB}{B + (\gamma - \psi)/\alpha} - \int_{B(t,T)}^{0} \frac{dB}{B + (\gamma + \psi)/\alpha}\right] \\
&= \frac{\alpha}{2\psi}\left[\ln \frac{(\gamma - \psi)/\alpha}{B + (\gamma - \psi)/\alpha} - \ln \frac{(\gamma + \psi)/\alpha}{B + (\gamma + \psi)/\alpha)}\right] \\
&= \frac{\alpha}{2}\int_t^T dt = \frac{\alpha}{2}(T - t).
\end{aligned}$$

From this we have

$$\frac{B + (\gamma + \psi)/\alpha}{B + (\gamma - \psi)/\alpha} = \frac{\gamma + \psi}{\gamma - \psi}\, \mathrm{e}^{\psi(T-t)}$$

or

$$\begin{aligned}
B &= \frac{\frac{\gamma + \psi}{\alpha}\, \mathrm{e}^{\psi(T-t)} - \frac{\gamma + \psi}{\alpha}}{1 - \frac{\gamma + \psi}{\gamma - \psi}\, \mathrm{e}^{\psi(T-t)}} \\
&= \frac{2\left[\mathrm{e}^{\psi(T-t)} - 1\right]}{(\gamma + \psi)\, \mathrm{e}^{\psi(T-t)} - (\gamma - \psi)},
\end{aligned} \tag{5.26}$$

where we have used the relation $\psi^2 - \gamma^2 = 2\alpha$. After we find B, from Eq. (5.22) we have

$$\begin{aligned}
\int_{A(t,T)}^{A(T,T)} dA &= A(T,T) - A(t,T) \\
&= \int_t^T \left(\frac{1}{2}\beta B^2 + \mu B\right) dt
\end{aligned}$$

or

$$A(t,T) = -\frac{1}{2}\beta \int_t^T B^2 dt - \mu \int_t^T B dt.$$

Using the relation (5.24), we can obtain the results of $\int_t^T B dt$ and $\int_t^T B^2 dt$ easily as follows:

$$\begin{aligned}
\int_t^T B dt &= \int_{B(t,T)}^0 \frac{2B/\alpha}{B^2 + 2\gamma B/\alpha - 2/\alpha} dB \\
&= \frac{2}{\alpha} \int_{B(t,T)}^0 \left[\frac{-(\gamma-\psi)/(2\psi)}{B + (\gamma-\psi)/\alpha} + \frac{(\gamma+\psi)/(2\psi)}{B + (\gamma+\psi)/\alpha} \right] dB \\
&= -\frac{\gamma-\psi}{\alpha\psi} \ln \frac{(\gamma-\psi)/\alpha}{B + (\gamma-\psi)/\alpha} + \frac{\gamma+\psi}{\alpha\psi} \ln \frac{(\gamma+\psi)/\alpha}{B + (\gamma+\psi)/\alpha}
\end{aligned}$$

and

$$\begin{aligned}
\int_t^T B^2 dt &= \frac{2}{\alpha} \int_{B(t,T)}^0 \frac{B^2}{B^2 + 2\gamma B/\alpha - 2/\alpha} dB \\
&= \frac{2}{\alpha} \int_{B(t,T)}^0 \left(1 - \frac{2\gamma B/\alpha}{B^2 + 2\gamma B/\alpha - 2/\alpha} + \frac{2/\alpha}{B^2 + 2\gamma B/\alpha - 2/\alpha} \right) dB \\
&= \frac{2}{\alpha} \int_{B(t,T)}^0 \left(1 + \frac{\gamma(\gamma-\psi)/(\alpha\psi)}{B + (\gamma-\psi)/\alpha} - \frac{\gamma(\gamma+\psi)/(\alpha\psi)}{B + (\gamma+\psi)/\alpha} \right. \\
&\quad \left. + \frac{1/\psi}{B + (\gamma-\psi)/\alpha} - \frac{1/\psi}{B + (\gamma+\psi)/\alpha} \right) dB \\
&= \frac{2}{\alpha} \left\{ -B + \left[\frac{\gamma(\gamma-\psi)}{\alpha\psi} + \frac{1}{\psi} \right] \ln \frac{(\gamma-\psi)/\alpha}{B + (\gamma-\psi)/\alpha} \right. \\
&\quad \left. - \left[\frac{\gamma(\gamma+\psi)}{\alpha\psi} + \frac{1}{\psi} \right] \ln \frac{(\gamma+\psi)/\alpha}{B + (\gamma+\psi)/\alpha} \right\} \\
&= \frac{2}{\alpha} \left\{ -B - \frac{\gamma-\psi}{(\gamma+\psi)\psi} \ln \frac{(\gamma-\psi)/\alpha}{B + (\gamma-\psi)/\alpha} \right. \\
&\quad \left. + \frac{(\gamma+\psi)}{(\gamma-\psi)\psi} \ln \frac{(\gamma+\psi)/\alpha}{B + (\gamma+\psi)/\alpha} \right\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
A = \frac{\beta}{\alpha} B &+ \left[\frac{\beta(\gamma-\psi)}{\alpha(\gamma+\psi)\psi} + \mu \frac{\gamma-\psi}{\alpha\psi} \right] \ln \frac{(\gamma-\psi)/\alpha}{B + (\gamma-\psi)/\alpha} \\
&- \left[\frac{\beta(\gamma+\psi)}{\alpha(\gamma-\psi)\psi} + \mu \frac{\gamma+\psi}{\alpha\psi} \right] \ln \frac{(\gamma+\psi)/\alpha}{B + (\gamma+\psi)/\alpha} \qquad (5.27)
\end{aligned}$$

and

$$V(r,t)=Z\left[\frac{B+(\gamma-\psi)/\alpha}{(\gamma-\psi)\alpha}\right]^{\beta(\psi-\gamma)/\alpha(\gamma+\psi)\psi+\mu(\psi-\gamma)/\alpha\psi} \times\left[\frac{B+(\gamma+\psi)/\alpha}{(\gamma+\psi)/\alpha}\right]^{\beta(\gamma+\psi)/\alpha(\gamma-\psi)\psi+\mu(\gamma+\psi)/\alpha\psi} \mathrm{e}^{(\beta/\alpha-r)B}. \quad (5.28)$$

This is a solution of a zero-coupon bond suitable for all the models (5.14) with constant $\alpha, \beta, \bar{\gamma}$, and $\bar{\mu}$ as long as we choose the market price of risk in the form $\lambda(r,t) = \bar{\lambda}\sqrt{\alpha r - \beta}$. The parameters $\alpha, \beta, \bar{\gamma}$, and $\bar{\mu}$ can be obtained from the data on the short rate on the market. However $\bar{\lambda}$, a parameter in the expression of the market price of risk, cannot be determined from the data on the short rate and should be obtained from the other data on the market. For example, $\bar{\lambda}$ can be determined from the yield function on the market by the least squares method, i.e., by choosing $\bar{\lambda}$ so that $\int_t^T \left[Y(t,T;\bar{\lambda}) - \tilde{Y}(t-T)\right]^2 dT$ is minimized, or

$$\int_t^T \left[Y\left(t,T;\bar{\lambda}\right) - \tilde{Y}(t,T)\right] \frac{\partial Y\left(t,T;\bar{\lambda}\right)}{\partial \bar{\lambda}} dT = 0. \quad (5.29)$$

Here, $\tilde{Y}(t,T)$ is the yield function observed on the market, whereas according to the expressions (5.10) and (5.19), the function $Y(t,T;\bar{\lambda})$ is given by

$$Y\left(t,T;\bar{\lambda}\right) = \frac{rB(t,T;\bar{\lambda}) - A(t,T;\bar{\lambda})}{T-t}, \quad (5.30)$$

where $A(t,T;\bar{\lambda})$ and $B(t,T;\bar{\lambda})$ are given by the expressions (5.26) and (5.27), respectively, but the dependence of A and B on $\bar{\lambda}$ is expressed explicitly here. If the value of yield is only available discretely on the market, then we can find a $\bar{\lambda}$ such that $\sum_i \left[Y(t,T_i;\bar{\lambda}) - \tilde{Y}(t,T_i)\right]^2$ is minimized, or

$$\sum_i \left[Y(t,T_i;\bar{\lambda}) - \tilde{Y}(t,T_i)\right] \frac{\partial Y(t,T_i;\bar{\lambda})}{\partial \bar{\lambda}} = 0. \quad (5.31)$$

As soon as we have $\bar{\lambda}$, the (5.16) with constant α, β, γ, and μ can be used to determine the value of any other short rate derivative. Generally speaking, it is impossible to fit the entire yield curve by choosing only one parameter. This is a drawback of such a model.

For some special models, for example, the Vasicek model (see [81]) and the Cox–Ingersoll–Ross model (see [23]), the expression can be simplified. Let us do this for these two models.

The Vasicek model is in the form

$$dr = (\bar{\mu} - \bar{\gamma}r)\,dt + \sqrt{-\beta}dX, \quad \beta < 0, \quad \bar{\gamma} > 0.$$

Therefore, the expressions (5.26) and (5.27) with

$$\alpha = 0, \quad \mu = \bar{\mu} + \bar{\lambda}\beta$$

and

$$\gamma = \bar{\gamma}$$

give B and A for this model. In this case, the expression (5.26) becomes[1]

$$B = \frac{\mathrm{e}^{\gamma(T-t)} - 1}{\gamma\, \mathrm{e}^{\gamma(T-t)}} = \frac{1}{\gamma}(1 - \mathrm{e}^{-\gamma(T-t)}). \tag{5.32}$$

However, the expression (5.27) cannot be used for calculation because of $\alpha = 0$. In order to have an expression for A that can be used for calculation, we need to find the limit of the expression (5.27) as $\alpha \to 0$ or solve Eq. (5.22) with B given by the expression (5.32) directly. Let us solve Eq. (5.22) directly. Putting the expression (5.32) into Eq. (5.22), we have:

$$\begin{aligned}
&A(T,T) - A(t,T) = \int_{A(t,T)}^{A(T,T)} dA \\
&= \int_t^T \left[\frac{\beta}{2\gamma^2}\left(1 - \mathrm{e}^{-\gamma(T-t)}\right)^2 + \frac{\mu}{\gamma}\left(1 - \mathrm{e}^{-\gamma(T-t)}\right)\right] dt \\
&= \int_t^T \left[\frac{\beta}{2\gamma^2}\left(1 - \mathrm{e}^{-\gamma(T-t)}\right)\left(-\mathrm{e}^{-\gamma(T-t)}\right)\right. \\
&\quad \left. + \left(\frac{\beta}{2\gamma^2} + \frac{\mu}{\gamma}\right)\left(1 - \mathrm{e}^{-\gamma(T-t)}\right)\right] dt \\
&= \left[\frac{\beta}{4\gamma^3}\left(1 - \mathrm{e}^{-\gamma(T-t)}\right)^2 + \left(\frac{\beta}{2\gamma^2} + \frac{\mu}{\gamma}\right)\left(t - \frac{1}{\gamma}\mathrm{e}^{-\gamma(T-t)}\right)\right]\Bigg|_t^T \\
&= -\frac{\beta}{4\gamma^3}\left(1 - \mathrm{e}^{-\gamma(T-t)}\right)^2 + \left(\frac{\beta}{2\gamma^2} + \frac{\mu}{\gamma}\right)(T-t) \\
&\quad - \left(\frac{\beta}{2\gamma^2} + \frac{\mu}{\gamma}\right)\frac{1}{\gamma}\left(1 - \mathrm{e}^{-\gamma(T-t)}\right).
\end{aligned}$$

Because of $A(T,T) = 0$, we obtain

$$A = -\left(\frac{\beta}{2\gamma^2} + \frac{\mu}{\gamma}\right)(T-t) + \left(\frac{\beta}{2\gamma^2} + \frac{\mu}{\gamma}\right)B + \frac{\beta B^2}{4\gamma}. \tag{5.33}$$

Consequently

$$\begin{aligned}
V(r,t) &= Z\mathrm{e}^{-(\beta/2\gamma^2+\mu/\gamma)(T-t)+(\beta/2\gamma^2+\mu/\gamma-r)B+\beta B^2/4\gamma} \\
&= Z\mathrm{e}^{-(\beta/2\gamma^2+\mu/\gamma)(T-t)+(\beta/2\gamma^2+\mu/\gamma-r)(1-\mathrm{e}^{-\gamma(T-t)})/\gamma+\beta\left(1-\mathrm{e}^{-\gamma(T-t)}\right)^2/4\gamma^3}.
\end{aligned} \tag{5.34}$$

[1]This expression can also be obtained by integrating Eq. (5.23) with $\alpha = 0$ directly, and for the case $\alpha = 0$, this direct way of finding the solution is easier.

This is the value of a zero-coupon bond if the Vasicek model is adopted. As pointed out above, the solution (5.34) can also be obtained by finding the limit of the solution (5.28). This is left to readers as Problem 7.

Noticing that B does not depend on $\bar{\lambda}$ in this case, we have

$$\begin{aligned}Y(t,T;\bar{\lambda}) &= \frac{rB\,(t,T) - A(t,T;\bar{\lambda})}{T-t}\\ &= \frac{\left(\dfrac{\beta}{2\gamma^2}+\dfrac{\mu}{\gamma}\right)(T-t) - \left(\dfrac{\beta}{2\gamma^2}+\dfrac{\mu}{\gamma}-r\right)B - \dfrac{\beta B^2}{4\gamma}}{T-t}\\ &= \frac{\left(\dfrac{\beta}{2\gamma^2}+\dfrac{\bar{\mu}}{\gamma}\right)(T-t) - \left(\dfrac{B}{2\gamma^2}+\dfrac{\bar{\mu}}{\gamma}-r\right)B - \dfrac{\beta B^2}{4\gamma}}{T-t}\\ &\quad+\frac{\bar{\lambda}\beta(T-t-B)}{\gamma\,(T-t)}\end{aligned}$$

and

$$\frac{\partial Y}{\partial \bar{\lambda}}(t,T;\bar{\lambda}) = \frac{\beta(T-t-B)}{\gamma(T-t)}.$$

Hence, Eq. (5.29) becomes a linear equation for $\bar{\lambda}$. From the linear equation, we see that $\bar{\lambda}$ is given by

$$\frac{\displaystyle\int_t^T \left[\frac{\left(\dfrac{\beta}{2\gamma^2}+\dfrac{\bar{\mu}}{\gamma}\right)(T-t) - \left(\dfrac{B}{2\gamma^2}+\dfrac{\bar{\mu}}{\gamma}-r\right)B - \dfrac{\beta B^2}{4\gamma}}{T-t} - \tilde{Y}\right]\frac{T-t-B}{(T-t)}dT}{\displaystyle -\frac{\beta}{\gamma}\int_t^T \frac{(T-t-B)^2}{(T-t)^2}dT}. \tag{5.35}$$

Because only $\bar{\lambda}$ is chosen, the yield curve cannot be fitted entirely. Another problem of this model is that r may be negative.

In order to rectify this problem, Cox, Ingersoll, and Ross (see [23]) proposed another model:

$$dr = (\bar{\mu} - \bar{\gamma} r)dt + \sqrt{\alpha r}dX. \tag{5.36}$$

This is also in the form (5.14) and $\beta = 0$ here. In this case, the solution for a zero-coupon bond is

$$V(r,t) = Z\left[\frac{B+(\gamma-\psi)\,/\alpha}{(\gamma-\psi)/\alpha}\right]^{\mu(\psi-\gamma)/\alpha\psi}\left[\frac{B+(\gamma+\psi)\,/\alpha}{(\gamma+\psi)\,/\alpha}\right]^{\mu(\gamma+\psi)/\alpha\psi}\mathrm{e}^{-rB}.$$

Here, B is given by the expression (5.26), i.e.,

$$B = \frac{2\left(e^{\psi(T-t)} - 1\right)}{(\gamma+\psi)\,e^{\psi(T-t)} - (\gamma-\psi)},$$

$$\mu = \bar{\mu}$$

and

$$\gamma = \bar{\gamma} + \bar{\lambda}\alpha,$$

where $\bar{\lambda}$ is a parameter in the expression (5.15) for the market price of risk. However, the solution can have another form. Because

$$\begin{aligned} A(T,T) - A(t,T) &= \int_{A(t,T)}^{A(T,T)} dA = \int_t^T \mu B dt \\ &= \mu \int_t^T \frac{2\left(e^{\psi(T-t)} - 1\right)}{(\gamma+\psi)\,e^{\psi(T-t)} - (\gamma-\psi)} dt, \end{aligned}$$

noticing $A(T,T) = 0$ and setting $\xi = e^{\psi(T-t)}$, we have

$$\begin{aligned} &A(t,T) \\ &= -\mu \int_t^T \frac{2\left(e^{\psi(T-t)} - 1\right)}{(\gamma+\psi)\,e^{\psi(T-t)} - (\gamma-\psi)} dt = \mu \int_\xi^1 \frac{2(\xi-1)d\xi}{[(\gamma+\psi)\xi - (\gamma-\psi)]\psi\xi} \\ &= \frac{2\mu}{\psi(\gamma+\psi)} \int_\xi^1 \left[\frac{-2\psi/(\gamma-\psi)}{\xi - (\gamma-\psi)/(\gamma+\psi)} + \frac{(\gamma+\psi)/(\gamma-\psi)}{\xi}\right] d\xi \\ &= \frac{-4\mu}{\gamma^2 - \psi^2} \left[\ln(\xi - (\gamma-\psi)/(\gamma+\psi)) - (\gamma+\psi)\ln\xi/2\psi\right]\Big|_\xi^1 \\ &= \frac{2\mu}{\alpha} \left[\ln(1 - (\gamma-\psi)/(\gamma+\psi)) - \ln(\xi - (\gamma-\psi)/(\gamma+\psi))\right. \\ &\quad \left. + (\gamma+\psi)\ln\xi/2\psi\right] \\ &= \ln\left(\frac{2\psi e^{(\gamma+\psi)(T-t)/2}}{(\gamma+\psi)e^{\psi(T-t)} - (\gamma-\psi)}\right)^{2\mu/\alpha}. \end{aligned}$$

Therefore, we have a solution

$$V(r,t) = Z\left[\frac{2\psi e^{(\gamma+\psi)(T-t)/2}}{(\gamma+\psi)e^{\psi(T-t)} - (\gamma-\psi)}\right]^{2\mu/\alpha} e^{-rB}, \tag{5.37}$$

which is the form given in the paper by Cox, Ingersoll, and Ross. It can be proved that the two expressions are identical. This is left to readers to prove as Problem 9.

In this case

$$Y(t,T;\bar{\lambda}) = \frac{\dfrac{2\left(e^{\psi(T-t)} - 1\right)r}{(\gamma+\psi)e^{\psi(T-t)} - (\gamma-\psi)} - \dfrac{2\mu}{\alpha}\ln\left(\dfrac{2\psi e^{(\gamma+\psi)(T-t)/2}}{(\gamma+\psi)e^{\psi(T-t)} - (\gamma-\psi)}\right)}{T-t},$$

where $\gamma = \bar{\gamma} + \bar{\lambda}\alpha$ and $\psi = \sqrt{\gamma^2 + 2\alpha}$, so the dependence of $Y(t, T; \bar{\lambda})$ on $\bar{\lambda}$ is quite complicated.

As we have stated, in order to use the partial differential equation (5.16) to price the value of other derivatives, we need to determine $\bar{\lambda}$ so that we can have $\gamma = \bar{\gamma} + \bar{\lambda}\alpha$. For example, we can obtain $\bar{\lambda}$ by solving Eq. (5.29). Because the dependence of $Y(t, T; \bar{\lambda})$ on $\bar{\lambda}$ in this case is quite complicated, Eq. (5.29) has to be solved numerically. Just like the Vasicek model, generally speaking, it is impossible to "build" the entire term structure of the short rate into a parameter $\bar{\lambda}$.

5.3.2 Explicit Solutions for the Ho–Lee and Hull–White Models

In order to fit the entire term structure of interest rates, it seems to be necessary to require $\bar{\lambda}$ to be dependent on t or r. If $\bar{\lambda}$ depends on t, then for some models in the form (5.14), the solution of a zero-coupon bond can explicitly be expressed by elemental functions and integrals with known integrands. We refer to such a solution as an explicit solution or a closed-form solution. The Ho–Lee model (see [41])

$$dr = \bar{\mu}(t)dt + \sqrt{-\beta}dX \tag{5.38}$$

and the Hull–White model (see [44])

$$dr = (\bar{\mu}(t) - \bar{\gamma}r)dt + \sqrt{-\beta}dX \tag{5.39}$$

are such models. We note that the Hull–White model is an extension of the Ho–Lee model and the Vasicek model. For the Hull–White model, $B(t, T)$ is the same as for the Vasicek model, given by the expression (5.32):

$$B(t, T) = \frac{1}{\gamma}(1 - \mathrm{e}^{-\gamma(T-t)}),$$

where

$$\gamma = \bar{\gamma}.$$

Let $\gamma \to 0$, we have

$$B(t, T) = T - t, \tag{5.40}$$

which is the expression of $B(t, T)$ for the Ho–Lee model. For both of them, Eq. (5.22) is in the form

$$\frac{dA}{dt} = \frac{1}{2}\beta B^2 + \mu(t)B,$$

where $\mu(t)$ is given by the expression (5.17):

$$\mu(t) = \bar{\mu}(t) + \bar{\lambda}(t)\beta.$$

Here, we assume that the market price of risk is $\lambda(r,t)=\bar{\lambda}(t)\sqrt{-\beta}$. From the ordinary differential equation above, we can find

$$A(t,T)=-\frac{1}{2}\beta\int_t^T B^2(\tau,T)d\tau-\int_t^T \mu(\tau)B(\tau,T)d\tau$$

and

$$V(r,t)=Z\mathrm{e}^{-\frac{1}{2}\beta\int_t^T B^2(\tau,T)d\tau-\int_t^T \mu(\tau)B(\tau,T)d\tau-rB(t,T)}. \tag{5.41}$$

Here, B is given by the expression (5.32) or the expression (5.40), depending on which model is used. Therefore, if $\bar{\lambda}$ is given, we can find $V(r,t)$ without any difficulties.

In practice, we need to find $\bar{\lambda}(t)$ from some data on the market, for example, a given yield function $Y(t,T)$. In order to do this, we rewrite the solution (5.41) as

$$\ln V(r,t)=\ln Z-\frac{1}{2}\beta\int_t^T B^2(\tau,T)d\tau-\int_t^T \mu(\tau)B(\tau,T)d\tau-rB(t,T)$$

or if we require that the solution (5.41) fits the yield function on the market, we furthermore have

$$\int_t^T \mu(\tau)B(\tau,T)d\tau=Y(t,T)(T-t)-\frac{1}{2}\beta\int_t^T B^2(\tau,T)d\tau-rB(t,T), \tag{5.42}$$

where we have used the definition of the yield (5.10). If we define

$$F_1(t,T)\equiv Y(t,T)(T-t)-\frac{1}{2}\beta\int_t^T B^2(\tau,T)d\tau-rB(t,T) \tag{5.43}$$

and substitute $(1-\mathrm{e}^{-\gamma(T-\tau)})/\gamma$ for B on the left-hand side of Eq. (5.42), it becomes

$$\frac{1}{\gamma}\int_t^T \mu(\tau)(1-\mathrm{e}^{-\gamma(T-\tau)})d\tau=F_1(t,T).$$

Differentiating both sides of this relation with respect to T twice yields

$$\mu(T)=\frac{\partial^2 F_1(t,T)}{\partial T^2}+\gamma\frac{\partial F_1(t,T)}{\partial T}, \tag{5.44}$$

which is the function $\mu(t)$ for the Hull–White model. After having the function $\mu(t)$, we can obtain $\bar{\lambda}(t)$ immediately by

$$\bar{\lambda}(t)=\frac{1}{\beta}\left[\mu(t)-\bar{\mu}(t)\right]$$

if we want. Therefore, for the Hull–White model, we can find a function for the market price of risk for r such that the entire term structure of interest rate can be fitted. For the Ho–Lee model, in order to do this, we can use the same formula with $\gamma = 0$, so in the expression (5.43) $B = T - t$. Because in these models the entire term structure of interest rate is built into the function $\bar{\lambda}(t)$, these two models are often referred to as no-arbitrage interest rate models. The difference between them is that the Hull–White model has the mean reversion property that an interest rate model should have, whereas the Ho–Lee model does not. However, even though the Hull-White model has the mean reversion property, r is still defined on $(-\infty, \infty)$ because the coefficient of dX in the model is a constant.

5.4 Inverse Problem on the Market Price of Risk

As we saw in Sect. 5.3, for some special interest rate models and some special function of the market price of risk, we can find an explicit solution for a zero-coupon bond and furthermore explicit expressions for the market price of risk for which the entire term structure of interest rate or the entire zero-coupon bond price curve is fitted. However, even though the model is in the form (5.14) and solving the partial differential equation (5.16) can be reduced to solving ordinary differential equations (5.22) and (5.23), we still may not be able to find an explicit expression for the market price of risk if $\alpha(t)$ really depends on t or even if α is a nonzero constant. In this case, the unknown function $\bar{\lambda}(t)$ appears in both Eqs. (5.22) and (5.23) and it may be necessary to use numerical methods.

Also, there are other models, for example, the Black–Derman–Toy model (see [9]):

$$d\ln r = \left[\bar{\mu}(t) - \frac{\sigma_r'(t)}{\sigma_r(t)}\ln r\right] dt + \sigma_r(t)dX$$

and the Black–Karasinski model (see [10]):

$$d\ln r = \left[\bar{\mu}(t) - \bar{\gamma}(t)\ln r\right] dt + \sigma_r(t)dX.$$

For these models, it might even be impossible to reduce solving a partial differential equation into solving two ordinary differential equations. In addition, it may be necessary to consider interest rate models (5.11):

$$dr = u(r,t)dt + w(r,t)dX$$

with more general functions $u(r,t)$ and $w(r,t)$. For example, a model might be more useful if $u(r,t)$ and $w(r,t)$ is determined from the data of the short rate on the market. Also in order for the model to be more realistic, the model should guarantee that the random variable r will be in a finite interval $[r_l, r_u]$ in the future if r is in the interval $[r_l, r_u]$ now. According to Sect. 2.4, if u and w satisfy

$$\begin{cases} u(r_l,t) - w(r_l,t)\dfrac{\partial}{\partial r}w(r_l,t) \geq 0, \\ w(r_l,t) = 0 \end{cases} \tag{5.45}$$

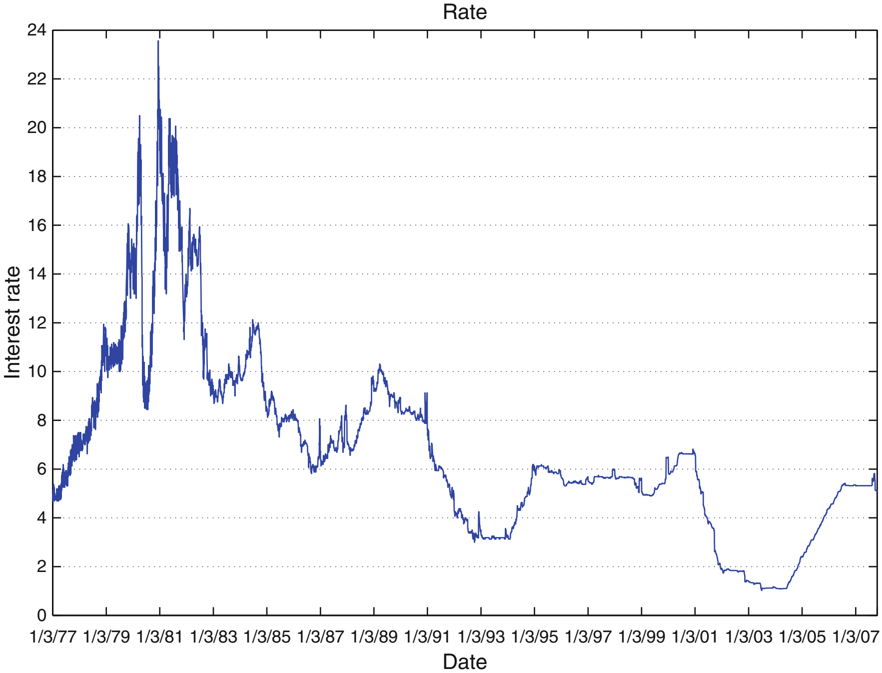

Fig. 5.1. One month LIBOR on US dollar during Jan 1977–Sep 2007

and

$$\begin{cases} u(r_u, t) - w(r_u, t)\dfrac{\partial}{\partial r} w(r_u, t) \le 0, \\ w(r_u, t) = 0, \end{cases} \tag{5.46}$$

then the random variable r is always in $[r_l, r_u]$. In what follows, we will describe a model having such properties.

The real data of the 1-month LIBOR (London Interbank Offer Rate) on U.S. dollar during January 1977–2010 is available and is shown as a curve in Fig. 5.1. From the data we know that the minimum interest rate $r_{\min}$ is 0.0022906 and the maximum interest rate $r_{\max}$ is 0.23562. Thus we assume that for the interval $[r_l, r_u]$ the lower bound r_l is 0.0 and the upper bound r_u is 0.24. From the data we can also determine the standard deviation of r for 40 values of r by statistics, which are shown as "$\circ$" in Fig. 5.2. Assuming

$$w(r) = (r - r_l)(r_u - r)(a_0 r^2 + b_0 r + c_0),$$

using the values of standard deviation of r obtained, and using the least squares method, we can find the values of a_0, b_0, and c_0, which are

$$a_0 = 4.1, \quad b_0 = -0.51, \quad c_0 = 0.0224.$$

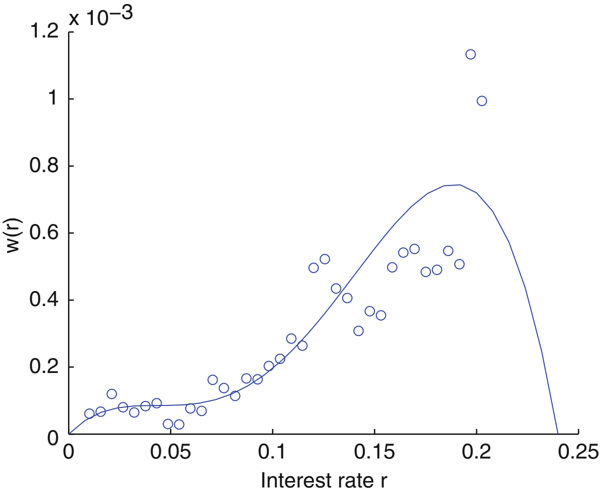

Fig. 5.2. $w(r)$ with $r_l = 0$ and $r_u = 0.24$

That is, the function $w(r)$ in the form above from the real data is:

$$w(r) = (r - r_l)(r_u - r)4.1r^2 - 0.51r + 0.0224$$

The curve of $w(r)$ is also given in Fig. 5.2. This function $w(r)$ satisfies the second conditions in the conditions (5.45) and (5.46). We can also find a function $u(r)$ satisfying the conditions (5.45) and (5.46), so r will be in $[r_l, r_u]$ if such a model is used. However here we do not discuss how to determine such a $u(r,t)$ from the real data. This is because, as we will see, we can choose $\lambda(r,t)$ so that $u(r,t)$ will not be used in order to do computation by using this model. If such a model is used, we have to solve PDE problems numerically in order to get market price of risk and the values of derivatives (for details, see the paper [72] by Shi). In what follows, we briefly discuss how to obtain the market price of risk numerically.

As pointed out in Sect. 5.2.2, if we use the model (5.11), then any interest rate derivative, $V(r,t)$, satisfies Eq. (5.12):

$$\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV = 0, \qquad 0 \leq t \leq T,$$

where we assume that there is no coupon related to the derivative, so $kZ = 0$. This parabolic partial differential equation degenerates to a hyperbolic partial differential equation or an ordinary differential equation at $r = r_l$ and r_u when $w\ (r_l, t) = 0$ and $w(r_u, t) = 0$. Moreover, if the condition (5.45) holds, from Sect. 2.4.2, we see that no extra boundary condition at $r = r_l$ is needed in order to find a unique solution. Similarly, if the condition (5.46) holds, then no extra boundary condition at $r = r_u$ is needed. Consequently, the final value problem without any boundary conditions

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}w^2\dfrac{\partial^2 V}{\partial r^2} + (u - \lambda w)\dfrac{\partial V}{\partial r} - rV = 0, & r_l \le r \le r_u, \quad t \le T, \\ V(r,T) = f(r), & r_l \le r \le r_u \end{cases}$$

has a unique solution if the conditions (5.45) and (5.46) hold. As we have discussed, u and w can be determined from the historical data of the short rate on the market. However, in order to use this equation to price any derivatives, we need to know $\lambda(r,t)$. As soon as such a $\lambda(r,t)$ is determined, an interest rate model (5.11) becomes a no-arbitrage interest rate model. Thus, it is important in practice. Suppose $\lambda(r,t)$ is a function of t plus $u(r,t)/w(r)$, i.e., $\lambda(r,t) = \bar{\lambda}(t) + u(r,t)/w(r)$.[2] Then $\bar{\lambda}(t)$, as the solution of the following inverse problem, can be determined numerically by the term structure of interest rates or, equivalently, by the zero-coupon bond price curve. Suppose that $t = 0$ corresponds to today and today's short rate is r^*. Let $V(r,t;T^*)$ be the solution of the problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}w^2\dfrac{\partial^2 V}{\partial r^2} + \left[u - (\bar{\lambda}(t) + u/w)w\right]\dfrac{\partial V}{\partial r} - rV = 0, \\ \qquad\qquad\qquad\qquad r_l \le r \le r_u, \quad 0 \le t \le T^*, \\ V(r,T^*;T^*) = 1, \qquad r_l \le r \le r_u. \end{cases} \tag{5.47}$$

Here T^* is a parameter. We need to find a function $\bar{\lambda}(t)$ defined on $[0, T^*_{\max}]$ such that $V(r^*,0;T^*)$ is equal to the today's value of the zero-coupon bond maturing at time T^* and with a face value of one dollar for any $T^* \in [0, T^*_{\max}]$, where $T^*_{\max}$ in the longest maturity of zero-coupon bonds on the market.

In this problem, the value of $\bar{\lambda}(t)$ for $t \in [0, T_1^*]$ is determined by the value of zero-coupon bonds maturing at time $T^* \in [0, T_1^*]$. If $\bar{\lambda}(t)$ for $t \in [0, T_1^*]$ has been obtained and $T_2^* > T_1^*$, then the value of $\bar{\lambda}(t)$ for $t \in [T_1^*, T_2^*]$ will be found by letting $V(r^*,0;T^*)$ be equal to the value of a zero-coupon bond maturing at time T^* for any $T^* \in [T_1^*, T_2^*]$. Therefore, the value of $\bar{\lambda}(t)$ at $t = T^*$ is determined by the value of a zero-coupon bond maturing at time T^* if the value of $\bar{\lambda}(t)$ for $t \in [0, T^*)$ has been obtained. In order to find the value of $\bar{\lambda}(T^*)$, we need to make a guess about it and solve the problem (5.47) from $t = T^*$ to $t = 0$ and then check if $V(r^*,0;T^*)$ is equal to the value of the zero-coupon bond maturing at time T^*. If T^* is 20 or 30 years, then the procedure of solving the problem (5.47) is quite long.

Actually the property of the function $\bar{\lambda}(t)$ has a close relation with the second derivative of the zero-coupon bond curve with respect to the maturity time T^* (see Sect. 10.1.1). If the zero-coupon bond curve is obtained by the cubic spline interpolation, then the second derivative is continuous, but the third derivative is discontinuous. The non-smoothness of $\bar{\lambda}(t)$ sometimes causes quite big oscillation of the solution of the bond equation if $T^*_{\max}$ is big.

[2]For such a choice of $\lambda(r,t)$, $u(r,t) - \lambda(r,t)w(r) = -\bar{\lambda}(t)w(r)$, so $u(r,t)$ disappears from the PDE. Thus we do not need $u(r,t)$ in order to solve the PDE.

5.5 Application of Bond Equations

The bond equation (5.12) can be applied to evaluating not only bonds but also bond options, options on bond futures contracts, swaps, caps, floors, collars, and even options on them. In what follows, we describe these applications.

5.5.1 Bond Options and Options on Bond Futures Contracts

A bond option is similar to an equity option except that the underlying asset is a bond. A bond depends on the interest r, and consequently, a bond option will also depend on r. Consider a T-year European option on a N-year bond. Suppose that the time today is zero. Then the bond should be issued on time T and will mature at time $T+N$. In what follows, let T_b denote $T+N$ and for simplicity, let the face value of the bond be equal to one. Thus, the bond price is the solution of the problem

$$\begin{cases} \dfrac{\partial V_b}{\partial t} + \dfrac{1}{2}w^2 \dfrac{\partial^2 V_b}{\partial r^2} + (u - \lambda w)\dfrac{\partial V_b}{\partial r} - rV_b + k = 0, \\ \qquad\qquad\qquad\qquad r_l \le r \le r_u, \quad T \le t \le T_b, \\ V_b(r, T_b; T_b) = 1, \quad r_l \le r \le r_u, \end{cases} \tag{5.48}$$

where we consider a coupon-bearing bond with a coupon payment $k(t)dt$ during a time period $[t, t+dt]$ and use V_b to represent the price of the bond. In practice the coupon is not paid continuously, the equation should be

$$\frac{\partial V_b}{\partial t} + \frac{1}{2}w^2 \frac{\partial^2 V_b}{\partial r^2} + (u - \lambda w)\frac{\partial V_b}{\partial r} - rV_b + \sum_i k_i \delta(t - t_i) = 0.$$

In this case V_b gives the quoted price (clean price). The price a purchaser needs to pay is the cash price (dirty price)—the clean price plus the accrued interest, which should be close to the price given by the model with a continuous coupon payment. Here, we assume that the conditions (5.45) and (5.46) hold, so at $r = r_l$ and $r = r_u$ the equation degenerates to a hyperbolic equation and does not require any boundary conditions. Every model can be modified locally, so the conditions (5.45) and (5.46) hold. Therefore, this assumption is realistic. We also assume that $\lambda(r, t)$ is known. A European call bond option is a contract whose holder has a right to purchase a bond at time T at a price E. Let $V(r, t)$ be the price of the option. Clearly, $V(r, t)$ should be the solution of the problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}w^2 \dfrac{\partial^2 V}{\partial r^2} + (u - \lambda w)\dfrac{\partial V}{\partial r} - rV = 0, \qquad r_l \le r \le r_u, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \le T, \\ V(r, T) = \max(V_b(r, T; T_b) - E, 0), \qquad r_l \le r \le r_u. \end{cases} \tag{5.49}$$

For a European put bond option, the final condition is

$$V(r,T) = \max\left(E - V_b(r,T;T_b), 0\right).$$

For American call and put bond options, we need to require

$$V(r,t) \geq \max\left(V_b\left(r,t;t+N\right) - E, 0\right)$$

and

$$V(r,t) \geq \max(E - V_b(r,t;t+N), 0)$$

for $t \in [0,t]$, respectively. For example, if the option is on a 3-year bond, then $N = 3$. In this case, in order to determine the solution, we need to solve a problem involving free boundaries, and the constraint is a function of t. Therefore, this free-boundary problem is more complicated than that in equity option cases.

We can also determine the value of an option on a bond futures contract, which is denoted by $V(r,t)$ in what follows. Again, let T_b be the maturity date of the bond and T be the expiry of the option and the date the futures contract is initiated. Also, suppose that the futures contract is matured at time $T_f \in (T, T_b)$ and that the delivery price given in the option—the exercise price of the option is K. When $V(r,T)$ is given, we can obtain the value of the option today by solving a problem similar to the problem (5.49). How do we find $V(r,T)$?

Let $V_{b0}(r,t;T_f)$ be the value of the zero-coupon bond with maturity date T_f, which is the solution of the following problem

$$\begin{cases} \dfrac{\partial V_{b0}}{\partial t} + \dfrac{1}{2}w^2\dfrac{\partial^2 V_{b0}}{\partial r^2} + (u - \lambda w)\dfrac{\partial V_{b0}}{\partial r} - rV_{b0} = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad r_l \leq r \leq r_u, \quad t \leq T_f, \\ V_{b0}(r, T_f; T_f) = 1, \qquad\qquad\qquad r_l \leq r \leq r_u. \end{cases} \tag{5.50}$$

Then the value of the bond futures contract with a delivery price K given in the option can be expressed as

$$V_f(r,t;T_f) = V_b(r,t;T_b) - KV_{b0}(r,t;T_f) \tag{5.51}$$

for any $t \leq T_f$. Let K^* be the futures price for the futures contract with maturity date T_f at time T. K^* should be determined by the condition that the value of the futures contract is equal to zero when it is initiated at time T, i.e.,

$$V_f(r,T;T_f) = V_b(r,T;T_b) - K^*V_{b0}(r,T;T_f) = 0.$$

From this condition, we immediately know that the futures price K^* is equal to $V_b(r,T;T_b)/V_{b0}(r,T;T_f)$. If

$$K < K^* = V_b(r,T;T_b)/V_{b0}(r,T;T_f),$$

the holder of the option will exercise the option because the value of the bond futures contract

$$V_f(r,T;T_f) = V_b(r,T;T_b) - KV_{b0}(r,T;T_f) > 0.$$

Actually this is the value of the option for this case. If

$$K \geq K^* = V_b(r,T;T_b)/V_{b0}(r,T;T_f),$$

the value of the bond futures contract

$$V_f(r,T;T_f) = V_b(r,T;T_b) - KV_b(r,T;T_f) \leq 0,$$

and the holder will not exercise the option, which means $V(r,T) = 0$. Putting the two cases together, for $V(r,T)$ we have the following expression

$$V(r,T) = \max\left(V_b(r,T;T_b) - KV_{b0}(r,T;T_f), 0\right). \tag{5.52}$$

Therefore, we can first solve the problem (5.48) from T_b to T to get $V_b(r,T;T_b)$ and solve the problem (5.50) from T_f to T to get $V_{b0}(r,T;T_f)$, and then use the formula (5.52) in order to get $V(r,T)$. As soon as we have $V(r,T)$, we can solve the problem (5.49) with $V(r,T)$ as the final condition in order to find the price of the option on a bond futures contract today.

It is possible to consider V_b as a state variable and let the bond option price depend on V_b and t. For example, suppose

$$dV_b = \mu V_b dt + \sigma V_b dX,$$

where μ and σ is constant. In this case, we get the Black–Scholes equation with independent variables t and V_b, and use the Black–Scholes formulae to find the prices of European bond options. However, because the bond price must be equal to the face value at time T_b, which is often referred to as the pull-to-par phenomenon, a bond has different features from an equity, especially when $t \approx T_b$ (see Fig. 1.3). Therefore, even though a model in the form $dV_b = \mu V_b dt + \sigma V_b dX$ can describe the dynamics of an equity well, it could not state that of a bond. Consequently, the bond price obtained in this way is expected to have a large error, especially when $T \approx T_b$. If the model is in the form

$$dV_b = \alpha(t)(1 - V_b)dt + \sigma(t)V_b dX,$$

where $\alpha(t) \to \infty$ and $\sigma(t) \to 0$ as $t \to T_b$, then the result might be much better because such a model guarantees that V_b has a unique value one at time T_b. Of course, in this case it might be necessary to get solutions by numerical methods. Another problem of pricing a bond option in this way is to assume that the short rate is constant throughout the whole life of the option. If T is not small, it is not a good assumption.

Promising to pay an amount E at time T is equivalent to issuing a bond maturing at time T with a face value E. Thus, a right to pay E for a bond with a maturity date T_b at time T is the same right to exchange a bond of a face value E with a maturity date T for another bond with a maturity date T_b at time T. Therefore, a bond option can be understood as an exchange option that allows the holder to exchange a bond maturing at time T for another bond maturing at time T_b. If a bond option is dealt with in this way, it may be necessary to choose a model so that at least the random variable for the bond maturing at time T has the property of "pull-to-par."

5.5.2 Interest Rate Swaps and Swaptions

This subsection is devoted to plain vanilla interest rate swaps and options on such swaps—swaptions. As an example, let us look at the following N-year swap on a notional principal Q between a bank and a company.[3] In the swap, the bank and the company agree that during the next N years, the company will pay the bank the interest payment on the notional principal Q at a fixed rate $r_s(N)$ semiannually and in return, the bank will pay the company the interest payment on the same principal at a floating rate at the same times. Here, the floating rate in many interest rate swap agreements is the 6-month London Interbank Offer Rate (LIBOR) prevailing 6 months before the payment date. When the swap is initiated, both parties do not need to pay any money. Thus, the contract has no value at initiation. The fixed rate $r_s(N)$ is called the swap rate for an N year swap and determined through negotiation by the two parties. Clearly, the company wants $r_s(N)$ to be as small as possible, and the bank prefers a higher $r_s(N)$. What is the value of $r_s(N)$ both parties can accept? $r_s(N)$ should be a rate such that the value of the swap at initiation is zero. In order to know what equation $r_s(N)$ should satisfy, we need to find out how the value of the swap is related to r_s, where r_s denotes a swap rate that might not equal $r_s(N)$.

Suppose the swap is initiated at time T and today's time is $t^* \geq T$. The interest payments are exchanged semiannually at time

$$t_k = T + k/2,$$

$k = k^* + 1, k^* + 2, \cdots, 2N$, where k^* is the integer part of $2(t^* - T)$. Suppose today the price of the zero-coupon bond with a face value of one dollar and with maturity date t_k is $Z(t^*; t_k)$. In the swap given above, the company will pay cash $Qr_s/2$ at time $t_k, k = k^* + 1, k^* + 2, \cdots, 2N$. The present value of this cash flow is

$$\sum_{k=k^*+1}^{2N} \frac{Qr_s}{2} Z(t^*; t_k).$$

[3] A swap can also be between two companies.

At the same times, the bank will pay the company an amount of cash $\frac{Q}{2}\bar{f}(t_{k-1}, t_{k-1}, t_k)$ at time t_k, $k = k^*+1, k^*+2, \cdots, 2N$, where $\bar{f}(t_{k-1}, t_{k-1}, t_k)$ is the forward rate for the period $[t_{k-1}, t_k]$ determined at time t_{k-1} and we define $t_{k^*} = T + k^*/2$. Because $t_{k^*} \le t^*$, $\bar{f}(t_{k^*}, t_{k^*}, t_{k^*+1})$ is known today and the present value of the first payment is

$$\frac{Q}{2}Z(t^*; t_{k^*+1})\bar{f}(t_{k^*}, t_{k^*}, t_{k^*+1}).$$

What is the present value of the other payments? Suppose we deposit Q in the bank at time t_{k^*+1} for a period $[t_{k^*+1}, t_{k^*+2}]$ at a floating rate $f(t_{k^*+1}, t_{k^*+1}, t_{k^*+2})$. At time t_k, $k = k^* + 2, k^* + 3, \cdots, 2N - 1$, we take the interest payment away and still leave Q in the bank for the next half year. In this way, we can generate a cash flow $\frac{Q}{2}\bar{f}(t_{k-1}, t_{k-1}, t_k)$ at time $t_k, k = k^* + 2, k^* + 3, \cdots, 2N - 1$ and cash $\frac{Q}{2}\bar{f}(t_{2N-1}, t_{2N-1}, t_{2N}) + Q$ at time t_{2N}. Therefore, the value of the other payments is the difference between Q at time t_{k^*+1} and Q at time $t_{2N} = T + N$. Written mathematically, the present value of the other payments is

$$QZ(t^*; t_{k^*+1}) - QZ(t^*; T + N).$$

This result also can be obtained analytically. In fact, from the relation (5.6) we know that the forward interest rate compounded semiannually at time t_k during a period $[t_k, t_{k+1}]$ is

$$\bar{f}(t_k, t_k, t_{k+1}) = 2\left[\frac{Z(t_k; t_k)}{Z(t_k; t_{k+1})} - 1\right],$$

where $t_{k+1} = t_k + 1/2$. Therefore at time t^*, the value of the cash flow $\frac{Q}{2}$ $\bar{f}(t_k, t_k, t_{k+1})$ at time t_{k+1}, $k = k^* + 1, k^* + 2, \cdots, 2N - 1$, is

$$\begin{aligned}
&\sum_{k=k^*+1}^{2N-1} \frac{Q}{2}\bar{f}(t_k, t_k, t_{k+1})Z(t^*; t_{k+1}) \\
&= Q\sum_{k=k^*+1}^{2N-1}\left[\frac{Z(t_k; t_k)}{Z(t_k; t_{k+1})} - 1\right]Z(t^*; t_{k+1}) \\
&= Q\sum_{k=k^*+1}^{2N-1}\frac{[Z(t_k; t_k) - Z(t_k; t_{k+1})]Z(t^*; t_k)}{Z(t_k; t_{k+1})Z(t^*; t_k)}Z(t^*; t_{k+1}) \\
&= Q\sum_{k=k^*+1}^{2N-1}[Z(t^*; t_k) - Z(t^*; t_{k+1})] \\
&= Q[Z(t^*; t_{k^*+1}) - Z(t^*; t_{2N})] \\
&= QZ(t^*; t_{k^*+1}) - QZ(t^*; T + N).
\end{aligned}$$

Let $V_s(t^*, r_s)$ be the present value of the swap to the company, which is the present value of the cash flow the company will receive minus the present value of the cash flow it will pay. From previous results, we arrive at

$$\begin{aligned}
V_s(t^*; r_s) &= \frac{Q}{2} Z(t^*; t_{k^*+1}) \bar{f}(t_{k^*}, t_{k^*}, t_{k^*+1}) + QZ(t^*; t_{k^*+1}) - QZ(t^*; T+N) \\
&\quad - \sum_{k=k^*+1}^{2N} \frac{Qr_s}{2} Z(t^*; t_k) \\
&= QZ(t^*; t_{k^*+1}) \left[1 + \frac{1}{2} \bar{f}(t_{k^*}, t_{k^*}, t_{k^*+1}) \right] \\
&\quad - Q \left[Z(t^*; T+N) + \sum_{k=k^*+1}^{2N} \frac{r_s}{2} Z(t^*; t_k) \right]. \qquad (5.53)
\end{aligned}$$

The expression $Q\left[Z(t^*; T+N) + \sum_{k=k^*+1}^{2N} \frac{r_s}{2} Z(t^*; t_k)\right]$ can be understood as the present value of a coupon-bearing bond, and the expression $QZ(t^*; t_{k^*+1}) \times \left[1 + \frac{1}{2}\bar{f}(t_{k^*}, t_{k^*}, t_{k^*+1})\right]$ is the present value of another coupon-bearing bond. Therefore, a swap can be seen as a combination of a long position in one coupon-bearing bond with a short position in another coupon-bearing bond.

Here, we also need to point out that the values of a swap to two parties have the same magnitude but opposite signs. Thus, the value of the swap mentioned above to the bank is

$$\begin{aligned}
&Q \left[Z(t^*; T_s + N) + \sum_{k=k^*+1}^{2N} \frac{r_s}{2} Z(t^*; t_k) \right] \\
&-QZ(t^*; t_{k^*+1}) \left[1 + \frac{1}{2} \bar{f}(t_{k^*}, t_{k^*}, t_{k^*+1}) \right].
\end{aligned}$$

In the case $t^* = T$, we have $k^* = 0, t_{k^*+1} = T + 1/2$ and

$$\bar{f}(T, T, T+1/2) = 2 \left[\frac{1}{Z(T; T+1/2)} - 1 \right],$$

that is,

$$Z(T; T+1/2) \left[1 + \frac{1}{2} \bar{f}(T, T, T+1/2) \right] = 1, \qquad (5.54)$$

so we have

$$V_s(T; r_s) = Q \left[1 - Z(T; T+N) - \frac{r_s}{2} \sum_{k=1}^{2N} Z(T; T+k/2) \right]. \qquad (5.55)$$

As we have stated, when the swap is initiated, the value of the swap should be zero. Therefore, for the fixed rate in the contract we obtain

$$r_s(N) = 2\frac{1 - Z(T;T+N)}{\sum\limits_{k=1}^{2N} Z(T;T+k/2)}. \tag{5.56}$$

Therefore, between the swap rate for an N-year swap and $Z(T;T+k/2), k = 1, 2, \cdots, 2N$, there is a simple relation: $r_s(N)$ can be determined by $Z(T;T+k/2)$, $k = 1, 2, \cdots 2N$. This relation is true for $N = 1/2, 1, 3/2, \cdots$. Actually, $Z(T;T+k/2)$, $k = 1, 2, \cdots, 2N$, can also be obtained recursively by

$$Z(T;T+k/2) = \frac{1 - \dfrac{r_s(k/2)}{2}\sum\limits_{i=1}^{k-1} Z(T;T+i/2)}{1 + \dfrac{r_s(k/2)}{2}} \tag{5.57}$$

if $r_s(k/2), k = 1, 2, \cdots, 2N$ are given. Therefore, knowing $r_s(k/2)$ for different k is the same as knowing the yield curve.

As we have mentioned, a swap can be understood as the difference between two different coupon-bearing bonds. From the expression (5.53), we know that the face values of both bonds are Q. The expiration date of one bond is t_{k^*+1} and it pays a coupon $\dfrac{Q}{2}\bar{f}(t_{k^*}, t_{k^*}, t_{k^*+1})$ at $t = t_{k^*+1}$. Let V_i denote the value of this bond. The expiration date of the other bond is $T + N$, and it pays coupons $\dfrac{Qr_s}{2}$ semiannually starting at $t = t_{k^*+1}$. Let V_o represent the value of the other bond. The value of swap $V_s(t)$ is equal to $V_i - V_o$. Any bond can be priced by the bond equation. In fact, $V_i(r,t)$ is the solution of the problem

$$\begin{cases} \dfrac{\partial V_i}{\partial t} + \dfrac{1}{2}w^2\dfrac{\partial^2 V_i}{\partial r^2} + (u - \lambda w)\dfrac{\partial V_i}{\partial r} - rV_i = 0, & r_l \le r \le r_u,\ t^* \le t \le t_{k^*+1}, \\ V_i(r, t_{k^*+1}) = Q[1 + \bar{f}(t_{k^*}, t_{k^*}, t_{k^*+1})/2], & r_l \le r \le r_u \end{cases}$$

and $V_o(r,t)$ is the solution of the problem

$$\begin{cases} \dfrac{\partial V_o}{\partial t} + \dfrac{1}{2}w^2\dfrac{\partial^2 V_o}{\partial r^2} + (u - \lambda w)\dfrac{\partial V_o}{\partial r} - rV_o + \sum\limits_{k=k^*+1}^{2N}\dfrac{Qr_s}{2}\delta(t - t_k) = 0, \\ \qquad\qquad r_l \le r \le r_u,\ t^* \le t \le T+N, \\ V_o(r, T+N) = Q, \qquad\qquad r_l \le r \le r_u. \end{cases}$$

Let $r = r^*$ today and let $\lambda(r,t)$ be chosen so that $V(r^*, t^*; t_k) = Z(t^*; t_k)$, $k = k^*+1, k^*+2, \cdots, 2N$, where $V(r, t; t_k)$ is the solution of the following problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}w^2 \dfrac{\partial^2 V}{\partial r^2} + (u - \lambda w)\dfrac{\partial V}{\partial r} - rV = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad r_l \le r \le r_u,\ t^* \le t \le t_k, \\ V(r, t_k; t_k) = 1, \qquad\qquad r_l \le r \le r_u, \end{cases} \tag{5.58}$$

then

$$\begin{aligned} V_o(r^*, t^*; r_s) &= QV(r^*, t^*; t_{2N}) + \frac{Qr_s}{2} \sum_{k=k^*+1}^{2N} V(r^*, t^*; t_k) \\ &= Q\left[Z(t^*; t_{2N}) + \frac{r_s}{2} \sum_{k=k^*+1}^{2N} Z(t^*; t_k)\right] \end{aligned}$$

and

$$\begin{aligned} V_i(r^*, t^*) &= Q\left[1 + \bar{f}(t_{k^*}, t_{k^*}, t_{k^*+1})/2\right] V(r^*, t^*; t_{k^*+1}) \\ &= Q\left[1 + \bar{f}(t_{k^*}, t_{k^*}, t_{k^*+1})/2\right] Z(t^*; t_{k^*+1}). \end{aligned}$$

From these two expressions, we can see that

$$V_i(r^*, t^*) - V_o(r^*, t^*; r_s)$$

will have the same value as that given by the expression (5.53). When the bond equation is used, the value of the swap is not only given at $r = r^*$, and V_s is considered as a function of r and t, i.e., $V_s = V_s(r, t)$. The value of the swap is also dependent on the value of r_s. Therefore sometimes $V_s(r, t)$ is written as $V_s(r, t; r_s)$, where r_s is a parameter.

Indeed, in order to find $V_s(r, t)$, it is not necessary to find $V_i(r, t)$ and $V_o(r, t)$ separately; instead, we only need to solve

$$\begin{cases} \dfrac{\partial V_s}{\partial t} + \dfrac{1}{2}w^2 \dfrac{\partial^2 V_s}{\partial r^2} + (u - \lambda w)\dfrac{\partial V_s}{\partial r} - rV_s - \displaystyle\sum_{k=k^*+1}^{2N} \dfrac{Qr_s}{2}\delta(t - t_k) \\ \qquad\qquad + Q\left[1 + \dfrac{\bar{f}(t_{k^*}, t_{k^*}, t_{k^*+1})}{2}\right]\delta(t - t_{k^*+1}) = 0, \\ \qquad\qquad\qquad\qquad r_l \le r \le r_u, \quad t^* \le t \le T + N, \\ V_s(r, T + N) = -Q, \quad r_l \le r \le r_u. \end{cases} \tag{5.59}$$

It is not difficult to show this conclusion, and we leave this proof to the reader as problem 15. Now we can find the value of a swap either using the formula (5.53) or solving the problem (5.59) and get the same answer. Many people will choose to calculate the value of the swap by the expression (5.53) because it is simple. Why do we need to consider problem (5.59)? It can provide some information on $\dfrac{\partial V_s(r^*, t)}{\partial r}$ and the bond equation will be useful when pricing a swaption by solving bond equations.

An option on a swap, or a swaption, is a contract to give the holder the right to enter into a certain interest rate swap by a certain time in the future. Consider a European swaption. Its holder has the right to choose if he should have an N-year swap at time T under which he will pay interest at a fixed rate r_{se} (the so-called exercise swap rate) and receive interest payment at a floating rate. Let r_s' be the N-year swap rate at time T, which can have infinitely many possible values. If $r_{se} < r_s'$, then the holder will choose to exercise the swaption because the value of a swap with a swap rate r_s' at time T is 0 and the value of a swap with a swap rate $r_{se} < r_s'$ should be positive, but the holder can enter into such a swap without paying any money. If $r_{se} > r_s'$, then the holder will choose not to exercise the option because the swap rate is lower on the market.

Such an option interests companies who plan to enter into a swap as a fixed rate payer because the swaption provides the companies with a guarantee that the fixed rate of interest they will pay on a loan will not exceed r_{se}.

According to the result (5.55), at time T, the values of the swaps with swap rates r_s' and r_{se} to the company are

$$V_s(T;r_s') = Q\left[1 - Z(T;T+N) - \sum_{k=1}^{2N} \frac{r_s'}{2} Z(T;t_k)\right]$$

and

$$V_s(T;r_{se}) = Q\left[1 - Z(T;T+N) - \sum_{k=1}^{2N} \frac{r_{se}}{2} Z(T;t_k)\right]$$

respectively. If $r_{se} \le r_s'$, then the value of the swaption V at time T is

$$V(r_s',T) = V_s(T;r_{se}) - V_s(T;r_s') = Q\frac{r_s' - r_{se}}{2}\sum_{k=1}^{2N} Z(T;t_k);$$

while if $r_{se} > r_s'$, then $V(r_s',T) = 0$. Consequently, the payoff of the swaption is

$$V(r_s',T) = \frac{Q}{2}\sum_{k=1}^{2N} Z(T;t_k)\max(r_s' - r_{se}, 0). \tag{5.60}$$

Suppose that at time T, r_s' has a lognormal distribution with the following probability density function

$$G(r_s') = \frac{1}{r_s'\sigma\sqrt{2\pi(T-t)}}\mathrm{e}^{-\left[\ln(r_s'/r_s)+\sigma^2(T-t)/2\right]^2/2\sigma^2(T-t)},$$

where r_s is the swap rate at time t. This model is often referred to as Black's model (see [8]). This probability density function is the probability density function (2.85) with $r - D_0 = 0$. Thus, the expectation of $\max(r_s' - r_{se}, 0)$ is $\mathrm{e}^{r(T-t)}$ times the price of a call option with $r - D_0 = 0$. That is

$$\mathrm{E}\left[\max(r_s' - r_{se}, 0)\right] = r_s N\left(\frac{\ln(r_s/r_{se}) + \sigma^2(T-t)/2}{\sigma\sqrt{T-t}}\right) - r_{se} N\left(\frac{\ln(r_s/r_{se}) - \sigma^2(T-t)/2}{\sigma\sqrt{T-t}}\right).$$

Therefore at time t, the value of the payoff is

$$\begin{aligned}
&Z(t;T)\frac{Q}{2}\sum_{k=1}^{2N} Z(T;t_k) \\
&\quad\times\left[r_s N\left(\frac{\ln\dfrac{r_s}{r_{se}} + \dfrac{\sigma^2(T-t)}{2}}{\sigma\sqrt{T-t}}\right) - r_{se} N\left(\frac{\ln\dfrac{r_s}{r_{se}} - \dfrac{\sigma^2(T-t)}{2}}{\sigma\sqrt{T-t}}\right)\right] \\
&= \frac{Q}{2}\sum_{k=1}^{2N} Z(t;t_k) \\
&\quad\times\left[r_s N\left(\frac{\ln\dfrac{r_s}{r_{se}} + \dfrac{\sigma^2(T-t)}{2}}{\sigma\sqrt{T-t}}\right) - r_{se} N\left(\frac{\ln\dfrac{r_s}{r_{se}} - \dfrac{\sigma^2(T-t)}{2}}{\sigma\sqrt{T-t}}\right)\right],
\end{aligned}$$

where $Z(t;T)$ is the discounting factor between t and T and we have used the relation $Z(t;T)\,Z(T;t_k) = Z(t;t_k)$. European swaptions are frequently valued in this way. Obviously, it is an approximate method.

We may also evaluate the European swaption by solving bond equations. As is given by the formula (5.60), the payoff of the swaption is

$$\frac{Q}{2}\sum_{k=1}^{2N} Z(T;T+k/2)\max(r_s' - r_{se}, 0).$$

At time T, $r_s(N)$ is determined by the formula (5.56), i.e., r_s' is given by

$$2\frac{1 - Z(T;T+N)}{\sum\limits_{k=1}^{2N} Z(T;T+k/2)}.$$

Thus the payoff of the swaption can be rewritten as

$$\begin{aligned}
&\frac{Q}{2}\sum_{k=1}^{2N} Z(T;T+k/2)\max\left(2\frac{1 - Z(T;T+N)}{\sum\limits_{k=1}^{2N} Z(T;T+k/2)} - r_{se},\ 0\right) \\
&= Q\max\left(1 - Z(T;T+N) - \frac{r_{se}}{2}\sum_{k=1}^{2N} Z(T;T+k/2),\ 0\right).
\end{aligned}$$

Let $V_{so}(r,t)$ denote the value of the swaption. In this case the procedure of determining swaption price is divided into two steps. The first step is to determine the value of swap with r_{se} as the swap rate for all $r \in [r_l, r_u]$, $V_s(r,t;r_{se},T)$, where r_{se} and T are parameters and T denotes the starting date of the swap, and the second step is to obtain the payoff of swaption and to find the value of swaption. In order to get $V_s(r,t;r_{se},T)$, we need to solve the following problem

$$\begin{cases} \dfrac{\partial V_s}{\partial t} + \dfrac{1}{2}w^2 \dfrac{\partial^2 V_s}{\partial r^2} + (u-\lambda w)\dfrac{\partial V_s}{\partial r} - rV_s \\ \qquad\qquad -\dfrac{Qr_{se}}{2}\displaystyle\sum_{k=1}^{2N}\delta(t-T-k/2) + Q\delta(t-T) = 0, \\ \qquad\qquad r_l \le r \le r_u, \quad T \le t \le T+N, \\ V_s(r,T+N) = -Q, \ r_l \le r \le r_u. \end{cases} \tag{5.61}$$

This problem actually is Eq. (5.59) with $k^* = 0$, $t^* = T$, and replacing

$$Q\left[1 + \frac{\bar{f}(t_{k^*}, t_{k^*}, t_{k^*+1})}{2}\right]\delta(t - t_{k^*+1})$$

by $Q\delta(t-T)$ because of the relation (5.54). After we obtain $V_s(r,T;r_{se},T)$, we need to get the payoff of the swaption

$$V_{so}(r,T) = \max\left(V_s(r,T;r_{se},T), 0\right)$$

and then to solve the following bond equation from $t = T$ to $t = 0$ in order to find the value of swaption:

$$\begin{cases} \dfrac{\partial V_{so}}{\partial t} + \dfrac{1}{2}w^2 \dfrac{\partial^2 V_{so}}{\partial r^2} + (u-\lambda w)\dfrac{\partial V_{so}}{\partial r} - rV_{so} = 0, & r_l \le r \le r_u, \\ & 0 \le t \le T, \\ V_{so}(r,T) = \max\left(V_s(r,T;r_{se},T), 0\right), & r_l \le r \le r_u. \end{cases} \tag{5.62}$$

The value of the swaption today is $V_{so}(r^*, 0)$, where r^* is the interest rate today. For American swaptions we can use a similar treatment to find their prices.

5.5.3 Interest Rate Caps, Floors, and Collars

An interest rate cap is a contract whose holder will receive some money from the issuer if the floating rate exceeds a certain level r_c, where r_c is called the cap rate. Therefore, interest rate caps can be used to provide insurance against the rate of interest on a floating-rate loan rising above a certain level.

For example, someone is going to get an N-year floating-rate loan with a principal amount Q at time t^*. The borrower will pay interest quarterly at time

$$t_k = t^* + k/4, \quad k = 1, 2, \cdots, 4N.$$

The rate is a 3-month LIBOR determined at time t_{k-1} for the period $[t_{k-1}, t_k]$, where we define $t_0 = t^*$. The LIBOR is a forward interest rate. According to the notation given in Sect. 5.1, $\bar{f}(t_{k-1}, t_{k-1}, t_k)$ stands for this rate. In what follows, we use the notation $\bar{f}_{k-1}$ instead of $\bar{f}(t_{k-1}, t_{k-1}, t_k)$ for brevity. The borrower is worrying that he will pay too much interest if the 3-month LIBOR becomes very high during the period $[t^*, t^* + N]$. Therefore, he is interested in such a cap: it starts from t^* and lasts N years, and at time t_k, the issuer of the cap will pay the holder an amount of cash $Q \max(\bar{f}_{k-1} - r_c, 0)/4$. Suppose he purchases this cap. Then when $\bar{f}_{k-1} < r_c$, he will pay interest payment on the loan $Q\bar{f}_{k-1}/4$ and receive zero from the issuer of the cap; whereas $\bar{f}_{k-1} > r_c$, his actual payment is $Qr_c/4$ because he receives $Q \max(\bar{f}_{k-1} - r_c, 0)/4$ from the cap. Hence the cap provides insurance against the interest rate on the floating-rate loan rising above an upper bound r_c.

How much should be paid in order to obtain such an insurance? The present value of the payment $Q \max(\bar{f}_{k-1} - r_c, 0)/4$ at time t_k is actually the value of a call option with expiry t_k. This call option is usually called the kth caplet. The LIBOR $\bar{f}_{k-1}$ is a forward rate determined at time t_{k-1} for the period $[t_{k-1}, t_k]$, so an amount $Q \max(\bar{f}_{k-1} - r_c, 0)/4$ at time t_k is equivalent to the amount

$$\frac{Q}{4(1 + \bar{f}_{k-1}/4)} \max(\bar{f}_{k-1} - r_c, 0) = \max\left(Q - Q\frac{1 + r_c/4}{1 + \bar{f}_{k-1}/4}, 0\right)$$

at time t_{k-1}. A loan with a face value $Q\,(1 + r_c/4)$ and maturity t_k is worth $Q(1 + r_c/4)/(1 + \bar{f}_{k-1}/4)$ at time t_{k-1} for any $\bar{f}_{k-1}$. Therefore, a caplet with a payoff $Q \max(\bar{f}_{k-1} - r_c, 0)/4$ at time t_k is equivalent to a put option with maturity t_{k-1} and a strike price Q on a zero-coupon bond with maturity t_k and a face value $Q\,(1 + r_c/4)$. At time $t^* (= t_0)$, the value of the first caplet is equal to a known value $\dfrac{Q}{4(1 + \bar{f}_0/4)} \max(\bar{f}_0 - r_c, 0)$. Usually, this value is excluded from the premium and there is no payment at time t_1 even if the LIBOR is greater than r_c. Thus, a cap comprises $4N - 1$ put options on zero-coupon bonds. Because a bond or an option on a bond can be seen as a derivative on the short rate r, their values can be calculated by the bond equation. Let the value of the bond with maturity t_k be $V_{bk}(r, t)$. Then, $V_{bk}(r, t)$ is the solution of the problem

$$\begin{cases} \dfrac{\partial V_{bk}}{\partial t} + \dfrac{1}{2}w^2 \dfrac{\partial^2 V_{bk}}{\partial r^2} + (u - \lambda w)\dfrac{\partial V_{bk}}{\partial r} - rV_{bk} = 0, & r_l \le r \le r_u, \\ & t_{k-1} \le t \le t_k, \\ V_{bk}(r, t_k) = (1 + r_c/4)\, Q, & r_l \le r \le r_u, \end{cases} \tag{5.63}$$

where $k = 2, 3, \cdots, 4N$. Let $V_c(r,t)$ be the solution of the problem

$$\begin{cases} \dfrac{\partial V_c}{\partial t} + \dfrac{1}{2}w^2 \dfrac{\partial^2 V_c}{\partial r^2} + (u - \lambda w)\dfrac{\partial V_c}{\partial r} - rV_c \\ \qquad + \sum\limits_{k=2}^{4N} \max(Q - V_{bk}(r, t_{k-1}), 0)\delta(t - t_{k-1}) = 0, \\ \qquad\qquad r_l \le r \le r_u, \quad t^* \le t \le t_{4N-1}, \\ V_c(r, t_{4N-1}) = 0, \qquad r_l \le r \le r_u. \end{cases} \tag{5.64}$$

Then, $V_c(r,t^*)$ gives the value of the cap — the total value of the $4N-1$ put options at time t^* and the premium of the cap is given by

$$V_c(r^*, t^*),$$

where r^* is the short rate at time t^*.

There are some other derivatives analogous to interest rate caps, such as interest rate floors and collars. A holder of a floor will receive some money from the issuer if the floating rate is below a certain level r_f, which is called the floor rate. If a borrower of a floating-rate loan believes that the floating rate will never be less than the lower bound r_f, then he may want to write such a floor. This is because he will get some money from writing a floor but, according to his opinion, he will not actually pay any money to the holder of the floor. Therefore, he hopes that he can reduce his expenses on the loan through writing a floor. If we assume that the floor starts at t^* and lasts N years, that the floating rate is 3-month LIBOR, and that the money will be paid quarterly at time t_k, $k = 2, 3, \cdots, 4N$, then the value of the floor is the sum of $4N-1$ floorlets that are call options on zero-coupon bonds. In order to determine the premium, we can first solve the problem

$$\begin{cases} \dfrac{\partial V_{bk}}{\partial t} + \dfrac{1}{2}w^2 \dfrac{\partial^2 V_{bk}}{\partial r^2} + (u - \lambda w)\dfrac{\partial V_{bk}}{\partial r} - rV_{bk} = 0, \quad r_l \le r \le r_u, \\ \qquad\qquad t_{k-1} \le t \le t_k, \\ V_{bk}(r, t_k) = (1 + r_f/4)\, Q, \qquad r_l \le r \le r_u \end{cases}$$

and get $V_{bk}(r, t_{k-1})$. Based on $V_{bk}(r, t_{k-1})$, we then can determine the solution of the problem

$$\begin{cases} \dfrac{\partial V_f}{\partial t} + \dfrac{1}{2}w^2 \dfrac{\partial^2 V_f}{\partial r^2} + (u - \lambda w)\dfrac{\partial V_f}{\partial r} - rV_f \\ \qquad + \sum\limits_{k=2}^{4N} \max(V_{bk}(r, t_{k-1}) - Q, 0)\delta(t - t_{k-1}) = 0, \\ \qquad\qquad r_l \le r \le r_u, \quad t^* \le t \le t_{4N-1}, \\ V_f(r, t_{4N-1}) = 0, \qquad r_l \le r \le r_u \end{cases}$$

and the value $V_f(r^*, t^*)$ gives the premium of the floor. The derivation of this conclusion is left for readers as Problem 17.

A collar specifies both the upper bound r_c and the lower bound r_f. It may be understood as a combination of a long position in a cap with a short position in a floor. The value of a collar V_{co} is

$$V_{co} = V_c - V_f.$$

Usually, we choose r_c and r_f such that

$$V_c = V_f \quad \text{or} \quad V_{co} = 0.$$

It is clear that a portfolio of a collar and the original floating-rate loan is equivalent to a new loan with a floating rate in $[r_c, r_f]$. If

$$r_c = r_f,$$

then the collar actually becomes a swap based on 3-month LIBOR and with $4N - 1$ exchanges of payments. There exist other interest rate derivatives such as captions and floortions. Their evaluations are similar to what we have discussed.

5.6 Multi-Factor Interest Rate Models

5.6.1 Brief Description of Several Multi-Factor Interest Rate Models

Sometimes, it is necessary to assume that interest rate derivatives depend on not only the short rate r, but also some other random state variables. Because volatility is always a dominant factor in determining the prices of bonds and options, we need to have a more accurate model for volatility. It may be necessary to consider the interest rate volatility as a random variable. Fong and Vasicek [30] proposed such a two-factor model. In their model, they postulated that both the short rate r and the variance υ of the short rate are stochastic state variables and assumed

$$\begin{aligned} dr &= (\bar{\mu} - \gamma r)dt + \sqrt{\upsilon}dX, \\ d\upsilon &= (\nu - \eta\upsilon)dt + \xi\sqrt{\upsilon}dX_\upsilon, \\ \mathrm{E}\left[dX dX_\upsilon\right] &= \rho dt, \end{aligned}$$

where $\bar{\mu}, \gamma, \nu, \eta, \xi$ are constants and dX and dX_υ are two standard Wiener processes. As we can see in this model, the stochastic equation for r is the same as that in the Vasicek model, and r could become negative. Here, not only the short rate but also the variance possess the mean reversion property. In this case, Eq. (2.34) can be written as

$$\frac{\partial V}{\partial t}+\frac{1}{2}\upsilon\frac{\partial^2 V}{\partial r^2}+\rho\xi\upsilon\frac{\partial^2 V}{\partial r\partial \upsilon}+\frac{1}{2}\xi^2\upsilon\frac{\partial^2 V}{\partial \upsilon^2}+(\bar{\mu}-\gamma r-\bar{\lambda}\upsilon)\frac{\partial V}{\partial r}$$
$$+\left[\nu-(\eta+\bar{\lambda}_\upsilon\xi)\upsilon\right]\frac{\partial V}{\partial \upsilon}-rV=0,$$

where the market prices of risk for r and υ are $\bar{\lambda}\sqrt{\upsilon}$ and $\bar{\lambda}_\upsilon\sqrt{\upsilon}$, respectively, $\bar{\lambda}$ and $\bar{\lambda}_\upsilon$ being constants.

Brenman and Schwartz [13] considered another two-factor model. In their model, the two random state variables are the short-term interest rate r and the long-term interest rate l. They assumed

$$dr=u(r,l,t)dt+w(r,l,t)dX,$$

$$dl=u_l(r,l,t)dt+w_l(r,l,t)dX_l,$$

$$\mathrm{E}\left[dX\,dX_l\right]=\rho(r,l,t)dt,$$

where dX and dX_l are the standard Wiener processes. According to Eq. (2.34), any derivative dependent on r and l should satisfy

$$\frac{\partial V}{\partial t}+\frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2}+\rho w w_l\frac{\partial^2 V}{\partial r\partial l}+\frac{1}{2}w_l^2\frac{\partial^2 V}{\partial l^2}+(u-\lambda w)\frac{\partial V}{\partial r}+(u_l-\lambda_l w_l)\frac{\partial V}{\partial l}-rV=0.$$

For other models, for example, see [2, 19, 57]. From these models, we can have the corresponding partial differential equations. Any reader who is interested in knowing more about these models and other models is suggested to consult these papers and the book [47] by James and Webber.

In order to use these models to price derivatives, we need to determine these market prices of risk, which is similar to what we have done for one-factor models. Also, if we make some modifications on these models so that some conditions similar to the conditions (5.45) and (5.46) hold, then unique solutions of these equations can be obtained only by requiring final conditions.

Not only can the interest rates and their variances be taken as state variables. Heath et al. [38, 39, 40] suggested a model where the driving state variable of the model is $F(t,T)$, the forward rate at time t for instantaneous borrowing at a later time T. They assume

$$dF(t,T)=\alpha_F(t,T)dt+\sum_{i=1}^{n}\sigma_F^i(t,T)dX_i,$$

where dX_i is the ith Wiener process, and the n Wiener processes are independent. In this sense, it can be called a multi-factor model. Jarrow wrote a monograph on this method in 1996 (see [48]). Any reader who wants to know its details is referred to that book.

5.6.2 Reducing the Randomness of a Zero-Coupon Bond Curve to That of a Few Zero-Coupon Bonds

As we know, if we have an effective way to describe the randomness of a zero-coupon bond curve, then we can have an effective model for interest rate derivatives such as bond options or swaptions. In this and the next subsections, we discuss a three-factor model, which can be easily used in practice and generalized to the cases with more factors without any difficulty.

As we have done in Sect. 5.1, let $Z(t; t+T)$ denote the price of a T-year zero-coupon bond with a face value of one dollar at time t, and we use the notation $Z_i(t) = Z(t; t+T_i)$ for any T_i, $i = 0, 1, \cdots, N$. Here, we also assume $T_i < T_{i+1}$, for $i = 0, 1, \cdots, N-1$, and $T_0 = 0$. According to $Z_i(t), i = 0, 1, \cdots, N$, we can have an interpolation function $\bar{Z}(T; t)$ for $T \in [0, T_N]$ by requiring $\bar{Z}(T; t)$ to be a continuous function with continuous first and second derivatives in the form:

$$\bar{Z}(T;t) = \begin{cases} a_{0,1} + a_{1,1}T + a_{2,1}T^2, & 0 \le T \le T_1, \\ a_{0,i} + a_{1,i}T + a_{2,i}T^2 + a_{3,i}T^3, & T_{i-1} \le T \le T_i, \\ & i = 2, \cdots, N-1, \\ a_{0,N} + a_{1,N}T + a_{2,N}T^2, & T_{N-1} \le T \le T_N. \end{cases} \tag{5.65}$$

In this function, there are $4(N-2)+6 = 4N-2$ coefficients. Because we have $N+1$ conditions on the value of the function

$$\bar{Z}(T_i; t) = Z_i(t), \quad i = 0, 1, \cdots, N$$

and $3(N-1)$ continuity conditions on the function, first and second derivatives at $T_1, T_2, \cdots, T_{N-1}$, the total number of conditions is also $4N-2$. Therefore, it is possible that those coefficients in the expression (5.65) can be determined by these conditions uniquely. This interpolation method is called a cubic spline interpolation, and the way of determining the coefficients in the expression (5.65) will be given in Sect. 6.1.1. A zero-coupon bond curve is a monotone function with respect to T. If for a set of $Z_i(t)$, $i = 0, 1, \cdots, N$, the expression (5.65) does not possess this property, the approximation needs to be modified so that the monotonicity is guaranteed. This is important in practice.

We assume that $\bar{Z}(T; t)$ is a very good approximation to the zero-coupon bond curve $Z(t; t+T)$. In this way, a random curve is reduced to N random variables with a small error.

Now let us reduce the number of random variables from N to K by the principal component analysis. Suppose that we have N random variables

$$S_i, \quad i = 1, 2, \cdots, N$$

and the covariance between S_i and S_j is

$$\mathrm{Cov}[S_i S_j] = b_i b_j \rho_{i,j}, \quad i, j = 1, 2, \cdots, N,$$

where $-1 \le \rho_{i,j} = \rho_{j,i} \le 1$ and $\rho_{i,i} = 1$. Let

$$c_i^2 \quad \text{and} \quad \mathbf{a}_i = \begin{bmatrix} a_{i,1} \\ a_{i,2} \\ \vdots \\ a_{i,N} \end{bmatrix}, \quad i = 1, 2, \cdots, N,$$

be the eigenvalues and unit eigenvectors of the covariance matrix

$$\mathbf{B} = \begin{bmatrix} b_1^2 & b_1 b_2 \rho_{1,2} & \cdots & b_1 b_N \rho_{1,N} \\ b_2 b_1 \rho_{2,1} & b_2^2 & \cdots & b_2 b_N \rho_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ b_N b_1 \rho_{N,1} & b_N b_2 \rho_{N,2} & \cdots & b_N^2 \end{bmatrix}.$$

That is, there is the following relation:

$$\mathbf{B}\mathbf{A}^T = \mathbf{A}^T\mathbf{C} \quad \text{or} \quad \mathbf{A}\mathbf{B}\mathbf{A}^T = \mathbf{C},$$

where $\mathbf{A}^T$ is the transpose of $\mathbf{A}$ and

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,N} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N,1} & a_{N,2} & \cdots & a_{N,N} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} c_1^2 & 0 & \cdots & 0 \\ 0 & c_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_N^2 \end{bmatrix}.$$

Here $\mathbf{A}$ is an orthogonal matrix, i.e., $\mathbf{A}\mathbf{A}^T = \mathbf{I}$ because $\mathbf{B}$ is a symmetric matrix.

Let $\bar{S}_1, \bar{S}_2, \cdots, \bar{S}_N$ be N other random variables defined by

$$\begin{bmatrix} \bar{S}_1 \\ \bar{S}_2 \\ \vdots \\ \bar{S}_N \end{bmatrix} = \mathbf{A} \begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_N \end{bmatrix}.$$

For simplicity, this relation can be written as

$$\bar{\mathbf{S}} = \mathbf{A}\mathbf{S},$$

where

$$\bar{\mathbf{S}} = \begin{bmatrix} \bar{S}_1 \\ \bar{S}_2 \\ \vdots \\ \bar{S}_N \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_N \end{bmatrix}.$$

Then

$$\begin{aligned}
\mathrm{Cov}\left[\bar{S}_i\bar{S}_j\right] &= \mathrm{E}\left[\left(\bar{S}_i - \mathrm{E}\left[\bar{S}_i\right]\right)\left(\bar{S}_j - \mathrm{E}\left[\bar{S}_j\right]\right)\right] \\
&= \mathrm{E}\left[\left(\sum_{k=1}^{N} a_{ik}\left(S_k - \mathrm{E}\left[S_k\right]\right)\right)\left(\sum_{l=1}^{N} a_{jl}\left(S_l - \mathrm{E}\left[S_l\right]\right)\right)\right] \\
&= \sum_{k=1}^{N}\sum_{l=1}^{N} a_{ik}a_{jl}\ \mathrm{Cov}\left[S_k S_l\right] \\
&= \begin{cases} 0, & i \neq j, \\ c_i^2, & i = j. \end{cases}
\end{aligned}$$

That is, $\mathbf{C}$ is the covariance matrix of the random vector $\bar{\mathbf{S}}$. We furthermore suppose that

$$c_i^2 \geq c_j^2 \quad \text{for} \quad i < j$$

and

$$c_i^2 \ll c_K^2, \quad i = K+1, \cdots, N.$$

Assume that on some day

$$\mathbf{S} = \begin{bmatrix} S_1^* \\ S_2^* \\ \vdots \\ S_N^* \end{bmatrix} \equiv \mathbf{S}^*$$

and

$$\bar{\mathbf{S}} = \mathbf{A}\begin{bmatrix} S_1^* \\ S_2^* \\ \vdots \\ S_N^* \end{bmatrix} = \begin{bmatrix} \bar{S}_1^* \\ \bar{S}_2^* \\ \vdots \\ \bar{S}_N^* \end{bmatrix} \equiv \bar{\mathbf{S}}^*.$$

Because c_i^2, $i = K+1, \cdots, N$ are very small, for a period starting from that day, we neglect the uncertainty caused by the last $N-K$ components of $\bar{\mathbf{S}}$. That is, we assume that in this period $\bar{\mathbf{S}}$ has the following form:

$$\bar{\mathbf{S}} = \begin{bmatrix} \bar{S}_1 \\ \vdots \\ \bar{S}_K \\ \bar{S}_{K+1}^* \\ \vdots \\ \bar{S}_N^* \end{bmatrix},$$

where $\bar{S}_1, \cdots, \bar{S}_K$ can take all possible values. In this case

$$\mathbf{S} = \mathbf{A}^T \begin{bmatrix} \bar{S}_1 \\ \vdots \\ \bar{S}_K \\ \bar{S}^*_{K+1} \\ \vdots \\ \bar{S}^*_N \end{bmatrix}. \tag{5.66}$$

Under this assumption, among $S_1, S_2, \cdots, S_N$, only K components are independent. Suppose

$$\begin{vmatrix} a_{1,1} & a_{2,1} & \cdots & a_{K,1} \\ a_{1,2} & a_{2,2} & \cdots & a_{K,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,K} & a_{2,K} & \cdots & a_{K,K} \end{vmatrix} \neq 0.$$

Then, we can choose $S_1, S_2, \cdots, S_K$ as independent components. Rewrite Eq. (5.66) as

$$\begin{bmatrix} S_1 \\ \vdots \\ S_K \end{bmatrix} = \mathbf{A}_1^T \begin{bmatrix} \bar{S}_1 \\ \vdots \\ \bar{S}_K \end{bmatrix} + \mathbf{A}_2^T \begin{bmatrix} \bar{S}^*_{K+1} \\ \vdots \\ \bar{S}^*_N \end{bmatrix},$$

$$\begin{bmatrix} S_{K+1} \\ \vdots \\ S_N \end{bmatrix} = \mathbf{A}_3^T \begin{bmatrix} \bar{S}_1 \\ \vdots \\ \bar{S}_K \end{bmatrix} + \mathbf{A}_4^T \begin{bmatrix} \bar{S}^*_{K+1} \\ \vdots \\ \bar{S}^*_N \end{bmatrix},$$

where

$$\mathbf{A}_1^T = \begin{bmatrix} a_{1,1} & \cdots & a_{K,1} \\ \vdots & \ddots & \vdots \\ a_{1,K} & \cdots & a_{K,K} \end{bmatrix}, \quad \mathbf{A}_2^T = \begin{bmatrix} a_{K+1,1} & \cdots & a_{N,1} \\ \vdots & \ddots & \vdots \\ a_{K+1,K} & \cdots & a_{N,K} \end{bmatrix},$$

$$\mathbf{A}_3^T = \begin{bmatrix} a_{1,K+1} & \cdots & a_{K,K+1} \\ \vdots & \ddots & \vdots \\ a_{1,N} & \cdots & a_{K,N} \end{bmatrix}, \quad \mathbf{A}_4^T = \begin{bmatrix} a_{K+1,K+1} & \cdots & a_{N,K+1} \\ \vdots & \ddots & \vdots \\ a_{K+1,N} & \cdots & a_{N,N} \end{bmatrix}.$$

Then, for $S_{K+1}, \cdots, S_N$, we have

$$\begin{bmatrix} S_{K+1} \\ \vdots \\ S_N \end{bmatrix} = \mathbf{A}_3^T \left(\mathbf{A}_1^T\right)^{-1} \left(\begin{bmatrix} S_1 \\ \vdots \\ S_K \end{bmatrix} - \mathbf{A}_2^T \begin{bmatrix} \bar{S}^*_{K+1} \\ \vdots \\ \bar{S}^*_N \end{bmatrix} \right) + \mathbf{A}_4^T \begin{bmatrix} \bar{S}^*_{K+1} \\ \vdots \\ \bar{S}^*_N \end{bmatrix}. \tag{5.67}$$

Thus, for given $S_1, \cdots, S_K$, using the relation (5.67) we can get all other components of a vector $\mathbf{S}$. Consequently, the relation (5.67) defines a class of vectors with K parameters. That is, by the relation (5.67), we actually determine a class of $\mathbf{S}$, where only $S_1, \cdots, S_K$ are independent. Here, we take $S_1, \cdots, S_K$ as independent components. However, it is also possible to choose other K components as independent components.

Letting $S_i = Z_i/T_i$, $i = 1, 2, \cdots, N$, by the principal component analysis described above, we can find a class of vectors $[Z_1/T_1, \cdots, Z_N/T_N]^T$ with K parameters[4] and using the cubic spline interpolation given at the beginning of this subsection, we can further determine the curve $\bar{Z}(T; t)$ for $T \in [0, T_N]$. From the books by Jarrow [48], Hull [43], James and Webber [47], and Wilmott [83], we know that K usually is equal to three or four for the random curves related to interest rates. Thus, all the curves determined by the relation (5.67) form a class of curves with three or four parameters. The zero-coupon bond curve at that day is one of such curves, and the projections of any vector $\mathbf{S}$ determined by the relation (5.67) on the eigenvectors corresponding to the eigenvalues $c_{K+1}, \cdots, c_N$ are the same as those of $\mathbf{S}^*$. Those projections are different for different $\mathbf{S}^*$, so this is a feature belonging to $\mathbf{S}^*$. It is clear that the class of curves with such a feature needs to be considered most for derivative-pricing problems. Hence, when $K = 3$ or 4, the class contains all possible and need-to-be-considered-most zero-coupon bond curves. As soon as we have a zero-coupon bond curve, we can determine various interest rates at t, including the short rate $r(Z_1, \cdots, Z_K, t)$ at time t. For example, for $r(Z_1, \cdots, Z_K, t)$, we have

$$r(Z_1, \cdots, Z_K, t) = -\left.\frac{\partial \bar{Z}(T; t)}{\partial T}\right|_{T=0}. \tag{5.68}$$

5.6.3 A Three-Factor Interest Rate Model and the Equation for Interest Rate Derivatives

Suppose Z_1, Z_2 and Z_3 are prices of zero-coupon bonds with maturities T_1, T_2, and T_3, respectively. Assume $T_1 < T_2 < T_3$, which implies the relations $1 \geq Z_1 \geq Z_2 \geq Z_3$. Furthermore, we assume $Z_1 \geq Z_{1,l}$, $Z_2 \geq Z_{2,l}$ and $Z_3 \geq Z_{3,l}$, where $Z_{1,l} \geq Z_{2,l} \geq Z_{3,l} \geq 0$. Z_1, Z_2 and Z_3 are random variables and satisfy the system of stochastic differential equations:

$$dZ_i = \mu_i \left(Z_1, Z_2, Z_3, t\right) dt + \sigma_i \left(Z_1, Z_2, Z_3, t\right) dX_i, \quad i = 1, 2, 3$$

on the domain Ω: $\{Z_{1,l} \leq Z_1 \leq 1,\ Z_{2,l} \leq Z_2 \leq Z_1,\ Z_{3,l} \leq Z_3 \leq Z_2\}$. dX_i are the Wiener processes and $\mathrm{E}\left[dX_i dX_j\right] = \rho_{i,j} dt$ with $-1 \leq \rho_{i,j} \leq 1$. The coefficients μ_i, σ_i and their first- and second-order derivatives are assumed to be bounded on the domain Ω. On the six boundaries of Ω, the following conditions hold:

[4] If the conditions $Z_i \geq Z_{i+1}$, $i = 0, 1, \cdots N-1$ are not satisfied, then some modification needs to be done in order to guarantee the monotonicity.

(i) On surface I: $\{Z_1 = Z_{1,l},\ Z_{2,l} \le Z_2 \le Z_1,\ Z_{3,l} \le Z_3 \le Z_2\}$,

$$\begin{cases} \mu_1 (Z_{1,l}, Z_2, Z_3, t) \ge 0, \\ \sigma_1 (Z_{1,l}, Z_2, Z_3, t) = 0; \end{cases} \tag{5.69}$$

(ii) On surface II: $\{Z_1 = 1,\ Z_{2,l} \le Z_2 \le Z_1,\ Z_{3,l} \le Z_3 \le Z_2\}$,

$$\begin{cases} \mu_1 (1, Z_2, Z_3, t) \le 0, \\ \sigma_1 (1, Z_2, Z_3, t) = 0; \end{cases} \tag{5.70}$$

(iii) On surface III: $\{Z_{1,l} \le Z_1 \le 1,\ Z_2 = Z_{2,l},\ Z_{3,l} \le Z_3 \le Z_2\}$,

$$\begin{cases} \mu_2 (Z_1, Z_{2,l}, Z_3, t) \ge 0, \\ \sigma_2 (Z_1, Z_{2,l}, Z_3, t) = 0; \end{cases} \tag{5.71}$$

(iv) On surface IV: $\{Z_{1,l} \le Z_1 \le 1,\ Z_2 = Z_1,\ Z_{3,l} \le Z_3 \le Z_2\}$,

$$\begin{cases} -\mu_1 (Z_1, Z_1, Z_3, t) + \mu_2 (Z_1, Z_1, Z_3, t) \le 0, \\ \sigma_1 (Z_1, Z_1, Z_3, t) = \sigma_2 (Z_1, Z_1, Z_3, t), \quad \rho_{1,2} (Z_1, Z_1, Z_3, t) = 1; \end{cases} \tag{5.72}$$

(v) On surface V: $\{Z_{1,l} \le Z_1 \le 1,\ Z_{2,l} \le Z_2 \le Z_1,\ Z_3 = Z_{3,l}\}$,

$$\begin{cases} \mu_3 (Z_1, Z_2, Z_{3,l}, t) \ge 0, \\ \sigma_3 (Z_1, Z_2, Z_{3,l}, t) = 0; \end{cases} \tag{5.73}$$

(vi) On surface VI: $\{Z_{1,l} \le Z_1 \le 1,\ Z_{2,l} \le Z_2 \le Z_1,\ Z_3 = Z_2\}$,

$$\begin{cases} -\mu_2 (Z_1, Z_2, Z_2, t) + \mu_3 (Z_1, Z_2, Z_2, t) \le 0, \\ \sigma_2 (Z_1, Z_2, Z_2, t) = \sigma_3 (Z_1, Z_2, Z_2, t), \quad \rho_{2,3} (Z_1, Z_2, Z_2, t) = 1. \end{cases} \tag{5.74}$$

This model will be called the three-factor interest rate model in this book.

As you can see, conditions (5.69)–(5.71) and (5.73) have the same form as the condition (5.45) or the condition (5.46), and the conditions (5.72) and (5.74) are in a similar form. They are the weak-form reversion conditions on the non-rectangular domain Ω. In order to guarantee that if a point is in Ω at time t^*, then the point is still in Ω at $t = t^* + dt$ for a positive dt, it is necessary to require that

$$n_1 dZ_1 + n_2 dZ_2 + n_3 dZ_3 \le 0 \tag{5.75}$$

holds at any point on the boundary of the domain Ω, where n_1, n_2, and n_3 are the three components of the outer normal vector of the boundary at the

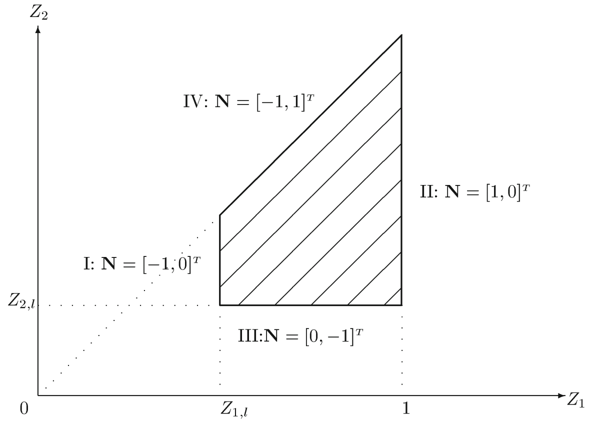

Fig. 5.3. Projection of the domain Ω on the (Z_1, Z_2)-plane

point. This is called the weak-form reversion conditions on a general domain. The condition of the condition (5.75) holding at every point on the boundary of the domain Ω is equivalent to the conditions (5.69)–(5.74). For example, on surface I (see Fig. 5.3), $n_1 = -1$, $n_2 = 0$ and $n_3 = 0$, so the condition (5.75) can be written as

$$n_1 dZ_1 + n_2 dZ_2 + n_3 dZ_3 = -dZ_1 = -\mu_1 dt - \sigma_1 dX_1 \leq 0.$$

This holds if and only if $\sigma_1 = 0$ and $\mu_1 \geq 0$. On surface IV, $n_1 = -1$, $n_2 = 1$, and $n_3 = 0$ (see Fig. 5.3). In this case the condition (5.75) can be written as

$$\begin{aligned} n_1 dZ_1 + n_2 dZ_2 + n_3 dZ_3 &= -dZ_1 + dZ_2 \\ &= -\mu_1 dt + \mu_2 dt - \sigma_1 dX_1 + \sigma_2 dX_2 \\ &= (-\mu_1 + \mu_2) dt + \sigma_{12} dX_{12} \leq 0, \end{aligned}$$

where we define

$$\sigma_{12} dX_{12} = -\sigma_1 dX_1 + \sigma_2 dX_2$$

and dX_{12} is another Wiener process. Using Itô's lemma, we know

$$\sigma_{12} = \sqrt{\sigma_1^2 - 2\rho_{1,2}\sigma_1\sigma_2 + \sigma_2^2}.$$

Thus in this case the condition (5.75) holds if and only if

$$-\mu_1 + \mu_2 \leq 0 \quad \text{and} \quad \sigma_{12} = 0.$$

$\sigma_{12} = 0$ is equivalent to

$$\sigma_1^2 - 2\rho_{1,2}\sigma_1\sigma_2 + \sigma_2^2 = (\sigma_1 - \sigma_2)^2 + 2(1 - \rho_{1,2})\sigma_1\sigma_2 = 0$$

or

$$\sigma_1 = \sigma_2 \quad \text{and} \quad \rho_{1,2} = 1.$$

Thus in this case the condition (5.75) is equivalent to $-\mu_1 + \mu_2 \le 0$, $\sigma_1 = \sigma_2$, and $\rho_{1,2} = 1$. If the derivatives of $\sigma_i(Z_1, Z_2, Z_3, t)$ with respect to Z_1, Z_2, and Z_3 are bounded, then it is expected that the condition (5.75) or the conditions (5.69)–(5.74) guarantee that a point (Z_1, Z_2, Z_3) will never move from inside of the domain Ω to its outside. This is a natural property of a stochastic model for interest rates when $Z_{1,l}$, $Z_{2,l}$ and $Z_{3,l}$ are given properly.

Let $V(Z_1, Z_2, Z_3, t)$ be the value of a derivative security depending on Z_1, Z_2, Z_3, t. According to Sect. 2.3.2, $V(Z_1, Z_2, Z_3, t)$ should satisfy

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sigma_i\sigma_j\rho_{i,j}\frac{\partial^2 V}{\partial Z_i \partial Z_j} + r\sum_{i=1}^{3} Z_i \frac{\partial V}{\partial Z_i} - rV = 0.$$

As we pointed out in Sect. 2.3, in this case in the PDE there is no market price of risk, or because zero-coupon bonds can be traded on the market, the market prices of risk for these bonds can be determined by the relation (2.36) with $D_{0i} = 0$:

$$\mu_i(Z_1, Z_2, Z_3, t) - \lambda_i(Z_1, Z_2, Z_3, t)\sigma_i(Z_1, Z_2, Z_3, t) = r(Z_1, Z_2, Z_3, t)Z_i, \quad i = 1, 2, 3.$$

Let

$$\mathbf{L}_{3z} = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sigma_i\sigma_j\rho_{i,j}\frac{\partial^2}{\partial Z_i \partial Z_j} + r\sum_{i=1}^{3} Z_i \frac{\partial}{\partial Z_i} - r. \tag{5.76}$$

The equation above can be written as

$$\frac{\partial V}{\partial t} + \mathbf{L}_{3z} V = 0.$$

For a derivative security, at the maturity date T, its price should be equal to its payoff $V_T(Z_1, Z_2, Z_3)$. Therefore, any European interest rate derivatives under this model should be solutions of the problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \mathbf{L}_{3z} V = 0 \quad \text{on} \ \ \Omega \times [0, T], \\ V(Z_1, Z_2, Z_3, T) = V_T(Z_1, Z_2, Z_3) \quad \text{on} \ \ \Omega. \end{cases} \tag{5.77}$$

Introduce the following transformation:

$$\begin{cases} \xi_1 = \dfrac{Z_1 - Z_{1,l}}{1 - Z_{1,l}}, \\ \xi_2 = \dfrac{Z_2 - Z_{2,l}}{Z_1 - Z_{2,l}}, \\ \xi_3 = \dfrac{Z_3 - Z_{3,l}}{Z_2 - Z_{3,l}}. \end{cases} \tag{5.78}$$

Through this transformation, the domain Ω in the (Z_1, Z_2, Z_3)-space is transformed into the domain $\tilde{\Omega}$: $[0,1]\times[0,1]\times[0,1]$ in the (ξ_1, ξ_2, ξ_3)-space. Because

$$
\begin{aligned}
&\frac{\partial \xi_1}{\partial Z_1} = \frac{1}{1 - Z_{1,l}}, \\
&\frac{\partial \xi_2}{\partial Z_1} = \frac{-\xi_2}{Z_1 - Z_{2,l}}, \quad \frac{\partial \xi_2}{\partial Z_2} = \frac{1}{Z_1 - Z_{2,l}}, \\
&\frac{\partial \xi_3}{\partial Z_2} = \frac{-\xi_3}{Z_2 - Z_{3,l}}, \quad \frac{\partial \xi_3}{\partial Z_3} = \frac{1}{Z_2 - Z_{3,l}},
\end{aligned}
$$

we have

$$
\begin{aligned}
\frac{\partial V}{\partial Z_1} &= \frac{1}{1 - Z_{1,l}} \frac{\partial V}{\partial \xi_1} - \frac{\xi_2}{Z_1 - Z_{2,l}} \frac{\partial V}{\partial \xi_2}, \\
\frac{\partial V}{\partial Z_2} &= \frac{1}{Z_1 - Z_{2,l}} \frac{\partial V}{\partial \xi_2} - \frac{\xi_3}{Z_2 - Z_{3,l}} \frac{\partial V}{\partial \xi_3}, \\
\frac{\partial V}{\partial Z_3} &= \frac{1}{Z_2 - Z_{3,l}} \frac{\partial V}{\partial \xi_3}, \\
\frac{\partial^2 V}{\partial Z_1^2} &= \frac{1}{(1 - Z_{1,l})^2} \frac{\partial^2 V}{\partial \xi_1^2} - \frac{2\xi_2}{(1 - Z_{1,l})(Z_1 - Z_{2,l})} \frac{\partial^2 V}{\partial \xi_1 \partial \xi_2} \\
&\quad + \frac{\xi_2^2}{(Z_1 - Z_{2,l})^2} \frac{\partial^2 V}{\partial \xi_2^2} + \frac{2\xi_2}{(Z_1 - Z_{2,l})^2} \frac{\partial V}{\partial \xi_2},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 V}{\partial Z_2^2} &= \frac{1}{(Z_1 - Z_{2,l})^2} \frac{\partial^2 V}{\partial \xi_2^2} - \frac{2\xi_3}{(Z_1 - Z_{2,l})(Z_2 - Z_{3,l})} \frac{\partial^2 V}{\partial \xi_2 \partial \xi_3} \\
&\quad + \frac{\xi_3^2}{(Z_2 - Z_{3,l})^2} \frac{\partial^2 V}{\partial \xi_3^2} + \frac{2\xi_3}{(Z_2 - Z_{3,l})^2} \frac{\partial V}{\partial \xi_3}, \\
\frac{\partial^2 V}{\partial Z_3^2} &= \frac{1}{(Z_2 - Z_{3,l})^2} \frac{\partial^2 V}{\partial \xi_3^2}, \\
\frac{\partial^2 V}{\partial Z_1 \partial Z_2} &= \frac{-1}{(Z_1 - Z_{2,l})^2} \frac{\partial V}{\partial \xi_2} + \frac{1}{Z_1 - Z_{2,l}} \left(\frac{1}{1 - Z_{1,l}} \frac{\partial^2 V}{\partial \xi_1 \partial \xi_2} - \frac{\xi_2}{Z_1 - Z_{2,l}} \frac{\partial^2 V}{\partial \xi_2^2} \right) \\
&\quad - \frac{\xi_3}{Z_2 - Z_{3,l}} \left(\frac{1}{1 - Z_{1,l}} \frac{\partial^2 V}{\partial \xi_1 \partial \xi_3} - \frac{\xi_2}{Z_1 - Z_{2,l}} \frac{\partial^2 V}{\partial \xi_2 \partial \xi_3} \right), \\
\frac{\partial^2 V}{\partial Z_1 \partial Z_3} &= \frac{1}{Z_2 - Z_{3,l}} \left(\frac{1}{1 - Z_{1,l}} \frac{\partial^2 V}{\partial \xi_1 \partial \xi_3} - \frac{\xi_2}{Z_1 - Z_{2,l}} \frac{\partial^2 V}{\partial \xi_2 \partial \xi_3} \right), \\
\frac{\partial^2 V}{\partial Z_2 \partial Z_3} &= \frac{-1}{(Z_2 - Z_{3,l})^2} \frac{\partial V}{\partial \xi_3} + \frac{1}{Z_2 - Z_{3,l}} \left(\frac{1}{Z_1 - Z_{2,l}} \frac{\partial^2 V}{\partial \xi_2 \partial \xi_3} - \frac{\xi_3}{Z_2 - Z_{3,l}} \frac{\partial^2 V}{\partial \xi_3^2} \right).
\end{aligned}
$$

Therefore, the operator $\mathbf{L}_{3z}$ defined by the expression (5.76) can be rewritten as

$$\begin{aligned}\mathbf{L}_{3\xi} = &\frac{1}{2}\tilde{\sigma}_1^2\frac{\partial^2}{\partial\xi_1^2} + \frac{1}{2}\tilde{\sigma}_2^2\frac{\partial^2}{\partial\xi_2^2} + \frac{1}{2}\tilde{\sigma}_3^2\frac{\partial^2}{\partial\xi_3^2}\\ &+\tilde{\sigma}_1\tilde{\sigma}_2\tilde{\rho}_{1,2}\frac{\partial^2}{\partial\xi_1\partial\xi_2} + \tilde{\sigma}_1\tilde{\sigma}_3\tilde{\rho}_{1,3}\frac{\partial^2}{\partial\xi_1\partial\xi_3} + \tilde{\sigma}_2\tilde{\sigma}_3\tilde{\rho}_{2,3}\frac{\partial^2}{\partial\xi_2\partial\xi_3}\\ &+b_1\frac{\partial}{\partial\xi_1} + b_2\frac{\partial}{\partial\xi_2} + b_3\frac{\partial}{\partial\xi_3} - r,\end{aligned} \tag{5.79}$$

where

$$\begin{cases}\dfrac{1}{2}\tilde{\sigma}_1^2 = \dfrac{\frac{1}{2}\sigma_1^2}{\left(1 - Z_{1,l}\right)^2},\\ \dfrac{1}{2}\tilde{\sigma}_2^2 = \dfrac{\frac{1}{2}\left(\sigma_1^2\xi_2^2 - 2\sigma_1\sigma_2\xi_2\rho_{1,2} + \sigma_2^2\right)}{\left(Z_1 - Z_{2,l}\right)^2},\\ \dfrac{1}{2}\tilde{\sigma}_3^2 = \dfrac{\frac{1}{2}\left(\sigma_2^2\xi_3^2 - 2\sigma_2\sigma_3\xi_3\rho_{2,3} + \sigma_3^2\right)}{\left(Z_2 - Z_{3,l}\right)^2},\end{cases} \tag{5.80}$$

$$\begin{cases}\tilde{\sigma}_1\tilde{\sigma}_2\tilde{\rho}_{1,2} = \dfrac{\sigma_1\left(\sigma_2\rho_{1,2} - \sigma_1\xi_2\right)}{\left(1 - Z_{1,l}\right)\left(Z_1 - Z_{2,l}\right)},\\ \tilde{\sigma}_1\tilde{\sigma}_3\tilde{\rho}_{1,3} = \dfrac{\sigma_1\left(\sigma_3\rho_{1,3} - \sigma_2\rho_{1,2}\xi_3\right)}{\left(1 - Z_{1,l}\right)\left(Z_2 - Z_{3,l}\right)},\\ \tilde{\sigma}_2\tilde{\sigma}_3\tilde{\rho}_{2,3} = \dfrac{\sigma_1\xi_2\left(\sigma_2\rho_{1,2}\xi_3 - \sigma_3\rho_{1,3}\right) + \sigma_2\left(\sigma_3\rho_{2,3} - \sigma_2\xi_3\right)}{\left(Z_1 - Z_{2,l}\right)\left(Z_2 - Z_{3,l}\right)},\end{cases} \tag{5.81}$$

and

$$\begin{cases}b_1 = \dfrac{rZ_1}{1 - Z_{1,l}},\\ b_2 = \dfrac{r\left(Z_2 - Z_1\xi_2\right)}{Z_1 - Z_{2,l}} + \dfrac{\sigma_1\left(\sigma_1\xi_2 - \sigma_2\rho_{1,2}\right)}{\left(Z_1 - Z_{2,l}\right)^2},\\ b_3 = \dfrac{r\left(Z_3 - Z_2\xi_3\right)}{Z_2 - Z_{3,l}} + \dfrac{\sigma_2\left(\sigma_2\xi_3 - \sigma_3\rho_{2,3}\right)}{\left(Z_2 - Z_{3,l}\right)^2}.\end{cases} \tag{5.82}$$

Consequently, the problem (5.77) can be rewritten as

$$\begin{cases}\dfrac{\partial V}{\partial t} + \mathbf{L}_{3\xi}V = 0 \quad \text{on } \tilde{\Omega}\times[0,T],\\ V(\xi_1,\xi_2,\xi_3,T) = V_T\left(Z_1(\xi_1), Z_2(\xi_1,\xi_2), Z_3(\xi_1,\xi_2,\xi_3)\right) \quad \text{on } \tilde{\Omega},\end{cases} \tag{5.83}$$

where $\mathbf{L}_{3\xi}$ is defined by Eq. (5.79) and

$$\begin{cases}Z_1\left(\xi_1\right) = Z_{1,l} + \xi_1\left(1 - Z_{1,l}\right),\\ Z_2\left(\xi_1,\xi_2\right) = Z_{2,l} + \xi_2\left[Z_{1,l} + \xi_1\left(1 - Z_{1,l}\right) - Z_{2,l}\right],\\ Z_3\left(\xi_1,\xi_2,\xi_3\right) = Z_{3,l} + \xi_3\left\{Z_{2,l} + \xi_2\left[Z_{1,l} + \xi_1\left(1 - Z_{1,l}\right) - Z_{2,l}\right] - Z_{3,l}\right\}.\end{cases} \tag{5.84}$$

This is a final-value problem on a rectangular domain. Thus, when the three-factor interest rate model is used, evaluating an interest rate derivative is reduced to solving a final-value problem on a rectangular domain.

We would like to point out the relations among $\tilde{\sigma}_1$, $\tilde{\sigma}_2$, $\tilde{\sigma}_3$, $\tilde{\rho}_{1,2}$, $\tilde{\rho}_{1,3}$, $\tilde{\rho}_{2,3}$ and $d\xi_1$, $d\xi_2$, $d\xi_3$. Using Itô's lemma, from the definitions of ξ_1, ξ_2, ξ_3, we can have

$$d\xi_1 = \tilde{\mu}_1 dt + \tilde{\sigma}_1 dX_1, \quad d\xi_2 = \tilde{\mu}_2 dt + \tilde{\sigma}_2 d\tilde{X}_2, \quad d\xi_3 = \tilde{\mu}_3 dt + \tilde{\sigma}_3 d\tilde{X}_3,$$

where $d\tilde{X}_2$ and $d\tilde{X}_3$ are two new Wiener processes. Therefore

$$\tilde{\sigma}_i^2 = \mathrm{Var}[d\xi_i]/dt, \quad j = 1, 2, 3.$$

It can also be shown that

$$\mathrm{Cov}[dX_1 d\tilde{X}_2]/dt = \tilde{\rho}_{1,2}, \quad \mathrm{Cov}[dX_1 d\tilde{X}_3]/dt = \tilde{\rho}_{1,3}$$

and

$$\mathrm{Cov}[d\tilde{X}_2 d\tilde{X}_3]/dt = \tilde{\rho}_{2,3}.$$

These are left for readers to prove as Problem 24.

From Eq. (5.80), it is easy to see that the equality conditions in the conditions (5.69)–(5.74) can be rewritten as

$$\begin{cases} \tilde{\sigma}_1(0, \xi_2, \xi_3, t) = \tilde{\sigma}_1(1, \xi_2, \xi_3, t) = 0, & 0 \le \xi_2 \le 1,\ 0 \le \xi_3 \le 1, \\ \tilde{\sigma}_2(\xi_1, 0, \xi_3, t) = \tilde{\sigma}_2(\xi_1, 1, \xi_3, t) = 0, & 0 \le \xi_1 \le 1,\ 0 \le \xi_3 \le 1, \\ \tilde{\sigma}_3(\xi_1, \xi_2, 0, t) = \tilde{\sigma}_3(\xi_1, \xi_2, 1, t) = 0, & 0 \le \xi_1 \le 1,\ 0 \le \xi_2 \le 1. \end{cases} \tag{5.85}$$

Therefore, in order for the equality conditions in the conditions (5.69)–(5.74) to hold, we just require that the volatilities of $d\xi_1$, $d\xi_2$, and $d\xi_3$ satisfy the condition (5.85), which is easier to be implemented than the equality conditions in the conditions (5.69)–(5.74). Suppose that $\tilde{\sigma}_1$, $\tilde{\sigma}_2$, $\tilde{\sigma}_3$, $\tilde{\rho}_{1,2}$, $\tilde{\rho}_{1,3}$, and $\tilde{\rho}_{2,3}$ are given functions. In this case, in order to be able to use the expressions of b_1, b_2, and b_3 conveniently, we express $\sigma_1 (\sigma_1 \xi_2 - \sigma_2 \rho_{1,2})$ and $\sigma_2 (\sigma_2 \xi_3 - \sigma_3 \rho_{2,3})$ in the expressions of b_2 and b_3 in terms of $\tilde{\sigma}_1$, $\tilde{\sigma}_2$, $\tilde{\sigma}_3$, $\tilde{\rho}_{1,2}$, $\tilde{\rho}_{1,3}$, and $\tilde{\rho}_{2,3}$. From the expression (5.81) we have

$$\begin{aligned} \tilde{\sigma}_1 \tilde{\sigma}_2 \tilde{\rho}_{1,2} (1 - Z_{1,l}) (Z_1 - Z_{2,l}) &= \sigma_1 (\sigma_2 \rho_{1,2} - \sigma_1 \xi_2), \\ \tilde{\sigma}_1 \tilde{\sigma}_3 \tilde{\rho}_{1,3} (1 - Z_{1,l}) (Z_2 - Z_{3,l}) &= \sigma_1 (\sigma_3 \rho_{1,3} - \sigma_2 \rho_{1,2} \xi_3), \\ \tilde{\sigma}_2 \tilde{\sigma}_3 \tilde{\rho}_{2,3} (Z_1 - Z_{2,l}) (Z_2 - Z_{3,l}) &= \sigma_1 \xi_2 (\sigma_2 \rho_{1,2} \xi_3 - \sigma_3 \rho_{1,3}) \\ &\quad + \sigma_2 (\sigma_3 \rho_{2,3} - \sigma_2 \xi_3), \end{aligned}$$

and from the second and third relations we further obtain

$$\begin{aligned} \sigma_2 (\sigma_3 \rho_{2,3} - \sigma_2 \xi_3) = {} & \tilde{\sigma}_1 \tilde{\sigma}_3 \tilde{\rho}_{1,3} \xi_2 (1 - Z_{1,l}) (Z_2 - Z_{3,l}) \\ & + \tilde{\sigma}_2 \tilde{\sigma}_3 \tilde{\rho}_{2,3} (Z_1 - Z_{2,l}) (Z_2 - Z_{3,l}). \end{aligned}$$

Therefore, the expressions of b_1, b_2 and b_3 can be rewritten as

$$\begin{cases} b_1 = \dfrac{rZ_1}{1-Z_{1,l}}, \\ b_2 = \dfrac{r(Z_2 - Z_1\xi_2)}{Z_1 - Z_{2,l}} - \dfrac{\tilde{\sigma}_1\tilde{\sigma}_2\tilde{\rho}_{1,2}(1-Z_{1,l})}{Z_1 - Z_{2,l}}, \\ b_3 = \dfrac{r(Z_3 - Z_2\xi_3)}{Z_2 - Z_{3,l}} - \dfrac{\tilde{\sigma}_1\tilde{\sigma}_3\tilde{\rho}_{1,3}\xi_2(1-Z_{1,l})}{Z_2 - Z_{3,l}} \\ \qquad -\dfrac{\tilde{\sigma}_2\tilde{\sigma}_3\tilde{\rho}_{2,3}(Z_1 - Z_{2,l})}{Z_2 - Z_{3,l}}. \end{cases} \tag{5.86}$$

By this relation, we can easily calculate b_1, b_2, and b_3 when the values of $\tilde{\sigma}_1$, $\tilde{\sigma}_2$, $\tilde{\sigma}_3$, $\tilde{\rho}_{1,2}$, $\tilde{\rho}_{1,3}$, and $\tilde{\rho}_{2,3}$ are given. Because of the condition (5.85) and $r = 0$ for $Z_1 = 1$ [see the expression (5.68)], we can easily show

$$\begin{cases} b_1(0,\xi_2,\xi_3,t) \ge 0, \quad b_1(1,\xi_2,\xi_3,t) = 0, \quad 0 \le \xi_2 \le 1,\ 0 \le \xi_3 \le 1, \\ b_2(\xi_1,0,\xi_3,t) \ge 0, \quad b_2(\xi_1,1,\xi_3,t) = 0, \quad 0 \le \xi_1 \le 1,\ 0 \le \xi_3 \le 1, \\ b_3(\xi_1,\xi_2,0,t) \ge 0, \quad b_3(\xi_1,\xi_2,1,t) = 0, \quad 0 \le \xi_1 \le 1,\ 0 \le \xi_2 \le 1. \end{cases}$$

It can be proved that when these inequalities and the condition (5.85) hold, the problem (5.83) has a unique solution (see [91]). Thus the problem (5.83) can be solved by numerical methods without any difficulty. In this subsection, the PDE in the problem (5.83) is derived through two steps: first it is obtained from the result given in Sect. 2.3.2 and then a new equation is gotten by means of a transformation. Actually this equation can be obtained directly by setting a portfolio and using Itô's lemma just like what we did in Sect. 2.3.4 for two-factor case. Readers are asked to derive the PDE in the problem (5.83) in this way as Problem 25.

Finally, we say a few words about how to use this model to evaluate interest rate derivatives. First, we need to choose Z_1, Z_2, and Z_3 and find $\tilde{\sigma}_1$, $\tilde{\sigma}_2$, $\tilde{\sigma}_3$ satisfying conditions (5.85), and $\tilde{\rho}_{1,2}$, $\tilde{\rho}_{1,3}$, $\tilde{\rho}_{2,3}$. Finding these functions can be done from the data on markets by statistics. After that, the problem (5.83) needs to be solved. Let $t = 0$ denote today, and suppose the derivative security is European style. On the maturity date T, for each point (ξ_1, ξ_2, ξ_3) in $\tilde{\Omega}$, we can have Z_1, Z_2, and Z_3 by the relation (5.84). Then, we determine a zero-coupon bond curve by using the method given in Sect. 5.6.2. When we obtain such a curve, the value of the payoff and r for the point can be determined. This can be done for all points (ξ_1, ξ_2, ξ_3) in the domain $\tilde{\Omega}$ for $t = T$. When we have the final value and all the coefficients of the partial differential equation in the problem (5.83), we can solve the final-value problem (5.83) from $t = T$ to $t = 0$ and get the value of the derivative security today for all the points in $\tilde{\Omega}$.

For American-style derivatives, the situation is similar. The only difference is that the value of derivative must be greater than the constraint. Because the value of the constraint can be obtained by the zero-coupon bond curves at all points in $\tilde{\Omega} \times [0, T]$, the value of an American-style derivative can be determined without any difficulty. However, free boundaries will usually appear in this case.

From what we have described, we see that this model has the following features:

- The state variables are prices of three zero-coupon bonds with different maturities that can be traded on markets, so the coefficients of the first derivatives with respect to the bond prices Z_i in the partial differential equation simply are rZ_i.
- The volatilities of these zero-coupon bonds and their correlation coefficients can be found directly from the real markets by statistics, so the model will have the real major feature of the markets.
- All the zero-coupon bond curves having appeared in the real market can be reproduced quite accurately. This is the basis of a model giving correct results. If taking three random variables is not good enough, four-factor models can be adopted. Generalizing three-factor models to four-factor models is straightforward.
- In other models, the partial differential equation is defined on an infinite domain. For this model, the corresponding partial differential equation is defined on a finite domain. It has been proved that no boundary condition is needed in order for its final-value problem to have a unique solution. Thus, it is not difficult to design correct and efficient numerical methods to price interest rate derivatives.

For the details on how to determine models from the market data and how to solve the final-value problem of the partial differential equation, see Sect. 10.3 and [96]. There, some numerical results are also given.

5.7 Two-Factor Convertible Bonds

Until now, we discussed derivatives depending on either equities or interest rates. This section deals with a derivative dependent on both equity prices and interest rates. This derivative security is a bond that may, at any time chosen by the holder, be converted into n shares of stocks of the company who issues the bond. Such a bond is commonly known as a convertible bond. As a bond, its price depends on the short rate r. It can be exchanged for n shares of stocks, so its value is also a function of the stock price S. Because its typical life span is about 3–10 years, both S and r are considered as random state variables. Therefore, this bond is called a two-factor convertible bond. In this section, we discuss how to price such a bond.

Consider a bond issued by a company and its payoff depends not only on r but also on the price of the stock of the company. In this case the value of this bond depends on both r and S. Let $B(S, r, t)$ be the value of such a bond. As usual, we assume that S is governed by

$$dS = \mu(S, t)Sdt + \sigma(S, t)SdX_1, \quad 0 \le S \tag{5.87}$$

and the interest rate by

$$dr = u(r, t)dt + w(r, t)dX_2, \quad r_l \le r \le r_u, \tag{5.88}$$

where dX_1 and dX_2 are different Wiener processes though they can be correlated. Suppose that

$$\mathrm{E}\left[dX_1 dX_2\right] = \rho dt,$$

where ρ is a constant belonging to $[-1,\ 1]$ and for S and r, ρ usually is a negative number. According to Sect. 2.3, such a derivative satisfies

$$\frac{\partial B}{\partial t} + \mathbf{L_{s,r}}B + kZ = 0, \quad 0 \le S,\ r_l \le r \le r_u,\ 0 \le t \le T, \tag{5.89}$$

where

$$\mathbf{L_{s,r}} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + \rho\sigma S w \frac{\partial^2}{\partial S \partial r} + \frac{1}{2}w^2 \frac{\partial^2}{\partial r^2} + (r - D_0)S\frac{\partial}{\partial S} + (u - \lambda w)\frac{\partial}{\partial r} - r.$$

Here, D_0 is the dividend yield a holder of the stock receives per unit time, and kZ is the coupon payment a holder of the bond receives per unit time, Z being the face value of the bond. λ is the market price of risk for the short rate. T is the maturity date of the bond.

We assume that at maturity time T, the holder of the bond can choose to get the face value Z or n shares of stocks. Therefore, the payoff is

$$B(S, r, T) = \max(Z, nS), \quad 0 \le S, \quad r_l \le r \le r_u. \tag{5.90}$$

This is the final condition for this bond. We assume that for the interest rate, the conditions (5.45) and (5.46) hold, i.e.,

$$\begin{cases} u(r_l, t) - w(r_l, t)\dfrac{\partial}{\partial r}w(r_l, t) \ge 0, \\ w(r_l, t) = 0, \end{cases}$$

and

$$\begin{cases} u(r_u, t) - w(r_u, t)\dfrac{\partial}{\partial r}w(r_u, t) \le 0, \\ w(r_u, t) = 0. \end{cases}$$

Because $w^2(r, t) \ge 0$ and $w(r_l, t) = 0$, on $[r_l, r_u]$ we conclude $w(r_l, t)\dfrac{\partial}{\partial r}w(r_l, t)$ $= \dfrac{1}{2}\dfrac{\partial}{\partial r}w^2(r_l, t) \ge 0$. Similarly, $w(r_u, t)\dfrac{\partial}{\partial r}w(r_u, t) \le 0$. Therefore, the conditions above can be rewritten as

$$\begin{cases} u(r_l,t) \geq w(r_l,t)\dfrac{\partial}{\partial r}w(r_l,t) \geq 0, \\ w(r_l,t) = 0, \end{cases}$$

$$\begin{cases} u(r_u,t) \leq w(r_u,t)\dfrac{\partial}{\partial r}w(r_u,t) \leq 0, \\ w(r_u,t) = 0. \end{cases}$$

Because of $w(r_l,t)=0$, Eq. (5.89) at $r=r_l$ degenerates into

$$\frac{\partial B}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 B}{\partial S^2} + (r - D_0)S\frac{\partial B}{\partial S} + u\frac{\partial B}{\partial r} - rB + kZ = 0.$$

This equation has hyperbolic properties in the r-direction. Thus, if $u(r_l,t) \geq 0$, then the value $B(S,r_l,t)$ is determined by the value $B(S,r,t)$ in the domain $[0,\infty) \times [r_l,r_u] \times [t,T]$ and no extra boundary condition at $r=r_l$ is needed. Similarly, no boundary condition should be required at $r=r_u$ because $u(r_u,t) \leq 0$ and $w(r_u,t)=0$. At $S=0$, Eq. (5.89) becomes

$$\frac{\partial B}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 B}{\partial r^2} + (u - \lambda w)\frac{\partial B}{\partial r} - rB + kZ = 0.$$

This is the bond equation, and the value $B(0,r,t)$ is determined by this equation and the final condition at $S=0$. Just like the Black–Scholes equation, there is no need for specifying a condition as $S \to \infty$. Therefore, if the conditions (5.45) and (5.46) hold, then we could expect that the problem

$$\begin{cases} \dfrac{\partial B}{\partial t} + \mathbf{L}_{\mathbf{s},\mathbf{r}}B + kZ = 0, & 0 \leq S, \quad r_l \leq r \leq r_u, \quad 0 \leq t \leq T, \\ B(S,r,T) = \max(Z, nS), & 0 \leq S, \quad r_l \leq r \leq r_u \end{cases} \tag{5.91}$$

has a unique solution. If $\dfrac{\partial}{\partial r}w(r_l,t)$ and $\dfrac{\partial}{\partial r}w(r_u,t)$ are bounded, which usually is true, then the uniqueness of solution of the problem (5.91) can be obtained from the results given in the paper by Zhu and Li (see [94]).

If this bond can be exchanged for n shares of stocks at any time, then this bond is called a convertible bond and let us denote its value by $B_c(S,r,t)$. It is clear that the value $B_c(S,r,t)$ must satisfy the following constraint

$$B_c(S,r,t) \geq nS, \quad 0 \leq S, \quad 0 \leq t \leq T. \tag{5.92}$$

This condition is called the constraint on convertible bonds. Sometimes, the solution of the problem (5.91) satisfies the constraint (5.92), so the problem (5.91) determines the solution of a convertible bond. For example, if $D_0 = 0$, then the problem (5.91) gives the price of a convertible bond, which will be explained later. If

$$D_0 > 0,$$

then the price of a convertible bond should be the solution of the following linear complementarity problem on the domain $[0,\infty) \times [r_l,r_u] \times [0,T]$:

$$\begin{cases} \min\left(-\dfrac{\partial B_c}{\partial t} - \mathbf{L_{S,r}} B_c - kZ,\ B_c(S,r,t) - nS\right) = 0, \\ B_c(S,r,T) = \max(Z, nS) \geq nS. \end{cases}$$

Let us reformulate this problem as a free-boundary problem if $D_0 > 0$. We cannot directly apply Theorem 3.1 in Sect. 3.1 to this case because there are two major differences between the problem in the theorem and the problem here. Here, the operator $\mathbf{L_{S,r}}$ is two-dimensional and there is a nonhomogeneous term kZ. However, the main idea is still true. For $S < Z/n$, $B_c(S,r,T) = Z > nS$. Therefore, on $[0, Z/n)$, $B_c(S,r,T-\Delta t)$ must be greater than nS if Δt is small enough, and no free boundary can appear in that region at time T. Now let us check the region $(Z/n, \infty)$. In this case, we need to check where

$$\left(\frac{\partial}{\partial t} + \mathbf{L_{S,r}}\right) nS + kZ \geq 0$$

and where

$$\left(\frac{\partial}{\partial t} + \mathbf{L_{S,r}}\right) nS + kZ < 0.$$

Because

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \mathbf{L_{S,r}}\right) nS + kZ &= (r - D_0)nS - rnS + kZ \\ &= kZ - D_0 nS, \end{aligned}$$

when $S > Z/n$ and $S > kZ/D_0 n$, namely, $S > \max(Z/n, kZ/D_0 n)$,

$$\left(\frac{\partial}{\partial t} + \mathbf{L_{S,r}}\right) nS + kZ < 0$$

and the solution is nS. Otherwise, we can use the partial differential equation to determine the solution. Therefore, there is a free boundary starting at $S = \max(Z/n, kZ/D_0 n)$ and $t = T$. Let $S_f(r,t)$ be the location of the free boundary, then

$$S_f(r,T) = \max\left(\frac{Z}{n}, \frac{kZ}{D_0 n}\right), \qquad r_l \leq r \leq r_u. \tag{5.93}$$

We assume that there is only one free boundary. From numerical solutions, we know that it is true at least for some cases. Thus, when $D_0 > 0$, the domain $[0, \infty) \times [r_l, r_u] \times [0, T]$ in (S, r, t)-space is divided into subdomains

$$I : [0, S_f(r,t)] \times [r_l, r_u] \times [0, T]$$

and

$$II : (S_f(r,t), \infty) \times [r_l, r_u] \times [0,T]\,.$$

The free boundary is between them. At the free boundary, the solution and its derivatives are continuous. In the subdomain II where $B_c = nS$,

$$\frac{\partial B_c}{\partial S} = n$$

and

$$\frac{\partial B_c}{\partial r} = 0.$$

Thus, it seems that in the subdomain I where the partial differential equation is used, we need to require

$$B_c\left(S_f(r,t), r, t\right) = nS_f(r,t), \qquad r_l \le r \le r_u, \qquad 0 \le t \le T, \tag{5.94}$$

$$\frac{\partial B_c}{\partial S}\left(S_f(r,t), r, t\right) = n, \qquad r_l \le r \le r_u, \qquad 0 \le t \le T, \tag{5.95}$$

and

$$\frac{\partial B_c}{\partial r}\left(S_f(r,t), r, t\right) = 0, \qquad r_l \le r \le r_u, \qquad 0 \le t \le T$$

on the free boundary. Differentiating both sides of the condition (5.94) with respect to r in subdomain I yields

$$\frac{\partial B_c}{\partial S}\left(S_f(r,t), r, t\right)\frac{\partial S_f}{\partial r}(r,t) + \frac{\partial B_c}{\partial r}\left(S_f(r,t), r, t\right) = n\frac{\partial S_f}{\partial r}(r,t).$$

Using the condition (5.95), we arrive at

$$\frac{\partial B_c}{\partial r}\left(S_f(r,t), r, t\right) = 0.$$

Consequently, the conditions (5.94) and (5.95) guarantee that all the first derivatives are continuous at the free boundary and we only need to impose the conditions (5.94) and (5.95) on the solution in subdomain I.

Thus in subdomain I, the solution $B_c(S, r, t)$ and the location of the free boundary $S = S_f(r,t)$ are obtained by solving the following problem:

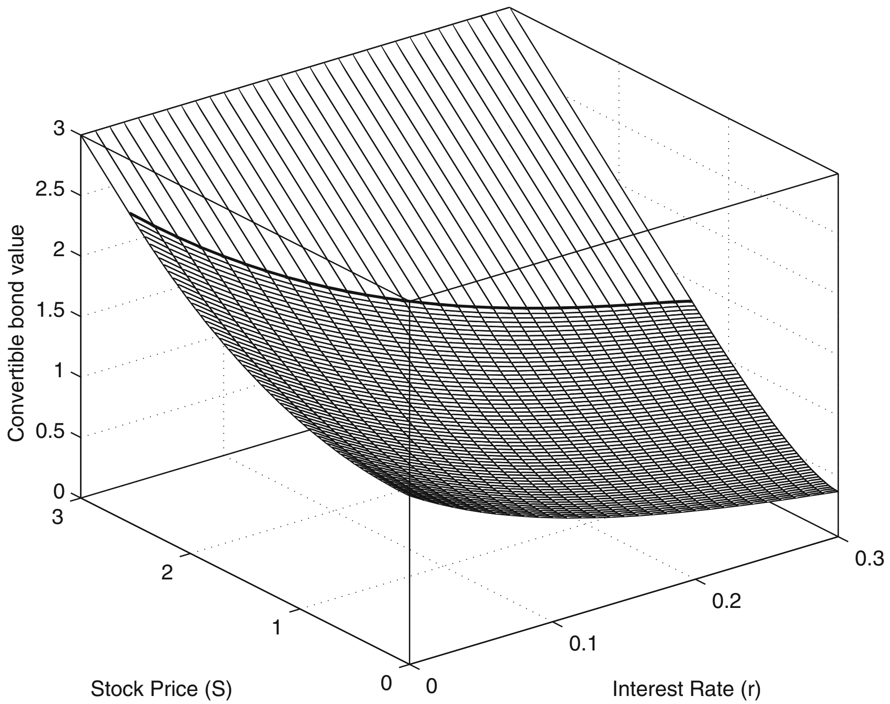

Fig. 5.4. The price of a two-factor convertible bond with $T = 30$ years

$$\begin{cases} \dfrac{\partial B_c}{\partial t} + \mathbf{L_{s,r}} B_c + kZ = 0, & 0 \le S \le S_f(r,t), \quad r_l \le r \le r_u, \\ & 0 \le t \le T, \\ B_c(S,r,T) = \max(Z, nS), & 0 \le S \le S_f(r,T), \quad r_l \le r \le r_u, \\ B_c(S_f(r,t),r,t) = nS_f(r,t), & r_l \le r \le r_u, \quad 0 \le t \le T, \\ \dfrac{\partial B_c}{\partial S}(S_f(r,t),r,t) = n, & r_l \le r \le r_u, \quad 0 \le t \le T, \\ S_f(r,T) = \max\left(\dfrac{Z}{n}, \dfrac{kZ}{D_0 n}\right), & r_l \le r \le r_u; \end{cases} \tag{5.96}$$

whereas in subdomain II, $B_c(S,r,t) = nS$. The problem (5.96) is usually called a free-boundary problem for convertible bonds.

The starting location of the free boundary

$$S_f(r,T) = \max\left(\frac{Z}{n}, \frac{kZ}{D_0 n}\right)$$

goes to infinity as D_0 tends to zero, so there is no free boundary at time T. Because

$$\left(\frac{\partial}{\partial t} + \mathbf{L_{S,r}}\right) nS + kZ \ge 0$$

always holds when $D_0 = 0$, no free boundary can appear at any time. This means that there is no free boundary when $D_0 = 0$. Thus, the value of a convertible bond in this case is determined by the problem (5.91).

In Fig. 5.4, the price of a two-factor convertible bond with $D_0 = 0.05$ is shown. For this case, there is only one free boundary, which confirms our assumption. The result there is obtained by the singularity-separating finite-difference method, which will be described in Chap. 9.

A convertible bond can also have a call feature that gives the company the right to purchase back the bond at any time (or during specified periods) for a fixed amount M_1. In this case, the price of the bond must not exceed M_1 because no one will spend an amount more than M_1 to buy a bond that can be purchased back for an amount M_1 at any time. When we evaluate the price of such a bond, the constraint

$$B_c(S, r, t) \le M_1, \; 0 \le S, \quad r_l \le r \le r_u, \quad 0 \le t \le T \tag{5.97}$$

is required. Because of this condition, the price of a convertible bond with a call feature can be less than a convertible bond without this feature. Because the company gets more rights, the buyer of the bond should be asked to pay less money.

A convertible bond can also incorporate a put feature, which means that the owner of the convertible bond can return the bond to the company for an amount M_2 at any time. Now we must impose the constraint

$$B_c(S, r, t) \ge M_2, \quad 0 \le S, \quad r_l \le r \le r_u, \quad 0 \le t \le T. \tag{5.98}$$

This condition might increase the value of the bond. The owner of the bond has more rights, so he usually needs to pay more money in order to purchase such a bond.

Just like the constraint (5.92), the constraint (5.97) or the constraint (5.98) may induce a free boundary or make the free boundary more complicated. For example, for a convertible bond with a call feature, the location of the free boundary at $t = T$ is

$$S_f(r, T) = \min\left(\frac{M_1}{n}, \; \max\left(\frac{Z}{n}, \frac{kZ}{D_0 n}\right)\right), \quad r_l \le r \le r_u. \tag{5.99}$$

If we assume that r is a given function of t, then the bond is a one-factor convertible bond. When $D_0 > 0$, the free-boundary problem is

$$\begin{cases} \dfrac{\partial B_c}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 B_c}{\partial S^2} + [r(t) - D_0]S\dfrac{\partial B_c}{\partial S} - r(t)B_c + kZ = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 \le S \le S_f(t), \quad 0 \le t \le T, \\ B_c(S,T) = \max(Z, nS), \qquad\qquad 0 \le S \le S_f(T), \\ B_c(S_f(t),t) = nS_f(t), \qquad\qquad 0 \le t \le T, \\ \dfrac{\partial B_c}{\partial S}(S_f(t),t) = n, \qquad\qquad 0 \le t \le T, \\ S_f(T) = \max\left(\dfrac{Z}{n}, \dfrac{kZ}{D_0 n}\right). \end{cases} \tag{5.100}$$

This problem is only a little different from an American call option problem. Similar to an American option, it can be proved rigorously that there is no free boundary if $D_0 = 0$ for the case $r = \mathit{constant}$. This is left as Problem 29 for readers.

Problems

Table 5.1. Problems and sections

Problems	Sections	Problems	Sections	Problems	Sections
1–3	5.2	4–9	5.3	10	5.4
11–17	5.5	18–25	5.6	26–30	5.7

1. (a) *Suppose the short rate is a known function $r(t)$. Consider a bond with a face value Z and assume that it pays a coupon with a coupon rate $k(t)$, that is, during a time interval $(t, t+dt]$, the coupon payment is $Zk(t)dt$. Show that the value of the bond is

$$V(t) = Z\mathrm{e}^{-\int_t^T r(\tau)d\tau}\left[1 + \int_t^T k(\bar{\tau})\mathrm{e}^{\int_{\bar{\tau}}^T r(\tau)d\tau} d\bar{\tau}\right].$$

(b) Suppose that $r(t)$ and $k(t)$ are equal to constants r and k, respectively. Show that in this case,

$$V(t) = Z\mathrm{e}^{-r(T-t)}[1 + k(\mathrm{e}^{r(T-t)} - 1)/r].$$

(c) Suppose that the bond pays coupon payments at two specified dates T_1 and T_2 before the maturity date T and the payments are Zk_1 and Zk_2, respectively. According to the formula given in part (a), and assuming

$T_1 < T_2$, find the values of the bond for $t \in [0, T_1), t \in (T_1, T_2)$, and $t \in (T_2, T)$, respectively, and give a financial interpretation of these expressions.

2. Suppose that the short rate r satisfies

$$dr = udt + w(t)dX,$$

where dX is a Wiener process. Assume that during the time period $[0, t^*]$, for example, t^* being 1 or 3 months, the interest rate is equal to the short rate r. Thus the price of a zero-coupon bond at $t = 0$ with face value one and maturity date t^* is e^{-rt^*}. Because the zero-coupon bond can be traded on the market, we can take $\Pi = V(r,t) - \Delta \mathrm{e}^{-rt^*}$ as the portfolio in order to derive the PDE for $V(r,t)$, the price of an interest rate derivative. Derive the PDE for $V(r,t)$ in this way.

3. Suppose that the short rate r satisfies

$$dr = udt + w(t)dX,$$

where dX is a Wiener process.

(a) Find the stochastic equation for $B(r) = \mathrm{e}^{-rt^*}$ by using Itô's lemma, where t^* is equal to, for example, 1 or 3 months.

(b) As we know, $B(r)$ is the price of a zero-coupon bond at $t = 0$ with face value one and maturity date t^* if during the time period $[0, t^*]$ the interest rate is a constant. $\overline{V}(B,t)$ is any derivative on the zero-coupon bond. Derive the PDE for $\overline{V}(B,t)$ by using Itô's lemma directly.

(c) As we know, if $dr = udt + w(t)dX$, then the price of any derivative security on r, $V(r,t)$, should satisfy the equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}w^2(t)\frac{\partial^2 V}{\partial r^2} + [u - \lambda w(t)]\frac{\partial V}{\partial r} - rV + kZ = 0,$$

where kZ is the coupon of the derivative. Define $V(r,t) = \overline{V}(B(r),t)$. Find the PDE for $V(r,t)$ from PDE obtained in part (b) by using transformation. This equation should be the same as the equation given here. Based on this fact, determine λ.

4. Suppose that $a(r,t) = \sum_{i=0}^{\infty} a_i(t)r^i$ and $b(r,t) = \sum_{i=0}^{\infty} b_i(t)r^i$ and require that the problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + a(r,t)\dfrac{\partial^2 V}{\partial r^2} + b(r,t)\dfrac{\partial V}{\partial r} - rV = 0, & 0 \le t \le T, \\ V(r,T) = 1 \end{cases}$$

has a solution in the form

$$V(r,t) = \mathrm{e}^{A(t) - rB(t)}.$$

Show that in order to fulfill this requirement, between a_i and $b_i, i = 2, 3, \cdots$, there must exist the following relations:

$$a_i B - b_i = 0, \quad i = 2, 3, \cdots .$$

This means that in order to choose $a(r,t)$ and $b(r,t)$ independently and for the solution to be in the form $\mathrm{e}^{A(t)-rB(t)}$, we have to assume $a(r,t) = a_0(t) + a_1(t)r$ and $b(r,t) = b_0(t) + b_1(t)r$.

5. Suppose that $a(r,t) = a_0(t) + a_1(t)r$ and $b(r,t) = b_0(t) + b_1(t)r$. Show that the problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + a(r,t)\dfrac{\partial^2 V}{\partial r^2} + b(r,t)\dfrac{\partial V}{\partial r} - rV = 0, \qquad 0 \le t \le T, \\ V(r,T) = 1 \end{cases}$$

has a solution in the form

$$V(r,t) = \mathrm{e}^{A(t)-rB(t)}$$

with $A(T) = B(T) = 0$ and determine the system of ordinary differential equations the functions $A(t)$ and $B(t)$ should satisfy.

6. *In the Vasicek model, the short rate is assumed to satisfy

$$dr = (\bar{\mu} - \gamma r)dt + \sqrt{-\beta}dX, \qquad \beta < 0, \qquad \gamma > 0,$$

where $\bar{\mu}$, γ, and β are constants, and dX is a Wiener process. Let the market price of risk $\lambda(r,t) = \bar{\lambda}\sqrt{-\beta}$. Then, the price $V(r,t;T)$ of a zero-coupon bond maturing at time T with a face value Z is the solution of the problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}(-\beta)\dfrac{\partial^2 V}{\partial r^2} + (\mu - \gamma r)\dfrac{\partial V}{\partial r} - rV = 0, \\ \qquad\qquad\qquad\qquad -\infty < r < \infty, \quad 0 \le t \le T, \\ V(r,T;T) = Z, \qquad -\infty < r < \infty, \quad 0 \le t \le T, \end{cases}$$

where

$$\mu = \bar{\mu} + \bar{\lambda}\beta.$$

(a) Show that this problem has a solution in the form

$$V(r,t;T) = Z\mathrm{e}^{A(t,T)-rB(t,T)}$$

and A and B are the solution of the system of ordinary differential equations

$$\begin{cases} \dfrac{dA}{dt} = \dfrac{1}{2}\beta B^2 + \mu B, \\ \dfrac{dB}{dt} = \gamma B - 1 \end{cases}$$

with the conditions

$$\begin{aligned} A(T,T) &= 0, \\ B(T,T) &= 0. \end{aligned}$$

(b) Find the solution of the above problem of ordinary differential equations by solving the two ODEs and show that the expressions of A and B can be rewritten as

$$\begin{cases} A = -\left(\dfrac{\beta}{2\gamma^2} + \dfrac{\mu}{\gamma}\right)(T-t) + \left(\dfrac{\beta}{2\gamma^2} + \dfrac{\mu}{\gamma}\right) B + \dfrac{\beta}{4\gamma} B^2, \\ B = \dfrac{1}{\gamma}\left(1 - \mathrm{e}^{-\gamma(T-t)}\right) \end{cases}$$

if the solution obtained is not in this form.

7. Show

$$\begin{aligned} &\lim_{\alpha\to 0}\left\{\frac{\beta}{\alpha}B + \left[\frac{\beta(\gamma-\psi)}{\alpha(\gamma+\psi)\psi} + \mu\frac{\gamma-\psi}{\alpha\psi}\right]\ln\frac{(\gamma-\psi)/\alpha}{B+(\gamma-\psi)/\alpha}\right. \\ &\quad \left. - \left[\frac{\beta(\gamma+\psi)}{\alpha(\gamma-\psi)\psi} + \mu\frac{\gamma+\psi}{\alpha\psi}\right]\ln\frac{(\gamma+\psi)/\alpha}{B+(\gamma+\psi)/\alpha}\right\} \\ &= -\left(\frac{\beta}{2\gamma^2}+\frac{\mu}{\gamma}\right)(T-t) + \left(\frac{\beta}{2\gamma^2}+\frac{\mu}{\gamma}\right)B + \frac{\beta B^2}{4\gamma}, \end{aligned}$$

where

$$B(t,T) = \frac{1}{\gamma}\left(1 - \mathrm{e}^{-\gamma(T-t)}\right) \quad \text{and} \quad \psi = \sqrt{\gamma^2 + 2\alpha}.$$

(The two sides are two expressions for $A(t,T)$ associated with the Vasicek model obtained by different approaches. This confirms that the two different approaches give the same answer.)

8. *In the Cox–Ingersoll–Ross model, the short rate is assumed to satisfy

$$dr = (\mu - \bar{\gamma}r)dt + \sqrt{\alpha r}dX,$$

where μ, $\bar{\gamma}$, and α are constants, and dX is a Wiener process. Let the market price of risk $\lambda(r,t)$ be $\bar{\lambda}\sqrt{\alpha r}$. Then, the price $V(r,t;T)$ of a zero-coupon bond maturing at time T with a face value Z is the solution of the problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\alpha r\dfrac{\partial^2 V}{\partial r^2} + (\mu - \gamma r)\dfrac{\partial V}{\partial r} - rV = 0, & 0 \le r, \quad 0 \le t \le T, \\ V(r,T;T) = Z, & 0 \le r, \end{cases}$$

where $\gamma = \bar{\gamma} + \bar{\lambda}\alpha$.

(a) Show that this problem has a solution in the form

$$V(r,t;T) = Z\mathrm{e}^{A(t,T)-rB(t,T)}$$

and A and B are the solutions of the system of ordinary differential equations

$$\begin{cases} \dfrac{dA}{dt} = \mu B, \\ \dfrac{dB}{dt} = \dfrac{1}{2}\alpha B^2 + \gamma B - 1 \end{cases}$$

with the conditions

$$A(T,T) = 0$$

and

$$B(T,T) = 0.$$

(b) Find the solution of the above problem of ordinary differential equations by solving the two ODEs and show that the expressions of A and B can be rewritten as

$$\begin{cases} A = \ln\left(\dfrac{2\psi \mathrm{e}^{(\gamma+\psi)(T-t)/2}}{(\gamma+\psi)\mathrm{e}^{\psi(T-t)} - (\gamma-\psi)}\right)^{2\mu/\alpha}, \\ B = \dfrac{2(\mathrm{e}^{\psi(T-t)}-1)}{(\gamma+\psi)\mathrm{e}^{\psi(T-t)} - (\gamma-\psi)} \\ \qquad \text{with } \psi = \sqrt{\gamma^2 + 2\alpha} \end{cases}$$

if the solution obtained is not in this form.

9. Show

$$Z\left[\frac{B+(\gamma-\psi)/\alpha}{(\gamma-\psi)/\alpha}\right]^{\mu(\psi-\gamma)/\alpha\psi} \left[\frac{B+(\gamma+\psi)/\alpha}{(\gamma+\psi)/\alpha}\right]^{\mu(\gamma+\psi)/\alpha\psi} \mathrm{e}^{-rB}$$
$$\equiv Z\left[\frac{2\psi \mathrm{e}^{(\gamma+\psi)(T-t)/2}}{(\gamma+\psi)\mathrm{e}^{\psi(T-t)} - (\gamma-\psi)}\right]^{2\mu/\alpha} \mathrm{e}^{-rB},$$

where

$$B = \frac{2\left(\mathrm{e}^{\psi(T-t)}-1\right)}{(\gamma+\psi)\,\mathrm{e}^{\psi(T-t)} - (\gamma-\psi)}.$$

(The two sides are two expressions for the zero-coupon bond price associated with the Cox–Ingersoll–Ross model obtained by different approaches. This confirms that the two different approaches give the same answer.)

10. *Describe a way to determine the market price of risk for the short rate.
11. *Suppose that any European-style interest rate derivative with a continuous coupon satisfies the equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + (u-\lambda w)\frac{\partial V}{\partial r} - rV + k = 0, \quad r_l \le r \le r_u, \quad t \le T,$$

where k is the coupon rate corresponding to the derivative, the coefficients u and w satisfy the reversion conditions on the boundaries $r = r_l$, $r = r_u$, and λ is a given bounded function. Describe how to evaluate the price of a European call option on a bond with coupon by using this equation.

12. (a) Let $Z(t;T^*)$ be the price of a zero-coupon bond with a face value of one dollar and with maturity date T^* at time t and let $\bar{f}(t,T,T+\frac{1}{2})$ be the forward interest rate compounded semiannually at time t for the period $(T,T+\frac{1}{2})$. Show

$$\bar{f}\left(t,T,T+\frac{1}{2}\right) = 2\left[\frac{Z(t;T)}{Z(t;T+1/2)} - 1\right].$$

(b) There is a cash flow $\frac{1}{2}\bar{f}(t_{k-1},t_{k-1},t_k)$, t_k being $t+k/2$, $k=1, 2, \cdots, 2N$ and t_0 being t. Find the value of the cash flow at time t.

(c) *Show that the value of an N-year swap with swap rate r_s and with notional principal Q is

$$V_s(T;r_s) = Q\left[1 - Z(T;T+N) - \frac{r_s}{2}\sum_{k=1}^{2N} Z(T;T+k/2)\right],$$

where T is the time the swap initiates.

13. Show that the solution of the problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}w^2\dfrac{\partial^2 V}{\partial r^2} + (u-\lambda w)\dfrac{\partial V}{\partial r} - rV = 0, \\ \qquad\qquad\qquad\qquad r_l \le r \le r_u, \quad t \le T, \\ V(r,T;T) = 1, \qquad r_l \le r \le r_u \end{cases}$$

is the same as that of the problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}w^2\dfrac{\partial^2 V}{\partial r^2} + (u-\lambda w)\dfrac{\partial V}{\partial r} - rV + \delta(t-T) = 0, \\ \qquad\qquad\qquad\qquad r_l \le r \le r_u, \quad t \le T, \\ V(r,T^+;T) = 0, \qquad r_l \le r \le r_u \end{cases}$$

for any $t < T$.

14. Let $V_{s1k}(r,T)$ denote the price of a $(k/2)$-year zero-coupon bond, $k = 1, 2, \cdots, 2N$, and we want to get $\sum\limits_{k=1}^{2N} V_{s1k}(r,T)$. Consider the following procedures. The first one is to solve the following problems

$$\begin{cases} \dfrac{\partial V_{s1k}}{\partial t} + \dfrac{1}{2}w^2\dfrac{\partial^2 V_{s1k}}{\partial r^2} + (u-\lambda w)\dfrac{\partial V_{s1k}}{\partial r} - rV_{s1k} = 0, \quad r_l \le r \le r_u, \\ \qquad\qquad\qquad\qquad T \le t \le T+k/2, \\ V_{s1k}(r,T+k/2) = 1, \qquad r_l \le r \le r_u, \end{cases}$$

$k = 1, 2, \cdots, 2N$, and then obtain $\sum_{k=1}^{2N} V_{s1k}(r, T)$ by adding $V_{s1k}(r, T)$, $k = 1, 2, \cdots, 2N$, together. The second one is to solve the problem:

$$\begin{cases} \dfrac{\partial V_{s1}}{\partial t} + \dfrac{1}{2}w^2 \dfrac{\partial^2 V_{s1}}{\partial r^2} + (u - \lambda w)\dfrac{\partial V_{s1}}{\partial r} - rV_{s1} \\ \qquad + \sum_{k=1}^{2N} \delta(t - T - k/2) = 0, \\ \qquad\qquad r_l \le r \le r_u, \quad T \le t \le T + N, \\ V_{s1}(r, T + N) = 0, \quad r_l \le r \le r_u. \end{cases}$$

(a) Show $V_{s1}(r, T) = \sum_{k=1}^{2N} V_{s1k}(r, T)$ holds.

(b) In order to get $\sum_{k=1}^{2N} V_{s1k}(r, T)$, which procedure is better and why?

15. Suppose that the solution of

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}w^2 \dfrac{\partial^2 V}{\partial r^2} + (u - \lambda w)\dfrac{\partial V}{\partial r} - rV = 0, & r_l \le r \le r_u, \ \ t^* \le t \le t_k, \\ V(r, t_k; t_k) = 1, & r_l \le r \le r_u \end{cases}$$

is $V(r, t; t_k)$ and that $V(r^*, t^*; t_k) = Z(t^*; t_k)$. Also assume that $V_s(r, t; r_s)$ is the solution of

$$\begin{cases} \dfrac{\partial V_s}{\partial t} + \dfrac{1}{2}w^2 \dfrac{\partial^2 V_s}{\partial r^2} + (u - \lambda w)\dfrac{\partial V_s}{\partial r} - rV_s - \sum_{k=k^*+1}^{2N} \dfrac{Qr_s}{2}\delta(t - t_k) \\ \qquad + Q\left[1 + \dfrac{\bar{f}(t_{k^*}, t_{k^*}, t_{k^*+1})}{2}\right]\delta(t - t_{k^*+1}) = 0, \\ \qquad\qquad r_l \le r \le r_u, \quad t^* \le t \le T + N, \\ V_s(r, T + N; r_s) = -Q, \quad r_l \le r \le r_u. \end{cases}$$

Here, $V_s(r, t; r_s)$ actually is the value of a swap. Q and r_s are the notional principal and the swap rate, respectively. t^*, T, and N denote the time today, the time the swap is initiated, and the duration of the swap with the relation $T \le t^* < T + N$. k^* is the integer part of $(t^* - T)/2$, and $t_k = T + k/2$, $k = k^* + 1, k^* + 2, \cdots, 2N$. $\bar{f}(t_{k^*}, t_{k^*}, t_{k^*+1})$ is the 6-month LIBOR for the period $[t_{k^*}, t_{k^*+1}]$ determined at time t_{k^*}. Show

$$V_s(r^*, t^*; r_s) = QZ(t^*; t_{k^*+1})\left[1 + \frac{1}{2}\bar{f}(t_{k^*}, t_{k^*}, t_{k^*+1})\right] - Q\left[\sum_{k=k^*+1}^{2N} \frac{r_s}{2}Z(t^*; t_k) + Z(t^*; T + N)\right].$$

16. *Suppose that any European-style interest rate derivative satisfies the equation:

$$\frac{\partial V}{\partial t}+\frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2}+(u-\lambda w)\frac{\partial V}{\partial r}-rV+f(t)=0, \quad r_l\le r\le r_u,$$

where all the coefficients in the equation are known. Let $V_{so}(r,t)$ be the value of a T-year swaption on a N-year swap. Its payoff is

$$Q\max\left(1-Z(T;T+N)-\frac{r_{se}}{2}\sum_{k=1}^{2N}Z(T;T+k/2),\, 0\right),$$

where Q is the notional principal, r_{se} is the exercise swap rate, and $Z(T;T+k/2)$ is the value of zero-coupon bond with maturity $k/2$ at time T. Describe how to find the price of the swaption, including to find the payoff of the swaption, by solving this equation from $T+N$ to T and from T to 0.

17. Consider an N-year floor with a floor rate r_f. Suppose that the money will be paid quarterly at time $t_k=t^*+k/4$, $k=2,3,\cdots,4N$, and the floating rate is the 3-month LIBOR. Suppose that $V_{bk}(r,t)$ is the solution of the problem

$$\begin{cases}\dfrac{\partial V_{bk}}{\partial t}+\dfrac{1}{2}w^2\dfrac{\partial^2 V_{bk}}{\partial r^2}+(u-\lambda w)\dfrac{\partial V_{bk}}{\partial r}-rV_{bk}=0,\\ \qquad\qquad r_l\le r\le r_u,\quad t_{k-1}\le t\le t_k,\\ V_{bk}(r,t_k)=Q\left(1+\dfrac{r_f}{4}\right), \qquad r_l\le r\le r_u,\end{cases}$$

where $k=2,3,\cdots,4N$ and $V_f(r,t)$ is the solution of the problem

$$\begin{cases}\dfrac{\partial V_f}{\partial t}+\dfrac{1}{2}w^2\dfrac{\partial^2 V_f}{\partial r^2}+(u-\lambda w)\dfrac{\partial V_f}{\partial r}-rV_f\\ \qquad +\displaystyle\sum_{k=2}^{4N}\max(V_{bk}(r,t_{k-1})-Q,0)\delta(t-t_{k-1})=0,\\ \qquad\qquad r_l\le r\le r_u,\quad t^*\le t\le t_{4N-1},\\ V_f(r,t_{4N-1})=0, \qquad r_l\le r\le r_u.\end{cases}$$

Show that the premium of the floor should be

$$V_f(r^*,t^*),$$

where r^* is the short rate at time t^*.

18. (a) $\mathbf{S}$ is a random vector and its covariance matrix is $\mathbf{B}$. Let $\bar{\mathbf{S}}=\mathbf{AS}$, $\mathbf{A}$ being a constant matrix, and its covariance matrix be $\mathbf{C}$. Find the relation among $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$.

(b) How do we choose **A** so that **C** will be a diagonal matrix?
(c) *Suppose that $\bar{S}_1, \bar{S}_2, \cdots, \bar{S}_K$ are variables and $\bar{S}_{K+1}, \bar{S}_{K+2}, \cdots, \bar{S}_N$ are fixed numbers. Find the dependence of $S_{K+1}, S_{K+2}, \cdots, S_N$ on S_1, S_2, $\cdots$, S_K.

19. (a) Suppose that there is a domain Ω on the (Z_1, Z_2)-plane, the boundary of Ω is Γ, and $(n_1, n_2)^T$ is the outer normal vector of the boundary Γ. Assume that Z_1 and Z_2 are two stochastic processes and satisfy the system of stochastic differential equations:

$$dZ_i = \mu_i(Z_1, Z_2, t)dt + \sigma_i(Z_1, Z_2, t)dX_i \quad \text{with} \quad \sigma_i \geq 0, \quad i = 1, 2,$$

where $dX_i, i = 1, 2$, are the Wiener processes and $\mathrm{E}\,[dX_1 dX_2] = \rho_{12}dt$ with $\rho_{12} \in [-1, 1]$. Suppose that at $t = 0$, $(Z_1, Z_2) \in \Omega$. Show that in order to guarantee $(Z_1, Z_2) \in \Omega$ for any time $t \in [0, T]$, we need to require, for any $t \in [0, T]$ and for any point on Γ, the following condition to be held:

(i) if $n_1 \neq 0$ and $n_2 = 0$, then

$$\begin{cases} n_1\mu_1 \leq 0, \\ \sigma_1 = 0; \end{cases}$$

(ii) if $n_1 = 0$ and $n_2 \neq 0$, then

$$\begin{cases} n_2\mu_2 \leq 0, \\ \sigma_2 = 0; \end{cases}$$

(iii) if $n_1 \neq 0$ and $n_2 \neq 0$, then

$$\begin{cases} n_1\mu_1 + n_2\mu_2 \leq 0, \\ n_1\sigma_1 - \mathrm{sign}(n_1 n_2) n_2 \sigma_2 = 0, \quad \text{and} \quad \rho_{12} = -\mathrm{sign}(n_1 n_2), \end{cases}$$

where

$$\mathrm{sign}(n_1 n_2) = \begin{cases} 1, & \text{if } n_1 n_2 > 0, \\ -1, & \text{if } n_1 n_2 < 0. \end{cases}$$

If a point is a corner point, then there are two normals and we need to require this condition to be held for the two outer normal vectors.

(b) Suppose that the domain Ω is $Z_{1l} \leq Z_1 \leq 1$ and $Z_{2l} \leq Z_2 \leq Z_1$, where Z_{1l} and Z_{2l} are constants, and $Z_{1l} \geq Z_{2l}$. Find the concrete condition for each segment of the boundary according to the condition given in part (a).

20. Assume that Z_1, Z_2, Z_3 are random variables and satisfy the system of stochastic differential equations:

$$dZ_i = \mu_i \left(Z_1, Z_2, Z_3, t\right) dt + \sigma_i \left(Z_1, Z_2, Z_3, t\right) dX_i, \quad i = 1, 2, 3,$$

where dX_i are the Wiener processes and $\mathrm{E}\left[dX_i dX_j\right] = \rho_{i,j} dt$ with $\rho_{i,j} \in [-1, 1]$. In order to guarantee that if a point is in a domain Ω at time t^*, then the point is still in the domain Ω at $t = t^* + dt$ for a positive dt, it is necessary to require that the condition

$$n_1 dZ_1 + n_2 dZ_2 + n_3 dZ_3 \le 0$$

holds at any point on the boundary of the domain Ω, where n_1, n_2, and n_3 are the three components of the outer normal vector of the boundary at the point. Suppose that the domain Ω is $\{Z_{1,l} \le Z_1 \le 1, \quad Z_{2,l} \le Z_2 \le Z_1, \quad Z_{3,l} \le Z_3 \le Z_2\}$. Show that on the surfaces $Z_1 = 1$, $Z_2 = Z_{2,l}$, and $Z_3 = Z_2$, the condition is equivalent to $\{\mu_1 \le 0,\ \sigma_1 = 0\}$, $\{\mu_2 \ge 0,\ \sigma_2 = 0\}$, and $\{-\mu_2 + \mu_3 \le 0,\ \sigma_2 = \sigma_3,\ \rho_{2,3} = 1\}$, respectively.

21. Suppose that $\sigma_1 \left(Z_l, Z_2, Z_3, t\right)$, $\sigma_2 \left(Z_l, Z_2, Z_3, t\right)$, and $\sigma_3 \left(Z_l, Z_2, Z_3, t\right)$ are defined on $\Omega : \{Z_{1,l} \le Z_1 \le 1,\ Z_{2,l} \le Z_2 \le Z_1,\ Z_{3,l} \le Z_3 \le Z_2\}$. Assume that

    (i) $\sigma_1 \left(Z_{1,l}, Z_2, Z_3, t\right) = 0$ on surface I: $\{Z_1 = Z_{1,l},\ Z_{2,l} \le Z_2 \le Z_1,\ Z_{3,l} \le Z_3 \le Z_2\}$.

    (ii) $\sigma_1 \left(1, Z_2, Z_3, t\right) = 0$ on surface II: $\{Z_1 = 1,\ Z_{2,l} \le Z_2 \le Z_1,\ Z_{3,l} \le Z_3 \le Z_2\}$.

    (iii) $\sigma_2 \left(Z_1, Z_{2,l}, Z_3, t\right) = 0$ on surface III: $\{Z_{1,l} \le Z_1 \le 1,\ Z_2 = Z_{2,l},\ Z_{3,l} \le Z_3 \le Z_2\}$.

    (iv) $\sigma_1 \left(Z_1, Z_1, Z_3, t\right) = \sigma_2 \left(Z_1, Z_1, Z_3, t\right), \quad \rho_{1,2} \left(Z_1, Z_1, Z_3, t\right) = 1$ on surface IV: $\{Z_{1,l} \le Z_1 \le 1,\ Z_2 = Z_1,\ Z_{3,l} \le Z_3 \le Z_2\}$.

    (v) $\sigma_3 \left(Z_1, Z_2, Z_{3,l}, t\right) = 0$ on surface V: $\{Z_{1,l} \le Z_1 \le 1,\ Z_{2,l} \le Z_2 \le Z_1,\ Z_3 = Z_{3,l}\}$.

    (vi) $\sigma_2 \left(Z_1, Z_2, Z_2, t\right) = \sigma_3 \left(Z_1, Z_2, Z_2, t\right), \quad \rho_{2,3} \left(Z_1, Z_2, Z_2, t\right) = 1$ on surface VI: n $\{Z_{1,l} \le Z_1 \le 1,\ Z_{2,l} \le Z_2 \le Z_1,\ Z_3 = Z_2\}$.

    Define

$$\begin{cases} \xi_1 = \dfrac{Z_1 - Z_{1,l}}{1 - Z_{1,l}}, \\ \xi_2 = \dfrac{Z_2 - Z_{2,l}}{Z_1 - Z_{2,l}}, \\ \xi_3 = \dfrac{Z_3 - Z_{3,l}}{Z_2 - Z_{3,l}}, \end{cases}$$

and

$$\begin{cases} \tilde{\sigma}_1^2(\xi_1, \xi_2, \xi_3, t) = \dfrac{\sigma_1^2}{(1 - Z_{1,l})^2}, \\ \tilde{\sigma}_2^2(\xi_1, \xi_2, \xi_3, t) = \dfrac{\sigma_1^2\xi_2^2 - 2\sigma_1\sigma_2\xi_2\rho_{1,2} + \sigma_2^2}{(Z_1 - Z_{2,l})^2}, \\ \tilde{\sigma}_3^2(\xi_1, \xi_2, \xi_3, t) = \dfrac{\sigma_2^2\xi_3^2 - 2\sigma_2\sigma_3\xi_3\rho_{2,3} + \sigma_3^2}{(Z_2 - Z_{3,l})^2}. \end{cases}$$

Show that the assumption on σ_1, σ_2, and σ_3 is equivalent to

$$\begin{cases} \tilde{\sigma}_1(0, \xi_2, \xi_3, t) = \tilde{\sigma}_1(1, \xi_2, \xi_3, t) = 0, & 0 \le \xi_2 \le 1,\ 0 \le \xi_3 \le 1, \\ \tilde{\sigma}_2(\xi_1, 0, \xi_3, t) = \tilde{\sigma}_2(\xi_1, 1, \xi_3, t) = 0, & 0 \le \xi_1 \le 1,\ 0 \le \xi_3 \le 1, \\ \tilde{\sigma}_3(\xi_1, \xi_2, 0, t) = \tilde{\sigma}_3(\xi_1, \xi_2, 1, t) = 0, & 0 \le \xi_1 \le 1,\ 0 \le \xi_2 \le 1. \end{cases}$$

22. (a) Show that under the transformation

$$\begin{cases} \xi_1 = \dfrac{Z_1 - Z_{1,l}}{1 - Z_{1,l}}, \\ \xi_2 = \dfrac{Z_2 - Z_{2,l}}{Z_1 - Z_{2,l}}, \end{cases}$$

the partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_i\sigma_j\rho_{i,j}\frac{\partial^2 V}{\partial Z_i \partial Z_j} + r\sum_{i=1}^{2} Z_i \frac{\partial V}{\partial Z_i} - rV = 0$$

becomes

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\tilde{\sigma}_i\tilde{\sigma}_j\tilde{\rho}_{i,j}\frac{\partial^2 V}{\partial \xi_i \partial \xi_j} + \sum_{i=1}^{2} b_i \frac{\partial V}{\partial \xi_i} - rV = 0,$$

where

$$\begin{cases} b_1 = \dfrac{rZ_1}{1 - Z_{1,l}}, \\ b_2 = \dfrac{r(Z_2 - Z_1\xi_2)}{Z_1 - Z_{2,l}} + \dfrac{\sigma_1(\sigma_1\xi_2 - \sigma_2\rho_{1,2})}{(Z_1 - Z_{2,l})^2}, \end{cases}$$

and $\tilde{\sigma}_1$, $\tilde{\sigma}_2$, $\tilde{\rho}_{1,2}$, are determined by

$$\begin{cases} \dfrac{1}{2}\tilde{\sigma}_1^2 = \dfrac{\frac{1}{2}\sigma_1^2}{(1 - Z_{1,l})^2}, \\ \dfrac{1}{2}\tilde{\sigma}_2^2 = \dfrac{\frac{1}{2}\left(\sigma_1^2\xi_2^2 - 2\sigma_1\sigma_2\xi_2\rho_{1,2} + \sigma_2^2\right)}{(Z_1 - Z_{2,l})^2}, \\ \tilde{\sigma}_1\tilde{\sigma}_2\tilde{\rho}_{1,2} = \dfrac{\sigma_1(\sigma_2\rho_{1,2} - \sigma_1\xi_2)}{(1 - Z_{1,l})(Z_1 - Z_{2,l})}. \end{cases}$$

(b) Show further that the expression of b_2 can be rewritten as

$$b_2 = \frac{r\left(Z_2 - Z_1\xi_2\right)}{Z_1 - Z_{2,l}} - \frac{\tilde{\sigma}_1\tilde{\sigma}_2\tilde{\rho}_{1,2}\left(1 - Z_{1,l}\right)}{Z_1 - Z_{2,l}}.$$

(c) $\tilde{\sigma}_i$ and b_i given above are functions of ξ_1, ξ_2, t and let $\tilde{\sigma}_i(\xi_1, \xi_2, t)$ and $b_i(\xi_1, \xi_2, t)$ denote these functions, $i = 1$ and 2. Show that if

$$\begin{cases} \tilde{\sigma}_1(0, \xi_2, t) = \tilde{\sigma}_1(1, \xi_2, t) = 0, \quad 0 \le \xi_2 \le 1, \\ \tilde{\sigma}_2(\xi_1, 0, t) = \tilde{\sigma}_2(\xi_1, 1, t) = 0, \quad 0 \le \xi_1 \le 1, \end{cases}$$

then

$$\begin{cases} b_1(0, \xi_2, t) \ge 0, \quad b_1(1, \xi_2, t) = 0, \quad 0 \le \xi_2 \le 1, \\ b_2(\xi_1, 0, t) \ge 0, \quad b_2(\xi_1, 1, t) = 0, \quad 0 \le \xi_1 \le 1. \end{cases}$$

(Hint: $r(\xi_1, \xi_2, t)|_{\xi_1=1} = 0$. This can be explained as follows. $\xi_1 = 1$ means $Z_1 = 1$, thus the zero-coupon bond curve must be flat near $T = 0$ and its derivative with respect to T at $T = 0$, $r(\xi_1, \xi_2, t)|_{\xi_1=1}$ equals 0. When $\tilde{\sigma}_i$, b_i, $i = 1, 2$, satisfy these conditions here, it can be proved that the final value problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sum\limits_{i=1}^{2}\sum\limits_{j=1}^{2} \tilde{\sigma}_i\tilde{\sigma}_j\tilde{\rho}_{i,j}\dfrac{\partial^2 V}{\partial \xi_i \partial \xi_j} + \sum\limits_{i=1}^{2} b_i \dfrac{\partial V}{\partial \xi_i} - rV = 0 \\ \qquad\qquad\qquad\qquad \text{on } [0,1] \times [0,1] \times [0,T], \\ V(\xi_1, \xi_2, T) = V_T(\xi_1, \xi_2) \qquad \text{on } [0,1] \times [0,1] \end{cases}$$

has a unique solution.)

23. (a) *Show that under the transformation

$$\begin{cases} \xi_1 = \dfrac{Z_1 - Z_{1,l}}{1 - Z_{1,l}}, \\ \xi_2 = \dfrac{Z_2 - Z_{2,l}}{Z_1 - Z_{2,l}}, \\ \xi_3 = \dfrac{Z_3 - Z_{3,l}}{Z_2 - Z_{3,l}}, \end{cases}$$

the partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} \sigma_i\sigma_j\rho_{i,j}\frac{\partial^2 V}{\partial Z_i \partial Z_j} + r\sum_{i=1}^{3} Z_i\frac{\partial V}{\partial Z_i} - rV = 0$$

becomes

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} \tilde{\sigma}_i\tilde{\sigma}_j\tilde{\rho}_{i,j}\frac{\partial^2 V}{\partial \xi_i \partial \xi_j} + \sum_{i=1}^{3} b_i\frac{\partial V}{\partial \xi_i} - rV = 0,$$

and find the expressions of $\tilde{\sigma}_1$, $\tilde{\sigma}_2$, $\tilde{\sigma}_3$, $\tilde{\rho}_{1,2}$, $\tilde{\rho}_{1,3}$, $\tilde{\rho}_{2,3}$, b_1, b_2, and b_3.

(b) Show

$$\begin{cases} \tilde{\sigma}_1(0,\xi_2,\xi_3,t) = \tilde{\sigma}_1(1,\xi_2,\xi_3,t) = 0, & 0 \le \xi_2 \le 1,\ 0 \le \xi_3 \le 1, \\ \tilde{\sigma}_2(\xi_1,0,\xi_3,t) = \tilde{\sigma}_2(\xi_1,1,\xi_3,t) = 0, & 0 \le \xi_1 \le 1,\ 0 \le \xi_3 \le 1, \\ \tilde{\sigma}_3(\xi_1,\xi_2,0,t) = \tilde{\sigma}_3(\xi_1,\xi_2,1,t) = 0, & 0 \le \xi_1 \le 1,\ 0 \le \xi_2 \le 1, \end{cases}$$

and

$$\begin{cases} b_1(0,\xi_2,\xi_3,t) \ge 0, & b_1(1,\xi_2,\xi_3,t) = 0, & 0 \le \xi_2 \le 1,\ 0 \le \xi_3 \le 1, \\ b_2(\xi_1,0,\xi_3,t) \ge 0, & b_2(\xi_1,1,\xi_3,t) = 0, & 0 \le \xi_1 \le 1,\ 0 \le \xi_3 \le 1, \\ b_3(\xi_1,\xi_2,0,t) \ge 0, & b_3(\xi_1,\xi_2,1,t) = 0, & 0 \le \xi_1 \le 1,\ 0 \le \xi_2 \le 1. \end{cases}$$

(When $\tilde{\sigma}_i$, b_i, $i = 1,2,3$, satisfy these conditions here, it can be proved that the final value problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} \tilde{\sigma}_i\tilde{\sigma}_j\tilde{\rho}_{i,j}\dfrac{\partial^2 V}{\partial \xi_i \partial \xi_j} + \sum_{i=1}^{3} b_i \dfrac{\partial V}{\partial \xi_i} - rV = 0 \\ \qquad\qquad \text{on } [0,1]\times[0,1]\times[0,1]\times[0,T], \\ V(\xi_1,\xi_2,\xi_3,T) = V_T(\xi_1,\xi_2,\xi_3) \quad \text{on } [0,1]\times[0,1]\times[0,1] \end{cases}$$

has a unique solution.)

24. Assume that Z_1, Z_2, Z_3 are random variables and satisfy the system of stochastic differential equations:

$$dZ_i = \mu_i\left(Z_1, Z_2, Z_3, t\right)dt + \sigma_i\left(Z_1, Z_2, Z_3, t\right)dX_i, \quad i = 1,2,3,$$

where dX_i are the Wiener processes and $\mathrm{E}\left[dX_i dX_j\right] = \rho_{ij}dt$ with $-1 \le \rho_{ij} \le 1$, and that ξ_1, ξ_2 and ξ_3 are governed by

$$d\xi_i = \tilde{\mu}_i(\xi_1,\xi_2,\xi_3,t)dt + \tilde{\sigma}_i(\xi_1,\xi_2,\xi_3,t)d\tilde{X}_i, \quad i = 1,2,3,$$

where $d\tilde{X}_i$ are the Wiener processes and $\mathrm{E}\left[d\tilde{X}_i d\tilde{X}_j\right] = \tilde{\rho}_{ij}dt$ with $-1 \le \tilde{\rho}_{ij} \le 1$. Furthermore, we suppose that ξ_1, ξ_2 and ξ_3 are defined by

$$\begin{cases} \xi_1 = \dfrac{Z_1 - Z_{1,l}}{1 - Z_{1,l}}, \\ \xi_2 = \dfrac{Z_2 - Z_{2,l}}{Z_1 - Z_{2,l}}, \\ \xi_3 = \dfrac{Z_3 - Z_{3,l}}{Z_2 - Z_{3,l}}, \end{cases}$$

where $Z_{1,l}$, $Z_{2,l}$, and $Z_{3,l}$ are constants. Find the expressions of $\tilde{\sigma}_1$, $\tilde{\sigma}_2$, $\tilde{\sigma}_3$, $\tilde{\rho}_{12}$, $\tilde{\rho}_{13}$, $\tilde{\rho}_{23}$ as functions of σ_1, σ_2, σ_3, $\rho_{1,2}$, $\rho_{1,3}$, $\rho_{2,3}$, Z_1, Z_2, and Z_3 by using Itô's lemma.

25. Suppose that ξ_1, ξ_2 and ξ_3 satisfy the system of stochastic differential equations:

$$d\xi_i = \tilde{\mu}_i(\xi_1, \xi_2, \xi_3, t)dt + \tilde{\sigma}_i(\xi_1, \xi_2, \xi_3, t)d\tilde{X}_i, \quad i = 1, 2, 3,$$

where $d\tilde{X}_i$ are the Wiener processes and $\mathrm{E}\left[d\tilde{X}_i d\tilde{X}_j\right] = \tilde{\rho}_{ij}dt$ with $-1 \leq \tilde{\rho}_{ij} \leq 1$. Define

$$\begin{cases} Z_1(\xi_1) &= Z_{1,l} + \xi_1(1 - Z_{1,l}), \\ Z_2(\xi_1, \xi_2) &= Z_{2,l} + \xi_2[Z_1(\xi_1) - Z_{2,l}] \\ &= Z_{2,l} + \xi_2[Z_{1,l} + \xi_1(1 - Z_{1,l}) - Z_{2,l}], \\ Z_3(\xi_1, \xi_2, \xi_3) &= Z_{3,l} + \xi_3\{Z_2(\xi_1, \xi_2) - Z_{3,l}\} \\ &= Z_{3,l} + \xi_3\{Z_{2,l} + \xi_2[Z_{1,l} + \xi_1(1 - Z_{1,l}) - Z_{2,l}] - Z_{3,l}\}. \end{cases}$$

Assume that $Z_1(\xi_1)$, $Z_2(\xi_1, \xi_2)$, and $Z_3(\xi_1, \xi_2, \xi_3)$ represent prices of three securities. Let $V(\xi_1, \xi_2, \xi_3, t)$ be the value of a derivative security. Setting a portfolio $\Pi = V - \Delta_1 Z_1(\xi_1) - \Delta_2 Z_2(\xi_1, \xi_2) - \Delta_3 Z_3(\xi_1, \xi_2, \xi_3)$ and using Itô's lemma, show that $V(\xi_1, \xi_2, \xi_3, t)$ satisfies the following PDE:

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\tilde{\sigma}_i\tilde{\sigma}_j\tilde{\rho}_{i,j}\frac{\partial^2 V}{\partial \xi_i \partial \xi_j} + \frac{rZ_1}{1 - Z_{1,l}}\frac{\partial V}{\partial \xi_1} \\ &+ \left[\frac{r(Z_2 - Z_1\xi_2)}{Z_1 - Z_{2,l}} - \frac{\tilde{\sigma}_1\tilde{\sigma}_2\tilde{\rho}_{1,2}(1 - Z_{1,l})}{Z_1 - Z_{2,l}}\right]\frac{\partial V}{\partial \xi_2} \\ &+ \left[\frac{r(Z_3 - Z_2\xi_3)}{Z_2 - Z_{3,l}} - \frac{\tilde{\sigma}_1\tilde{\sigma}_3\tilde{\rho}_{1,3}\xi_2(1 - Z_{1,l}) + \tilde{\sigma}_2\tilde{\sigma}_3\tilde{\rho}_{2,3}(Z_1 - Z_{2,l})}{Z_2 - Z_{3,l}}\right]\frac{\partial V}{\partial \xi_3} \\ &-rV = 0. \end{aligned}$$

26. Consider a two-factor convertible bond paying coupons with a rate k. For such a convertible bond, derive directly the partial differential equation that contains only the unknown market price of risk for the short rate. "Directly" means "without using the general PDE for derivatives." (Hint: Take a portfolio in the form $\Pi = \Delta_1 V_1 + \Delta_2 V_2 + S$, where V_1 and V_2 are two different convertible bonds.)
27. *Formulate the two-factor convertible coupon-paying bond problem as a linear complementarity problem.
28. Consider two-factor convertible coupon-paying bond problems.
    (a) Show that if $D_0 \leq 0$, then there is no free boundary; if $D_0 > 0$, then there exists at least one free boundary.
    (b) *Formulate a two-factor convertible coupon-paying bond problem as a free-boundary problem if $D_0 > 0$. (Suppose it is known that on the

free boundary, the price of the convertible bond and its derivative are continuous, and assume that there exists only one free boundary.)

29. Consider the problem

$$\begin{cases} \dfrac{\partial B_c}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 B_c}{\partial S^2} + (r - D_0)S\dfrac{\partial B_c}{\partial S} - rB_c + kZ = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 \le S, \quad 0 \le t \le T, \\ B_c(S,T) = \max(Z, nS), \qquad\qquad 0 \le S, \end{cases}$$

where σ, r, D_0, k, Z, and n are constants. Show that if $D_0 \le 0$, then

$$B_c(S,t) \ge nS \quad \text{for} \quad 0 \le t \le T.$$

(Hint: Define $\overline{B}_c(S,t) = B_c(S,t) - b_0(t)$, where $b_0(t)$ is the solution of the problem:

$$\begin{cases} \dfrac{db_0}{dt} - rb_0 + kZ = 0, \quad 0 \le t \le T, \\ b_0(T) = 0. \end{cases}$$

Show $\overline{B}_c(S,t) \ge nS$ and $b_0(t) \ge 0$, and then show $B_c(S,t) \ge nS$.)
(**Remark**: If the solution of this problem fulfills the constraint condition $B_c(S,t) \ge nS$ for $0 \le t \le T$, then the solution of the problem above represents the price of a one-factor convertible bond. In this case, the solution of a one-factor convertible bond does not involve any free boundary. Therefore, no free boundary will be encountered when one prices a one-factor convertible bond with $D_0 \le 0$.)

30. Consider the problem

$$\begin{cases} \dfrac{\partial B_c}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 B_c}{\partial S^2} + (r - D_0)S\dfrac{\partial B_c}{\partial S} - rB_c + kZ = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 \le S, \quad 0 \le t \le T, \\ B_c(S,T) = \max(Z, nS) = n\max(S - Z/n, 0) + Z, \qquad 0 \le S, \end{cases}$$

where σ, r, D_0, k, Z, and n are constants. Show that its solution is

$$nc(S,t;Z/n) + Z\mathrm{e}^{-r(T-t)}\left[1 + k\left(\mathrm{e}^{r(T-t)} - 1\right)/r\right],$$

where $c(S,t;Z/n)$ is the price of a European call option with an exercise price $E = Z/n$. This means that the problem can be understood as a problem to determine the value of an investment consisting of n units of European call options with $E = Z/n$ and a bond with face value Z and coupon rate k [see the result of Problem 1 part (b)]. According to the result of Problem 29, if $D_0 \le 0$, then it is the price of a convertible bond. Therefore when $D_0 \le 0$, the value of a one-factor convertible bond is equal to the price of n units of European call options with $E = Z/n$ plus the price of a bond with face value Z and coupon rate k.

Part II

Numerical Methods for Derivative Securities

6

Basic Numerical Methods

This chapter is devoted to the basic numerical methods. We first discuss various approximations, solution of systems, and eigenvalue problems. Then, we describe how to determine the parameters in stochastic models.

6.1 Approximations

6.1.1 Interpolation

Linear Interpolation. Suppose that the values of a function $f(x)$ are given on the grid points x_m, $m = 0, 1, \cdots, M$, where $x_0 < x_1 < \cdots < x_M$. Sometimes, we may need to find the value of the function at other points. A simple way to do this is to interpolate the function by using the known values of the function. Let f_m denote the value of the function $f(x)$ at a point x_m, $m = 0, 1, \cdots, M$. We want to approximate the value $f(x^*)$ for $x^* \in (x_m, x_{m+1})$. The simplest interpolation is to use a linear function to approximate the function $f(x)$ on the subinterval $[x_m, x_{m+1}]$. Let

$$p_1(x) = a_0 + a_1 x.$$

Using the conditions

$$p_1(x_m) = f_m, \quad p_1(x_{m+1}) = f_{m+1},$$

we find

$$a_0 = \frac{x_{m+1} f_m - x_m f_{m+1}}{x_{m+1} - x_m}, \quad a_1 = \frac{f_{m+1} - f_m}{x_{m+1} - x_m}.$$

Then, we have

$$p_1(x) = \frac{x_{m+1} - x}{x_{m+1} - x_m} f_m + \frac{x - x_m}{x_{m+1} - x_m} f_{m+1}.$$

Y.-l. Zhu et al., *Derivative Securities and Difference Methods*, Springer Finance,
DOI 10.1007/978-1-4614-7306-0_6,

Thus, we have the approximate value:

$$f(x^*) \approx p_1(x^*) = \frac{x_{m+1} - x^*}{x_{m+1} - x_m} f_m + \frac{x^* - x_m}{x_{m+1} - x_m} f_{m+1}.$$

This is called the linear interpolation. If we do the interpolation for all subintervals, then we obtain a piecewise linear function on the interval $[x_0, x_M]$.

Higher Order Interpolation. If the function data indicates that the function is smooth, then we can use a quadratic or Nth order interpolation to get a better approximation. Assume that we have obtained the values f_{m-1}, f_m, and f_{m+1}. Let

$$p_2(x) = a_0 + a_1 x + a_2 x^2.$$

Using the conditions

$$p_2(x_{m-1}) = f_{m-1}, \quad p_2(x_m) = f_m, \quad p_2(x_{m+1}) = f_{m+1},$$

we find

$$p_2(x) = \frac{(x_m - x)(x_{m+1} - x)}{(x_m - x_{m-1})(x_{m+1} - x_{m-1})} f_{m-1} + \frac{(x - x_{m-1})(x_{m+1} - x)}{(x_m - x_{m-1})(x_{m+1} - x_m)} f_m + \frac{(x - x_{m-1})(x - x_m)}{(x_{m+1} - x_{m-1})(x_{m+1} - x_m)} f_{m+1}.$$

Then, for any $x^* \in (x_{m-1}, x_{m+1})$, $f(x^*)$ can be approximated by $p_2(x^*)$. This is called the quadratic interpolation.

In general, if f_m, $m = i, i+1, \cdots, i+N$, are known for an integer i, then an Nth Lagrange interpolating polynomial can be obtained. For simplicity, let $i = 0$ and write down the polynomial as follows:

$$p_N(x) = \varphi_0(x) f_0 + \varphi_1(x) f_1 + \cdots + \varphi_N(x) f_N,$$

where

$$\varphi_k(x) = \frac{(x - x_0)(x - x_1) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_N)}{(x_k - x_0)(x_k - x_1) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_N)}$$

for $k = 0, 1, \cdots, N$. This approximation can be used for any $x \in (x_0, x_N)$. It is clear that the linear and quadratic interpolating polynomials are the Lagrange interpolating polynomials with $N = 1$ and 2, respectively. For an Nth Lagrange interpolating polynomial, the error is given by the following theorem:

Theorem 6.1 *If x_m, $m = 0, 1, \cdots, N$, are distinct numbers and $f(x)$ is $N+1$ times continuous differentiable on $[x_0, x_N]$, then for any $x \in [x_0, x_N]$, there exists a $\xi \in [x_0, x_N]$, such that*

$$f(x) - p_N(x) = \frac{f^{(N+1)}(\xi)}{(N+1)!} (x - x_0)(x - x_1) \cdots (x - x_N).$$

Therefore, the error of linear interpolation is $O(\Delta x^2)$, and the error of quadratic interpolation is $O(\Delta x^3)$, where $\Delta x = \max_m (x_{m+1} - x_m)$.

Cubic Spline Interpolation. As we can see, linear interpolations result in piecewise linear functions on the interval $[x_0, x_M]$: the function is smooth in each subinterval $[x_m, x_{m+1}]$, continuous in $[x_0, x_M]$, but may not be smooth in $[x_0, x_M]$. For quadratic interpolations, the situation is similar. Cubic spline interpolation is the most commonly used piecewise polynomial approximation, which is a cubic polynomial on each subinterval $[x_m, x_{m+1}]$ and has a continuous second derivative on the whole interval. The cubic spline interpolation $S(x)$ satisfies the following conditions:

(A) On the subinterval $[x_m, x_{m+1}]$, $S(x) = S_m(x)$ is a cubic polynomial, $m = 0, 1, \cdots, M-1$;
(B) $S(x_m) = f_m$, $m = 0, 1, \cdots, M$;
(C) $S_m(x_m) = S_{m-1}(x_m)$, $S'_m(x_m) = S'_{m-1}(x_m)$, $S''_m(x_m) = S''_{m-1}(x_m)$, $m = 1, 2, \cdots, M-1$;
(D) $S''(x_0) = S''(x_M) = 0$, or other two conditions.

Let

$$S_m(x) = a_m + b_m(x - x_m) + c_m(x - x_m)^2 + d_m(x - x_m)^3, \quad m = 0, 1, \cdots, M-1.$$

Condition B, $m = 0, 1, \cdots, M-1$, can be written as

$$a_m = S_m(x_m) = f_m, \quad m = 0, 1, \cdots, M-1.$$

Using condition C, we get

$$\begin{cases} a_m = a_{m-1} + b_{m-1}h_{m-1} + c_{m-1}h_{m-1}^2 + d_{m-1}h_{m-1}^3, \\ b_m = b_{m-1} + 2c_{m-1}h_{m-1} + 3d_{m-1}h_{m-1}^2, \\ c_m = c_{m-1} + 3d_{m-1}h_{m-1}, \\ \qquad\qquad m = 1, 2, \cdots, M-1, \end{cases} \tag{6.1}$$

where $h_{m-1} = x_m - x_{m-1}$. Define

$$a_M = f_M$$

and

$$c_M = S''(x_M)/2.$$

Then, from the expression $S_{M-1}(x)$ and Condition B with $m = M$, we further have

$$\begin{cases} a_M = a_{M-1} + b_{M-1}h_{M-1} + c_{M-1}h_{M-1}^2 + d_{M-1}h_{M-1}^3, \\ c_M = c_{M-1} + 3d_{M-1}h_{M-1}. \end{cases} \tag{6.2}$$

Rewrite the last relations in the sets of relations (6.1) and (6.2) as

$$d_{m-1} = \frac{c_m - c_{m-1}}{3h_{m-1}}, \quad m = 1, 2, \cdots, M, \tag{6.3}$$

and the first relations in the sets of relations (6.1) and (6.2) as

$$\begin{aligned} b_{m-1} &= \frac{a_m - a_{m-1}}{h_{m-1}} - c_{m-1}h_{m-1} - d_{m-1}h_{m-1}^2 \\ &= \frac{a_m - a_{m-1}}{h_{m-1}} - c_{m-1}h_{m-1} - \frac{c_m - c_{m-1}}{3}h_{m-1}, \\ m &= 1, 2, \cdots, M. \end{aligned} \tag{6.4}$$

Substituting them into the second relation in the set of relations (6.1) yields

$$\begin{aligned} &\frac{a_{m+1} - a_m}{h_m} - c_m h_m - \frac{c_{m+1} - c_m}{3}h_m \\ &= \frac{a_m - a_{m-1}}{h_{m-1}} - c_{m-1}h_{m-1} - \frac{c_m - c_{m-1}}{3}h_{m-1} \\ &\quad +2c_{m-1}h_{m-1} + (c_m - c_{m-1})h_{m-1}, \\ &\qquad m = 1, 2, \cdots, M-1, \end{aligned}$$

or

$$\begin{aligned} u_m c_{m-1} + 2c_m + v_m c_{m+1} &= \frac{1}{h_{m-1} + h_m}\left[\frac{3(a_{m+1} - a_m)}{h_m} - \frac{3(a_m - a_{m-1})}{h_{m-1}}\right], \\ m &= 1, 2, \cdots, M-1, \end{aligned}$$

where $u_m = h_{m-1}/(h_{m-1} + h_m)$ and $v_m = h_m/(h_{m-1} + h_m)$. This system is equivalent to Conditions A–C. If Condition D is $S''(x_0) = S''(x_M) = 0$, we have two other equations $c_0 = 0$ and $c_M = 0$. In this case the entire system can be written in the following matrix form:

$$A\mathbf{c} = \mathbf{h}, \tag{6.5}$$

where

$$A = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ u_1 & 2 & v_1 & 0 & \cdots & 0 \\ 0 & u_2 & 2 & v_2 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & u_{M-1} & 2 & v_{M-1} \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ \vdots \\ c_M \end{bmatrix},$$

$$\mathbf{h} = \begin{bmatrix} 0 \\ \dfrac{1}{h_0 + h_1}\left(\dfrac{3(a_2 - a_1)}{h_1} - \dfrac{3(a_1 - a_0)}{h_0}\right) \\ \vdots \\ \dfrac{1}{h_{M-2} + h_{M-1}}\left(\dfrac{3(a_M - a_{M-1})}{h_{M-1}} - \dfrac{3(a_{M-1} - a_{M-2})}{h_{M-2}}\right) \\ 0 \end{bmatrix}.$$

Solving this linear system, we obtain c_m, $m = 0, 1, \cdots, M$. Then d_m, $m = 0, 1, \cdots, M-1$, can be obtained from the set of relations (6.3) and b_m, $m = 0, 1, \cdots, M-1$, from the set of relations (6.4).

The condition $S''(x_0) = 0$ could be replaced by $S'(x_0) = f'(x_0)$ or $d_0 = 0$, namely, assuming $S_0(x) = a_0 + b_0(x-x_0) + c_0(x-x_0)^2$. At $x = x_M$, the situation is similar. If such a case appears, then the way to determine these coefficients needs to be changed slightly. Here, assuming $S'(x_0) = f'(x_0)$ and $d_{M-1} = 0$, we explain how to modify the way to determine these coefficients. Because $S_0'(x_0) = b_0$, the coefficient b_0 is known in this case, namely, $b_0 = f'(x_0)$. From

$$\begin{cases} a_1 = a_0 + b_0 h_0 + c_0 h_0^2 + d_0 h_0^3, \\ c_1 = c_0 + 3 d_0 h_0, \end{cases}$$

we eliminate d_0 and obtain

$$2c_0 + c_1 = 3\left(\frac{a_1 - a_0}{h_0^2} - \frac{b_0}{h_0}\right) = 3\left(\frac{a_1 - a_0}{h_0^2} - \frac{f'(x_0)}{h_0}\right).$$

This equation should replace the first equation in the system (6.5). From $d_{M-1} = 0$ and the second equation in the set of relations (6.2), we have

$$c_{M-1} - c_M = 0.$$

This equation should replace the last equation in the system (6.5). Solving the modified system (6.5) yields c_m, $m = 0, 1, \cdots, M$, for this case. As soon as all the c_m are obtained, d_m, $m = 0, 1, \cdots, M-1$, can be obtained from the set of relations (6.3) and b_m, $m = 0, 1, \cdots, M-1$, from the set of relations (6.4). For more about cubic spline interpolation, see books on numerical methods.

When we write a code to calculate the approximate value $f(x^*)$ by quadratic interpolation, in order to guarantee to use an interpolation, we need to find a number m such that $x^* \in [x_{m-1}, x_{m+1}]$. This can be realized by using a loop statement. If $x_m = m\Delta x, m = 0, 1, \cdots, M$, then the expression

$$m = \max\left(1, \min\left(\operatorname{int}\left(\frac{x^*}{\Delta x} + 0.5\right), M-1\right)\right)$$

will also always give such a number.

6.1.2 Approximation of Partial Derivatives

Finite-Difference Approximation. Here, we will discuss how derivatives of a function $u(x,t)$ at a point can be approximated by a linear combination of values of the function at adjacent points. Let $x_m = a + m\Delta x$ and $\tau^n = n\Delta\tau$, where m is an integer and n is an integer or an integer plus a half.

Using the Taylor expansion, we have[1]

$$u(x_m, \tau^{n+1}) = u(x_m, \tau^n) + \Delta\tau \frac{\partial u}{\partial \tau}(x_m, \tau^n) + \frac{\Delta\tau^2}{2}\frac{\partial^2 u}{\partial \tau^2}(x_m, \eta),$$

[1] In this book $\Delta\tau^2$ stands for $(\Delta\tau)^2$. For $\Delta\tau^3$, Δx^2, Δx^3 etc., the situation is similar.

where $\tau^n < \eta < \tau^{n+1}$. Then,

$$\frac{\partial u}{\partial \tau}(x_m, \tau^n) = \frac{u(x_m, \tau^{n+1}) - u(x_m, \tau^n)}{\Delta\tau} - \frac{\Delta\tau}{2}\frac{\partial^2 u}{\partial \tau^2}(x_m, \eta).$$

If $\Delta\tau$ is small, we have

$$\frac{\partial u}{\partial \tau}(x_m, \tau^n) \approx \frac{u(x_m, \tau^{n+1}) - u(x_m, \tau^n)}{\Delta\tau}.$$

This approximation is called the forward finite-difference approximation or the **forward difference** for $\frac{\partial u}{\partial \tau}$. Similarly, we can obtain the **backward difference**

$$\frac{\partial u}{\partial \tau}(x_m, \tau^n) \approx \frac{u(x_m, \tau^n) - u(x_m, \tau^{n-1})}{\Delta\tau}.$$

Both forward and backward finite-difference approximations have errors of first order in $\Delta\tau$ (first-order accurate). To obtain a second-order accurate finite-difference approximation, we use the following Taylor expansions:

$$\begin{aligned} u(x_m, \tau^{n+1}) &= u(x_m, \tau^{n+1/2}) + \frac{\Delta\tau}{2}\frac{\partial u}{\partial \tau}(x_m, \tau^{n+1/2}) \\ &\quad + \frac{\Delta\tau^2}{8}\frac{\partial^2 u}{\partial \tau^2}(x_m, \tau^{n+1/2}) + \frac{\Delta\tau^3}{48}\frac{\partial^3 u}{\partial \tau^3}(x_m, \eta_1), \\ u(x_m, \tau^n) &= u(x_m, \tau^{n+1/2}) - \frac{\Delta\tau}{2}\frac{\partial u}{\partial \tau}(x_m, \tau^{n+1/2}) \\ &\quad + \frac{\Delta\tau^2}{8}\frac{\partial^2 u}{\partial \tau^2}(x_m, \tau^{n+1/2}) - \frac{\Delta\tau^3}{48}\frac{\partial^3 u}{\partial \tau^3}(x_m, \eta_2), \end{aligned}$$

where $\tau^{n+1/2} < \eta_1 < \tau^{n+1}$ and $\tau^n < \eta_2 < \tau^{n+1/2}$. Subtracting the second equation from the first one, we get

$$\frac{\partial u}{\partial \tau}(x_m, \tau^{n+1/2}) = \frac{u(x_m, \tau^{n+1}) - u(x_m, \tau^n)}{\Delta\tau} - \frac{\Delta\tau^2}{24}\frac{\partial^3 u}{\partial \tau^3}(x_m, \eta_3),$$

where $\tau^n < \eta_3 < \tau^{n+1}$. Then, we have

$$\frac{\partial u}{\partial \tau}(x_m, \tau^{n+1/2}) \approx \frac{u(x_m, \tau^{n+1}) - u(x_m, \tau^n)}{\Delta\tau}.$$

This approximation is called the central finite-difference approximation or a **central difference** for $\frac{\partial u}{\partial \tau}$.

Similarly, for $\frac{\partial u}{\partial x}(x_m, \tau^n)$ we can have the following approximations

$$\frac{\partial u}{\partial x}(x_m,\tau^n) \approx \frac{u(x_{m+1},\tau^n)-u(x_{m-1},\tau^n)}{2\Delta x},$$
$$\frac{\partial u}{\partial x}(x_m,\tau^n) \approx \frac{u(x_m,\tau^n)-u(x_{m-1},\tau^n)}{\Delta x}$$

and

$$\frac{\partial u}{\partial x}(x_m,\tau^n) \approx \frac{u(x_{m+1},\tau^n)-u(x_m,\tau^n)}{\Delta x}.$$

The first one is second order and called the **second-order central difference** for first derivatives. The second and third approximations are first order and called the **first-order one-sided difference**. Sometimes, we also need the following **second-order one-sided differences**:

$$\frac{\partial u}{\partial x}(x_m,\tau^n) \approx \frac{3u(x_m,\tau^n)-4u(x_{m-1},\tau^n)+u(x_{m-2},\tau^n)}{2\Delta x}$$

and

$$\frac{\partial u}{\partial x}(x_m,\tau^n) \approx \frac{-3u(x_m,\tau^n)+4u(x_{m+1},\tau^n)-u(x_{m+2},\tau^n)}{2\Delta x}.$$

For the approximation of the second-order partial derivative with respect to x, we use the following Taylor expansions:

$$\begin{aligned}
u(x_{m+1},\tau^n) &= u(x_m,\tau^n) + \Delta x\frac{\partial u}{\partial x}(x_m,\tau^n) + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}(x_m,\tau^n)\\
&\quad + \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3}(x_m,\tau^n) + \frac{\Delta x^4}{24}\frac{\partial^4 u}{\partial x^4}(\xi_1,\tau^n),\\
u(x_{m-1},\tau^n) &= u(x_m,\tau^n) - \Delta x\frac{\partial u}{\partial x}(x_m,\tau^n) + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}(x_m,\tau^n)\\
&\quad - \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3}(x_m,\tau^n) + \frac{\Delta x^4}{24}\frac{\partial^4 u}{\partial x^4}(\xi_2,\tau^n),
\end{aligned}$$

where $x_m < \xi_1 < x_{m+1}$ and $x_{m-1} < \xi_2 < x_m$. Adding these two equations, we obtain

$$\frac{\partial^2 u}{\partial x^2}(x_m,\tau^n) = \frac{u(x_{m+1},\tau^n)-2u(x_m,\tau^n)+u(x_{m-1},\tau^n)}{\Delta x^2} - \frac{\Delta x^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_3,\tau^n),$$

where $x_{m-1} < \xi_3 < x_{m+1}$. Thus, we have the **second-order central difference** for second derivatives:

$$\frac{\partial^2 u}{\partial x^2}(x_m,\tau^n) \approx \frac{u(x_{m+1},\tau^n)-2u(x_m,\tau^n)+u(x_{m-1},\tau^n)}{\Delta x^2}.$$

Sometimes, we also need to have an approximation to mixed second-order partial derivatives. For $\dfrac{\partial^2 u}{\partial x \partial y}$, we have

$$\frac{\partial^2 u}{\partial x \partial y}(x_m, y_l, \tau^n) \approx \frac{1}{2\Delta x}\left[\frac{u(x_{m+1}, y_{l+1}, \tau^n) - u(x_{m+1}, y_{l-1}, \tau^n)}{2\Delta y}\right.$$
$$\left. - \frac{u(x_{m-1}, y_{l+1}, \tau^n) - u(x_{m-1}, y_{l-1}, \tau^n)}{2\Delta y}\right],$$

where $y_l = b + l\Delta y$, Δy being a small number. It is clear that this is a second-order scheme, and this formula is called the **second-order central difference** for mixed second-order partial derivatives.

Pseudo-Spectral Approximation. By using more points, we can also construct higher order finite-difference approximations for the partial derivatives. An alternative way to obtain higher order approximations for partial derivatives is to use a pseudo-spectral method. To illustrate the method, we consider the approximation to the partial derivatives with respect to x for a fixed τ. Assume that we want to find the solution u in the interval $0 \leq x \leq 1$. Suppose we use non-equidistant nodes. For example, we can use the following grid points

$$x_m = \frac{1}{2}\left(1 - \cos\frac{m\pi}{M}\right), \quad m = 0, 1, \cdots, M. \tag{6.6}$$

These points are in $[0, 1]$ and equal to $(1 - x_m^*)/2$, x_m^* being the extrema of the Mth order Chebyshev polynomial $T_M(x)$. Here, the Mth order Chebyshev polynomial is defined by $T_M(x) = \cos(M \cos^{-1} x)$. Assume that the solution for a fixed τ is a polynomial in x with degree M. If we require the polynomial to have a value $u(x_m)$ at $x = x_m$, then we can determine the coefficients of the polynomial, each of which is a linear combination of $u(x_i)$, $i = 0, 1, \cdots, M$. Thus, the derivatives of the polynomial at the point x_m is also a linear combination of $u(x_i)$, $i = 0, 1, \cdots, M$, with coefficients depending on x_m and x_i. Therefore,

$$\frac{\partial u}{\partial x}(x_m) = \sum_{i=0}^{M} D_{x,m,i}\, u(x_i). \tag{6.7}$$

This is an **Mth order approximation to the derivative** with respect to x used in the pseudo-spectral method. If the grid points are given by the expression (6.6), then $D_{x,m,i}$ has the following expression:

$$D_{x,m,i} = \begin{cases} \dfrac{c_m(-1)^{m+i}}{c_i(x_m - x_i)}, \ m \neq i, \\ -\dfrac{2M^2+1}{3}, \quad m = i = 0, \\ \dfrac{1-2x_i}{4x_i(1-x_i)}, \quad m = i = 1, 2, \cdots, M-1, \\ \dfrac{2M^2+1}{3}, \qquad m = i = M, \end{cases} \tag{6.8}$$

where $c_0 = c_M = 2$ and $c_i = 1$, $i = 1, 2, \cdots, M - 1$ (see [36]). Similarly, we have

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2}(x_m) &= \sum_{j=0}^{M} D_{x,m,j} \frac{\partial u}{\partial x}(x_j) \\ &= \sum_{j=0}^{M} D_{x,m,j} \left[\sum_{i=0}^{M} D_{x,j,i} u(x_i) \right] \\ &= \sum_{i=0}^{M} \left(\sum_{j=0}^{M} D_{x,m,j} D_{x,j,i} \right) u(x_i) \\ &= \sum_{i=0}^{M} D_{xx,m,i} u(x_i), \end{aligned} \tag{6.9}$$

where

$$D_{xx,m,i} = \sum_{j=0}^{M} D_{x,m,j} D_{x,j,i}. \tag{6.10}$$

When the solution is very smooth, only a small M may be needed in order to get a satisfying result. In such a case, its performance could be better than the finite-difference approximations.

6.1.3 Approximate Integration

Trapezoidal Rule. The approximation of the integral

$$\int_a^b f(x)dx$$

is needed in the numerical solution of integro-differential equations and sometimes in the numerical solution of partial differential equations. The simplest method for the approximation is called the **trapezoidal rule.** Let $h = (b - a)/M$, and $x_m = a + mh$, $m = 0, 1, \cdots, M$. In the subinterval $[x_m, x_{m+1}]$, we use the linear function

$$p_1(x) = \frac{x_{m+1} - x}{h} f(x_m) + \frac{x - x_m}{h} f(x_{m+1})$$

to approximate $f(x)$. Thus,

$$\begin{aligned}
\int_{x_m}^{x_{m+1}} f(x)dx &\approx \frac{1}{h} \int_{x_m}^{x_{m+1}} [(x_{m+1} - x) f(x_m) + (x - x_m) f(x_{m+1})] dx \\
&= \frac{h}{2} \left[f(x_m) + f(x_{m+1}) \right].
\end{aligned}$$

Using this for all subintervals, we obtain

$$\begin{aligned}
\int_a^b f(x)dx &= \sum_{m=0}^{M-1} \int_{x_m}^{x_{m+1}} f(x)dx \\
&\approx \sum_{m=0}^{M-1} \frac{h}{2} \left[f(x_m) + f(x_{m+1}) \right] \\
&= \frac{h}{2} \left[f(a) + 2 \sum_{m=1}^{M-1} f(x_m) + f(b) \right].
\end{aligned}$$

The error of the trapezoidal rule is

$$-\frac{(b-a)h^2}{12} f''(\xi),$$

where $\xi \in (a, b)$.

Simpson's Rule. Simpson's rule is a better approximation for the integral by using the quadratic interpolation polynomial. In the subinterval $[x_{m-1}, x_{m+1}]$, we use

$$\begin{aligned}
p_2(x) = &\frac{(x_m - x)(x_{m+1} - x)}{(x_m - x_{m-1})(x_{m+1} - x_{m-1})} f(x_{m-1}) \\
&+ \frac{(x - x_{m-1})(x_{m+1} - x)}{(x_m - x_{m-1})(x_{m+1} - x_m)} f(x_m) \\
&+ \frac{(x - x_{m-1})(x - x_m)}{(x_{m+1} - x_{m-1})(x_{m+1} - x_m)} f(x_{m+1})
\end{aligned}$$

to approximate $f(x)$. Thus

$$\begin{aligned}
\int_{x_{m-1}}^{x_{m+1}} f(x)dx \approx \frac{1}{2h^2} \int_{x_{m-1}}^{x_{m+1}} &[(x_m - x)(x_{m+1} - x) f(x_{m-1}) \\
&+ 2(x - x_{m-1})(x_{m+1} - x) f(x_m) \\
&+ (x - x_{m-1})(x - x_m) f(x_{m+1})] dx \\
= \frac{h}{3} [f(x_{m-1}) + 4f(x_m) + f(x_{m+1})].
\end{aligned}$$

Suppose that M is an even number and using this for all subintervals, we obtain

$$\int_a^b f(x)dx \approx \frac{h}{3}\left[f(a) + 2\sum_{m=1}^{M/2-1} f(x_{2m}) + 4\sum_{m=1}^{M/2} f(x_{2m-1}) + f(b)\right].$$

The error of the Simpson's rule is

$$-\frac{(b-a)h^4}{180} f^{(4)}(\xi),$$

where $\xi \in (a, b)$.

6.1.4 Least Squares Approximation

In Sect. 6.1.1 we discussed various interpolations. In those cases, all the given points (x_m, f_m) are on the interpolation function. Here, we will discuss how to find an approximate function satisfying the following two conditions:

(A) The number of parameters in the function is less than the number of given points.
(B) Let the function have the "best fit" to those given points (x_m, f_m) in some sense.

Let x_m, $m = 0, 1, \cdots, M$, be distinct, and let $M + 1$ points (x_m, f_m) be given. We want to find a product of a given function $g(x)$ and a polynomial of degree $N < M$

$$g(x)\sum_{n=0}^{N} a_n x^n$$

such that the value of the total least squares error

$$\sum_{m=0}^{M} b_m \left[f_m - g(x_m)\sum_{n=0}^{N} a_n x_m^n\right]^2$$

has a minimum, where b_m, $m = 0, 1, \cdots, M$, are given positive numbers called the weights. In order to minimize the least squares error, the necessary conditions are

$$\frac{\partial}{\partial a_i}\left\{\sum_{m=0}^{M} b_m \left[f_m - g(x_m)\sum_{n=0}^{N} a_n x_m^n\right]^2\right\}$$
$$= -2\sum_{m=0}^{M} b_m \left[f_m - g(x_m)\sum_{n=0}^{N} a_n x_m^n\right] g(x_m)x_m^i = 0,$$
$$i = 0, 1, \cdots, N.$$

This system can be written as

$$\sum_{n=0}^{N}\left[\sum_{m=0}^{M} b_m g^2(x_m)x_m^{n+i}\right]a_n = \sum_{m=0}^{M} b_m f_m g(x_m)x_m^i, \qquad (6.11)$$
$$i = 0, 1, \cdots, N.$$

It is a linear system for $a_0, a_1, \cdots, a_N$, which is usually known as the system of normal equations. When x_0, x_1, $\cdots$, x_M are distinct, we can find $a_0, a_1, \cdots, a_N$ without any difficulty. This method is usually referred to as the least squares method with weights.

6.2 Solution of Systems and Eigenvalue Problems

6.2.1 LU Decomposition of Linear Systems

One efficient method for solving linear systems is the LU decomposition, which decomposes the matrix **A** into a product of a unit lower triangular matrix[2] **L** and an upper triangular matrix **U**, i.e., **A** = **LU**. In this subsection, we will give the details of the method for linear tridiagonal systems. For more complicated systems, the procedure is similar.

Consider a general linear tridiagonal system

$$\mathbf{Ax} = \mathbf{q}, \qquad (6.12)$$

where

$$\mathbf{A} = \begin{bmatrix} b_1 & c_1 & & & & \\ a_2 & b_2 & c_2 & & 0 & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & a_{m-1} & b_{m-1} & c_{m-1} \\ & & & & a_m & b_m \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_m \end{bmatrix}.$$

Let us discuss how to find the solution of this system. The first equation of the system is

$$b_1 x_1 + c_1 x_2 = q_1.$$

Let

$$u_1 = b_1, \quad y_1 = q_1,$$

the equation above can be written as

$$u_1 x_1 + c_1 x_2 = y_1. \qquad (6.13)$$

[2]A matrix **A** is called a unit lower triangular matrix if $a_{ii} = 1$ and $a_{ij} = 0$ for $i < j$. If $a_{ij} = 0$ for $i > j$, then the matrix **A** is called an upper triangular matrix.

Now suppose we have a relation in the form

$$u_{i-1}x_{i-1} + c_{i-1}x_i = y_{i-1}.$$

We put this relation and the ith equation of the system together and obtain

$$\begin{cases} u_{i-1}x_{i-1} + c_{i-1}x_i = y_{i-1}, \\ a_i x_{i-1} + b_i x_i + c_i x_{i+1} = q_i. \end{cases}$$

Subtracting the first relation multiplied by a_i/u_{i-1} from the second equation, we can eliminate x_{i-1} and have another relation in the same form:

$$\left(b_i - c_{i-1}\frac{a_i}{u_{i-1}}\right)x_i + c_i x_{i+1} = q_i - y_{i-1}\frac{a_i}{u_{i-1}}$$

or

$$u_i x_i + c_i x_{i+1} = y_i,$$

where

$$u_i = b_i - \frac{c_{i-1}a_i}{u_{i-1}},$$
$$y_i = q_i - \frac{y_{i-1}a_i}{u_{i-1}}.$$

Because we have Eq. (6.13) that is in this form, this procedure can be done for $i = 2, 3, \cdots, m$ successively and generates

$$u_i x_i + c_i x_{i+1} = y_i, \quad i = 2, 3, \cdots, m-1 \tag{6.14}$$

and

$$u_m x_m = y_m. \tag{6.15}$$

Because in the last equation of the system x_{m+1} does not appear, which means $c_m = 0$, the last relation is in the form Eq. (6.15) instead of the set of equations (6.14). This procedure can be called elimination or forward substitution.

When we obtain Eq. (6.15), we can have

$$x_m = \frac{y_m}{u_m}.$$

Furthermore, from Eq. (6.13) and the set of equations (6.14) we can get

$$x_i = \frac{y_i - c_i x_{i+1}}{u_i}, \quad i = m-1, \cdots, 1$$

successively. This procedure is called back substitution. Through these two procedures, we can obtain the solution of this system.

The elimination procedure can be written in matrix form

$$\mathbf{L}_{m-1}\cdots\mathbf{L}_2\mathbf{L}_1\mathbf{A}\mathbf{x} = \mathbf{U}\mathbf{x} = \mathbf{L}_{m-1}\cdots\mathbf{L}_1\mathbf{q} = \mathbf{y}, \tag{6.16}$$

where

$$\mathbf{L}_i = \begin{bmatrix} 1 & 0 & & \cdots & & & & 0 \\ 0 & 1 & \ddots & & & & & \\ & \ddots & \ddots & \ddots & & & & \\ \vdots & & 0 & 1 & \ddots & & & \vdots \\ & & & -l_i & 1 & \ddots & & \\ & & & & 0 & \ddots & \ddots & \\ & & & & & \ddots & \ddots & 0 \\ 0 & & & \cdots & & & 0 & 1 \end{bmatrix},$$

$$\mathbf{U} = \begin{bmatrix} u_1 & c_1 & 0 & \cdots & 0 \\ 0 & u_2 & c_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & u_{m-1} & c_{m-1} \\ 0 & \cdots & \cdots & 0 & u_m \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix},$$

l_i equalling a_{i+1}/u_i and being in the ith column and the $(i+1)$th row of $\mathbf{L}_i$, $i = 1, 2, \cdots, m-1$.

Let

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_1 & 1 & 0 & & \vdots \\ 0 & l_2 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & l_{m-1} & 1 \end{bmatrix}.$$

It is easy to see

$$\mathbf{L}_{m-1}\mathbf{L}_{m-2}\cdots\mathbf{L}_1\mathbf{L} = \mathbf{I}.$$

Thus, we have

$$\mathbf{L}_{m-1}\mathbf{L}_{m-2}\cdots\mathbf{L}_1 = \mathbf{L}^{-1}$$

and Eq. (6.16) can be written as

$$\mathbf{L}^{-1}\mathbf{A}\mathbf{x} = \mathbf{U}\mathbf{x} = \mathbf{L}^{-1}\mathbf{q} = \mathbf{y}.$$

Consequently,

$$\mathbf{A}\mathbf{x} = \mathbf{L}\mathbf{U}\mathbf{x} = \mathbf{q}.$$

This means that $\mathbf{A}$ can be decomposed into a unit lower triangular matrix $\mathbf{L}$ multiplied by an upper triangular matrix $\mathbf{U}$. The procedure of solving the system is as follows. We first multiply the equation by $\mathbf{L}^{-1}$ so that the equation becomes $\mathbf{Ux} = \mathbf{L}^{-1}\mathbf{q} = \mathbf{y}$ and then solve $\mathbf{Ux} = \mathbf{y}$ to get $\mathbf{x} = \mathbf{U}^{-1}\mathbf{y}$. Because these two procedures are easy to perform, the method is quite popular.

6.2.2 Iteration Methods for Linear Systems

An alternative to LU decomposition is iteration. Iteration methods are especially effective for large systems with sparse coefficient matrices. Consider the linear system

$$\mathbf{Ax} = \mathbf{q}.$$

$\mathbf{A}$ may be decomposed as

$$\mathbf{A} = \mathbf{D} + \mathbf{L} + \mathbf{U}$$

where

$$\mathbf{L} = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ a_{2,1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,m-1} & 0 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} 0 & a_{1,2} & \cdots & a_{1,m} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & a_{m-1,m} \\ 0 & \cdots & \cdots & 0 \end{bmatrix},$$

and $\mathbf{D} = \mathrm{diag}\{a_{1,1}, a_{2,2}, \cdots, a_{m,m}\}$. Then, the linear system can be rewritten as the following system,

$$\mathbf{x} = \mathbf{D}^{-1}\left[\mathbf{q} - (\mathbf{L} + \mathbf{U})\mathbf{x}\right],$$

where we assume that $\mathbf{D}$ is invertible.

Jacobi Iteration. A simple way to find the solution is to use the following iteration:

$$\mathbf{x}^{(k+1)} = \mathbf{D}^{-1}\left[\mathbf{q} - (\mathbf{L} + \mathbf{U})\mathbf{x}^{(k)}\right], \quad k = 0, 1, \cdots,$$

or in component form

$$\begin{aligned} x_1^{(k+1)} &= \frac{1}{a_{1,1}}[q_1 - (a_{1,2}x_2^{(k)} + \cdots + a_{1,m}x_m^{(k)})], \\ x_2^{(k+1)} &= \frac{1}{a_{2,2}}[q_2 - (a_{2,1}x_1^{(k)} + a_{2,3}x_3^{(k)} + \cdots + a_{2,m}x_m^{(k)})], \\ &\vdots \\ x_m^{(k+1)} &= \frac{1}{a_{m,m}}[q_m - (a_{m,1}x_1^{(k)} + \cdots + a_{m,m-1}x_{m-1}^{(k)})]. \end{aligned}$$

It is clear that in order to implement this iteration, an initial guess $\mathbf{x}^{(0)}$ should be given. This method is called the Jacobi iteration.

Gauss–Seidel Iteration. In the Jacobi iteration, at the iteration step for $x_i^{(k+1)}$, all the solutions $x_1^{(k+1)}, \cdots, x_{i-1}^{(k+1)}$ have been obtained. Therefore, new variables can be used in the iteration, namely, we can have the following iteration:

$$\begin{aligned}
x_1^{(k+1)} &= \frac{1}{a_{1,1}}[q_1 - (a_{1,2}x_2^{(k)} + \cdots + a_{1,m}x_m^{(k)})], \\
x_2^{(k+1)} &= \frac{1}{a_{2,2}}[q_2 - (a_{2,1}x_1^{(k+1)} + a_{2,3}x_3^{(k)} + \cdots + a_{2,m}x_m^{(k)})], \\
x_3^{(k+1)} &= \frac{1}{a_{3,3}}[q_3 - (a_{3,1}x_1^{(k+1)} + a_{3,2}x_2^{(k+1)} + a_{3,4}x_4^{(k)} + \cdots + a_{3,m}x_m^{(k)})], \\
&\vdots \\
x_m^{(k+1)} &= \frac{1}{a_{m,m}}[q_m - (a_{m,1}x_1^{(k+1)} + \cdots + a_{m,m-1}x_{m-1}^{(k+1)})]
\end{aligned}$$

or in matrix form

$$\mathbf{x}^{(k+1)} = \mathbf{D}^{-1}\left[\mathbf{q} - \mathbf{L}\mathbf{x}^{(k+1)} - \mathbf{U}\mathbf{x}^{(k)}\right].$$

This method is called the Gauss–Seidel iteration.

SOR (Successive Over Relaxation). The Gauss–Seidel iteration can be modified in the following way: Take a combination of the previous value of $\mathbf{x}$ and the current update (from the Gauss–Seidel method) as the next approximation:

$$\mathbf{x}^{(k+1)} = (1-\omega)\mathbf{x}^{(k)} + \omega\mathbf{D}^{-1}\left[\mathbf{q} - \mathbf{L}\mathbf{x}^{(k+1)} - \mathbf{U}\mathbf{x}^{(k)}\right],$$

or in component form

$$\begin{aligned}
x_1^{(k+1)} &= (1-\omega)x_1^{(k)} + \frac{\omega}{a_{1,1}}[q_1 - (a_{1,2}x_2^{(k)} + \cdots + a_{1,m}x_m^{(k)})], \\
x_2^{(k+1)} &= (1-\omega)x_2^{(k)} + \frac{\omega}{a_{2,2}}[q_2 - (a_{2,1}x_1^{(k+1)} + a_{2,3}x_3^{(k)} + \cdots + a_{2,m}x_m^{(k)})], \\
x_3^{(k+1)} &= (1-\omega)x_3^{(k)} + \frac{\omega}{a_{3,3}}[q_3 - (a_{3,1}x_1^{(k+1)} + a_{3,2}x_2^{(k+1)} + a_{3,4}x_4^{(k)} + \cdots + \\
&\qquad a_{3,m}x_m^{(k)})], \\
&\vdots \\
x_m^{(k+1)} &= (1-\omega)x_m^{(k)} + \frac{\omega}{a_{m,m}}[q_m - (a_{m,1}x_1^{(k+1)} + \cdots + a_{m,m-1}x_{m-1}^{(k+1)})].
\end{aligned}$$

Here, ω is a real number. This method usually is called the method of successive over relaxation (SOR). When $\omega = 1$, it is the Gauss–Seidel iteration. The parameter ω should be chosen so that the method will converge and work better than the Gauss–Seidel iteration. The following result has been proved:

Theorem 6.2 *If* $\mathbf{A}$ *is a symmetric positive definite matrix and* $0 < \omega < 2$*, then the method of successive over relaxation will converge for any initial vector* $\mathbf{x}$*.*

Practical computation shows that this method also works for some nonsymmetric linear systems if ω is chosen properly. For many cases, this method gives faster convergence than the Gauss–Seidel iteration if $\omega \in (1, 2)$. We would like to point out that in the books by Golub and Loan [35] and Saad [71], there are some other iteration methods that can also be used for solving linear systems in Chaps. 8–10. Interested readers are referred to these books.

6.2.3 Iteration Methods for Nonlinear Systems

In the numerical solution of partial differential equations, the resulting algebraic systems are sometimes nonlinear. In this section, we discuss three iteration methods for the nonlinear systems.

Newton's Method. Consider the following nonlinear system,

$$\begin{aligned} f_1(x_1, x_2, \cdots, x_n) &= 0, \\ f_2(x_1, x_2, \cdots, x_n) &= 0, \\ &\vdots \\ f_n(x_1, x_2, \cdots, x_n) &= 0. \end{aligned}$$

Let

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x}) \end{bmatrix}.$$

Then, the nonlinear system has the form

$$\mathbf{f}(\mathbf{x}) = 0.$$

Suppose $\mathbf{x}^{(0)} = [x_1^0, x_2^0, \cdots, x_n^0]^T$ is a good initial guess to the true solution $\mathbf{x}^* = [x_1^*, x_2^*, \cdots, x_n^*]^T$, i.e.,

$$\delta\mathbf{x} = \mathbf{x}^* - \mathbf{x}^{(0)} = [\delta x_1, \delta x_2, \cdots, \delta x_n]^T$$

is small in norm. Then, for $i = 1, 2, \cdots, n$

$$\begin{aligned} 0 = f_i(x_1^*, x_2^*, \cdots, x_n^*) &= f_i(x_1^0 + \delta x_1, x_2^0 + \delta x_2, \cdots, x_n^0 + \delta x_n) \\ &\approx f_i(x_1^0, x_2^0, \cdots, x_n^0) + \sum_{k=1}^{n} \frac{\partial f_i(x_1^0, x_2^0, \cdots, x_n^0)}{\partial x_k} \delta x_k. \end{aligned}$$

In matrix form, we have

$$\begin{bmatrix} \frac{\partial f_1(\mathbf{x}^{(0)})}{\partial x_1} & \frac{\partial f_1(\mathbf{x}^{(0)})}{\partial x_2} & \cdots & \frac{\partial f_1(\mathbf{x}^{(0)})}{\partial x_n} \\ \frac{\partial f_2(\mathbf{x}^{(0)})}{\partial x_1} & \frac{\partial f_2(\mathbf{x}^{(0)})}{\partial x_2} & \cdots & \frac{\partial f_2(\mathbf{x}^{(0)})}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n(\mathbf{x}^{(0)})}{\partial x_1} & \frac{\partial f_n(\mathbf{x}^{(0)})}{\partial x_2} & \cdots & \frac{\partial f_n(\mathbf{x}^{(0)})}{\partial x_n} \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \vdots \\ \delta x_n \end{bmatrix} + \begin{bmatrix} f_1(\mathbf{x}^{(0)}) \\ f_2(\mathbf{x}^{(0)}) \\ \vdots \\ f_n(\mathbf{x}^{(0)}) \end{bmatrix} \approx 0,$$

or

$$\mathbf{J_f}(\mathbf{x}^{(0)}) \cdot \delta\mathbf{x} + \mathbf{f}(\mathbf{x}^{(0)}) \approx 0,$$

where $\mathbf{J_f}(\mathbf{x}^{(0)})$ denotes the above Jacobian matrix. Solving for $\delta\mathbf{x}$ we get

$$\delta\mathbf{x} \approx -[\mathbf{J_f}(\mathbf{x}^{(0)})]^{-1}\mathbf{f}(\mathbf{x}^{(0)})$$

or

$$\mathbf{x}^* \approx \mathbf{x}^{(0)} - [\mathbf{J_f}(\mathbf{x}^{(0)})]^{-1}\mathbf{f}(\mathbf{x}^{(0)}).$$

This means that the vector

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} - [\mathbf{J_f}(\mathbf{x}^{(0)})]^{-1}\mathbf{f}(\mathbf{x}^{(0)})$$

will be a better approximation to the solution $\mathbf{x}^*$. In general, suppose $\mathbf{x}^{(k)}$ has been obtained, then

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - [\mathbf{J_f}(\mathbf{x}^{(k)})]^{-1}\mathbf{f}(\mathbf{x}^{(k)}). \tag{6.17}$$

When $n = 1$, it is an equation, not a system:

$$x^{(k+1)} = x^{(k)} - f(x^{(k)})/f'(x^{(k)}). \tag{6.18}$$

This iteration method is called Newton's method. Because finding an inverse of a matrix is time consuming, in the real computation, Newton's method has the form

$$\begin{cases} \mathbf{J_f}(\mathbf{x}^{(k)})\mathbf{y} = -\mathbf{f}(\mathbf{x}^{(k)}), \\ \mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \mathbf{y}. \end{cases}$$

Newton's method converges locally with second order. More precisely, it can be proved that the following result holds.

Theorem 6.3 *Let* $\mathbf{x}^*$ *be a solution of* $\mathbf{f}(\mathbf{x}) = 0$. *Assume that* $\mathbf{J_f}(\mathbf{x}^*)$ *is not singular, and that* $f_i(\mathbf{x})$ *has continuous second-order partial derivatives near* $\mathbf{x}^*$. *Then, if* $\mathbf{x}^{(0)}$ *is close enough to* $\mathbf{x}^*$, *Newton's method converges and*

$$\left\|\mathbf{x}^{(k+1)} - \mathbf{x}^*\right\|_\infty \le C\left\|\mathbf{x}^{(k)} - \mathbf{x}^*\right\|_\infty^2.$$

Generalized Secant Method. One of weaknesses of Newton's method for solving nonlinear systems is that the Jacobian matrix must be computed at each iteration. The Jacobian matrix associated with a system $\mathbf{f}(\mathbf{x}) = 0$ requires n^2 partial derivatives to be evaluated. In many situations, the exact evaluation of the partial derivatives is inconvenient. This difficulty can be overcome by using finite-difference approximations to the partial derivatives. For example,

$$\begin{aligned}&\frac{\partial f_i(x_1, x_2, \cdots, x_n)}{\partial x_k} \\ &\approx \frac{1}{\Delta x_k}[f_i(x_1, \cdots, x_k + \Delta x_k, \cdots, x_n) - f_i(x_1, \cdots, x_k, \cdots, x_n)], \\ &\qquad\qquad k = 1, 2, \cdots, n,\end{aligned}$$

where Δx_k is small in absolute value. This approximation, however, still requires at least n^2 function evaluations to be performed in order to approximate the Jacobian and does not decrease the amount of calculations. Actually, if we have $\mathbf{f}(\mathbf{x})$ at $n+1$ points, then we usually can have an approximate Jacobian at some point. Suppose that we have $\mathbf{x}^{(l)}$ and $\mathbf{f}(\mathbf{x}^{(l)})$, $l = k-n, k-n+1, \cdots, k$. Because

$$\begin{aligned}&\left[\mathbf{f}(\mathbf{x}^{(k-n)}) - \mathbf{f}(\mathbf{x}^{(k)}), \mathbf{f}(\mathbf{x}^{(k-n+1)}) - \mathbf{f}(\mathbf{x}^{(k)}), \cdots, \mathbf{f}(\mathbf{x}^{(k-1)}) - \mathbf{f}(\mathbf{x}^{(k)})\right] \\ &\approx \mathbf{J}_{\mathbf{f}}(\mathbf{x}^{(k)}) \left[\mathbf{x}^{(k-n)} - \mathbf{x}^{(k)}, \mathbf{x}^{(k-n+1)} - \mathbf{x}^{(k)}, \cdots, \mathbf{x}^{(k-1)} - \mathbf{x}^{(k)}\right],\end{aligned}$$

we have

$$\begin{aligned}&\mathbf{J}_{\mathbf{f}}(\mathbf{x}^{(k)}) \\ &\approx \left[\mathbf{f}(\mathbf{x}^{(k-n)}) - \mathbf{f}(\mathbf{x}^{(k)}), \mathbf{f}(\mathbf{x}^{(k-n+1)}) - \mathbf{f}(\mathbf{x}^{(k)}), \cdots, \mathbf{f}(\mathbf{x}^{(k-1)}) - \mathbf{f}(\mathbf{x}^{(k)})\right] \\ &\quad \times \left[\mathbf{x}^{(k-n)} - \mathbf{x}^{(k)}, \mathbf{x}^{(k-n+1)} - \mathbf{x}^{(k)}, \cdots, \mathbf{x}^{(k-1)} - \mathbf{x}^{(k)}\right]^{-1}.\end{aligned}$$

Therefore, Newton's method can be modified to

$$\begin{aligned}\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - &\left[\mathbf{x}^{(k-n)} - \mathbf{x}^{(k)}, \mathbf{x}^{(k-n+1)} - \mathbf{x}^{(k)}, \cdots, \mathbf{x}^{(k-1)} - \mathbf{x}^{(k)}\right] \\ &\times \left[\mathbf{f}(\mathbf{x}^{(k-n)}) - \mathbf{f}(\mathbf{x}^{(k)}), \mathbf{f}(\mathbf{x}^{(k-n+1)}) - \mathbf{f}(\mathbf{x}^{(k)}), \right. \\ &\qquad \left. \cdots, \mathbf{f}(\mathbf{x}^{(k-1)}) - \mathbf{f}(\mathbf{x}^{(k)})\right]^{-1} \mathbf{f}(\mathbf{x}^{(k)}).\end{aligned} \tag{6.19}$$

Consequently, if we have $n+1$ guesses $\mathbf{x}^{(l)}$, $l = 0, 1, \cdots, n$, and the values of the function $\mathbf{f}(\mathbf{x})$ at these points, then we can do the iteration (6.19) for $k = n, n+1, \cdots$ and at each iteration we spend very little time to calculate a Jacobian. Of course, it needs to be guaranteed that the matrix

$$\left[\mathbf{f}(\mathbf{x}^{(k-n)}) - \mathbf{f}(\mathbf{x}^{(k)}), \cdots, \mathbf{f}(\mathbf{x}^{(k-1)}) - \mathbf{f}(\mathbf{x}^{(k)})\right]$$

is invertible. If during the iteration this matrix is not invertible, we need to find the guess $\mathbf{x}^{(k+1)}$ that is close to $\mathbf{x}^{(k)}$ in another way, for example, by changing a component of $\mathbf{x}^{(k)}$ a little bit. In practice, it happens very seldom.

If $n=1$, then the vectors $\mathbf{x}$ and $\mathbf{f}$ become scalars x and f and the iteration (6.19) becomes

$$x^{(k+1)} = x^{(k)} - \frac{(x^{(k-1)} - x^{(k)})f(x^{(k)})}{f(x^{(k-1)}) - f(x^{(k)})}. \tag{6.20}$$

Thus, if we have two initial guesses $x^{(0)}$ and $x^{(1)}$, we can do this iteration starting from $k=1$. This method is called the secant method, and the iteration (6.19) is referred to as the generalized secant method. Under some conditions, for the iteration (6.19) we can prove that the following relation holds:

$$\left\|\mathbf{x}^{(k+1)} - \mathbf{x}^*\right\|_\infty \le C\left\|\mathbf{x}^{(k)} - \mathbf{x}^*\right\|_\infty^2 + C\sup_{1\le l\le n}\left\|\mathbf{x}^{(k-l)} - \mathbf{x}^{(k)}\right\|_\infty \left\|\mathbf{f}(\mathbf{x}^{(k)})\right\|_\infty$$

for $k = n, n+1, \cdots$, where C is a constant (see [97]).

Bisection Method and Modified Secant Method. Consider the case $n = 1$ and suppose $x_1^{(k-1)}$ and $x_2^{(k-1)}$ be a pair of guess for the $(k-1)$th iteration with the property $f\left(x_1^{(k-1)}\right)\cdot f\left(x_2^{(k-1)}\right) < 0$. Set $\bar{x}^{(k)} = \frac{1}{2}\left(x_1^{(k-1)} + x_2^{(k-1)}\right)$. If $f\left(\bar{x}^{(k)}\right)\cdot f\left(x_1^{(k-1)}\right) > 0$, then let $x_1^{(k)} = \bar{x}^{(k)}$ and $x_2^{(k)} = x_2^{(k-1)}$; otherwise, let $x_1^{(k)} = x_1^{(k-1)}$ and $x_2^{(k)} = \bar{x}^{(k)}$. Because $\bar{x}^{(k)}$ always replaces the component $x_i^{(k-1)}$ with the condition $f\left(\bar{x}^{(k)}\right)\cdot f\left(x_i^{(k-1)}\right) > 0,\ i = 1$ or $2,\ f\left(x_1^{(k)}\right)\cdot f\left(x_2^{(k)}\right) < 0$ still holds. It is clear that

$$\left|x_2^{(k)} - x_1^{(k)}\right| = \frac{1}{2}\left|x_2^{(k-1)} - x_1^{(k-1)}\right|.$$

Thus the method is always convergent. For the secant method, if $f\left(x_1^{(k-1)}\right)\cdot f\left(x_2^{(k-1)}\right) < 0$ holds, then we can make a modification on choosing a pair of guess, so that $f\left(x_1^{(k)}\right)\cdot f\left(x_2^{(k)}\right) < 0$. In this way the convergence of the modified secant method is also guaranteed.

Broyden's Method. There are some other ways to avoid calculating the Jacobian for each iteration except for the first iteration. Another weakness of Newton's method is that an $n\times n$ linear system has to be solved at each iteration, which usually requires $O(n^3)$ arithmetic calculations. Here, we introduce Broyden's method, which avoids calculating the Jacobian at each iteration and reduces the number of arithmetic calculations to $O(n^2)$ at each iteration if we get the inverse of the matrix for the first iteration.

Suppose that an initial approximation $\mathbf{x}^{(0)}$ is given, and $\mathbf{x}^{(1)}$ is computed by Newton's method

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} - [\mathbf{J_f}(\mathbf{x}^{(0)})]^{-1}\mathbf{f}(\mathbf{x}^{(0)}).$$

In order to get $\mathbf{x}^{(2)}$, we replace the matrix $\mathbf{J_f}(\mathbf{x}^{(1)})$ in Newton's method by a matrix $\mathbf{A}_1$ satisfying

$$\mathbf{A}_1(\mathbf{x}^{(1)} - \mathbf{x}^{(0)}) = \mathbf{f}(\mathbf{x}^{(1)}) - \mathbf{f}(\mathbf{x}^{(0)})$$

and

$$\mathbf{A}_1\mathbf{z} = \mathbf{J_f}(\mathbf{x}^{(0)})\mathbf{z} \quad \text{whenever} \quad (\mathbf{x}^{(1)} - \mathbf{x}^{(0)})^T\mathbf{z} = 0.$$

From these conditions, it can be proved that

$$\mathbf{A}_1 = \mathbf{J_f}(\mathbf{x}^{(0)}) + \frac{\mathbf{f}(\mathbf{x}^{(1)}) - \mathbf{f}(\mathbf{x}^{(0)}) - \mathbf{J_f}(\mathbf{x}^{(0)})(\mathbf{x}^{(1)} - \mathbf{x}^{(0)})}{\left\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\right\|_2^2}(\mathbf{x}^{(1)} - \mathbf{x}^{(0)})^T.$$

Using this matrix in place of $\mathbf{J_f}(\mathbf{x}^{(1)})$, we have

$$\mathbf{x}^{(2)} = \mathbf{x}^{(1)} - \mathbf{A}_1^{-1}\mathbf{f}(\mathbf{x}^{(1)}).$$

In general, suppose we have $\mathbf{x}^{(i-1)}$, $\mathbf{x}^{(i)}$ and $\mathbf{A}_{i-1}$, then we can have $\mathbf{x}^{(i+1)}$ by

$$\mathbf{A}_i = \mathbf{A}_{i-1} + \frac{\mathbf{y}^{(i)} - \mathbf{A}_{i-1}\mathbf{s}^{(i)}}{\left\|\mathbf{s}^{(i)}\right\|_2^2}(\mathbf{s}^{(i)})^T,$$

and

$$\mathbf{x}^{(i+1)} = \mathbf{x}^{(i)} - \mathbf{A}_i^{-1}\mathbf{f}(\mathbf{x}^{(i)}),$$

where $\mathbf{y}^{(i)} = \mathbf{f}(\mathbf{x}^{(i)}) - \mathbf{f}(\mathbf{x}^{(i-1)})$ and $\mathbf{s}^{(i)} = \mathbf{x}^{(i)} - \mathbf{x}^{(i-1)}$. However, at each iteration step, the linear system

$$\mathbf{A}_i\mathbf{s}^{(i+1)} = -\mathbf{f}(\mathbf{x}^{(i)})$$

still needs to be solved. To further improve the method, we need the following theorem.

Theorem 6.4 *If $\mathbf{A} \in \mathbb{R}^{n\times n}$ is nonsingular, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, and $\mathbf{y}^T\mathbf{A}^{-1}\mathbf{x} \neq -1$, then $\mathbf{A} + \mathbf{x}\mathbf{y}^T$ is also nonsingular, moreover,*

$$(\mathbf{A} + \mathbf{x}\mathbf{y}^T)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{x}\mathbf{y}^T\mathbf{A}^{-1}}{1 + \mathbf{y}^T\mathbf{A}^{-1}\mathbf{x}}.$$

This theorem suggests a simple way to find the inverse of $\mathbf{A}_i$. By setting

$$\begin{aligned}
\mathbf{A} &= \mathbf{A}_{i-1}, \\
\mathbf{x} &= \frac{\mathbf{y}^{(i)} - \mathbf{A}_{i-1}\mathbf{s}^{(i)}}{\left\|\mathbf{s}^{(i)}\right\|_2^2}, \\
\mathbf{y} &= \mathbf{s}^{(i)}, \\
\mathbf{A} + \mathbf{x}\mathbf{y}^T &= \mathbf{A}_i
\end{aligned}$$

in the above theorem, we have

$$\begin{aligned}\mathbf{A}_i^{-1} &= \mathbf{A}_{i-1}^{-1} - \frac{\mathbf{A}_{i-1}^{-1}\left(\dfrac{\mathbf{y}^{(i)} - \mathbf{A}_{i-1}\mathbf{s}^{(i)}}{\left\|\mathbf{s}^{(i)}\right\|_2^2}(\mathbf{s}^{(i)})^T\right)\mathbf{A}_{i-1}^{-1}}{1 + (\mathbf{s}^{(i)})^T\mathbf{A}_{i-1}^{-1}\left(\dfrac{\mathbf{y}^{(i)} - \mathbf{A}_{i-1}\mathbf{s}^{(i)}}{\left\|\mathbf{s}^{(i)}\right\|_2^2}\right)} \\ &= \mathbf{A}_{i-1}^{-1} - \frac{(\mathbf{A}_{i-1}^{-1}\mathbf{y}^{(i)} - \mathbf{s}^{(i)})(\mathbf{s}^{(i)})^T\mathbf{A}_{i-1}^{-1}}{\left\|\mathbf{s}^{(i)}\right\|_2^2 + (\mathbf{s}^{(i)})^T\mathbf{A}_{i-1}^{-1}\mathbf{y}^{(i)} - \left\|\mathbf{s}^{(i)}\right\|_2^2} \\ &= \mathbf{A}_{i-1}^{-1} + \frac{(\mathbf{s}^{(i)} - \mathbf{A}_{i-1}^{-1}\mathbf{y}^{(i)})(\mathbf{s}^{(i)})^T\mathbf{A}_{i-1}^{-1}}{(\mathbf{s}^{(i)})^T\mathbf{A}_{i-1}^{-1}\mathbf{y}^{(i)}}.\end{aligned}$$

This computation requires only $O(n^2)$ arithmetic calculations because it involves only matrix-vector multiplications. Therefore, we have the following Broyden's method:

- Given initial guess $\mathbf{x}^{(0)}$, compute $\mathbf{A}_0^{-1} = [\mathbf{J}_\mathbf{f}(\mathbf{x}^{(0)})]^{-1}$ and $\mathbf{x}^{(1)}$.
- For $i = 1, 2, \cdots$, do the following:

$$\begin{cases}\mathbf{y}^{(i)} = \mathbf{f}(\mathbf{x}^{(i)}) - \mathbf{f}(\mathbf{x}^{(i-1)}), \ \ \mathbf{s}^{(i)} = \mathbf{x}^{(i)} - \mathbf{x}^{(i-1)}, \\ \mathbf{A}_i^{-1} = \mathbf{A}_{i-1}^{-1} + \dfrac{(\mathbf{s}^{(i)} - \mathbf{A}_{i-1}^{-1}\mathbf{y}^{(i)})(\mathbf{s}^{(i)})^T\mathbf{A}_{i-1}^{-1}}{(\mathbf{s}^{(i)})^T\mathbf{A}_{i-1}^{-1}\mathbf{y}^{(i)}}, \\ \mathbf{x}^{(i+1)} = \mathbf{x}^{(i)} - \mathbf{A}_i^{-1}\mathbf{f}(\mathbf{x}^{(i)}).\end{cases} \tag{6.21}$$

Broyden's method reduces a large amount of work from Newton's method. However, the quadratic convergence of Newton's method is lost. For Broyden's method, we have

$$\lim_{i\to\infty} \frac{\left\|\mathbf{x}^{(i+1)} - \mathbf{x}^*\right\|}{\left\|\mathbf{x}^{(i)} - \mathbf{x}^*\right\|} = 0.$$

This type of convergence is called **superlinear**. For more about Broyden's method, see books on numerical methods.

6.2.4 Obtaining Eigenvalues and Eigenvectors

In this subsection, we will discuss how to get eigenvalues and eigenvectors of a square matrix, especially, a symmetric matrix. Before that, we introduce some basic tools we will need.

Consider an $m \times m$ matrix in the form

$$\mathbf{H}_m = \mathbf{I}_m - \alpha \mathbf{v}\mathbf{v}^T,$$

where $\mathbf{I}_m$ is an $m \times m$ identity matrix, $\mathbf{v}$ is an m-dimensional vector, and α is a number. Obviously, $\mathbf{H}_m$ is a symmetric matrix. We also want $\mathbf{H}_m$ to be orthogonal, namely,

$$\begin{aligned} \mathbf{H}_m^T \mathbf{H}_m &= \left(\mathbf{I}_m - \alpha \mathbf{v}\mathbf{v}^T\right)\left(\mathbf{I}_m - \alpha \mathbf{v}\mathbf{v}^T\right) \\ &= \mathbf{I}_m - 2\alpha \mathbf{v}\mathbf{v}^T + \alpha^2 \mathbf{v}\mathbf{v}^T\mathbf{v}\mathbf{v}^T \\ &= \mathbf{I}_m - \left(2\alpha - \alpha^2 \mathbf{v}^T\mathbf{v}\right)\mathbf{v}\mathbf{v}^T = \mathbf{I}_m. \end{aligned}$$

Therefore, we require

$$\alpha = \frac{2}{\mathbf{v}^T\mathbf{v}}$$

and

$$\mathbf{H}_m = \mathbf{I}_m - \frac{2}{\mathbf{v}^T\mathbf{v}} \mathbf{v}\mathbf{v}^T. \tag{6.22}$$

The matrix defined by the expression (6.22) is called a Householder matrix. We are especially interested in the Householder matrix satisfying

$$\mathbf{H}_m \mathbf{x} = \beta \mathbf{e}_1, \tag{6.23}$$

where $\mathbf{x} = [x_1, x_2, \cdots, x_m]^T$ is an m-dimensional vector, β is a number whose value may depend on the components of $\mathbf{x}$, and $\mathbf{e}_1 = [1, 0, \cdots, 0]^T$. Because

$$\mathbf{H}_m \mathbf{x} = \mathbf{x} - \frac{2}{\mathbf{v}^T\mathbf{v}} \mathbf{v}\mathbf{v}^T\mathbf{x} = \mathbf{x} - \frac{2\mathbf{v}^T\mathbf{x}}{\mathbf{v}^T\mathbf{v}} \mathbf{v} = \beta \mathbf{e}_1,$$

we have

$$\mathbf{u} \equiv \frac{2\mathbf{v}^T\mathbf{x}}{\mathbf{v}^T\mathbf{v}} \mathbf{v} = \mathbf{x} - \beta \mathbf{e}_1 \tag{6.24}$$

and

$$\mathbf{u}^T\mathbf{x} = \mathbf{x}^T\mathbf{x} - \beta x_1, \quad \mathbf{u}^T\mathbf{u} = \mathbf{x}^T\mathbf{x} - 2\beta x_1 + \beta^2.$$

Therefore, we further obtain

$$\begin{aligned} \mathbf{H}_m \mathbf{x} &= \left(\mathbf{I}_m - \frac{2}{\mathbf{v}^T\mathbf{v}} \mathbf{v}\mathbf{v}^T\right)\mathbf{x} = \left(\mathbf{I}_m - \frac{2}{\mathbf{u}^T\mathbf{u}} \mathbf{u}\mathbf{u}^T\right)\mathbf{x} \\ &= \mathbf{x} - \frac{2\mathbf{u}^T\mathbf{x}}{\mathbf{u}^T\mathbf{u}} \mathbf{u} = \left(1 - \frac{2\mathbf{u}^T\mathbf{x}}{\mathbf{u}^T\mathbf{u}}\right)\mathbf{x} + \frac{2\mathbf{u}^T\mathbf{x}}{\mathbf{u}^T\mathbf{u}} \beta \mathbf{e}_1. \end{aligned}$$

Because we want the relation (6.23) to hold, we require

$$1 - \frac{2\mathbf{u}^T\mathbf{x}}{\mathbf{u}^T\mathbf{u}} = 1 - \frac{2\left(\mathbf{x}^T\mathbf{x} - \beta x_1\right)}{\mathbf{x}^T\mathbf{x} - 2\beta x_1 + \beta^2} = 0$$

or

$$\beta = \pm\sqrt{\mathbf{x}^T\mathbf{x}}. \tag{6.25}$$

Usually, we take the + sign so that the first component of the vector $\mathbf{H}_m\mathbf{x}$ is nonnegative. In this case

$$\mathbf{H}_m = \mathbf{I}_m - \frac{2}{\mathbf{u}^T\mathbf{u}}\mathbf{u}\mathbf{u}^T = \mathbf{I}_m - \frac{1}{\beta(\beta - x_1)}\mathbf{u}\mathbf{u}^T, \tag{6.26}$$

where $\mathbf{u}$ and β are given by the expressions (6.24) and (6.25).

An $n \times n$ matrix

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix}$$

is called an upper triangular matrix if $a_{ij} = 0$ for $i > j$ and an upper Hessenberg matrix if $a_{ij} = 0$ for $i > j + 1$. Because a Householder matrix defined by the expression (6.26) has the property (6.23), it can be used to reduce a matrix $\mathbf{A}$ to an upper triangular matrix or an upper Hessenberg matrix, which will be described below. Based on this fact, we can have the so-called **QR** algorithm for finding the eigenvalues of a matrix.

The first step of the **QR** algorithm for finding the eigenvalues of a matrix $\mathbf{A}$ is to reduce the matrix to an upper Hessenberg matrix. Let $\mathbf{P}_k$ be an $n \times n$ matrix in the form:

$$\mathbf{P}_k = \begin{bmatrix} \mathbf{I}_k & 0 \\ 0 & \mathbf{H}_{n-k} \end{bmatrix},$$

where k is equal to $0, 1, \cdots$, or $n-2$, $\mathbf{H}_{n-k}$ is an $(n-k) \times (n-k)$ matrix defined by the expression (6.26). Clearly, $\mathbf{P}_k$ is a Householder matrix. Suppose that after using $k-1$ Householder transformations, $\mathbf{A}$ is changed to

$$\mathbf{A}_{k-1} = (\mathbf{P}_1 \cdots \mathbf{P}_{k-1})^T \mathbf{A} (\mathbf{P}_1 \cdots \mathbf{P}_{k-1}) = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} & \mathbf{C}_{13} \\ \mathbf{C}_{21} & \mathbf{C}_{22} & \mathbf{C}_{23} \\ 0 & \mathbf{C}_{32} & \mathbf{C}_{33} \end{bmatrix},$$

where

$$\begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix}$$

is a $k \times k$ upper Hessenberg matrix and $\mathbf{C}_{32}$ is a column vector. Now let us define $\mathbf{A}_k = \mathbf{P}_k^T \mathbf{A}_{k-1} \mathbf{P}_k$, and from the forms of $\mathbf{A}_{k-1}$ and $\mathbf{P}_k$ we have

$$\mathbf{A}_k = \mathbf{P}_k^T \mathbf{A}_{k-1} \mathbf{P}_k = \mathbf{P}_k \mathbf{A}_{k-1} \mathbf{P}_k = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} & \mathbf{C}_{13}\mathbf{H}_{n-k} \\ \mathbf{C}_{21} & \mathbf{C}_{22} & \mathbf{C}_{23}\mathbf{H}_{n-k} \\ 0 & \mathbf{H}_{n-k}\mathbf{C}_{32} & \mathbf{H}_{n-k}\mathbf{C}_{33}\mathbf{H}_{n-k} \end{bmatrix}.$$

If we take $\mathbf{C}_{32}$ as $\mathbf{x}$ and determine $\mathbf{H}_{n-k}$, then we arrive at

$$\mathbf{H}_{n-k}\mathbf{C}_{32} = \left[\beta, 0, \cdots, 0\right]^T.$$

Therefore, the $(k+1)\times(k+1)$ submatrix at the upper-left corner of $\mathbf{A}_k$ is an upper Hessenberg matrix, and the procedure can continue for $k+1$. For $k=1$, this procedure can be done. Consequently, we can do this procedure from $k=1$ to $n-2$, and finally obtain an upper Hessenberg matrix:

$$\mathbf{A}_{n-2} = (\mathbf{P}_1\cdots\mathbf{P}_{n-2})^T\mathbf{A}(\mathbf{P}_1\cdots\mathbf{P}_{n-2}). \tag{6.27}$$

Now let us discuss the second step. If the procedure above starts from $\mathbf{P}_0$ and a matrix is multiplied only by $\mathbf{P}_k^T$ from the left-hand side, then we will obtain an upper triangular matrix with nonnegative main diagonal entries. Therefore, for any matrix $\mathbf{B}$, we can find an orthogonal matrix $\mathbf{Q}^T$ such that $\mathbf{Q}^T\mathbf{B}=\mathbf{R}$ or $\mathbf{B}=\mathbf{QR}$, where $\mathbf{R}$ is an upper triangular matrix with nonnegative main diagonal entries. This procedure is called QR factorization. Using the QR factorization and letting $\mathbf{B}_1=\mathbf{A}_{n-2}$, we have the following iteration:

$$\begin{aligned}\mathbf{B}_k &= \mathbf{Q}_k\mathbf{R}_k,\\ \mathbf{B}_{k+1} &= \mathbf{R}_k\mathbf{Q}_k = \mathbf{Q}_k^T\mathbf{B}_k\mathbf{Q}_k\end{aligned} \tag{6.28}$$

for $k=1,2,\cdots$. That is, first get $\mathbf{Q}_k$ and $\mathbf{R}_k$ from $\mathbf{B}_k$ and then multiplying $\mathbf{R}_k$ by $\mathbf{Q}_k$ from the right-hand side yields $\mathbf{B}_{k+1}$. For this iteration, we have the following relation

$$\begin{aligned}\mathbf{B}_{k+1} &= \mathbf{Q}_k^T\mathbf{B}_k\mathbf{Q}_k = \mathbf{Q}_k^T\cdots\mathbf{Q}_1^T\mathbf{A}_{n-2}\mathbf{Q}_1\cdots\mathbf{Q}_k\\ &= (\mathbf{Q}_1\cdots\mathbf{Q}_k)^T\,\mathbf{A}_{n-2}\,(\mathbf{Q}_1\cdots\mathbf{Q}_k)\\ &= (\mathbf{P}_1\cdots\mathbf{P}_{n-2}\mathbf{Q}_1\cdots\mathbf{Q}_k)^T\,\mathbf{A}\,(\mathbf{P}_1\cdots\mathbf{P}_{n-2}\mathbf{Q}_1\cdots\mathbf{Q}_k)\,,\end{aligned}$$

or

$$\mathbf{B}_{k+1} = \mathbf{S}_k^T\mathbf{A}\mathbf{S}_k,$$

where

$$\mathbf{S}_k = \mathbf{P}_1\cdots\mathbf{P}_{n-2}\mathbf{Q}_1\cdots\mathbf{Q}_k.$$

Let $\mathbf{B}$ and $\mathbf{S}$ be the limits of $\mathbf{B}_{k+1}$ and $\mathbf{S}_k$ as $k\to\infty$ respectively, then we have

$$\mathbf{B} = \mathbf{S}^T\mathbf{A}\mathbf{S}.$$

The goal of the iteration is to find an upper triangular matrix that is similar to $\mathbf{A}$, so that we can have the eigenvalues of $\mathbf{A}$ from the main diagonal entries of the upper triangular matrix. From the relation (6.28), we can see as follows. First, we get an upper triangular matrix by multiplying an orthogonal matrix from the left-hand side, but in order to let the new matrix be similar to the old one, multiplying the same orthogonal matrix from the right-hand side is needed, which may destroy the goal of finding an upper triangular matrix. However, under certain conditions it will be proved that the limit $\mathbf{B}$ is an upper triangular matrix. Therefore, we may reach our goal at the end of the iteration.

In order to find the eigenvectors of $\mathbf{A}$, we first need to find the eigenvectors of $\mathbf{B}$. As soon as we find the eigenvectors of $\mathbf{B}$, the eigenvectors of $\mathbf{A}$ can be obtained through multiplying the eigenvectors of $\mathbf{B}$ from the left-hand side by $\mathbf{S}$. If $\mathbf{A}$ is symmetric, then $\mathbf{B}$ is diagonal and every column of $\mathbf{S}$ is an eigenvector of $\mathbf{A}$.

For the convergence of the iteration we have

Theorem 6.5 *Assume that the eigenvalues of* $\mathbf{B}_1$ *have distinct absolute values, and* $\mathbf{X}^{-1}$ *has an* $\mathbf{LU}$ *decomposition, where* $\mathbf{X}$ *is the matrix of eigenvectors. Then,* $\mathbf{B}_k$ *converges to an upper triangular matrix.*

Proof. Suppose $\mathbf{X}$ has the decomposition

$$\mathbf{X} = \mathbf{Q}_x\mathbf{R}_x,$$

where $\mathbf{Q}_x$ is orthogonal and $\mathbf{R}_x$ is upper triangular with positive main diagonal entries. Then, we have

$$\begin{aligned}\mathbf{B}_1^k &= \mathbf{X}\boldsymbol{\Lambda}^k\mathbf{X}^{-1} = \mathbf{X}(\boldsymbol{\Lambda}^k\mathbf{L}\boldsymbol{\Lambda}^{-k})\boldsymbol{\Lambda}^k\mathbf{U}\\ &= \mathbf{Q}_x\mathbf{R}_x(\mathbf{I}+\mathbf{E}_k)\boldsymbol{\Lambda}^k\mathbf{U}\\ &= \mathbf{Q}_x(\mathbf{I}+\mathbf{R}_x\mathbf{E}_k\mathbf{R}_x^{-1})\mathbf{R}_x\boldsymbol{\Lambda}^k\mathbf{U},\end{aligned}$$

where $\boldsymbol{\Lambda}$ is the Jordan canonical matrix of $\mathbf{B}_1$ and $\mathbf{E}_k = \boldsymbol{\Lambda}^k\mathbf{L}\boldsymbol{\Lambda}^{-k} - \mathbf{I} \to 0$ as $k \to \infty$ because we assume $|\lambda_1| > |\lambda_2| > \cdots |\lambda_n|$, $|\lambda_i|$ being an eigenvalue of $\mathbf{B}_1$. Let

$$\mathbf{I} + \mathbf{R}_x\mathbf{E}_k\mathbf{R}_x^{-1} = \mathbf{Q}^{(k)}\mathbf{R}^{(k)},$$

where $\mathbf{Q}^{(k)}$ is orthogonal and $\mathbf{R}^{(k)}$ is upper triangular with positive main diagonal entries. Obviously,

$$\mathbf{Q}^{(k)} \to \mathbf{I},\ \ \mathbf{R}^{(k)} \to \mathbf{I}.$$

Let $\mathbf{D}$ and $\mathbf{D}_u$ be diagonal matrices defined by

$$\begin{aligned}\mathbf{D} &= \mathrm{diag}(\lambda_1/|\lambda_1|, \cdots, \lambda_n/|\lambda_n|),\\ \mathbf{D}_u &= \mathrm{diag}(u_{11}/|u_{11}|, \cdots, u_{nn}/|u_{nn}|),\end{aligned}$$

where u_{ii}, $i = 1, \cdots, n$, are the main diagonal entries of $\mathbf{U}$. Then, we have

$$\begin{aligned}\mathbf{B}_1^k &= \mathbf{Q}_x\mathbf{Q}^{(k)}\mathbf{R}^{(k)}\mathbf{R}_x\boldsymbol{\Lambda}^k\mathbf{U}\\ &= (\mathbf{Q}_x\mathbf{Q}^{(k)}\mathbf{D}_u\mathbf{D}^k)(\mathbf{D}^{-k}\mathbf{D}_u^{-1}\mathbf{R}^{(k)}\mathbf{R}_x\boldsymbol{\Lambda}^k\mathbf{U}).\end{aligned}$$

Because a product of two upper triangular matrices is an upper triangular matrix, and because a main diagonal entry of the new matrix is the product of the corresponding main diagonal entries in each original matrix, this is a QR decomposition of $\mathbf{B}_1^k$ and the upper triangular matrix $\mathbf{D}^{-k}\mathbf{D}_u^{-1}\mathbf{R}^{(k)}\mathbf{R}_x\boldsymbol{\Lambda}^k\mathbf{U}$ has positive main diagonal entries. On the other hand, it can be shown that

$$\mathbf{B}_1^k = \hat{\mathbf{Q}}_k\hat{\mathbf{R}}_k,$$

where

$$\hat{\mathbf{Q}}_k = \mathbf{Q}_1 \cdots \mathbf{Q}_k, \ \hat{\mathbf{R}}_k = \mathbf{R}_k \cdots \mathbf{R}_1.$$

In fact

$$\mathbf{B}_k = \hat{\mathbf{Q}}_{k-1}^T \mathbf{B}_1 \hat{\mathbf{Q}}_{k-1}$$

or

$$\mathbf{B}_1 \hat{\mathbf{Q}}_{k-1} = \hat{\mathbf{Q}}_{k-1} \mathbf{B}_k = \hat{\mathbf{Q}}_{k-1} \mathbf{Q}_k \mathbf{R}_k = \hat{\mathbf{Q}}_k \mathbf{R}_k.$$

Multiplying $\hat{\mathbf{R}}_{k-1}$ from the right-hand side on both sides of the relation $\hat{\mathbf{Q}}_k \mathbf{R}_k = \mathbf{B}_1 \hat{\mathbf{Q}}_{k-1}$, we get

$$\hat{\mathbf{Q}}_k \hat{\mathbf{R}}_k = \mathbf{B}_1 \hat{\mathbf{Q}}_{k-1} \hat{\mathbf{R}}_{k-1}$$

and furthermore we obtain

$$\hat{\mathbf{Q}}_k \hat{\mathbf{R}}_k = \mathbf{B}_1 \hat{\mathbf{Q}}_{k-1} \hat{\mathbf{R}}_{k-1} = \mathbf{B}_1^2 \hat{\mathbf{Q}}_{k-2} \hat{\mathbf{R}}_{k-2} = \cdots = \mathbf{B}_1^k.$$

Therefore, we have another QR decomposition of $\mathbf{B}_1^k$. Because the QR decomposition is unique, we have

$$\hat{\mathbf{Q}}_k = \mathbf{Q}_x \mathbf{Q}^{(k)} \mathbf{D}_u \mathbf{D}^k, \ \hat{\mathbf{R}}_k = \mathbf{D}^{-k} \mathbf{D}_u^{-1} \mathbf{R}^{(k)} \mathbf{R}_x \mathbf{\Lambda}^k \mathbf{U}.$$

Therefore,

$$\begin{aligned}
\mathbf{B}_{k+1} &= (\mathbf{D}^T)^k \mathbf{D}_u^T (\mathbf{Q}^{(k)})^T \mathbf{Q}_x^T \mathbf{B}_1 \mathbf{Q}_x \mathbf{Q}^{(k)} \mathbf{D}_u \mathbf{D}^k \\
&= (\mathbf{D}^T)^k \mathbf{D}_u^T (\mathbf{Q}^{(k)})^T \mathbf{Q}_x^T \mathbf{Q}_x \mathbf{R}_x \mathbf{\Lambda} \mathbf{R}_x^{-1} \mathbf{Q}_x^{-1} \mathbf{Q}_x \mathbf{Q}^{(k)} \mathbf{D}_u \mathbf{D}^k \\
&= (\mathbf{D}^T)^k \mathbf{D}_u^T (\mathbf{Q}^{(k)})^T \mathbf{R}_x \mathbf{\Lambda} \mathbf{R}_x^{-1} \mathbf{Q}^{(k)} \mathbf{D}_u \mathbf{D}^k.
\end{aligned}$$

Because $\mathbf{Q}^{(k)} \to \mathbf{I}$ and an inverse of an upper triangular matrix is still an upper triangular matrix, $\mathbf{B}_{k+1}$ converges to an upper triangular matrix. □

From the proof, we can see that it is not necessary for $\mathbf{B}_1$ to be an upper Hessenberg matrix. Having a Hessenberg matrix at the first step is for the practical reason of reducing computational cost. If $\mathbf{B}_k$ is in upper Hessenberg form, then $\mathbf{B}_{k+1}$ is also in upper Hessenberg form. Thus, in the entire iteration process, we deal with upper Hessenberg matrices. For an upper Hessenberg matrix, the amount of computational work at each step of the QR factorization is $O(n^2)$, which is much smaller than $O(n^3)$ for a full matrix. In order to make computation faster, we can also speed up the convergence of the **QR** algorithm by combining the shifting technique. In addition, there are some other methods for finding eigenvalues of a matrix, for example, the Jacobi algorithm. For more about the **QR** algorithm, the details of the shifting technique, and other methods, see books on matrix computation, for example, the book [35] by Golub and Loan.

6.3 Determination of Parameters in Models

In order to price an option on a specified underlying asset, we must have a model for the asset. We can have various models, and we have to determine the parameters in the model before pricing. In this section, we will discuss how to determine the parameters in models from the market data.

6.3.1 Constant Variances and Covariances

Assume that the stochastic process of an asset price S can be described by

$$dS = adt + bdX,$$

where a and b are constants and dX is a Wiener process. Because we assume that the parameters in the stochastic process do not depend on time, we can determine a and b according to the historical data. Clearly,

$$\mathrm{E}\left[dS\right] = adt$$

and

$$\mathrm{Var}\left[dS\right] = \mathrm{E}\left[(dS - adt)^2\right] = \mathrm{E}\left[(bdX)^2\right] = b^2 dt,$$

that is,

$$a = \frac{1}{dt}\mathrm{E}\left[dS\right]$$

and

$$b^2 = \frac{1}{dt}\mathrm{Var}\left[dS\right].$$

Suppose that from the market, we have the values of the asset price S at time $t^i = T_1 + (i-1)dt$, $i = 1, 2, \cdots, I+1$. From any statistics textbook, we know that the mean and variance of dS can be approximated by

$$\mathrm{E}\left[dS\right] \approx \frac{1}{I}\sum_{i=1}^{I} dS_i = \frac{1}{I}\sum_{i=1}^{I}(S_{i+1} - S_i)$$

and

$$\mathrm{Var}\left[dS\right] \approx \frac{1}{I-1}\sum_{i=1}^{I}\left[S_{i+1} - S_i - \frac{1}{I}\sum_{i=1}^{I}(S_{i+1} - S_i)\right]^2.$$

Thus, we have the estimates for a and b^2 as follows:

$$a \approx \frac{1}{Idt}\sum_{i=1}^{I}(S_{i+1} - S_i) \tag{6.29}$$

and

$$b^2 \approx \frac{1}{(I-1)dt}\sum_{i=1}^{I}\left[S_{i+1}-S_i-\frac{1}{I}\sum_{i=1}^{I}(S_{i+1}-S_i)\right]^2$$
$$=\frac{1}{(I-1)dt}\left[\sum_{i=1}^{I}(S_{i+1}-S_i)^2-\frac{1}{I}\left(\sum_{i=1}^{I}(S_{i+1}-S_i)\right)^2\right]. \quad (6.30)$$

Now suppose

$$dS=\mu Sdt+\sigma SdX$$

and let us discuss how to find μ and σ from the market data. Because $dS = \mu Sdt + \sigma SdX$ can be written as

$$d\ln S=(\mu-\sigma^2/2)dt+\sigma dX,$$

then we can estimate μ and σ^2 by

$$\sigma^2 \approx \frac{1}{(I-1)dt}\left[\sum_{i=1}^{I}(\ln S_{i+1}-\ln S_i)^2-\frac{1}{I}\left(\sum_{i=1}^{I}(\ln S_{i+1}-\ln S_i)\right)^2\right]$$
$$\approx \frac{1}{(I-1)dt}\left[\sum_{i=1}^{I}\left(\frac{S_{i+1}-S_i}{S_i}\right)^2-\frac{1}{I}\left(\sum_{i=1}^{I}\frac{S_{i+1}-S_i}{S_i}\right)^2\right] \quad (6.31)$$

and

$$\mu-\sigma^2/2 \approx \frac{1}{Idt}\sum_{i=1}^{I}(\ln S_{i+1}-\ln S_i) \approx \frac{1}{Idt}\sum_{i=1}^{I}\frac{S_{i+1}-S_i}{S_i}$$

or

$$\mu \approx \frac{1}{Idt}\sum_{i=1}^{I}\frac{S_{i+1}-S_i}{S_i}+\sigma^2/2. \quad (6.32)$$

Here, we have used the approximate relation

$$\ln S_{i+1}-\ln S_i \approx \frac{S_{i+1}-S_i}{S_i}.$$

Suppose that there are two stochastic processes:

$$dS_1=a_1dt+b_1dX_1$$

and

$$dS_2=a_2dt+b_2dX_2,$$

where a_1, b_1, a_2, and b_2 are constants, dX_1, dX_2 are two Wiener processes correlated with $\mathrm{E}\left[dX_1dX_2\right]=\rho dt$. Assume that we have the values of the asset prices S_1 and S_2 at time $t^i=T_1+(i-1)dt$, which are denoted by $S_{1,i}$ and $S_{2,i}$, $i=1,2,\cdots,I+1$. We can have estimates for a_1, b_1, a_2, and b_2 by the formulae (6.29) and (6.30). Now let us discuss how to estimate ρ from $S_{1,i}$ and $S_{2,i}$, $i = 1, 2, \cdots, I+1$. Because

$$\mathrm{E}\left[dX_1 dX_2\right] = \mathrm{E}\left[\frac{dS_1 - a_1 dt}{b_1} \times \frac{dS_2 - a_2 dt}{b_2}\right] = \frac{1}{b_1 b_2}\left\{\mathrm{E}\left[dS_1 dS_2\right] - a_1 a_2 dt^2\right\},$$

we have

$$\rho = \frac{1}{b_1 b_2 dt}\left\{\mathrm{E}\left[dS_1 dS_2\right] - a_1 a_2 dt^2\right\}.$$

From statistics, we know

$$\mathrm{E}\left[dS_1 dS_2\right] \approx \frac{1}{I-1}\sum_{i=1}^{I}\left(S_{1,i+1} - S_{1,i}\right)\left(S_{2,i+1} - S_{2,i}\right),$$

so we have

$$\rho \approx \frac{1}{b_1 b_2 dt}\left[\frac{1}{I-1}\sum_{i=1}^{I}\left(S_{1,i+1} - S_{1,i}\right)\left(S_{2,i+1} - S_{2,i}\right) - a_1 a_2 dt^2\right]. \tag{6.33}$$

On the market, the data are given hourly, daily, and so forth, and only on workdays. Suppose we use the data given daily and the adopted time unit is year. When doing the computation, we should think that dt between two successive workdays is always equal to $1/I_w$, where I_w is the number of workdays per year.

6.3.2 Variable Parameters

From Figs. 1.1–1.7, we can see that the assumption of the volatility being constant might not be a good assumption. For example, Figs. 1.1 and 1.2 show that the prices of IBM and GE stocks have less volatilities if the price is lower. Therefore, we assume that volatilities are functions of stock prices S. That is, the stochastic process of S is described by

$$dS = a\left(S\right)dt + b\left(S\right)dX,$$

where $a\left(S\right)$ and $b\left(S\right)$ are functions of S to be determined. Because we do not assume the dependence of the parameters on time t, we can still determine $a\left(S\right)$ and $b\left(S\right)$ from the historical data.

Again, suppose that we have $I+1$ prices of an asset from the market: S_i, $i = 1, 2, \cdots, I+1$. Let $S_{\max}$ and $S_{\min}$ be the maximum and minimum values among them. Set $S_{(m)} = S_{\min} - \varepsilon + m\left(S_{\max} - S_{\min} + \varepsilon\right)/\left(M+1\right)$, $m = 0, 1, \cdots, M+1$, where ε is a small positive number. Clearly, $S_{(0)} = S_{\min} - \varepsilon$ and $S_{(M+1)} = S_{\max}$. The entire interval $\left(S_{(0)}, S_{(M+1)}\right]$ is divided into $M+1$ subintervals $\left(S_{(m-1)}, S_{(m)}\right], m = 1, 2, \cdots, M+1$. Every S_i belongs to one of these subintervals. Consider S_i, $i = 1, 2, \cdots, I$. If $S_i \in \left(S_{(m-1)}, S_{(m)}\right]$, then we say that S_i belongs to the set $\mathcal{S}_{(m)}$. Let I_m be the number of elements in the set $\mathcal{S}_{(m)}$. It is clear that $\sum\limits_{m=1}^{M+1} I_m = I$. For each set $\mathcal{S}_{(m)}$, we can have a mean $a_{(m)}$ and a variance $b^2_{(m)}$ by the formulae (6.29) and (6.30).

The variance $b^2_{(m)}$ is an approximate variance of the random variable S at $S = (S_{(m-1)} + S_{(m)})/2$, $m = 1, 2, \cdots, M+1$. We define $S_{(m-1/2)} = \left(S_{(m-1)} + S_{(m)}\right)/2$, so $b(S_{(m-1/2)}) \approx b_{(m)}$. Because S is defined on $[0, \infty)$, $b(S)$ is a function on $[0, \infty)$. However, it is not convenient to approximate the function $b(S)$ defined on an infinite interval. Hence we introduce a transformation

$$\xi = \frac{S}{S + P_m},$$

where P_m is a positive number. This transformation maps $[0, \infty)$ to $[0, 1)$. Therefore, we assume that $b\,(S)$ is in the form $\bar{b}(\xi)$ and find $\bar{b}(\xi)$ on the interval $[0, 1)$. It is clear that $b_{(m)}$ should be an approximation to $\bar{b}\left(\xi_{(m-1/2)}\right)$, where $\xi_{(m-1/2)} = \dfrac{S_{(m-1/2)}}{S_{(m-1/2)} + P_m}$. Now the problem is reduced to finding a function $\bar{b}(\xi)$ such that the points $\left(\xi_{(m-1/2)}, b_{(m)}\right)$, $m = 1, 2, \cdots, M+1$, are as close to $\bar{b}(\xi)$ as possible. Assume

$$\bar{b}\,(\xi) = g\,(\xi) \sum_{n=0}^{N} a_n \xi^n,$$

where $N < M$ and $g\,(\xi)$ is a given function, for example, $g\,(\xi) = 1$ or $\dfrac{P_m \xi}{1 - \xi}$. Under this assumption, using the points $\left(\xi_{\left(m-\frac{1}{2}\right)}, b_{(m)}\right)$, $m = 1, 2, \cdots, M+1$ and taking the weights $b_m = I_m/I$, we can find $a_0, a_1, \cdots, a_N$ by the least squares method with weights in Sect. 6.1.4. As soon as we find $\bar{b}\,(\xi)$, we have $b\,(S)$ by

$$b\,(S) = g\left(\frac{S}{S + P_m}\right) \sum_{n=0}^{N} a_n \left(\frac{S}{S + P_m}\right)^n .$$

If $b(S) < 0$ in some small regions, then a local modification is needed in order to guarantee $b(S) \geq 0$ for all $S \in [0, \infty)$. For $a\,(S)$, the method is similar.

Now let us discuss the case involving several stochastic processes. For simplicity, suppose we have two stochastic processes governed by

$$dS_1 = a_1(S_1)dt + b_1(S_1)dX_1$$

and

$$dS_2 = a_2(S_2)dt + b_2(S_2)dX_2$$

with $\mathrm{E}[dX_1 dX_2] = \rho dt$, ρ being a constant. Using the method given above, we can find $a_1(S_1)$, $b_1(S_1)$, $a_2(S_2)$, and $b_2(S_2)$. Because we assume that ρ is a constant, it can be determined by

$$\begin{aligned}\rho &= \frac{1}{dt}\mathrm{E}[dX_1 dX_2] \\ &= \frac{1}{dt}\mathrm{E}\left[\frac{dS_1 - a_1(S_1)dt}{b_1(S_1)} \times \frac{dS_2 - a_2(S_2)dt}{b_2(S_2)}\right] \\ &\approx \frac{1}{(I-1)dt}\sum_{i=1}^{I}\left[\frac{S_{1,i+1} - S_{1,i} - a_1(S_{1,i})dt}{b_1(S_{1,i})} \times \frac{S_{2,i+1} - S_{2,i} - a_2(S_{2,i})dt}{b_2(S_{2,i})}\right].\end{aligned}$$

Problems

Table 6.1. Problems and Sections

Problems	Sections	Problems	Sections	Problems	Sections
1–6	6.1	7–14	6.2	15	6.3

1. Suppose $x_m = m\Delta x$.
   (a) Find the order of the error of the following approximate function
   $$u(x) \approx \frac{x_{m+1} - x}{\Delta x}u(x_m) + \frac{x - x_m}{\Delta x}u(x_{m+1})$$
   by the Taylor expansion. Here $x \in [x_m, x_{m+1}]$.
   (b) Find the order of the error of the following approximate function
   $$\begin{aligned}u(x) \approx\ & \frac{(x - x_m)(x - x_{m+1})}{2\Delta x^2}u(x_{m-1}) \\ & - \frac{(x - x_{m-1})(x - x_{m+1})}{\Delta x^2}u(x_m) \\ & + \frac{(x - x_{m-1})(x - x_m)}{2\Delta x^2}u(x_{m+1})\end{aligned}$$
   by the Taylor expansion. Here $x \in [x_{m-1}, x_{m+1}]$ and $x_{m-1} < x_m < x_{m+1}$.
2. *Show that from
$$\begin{cases} a_m = a_{m-1} + b_{m-1}h_{m-1} + c_{m-1}h_{m-1}^2 + d_{m-1}h_{m-1}^3, \\ b_m = b_{m-1} + 2c_{m-1}h_{m-1} + 3d_{m-1}h_{m-1}^2, \\ c_m = c_{m-1} + 3d_{m-1}h_{m-1}, \qquad m = 1, 2, \cdots, M-1, \end{cases}$$

and

$$\begin{cases} a_M = a_{M-1} + b_{M-1}h_{M-1} + c_{M-1}h_{M-1}^2 + d_{M-1}h_{M-1}^3, \\ c_M = c_{M-1} + 3d_{M-1}h_{M-1}, \end{cases}$$

the following relation can be derived:

$$\begin{aligned} &\frac{h_{m-1}}{h_{m-1}+h_m}c_{m-1} + 2c_m + \frac{h_m}{h_{m-1}+h_m}c_{m+1} \\ &= \frac{1}{h_{m-1}+h_m}\left[\frac{3(a_{m+1}-a_m)}{h_m} - \frac{3(a_m-a_{m-1})}{h_{m-1}}\right], \\ &m = 1, 2, \cdots, M-1. \end{aligned}$$

3. Consider the cubic spline problem. Suppose that the derivative is given at $x = x_M$, instead of assuming $c_M = 0$. Derive the equation which should replace the equation $c_M = 0$ in the system for $c_0, c_1, \cdots, c_M$.
4. Suppose $x_m = m\Delta x$, $y_l = l\Delta y$, and $\tau^n = n\Delta\tau$. Find the expression of the error of each of the following approximations:

(a) $u(x_m, \tau^{n+1/2}) \approx \dfrac{u(x_m, \tau^{n+1}) + u(x_m, \tau^n)}{2}$;

(b) $\dfrac{\partial u}{\partial \tau}(x_m, \tau^n) \approx \dfrac{u(x_m, \tau^{n+1}) - u(x_m, \tau^n)}{\Delta\tau}$;

(c) $\dfrac{\partial u}{\partial \tau}(x_m, \tau^{n+1/2}) \approx \dfrac{u(x_m, \tau^{n+1}) - u(x_m, \tau^n)}{\Delta\tau}$;

(d) $\dfrac{\partial u}{\partial x}(x_m, \tau^n) \approx \dfrac{u(x_{m+1}, \tau^n) - u(x_m, \tau^n)}{\Delta x}$;

(e) $\dfrac{\partial u}{\partial x}(x_m, \tau^n) \approx \dfrac{u(x_{m+1}, \tau^n) - u(x_{m-1}, \tau^n)}{2\Delta x}$;

(f) $\dfrac{\partial u}{\partial x}(x_m, \tau^n) \approx \dfrac{3u(x_m, \tau^n) - 4u(x_{m-1}, \tau^n) + u(x_{m-2}, \tau^n)}{2\Delta x}$;

(g) $\dfrac{\partial^2 u}{\partial x^2}(x_m, \tau^n) \approx \dfrac{u(x_{m+1}, \tau^n) - 2u(x_m, \tau^n) + u(x_{m-1}, \tau^n)}{\Delta x^2}$;

(h)

$$\begin{aligned} \frac{\partial^2 u}{\partial x \partial y}(x_m, y_l, \tau^n) \approx \frac{1}{2\Delta x}\Bigg[&\frac{u(x_{m+1}, y_{l+1}, \tau^n) - u(x_{m+1}, y_{l-1}, \tau^n)}{2\Delta y} \\ &- \frac{u(x_{m-1}, y_{l+1}, \tau^n) - u(x_{m-1}, y_{l-1}, \tau^n)}{2\Delta y}\Bigg]. \end{aligned}$$

5. The Chebyshev polynomial of first kind with degree N is defined by

$$T_N(y) = \cos\left(N\cos^{-1} y\right),$$

where N is an integer and $y \in [-1, 1]$. Let

$$y_j = \cos\frac{j\pi}{N}, \quad j = 0, 1, \cdots, N.$$

Show

(a) $T_{k+1}(y) - 2yT_k(y) + T_{k-1}(y) = 0,\ k \geq 1.$

(b) $T_N(y)$ is a polynomial of degree N for any nonnegative integer.

(c) $$\frac{dT_N(y_j)}{dy} = \begin{cases} N^2, & j = 0, \\ 0, & j = 1, 2, \cdots, N-1, \\ (-1)^{N+1} N^2, & j = N; \end{cases}$$

(d) $$\frac{d^2T_N(y_j)}{dy^2} = \begin{cases} \dfrac{N^2\left(N^2-1\right)}{3}, & j = 0, \\ \dfrac{(-1)^{j+1} N^2}{\left(1-y_j^2\right)}, & j = 1, 2, \cdots, N-1, \\ \dfrac{(-1)^N N^2\left(N^2-1\right)}{3}, & j = N; \end{cases}$$

(e) $$\frac{d^3T_N(y_j)}{dy^3} = \frac{(-1)^{j+1} 3N^2 y_j}{\left(1-y_j^2\right)^2}, \quad j = 1, 2, \cdots, N-1.$$

6. Let

$$h_j(y) = \frac{(-1)^{j+1}\left(1-y^2\right)T_N'(y)}{c_j N^2 (y-y_j)}, \quad j = 0, 1, \cdots, N,$$

where $T_N(y)$ is the Chebyshev polynomial of first kind with degree N, $y_j = \cos\frac{j\pi}{N}$, $j = 0, 1, \cdots, N$, and

$$c_j = \begin{cases} 2, & j = 0, \\ 1, & j = 1, 2, \cdots, N-1, \\ 2, & j = N. \end{cases}$$

(a) Show

$$h_j(y_i) = \frac{(-1)^{j+1}\left(1-y_i^2\right)T_N'(y_i)}{c_j N^2 (y_i-y_j)} = \delta_{ij}, \quad i, j = 0, 1, \cdots, N,$$

where δ_{ij} is the Kronecker delta.

(b) Define

$$d_{ij} = \frac{dh_j(y_i)}{dy}, \quad i, j = 0, 1, \cdots, N.$$

Show that

$$d_{ij} = \begin{cases} \dfrac{(-1)^{i+j} c_i}{c_j (y_i - y_j)}, & i \neq j, \\ \dfrac{2N^2+1}{6}, & i = j = 0, \\ -\dfrac{y_j}{2\left(1 - y_j^2\right)}, & i = j = 1, 2, \cdots, N-1, \\ -\dfrac{2N^2+1}{6}, & i = j = N. \end{cases}$$

(c) Let $f_1(y_j)$ denote the values of the function $f_1(y)$ at $y = y_j$, $j = 0, 1, \cdots, N$. Show that

$$p_{N1}(y) = \sum_{j=0}^{N} h_j(y) f_1(y_j)$$

is an interpolation polynomial with degree N for $f_1(y)$ on $[-1, 1]$ and

$$\frac{dp_{N1}(y_i)}{dy} = \sum_{j=0}^{N} d_{ij} f_1(y_j).$$

(d) Define $x = (1-y)/2$ or $y = 1 - 2x$. Let $f(x_j)$ denote the values of the function $f(x)$ at $x = x_j$, $j = 0, 1, \cdots, N$. Show that

$$p_N(x) = \sum_{j=0}^{N} h_j(1 - 2x) f(x_j)$$

is an interpolation polynomial with degree N for $f(x)$ on $[0, 1]$ and

$$\frac{dp_N(x_i)}{dx} = \sum_{j=0}^{N} D_{ij} f(x_j),$$

where

$$D_{ij} = \begin{cases} \dfrac{(-1)^{i+j} c_i}{c_j (x_i - x_j)}, & i \neq j, \\ -\dfrac{2N^2+1}{3}, & i = j = 0, \\ \dfrac{1 - 2x_j}{4x_j (1 - x_j)}, & i = j = 1, 2, \cdots, N-1, \\ \dfrac{2N^2+1}{3}, & i = j = N. \end{cases}$$

7. Derive the formulae of the LU decomposition method for the following almost tridiagonal system

$$\mathbf{Ax} = \mathbf{q},$$

where

$$\mathbf{A} = \begin{bmatrix} b_1 & c_1 & & & & d_1 \\ a_2 & b_2 & c_2 & & 0 & d_2 \\ & \ddots & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & \ddots & \vdots \\ & 0 & & a_{m-1} & b_{m-1} & d_{m-1} \\ & & & & a_m & d_m \end{bmatrix},$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}, \qquad \mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_m \end{bmatrix}.$$

8. Suppose that we already have a solver for solving tridiagonal system:

$$\mathbf{Ax} = \mathbf{q},$$

where

$$\mathbf{A} = \begin{bmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & 0 & \\ & \ddots & \ddots & \ddots & \\ & 0 & a_{m-1} & b_{m-1} & c_{m-1} \\ & & & a_m & b_m \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m-1} \\ x_m \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{m-1} \\ q_m \end{bmatrix}.$$

In order to solve the following almost tridiagonal system

$$\mathbf{Ax} = \mathbf{q},$$

where

$$\mathbf{A} = \begin{bmatrix} a_1 & b_1 & c_1 & & \\ a_2 & b_2 & c_2 & 0 & \\ & \ddots & \ddots & \ddots & \\ & 0 & a_{m-1} & b_{m-1} & c_{m-1} \\ & & a_m & b_m & c_m \end{bmatrix} \quad \text{or} \quad \mathbf{A} = \begin{bmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & 0 & \\ & \ddots & \ddots & \ddots & \\ & 0 & a_{m-1} & b_{m-1} & c_{m-1} \\ & & a_m & b_m & c_m \end{bmatrix},$$

we can convert it to a tridiagonal system and solve the new system by the existing solver. Design such a method.

9. *Describe the Jacobi iteration, the Gauss–Seidel iteration, and the method of successive over relaxation for an $n \times n$ system of linear equations.

10. *Suppose $f(x) = 0$ is a nonlinear equation. Derive the iteration formulae of Newton's method and the secant method for solving the nonlinear equation.
11. (a) For each of the following methods, describe the details of the method and its advantage and disadvantage:
    i. The secant method;
    ii. The bisection method;
    iii. The modified secant method.
    (b) Based on the methods in part a), design an efficient and robust method of finding a root of the equation $f(x) = 0$.
12. Suppose

$$\mathbf{A}_1 = \mathbf{J}_\mathbf{f}(\mathbf{x}^{(0)}) + \frac{\mathbf{f}(\mathbf{x}^{(1)}) - \mathbf{f}(\mathbf{x}^{(0)}) - \mathbf{J}_\mathbf{f}(\mathbf{x}^{(0)})(\mathbf{x}^{(1)} - \mathbf{x}^{(0)})}{\left\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\right\|_2^2}(\mathbf{x}^{(1)} - \mathbf{x}^{(0)})^T.$$

Show that the following relations hold:

$$\mathbf{A}_1(\mathbf{x}^{(1)} - \mathbf{x}^{(0)}) = \mathbf{f}(\mathbf{x}^{(1)}) - \mathbf{f}(\mathbf{x}^{(0)})$$

and

$$\mathbf{A}_1\mathbf{z} = \mathbf{J}_\mathbf{f}(\mathbf{x}^{(0)})\mathbf{z} \quad \text{whenever} \quad (\mathbf{x}^{(1)} - \mathbf{x}^{(0)})^T\mathbf{z} = 0.$$

13. Prove that if $\mathbf{A} \in \mathbb{R}^{n\times n}$ is nonsingular, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, and $\mathbf{y}^T\mathbf{A}^{-1}\mathbf{x} \neq -1$, then $\mathbf{A} + \mathbf{x}\mathbf{y}^T$ is also nonsingular, moreover,

$$(\mathbf{A} + \mathbf{x}\mathbf{y}^T)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{x}\mathbf{y}^T\mathbf{A}^{-1}}{1 + \mathbf{y}^T\mathbf{A}^{-1}\mathbf{x}}.$$

14. (a) *Show

$$\mathbf{H}_m\mathbf{x} = \beta\mathbf{e}_1,$$

where

$$\begin{aligned} \mathbf{x} &= [x_1, x_2, \cdots, x_m]^T, \\ \beta &= \sqrt{\mathbf{x}^T\mathbf{x}}, \\ \mathbf{H}_m &= \mathbf{I}_m - \frac{1}{\beta(\beta - x_1)}\mathbf{u}\mathbf{u}^T, \\ &\mathbf{u} \text{ being } \mathbf{x} - \beta\mathbf{e}_1. \end{aligned}$$

    (b) *Using the result in part a), design a method to obtain an orthogonal matrix $\mathbf{Q}$ from $\mathbf{A}$ such that $\mathbf{A} = \mathbf{Q}\mathbf{R}$, where $\mathbf{R}$ is an upper triangular matrix with nonnegative diagonals.
15. *Assume that the volatility of a stock is a function of the stock price. Describe a method determining the function from the market data.

Projects

General Requirements

(A) Submit a code or codes in C or C^{++} that will work on a computer the instructor can get access to. At the beginning of the code, write down the name of the student and indicate on which computer it works and the procedure to make it work.

(B) Each code should use an input file to specify all the problem parameters and the computational parameters and an output file to store all the results. In an output file, the name of the problem, all the problem parameters, and the computational parameters should be given, so that one can know what the results are and how they were obtained. The input file should be submitted with the code.

(C) Submit results in form of tables. When a result is given, always provide the problem parameters and the computational parameters.

1. **Cumulative Distribution Functions and Black–Scholes Formulae**.

   Write five functions:

   (a) **double** N(double z)

   for computing approximate values of the cumulative distribution function for the standardized normal variable by using the expression given in a footnote of Sect. 2.6.3, where z is the independent variable.
   - Give the values of $N(z)$ for $z = -2, -1, 0, 1, 2$.

   (b) **double** BS(double S, double E, double tau, double r, double D0, double sigma, char option),

   which gives prices of the European options by using Black–Scholes formulae (see Sect. 2.6.5). When the value of the character 'option' is equal to 'c' or 'C', the value of the European call needs to be evaluated. Otherwise, the value of the European put needs to be evaluated.
   - For European call and put options, give the results for the cases: $S = 100$, $E = 95, 100, 105$, $T = 1$, $r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$.
   - For European call and put options, give the results for the cases: $S = 100$, $E = 95, 100, 105$, $T = 1$, $r = 0.05$, $D_0 = 0.1$, $\sigma = 0.2$.

   (c) **double** BS_bar(double xi, double E, double tau, double r, double D0, double sigma, char option)

   This function gives the value of $\bar{c}(\xi, \tau) = c(S, t)/(S + E)$ or $\bar{p}(\xi, \tau) = p(S, t)/(S + E)$.

- For $\xi = 0.5128, 0.5000, 0.4878$, $E = 95, 100, 105$, $\tau = 1$, $r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$, calculate the results of $\bar{c}(\xi, \tau)$ and $\bar{p}(\xi, \tau)$ by this function.

(d) **double** N_2(double x1, double x2, double rho)

for computing approximate values of the cumulative distribution function for the bivariate standard normal distribution by using the expression given in a footnote of Sect. 4.5.3, where x1, x2 and rho are parameters.

- Give the values of $N_2(x_1, x_2, \rho)$ for the following sets of (x_1, x_2, ρ): $(0.6, 0.5, 0.6)$, $(0.4, 0.5, 0.8)$, $(0.3, 0.4, -0.6)$, $(0.5, 0.7, -0.8)$.

(e) **double** BS_2(double S1, double S2, double E, double tau, double r, double D01, double D02, double sigma1, double sigma2, double rho, char option)

which gives prices of the European call option on the maximum of two assets and the European put option on the minimum of two assets by using the closed-form solutions (4.76) and (4.77) given in Sect. 4.5.3. When the value of the character 'option' is equal to 'c' or 'C', the value of the European call needs to be evaluated. Otherwise, the value of the European put needs to be evaluated.

- Find the prices of the European call option on the maximum of two assets for the following parameter sets of $(S_1, S_2, E, \tau, r, D_{01}, D_{02}, \sigma_1, \sigma_2, \rho, \text{option})$:

$$
\begin{aligned}
&(100, 100, 100, 1.0, 0.02, 0.01, 0.01, 0.20, 0.20, 0.8, \text{c}),\\
&(100, 105, 100, 1.0, 0.02, 0.01, 0.01, 0.20, 0.15, 0.8, \text{c}),\\
&(100, 105, 100, 1.0, 0.02, 0.01, 0.01, 0.15, 0.20, 0.8, \text{c}),\\
&(100, \quad 95, 100, 1.0, 0.02, 0.01, 0.01, 0.20, 0.15, 0.8, \text{c}),\\
&(100, \quad 95, 100, 1.0, 0.02, 0.01, 0.01, 0.15, 0.20, 0.8, \text{c}).
\end{aligned}
$$

- Find the prices of the European put option on the minimum of two assets for the following parameter sets of $(S_1, S_2, E, \tau, r, D_{01}, D_{02}, \sigma_1, \sigma_2, \rho, \text{option})$:

$$
\begin{aligned}
&(100, 100, 100, 1.0, 0.02, 0.01, 0.01, 0.20, 0.20, 0.8, \text{p}),\\
&(100, 105, 100, 1.0, 0.02, 0.01, 0.01, 0.20, 0.15, 0.8, \text{p}),\\
&(100, 105, 100, 1.0, 0.02, 0.01, 0.01, 0.15, 0.20, 0.8, \text{p}),\\
&(100, \quad 95, 100, 1.0, 0.02, 0.01, 0.01, 0.20, 0.15, 0.8, \text{p}),\\
&(100, \quad 95, 100, 1.0, 0.02, 0.01, 0.01, 0.15, 0.20, 0.8, \text{p}).
\end{aligned}
$$

2. **Quadratic Interpolation and LU Decomposition of a Tridiagonal System**.
For the quadratic interpolation method (see Sect. 6.1.1), write a function

(a) **double** Interpolation(double x, int M, double *y)

Suppose that x, M, and $y_m = y(x_m), m = 0, 1, \ldots, M$, are given, where $x_m = m/M$. This function gives an approximate value of y(x) by quadratic interpolation. The concrete method is as follows. If $x < 1/2M$, then interpolate or extrapolate $y(x)$ by (x_0, y_0), (x_1, y_1), (x_2, y_2), if $x_m - 1/2M \le x < x_m + 1/2M$, $m = 1, 2, \cdots, M-1$, then interpolate $y(x)$ by (x_{m-1}, y_{m-1}), (x_m, y_m), (x_{m+1}, y_{m+1}), and if $x_M - 1/2M \le x$, then interpolate or extrapolate $y(x)$ by (x_{M-2}, y_{M-2}), (x_{M-1}, y_{M-1}), (x_M, y_M).

- Let $M = 5$ and the six components from y_0 to y_5 are 0.0, 0.008, 0.064, 0.216, 0.512, 1.0. Calculate the values of $y(x)$ for $x = -0.1, 0.45, 1.01$ by this function.

For LU decomposition (see Sect. 6.2.1), write two functions:

(b) **int** LUT(int m, double *a, double *b, double *c, double *q, double *x).

Suppose that we have a tridiagonal system (6.12). The number of unknowns is given in the integer 'm'. The nonhomogeneous term q_i is given in q[i-1] (the ith component of the array 'q'). The coefficients a_i, b_i, and c_i are given in the ith component of the arrays 'a', 'b', and 'c', respectively. Write a function to solve the system by using the method described in Sect. 6.2.1. If all the u_i are not equal to zero, then the code should return an integer number 0 and gives the value of the ith unknown in the ith component of the array **x**. If one of u_i is equal to zero, then the solution(s) of the system cannot be found by the method (or the system has no solution), and the code should return an integer number 1. The values in the arrays 'a', 'b', 'c', and 'q' are required unchanged.

- Let $m = 4$, $a_2 = a_3 = a_4 = -0.48$, $b_1 = b_2 = b_3 = b_4 = 1$, $c_1 = c_2 = c_3 = -0.49$, $q_1 = 0.02$, $q_2 = 0.05$, $q_3 = 0.08$, and $q_4 = 2.56$. Find the solution of the system (6.12).

(c) **int** LUAT(int m, double *a, double *b, double *c, double *q, double *x)

This is a solver for an almost tridiagonal system by LU decomposition. The almost tridiagonal system is in the following form:

$$\mathbf{Ax} = \mathbf{q},$$

where

$$\mathbf{A} = \begin{bmatrix} a_1 & b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & 0 & \\ & \ddots & \ddots & & \ddots & \\ & 0 & a_{m-1} & b_{m-1} & c_{m-1} & \\ & & a_m & b_m & c_m & \end{bmatrix}.$$

This function calculates $\mathbf{x}$ if m, $\mathbf{a}, \mathbf{b}, \mathbf{c}$, and $\mathbf{q}$ are given. Require m, $\mathbf{a}, \mathbf{b}, \mathbf{c}$, and $\mathbf{q}$ unchanged.

- Let $m = 5$, $\mathbf{a} = \{1.75, -0.48, -0.48, -0.48, 0.25\}$, $\mathbf{b} = \{-0.5, 1, 1, 1, -0.5\}$, $\mathbf{c} = \{0.25, -0.49, -0.49, -0.49, 1.75\}$, $\mathbf{q} = \{1.5, 0.05, 0.08, 0.11, 7.5\}$, calculate the result of $\mathbf{x}$ by this function.

7

Finite-Difference Methods

In this chapter, we deal with finite-difference methods for parabolic partial differential equations, including algorithms, stability and convergence analysis, and extrapolation techniques of numerical solutions.

7.1 Finite-Difference Schemes

In this section, we will discuss the finite-difference methods for parabolic partial differential equation problems (parabolic PDE problems). Usually, a parabolic partial differential equation problem is formulated as follows:

$$\begin{cases} \dfrac{\partial u}{\partial \tau} = a(x,\tau)\dfrac{\partial^2 u}{\partial x^2} + b(x,\tau)\dfrac{\partial u}{\partial x} + c(x,\tau)u + g(x,\tau), \\ \qquad\qquad\qquad\qquad x_l \le x \le x_u, \quad 0 \le \tau \le T, \\ u(x,0) = f(x), \qquad x_l \le x \le x_u, \\ u(x_l,\tau) = f_l(\tau), \qquad 0 \le \tau \le T, \\ u(x_u,\tau) = f_u(\tau), \qquad 0 \le \tau \le T, \end{cases} \tag{7.1}$$

where $a(x,\tau) > 0$ on the domain $[x_l, x_u] \times [0,T]$ and the compatibility conditions $f(x_l) = f_l(0)$ and $f(x_u) = f_u(0)$ hold. Though sometimes, a European option problem can be approximately formulated in such a way after giving some approximate boundary condition on certain artificial boundary. However, for most of the European option problems, the problems are in or can be transformed into the following degenerate parabolic partial differential equation problem:

$$\begin{cases} \dfrac{\partial u}{\partial \tau} = a(x,\tau)\dfrac{\partial^2 u}{\partial x^2} + b(x,\tau)\dfrac{\partial u}{\partial x} + c(x,\tau)u + g(x,\tau), \\ \qquad\qquad\qquad x_l \le x \le x_u, \quad 0 \le \tau \le T, \\ u(x,0) = f(x), \quad x_l \le x \le x_u, \end{cases} \tag{7.2}$$

where $a(x,\tau) \ge 0$ on the domain $[x_l, x_u] \times [0,T]$,

Y.-l. Zhu et al., *Derivative Securities and Difference Methods*, Springer Finance,
DOI 10.1007/978-1-4614-7306-0_7,

$$\begin{cases} b(x_l,\tau) - \dfrac{\partial a}{\partial x}(x_l,\tau) \ge 0, \quad 0 \le \tau \le T, \\ a\,(x_l,\tau) = 0, \quad 0 \le \tau \le T, \end{cases} \tag{7.3}$$

and

$$\begin{cases} b(x_u,\tau) - \dfrac{\partial a}{\partial x}(x_u,\tau) \le 0, \quad 0 \le \tau \le T, \\ a\,(x_u,\tau) = 0, \quad 0 \le \tau \le T. \end{cases} \tag{7.4}$$

For example, the prices of vanilla European call/put options are solutions of the problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2(S)S^2\dfrac{\partial^2 V}{\partial S^2} + (r - D_0)S\dfrac{\partial V}{\partial S} - rV = 0\ ,\ 0 \le S,\ \ 0 \le t \le T, \\ V(S,t) = \max(\pm(S-E),0),\ \ 0 \le S. \end{cases}$$

Through the transformation

$$\begin{cases} \xi = \dfrac{S}{S+E}, \\ \tau = T - t, \\ V(S,t) = (S+E)\overline{V}(\xi,\tau), \end{cases}$$

the problem is converted into

$$\begin{cases} \dfrac{\partial \overline{V}}{\partial \tau} = \dfrac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1-\xi)^2\dfrac{\partial^2 \overline{V}}{\partial \xi^2} + (r-D_0)\xi(1-\xi)\dfrac{\partial \overline{V}}{\partial \xi} - [r(1-\xi) + D_0\xi]\overline{V}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le \xi \le 1, \quad 0 \le \tau \le T, \\ \overline{V}(\xi,0) = \max(\pm(2\xi-1),0), \qquad\qquad\qquad 0 \le \xi \le 1, \end{cases}$$

where $\bar{\sigma}(\xi) = \sigma(E\xi/(1-\xi))$. (For details, see Sect. 2.2.5.) Clearly, this problem is in the form (7.2). Moreover, if a stochastic model

$$dS = udt + wdX$$

is defined on $[S_l, S_u]$, and the conditions

$$\begin{cases} u\,(S_l,t) - w(S_l,t)\dfrac{\partial}{\partial S}w(S_l,t) \ge 0, \\ w\,(S_l,t) = 0 \end{cases}$$

and

$$\begin{cases} u\,(S_u,t) - w(S_u,t)\dfrac{\partial}{\partial S}w(S_u,t) \le 0, \\ w\,(S_u,t) = 0 \end{cases}$$

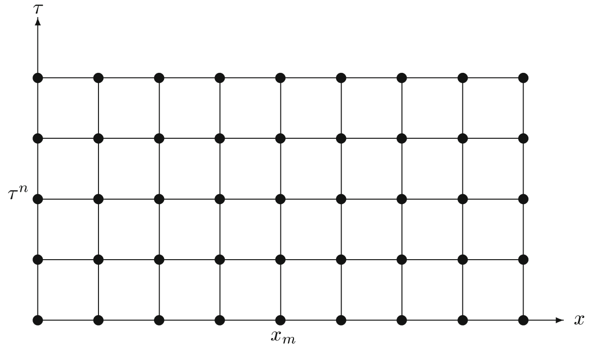

Fig. 7.1. A mesh for finite-difference methods

hold, then prices of European-style derivatives on this random variable also are solutions of the problem (7.2). (For details, see Sect. 2.4.)

To find an approximate solution of a partial differential equation problem by finite-difference methods, we first divide the domain $[x_l, x_u] \times [0, T]$ into small subdomains using lines $x_m = x_l + m\Delta x$ and $\tau^n = n\Delta\tau$, where $\Delta x = (x_u - x_l)/M$, $\Delta\tau = T/N$ and M, N are positive integers. These lines form a grid, and these points (x_m, τ^n) are called grid points (see Fig. 7.1). We want to find the approximate values of the solution on these grid points.

Let us look at the problem (7.2). First consider the case[1]

$$b(x_l, \tau) = 0, \quad 0 \le \tau \le T$$

and

$$b(x_u, \tau) = 0, \quad 0 \le \tau \le T.$$

In this case, the partial differential equation in the problem (7.2) degenerates into an ordinary differential equation at each boundary, and the degenerate parabolic problem (7.2) can be discretized in the following way.

Using forward difference for $\dfrac{\partial u}{\partial \tau}(x_m, \tau^n)$, second-order central difference for $\dfrac{\partial u}{\partial x}(x_m, \tau^n)$ and $\dfrac{\partial^2 u}{\partial x^2}(x_m, \tau^n)$ in the problem (7.2) at the point (x_m, τ^n), we have

[1]Because $a(x, \tau) \ge 0$ on $[x_l, x_u]$ and $a(x_l, \tau) = a(x_u, \tau) = 0$, we have $\dfrac{\partial a}{\partial x}(x_l, \tau) \ge 0$ and $\dfrac{\partial a}{\partial x}(x_u, \tau) \le 0$. Thus the inequality conditions in the conditions (7.3) and (7.4) can be rewritten as $b(x_l, \tau) \ge \dfrac{\partial a}{\partial x}(x_l, \tau) \ge 0$ and $b(x_u, \tau) \le \dfrac{\partial a}{\partial x}(x_u, \tau) \le 0$. Consequently, the two conditions below imply $\dfrac{\partial a}{\partial x}(x_l, \tau) = \dfrac{\partial a}{\partial x}(x_u, \tau) = 0$.

$$\begin{aligned}&\frac{u(x_m,\tau^{n+1})-u(x_m,\tau^n)}{\Delta\tau}-\frac{\Delta\tau}{2}\frac{\partial^2 u}{\partial\tau^2}(x_m,\eta)\\
&=a_m^n\left[\frac{u(x_{m+1},\tau^n)-2u(x_m,\tau^n)+u(x_{m-1},\tau^n)}{\Delta x^2}-\frac{\Delta x^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi,\tau^n)\right]\\
&\quad+b_m^n\left[\frac{u(x_{m+1},\tau^n)-u(x_{m-1},\tau^n)}{2\Delta x}-\frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3}(\bar\xi,\tau^n)\right]\\
&\quad+c_m^n u(x_m,\tau^n)+g_m^n,\end{aligned}$$

where

$$\eta\in(\tau^n,\tau^{n+1}),\quad \xi\in(x_{m-1},x_{m+1}),\quad \bar\xi\in(x_{m-1},x_{m+1}),$$

and a_m^n, b_m^n, c_m^n, and g_m^n denote $a(x_m,\tau^n), b(x_m,\tau^n), c(x_m,\tau^n)$, and $g(x_m,\tau^n)$, respectively. Dropping the term $-\dfrac{\Delta\tau}{2}\dfrac{\partial^2 u}{\partial\tau^2}(x_m,\eta)$ from the left-hand side and the two terms $-a_m^n\dfrac{\Delta x^2}{12}\dfrac{\partial^4 u}{\partial x^4}(\xi,\tau^n)$ and $-b_m^n\dfrac{\Delta x^2}{6}\dfrac{\partial^3 u}{\partial x^3}(\bar\xi,\tau^n)$ from the right-hand side, and denoting the approximate solution of $u(x_m,\tau^n)$ by u_m^n, we obtain the following approximation to the partial differential equation in the problem (7.2):

$$\frac{u_m^{n+1}-u_m^n}{\Delta\tau}=a_m^n\frac{u_{m+1}^n-2u_m^n+u_{m-1}^n}{\Delta x^2}+b_m^n\frac{u_{m+1}^n-u_{m-1}^n}{2\Delta x}+c_m^n u_m^n+g_m^n,$$
$$m=0,1,\cdots,M,\quad n=0,1,\cdots,N-1.$$

From the initial condition in problem (7.2), we have $u_m^0=f(x_m)$, $m=0,1,\cdots,M$. Therefore, the degenerate parabolic problem (7.2) can be discretized by

$$\begin{cases}u_m^{n+1}=\left(\dfrac{a_m^n\Delta\tau}{\Delta x^2}+\dfrac{b_m^n\Delta\tau}{2\Delta x}\right)u_{m+1}^n+\left(1-2\dfrac{a_m^n\Delta\tau}{\Delta x^2}+c_m^n\Delta\tau\right)u_m^n\\
\qquad\qquad+\left(\dfrac{a_m^n\Delta\tau}{\Delta x^2}-\dfrac{b_m^n\Delta\tau}{2\Delta x}\right)u_{m-1}^n+g_m^n\Delta\tau,\\
\qquad\qquad m=0,1,\cdots,M,\quad n=0,1,\cdots,N-1,\\
u_m^0=f(x_m),\quad m=0,1,\cdots,M.\end{cases}\tag{7.5}$$

Here, we need to point out that because we discretize ordinary differential equations at the boundaries, only u_0^n appears in the equation for $m=0$ and only u_M^n for $m=M$. That is, because $a_0^n=b_0^n=a_M^n=b_M^n=0$, u_{-1}^n and u_{M+1}^n actually do not appear in the equations above.

When u_m^n, $m=0,1,\cdots,M$ are known, we can find u_m^{n+1}, $m=0,1,\cdots,M$ by difference scheme (7.5). Because u_m^0, $m=0,1,\cdots,M$ are given in the scheme (7.5), this procedure can be done for $n=0,1,\cdots,N-1$ successively, and the approximate solution on all the grid points can be obtained. This method is called an **explicit finite-difference method**. This is because when u_m^n has been obtained, one equation involves only one unknown, so the unknown u_m^{n+1} can be computed from u_{m-1}^n, u_m^n and u_{m+1}^n explicitly.

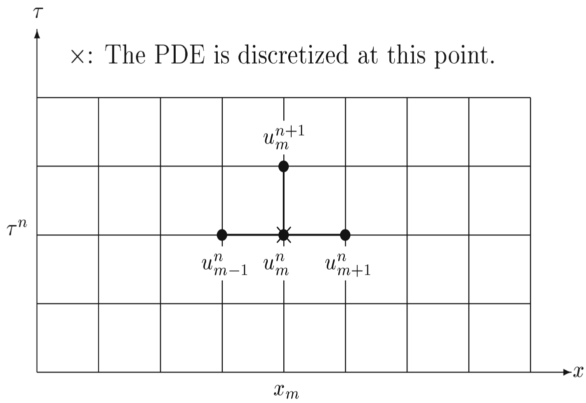

Fig. 7.2. An explicit finite-difference discretization

Figure 7.2 gives a diagram for this procedure. When we have the approximation (7.5), we have dropped the terms

$$\frac{\Delta\tau}{2}\frac{\partial^2 u}{\partial\tau^2}(x_m,\eta) - a_m^n\frac{\Delta x^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi,\tau^n) - b_m^n\frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3}(\bar{\xi},\tau^n)$$

from the equations. These terms as a whole are called the **truncation error** for scheme (7.5). Because the truncation error can be rewritten as $O(\Delta x^2, \Delta\tau)$, we say that for scheme (7.5), the truncation error is second order in Δx and first order in $\Delta\tau$.

Now let us discretize the problem (7.2) at the point $(x_m, \tau^{n+1/2})$. For $\dfrac{\partial u}{\partial\tau}(x_m, \tau^{n+1/2})$, we use the central scheme. The derivative $\dfrac{\partial u}{\partial x}(x_m, \tau^{n+1/2})$ is approximated first by the average of the values at the points (x_m, τ^n) and (x_m, τ^{n+1}), and then the derivatives at these two points are discretized by the central difference. The second derivative $\dfrac{\partial^2 u}{\partial x^2}(x_m, \tau^{n+1/2})$ is dealt with

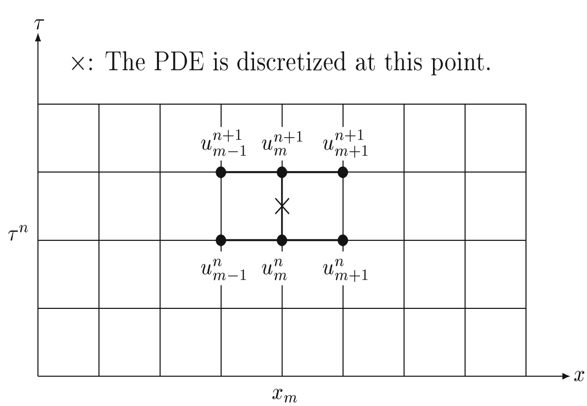

Fig. 7.3. An implicit finite-difference discretization

similarly. Using this way, the degenerate parabolic problem (7.2) can be approximated by the implicit finite-difference method:

$$\begin{cases} \dfrac{u_m^{n+1}-u_m^n}{\Delta\tau} = \dfrac{a_m^{n+1/2}}{2}\left(\dfrac{u_{m+1}^{n+1}-2u_m^{n+1}+u_{m-1}^{n+1}}{\Delta x^2} \right. \\ \qquad\qquad \left. + \dfrac{u_{m+1}^{n}-2u_m^{n}+u_{m-1}^{n}}{\Delta x^2}\right) \\ \qquad\qquad + \dfrac{b_m^{n+1/2}}{2}\left(\dfrac{u_{m+1}^{n+1}-u_{m-1}^{n+1}}{2\Delta x} + \dfrac{u_{m+1}^{n}-u_{m-1}^{n}}{2\Delta x}\right) \\ \qquad\qquad + \dfrac{c_m^{n+1/2}}{2}(u_m^{n+1}+u_m^n) + g_m^{n+1/2}, \\ \qquad\qquad m = 0,1,\cdots,M, \quad n = 0,1,\cdots,N-1, \\ u_m^0 = f(x_m), \qquad m = 0,1,\cdots,M. \end{cases} \tag{7.6}$$

From here, we see that each equation involves six grid points (see Fig. 7.3) and that there are three unknowns. As we know, the error of a central difference is second order. For a function, the average of the values at the points (x_m, τ^n) and (x_m, τ^{n+1}) is an approximate value at the point $(x_m, \tau^{n+1/2})$ with an error of $O(\Delta\tau^2)$ because it actually is the result obtained by the linear interpolation. Therefore, the truncation error of this scheme is $O(\Delta x^2, \Delta\tau^2)$.

Similar to the scheme (7.5), because we actually discretize ordinary differential equations at the boundaries, the equations for $m=0$ and $m=M$ can be written as

$$\frac{u_m^{n+1}-u_m^n}{\Delta\tau} = \frac{c_m^{n+1/2}}{2}(u_m^{n+1}+u_m^n) + g_m^{n+1/2},$$
$$m = 0, M, \quad n = 0,1,\cdots,N-1.$$

Consequently, these equations actually do not involve u_{-1}^n and u_{M+1}^n. Furthermore, the equations for $m=0$ alone can determine u_0^n, $n = 1,2,\cdots,N$ from u_0^0. For u_M^n, the situation is similar. However, for u_m^n, $m \neq 0$ and M, the situation is different. We cannot determine u_m^{n+1} only from a few equations. In order to obtain u_m^{n+1}, $m = 1,2,\cdots,M-1$, we have to solve a tridiagonal system of linear equations, and each of u_m^{n+1} is determined by all the u_m^n. Consequently, this method is called an **implicit finite-difference method**.

The problem (7.1) can be discretized similarly. The only difference is that the partial differential equation should not be discretized for $m=0$ and $m=M$ because the boundary conditions

$$u(x_l, \tau) = f_l(\tau)$$

and

$$u(x_u, \tau) = f_u(\tau)$$

provide the equations we need. When $a(x,\tau)$ is equal to a positive constant a, $b(x,\tau) = 0$, $c(x,\tau) = 0$, and $g(x,\tau) = 0$, i.e., for the heat conductivity problem

$$\begin{cases} \dfrac{\partial u}{\partial \tau} = a\dfrac{\partial^2 u}{\partial x^2}, & x_l \le x \le x_u, \quad 0 \le \tau \le T, \\ u(x,0) = f(x), & x_l \le x \le x_u, \\ u(x_l,\tau) = f_l(\tau), & 0 \le \tau \le T, \\ u(x_u,\tau) = f_u(\tau), & 0 \le \tau \le T, \end{cases} \tag{7.7}$$

corresponding to the explicit scheme (7.5), (7.7) can be approximated by

$$\begin{cases} u_m^{n+1} = \alpha u_{m+1}^n + (1-2\alpha)u_m^n + \alpha u_{m-1}^n, & \\ \qquad\qquad\qquad\qquad m = 1, 2, \cdots, M-1, & \\ \qquad\qquad\qquad\qquad n = 0, 1, \cdots, N-1, & \\ u_0^{n+1} = f_l(\tau^{n+1}), & n = 0, 1, \cdots, N-1, \\ u_M^{n+1} = f_u(\tau^{n+1}), & n = 0, 1, \cdots, N-1, \\ u_m^0 = f(x_m), & m = 0, 1, \cdots, M, \end{cases} \tag{7.8}$$

where

$$\alpha = \frac{a\Delta\tau}{\Delta x^2}.$$

Similar to the implicit scheme (7.6), (7.7) can also be approximated by

$$\begin{cases} \dfrac{u_m^{n+1} - u_m^n}{\Delta\tau} = \dfrac{a}{2}\left(\dfrac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{\Delta x^2} \right. \\ \qquad\qquad\qquad \left. + \dfrac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2}\right), \\ \quad m = 1, 2, \cdots, M-1, & n = 0, 1, \cdots, N-1, \\ u_0^{n+1} = f_l(\tau^{n+1}), & n = 0, 1, \cdots, N-1, \\ u_M^{n+1} = f_u(\tau^{n+1}), & n = 0, 1, \cdots, N-1, \\ u_m^0 = f(x_m), & m = 0, 1, \cdots, M, \end{cases} \tag{7.9}$$

which is called the Crank–Nicolson scheme.

Since $u(x_l, \tau)$ and $u(x_u, \tau)$ are given, there are only $M-1$ unknowns for each time level, and the $M-1$ equations in the difference scheme (7.9) can be written together in matrix form:

$$\mathbf{A}\mathbf{u}^{n+1} = \mathbf{B}\mathbf{u}^n + \mathbf{b}^n, \tag{7.10}$$

where

$$\mathbf{A} = \begin{bmatrix} 1+\alpha & -\frac{\alpha}{2} & 0 & \cdots & 0 \\ -\frac{\alpha}{2} & 1+\alpha & -\frac{\alpha}{2} & \ddots & \vdots \\ 0 & -\frac{\alpha}{2} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\frac{\alpha}{2} \\ 0 & \cdots & 0 & -\frac{\alpha}{2} & 1+\alpha \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 1-\alpha & \frac{\alpha}{2} & 0 & \cdots & 0 \\ \frac{\alpha}{2} & 1-\alpha & \frac{\alpha}{2} & \ddots & \vdots \\ 0 & \frac{\alpha}{2} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \frac{\alpha}{2} \\ 0 & \cdots & 0 & \frac{\alpha}{2} & 1-\alpha \end{bmatrix},$$

$$\mathbf{u}^n = \begin{bmatrix} u_1^n \\ u_2^n \\ \vdots \\ u_{M-2}^n \\ u_{M-1}^n \end{bmatrix} \quad \text{and} \quad \mathbf{b}^n = \begin{bmatrix} \frac{1}{2}\alpha u_0^n + \frac{1}{2}\alpha u_0^{n+1} \\ 0 \\ \vdots \\ 0 \\ \frac{1}{2}\alpha u_M^n + \frac{1}{2}\alpha u_M^{n+1} \end{bmatrix}.$$

Now we consider the problem (7.2) for the case

$$b(x_l, \tau) > 0, \quad 0 \le \tau \le T$$

and

$$b(x_u, \tau) < 0, \quad 0 \le \tau \le T.$$

In this case, the PDE degenerates into hyperbolic partial differential equations at the boundaries, and the first derivative there has to be discretized by a one-sided difference. For example, if in the scheme (7.5) or (7.6), we use a one-sided difference for the first derivative in the equations for $m = 0$ and $m = M$, we can have the approximation we need. We call them the modified schemes (7.5) and (7.6). However, here the way of discretizing the first derivative at $m = 0$ is different from that at $m = 1$, namely, the discretization "jumps" from $m = 0$ to $m = 1$, so from the finite-difference equation at $m = 0$ to $m = 1$, the coefficients do not satisfy the Lipschitz condition. This causes some problems when doing stability analysis. A similar situation occurs from $m = M - 1$ to $m = M$. In order to avoid the "jump," we can approximate the degenerate parabolic problem (7.2) by the explicit finite-difference method:

$$\begin{cases} \dfrac{u_m^{n+1} - u_m^n}{\Delta\tau} = a_m^n \dfrac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2} + \varPhi_m^n + c_m^n u_m^n + g_m^n, \\ \qquad\qquad\qquad m = 0, 1, \cdots, M, \quad n = 0, 1, \cdots, N-1, \\ u_m^0 = f(x_m), \qquad m = 0, 1, \cdots, M, \end{cases} \tag{7.11}$$

where

$$\Phi_m^n = \begin{cases} b_m^n \dfrac{-u_{m+2}^n + 4u_{m+1}^n - 3u_m^n}{2\Delta x}, & \text{if} \quad b_m^n > 0, \\ 0, & \text{if} \quad b_m^n = 0, \\ b_m^n \dfrac{3u_m^n - 4u_{m-1}^n + u_{m-2}^n}{2\Delta x}, & \text{if} \quad b_m^n < 0 \end{cases}$$

or by the implicit finite-difference method:

$$\begin{cases} \dfrac{u_m^{n+1} - u_m^n}{\Delta\tau} = \dfrac{a_m^{n+1/2}}{2}\left(\dfrac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{\Delta x^2}\right. \\ \qquad\qquad \left. + \dfrac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2}\right) \\ \qquad\qquad +\Phi_m^{n+1/2} + \dfrac{c_m^{n+1/2}}{2}(u_m^{n+1} + u_m^n) + g_m^{n+1/2}, \\ \qquad\qquad m = 0, 1, \cdots, M, \quad n = 0, 1, \cdots, N-1, \\ u_m^0 = f(x_m), \quad m = 0, 1, \cdots, M, \end{cases} \tag{7.12}$$

where

$$\Phi_m^{n+1/2} = \begin{cases} \dfrac{b_m^{n+1/2}}{2}\left(\dfrac{-u_{m+2}^{n+1} + 4u_{m+1}^{n+1} - 3u_m^{n+1}}{2\Delta x}\right. \\ \qquad \left. + \dfrac{-u_{m+2}^n + 4u_{m+1}^n - 3u_m^n}{2\Delta x}\right), & \text{if} \quad b_m^{n+1/2} > 0, \\ 0, & \text{if} \quad b_m^{n+1/2} = 0, \\ \dfrac{b_m^{n+1/2}}{2}\left(\dfrac{3u_m^{n+1} - 4u_{m-1}^{n+1} + u_{m-2}^{n+1}}{2\Delta x}\right. \\ \qquad \left. + \dfrac{3u_m^n - 4u_{m-1}^n + u_{m-2}^n}{2\Delta x}\right), & \text{if} \quad b_m^{n+1/2} < 0. \end{cases}$$

Scheme (7.12) usually involves eight points, among them there are four unknowns (see Fig. 7.4). However, at boundaries there are three unknowns because $a_0^{n+1/2} = a_M^{n+1/2} = 0$. When the partial differential equation is discretized in this way, the stability analysis can be done much easier. In the paper [79] by Sun, Yan, and Zhu, the stability problem of scheme (7.12) has been carefully studied. Clearly, the truncation error of the scheme (7.11) is $O(\Delta x^2, \Delta\tau)$ and that of the scheme (7.12) is $O(\Delta x^2, \Delta\tau^2)$.

Therefore, in order to find a solution, we can use either an explicit finite-difference method or an implicit finite-difference method. From the next section, we will see that for an explicit method, the step size $\Delta\tau$ must be less than a constant times Δx^2 for a stable computation. Thus, if a small Δx must be adopted in order to have satisfying results, the computation could take quite a long time. However, there is no restriction on the step size $\Delta\tau$

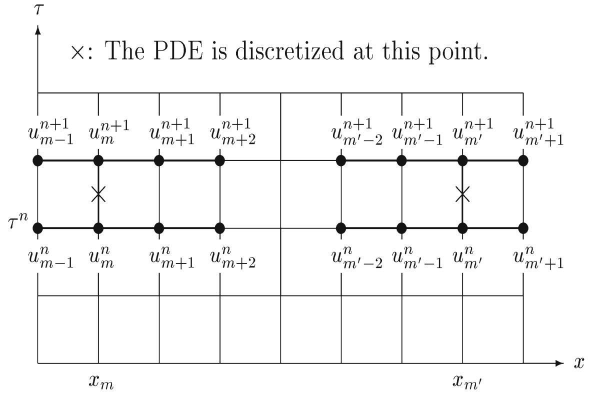

Fig. 7.4. Implicit eight-point finite-difference discretizations

for implicit finite-difference methods. This is the main advantage of implicit methods over explicit methods.

A European-style derivative could involve several random state variables. In this case, we need to discretize a multi-dimensional problem, which will be dealt with in Chaps. 8 and 10. Usually, an American-style derivative problem can be formulated as a free boundary problem. Discretization of such a problem will be discussed in Chap. 9.

7.2 Stability and Convergence Analysis

7.2.1 Stability

Stability is concerned with the propagation of errors. During the computation, truncation errors are brought into approximate solutions at each step. Also rounding errors are introduced into solutions all the time because any computer has a finite number of digits for numbers. If for a given finite-difference method, the errors are not magnified at each step in some norm, then we say that the finite-difference method is stable. There are two different norms that are often used in studying stability. Suppose

$$\mathbf{x} = (x_1, x_2, \cdots, x_{M-1})^T$$

is a vector with $M-1$ components. The L_∞ and L_2 norms of the vector $\mathbf{x}$ are defined as follows:

$$||\mathbf{x}||_{L_\infty} = \max_{1 \le m \le M-1} |x_m|$$

and

$$||\mathbf{x}||_{L_2} = \left(\frac{1}{M-1} \sum_{m=1}^{M-1} x_m^2 \right)^{1/2} .$$

Here, $M-1$ could be any positive integer and is allowed to go to infinity.

Stability of Explicit Finite-Difference Methods for the Heat Equation. Consider the explicit finite-difference method (7.8) for the heat conductivity problem. Suppose an initial error e_m^0 appears in computing $f(x_m)$ for $m = 1, 2, \cdots, M-1$. That is, instead of $f(x_m)$, $f(x_m) + e_m^0$ is given as the initial value. We assume that there is no error from boundary conditions, that is, $e_0^0 = e_M^0 = 0$. Let $\tilde{u}_m^n, m = 0, 1, \cdots, M, n = 0, 1, \cdots, N$, be the computed solution. We want to study how $\tilde{u}_m^n$ is affected by e_m^0. This is usually referred to as studying the stability of schemes with respect to initial values. Clearly, $\tilde{u}_m^n$ satisfies

$$\begin{cases} \tilde{u}_m^{n+1} = \alpha\tilde{u}_{m+1}^n + (1-2\alpha)\tilde{u}_m^n + \alpha\tilde{u}_{m-1}^n, & \\ \qquad m = 1, 2, \cdots, M-1, & n = 0, 1, \cdots, N-1, \\ \tilde{u}_0^{n+1} = f_l(\tau^{n+1}), & n = 0, 1, \cdots, N-1, \\ \tilde{u}_M^{n+1} = f_u(\tau^{n+1}), & n = 0, 1, \cdots, N-1, \\ \tilde{u}_m^0 = f(x_m) + e_m^0, & m = 0, 1, \cdots, M. \end{cases}$$

Let

$$e_m^n = \tilde{u}_m^n - u_m^n, \quad m = 0, 1, \cdots, M, \quad n = 0, 1, \cdots, N.$$

Taking the difference of the scheme (7.8) and this system, we get

$$\begin{cases} e_m^{n+1} = \alpha e_{m+1}^n + (1-2\alpha)e_m^n + \alpha e_{m-1}^n, & \\ \qquad m = 1, 2, \cdots, M-1, & n = 0, 1, \cdots, N-1, \\ e_0^{n+1} = 0, & n = 0, 1, \cdots, N-1, \\ e_M^{n+1} = 0, & n = 0, 1, \cdots, N-1, \\ e_m^0 = e_m^0, & m = 0, 1, \cdots, M. \end{cases} \tag{7.13}$$

For this scheme, we can analyze its stability in two ways. First, we show that this scheme is stable in the maximum norm if $\alpha \le 1/2$. In this case, all the coefficients in the right-hand side of the finite-difference equation, α, $1-2\alpha$, α, are nonnegative, so

$$\begin{aligned} |e_m^{n+1}| &= |\alpha e_{m+1}^n + (1-2\alpha)e_m^n + \alpha e_{m-1}^n| \\ &\le \alpha|e_{m+1}^n| + (1-2\alpha)|e_m^n| + \alpha|e_{m-1}^n| \\ &\le \max_{1\le m\le M-1} |e_m^n|, \quad m = 1, 2, \cdots, M-1, \end{aligned}$$

or

$$\max_{1\le m\le M-1} |e_m^{n+1}| \le \max_{1\le m\le M-1} |e_m^n|,$$

where we have used the fact $e_0^n = e_M^n = 0$, $n = 0, 1, \cdots, N$. This is true for any n. Therefore,

$$\max_{1\le m\le M-1} |e_m^n| \le \max_{1\le m\le M-1} |e_m^0|$$

or

$$||\mathbf{e}^n||_{L_\infty} \le ||\mathbf{e}^0||_{L_\infty}.$$

Consequently, the difference scheme (7.8) is stable with respect to initial value in the maximum norm. This method of analyzing stability is very simple. Unfortunately, it seems that this method works only for explicit schemes with positive coefficients on the right-hand side.

Now let us study the stability of scheme (7.8) in another way. Set

$$\mathbf{A}_1 = \begin{bmatrix} 1-2\alpha & \alpha & 0 & \cdots & 0 \\ \alpha & 1-2\alpha & \alpha & \ddots & \vdots \\ 0 & \alpha & 1-2\alpha & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \alpha \\ 0 & \cdots & 0 & \alpha & 1-2\alpha \end{bmatrix}, \quad \mathbf{e}^n = \begin{bmatrix} e_1^n \\ e_2^n \\ \vdots \\ \vdots \\ e_{M-1}^n \end{bmatrix}. \tag{7.14}$$

From the system (7.13), we see that between $\mathbf{e}^{n+1}$ and $\mathbf{e}^n$ there is the following relation:

$$\mathbf{e}^{n+1} = \mathbf{A}_1\mathbf{e}^n.$$

Suppose λ is an eigenvalue of $\mathbf{A}_1$ and $\mathbf{x} = (x_1, x_2, \cdots, x_{M-1})^T$ is an associated eigenvector, i.e., we assume that λ and $\mathbf{x}$ satisfy the equation

$$\mathbf{A}_1\mathbf{x} = \lambda\mathbf{x}.$$

Now let us find $M-1$ linearly independent eigenvectors of $\mathbf{A}_1$ and their associated eigenvalues. Define

$$x_0 = x_M = 0.$$

Then the equation above can be rewritten as

$$\alpha x_{m-1} + (1-2\alpha)x_m + \alpha x_{m+1} = \lambda x_m, \quad 1 \le m \le M-1, \tag{7.15}$$

or

$$\alpha x_{m-1} + (1-2\alpha-\lambda)x_m + \alpha x_{m+1} = 0, \quad 1 \le m \le M-1. \tag{7.16}$$

For the system (7.16) with arbitrary x_0 and x_M, let us try to find a solution in the form

$$x_m = \mu^m, \quad 0 \le m \le M. \tag{7.17}$$

Substituting it into system (7.16), we have

$$\left[\alpha + (1-2\alpha-\lambda)\mu + \alpha\mu^2\right]\mu^{m-1} = 0, \quad 1 \le m \le M-1,$$

which can be reduced to one equation:

$$\alpha\mu^2 + (1-2\alpha-\lambda)\mu + \alpha = 0. \tag{7.18}$$

Denote the two roots of Eq. (7.18) by μ_1 and μ_2. It is clear that μ_1 and μ_2 should satisfy the following conditions:

$$\mu_1 + \mu_2 = -\frac{1}{\alpha}(1 - 2\alpha - \lambda), \qquad \mu_1\mu_2 = 1.$$

Case one: $\mu_1 = \mu_2 = \mu_*$. In this case,

$$x_m = m\mu_*^m, \quad 0 \le m \le M,$$

also is a solution of the system (7.16). Substituting it into system (7.16) yields

$$\begin{aligned}&\alpha(m-1)\mu_*^{m-1} + (1 - 2\alpha - \lambda)m\mu_*^m + \alpha(m+1)\mu_*^{m+1}\\ &= -\alpha\mu_*^{m-1} + \alpha\mu_*^{m+1} = \alpha\mu_*^{m-1}(\mu_*^2 - 1) = 0, \quad 1 \le m \le M-1,\end{aligned}$$

because of $\mu_1\mu_2 = \mu_*^2 = 1$, so it is true that $x_m = m\mu_*^m, \quad 0 \le m \le M$, is another solution of the system (7.16) besides the solution (7.17) with $\mu = \mu_*$. Thus for any c_1 and c_2,

$$x_m = (\,c_1 + c_2 m\,)\mu_*^m, \quad 0 \le m \le M,$$

should be a solution of the system (7.16). It follows from $x_0 = x_M = 0$ that $c_1 = c_2 = 0$. Consequently, $x_m \equiv 0$, $1 \le m \le M-1$, which contradicts that $\mathbf{x} = (x_1, x_2, \cdots, x_{M-1})^T$ is an eigenvector.

Case two: $\mu_1 \ne \mu_2$. In this case for any c_1 and c_2,

$$x_m = c_1\mu_1^m + c_2\mu_2^m, \quad 0 \le m \le M,$$

should be a solution of the system (7.16). It follows from $x_0 = x_M = 0$ that

$$c_1 + c_2 = 0, \quad c_1\mu_1^M + c_2\mu_2^M = 0.$$

From these two relations we can obtain

$$\left(\frac{\mu_1}{\mu_2}\right)^M = -\frac{c_2}{c_1} = 1 = \mathrm{e}^{\mathrm{i}2k\pi}, \quad k \text{ being any integer.}$$

Consequently,

$$\frac{\mu_1}{\mu_2} = \mathrm{e}^{\mathrm{i}2\omega_k}, \quad \omega_k = \frac{k\pi}{M}, \quad k \text{ being any integer.}$$

It is clear that $k = k^*$ and $k = k^* + M$ give the same solution. Thus we need to set $k = 0, 1, \cdots, M-1$ only. For $k = 0$, we have $\mu_1 = \mu_2$. As we have pointed out, in this case we could not find any eigenvector. For $k = 1, 2, \cdots,$ or $M-1$, we have $\dfrac{\mu_1}{\mu_2} = \mathrm{e}^{\mathrm{i}2\omega_k}$. Combining this relation with $\mu_1\mu_2 = 1$ yields

$$\mu_1^{(k)} = \mathrm{e}^{\mathrm{i}\omega_k}, \quad \mu_2^{(k)} = \mathrm{e}^{-\mathrm{i}\omega_k}.$$

For such a k, taking $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}$, we have the following eigenvector

$$\mathbf{x}_{\omega_k} = \begin{bmatrix} \frac{1}{2}e^{i\omega_k} - \frac{1}{2}e^{-i\omega_k} \\ \frac{1}{2}e^{i2\omega_k} - \frac{1}{2}e^{-i2\omega_k} \\ \vdots \\ \vdots \\ \frac{1}{2}e^{i(M-1)\omega_k} - \frac{1}{2}e^{-i(M-1)\omega_k} \end{bmatrix} = \begin{bmatrix} \sin\omega_k \\ \sin 2\omega_k \\ \vdots \\ \vdots \\ \sin(M-1)\omega_k \end{bmatrix}. \tag{7.19}$$

The corresponding eigenvalue λ_{ω_k} satisfies system (7.15), i.e.,

$$\begin{aligned} \lambda_{\omega_k} &= \frac{\alpha\sin(m-1)\omega_k + (1-2\alpha)\sin m\omega_k + \alpha\sin(m+1)\omega_k}{\sin m\omega_k} \\ &= \frac{\alpha\sin m\omega_k\cos\omega_k + (1-2\alpha)\sin m\omega_k + \alpha\sin m\omega_k\cos\omega_k}{\sin m\omega_k} \\ &= 1-2\alpha+2\alpha\cos\omega_k = 1-4\alpha\sin^2(\omega_k/2). \end{aligned}$$

Here $k = 1, 2, \cdots, M-1$, i.e., we have found $M-1$ eigenvalues of $\mathbf{A}_1$ and their associated eigenvectors. Because λ_{ω_k}, $k = 1, 2, \cdots, M-1$, are distinct eigenvalues of the symmetric matrix $\mathbf{A}_1$, the $M-1$ associated eigenvectors, $\mathbf{x}_{\omega_k}$, $k = 1, 2, \cdots, M-1$, are linearly independent.

As a consequence, any vector with $M-1$ components can be expressed as linear combination of $\mathbf{x}_{\omega_k}$, which means that an error $\mathbf{e}^0$ can be expressed as

$$\mathbf{e}^0 = \sum_{k=1}^{M-1} \varepsilon_{\omega_k}\mathbf{x}_{\omega_k}.$$

Substituting this expression into $\mathbf{e}^{n+1} = \mathbf{A}_1\mathbf{e}^n$, we have

$$\mathbf{e}^1 = \mathbf{A}_1\mathbf{e}^0 = \sum_{k=1}^{M-1} \varepsilon_{\omega_k}\lambda_{\omega_k}\mathbf{x}_{\omega_k}$$

and furthermore

$$\mathbf{e}^n = \sum_{k=1}^{M-1} \varepsilon_{\omega_k}\lambda_{\omega_k}^n\mathbf{x}_{\omega_k}$$

or in component form

$$e_m^n = \sum_{k=1}^{M-1} \varepsilon_{\omega_k}\lambda_{\omega_k}^n \sin m\omega_k, \quad m = 1, 2, \cdots, M-1.$$

As eigenvectors of a symmetric matrix $\mathbf{A}_1$, $\mathbf{x}_{\omega_k}$, $k = 1, 2, \cdots, M-1$ are orthogonal. Thus, from the expressions of $\mathbf{e}^0$ and $\mathbf{e}^n$ above, we have

$$||\mathbf{e}^0||_{L_2} = \left(\frac{1}{M-1}\sum_{m=1}^{M-1} \varepsilon_{\omega_k}^2 ||\mathbf{x}_{\omega_k}||_{L_2}^2\right)^{1/2}$$

and

$$||\mathbf{e}^n||_{\mathrm{L}_2} = \left(\frac{1}{M-1} \sum_{m=1}^{M-1} \varepsilon_{\omega_k}^2 \lambda_{\omega_k}^{2n} ||\mathbf{x}_{\omega_k}||_{\mathrm{L}_2}^2 \right)^{1/2} .$$

Consequently, we obtain

$$||\mathbf{e}^n||_{\mathrm{L}_2} \le ||\mathbf{e}^0||_{\mathrm{L}_2}$$

if all the eigenvalues of $\mathbf{A}_1$ are in $[-1, 1]$. From what we have gotten the following conclusion is obtained: if

$$0 \le \alpha \le 1/2,$$

then we have the following inequality

$$-1 \le 1 - 4\alpha \le \lambda_{\omega_k} = 1 - 4\alpha \sin^2(\omega_k/2) \le 1, \quad k = 1, 2, \cdots, M-1,$$

which means that the computation is stable with respect to the initial value. If $\alpha > 1/2$, then when M is large enough, some of the eigenvalues of $\mathbf{A}_1$ must be less than -1. Hence, if a component of $\mathbf{e}^0$ associated with such an eigenvalue is not zero, then the corresponding component of $\mathbf{e}^n$ will be greater than the component of $\mathbf{e}^0$ and go to infinity as n goes to infinity. Because the errors are random variables, the ε_{ω_k} corresponding to such an eigenvalue λ_{ω_k} might not be zero. Thus, the computation is unstable. This can be summarized as: scheme (7.8) is stable if

$$\alpha = \frac{a\Delta\tau}{\Delta x^2} \le 1/2;$$

whereas the scheme is unstable if

$$\alpha = \frac{a\Delta\tau}{\Delta x^2} > 1/2.$$

Stability of Implicit Finite-Difference Methods for the Heat Equation. The second method used above to analyze stability can be applied to other cases, for example, implicit finite-difference methods. For an implicit finite-difference scheme, suppose $\mathbf{e}^n$ satisfies

$$\mathbf{A}\mathbf{e}^{n+1} = \mathbf{B}\mathbf{e}^n,$$

where $\mathbf{A}$ and $\mathbf{B}$ are two matrices, and $\mathbf{A}$ is invertible. Also, assume that the following relations hold:

$$\lambda_{\omega_k} \mathbf{A}\mathbf{x}_{\omega_k} = \mathbf{B}\mathbf{x}_{\omega_k}, \quad k = 1, 2, \cdots, M-1, \tag{7.20}$$

where $\mathbf{x}_{\omega_k}$, $k = 1, 2, \cdots, M-1$ are linear independent vectors. In this case, this method still works: if all the $\lambda_{\omega_k} \in [-1, 1]$, then the scheme is stable; if certain λ_{ω_k} does not belong to $[-1, 1]$, then the scheme is unstable. In fact, any initial error can be expressed as

$$\mathbf{e}^0 = \sum_{k=1}^{M-1} \varepsilon_{\omega_k} \mathbf{x}_{\omega_k}$$

and because of the set of relations (7.20), we have

$$\mathbf{e}^n = \sum_{k=1}^{M-1} \varepsilon_{\omega_k} \lambda_{\omega_k}^n \mathbf{x}_{\omega_k}$$

for any n. Therefore, the scheme is stable if and only if

$$|\lambda_{\omega_k}| \le 1$$

for all the ω_k.

For the Crank–Nicolson scheme (7.9), $\mathbf{A}$ and $\mathbf{B}$ are given in Sect. 7.1. As pointed out above, in order to study the stability, we need to find the solution of

$$\lambda \mathbf{A}\mathbf{x} = \mathbf{B}\mathbf{x}.$$

In Problem 7, for more general equations, readers are asked to find the eigenvectors and the eigenvalues. Here we only give the result. The result is as follows. For this case, there are $M-1$ linearly independent vectors given by the expression (7.19) and the corresponding eigenvalues are

$$\begin{aligned} \lambda_{\omega_k} &= \frac{\frac{1}{2}\alpha \sin{(m+1)\omega_k} + (1-\alpha)\sin{m\omega_k} + \frac{1}{2}\alpha\sin{(m-1)\omega_k}}{-\frac{1}{2}\alpha \sin{(m+1)\omega_k} + (1+\alpha)\sin{m\omega_k} - \frac{1}{2}\alpha\sin{(m-1)\omega_k}} \\ &= \frac{(1-\alpha)\sin{m\omega_k} + \alpha \sin{m\omega_k}\cos{\omega_k}}{(1+\alpha)\sin{m\omega_k} - \alpha\sin{m\omega_k}\cos{\omega_k}} \\ &= \frac{1-2\alpha\sin^2\dfrac{\omega_k}{2}}{1+2\alpha\sin^2\dfrac{\omega_k}{2}}, \quad k=1,2,\cdots,M-1, \end{aligned}$$

where $\omega_k = k\pi/M$. Because $|\lambda_{\omega_k}| \le 1$ for any ω_k, the difference scheme (7.9) is stable in the L_2 norm.

Stability for Periodic Problems. In schemes (7.8) and (7.9), the values are given at both boundaries, and during stability analysis, we assume that there is no error at the boundaries. It is clear that this is not always the case. Consider problems satisfying periodic conditions and assume $u_m^n = u_{m+M}^n$. In this case, we only need to find u_m^n, $m = 0, 1, \cdots, M-1$ for each time level. If the coefficients of the problem are constant, then we can analyze the stability in a similar way. Let us further assume that the solution satisfies the system:

$$a_1 u_{m+1}^{n+1} + a_0 u_m^{n+1} + a_{-1} u_{m-1}^{n+1} = b_1 u_{m+1}^n + b_0 u_m^n + b_{-1} u_{m-1}^n, \; m = 0, 1, \cdots, M-1.$$

If e_m^n is the error of u_m^n, then e_m^n satisfy the same system. Thus, the system for e_m^n can be written as

$$\mathbf{A}_2\mathbf{e}^{n+1} = \mathbf{B}_2\mathbf{e}^n,$$

where we have used the conditions

$$e_{-1}^n = e_{M-1}^n, \quad e_M^n = e_0^n$$

and adopted the following notation:

$$\mathbf{A}_2 = \begin{bmatrix} a_0 & a_1 & 0 & \cdots & a_{-1} \\ a_{-1} & a_0 & a_1 & \ddots & \vdots \\ 0 & a_{-1} & a_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_1 \\ a_1 & \cdots & 0 & a_{-1} & a_0 \end{bmatrix}, \quad \mathbf{e}^n = \begin{bmatrix} e_0^n \\ e_1^n \\ \vdots \\ \vdots \\ e_{M-1}^n \end{bmatrix}$$

and

$$\mathbf{B}_2 = \begin{bmatrix} b_0 & b_1 & 0 & \cdots & b_{-1} \\ b_{-1} & b_0 & b_1 & \ddots & \vdots \\ 0 & b_{-1} & b_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_1 \\ b_1 & \cdots & 0 & b_{-1} & b_0 \end{bmatrix}.$$

In order to study stability, we need to find the solution of the equation

$$\lambda\mathbf{A}_2\mathbf{x} = \mathbf{B}_2\mathbf{x}.$$

This is left for readers to do as Problem 8. The result is as follows. For this equation, the eigenvectors are

$$\mathbf{x}_{\theta_k} = \begin{bmatrix} 1 \\ \mathrm{e}^{\mathrm{i}\theta_k} \\ \vdots \\ \vdots \\ \mathrm{e}^{\mathrm{i}(M-1)\theta_k} \end{bmatrix}, \quad k = 0, 1, \cdots, M-1,$$

where $\theta_k = 2k\pi/M$ and the eigenvalues are

$$\lambda_{\theta_k} = \frac{b_1\mathrm{e}^{\mathrm{i}\theta_k} + b_0 + b_{-1}\mathrm{e}^{-\mathrm{i}\theta_k}}{a_1\mathrm{e}^{\mathrm{i}\theta_k} + a_0 + a_{-1}\mathrm{e}^{-\mathrm{i}\theta_k}}, \quad k = 0, 1, \cdots, M-1.$$

By using the relations $\mathrm{e}^{-\mathrm{i}\theta_k} = \mathrm{e}^{\mathrm{i}(M-1)\theta_k}$ and $\mathrm{e}^{\mathrm{i}M\theta_k} = 1$, this result can be shown by a straightforward calculation. If $|\lambda_{\theta_k}| \le 1, \quad k = 0, 1, \cdots, M-1$, then the method is stable. If $|\lambda_{\theta_k}| > 1$ for some k, then the method is unstable.

Because M can go to infinity, θ_k indeed can be any number in the interval $[0, 2\pi]$. Therefore, if for any $\theta \in [0, 2\pi]$,

$$|\lambda_\theta| = \left| \frac{b_1 \mathrm{e}^{\mathrm{i}\theta} + b_0 + b_{-1} \mathrm{e}^{-\mathrm{i}\theta}}{a_1 \mathrm{e}^{\mathrm{i}\theta} + a_0 + a_{-1} \mathrm{e}^{-\mathrm{i}\theta}} \right| \le 1, \tag{7.21}$$

then the scheme is stable. Otherwise, the method is unstable. Such a method of analyzing stability is usually called the von Neumann method and λ_θ is called the amplification factor. This method gives a complete stability analysis for periodic initial value problems with constant coefficients. For more general case, this method can be performed in the following way. Assume

$$e_m^n = \lambda_\theta^n \mathrm{e}^{\mathrm{i}m\theta}, \tag{7.22}$$

where θ can be any real number in the interval $[0, 2\pi]$. Substituting this expression into the finite-difference equation, we can find λ_θ. If all $|\lambda_\theta| \le 1$, then the scheme is stable; if some $|\lambda_\theta| > 1$, then the scheme is unstable. For more about this method, see the book [67] by Richtmyer and Morton and many other books.

Stability Analysis in Practice. In practice, most problems have variable coefficients. Therefore, the von Neumann method does not give a complete stability analysis. However, it is still very useful. The von Neumann method can be applied in practice in the following way.

Consider the following scheme with variable coefficients:

$$\begin{aligned} &a_{1,m}^n u_{m+1}^{n+1} + a_{0,m}^n u_m^{n+1} + a_{-1,m}^n u_{m-1}^{n+1} \\ &= b_{1,m}^n u_{m+1}^n + b_{0,m}^n u_m^n + b_{-1,m}^n u_{m-1}^n, \end{aligned} \tag{7.23}$$

where for simplicity, we assume that only three points in the x direction are involved. If more points are involved, the procedure is still the same. Suppose

$$|f_{m+1}^n - f_m^n| < c\Delta x, \qquad |f_{m+1}^n - 2f_m^n + f_{m-1}^n| < c\Delta x^2,$$

and

$$|f_m^{n+1} - f_m^n| < c\Delta\tau$$

for $f = a_1, a_0, a_{-1}, b_1, b_0$, and b_{-1}. Assume that e_m^n has the form (7.22). Substituting this expression into the finite-difference equation (7.23) yields

$$\lambda_\theta(x_m, \tau^n) = \frac{b_{1,m}^n \mathrm{e}^{\mathrm{i}(m+1)\theta} + b_{0,m}^n \mathrm{e}^{\mathrm{i}m\theta} + b_{-1,m}^n \mathrm{e}^{\mathrm{i}(m-1)\theta}}{a_{1,m}^n \mathrm{e}^{\mathrm{i}(m+1)\theta} + a_{0,m}^n \mathrm{e}^{\mathrm{i}m\theta} + a_{-1,m}^n \mathrm{e}^{\mathrm{i}(m-1)\theta}}.$$

If for the amplification factor, we have

$$|\lambda_\theta(x_m, \tau^n)| \le 1$$

for every point and the treatment of boundary conditions is reasonable, then we can expect the scheme to be stable. Clearly, the condition $|\lambda_\theta(x_m, \tau^n)| \le 1$ is equivalent to

$$|b^n_{1,m}\mathrm{e}^{\mathrm{i}\theta} + b^n_{0,m} + b^n_{-1,m}\mathrm{e}^{-\mathrm{i}\theta}|^2 - |a^n_{1,m}\mathrm{e}^{\mathrm{i}\theta} + a^n_{0,m} + a^n_{-1,m}\mathrm{e}^{-\mathrm{i}\theta}|^2 \le 0 \qquad (7.24)$$

if $|a^n_{1,m}\mathrm{e}^{\mathrm{i}\theta} + a^n_{0,m} + a^n_{-1,m}\mathrm{e}^{-\mathrm{i}\theta}|^2 \ge \tilde{c} > 0$, $\tilde{c}$ being a constant. The latter is easier to use in practice than the former.

Let us analyze the stability of scheme (7.6) in this way. This scheme has the form (7.23) with

$$\begin{aligned}
a^n_{1,m} &= -\left(\frac{a^{n+1/2}_m}{2\Delta x^2} + \frac{b^{n+1/2}_m}{4\Delta x}\right)\Delta\tau,\\
a^n_{0,m} &= 1 + \frac{a^{n+1/2}_m}{\Delta x^2}\Delta\tau,\\
a^n_{-1,m} &= -\left(\frac{a^{n+1/2}_m}{2\Delta x^2} - \frac{b^{n+1/2}_m}{4\Delta x}\right)\Delta\tau,\\
b^n_{1,m} &= -a^n_{1,m},\\
b^n_{0,m} &= 2 - a^n_{0,m},\\
b^n_{-1,m} &= -a^n_{-1,m}.
\end{aligned}$$

Here, we assume

$$g^{n+1/2}_m = c^{n+1/2}_m = 0$$

because we analyze the stability with respect to initial values only and ignoring a term of $O(\Delta\tau)$ in coefficients will have no effect on the conclusion on stability. The left-hand side of the condition (7.24) for this scheme is

$$\begin{aligned}
&\left[-a^n_{1,m}\mathrm{e}^{\mathrm{i}\theta} + (2 - a^n_{0,m}) - a^n_{-1,m}\mathrm{e}^{-\mathrm{i}\theta}\right]\left[-a^n_{1,m}\mathrm{e}^{-\mathrm{i}\theta} + (2 - a^n_{0,m}) - a^n_{-1,m}\mathrm{e}^{\mathrm{i}\theta}\right]\\
&-(a^n_{1,m}\mathrm{e}^{\mathrm{i}\theta} + a^n_{0,m} + a^n_{-1,m}\mathrm{e}^{-\mathrm{i}\theta})(a^n_{1,m}\mathrm{e}^{-\mathrm{i}\theta} + a^n_{0,m} + a^n_{-1,m}\mathrm{e}^{\mathrm{i}\theta})\\
={}& (a^n_{1,m})^2 + (a^n_{0,m} - 2)^2 + (a^n_{-1,m})^2 + 2a^n_{1,m}(a^n_{0,m} - 2)\cos\theta\\
&+2(a^n_{0,m} - 2)a^n_{-1,m}\cos\theta + 2a^n_{1,m}a^n_{-1,m}\cos 2\theta\\
&-\left[(a^n_{1,m})^2 + (a^n_{0,m})^2 + (a^n_{-1,m})^2 + 2a^n_{1,m}a^n_{0,m}\cos\theta + 2a^n_{0,m}a^n_{-1,m}\cos\theta\right.\\
&\left.+2a^n_{1,m}a^n_{-1,m}\cos 2\theta\right]\\
={}& (a^n_{0,m} - 2)^2 - (a^n_{0,m})^2 - 4a^n_{1,m}\cos\theta - 4a^n_{-1,m}\cos\theta\\
={}& -\frac{4a^{n+1/2}_m}{\Delta x^2}\Delta\tau + \frac{4a^{n+1/2}_m}{\Delta x^2}\Delta\tau\cos\theta\\
={}& \frac{4a^{n+1/2}_m}{\Delta x^2}\Delta\tau(\cos\theta - 1).
\end{aligned}$$

This expression is always nonpositive. Therefore, the condition (7.24) is satisfied at every grid point. For scheme (7.6), there is no other boundary condition. Consequently, the scheme is expected to be stable.

So far, we say that a scheme is stable with respect to initial values if the error of the solution caused by the error in the initial condition is less than or equal to the error in the initial condition. However, generally speaking, we say that a scheme is stable with respect to initial values if the error of the solution caused by the error in the initial condition is less than c times the error in the initial condition. c is a constant independent of Δx and $\Delta\tau$, but is allowed to be greater than one. That is, the error is allowed to increase by a certain factor, but the factor must be bounded and independent of Δx and $\Delta\tau$. Therefore, we can take

$$|\lambda_\theta(x_m, \tau^n)| \le 1 + \bar{c}\Delta\tau \tag{7.25}$$

as a criterion for stability.[2] In fact, if the inequality (7.25) holds for any θ, then usually we can have

$$||\mathbf{e}^n||_{\mathrm{L}_2} \le (1+\bar{c}\Delta\tau)||\mathbf{e}^{n-1}||_{\mathrm{L}_2} \le (1+\bar{c}\Delta\tau)^n||\mathbf{e}^0||_{\mathrm{L}_2} \le \mathrm{e}^{\bar{c}nT/N}||\mathbf{e}^0||_{\mathrm{L}_2}$$

for any $n \le N$, so the error increases at most by a factor $\mathrm{e}^{\bar{c}T}$. Here we have used the relation $(1+\bar{c}\Delta\tau)^{\frac{1}{\bar{c}\Delta\tau}} \le \mathrm{e}$ for any positive $\Delta\tau$.

Now let us study the stability of the difference scheme (7.5) by using the criterion (7.25). We consider the stability with respect to initial values only, so we can set $g_m^n = 0$. In this case, the scheme has the form (7.23) with $a_{1,m}^n = 0$, $a_{0,m}^n = 1$, $a_{-1,m}^n = 0$ and

$$\begin{aligned} b_{1,m}^n &= \frac{a_m^n\Delta\tau}{\Delta x^2} + \frac{b_m^n\Delta\tau}{2\Delta x}, \\ b_{0,m}^n &= 1 - 2\frac{a_m^n\Delta\tau}{\Delta x^2} + c_m^n\Delta\tau, \\ b_{-1,m}^n &= \frac{a_m^n\Delta\tau}{\Delta x^2} - \frac{b_m^n\Delta\tau}{2\Delta x}. \end{aligned}$$

Therefore,

$$\begin{aligned} \lambda_\theta(x_m,\tau^n) &= b_{1,m}^n\mathrm{e}^{\mathrm{i}\theta} + b_{0,m}^n + b_{-1,m}^n\mathrm{e}^{-\mathrm{i}\theta} \\ &= b_{0,m}^n + \left(b_{1,m}^n + b_{-1,m}^n\right)\cos\theta + \mathrm{i}\left(b_{1,m}^n - b_{-1,m}^n\right)\sin\theta \\ &= 1 - 2\frac{a_m^n\Delta\tau}{\Delta x^2} + c_m^n\Delta\tau + 2\frac{a_m^n\Delta\tau}{\Delta x^2}\cos\theta + \mathrm{i}\frac{b_m^n\Delta\tau}{\Delta x}\sin\theta \\ &= 1 - 4\frac{a_m^n\Delta\tau}{\Delta x^2}\sin^2\frac{\theta}{2} + c_m^n\Delta\tau + \mathrm{i}\frac{b_m^n\Delta\tau}{\Delta x}\sin\theta. \end{aligned}$$

If

$$\max\frac{a_m^n\Delta\tau}{\Delta x^2} \le \frac{1}{2} \quad \text{or} \quad \frac{\Delta\tau}{\Delta x^2} \le \frac{1}{2\max a_m^n}, \tag{7.26}$$

[2]This criterion is equivalent to

$$|b_{1,m}^n\mathrm{e}^{\mathrm{i}\theta} + b_{0,m}^n + b_{-1,m}^n\mathrm{e}^{-\mathrm{i}\theta}|^2 - |a_{1,m}^n\mathrm{e}^{\mathrm{i}\theta} + a_{0,m}^n + a_{-1,m}^n\mathrm{e}^{-\mathrm{i}\theta}|^2 \le \bar{\bar{c}}\Delta\tau$$

if $|a_{1,m}^n\mathrm{e}^{\mathrm{i}\theta} + a_{0,m}^n + a_{-1,m}^n\mathrm{e}^{-\mathrm{i}\theta}|^2 \ge \tilde{c} > 0$, $\tilde{c}$ being a constant, which is easier to use in practice than the criterion (7.25).

then

$$\begin{aligned}
|\lambda_\theta(x_m,\tau^n)|^2 &\le (1+|c_m^n|\,\Delta\tau)^2+\left(\frac{b_m^n\Delta\tau}{\Delta x}\right)^2\\
&\le (1+|c_m^n|\,\Delta\tau)^2+\frac{(b_m^n)^2}{2\max a_m^n}\Delta\tau\\
&\le (1+|c_m^n|\,\Delta\tau)^2+2\,(1+|c_m^n|\,\Delta\tau)\,\frac{(b_m^n)^2}{4\max a_m^n}\Delta\tau\\
&\quad +\left[\frac{(b_m^n)^2}{4\max a_m^n}\Delta\tau\right]^2\\
&=\left[1+|c_m^n|\,\Delta\tau+\frac{(b_m^n)^2}{4\max a_m^n}\Delta\tau\right]^2.
\end{aligned}$$

Thus, let $\bar{c}=|c_m^n|+(b_m^n)^2/(4\max a_m^n)$, we have

$$|\lambda_\theta(x_m,\tau^n)|\le 1+\bar{c}\Delta\tau$$

and we can expect this scheme to be stable if inequality (7.26) holds.

In fact, the stability of scheme (7.6) with variable coefficients has been proved rigorously in the paper [79] by Sun, Yan, and Zhu. By a similar method, the stability of scheme (7.5) with variable coefficients can also be shown when inequality (7.26) holds. If readers are interested in such a subject, please see that paper and the book [97] by Zhu, Zhong, Chen, and Zhang.

7.2.2 Convergence

If a scheme is stable with respect to initial values, and the truncation error of the scheme goes to zero as Δx and $\Delta\tau$ tend to zero, then the approximate solution will usually go to the exact solution. Such a result is usually referred to as the Lax equivalence theorem (see the book [67] by Richtmyer and Morton). We are not going to prove this conclusion for general cases but explain this result intuitively through proving this result for special cases.

Consider the explicit finite-difference method (7.8). We know that the exact solution $u(x,\tau)$ satisfies the equation

$$\begin{aligned}
&u(x_m,\tau^{n+1})\\
&=\alpha u(x_{m+1},\tau^n)+(1-2\alpha)u(x_m,\tau^n)+\alpha u(x_{m-1},\tau^n)+\Delta\tau R_m^n(\Delta x^2,\Delta\tau),\\
&\qquad m=1,2,\cdots,M-1,\quad n=0,1,\cdots,N-1,
\end{aligned}$$

where

$$R_m^n(\Delta x^2,\Delta\tau)=\frac{\Delta\tau}{2}\frac{\partial^2 u}{\partial\tau^2}(x_m,\eta)-a\frac{\Delta x^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi,\tau^n).$$

Let e_m^n be the error of the approximate solution on the point (x_m, τ^n), that is,

$$e_m^n = u(x_m, \tau^n) - u_m^n, \quad m = 0, 1, \cdots, M, \; n = 0, 1, \cdots, N.$$

Then, e_m^n is the solution of the problem

$$\begin{cases} e_m^{n+1} = \alpha e_{m+1}^n + (1-2\alpha) e_m^n + \alpha e_{m-1}^n + \Delta\tau R_m^n(\Delta x^2, \Delta\tau), \\ \qquad\qquad m = 1, 2, \cdots, M-1, \qquad n = 0, 1, \cdots, N-1, \\ e_0^{n+1} = 0, \qquad\qquad\qquad\qquad\qquad n = 0, 1, \cdots, N-1, \\ e_M^{n+1} = 0, \qquad\qquad\qquad\qquad\qquad n = 0, 1, \cdots, N-1, \\ e_m^0 = 0, \qquad\qquad\qquad\qquad\qquad\quad m = 0, 1, \cdots, M. \end{cases}$$

Because $e_0^n = e_M^n = 0$ for any n, the system can be written as

$$\begin{cases} \mathbf{e}^{n+1} = \mathbf{A}_1 \mathbf{e}^n + \Delta\tau \mathbf{R}^n(\Delta x^2, \Delta\tau), \quad n = 0, 1, \cdots, N-1, \\ \mathbf{e}^0 = 0, \end{cases}$$

where $\mathbf{e}^n$ is a vector with $M-1$ components e_m^n, $m = 1, 2, \cdots, M-1$ and

$$\mathbf{R}^n(\Delta x^2, \Delta\tau) = \begin{bmatrix} R_1^n(\Delta x^2, \Delta\tau) \\ R_2^n(\Delta x^2, \Delta\tau) \\ \vdots \\ R_{M-1}^n(\Delta x^2, \Delta\tau) \end{bmatrix}.$$

Actually, $\mathbf{e}^n$ can be written as $\sum_{k=1}^n \mathbf{e}_{(k)}^n$. Here, for $k = n$,

$$\mathbf{e}_{(n)}^n = \Delta\tau \mathbf{R}^{n-1}(\Delta x^2, \Delta\tau)$$

and for $k = 1, 2, \cdots, n-1$, $\mathbf{e}_{(k)}^n$ is the solution of the following problem

$$\begin{cases} \mathbf{e}_{(k)}^{\bar{n}+1} = \mathbf{A}_1 \mathbf{e}_{(k)}^{\bar{n}}, \quad \bar{n} = k, k+1, \cdots, n-1, \\ \mathbf{e}_{(k)}^k = \Delta\tau \mathbf{R}^{k-1}(\Delta x^2, \Delta\tau). \end{cases}$$

Because the error does not increase for the scheme (7.8) if $\alpha \le 1/2$, $||\mathbf{e}^n||_{\mathrm{L}_2}$ should not be greater than $\sum_{k=1}^n \Delta\tau ||\mathbf{R}^{k-1}(\Delta x^2, \Delta\tau)||_{\mathrm{L}_2}$. Noticing $n \le T/\Delta\tau$, we see that e_m^n goes to zero as $R_m^{k-1}(\Delta x^2, \Delta\tau)$ tends to zero for $k = 1, 2, \cdots, n$ and $m = 1, 2, \cdots, M-1$. Hence, the approximate solution converges to the exact solution as Δx and $\Delta\tau$ tend to zero and α stays less than $1/2$ and $||\mathbf{e}^n||_{\mathrm{L}_2}$ has an order of $O(\Delta x^2, \Delta\tau)$. Usually, $\alpha = a\Delta\tau/\Delta x^2$ stays constant as Δx and $\Delta\tau$ tend to zero. Therefore, $||\mathbf{e}^n||_{\mathrm{L}_2} = O(\Delta\tau)$, and we say that the scheme (7.8) converges with order of $\Delta\tau$.

For implicit schemes, the situation is similar. Consider the Crank–Nicolson scheme (7.9). The exact solution satisfies

$$
\begin{aligned}
&\frac{u(x_m, \tau^{n+1}) - u(x_m, \tau^n)}{\Delta\tau} \\
= &\frac{a}{2}\left[\frac{u(x_{m+1}, \tau^{n+1}) - 2u(x_m, \tau^{n+1}) + u(x_{m-1}, \tau^{n+1})}{\Delta x^2}\right. \\
&\left.+\frac{u(x_{m+1}, \tau^n) - 2u(x_m, \tau^n) + u(x_{m-1}, \tau^n)}{\Delta x^2}\right] + R_m^n(\Delta x^2, \Delta\tau^2), \\
&\qquad m = 1, 2, \cdots, M-1,
\end{aligned}
$$

where

$$
\begin{aligned}
&R_m^n(\Delta x^2, \Delta\tau^2) \\
= &\Delta\tau^2\left[\frac{1}{24}\frac{\partial^3 u}{\partial\tau^3}(x_m, \eta^{(1)}) - \frac{a}{8}\frac{\partial^4 u}{\partial x^2\tau^2}(x_m, \eta^{(2)})\right] - \frac{\Delta x^2 a}{12}\frac{\partial^4 u}{\partial x^4}(\xi, \eta^{(3)}).
\end{aligned}
$$

In this case, the error satisfies

$$
\mathbf{A}\mathbf{e}^{n+1} = \mathbf{B}\mathbf{e}^n + \Delta\tau\mathbf{R}^n(\Delta x^2, \Delta\tau^2),
$$

where $\mathbf{e}^n$ and $\mathbf{R}^n(\Delta x^2, \Delta\tau^2)$ are two $(M-1)$-dimensional vectors with e_m^n and $R_m^n(\Delta x^2, \Delta\tau^2)$ as components, respectively, and $\mathbf{A}$ and $\mathbf{B}$ are given in the difference scheme (7.10). Just like in the case of the scheme (7.8), $\mathbf{e}^n$ can also be written as $\sum_{k=1}^n \mathbf{e}_{(k)}^n$. Here, for $k = n$,

$$
\mathbf{e}_{(n)}^n = \Delta\tau\mathbf{A}^{-1}\mathbf{R}^{n-1}(\Delta x^2, \Delta\tau^2)
$$

and for $k = 1, 2, \cdots, n-1$, $\mathbf{e}_{(k)}^n$ is the solution of the following problem:

$$
\begin{cases}
\mathbf{A}\mathbf{e}_{(k)}^{\bar{n}+1} = \mathbf{B}\mathbf{e}_{(k)}^{\bar{n}}, \quad \bar{n} = k, k+1, \cdots, n-1, \\
\mathbf{e}_{(k)}^k = \Delta\tau\mathbf{A}^{-1}\mathbf{R}^{k-1}(\Delta x^2, \Delta\tau^2).
\end{cases}
$$

The Crank–Nicolson scheme is stable with respect to the initial value. Thus, $||\mathbf{e}^n||_{\mathrm{L}_2}$ does not exceed $\sum_{k=1}^n \Delta\tau||\mathbf{A}^{-1}\mathbf{R}^{k-1}(\Delta x^2, \Delta\tau^2)||_{\mathrm{L}_2}$. Because

$$
\mathbf{A}\mathbf{e}_{\omega_k} = \left(1 + 2\alpha\sin^2\frac{\omega_k}{2}\right)\mathbf{e}_{\omega_k},
$$

we see that $1+2\alpha\sin^2(\omega_k/2)$ is an eigenvalue of $\mathbf{A}$. Thus, $1/[1+2\alpha\sin^2(\omega_k/2)]$ is an eigenvalue of $\mathbf{A}^{-1}$. This means that $\mathbf{A}^{-1}$ always exists and that its norm is bounded for any case. Consequently, $||\mathbf{e}^n||_{\mathrm{L}_2}$ goes to zero as Δx and $\Delta\tau$ tend to zero. In this case, we say that this scheme is convergent. Furthermore, because $||\mathbf{e}^n||_{\mathrm{L}_2}$ is of the order $O(\Delta x^2, \Delta\tau^2)$, we say that the scheme has a second-order convergence or possesses a second-order accuracy.

For schemes with variable coefficients, from the stability with respect to initial values and the consistency of a scheme, we also can have its convergence. Here, we say that a scheme is consistent with the partial differential equation if the truncation error of the scheme goes to zero as Δx and $\Delta\tau$ tend to zero. In the paper [79] by Sun, Yan, and Zhu, some results on this issue are given.

7.3 Extrapolation of Numerical Solutions

When a partial differential equation problem is discretized, a truncation error is introduced that causes the numerical solution to have an error. What is the relation between the truncation error and the error of the numerical solution? Intuitively, the answer should be that a term of $O(\Delta x^{k_1}, \Delta\tau^{k_2})$ in the truncation error causes an error of $O(\Delta x^{k_1}, \Delta\tau^{k_2})$ in the numerical solution. Here $O(\Delta x^{k_1}, \Delta\tau^{k_2})$ denotes a term less than $C\left(\Delta x^{k_1} + \Delta x^{k_2}\right)$, where C is a constant. Let us illustrate this fact.

Consider the following problem

$$\begin{cases} \dfrac{\partial u}{\partial \tau} = a(x,\tau)\dfrac{\partial^2 u}{\partial x^2} + b(x,\tau)\dfrac{\partial u}{\partial x} + c(x,\tau)u + g(x,\tau), \\ \qquad\qquad\qquad\qquad 0 \le x \le 1, \quad 0 \le \tau \le T, \\ u(x,0) = f(x), \quad 0 \le x \le 1, \end{cases}$$

where $b(0,\tau) = a(0,\tau) = a_x(0,\tau) = b(1,\tau) = a(1,\tau) = a_x(1,\tau) = 0$ and $a(x,\tau) \ge 0$. This problem can be approximated by

$$\begin{cases} \delta_\tau u_m^{n+1/2} = a_m^{n+1/2}\delta_x^2 u_m^{n+1/2} + b_m^{n+1/2}\delta_{0x} u_m^{n+1/2} + c_m^{n+1/2}u_m^{n+1/2} + g_m^{n+1/2}, \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 \le m \le M, \quad 0 \le n \le N-1, \\ u_m^0 = f(x_m), \qquad\qquad\qquad\qquad 0 \le m \le M. \end{cases} \tag{7.27}$$

Here,

$$\begin{aligned} \delta_\tau u_m^{n+1/2} &= \frac{u_m^{n+1} - u_m^n}{\Delta\tau}, \\ \delta_x^2 u_m^{n+1/2} &= \frac{1}{2}\left(\frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{\Delta x^2} + \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2}\right), \\ \delta_{0x} u_m^{n+1/2} &= \frac{1}{2}\left(\frac{u_{m+1}^{n+1} - u_{m-1}^{n+1}}{2\Delta x} + \frac{u_{m+1}^n - u_{m-1}^n}{2\Delta x}\right), \\ f_m^{n+1/2} &= \frac{1}{2}\left(f_m^{n+1} + f_m^n\right), \quad f \text{ being } u, a, b, c, g, \end{aligned}$$

and the same notation will be used for other functions in what follows. The truncation error of this scheme is $O(\Delta x^2) + O(\Delta\tau^2)$ everywhere; more accurately, it is in the form

$$P_m^{n+1/2}\Delta x^2 + R_m^{n+1/2}\Delta\tau^2 + O(\Delta x^4 + \Delta\tau^4),$$

where $P_m^{n+1/2}$ and $R_m^{n+1/2}$ denote the values of two functions $P(x,\tau)$ and $R(x,\tau)$ at $x = x_m$ and $\tau = \tau^{n+1/2}$. That is, the exact solution satisfies the following equation:

$$\begin{cases} \delta_\tau U_m^{n+1/2} = a_m^{n+1/2}\delta_x^2 U_m^{n+1/2} + b_m^{n+1/2}\delta_{0x} U_m^{n+1/2} + c_m^{n+1/2} U_m^{n+1/2} + g_m^{n+1/2} \\ \qquad + P_m^{n+1/2}\Delta x^2 + R_m^{n+1/2}\Delta\tau^2 + O(\Delta x^4 + \Delta\tau^4), \\ \qquad\qquad 0 \le m \le M, \quad 0 \le n \le N-1, \\ u_m^0 = f(x_m), \qquad 0 \le m \le M, \end{cases}$$

where U_m^n stands for $u(x_m, \tau^n)$. Suppose v_1 and v_2 are the solutions of the problems

$$\begin{cases} \dfrac{\partial v_1}{\partial \tau} = a(x,\tau)\dfrac{\partial^2 v_1}{\partial x^2} + b(x,\tau)\dfrac{\partial v_1}{\partial x} + c(x,\tau)v_1 + P(x,\tau), \\ \qquad\qquad 0 \le x \le 1, \quad 0 \le \tau \le T, \\ v_1(x,0) = 0, \qquad 0 \le x \le 1 \end{cases}$$

and

$$\begin{cases} \dfrac{\partial v_2}{\partial \tau} = a(x,\tau)\dfrac{\partial^2 v_2}{\partial x^2} + b(x,\tau)\dfrac{\partial v_2}{\partial x} + c(x,\tau)v_2 + R(x,\tau), \\ \qquad\qquad 0 \le x \le 1, \quad 0 \le \tau \le T, \\ v_2(x,0) = 0, \qquad 0 \le x \le 1, \end{cases}$$

respectively. Let $V_{1,m}^n$ and $V_{2,m}^n$ denote $v_1(x_m, \tau^n)$ and $v_2(x_m, \tau^n)$. Then,

$$\begin{cases} \delta_\tau V_{1,m}^{n+1/2} = a_m^{n+1/2}\delta_x^2 V_{1,m}^{n+1/2} + b_m^{n+1/2}\delta_{0x} V_{1,m}^{n+1/2} + c_m^{n+1/2} V_{1,m}^{n+1/2} + P_m^{n+1/2} \\ \qquad + O(\Delta x^2 + \Delta\tau^2), \qquad 0 \le m \le M, \quad 0 \le n \le N-1, \\ V_{1,m}^0 = 0, \qquad 0 \le m \le M, \end{cases}$$

and

$$\begin{cases} \delta_\tau V_{2,m}^{n+1/2} = a_m^{n+1/2}\delta_x^2 V_{2,m}^{n+1/2} + b_m^{n+1/2}\delta_{0x} V_{2,m}^{n+1/2} + c_m^{n+1/2} V_{2,m}^{n+1/2} + R_m^{n+1/2} \\ \qquad + O(\Delta x^2 + \Delta\tau^2), \qquad 0 \le m \le M, \quad 0 \le n \le N-1, \\ V_{2,m}^0 = 0, \qquad 0 \le m \le M. \end{cases}$$

Let us define

$$W_m^n = U_m^n - u_m^n - V_{1,m}^n \Delta x^2 - V_{2,m}^n \Delta\tau^2.$$

It is clear that W_m^n satisfies

$$\begin{cases} \delta_\tau W_m^{n+1/2} = a_m^{n+1/2}\delta_x^2 W_m^{n+1/2} + b_m^{n+1/2}\delta_{0x} W_m^{n+1/2} + c_m^{n+1/2} W_m^{n+1/2} \\ \qquad + O(\Delta x^4 + \Delta x^2\Delta\tau^2 + \Delta\tau^4), \quad 0 \le m \le M, \quad 0 \le n \le N-1, \\ W_m^0 = 0, \qquad 0 \le m \le M. \end{cases}$$

Because the scheme is stable with respect to the initial value and the nonhomogeneous term (see the paper [76] by Sun and the paper [79] by Sun, Yan, and Zhu for the details of the proof) and $O(\Delta x^2 \Delta\tau^2)$ can be expressed as $O(\Delta x^4 + \Delta\tau^4)$, we have

$$|U_m^n - u_m^n - V_{1,m}^n \Delta x^2 - V_{2,m}^n \Delta\tau^2| \le O(\Delta x^4 + \Delta\tau^4),$$

or we can write this relation as

$$u(x_m, \tau^n) - u_m^n(\Delta x, \Delta\tau) = v_1(x_m, \tau^n)\Delta x^2 + v_2(x_m, \tau^n)\Delta\tau^2 + O(\Delta x^4 + \Delta\tau^4),$$

that is,

$$\begin{aligned} u_m^n(\Delta x, \Delta\tau) = {} & u(x_m, \tau^n) - v_1(x_m, \tau^n)\Delta x^2 - v_2(x_m, \tau^n)\Delta\tau^2 \\ & + O(\Delta x^4 + \Delta\tau^4). \end{aligned} \tag{7.28}$$

Here, we write u_m^n as $u_m^n(\Delta x, \Delta\tau)$ in order to indicate that the approximate solution is obtained on a mesh with mesh sizes Δx and $\Delta\tau$. For this case, the error of a numerical solution is in the form

$$v_1(x_m, \tau^n)\Delta x^2 + v_2(x_m, \tau^n)\Delta\tau^2 + O(\Delta x^4 + \Delta\tau^4),$$

which has the same form as the truncation error given above. Similarly, if the truncation error of a numerical scheme, including the algorithms for boundary conditions, is

$$P\Delta x^2 + Q\Delta x\Delta\tau + R\Delta\tau^2 + O(\Delta\tau^3),$$

i.e., the scheme is second order and stable, then the numerical solution can be expressed as

$$\begin{aligned} u_m^n(\Delta x, \Delta\tau) = {} & u(x_m, \tau^n) - v_1(x_m, \tau^n)\Delta x^2 - v_{12}(x_m, \tau^n)\Delta x\Delta\tau \\ & - v_2(x_m, \tau^n)\Delta x^2 + O(\Delta\tau^3), \end{aligned} \tag{7.29}$$

where $O(\Delta\tau^3)$ means $O(\Delta x^3 + \Delta x^2\Delta\tau + \Delta x\Delta\tau^2 + \Delta\tau^3)$ for simplicity.

Here, the approximate value is given only at the nodes. Now let us generate a function defined on the domain $[0, 1] \times [0, T]$ by some type of interpolation. We assume that the interpolation function generated from the values on the nodes by an interpolation method is an approximation to $f(x, \tau)$ with an error of $O(\Delta\tau^3)$ for any smooth enough function $f(x, \tau)$. For example, if we use quadratic interpolation, then the interpolation function generated has such a property. Let $u(x, \tau; \Delta x, \Delta\tau)$ denote such a function generated by $u(x_m, \tau^n; \Delta x, \Delta\tau)$. Because $u(x_m, \tau^n; \Delta x, \Delta\tau)$ consists of $u(x_m, \tau^n) - v_1(x_m, \tau^n)\Delta x^2 - v_{12}(x_m, \tau^n)\Delta x\Delta\tau - v_2(x_m, \tau^n)\Delta\tau^2$ and $O(\Delta\tau^3)$, the interpolation function also has two parts. One part is the interpolation function generated by $u(x_m, \tau^n) - v_1(x_m, \tau^n)\Delta x^2 - v_{12}(x_m, \tau^n)\Delta x\Delta\tau - v_2(x_m, \tau^n)\Delta\tau^2$, which we call $u_1(x, \tau; \Delta x, \Delta\tau)$. The other part is generated by the term $O(\Delta\tau^3)$, which is denoted by $u_2(x, \tau; \Delta x, \Delta\tau)$. Clearly,

$$u_1(x, \tau; \Delta x, \Delta\tau) - u(x, \tau) + v_1(x, \tau)\Delta x^2 + v_{12}(x, \tau)\Delta x\Delta\tau + v_2(x, \tau)\Delta\tau^2$$

is a term of $O(\Delta\tau^3)$. The function $u_2(x, \tau; \Delta x, \Delta\tau)$ is also a term of $O(\Delta\tau^3)$. Consequently, we have

$$\begin{aligned}u(x,\tau;\Delta x,\Delta\tau) &= u_1(x,\tau;\Delta x,\Delta\tau)+u_2(x,\tau;\Delta x,\Delta\tau)\\ &= u(x,\tau)-v_1(x,\tau)\Delta x^2-v_{12}(x,\tau)\Delta x\Delta\tau-v_2(x,\tau)\Delta\tau^2\\ &\quad +O(\Delta\tau^3).\end{aligned}$$

In this case, we can use the following technique to eliminate the error of $O(\Delta x^2+\Delta x\Delta\tau+\Delta\tau^2)$ if we have numerical solutions on a mesh with mesh sizes Δx and $\Delta\tau$ and on a mesh with mesh sizes $2\Delta x$ and $2\Delta\tau$. Let us consider a linear combination of the solutions on the two different meshes, which are denoted by $u(x,\tau;\Delta x,\Delta\tau)$ and $u(x,\tau;\,2\Delta x,2\Delta\tau)$:

$$\begin{aligned}&(1-d)\times u(x,\tau;\Delta x,\Delta\tau)+d\times u(x,\tau;2\Delta x,2\Delta\tau)\\ &= u(x,\tau)-v_1(x,\tau)(1-d+4d)\Delta x^2-v_{12}(x,\tau)(1-d+4d)\Delta x\Delta\tau\\ &\quad -v_2(x,\tau)(1-d+4d)\Delta\tau^2+O(\Delta\tau^3).\end{aligned}$$

If we choose d such that $1-d+4d=0$, that is, $d=-\dfrac{1}{3}$, then

$$(1-d)\times u(x,\tau;\Delta x,\Delta\tau)+d\times u(x,\tau;2\Delta x,2\Delta\tau)=u(x,\tau)+O(\Delta\tau^3).$$

Therefore,

$$\frac{1}{3}[4u(x,\tau;\Delta x,\Delta\tau)-u(x,\tau;2\Delta x,2\Delta\tau)] \tag{7.30}$$

is an approximate to $u(x,\tau)$ with an error of $O(\Delta\tau^3)$.

However, for the approximation (7.27), the expression of the numerical solution is in the form (7.28), and the extrapolation formula of numerical solutions (7.30) gives an approximation to $u(x,\tau)$ with an error of $O(\Delta\tau^4)$. This is a special case. Generally speaking, if for a second-order scheme we have three solutions $u_m^n(\Delta x,\Delta\tau), u_m^n(2\Delta x,2\Delta\tau)$, and $u_m^n(4\Delta x,4\Delta\tau)$, then we can have an approximation with an error of $O(\Delta\tau^4)$. In order to do that, we first generate an interpolation function from the values at these nodes and require the interpolation with an error of $O(\Delta\tau^4)$. This can be done, for example, by cubic interpolation. Let $u(x,\tau;\Delta x,\Delta\tau)$, $u(x,\tau;2\Delta x,2\Delta\tau)$, and $u(x,\tau;4\Delta x,4\Delta\tau)$ represent these functions. Then, consider a linear combination of them:

$$(1-d_1-d_2)u(x,\tau;\Delta x,\Delta\tau)+d_1u(x,\tau;2\Delta x,2\Delta\tau)+d_2u(x,\tau;4\Delta x,4\Delta\tau).$$

If we choose $d_1$and d_2 such that

$$\begin{cases}1-d_1-d_2+2^2d_1+4^2d_2=0,\\ 1-d_1-d_2+2^3d_1+4^3d_2=0,\end{cases}$$

which gives

$$\begin{cases}d_1=-\dfrac{12}{21},\\ d_2=\dfrac{1}{21},\end{cases}$$

then all the terms of $O(\Delta\tau^2)$ and the terms of $O(\Delta\tau^3)$ in

$$(1-d_1-d_2)u(x,\tau;\Delta x,\Delta\tau)+d_1u(x,\tau;2\Delta x,2\Delta\tau)+d_2u(x,\tau;4\Delta x,4\Delta\tau)$$

are eliminated. Therefore

$$\frac{1}{21}[32u(x,\tau;\Delta x,\Delta\tau)-12u(x,\tau;2\Delta x,2\Delta\tau)+u(x,\tau;4\Delta x,4\Delta\tau)] \qquad (7.31)$$

gives an approximation to $u(x,\tau)$ with an error of $O(\Delta\tau^4)$ for any second-order scheme.

Here, we need to point out that in order to obtain an approximate solution with an error of $O(\Delta\tau^3)$, it is not necessary for both $\Delta x_1/\Delta x_2$ and $\Delta\tau_1/\Delta\tau_2$ to equal two, where $\Delta x_1, \Delta\tau_1$ are mesh sizes for one mesh and $\Delta x_2, \Delta\tau_2$ for the other. For example, if we have a solution on a 12×16 mesh and a solution on a 9×12 mesh, then we still can obtain an approximate solution with an error of $O(\Delta\tau^3)$ by using extrapolation. Furthermore, if there exist solutions on 15×20, 12×16, and 9×12 meshes, then we can have an approximate solution with an error of $O(\Delta\tau^4)$ by using extrapolation. These are left as a problem for the reader to prove. Generally speaking, when a scheme has an error of Δx^{k_1} and $\Delta\tau^{k_2}$ and we know solutions on two meshes, the extrapolation can be used if $\dfrac{\Delta x_1^{k_1}}{\Delta\tau_1^{k_2}}=\dfrac{\Delta x_2^{k_1}}{\Delta\tau_2^{k_2}}$, where Δx_i and $\Delta\tau_i$, $i=1,2$, are mesh sizes used in order to obtain the two solutions. For example, if $k_1=2$ and $k_2=1$, then when solutions on a 20×20 mesh and a 40×80 mesh are obtained, this technique can also be used because $\dfrac{\left(\frac{1}{20}\right)^2}{\frac{1}{20}}=\dfrac{\left(\frac{1}{40}\right)^2}{\frac{1}{80}}$ (see Problem 16).

The technique of generating more accurate results by combining several numerical results, which is similar to Richardson's extrapolation in numerical methods for ordinary differential equations, is referred to as the extrapolation technique of numerical solutions in next few chapters. Finally we need to point out that this technique works if the solution is smooth, but may not work if the solution is not smooth enough.

7.4 Two-Dimensional Degenerate Parabolic Equations

Generally speaking, the coefficients of PDEs are variable, and so the difference equations also have variable coefficients. For such a case, the theoretical analysis of numerical methods is more complicated. In this section, for some type of two-dimensional degenerate parabolic equations and for a special but popular scheme, a complete theoretical analysis of numerical methods is given.

Consider the following two-dimensional degenerate parabolic partial differential equation:

$$\frac{\partial u}{\partial \tau} = a_{11}(x,y,\tau)\frac{\partial^2 u}{\partial x^2} + 2a_{12}(x,y,\tau)\frac{\partial^2 u}{\partial x \partial y} + a_{22}(x,y,\tau)\frac{\partial^2 u}{\partial y^2} + b_1(x,y,\tau)\frac{\partial u}{\partial x}$$
$$+b_2(x,y,\tau)\frac{\partial u}{\partial y}+c(x,y,\tau)u+g(x,y,\tau), \quad (x,y)\in\Omega,\ 0\le\tau\le T, \tag{7.32}$$

with the initial condition

$$u(x,y,0) = f(x,y), \quad (x,y) \in \Omega, \tag{7.33}$$

where

$$\Omega = \{(x,y) \mid x_l \le x \le x_u, y_l \le y \le y_u\},$$

$$a_{11}(x,y,\tau)\Big|_{x=x_l \text{ or } x_u} = 0, \qquad a_{22}(x,y,\tau)\Big|_{y=y_l \text{ or } y_u} = 0, \tag{7.34}$$

$$b_1(x,y,\tau)\Big|_{x=x_l \text{ or } x_u} = 0, \qquad b_2(x,y,\tau)\Big|_{y=y_l \text{ or } y_u} = 0, \tag{7.35}$$

$$\frac{\partial a_{11}(x,y,\tau)}{\partial x}\Big|_{x=x_l \text{ or } x_u} = 0, \qquad \frac{\partial a_{22}(x,y,\tau)}{\partial y}\Big|_{y=y_l \text{ or } ,y_u} = 0, \tag{7.36}$$

and the matrix

$$\begin{pmatrix} a_{11}(x,y,\tau) & a_{12}(x,y,\tau) \\ a_{12}(x,y,\tau) & a_{22}(x,y,\tau) \end{pmatrix}$$

is semi-positive (nonnegative); i.e., for any $X \in \mathcal{R}$ and $Y \in \mathcal{R}$, we have

$$a_{11}(x,y,\tau)X^2 + 2a_{12}(x,y,\tau)XY + a_{22}(x,y,\tau)Y^2 \ge 0. \tag{7.37}$$

The matrix of the coefficients of second derivatives is semi-positive, so $a_{12}^2 \le a_{11}a_{22}$. Thus, when $a_{11} = 0$ or $a_{22} = 0$, we have $a_{12} = 0$. Thus, from the expression (7.34), we have

$$a_{12}(x,y,\tau)\Big|_{x=x_l \text{ or } x_u} = 0, \qquad a_{12}(x,y,\tau)\Big|_{y=y_l \text{ or } y_u} = 0. \tag{7.38}$$

Taking the partial derivative of the first and second relations in the result (7.38) with respect to y and x, respectively, we can further have

$$\frac{\partial a_{12}(x,y,\tau)}{\partial y}\Big|_{x=x_l \text{ or } x_u} = 0, \qquad \frac{\partial a_{12}(x,y,\tau)}{\partial x}\Big|_{y=y_l \text{ or } y_u} = 0. \tag{7.39}$$

Denote

$$c_1 = \max_{(x,y,\tau)\in\Omega\times[0,T]} \left|\frac{\partial^2 a_{11}(x,y,\tau)}{\partial x^2}\right|, \qquad c_2 = \max_{(x,y,\tau)\in\Omega\times[0,T]} \left|\frac{\partial^2 a_{12}(x,y,\tau)}{\partial x \partial y}\right|,$$

$$c_3 = \max_{(x,y,\tau)\in\Omega\times[0,T]} \left|\frac{\partial^2 a_{22}(x,y,\tau)}{\partial y^2}\right|, \qquad c_4 = \max_{(x,y,\tau)\in\Omega\times[0,T]} \left|\frac{\partial b_1(x,y,\tau)}{\partial x}\right|,$$

$$c_5 = \max_{(x,y,\tau)\in\Omega\times[0,T]} \left|\frac{\partial b_2(x,y,\tau)}{\partial y}\right|, \qquad c_6 = \max_{(x,y,\tau)\in\Omega\times[0,T]} |c(x,y,\tau)|,$$

and set

$$c = c_1 + 2c_2 + c_3 + c_4 + c_5 + 2c_6. \tag{7.40}$$

In Sect. 2.4.3, for more general problems we have obtained the following inequality:

$$\begin{aligned}\iint_\Omega u^2(x,y,\tau)dxdy \le e^{\bar{c}T}\Bigg[&\iint_\Omega f^2(x,y)dxdy \\ &+ \int_0^\tau \Big(\iint_\Omega g^2(x,y,s)dxdy\Big)ds\Bigg], \quad 0 \le \tau \le T,\end{aligned}$$

where $\bar{c}$ is a constant determined by the bounds of the coefficients of the PDE and their derivatives. Of course, for the problem here, such an inequality holds. In this section, we are going to prove that for the numerical solutions obtained by a special but popular scheme, such an inequality still holds.

7.4.1 The Crank–Nicolson Difference Scheme and a Preliminary Lemma

Take three positive integers M, N, and K. Set $h_1 = (x_u - x_l)/M, h_2 = (y_u - y_l)/N, \Delta\tau = T/K$, and denote

$$\begin{aligned}&x_m = x_l + mh_1, \quad 0 \le m \le M,\\ &y_n = y_l + nh_2, \quad 0 \le n \le N,\\ &\tau^k = k\Delta\tau, \quad 0 \le k \le K,\\ &\Omega_h = \{(x_m, y_n) \mid 0 \le m \le M, 0 \le n \le N\},\\ &\Omega_{\Delta\tau} = \{\tau^k \mid 0 \le k \le K\}.\end{aligned}$$

Let $\mathcal{V} = \{u \mid u = \{u_{mn}, 0 \le m \le M, 0 \le n \le N\}\}$ be the grid function space on Ω_h. If $u \in \mathcal{V}$, we introduce the following notation:

$$\begin{aligned}\delta_x u_{m+\frac{1}{2},n} &= \tfrac{1}{h_1}(u_{m+1,n} - u_{mn}), &\quad \Delta_x u_{mn} &= \tfrac{1}{2h_1}(u_{m+1,n} - u_{m-1,n}),\\ \delta_y u_{m,n+\frac{1}{2}} &= \tfrac{1}{h_2}(u_{m,n+1} - u_{mn}), &\quad \Delta_y u_{mn} &= \tfrac{1}{2h_2}(u_{m,n+1} - u_{m,n-1}),\\ \delta_x^2 u_{mn} &= \tfrac{1}{h_1^2}(u_{m+1,n} - 2u_{mn} + u_{m-1,n}),\\ \delta_y^2 u_{mn} &= \tfrac{1}{h_2^2}(u_{m,n+1} - 2u_{mn} + u_{m,n-1}).\end{aligned}$$

It is obvious that

$$\begin{aligned}\Delta_x u_{mn} &= \frac{1}{2}(\delta_x u_{m+\frac{1}{2},n} + \delta_x u_{m-\frac{1}{2},n}),\ \delta_x^2 u_{mn} = \frac{1}{h_1}(\delta_x u_{m+\frac{1}{2},n} - \delta_x u_{m-\frac{1}{2},n}),\\ \Delta_y u_{mn} &= \frac{1}{2}(\delta_y u_{m,n+\frac{1}{2}} + \delta_y u_{m,n-\frac{1}{2}}),\ \delta_y^2 u_{mn} = \frac{1}{h_2}(\delta_y u_{m,n+\frac{1}{2}} - \delta_y u_{m,n-\frac{1}{2}}).\end{aligned}$$

For any $u \in \mathcal{V}$, and $v \in \mathcal{V}$, their inner product is defined by

$$(u,v) = h_1 h_2 \left[\sum_{m=1}^{M-1}\sum_{n=1}^{N-1} u_{mn}v_{mn} + \frac{1}{2}\sum_{m=1}^{M-1}(u_{m0}v_{m0} + u_{mN}v_{mN}) \right.$$
$$\left. +\frac{1}{2}\sum_{n=1}^{N-1}(u_{0n}v_{0n} + u_{Mn}v_{Mn}) + \frac{1}{4}(u_{00}v_{00} + u_{M0}v_{M0} + u_{0N}v_{0N} + u_{MN}v_{MN})\right] \tag{7.41}$$

and the norm of a grid function is defined by

$$\|u\| = \sqrt{(u,u)}.$$

It is also obvious that the definition of the inner product can also be written in another form:

$$(u,v) = \frac{1}{4}h_1 h_2 \sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\Big(u_{mn}v_{mn} + u_{m+1,n}v_{m+1,n}$$
$$+u_{m,n+1}v_{m,n+1} + u_{m+1,n+1}v_{m+1,n+1}\Big). \tag{7.42}$$

We also define the grid function U on $\Omega_h \times \Omega_{\Delta\tau}$ as follows:

$$U_{mn}^k = u(x_m, y_n, \tau^k), \quad 0 \le m \le M, \quad 0 \le n \le N, \quad 0 \le k \le K.$$

In what follows, we use the following notations:

$$U_{mn}^{k+\frac{1}{2}} = \frac{1}{2}(U_{mn}^{k+1} + U_{mn}^k), \quad \tau^{k+\frac{1}{2}} = \frac{1}{2}(\tau^k + \tau^{k+1})$$

and

$$\begin{aligned}
&(a_{11})_{mn}^{k+\frac{1}{2}} = a_{11}(x_m, y_n, \tau^{k+\frac{1}{2}}), && (a_{12})_{mn}^{k+\frac{1}{2}} = a_{12}(x_m, y_n, \tau^{k+\frac{1}{2}}),\\
&(a_{22})_{mn}^{k+\frac{1}{2}} = a_{22}(x_m, y_n, \tau^{k+\frac{1}{2}}), && (b_1)_{mn}^{k+\frac{1}{2}} = b_1(x_m, y_n, \tau^{k+\frac{1}{2}}),\\
&(b_2)_{mn}^{k+\frac{1}{2}} = b_2(x_m, y_n, \tau^{k+\frac{1}{2}}), && c_{mn}^{k+\frac{1}{2}} = c(x_m, y_n, \tau^{k+\frac{1}{2}}),\\
&g_{mn}^{k+\frac{1}{2}} = g(x_m, y_n, \tau^{k+\frac{1}{2}}), && f_{mn} = f(x_m, y_n).
\end{aligned}$$

Suppose problem (7.32)–(7.33) has a smooth solution $u(x, y, \tau)$. Applying the Taylor expansion, we can obtain

$$\begin{aligned}
\frac{1}{\Delta\tau}(U_{mn}^{k+1} - U_{mn}^k) &= (a_{11})_{mn}^{k+\frac{1}{2}}\delta_x^2 U_{mn}^{k+\frac{1}{2}} + 2(a_{12})_{mn}^{k+\frac{1}{2}}\Delta_x\Delta_y U_{mn}^{k+\frac{1}{2}}\\
&+(a_{22})_{mn}^{k+\frac{1}{2}}\delta_y^2 U_{mn}^{k+\frac{1}{2}} + (b_1)_{mn}^{k+\frac{1}{2}}\Delta_x U_{mn}^{k+\frac{1}{2}} + (b_2)_{mn}^{k+\frac{1}{2}}\Delta_y U_{mn}^{k+\frac{1}{2}}\\
&+c_{mn}^{k+\frac{1}{2}} U_{mn}^{k+\frac{1}{2}} + g_{mn}^{k+\frac{1}{2}} + R_{mn}^{k+\frac{1}{2}},\\
&\quad 0 \le m \le M, \quad 0 \le n \le N, \quad 0 \le k \le K-1
\end{aligned} \tag{7.43}$$

and there exists a constant c_0 such that

$$\begin{aligned}|R_{mn}^{k+\frac{1}{2}}| &\le c_0(h_1^2+h_2^2+\Delta\tau^2),\\ 0\le m\le M,\quad 0&\le n\le N,\quad 0\le k\le K-1.\end{aligned}\tag{7.44}$$

Omitting the small term $R_{mn}^{k+\frac{1}{2}}$ in the expression (7.43) and writing down the initial condition on Ω_h:

$$U_{mn}^0=f_{mn},\quad 0\le m\le M,\quad 0\le n\le N,\tag{7.45}$$

we have for the problem (7.32)–(7.33) the following difference scheme:

$$\begin{aligned}\frac{1}{\Delta\tau}(u_{mn}^{k+1}-u_{mn}^k) &= (a_{11})_{mn}^{k+\frac{1}{2}}\delta_x^2 u_{mn}^{k+\frac{1}{2}}+2(a_{12})_{mn}^{k+\frac{1}{2}}\Delta_x\Delta_y u_{mn}^{k+\frac{1}{2}}\\ &\quad+(a_{22})_{mn}^{k+\frac{1}{2}}\delta_y^2 u_{mn}^{k+\frac{1}{2}}+(b_1)_{mn}^{k+\frac{1}{2}}\Delta_x u_{mn}^{k+\frac{1}{2}}+(b_2)_{mn}^{k+\frac{1}{2}}\Delta_y u_{mn}^{k+\frac{1}{2}}+c_{mn}^{k+\frac{1}{2}}u_{mn}^{k+\frac{1}{2}}\\ &\quad+g_{mn}^{k+\frac{1}{2}},\quad 0\le m\le M,\quad 0\le n\le N,\quad 0\le k\le K-1,\end{aligned}\tag{7.46}$$

$$u_{mn}^0=f_{mn},\quad 0\le m\le M,\quad 0\le n\le N.\tag{7.47}$$

The following lemma will be used for the analysis of the difference scheme.

Lemma 7.1. *Let $u\in\mathcal{V}$. Then we have*

$$\begin{aligned}&\left(a_{11}^{k+\frac{1}{2}}\delta_x^2 u,u\right)+2\left(a_{12}^{k+\frac{1}{2}}\Delta_x\Delta_y u,u\right)+\left(a_{22}^{k+\frac{1}{2}}\delta_y^2 u,u\right)\\ &+\left(b_1^{k+\frac{1}{2}}\Delta_x u,u\right)+\left(b_2^{k+\frac{1}{2}}\Delta_y u,u\right)+\left(c^{k+\frac{1}{2}}u,u\right)\le\frac{c}{2}\|u\|^2,\end{aligned}\tag{7.48}$$

where c is defined by the expression (7.40).

Section 7.4.2 is devoted to the proof of this lemma.

7.4.2 ‡The Proof of the Preliminary Lemma

We will estimate each term in the inequality (7.48). For simplicity, we omit the superscript.

Proposition 7.1 *For $\left(a_{11}\delta_x^2 u,u\right)$ and $\left(a_{22}\delta_y^2 u,u\right)$, we have the following inequalities:*

$$
\begin{aligned}
B_1 &\equiv \left(a_{11}\delta_x^2 u, u\right) \\
&\le -h_1 h_2 \left[\sum_{m=1}^{M-1}\sum_{n=1}^{N-1}(a_{11})_{mn}\frac{(\delta_x u_{m-\frac{1}{2},n})^2+(\delta_x u_{m+\frac{1}{2},n})^2}{2}\right. \\
&\qquad +\frac{1}{2}\sum_{m=1}^{M-1}(a_{11})_{m0}\frac{(\delta_x u_{m-\frac{1}{2},0})^2+(\delta_x u_{m+\frac{1}{2},0})^2}{2} \\
&\qquad \left.+\frac{1}{2}\sum_{m=1}^{M-1}(a_{11})_{mN}\frac{(\delta_x u_{m-\frac{1}{2},N})^2+(\delta_x u_{m+\frac{1}{2},N})^2}{2}\right] + \frac{1}{2}c_1\|u\|^2 \\
&\le -h_1 h_2 \sum_{m=1}^{M-1}\sum_{n=1}^{N-1}\left[(a_{11})_{mn}\frac{(\delta_x u_{m-\frac{1}{2},n})^2+(\delta_x u_{m+\frac{1}{2},n})^2}{2}\right] + \frac{1}{2}c_1\|u\|^2.
\end{aligned}
\tag{7.49}
$$

and

$$
\begin{aligned}
B_3 &\equiv \left(a_{22}\delta_y^2 u, u\right) \\
&\le -h_1 h_2 \left[\sum_{m=1}^{M-1}\sum_{n=1}^{N-1}(a_{22})_{mn}\frac{(\delta_y u_{m,n-\frac{1}{2}})^2+(\delta_y u_{m,n+\frac{1}{2}})^2}{2}\right. \\
&\qquad +\frac{1}{2}\sum_{n=1}^{N-1}(a_{22})_{0n}\frac{(\delta_y u_{0,n-\frac{1}{2}})^2+(\delta_y u_{0,n+\frac{1}{2}})^2}{2} \\
&\qquad \left.+\frac{1}{2}\sum_{n=1}^{N-1}(a_{22})_{Mn}\frac{(\delta_y u_{M,n-\frac{1}{2}})^2+(\delta_y u_{M,n+\frac{1}{2}})^2}{2}\right] + \frac{1}{2}c_3\|u\|^2 \\
&\le -h_1 h_2 \sum_{m=1}^{M-1}\sum_{n=1}^{N-1}\left[(a_{22})_{mn}\frac{(\delta_y u_{m,n-\frac{1}{2}})^2+(\delta_y u_{m,n+\frac{1}{2}})^2}{2}\right] + \frac{1}{2}c_3\|u\|^2.
\end{aligned}
\tag{7.50}
$$

Proof. Because $(a_{11})_{0n} = (a_{11})_{Mn} = 0$ for $n = 0, 1, \cdots, N$, some terms in the inner product are zero. Thus, the expression of $\left(a_{11}\delta_x^2 u, u\right)$ is

$$
\begin{aligned}
B_1 = \left(a_{11}\delta_x^2 u, u\right) = h_1 h_2 &\left[\sum_{m=1}^{M-1}\sum_{n=1}^{N-1}(a_{11})_{mn}\;\delta_x^2 u_{mn}\;u_{mn}\right. \\
&\left.+\frac{1}{2}\sum_{m=1}^{M-1}(a_{11})_{m0}\;\delta_x^2 u_{m0}\;u_{m0} + \frac{1}{2}\sum_{m=1}^{M-1}(a_{11})_{mN}\;\delta_x^2 u_{mn}\;u_{mN}\right].
\end{aligned}
\tag{7.51}
$$

Averaging the following two equalities:

$$\begin{aligned}
& h_1 \sum_{m=1}^{M-1} (a_{11})_{mn}\, \delta_x^2 u_{mn}\, u_{mn} \\
&= \sum_{m=1}^{M-1} (a_{11})_{mn} (\delta_x u_{m+\frac{1}{2},n} - \delta_x u_{m-\frac{1}{2},n}) u_{mn} \\
&= \sum_{m=1}^{M-1} (a_{11})_{mn}\, \delta_x u_{m+\frac{1}{2},n}\, u_{mn} - \sum_{m=0}^{M-2} (a_{11})_{m+1,n}\, \delta_x u_{m+\frac{1}{2},n}\, u_{m+1,n} \\
&= \sum_{m=0}^{M-1} (a_{11})_{mn}\, \delta_x u_{m+\frac{1}{2},n}\, u_{mn} - \sum_{m=0}^{M-1} (a_{11})_{m+1,n}\, \delta_x u_{m+\frac{1}{2},n}\, u_{m+1,n} \\
&= \sum_{m=0}^{M-1} (a_{11})_{mn}\, \delta_x u_{m+\frac{1}{2},n}\, (u_{mn} - u_{m+1,n}) \\
&\quad + \sum_{m=0}^{M-1} [(a_{11})_{mn} - (a_{11})_{m+1,n}]\, \delta_x u_{m+\frac{1}{2},n}\, u_{m+1,n} \\
&= -h_1 \sum_{m=1}^{M-1} (a_{11})_{mn} (\delta_x u_{m+\frac{1}{2},n})^2 - h_1 \sum_{m=0}^{M-1} (\delta_x a_{11})_{m+\frac{1}{2},n}\, \delta_x u_{m+\frac{1}{2},n}\, u_{m+1,n}
\end{aligned}$$

and

$$\begin{aligned}
& h_1 \sum_{m=1}^{M-1} (a_{11})_{mn}\, \delta_x^2 u_{mn}\, u_{mn} \\
&= \sum_{m=1}^{M-1} (a_{11})_{mn} (\delta_x u_{m+\frac{1}{2},n} - \delta_x u_{m-\frac{1}{2},n}) u_{mn} \\
&= \sum_{m=2}^{M} (a_{11})_{m-1,n}\, \delta_x u_{m-\frac{1}{2},n}\, u_{m-1,n} - \sum_{m=1}^{M-1} (a_{11})_{mn}\, \delta_x u_{m-\frac{1}{2},n}\, u_{mn} \\
&= \sum_{m=1}^{M} (a_{11})_{m-1,n}\, \delta_x u_{m-\frac{1}{2},n}\, u_{m-1,n} - \sum_{m=1}^{M} (a_{11})_{mn}\, \delta_x u_{m-\frac{1}{2},n}\, u_{mn} \\
&= \sum_{m=1}^{M} (a_{11})_{mn}\, \delta_x u_{m-\frac{1}{2},n}\, (u_{m-1,n} - u_{mn}) \\
&\quad + \sum_{m=1}^{M} [(a_{11})_{m-1,n} - (a_{11})_{mn}]\, \delta_x u_{m-\frac{1}{2},n}\, u_{m-1,n} \\
&= -h_1 \sum_{m=1}^{M-1} (a_{11})_{mn} (\delta_x u_{m-\frac{1}{2},n})^2 - h_1 \sum_{m=0}^{M-1} (\delta_x a_{11})_{m+\frac{1}{2},n}\, \delta_x u_{m+\frac{1}{2},n}\, u_{mn},
\end{aligned}$$

we have

$$\begin{aligned}
& h_1 \sum_{m=1}^{M-1} (a_{11})_{mn}\, \delta_x^2 u_{mn}\, u_{mn} \\
= & -h_1 \sum_{m=1}^{M-1} (a_{11})_{mn} \frac{(\delta_x u_{m-\frac{1}{2},n})^2 + (\delta_x u_{m+\frac{1}{2},n})^2}{2} \\
& -h_1 \sum_{m=0}^{M-1} (\delta_x a_{11})_{m+\frac{1}{2},n}\, \delta_x u_{m+\frac{1}{2},n}\, u_{m+\frac{1}{2},n} \\
= & -h_1 \sum_{m=1}^{M-1} (a_{11})_{mn} \frac{(\delta_x u_{m-\frac{1}{2},n})^2 + (\delta_x u_{m+\frac{1}{2},n})^2}{2} \\
& -\frac{1}{2} \sum_{m=0}^{M-1} (\delta_x a_{11})_{m+\frac{1}{2},n} \left(u_{m+1,n}^2 - u_{m,n}^2\right) \\
= & -h_1 \sum_{m=1}^{M-1} (a_{11})_{mn} \frac{(\delta_x u_{m-\frac{1}{2},n})^2 + (\delta_x u_{m+\frac{1}{2},n})^2}{2} \\
& +\frac{1}{2} \left[\sum_{m=1}^{M-1} \left((\delta_x a_{11})_{m+\frac{1}{2},n} - (\delta_x a_{11})_{m-\frac{1}{2},n} \right) u_{mn}^2 \right. \\
& \qquad \left. + (\delta_x a_{11})_{\frac{1}{2},n} u_{0n}^2 - (\delta_x a_{11})_{M-\frac{1}{2},n} u_{Mn}^2 \right] \\
\le & -h_1 \sum_{m=1}^{M-1} (a_{11})_{mn} \frac{(\delta_x u_{m-\frac{1}{2},n})^2 + (\delta_x u_{m+\frac{1}{2},n})^2}{2} \\
& +\frac{1}{2} c_1 h_1 \left(\frac{1}{2} u_{0n}^2 + \sum_{m=1}^{M-1} u_{mn}^2 + \frac{1}{2} u_{Mn}^2 \right).
\end{aligned}$$

Here we have used the relations

$$\left| (\delta_x a_{11})_{m+\frac{1}{2},n} - (\delta_x a_{11})_{m-\frac{1}{2},n} \right| \le c_1 h_1,$$

$$|(\delta_x a_{11})_{\frac{1}{2},n}| \le \frac{1}{2} c_1 h_1, \quad |(\delta_x a_{11})_{M-\frac{1}{2},n}| \le \frac{1}{2} c_1 h_1,$$

which hold because of

$$c_1 = \max_{(x,y,\tau)\in\Omega\times[0,T]} \left| \frac{\partial^2 a_{11}(x,y,\tau)}{\partial x^2} \right|_t \quad \text{and} \quad \left. \frac{\partial a_{11}(x,y,\tau)}{\partial x} \right|_{x=x_l \text{ or } x_u} = 0.$$

Inserting the above equality into the equality (7.51), we obtain the inequality (7.49).

It is clear that for the second inequality in Proposition 7.1, the proof is almost the same as the proof for the first one. The concrete proof is omitted here. ∎

Proposition 7.2

$$\begin{aligned}
&B_2 \equiv (a_{12}\Delta_x\Delta_y u, u) \\
\le& -\frac{1}{4}h_1h_2\sum_{m=1}^{M-1}\sum_{n=1}^{N-1}(a_{12})_{mn}\Big[\delta_x u_{m+\frac{1}{2},n}\ \delta_y u_{m,n-\frac{1}{2}} + \delta_x u_{m-\frac{1}{2},n}\ \delta_y u_{m,n-\frac{1}{2}} \\
&+\delta_x u_{m+\frac{1}{2},n}\ \delta_y u_{m,n+\frac{1}{2}} + \delta_x u_{m-\frac{1}{2},n}\ \delta_y u_{m,n+\frac{1}{2}}\Big] + \frac{1}{2}c_2\|u\|^2. \qquad (7.52)
\end{aligned}$$

Proof. Because $a_{12} = 0$ on all the boundary points, the expression of $(a_{12}\Delta_x\Delta_y u, u)$ can be written as follows:

$$\begin{aligned}
B_2 &= h_1h_2\sum_{m=1}^{M-1}\sum_{n=1}^{N-1}(a_{12})_{mn}(\Delta_x\Delta_y u)_{mn}u_{mn} \\
&= \frac{1}{4}h_1h_2\sum_{m=1}^{M-1}\sum_{n=1}^{N-1}(a_{12})_{mn}\Big(\delta_x\delta_y u_{m-\frac{1}{2},n-\frac{1}{2}} + \delta_x\delta_y u_{m+\frac{1}{2},n-\frac{1}{2}} \\
&\qquad +\delta_x\delta_y u_{m-\frac{1}{2},n+\frac{1}{2}} + \delta_x\delta_y u_{m+\frac{1}{2},n+\frac{1}{2}}\Big)u_{mn} \\
&= \frac{1}{4}\Big[h_1h_2\sum_{m=1}^{M-1}\sum_{n=1}^{N-1}(a_{12})_{mn}\ \delta_x\delta_y u_{m-\frac{1}{2},n-\frac{1}{2}}\ u_{mn} \\
&\quad +h_1h_2\sum_{m=1}^{M-1}\sum_{n=1}^{N-1}(a_{12})_{mn}\ \delta_x\delta_y u_{m+\frac{1}{2},n-\frac{1}{2}}\ u_{mn} \\
&\quad +h_1h_2\sum_{m=1}^{M-1}\sum_{n=1}^{N-1}(a_{12})_{mn}\ \delta_x\delta_y u_{m-\frac{1}{2},n+\frac{1}{2}}\ u_{mn} \\
&\quad +h_1h_2\sum_{m=1}^{M-1}\sum_{n=1}^{N-1}(a_{12})_{mn}\ \delta_x\delta_y u_{m+\frac{1}{2},n+\frac{1}{2}}\ u_{mn}\Big] \\
&\equiv \frac{1}{4}(B_{21}+B_{22}+B_{23}+B_{24}). \qquad (7.53)
\end{aligned}$$

For B_{21}, we have

$$\begin{aligned}
B_{21} &= h_1h_2\sum_{m=1}^{M-1}\sum_{n=1}^{N-1}(a_{12})_{mn}\ \delta_x\delta_y u_{m-\frac{1}{2},n-\frac{1}{2}}\ u_{mn} \\
&= h_2\sum_{n=1}^{N-1}\sum_{m=1}^{M-1}(a_{12})_{mn}(\delta_y u_{m,n-\frac{1}{2}} - \delta_y u_{m-1,n-\frac{1}{2}})u_{mn} \\
&= h_2\sum_{n=1}^{N-1}\left[\sum_{m=1}^{M-1}(a_{12})_{mn}\ \delta_y u_{m,n-\frac{1}{2}}\ u_{mn} - \sum_{m=0}^{M-2}(a_{12})_{m+1,n}\ \delta_y u_{m,n-\frac{1}{2}}\ u_{m+1,n}\right] \\
&= h_2\sum_{n=1}^{N-1}\left[\sum_{m=0}^{M-1}(a_{12})_{mn}\ \delta_y u_{m,n-\frac{1}{2}}\ u_{mn} - \sum_{m=0}^{M-1}(a_{12})_{m+1,n}\ \delta_y u_{m,n-\frac{1}{2}}\ u_{m+1,n}\right]
\end{aligned}$$

$$
\begin{aligned}
&= h_2 \sum_{n=1}^{N-1} \Big[\sum_{m=0}^{M-1} (a_{12})_{mn}\ \delta_y u_{m,n-\frac{1}{2}}\ (u_{mn} - u_{m+1,n}) \\
&\qquad + \sum_{m=0}^{M-1} \left[(a_{12})_{mn} - (a_{12})_{m+1,n}\right]\ \delta_y u_{m,n-\frac{1}{2}}\ u_{m+1,n} \Big] \\
&= -h_1 h_2 \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} (a_{12})_{mn}\ \delta_x u_{m+\frac{1}{2},n}\ \delta_y u_{m,n-\frac{1}{2}} \\
&\quad - h_1 h_2 \sum_{m=0}^{M-1} \sum_{n=1}^{N-1} (\delta_x a_{12})_{m+\frac{1}{2},n}\ \delta_y u_{m,n-\frac{1}{2}}\ u_{m+1,n}\ . \qquad (7.54)
\end{aligned}
$$

For B_{22}, we have

$$
\begin{aligned}
B_{22} &= h_1 h_2 \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} (a_{12})_{mn}\ \delta_x \delta_y u_{m+\frac{1}{2},n-\frac{1}{2}}\ u_{mn} \\
&= h_2 \sum_{n=1}^{N-1} \sum_{m=1}^{M-1} (a_{12})_{mn} (\delta_y u_{m+1,n-\frac{1}{2}} - \delta_y u_{m,n-\frac{1}{2}}) u_{mn} \\
&= h_2 \sum_{n=1}^{N-1} \left[\sum_{m=0}^{M-1} (a_{12})_{mn}\ \delta_y u_{m+1,n-\frac{1}{2}}\ u_{mn} \right. \\
&\qquad \left. - \sum_{m=1}^{M} (a_{12})_{mn}\ \delta_y u_{m,n-\frac{1}{2}}\ u_{m,n} \right] \\
&= h_2 \sum_{n=1}^{N-1} \left[\sum_{m=1}^{M} \left[(a_{12})_{m-1,n} - (a_{12})_{m,n}\right]\ \delta_y u_{m,n-\frac{1}{2}}\ u_{m-1,n} \right. \\
&\qquad \left. - \sum_{m=1}^{M-1} (a_{12})_{mn}\ \delta_y u_{m,n-\frac{1}{2}}\ (u_{m,n} - u_{m-1,n}) \right] \\
&= h_2 \sum_{n=1}^{N-1} \left[-h_1 \sum_{m=1}^{M} (\delta_x a_{12})_{m-\frac{1}{2},n}\ \delta_y u_{m,n-\frac{1}{2}}\ u_{m-1,n} \right. \\
&\qquad \left. - h_1 \sum_{m=1}^{M-1} (a_{12})_{mn}\ \delta_x u_{m-\frac{1}{2},n}\ \delta_y u_{m,n-\frac{1}{2}} \right] \\
&= -h_1 h_2 \sum_{n=1}^{N-1} \left[\sum_{m=1}^{M-1} (a_{12})_{mn}\ \delta_x u_{m-\frac{1}{2},n}\ \delta_y u_{m,n-\frac{1}{2}} \right. \\
&\qquad \left. + \sum_{m=0}^{M-1} (\delta_x a_{12})_{m+\frac{1}{2},n}\ \delta_y u_{m+1,n-\frac{1}{2}}\ u_{m,n} \right]
\end{aligned}
$$

$$= -h_1 h_2 \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} (a_{12})_{mn}\ \delta_x u_{m-\frac{1}{2},n}\ \delta_y u_{m,n-\frac{1}{2}}$$
$$-h_1 h_2 \sum_{m=0}^{M-1} \sum_{n=1}^{N-1} (\delta_x a_{12})_{m+\frac{1}{2},n}\ \delta_y u_{m+1,n-\frac{1}{2}}\ u_{m,n}\ . \tag{7.55}$$

We can see that during deriving the equalities (7.54) and (7.55), the subscripts n and $n-\frac{1}{2}$ are unchanged. Thus, from the equalities (7.54) and (7.55), for B_{23} and B_{24}, we can have

$$B_{23} = -h_1 h_2 \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} (a_{12})_{mn}\ \delta_x u_{m+\frac{1}{2},n}\ \delta_y u_{m,n+\frac{1}{2}}$$
$$-h_1 h_2 \sum_{m=0}^{M-1} \sum_{n=1}^{N-1} (\delta_x a_{12})_{m+\frac{1}{2},n}\ \delta_y u_{m,n+\frac{1}{2}}\ u_{m+1,n}\ ; \tag{7.56}$$

$$B_{24} = -h_1 h_2 \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} (a_{12})_{mn}\ \delta_x u_{m-\frac{1}{2},n}\ \delta_y u_{m,n+\frac{1}{2}}$$
$$-h_1 h_2 \sum_{m=0}^{M-1} \sum_{n=1}^{N-1} (\delta_x a_{12})_{m+\frac{1}{2},n}\ \delta_y u_{m+1,n+\frac{1}{2}}\ u_{mn}\ . \tag{7.57}$$

Putting the second terms in the last expressions of B_{21}, B_{22}, B_{23}, and B_{24} in the expressions (7.54)–(7.57) together yields

$$-h_1 h_2 \sum_{m=0}^{M-1} \sum_{n=1}^{N-1} (\delta_x a_{12})_{m+\frac{1}{2},n} (\delta_y u_{m,n-\frac{1}{2}} + \delta_y u_{m,n+\frac{1}{2}}) u_{m+1,n}$$
$$-h_1 h_2 \sum_{m=0}^{M-1} \sum_{n=1}^{N-1} (\delta_x a_{12})_{m+\frac{1}{2},n} (\delta_y u_{m+1,n-\frac{1}{2}} + \delta_y u_{m+1,n+\frac{1}{2}}) u_{mn}$$
$$= -h_1 \sum_{m=0}^{M-1} \sum_{n=1}^{N-1} (\delta_x a_{12})_{m+\frac{1}{2},n} [(u_{m,n+1} - u_{m,n-1}) u_{m+1,n}$$
$$+(u_{m+1,n+1} - u_{m+1,n-1}) u_{mn}]$$
$$= -h_1 \sum_{m=0}^{M-1} \sum_{n=1}^{N-1} (\delta_x a_{12})_{m+\frac{1}{2},n} (u_{m+1,n+1} u_{mn} + u_{m,n+1} u_{m+1,n}$$
$$-u_{m+1,n-1} u_{mn} - u_{m,n-1} u_{m+1,n})$$
$$= -h_1 \sum_{m=0}^{M-1} \left[\sum_{n=0}^{N-1} (\delta_x a_{12})_{m+\frac{1}{2},n} (u_{m+1,n+1} u_{mn} + u_{m,n+1} u_{m+1,n}) \right.$$
$$\left. - \sum_{n=0}^{N-1} (\delta_x a_{12})_{m+\frac{1}{2},n+1} (u_{m+1,n} u_{m,n+1} + u_{mn} u_{m+1,n+1}) \right]$$

$$\begin{aligned}
&= h_1 h_2 \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} (\delta_y \delta_x a_{12})_{m+\frac{1}{2},n+\frac{1}{2}} \left(u_{m+1,n+1} u_{mn} + u_{m,n+1} u_{m+1,n}\right) \\
&\leq \frac{1}{2} c_2 h_1 h_2 \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \left(u_{m+1,n+1}^2 + u_{mn}^2 + u_{m,n+1}^2 + u_{m+1,n}^2\right) \\
&= 2c_2 \|u\|^2. \qquad (7.58)
\end{aligned}$$

Here we have used $(\delta_x a_{12})_{m+\frac{1}{2},0} = (\delta_x a_{12})_{m+\frac{1}{2},N} = 0$ and another form of the definition of inner product (7.42).

Thus, inserting the equalities (7.54)–(7.57) into the expression (7.53) and using the inequality (7.58), we get

$$\begin{aligned}
B_2 = &-\frac{1}{4} h_1 h_2 \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} (a_{12})_{mn} \Big(\delta_x u_{m+\frac{1}{2},n}\, \delta_y u_{m,n-\frac{1}{2}} + \delta_x u_{m-\frac{1}{2},n}\, \delta_y u_{m,n-\frac{1}{2}} \\
&\qquad + \delta_x u_{m+\frac{1}{2},n}\, \delta_y u_{m,n+\frac{1}{2}} + \delta_x u_{m-\frac{1}{2},n}\, \delta_y u_{m,n+\frac{1}{2}}\Big) \\
&-\frac{1}{4} h_1 h_2 \sum_{m=0}^{M-1} \sum_{n=1}^{N-1} (\delta_x a_{12})_{m+\frac{1}{2},n} (\delta_y u_{m,n-\frac{1}{2}} + \delta_y u_{m,n+\frac{1}{2}}) u_{m+1,n} \\
&-\frac{1}{4} h_1 h_2 \sum_{m=0}^{M-1} \sum_{n=1}^{N-1} (\delta_x a_{12})_{m+\frac{1}{2},n} (\delta_y u_{m+1,n-\frac{1}{2}} + \delta_y u_{m+1,n+\frac{1}{2}}) u_{mn} \\
\leq &-\frac{1}{4} h_1 h_2 \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} (a_{12})_{mn} \Big(\delta_x u_{m+\frac{1}{2},n}\, \delta_y u_{m,n-\frac{1}{2}} + \delta_x u_{m-\frac{1}{2},n}\, \delta_y u_{m,n-\frac{1}{2}} \\
&+ \delta_x u_{m+\frac{1}{2},n}\, \delta_y u_{m,n+\frac{1}{2}} + \delta_x u_{m-\frac{1}{2},n}\, \delta_y u_{m,n+\frac{1}{2}}\Big) + \frac{1}{2} c_2 \|u\|^2. \qquad \blacksquare
\end{aligned}$$

Proposition 7.3 *For* $(b_1 \varDelta_x u, u)$ *and* $(b_2 \varDelta_y u, u)$*, we have*

$$B_4 \equiv (b_1 \varDelta_x u, u) \leq \frac{1}{2} c_4 \|u\|^2 \qquad (7.59)$$

and

$$B_5 \equiv (b_2 \varDelta_y u, u) \leq \frac{1}{2} c_5 \|u\|^2. \qquad (7.60)$$

Proof. Because $(b_1)_{0,n} = (b_1)_{M,n}$ for $n = 0, 1, \cdots, N$, the concrete expression for $(b_1 \varDelta_x u, u)$ is

$$\begin{aligned}
B_4 = h_1 h_2 \Bigg[&\sum_{m=1}^{M-1} \sum_{n=1}^{N-1} (b_1)_{mn}\, \varDelta_x u_{mn}\, u_{mn} + \frac{1}{2} \sum_{m=1}^{M-1} (b_1)_{m0}\, \varDelta_x u_{m0}\, u_{m0} \\
&+ \frac{1}{2} \sum_{m=1}^{M-1} (b_1)_{mN}\, \varDelta_x u_{mN}\, u_{mN}\Bigg].
\end{aligned}$$

For any n, we have

$$\begin{aligned}
& h_1 \sum_{m=1}^{M-1} (b_1)_{mn} \, \Delta_x u_{mn} \, u_{mn} \\
&= \frac{1}{2} \sum_{m=1}^{M-1} (b_1)_{mn} (u_{m+1,n} - u_{m-1,n}) u_{mn} \\
&= \frac{1}{2} \left(\sum_{m=1}^{M-1} (b_1)_{mn} u_{mn} u_{m+1,n} - \sum_{m=0}^{M-2} (b_1)_{m+1,n} u_{mn} u_{m+1,n} \right) \\
&= -\frac{1}{2} h_1 \sum_{m=0}^{M-1} (\delta_x b_1)_{m+\frac{1}{2},n} \, u_{mn} u_{m+1,n} \\
&\le \frac{1}{2} c_4 h_1 \Big(\frac{1}{2} \, u_{0n}^2 + \sum_{m=1}^{M-1} u_{mn}^2 + \frac{1}{2} \, u_{Mn}^2 \Big).
\end{aligned}$$

Adding them together yields

$$B_4 \le \frac{1}{2} c_4 \|u\|^2.$$

It is easy to see that changing x to y and m to n during the derivation above, we can prove the second inequality in Proposition 7.3. Thus, we have proved the conclusion we need. ■

Proposition 7.4

$$B_6 \equiv (cu, u) \le c_6 \|u\|^2. \tag{7.61}$$

Proof. Since $|c_{mn}^k| \le c_6$, it is easy to see the validity of the inequality (7.61). ■

The proof of Lemma 7.1 Based on these inequalities and noticing the matrix

$$\begin{pmatrix} a_{11}(x, y, \tau) & a_{12}(x, y, \tau) \\ a_{12}(x, y, \tau) & a_{22}(x, y, \tau) \end{pmatrix}$$

is semi-positive, we can prove the lemma immediately. Adding the relations (7.49), (7.52), (7.50), (7.59), (7.60), and (7.61), then using the inequality (7.37), we get

$$\begin{aligned}
& B_1 + 2B_2 + B_3 + B_4 + B_5 + B_6 \\
\le\ & \frac{1}{2} (c_1 + 2c_2 + c_3 + c_4 + c_5 + 2c_6) \|u\|^2 \\
& -\frac{1}{4} h_1 h_2 \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} \Big\{ (a_{11})_{mn} \Big[2(\delta_x u_{m-\frac{1}{2},n})^2 + 2(\delta_x u_{m+\frac{1}{2},n})^2 \Big]
\end{aligned}$$

$$
\begin{aligned}
&+2(a_{12})_{mn}\Big[\delta_x u_{m+\frac{1}{2},n}\ \delta_y u_{m,n-\frac{1}{2}} + \delta_x u_{m-\frac{1}{2},n}\ \delta_y u_{m,n-\frac{1}{2}} \\
&+\delta_x u_{m+\frac{1}{2},n}\ \delta_y u_{m,n+\frac{1}{2}} + \delta_x u_{m-\frac{1}{2},n}\ \delta_y u_{m,n+\frac{1}{2}}\Big] \\
&+(a_{22})_{mn}\Big[2(\delta_y u_{m,n-\frac{1}{2}})^2 + 2(\delta_y u_{m,n+\frac{1}{2}})^2\Big]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
= {} & \frac{c}{2}\|u\|^2 - \frac{1}{4}h_1 h_2 \sum_{m=1}^{M-1}\sum_{n=1}^{N-1}\Big\{ \\
& \Big[(a_{11})_{mn}(\delta_x u_{m+\frac{1}{2},n})^2 + 2(a_{12})_{mn}\ \delta_x u_{m+\frac{1}{2},n}\ \delta_y u_{m,n-\frac{1}{2}} \\
& \qquad + (a_{22})_{mn}(\delta_y u_{m,n-\frac{1}{2}})^2\Big] \\
& + \Big[(a_{11})_{mn}(\delta_x u_{m-\frac{1}{2},n})^2 + 2(a_{12})_{mn}\ \delta_x u_{m-\frac{1}{2},n}\ \delta_y u_{m,n-\frac{1}{2}} \\
& \qquad + (a_{22})_{mn}(\delta_y u_{m,n-\frac{1}{2}})^2\Big] \\
& + \Big[(a_{11})_{mn}(\delta_x u_{m+\frac{1}{2},n})^2 + 2(a_{12})_{mn}\ \delta_x u_{m+\frac{1}{2},n}\ \delta_y u_{m,n+\frac{1}{2}} \\
& \qquad + (a_{22})_{mn}(\delta_y u_{m,n+\frac{1}{2}})^2\Big] \\
& + \Big[(a_{11})_{mn}(\delta_x u_{m-\frac{1}{2},n})^2 + 2(a_{12})_{mn}\ \delta_x u_{m-\frac{1}{2},n}\ \delta_y u_{m,n+\frac{1}{2}} \\
& \qquad + (a_{22})_{mn}(\delta_y u_{m,n+\frac{1}{2}})^2\Big]\Big\} \\
\le {} & \frac{c}{2}\|u\|^2.
\end{aligned}
$$

This completes the proof of Lemma 7.1. □

7.4.3 ‡Solvability and Stability

In this subsection, we will prove the solvability and stability of the two-dimensional finite-difference scheme (7.46)–(7.47).

Theorem 7.1 *If $\Delta\tau < 1/c$, then the difference scheme (7.46)–(7.47) is uniquely solvable.*

Proof. Suppose $\{u^k_{mn} \mid 0 \le m \le M, 0 \le n \le N\}$ has been determined. Then the difference scheme (7.46) is a linear system about $\{u^{k+1}_{mn} \mid 0 \le m \le M, 0 \le n \le N\}$. Consider its homogeneous system

$$
\begin{aligned}
\frac{1}{\Delta\tau}u^{k+1}_{mn} = {} & \frac{1}{2}(a_{11})^{k+\frac{1}{2}}_{mn}\delta_x^2 u^{k+1}_{mn} + (a_{12})^{k+\frac{1}{2}}_{mn}\Delta_x\Delta_y u^{k+1}_{mn} + \frac{1}{2}(a_{22})^{k+\frac{1}{2}}_{mn}\delta_y^2 u^{k+1}_{mn} \\
& + \frac{1}{2}(b_1)^{k+\frac{1}{2}}_{mn}\Delta_x u^{k+1}_{mn} + \frac{1}{2}(b_2)^{k+\frac{1}{2}}_{mn}\Delta_y u^{k+1}_{mn} + \frac{1}{2}c^{k+\frac{1}{2}}_{mn}u^{k+1}_{mn}, \\
& 0 \le m \le M, \quad 0 \le n \le N.
\end{aligned}
\tag{7.62}
$$

Taking the inner product of equality (7.62) with $2u^{k+1}$ and using Lemma 7.1, we have

$$\begin{aligned}\frac{2}{\Delta\tau}\|u^{k+1}\|^2 &= \left((a_{11})^{k+\frac{1}{2}}\delta_x^2 u^{k+1}, u^{k+1}\right) + 2\left((a_{12})^{k+\frac{1}{2}}\Delta_x\Delta_y u^{k+1}, u^{k+1}\right)\\ &\quad + \left((a_{22})^{k+\frac{1}{2}}\delta_y^2 u^{k+1}, u^{k+1}\right) + \left((b_1)^{k+\frac{1}{2}}\Delta_x u^{k+1}, u^{k+1}\right)\\ &\quad + \left((b_2)^{k+\frac{1}{2}}\Delta_y u^{k+1}, u^{k+1}\right) + \left(c^{k+\frac{1}{2}}u^{k+1}, u^{k+1}\right)\\ &\le \frac{c}{2}\|u^{k+1}\|^2. \end{aligned} \tag{7.63}$$

If $\Delta\tau < 1/c$, then $\|u^{k+1}\| = 0$. This completes the proof. ■

Theorem 7.2 *If $\Delta\tau \le 2/[3(1+c)]$, then the solution to the difference scheme (7.46)–(7.47) satisfies*

$$\|u^{k+1}\|^2 \le \mathrm{e}^{3(c+1)T/2}\left(\|u^0\|^2 + \frac{3}{2}\Delta\tau\sum_{l=0}^{k}\|g^{l+\frac{1}{2}}\|^2\right), \quad 0 \le k \le K-1. \tag{7.64}$$

Proof. Taking the inner product of Eq. (7.46) with $u^{k+\frac{1}{2}}$ and using Lemma 7.1, we have

$$\begin{aligned}&\frac{1}{2\Delta\tau}\left(\|u^{k+1}\|^2 - \|u^k\|^2\right)\\ &= \left((a_{11})^{k+\frac{1}{2}}\delta_x^2 u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}}\right) + 2\left((a_{12})^{k+\frac{1}{2}}\Delta_x\Delta_y u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}}\right)\\ &\quad + \left((a_{22})^{k+\frac{1}{2}}\delta_y^2 u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}}\right) + \left((b_1)^{k+\frac{1}{2}}\Delta_x u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}}\right)\\ &\quad + \left((b_2)^{k+\frac{1}{2}}\Delta_y u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}}\right) + \left(c^{k+\frac{1}{2}}u^{k+1}, u^{k+\frac{1}{2}}\right) + \left(g^{k+\frac{1}{2}}, u^{k+\frac{1}{2}}\right)\\ &\le \frac{c}{2}\|u^{k+\frac{1}{2}}\|^2 + \frac{1}{2}\|g^{k+\frac{1}{2}}\|^2 + \frac{1}{2}\|u^{k+\frac{1}{2}}\|^2, \quad 0 \le k \le K-1,\end{aligned}$$

from which we further obtain

$$\begin{aligned}\|u^{k+1}\|^2 &\le \|u^k\|^2 + (1+c)\Delta\tau\|u^{k+\frac{1}{2}}\|^2 + \Delta\tau\|g^{k+\frac{1}{2}}\|^2\\ &\le \|u^k\|^2 + \frac{1+c}{2}\Delta\tau\left(\|u^k\|^2 + \|u^{k+1}\|^2\right) + \Delta\tau\|g^{k+\frac{1}{2}}\|^2,\\ &\qquad 0 \le k \le K-1.\end{aligned}$$

If $1 - \dfrac{1+c}{2}\Delta\tau > 0$, then the inequality can be rewritten as

$$\|u^{k+1}\|^2 \le \frac{1 + \frac{1+c}{2}\Delta\tau}{1 - \frac{1+c}{2}\Delta\tau}\|u^k\|^2 + \frac{\Delta\tau}{1 - \frac{1+c}{2}\Delta\tau}\|g^{k+\frac{1}{2}}\|^2.$$

It is clear that for $\bar{C} > 2$, when $\Delta\tau$ is small enough, we can have $\dfrac{1 + \frac{1+c}{2}\Delta\tau}{1 - \frac{1+c}{2}\Delta\tau} \le 1 + \bar{C}\frac{1+c}{2}\Delta\tau$. Let us take $\bar{C} = 3$; then we can easily find that the corresponding condition for $\Delta\tau$ is $\Delta\tau \le 2/[3(c+1)]$ and that in this case $1 - \frac{1+c}{2}\Delta\tau \ge \frac{2}{3}$. Thus, when $\Delta\tau \le 2/[3(c+1)]$, we have

$$\|u^{k+1}\|^2 \le \left(1 + \frac{3(c+1)}{2}\Delta\tau\right)\|u^k\|^2 + \frac{3}{2}\Delta\tau\|g^{k+\frac{1}{2}}\|^2, \quad 0 \le k \le K-1.$$

From this discrete Gronwall inequality, we finally arrive at

$$\|u^{k+1}\|^2 \le \mathrm{e}^{3(c+1)T/2}\left[\|u^0\|^2 + \frac{3}{2}\Delta\tau\sum_{l=0}^{k}\|g^{l+\frac{1}{2}}\|^2\right], \quad 0 \le k \le K-1.$$

This completes the proof. ■

The method used here to prove the stability is usually called the energy method for stability analysis.

7.4.4 ‡Convergence

For the convergence of the finite-difference scheme (7.46)–(7.47), we have

Theorem 7.3 *Let* $\{U_{mn}^k\}$ *be the solution of the problem (7.32)–(7.33) and* $\{u_{mn}^k\}$ *be the solution of Eqs. (7.46)–(7.47). Denote*

$$e_{mn}^k = U_{mn}^k - u_{mn}^k, \quad 0 \le m \le M, \quad 0 \le n \le N, \quad 0 \le k \le K.$$

If $\Delta\tau \le 2/[3(c+1)]$, *then we have*

$$\|e^{k+1}\| \le \mathrm{e}^{3(c+1)T/4}\sqrt{\frac{3(x_u - x_l)(y_u - y_l)T}{2}}\, c_0\left(h_1^2 + h_2^2 + \Delta\tau^2\right),$$
$$0 \le k \le K-1.$$

Proof. Subtracting the equalities (7.46) and (7.47) from the equalities (7.43) and (7.45), respectively, we obtain the error equations

$$\begin{aligned}\frac{1}{\Delta\tau}(e_{mn}^{k+1} - e_{mn}^k) = {} & (a_{11})_{mn}^{k+\frac{1}{2}}\delta_x^2 e_{mn}^{k+\frac{1}{2}} + 2(a_{12})_{mn}^{k+\frac{1}{2}}\Delta_x\Delta_y e_{mn}^{k+\frac{1}{2}} \\ & +(a_{22})_{mn}^{k+\frac{1}{2}}\delta_y^2 e_{mn}^{k+\frac{1}{2}} + (b_1)_{mn}^{k+\frac{1}{2}}\Delta_x e_{mn}^{k+\frac{1}{2}} \\ & +(b_2)_{mn}^{k+\frac{1}{2}}\Delta_y e_{mn}^{k+\frac{1}{2}} + c_{mn}^{k+\frac{1}{2}} e_{mn}^{k+\frac{1}{2}} + R_{mn}^{k+\frac{1}{2}}, \\ & 0 \le m \le M, \quad 0 \le n \le N, \quad 0 \le k \le K-1, \end{aligned} \tag{7.65}$$

$$e_{mn}^0 = 0, \quad 0 \le m \le M, \quad 0 \le n \le N. \tag{7.66}$$

Taking the inner product of the system (7.65) with $e^{k+\frac{1}{2}}$ and using Lemma 7.1, we have

$$\begin{aligned}
&\frac{1}{2\Delta\tau}\left(\|e^{k+1}\|^2-\|e^k\|^2\right)\\
&=\left((a_{11})^{k+\frac{1}{2}}\delta_x^2 e^{k+\frac{1}{2}},e^{k+\frac{1}{2}}\right)+2\left((a_{12})^{k+\frac{1}{2}}\Delta_x\Delta_y e^{k+\frac{1}{2}},e^{k+\frac{1}{2}}\right)\\
&\quad+\left((a_{22})^{k+\frac{1}{2}}\delta_y^2 e^{k+\frac{1}{2}},e^{k+\frac{1}{2}}\right)+\left((b_1)^{k+\frac{1}{2}}\Delta_x e^{k+\frac{1}{2}},e^{k+\frac{1}{2}}\right)\\
&\quad+\left((b_2)^{k+\frac{1}{2}}\Delta_y e^{k+\frac{1}{2}},e^{k+\frac{1}{2}}\right)+\left(c^{k+\frac{1}{2}}e^{k+1},e^{k+\frac{1}{2}}\right)+\left(R^{k+\frac{1}{2}},e^{k+\frac{1}{2}}\right)\\
&\le\frac{c}{2}\|e^{k+\frac{1}{2}}\|^2+\frac{1}{2}\|R^{k+\frac{1}{2}}\|^2+\frac{1}{2}\|e^{k+\frac{1}{2}}\|^2,\quad 0\le k\le K-1,
\end{aligned}$$

from which we further get

$$\begin{aligned}
\|e^{k+1}\|^2&\le\|e^k\|^2+(1+c)\Delta\tau\|e^{k+\frac{1}{2}}\|^2+\Delta\tau\|R^{k+\frac{1}{2}}\|^2\\
&\le\|e^k\|^2+\frac{1+c}{2}\Delta\tau\left(\|e^k\|^2+\|e^{k+1}\|^2\right)+\Delta\tau\|R^{k+\frac{1}{2}}\|^2,\\
&\qquad 0\le k\le K-1.
\end{aligned}$$

Using the condition (7.44) and when $\Delta\tau\le 2/[3(c+1)]$, we can rewrite this inequality as

$$\begin{aligned}
\|e^{k+1}\|^2&\le\left(1+\frac{3(c+1)}{2}\Delta\tau\right)\|e^k\|^2+\frac{3}{2}\Delta\tau\|R^{k+\frac{1}{2}}\|^2\\
&\le\left(1+\frac{3(c+1)}{2}\Delta\tau\right)\|e^k\|^2\\
&\quad+\frac{3}{2}(x_u-x_l)(y_u-y_l)c_0^2\Delta\tau\left(h_1^2+h_2^2+\Delta\tau^2\right)^2,\\
&\qquad 0\le k\le K-1.
\end{aligned}$$

The Gronwall inequality gives

$$\|e^{k+1}\|^2\le \mathrm{e}^{3(c+1)T/2}\frac{3(x_u-x_l)(y_u-y_l)T}{2}c_0^2\left(h_1^2+h_2^2+\Delta\tau^2\right)^2,\ 0\le k\le K-1,$$

or

$$\|e^{k+1}\|\le \mathrm{e}^{3(c+1)T/4}\sqrt{3\frac{(x_u-x_l)(y_u-y_l)T}{2}}\,c_0\left(h_1^2+h_2^2+\Delta\tau^2\right),\quad 0\le k\le K-1.$$

This completes the proof. ■

For the solution to the difference scheme (7.46)–(7.47), we can also use the extrapolation technique to improve the accuracy of the numerical solutions when solutions are smooth. The idea is the same as what is described in Sect. 7.3. Based on the results given in this subsection, some theoretical conclusions on the extrapolation technique can be obtained. For details, see the paper [78] by Sun and Zhu.

Problems

Table 7.1. Problems and Sections

Problems	Sections	Problems	Sections	Problems	Sections
1–5	7.1	6–15	7.2	16–18	7.3
19–21	7.4				

1. *Let f_m^n denote $f(m\Delta x, n\Delta\tau)$. Find the truncation error of the explicit difference scheme

$$\frac{u_m^{n+1}-u_m^n}{\Delta\tau} = a_m^n \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2} + b_m^n \frac{u_{m+1}^n - u_{m-1}^n}{2\Delta x} + c_m^n u_m^n$$

to the parabolic partial differential equation

$$\frac{\partial u}{\partial \tau} = a(x,\tau)\frac{\partial^2 u}{\partial x^2} + b(x,\tau)\frac{\partial u}{\partial x} + c(x,\tau)u.$$

2. Show that the truncation error of the Crank–Nicolson scheme for the heat equation at the point $(x_m, \tau^{n+1/2})$ is in the following form:

$$\Delta\tau^2\left[\frac{1}{24}\frac{\partial^3 u}{\partial \tau^3}(x_m,\eta^{(1)}) - \frac{a}{8}\frac{\partial^4 u}{\partial x^2 \partial \tau^2}(x_m,\eta^{(2)})\right] - \frac{\Delta x^2 a}{12}\frac{\partial^4 u}{\partial x^4}(\xi,\eta^{(3)}),$$

where $\xi \in (x_{m-1}, x_{m+1})$, $\eta^{(k)} \in (\tau^n, \tau^{n+1})$, $k = 1,2,3$, and a is the conductivity coefficient in the heat equation.

3. *Let f_m^n denote $f(m\Delta x, n\Delta\tau)$. Find the truncation error of the implicit difference scheme

$$\frac{u_m^{n+1}-u_m^n}{\Delta\tau} = \frac{a_m^{n+1/2}}{2}\left(\frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{\Delta x^2} + \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2}\right) + \frac{b_m^{n+1/2}}{2}\left(\frac{u_{m+1}^{n+1} - u_{m-1}^{n+1}}{2\Delta x} + \frac{u_{m+1}^n - u_{m-1}^n}{2\Delta x}\right) + \frac{c_m^{n+1/2}}{2}(u_m^{n+1} + u_m^n)$$

to the parabolic partial differential equation

$$\frac{\partial u}{\partial \tau} = a(x,\tau)\frac{\partial^2 u}{\partial x^2} + b(x,\tau)\frac{\partial u}{\partial x} + c(x,\tau)u.$$

4. The heat equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

can also be discretized by

$$\frac{u_m^{n+1} - u_m^n}{\Delta\tau} = \theta\left(\frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{\Delta x^2}\right) + (1-\theta)\left(\frac{u_{m+1}^{n} - 2u_m^{n} + u_{m-1}^{n}}{\Delta x^2}\right)$$

or

$$u_m^{n+1} - \theta\alpha(u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}) = u_m^n + (1-\theta)\alpha(u_{m+1}^n - 2u_m^n + u_{m-1}^n),$$

where $0 \le \theta \le 1$ and $\alpha = \Delta\tau/\Delta x^2$. This scheme is called the θ–scheme. It is clear that when $\theta = 0$, the scheme reduces to the explicit scheme and when $\theta = 1/2$, the scheme becomes the Crank–Nicolson scheme. Show that the order of truncation error of the θ–scheme is

$$O\left((1-2\theta)\,\Delta\tau + \Delta\tau^2 + \Delta x^2\right).$$

(Hint: Discretize the partial differential equation at $x = x_m$ and $\tau = \tau^{n+\theta}$.)

5. Consider the parabolic partial differential equation

$$\frac{\partial u}{\partial \tau} = a(x,\tau)\frac{\partial^2 u}{\partial x^2} + b(x,\tau)\frac{\partial u}{\partial x} + c(x,\tau)u,$$

which is defined for $x \in [0,1]$ and $\tau \ge 0$. Here $a(x,\tau) \ge 0$ holds and we suppose that $\dfrac{\partial a}{\partial x}$ is bounded. Assuming that $u(x,\tau)$ is given, we want to determine $u(x,\tau+\Delta\tau)$ with $\Delta\tau > 0$ for $x \in [0,1]$.
   (a) Under what conditions on $a(x,\tau)$ and $b(x,\tau)$ a boundary condition is needed and under what conditions no boundary condition is needed at $x = 0$ and $x = 1$?
   (b) Suppose that an explicit scheme will be used. How do we determine $u(0,\tau+\Delta\tau)$ and $u(1,\tau+\Delta\tau)$ if no boundary condition should be given?

6. *Consider the three-point explicit finite-difference scheme:

$$u_m^{n+1} = a_m u_{m-1}^n + b_m u_m^n + c_m u_{m+1}^n, \quad m = 1, 2, \cdots, M-1,$$

where $a_m \ge 0$, $b_m = 1 - a_m - c_m \ge 0$, $c_m \ge 0$ and $a_0 = c_M = 0$. Show

$$\max_{1\le m\le M-1} |u_m^{n+1}| \le \max_{1\le m\le M-1} |u_m^n|.$$

This means that the numerical procedure is stable under the maximum norm.

7. Consider the equation

$$\lambda \mathbf{A}\mathbf{x} = \mathbf{B}\mathbf{x} \quad \text{or} \quad \mathbf{A}^{-1}\mathbf{B}\mathbf{x} = \lambda \mathbf{x},$$

where $\mathbf{A}$ and $\mathbf{B}$ are $(M-1) \times (M-1)$ matrices and their concrete expressions are

$$\mathbf{A} = \begin{bmatrix} a_0 & a_1 & 0 & \cdots & 0 \\ a_1 & a_0 & a_1 & \ddots & \vdots \\ 0 & a_1 & a_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_1 \\ 0 & \cdots & 0 & a_1 & a_0 \end{bmatrix}$$

and

$$\mathbf{B} = \begin{bmatrix} b_0 & b_1 & 0 & \cdots & 0 \\ b_1 & b_0 & b_1 & \ddots & \vdots \\ 0 & b_1 & b_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_1 \\ 0 & \cdots & 0 & b_1 & b_0 \end{bmatrix}.$$

Find $M-1$ linearly independent eigenvectors of $\mathbf{A}^{-1}\mathbf{B}$ and their associated eigenvalues.

8. Consider the equation

$$\lambda \mathbf{A}_2\mathbf{x} = \mathbf{B}_2\mathbf{x}$$

or

$$\mathbf{A}_2^{-1}\mathbf{B}_2\mathbf{x} = \lambda \mathbf{x},$$

where $\mathbf{A}_2$ and $\mathbf{B}_2$ are $M \times M$ matrices and their concrete expressions are

$$\mathbf{A}_2 = \begin{bmatrix} a_0 & a_1 & 0 & \cdots & a_{-1} \\ a_{-1} & a_0 & a_1 & \ddots & \vdots \\ 0 & a_{-1} & a_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_1 \\ a_1 & \cdots & 0 & a_{-1} & a_0 \end{bmatrix}$$

and

$$\mathbf{B}_2 = \begin{bmatrix} b_0 & b_1 & 0 & \cdots & b_{-1} \\ b_{-1} & b_0 & b_1 & \ddots & \vdots \\ 0 & b_{-1} & b_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_1 \\ b_1 & \cdots & 0 & b_{-1} & b_0 \end{bmatrix}.$$

Find M linearly independent eigenvectors of ${\mathbf{A}_2}^{-1}\mathbf{B}_2$ and their associated eigenvalues.

9. (a) Consider an $M \times M$ matrix

$$\mathbf{A} = \begin{pmatrix} a & b & 0 & \cdots & \cdots & 0 & b \\ b & a & b & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & b & a & b \\ b & 0 & \cdots & \cdots & 0 & b & a \end{pmatrix}.$$

Suppose $a = q + 2/h^2$ and $b = -1/h^2$. Show that its eigenvalues are $\lambda_j = q + \dfrac{4}{h^2}\sin^2\dfrac{\theta_j}{2}$, $j = 0, 1, \cdots, M-1$, where $\theta_j = j\frac{2\pi}{M}$, and the corresponding eigenvectors are

$$\mathbf{v}_j = \begin{pmatrix} 1 \\ \cos\theta_j \\ \cos 2\theta_j \\ \vdots \\ \cos(M-1)\theta_j \end{pmatrix}, \qquad j = 0, 1, \cdots, \text{int}\left(\frac{M}{2}\right),$$

and

$$\mathbf{v}_j = \begin{pmatrix} 0 \\ \sin\theta_j \\ \sin 2\theta_j \\ \vdots \\ \sin(M-1)\theta_j \end{pmatrix}, \qquad j = \text{int}\left(\frac{M}{2}\right) + 1, \cdots, M-1,$$

respectively, where $\text{int}\left(\dfrac{M}{2}\right)$ is the integer part of $\dfrac{M}{2}$.

(b) Find the eigenvalues and eigenvectors of $\mathbf{A}^{-1}$.

(c) Suppose $a = \dfrac{q}{2} + \dfrac{2}{h^2}$ and $b = \dfrac{q}{4} - \dfrac{1}{h^2}$, find the eigenvalues and eigenvectors of $\mathbf{A}$ and $\mathbf{A}^{-1}$.

10. *Consider the explicit scheme

$$\frac{u_m^{n+1} - u_m^n}{\varDelta\tau} = a\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\varDelta x^2}, \quad m = 1, 2, \cdots, M-1$$

with $u_0^{n+1} = f_l(\tau^{n+1})$ and $u_M^{n+1} = f_u(\tau^{n+1})$. Determine when it is stable with respect to initial values in L_2 norm and when it is unstable. (Suppose $a > 0$.)

11. *Consider the implicit scheme

$$\frac{u_m^{n+1} - u_m^n}{\varDelta\tau} = \frac{a}{2}\left(\frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{\varDelta x^2} + \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\varDelta x^2}\right),$$
$$m = 1, 2, \cdots, M-1$$

with $u_0^{n+1} = f_l(\tau^{n+1})$ and $u_M^{n+1} = f_u(\tau^{n+1})$. Show that it is always stable with respect to initial values in L_2 norm. (Suppose $a > 0$.)

12. By using the von Neumann method, show that for periodic problems, the θ–scheme for the heat equation

$$\begin{aligned} &u_m^{n+1} - \theta\alpha\left(u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}\right) \\ &= u_m^n + (1-\theta)\,\alpha\left(u_{m+1}^n - 2u_m^n + u_{m-1}^n\right) \end{aligned}$$

is stable for all $\alpha > 0$ if $\dfrac{1}{2} \le \theta \le 1$ and that it is stable for $0 < \alpha \le \dfrac{1}{2(1-2\theta)}$ if $0 < \theta < \dfrac{1}{2}$.

13. Consider the following parabolic partial differential equation:

$$\frac{\partial u}{\partial \tau} = a_{11}\frac{\partial^2 u}{\partial x^2} + 2a_{12}\frac{\partial^2 u}{\partial x \partial y} + a_{22}\frac{\partial^2 u}{\partial y^2} + b_1\frac{\partial u}{\partial x} + b_2\frac{\partial u}{\partial y},$$

where $a_{11}(x, y, \tau) \ge 0, a_{22}(x, y, \tau) \ge 0, a_{12}(x, y, \tau) = \rho_{12}(x, y, \tau)\sqrt{a_{11}a_{22}}$ with $\rho_{12} \in [-1, 1]$, and b_1, b_2 are any functions of x, y, τ. This equation can be approximated by

(i)

$$\begin{aligned} &\frac{u_{m,n}^{k+1} - u_{m,n}^k}{\varDelta\tau} \\ &= \frac{a_{11,m,n}^{k+\frac{1}{2}}}{2}\left(\frac{u_{m+1,n}^{k+1} - 2u_{m,n}^{k+1} + u_{m-1,n}^{k+1}}{\varDelta x^2} + \frac{u_{m+1,n}^k - 2u_{m,n}^k + u_{m-1,n}^k}{\varDelta x^2}\right) \end{aligned}$$

$$+a_{12,m,n}^{k+\frac{1}{2}}\left(\frac{u_{m+1,n+1}^{k+1}-u_{m+1,n-1}^{k+1}-u_{m-1,n+1}^{k+1}+u_{m-1,n-1}^{k+1}}{4\Delta x\Delta y}\right.$$
$$\left.+\frac{u_{m+1,n+1}^{k}-u_{m+1,n-1}^{k}-u_{m-1,n+1}^{k}+u_{m-1,n-1}^{k}}{4\Delta x\Delta y}\right)$$
$$+\frac{a_{22,m,n}^{k+\frac{1}{2}}}{2}\left(\frac{u_{m,n+1}^{k+1}-2u_{m,n}^{k+1}+u_{m,n-1}^{k+1}}{\Delta y^2}+\frac{u_{m,n+1}^{k}-2u_{m,n}^{k}+u_{m,n-1}^{k}}{\Delta y^2}\right)$$
$$+\frac{b_{1,m,n}^{k+\frac{1}{2}}}{2}\left(\frac{u_{m+1,n}^{k+1}-u_{m-1,n}^{k+1}}{2\Delta x}+\frac{u_{m+1,n}^{k}-u_{m-1,n}^{k}}{2\Delta x}\right)$$
$$+\frac{b_{2,m,n}^{k+\frac{1}{2}}}{2}\left(\frac{u_{m,n+1}^{k+1}-u_{m,n-1}^{k+1}}{2\Delta y}+\frac{u_{m,n+1}^{k}-u_{m,n-1}^{k}}{2\Delta y}\right)\quad\text{or}$$

(ii)

$$\frac{u_{m,n}^{k+1}-u_{m,n}^{k}}{\Delta\tau}$$
$$=\frac{a_{11,m,n}^{k+\frac{1}{2}}}{2}\left(\frac{u_{m+1,n}^{k+1}-2u_{m,n}^{k+1}+u_{m-1,n}^{k+1}}{\Delta x^2}+\frac{u_{m+1,n}^{k}-2u_{m,n}^{k}+u_{m-1,n}^{k}}{\Delta x^2}\right)$$
$$+a_{12,m,n}^{k+\frac{1}{2}}\left(\frac{u_{m+1,n+1}^{k+1}-u_{m+1,n-1}^{k+1}-u_{m-1,n+1}^{k+1}+u_{m-1,n-1}^{k+1}}{4\Delta x\Delta y}\right.$$
$$\left.+\frac{u_{m+1,n+1}^{k}-u_{m+1,n-1}^{k}-u_{m-1,n+1}^{k}+u_{m-1,n-1}^{k}}{4\Delta x\Delta y}\right)$$
$$+\frac{a_{22,m,n}^{k+\frac{1}{2}}}{2}\left(\frac{u_{m,n+1}^{k+1}-2u_{m,n}^{k+1}+u_{m,n-1}^{k+1}}{\Delta y^2}+\frac{u_{m,n+1}^{k}-2u_{m,n}^{k}+u_{m,n-1}^{k}}{\Delta y^2}\right)$$
$$+\frac{b_{1,m,n}^{k+\frac{1}{2}}}{2}\left(\frac{-u_{m+2,n}^{k+1}+4u_{m+1,n}^{k+1}-3u_{m,n}^{k+1}}{2\Delta x}\right.$$
$$\left.+\frac{-u_{m+2,n}^{k}+4u_{m+1,n}^{k}-3u_{m,n}^{k}}{2\Delta x}\right)$$
$$+\frac{b_{2,m,n}^{k+\frac{1}{2}}}{2}\left(\frac{3u_{m,n}^{k+1}-4u_{m,n-1}^{k+1}+u_{m,n-2}^{k+1}}{2\Delta y}+\frac{3u_{m,n}^{k}-4u_{m,n-1}^{k}+u_{m,n-2}^{k}}{2\Delta y}\right)$$

if $b_1(x,y,\tau)\geq 0$ and $b_2(x,y,\tau)\leq 0$. By the von Neumann method, show that they are stable.

(Hint:

(a) First show that the amplification factor λ can be written as $\lambda = \dfrac{1+a+ib}{1-a-ib}$.

(b) Then show that $|\lambda|^2 \le 1$ is equivalent to $|1-a-ib|^2 - |1+a+ib|^2 = -4a \ge 0$.

(c) Finally show $-4a \ge 0$ by using the following inequalities: (i) $A^2 + B^2 + 2\rho AB = (A+\rho B)^2 + B^2\left(1-\rho^2\right) \ge 0$ if $|\rho| \le 1$; (ii) $\cos 2\theta - 4\cos\theta + 3 = 2\left(\cos\theta - 1\right)^2 \ge 0$.)

14. *Show that if

$$\max_{0\le m\le M} \frac{x_m^2(1-x_m)^2\bar{\sigma}_m^2}{2}\frac{\Delta\tau}{\Delta x^2} \le \frac{1}{2},$$

then for the scheme with variable coefficients

$$\begin{aligned}\frac{u_m^{n+1}-u_m^n}{\Delta\tau} =& \frac{1}{2}[x_m(1-x_m)\bar{\sigma}_m]^2\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2}\\ &+ (r-D_0)x_m(1-x_m)\frac{u_{m+1}^n - u_{m-1}^n}{2\Delta x}\\ &- \left[r\left(1-x_m\right) + D_0 x_m\right]u_m^n,\end{aligned}$$

the condition $|\lambda_\theta(x_m,\tau^n)| \le 1 + O(\Delta\tau)$ is satisfied for any $x_m = m/M \in [0,1]$. (When you prove this result, you should derive the stability condition for explicit schemes by yourself.)

15. For the scheme with variable coefficients

$$\begin{aligned}&\frac{u_m^{n+1}-u_m^n}{\Delta\tau}\\ =&\frac{1}{4}[x_m(1-x_m)\bar{\sigma}_m]^2\left(\frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{\Delta x^2} + \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2}\right)\\ &+ \frac{1}{2}(r-D_0)x_m(1-x_m)\left(\frac{u_{m+1}^{n+1} - u_{m-1}^{n+1}}{2\Delta x} + \frac{u_{m+1}^n - u_{m-1}^n}{2\Delta x}\right)\\ &- \frac{1}{2}\left[r\left(1-x_m\right) + D_0 x_m\right]\left(u_m^{n+1} + u_m^n\right),\end{aligned}$$

show that the condition $|\lambda_\theta(x_m,\tau^n)| \le 1 + O(\Delta\tau)$ is satisfied for any $x_m \in [0,1]$.

16. (a) Consider the explicit difference scheme

$$\frac{u_m^{n+1}-u_m^n}{\Delta\tau} = a_m^n\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2} + b_m^n\frac{u_{m+1}^n - u_{m-1}^n}{2\Delta x} + c_m^n u_m^n$$

to the parabolic partial differential equation

$$\frac{\partial u}{\partial\tau} = a(x,\tau)\frac{\partial^2 u}{\partial x^2} + b(x,\tau)\frac{\partial u}{\partial x} + c(x,\tau)u.$$

Assume that its stability with respect to initial value and nonhomogeneous term is proved under certain conditions. Show that for its solution, under these conditions there is the following relation: $u\left(x,\tau;\Delta x,\Delta\tau\right)=u(x,\tau)+a\left(x,\tau;\dfrac{\Delta x^2}{\Delta\tau}\right)\Delta\tau+O(\Delta\tau^2)$, where $\left|O(\Delta\tau^2)\right|\le c\Delta\tau^2$, c being bounded as $\Delta\tau\to 0$ with $\dfrac{\Delta x^2}{\Delta\tau}=constant.$

(b) Suppose we have two such approximate solutions $u\left(x,\tau;\Delta x,\Delta\tau\right)$ and $u\left(x,\tau;\Delta x/2,\Delta\tau/4\right)$. Find a linear combination

$$(1-d)\times u\left(x,\tau;\Delta x,\Delta\tau\right)+d\times u\left(x,\tau;\Delta x/2,\Delta\tau/4\right)$$

such that it is an approximate solution with an error of $O(\Delta\tau^2)$.

17. (a) Assume that an approximate solution $u\left(x,\tau;\Delta x,\Delta\tau\right)$ has the following expression:

$$\begin{aligned}&u\left(x,\tau;\Delta x,\Delta\tau\right)\\&=u\left(x,\tau\right)+a\left(x,\tau;\frac{\Delta x}{\Delta\tau}\right)\Delta\tau^2+b\left(x,\tau;\frac{\Delta x}{\Delta\tau}\right)\Delta\tau^3+O\left(\Delta\tau^4\right),\end{aligned}$$

where $u\left(x,\tau\right)$ is the exact solution. Suppose that we have two approximate solutions: $u\left(x,\tau;\dfrac{1}{12},\dfrac{T}{16}\right)$ and $u\left(x,\tau;\dfrac{1}{9},\dfrac{T}{12}\right)$. Find a linear combination

$$(1-d)\times u\left(x,\tau;\frac{1}{12},\frac{T}{16}\right)+d\times u\left(x,\tau;\frac{1}{9},\frac{T}{12}\right)$$

such that it is an approximate solution with an error of $O(\Delta\tau^3)$.

(b) Suppose that there is another approximate solution $u\left(x,\tau;\dfrac{1}{15},\dfrac{T}{20}\right)$. Find a linear combination

$$d_0\times u\left(x,\tau;\frac{1}{15},\frac{T}{20}\right)+d_1\times u\left(x,\tau;\frac{1}{12},\frac{T}{16}\right)+d_2\times u\left(x,\tau;\frac{1}{9},\frac{T}{12}\right)$$

such that it is an approximate solution with an error of $O(\Delta\tau^4)$, where $d_0=1-d_1-d_2$.

18. *Explain why, how and when the extrapolation technique will improve the accuracy of numerical solutions.

19. Let $\mathcal{V}=\{u\,|\,u=(u_0,u_1,\cdots,u_{M-1},u_M)\}$ be the grid function space on $\Omega_h=\{x_m\mid x_m=x_l+mh,0\le m\le M,h=(x_u-x_l)/M\}$. For any $u\in\mathcal{V}$, and $v\in\mathcal{V}$, introduce the inner product

$$(u,v)=h\left(\frac{1}{2}u_0v_0+\sum_{m=1}^{M-1}u_mv_m+\frac{1}{2}u_Mv_M\right)$$

and norm

$$\|u\| = \sqrt{(u, u)}.$$

In addition, denote

$$\Delta_x u_m = \frac{1}{2h}(u_{m+1} - u_{m-1}), \quad \delta_x^2 u_m = \frac{1}{h^2}(u_{m+1} - 2u_m + u_{m-1}).$$

(a) Suppose

$$a(x) \in C^{(2)}[x_l, x_u], \quad a(x) \ge 0, \quad a(x_l) = a(x_u) = a'(x_l) = a'(x_u) = 0$$

and

$$\max_{x_l \le x \le x_u} |a''(x)| = c_1.$$

Prove

$$\left(a\delta_x^2 u, u\right) \le \frac{1}{2} c_1 \|u\|^2.$$

(b) Suppose

$$b(x) \in C^{(1)}[x_l, x_u], \qquad b(x_l) = b(x_u) = 0, \qquad \max_{x_l \le x \le x_u} |b'(x)| = c_2.$$

Prove

$$(b\Delta_x u, u) \le \frac{1}{2} c_2 \|u\|^2.$$

20. Suppose that $(a_{12})_{0n} = (a_{12})_{Mn} = (a_{12})_{m0} = (a_{12})_{mN} = 0$. Show

$$\begin{aligned}
&h_1 h_2 \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} (a_{12})_{mn}\, \delta_x \delta_y u_{m-\frac{1}{2}, n+\frac{1}{2}}\, u_{mn} \\
&= -h_1 h_2 \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} (a_{12})_{mn}\, \delta_x u_{m+\frac{1}{2}, n}\, \delta_y u_{m, n+\frac{1}{2}} \\
&\quad - h_1 h_2 \sum_{m=0}^{M-1} \sum_{n=1}^{N-1} (\delta_x a_{12})_{m+\frac{1}{2}, n}\, \delta_y u_{m, n+\frac{1}{2}}\, u_{m+1, n}
\end{aligned}$$

and

$$\begin{aligned}
&h_1 h_2 \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} (a_{12})_{mn}\, \delta_x \delta_y u_{m+\frac{1}{2}, n+\frac{1}{2}}\, u_{mn} \\
&= -h_1 h_2 \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} (a_{12})_{mn}\, \delta_x u_{m-\frac{1}{2}, n}\, \delta_y u_{m, n+\frac{1}{2}} \\
&\quad - h_1 h_2 \sum_{m=0}^{M-1} \sum_{n=1}^{N-1} (\delta_x a_{12})_{m+\frac{1}{2}, n}\, \delta_y u_{m+1, n+\frac{1}{2}}\, u_{mn}
\end{aligned}$$

by a direct calculation.

21. Suppose $\{u_m^k\}$ is the solution of the difference scheme

$$\frac{1}{\Delta\tau}(u_m^{k+1} - u_m^k) = a(x_m)\delta_x^2 u_m^{k+\frac{1}{2}} + b(x_m)\Delta_x u_m^{k+\frac{1}{2}} + c(x_m)u_m^{k+\frac{1}{2}}$$
$$+ g(x_m, \tau^{k+\frac{1}{2}}), \quad 0 \le m \le M, \quad 0 \le k \le K-1,$$
$$u_m^0 = f(x_m), \quad 0 \le m \le M,$$

where $u_m^{k+\frac{1}{2}} = \frac{1}{2}\left(u_m^k + u_m^{k+1}\right)$ and

$$a(x) \in C^{(2)}[x_l, x_u], \qquad b(x) \in C^{(1)}[x_l, x_u],$$
$$a(x) \ge 0, \quad a(x_l) = a(x_u) = a'(x_l) = a'(x_u) = b(x_l) = b(x_u) = 0,$$
$$\max_{x_l \le x \le x_u} |a''(x)| = c_1, \qquad \max_{x_l \le x \le x_u} |b'(x)| = c_2, \qquad \max_{x_l \le x \le x_u} |c(x)| = c_3,$$
$$c = c_1 + c_2 + 2c_3, \qquad \Delta\tau \le 2/[3(c+1)].$$

Prove

$$\|u^{k+1}\|^2 \le \mathrm{e}^{3(c+1)T/2}\left(\|f\|^2 + \frac{3}{2}\Delta t \sum_{l=0}^{k} \|g^{l+\frac{1}{2}}\|^2\right), \quad 0 \le k \le K-1.$$

8

Initial-Boundary Value and LC Problems

Evaluation of European-style derivatives can be reduced to solving initial value or initial-boundary value problems of parabolic partial differential equations. This chapter discusses numerical methods for such problems. If an American option problem is formulated as a linear complementarity problem, then the only difference between solving a European option and an American option is that if the solution obtained by the partial differential equation does not satisfy the constraint at some point, then the solution of the PDE at the point should be replaced by the value determined from the constraint condition. Such methods are usually referred to as projected methods for American-style derivatives. Therefore, the two methods are very close, and we also study the projected methods in this chapter.

In this chapter, there are four sections. The first two sections are devoted to explicit and implicit schemes, respectively. As we know, the derivative of the function representing the payoff of an option usually is discontinuous. This fact makes numerical methods inefficient. In many cases, an option problem can be reduced to another problem that has either a smooth solution or a solution with a weaker singularity than the solution of the option problem itself, and the numerical solution of the new problem can be obtained efficiently. We call such a method the singularity-separating method. In Sect. 8.3, we give several examples to illustrate how such a method works. In the final section, we discuss the pseudo-spectral method, which is very efficient if the solution is smooth. Examples are given to explain this fact.

8.1 Explicit Methods

8.1.1 Pricing European Options by Using $\overline{V}$, ξ, τ or u, x, $\bar{\tau}$ Variables

In Sect. 2.2.5, we obtained the formulation of the problem satisfied by a call/put option on a finite domain:

Y.-l. Zhu et al., *Derivative Securities and Difference Methods*, Springer Finance, DOI 10.1007/978-1-4614-7306-0_8,

$$\begin{cases} \dfrac{\partial \overline{V}}{\partial \tau} = \dfrac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1-\xi)^2\dfrac{\partial^2 \overline{V}}{\partial \xi^2} + (r - D_0)\xi(1-\xi)\dfrac{\partial \overline{V}}{\partial \xi} \\ \qquad -[r(1-\xi) + D_0\xi]\overline{V}, \qquad 0 \le \xi \le 1, \quad 0 \le \tau \le T, \\ \overline{V}(\xi, 0) = \max(\pm(2\xi - 1), 0), \qquad 0 \le \xi \le 1, \end{cases} \tag{8.1}$$

where the sign "+" in $\pm$ corresponds to the call option and the sign "−" in $\pm$ the put option. Here, we assume that the volatility depends on S, so $\bar{\sigma}$ is a function of ξ in the equation. Let

$$\begin{cases} \xi_m = m\Delta\xi, \; m = 0, 1, \cdots, M, \\ \tau^n = n\Delta\tau, \;\; n = 0, 1, \cdots, N, \end{cases} \tag{8.2}$$

where M and N are given integers, and $\Delta\xi = 1/M$ and $\Delta\tau = T/N$. This means that we use an $M \times N$ equidistant mesh on the domain $[0, 1] \times [0, T]$. Let v_m^n denote the approximate value of $\overline{V}(\xi, \tau)$ at $\xi = \xi_m$ and $\tau = \tau^n$, and $\{v_m^n\}$ represent the set v_m^n, $m = 0, 1, \cdots, M$. Discretizing the partial differential equation in the problem (8.1) at the point $(\xi_m, \; \tau^n)$ by scheme (7.5), i.e., by using the forward difference for $\dfrac{\partial \overline{V}}{\partial \tau}$ and the central difference for $\dfrac{\partial^2 \overline{V}}{\partial \xi^2}$ and $\dfrac{\partial \overline{V}}{\partial \xi}$, we get

$$\begin{aligned} \frac{v_m^{n+1} - v_m^n}{\Delta\tau} &= \frac{1}{2}\bar{\sigma}_m^2\xi_m^2(1-\xi_m)^2\frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{\Delta\xi^2} \\ &\quad + (r - D_0)\xi_m(1-\xi_m)\frac{v_{m+1}^n - v_{m-1}^n}{2\Delta\xi} \\ &\quad - [r(1-\xi_m) + D_0\xi_m]v_m^n \end{aligned}$$

or

$$\begin{aligned} v_m^{n+1} &= \frac{1}{2}\left[\bar{\sigma}_m^2\xi_m^2(1-\xi_m)^2 - (r - D_0)\xi_m(1-\xi_m)\Delta\xi\right]\alpha v_{m-1}^n \\ &\quad + [1 - \bar{\sigma}_m^2\xi_m^2(1-\xi_m)^2\alpha - (r(1-\xi_m) + D_0\xi_m)\Delta\tau]v_m^n \\ &\quad + \frac{1}{2}\left[\bar{\sigma}_m^2\xi_m^2(1-\xi_m)^2 + (r - D_0)\xi_m(1-\xi_m)\Delta\xi\right]\alpha v_{m+1}^n, \\ &\quad m = 0, 1, \cdots, M, \quad n = 0, 1, \cdots, N-1, \end{aligned} \tag{8.3}$$

where

$$\alpha = \frac{\Delta\tau}{\Delta\xi^2}.$$

In order for scheme (8.3) to be stable, we require

$$\max_{0 \le m \le M}[\bar{\sigma}_m^2\xi_m^2(1-\xi_m)^2]\frac{\Delta\tau}{2\Delta\xi^2} \le \frac{1}{2}$$

because if this is true, then $|\lambda_\theta| \le 1 + O(\Delta\tau)$ (see Problem 14 in Chap. 7). In practice, we can replace this condition by a slightly stronger condition as follows:

$$\Delta\tau \le \frac{16\Delta\xi^2}{\max\limits_{0\le m\le M} \bar{\sigma}_m^2}. \tag{8.4}$$

Sometimes, for example, when a lookback option needs to be priced, the value at a boundary is determined by a boundary condition which involves a derivative. In such cases, $\dfrac{\partial}{\partial\xi}$ needs to be discretized by a one-sided first or second order scheme.

From the difference scheme (8.3), we know that when the values v_{m-1}^n, v_m^n, and v_{m+1}^n are given, v_m^{n+1} can be obtained immediately. At a glance, it appears that v_{-1}^n and v_{M+1}^n are needed when v_0^{n+1} and v_M^{n+1} are calculated. As pointed out in Sect. 7.1, because the coefficients of v_{-1}^n and v_{M+1}^n equal zero, the values of v_{-1}^n and v_{M+1}^n will not be used. Therefore, if $\{v_m^n\}$ is given, then $\{v_m^{n+1}\}$ can be obtained by the difference scheme (8.3). According to the initial condition given in the problem (8.1), we have

$$v_m^0 = \max(\pm(2\xi_m - 1), 0).$$

Therefore, from $\{v_m^0\}$, we can get $\{v_m^n\}$, $n = 1, 2, \cdots, N$ successively. Usually, we need the value of V at a certain point S^* at time zero. After $\{v_m^N\}$ have been obtained, $V(S^*, 0)$ can be found in the following way. First, we need to find $v(\xi^*, T)$ by using the quadratic interpolation given in Sect. 6.1, where $\xi^* = \dfrac{S^*}{S^* + E}$. Then, we can obtain $V(S^*, 0)$ from $v(\xi^*, T)$ by

$$V(S^*, 0) = (S^* + E)v(\xi^*, T).$$

This method works not only for a constant σ but also for a variable σ, namely, $\sigma = \sigma(S)$, even $\sigma = \sigma(S, t)$. In what follows, this scheme is referred to as the explicit finite-difference method I, and its abbreviation is EFDI.

If σ is a constant, then an alternative way to find the approximate solution of the European options is to use u, x, $\bar{\tau}$ variables. From Sect. 2.6.1, we know that if $E = 1$, i.e., if the stock price and the option price has been divided by the exercise price, then pricing a call/put option can be reduced to finding $u(x, \bar{\tau})$, which is the solution of the problem:

$$\begin{cases} \dfrac{\partial u}{\partial \bar{\tau}} = \dfrac{\partial^2 u}{\partial x^2}, & -\infty < x < \infty, \quad 0 \le \bar{\tau} \le \frac{1}{2}\sigma^2 T, \\ u(x, 0) = \max(\pm(\mathrm{e}^x - 1), 0), & -\infty < x < \infty. \end{cases} \tag{8.5}$$

Here,

$$x = \ln S + (r - D_0 - \sigma^2/2)(T - t), \quad \bar{\tau} = \sigma^2(T - t)/2$$

and

$$u(x, \bar{\tau}) = \mathrm{e}^{r(T-t)} V(S, t).$$

Let $x_m = a + m\Delta x$, a being a given number and $\bar{\tau}^n = n\Delta\bar{\tau}$, and let u_m^n denote the approximate value of $u(x_m, \bar{\tau}^n)$. Then, the partial differential equation can be discretized by the difference scheme (7.8):

$$u_m^{n+1} = \bar{\alpha} u_{m+1}^n + (1 - 2\bar{\alpha}) u_m^n + \bar{\alpha} u_{m-1}^n, \tag{8.6}$$

where

$$\bar{\alpha} = \frac{\Delta\bar{\tau}}{\Delta x^2}.$$

From Sect. 7.2.1, we know that in order for the scheme to be stable, we need to require

$$\bar{\alpha} = \frac{\Delta\bar{\tau}}{\Delta x^2} \leq \frac{1}{2}. \tag{8.7}$$

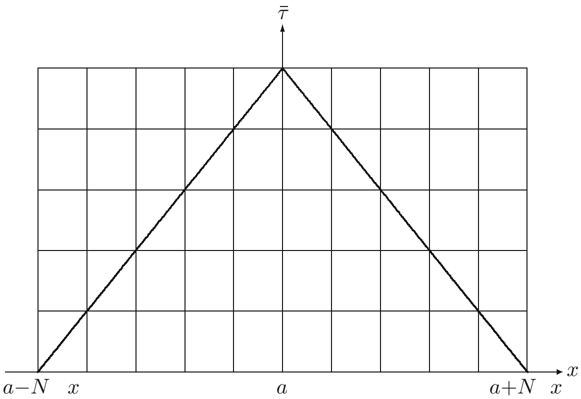

Fig. 8.1. A triangle mesh ($N = 5$)

Suppose again that we need to find $V(S^*, 0)$, i.e., we need to know $u\left(\ln S^* + (r - D_0 - \sigma^2/2)T, \sigma^2 T/2\right)$. Assume that we will use N steps in $\bar{\tau}$ direction, i.e., $\Delta\bar{\tau} = \dfrac{\sigma^2 T}{2N}$. In order to find $u\left(\ln S^* + (r - D_0 - \sigma^2/2)T, \sigma^2 T/2\right)$, we can use a triangle mesh (see Fig. 8.1): $\bar{\tau}^n = n\Delta\bar{\tau}, n = 0, 1, \cdots, N$ and for each n, $x_m = \ln S^* + (r - D_0 - \sigma^2/2)T + m\Delta x$, $m = -N + n, -N + n + 1, \cdots, N - n - 1, N - n$. From the initial condition at $\bar{\tau} = 0$, we have

$$u_m^0 = \max\left(\pm(\mathrm{e}^{x_m} - 1), 0\right), \quad m = -N, -N + 1, \cdots, N - 1, N.$$

It is clear that when $u_m^n, m = -N+n, -N+n+1, \cdots, N-n-1, N-n$ are given, we can obtain $u_m^{n+1}, m = -N + n + 1, -N + n + 2, \cdots, N - n - 2, N - n - 1$.

Therefore, starting from $u_m^0, m = -N, -N+1, \cdots, N-1, N$, we can find $u_m^n, m = -N+n, -N+n+1, \cdots, N-n-1, N-n$ for $n = 1, 2, \cdots, N$ successively. When we get u_0^N, $V(S^*, 0)$ can be calculated by

$$V(S^*, 0) = \mathrm{e}^{-rT} u_0^N$$

because $V(S,t) = \mathrm{e}^{-r(T-t)} u(\ln S + (r - D_0 - \sigma^2/2)(T-t), \sigma^2(T-t)/2)$.

Table 8.1. Values of European put options (EFDI)

($E = 50$, $S = 48$, $r = 0.05$, $\sigma = 0.20$, and $D_0 = 0$)

$\Delta\tau$	$T = 0.25$	$T = 0.50$	$T = 0.75$	$T = 1.00$
0.01	2.7220	3.1163	3.4045	3.5852
0.001	2.7087	3.1275	3.3989	3.5910
0.0001	2.7083	3.1272	3.3986	3.5907
Exact	$2.708349\cdots$	$3.127199\cdots$	$3.398586\cdots$	$3.590738\cdots$

Assume that we want to calculate the value of an option on a stock when the stock price is \$100 and the exercise price is \$90. In this method above, the stock price and the option price has been divided by E, so S^* should be 100/90, and the real option price should be obtained by $90 \times V(S^*, 0)$. This method is referred to as the explicit finite-difference method II, and its abbreviation is EFDII.

Example 1: Using EFDI with

$$\Delta\xi \approx \sqrt{\max_{0 \le m \le M} \bar{\sigma}_m^2 \Delta\tau}/4,$$

we have solved European put problems using different $\Delta\tau$. Numerical results for $T = 0.25$, 0.5, 0.75, and 1.00 are listed in Table 8.1, and the other problem parameters are also shown there. From the table we see that for $\Delta\tau = 0.01, 0.001$, and 0.0001, the error is about on the second, third, and fourth decimal places.

8.1.2 Projected Methods for LC Problems

In Sect. 3.2, we saw that an American option problem could be formulated as a linear complementarity problem. When the variables $\overline{V}, \xi, \tau$ are adopted, the linear complementarity problem is

$$\begin{cases} \min\left(\dfrac{\partial \overline{V}}{\partial \tau} - \mathbf{L}_\xi \overline{V}, \overline{V}(\xi, \tau) - \max(\pm(2\xi - 1), 0)\right) = 0, \\ \overline{V}(\xi, 0) = \max(\pm(2\xi - 1), 0), \end{cases} \tag{8.8}$$

where

$$\mathbf{L}_\xi = \frac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1-\xi)^2\frac{\partial^2}{\partial \xi^2} + (r-D_0)\xi(1-\xi)\frac{\partial}{\partial \xi} - [r(1-\xi)+D_0\xi];$$

whereas if the variables $u, x, \bar{\tau}$ are used, the linear complementarity problem is

$$\begin{cases} \min\left(\dfrac{\partial u}{\partial \bar{\tau}} - \dfrac{\partial^2 u}{\partial x^2},\, u(x,\bar{\tau}) - g(x,\bar{\tau})\right) = 0, \\ u(x,0) = g(x,0), \end{cases} \tag{8.9}$$

where

$$g(x,\bar{\tau}) = \max\left(\pm(\mathrm{e}^{x+(2D_0/\sigma^2+1)\bar{\tau}} - \mathrm{e}^{2r\bar{\tau}/\sigma^2}), 0\right).$$

Such a formulation can be described in another way. Let us take the problem (8.9) as an example in order to explain it. Suppose that we have obtained the solution at $\bar{\tau} = \bar{\tau}^*$, $u(x,\bar{\tau}^*)$. Starting from $u(x,\bar{\tau}^*)$, we can find the solution $u(x,\bar{\tau}^* + \Delta\bar{\tau})$ in the following way. Let $\tilde{u}(x,\bar{\tau}^* + \Delta\bar{\tau})$ be the solution determined by an approximation to the equation

$$\frac{\partial \tilde{u}}{\partial \bar{\tau}} - \frac{\partial^2 \tilde{u}}{\partial x^2} = 0.$$

If

$$\tilde{u}(x,\bar{\tau}^* + \Delta\bar{\tau}) \geq g(x,\bar{\tau}^* + \Delta\bar{\tau})$$

at a point, then

$$u(x,\bar{\tau}^* + \Delta\bar{\tau}) = \tilde{u}(x,\bar{\tau}^* + \Delta\bar{\tau});$$

otherwise

$$u(x,\bar{\tau}^* + \Delta\bar{\tau}) = g(x,\bar{\tau}^* + \Delta\bar{\tau}).$$

That is, for each x,

$$u(x,\bar{\tau}^* + \Delta\bar{\tau}) = \max\left(\tilde{u}(x,\bar{\tau}^* + \Delta\bar{\tau}),\ g(x,\bar{\tau}^* + \Delta\bar{\tau})\right).$$

Does the solution determined in this way satisfy all the requirements in the problem (8.9)? When $\tilde{u}(x,\bar{\tau}^* + \Delta\bar{\tau}) \geq g(x,\bar{\tau}^* + \Delta\bar{\tau})$, we have $u(x,\bar{\tau}^* + \Delta\bar{\tau}) = \tilde{u}(x,\bar{\tau}^* + \Delta\bar{\tau})$, $u(x,\bar{\tau}^* + \Delta\bar{\tau}) \geq g(x,\bar{\tau}^* + \Delta\bar{\tau})$ and $\dfrac{\partial u}{\partial \bar{\tau}} - \dfrac{\partial^2 u}{\partial x^2} = 0$, so the first relation in the problem (8.9) holds; when $\tilde{u}(x,\bar{\tau}^* + \Delta\bar{\tau}) < g(x,\bar{\tau}^* + \Delta\bar{\tau})$, we have $u(x,\bar{\tau}^* + \Delta\bar{\tau}) = g(x,\bar{\tau}^* + \Delta\bar{\tau})$ and $\dfrac{\partial u}{\partial \bar{\tau}} - \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial g}{\partial \bar{\tau}} - \dfrac{\partial^2 g}{\partial x^2} > 0$, so the first relation in the problem (8.9) also holds. Thus the first relation in the problem (8.9) holds at any point. If the problem is formulated in the form (8.8), the situation is the same.

Therefore, if an American option is formulated as a linear complementarity problem, the difference between the numerical methods for European options and American options is not big. In fact, if the formulation (8.8) is used, then we can compute the value of American options by

$$v_m^{n+1} = \max(\tilde{v}_m^{n+1},\ \pm(2\xi - 1),\ 0), \tag{8.10}$$

where

$$\begin{aligned}\tilde{v}_m^{n+1} = & \frac{1}{2}\left[\bar{\sigma}_m^2 \xi_m^2 (1-\xi_m)^2 - (r - D_0)\xi_m(1-\xi_m)\Delta\xi\right]\alpha v_{m-1}^n \\ & +[1 - \bar{\sigma}_m^2 \xi_m^2 (1-\xi_m)^2 \alpha - (r(1-\xi_m) + D_0\xi)\Delta\tau] v_m^n \\ & +\frac{1}{2}\left[\bar{\sigma}_m^2 \xi_m^2 (1-\xi_m)^2 + (r - D_0)\xi_m(1-\xi_m)\Delta\xi\right]\alpha v_{m+1}^n .\end{aligned}$$

If the formulation (8.9) is adopted, then the computation is done by

$$u_m^{n+1} = \max\left(\bar{\alpha} u_{m+1}^n + (1-2\bar{\alpha})u_m^n + \bar{\alpha} u_{m-1}^n,\ g(x_m, \bar{\tau}^{n+1})\right). \tag{8.11}$$

Table 8.2. American call option (PEFDII)

($r = 0.1$, $\sigma = 0.2$, $D_0 = 0.05$, $S = E = 100$, $T = 1$, and the exact solution is $C = 9.94092345\cdots$)

Numbers of time steps	Results	|Errors|	CPU(s)
50	9.902768	0.038156	0.0003
100	9.921822	0.019102	0.0013
200	9.931367	0.009557	0.0053
400	9.936144	0.004780	0.0220
800	9.938533	0.002390	0.0880

Table 8.3. American put option (PEFDII)

($r = 0.1$, $\sigma = 0.2$, $D_0 = 0.05$, $S = E = 100$, $T = 1$, and the exact solution is $P = 5.92827717\cdots$)

Numbers of time steps	Results	|Errors|	CPU(s)
50	5.911829	0.016448	0.0003
100	5.920472	0.007805	0.0013
200	5.924476	0.003801	0.0054
400	5.926424	0.001853	0.0220
800	5.927360	0.000917	0.0880

Finding the prices of American options in such a way is referred to as a projected method in the book [84] by Wilmott, Dewynne, and Howison. We call Eqs. (8.10) and (8.11) projected explicit finite-difference methods I and II, respectively, and their abbreviations are PEFDI and PEFDII. Clearly, PEFDI can be applied to the cases with both a constant σ and a variable σ, and PEFDII is suitable only for the case that σ is a constant. In Tables 8.2 and 8.3, the results of call and put options on several meshes are given. The method used is PEFDII. The error and the CPU time needed are also shown. In order to have an error, we must have the exact solutions. The exact solution for the American call and put option problems with these parameters are $C = 9.94092345\cdots$ and $P = 5.92827717\cdots$, which are obtained by the SSM given in Chap. 9. Here, the first nine digits are given, and it is enough to determine the first few digits of the errors given in these tables. Computation is done on a Space Ultra 10 computer. In this book, when a CPU time is mentioned, the computation is done on such a computer if no other explanation is given.

8.1.3 Binomial and Trinomial Methods

This subsection is devoted to the binomial and trinomial methods. In these methods, there is a lattice of possible asset prices. Thus, such methods are also called lattice methods.

Binomial Methods. The binomial method is a simple and very effective method for computing the option prices.

When the Black–Scholes equation is derived, a risk-free portfolio is established. This idea can also be used to design numerical methods. Let S_n be the given stock price at time t^n, S_{n+1} be the stock price at time $t^{n+1} = t^n + \Delta t$, and the possible values of S_{n+1} be $S_{n+1,0}$ and $S_{n+1,1}$. Assume that the stock pays dividends continuously and the dividend yield is D_0. Therefore one share of stock at time t^n becomes $\mathrm{e}^{D_0 \Delta t}$ shares at time t^{n+1}. Let V_n be the price of a derivative at time t^n, and $V_{n+1,i}$ be the price of the derivative at time t^{n+1} if the stock price is $S_{n+1,i}$, $i = 0$ and 1. That the portfolio

$$V - \Delta S$$

is risk-free means that

$$V_{n+1,0} - \Delta \mathrm{e}^{D_0 \Delta t} S_{n+1,0} = V_{n+1,1} - \Delta \mathrm{e}^{D_0 \Delta t} S_{n+1,1} = (V_n - \Delta S_n)\, \mathrm{e}^{r \Delta t}.$$

Therefore

$$\Delta = \frac{V_{n+1,1} - V_{n+1,0}}{S_{n+1,1} - S_{n+1,0}} \mathrm{e}^{-D_0 \Delta t}$$

and

$$\begin{aligned}
V_n &= \mathrm{e}^{-r\Delta t} \left(V_{n+1,0} - \Delta \mathrm{e}^{D_0 \Delta t} S_{n+1,0} \right) + \Delta S_n \\
&= \mathrm{e}^{-r\Delta t} \left(V_{n+1,0} - \frac{V_{n+1,1} - V_{n+1,0}}{S_{n+1,1} - S_{n+1,0}} S_{n+1,0} \right) + \frac{V_{n+1,1} - V_{n+1,0}}{S_{n+1,1} - S_{n+1,0}} \mathrm{e}^{-D_0 \Delta t} S_n \\
&= \mathrm{e}^{-r\Delta t} \left[\frac{S_n \mathrm{e}^{(r-D_0)\Delta t} - S_{n+1,0}}{S_{n+1,1} - S_{n+1,0}} V_{n+1,1} \right. \\
&\qquad \left. + \left(1 - \frac{S_n \mathrm{e}^{(r-D_0)\Delta t} - S_{n+1,0}}{S_{n+1,1} - S_{n+1,0}} \right) V_{n+1,0} \right].
\end{aligned}$$

Let

$$p = \frac{S_n \mathrm{e}^{(r-D_0)\Delta t} - S_{n+1,0}}{S_{n+1,1} - S_{n+1,0}}, \tag{8.12}$$

then we have

$$V_n = \mathrm{e}^{-r\Delta t} \left[pV_{n+1,1} + (1-p)\, V_{n+1,0} \right]. \tag{8.13}$$

Suppose that in the real world, the stock price satisfies

$$dS = \mu S dt + \sigma S dX = \mu S dt + \sigma S \phi \sqrt{dt},$$

or

$$S_{n+1} - S_n = \mu S_n \Delta t + \sigma S_n \phi \sqrt{\Delta t},$$

where ϕ is the standardized normal random variable. Using Itô's lemma, this model can be rewritten as

$$d \ln S = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dX = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma \phi \sqrt{dt},$$

or

$$\ln S_{n+1} - \ln S_n = \left(\mu - \frac{\sigma^2}{2} \right) \Delta t + \sigma \phi \sqrt{\Delta t}. \tag{8.14}$$

According to this model, the number of possible prices of the stock at time t_{n+1} is infinity. In the derivation above, we think that there are only two possible values the price of the stock can take at time t_{n+1}. Thus the random variable ϕ is approximated by a binomial random variable. Let ψ denote this binomial random variable. Because $\mathrm{E}[\phi] = 0$ and $\mathrm{E}[\phi^2] = \mathrm{Var}[\phi] + \mathrm{E}^2[\phi] = 1$, it is natural to require $\mathrm{E}[\psi] = 0$ and $\mathrm{E}[\psi^2] = 1$. Suppose that the two values of ψ are ψ_0 and ψ_1 and that the probabilities of taking ψ_0 and ψ_1 are $1-q$ and q, respectively. Then the two conditions can be written as

$$\begin{cases} (1-q)\, \psi_0 + q\psi_1 = 0, \\ (1-q)\, \psi_0^2 + q\psi_1^2 = 1. \end{cases}$$

From these two equations we can have

$$\begin{cases} q = \dfrac{-\psi_0}{\psi_1 - \psi_0}, \\ q = \dfrac{1 - \psi_0^2}{\psi_1^2 - \psi_0^2}. \end{cases}$$

Hence

$$-\psi_0 = \frac{1 - \psi_0^2}{\psi_1 + \psi_0}$$

or

$$\psi_0 \psi_1 = -1.$$

Therefore $\psi_0\psi_1 = -1$ is a necessary condition for $\mathrm{E}[\psi^2] = 1$ and $\mathrm{E}[\psi] = 0$. From the procedure of deriving this condition, it is easy to see that this condition is also a sufficient condition for $\mathrm{E}[\psi^2] = 1$ if $\mathrm{E}[\psi] = 0$. It is clear, if we choose ψ_0 and ψ_1 so that

$$\psi_0\psi_1 = -1 + O(\Delta t)$$

and require $\mathrm{E}[\psi] = 0$, then ψ is still a good approximate to ϕ.

Suppose that ψ_i is related to $S_{n+1,i}, i = 0, 1$. Thus we have

$$\begin{cases} \ln S_{n+1,0} = \ln S_n + \left(\mu - \dfrac{\sigma^2}{2}\right)\Delta t + \sigma\psi_0\sqrt{\Delta t}, \\ \ln S_{n+1,1} = \ln S_n + \left(\mu - \dfrac{\sigma^2}{2}\right)\Delta t + \sigma\psi_1\sqrt{\Delta t}. \end{cases}$$

Let us choose

$$\begin{cases} \psi_0 = -1 - \left(\mu - \dfrac{\sigma^2}{2}\right)\sqrt{\Delta t}/\sigma, \\ \psi_1 = 1 - \left(\mu - \dfrac{\sigma^2}{2}\right)\sqrt{\Delta t}/\sigma. \end{cases} \tag{8.15}$$

Because $\psi_0\psi_1 = -1 + \left(\mu - \dfrac{\sigma^2}{2}\right)^2 \Delta t/\sigma^2$, ψ is an approximate to ϕ. In this case

$$\begin{cases} \ln S_{n+1,0} = \ln S_n - \sigma\sqrt{\Delta t}, \\ \ln S_{n+1,1} = \ln S_n + \sigma\sqrt{\Delta t}, \end{cases}$$

or

$$\begin{cases} S_{n+1,0} = S_n \mathrm{e}^{-\sigma\sqrt{\Delta t}}, \\ S_{n+1,1} = S_n \mathrm{e}^{\sigma\sqrt{\Delta t}}. \end{cases} \tag{8.16}$$

Using the formulae (8.12), (8.13) and (8.16), we can evaluate the price of a derivative if the stock price satisfies Eq. (8.14). This is called the binomial method which was proposed by Cox, Ross, and Rubinstein in 1979 [22].

For ψ_0 and ψ_1, we can choose other expressions. For example (see the book by McDonald [61]), let

$$\begin{cases} \psi_0 = -1 - \left(\mu - r + D_0 - \dfrac{\sigma^2}{2}\right)\sqrt{\Delta t}/\sigma, \\ \psi_1 = 1 - \left(\mu - r + D_0 - \dfrac{\sigma^2}{2}\right)\sqrt{\Delta t}/\sigma. \end{cases} \tag{8.17}$$

Because $\psi_0\psi_1 = -1 + \left(\mu - r - D_0 - \frac{\sigma^2}{2}\right)^2 \Delta t/\sigma^2$, ψ is an approximate to ϕ. In this case

$$\begin{cases} S_{n+1,0} = S_n \mathrm{e}^{(r-D_0)\Delta t - \sigma\sqrt{\Delta t}}, \\ S_{n+1,1} = S_n \mathrm{e}^{(r-D_0)\Delta t + \sigma\sqrt{\Delta t}}. \end{cases} \tag{8.18}$$

Generally, we can choose

$$\begin{cases} \psi_0 = -1 - \left(\mu - c - \frac{\sigma^2}{2}\right)\sqrt{\Delta t}/\sigma, \\ \psi_1 = 1 - \left(\mu - c - \frac{\sigma^2}{2}\right)\sqrt{\Delta t}/\sigma. \end{cases} \tag{8.19}$$

In this case

$$\begin{cases} S_{n+1,0} = S_n \mathrm{e}^{c\Delta t - \sigma\sqrt{\Delta t}}, \\ S_{n+1,1} = S_n \mathrm{e}^{c\Delta t + \sigma\sqrt{\Delta t}}, \end{cases} \tag{8.20}$$

and both the formulae (8.16) and (8.18) are in this form.

If p is determined by the formula (8.12), then we have

$$\begin{aligned} & pS_{n+1,1} + (1-p)\, S_{n+1,0} \\ = & \frac{S_n \mathrm{e}^{(r-D_0)\Delta t} - S_{n+1,0}}{S_{n+1,1} - S_{n+1,0}} S_{n+1,1} + \frac{S_{n+1,1} - S_n \mathrm{e}^{(r-D_0)\Delta t}}{S_{n+1,1} - S_{n+1,0}} S_{n+1,0} \\ = & \mathrm{e}^{(r-D_0)\Delta t} S_n. \end{aligned}$$

When $0 \le p \le 1$, this relation can be interpreted as follows. When a derivative is priced, the probability of the price at t^{n+1} being $S_{n+1,1}$ is p and the probability of the price at t^{n+1} being $S_{n+1,0}$ is $1-p$, and the expectation of the stock price at t^{n+1} is $\mathrm{e}^{(r-D_0)\Delta t} S_n$:

$$\mathrm{E}_D\left[S_{n+1}\right] = pS_{n+1,1} + (1-p)\, S_{n+1,0} = \mathrm{e}^{(r-D_0)\Delta t} S_n = \mathrm{e}^{r\Delta t}\mathrm{e}^{-D_0\Delta t} S_n, \tag{8.21}$$

where we use E_D as the notation for expectation in the case a derivative is priced. In the front of S_n there is a factor $\mathrm{e}^{-D_0\Delta t}$ because the expectation of the stock price should go down by a factor of $\mathrm{e}^{-D_0\Delta t}$ as one share of stock at time t^n becomes $\mathrm{e}^{D_0\Delta t}$ shares of stock at time t^{n+1}, and there is another factor $\mathrm{e}^{r\Delta t}$ because the expectation of the stock price should go up by a factor of $\mathrm{e}^{r\Delta t}$ just like any risk-free investment. Because of this, we usually say that $\mathrm{E}_D\left[S_{n+1}\right]$ is the expectation of S_{n+1} in the "risk-neutral" world. According to the model for the stock price, we have

$$\mathrm{E}[S_{n+1}] = S_n + \mu S_n \Delta t = \left(\mathrm{e}^{\mu\Delta t} + O(\Delta t^2)\right) S_n.$$

That is, in the expression for the expectation of the stock price at time t_{n+1} in the real world, there is a factor about $\mathrm{e}^{\mu\Delta t}$ in the front of S_n, which is completely different from the case when we price derivatives.

When $S_{n+1,0}$ and $S_{n+1,1}$ are given by Eq. (8.16), then

$$p=\frac{S_n\mathrm{e}^{(r-D_0)\varDelta t}-S_n\mathrm{e}^{-\sigma\sqrt{\varDelta t}}}{S_n\mathrm{e}^{\sigma\sqrt{\varDelta t}}-S_n\mathrm{e}^{-\sigma\sqrt{\varDelta t}}}=\frac{\mathrm{e}^{(r-D_0)\varDelta t}-\mathrm{e}^{-\sigma\sqrt{\varDelta t}}}{\mathrm{e}^{\sigma\sqrt{\varDelta t}}-\mathrm{e}^{-\sigma\sqrt{\varDelta t}}} \tag{8.22}$$

and $0\le p\le 1$ is equivalent to $\mathrm{e}^{-\sigma\sqrt{\varDelta t}}\le \mathrm{e}^{(r-D_0)\varDelta t}\le \mathrm{e}^{\sigma\sqrt{\varDelta t}}$. The inequality $\mathrm{e}^{(r-D_0)\varDelta t}\le \mathrm{e}^{\sigma\sqrt{\varDelta t}}$ might not hold for large $\varDelta t$ and p does not represent a probability in this case. However this case usually does not occur in practice because $\varDelta t$ would be small in real computation. When $S_{n+1,0}$ and $S_{n+1,1}$ are given by the formula (8.18), then

$$p=\frac{S_n\mathrm{e}^{(r-D_0)\varDelta t}-S_n\mathrm{e}^{(r-D_0)\varDelta t-\sigma\sqrt{\varDelta t}}}{S_n\mathrm{e}^{(r-D_0)\varDelta t+\sigma\sqrt{\varDelta t}}-S_n\mathrm{e}^{(r-D_0)\varDelta t-\sigma\sqrt{\varDelta t}}}=\frac{1-\mathrm{e}^{-\sigma\sqrt{\varDelta t}}}{\mathrm{e}^{\sigma\sqrt{\varDelta t}}-\mathrm{e}^{-\sigma\sqrt{\varDelta t}}} \tag{8.23}$$

and $0\le p\le 1$ always holds. Hence in this case p can always be interpreted as the probability of the price being $S_{n+1,1}$ at t_{n+1}.

In the "risk-neutral" world, the variance of S_{n+1} is

$$\begin{aligned}
&\mathrm{Var}_D\left[S_{n+1}\right]\\
&=\frac{S_n\mathrm{e}^{(r-D_0)\varDelta t}-S_{n+1,0}}{S_{n+1,1}-S_{n+1,0}}\left(S_{n+1,1}-\mathrm{e}^{(r-D_0)\varDelta t}S_n\right)^2\\
&\quad+\frac{S_{n+1,1}-S_n\mathrm{e}^{(r-D_0)\varDelta t}}{S_{n+1,1}-S_{n+1,0}}\left(S_{n+1,0}-\mathrm{e}^{(r-D_0)\varDelta t}S_n\right)^2\\
&=\left(S_n\mathrm{e}^{(r-D_0)\varDelta t}-S_{n+1,0}\right)\left(S_{n+1,1}-S_n\mathrm{e}^{(r-D_0)\varDelta t}\right)\\
&=S_n^2\mathrm{e}^{2(r-D_0)\varDelta t}\cdot\left(1-\frac{S_{n+1,0}}{S_n\mathrm{e}^{(r-D_0)\varDelta t}}\right)\left(\frac{S_{n+1,1}}{S_n\mathrm{e}^{(r-D_0)\varDelta t}}-1\right)\\
&=S_n^2\mathrm{e}^{2(r-D_0)\varDelta t}\cdot\left(\frac{S_{n+1,0}}{S_n\mathrm{e}^{(r-D_0)\varDelta t}}+\frac{S_{n+1,1}}{S_n\mathrm{e}^{(r-D_0)\varDelta t}}-\frac{S_{n+1,0}S_{n+1,1}}{S_n^2\mathrm{e}^{2(r-D_0)\varDelta t}}-1\right).
\end{aligned}$$

When $S_{n+1,0}$ and $S_{n+1,1}$ are given by the expression (8.20), both the formulae (8.16) and (8.18) being in this form, the expression above can further be written as:

$$\begin{aligned}
&\mathrm{Var}_D\left[S_{n+1}\right]\\
&=S_n^2\mathrm{e}^{2(r-D_0)\varDelta t}\left(\mathrm{e}^{-(r-D_0-c)\varDelta t-\sigma\sqrt{\varDelta t}}+\mathrm{e}^{-(r-Dt_0-c)\varDelta t+\sigma\sqrt{\varDelta t}}\right.\\
&\qquad\qquad\left.-\mathrm{e}^{-2(r-D_0-c)\varDelta t}-1\right)\\
&=S_n^2\mathrm{e}^{(r-D_0+c)\varDelta t}\left(\mathrm{e}^{-\sigma\sqrt{\varDelta t}}+\mathrm{e}^{\sigma\sqrt{\varDelta t}}-\mathrm{e}^{-(r-D_0-c)\varDelta t}-\mathrm{e}^{(r-D_0-c)\varDelta t}\right)\\
&=S_n^2\mathrm{e}^{(r-D_0+c)\varDelta t}\left[1-\sigma\sqrt{\varDelta t}+\frac{1}{2}\sigma^2\varDelta t-\frac{1}{6}\sigma^3\varDelta t^{3/2}+1+\sigma\sqrt{\varDelta t}+\frac{1}{2}\sigma^2\varDelta t\right.\\
&\qquad\left.+\frac{1}{6}\sigma^3\varDelta t^{3/2}-1+(r-D_0-c)\varDelta t-1-(r-D_0-c)\varDelta t+O\left(\varDelta t^2\right)\right]\\
&=S_n^2\mathrm{e}^{(r-D_0+c)\varDelta t}\left[\sigma^2\varDelta t+O\left(\varDelta t^2\right)\right]\\
&=S_n^2\sigma^2\varDelta t+O\left(\varDelta t^2\right).
\end{aligned} \tag{8.24}$$

In the real world,

$$\mathrm{Var}[S_{n+1}] = \mathrm{Var}\left[S_n + \mu S_n \Delta t + \sigma S_n \phi \sqrt{\Delta t}\right] = \sigma^2 S_n^2 \Delta t.$$

Therefore as $\Delta t \to 0$ the variance of S_{n+1} in the "risk-neutral" world will tend to the variance of S_{n+1} in the real world.

Now let us describe the complete method proposed by Cox, Ross, and Rubinstein [22]. Define

$$d = \mathrm{e}^{-\sigma\sqrt{\Delta t}} \tag{8.25}$$

and

$$u = \frac{1}{d} = \mathrm{e}^{\sigma\sqrt{\Delta t}}, \tag{8.26}$$

then $S_{n+1,1} = S_n u$, $S_{n+1,0} = S_n d$, and Eqs. (8.22) and (8.13) can be rewritten as

$$p = \frac{\mathrm{e}^{(r-D_0)\Delta t} - d}{u - d} \tag{8.27}$$

and

$$\begin{aligned} &V(S_n, n\Delta t) \\ &= \mathrm{e}^{-r\Delta t}\left[pV\left(S_{n+1,1}, (n+1)\Delta t\right) + (1-p)V\left(S_{n+1,0}, (n+1)\Delta t\right)\right]. \end{aligned} \tag{8.28}$$

Here $V(S, t)$ is the value of an option.

Suppose the asset price at the current time t to be S, and we divide the remaining life of the derivative security into N equal time subintervals with time step $\Delta t = (T - t)/N$. At the first time level $t + \Delta t$, there are two possible asset prices Su and $Sd = Su^{-1}$. At the second time level $t + 2\Delta t$, there are three possible asset prices, Su^2, $Sud = Sdu = S$, and $Sd^2 = Su^{-2}$. At the third time level $t + 3\Delta t$, there are four possible asset prices, Su^3, $Su^2d = Su$, $Sud^2 = Su^{-1}$, and $Sd^3 = Su^{-3}$. In general, at the n-th time level $t + n\Delta t$, there are $n + 1$ possible values of the asset price. Originally, at the n-th time level, there should be 2^n possible values of the asset price. However since $d = 1/u$ is required, many points are the same. For example, S, Su^2d^2, Su^4d^4, $\cdots$ are the same point because $d = 1/u$. Hence the number of possible values is greatly reduced. Let $S_{n,m}, m = 0, 1, \cdots, n$, denote the $n+1$ possible values of the asset price at the n-th time level from the smallest to the largest. Then

$$S_{n,m} = Su^{2m-n}, \; m = 0, 1, \cdots, n. \tag{8.29}$$

For $N = 4$, all the possible prices for each n are given in Fig. 8.2. This plot is usually referred to as a tree or lattice of possible asset prices.

Assuming that we know the payoff function for our derivative security and that it depends only on the values of the underlying asset at expiry, this

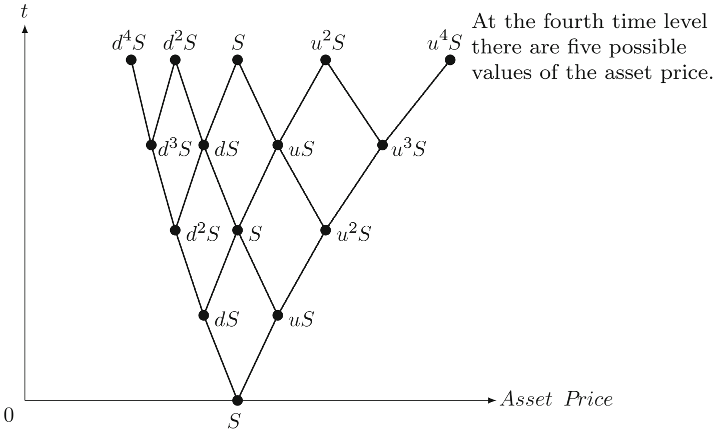

Fig. 8.2. Tree of asset prices for a binomial model

enables us to value it at expiry, the N-th time level. If we are considering a call, for example, we find

$$c_{N,m} = \max(S_{N,m} - E, 0), \quad m = 0, 1, \cdots, N, \tag{8.30}$$

where E is the exercise price and $c_{N,m}$ denotes the value of the call for the m-th possible asset value $S_{N,m}$ at time-step N. For a put, we know that

$$p_{N,m} = \max(E - S_{N,m}, 0), \quad m = 0, 1, \cdots, N, \tag{8.31}$$

where $p_{N,m}$ denotes the value of the put for the m-th possible asset value $S_{N,m}$ at expiry.

We can now find the expected value of the derivative security at the $(N-1)$-th time level and for possible asset prices $S_{N-1,m}$, $m = 0, 1, \ldots, N-1$ because we know that the probability of an asset price moving from $S_{N-1,m}$ to $S_{N,m+1}$ during a time step is p and that the probability of it moving to $S_{N,m}$ is $(1-p)$. Using the discounting factor $\mathrm{e}^{-r\Delta t}$, we can obtain the value of the security at each possible asset price for the $(N-1)$-th time level. This procedure can be applied to the n-th time level if the values of the option for the $(n+1)$-th time level have been obtained, and the computational formula is Eq. (8.28) or, in a general form,

$$V_{n,m} = \mathrm{e}^{-r\Delta t}(pV_{n+1,m+1} + (1-p)V_{n+1,m}), \quad m = 0, 1, \cdots, n. \tag{8.32}$$

Here, $V_{n,m}$ denotes the value of a European option at the n-th time level and corresponding to asset price $S_{n,m}$. According to this formula, starting from the payoff function, $V_{N,m}$, $m = 0, 1, \cdots, N$, we can recursively determine $V_{n,m}, m = 0, 1, \cdots, n$ for $n = N-1, N-2, \cdots, 0$, and the final value $V_{0,0}$ is the current value of the option.

For American options, we can easily incorporate the possibility of early exercise of an option into the binomial model. Because the price of an American call option must be greater than or equal to

$$\max(S_{n,m} - E, 0), \tag{8.33}$$

when calculating the price of an American call option, we need to replace the formula (8.32) by

$$C_{n,m} = \max\left(\mathrm{e}^{-r\Delta t}\left[pC_{n+1,m+1} + (1-p)C_{n+1,m}\right],\ S_{n,m} - E,\ 0\right) \tag{8.34}$$

at each point. Similarly, for an American put option, the formula is

$$P_{n,m} = \max\left(\mathrm{e}^{-r\Delta t}\left[pP_{n+1,m+1} + (1-p)P_{n+1,m}\right],\ E - S_{n,m},\ 0\right) \tag{8.35}$$

because the price of an American put option has to be at least

$$\max(E - S_{n,m}, 0). \tag{8.36}$$

From what has been described, we see that the entire computation can be done in two steps. In the first step, we calculate all the $S_{n,m}$ to be used. Then, we find $V_{N,m}, m = 0, 1, \cdots, N$ and calculate $V_{n,m}, m = 0, 1, \cdots, n$ for $n = N-1, N-2, \cdots, 0$ successively. When a European option is calculated, only the $S_{N,m}, m = 0, 1, \cdots, N$, are used in order to find $V_{N,m}$. When an American option is evaluated, all the $S_{n,m}$ are needed. However, because $S_{n,m} = Su^{2m-n} = Su^{2(m-1)-(n-2)} = S_{n-2,m-1}$, we indeed only need to calculate $S_{N,m}, m = 0, 1, \cdots, N$ and $S_{N-1,m}, m = 0, 1, \cdots, N-1$, i.e., $Su^m, m = -N, -N+1, \ldots, N$. For this method, the total number of nodes is $(N+2)(N+1)/2$, so the execution time for computing all the $V_{n,m}$ is $O(N^2)$.

If the method given in the book by McDonald [61] wants to be adopted, instead of the formulae (8.25)–(8.27), (8.18) and (8.23) should be used. Also the tree of asset prices is different. In this case we should define

$$S_{n,m} = S\mathrm{e}^{n(r-D_0)\Delta t}u^{2m-n},\ m = 0, 1, \cdots, n$$

with $u = \mathrm{e}^{\sigma\sqrt{\Delta t}}$.

Trinomial Methods. If σ depends on S, then u is not a constant. In this case, generally speaking, at the n-th time level, there are 2^n possible values of the asset prices that need to be considered, and the total nodes and the execution time will be very large if a binomial method is used. In order to reduce the nodes for a problem with variable σ, we can use trinomial methods. In a trinomial method, given a current asset value S, the asset value after a time-step Δt can take any of the three values

$$Su,\ Sq,\ Sd,$$

where $0 \le d < q < u$. Let p_u be the probability of the value of the asset after a time-step Δt being Su, p_q be the probability of the value being Sq, and p_d

be the probability of the value being Sd. Because there are only three possible cases, we must have

$$p_u + p_q + p_d = 1, \quad 0 \le p_u \le 1, \quad 0 \le p_q \le 1, \quad 0 \le p_u \le 1.$$

When the binomial method is used for pricing call/put options, from the expressions (8.21) and (8.24) we have

$$\mathrm{E}_D\left[S_{n+1}\right] = \mathrm{e}^{(r-D_0)\Delta t} S_n$$

and

$$\begin{aligned}\mathrm{E}_D\left[S_{n+1}^2\right] &= \mathrm{Var}_D\left[S_{n+1}\right] + \left(\mathrm{E}_D\left[S_{n+1}\right]\right)^2 \\ &= S_n^2\sigma^2\Delta t + O(\Delta t^2) + \mathrm{e}^{2(r-D_0)\Delta t} S_n^2 \\ &= \mathrm{e}^{[2(r-D_0)+\sigma^2]\Delta t} S_n^2 + O(\Delta t^2).\end{aligned}$$

Thus for p_u, p_q and p_d, we require[1]

$$\begin{aligned}p_u u + p_q q + p_d d &= \mathrm{e}^{(r-D_0)\Delta t}, \\ p_u u^2 + p_q q^2 + p_d d^2 &= \mathrm{e}^{(2(r-D_0)+\sigma^2)\Delta t}.\end{aligned}$$

Because there are three equations above for six unknowns, u, q, d, p_u, p_q, p_d, we can choose three parameters. In order that the number of the possible asset prices is not 3^n at the n-th time level, we can choose

$$d = 1/u \quad \text{and} \quad q = 1. \tag{8.37}$$

Now there are only four parameters u, p_u, p_q, p_d left. They should satisfy the three conditions above. If u is given, then this is a linear system for p_u, p_q, p_d and can be solved for them easily. Its solution is

$$\begin{cases} p_u = \dfrac{\mathrm{e}^{(2(r-D_0)+\sigma^2)\Delta t} - \mathrm{e}^{(r-D_0)\Delta t}(q+d) + qd}{(u-q)(u-d)}, \\[2ex] p_q = \dfrac{\mathrm{e}^{(2(r-D_0)+\sigma^2)\Delta t} - \mathrm{e}^{(r-D_0)\Delta t}(d+u) + du}{(q-d)(q-u)}, \\[2ex] p_d = \dfrac{\mathrm{e}^{(2(r-D_0)+\sigma^2)\Delta t} - \mathrm{e}^{(r-D_0)\Delta t}(u+q) + uq}{(d-u)(d-q)}. \end{cases} \tag{8.38}$$

Because they represent probabilities, we need to choose such a u that p_u, p_q and p_d all are nonnegative. If σ depends on S and t, then p_u, p_q and p_d will be different at different points. In this case, we need to choose such a u that at all the points p_u, p_q and p_d are nonnegative and the set of formulae (8.38) can still be used.

[1] We also know that because the Black–Scholes equation holds, $\mathrm{E}_D\left[S_{n+1}\right] = \mathrm{e}^{(r-D_0)\Delta t} S_n$ and $\mathrm{E}_D\left[S_{n+1}^2\right] = \mathrm{e}^{[2(r-D_0)+\sigma^2]\Delta t} S_n^2$ should be true (see Problem 39 of Chap. 2).

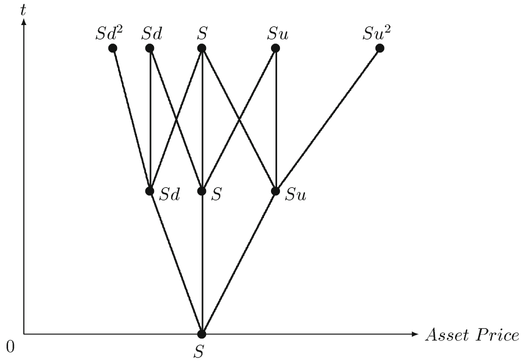

Fig. 8.3. Lattice generated by a trinomial model

The details for evaluating derivative securities using a trinomial method are nearly identical to the binomial method. The only major difference is that the expected value of the security at the n-th time level depends on the three possible values at the $(n+1)$-th time level, and that at the n-th time level, there are $2n+1$ possible asset prices, which are

$$S_{n,m} = Su^m, m = -n, -n+1, \cdots, n.$$

In this case, the corresponding lattice is illustrated in Fig. 8.3. Let $V_{n,m}$ be the security price at $S_{n,m}$. Then, the formula for finding the expected value of a security at time level $n+1$ is

$$\mathrm{E}_D\left[V_{n+1,m}\right] = p_u V_{n+1,m+1} + p_q V_{n+1,m} + p_d V_{n+1,m-1}$$

and the value of a European derivative security for $S_{n,m}$ is

$$V_{n,m} = \mathrm{e}^{-r\Delta t}(p_u V_{n+1,m+1} + p_q V_{n+1,m} + p_d V_{n+1,m-1}),$$

and for American puts and calls we have

$$P_{n,m} = \max\left(\mathrm{e}^{-r\Delta t}\left[p_u P_{n+1,m+1} + p_q P_{n+1,m} + p_d P_{n+1,m-1}\right],\ E - S_{n,m},\ 0\right), \tag{8.39}$$

$$C_{n,m} = \max\left(\mathrm{e}^{-r\Delta t}\left[p_u C_{n+1,m+1} + p_q C_{n+1,m} + p_d C_{n+1,m-1}\right],\ S_{n,m} - E,\ 0\right). \tag{8.40}$$

In Tables 8.4 and 8.5, we give binomial lattice approximations to American call and put options when the formulae (8.25)–(8.28) are used. The errors and the CPU times on a computer are also shown.

Table 8.4. American call option [binomial method (8.25)–(8.28)]

($r = 0.1$, $\sigma = 0.2$, $D_0 = 0.05$, $T = 1\,\text{year}$, $S = E = 100$,
and the exact solution is $C = 9.94092345\cdots$)

Numbers of time steps	Results	|Errors|	CPU(s)
50	9.902969	0.037955	0.0004
100	9.921921	0.019002	0.0013
200	9.931416	0.009507	0.0053
400	9.936168	0.004755	0.0220
800	9.938546	0.002378	0.0890

Table 8.5. American put option [binomial method (8.25)–(8.28)]

($r = 0.1$, $\sigma = 0.2$, $D_0 = 0.05$, $T = 1\,\text{year}$, $S = E = 100$,
and the exact solution is $P = 5.92827717\cdots$)

Numbers of time steps	Results	|Errors|	CPU(s)
50	5.911020	0.017257	0.0004
100	5.920066	0.008211	0.0014
200	5.924273	0.004005	0.0053
400	5.926323	0.001955	0.0210
800	5.927309	0.000968	0.0880

8.1.4 Relations Between the Lattice Methods and the Explicit Finite-Difference Methods

From the view point of PDEs, the procedure given by the formulae (8.12), (8.13), and (8.20) can be understood in the following way. The value of any derivative, V, satisfies

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0)\, S \frac{\partial V}{\partial S} - rV = 0.$$

Let $\bar{S} = S\mathrm{e}^{-ct}$ and $\overline{V}\left(\bar{S}, t\right) = V\left(S, t\right)$. Since

$$\frac{\partial V}{\partial S} = \frac{\partial \overline{V}}{\partial \bar{S}}\mathrm{e}^{-ct},$$
$$\frac{\partial^2 V}{\partial S^2} = \frac{\partial^2 \overline{V}}{\partial \bar{S}^2}\mathrm{e}^{-2ct},$$

and

$$\frac{\partial V}{\partial t} = \frac{\partial \overline{V}}{\partial t} + \frac{\partial \overline{V}}{\partial \bar{S}} S\mathrm{e}^{-ct} \cdot (-c)\,,$$

we have

$$\frac{\partial \overline{V}}{\partial t} + \frac{1}{2}\sigma^2 \bar{S}^2 \frac{\partial^2 \overline{V}}{\partial \bar{S}^2} + (r - D_0 - c)\, \bar{S} \frac{\partial \overline{V}}{\partial \bar{S}} - r\overline{V} = 0.$$

Furthermore let us set $x = \ln \bar{S}$ and $\tilde{V}(x,t) = \overline{V}(\bar{S},t)$. Noticing

$$\frac{\partial \overline{V}}{\partial \bar{S}} = \frac{\partial \tilde{V}}{\partial x}\frac{1}{\bar{S}},$$
$$\frac{\partial^2 \overline{V}}{\partial \bar{S}^2} = \frac{\partial^2 \tilde{V}}{\partial x^2}\frac{1}{\bar{S}^2} - \frac{1}{\bar{S}^2}\frac{\partial \tilde{V}}{\partial x},$$

and

$$\frac{\partial \overline{V}}{\partial t} = \frac{\partial \tilde{V}}{\partial t},$$

we arrive at

$$\frac{\partial \tilde{V}}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 \tilde{V}}{\partial x^2} + \left(r - D_0 - c - \sigma^2/2\right)\frac{\partial \tilde{V}}{\partial x} - r\tilde{V} = 0. \tag{8.41}$$

For this equation, we can have the following finite-difference scheme

$$\frac{\tilde{V}_m^{n+1} - \tilde{V}_m^n}{\Delta t} + \frac{1}{2}\sigma^2\frac{\tilde{V}_{m+1}^{n+1} - 2\tilde{V}_m^{n+1} + \tilde{V}_{m-1}^{n+1}}{\Delta x^2}$$
$$+ \left(r - D_0 - c - \sigma^2/2\right)\frac{\tilde{V}_{m+1}^{n+1} - \tilde{V}_{m-1}^{n+1}}{2\Delta x} - r\tilde{V}_m^n = 0,$$

or

$$\begin{aligned}\tilde{V}_m^n = \frac{1}{1 + r\Delta t}\Bigg[&\left(\frac{\sigma^2}{2}\frac{\Delta t}{\Delta x^2} + \frac{r - D_0 - c - \sigma^2/2}{2}\frac{\Delta t}{\Delta x}\right)\tilde{V}_{m+1}^{n+1} \\ &+ \left(1 - \frac{\sigma^2\Delta t}{\Delta x^2}\right)\tilde{V}_m^{n+1} \\ &+ \left(\frac{\sigma^2}{2}\frac{\Delta t}{\Delta x^2} - \frac{r - D_0 - c - \sigma^2/2}{2}\frac{\Delta t}{\Delta x}\right)\tilde{V}_{m-1}^{n+1}\Bigg].\end{aligned} \tag{8.42}$$

Here $\tilde{V}_m^n$ denotes the value of $\tilde{V}$ at $x_m = \bar{x} + m\Delta x$ and $t^n = n\Delta t$. If we choose

$$\Delta x = \sigma\sqrt{\Delta t}, \tag{8.43}$$

then we have

$$\begin{aligned}\tilde{V}_m^n = \frac{1}{1 + r\Delta t}\Bigg[&\left(\frac{1}{2} + \frac{r - D_0 - c - \frac{1}{2}\sigma^2}{2\sigma}\sqrt{\Delta t}\right)\tilde{V}_{m+1}^{n+1} \\ &+ \left(\frac{1}{2} - \frac{r - D_0 - c - \frac{1}{2}\sigma^2}{2\sigma}\sqrt{\Delta t}\right)\tilde{V}_{m-1}^{n+1}\Bigg].\end{aligned} \tag{8.44}$$

Now we show that a trinomial method (a binomial method) is close to an explicit method (8.42) [an explicit method (8.44)]. First we will show that the mesh here can overlap the lattices of trinomial and binomial methods. Consider the case $c = 0$ and let $\bar{x} = \ln S^*$, S^* being the asset price at the current time. In this case

$$S(x_m) = \mathrm{e}^{\bar{x}+m\Delta x} = S^* \left(\mathrm{e}^{\Delta x}\right)^m .$$

Therefore, a uniform mesh on (x,t)-plane (see Fig. 8.4) corresponds to a non-uniform mesh on (S,t)-plane (see Fig. 8.5), which overlaps the lattices in Figs. 8.2 and 8.3 with $u = \mathrm{e}^{\Delta x}$ and $S = S^*$. Consequently, this explicit difference method can be understood as a trinomial method with a lattice in Fig. 8.3 and as a binomial method with a lattice in Fig. 8.2 if the expression (8.43) holds.

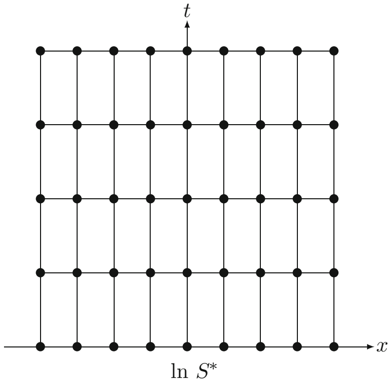

Fig. 8.4. A uniform mesh on (x,t)-plane

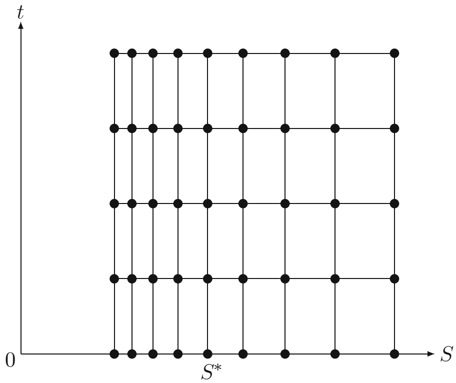

Fig. 8.5. The mesh on (S,t)-plane corresponding to a uniform mesh on (x,t)-plane

Now let show that the difference between the formulae (8.13) and (8.44) is very small. Let x_m, S_m^n, and $\bar{S}_m^n$ denote the x-coordinates, S-coordinates, and $\bar{S}$-coordinates of the m-point at time t^n, respectively. Because

$$x_{m+1} = x_m + \Delta x = x_m + \sigma\sqrt{\Delta t},$$

which means

$$\ln \bar{S}_{m+1}^{n+1} = \ln \bar{S}_m^n + \sigma\sqrt{\Delta t}$$

or

$$\ln\left(S_{m+1}^{n+1}\mathrm{e}^{-ct^{n+1}}\right) = \ln\left(S_m^n\mathrm{e}^{-ct^n}\right) + \sigma\sqrt{\Delta t},$$

we have

$$S_{m+1}^{n+1} = S_m^n\mathrm{e}^{c\left(t^{n+1}-t^n\right)+\sigma\sqrt{\Delta t}} = S_m^n\mathrm{e}^{c\Delta t+\sigma\sqrt{\Delta t}}. \tag{8.45}$$

Similarly,

$$S_{m-1}^{n+1} = S_m^n\mathrm{e}^{c\Delta t-\sigma\sqrt{\Delta t}}. \tag{8.46}$$

Noticing that $S_{m+1}^{n+1}, S_{m-1}^{n+1}$ and S_m^n correspond to $S_{n+1,1}, S_{n+1,0}$ and S_n, we have the relations (8.20). Therefore from the expression (8.12), we have

$$\begin{aligned}
p &= \frac{S_n\mathrm{e}^{(r-D_0)\Delta t} - S_n\mathrm{e}^{c\Delta t-\sigma\sqrt{\Delta t}}}{S_n\mathrm{e}^{c\Delta t+\sigma\sqrt{\Delta t}} - S_n\mathrm{e}^{c\Delta t-\sigma\sqrt{\Delta t}}} = \frac{\mathrm{e}^{(r-D_0-c)\Delta t} - \mathrm{e}^{-\sigma\sqrt{\Delta t}}}{\mathrm{e}^{\sigma\sqrt{\Delta t}} - \mathrm{e}^{-\sigma\sqrt{\Delta t}}} \\
&= \frac{1 + (r-D_0-c)\,\Delta t - \left(1 - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t - \frac{1}{6}\sigma^3\Delta t^{3/2}\right) + O\left(\Delta t^2\right)}{2\sigma\sqrt{\Delta t} + \frac{1}{3}\sigma^3\Delta t^{3/2} + O\left(\Delta t^2\right)} \\
&= \frac{\sigma\sqrt{\Delta t}\left[1 + \left(r-D_0-c-\sigma^2/2\right)\sqrt{\Delta t}/\sigma + \frac{1}{6}\sigma^2\Delta t + O\left(\Delta t^{3/2}\right)\right]}{2\sigma\sqrt{\Delta t}\left[1 + \frac{1}{6}\sigma^2\Delta t + O\left(\Delta t^{3/2}\right)\right]} \\
&= \frac{1}{2}\left[1 + \left(r-D_0-c-\sigma^2/2\right)\sqrt{\Delta t}/\sigma + \frac{1}{6}\sigma^2\Delta t + O\left(\Delta t^{3/2}\right)\right] \\
&\quad\times\left[1 - \frac{1}{6}\sigma^2\Delta t + O\left(\Delta t^{3/2}\right)\right] \\
&= \frac{1}{2}\left[1 + \frac{r-D_0-c-\frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right] + O\left(\Delta t^{3/2}\right).
\end{aligned}$$

Also the difference between $\mathrm{e}^{-r\Delta t}$ and $\frac{1}{1+r\Delta t}$ is $O(\Delta t^2)$. Thus the formula (8.13) is almost the same as the formula (8.44). Consequently, the method given by the formulae (8.12), (8.13), and (8.20) is almost an explicit scheme (8.44). Therefore, the binomial method and the trinomial method can be understood as explicit finite-difference methods in some sense.

Finally we point out that because the convergence of the explicit scheme here with $\Delta t/\Delta x^2 = \sigma^{-2}$ can be easily proved, the difference between $\frac{1}{2}\left[1 + \frac{r-D_0-c-\frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right]$ and p is $O\left(\Delta t^{3/2}\right)$, and the difference between $\mathrm{e}^{-r\Delta t}$ and $\frac{1}{1+r\Delta t}$ is $O(\Delta t^2)$, the convergence of the binomial method can also be proved.

The formulae (8.45) and (8.46) actually are the formula (8.20), so the conclusion given here can be used for both the Cox–Ross–Rubinstein method (See [22]) and the McDonald method (See [61]).

8.1.5 Examples of Unstable Schemes

As has been pointed out in Sect. 8.1.1, when the scheme (8.3) or (8.6) is used, stability condition (8.4) or (8.7) is required. What will happen if these conditions are violated?

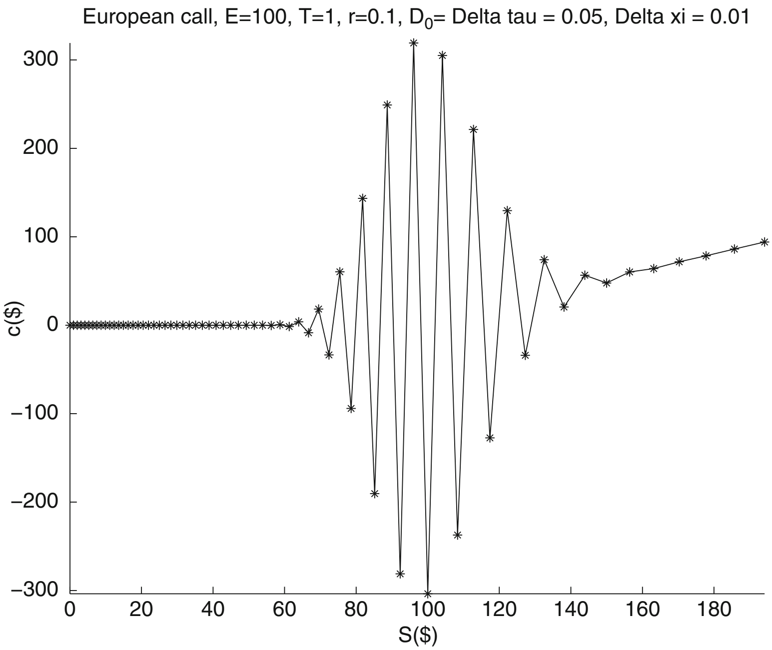

Fig. 8.6. A unstable solution of EFDI
(The solution appears when Eq. (8.4) is violated. $E = 100$, $T = 1$, $r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$, $\Delta\tau = 0.05$, and $\Delta\xi = 0.01$.)

Let us try scheme (8.3) for a European call option with parameters $E = 100$, $T = 1$, $r = 0.1$, $D_0 = 0.05$, and $\sigma = 0.2$. Take $\Delta\tau = 0.05$ and $\Delta\xi = 0.01$. The solution at $t = 0$ is shown in Fig. 8.6, where we see that rather large oscillations develop. In this case, $\Delta\tau = 0.05$ and $16\Delta\xi^2/\sigma^2 = 0.04$, so condition (8.4) does not hold, and the scheme is unstable. We cannot get a useful solution if such a set of $\Delta\tau$ and $\Delta\xi$ is adopted.

8.2 Implicit Methods

8.2.1 Pricing European Options by Using $\overline{V}$, ξ, τ Variables

The problem (8.1) can also be discretized by an implicit scheme, for example, by scheme (7.6). In this case, the difference equations are

$$\begin{aligned}
&\frac{v_m^{n+1}-v_m^n}{\Delta\tau}\\
&=\frac{1}{4}\bar{\sigma}_m^2\xi_m^2(1-\xi_m)^2\left(\frac{v_{m+1}^{n+1}-2v_m^{n+1}+v_{m-1}^{n+1}}{\Delta\xi^2}+\frac{v_{m+1}^{n}-2v_m^{n}+v_{m-1}^{n}}{\Delta\xi^2}\right)\\
&\quad+\frac{1}{2}(r-D_0)\xi_m(1-\xi_m)\left(\frac{v_{m+1}^{n+1}-v_{m-1}^{n+1}}{2\Delta\xi}+\frac{v_{m+1}^{n}-v_{m-1}^{n}}{2\Delta\xi}\right)\\
&\quad-\frac{1}{2}[r(1-\xi_m)+D_0\xi_m](v_m^{n+1}+v_m^n),\\
&\qquad\qquad m=0,1,\cdots,M,\quad n=0,1,\cdots,N-1.
\end{aligned}\tag{8.47}$$

Like the explicit scheme case, v_{-1}^n and v_{M+1}^n do not appear in the equation. For a fixed n, there are $M+1$ equations in the following form:

$$\begin{aligned}
a_m v_{m-1}^{n+1}+b_m v_m^{n+1}+c_m v_{m+1}^{n+1}&=-a_m v_{m-1}^n+(2-b_m)v_m^n-c_m v_{m+1}^n,\\
m&=0,1,\cdots,M,
\end{aligned}$$

where

$$\begin{aligned}
a_m&=\left[(r-D_0)m(1-\xi_m)-\bar{\sigma}_m^2m^2(1-\xi_m)^2\right]\Delta\tau/4,\\
b_m&=1+[\bar{\sigma}_m^2m^2(1-\xi_m)^2+r(1-\xi_m)+D_0\xi_m]\Delta\tau/2,\\
c_m&=\left[-(r-D_0)m(1-\xi_m)-\bar{\sigma}_m^2m^2(1-\xi_m)^2\right]\Delta\tau/4,\\
&\qquad m=0,1,\cdots,M.
\end{aligned}$$

In matrix form, we have

$$A\mathbf{v}^{n+1}=\mathbf{q}^n,$$

where

$$A=\begin{bmatrix}
b_0 & c_0 & & & & \\
a_1 & b_1 & c_1 & & 0 & \\
 & \ddots & \ddots & \ddots & & \\
 & & \ddots & \ddots & \ddots & \\
 & 0 & & a_{M-1} & b_{M-1} & c_{M-1}\\
 & & & & a_M & b_M
\end{bmatrix},$$

$$\mathbf{v}^n=\begin{bmatrix} v_0^n\\ v_1^n\\ \vdots\\ v_M^n\end{bmatrix},\quad
\mathbf{q}^n=\begin{bmatrix}(2-b_0)v_0^n-c_0v_1^n\\ -a_1v_0^n+(2-b_1)v_1^n-c_1v_2^n\\ \vdots\\ -a_Mv_{M-1}^n+(2-b_M)v_M^n\end{bmatrix}.$$

The difference between an implicit method and an explicit method is that for an implicit method, a linear system needs to be solved in order to get $\mathbf{v}^{n+1}$ from $\mathbf{v}^n$. This can be done by the LU decomposition or an iteration method given in Sects. 6.2.1 and 6.2.2. The linear system here has a variable coefficient matrix, however, it does not depend on time if σ does not depend on t. Thus, the linear system can be solved with only slightly more cost compared to a linear system with a constant coefficient matrix. It is clear that scheme (8.47) can even be applied to the case when σ depends on S and t. We will refer to this scheme as the implicit finite-difference scheme. From Problem 15 in Chap. 7, we can expect this scheme to be stable without any condition on the ratio $\Delta\tau/\Delta\xi$. In fact, in the paper by Sun, Yan, and Zhu [79], it is rigorously proved that this scheme with variable coefficients is unconditionally stable.

When σ is a constant, we can also use the variables u, x and $\bar{\tau}$. In this case, the difference scheme (7.9) can be applied to the equation in problem (8.5). However, when the scheme (7.9) is used for problem (8.5), we have to modify the problem formulation slightly. Let the problem be defined on a finite domain $[x_l, x_u]$ and give an artificial boundary condition on each boundary. From the expressions (2.19) and (2.23) in Sect. 2.2.5, we know at $S = 0, V(0,t) = V(0,T)\mathrm{e}^{-r(T-t)}$ and for $S \approx \infty$, $V(S,t) \approx V(S,T)\mathrm{e}^{-D_0(T-t)}$. Therefore, noticing $u(x,\bar{\tau}) = \mathrm{e}^{r(T-t)}V(S,t)$, for $S \approx 0$, i.e., $x \approx -\infty$ we have

$$u(x,\bar{\tau}) \approx V(S,T)$$

and for $S \approx \infty$, i.e., $x \approx \infty$,

$$u(x,\bar{\tau}) \approx V(S,T)\mathrm{e}^{(r-D_0)(T-t)},$$

where $x = \ln S + (r - D_0 - \sigma^2/2)(T-t)$ and $\bar{\tau} = \sigma^2(T-t)/2$. These two relations can be taken as artificial boundary conditions at $x = x_l$ and $x = x_u$, respectively, if x_l is small enough and x_u is large enough. For example, in order to calculate a call option,

$$u(x_l,\bar{\tau}) = 0 \quad \text{and} \quad u(x_u,\bar{\tau}) = (\mathrm{e}^{x_u-(2(r-D_0)/\sigma^2-1)\bar{\tau}} - E)\mathrm{e}^{2(r-D_0)\bar{\tau}/\sigma^2}$$

can be adopted as artificial boundary conditions. If the call option has parameters $r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$, $E = 1$, and $T = 1$, we can let $x_l = \ln 0.2$ and $x_u = \ln 2.3$.

The method for solving European average strike and double average options with continuous sampling is similar. However the transformations will be different for the two different cases.

8.2.2 European Options with Discrete Dividends and Asian and Lookback Options with Discrete Sampling

A holder of a stock usually obtains dividends on certain days, not continuously. Thus, in practice, it is important to know how to price options on stocks with discrete dividends. For Asian and lookback options, sampling is usually done discretely even though the time interval between two samples is very small so it can be seen as being continuously. This subsection is devoted to discussing how to evaluate European options with discrete dividends and European-style Asian and lookback options with discrete sampling. We give details here only for European options with discrete dividends and European average price options with discrete sampling. For other cases, the prices can be obtained in a similar way. Some results on such options are also given here.

European Options with Discrete Dividends. First, we work on options on stocks with discrete dividends. Let $V(S,t)$ be the price of an option on stocks with discrete dividends. From Sect. 2.2.2, we know that $V(S,t)$ is the solution to the following problem:

$$\begin{cases} \dfrac{\partial V}{\partial t}+\dfrac{1}{2}\sigma^2(S)S^2\dfrac{\partial^2 V}{\partial S^2}+[rS-D(S,t)]\dfrac{\partial V}{\partial S}-rV=0, \quad 0\le S, \quad t\le T, \\ V(S,T)=V_T(S), \qquad 0\le S, \end{cases} \tag{8.48}$$

where $D(S,t) = \sum_{i=1}^{I} D_i(S)\delta(t-t_i)$ and $D_i(S) \le S$ for any S. The meaning of the condition $D_i(S) \le S$ here is that the price of a stock at any time should be greater than or equal to the dividend paid at that time. From the problem (8.48), we know the following: At $t \ne t_i, i=1,2,\cdots,I$, V satisfies

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2(S)S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0, \quad 0\le S \tag{8.49}$$

and at $t=t_i, i=1,2,\cdots,$ or I, the equation

$$\frac{\partial V}{\partial t}-D_i(S)\delta(t-t_i)\frac{\partial V}{\partial S}=0, \quad 0\le S$$

holds. From Sect. 2.5.2, we see that this equation gives

$$V(S,t_i^-)=V\left(S-D_i(S),t_i^+\right). \tag{8.50}$$

As we know from Sect. 2.2.5, through the transformation

$$\begin{cases} \xi=\dfrac{S}{S+P_m}, \\ \tau=T-t, \\ \overline{V}(\xi,\tau)=\dfrac{V(S,t)}{S+P_m}, \end{cases} \tag{8.51}$$

Eq. (8.49) becomes

$$\frac{\partial \overline{V}}{\partial \tau}=\frac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1-\xi)^2\frac{\partial^2 \overline{V}}{\partial \xi^2}+r\xi(1-\xi)\frac{\partial \overline{V}}{\partial \xi}-r(1-\xi)\overline{V}, \quad 0\le\xi\le 1, \tag{8.52}$$

where $\bar{\sigma}(\xi)=\sigma\left(\dfrac{P_m\xi}{1-\xi}\right)$, the final condition in the problem (8.48) is converted into an initial condition of the form

$$\overline{V}(\xi,0)=\frac{1-\xi}{P_m}V_T\left(\frac{P_m\xi}{1-\xi}\right), \quad 0\le\xi\le1 \tag{8.53}$$

and the condition (8.50) is transferred to

$$\overline{V}(\xi,\ \tau_i^+)=\left[1-D_i\left(\frac{\xi P_m}{1-\xi}\right)\frac{1-\xi}{P_m}\right]\overline{V}\left(\frac{P_m\xi-D_i\left(\frac{\xi P_m}{1-\xi}\right)(1-\xi)}{P_m-D_i\left(\frac{\xi P_m}{1-\xi}\right)(1-\xi)},\ \tau_i^-\right). \tag{8.54}$$

Table 8.6. European and American options on stocks with discrete dividends

($r=0.09$, $\sigma=0.3$, $T=0.5$ year and $E=40$.
There are two dividend payments and each pays 0.50.)

		$t_1=1/12$ and $t_2=4/12$		$t_1=2/12$ and $t_2=5/12$	
	S	European	American	European	American
Call	38	2.64	2.64	2.66	2.69
	40	3.70	3.70	3.72	3.77
	42	4.95	4.95	4.97	5.03
Put	38	3.86	4.08	3.87	4.02
	40	2.92	3.08	2.93	3.04
	42	2.17	2.28	2.18	2.26

We solve the problem here using the following mesh. The mesh is still uniform in ξ with $\Delta\xi=1/M$, but in the τ direction, the interval $[0,T]$ is divided into N subintervals with $\tau=\tau_n,\ n=0,1,\cdots,N$, where $\tau_0=0<\tau_1<\cdots<\tau_{N-1}<\tau_N=T$, and suppose t_i corresponds to τ_{n_i}, $i=1,2,\cdots,I$. Furthermore, define $n_0=0$ and $n_{I+1}=N$. Just like before, let v_m^n be an approximate value of $\overline{V}$ at $\xi=\xi_m$ and $\tau=\tau_n$ and $\{v_m^n\}$ denote v_m^n, $m=0,1,\cdots,M$. The problem can be solved in the following way. When $\{v_m^{n_i^+}\}$ are known at $\tau_{n_i}^+$, we can obtain $\{v_m^{n_{i+1}^-}\}$ at $\tau_{n_{i+1}}^-$ by a scheme approximating Eq. (8.52), for example, the scheme (8.47). Then we use condition (8.54) to interpolate $\{v_m^{n_{i+1}^+}\}$ from $\{v_m^{n_{i+1}^-}\}$. At $t=0$, the option values are the same for $t=0^-$ and $t=0^+$. Thus, from the initial condition (8.53), we can have $\{v_m^{n_0^+}\}$. Consequently, we can do the procedure of getting $\{v_m^{n_{i+1}^+}\}$ from $\{v_m^{n_i^+}\}$ for $i=0,1,\cdots,I-1$ successively. As soon as we have $\{v_m^{n_I^+}\}$, we can find $\{v_m^{n_{I+1}^-}\}$, that is, $\{v_m^N\}$ by scheme (8.47). For American options, the maximum between $v_m^{n_{i+1}^+}$ and the constraint condition should be taken as the value of the American option at $\tau=\tau_{n_{i+1}}^+$, $i=0,1,\cdots,I-1$.

In Table 8.6, we give some values of half-year European and American options with two dividend payments. Each time, the dividend payment is 0.50 if the price of stock is greater than or equal to 0.50. If $S<0.50$, we let

$D_i(S) = S$ in the computation. The payments are given at times 1/12 and 4/12 or 2/12 and 5/12. In order to check if the results of European options are correct, we can check if the put–call parity relation holds. For European options on stocks with discrete dividends, the put–call parity relation is in the form (3.44) in Chap. 3. For the case with $S = 40$ and the payment dates $t_1 = 2/12$ and $t_2 = 5/12$, this relation is $c(40,0) + E\mathrm{e}^{-rT} = p(40,0) + 40 - 0.5\left(\mathrm{e}^{-r\cdot 2/12} + \mathrm{e}^{-r\cdot 5/12}\right)$. From the data given in Table 8.6, we have

$$\begin{aligned} &c(40,0) + E\mathrm{e}^{-rT} = 3.72 + 40 \cdot \mathrm{e}^{-0.09\cdot 0.5} = 3.72 + 38.24 = 41.96, \\ &p(40,0) + 40 - 0.5\left(\mathrm{e}^{-r\cdot 2/12} + \mathrm{e}^{-r\cdot 5/12}\right) = 2.93 + 40 \\ &\qquad\qquad -0.5\left(\mathrm{e}^{-0.09\cdot 2/12} + \mathrm{e}^{-0.09\cdot 5/12}\right) = 42.93 - 0.97 = 41.96. \end{aligned}$$

Thus, the put–call parity relation holds. In Hull's book [43], an approximate method to get $c(S,t)$ is provided. It gives $c(40,0) = 3.67$ for this case. The numerical result here is 3.72, so it gives a very good estimate. From Table 8.6, we know that for the case $t_1 = 1/12$ and $t_2 = 4/12$, the values of European and American call options are the same. This is because $E(1 - \mathrm{e}^{-r(T-t_2)}) = 40 \cdot (1 - \mathrm{e}^{-0.09/6}) = 0.60 > 0.5$ and $E(1 - \mathrm{e}^{-r(t_2-t_1)}) = 40 \cdot (1 - \mathrm{e}^{-0.09/4}) = 0.89 > 0.5$, where 0.5 is the dividend payment. When such inequalities hold, it is impossible to have an optimal exercise price and the value of the American option must be equal to the value of the European option (see Problem 15 in Chap. 3 or the book [43] by Hull).

European Average Price Options with Discrete Sampling. Now we give some details on how to price European average price options. Suppose that sampling is done at $t = t_1, t_2, \cdots, t_K$, where $0 \le t_1 < t_2 < \cdots < t_K \le T$. Define

$$I = \frac{1}{K}\int_0^t S(\tau) f(\tau)\, d\tau,$$

where $f(\tau) = \sum\limits_{i=1}^{K} \delta(\tau - t_i)$. It is clear that at $t = T$, $I = A$. Let the price of a European average price option be $V(S,I,t)$ and let E be the exercise price. Then $V(S,I,t)$ is the solution of the problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r - D_0) S \dfrac{\partial V}{\partial S} \\ \quad + \dfrac{S}{K}\sum\limits_{i=1}^{K} \delta(t - t_i) \dfrac{\partial V}{\partial I} - rV = 0, \quad 0 \le S < \infty, \quad 0 \le I < \infty, \quad t \le T, \\ V(S,I,T) = \max(\pm(A - E), 0) \\ \quad = \max(\pm(I - E), 0), \quad 0 \le S < \infty, \quad 0 \le I < \infty, \end{cases}$$

where the "+" and "−" in $\pm$ correspond to the call and put options, respectively. Let $\eta = \dfrac{I - E}{S}$, $W = \dfrac{V}{S}$. In this case, the first three relations in the set of expressions (4.24) are still true and $\dfrac{\partial V}{\partial I} = \dfrac{\partial W}{\partial \eta}$. Also, from $V(S, I, T) = \max(\pm(I - E), 0)$, we have

$$W(\eta, T) = \max(\pm\eta, 0).$$

Therefore, $W(\eta, t)$ satisfies

$$\begin{cases} \dfrac{\partial W}{\partial t} + \dfrac{1}{2}\sigma^2\eta^2\dfrac{\partial^2 W}{\partial \eta^2} + \left[(D_0 - r)\eta + \dfrac{1}{K}\sum_{i=1}^{K}\delta(t - t_i)\right]\dfrac{\partial W}{\partial \eta} \\ -D_0 W = 0, \qquad -\infty < \eta < \infty, \quad t \le T, \\ W(\eta, T) = \max(\pm\eta, 0), \qquad -\infty < \eta < \infty. \end{cases} \tag{8.55}$$

Suppose $t_1 = 0$ and let $t_{K+1} = T > t_K$; then the problem can be solved as follows. Starting with $f_{K+1,w} = \max(\pm\eta, 0)$, for $i = K+1, K, \cdots, 2$, successively, solve the following problem:

$$\begin{cases} \dfrac{\partial W}{\partial t} + \frac{1}{2}\sigma^2\eta^2\dfrac{\partial^2 W}{\partial \eta^2} + (D_0 - r)\eta\dfrac{\partial W}{\partial \eta} - D_0 W = 0, \\ \qquad -\infty < \eta < \infty, \quad t_{i-1}^+ < t < t_i^-, \\ W(\eta, t_i^-) = f_{i,w}(\eta), \qquad -\infty < \eta < \infty \end{cases} \tag{8.56}$$

and obtain $W(\eta, t_{i-1}^-)$ from $W(\eta, t_{i-1}^+)$ by the jump condition

$$W\left(\eta, t_i^-\right) = W\left(\eta + \frac{1}{K}, t_i^+\right). \tag{8.57}$$

We want to solve this problem as an initial-value problem on a finite domain. Thus, we introduce the following transformation:

$$\begin{cases} \xi = \dfrac{\eta}{|\eta| + P_m}, \\ \tau = T - t, \\ W(\eta, t) = (|\eta| + P_m)\overline{u}(\xi, \tau), \end{cases} \tag{8.58}$$

where $P_m > 0$. From the expression (8.58), we have

$$\mathrm{sign}(\xi) = \mathrm{sign}(\eta), \ |\xi| \le 1, \ |\eta| = \frac{P_m|\xi|}{1 - |\xi|}, \ \eta = \frac{P_m\xi}{1 - |\xi|}, \ |\eta| + P_m = \frac{P_m}{1 - |\xi|},$$

and

$$\frac{d\xi}{d\eta} = \frac{|\eta| + P_m - \eta \cdot \mathrm{sign}(\eta)}{(|\eta| + P_m)^2} = \frac{P_m}{(|\eta| + P_m)^2} = \frac{(1 - |\xi|)^2}{P_m}.$$

Because

$$\begin{aligned}
\frac{\partial W}{\partial t} &= -(|\eta| + P_m)\frac{\partial \overline{u}}{\partial \tau} = -\frac{P_m}{1-|\xi|}\frac{\partial \overline{u}}{\partial \tau}, \\
\frac{\partial W}{\partial \eta} &= \frac{\partial}{\partial \eta}\left[(|\eta| + P_m)\overline{u}\right] = \operatorname{sign}(\eta)\overline{u} + (\eta + |P_m|)\frac{\partial \overline{u}}{\partial \xi}\frac{d\xi}{d\eta} \\
&= \operatorname{sign}(\xi)\overline{u} + (1-|\xi|)\frac{\partial \overline{u}}{\partial \xi}, \\
\frac{\partial^2 W}{\partial \eta^2} &= \frac{\partial}{\partial \xi}\left[(1-|\xi|)\frac{\partial \overline{u}}{\partial \xi} + \operatorname{sign}(\xi)\overline{u}\right]\frac{d\xi}{d\eta} = \frac{(1-|\xi|)^3}{P_m}\frac{\partial^2 \overline{u}}{\partial \xi^2},
\end{aligned}$$

from the PDE for W we have

$$\begin{aligned}
\frac{P_m}{1-|\xi|}\frac{\partial \overline{u}}{\partial \tau} &= \frac{\sigma^2 P_m \xi^2 (1-|\xi|)}{2}\frac{\partial^2 \overline{u}}{\partial \xi^2} \\
&\quad + \left[(D_0 - r)\frac{P_m \xi}{1-|\xi|}\right]\left[\operatorname{sign}(\xi)\overline{u} + (1-|\xi|)\frac{\partial \overline{u}}{\partial \xi}\right] - D_0 \frac{P_m}{1-|\xi|}\overline{u}
\end{aligned}$$

or

$$\begin{aligned}
\frac{\partial \overline{u}}{\partial \tau} &= \frac{\sigma^2 \xi^2 (1-|\xi|)^2}{2}\frac{\partial^2 \overline{u}}{\partial \xi^2} + (D_0 - r)\xi(1-|\xi|)\frac{\partial \overline{u}}{\partial \xi} \\
&\quad + \left[(D_0 - r)|\xi| - D_0\right]\overline{u}, \quad -1 < \xi < 1, \quad 0 \le \tau.
\end{aligned}$$

Thus, under this transformation, the problem (8.56) becomes

$$\begin{cases}
\dfrac{\partial \overline{u}}{\partial \tau} = \dfrac{1}{2}\sigma^2 \xi^2 (1-|\xi|)^2 \dfrac{\partial^2 \overline{u}}{\partial \xi^2} + (D_0 - r)\xi(1-|\xi|)\dfrac{\partial \overline{u}}{\partial \xi} \\
-\left[r|\xi| + D_0(1-|\xi|)\right]\overline{u}, \qquad -1 \le \xi \le 1, \quad \tau_i^+ < \tau < \tau_{i-1}^-, \\
\overline{u}(\xi, \tau_i^+) = \dfrac{1-|\xi|}{P_m} f_{i,w}\left(\dfrac{P_m \xi}{1-|\xi|}\right), -1 \le \xi \le 1.
\end{cases} \tag{8.59}$$

Here we have used the following relation:

$$\overline{u}(\xi, \tau_i^+) = \frac{W\left(\eta, t_i^-\right)}{|\eta| + P_m} = \frac{f_{i,w}(\eta)}{|\eta| + P_m} = \frac{1-|\xi|}{P_m} f_{i,w}\left(\frac{P_m \xi}{1-|\xi|}\right).$$

At $\xi = 0$, the PDE in the problem (8.59) degenerates into

$$\frac{\partial \overline{u}}{\partial \tau} = -D_0 \overline{u}.$$

Thus, the solution at $\xi = 0$ can be determined alone. Therefore, the problem (8.59) can be divided into two problems:

$$\begin{cases} \dfrac{\partial \overline{u}}{\partial \tau} = \dfrac{1}{2}\sigma^2\xi^2(1-\xi)^2\dfrac{\partial^2 \overline{u}}{\partial \xi^2} + (D_0 - r)\xi(1-\xi)\dfrac{\partial \overline{u}}{\partial \xi} \\ -\left[r\xi + D_0(1-\xi)\right]\overline{u}, \qquad 0 \le \xi \le 1, \quad \tau_i^+ < \tau < \tau_{i-1}^-, \\ \overline{u}(\xi, \tau_i^+) = \dfrac{1-\xi}{P_m} f_{i,w}\left(\dfrac{P_m\xi}{1-\xi}\right), 0 \le \xi \le 1 \end{cases} \tag{8.60}$$

and

$$\begin{cases} \dfrac{\partial \overline{u}}{\partial \tau} = \dfrac{1}{2}\sigma^2\xi^2(1-|\xi|)^2\dfrac{\partial^2 \overline{u}}{\partial \xi^2} + (D_0 - r)\xi(1-|\xi|)\dfrac{\partial \overline{u}}{\partial \xi} \\ -\left[r|\xi| + D_0(1-|\xi|)\right]\overline{u}, \qquad -1 \le \xi \le 0, \quad \tau_i^+ < \tau < \tau_{i-1}^-, \\ \overline{u}(\xi, \tau_i^+) = \dfrac{1-|\xi|}{P_m} f_{i,w}\left(\dfrac{P_m\xi}{1-|\xi|}\right), -1 \le \xi \le 0. \end{cases} \tag{8.61}$$

Letting $\xi_1 = -\xi$ and $\overline{u}_1(\xi_1, \tau) = \overline{u}(\xi, \tau)$, we have $|\xi| = \xi_1$ for any $\xi \in [-1, 0]$ and $\xi\dfrac{\partial \overline{u}}{\partial \xi} = \xi_1\dfrac{\partial \overline{u}_1}{\partial \xi_1}$. Thus, the problem (8.61) can be rewritten as

$$\begin{cases} \dfrac{\partial \overline{u}_1}{\partial \tau} = \dfrac{1}{2}\sigma^2\xi_1^2(1-\xi_1)^2\dfrac{\partial^2 \overline{u}_1}{\partial \xi_1^2} + (D_0 - r)\xi_1(1-\xi_1)\dfrac{\partial \overline{u}_1}{\partial \xi_1} \\ -\left[r\xi_1 + D_0(1-\xi_1)\right]\overline{u}_1, \qquad 0 \le \xi_1 \le 1, \quad \tau_i^+ < \tau < \tau_{i-1}^-, \\ \overline{u}_1(\xi_1, \tau_i^+) = \dfrac{1-\xi_1}{P_m} f_{i,w}\left(\dfrac{-P_m\xi_1}{1-\xi_1}\right), \quad 0 \le \xi_1 \le 1. \end{cases} \tag{8.62}$$

The formulation of the two problems are the same as the problem (8.1). Thus, using the scheme (8.47), we can obtain $\overline{u}(\xi, \tau_{i-1}^-)$ from $\overline{u}(\xi, \tau_i^+)$ for $-1 \le \xi \le 1$. In order to have $\overline{u}(\xi, \tau_{i-1}^+)$ from $\overline{u}(\xi, \tau_{i-1}^-)$ for $-1 \le \xi \le 1$, we need to use the jump condition:

$$\overline{u}\left(\xi,\ \tau_i^+\right) = \frac{\left|\frac{P_m\xi}{1-|\xi|} + \frac{1}{K}\right| + P_m}{\left|\frac{P_m\xi}{1-|\xi|}\right| + P_m}\overline{u}\left(\frac{\frac{P_m\xi}{1-|\xi|} + \frac{1}{K}}{\left|\frac{P_m\xi}{1-|\xi|} + \frac{1}{K}\right| + P_m}, \qquad \tau_i^-\right), \tag{8.63}$$

which is another version of the jump condition (8.57) if the function $\overline{u}(\xi, \tau)$ is used instead of the function $W(\eta, t)$. It is not difficult to rewrite the jump condition (8.57) into the jump condition (8.63), which is left as a portion of Problem 9. As soon as we have $\overline{u}(\xi, T^+)$ when $\tau_1 = T$, that is, $t_1 = 0$, we can find

$$V(S, 0, 0) = SW\left(-E/S, 0\right) = S\left(E/S + P_m\right)\overline{u}\left(\frac{-E/S}{E/S + P_m}, T\right).$$

Because $\dfrac{d\xi}{d\eta} = \dfrac{(1-|\xi|)^2}{P_m} > 0$, when η varies from $-\infty$ to ∞, ξ varies from -1 to 1 monotonically. Thus, $\xi(\eta) < \xi(\eta + 1/K)$ for any η; that is,

$$\xi(\eta) = \frac{\eta}{|\eta| + P_m} < \xi\left(\eta + \frac{1}{K}\right) = \frac{\eta + \dfrac{1}{K}}{|\eta + \dfrac{1}{K}| + P_m} = \frac{\frac{P_m \xi}{1-|\xi|} + \frac{1}{K}}{\left|\frac{P_m \xi}{1-|\xi|} + \frac{1}{K}\right| + P_m}.$$

Consequently, when ξ varies from 0 to 1, $\frac{\frac{P_m \xi}{1-|\xi|} + \frac{1}{K}}{\left|\frac{P_m \xi}{1-|\xi|} + \frac{1}{K}\right| + P_m}$ varies from $\dfrac{1/K}{1/K + P_m}$ to 1, and when ξ varies from -1 to 0, $\frac{\frac{P_m \xi}{1-|\xi|} + \frac{1}{K}}{\left|\frac{P_m \xi}{1-|\xi|} + \frac{1}{K}\right| + P_m}$ varies from -1 to $\dfrac{1/K}{1/K + P_m}$. For an average price call option, we need to solve problems (8.60) and (8.62) from τ_i^+ to τ_{i-1}^- and then use condition (8.63) for $\xi \in [-1, 1]$, $i = K+1, K, \cdots, 2$, successively.[2] For an average rate put option, $\overline{u}(\xi, 0) = 0$ for $\xi \in [0, 1]$, and so the solution of the problem (8.60) with the jump condition

Table 8.7. Prices of average price put options with discrete sampling

$(T = 1,\ S = 100,\ r = 0.05,\ D_0 = 0,\ \sigma = 0.2)$

E	Monthly	Weekly	Daily
90.0000	0.7861	0.6929	0.6694
92.5000	1.2239	1.1092	1.0800
95.0000	1.8162	1.6840	1.6501
97.5000	2.5823	2.4392	2.4023
100.0000	3.5345	3.3888	3.3512
102.5000	4.6771	4.5378	4.5020
105.0000	6.0068	5.8823	5.8506
107.5000	7.5132	7.4107	7.3850
110.0000	9.1810	9.1055	9.0871

Table 8.8. Prices of average price call options with discrete sampling

$(T = 1,\ S = 100,\ r = 0,\ D_0 = 0,\ \sigma = 0.2)$

E	Monthly	Weekly	Daily	Continuously
90.0000	11.2304	11.0853	11.0487	11.0426
92.5000	9.3506	9.1760	9.1315	9.1240
95.0000	7.6595	7.4610	7.4102	7.4016
97.5000	6.1708	5.9566	5.9015	5.8922
100.0000	4.8888	4.6685	4.6118	4.6022
102.5000	3.8091	3.5922	3.5365	3.5271
105.0000	2.9194	2.7143	2.6618	2.6529
107.5000	2.2019	2.0148	1.9672	1.9592
110.0000	1.6350	1.4701	1.4284	1.4215

[2] In this case, the problem (8.60) with the jump condition (8.63) can be solved independently and have an analytic solution (see Andreasen [3], Zhu [90], or Problem 32 in Chap. 2).

Table 8.9. Comparison between two sampling-daily-average price call options

($T = 1$, $S = 100$, $r = 0$, $D_0 = 0$, $\sigma = 0.2$)

	Money spent in the case with $E = 100$		Money spent in the case with $E = 90$
$A \geq 100$	104.61	$>$	101.05
$100 > A > 96.44$	$4.61 + A$	$>$	101.05
$A = 96.44$	101.05	$=$	101.05
$96.44 > A \geq 90$	$4.61 + A$	$<$	101.05
$90 > A$	$4.61 + A$	$<$	$11.05 + A$

(8.63) is zero. Thus, in order to obtain $\overline{u}\left(\dfrac{-E/S}{E/S + P_m}, T\right)$, we only need to solve the problem (8.62) and to use the jump condition (8.63) alternatively.

In Sect. 4.3.7, we have given some results on European average price options with discrete sampling. Here we give more results for the European average rate call and put options obtained by the method described here. In Table 8.7 for the cases with sampling monthly, weekly, or daily, for $S = 100$, the values of the average price put options with $T = 1, r = 0.05, D_0 = 0, \sigma = 0.2$ are listed. In Table 8.8 for the cases with sampling monthly, weekly, or daily, for $S = 100$, the values of the average price call options with $T = 1, r = 0, D_0 = 0, \sigma = 0.2$ are given. Here we assume that there are 12 months, 52 weeks, 360 days per year, which are not real. The error of the results given in the table should be around 0.0001 because when a finer mesh is used, the difference between the new value and the value given here is less than 0.0001. In Table 8.8, the results of options with continuous sampling are also given. From that table, we can see that the difference between the option price with sampling daily and the option price with sampling continuously is about 0.01.

Suppose that a company will buy a certain amount of some raw material every day during the next year. Let A be the average price of the raw material the company paid during this period. Usually, the company does not want A to be much higher than the price today S. It is clear that the company cannot control the price on the market. However, if the company purchases certain units of sampling-daily-average price call options on such a raw material, then the company will get some money from exercising these call options when A is higher than E, so it will be guaranteed that the money spent on this raw material will be less than a certain level. From Table 8.8, we can see that when today's price of the raw material is \$100, the company needs to pay \$4.61 in order to buy a sampling-daily-average price call option with $E = 100$. Thus, the money spent on each unit of the raw material is \$ $4.61 + 100 =$ \$104.61 if $A \geq 100$ or \$ $4.61 + A$ if $A < 100$, which means that the money spent on each unit of the raw material is not greater than \$104.61. When an option with $E = 90$ is purchased, the money spent on each unit of the raw material is not greater than \$11.05 + 90 =\$101.05 because the premium for the call option for this case is \$11.05 (see Table 8.8). Which choice is better? This is determined by what you want. When the option with $E = 90$ is purchased, the maximum money spent is lower than that for the case with $E = 100$, but the money spent for lower A is higher than that for the case with $E = 100$. Table 8.9 shows you this fact.

Table 8.10. Double average call option prices on four meshes ($D_0 < r$)
($T = 1$, $S = 100$, $r = 0.05$, $D_0 = 0$, $\sigma = 0.2$,
$T_{1s} = 0.1$, $T_{1e} = 0.5$, $K_1 = 5$, $T_s = 0.6$, $T_e = 1.0$, $K = 5$, $P_m = 0.4$,
the payoff $= \max\left(\frac{I}{K} - \frac{I_1}{K_1}, 0\right)$, and the exact solution $= 5.872133\cdots$)

Mesh sizes	Results	|Errors|	CPU times	Results without extrapolation	|Errors|	CPU times
200×20	5.870320	0.001813	0.0042	5.869883	0.002250	0.0020
400×40	5.871861	0.000272	0.0094	5.871367	0.000766	0.0077
800×80	5.872133	0.000000	0.0282	5.871942	0.000191	0.0203
$1,600 \times 160$	5.872126	0.000007	0.0928	5.872080	0.000053	0.0745

Table 8.11. Double average call option prices on four meshes ($D_0 > r$)
($T = 1$, $S = 100$, $r = 0.05$, $D_0 = 0.1$, $\sigma = 0.2$,
$T_{1s} = 0.1$, $T_{1e} = 0.5$, $K_1 = 5$, $T_s = 0.6$, $T_e = 1.0$, $K = 5$, $P_m = 0.2$,
the payoff $= \max\left(\frac{I}{K} - \frac{I_1}{K_1}, 0\right)$, and the exact solution $= 3.244201\cdots$)

Mesh sizes	Results	|Errors|	CPU times	Results without extrapolation	|Errors|	CPU times
200×20	3.241122	0.003079	0.0052	3.235091	0.009110	0.0030
400×40	3.244162	0.000039	0.0116	3.241894	0.002307	0.0084
800×80	3.244263	0.000062	0.0321	3.243671	0.000530	0.0217
$1,600 \times 160$	3.244196	0.000005	0.1009	3.244064	0.000137	0.0813

Some Results of Double Average Call Options. For European-style other Asian and lookback options with discrete sampling, the method is similar. That is, the problem is solved by numerical schemes for partial differential equations and interpolation alternately. For details of the methods, see the papers by Andreasen [3] and Zhu [90]. Some results for such options are given in Sects. 4.3.7 and 4.4.7. Here we give some results for two double average call options, to show the effect of the extrapolation technique and how the approximate solutions converge to exact solutions in Tables 8.10 and 8.11. In Table 8.10 $D_0 = 0.1 > r = 0.05$, and in Table 8.11 $D_0 = 0 < r = 0.05$. There are 10 samplings at $t = 0.1, 0.2, \cdots, 1.0$. From these tables, we can see that the extrapolation technique greatly improves the rate of convergence and the accuracy, with about 25 % extra CPU time.

8.2.3 Projected Direct Methods for the LC Problem

As seen in Sect. 8.2.1, using implicit finite-difference methods for European options is straightforward. From Sect. 8.1.2, if an American option is formulated as a linear complementarity problem, then there is not a big difference between explicit finite-difference methods for European and American options. The implicit methods for American options are also only a little more complicated than the methods for European options.

Suppose we use a direct method to solve the system related to an American call option, which is formulated as the problem (8.8). Assuming that the partial differential equation holds everywhere and using scheme (8.47), we have a system in the form:

$$a_m \bar{v}_{m-1}^{n+1} + b_m \bar{v}_m^{n+1} + c_m \bar{v}_{m+1}^{n+1} = q_m^n, \quad m = 0, 1, 2, \cdots, M. \tag{8.64}$$

Actually, $\bar{v}_{-1}^{n+1}$ and $\bar{v}_{M+1}^{n+1}$ do not appear in the system because

$$a_0 = c_M = 0.$$

It is clear that the solution of the system (8.64) may not be the solution of the American option. However, we can find the solution of the American option with the aid of the system (8.64).

Similar to what we did in Sect. 6.2.1, if we let

$$u_0 = b_0, \quad y_0 = q_0^n, \tag{8.65}$$

and

$$u_m = b_m - \frac{c_{m-1} a_m}{u_{m-1}}, \quad y_m = q_m^n - \frac{y_{m-1} a_m}{u_{m-1}}, \quad m = 1, 2, \cdots, M, \tag{8.66}$$

then the equations in system (8.64) can be rewritten as

$$\bar{v}_m^{n+1} = \frac{y_m - c_m \bar{v}_{m+1}^{n+1}}{u_m}, \quad m = M, M-1, \cdots, 0, \tag{8.67}$$

where the relation with $m = M$ actually is

$$\bar{v}_M^{n+1} = \frac{y_M}{u_M}$$

because $c_M = 0$. From the derivation, we know that the relations in the system (8.67) with $m = 0, 1, \cdots, M_f$ are equivalent to the equations in the system (8.64) with $m = 0, 1, \cdots, M_f$, where M_f is any positive integer less than or equal to M. Obviously, $\bar{v}_m^{n+1}$ may not be greater than or equal to $\max(2\xi_m - 1, 0)$. Therefore, we need to find the value of the American option by

$$v_m^{n+1} = \max\left(\bar{v}_m^{n+1}, 2\xi_m - 1, 0\right), \quad m = 0, 1, \cdots, M \tag{8.68}$$

Table 8.12. American call option (PIFDI)

($r = 0.05$, $\sigma = 0.2$, $D_0 = 0.1$, $S = E = 100$, $T = 1$,
and the exact solution is $C = 5.92827717\cdots$)

Meshes	Results by Eq. (8.68)	|Errors|	Results by Eq. (8.69)	|Errors|
50×50	5.752424	0.175853	5.760096	0.168181
100×100	5.878708	0.049569	5.884210	0.044067
200×200	5.914582	0.013695	5.917403	0.010874
400×400	5.924045	0.004132	5.925541	0.002736
800×800	5.926810	0.001467	5.927574	0.000703
$1,600 \times 1,600$	5.927706	0.000571	5.928097	0.000180
$3,200 \times 3,200$	5.928032	0.000245	5.928230	0.000047

or by

$$v_m^{n+1} = \max\left(\frac{y_m - c_m v_{m+1}^{n+1}}{u_m}, 2\xi_m - 1, 0\right), \quad m = M, M-1, \cdots, 0, \quad (8.69)$$

successively. This method is referred to as the projected implicit finite-difference method I (PIFDI).

Is there any difference between the formulae (8.68) and (8.69)? The answer is yes. Let us explain this. As we know from Sect. 3.3.1, there is only one free boundary for a call option. It is natural to expect that when the formula (8.68) is used, there exists an M_f so that $v_m^{n+1} = \bar{v}_m^{n+1}$ for $m = 0, 1, \cdots, M_f$ and $v_m^{n+1} = \max(2\xi_m - 1, 0)$ for $m = M_f + 1, M_f + 2, \cdots, M$. When $\bar{v}_m^{n+1}$ are determined, we assume all the equations in the system (8.64) to hold. Even though for $m = M_f+1, M_f+2, \cdots, M$ we do not take $\bar{v}_m^{n+1}$ as solutions so that the constraint condition is satisfied, v_m^{n+1}, $m = 0, 1, \cdots, M_f$ are determined under the assumption of all the equations in the system (8.64) holding. For the formula (8.69), the situation is different. We assume that for $m = M, M-1, \cdots, M_f + 1$, $v_m^{n+1} = \max(2\xi_m - 1, 0)$ and for $m = M_f, M_f - 1, \cdots, 0$,

$$v_m^{n+1} = \frac{y_m - c_m v_{m+1}^{n+1}}{u_m}.$$

In this case, we only use the relations in the expression (8.67) with $m = M_f, M_f - 1, \cdots, 0$, which are equivalent to the equations in the system (8.64) with $m = M_f, M_f - 1, \cdots, 0$. Therefore, we only assume that the equations in the system (8.64) hold for $m = M_f, M_f - 1, \cdots, 0$. Consequently, this is closer to what the situation should be. In Table 8.12, results obtained by the formulae (8.68) and (8.69) and their errors are listed. You can see that on the same mesh, the error of the results obtained by the formula (8.68) is greater than the formula (8.69) and that the smaller the mesh size, the greater the difference. Even though the formula (8.68) can be used to obtain the price of American options, it brings some error that can be avoided if the formula (8.69) is used. However, if the free boundary is far away from $S = E$, then in the region $S \approx E$, the difference of the solutions obtained by the two direct methods is very small.

When an implicit scheme is used to solve problem (8.9), we need to choose the lower and upper bounds of the computational domain and give some artificial boundary conditions at these two boundaries because we cannot do computation on an infinite domain. Let the lower and upper bounds be x_l and x_u. For a call option, we assume $u(x_l, \bar{\tau}) = 0$ and $u(x_u, \bar{\tau}) = g(x_u, \bar{\tau})$, and for a put option, $u(x_l, \bar{\tau}) = g(x_l, \bar{\tau})$ and $u(x_u, \bar{\tau}) = 0$. As soon as we set these conditions, the problem (8.9) can be discretized and solved in the same way as described above for the problem (8.8). This method is referred to as the projected implicit finite-difference method II (PIFDII).

In Tables 8.13 and 8.14, the values of American call and put options obtained by PIFDII are given. When we do computation, we take

$$x_l = \ln(S_l/E) - |(r - D_0 - \sigma^2/2)T|$$

and

$$x_u = \ln(S_u/E) + |(r - D_0 - \sigma^2/2)T|.$$

For the call option, $S_l = 20$ and $S_u = 230$, and for the put, $S_l = 80$ and $S_u = 350$. There, we also give a solution with an error less than 10^{-8} in each table, which is obtained by the SSM given in Chap. 9. Therefore, we can have the errors of the solutions on different meshes. The CPU time used is also given, so you can have a notion about the performance of the method.

Table 8.13. American call option (PIFDII)

($r = 0.1$, $\sigma = 0.2$, $D_0 = 0.05$, $S = E = 100$, $T = 1$, and the exact solution is $C = 9.94092345 \cdots$)

Meshes	Results	|Errors|	CPU(s)
100×25	9.928528	0.012396	0.0025
200×50	9.937831	0.003093	0.0096
400×100	9.940151	0.000773	0.0400
800×200	9.940729	0.000194	0.1700
$1,600 \times 400$	9.940875	0.000048	0.6700

Table 8.14. American put option (PIFDII)

($r = 0.1$, $\sigma = 0.2$, $D_0 = 0.05$, $S = E = 100$, $T = 1$, and the exact solution is $P = 5.92827717 \cdots$)

Meshes	Results	|Errors|	CPU(s)
100×25	5.922275	0.006002	0.0025
200×50	5.926394	0.001883	0.0094
400×100	5.927654	0.000623	0.0400
800×200	5.928050	0.000227	0.1700
$1,600 \times 400$	5.928188	0.000089	0.6700

8.2.4 Projected Iteration Methods for the LC Problem

As we know, there are two types of methods to solve a linear system: iteration methods and direct methods. Similarly, there are two ways to solve the system related to American options. We discussed direct methods in the last subsection. Now let us study an iteration method. We still consider call options and use the system (8.64). This problem can be solved by a method similar to the SOR method for a system of linear equations given in Sect. 6.2.2. Any equation in the system (8.64) can be rewritten as

$$\bar{v}_m^{n+1} = (1-\omega)\bar{v}_m^{n+1} + \frac{\omega}{b_m}\left(q_m^n - a_m\bar{v}_{m-1}^{n+1} - c_m\bar{v}_{m+1}^{n+1}\right),$$

where ω is a constant. The value of the American option v_m^{n+1} satisfies the relation above if $\bar{v}_m^{n+1} > \max(2\xi_m - 1, 0)$ or equal to $\max(2\xi_m - 1, 0)$ otherwise. Therefore, for v_m^{n+1} we have the following relations:

$$v_m^{n+1} = \max\left((1-\omega)v_m^{n+1} + \frac{\omega}{b_m}\left(q_m^n - a_m v_{m-1}^{n+1} - c_m v_{m+1}^{n+1}\right), 2\xi_m - 1, 0\right),$$
$$m = 0, 1, \cdots, M.$$

We use an iteration method for finding its solution. Let $v_m^{(k)}$ be the k-th iteration of v_m^{n+1}, and the relation above can be rewritten in the following iteration form:

$$v_m^{(k+1)} = \max\left((1-\omega)v_m^{(k)} + \frac{\omega}{b_m}\left(q_m^n - a_m v_{m-1}^{(k+1)} - c_m v_{m+1}^{(k)}\right), 2\xi_m - 1, 0\right), \tag{8.70}$$

where $\omega \in (0, 2)$. Let $v_m^{(0)} = v_m^n$ for $m = 0, 1, \cdots, M$. As soon as we have $v_m^{(k)}$ for all m, the $(k+1)$-th iterative value of v_m^{n+1} can be obtained by equality (8.70) for $m = 0, 1, \cdots, M$ successively, starting from $k = 0$. When

$$\frac{1}{M+1}\sum_{m=0}^{M}\left(v_m^{(k)} - v_m^{(k+1)}\right)^2 \le \epsilon^2,$$

where ϵ^2 is a small number given according to the required accuracy, we can stop the iteration because for any m, $v_m^{(k)}$ and $v_m^{(k+1)}$ are very close to each other. This method is referred to as the projected successive over relaxation method I (PSORI). If the formulation (8.9) is adopted, after setting the values of x_l, x_u and the artificial boundary conditions, we can have a similar method and the corresponding method is referred to as PSORII. The details of the PSORII are left for readers to write as Problem 14.

In Tables 8.15 and 8.16, the prices of American call and put options on several meshes obtained by PSORII are given. The corresponding errors, CPU times, and ϵ^2 are also listed. All the parameters are the same as those given in

Table 8.15. American call option (PSORII)

($r = 0.1$, $\sigma = 0.2$, $D_0 = 0.05$, $S = E = 100$,
and the exact solution is $C = 9.94092345\cdots$)

Meshes	Results	|Errors|	CPU(s)	ϵ^2
100×25	9.929351	0.011573	0.0240	10^{-8}
200×50	9.938037	0.002887	0.1100	$0.5 \cdot 10^{-9}$
400×100	9.940202	0.000721	0.5300	$0.25 \cdot 10^{-10}$
800×200	9.940743	0.000181	2.7500	$0.125 \cdot 10^{-11}$
$1,600 \times 400$	9.940878	0.000046	20.000	$0.6125 \cdot 10^{-13}$

Table 8.16. American put option (PSORII)

($r = 0.1$, $\sigma = 0.2$, $D_0 = 0.05$, $S = E = 100$,
and the exact solution is $P = 5.92827717\cdots$)

Meshes	Results	|Errors|	CPU(s)	ϵ^2
100×25	5.922349	0.005928	0.0180	10^{-8}
200×50	5.926410	0.001867	0.0960	$0.5 \cdot 10^{-9}$
400×100	5.927651	0.000626	0.6100	$0.25 \cdot 10^{-10}$
800×200	5.928048	0.000230	5.2200	$0.125 \cdot 10^{-11}$
$1,600 \times 400$	5.928188	0.000089	46.300	$0.6125 \cdot 10^{-13}$

Tables 8.13 and 8.14. The only difference between the results here and there is the way we solved the system.

Comparing Tables 8.13 and 8.14 with Tables 8.15 and 8.16 shows that the CPU time here is longer. This implies that the cost of PSORII method is greater than the PIFDII method for this case. However, we need to point out that for most of multi-dimensional problems, the iteration methods may be better than the direct methods even though here we show that the direct method is better than the iteration method for one-dimensional problems.

8.2.5 Comparison with Explicit Methods

Explicit methods are usually very simple and very easy to use. The main problem of explicit methods is the stability requirement. For the explicit method (8.6), the stability requirement is

$$\bar{\alpha} \le \frac{1}{2} \quad \text{or} \quad \Delta\bar{\tau} \le \frac{1}{2}\Delta x^2.$$

Thus, if the accuracy of the solution requires a small Δx, then a much smaller $\Delta\bar{\tau}$ must be taken in order to satisfy the stability condition, which slows down the computation. For implicit methods, no such restrictions are needed, and we can let $\Delta\bar{\tau}/\Delta x = constant$. Therefore, if we require higher accuracy, an implicit scheme will give a better performance. This can be seen in the following way.

Suppose we solve the problem (8.5) by the explicit scheme (8.6) and the implicit scheme (7.9). Assume that for the scheme (8.6) $\Delta\bar{\tau} = \alpha\Delta x^2$, where

α is a constant not greater than 1/2 and that for the scheme (7.9), $\Delta\bar{\tau} = \beta\Delta x$, where β is a constant. For the explicit scheme (8.6), the amount of computational work is

$$W_e = \frac{a_e}{\Delta\bar{\tau}\Delta x} = \frac{a_e}{\alpha\Delta x^3},$$

and the error is

$$E = b_{e\bar{\tau}}\Delta\bar{\tau} + b_{ex}\Delta x^2 = (b_{e\bar{\tau}}\alpha + b_{ex})\Delta x^2,$$

where a_e, $b_{e\bar{\tau}}$, and b_{ex} are three parameters related to scheme (8.6) and the solution. From these two relations for the scheme (8.6), we have the relation between the amount of work and the error required:

$$W_e = \frac{a_e[b_{e\bar{\tau}}\alpha + b_{ex}]^{3/2}}{\alpha}E^{-3/2}.$$

For the scheme (7.9),

$$W_i = \frac{a_i}{\Delta\bar{\tau}\Delta x} = \frac{a_i}{\beta\Delta x^2}$$

and

$$E = b_{i\bar{\tau}}\Delta\bar{\tau}^2 + b_{ix}\Delta x^2 = (b_{i\bar{\tau}}\beta^2 + b_{ix})\Delta x^2,$$

where a_i, $b_{i\bar{\tau}}$, and b_{ix} are three parameters related to scheme (7.9) and the solution. Here, we assume that a direct method is used for solving the linear system. Therefore, the relation between the amount of work and the error required is

$$W_i = \frac{a_i(b_{i\bar{\tau}}\beta^2 + b_{ix})}{\beta}E^{-1}.$$

Usually, a_i is greater than a_e because for the scheme (7.9) a linear system needs to be solved at each time step. Consequently, when E is not too small, it is possible that W_i is greater than W_e for the same E, which means that the scheme (8.6) is better than the scheme (7.9). When the solution is much smoother in the $\bar{\tau}$-direction than in the x-direction, the scheme (7.9) might be better than the scheme (8.6) even if E is not very small. This is because in this case for the scheme (7.9) we can choose a big β such that $b_{i\bar{\tau}}\beta^2$ is close to b_{ix}, which makes W_i smaller, but for the scheme (8.6) we cannot take this advantage because of the stability requirement. However, when E is small enough, then W_i must be less than W_e. This can be seen from comparing Tables 8.2 and 8.3 with Tables 8.13 and 8.14. The tables show that for the American call problem with the parameters given there, in order to reach an error about 0.003, the CPU time for the scheme (8.6) is about 0.06 and the CPU time for the scheme (7.9) is about 0.01.

8.2.6 Two-Asset Options

Sometimes two assets are involved in an option problem. In this case, usually a two-dimensional problem needs to be solved. As shown in Sect. 4.5.4, pricing a two-asset option can be reduced to solving Eq. (4.79) with final condition (4.80). This problem is a final-value problem. In order to use the scheme (7.46), we need to introduce a new variable $\tau = T - t$ and modify Eq. (4.79) into an equation with independent variables ξ, θ and τ. Let us call the new equation the modified Eq. (4.79). The modified Eq. (4.79) can be discretized by scheme (7.46). For a two-asset call option, the final condition is

$$V(S_1, S_2, T) = \max(E_1 - S_1, E_2 - S_2, 0),$$

and for a two-asset put option, the final condition is

$$V(S_1, S_2, T) = \max(S_1 - E_1, S_2 - E_2, 0).$$

Under the coordinate system (ξ, θ, t) introduced in Sect. 4.5.4, letting $\tau = T - t$, and instead of V, using $w = \dfrac{V}{S + P_m}$ as a dependent variable, S being $\dfrac{\xi P_m}{1-\xi}$, these two conditions become

$$w(\xi, \theta, 0) = \frac{1}{\sqrt{\left(\dfrac{S_1}{P_1}\right)^2 + \left(\dfrac{S_2}{P_2}\right)^2} + P_m} \max\left(E_1 - S_1, E_2 - S_2, 0\right) \qquad (8.71)$$

Table 8.17. Prices of a European two-asset call option

($r = 0.02, D_{01} = 0.01, D_{02} = 0.01, \sigma_1 = 0.2,$
$\sigma_2 = 0.15, \rho = 0.8, E_1 = 100, E_2 = 95$, and $T = 1$)

S_1	S_2	Price
95.0	90.0	6.76
97.5	92.5	8.22
100.0	95.0	9.84
102.5	97.5	11.61
105.0	100.0	13.52

Table 8.18. Prices of a European two-asset put option

($r = 0.02, D_{01} = 0.01, D_{02} = 0.01, \sigma_1 = 0.2,$
$\sigma_2 = 0.15, \rho = 0.8, E_1 = 100, E_2 = 95$, and $T = 1$)

S_1	S_2	Price
95.0	90.0	11.29
97.5	92.5	9.78
100.0	95.0	8.41
102.5	97.5	7.19
105.0	100.0	6.11

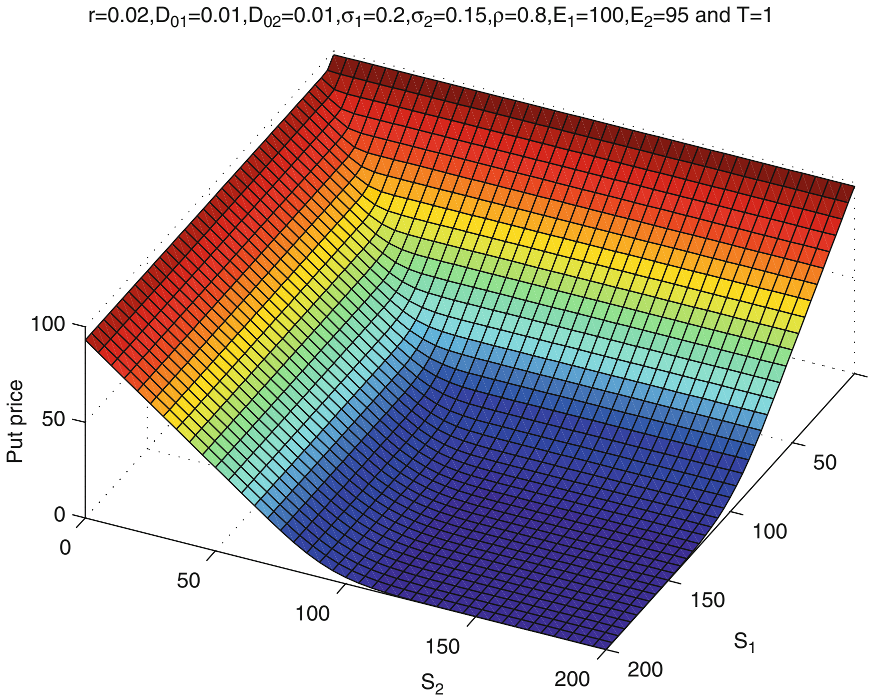

Fig. 8.7. Values of a European two-asset put option ($r = 0.02$, $D_{01} = 0.01$, $D_{02} = 0.01$, $\sigma_1 = 0.2$, $\sigma_2 = 0.15$, $\rho = 0.8$, $E_1 = 100$, $E_2 = 95$, and $T = 1$)

and

$$w(\xi, \theta, 0) = \frac{1}{\sqrt{\left(\frac{S_1}{P_1}\right)^2 + \left(\frac{S_2}{P_2}\right)^2} + P_m} \max\left(S_1 - E_1, S_2 - E_2, 0\right), \quad (8.72)$$

respectively. Here P_1, P_2, and P_m are parameters, and

$$S_1 = P_1 \frac{\xi P_m}{1-\xi} \cos\theta,$$

$$S_2 = P_2 \frac{\xi P_m}{1-\xi} \sin\theta.$$

About the value of the parameters P_1, P_2, P_m, we can let

$$P_1 = E_1, \quad P_2 = E_2, \quad P_m = 1.$$

Using the initial condition (8.71) or (8.72) and scheme (7.46) obtained by discretizing the modified Eq. (4.79), we can get the price of a European two-asset call or put option. Some values of such options are listed in Tables 8.17

and 8.18. These results are obtained by a $400 \times 600 \times 400$ mesh, which means that

$$\Delta\xi = 1/400, \quad \Delta\theta = 1/600, \quad \Delta\tau = 1/400.$$

Computation is also done on the $800 \times 1,200 \times 800$ mesh; the results to two decimal places are the same except for the case of the put option with $S_1 = 95$ and $S_2 = 90$. For this case, on the $800 \times 1,200 \times 800$ mesh, the result is 11.28, and on the $400 \times 600 \times 400$ mesh, the result is 11.29. In order to give readers an idea as to what solutions of two-asset put options look like, the value of a two-asset put option for $(S_1, S_2) \in [0, 200] \times [0, 200]$ is shown in Fig. 8.7.

8.3 Singularity-Separating Method

In this section, we will discuss how to make numerical methods more efficient. Generally speaking, the smoother the solution, the smaller the truncation error. Therefore, if the solution is smooth, even on a coarse mesh, the numerical result is still quite good. Suppose that the solution we need to find is not very smooth but has a certain type of singularity caused by the final condition. Also, we assume that there is an analytic expression that satisfies the same final condition and the same equation or a similar equation. If both the final conditions and the equations are the same, their singularities caused by the final conditions are the same, and the difference between them is a smooth function; if only the final conditions are the same, they possess similar singularities, and the difference between them is usually smoother than the solution we need to find. In both cases, we can first compute the difference using numerical methods and then have our solution by adding the analytic expression and the difference together. Such a method or technique will be referred to as singularity-separating method (SSM), or singularity-separating technique, in this book. Because computing the difference is quite efficient, we can have the solution quite efficiently. Of course, there is some extra work in order to compute the difference. However, from the examples we are going to show, such a way can truly make numerical methods more efficient. In this section, we will give some details of the method for European double moving barrier options, European vanilla option with variable volatilities, Bermudan options, European Parisian options, European average price options, two-factor vanilla options, and two-factor convertible bonds with $D_0 = 0$. Indeed, the method can be used for many more cases, including multi-factor derivative securities.

8.3.1 Barrier Options

If the option has a fixed barrier and σ, r, and D_0 are constants, we can find analytic solutions of barrier options (see Sect. 4.2). However, if the option has two moving barriers, analytic solutions may not exist even if σ, r, and D_0 are constants, and we may need to rely on numerical methods for pricing such an option. Here, we discuss how to make numerical methods more efficient.

The price $V(S,t)$ of a double moving barrier call option with rebates satisfies the equation

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r - D_0)S\dfrac{\partial V}{\partial S} - rV = 0, \\ \qquad\qquad\qquad\qquad f(t) \le S \le g(t), \quad 0 \le t \le T, \\ V(S,T) = \max(S - E, 0), \quad f(T) \le S \le g(T), \\ V(f(t),t) = 0, \quad 0 \le t \le T, \\ V(g(t),t) = g(t) - E, \quad 0 \le t \le T, \end{cases} \tag{8.73}$$

where $f(t)$ and $g(t)$ are the locations of the lower and upper barriers with

$$f(t) < E \quad \text{and} \quad g(t) > E,$$

and we assume that at the lower barrier, there is no rebate and at the upper barrier, the rebate is

$$g(t) - E.$$

Because the derivative of the payoff function $\max(S-E,0)$ is discontinuous at $S = E$, the solution $V(S,t)$ at $t \approx T$ and $S \approx E$ is not very smooth. Therefore, the error of numerical solutions in the region around $t = T$ and $S = E$ is relatively large compared with that in the region far away from this point. In order to make the numerical solution better, we introduce a new function

$$\overline{V}(S,t) = V(S,t) - c(S,t),$$

where $c(S,t)$ is the price of the vanilla call option. Because $c(S,t)$ also satisfies the partial differential equation and the final condition in problem (8.73), $\overline{V}(S,t)$ satisfies

$$\begin{cases} \dfrac{\partial \overline{V}}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 \overline{V}}{\partial S^2} + (r - D_0)S\dfrac{\partial \overline{V}}{\partial S} - r\overline{V} = 0, \\ \qquad\qquad\qquad\qquad f(t) \le S \le g(t), \quad 0 \le t \le T, \\ \overline{V}(S,T) = 0, \quad f(T) \le S \le g(T), \\ \overline{V}(f(t),t) = -c(f(t),t), \quad 0 \le t \le T, \\ \overline{V}(g(t),t) = g(t) - E - c(g(t),t), \quad 0 \le t \le T. \end{cases} \tag{8.74}$$

The derivative of $\overline{V}(S,t)$ at $t \approx T$ and $S \approx E$ is very smooth, so the error of the numerical solution of $\overline{V}(S,t)$ is usually smaller than that of $V(S,t)$.

Therefore, in order to get a better $V(S,t)$, we can first obtain the numerical solution of $\overline{V}(S,t)$ and then have $V(S,t)$ by adding $\overline{V}(S,t)$ and $c(S,t)$ together. We refer to this procedure as the singularity-separating method (SSM) or the singularity-separating technique for European barrier options. The reason is as follows. The derivative of $V(S,t)$ is discontinuous at $t=T$ and $S=E$. Thus, we say that $V(S,t)$ has some weak singularity. The function $\overline{V}(S,t)$, which will be determined numerically, is smooth. Therefore, the weak singularity has been "separated" from the numerical computation. The CPU time of getting $\overline{V}(S,t)$ is slightly longer than that of getting $V(S,t)$ directly because $c(f(t),t)$ and $c(g(t),t)$ need to be computed in order to get $\overline{V}(S,t)$. Because the error is smaller, we can usually expect better performance, i.e., we can usually expect to have the same accuracy by spending less CPU time or to spend the same CPU time for a better accuracy. Consequently, the singularity-separating technique can usually improve the performance.

Both $V(S,t)$ and $\overline{V}(S,t)$ are solutions of the following problem

$$\begin{cases} \dfrac{\partial \bar{u}}{\partial t}+\dfrac{1}{2}\sigma^2 S^2\dfrac{\partial^2 \bar{u}}{\partial S^2}+(r-D_0)S\dfrac{\partial \bar{u}}{\partial S}-r\bar{u}=0, & \\ \qquad\qquad\qquad\qquad\qquad\qquad f(t)\le S\le g(t), \quad 0\le t\le T, & \\ \bar{u}(S,T)=\bar{f}_1(S), & f(T)\le S\le g(T), \\ \bar{u}(f(t),t)=\bar{b}_l(t), & 0\le t\le T, \\ \bar{u}(g(t),t)=\bar{b}_u(t), & 0\le t\le T. \end{cases} \tag{8.75}$$

The only difference between the two cases is the functions in the final condition and in the boundary conditions. Thus, no matter whether the singularity-separating technique is used, we need a numerical method for problem (8.75) in order to have $V(S,t)$.

Problem (8.75) is a typical moving boundary problem. In order to convert it into a problem with fixed boundaries and transfer the final condition to an initial condition, we use the following transformation:

$$\begin{cases} \eta=\dfrac{S-f(t)}{g(t)-f(t)}, \\ \tau=T-t. \end{cases} \tag{8.76}$$

Let

$$\begin{aligned} &u(\eta,\ \tau)=u(\eta(S,t),\ T-t)=\bar{u}(S,\ t), \\ &F(\tau)=F(T-t)=f(t), \\ &G(\tau)=G(T-t)=g(t). \end{aligned}$$

Because

$$\begin{aligned}\frac{\partial \bar{u}}{\partial t} &= \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial \tau}\frac{\partial \tau}{\partial t} \\ &= -\frac{1}{g-f}\left[\frac{df}{dt} + \eta\left(\frac{dg}{dt} - \frac{df}{dt}\right)\right]\frac{\partial u}{\partial \eta} - \frac{\partial u}{\partial \tau} \\ &= \frac{1}{G-F}\left[\frac{dF}{d\tau} + \eta\left(\frac{dG}{d\tau} - \frac{dF}{d\tau}\right)\right]\frac{\partial u}{\partial \eta} - \frac{\partial u}{\partial \tau}, \\ \frac{\partial \bar{u}}{\partial S} &= \frac{1}{G(\tau)-F(\tau)}\frac{\partial u}{\partial \eta}, \\ \frac{\partial^2 \bar{u}}{\partial S^2} &= \frac{1}{[G(\tau)-F(\tau)]^2}\frac{\partial^2 u}{\partial \eta^2},\end{aligned}$$

$u(\eta,\tau)$ is the solution of the problem

$$\begin{cases}\dfrac{\partial u}{\partial \tau} = \mathbf{L}_{\eta 1} u, & 0 \le \eta \le 1 \quad 0 \le \tau \le T, \\ u(\eta,0) = f_1(\eta), & 0 \le \eta \le 1, \\ u(0,\tau) = b_l(\tau), & 0 \le \tau \le T, \\ u(1,\tau) = b_u(\tau), & 0 \le \tau \le T,\end{cases} \tag{8.77}$$

where

$$\begin{aligned}\mathbf{L}_{\eta 1} &= \frac{1}{2}\left(\frac{S\sigma}{G-F}\right)^2\frac{\partial^2}{\partial \eta^2} + \left\{\frac{S}{G-F}(r-D_0)\right. \\ &\qquad \left. + \frac{1}{G-F}\left[\frac{dF}{d\tau} + \eta\left(\frac{dG}{d\tau} - \frac{dF}{d\tau}\right)\right]\right\}\frac{\partial}{\partial \eta} - r, \\ f_1(\eta) &= \bar{f}_1\left(F(0) + \eta[G(0)-F(0)]\right), \\ b_l(\tau) &= \bar{b}_l(T-\tau), \\ b_u(\tau) &= \bar{b}_u(T-\tau).\end{aligned}$$

The problem (8.77) can be solved by explicit finite-difference schemes or implicit finite-difference schemes and even by pseudo-spectral methods. Here, we give some results to explain the effect of this technique if implicit finite-difference methods are used.

We have solved an identical problem by scheme (7.6) in two different ways: with and without SSM. In Table 8.19, the results, the errors, and the CPU time in seconds for four meshes are given. There, $N \times M$ in the column "Meshes" stands for a mesh that has $N+1$ nodes in the t-direction (the τ-direction) and $M+1$ nodes in the S-direction (the η-direction). The lower and upper knock-out boundaries are

$$f(t) = 0.9E\mathrm{e}^{-0.1t} \quad \text{and} \quad g(t) = 1.6E\mathrm{e}^{0.1t}.$$

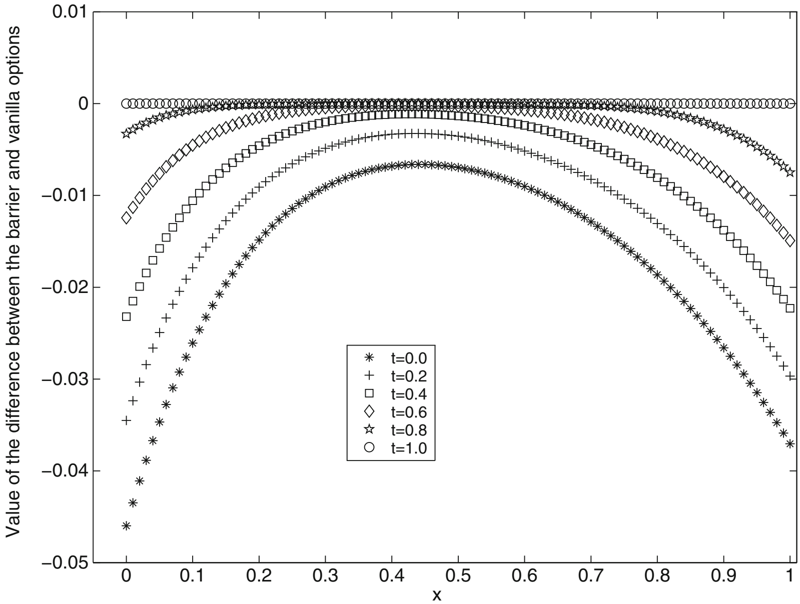

Fig. 8.8. Variation of the difference between the barrier and vanilla option values

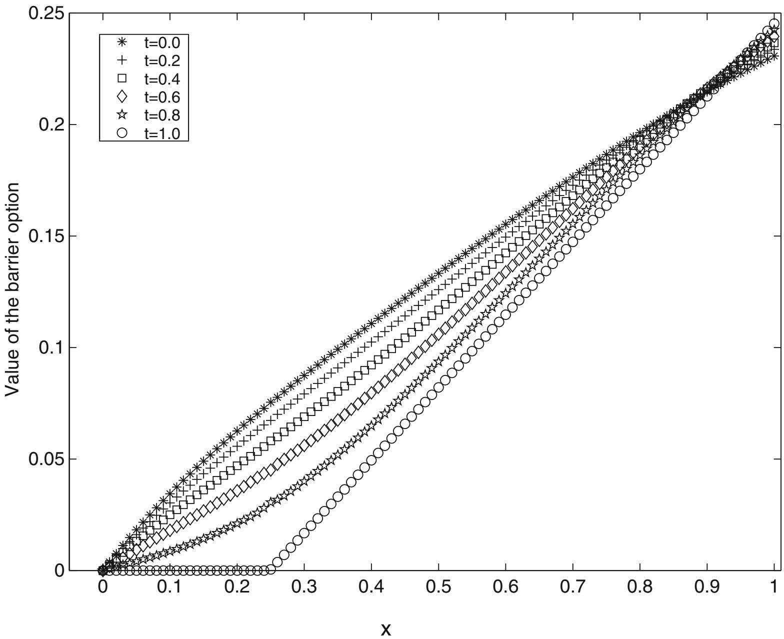

Fig. 8.9. Variation of the barrier option value

Table 8.19. Implicit methods with and without the SSM

($S = 95$, $T = 1$, $E = 100$, $\sigma = 0.25$, $r = 0.1$, $D_0 = 0$,
$f(t) = 0.9E\mathrm{e}^{-0.1t}$, $g(t) = 1.6E\mathrm{e}^{0.1t}$, the rebate $= g(t) - E$,
and the exact solution is $6.8441468\cdots$)

	Without SSM			With SSM		
Meshes	Solution	|Errors|	CPU	Solution	|Errors|	CPU
12×48	6.845973	0.001826	0.00039	6.843292	0.000855	0.00049
25×100	6.844623	0.000476	0.0019	6.844205	0.000058	0.0019
50×200	6.844187	0.000040	0.0062	6.844163	0.000016	0.0063
100×400	6.844167	0.000020	0.0221	6.844150	0.000003	0.0221

There, the results both with and without SSM are given. In order to give errors, we have to find the exact solution. To our knowledge, no analytic solution for such a problem has been found. Therefore, we take a very accurate approximate solution as an exact solution. For this case, the exact solution is $6.8441468\cdots$ (here the eight digits are correct). From there, we can see that the result with SSM is clearly better than without SSM on the same mesh whereas the CPU time difference between the two cases is very small. Therefore, the advantage of the singularity-separating technique is obvious for this case. As we know, if the error $\approx a\Delta\tau^\alpha = a(T/N)^\alpha$ (suppose $\Delta\tau/\Delta\eta = constant$), then we say that the convergence rate is $O(\Delta\tau^\alpha)$. From Table 8.19, we can see that when N is doubled, the error of the implicit finite-difference method with the singularity-separating technique decreases by a factor of about 4. This implies that the convergence rate of this method is $O(\Delta\tau^2)$.

In what follows, we give an intuitive explanation on why the singularity-separating method can improve the numerical results. The functions computed numerically for the methods with and without the singularity-separating technique are plotted in Figs. 8.8 and 8.9 respectively. In each figure, there are six curves, which correspond to $t = 0$, 0.2, 0.4, 0.6, 0.8, 1.0. In Fig. 8.9, the functions are not as smooth as those in Fig. 8.8, especially, the derivative of the function for $t = 1$ in Fig. 8.9 is discontinuous. Therefore, when the singularity-separating technique is used, the truncation is smaller.

When there is no rebate at the upper barrier, such a method can still improve the performance. This is left for the reader to study (see Problem 16). For the case discussed in this subsection, the singularity is removed completely. For the European options with discrete dividends and some other cases, the singularity can also be completely removed in the same way. In many other cases, the singularity cannot be completely separated but can be made much weaker. In the next several subsections, we will discuss how the SSM works for other cases.

8.3.2 European Vanilla Options with Variable Volatilities

When σ is a constant, for European vanilla options we can get their prices by the Black–Scholes formulae. However, it seems that the assumption of σ being a constant needs to be modified. One of the modifications is to let σ be a function of S. In this case, in order to evaluate an option, we usually need to solve a partial differential equation problem numerically. In order to overcome the problem caused by the discontinuous derivative in the payoff, we can do the following.

Let us consider call options. Their prices $c(S,t)$ are solutions of the problem:

$$\begin{cases} \dfrac{\partial c}{\partial t} + \dfrac{1}{2}\sigma^2(S)S^2\dfrac{\partial^2 c}{\partial S^2} + (r-D_0)S\dfrac{\partial c}{\partial S} - rc = 0, \quad 0 \le S, \quad t \le T, \\ c(S,T) = \max(S-E,0), \quad 0 \le S. \end{cases}$$

Suppose that $c_E(S,t;\sigma(E))$ is the price of the option with the volatility at $S=E$, $\sigma(E)$, i.e., $c_E(S,t;\sigma(E))$ satisfies

$$\begin{cases} \dfrac{\partial c_E}{\partial t} + \dfrac{1}{2}\sigma^2(E)S^2\dfrac{\partial^2 c_E}{\partial S^2} + (r-D_0)S\dfrac{\partial c_E}{\partial S} - rc_E = 0, \quad 0 \le S, \quad t \le T, \\ c_E(S,T) = \max(S-E,0), \quad 0 \le S. \end{cases}$$

Let $\bar{c}(S,t) = c(S,t) - c_E(S,t;\sigma(E))$. Then, $\bar{c}(S,t)$ is the solution of the problem

$$\begin{cases} \dfrac{\partial \bar{c}}{\partial t} + \dfrac{1}{2}\sigma^2(S)S^2\dfrac{\partial^2 \bar{c}}{\partial S^2} + (r-D_0)S\dfrac{\partial \bar{c}}{\partial S} - r\bar{c} = f(S,t), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le S, \quad t \le T, \\ \bar{c}(S,T) = 0, \qquad\qquad\qquad\qquad\quad 0 \le S, \end{cases} \tag{8.78}$$

where

$$\begin{aligned} f(S,t) &= \frac{1}{2}\left[\sigma^2(E) - \sigma^2(S)\right] S^2 \frac{\partial^2 c_E}{\partial S^2} \\ &= \frac{1}{2\sigma(E)\sqrt{2\pi(T-t)}}\left[\sigma^2(E) - \sigma^2(S)\right] S \mathrm{e}^{-(D_0(T-t)+d_1^2/2)} \end{aligned} \tag{8.79}$$

and

$$d_1 = \left\{\ln(S/E) + \left[r - D_0 + \frac{1}{2}\sigma^2(E)\right](T-t)\right\} \Big/ \left[\sigma(E)\sqrt{T-t}\right].$$

This problem is defined on an infinite domain. In order to convert it into a problem on a finite domain with a bounded solution, we use the following transformation:

$$\begin{cases} \xi = \dfrac{S}{S+E}, \\ \tau = T - t, \\ \overline{c}(S,t) = (S+E)\overline{V}(\xi,\tau). \end{cases}$$

Finally, we have

$$\begin{cases} \dfrac{\partial \overline{V}}{\partial \tau} = \dfrac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1-\xi)^2\dfrac{\partial^2 \overline{V}}{\partial \xi^2} + (r - D_0)\xi(1-\xi)\dfrac{\partial \overline{V}}{\partial \xi} \\ \qquad - [r(1-\xi) + D_0\xi]\overline{V} + \bar{f}(\xi,\tau), \qquad 0 \le \xi \le 1, \quad 0 \le \tau, \\ \overline{V}(\xi, 0) = 0, \qquad\qquad 0 \le \xi \le 1, \end{cases} \tag{8.80}$$

where

$$\bar{\sigma}(\xi) = \sigma(\xi E/(1-\xi)),$$

$$\bar{f}(\xi,\tau) = \frac{-f(S,t)}{S+E} = \frac{1}{2\sigma(E)\sqrt{2\pi\tau}}\left[\bar{\sigma}^2(\xi) - \sigma^2(E)\right]\xi \mathrm{e}^{-(D_0\tau + d_1^2/2)}$$

and

$$d_1 = \left\{\ln\frac{\xi}{1-\xi} + \left[r - D_0 + \frac{1}{2}\sigma^2(E)\right]\tau\right\} \Big/ \left[\sigma(E)\sqrt{\tau}\right].$$

In order to do some computation, we need the function $\sigma(S)$ or $\bar{\sigma}(\xi)$. For the Japanese yen–U.S. dollar exchange rate, we determine the function by the method in Sect. 6.3.2. In order to avoid approximating a function on an infinite domain, a new variable $\xi = S/(S + P_m)$ is introduced. Because the exchange rate is around 0.01, we set $P_m = 0.01$. Using the data of 1990–2000 from the market (see the curve in Fig. 1.5), we find the maximum and minimum values, $S_{\max} = 0.01232741616$ and $S_{\min} = 0.00625390870$. The corresponding values of ξ are

$$\xi_l = \frac{S_{\min}}{S_{\min} + P_m} = 0.384763371, \quad \xi_u = \frac{S_{\max}}{S_{\max} + P_m} = 0.552120141.$$

Assume that the function $\bar{\sigma}(\xi)$ is in the form:

$$\bar{\sigma}(\xi) = \begin{cases} c_l + a_l\left[1 - \left(\dfrac{\xi}{\xi_l}\right)^{200}\right], & 0 \le \xi < \xi_l, \\ a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3, & \xi_l \le \xi \le \xi_u, \\ c_u + a_u\left[1 - \left(\dfrac{1-\xi}{1-\xi_u}\right)^{200}\right], & \xi_u \le \xi \le 1, \end{cases}$$

where c_l, a_l, a_0, a_1, a_2, a_3, c_u, a_u are eight parameters to be determined. Taking the data of 1990–2000 from the market, using the method described in Sect. 6.3.2 with $g(\xi) \equiv 1$ and setting $M = 7$, we find the values of a_0, a_1, a_2, a_3:

$$a_0 = -10.7848, \quad a_1 = 72.8005, \quad a_2 = -161.134, \quad a_3 = 118.208.$$

Then, requiring the continuity of the function at $\xi = \xi_l$ and $\xi = \xi_u$ up to the first derivative yields

$$c_l = 0.104667, \quad a_l = -0.00250664, \quad c_u = 0.185335, \quad a_u = 0.00665520.$$

In Fig. 8.10, this function is plotted as a solid line, and the circles are the volatilities for different S obtained by statistics.

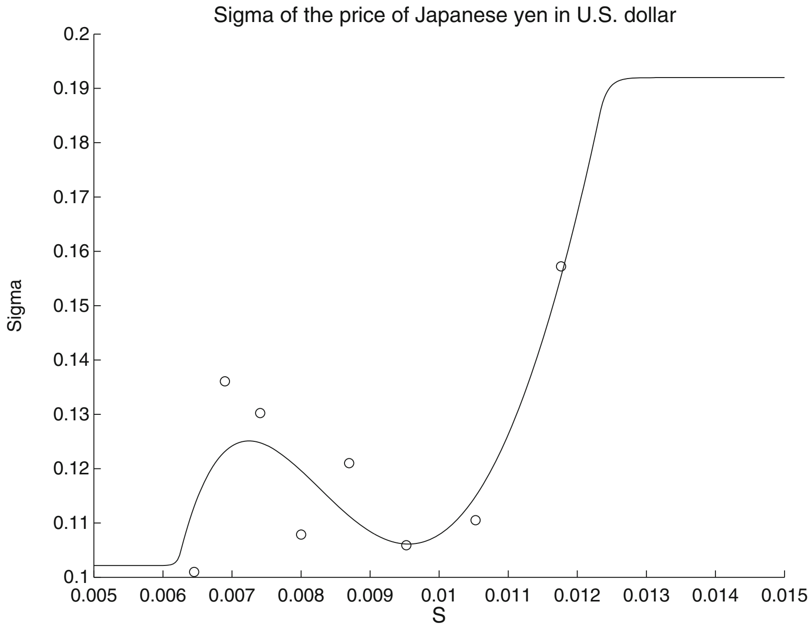

Fig. 8.10. The volatility function for Japanese yen–U.S. dollar exchange rate

As soon as we have this function, we can evaluate the price of options on the Japanese yen–U.S. dollar exchange rate. Discretizing problem (8.80) by the difference scheme (7.6) and solving the linear system by the LU decomposition, we can find the price. In Fig. 8.11, the solid line gives the value of the European call option. There, we also compare different models. Another model is to let the volatility be a constant. Using the same data, we find $\sigma = 0.1165$. The dashed line in Fig. 8.11 gives the option price for this model obtained by the Black–Scholes formula. The maximum difference of the results between

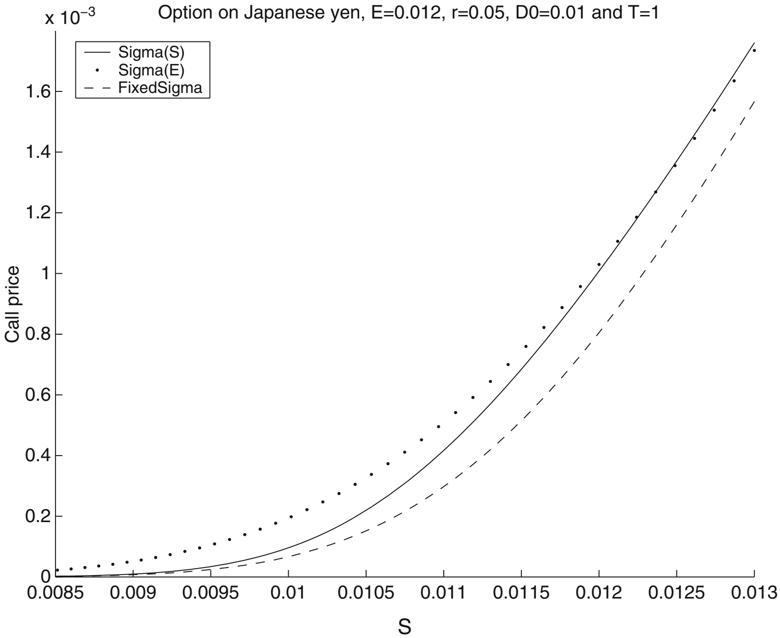

Fig. 8.11. The value of a European call option with a variable volatility with $E = 0.012$, $r = 0.05$, $D_0 = 0.01$, and $T = 1$ year

the two models is more than 30 % if $S \in [0.0115, 0.0125]$. If we assume σ to take the value of $\sigma(E)$ (the result for this case is given by the dotted line in Fig. 8.11), the maximum difference is more than 8 % for $S \in [0.0115, 0.0125]$. Therefore, among the results obtained by using different models, there is quite a big difference. In our computation for the model with variable volatility, the numerical method is quite efficient because we are calculating the difference numerically. For this example problem, on a 60×4 mesh for the option price at $S = E$, the error is $6 \times 10^{-5}E$ when the SSM is used and $1 \times 10^{-3}E$ when the SSM is not used.

Finally, we would like to point out that unlike the barrier options, in this case the weak singularity is not removed completely. However, the singularity is weakened so the SSM still succeeds as shown above. Let us explain this matter as follows. Because $\dfrac{\partial^2 c_E}{\partial S^2}$ has some singularity at the point $T = t$ and $S = E$, the function $f(S,t) = \dfrac{1}{2}\left[\sigma^2(E) - \sigma^2(S)\right] S^2 \dfrac{\partial^2 c_E}{\partial S^2}$ also has some singularity. However, because the term $\sigma^2(E) - \sigma^2(S)$ is equal to zero at $S = E$, the singularity of $f(S,t)$ at that point is much weaker than that of $\dfrac{\partial^2 c_E}{\partial S^2}$. In Figs. 8.12 and 8.13, $f(S,t)$ used in this example and $\dfrac{\partial^2 c_E}{\partial S^2}$ for $t = T - 0.01$, $T - 0.001$, $T - 0.0001$ are plotted, respectively. Noticing the maximum value

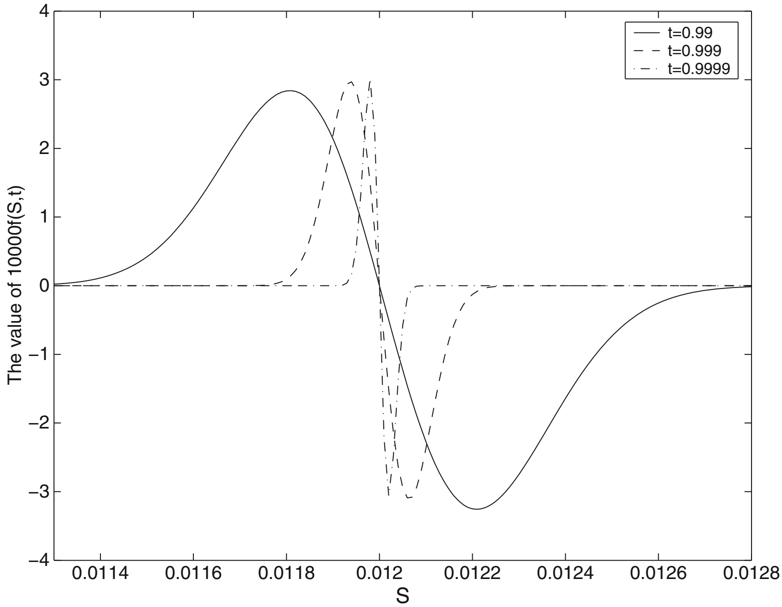

Fig. 8.12. The value of the function $10,000f(S,t)$ at $t \approx T$ $(T = 1)$

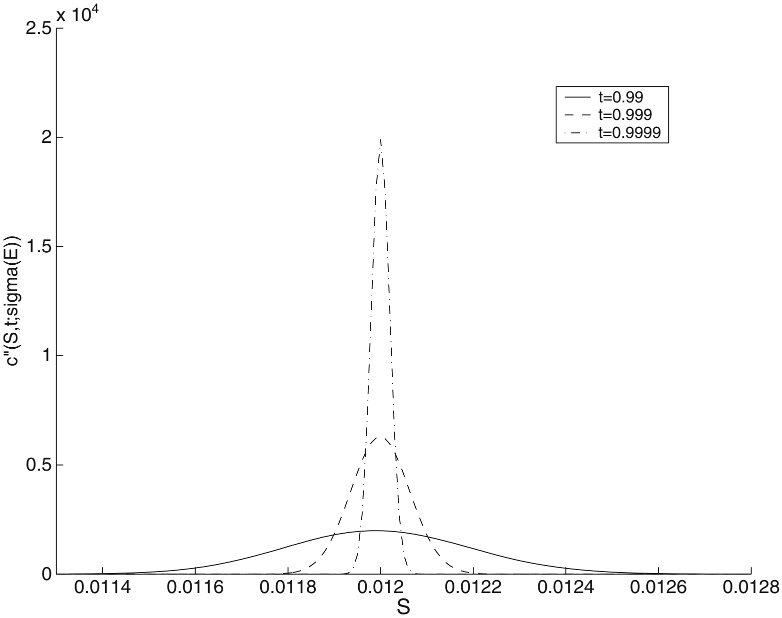

Fig. 8.13. The value of $\dfrac{\partial^2 c_E}{\partial S^2}$ at $t \approx T$ $(T = 1)$

of $|f(S,t)|$ is about 3.5×10^{-4} and the value of $\dfrac{\partial^2 c_E}{\partial S^2}$ could be very large, reaching 2×10^4 at $t = T - 0.0001$, we can see that the singularity of $f(S,t)$ at that point is truly weaker than that of $\dfrac{\partial^2 c_E}{\partial S^2}$. Because the singularity of $f(S,t)$ is quite weak and the singularity of $\bar{c}(S,t)$ is weaker than $f(S,t)$, the function $\bar{c}(S,t)$ is quite smooth. This is an important reason to guarantee the success of the SSM.

8.3.3 Bermudan Options

A Bermudan option is an option that can be exercised early, but only on predetermined dates. It is clear that the holder of a Bermudan option has more rights than the holder of a European option and less rights than the holder of an American option, just like the fact that Bermuda is situated between America and Europe. This is how the option got its name. If we use projected methods, it is easy to price. Here, we suggest some more efficient methods. Assume the expiry of the option to be T and suppose the option can be exercised at time $t = T_1, T_2, \cdots, T_K = T$, where $T_k = kT/K$, $k = 1, 2, \cdots, K$.

Let us consider a Bermudan call option with $D_0 > 0$ and a variable $\sigma(S)$, and denote its value by $C_b(S,t)$. Define $T_0 = 0$ and assume $T_0 < T_1 < \cdots < T_K$. Then, $C_b(S,t)$ is a solution of K successive problems:

$$\begin{cases} \dfrac{\partial C_b}{\partial t} + \dfrac{1}{2}\sigma^2(S)S^2\dfrac{\partial^2 C_b}{\partial S^2} + (r - D_0)S\dfrac{\partial C_b}{\partial S} - rC_b = 0, \\ \qquad\qquad\qquad 0 \le S, \quad T_{k-1} < t < T_k, \\ C_b(S, T_k^-) = \max(C_b(S, T_k^+), \max(S - E, 0)), \quad 0 \le S, \end{cases} \tag{8.81}$$

$$k = K, K-1, \cdots, 1$$

with $C_b(S, T_K^+) = \max(S - E, 0)$. Clearly, at $t = T_K$, $C_b(S, T_K^-) = \max(S - E, 0)$ for $S \in [0, \infty)$. At $t = T_k$, $k = K-1, K-2, \cdots, 1$, the whole interval $[0, \infty)$ is divided into two parts $[0, S_k^*]$ and (S_k^*, ∞). On $[0, S_k^*]$, $C_b(S, T_k^+) \ge \max(S - E, 0)$ and on (S_k^*, ∞), $C_b(S, T_k^+) < \max(S - E, 0)$. Because these functions are nonnegative and continuous, $S_k^* \ge E$ and $C_b(S_k^*, T_k^+) = S_k^* - E$. Therefore, the final condition of each problem above can be written as

$$C_b(S, T_k^-) = \begin{cases} C_b(S, T_k^+), & \text{if } 0 \le S \le S_k^*, \\ S - E, & \text{if } S_k^* < S. \end{cases}$$

Because a European call option with a constant volatility has a closed-form solution, just like what we did in the last subsection, we consider the difference between the Bermudan call option and the European call option with a constant volatility $\sigma(E)$ and denote the difference by

$$\tilde{C}_b = C_b - c_E(S, t; \sigma(E)).$$

It is clear that $\tilde{C}_b$ satisfies the partial differential equation in problem (8.78). At $t = T_k$, we have

$$\tilde{C}_b(S, T_k^-) = \begin{cases} \tilde{C}_b(S, T_k^+), & \text{if } 0 \le S \le S_k^*, \\ S - E - c_E\left(S, T_k; \sigma(E)\right), & \text{if } S_k^* < S. \end{cases}$$

Therefore, $\tilde{C}_b$ is the solution of the following K successive problems:

$$\begin{cases} \dfrac{\partial \tilde{C}_b}{\partial t} + \dfrac{1}{2}\sigma^2(S)S^2 \dfrac{\partial^2 \tilde{C}_b}{\partial S^2} + (r - D_0)S\dfrac{\partial \tilde{C}_b}{\partial S} - r\tilde{C}_b = f(S, t), \\ \qquad\qquad\qquad\qquad 0 \le S, \quad T_{k-1} < t < T_k, \\ \tilde{C}_b(S, T_k^-) = \max\left(\tilde{C}_b(S, T_k^+), \max(S - E, 0) - c_E(S, T_k; \sigma(E))\right), \\ \qquad\qquad\qquad\qquad 0 \le S, \end{cases} \tag{8.82}$$

$$k = K, K-1, \cdots, 1$$

with $\tilde{C}_b(S, T_K^+) = 0$. This problem can be solved in a way similar to what we have used to find the solution of a European option with discrete dividends in Sect. 8.2.2. The only difference is that using jump conditions should be replaced by taking the maximum between $\tilde{C}_b(S, T_k^+)$ and $\max(S - E, 0) - c_E(S, t; \sigma(E))$ at these specified times.

In many cases, this method can be further improved by doing the following. For $k = K-1, K-2, \cdots, 1$, let us define $K-1$ polynomials of degree J: $f_k(S) = a_{0,k} + a_{1,k}S + \cdots + a_{J,k}S^J$ on $[S_k^{**}, S_k^*]$, which satisfies the conditions $f_k(S_k^*) = S_k^* - E$ and $f_k(S_k^{**}) = 0$. Besides satisfying these two conditions, we choose these coefficients $a_{0,k}$, $a_{1,k}$, $\cdots$, $a_{J,k}$ and $S_k^{**} \in [0, S_k^*]$ such that the norm of the function

$$\begin{cases} C_b(S, T_k^-), & \text{if } 0 \le S < S_k^{**}, \\ C_b(S, T_k^-) - f_k(S), & \text{if } S_k^{**} \le S < S_k^* \end{cases}$$

is as small as possible. It is clear that the function

$$\begin{cases} 0, & \text{if } 0 \le S < S_k^{**}, \\ f_k(S), & \text{if } S_k^{**} \le S < S_k^*, \\ S - E, & \text{if } S_k^* \le S \end{cases}$$

is a good approximation to $C_b(S, T_k^-)$. For $k = K$, if we define $S_k^* = S_k^{**} = E$, then the function defined above is equal to $C_b(S, T_K^-)$. Therefore, we assume the function above to be defined for $k = K, K-1, \cdots, 1$.

Consider the problems

$$\begin{cases} \dfrac{\partial c_b}{\partial t} + \dfrac{1}{2}\sigma^2(S_k^*)S^2\dfrac{\partial^2 c_b}{\partial S^2} + (r - D_0)S\dfrac{\partial c_b}{\partial S} - rc_b = 0, \\ \qquad\qquad 0 \le S, \ \ T_{k-1} < t < T_k, \\ c_b(S, T_k^-) = \begin{cases} 0, & \text{if } 0 \le S < S_k^{**}, \\ f_k(S), & \text{if } S_k^{**} \le S < S_k^*, \\ S - E, & \text{if } S_k^* \le S. \end{cases} \end{cases} \tag{8.83}$$

Noticing that for any integer n, we have (see Problem 39 in Chap. 2)

$$\frac{1}{\sqrt{2\pi}b}\int_c^d S^n \mathrm{e}^{-\left(\ln(S/a)+b^2/2\right)^2/2b^2}\frac{dS}{S}$$
$$= a^n \mathrm{e}^{(n^2-n)b^2/2}\left[N\left(\frac{\ln(d/a)+(1/2-n)b^2}{b}\right) - N\left(\frac{\ln(c/a)+(1/2-n)b^2}{b}\right)\right],$$

we can find a closed-form solution of problem (8.83) (see Problem 48 in Chap. 2)

$$\begin{aligned} c_b(S,t) = \sum_{n=0}^{J}\Big\{ & a_{n,k}S^n \mathrm{e}^{[(n-1)r - nD_0 + (n-1)n\sigma^2(S_k^*)/2](T_k - t)} \\ & \times \left[N\left(d_k^* - n\sigma(S_k^*)\sqrt{T_k - t}\right) - N\left(d_k^{**} - n\sigma(S_k^*)\sqrt{T_k - t}\right)\right]\Big\} \\ + S\mathrm{e}^{-D_0(T_k - t)} & \left[1 - N\left(d_k^* - \sigma(S_k^*)\sqrt{T_k - t}\right)\right] - E\mathrm{e}^{-r(T_k - t)}[1 - N(d_k^*)], \end{aligned} \tag{8.84}$$

where $t \in (T_{k-1}, T_k)$ and

$$d_k^* = \left[\ln(S_k^*/S) - \left(r - D_0 - \frac{1}{2}\sigma^2(S_k^*)\right)(T_k - t)\right] \Big/ \left(\sigma(S_k^*)\sqrt{T_k - t}\right),$$
$$d_k^{**} = \left[\ln(S_k^{**}/S) - \left(r - D_0 - \frac{1}{2}\sigma^2(S_k^*)\right)(T_k - t)\right] \Big/ \left(\sigma(S_k^*)\sqrt{T_k - t}\right).$$

It is easy to see that for $t \in (T_{K-1}, T_K]$, c_b represents the price of the European option with a constant volatility $\sigma(S^*) = \sigma(E)$ because $S^* = E$ at time $t = T$, that is, $c_b(S,t)$ is equal to $c_E(S,t;\sigma(E))$ for this period.

At the point $S = S_k^*$ and $t = T_k$, the singularity of the solution of the problem (8.83) is very close to that of the problem (8.81). Therefore, the function

$$\overline{C}_b = C_b - c_b$$

is smooth near this point for $t \in (T_{k-1}, T_k)$ and its value is quite small if $T_k - T_{k-1}$ is not big. This function satisfies the following equation and condition:

$$\begin{cases} \dfrac{\partial \overline{C}_b}{\partial t} + \dfrac{1}{2}\sigma^2(S)S^2 \dfrac{\partial^2 \overline{C}_b}{\partial S^2} + (r - D_0)S\dfrac{\partial \overline{C}_b}{\partial S} - r\overline{C}_b = \\ \qquad \dfrac{1}{2}(\sigma^2(S_k^*) - \sigma^2(S))S^2 \dfrac{\partial^2 c_b}{\partial S^2}, \quad 0 \le S, \quad T_{k-1} < t < T_k, \\ \overline{C}_b(S, T_k^-) = \begin{cases} C_b(S, T_k^-), & \text{if } 0 \le S < S_k^{**}, \\ C_b(S, T_k^-) - f_k(S), & \text{if } S_k^{**} \le S < S_k^*, \\ 0, & \text{if } S_k^* \le S. \end{cases} \end{cases} \tag{8.85}$$

Therefore, in order to have $C_b(S, T_{k-1}^+)$, we can first find $\overline{C}_b(S, T_{k-1}^+)$ by solving the problem (8.85) from $t = T_k$ to T_{k-1} and then obtain $C_b(S, T_{k-1}^+)$ by

$$C_b(S, T_{k-1}^+) = \overline{C}_b(S, T_{k-1}^+) + c_b(S, T_{k-1}^+).$$

Because for a variable σ the partial differential equation in the problem (8.85) is nonhomogeneous and the right-hand side is quite complicated, the amount of computation of solving the problem (8.85) is greater than solving the problem (8.81) on the same mesh. However, in order to have a solution with the same accuracy, the number of mesh points needed for the problem (8.85) is much smaller than the problem (8.81). It is expected that in order to reach the same accuracy, solving the problem (8.85) is better. If $\sigma = constant$, then the problem (8.85) becomes

$$\begin{cases} \dfrac{\partial \overline{C}_b}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 \overline{C}_b}{\partial S^2} + (r - D_0)S\dfrac{\partial \overline{C}_b}{\partial S} - r\overline{C}_b = 0, \\ \qquad 0 \le S, \quad T_{k-1} < t < T_k, \\ \overline{C}_b(S, T_k^-) = \begin{cases} C_b(S, T_k^-), & \text{if } 0 \le S < S_k^{**}, \\ C_b(S, T_k^-) - f_k(S), & \text{if } S_k^{**} \le S < S_k^*, \\ 0, & \text{if } S_k^* \le S. \end{cases} \end{cases} \tag{8.86}$$

The partial differential equation in the problem (8.86) is a homogeneous equation. Hence, the amount of computation of solving the problem (8.86) is very close to that of solving the original problem (8.81).

Sometimes, the singularities at the points $S = S_k^*$ and $t = T_k$, $k = K-1, K-2, \cdots, 1$, are quite weak and far away from the region $S \approx E$. Therefore, these singularities only cause small errors in the region $S \approx E$. Also, $[S_k^{**}, S_k^*]$ is not a small interval, so $f_k(S)$ may not be a good approximation to $C_b(S, T_k^-)$. In this case, using the method described at the beginning of this subsection might be better.

Table 8.20. Bermudan call option prices ($r < D_0$)

($S = 100$, $E = 100$, $T = 1$, $r = 0.05$, $D_0 = 0.1$, $\sigma = 0.2$, $K = 4$, and the exact solution $= 5.77654\cdots$)

	Implicit method			SSM		
Mesh sizes	Results	|Errors|	CPU(s)	Results	|Errors|	CPU(s)
24×24	5.0474	0.7291	0.0002	5.8564	0.0799	0.0014
36×36	5.4507	0.3258	0.0005	5.7788	0.0023	0.0019
48×48	5.6143	0.1622	0.0008	5.7881	0.0116	0.0028
60×60	5.6732	0.1033	0.0013	5.7845	0.0080	0.0037
72×72	5.7069	0.0696	0.0018	5.7833	0.0068	0.0048
84×84	5.7332	0.0433	0.0024	5.7833	0.0068	0.0061
96×96	5.7362	0.0403	0.0032	5.7809	0.0044	0.0073
108×108	5.7479	0.0286	0.0039	5.7807	0.0042	0.0086
120×120	5.7543	0.0222	0.0049	5.7797	0.0032	0.0101
132×132	5.7599	0.0166	0.0059	5.7804	0.0039	0.0119
144×144	5.7592	0:0173	0.0073	5.7800	0.0035	0.0134
156×156	5.7649	0.0116	0.0082	5.7799	0.0034	0.0152
168×168	5.7674	0.0091	0.0096	5.7790	0.0025	0.0172
180×180	5.7659	0.0106	0.0109	5.7784	0.0019	0.0190

Table 8.21. Bermudan call option prices ($r > D_0$)

($S = 100$, $E = 100$, $T = 1$, $r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$, $K = 12$, and the exact solution $= 9.940918\cdots$)

	Implicit method			SSM		
Mesh sizes	Results	|Errors|	CPU(s)	Results	|Errors|	CPU(s)
24×24	9.1488	0.7922	0.0003	9.9411	0.0002	0.0017
36×36	9.6261	0.3148	0.0006	9.9410	0.0001	0.0026
48×48	9.7704	0.1705	0.0011	9.9410	0.0001	0.0037
60×60	9.8333	0.1076	0.0015	9.9409	0.0000	0.0049
72×72	9.8667	0.0742	0.0020	9.9409	0.0000	0.0062
84×84	9.8866	0.0543	0.0027	9.9409	0.0000	0.0075
96×96	9.8995	0.0414	0.0034	9.9409	0.0000	0.0090
108×108	9.9082	0.0327	0.0043	9.9409	0.0000	0.0105
120×120	9.9145	0.0264	0.0052	9.9409	0.0000	0.0121

In what follows, we give some examples. Consider a Bermudan call option with $r = 0.05$, $D_0 = 0.1$, and $T_k = k/4$, $k = 1, 2, 3, 4$. The price of the option is evaluated by two different ways. One is to solve problem (8.81) by the implicit method (7.6) and the other is to take $J = 6$ and solve problem (8.86) by difference scheme (7.6). For $S = 100$, the results obtained by the two ways, the errors, and CPU times needed on a Pentium III 800 MHz computer are given in Table 8.20. From there, we can see that for this case in order to have a result with an error about 10^{-2} (the corresponding relative error to E is 10^{-4}), CPU time needed is about 0.003 s if the singularity-separating method described here is used, and CPU time needed is about 0.01 s if the singularity-separating method is not used. Therefore, even though on an identical mesh, the CPU time needed for the SSM is much longer, overall the SSM is still fast for a fixed accuracy.

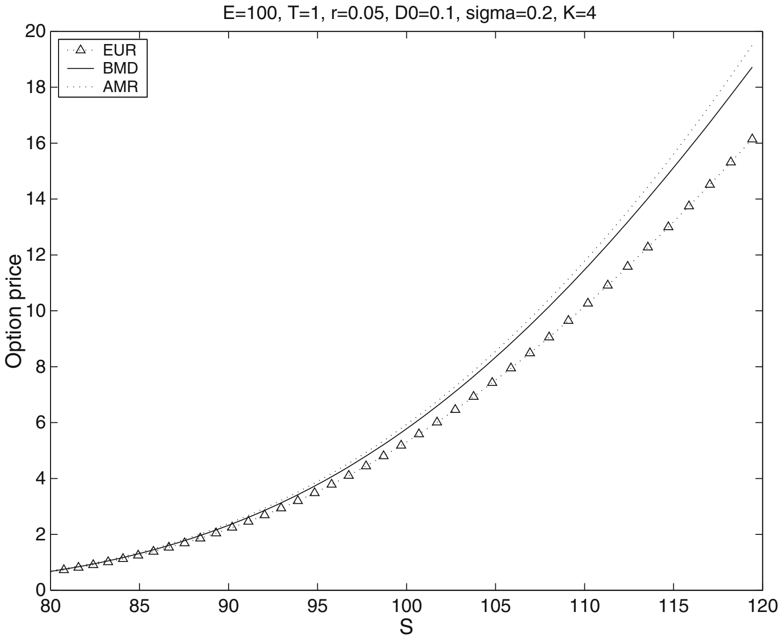

Fig. 8.14. Prices of American, Bermudan, and European call options ($E = 100$, $T = 1$, $r = 0.05$, $D_0 = 0.1$, $\sigma = 0.2$, and $K = 4$)

The next example is a Bermudan option with $r = 0.1$, $D_0 = 0.05$, and $T_k = k/12$, $k = 1, 2, \cdots, 12$. The other parameters are the same as those for the first example. In this case, the singularities at the points $S = S_k^*$ and $t = T_k$, $k = K-1, K-2, \cdots, 1$, are weak and we choose $c_E(S, t; \sigma(E))$ as c_b and solve problem (8.82) by the difference scheme (7.6). The results for $S = 100$ are given in Table 8.21. When the SSM is not used, the errors are close to those in the first example. However, when the SSM is used, the errors are even much less than those in the first example due to the very small value of $\overline{C}_b$.

In Fig. 8.14, the price of the first Bermudan call option is given as a function of S. The prices of the American and European call options are also given there. The figure shows that the price of the Bermudan option is less than the price of the American option and greater than the price of the European option, and it is quite close to the price of the corresponding American option. The financial reason of this fact is as follows. As has been pointed out at the beginning of this subsection, the holder of a Bermudan option has more rights than a holder of a European option and less rights than a holder of an American option. Thus, the money paid by the holder of the Bermudan option should be greater than the price of a European option and less than the price of an American option.

The symmetry relations also hold for Bermudan options, which is left for readers to prove. Therefore, we only need to study numerical methods for Bermudan call options. In order to obtain the price of a put option, we first solve a corresponding call option problem and then find the price of the put option by the symmetry relation.

8.3.4 European Parisian Options

Let us take a European Parisian up-and-out call option with continuous sampling as an example to show how the singularity-separating method works for Parisian options.

Suppose c_p is the price of the Parisian up-and-out call option. From Sect. 4.2.4, we see that $c_p(S, t_d, t)$ is the solution of problem (4.6):

$$\begin{cases} \dfrac{\partial c_p}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 c_p}{\partial S^2} + (r - D_0)\, S \dfrac{\partial c_p}{\partial S} + H(S - B_u) \dfrac{\partial c_p}{\partial t_d} - rc_p = 0, \\ \quad 0 \le S, \quad t_d = 0 \quad \text{and} \quad B_u \le S, \quad 0 < t_d \le T_d, \quad 0 \le t \le T, \\ c_p(S, t_d, T) = \begin{cases} \max(S - E, 0), & 0 \le S < B_u, \quad t_d = 0, \\ S - E, & B_u \le S, \quad 0 \le t_d < T_d, \\ 0, & B_u \le S, \quad t_d = T_d, \end{cases} \\ c_p(B_u, t_d, t) = c_p(B_u, 0, t), \qquad t_d \in (0, T_d), \quad 0 \le t \le T, \\ c_p(S, T_d, t) = 0, \qquad B_u \le S, \quad 0 \le t \le T. \end{cases}$$

Let $c(S, t)$ be the price of the European vanilla call option and define

$$\bar{c}_p(S, t_d, t) = c_p(S, t_d, t) - c(S, t).$$

Because $c(S, t)$ does not depend on t_d, it is clear that $c(S, t)$ also satisfies the partial differential equation in the problem (4.6). Therefore, $\bar{c}_p(S, t_d, t)$ is the solution of the following problem:

$$\begin{cases} \dfrac{\partial \bar{c}_p}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 \bar{c}_p}{\partial S^2} + (r - D_0)\, S \dfrac{\partial \bar{c}_p}{\partial S} + H(S - B_u) \dfrac{\partial \bar{c}_p}{\partial t_d} - r\bar{c}_p = 0, \\ \quad 0 \le S, \quad t_d = 0 \quad \text{and} \quad B_u \le S, \quad 0 < t_d \le T_d, \quad 0 \le t \le T, \\ \bar{c}_p(S, t_d, T) = \begin{cases} 0, & 0 \le S < B_u, \quad t_d = 0, \\ 0, & B_u \le S, \quad 0 \le t_d < T_d, \\ E - S, & B_u \le S, \quad t_d = T_d, \end{cases} \\ \bar{c}_p(B_u, t_d, t) = \bar{c}_p(B_u, 0, t), \quad t_d \in (0, T_d), \quad 0 \le t \le T, \\ \bar{c}_p(S, T_d, t) = -c(S, t), \qquad B_u \le S, \quad 0 \le t \le T. \end{cases} \tag{8.87}$$

Because $c_p(S, t_d, t)$ and $c(S, t)$ have the same singularity at the point $S = E$ and $t = T$, $\bar{c}_p$ is quite smooth near $S = E$ and $t = T$, that is, the singularity has been separated. Therefore, it is expected that on the same mesh, the error of the numerical results obtained by solving the problem (8.87) is smaller than that obtained by solving the problem (4.6). Tables 8.22 and 8.23 (see [58]) give the results and the relative errors when the SSM is not used and when it is used, respectively. From there, we can see that the results with the SSM are much better than the results without the SSM.

Problem (8.87) is a two-dimensional problem. However, it can be solved by a modified one-dimensional method. Let us explain why this problem can be solved like a one-dimensional problem. Because there is no second derivative in the t_d-direction, the coefficient of $\dfrac{\partial \bar{c}_p}{\partial t_d}$ is positive or zero, and the boundary condition is given at $t_d = T_d$, for a fixed time t^* the solution of the problem can be obtained from $t_d = T_d$ to $t_d = 0$ successively. Suppose the value of $\bar{c}_p$ for $t = t^*$ and $t_d \ge t_d^*$ has been obtained. We want to find the value of $\bar{c}_p$ for $t = t^*$ and $t_d = t_d^* - \Delta t_d$ with a positive Δt_d. Because the value at $t = t^*$ and $t_d = t_d^*$ is known, the value at $t = t^*$ and $t_d = t_d^* - \Delta t_d$ can be found by solving a one-dimensional problem on an (S, t)-plane. This can be done by various methods. After transforming the problem to one defined on

Table 8.22. Numerical solutions for Parisian up-and-out call options

($r = 0.1$, $D_0 = 0.05$, $\sigma = 0.25$, $E = 100$, $T = 0.5$, $B_u = 150$, and $T_d = 0.02$)

	$S = 100$		$S = 120$		$S = 150$	
Meshes	Solutions	|Errors|	Solution	|Errors|	Solution	|Errors|
200×100	7.4139	$1.08 \cdot 10^{-3}$	15.3107	$7.79 \cdot 10^{-3}$	5.0574	$3.73 \cdot 10^{-2}$
300×150	7.4067	$1.08 \cdot 10^{-4}$	15.2886	$6.33 \cdot 10^{-3}$	4.9389	$1.30 \cdot 10^{-2}$
400×200	7.4059	–	15.1924	–	4.8754	–

Table 8.23. Numerical solutions for Parisian up-and-out call options (with SSM)

($r = 0.1$, $D_0 = 0.05$, $\sigma = 0.25$, $E = 100$, $T = 0.5$, $B_u = 150$, and $T_d = 0.02$)

	$S = 100$		$S = 120$		$S = 150$	
Meshes	Solutions	|Errors|	Solution	|Errors|	Solution	|Errors|
200×100	7.3943	$1.76 \cdot 10^{-4}$	15.2016	$5.13 \cdot 10^{-4}$	4.9232	$2.09 \cdot 10^{-2}$
300×150	7.3936	$8.16 \cdot 10^{-5}$	15.1947	$5.92 \cdot 10^{-5}$	4.8251	$5.18 \cdot 10^{-4}$
400×200	7.3930	–	15.1938	–	4.8226	–

a finite domain by the transformation (2.17), the partial differential equation can be discretized by scheme (7.6) at interior points, and the right boundary point and the solution can be found from these finite-difference equations. The results given in this subsection are obtained by using a method that is a little different from what we have described here. For details, see the paper [58] by Luo and Wu.

When σ is a function of S, the SSM method can still be used. However, a European vanilla call option has a closed-form solution only when σ is a constant. Therefore, we do not have a closed-form solution for the corresponding European vanilla call option. In this case, we can consider the difference between the Parisian call option and the vanilla call option with a constant volatility $\sigma(E)$. This difference satisfies a nonhomogeneous equation (for details, see Sect. 8.3.2), but we still can expect that the SSM will make the computation more efficient.

8.3.5 European Average Price Options

In the last few subsections, we always computed the difference between an option and the corresponding vanilla option with a constant volatility. However, other functions can also be used as long as they have a similar singularity, and even they may be better. In this subsection, we give such an example.

From Eq. (4.20), we know that if sampling is done continuously, then the European-style Asian option may be modeled by the following partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0)S\frac{\partial V}{\partial S} + \frac{S}{T}\frac{\partial V}{\partial I} - rV = 0, \tag{8.88}$$

where

$$I = \frac{1}{T}\int_0^t S(\tau)d\tau.$$

Let us consider an average price call option whose final condition is

$$V(S, I, T) = \max(I - E, 0). \tag{8.89}$$

Zhang[3] in his paper [88] proposed to solve the problem in the following way. By letting (see Sect. 4.3.4)

$$\eta = \frac{I-E}{S} \quad \text{and} \quad W(\eta,t) = \frac{V(S,I,t)}{S},$$

the two-dimensional equation (8.88) can be converted into a one-dimensional equation:

$$\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2\eta^2\frac{\partial^2 W}{\partial \eta^2} + \left[(D_0-r)\eta + \frac{1}{T}\right]\frac{\partial W}{\partial \eta} - D_0 W = 0$$

and the final condition becomes

$$W(\eta,T) = \max(\eta,0).$$

Because the equation

$$\frac{d\eta}{dt} = (D_0-r)\eta + \frac{1}{T}$$

has solutions in the form

$$\eta e^{-(r-D_0)(T-t)} + \frac{1}{(r-D_0)T}\left(1-e^{-(r-D_0)(T-t)}\right) = constant,$$

introducing the transformation

$$\begin{cases} \xi = \eta e^{-(r-D_0)(T-t)} + \dfrac{1}{(r-D_0)T}\left(1-e^{-(r-D_0)(T-t)}\right), \\ \tau = T-t, \\ W(\eta,t) = e^{-D_0\tau} f(\xi,\tau), \end{cases} \tag{8.90}$$

we can get rid of the first derivative of W and the function W, and we arrive at an initial value problem of a heat equation

$$\begin{cases} \dfrac{\partial f}{\partial \tau} - \dfrac{1}{2}\sigma^2\left[\xi - \dfrac{1}{(r-D_0)T}(1-e^{-(r-D_0)\tau})\right]^2\dfrac{\partial^2 f}{\partial \xi^2} = 0, \\ \qquad\qquad -\infty < \xi < \infty, \quad 0 \le \tau \le T, \\ f(\xi,0) = \max(\xi,0), \qquad -\infty < \xi < \infty. \end{cases} \tag{8.91}$$

The initial condition $f(\xi,0) = \max(\xi,0)$ is not smooth at the point $\xi = 0$. To separate the singularity, the problem that is obtained by setting $\xi = 0$ in the above equation

$$\begin{cases} \dfrac{\partial \tilde{f}_0}{\partial \tau} - \dfrac{\sigma^2}{2(r-D_0)^2T^2}\left(1-e^{-(r-D_0)\tau}\right)^2\dfrac{\partial^2 \tilde{f}_0}{\partial \xi^2} = 0, \\ \qquad\qquad -\infty < \xi < \infty, \quad 0 \le \tau \le T, \\ \tilde{f}_0(\xi,0) = \max(\xi,0), \qquad -\infty < \xi < \infty \end{cases} \tag{8.92}$$

[3] In his paper, he assumes $D_0 = 0$. However, it is not difficult to generalize that result to the case with $D_0 \neq 0$.

is considered. Introducing a new variable $\tau_1(\tau)$ by

$$d\tau_1 = \frac{\sigma^2}{2(r-D_0)^2T^2}\left(1-\mathrm{e}^{-(r-D_0)\tau}\right)^2 d\tau \quad \text{with} \quad \tau_1(0)=0,$$

which gives

$$\begin{aligned}\tau_1(\tau) &= \int_0^\tau \frac{\sigma^2}{2(r-D_0)^2T^2}\left(1-\mathrm{e}^{-(r-D_0)\tau}\right)^2 d\tau \\ &= \frac{\sigma^2}{4(r-D_0)^3T^2}\left[2(r-D_0)\tau + 4\mathrm{e}^{-(r-D_0)\tau} - \mathrm{e}^{-2(r-D_0)\tau} - 3\right],\end{aligned} \tag{8.93}$$

and letting $f_0(\xi,\tau_1) = \tilde{f}_0\left(\xi,\tau(\tau_1)\right)$, we obtain the following parabolic problem

$$\begin{cases} \dfrac{\partial f_0}{\partial \tau_1} - \dfrac{\partial^2 f_0}{\partial \xi^2} = 0, & -\infty < \xi < \infty, \quad 0 \le \tau_1 \le \tau_1(T), \\ f_0(\xi,0) = \max(\xi,0), & -\infty < \xi < \infty. \end{cases} \tag{8.94}$$

The solution of this problem is given by

$$f_0(\xi,\tau_1) = \int_0^\infty \frac{\xi_T}{2\sqrt{\pi\tau_1}}\mathrm{e}^{-(\xi_T-\xi)^2/4\tau_1} d\xi_T = \xi N\left(\frac{\xi}{\sqrt{2\tau_1}}\right) + \sqrt{\frac{\tau_1}{\pi}}\mathrm{e}^{-\xi^2/4\tau_1}. \tag{8.95}$$

This analytic formula gives quite a good approximation to the prices of European average price call options. That is, the value of the difference between $f(\xi,\tau)$ and $f_0\left(\xi,\tau_1(\tau)\right)$,

$$f_1(\xi,\tau) = f(\xi,\tau) - f_0\left(\xi,\tau_1(\tau)\right), \tag{8.96}$$

is quite small. If we want to have more accurate results, we need to find $f_1(\xi,\tau)$. This function satisfies the following equation and initial condition:

$$\begin{cases} \dfrac{\partial f_1}{\partial \tau} - \dfrac{1}{2}\sigma^2\left[\xi - \dfrac{1}{(r-D_0)T}\left(1-\mathrm{e}^{-(r-D_0)\tau}\right)\right]^2 \dfrac{\partial^2 f_1}{\partial \xi^2} = \dfrac{\sigma^2\xi\,\mathrm{e}^{-\xi^2/4\tau_1}}{4\sqrt{\pi\tau_1}} \\ \times\left[\xi - \dfrac{2}{(r-D_0)T}\left(1-\mathrm{e}^{-(r-D_0)\tau}\right)\right], \quad -\infty < \xi < \infty, \quad 0 \le \tau \le T, \\ f_1(\xi,0) = 0, \quad -\infty < \xi < \infty. \end{cases} \tag{8.97}$$

The function $f_1(\xi,\tau)$ is smooth, and its value is quite small compared with $f(\xi,\tau)$, so in order to get a very good numerical solution, we need only a very coarse mesh. In this way, we can find quite accurate solutions very fast. The problem (8.97) is defined on an infinite domain. In order to convert the infinite domain into a finite domain, we can introduce the following transformation:

$$\xi_1 = \frac{1}{2}\left(\frac{\xi}{|\xi| + P_m} + 1\right) \quad \text{and} \quad u(\xi_1, \tau) = \frac{f_1(\xi, \tau)}{|\xi| + P_m}.$$

After this transformation, the problem for $u(\xi_1, \tau)$ is defined on $[0, 1] \times [0, T]$ in the (ξ_1, τ)-space and can be solved by scheme (7.6).

We can also take the difference between the price of a European-style Asian option and the price of a European vanilla option and do the numerical computation. However, the performance might not be as good as the method here. The reason is that the difference in the method given here is smaller than the difference between the price of a European-style Asian option and the price of a European vanilla option. This can be roughly explained as follows. Consider the following linear parabolic problem:

$$\begin{cases} \dfrac{\partial u}{\partial \tau} = a_2 \dfrac{\partial^2 u}{\partial \xi^2} + a_1 \dfrac{\partial u}{\partial \xi} + a_0 u + g(\xi, \tau), & -\infty < \xi < \infty, \quad 0 \le \tau \le T, \\ u(\xi, 0) = f(\xi), & -\infty < \xi < \infty. \end{cases}$$

Suppose that $\tilde{u}$ is an approximate solution by a numerical method on a certain mesh. It is clear that $v = u/10$ is the solution of the problem:

$$\begin{cases} \dfrac{\partial v}{\partial \tau} = a_2 \dfrac{\partial^2 v}{\partial \xi^2} + a_1 \dfrac{\partial v}{\partial \xi} + a_0 v + g(\xi, \tau)/10, & -\infty < \xi < \infty, \quad 0 \le \tau \le T, \\ v(\xi, 0) = f(\xi)/10, & -\infty < \xi < \infty. \end{cases}$$

Let $\tilde{v}$ be the approximate solution of this problem by using the same method on the same mesh. Just like the relation between u and v, we have $\tilde{v} = \tilde{u}/10$. Thus, $v - \tilde{v} = (u - \tilde{u})/10$, which means that the smaller the solution, the smaller the error of approximate solutions. Therefore, when we choose an analytic solution, we should let the analytic solution be as close to the desired solution as possible. In this way, we can have a better performance.

8.3.6 European Two-Factor Options

In Sect. 8.3.2, we pointed out that the assumption of the volatility being constant might need to be modified. One possible modification is to let the volatility be a given function of S. In Sect. 8.3.2, we discussed how to solve such a problem. Another possible modification is to allow the volatility to be a random variable, i.e., the volatility is stochastic. This subsection is devoted to studying how to solve this problem. In this case, option prices depend on two random variables. In what follows, such an option will be referred to as a two-factor option, and we will call an option a one-factor option if only the stock price is considered as a random variable.

Now let us discuss how to evaluate quickly such a European vanilla option or American vanilla option with $D_0 = 0$. We assume that the asset price S and the stochastic volatility are governed by the following two stochastic processes

$$\begin{cases} dS = \mu S dt + \sigma S dX_1, \quad 0 \le S, \\ \\ d\sigma = p(\sigma, t)dt + q(\sigma, t)dX_2, \quad \sigma_l \le \sigma \le \sigma_u, \end{cases} \tag{8.98}$$

where dX_1 and dX_2 are two Wiener processes. These two random variables could be correlated and $\mathrm{E}[dX_1 dX_2] = \rho dt$.

As we have seen in Sect. 2.4.1, in order to guarantee $\sigma \in [\sigma_l, \sigma_u]$, p and q in the model for the volatility need to satisfy the following reversion conditions:

$$\begin{cases} p(\sigma_l, t) - q(\sigma_l, t)\dfrac{\partial q(\sigma_l, t)}{\partial \sigma} \ge 0, \\ \\ q(\sigma_l, t) = 0 \end{cases} \tag{8.99}$$

and

$$\begin{cases} p(\sigma_u, t) - q(\sigma_u, t)\dfrac{\partial q(\sigma_u, t)}{\partial \sigma} \le 0, \\ \\ q(\sigma_u, t) = 0. \end{cases} \tag{8.100}$$

It is clear that if $\dfrac{\partial q(\sigma_l, t)}{\partial \sigma}$ and $\dfrac{\partial q(\sigma_u, t)}{\partial \sigma}$ are bounded, then the conditions (8.99) and (8.100) are simplified into

$$\begin{cases} p(\sigma_l, t) \ge 0, \\ \\ q(\sigma_l, t) = 0 \end{cases} \tag{8.101}$$

and

$$\begin{cases} p(\sigma_u, t) \le 0, \\ \\ q(\sigma_u, t) = 0. \end{cases} \tag{8.102}$$

Suppose $V(S, \sigma, t)$ is the value of an option depending on two random variables S and σ. From Sect. 2.3, such an option satisfies the following equation:

$$\frac{\partial V}{\partial t} + \mathbf{L}_{\mathbf{s},\sigma} V = 0, \tag{8.103}$$

where $\mathbf{L}_{S,\sigma}$ is an operator defined by

$$\begin{aligned} \mathbf{L}_{\mathbf{s},\sigma} = & \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + \rho\sigma S q \frac{\partial^2}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2}{\partial \sigma^2} \\ & + (r - D_0) S \frac{\partial}{\partial S} + (p - \lambda q) \frac{\partial}{\partial \sigma} - r. \end{aligned} \tag{8.104}$$

Consider a two-factor European vanilla call option problem, and let its value be $c(S, \sigma, t)$. Because the volatility model satisfies the reversion conditions, no boundary conditions need to be given at the boundaries $\sigma = \sigma_l$ and $\sigma = \sigma_u$. Therefore, the value of the two-factor European vanilla call option is the solution of the following final-value problem:

$$\begin{cases} \dfrac{\partial c}{\partial t} + \mathbf{L}_{S,\sigma} c = 0, & 0 \le S, \quad \sigma_l \le \sigma \le \sigma_u, \quad t \le T, \\ c(S,\sigma,T) = \max(S-E,0), & 0 \le S, \quad \sigma_l \le \sigma \le \sigma_u. \end{cases} \tag{8.105}$$

In order to make the computed solution smoother, which will make numerical methods more efficient, we let

$$\overline{c}(S,\sigma,t) = c(S,\sigma,t) - c_1(S,\sigma,t) \tag{8.106}$$

on the entire computational domain. $c_1(S,\sigma,t)$ is the price of the one-factor European vanilla call option, that is, the price of the European vanilla call option with a parameter σ. Here, we denote the value of this option by $c_1(S,\sigma,t)$ instead of $c(S,t)$ in order to indicate explicitly its dependence on σ and to explain that it is the price of the one-factor model. From Sect. 2.6.5, we know that its expression is given by

$$c_1(S,\sigma,t) = S\mathrm{e}^{-D_0(T-t)}N(d_1) - E\mathrm{e}^{-r(T-t)}N(d_2),$$

where

$$\begin{aligned} N(z) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \mathrm{e}^{-\xi^2/2} d\xi, \\ d_1 &= \left[\ln \frac{S\mathrm{e}^{-D_0(T-t)}}{E\mathrm{e}^{-r(T-t)}} + \frac{1}{2}\sigma^2(T-t) \right] \Big/ \left(\sigma\sqrt{T-t} \right), \\ d_2 &= d_1 - \sigma\sqrt{T-t}. \end{aligned}$$

Because $c_1(S,\sigma,t)$ satisfies the Black–Scholes equation, the difference $\overline{c}$ is the solution of the following final-value problem:

$$\begin{cases} \dfrac{\partial \overline{c}}{\partial t} + \mathbf{L}_{\mathbf{s},\sigma} \overline{c} = f(S,\sigma,t), & 0 \le S, \quad \sigma_l \le \sigma \le \sigma_u, \quad 0 \le t \le T, \\ \overline{c}(S,\sigma,T) = 0, & 0 \le S, \quad \sigma_l \le \sigma \le \sigma_u, \end{cases} \tag{8.107}$$

where

$$f(S,\sigma,t) = -\rho\sigma S q \frac{\partial^2 c_1}{\partial S \partial \sigma} - \frac{1}{2} q^2 \frac{\partial^2 c_1}{\partial \sigma^2} - (p - \lambda q) \frac{\partial c_1}{\partial \sigma}.$$

From the expressions of $c_1(S,\sigma,t)$, noticing

$$\begin{aligned} \frac{\partial c_1}{\partial S} &= \mathrm{e}^{-D_0(T-t)}N(d_1), \\ \frac{\partial d_1}{\partial \sigma} &= \sqrt{T-t} - \left[\ln \frac{S\mathrm{e}^{-D_0(T-t)}}{E\mathrm{e}^{-r(T-t)}} + \frac{1}{2}\sigma^2(T-t) \right] \Big/ (\sigma^2\sqrt{T-t}) \\ &= \sqrt{T-t} - \frac{d_1}{\sigma}, \\ \frac{\partial d_2}{\partial \sigma} &= \frac{\partial d_1}{\partial \sigma} - \sqrt{T-t} = -\frac{d_1}{\sigma}, \\ N'(z) &= \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-z^2/2}, \end{aligned}$$

we can easily find

$$\begin{cases} \dfrac{\partial c_1}{\partial \sigma} &= S\mathrm{e}^{-D_0(T-t)}N'(d_1)\dfrac{\partial d_1}{\partial \sigma} - E\mathrm{e}^{-r(T-t)}N'(d_2)\dfrac{\partial d_2}{\partial \sigma} \\ &= S\sqrt{T-t}\,\mathrm{e}^{-D_0(T-t)}N'(d_1), \\ \dfrac{\partial^2 c_1}{\partial \sigma^2} &= S\sqrt{T-t}\,\mathrm{e}^{-D_0(T-t)}N''(d_1)\dfrac{\partial d_1}{\partial \sigma} \\ &= -S\sqrt{T-t}\,\mathrm{e}^{-D_0(T-t)}d_1 N'(d_1)\dfrac{\partial d_1}{\partial \sigma}, \\ \dfrac{\partial^2 c_1}{\partial S \partial \sigma} &= \mathrm{e}^{-D_0(T-t)}N'(d_1)\dfrac{\partial d_1}{\partial \sigma}. \end{cases} \tag{8.108}$$

As we see from the problem (8.105), the derivative of $c(S,\sigma,t)$ with respect to S is discontinuous at $t=T$ and $S=E$. However, the problem (8.107) shows the derivative of $\overline{c}(S,\sigma,t)$ with respect to S to be identically equal to zero at $t=T$. Therefore, when a numerical method is used, the truncation error for the problem (8.107) will be much smaller than the problem (8.105). This is why we consider the formulation (8.107) instead of the formulation (8.105).

The final-value problem (8.107) is defined on an infinite domain. In order to convert it into a problem on a finite domain, we introduce the following transformation

$$\begin{cases} \xi = \dfrac{S}{S+P_m}, \\ \sigma = \sigma, \\ \tau = T - t, \\ u(\xi,\sigma,\tau) = \dfrac{\overline{c}(S,\sigma,t)}{S+P_m}. \end{cases} \tag{8.109}$$

In the $\{\xi,\sigma,\tau\}$-space, we need to solve a problem on the domain $[0,1] \times [\sigma_l,\sigma_u] \times [0,T]$. This is a finite domain, and it is easy to construct numerical methods to solve the problem on this domain. From the expression (8.109), we have

$$\overline{c}(S,\sigma,t) = (S+P_m)u(\xi,\sigma,\tau) = \frac{P_m}{1-\xi}u(\xi,\sigma,\tau) \quad \text{and} \quad \frac{d\xi}{dS} = \frac{(1-\xi)^2}{P_m}.$$

Therefore, among the derivatives of $\overline{c}$ and u, there are the following relations:

$$\begin{aligned} \frac{\partial \overline{c}}{\partial t} &= -\frac{P_m}{1-\xi}\frac{\partial u}{\partial \tau}, \\ \frac{\partial \overline{c}}{\partial S} &= (1-\xi)\frac{\partial u}{\partial \xi} + u, \\ \frac{\partial \overline{c}}{\partial \sigma} &= \frac{P_m}{1-\xi}\frac{\partial u}{\partial \sigma}, \end{aligned}$$

$$\frac{\partial^2 \overline{c}}{\partial S^2} = \frac{(1-\xi)^3}{P_m}\frac{\partial^2 u}{\partial \xi^2},$$
$$\frac{\partial^2 \overline{c}}{\partial S \partial \sigma} = (1-\xi)\frac{\partial^2 u}{\partial \xi \partial \sigma} + \frac{\partial u}{\partial \sigma},$$
$$\frac{\partial^2 \overline{c}}{\partial \sigma^2} = \frac{P_m}{1-\xi}\frac{\partial^2 u}{\partial \sigma^2}.$$

Substituting them into the partial differential equation in the problem (8.107) yields

$$\frac{\partial u}{\partial \tau} = a_1\frac{\partial^2 u}{\partial \xi^2} + a_2\frac{\partial^2 u}{\partial \xi \partial \sigma} + a_3\frac{\partial^2 u}{\partial \sigma^2} + a_4\frac{\partial u}{\partial \xi} + a_5\frac{\partial u}{\partial \sigma} + a_6 u + a_7,$$

where

$$\begin{aligned}
a_1 &= \frac{1}{2}\sigma^2\xi^2(1-\xi)^2, & a_2 &= \rho\sigma q\xi(1-\xi),\\
a_3 &= \frac{1}{2}q^2, & a_4 &= (r-D_0)\xi(1-\xi),\\
a_5 &= p-(\lambda-\rho\sigma\xi)q, & a_6 &= -[r(1-\xi)+D_0\xi],
\end{aligned}$$

$$\begin{aligned}
a_7 &= -f(\xi P_m/(1-\xi),\sigma,T-\tau)(1-\xi)/P_m\\
&= \rho\sigma\xi q\,\mathrm{e}^{-D_0(T-t)}N'(d_1)\frac{\partial d_1}{\partial \sigma} - \frac{1}{2}q^2\xi\sqrt{T-t}\,\mathrm{e}^{-D_0(T-t)}d_1 N'(d_1)\frac{\partial d_1}{\partial \sigma}\\
&\quad +(p-\lambda q)\xi\sqrt{T-t}\,\mathrm{e}^{-D_0(T-t)}N'(d_1)\\
&= \frac{1}{\sqrt{2\pi}}\xi\mathrm{e}^{-D_0\tau-d_1^2/2}\left[q(\sqrt{\tau}-d_1/\sigma)(\rho\sigma-q\sqrt{\tau}d_1/2)+(p-\lambda q)\sqrt{\tau}\right].
\end{aligned}$$

Therefore, the problem (8.107) becomes

$$\begin{cases}
\dfrac{\partial u}{\partial \tau} = a_1\dfrac{\partial^2 u}{\partial \xi^2} + a_2\dfrac{\partial^2 u}{\partial \xi \partial \sigma} + a_3\dfrac{\partial^2 u}{\partial \sigma^2} + a_4\dfrac{\partial u}{\partial \xi} + a_5\dfrac{\partial u}{\partial \sigma} + a_6 u + a_7,\\
\qquad\qquad 0 \le \xi \le 1, \quad \sigma_l \le \sigma \le \sigma_u, \quad 0 \le \tau \le T,\\
u(\xi,\sigma,0) = 0, \qquad 0 \le \xi \le 1, \quad \sigma_l \le \sigma \le \sigma_u.
\end{cases} \tag{8.110}$$

Once we have $u(\xi,\sigma,\tau)$, we can get the value of the two-factor European call option by

$$\begin{aligned}
c(S,\sigma,t) &= \overline{c}(S,\ \sigma,\ t) + c_1(S,\ \sigma,\ t)\\
&= (S+P_m)u\left(\frac{S}{S+P_m},\ \sigma,\ T-t\right) + c_1(S,\ \sigma,\ t).
\end{aligned} \tag{8.111}$$

As we have pointed out in Sect. 2.4.4, when the reversion conditions (8.99) and (8.100), and conditions (ii) and (iii) in Theorem 2.2 hold, it has been proved that the final value problem (8.110) has a unique solution. In this case it is not difficult to design a well-posed numerical method to solve this problem.

The following is such a numerical method for problem (8.110). Let $u_{m,i}^n$ be the approximate value of u at $\xi = m\Delta\xi$, $\sigma = \sigma_l + i\Delta\sigma$, and $\tau = n\Delta\tau$, where $\Delta\xi = 1/M$, $\Delta\sigma = (\sigma_u - \sigma_l)/I$, and $\Delta\tau = 1/N$, M, I, N being integers. This partial differential equation can be discretized by the following scheme. If $\sigma \neq \sigma_l$ and $\sigma \neq \sigma_u$, at a point $(\xi_m, \sigma_i, \tau^{n+1/2})$ the partial differential equation in the problem (8.110) can be discretized by the following second-order approximation:

$$\begin{aligned}
&\frac{u_{m,i}^{n+1} - u_{m,i}^n}{\Delta\tau} \\
&= \frac{a_1}{2\Delta\xi^2}\left(u_{m+1,i}^{n+1} - 2u_{m,i}^{n+1} + u_{m-1,i}^{n+1} + u_{m+1,i}^n - 2u_{m,i}^n + u_{m-1,i}^n\right) \\
&\quad + \frac{a_2}{8\Delta\sigma\Delta\xi}(u_{m+1,i+1}^{n+1} - u_{m+1,i-1}^{n+1} - u_{m-1,i+1}^{n+1} + u_{m-1,i-1}^{n+1} \\
&\qquad\qquad + u_{m+1,i+1}^n - u_{m+1,i-1}^n - u_{m-1,i+1}^n + u_{m-1,i-1}^n) \\
&\quad + \frac{a_3}{2\Delta\sigma^2}(u_{m,i+1}^{n+1} - 2u_{m,i}^{n+1} + u_{m,i-1}^{n+1} \\
&\qquad\qquad + u_{m,i+1}^n - 2u_{m,i}^n + u_{m,i-1}^n) \\
&\quad + \frac{a_4}{4\Delta\xi}(u_{m+1,i}^{n+1} - u_{m-1,i}^{n+1} + u_{m+1,i}^n - u_{m-1,i}^n) \\
&\quad + \frac{a_5}{4\Delta\sigma}(u_{m,i+1}^{n+1} - u_{m,i-1}^{n+1} + u_{m,i+1}^n - u_{m,i-1}^n) \\
&\quad + \frac{a_6}{2}(u_{m,i}^{n+1} + u_{m,i}^n) + a_7, \qquad m = 0, 1, \cdots, M, \quad i = 1, 2, \cdots, I-1.
\end{aligned} \tag{8.112}$$

Here, all the coefficients a_1–a_7 should be evaluated at the point $(\xi_m, \sigma_i, \tau^{n+1/2})$ in order to guarantee second-order accuracy.

At the boundaries $\sigma = \sigma_l$ and $\sigma = \sigma_u$, due to $q = 0$ the partial differential equation in the problem (8.110) becomes

$$\frac{\partial u}{\partial \tau} = a_1 \frac{\partial^2 u}{\partial \xi^2} + a_4 \frac{\partial u}{\partial \xi} + a_5 \frac{\partial u}{\partial \sigma} + a_6 u + a_7,$$

which possesses hyperbolic properties in the σ-direction. From the reversion conditions, we see $a_5 = p - (\lambda - \rho\sigma\xi)q = p \geq p - q\dfrac{\partial q}{\partial \sigma} \geq 0$ at the boundary $\sigma = \sigma_l$ and $a_5 = p - (\lambda - \rho\sigma\xi)q = p \leq p - q\dfrac{\partial q}{\partial \sigma} \leq 0$ at $\sigma = \sigma_u$. These facts tell us that the value of u on the boundaries $\sigma = \sigma_l$ and $\sigma = \sigma_u$ can be determined by the value of u inside the domain. Hence, we can approximate the partial differential equation in the problem (8.110) at the boundaries $\sigma = \sigma_l$ and $\sigma = \sigma_u$ by

$$
\begin{aligned}
&\frac{u_{m,0}^{n+1}-u_{m,0}^{n}}{\Delta\tau}\\
=&\frac{a_1}{2\Delta\xi^2}(u_{m+1,0}^{n+1}-2u_{m,0}^{n+1}+u_{m-1,0}^{n+1}+u_{m+1,0}^{n}-2u_{m,0}^{n}+u_{m-1,0}^{n})\\
&+\frac{a_4}{4\Delta\xi}(u_{m+1,0}^{n+1}-u_{m-1,0}^{n+1}+u_{m+1,0}^{n}-u_{m-1,0}^{n})\\
&+\frac{a_5}{4\Delta\sigma}(-u_{m,2}^{n+1}+4u_{m,1}^{n+1}-3u_{m,0}^{n+1}-u_{m,2}^{n}+4u_{m,1}^{n}-3u_{m,0}^{n})\\
&+\frac{a_6}{2}(u_{m,0}^{n+1}+u_{m,0}^{n})+a_7,\\
&\qquad\qquad m=0,1,\cdots,M
\end{aligned}
\tag{8.113}
$$

and

$$
\begin{aligned}
&\frac{u_{m,I}^{n+1}-u_{m,I}^{n}}{\Delta\tau}\\
=&\frac{a_1}{2\Delta\xi^2}(u_{m+1,I}^{n+1}-2u_{m,I}^{n+1}+u_{m-1,I}^{n+1}+u_{m+1,I}^{n}-2u_{m,I}^{n}+u_{m-1,I}^{n})\\
&+\frac{a_4}{4\Delta\xi}(u_{m+1,I}^{n+1}-u_{m-1,I}^{n+1}+u_{m+1,I}^{n}-u_{m-1,I}^{n})\\
&+\frac{a_5}{4\Delta\sigma}(3u_{m,I}^{n+1}-4u_{m,I-1}^{n+1}+u_{m,I-2}^{n+1}+3u_{m,I}^{n}-4u_{m,I-1}^{n}+u_{m,I-2}^{n})\\
&+\frac{a_6}{2}(u_{m,I}^{n+1}+u_{m,I}^{n})+a_7,\\
&\qquad\qquad m=0,1,\cdots,M
\end{aligned}
\tag{8.114}
$$

respectively. Here, $\dfrac{\partial u}{\partial\sigma}$ is discretized by one-sided second-order scheme in order for all the node points involved to be in the computational domain. a_1 and a_4–a_7 are also evaluated at the point $(\xi_m,\sigma_i,\tau^{n+1/2})$, $i=0$ or I. When $u_{m,i}^n$, $m=0,1,\cdots,M$, $i=0,1,\cdots,I$ are known, from the difference scheme (8.112)–(8.114) we can determine $u_{m,i}^{n+1}$, $m=0,1,\cdots,M$, $i=0,1,\cdots,I$. The initial condition gives $u_{m,i}^0$, $m=0,1,\cdots,M$, $i=0,1,\cdots,I$. Therefore, we can do this procedure for $n=0,1,\cdots,N-1$ successively and finally find $u_{m,i}^N$, $m=0,1,\cdots,M$, $i=0,1,\cdots,I$.

In Fig. 8.15, the price of a European call option obtained in this way is given. The mesh used is $20\times20\times20$, where the first, second, and third numbers are M, I, and N, respectively. The parameters of the problem are given in the figure and the parameter functions are

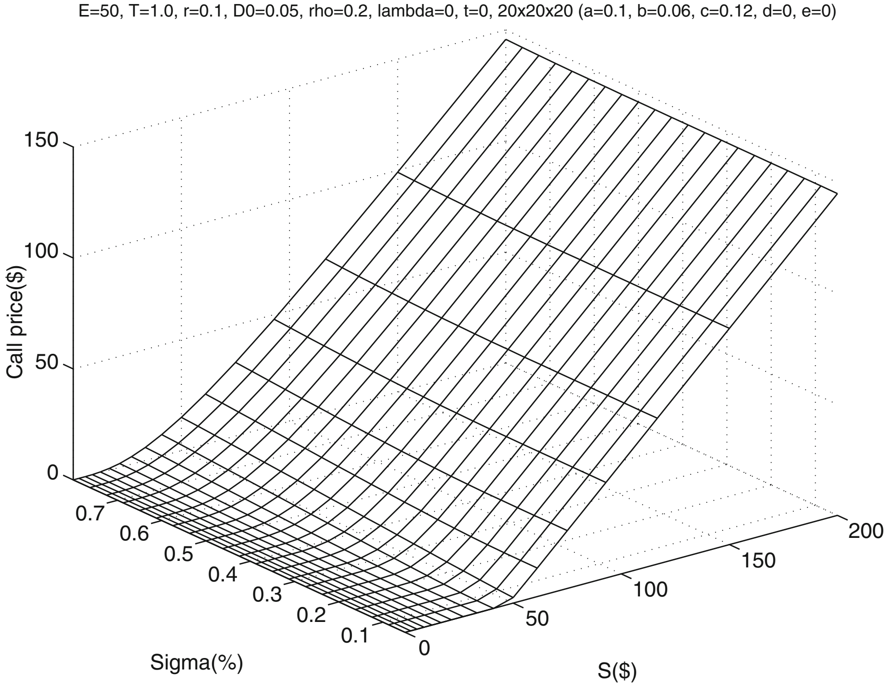

Fig. 8.15. The price of a two-factor European call option

$$\begin{cases} p = a(b-\sigma), & \sigma_l \le \sigma \le \sigma_u, \\ q = c\dfrac{1-\left(1-2\dfrac{\sigma-\sigma_l}{\sigma_u-\sigma_l}\right)^2}{1-0.975\left(1-2\dfrac{\sigma-\sigma_l}{\sigma_u-\sigma_l}\right)^2}\sigma, & \sigma_l \le \sigma \le \sigma_u, \\ \rho = 0.2, & \\ \lambda = d + e\sigma, & \sigma_l \le \sigma \le \sigma_u, \end{cases}$$

where $a = 0.1$, $b = 0.06$, $c = 0.12$, $d = 0$, $e = 0$, $\sigma_l = 0.05$, and $\sigma_u = 0.8$.

When the singularity-separating technique is not adopted, the scheme above can also be used. In that case,

$$a_7 = 0 \quad \text{and} \quad u(\xi, \sigma, 0) = \max(2\xi - 1, 0).$$

In order to give some idea about the performance of the method described in this subsection, we list the values of the option obtained by the method here with and without using extrapolation technique in Tables 8.24 and 8.25 for $S = 50$ and $\sigma = 0.2$. When these results were computed, for the first five coarser meshes, the linear systems were solved by the LU decomposition method and for the last three finer meshes, the Gauss–Seidel iteration was

Table 8.24. SSM with and without extrapolation technique

($S = 50$, $E = 50$, $T = 1$, $\sigma = 0.2$, $r = 0.1$, $D_0 = 0.05$, $a = 0.1$, $b = 0.06$, $c = 0.12$, $d = 0$, and $e = 0$. The exact solution is $4.848069 \cdots$.)

	Without extrapolation		With extrapolation	
Meshes	Solution	|Errors|	Solution	|Errors|
$10 \times 10 \times 10$	4.8143085	0.033761	–	–
$20 \times 20 \times 20$	4.8361039	0.011966	4.8433691	0.004700
$40 \times 40 \times 40$	4.8460151	0.002054	4.8493188	0.001249
$80 \times 80 \times 80$	4.8476154	0.000454	4.8481488	0.000079
$160 \times 160 \times 160$	4.8479592	0.000110	4.8480738	0.000004
$320 \times 320 \times 320$	4.8480421	0.000027	4.8480697	Less than 10^{-6}
$640 \times 640 \times 640$	4.8480626	0.000007	4.8480694	Less than 10^{-6}
$960 \times 960 \times 960$	4.8480664	0.000003	4.8480694	Less than 10^{-6}

Table 8.25. Implicit method with and without extrapolation technique

($S = 50$, $E = 50$, $T = 1$, $\sigma = 0.2$, $r = 0.1$, $D_0 = 0.05$, $a = 0.1$, $b = 0.06$, $c = 0.12$, $d = 0$, and $e = 0$. The exact solution is $4.848069 \cdots$.)

	Without extrapolation		With extrapolation	
Meshes	Solution	|Errors|	Solution	|Errors|
$10 \times 10 \times 10$	3.1774889	1.670580	–	–
$20 \times 20 \times 20$	4.2406270	0.607442	4.5950063	0.253063
$40 \times 40 \times 40$	4.7179697	0.130100	4.8770840	0.029015
$80 \times 80 \times 80$	4.8171183	0.030951	4.8501678	0.002098
$160 \times 160 \times 160$	4.8404088	0.007661	4.8481722	0.000103
$320 \times 320 \times 320$	4.8461590	0.001910	4.8480758	0.000006
$640 \times 640 \times 640$	4.8475923	0.000477	4.8480700	0.000001
$960 \times 960 \times 960$	4.8478575	0.000212	4.8480697	Less than 10^{-6}

used in order to solve the linear systems. From the tables, we see that the exact solution up to the sixth decimal place is 4.848069, which we obtained by a very fine mesh. Therefore, we can find out the errors of the results up to the sixth decimal place, which are also listed there. From the results without extrapolation in Table 8.24, it can be seen that this method has a second order accuracy because the error is reduced by a factor of about 1/4 when the mesh size is reduced by a factor of 1/2 (see the errors for the meshes $20 \times 20 \times 20, \cdots, 640 \times 640 \times 640$). Table 8.24 also shows that for a $20 \times 20 \times 20$ mesh with extrapolation, the error relative to E is $0.0047/50 \approx 10^{-4}$ and that the error relative to the option value is $0.0047/4.848069 \approx 10^{-3}$. In practice, requiring such accuracy is reasonable. The CPU time on a Pentium III 800 MHz computer is 0.07 s. If the singularity-separating technique is not used, in order to reach a similar accuracy, the mesh is between $40 \times 40 \times 40$ and $80 \times 80 \times 80$ and the CPU time is between 1 to 8 s and close to 8 s, respectively.

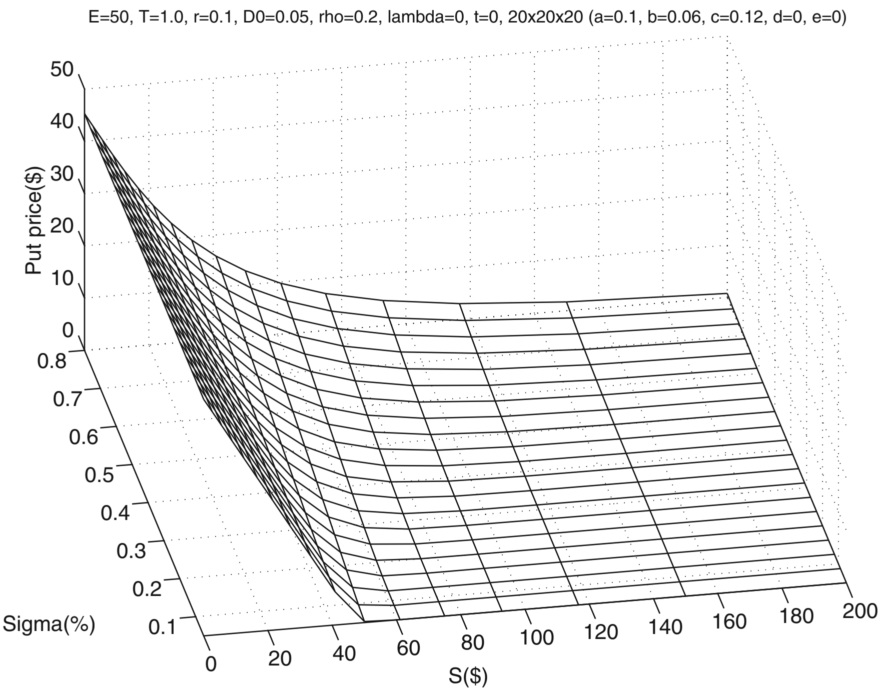

Fig. 8.16. The price of a two-factor European put option

Noticing

$$\frac{\partial p_1}{\partial \sigma} = \frac{\partial c_1}{\partial \sigma}, \quad \frac{\partial^2 p_1}{\partial \sigma^2} = \frac{\partial^2 c_1}{\partial \sigma^2}, \quad \frac{\partial^2 p_1}{\partial S \partial \sigma} = \frac{\partial^2 c_1}{\partial S \partial \sigma},$$

where p_1 is the price of the one-factor put option, we see that the difference between the two-factor and one-factor put options is also the solution of the problem (8.110). Therefore, in order to have the price of a European put option, we proceed as follows. First solving problem (8.110), then we can have the put price by

$$p(S, \sigma, t) = (S + P_m)u\left(\frac{S}{S + P_m},\ \sigma,\ T - t\right) + p_0(S,\ \sigma,\ t).$$

In Fig. 8.16, the price of a two-factor European put option obtained by this way is shown. The parameters of the problem and the parameter functions are the same as these for the two-factor European call option. Also, for European vanilla options, both the put–call parity relation and the put–call symmetric relation exist. The put–call parity relation still is

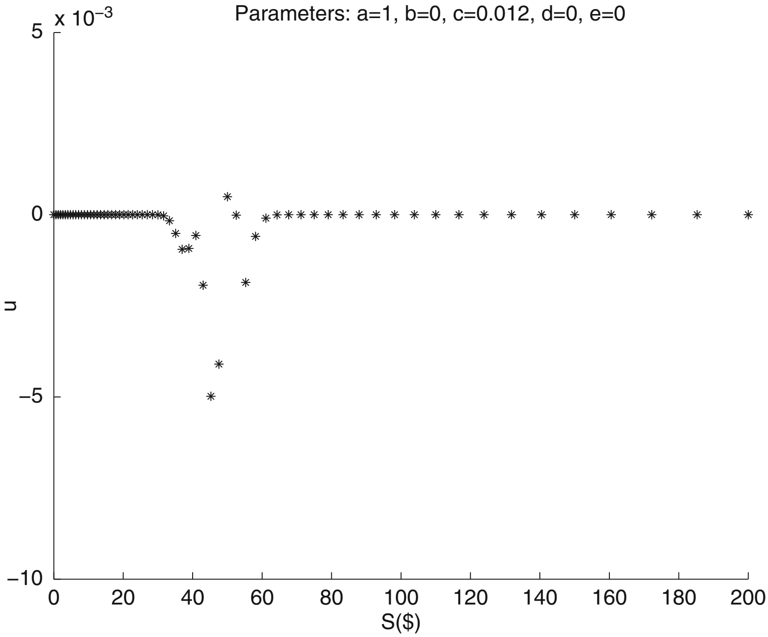

Fig. 8.17. An unstable solution of implicit schemes (Variation of u with respect to S on the line $\sigma = 0.05$ at $t = 0$. $E = 50$, $T = 2$, $r = 0.1$, and $D_0 = 0.05$. The solution is on a $80 \times 40 \times 80$ mesh.)

$$p(S, \sigma, t) = c(S, \sigma, t) - S\mathrm{e}^{-D_0(T-t)} + E\mathrm{e}^{-r(T-t)}. \quad (8.115)$$

When we calculate put option prices without using SSM, this relation can be used to check the correctness of the code to some extent. First, we compute the prices of a call option and a put option with the same parameters. Then, the results are substituted into the put–call parity relation to see if it holds. If it holds with a small error, then the code most likely gives correct results; if the relation does not hold, then the code must have some problems.

Finally, we give an example to explain that if the reversion conditions are not satisfied, then the final-value problem (8.110) is not well-posed and we cannot determine the solution using only the partial differential equation and the final condition in the problem (8.110). Consider a problem with $a = 1$, $b = 0$, $c = 0.012$, $d = 0$, $e = 0$, and $T = 2$. The other parameters are the same as before. We still use the numerical method above to find the numerical solution. In Fig. 8.17, we give the variation of u with S on the line $\sigma = \sigma_l$ at time $t = 0$. From there, we can see some "nonphysical" oscillations, which means that the computation is unstable even though an implicit scheme is used. This indicates that for this case, the solution is not determined only by the partial differential equation and final condition. The reason is that a proper boundary condition is needed at the boundary $\sigma = \sigma_l$ because the inequality condition in the condition (8.101) is not satisfied at $\sigma = \sigma_l$ due to $b = 0 < \sigma_l = 0.05$. If a reasonable condition cannot be given, then an artificial boundary condition has to be added. If the artificial boundary condition is not proper, then one will encounter some difficulty during computation.

8.3.7 Two-Factor Convertible Bonds with $D_0 = 0$

If $D_0 = 0$, then the convertible bond problem has no free boundary, and the problem has the same form as a European-style two-factor derivative problem does. The only difference is that the another random variable is the spot interest rate instead of the volatility. In order to make numerical methods more efficient, there are also two things we need to deal with. The first thing is the weak singularity generated by a discontinuous derivative of the payoff function. In order to separate this singularity, we can calculate numerically the difference between the values of two-factor and one-factor convertible bonds for the case $D_0 = 0$. We will not give the method here because it is similar to the method for two-factor options and the method for two-factor convertible bonds with $D_0 \neq 0$, which will be given in Sect. 9.1.2. The second thing is that the problem is defined on an infinite domain. Through a transformation similar to expression (8.109), the problem can be converted into a problem similar to problem (8.110) and the solution can be obtained by numerical methods efficiently. The details are similar to what we have done for two-factor options and left for readers to complete (Problem 26).

8.4 Pseudo-Spectral Methods

After the singularity-separating method is used, the solution to be computed numerically (the difference between the original unknown solution and a closed-form solution) is quite smooth. In this case, the pseudo-spectral method might be another good choice for computing the difference numerically. The basic principle of the method was discussed in Chap. 6. In this subsection, we give some details when the pseudo-spectral method is applied to problems (7.1) and (7.2).

Let us take $M+1$ grid points x_m, $m = 0, 1, \cdots, M$, on $[0, 1]$ and assume that the values of a function $u(x)$ for any x_m are given. Then, the values of the derivatives of $u(x)$ can be expressed as linear combinations of $u(x_m)$. Especially, if x_m is given by the expression (6.6), then the first derivative is approximated by the formula (6.7):

$$\frac{\partial u}{\partial x}(x_m) = \sum_{i=0}^{M} D_{x,m,i} u(x_i)$$

and the second derivative by expression (6.9):

$$\frac{\partial^2 u}{\partial x^2}(x_m) = \sum_{i=0}^{M} D_{xx,m,i} u(x_i),$$

where $D_{x,m,i}$ and $D_{xx,m,i}$ are given by the formulae (6.8) and (6.10), respectively. Consequently, the PDE in the problem (7.2) can be approximated by

$$\begin{aligned}
&\frac{u^{n+1}(x_m) - u^n(x_m)}{\Delta\tau} \\
=&\frac{1}{2}\left[a_m^{n+\frac{1}{2}} \sum_{i=0}^{M} D_{xx,m,i} u^{n+1}(x_i) + b_m^{n+\frac{1}{2}} \sum_{i=0}^{M} D_{x,m,i} u^{n+1}(x_i) + c_m^{n+\frac{1}{2}} u^{n+1}(x_m)\right] \\
&+\frac{1}{2}\left[a_m^{n+\frac{1}{2}} \sum_{i=0}^{M} D_{xx,m,i} u^{n}(x_i) + b_m^{n+\frac{1}{2}} \sum_{i=0}^{M} D_{x,m,i} u^{n}(x_i) + c_m^{n+\frac{1}{2}} u^{n}(x_m)\right] \\
&+g_m^{n+\frac{1}{2}}, \\
&\qquad m = 0, 1, \cdots, M,
\end{aligned} \tag{8.116}$$

where $u^{n+1}(x_m) = u\left(x_m, (n+1)\Delta\tau\right)$. Just like the implicit finite-difference method, if $u^n(x_m)$, $m = 0, 1, \cdots, M$ are given, we can determine $u^{n+1}(x_m)$, $m = 0, 1, \cdots, M$ by the linear system (8.116). However, the matrix of the current system is a full matrix, and the CPU time needed for solving this system is longer than the implicit finite-difference method if M is the same. When the solution is very smooth, only a small M might be needed in order to get a satisfying result. In such a case, its performance could be better than the implicit finite-difference method. This numerical method is referred to as the implicit pseudo-spectral method for one-dimensional problems.

Table 8.26. Pseudo-spectral methods

($S = 95$, $T = 1$, $E = 100$, $\sigma = 0.25$, $r = 0.1$, $D_0 = 0$,
$f(t) = (0.9 - 0.05t)E$, $g(t) = (1.6 + 0.05t)E$, and
the rebate $= g(t) - E$. The exact solution is $6.43129316 \cdots$.)

	Without SSM			With SSM		
Meshes	Solutions	|Errors|	CPU	Solution	|Errors|	CPU
7×50	6.454922	0.023629	0.0007	6.431842	0.000549	0.0014
7×100	6.454789	0.023596	0.0015	6.431438	0.000145	0.0022
7×200	6.454755	0.023462	0.0028	6.431426	0.000133	0.0043
8×50	6.438364	0.007071	0.0010	6.431351	0.000058	0.0014
8×100	6.438227	0.006934	0.0019	6.431305	0.000012	0.0028
8×200	6.438193	0.006900	0.0038	6.431293	0.0000005	0.0058
9×50	6.404701	0.026592	0.0013	6.431350	0.000057	0.0021
9×100	6.404555	0.026738	0.0024	6.431304	0.000011	0.0036
9×200	6.404518	0.002678	0.0044	6.431292	0.000001	0.0065

If we consider problem (7.1), the only difference is that instead of the partial differential equation being discretized at x_m, $m = 0, 1, \cdots, M$, now it is discretized at x_m, $m = 1, 2, \cdots, M-1$, and these equations and the boundary conditions given in the problem (7.1) form the entire system we need.

Table 8.26 gives some results obtained by the implicit pseudo-spectral method described above with $M = 7$, 8, 9. The corresponding time steps used are $\Delta\tau = 1/N$, $N = 50, 100, 200$, respectively. In the column "Meshes," $M \times N$ is given. The problem is a double barrier call option whose lower and upper knock-out boundaries are $f(t) = (0.9 - 0.05t)E$ and $g(t) = (1.6 + 0.05t)E$. The other parameters are given in the table. When the computation is done, the independent variable x is defined by

$$x = \frac{\dfrac{S}{E+S} - \dfrac{f(t)}{E+f(t)}}{\dfrac{g(t)}{E+g(t)} - \dfrac{f(t)}{E+f(t)}}.$$

The exact solution for this case is $6.43129316\cdots$, where the nine digits given are correct. When we have the exact solution, we can have the error of the solution, which is also given. The CPU time in seconds is also shown in order to see the performance.

In Table 8.26, both the results with and without the SSM are listed. From there, we can see that if the SSM is not used, the result obtained by using higher order polynomials might be worse than the results obtained by using lower order polynomials. However, it shows that when the pseudo-spectral method is combined with the singularity-separating technique, the higher the polynomial order, the better the result. Hence, the result of the pseudo-spectral method with the singularity-separating technique is much better than without it. Consequently, if the pseudo-spectral method is adopted, then combining it with SSM is essential. In Figs. 8.8 and 8.9, the functions computed when SSM is used and not used are shown, respectively. As pointed out, the functions in Fig. 8.9 are not as smooth as those in Fig. 8.8, especially, the derivative of the function for $t = 1$ in Fig. 8.9 is discontinuous. Therefore, the pseudo-spectral method does not provide a good performance for this case. However, if the singularity-separating technique is used, then the functions determined numerically are always very smooth, which can be seen from Fig. 8.8. In this case, the performance of the pseudo-spectral method is very good, and in certain cases it may even be better than the second-order implicit finite-difference methods because a pseudo-spectral method can be understood as a high-order difference method. In Fig. 8.18, the performances of the implicit finite-difference method and the implicit pseudo-spectral method with the singularity-separating technique are compared, which confirms this conclusion.

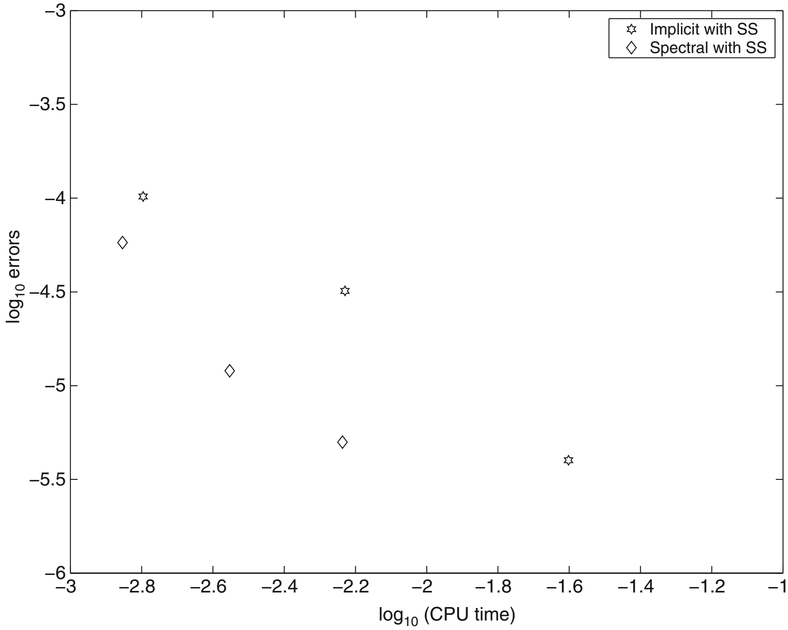

Fig. 8.18. Comparison between an implicit scheme and a pseudo-spectral method

The idea described here also works for double moving barrier put options with rebates and many other cases. For details, see the paper [92] by Zhu and Abifaker.

In Sect. 8.3, we pointed out that two-dimensional European-style derivative problems and American-style derivative problems that do not have free boundaries could be written in the form (8.110). The pseudo-spectral method can also be applied to such a problem. When this method is combined with the singularity-separating method, a good performance can be expected. For details of this method for two-dimensional case, see Chap. 9.

Problems

Table 8.27. Problems and sections

Problems	Sections	Problems	Sections	Problems	Sections
1–7	8.1	8–15	8.2	16–27	8.3
28–29	8.4				

1. *Suppose that we determine the price of an American vanilla call/put option through solving the following problem:

$$\begin{cases} \min\left(\dfrac{\partial u}{\partial \bar{\tau}} - \dfrac{\partial^2 u}{\partial x^2},\ u(x,\bar{\tau}) - g(x,\bar{\tau})\right) = 0, & -\infty < x < \infty,\ \bar{\tau} \ge 0, \\ u(x,0) = g(x,0), & -\infty < x < \infty, \end{cases}$$

where

$$g(x,\bar{\tau}) = \max\left(\pm(\mathrm{e}^{x+(2D_0/\sigma^2+1)\bar{\tau}} - \mathrm{e}^{2r\bar{\tau}/\sigma^2}), 0\right).$$

Describe a numerical method for solving this problem by using an explicit scheme.

2. As we know, an American lookback strike put option is the solution of the following linear complementarity problem:

$$\begin{cases} \min\left(-\dfrac{\partial W}{\partial t} - \mathbf{L}_\eta W,\ W - \max(\eta - \beta, 0)\right) = 0, & 1 \le \eta, \quad t \le T, \\ W(\eta, T) = \max(\eta - \beta, 0), & 1 \le \eta, \\ \dfrac{\partial W}{\partial \eta}(1,t) = 0, & t \le T, \end{cases}$$

where we assume that $\beta \ge 1$ and the operator $\mathbf{L}_\eta$ to be defined by

$$\mathbf{L}_\eta \equiv \frac{1}{2}\sigma^2\eta^2\frac{\partial^2}{\partial \eta^2} + (D_0 - r)\,\eta\frac{\partial}{\partial \eta} - D_0.$$

Convert this problem into a problem on $[0,1]$ and with an initial condition, and design an explicit method with a first-order accuracy in time and a second-order accuracy in space for solving the new problem.

3. Suppose that ψ is a binomial random variable and its two values are ψ_0 and ψ_1. Show the following:
   (a) If $\mathrm{E}[\psi] = 0$ and $\mathrm{E}[\psi^2] = 1$, then $\psi_0\psi_1 = -1$.
   (b) If $\mathrm{E}[\psi] = 0$ and $\psi_0\psi_1 = -1$, then $\mathrm{E}[\psi^2] = 1$.
   (c) If $\mathrm{E}[\psi] = 0$ and $\psi_0\psi_1 = -1 + O(\Delta t)$, then $\mathrm{E}[\psi^2] = 1 + O(\Delta t)$.
4. (a) *Derive the binomial methods proposed by Cox, Ross, and Rubinstein and by McDonald.
   (b) *Can the parameter p in the Cox–Ross–Rubinstein binomial method always represent a probability? Find out when it can and when it cannot. Can the parameter p given in the book by McDonald always represent a probability? Find out when it can and when it cannot.
5. From the Black–Scholes equation, we know that when a derivative security is priced, the value of the stock price at time t^n, S_n, and the value at time t^{n+1}, S_{n+1}, have the following relations:

$$\mathrm{E}_D\left[S_{n+1}\right] = \mathrm{e}^{(r-D_0)\Delta t}S_n$$

and

$$\mathrm{E}_D\left[S_{n+1}^2\right] = \mathrm{e}^{[2(r-D_0)+\sigma^2]\Delta t}S_n^2,$$

where $\Delta t = t^{n+1} - t^n$ (see Problem 39 of Chap. 2). Thus if the possible values of S_{n+1} are $S_{n+1,0} = S_n/u$ and $S_{n+1,1} = uS_n$, and the probabilities of S_{n+1} being $S_{n+1,0}$ and the probabilities of S_{n+1} being $S_{n+1,1}$ are $1-p$ and p, respectively, then in a binomial method, u and p should be determined by

$$\begin{cases} pu + (1-p)\,u^{-1} = \mathrm{e}^{(r-D_0)\Delta t}, \\ pu^2 + (1-p)\,u^{-2} = \mathrm{e}^{[2(r-D_0)+\sigma^2]\Delta t}. \end{cases}$$

If Δt is very small, this problem can be approximated by

$$\begin{cases} pu + (1-p)\,u^{-1} = 1 + (r-D_0)\Delta t, \\ pu^2 + (1-p)\,u^{-2} = 1 + [2(r-D_0)+\sigma^2]\Delta t. \end{cases}$$

(a) Find u and p for both cases (suppose $u > 1$).
(b) Consider a more general system

$$\begin{cases} pu + (1-p)\,u^{-1} = 1 + (r-D_0)\Delta t + O(\Delta t^2), \\ pu^2 + (1-p)\,u^{-2} = 1 + [2(r-D_0)+\sigma^2]\Delta t + O(\Delta t^2). \end{cases}$$

Show that if u and p are determined by such a system, then we always have

$$u = \mathrm{e}^{\sigma\sqrt{\Delta t}} + O\left(\Delta t^{3/2}\right)$$

and

$$p = \frac{1}{2}\left[1 + \frac{\sqrt{\Delta t}}{\sigma}\left(r - D_0 - \frac{1}{2}\sigma^2\right)\right] + O\left(\Delta t^{3/2}\right).$$

(Hint: When you derive the expression for p, write u as $1 + \sigma\sqrt{\Delta t} + \sigma^2\Delta t/2 + c\Delta t^{3/2} + O(\Delta t^2)$, c being a constant.)

6. *Describe the binomial methods for solving American vanilla call/put options
7. *Show that the Cox–Ross–Rubinstein binomial method for European options almost is an explicit difference scheme for the following problem:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2\dfrac{\partial^2 V}{\partial y^2} + \left(r - D_0 - \dfrac{1}{2}\sigma^2\right)\dfrac{\partial V}{\partial y} - rV = 0, \\ \qquad\qquad\qquad\qquad -\infty < y < \infty, \quad t \le T, \\ V(y,T) = \max(\pm(\mathrm{e}^y - 1), 0), \qquad -\infty < y < \infty, \end{cases}$$

where $y = \ln S$, S being the price of the stock and assume $E = 1$.

8. Show that the relation

$$V(S,t_i^-) = V(S - D_i(S), t_i^+)$$

becomes

$$\overline{V}(\xi,\ \tau_i^+)$$
$$= \left[1 - D_i\left(\frac{P_m\xi}{1-\xi}\right)\frac{1-\xi}{P_m}\right]\overline{V}\left(\frac{P_m\xi - D_i\left(\frac{P_m\xi}{1-\xi}\right)(1-\xi)}{P_m - D_i\left(\frac{P_m\xi}{1-\xi}\right)(1-\xi)},\ \tau_i^-\right)$$

under the transformation

$$\begin{cases} \xi = \dfrac{S}{S+P_m}, \\ \tau = T - t, \\ \overline{V}(\xi,\tau) = \dfrac{V(S,t)}{S+P_m}. \end{cases}$$

9. (a) Show that the jump condition

$$W\left(\eta, t_i^-\right) = W\left(\eta + J, t_i^+\right)$$

becomes

$$\overline{u}\left(\xi,\ \tau_i^+\right) = \frac{\left|\frac{P_m\xi}{1-|\xi|} + J\right| + P_m}{\left|\frac{P_m\xi}{1-|\xi|}\right| + P_m}\overline{u}\left(\frac{\frac{P_m\xi}{1-|\xi|} + J}{\left|\frac{P_m\xi}{1-|\xi|} + J\right| + P_m},\quad \tau_i^-\right)$$

under the transformation

$$\begin{cases} \xi = \dfrac{\eta}{|\eta| + P_m}, \\ \tau = T - t, \\ W(\eta,t) = (|\eta| + P_m)\overline{u}(\xi,\tau), \end{cases}$$

where $P_m > 0$.

(b) Suppose that the jump condition for $W(\eta,t)$ is

$$W\left(\eta, t_i^-\right) = W\left(\eta + J, t_i^+\right)$$

and introduce the transformation

$$\begin{cases} \xi = \dfrac{\eta}{|\eta| + P_m(\eta)}, \\ \tau = T - t, \\ W(\eta,t) = (|\eta| + P_m(\eta))\overline{u}(\xi,\tau), \end{cases}$$

where

$$P_m(\eta) = \begin{cases} P_{mr}, & \text{if } \eta > 0, \\ \\ P_{ml}, & \text{if } \eta < 0. \end{cases}$$

Here $P_{mr} > 0$ and $P_{ml} > 0$. Find the jump condition for $\overline{u}(\xi, \tau)$.

10. *Suppose that we determine the price of an American vanilla call/put option through solving the following problem:

$$\begin{cases} \min\left(\dfrac{\partial \overline{V}}{\partial \tau} - \mathbf{L}_\xi \overline{V},\ \overline{V}(\xi,\tau) - \max(\pm(2\xi-1),0)\right) = 0, & 0 \le \xi \le 1, \\ & \tau \ge 0, \\ \overline{V}(\xi,0) = \max(\pm(2\xi-1),0), & 0 \le \xi \le 1, \end{cases}$$

where

$$\mathbf{L}_\xi = \frac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1-\xi)^2\frac{\partial^2}{\partial \xi^2} + (r - D_0)\xi(1-\xi)\frac{\partial}{\partial \xi} - [r(1-\xi) + D_0\xi].$$

Describe a numerical method for solving this problem by using a second order implicit scheme. (Discuss the discretization of the problem only.)

11. As we know, an American average strike call option is the solution of the following linear complementarity problem:

$$\begin{cases} \min\left(-\dfrac{\partial W}{\partial t} - \mathbf{L}_{a,t}W,\ W(\eta,t) - \max(\alpha - \eta, 0)\right) = 0, & 0 \le \eta,\ t \le T, \\ W(\eta, T) = \max(\alpha - \eta, 0), & 0 \le \eta, \end{cases}$$

where $\alpha \approx 1$ and

$$\mathbf{L}_{a,t} = \frac{1}{2}\sigma^2\eta^2\frac{\partial^2}{\partial \eta^2} + \left[(D_0 - r)\,\eta + \frac{1-\eta}{t}\right]\frac{\partial}{\partial \eta} - D_0.$$

Convert this problem into a problem on a finite domain and with an initial condition, and design an implicit second-order method for solving this new problem. (Discuss the discretization of the problem only.)

12. Based on the partial differential equation

$$\frac{\partial \overline{V}}{\partial \tau} = \frac{1}{2}\bar{\sigma}^2(\xi)\xi^2(1-\xi)^2\frac{\partial^2 \overline{V}}{\partial \xi^2} + r\xi(1-\xi)\frac{\partial \overline{V}}{\partial \xi} - r(1-\xi)\overline{V},$$

design an implicit method for the LC problem of American options with discrete dividends.

13. *Suppose that the scheme

$$\begin{aligned}
&\frac{v_m^{n+1}-v_m^n}{\Delta\tau}\\
&=\frac{1}{4}\bar{\sigma}_m^2\xi_m^2(1-\xi_m)^2\left(\frac{v_{m+1}^{n+1}-2v_m^{n+1}+v_{m-1}^{n+1}}{\Delta\xi^2}+\frac{v_{m+1}^{n}-2v_m^{n}+v_{m-1}^{n}}{\Delta\xi^2}\right)\\
&\quad+\frac{1}{2}(r-D_0)\xi_m(1-\xi_m)\left(\frac{v_{m+1}^{n+1}-v_{m-1}^{n+1}}{2\Delta\xi}+\frac{v_{m+1}^{n}-v_{m-1}^{n}}{2\Delta\xi}\right)\\
&\quad-\frac{1}{2}[r(1-\xi_m)+D_0\xi_m](v_m^{n+1}+v_m^n)
\end{aligned}$$

is used for solving an American call option problem. Design a projected direct method, which you think is most accurate, to find the solution at each time step.

14. *Consider the following LC problem:

$$\begin{cases}\min\left(\dfrac{\partial u}{\partial\bar{\tau}}-\dfrac{\partial^2 u}{\partial x^2},\ u(x,\bar{\tau})-g(x,\bar{\tau})\right)=0, & -\infty<x<\infty,\ \bar{\tau}\ge 0,\\ u(x,0)=g(x,0), & -\infty<x<\infty,\end{cases}$$

where

$$g(x,\bar{\tau})=\max\left(\pm(\mathrm{e}^{x+(2D_0/\sigma^2+1)\bar{\tau}}-\mathrm{e}^{2r\bar{\tau}/\sigma^2}),0\right).$$

Suppose an implicit finite-difference method based on such a formulation is used for solving an American option problem. Design an iteration method similar to the SOR method for a linear system to find the solution of the problem at each time step.

15. *The heat equation

$$\frac{\partial u}{\partial\tau}=a\frac{\partial^2 u}{\partial x^2}$$

can be approximated by the explicit first-order scheme

$$\frac{u_m^{n+1}-u_m^n}{\Delta\tau}=a\frac{u_{m+1}^n-2u_m^n+u_{m-1}^n}{\Delta x^2}$$

or the implicit second-order scheme (the Crank–Nicolson scheme)

$$\frac{u_m^{n+1}-u_m^n}{\Delta\tau}=\frac{a}{2}\left(\frac{u_{m+1}^{n+1}-2u_m^{n+1}+u_{m-1}^{n+1}}{\Delta x^2}+\frac{u_{m+1}^n-2u_m^n+u_{m-1}^n}{\Delta x^2}\right).$$

When do we choose the explicit first-order scheme and when do we use the implicit second-order scheme? Why should we choose the implicit second-order scheme if we need highly accurate results?

16. (a) Find a closed-form solution of the problem:

$$\begin{cases} \dfrac{\partial c_u}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 c_u}{\partial S^2} + (r-D_0)S\dfrac{\partial c_u}{\partial S} - rc_u = 0, \quad 0 \le S, \quad 0 \le t \le T, \\ c_u(S,T) = \begin{cases} \max(S-E, 0), \text{ if} & 0 \le S < g(T), \\ 0, & \text{if} \quad g(T) \le S. \end{cases} \end{cases}$$

Here we assume $g(T) > E$.

(b) Consider the following European barrier option problem:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r-D_0)S\dfrac{\partial V}{\partial S} - rV = 0, \\ \qquad\qquad f(t) \le S \le g(t), \quad 0 \le t \le T, \\ V(S,T) = \max(S-E, 0), \quad f(T) \le S \le g(T), \\ V(f(t),t) = 0, \quad 0 \le t \le T, \\ V(g(t),t) = 0, \quad 0 \le t \le T, \end{cases}$$

where $S = f(t)$ and $S = g(t)$ are the locations of the lower and upper barriers with $f(t) < E$ and $g(t) > E$. Assume that we need to find the solution by numerical methods. Design a SSM for this problem based on the result given in part (a) . (Here the problem can be defined on a non-rectangular domain.)

17. Suppose that η_1, η_2, and p are given, where $0 < \eta_1 < \eta_2$, $2\eta_2 - \eta_1 < 1$, and $p > 1$. Set $\eta_3 = 2\eta_2 - \eta_1$ and let $f(\eta)$ be a function on $[0,1]$ satisfying the condition $f(0) = 0$ and its derivative be equal to

$$f_\eta(\eta) = \begin{cases} d, & 0 \le \eta < \eta_1, \\ a(\eta-\eta_2)^4 + b(\eta-\eta_2)^2 + c, & \eta_1 \le \eta < \eta_3, \\ d, & \eta_3 \le \eta \le 1. \end{cases}$$

Here $d = a(\eta_2-\eta_1)^4 + b(\eta_2-\eta_1)^2 + c$, which guarantees $f_\eta(\eta)$ is continuous at $\eta = \eta_1$. From the definition of η_3, we know $\eta_2 - \eta_1 = \eta_3 - \eta_2$, so $f_\eta(\eta)$ is also continuous at $\eta = \eta_3$.

(a) Assume that the following three conditions hold:

(i) $\dfrac{f_\eta(\eta_2)}{f_\eta(\eta_1)} = \dfrac{c}{a(\eta_2-\eta_1)^4 + b(\eta_2-\eta_1)^2 + c} = p,$

(ii) $f_{\eta\eta}(\eta_1) = 4a(\eta_1-\eta_2)^3 + 2b(\eta_1-\eta_2) = 0,$

(iii) $f(1) = 1.$

Find the expressions of $a, b,$ and c as functions of η_1, η_2, η_3, and p and show that $f(\eta)$ is an increasing function on $[0,1]$ in this case.

(b) Find the expression of $f(\eta)$.

(c) When solving a PDE/OPE problem, a variable mesh can be realized by using transformation. Suppose that the independent variable in a PDE/ODE problem is η and a new variable is introduced by setting $\xi = f(\eta)$. How should we choose the parameters in the function $f(\eta)$ if we want to let the mesh size in the region near the point $\eta = 0.4$ is about 1/10 of the mesh size in the regions $[0, 0.2]$ and $[0.6, 1]$?

18. Let $\bar{c}(\xi,\tau) = c(S,t)/(S+P_m)$ and $\bar{p}(\xi,\tau) = p(S,t)/(S+P_m)$, where $\xi = S/(S+P_m)$ and $\tau = T-t$. Derive the expressions of $\bar{c}(\xi,\tau)$ and $\bar{p}(\xi,\tau)$ and find the limits of $\bar{c}(\xi,\tau)$ and $\bar{p}(\xi,\tau)$ as ξ tends to 0 and 1. Also write down the formulae for the case $P_m = E$.

19. Suppose that $V(S,t)$ satisfies the following jump condition at $t = t_i$:

$$V(S,t_i^-) = V(S - D_i(S), t_i^+)$$

and that $V_0(S,t)$ is continuous at $t = t_i$. Define

$$\begin{cases} \xi = \dfrac{S}{S+P_m}, \\ \tau = T - t, \\ u(\xi,\tau) = \dfrac{V(S,t) - V_0(S,t)}{S+P_m}, \\ u_0(\xi,\tau) = \dfrac{V_0(S,t)}{S+P_m}, \end{cases}$$

where P_m is a positive number. Show that the following jump condition for $u(\xi,\tau)$ holds:

$$\begin{aligned} & u\left(\xi,\tau_i^+\right) \\ &= \left[1 - \frac{1-\xi}{P_m} D_i\left(\frac{\xi P_m}{1-\xi}\right)\right] \left[u\left(\frac{P_m\xi - D_i\left(\dfrac{\xi P_m}{1-\xi}\right)(1-\xi)}{P_m - D_i\left(\dfrac{\xi P_m}{1-\xi}\right)(1-\xi)}, \tau_i^-\right)\right. \\ &\quad \left. + u_0\left(\frac{P_m\xi - D_i\left(\dfrac{\xi P_m}{1-\xi}\right)(1-\xi)}{P_m - D_i\left(\dfrac{\xi P_m}{1-\xi}\right)(1-\xi)}, \tau_i\right)\right] - u_0\left(\xi,\tau_i\right). \end{aligned}$$

20. Design a SSM for European vanilla options with discrete dividends and a constant volatility, and formulate the problem as a problem defined on a finite domain and with an initial condition.

21. *Design a SSM for Bermudan options with variable volatilities and formulate the problem as a problem defined on a finite domain and with an initial condition.

22. Suppose r and D_0 are constant and $\sigma = \sigma(S)$. Derive the symmetry relations for Bermudan options.
23. *Find a transformation to convert an average price call option problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r - D_0)S\dfrac{\partial V}{\partial S} + \dfrac{S}{T}\dfrac{\partial V}{\partial I} - rV = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 \le S, \quad 0 \le I, \quad t \le T, \\ V(S, I, T) = \max(I - E, 0), \qquad 0 \le S, \quad 0 \le I, \end{cases}$$

where

$$I = \frac{1}{T}\int_0^t S(\tau)d\tau,$$

into the problem

$$\begin{cases} \dfrac{\partial f}{\partial \tau} - \dfrac{1}{2}\sigma^2 \left[\xi - \dfrac{1}{(r - D_0)T}(1 - \mathrm{e}^{-(r-D_0)\tau})\right]^2 \dfrac{\partial^2 f}{\partial \xi^2} = 0, \\ \qquad\qquad\qquad\qquad -\infty < \xi < \infty, \quad 0 \le \tau \le T, \\ f(\xi, 0) = \max(\xi, 0), \qquad -\infty < \xi < \infty. \end{cases}$$

24. Find a closed-form solution of the problem:

$$\begin{cases} \dfrac{\partial \tilde{f}_0}{\partial \tau} - \dfrac{\sigma^2}{2(r - D_0)^2 T^2}\left(1 - \mathrm{e}^{-(r-D_0)\tau}\right)^2 \dfrac{\partial^2 \tilde{f}_0}{\partial \xi^2} = 0, \\ \qquad\qquad\qquad\qquad -\infty < \xi < \infty, \quad 0 \le \tau \le T, \\ \tilde{f}_0(\xi, 0) = \max(\xi, 0), \qquad -\infty < \xi < \infty. \end{cases}$$

25. Convert the problem

$$\begin{cases} \dfrac{\partial f_1}{\partial \tau} - \dfrac{1}{2}\sigma^2 \left[\xi - \dfrac{1}{(r - D_0)T}\left(1 - \mathrm{e}^{-(r-D_0)\tau}\right)\right]^2 \dfrac{\partial^2 f_1}{\partial \xi^2} = \dfrac{\sigma^2 \xi\, \mathrm{e}^{-\xi^2/4\tau_1}}{4\sqrt{\pi\tau_1}} \\ \quad \times \left[\xi - \dfrac{2}{(r - D_0)T}\left(1 - \mathrm{e}^{-(r-D_0)\tau}\right)\right], \qquad -\infty < \xi < \infty, \quad 0 \le \tau \le T, \\ f_1(\xi, 0) = 0, \qquad -\infty < \xi < \infty. \end{cases}$$

into a problem defined on $[0, 1]$ and with an initial condition, and design an implicit second-order scheme for the new problem.
26. By using the transformation

$$\begin{cases} \xi = \dfrac{S}{S + P_m}, \\ r = r, \\ \tau = T - t, \\ u(\xi, r, \tau) = \dfrac{B_c(S, r, t)}{n\,(S + P_m)}, \end{cases}$$

the two-factor convertible bond problem for non-dividend stocks

$$\begin{cases} \dfrac{\partial B_c}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 B_c}{\partial S^2} + \rho\sigma S w \dfrac{\partial^2 B_c}{\partial S \partial r} + \dfrac{1}{2} w^2 \dfrac{\partial^2 B_c}{\partial r^2} + rS\dfrac{\partial B_c}{\partial S} \\ \qquad + (u - \lambda w)\dfrac{\partial B_c}{\partial r} - rB_c + kZ = 0, \\ \qquad\qquad\qquad\qquad 0 \le S, \quad r_l \le r \le r_u, \quad 0 \le t \le T, \\ B_c(S, r, T) = \max(Z, nS), \qquad 0 \le S, \quad r_l \le r \le r_u \end{cases}$$

can be converted into a problem on a finite domain with a bounded final condition. The one-factor convertible zero-coupon bond problem for non-dividend stocks

$$\begin{cases} \dfrac{\partial b_c}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 b_c}{\partial S^2} + rS\dfrac{\partial b_c}{\partial S} - rb_c = 0, & 0 \le S, \quad 0 \le t \le T, \\ b_c(S, r, T) = \max(Z, nS), & 0 \le S \end{cases}$$

has the following solution:

$$nc(S, t; Z/n) + \mathrm{e}^{-r(T-t)} Z,$$

where $c(S, t; Z/n)$ is the price of a call option with an exercise price Z/n. Find the partial differential equation and the final condition the difference between the two bonds should satisfy. Convert the derived problem into a problem on a finite domain and with an initial condition by using the transformation above, and briefly describe a second-order implicit scheme for the new problem.

27. Suppose that $c(S, \sigma, t)$ and $p(S, \sigma, t)$ are solutions of the following problems

$$\begin{cases} \dfrac{\partial c}{\partial t} + \mathbf{L}_{S,\sigma} c = 0, & 0 \le S, \quad \sigma_l \le \sigma \le \sigma_u, \quad t \le T, \\ c(S, \sigma, T) = \max(S - E, 0), & 0 \le S, \quad \sigma_l \le \sigma \le \sigma_u \end{cases}$$

and

$$\begin{cases} \dfrac{\partial p}{\partial t} + \mathbf{L}_{S,\sigma} p = 0, & 0 \le S, \quad \sigma_l \le \sigma \le \sigma_u, \quad t \le T, \\ p(S, \sigma, T) = \max(E - S, 0), & 0 \le S, \quad \sigma_l \le \sigma \le \sigma_u, \end{cases}$$

where $\mathbf{L}_{S,\sigma}$ is an operator defined by

$$\mathbf{L}_{S,\sigma} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + \rho\sigma S q \frac{\partial^2}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2}{\partial \sigma^2} + (r - D_0) S \frac{\partial}{\partial S} + (p - \lambda q)\frac{\partial}{\partial \sigma} - r.$$

Show that the following put–call parity relation

$$c(S, \sigma, t) - p(S, \sigma, t) = S\mathrm{e}^{-D_0(T-t)} - E\mathrm{e}^{-r(T-t)}$$

holds by the superposition principle. (Hint: Let u denote $c(S,\sigma,t) - p(S,\sigma,t)$. Show that u is the solution of the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathbf{L}_{S,\sigma} u = 0, & 0 \le S, \quad \sigma_l \le \sigma \le \sigma_u, \quad t \le T, \\ u(S,\sigma,T) = S - E, & 0 \le S, \quad \sigma_l \le \sigma \le \sigma_u \end{cases}$$

and that $S\mathrm{e}^{-D_0(T-t)} - E\mathrm{e}^{-r(T-t)}$ is also the solution of this problem.)

28. *Convert the following double moving barrier call option problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r - D_0)S\dfrac{\partial V}{\partial S} - rV = 0, & \\ & f(t) \le S \le g(t), \quad 0 \le t \le T, \\ V(S,T) = \max(S-E, 0), & f(T) \le S \le g(T), \\ V(f(t),t) = 0, & 0 \le t \le T, \\ V(g(t),t) = g(t) - E, & 0 \le t \le T \end{cases}$$

into a problem that has a smooth solution and an initial condition, and design an implicit pseudo-spectral method for the new problem.

29. For the new problem obtained in Problem 26, design an implicit pseudo-spectral method.

Projects

General Requirements

(A) Submit a code or codes in C or C^{++} that will work on a computer the instructor can get access to. At the beginning of the code, write down the name of the student and indicate on which computer it works and the procedure to make it work.

(B) Each code should use an input file to specify all the problem parameters and the computational parameters for each computation and an output file to store all the results. In an output file, the name of the problem, all the problem parameters, and the computational parameters should be given, so that one can know what the results are and how they were obtained. The input file should be submitted with the code.

(C) If not specified, for each case two results are required. For the first result, a 20×12 mesh should be used.(In this case, the error of the solution might be quite large.) For the second result, the accuracy required is 0.01, and the mesh used should be as coarse as possible.

(D) Submit results in form of tables or figures. When a result is given, always provide the problem parameters and the computational parameters.

1. **Explicit Method (8.3).** Suppose that σ, r are constants and the dividends are given discretely or continuously. Write a code for European, Bermudan, and American calls and puts.
   - For American call and put options, give the results for the case: $S = 100$, $E = 100$, $T = 0.75$, $r = 0.1$, $D_0 = 0.05$, and $\sigma = 0.3$.
   - For Bermudan call and put options, give the results for the case: $S = 100$, $E = 100$, $T = 1$, $r = 0.05$, $D_0 = 0.1$, $\sigma = 0.2$, and $K = 4$ (see Sect. 8.3.3).
   - For European call and put options, give the results for the cases: $S = 100$, $E = 95, 100, 105$, $T = 1$, $r = 0.1$, $\sigma = 0.2$, and two dividend payments of \$1.25 paid at $t = 2$ months and $t = 8$ months. $D(S)$ is defined by

     $$D(S) = \begin{cases} S & \text{if } S \le d, \\ d & \text{if } S > d, \end{cases}$$

     where d is the dividend payment.
   - Taking the European call option with $E = 100$, $T = 1$, $r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$ as an example, show that the explicit method (8.3) is unstable if $\Delta\tau$ is too large. For this problem, only one example is required. Plot the S-c curve with $t = 0$.
2. **Binomial Methods (8.28) with the formulae (8.25)–(8.27) and Eq. (8.28) with the formulae (8.18) and (8.23).** Suppose that σ, r, D_0 are constants. Write a code for European, Bermudan, and American calls and puts. For this problem, instead of the result on a 20×12 mesh, a result with $\Delta t = T/12$ is required.
   - For European call and put options, give the results for the cases: $S = 100$, $E = 95, 100, 105$, $T = 1$, $r = 0.1$, $D_0 = 0.025$, and $\sigma = 0.2$.
   - For Bermudan call and put options, give the results for the case: $S = 100$, $E = 100$, $T = 1$, $r = 0.05$, $D_0 = 0.1$, $\sigma = 0.2$, and $K = 4$.
   - For American call and put options, give the results for the case: $S = 100$, $E = 100$, $T = 0.75$, $r = 0.1$, $D_0 = 0.05$, and $\sigma = 0.3$.
3. **Implicit Method (8.47) for Vanilla Options (Solving the Corresponding System by Direct Methods).** Suppose that σ, r, and D_0 are constants. Write a code for European, Bermudan, and American calls and puts.
   - For European call and put options, give the results for the cases: $S = 100$, $E = 95, 100, 105$, $T = 1$, $r = 0.1$, $D_0 = 0.025$, and $\sigma = 0.2$.
   - For Bermudan call and put options, give the results for the case: $S = 100$, $E = 100$, $T = 1$, $r = 0.05$, $D_0 = 0.1$, $\sigma = 0.2$, and $K = 4$.
   - For American call and put options, give the results for the case: $S = 100$, $E = 100$, $T = 0.75$, $r = 0.1$, $D_0 = 0.05$, and $\sigma = 0.3$.
4. **Implicit Method (8.47) for European Average Price Options with Discrete Sampling (Solving the Corresponding System by Direct Methods).** Suppose that σ, r, and D_0 are constants. Write a

code for European average price call and put options with various discrete samplings.

- For European average price call and put options with sampling daily, give the results for the cases: $S = 100$, $E = 90, 95, 100, 105, 110$, $T = 1$, $r = 0.05$, $D_0 = 0.025$, and $\sigma = 0.2$. (The results on a 20×12 mesh are not required.)
- For European average price call and put options with sampling weekly, give the results for the cases: $S = 100$, $E = 90, 95, 100, 105, 110$, $T = 0.5$, $r = 0.05$, $D_0 = 0.0$, and $\sigma = 0.2$. (The results on a 20×12 mesh are not required.)
- For European average price call and put options with sampling monthly, give the results for the cases: $S = 100$, $E = 90, 95, 100, 105, 110$, $T = 1$, $r = 0.0$, $D_0 = 0.0$, and $\sigma = 0.3$.

5. **Singularity-Separating Implicit Method with Scheme (8.47).** Suppose that σ, r are constants and the dividends are given discretely or continuously. Write a code for Bermudan calls and puts with continuous dividends and a code for European vanilla calls and puts with discrete dividends. Calculate the difference between the value of the option and the closed-form solution of a corresponding European vanilla option numerically. In order to calculate the price of a Bermudan put, Compute a corresponding call first and then obtain the value of the Bermudan put by using the symmetry relation.
   - For Bermudan call options, give the results for the case: $S = 100$, $E = 100$, $T = 1$, $r = 0.05$, $D_0 = 0.1$, $\sigma = 0.2$, and $K = 4$. For Bermudan put options, give the results for the case: $S = 100$, $E = 100$, $T = 1$, $r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$, and $K = 12$.
   - For European call and put options, give the results for the cases: $S = 100$, $E = 95, 100, 105$, $T = 1$, $r = 0.1$, $\sigma = 0.2$, and two dividend payments of \$1.25 paid at $t = 2$ months and $t = 8$ months. $D(S)$ is defined by

$$D(S) = \begin{cases} S & \text{if } S \leq d, \\ d & \text{if } S > d, \end{cases}$$

where d is the dividend payment.

9

Free-Boundary Problems

As we know, a problem of pricing an American-style derivative can be formulated as a linear complementarity problem, and for most cases, it can also be written as a free-boundary problem. In Chap. 8, we have discussed how to solve a linear complementarity problem. Here, we study how to solve a free-boundary problem numerically. Many derivative security problems have a final condition with discontinuous derivatives at some point. In this case, their solutions are not very smooth in the domain near this point, and their numerical solutions will have relatively large error. In Chap. 8, we have suggested to deal with this problem in the following way: instead of calculating the price of the derivative security, a difference between the price and an expression with the same or almost the same weak singularity is solved numerically. Because the difference is smooth, the error of numerical solution will be smaller. This method can still be used for free-boundary problems. For them there is another problem. On one side of the free boundary, the price of an American-style derivative satisfies a partial differential equation, and on the other side, it is equal to a given function. Because of this, the second derivative of the price is usually discontinuous on the free boundary. If we can follow the free boundary and use the partial differential equation only on the domain where the equation holds, then we can have less error. Hence, in Sect. 9.1 we not only discuss how to separate the weak singularity caused by the discontinuous first derivative at expiry but also describe how to convert a free-boundary problem into a problem defined on a rectangular domain so that we can easily use the partial differential equation only on the domain where the equation holds. The method described in Sect. 9.1 is referred to as the singularity-separating method (SSM) for free-boundary problems. The next two sections are devoted to discussing how to solve this problem using implicit schemes and pseudo-spectral methods for one-dimensional and two-dimensional cases. There, we also give some results on American vanilla, barrier, Asian, and lookback options, two-factor American vanilla options, and two-factor convertible bonds.

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9.1 SSM for Free-Boundary Problems

9.1.1 One-Dimensional Cases

From Chaps. 3–5, we know that there are many American-style derivatives. Their major features are the same, but there are some differences among them. In this subsection, first taking an American vanilla call option as an example, we give the details of the singularity-separating method for free-boundary problems. Then, we briefly point out what modifications are needed in order to apply the method to other American-style derivatives.

From Sect. 3.3, we know that on the domain $[0, S_f(t)] \times [0, T]$, the price of an American call option, $C(S,t)$, is the solution of the free-boundary problem

$$\begin{cases} \dfrac{\partial C}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 C}{\partial S^2} + (r - D_0) S \dfrac{\partial C}{\partial S} - rC = 0, & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le S \le S_f(t), \quad 0 \le t \le T, & \\ C(S,T) = \max(S - E, 0), \qquad 0 \le S \le S_f(T), & \\ C\left(S_f(t), t\right) = S_f(t) - E, \qquad 0 \le t \le T, & \\ \dfrac{\partial C}{\partial S}\left(S_f(t), t\right) = 1, \qquad 0 \le t \le T, & \\ S_f(T) = \max\left(E, rE/D_0\right); & \end{cases} \tag{9.1}$$

whereas on the domain $(S_f(t), \infty) \times [0, T]$, $C(S,t) = S - E$. Here, we assume $D_0 \neq 0$. Therefore, as long as we have the solution of the free-boundary problem, we can determine $C(S,t)$ for any $S \ge 0$ and any $t \in [0, T]$. The function $C(S,T) = \max(S - E, 0)$ has a discontinuous derivative at $S = E$. Therefore, $C(S,t)$ is not very smooth in the region where $S \approx E$ and $t \approx T$. Because the second derivative of $C(S,T)$ at $S = E$ goes to infinity, the truncation error of numerical methods near $S = E$ and $t = T$ is relatively large. In order to avoid such a relatively large error, we first find the numerical result of the difference between the prices of the American call option and the European call option, and then add the difference and the price of the European call option together to get the price of the American call option. Similar to those cases given in Sect. 8.3, the function representing the difference is very smooth, so numerical solution can be obtained efficiently.

Now we give the details of the method. Let $c(S,t)$ represent the price of the European call option, whose closed-form expression is given by the formula (2.90). As we know, $c(S,t)$ is the solution of the problem

$$\begin{cases} \dfrac{\partial c}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 c}{\partial S^2} + (r - D_0) S \dfrac{\partial c}{\partial S} - rc = 0, & 0 \le S, \quad 0 \le t \le T \\ c(S,T) = \max(S - E, 0), & 0 \le S. \end{cases}$$

Define

$$\overline{C}(S,t) = C(S,t) - c(S,t)$$

on the domain $[0, S_f(t)] \times [0, T]$. Both $C(S,T)$ and $c(S,T)$ are equal to $\max(S-E,0)$, so $\overline{C}(S,T) = 0$. The functions $C(S,t)$ and $c(S,t)$ satisfy the same linear homogeneous partial differential equation, so the difference between them does the same. At the free boundary $S = S_f(t)$, we have

$$\overline{C}\left(S_f(t),t\right) = C\left(S_f(t),t\right) - c\left(S_f(t),t\right) = S_f(t) - E - c\left(S_f(t),t\right)$$

and

$$\frac{\partial \overline{C}}{\partial S}\left(S_f(t),t\right) = \frac{\partial C}{\partial S}\left(S_f(t),t\right) - \frac{\partial c}{\partial S}\left(S_f(t),t\right) = 1 - \frac{\partial c}{\partial S}\left(S_f(t),t\right).$$

Therefore, $\overline{C}(S,t)$ is the solution of the following free-boundary problem

$$\begin{cases} \dfrac{\partial \overline{C}}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 \overline{C}}{\partial S^2} + (r - D_0)S\dfrac{\partial \overline{C}}{\partial S} - r\overline{C} = 0, & 0 \le S \le S_f(t) \\ & 0 \le t \le T, \\ \overline{C}(S,T) = 0, & 0 \le S \le S_f(T), \\ \overline{C}\left(S_f(t),t\right) = S_f(t) - E - c\left(S_f(t),t\right), & 0 \le t \le T, \\ \dfrac{\partial \overline{C}}{\partial S}\left(S_f(t),t\right) = 1 - \dfrac{\partial c}{\partial S}\left(S_f(t),t\right), & 0 \le t \le T, \\ S_f(T) = \max(E, rE/D_0). \end{cases} \tag{9.2}$$

In the problem above, we need to determine $\overline{C}(S,t)$ on a non-rectangular domain, and one of its boundaries, $S = S_f(t)$, is also unknown.

In order to make discretization of the boundary conditions on the free boundary simple and convert the final-boundary value problem into an initial-boundary value problem, we introduce a new coordinate system $\{\xi, \tau\}$ through a transformation defined by

$$\begin{cases} \xi = \dfrac{S}{S_f(t)}, \\ \tau = T - t. \end{cases}$$

This transformation converts the four boundaries of the domain of the problem (9.2), $S = 0$, $S = S_f(t)$, $t = T$, and $t = 0$, into $\xi = 0$, $\xi = 1$, $\tau = 0$, and $\tau = T$, respectively (see Fig. 9.1). Now the problem is defined on a rectangular domain, and the value of the solution at $\tau = 0$ is given, that is, the problem now is an initial-boundary value problem on a rectangular domain.

Let

$$s_f(\tau) = \frac{1}{E} S_f(T - \tau)$$

and

$$u(\xi, \tau) = \frac{1}{E}\overline{C}(S,t) = \frac{1}{E}\overline{C}\left(\xi E s_f(\tau),\ T - \tau\right),$$

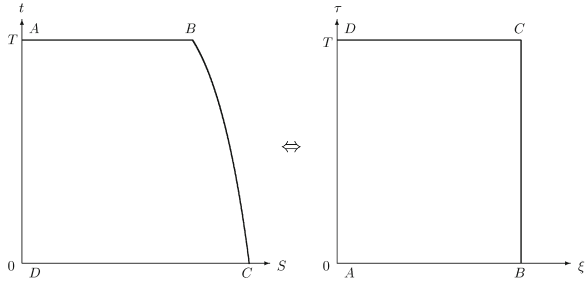

Fig. 9.1. Transforming a non-rectangular domain to a rectangular domain

that is,

$$S_f\left(t\right) = Es_f\left(T-t\right)$$

and

$$\overline{C}(S,t) = Eu\left(\frac{S}{Es_f(T-t)},\ T-t\right).$$

Since

$$\begin{aligned}
\frac{\partial \overline{C}}{\partial t} &= E\left[\frac{\xi}{s_f(\tau)}\frac{ds_f(\tau)}{d\tau}\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \tau}\right],\\
\frac{\partial \overline{C}}{\partial S} &= \frac{\partial u}{\partial \xi}\frac{1}{s_f(\tau)},\\
\frac{\partial^2 \overline{C}}{\partial S^2} &= \frac{1}{E}\frac{\partial^2 u}{\partial \xi^2}\left[\frac{1}{s_f(\tau)}\right]^2,
\end{aligned}$$

the problem (9.2) can be rewritten as

$$\begin{cases}
\dfrac{\partial u}{\partial \tau} = k_2\xi^2\dfrac{\partial^2 u}{\partial \xi^2} + \left(k_1 + \dfrac{1}{s_f}\dfrac{ds_f}{d\tau}\right)\xi\dfrac{\partial u}{\partial \xi} - k_0 u, & 0 \le \xi \le 1,\\
 & 0 \le \tau \le T,\\
u(\xi,0) = 0, & 0 \le \xi \le 1,\\
u(1,\tau) = g\left(s_f(\tau),\tau\right), & 0 \le \tau \le T,\\
\dfrac{\partial u}{\partial \xi}(1,\tau) = h\left(s_f(\tau),\tau\right), & 0 \le \tau \le T,\\
s_f(0) = \max(1, r/D_0),
\end{cases} \tag{9.3}$$

where $k_0 = r$, $k_1 = r - D_0$, $k_2 = \sigma^2/2$,

$$g\left(s_f(\tau),\tau\right) = s_f(\tau) - 1 - \frac{1}{E}c\left(Es_f(\tau),\ T-\tau\right)$$

and

$$h\left(s_f(\tau),\tau\right)=s_f(\tau)\left[1-\frac{\partial c\left(Es_f(\tau),\ T-\tau\right)}{\partial S}\right].$$

The differential equation in the problem (9.3) is a partial differential equation for u and can be understood as an ordinary differential equation for $s_f(\tau)$. This problem is a combination of an initial-boundary value problem for $u(\xi,\tau)$ on the domain $[0,1]\times[0,T]$ and an initial value problem for $s_f(\tau)$ on the interval $[0,T]$. It can be solved using explicit schemes, implicit schemes, or pseudo-spectral methods. After we obtain $u(\xi,\tau)$, we can get the price of the American call option on the domain $[0,S_f(t)]\times[0,T]$ by

$$C(S,t)=Eu\left(\frac{S}{Es_f(T-t)},T-t\right)+c(S,t).$$

From the expression of $C(S,t)$, we know that in order to computing $C(S,t)$, we need to write a code for computing $u(\xi,\tau)$ and also need to have a code for calculating $c(S,t)$. When the projects of Chap. 6 have been finished, the function **double** BS can be used for such a purpose.

The method described here is referred to as the singularity-separating method for American call options. The solution of the original American call option satisfies different equations in the two regions divided by the free boundary $S=S_f(t)$, and its solution has a discontinuous second derivative—a type of weak singularity—on the free boundary. In this method, the position of the free boundary is tracked accurately, so that we can use the different equations in each region exactly. Because the solution in the domain $(S_f(t),\infty)\times[0,T]$ is given by a known function, we only need to determine the solution in the region $[0,S_f(t)]\times[0,T]$. In this region, the second derivative near the free boundary is continuous, so the solution we want to get numerically is smoother than the original solution. Here, we also suggest to compute the difference between the American call option and the European call option numerically in the domain $[0,S_f(t)]\times[0,T]$, instead of directly computing the price of the American call option numerically. The derivative of solution of the American call option with respect to S at the point (E,T) is discontinuous if $S_f(T)\neq E$. The difference is much smoother than the solution of the American call option in the domain $[0,S_f(t)]\times[0,T]$, which make the truncation error smaller. Therefore, in the method described above, we use some techniques such that the solution we need to get numerically is much smoother than the original solution, which makes numerical methods more efficient. We refer to this as singularity-separating as we did in Sect. 8.3, because the solution becomes smoother than the original one after some singularities on the free boundary and at the point (E,T) have been "separated". Here, the singularity that has been "separated" is the discontinuity of the derivatives of the solution, which is weak. The idea of the method was originally developed for dealing with shock problems in fluid mechanics (see [97]) and the Stefan problem (see [86]), the solutions of which had, for most of the cases, stronger discontinuities than we have here. It might be more precise if we use "weak-singularity-separating" instead of singularity-separating. However, for simplicity we just keep the name of the method.

As pointed in Sect. 3.3.3, between American call and put options there exists the put–call symmetry relations. Using these relations, pricing a put option can be reduced to pricing a call option. There, the symmetry relations have been derived when American option problems are formulated as linear complementarity problems. Here, let us derive this conclusion when the problems are written as free-boundary problems. Let $P(S,t)$ stand for the price of an American put option. $P(S,t)$ should be the solution of the problem (3.16) on the domain $[S_f(t),\infty)\times[0,T]$ and equal $E-S$ on the domain $[0,S_f(t))\times[0,T]$. Let

$$\begin{cases}\eta=\dfrac{E^2}{S},\\[2mm] u(\eta,t)=\dfrac{EP(S,t)}{S},\end{cases}$$

then it is easy to see that $u(\eta,t)$ is the solution of the free-boundary problem

$$\begin{cases}\dfrac{\partial u}{\partial t}+\dfrac{1}{2}\sigma^2\eta^2\dfrac{\partial^2 u}{\partial\eta^2}+(D_0-r)\eta\dfrac{\partial u}{\partial\eta}-D_0u=0, & 0\le\eta\le\eta_f(t),\\ & 0\le t\le T,\\ u(\eta,T)=\max(\eta-E,0), & 0\le\eta\le\eta_f(T),\\ u\left(\eta_f(t),t\right)=\eta_f(t)-E, & 0\le t\le T,\\ \dfrac{\partial u}{\partial\eta}\left(\eta_f(t),t\right)=1, & 0\le t\le T,\\ \eta_f(T)=\max\left(E,D_0E/r\right) & \end{cases}\tag{9.4}$$

on the domain $[0,\eta_f(t)]\times[0,T]$; whereas on the domain $(\eta_f(t),\infty)\times[0,T]$,

$$u(\eta,t)=\eta-E.$$

As we can see, if the parameter r and the parameter D_0 in the problem (9.1) exchange their positions, then the problem (9.1) almost becomes the problem (9.4), except for the state variable. Therefore, $P(S,t)$ can be determined in the following way. First, understanding D_0 as r and r as D_0, we solve the problem (9.1) with the state variable η, instead of S, and get $u(\eta,t)$. Then, $P(S,t)$ is obtained by

$$P(S,t)=\frac{S}{E}u\left(\frac{E^2}{S},t\right).$$

That is, we find $P(S,t)$ by using one of the symmetry relations.

It is not always reasonable to assume the volatility to be a constant. If the volatility is thought as a function of S, namely, $\sigma=\sigma(S)$, then the formulation

(9.1) is still true after changing σ to $\sigma(S)$. Is the formulation (9.2) still true? The answer is no because in this case we do not have analytic solutions for European option. However, we can define

$$\overline{C}(S,t) = C(S,t) - c_{E}(S,t;\sigma(E))$$

on the domain $[0, S_f(t)] \times [0, T]$, where $c_{E}(S,t;\sigma(E))$ denotes the price of the European call option with $\sigma = \sigma(E)$. In this case, $\overline{C}(S,t)$ does not satisfy the Black–Scholes equation. Instead, it satisfies the following nonhomogeneous equation:

$$\frac{\partial \overline{C}}{\partial t} + \frac{1}{2}\sigma^2(S)S^2\frac{\partial^2 \overline{C}}{\partial S^2} + (r - D_0)S\frac{\partial \overline{C}}{\partial S} - r\overline{C} = f(S,t), \tag{9.5}$$

where $f(S,t)$ is given by the expression (8.79) in Sect. 8.3.2. For this case, the formulation is almost the same as the problem (9.2) except that the partial differential equation in problem (9.2) should be replaced by problem (9.5). Therefore, the singularity-separating method still works for American options with variable volatilities because the singularity is weakened.

The same idea still works for American barrier, Asian, and lookback options. In order to remove the weak singularity at $S = E$ and $t = T$, we can use the solutions of vanilla European options for American barrier, Asian, and lookback options. However, it will be better to compute numerically the differences between American and European barrier options and between American and European lookback options because the differences are smaller in these cases. Just like the vanilla option case, the partial differential equation that the differences satisfy in these cases is still the partial differential equation in the problem (9.2). For European Asian options, explicit solutions have not been found, and the partial differential equation for Asian options is different from vanilla options. Thus, when we apply the SSM, the resulting equation for Asian options differs slightly from barrier and lookback options. For average strike options with $\alpha = 1$, the singularity-separating method will still work, and the difference will be a solution of a nonhomogeneous partial differential equation problem with a weaker singularity. It is not difficult to derive the problem in this case, and we leave this as a problem for readers.

Consider put options on stocks paying dividends discretely. Suppose that the last dividend is paid at time t_{K}. This method can still be used from $t = T$ to $t = t_{K}$. From $t = t_{K}$ to $t = 0$, the solution is already smooth, so we can just compute the price of the American option directly. It is clear that in this way a quite good result still can be obtained on a coarse mesh.

9.1.2 Two-Dimensional Cases

Two-Factor Options. In the above, we have discussed the formulation of American options if the volatility is a constant or a function of S. Now let us look at the case both the price of asset and the volatility of the asset price are random variables. As we have done in Sect. 8.3.6, we call such an option a two-factor option. Here, we discuss how to formulate the American two-factor vanilla call option as a free boundary problem if $D_0 \neq 0$.

We still assume the asset price S and the stochastic volatility σ to follow the set of equations (8.98) and require the conditions (8.99) and (8.100) or the conditions (8.101) and (8.102) to hold.

Consider an American two-factor vanilla call option problem and let its value be $C(S, \sigma, t)$. As an American call option, it satisfies the condition:

$$C(S, \sigma, t) \geq \max(S - E, 0).$$

Because a European two-factor call option is a solution of the problem (8.105), the value of a two-factor vanilla American call option is a solution of the following linear complementarity problem:

$$\begin{cases} \min\left(-\dfrac{\partial C}{\partial t} - \mathbf{L}_{\mathbf{s},\sigma} C,\ C - G_c\right) = 0, & 0 \leq S,\ \sigma_l \leq \sigma \leq \sigma_u,\ t \leq T, \\ C(S, \sigma, T) = G_c(S, T), & 0 \leq S,\ \sigma_l \leq \sigma \leq \sigma_u, \end{cases} \tag{9.6}$$

where $\mathbf{L}_{\mathbf{s},\sigma}$ is given by the expression (8.104):

$$\begin{aligned} \mathbf{L}_{\mathbf{s},\sigma} = & \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + \rho\sigma S q \frac{\partial^2}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2}{\partial \sigma^2} \\ & + (r - D_0) S \frac{\partial}{\partial S} + (p - \lambda q) \frac{\partial}{\partial \sigma} - r, \end{aligned}$$

and

$$G_c(S, t) = \max(S - E, 0).$$

Consider the case $D_0 > 0$. Because

$$\frac{\partial G_c}{\partial t} + \mathbf{L}_{\mathbf{s},\sigma} G_c < 0 \quad \text{for} \quad S > \max(E, rE/D_0)$$

and

$$\frac{\partial G_c}{\partial t} + \mathbf{L}_{\mathbf{s},\sigma} G_c \geq 0 \quad \text{for} \quad S \leq \max(E, rE/D_0),$$

there exists a free boundary $S = S_f(\sigma, t)$ starting from the straight line $S = \max(E, rE/D_0)$ at $t = T$ in the (S, σ, t)-space, and the entire domain is divided into two regions by the free boundary. On the domain $(S_f(\sigma, t), \infty) \times [\sigma_l, \sigma_u] \times [0, T]$,

$$C(S, \sigma, t) = \max(S - E, 0);$$

whereas on $[0, S_f(\sigma, t)] \times [\sigma_l, \sigma_u] \times [0, T]$, $C(S, \sigma, t)$ is the solution of the following free-boundary problem:

$$\begin{cases} \dfrac{\partial C}{\partial t} + \mathbf{L}_{\mathbf{S},\sigma} C = 0, & 0 \le S \le S_f(\sigma,t), \\ & \sigma_l \le \sigma \le \sigma_u, \quad 0 \le t \le T, \\ C(S,\sigma,T) = \max(S-E,0), & 0 \le S \le S_f(\sigma,T), \\ & \sigma_l \le \sigma \le \sigma_u, \\ C\left(S_f(\sigma,t),\sigma,t\right) = S_f(\sigma,t) - E, & \sigma_l \le \sigma \le \sigma_u, \quad 0 \le t \le T, \\ \dfrac{\partial C\left(S_f(\sigma,t),\sigma,t\right)}{\partial S} = 1, & \sigma_l \le \sigma \le \sigma_u, \quad 0 \le t \le T, \\ S_f(\sigma,T) = \max(E, rE/D_0), & \sigma_l \le \sigma \le \sigma_u. \end{cases} \tag{9.7}$$

Just like the European two-factor option case, we let

$$\overline{C}(S,\sigma,t) = C(S,\sigma,t) - c_1(S,\sigma,t) \tag{9.8}$$

on the domain $[0, S_f(\sigma,t)] \times [\sigma_l, \sigma_u] \times [0,T]$. Here, $c_1(S,\sigma,t)$ is the same as the function $c(S,t)$ given by the formula (2.90) in Sect. 2.6.5, namely, the price of the vanilla European call option when σ is a constant. Thus, the difference $\overline{C}$ is the solution of the following free-boundary problem:

$$\begin{cases} \dfrac{\partial \overline{C}}{\partial t} + \mathbf{L}_{\mathbf{S},\sigma} \overline{C} = f(S,\sigma,t), & 0 \le S \le S_f(\sigma,t), & \sigma_l \le \sigma \le \sigma_u, \\ & & 0 \le t \le T, \\ \overline{C}(S,\sigma,T) = 0, & 0 \le S \le S_f(\sigma,T), & \sigma_l \le \sigma \le \sigma_u, \\ \overline{C}\left(S_f(\sigma,t),\sigma,t\right) = S_f(\sigma,t) - E - c_1\left(S_f(\sigma,t),\sigma,t\right), & & \sigma_l \le \sigma \le \sigma_u, \\ & & 0 \le t \le T, \\ \dfrac{\partial \overline{C}\left(S_f(\sigma,t),\sigma,t\right)}{\partial S} = 1 - \dfrac{\partial c_1\left(S_f(\sigma,t),\sigma,t\right)}{\partial S}, & & \sigma_l \le \sigma \le \sigma_u, \\ & & 0 \le t \le T, \\ S_f(\sigma,T) = \max(E, rE/D_0), & & \sigma_l \le \sigma \le \sigma_u, \end{cases} \tag{9.9}$$

where

$$f(S,\sigma,t) = -\rho\sigma Sq \frac{\partial^2 c_1}{\partial S \partial \sigma} - \frac{1}{2} q^2 \frac{\partial^2 c_1}{\partial \sigma^2} - (p - \lambda q) \frac{\partial c_1}{\partial \sigma},$$

$\dfrac{\partial c_1}{\partial \sigma}$, $\dfrac{\partial^2 c_1}{\partial \sigma^2}$, and $\dfrac{\partial^2 c}{\partial S \partial \sigma}$ being given by the set of expressions (8.108).

As we see from the problems (9.7) and (9.9), the derivative of $C(S,\sigma,t)$ with respect to S is discontinuous at the point $t = T$ and $S = E$, and the derivative of $\overline{C}(S,\sigma,t)$ with respect to S at $t = T$ is identically equal to zero. It is expected that $\overline{C}(S,\sigma,t)$ is smoother than $C(S,\sigma,t)$ even though in this case the singularity only becomes weaker but is not completely removed because of the term $\dfrac{\partial^2 c_1}{\partial S \partial \sigma}$ in $f(S, \sigma, t)$. Therefore, when a numerical method is used, the truncation error for the problem (9.9) will be smaller than the problem (9.7). This is why we consider the formulation (9.9) instead of the formulation (9.7).

The free-boundary problem (9.9) is defined on the domain $[0, S_f(\sigma, t)] \times [\sigma_l, \sigma_u] \times [0, T]$ and the free boundary $S_f(\sigma, t)$ is a moving and unknown boundary. In order to make the discretization simple, we introduce the following transformation

$$\begin{cases} \xi = \dfrac{S}{S_f(\sigma, t)}, \\ \sigma = \sigma, \\ \tau = T - t. \end{cases} \tag{9.10}$$

This transformation maps the domain

$$[0, S_f(\sigma, t)] \times [\sigma_l, \sigma_u] \times [0, T]$$

in the (S, σ, t)-space onto a new domain

$$[0, 1] \times [\sigma_l, \sigma_u] \times [0, T]$$

in the (ξ, σ, τ)-space and the moving boundary onto a plane under the new coordinate system. In the (ξ, σ, τ)-space, it is easy to construct numerical methods to solve the problem. Define

$$s_f(\sigma, \tau) = S_f(\sigma, t) = S_f(\sigma, T - \tau)$$

and

$$u(\xi, \sigma, \tau) = \overline{C}(S, \sigma, t) = \overline{C}\left(\xi s_f(\sigma, \tau), \sigma, T - \tau\right).$$

Among the derivatives of $\overline{C}$ and u, there are the following relations:

$$\begin{aligned}
\frac{\partial \overline{C}}{\partial t} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \tau}\frac{\partial \tau}{\partial t} = \frac{\xi}{s_f}\frac{\partial s_f}{\partial \tau}\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \tau}, \\
\frac{\partial \overline{C}}{\partial S} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial S} = \frac{1}{s_f}\frac{\partial u}{\partial \xi}, \\
\frac{\partial \overline{C}}{\partial \sigma} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial \sigma} + \frac{\partial u}{\partial \sigma} = -\left(\frac{\xi}{s_f}\frac{\partial s_f}{\partial \sigma}\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \sigma}\right), \\
\frac{\partial^2 \overline{C}}{\partial S^2} &= \frac{1}{s_f^2}\frac{\partial^2 u}{\partial \xi^2},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \overline{C}}{\partial S \partial \sigma} &= \frac{\partial}{\partial \sigma}\left(\frac{1}{s_f}\frac{\partial u}{\partial \xi}\right) = -\frac{1}{s_f^2}\frac{\partial s_f}{\partial \sigma}\frac{\partial u}{\partial \xi} + \frac{1}{s_f}\left(\frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial \sigma} + \frac{\partial^2 u}{\partial \xi \partial \sigma}\right) \\
&= -\frac{1}{s_f^2}\frac{\partial s_f}{\partial \sigma}\frac{\partial u}{\partial \xi} - \frac{\xi}{s_f^2}\frac{\partial s_f}{\partial \sigma}\frac{\partial^2 u}{\partial \xi^2} + \frac{1}{s_f}\frac{\partial^2 u}{\partial \xi \partial \sigma}, \\
\frac{\partial^2 \overline{C}}{\partial \sigma^2} &= -\left[\frac{\partial}{\partial \sigma}\left(\frac{\xi}{s_f}\frac{\partial s_f}{\partial \sigma}\right)\frac{\partial u}{\partial \xi} + \frac{\xi}{s_f}\frac{\partial s_f}{\partial \sigma}\left(\frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial \sigma} + \frac{\partial^2 u}{\partial \xi \partial \sigma}\right)\right. \\
&\quad \left. - \left(\frac{\partial^2 u}{\partial \sigma \partial \xi}\frac{\partial \xi}{\partial \sigma} + \frac{\partial^2 u}{\partial \sigma^2}\right)\right] \\
&= \left\{\left(\frac{\xi}{s_f}\right)^2\left(\frac{\partial s_f}{\partial \sigma}\right)^2\frac{\partial^2 u}{\partial \xi^2} - 2\frac{\xi}{s_f}\frac{\partial s_f}{\partial \sigma}\frac{\partial^2 u}{\partial \xi \partial \sigma} + \frac{\partial^2 u}{\partial \sigma^2}\right. \\
&\quad \left. + \left[2\frac{\xi}{s_f^2}\left(\frac{\partial s_f}{\partial \sigma}\right)^2 - \frac{\xi}{s_f}\frac{\partial^2 s_f}{\partial \sigma^2}\right]\frac{\partial u}{\partial \xi}\right\}.
\end{aligned}$$

Substituting them into the partial differential equation in the problem (9.9) yields

$$\frac{\partial u}{\partial \tau} = a_1 \xi^2 \frac{\partial^2 u}{\partial \xi^2} + a_2 \xi q \frac{\partial^2 u}{\partial \xi \partial \sigma} + a_3 q^2 \frac{\partial^2 u}{\partial \sigma^2} + a_4 \xi \frac{\partial u}{\partial \xi} + a_5 \frac{\partial u}{\partial \sigma} + a_6 u + a_7,$$

where

$$\begin{aligned}
a_1 &= \frac{1}{2}\sigma^2 - \frac{\rho \sigma q}{s_f}\frac{\partial s_f}{\partial \sigma} + \frac{1}{2}\left(\frac{q}{s_f}\frac{\partial s_f}{\partial \sigma}\right)^2, \\
a_2 &= \rho \sigma - \frac{q}{s_f}\frac{\partial s_f}{\partial \sigma}, \\
a_3 &= \frac{1}{2}, \\
a_4 &= \frac{1}{s_f}\frac{\partial s_f}{\partial \tau} + r - D_0 - (\rho \sigma q + p - \lambda q)\frac{1}{s_f}\frac{\partial s_f}{\partial \sigma} \\
&\quad + \left(\frac{q}{s_f}\frac{\partial s_f}{\partial \sigma}\right)^2 - \frac{1}{2}q^2\frac{1}{s_f}\frac{\partial^2 s_f}{\partial \sigma^2}, \\
a_5 &= p - \lambda q, \\
a_6 &= -r, \\
a_7 &= -f(S, \sigma, t) = -f\left(\xi s_f(\sigma, \tau), \sigma, T - \tau\right).
\end{aligned}$$

Therefore, noticing

$$\begin{cases}
c_1(S_f, \sigma, t) = S_f \mathrm{e}^{-D_0(T-t)} N(d_1) - E\mathrm{e}^{-r(T-t)} N(d_2), \\
\dfrac{\partial c_1(S_f, \sigma, t)}{\partial S} = \mathrm{e}^{-D_0(T-t)} N(d_1),
\end{cases}$$

we can rewrite the problem (9.9) as

$$\begin{cases} \dfrac{\partial u}{\partial \tau} = a_1\xi^2\dfrac{\partial^2 u}{\partial \xi^2} + a_2\xi q\dfrac{\partial^2 u}{\partial \xi \partial \sigma} + a_3 q^2\dfrac{\partial^2 u}{\partial \sigma^2} + a_4\xi\dfrac{\partial u}{\partial \xi} + a_5\dfrac{\partial u}{\partial \sigma} \\ \qquad + a_6 u + a_7, \qquad 0 \le \xi \le 1, \quad \sigma_l \le \sigma \le \sigma_u, \quad 0 \le \tau \le T, \\ u(\xi,\sigma,0) = 0, \qquad 0 \le \xi \le 1, \quad \sigma_l \le \sigma \le \sigma_u, \\ u(1,\sigma,\tau) = s_f(\sigma,\tau)\left[1 - \mathrm{e}^{-D_0\tau}N(d_1)\right] - E\left[1 - \mathrm{e}^{-r\tau}N(d_2)\right], \\ \qquad \sigma_l \le \sigma \le \sigma_u, \quad 0 \le \tau \le T, \\ \dfrac{\partial u(1,\sigma,\tau)}{\partial \xi} = s_f(\sigma,\tau)\left[1 - \mathrm{e}^{-D_0\tau}N(d_1)\right], \\ \qquad \sigma_l \le \sigma \le \sigma_u, \quad 0 \le \tau \le T, \\ s_f(\sigma,0) = \max\left(E, \dfrac{rE}{D_0}\right), \qquad \sigma_l \le \sigma \le \sigma_u, \end{cases} \tag{9.11}$$

where

$$d_1 = \left[\ln\frac{s_f\mathrm{e}^{-D_0\tau}}{E\mathrm{e}^{-r\tau}} + \frac{1}{2}\sigma^2\tau\right] \Big/ \left(\sigma\sqrt{\tau}\right) \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{\tau}.$$

Once we have the solution of the problem (9.11), $u(\xi,\sigma,\tau)$, we can get the value of the original American call option by

$$\begin{aligned} C(S,\sigma,t) &= \overline{C}(S,\ \sigma,\ t) + c_1(S,\ \sigma,\ t) \\ &= u\left(\frac{S}{s_f(\sigma,T-t)},\ \sigma,\ T-t\right) + c_1(S,\ \sigma,\ t). \end{aligned} \tag{9.12}$$

This method is called the singularity-separating method for American two-factor call options.

For two-factor vanilla American put options, the linear complementarity problem is

$$\begin{cases} \min\left(-\dfrac{\partial P}{\partial t} - \mathbf{L}_{\mathbf{S},\sigma}P,\ P - G_p\right) = 0, \quad 0 \le S,\ \sigma_l \le \sigma \le \sigma_u,\ t \le T, \\ P(S,\sigma,T) = G_p(S,T), \qquad 0 \le S,\ \sigma_l \le \sigma \le \sigma_u, \end{cases}$$

where

$$G_p(S,t) = \max(E - S, 0).$$

Introducing the transformation

$$\begin{cases} \eta = \dfrac{E^2}{S}, \\ \sigma = \sigma, \\ t = t, \\ u(\eta,\sigma,t) = \dfrac{EP(S,\sigma,t)}{S} \end{cases} \tag{9.13}$$

and noticing the following relations

$$
\begin{aligned}
\frac{\partial \eta}{\partial S} &= -\frac{E^2}{S^2}, & \frac{\partial P}{\partial t} &= \frac{S}{E}\frac{\partial u}{\partial t},\\
\frac{\partial P}{\partial S} &= \frac{u}{E} - \frac{E}{S}\frac{\partial u}{\partial \eta}, & \frac{\partial P}{\partial \sigma} &= \frac{S}{E}\frac{\partial u}{\partial \sigma},\\
\frac{\partial^2 P}{\partial S^2} &= \frac{E^3}{S^3}\frac{\partial^2 u}{\partial \eta^2}, & & \\
\frac{\partial^2 P}{\partial S \partial \sigma} &= \frac{1}{E}\frac{\partial u}{\partial \sigma} - \frac{E}{S}\frac{\partial^2 u}{\partial \eta \partial \sigma}, & & \\
\frac{\partial^2 P}{\partial \sigma^2} &= \frac{S}{E}\frac{\partial^2 u}{\partial \sigma^2}, & &
\end{aligned}
$$

we can convert the linear complementarity problem above into another linear complementarity problem

$$
\begin{cases}
\min\left(-\dfrac{\partial u}{\partial t} - \mathbf{L}_{\eta,\sigma} u,\ u - G_u\right) = 0, & 0 \le \eta,\ \sigma_l \le \sigma \le \sigma_u,\ t \le T,\\
u(\eta, \sigma, T) = G_u(\eta, T), & 0 \le \eta,\ \sigma_l \le \sigma \le \sigma_u,
\end{cases}
$$

where

$$G_u(\eta, t) = \max(\eta - E, 0)$$

and

$$
\begin{aligned}
\mathbf{L}_{\eta,\sigma} = {} & \frac{1}{2}\sigma^2\eta^2\frac{\partial^2}{\partial \eta^2} - \rho\sigma q\eta\frac{\partial^2}{\partial \eta \partial \sigma} + \frac{1}{2}q^2\frac{\partial^2}{\partial \sigma^2}\\
& + (D_0 - r)\eta\frac{\partial}{\partial \eta} + [p - (\lambda - \rho\sigma)q]\frac{\partial}{\partial \sigma} - D_0.
\end{aligned}
$$

This problem has the same form as the problem (9.6). The only difference is that r and D_0 are switched, and ρ and λ in the problem (9.6) are replaced by $-\rho$ and $\lambda - \rho\sigma$ here. Therefore, a put problem can be written as a call problem.

Let $C(S, \sigma, t; a, b, c, d)$ and $P(S, \sigma, t; a, b, c, d)$ denote the prices of American call and put options and $S_{cf}(\sigma, t; a, b, c, d)$ and $S_{pf}(\sigma, t; a, b, c, d)$ be their optimal exercise prices. Here, a, b, c, and d are parameters (or parameter functions) for the risk-free interest rate r, dividend yield rate D_0, correlation coefficient ρ, and market price of volatility risk λ, respectively. Then, what we have described above can be written as a relation between the American two-factor vanilla put and call options:

$$
\begin{cases}
P(S, \sigma, t; a, b, c, d) = \dfrac{S}{E} C\left(\dfrac{E^2}{S}, \sigma, t; b, a, -c, d - c\sigma\right),\\
S_{pf}(\sigma, t; a, b, c, d) = E^2 / S_{cf}(\sigma, t; b, a, -c, d - c\sigma).
\end{cases}
$$

If we let

$$\eta = E^2/S, \quad \bar{c} = -c$$

and

$$\bar{d} = d - c\sigma = d + \bar{c}\sigma,$$

then the first relation above can be written as

$$P\left(\frac{E^2}{\eta}, \sigma, t; a, b, -\bar{c}, \bar{d} - \bar{c}\sigma\right) = \frac{E}{\eta} C\left(\eta, \sigma, t; b, a, \bar{c}, \bar{d}\right)$$

or

$$C\left(S, \sigma, t; a, b, c, d\right) = \frac{S}{E} P\left(\frac{E^2}{S}, \sigma, t; b, a, -c, d - c\sigma\right).$$

The second relation can be written in a symmetric form

$$S_{pf}(\sigma, t; a, b, c, d) \times S_{cf}(\sigma, t; b, a, -c, d - c\sigma) = E^2.$$

Therefore, we can have the following relations:

$$\begin{cases} P(S, \sigma, t; a, b, c, d) = \dfrac{S}{E} C\left(\dfrac{E^2}{S}, \sigma, t; b, a, -c, d - c\sigma\right), \\ C\left(S, \sigma, t; a, b, c, d\right) = \dfrac{S}{E} P\left(\dfrac{E^2}{S}, \sigma, t; b, a, -c, d - c\sigma\right), \\ S_{pf}(\sigma, t; a, b, c, d) \times S_{cf}(\sigma, t; b, a, -c, d - c\sigma) = E^2, \end{cases} \tag{9.14}$$

which in this book are referred to as the call–put symmetry relations between American two-factor vanilla call and put options. Thus, if we have a code for one type of option, call or put, then in order to calculate another type of option, we only need to make a little change.

The free-boundary problem for a call option is defined on a finite domain and that for a put option is on an infinite domain. Consequently, it will be natural to write a code for call options and calculate a put option as a call option.

Two-Factor Convertible Bonds. Another example of American-style derivatives depending on two random variables is two-factor convertible bonds. Let $B_c(S, r, t)$ be the price of such a bond. As was pointed out in Sect. 5.7, the computational domain of a two-factor convertible bond problem can be divided into two parts. On the domain $(S_f(r, t), \infty) \times [r_l, r_u] \times [0, T]$,

$$B_c(S, r, t) = \max(Z, nS);$$

whereas on the domain $[0, S_f(r, t)] \times [r_l, r_u] \times [0, T]$, $B_c(S, r, t)$ is the solution of the free-boundary problem:

$$\begin{cases} \dfrac{\partial B_c}{\partial t} + \mathbf{L_{s,r}} B_c + kZ = 0, & 0 \le S \le S_f(r,t), \quad r_l \le r \le r_u, \\ & 0 \le t \le T, \\ B_c(S,r,T) = \max(Z, nS), & 0 \le S \le S_f(r,T), \quad r_l \le r \le r_u, \\ B_c\left(S_f(r,t), r, t\right) = nS_f(r,t), & r_l \le r \le r_u, \quad 0 \le t \le T, \\ \dfrac{\partial B_c}{\partial S}\left(S_f(r,t), r, t\right) = n, & r_l \le r \le r_u, \ \ 0 \le t \le T, \\ S_f(r,T) = \max\left(\dfrac{Z}{n}, \dfrac{kZ}{D_0 n}\right), & r_l \le r \le r_u. \end{cases} \tag{9.15}$$

where

$$\mathbf{L_{s,r}} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + \rho\sigma S w \frac{\partial^2}{\partial S \partial r} + \frac{1}{2} w^2 \frac{\partial^2}{\partial r^2} + (r - D_o) S \frac{\partial}{\partial S} + (u - \lambda w)\frac{\partial}{\partial r} - r.$$

Let $b_c(S, r, t)$ be the solution of the problem

$$\begin{cases} \dfrac{\partial b_c}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 b_c}{\partial S^2} + (r - D_0) S \dfrac{\partial b_c}{\partial S} - r b_c = 0, & 0 \le S, \\ & 0 \le t \le T, \\ b_c(S,T) = \max(Z, nS) = n \max(S - Z/n, 0) + Z, & 0 \le S, \end{cases} \tag{9.16}$$

where σ, r, and D_0 are constants. This problem has the following solution:

$$b_c(S, r, t) = nc(S, t; Z/n) + \mathrm{e}^{-r(T-t)} Z,$$

where $c(S, t; Z/n)$ is the price of a European call option with an exercise price $E = Z/n$. Define

$$\overline{B}_c(S, r, t) = B_c(S, r, t) - b_c(S, r, t). \tag{9.17}$$

For $\overline{B}_c(S, r, t)$, the free boundary problem is

$$\begin{cases} \dfrac{\partial \overline{B}_c}{\partial t} + \mathbf{L_{s,r}} \overline{B}_c + kZ = f(S,r,t), & 0 \le S \le S_f(r,t), \\ & r_l \le r \le r_u, \ \ 0 \le t \le T, \\ \overline{B}_c(S,r,T) = 0, & 0 \le S \le S_f(r,T), \ \ r_l \le r \le r_u, \\ \overline{B}_c\left(S_f(r,t), r, t\right) = nS_f(r,t) - b_c\left(S_f(r,t), r, t\right), & \\ & r_l \le r \le r_u, \ \ 0 \le t \le T, \\ \dfrac{\partial \overline{B}_c}{\partial S}\left(S_f(r,t), r, t\right) = n - \dfrac{\partial b_c\left(S_f(r,t), r, t\right)}{\partial S}, & \\ & r_l \le r \le r_u, \ \ 0 \le t \le T, \\ S_f(r,T) = \max\left(\dfrac{Z}{n}, \dfrac{kZ}{D_0 n}\right), & r_l \le r \le r_u, \end{cases} \tag{9.18}$$

where

$$f(S,r,t) = -\rho\sigma Sw\frac{\partial^2 b_c}{\partial S\partial r} - \frac{1}{2}w^2\frac{\partial^2 b_c}{\partial r^2} - (u-\lambda w)\frac{\partial b_c}{\partial r}.$$

In order to make the discretization easy, we introduce the following transformation

$$\begin{cases} \xi = \dfrac{S}{S_f(r,t)}, \\ \bar{r} = \dfrac{r-r_l}{r_u-r_l}, \\ \tau = T-t. \end{cases} \tag{9.19}$$

This transformation maps the domain

$$[0, S_f(r,t)] \times [r_l, r_u] \times [0,T]$$

in the (S,r,t)-space onto the domain

$$[0,1] \times [0,1] \times [0,T]$$

in the $(\xi,\bar{r},\tau)$-space. We also introduce two new variables u and s_f defined by

$$\begin{cases} u(\xi,\bar{r},\tau) = \dfrac{\overline{B}_c(S,r,t)}{Z}, \\ s_f(\bar{r},\tau) = \dfrac{S_f(r,t)}{Z/n} \end{cases} \tag{9.20}$$

and let

$$v(\xi,\bar{r},\tau) = b_c(S,r,t)/Z.$$

For v we have

$$\begin{aligned} v(\xi,\bar{r},\tau) &= nc(S,t;Z/n)/Z + \mathrm{e}^{-r(T-t)} \\ &= (nS/Z)\mathrm{e}^{-D_0(T-t)}N(d_1) - \mathrm{e}^{-r(T-t)}N(d_2) + \mathrm{e}^{-r(T-t)} \\ &= \xi s_f(\bar{r},\tau)\mathrm{e}^{-D_0\tau}N(d_1) + \mathrm{e}^{-r\tau}N(-d_2), \end{aligned}$$

where

$$\begin{aligned} d_1 &= \left[\ln\frac{S\mathrm{e}^{(r-D_0)(T-t)}}{Z/n} + \frac{1}{2}\sigma^2(T-t)\right] \Bigg/ \left(\sigma\sqrt{T-t}\right) \\ &= \left[\ln\left(\xi s_f(\bar{r},\tau)\mathrm{e}^{(r-D_0)\tau}\right) + \frac{1}{2}\sigma^2\tau\right] \Bigg/ \left(\sigma\sqrt{\tau}\right), \\ d_2 &= d_1 - \sigma\sqrt{\tau}. \end{aligned}$$

Thus, v can be expressed as a function of $\xi s_f(\bar{r},\tau)$ and τ. Because

$$\overline{B}_c(S,r,t) = Zu(\xi,\bar{r},\tau) = Zu\left(\frac{nS}{Zs_f\left(\dfrac{r-r_l}{r_u-r_l},T-t\right)},\ \frac{r-r_l}{r_u-r_l},\ T-t\right),$$

we have

$$\begin{aligned}
\frac{\partial \overline{B}_c}{\partial t} &= Z\left(-\frac{\partial u}{\partial \tau} + \frac{\partial u}{\partial \xi}\frac{\xi}{s_f}\frac{\partial s_f}{\partial \tau}\right),\\
\frac{\partial \overline{B}_c}{\partial S} &= \frac{\partial u}{\partial \xi}\frac{n}{s_f},\\
\frac{\partial \overline{B}_c}{\partial r} &= Z\left(-\frac{\partial u}{\partial \xi}\frac{\xi}{s_f}\frac{\partial s_f}{\partial \bar{r}} + \frac{\partial u}{\partial \bar{r}}\right)\frac{1}{r_u-r_l},\\
\frac{\partial^2 \overline{B}_c}{\partial S^2} &= \frac{1}{Z}\frac{\partial^2 u}{\partial \xi^2}\left(\frac{n}{s_f}\right)^2,\\
\frac{\partial^2 \overline{B}_c}{\partial S\partial r} &= \left(-\frac{\partial^2 u}{\partial \xi^2}\frac{n\xi}{s_f^2}\frac{\partial s_f}{\partial \bar{r}} + \frac{\partial^2 u}{\partial \xi\partial \bar{r}}\frac{n}{s_f} - \frac{\partial u}{\partial \xi}\frac{n}{s_f^2}\frac{\partial s_f}{\partial \bar{r}}\right)\frac{1}{r_u-r_l},\\
\frac{\partial^2 \overline{B}_c}{\partial r^2} &= Z\left\{\frac{\partial^2 u}{\partial \xi^2}\left(\frac{\xi}{s_f}\frac{\partial s_f}{\partial \bar{r}}\right)^2 - 2\frac{\partial^2 u}{\partial \xi\partial \bar{r}}\frac{\xi}{s_f}\frac{\partial s_f}{\partial \bar{r}}\right.\\
&\qquad \left. + \frac{\partial u}{\partial \xi}\left[2\frac{\xi}{s_f^2}\left(\frac{\partial s_f}{\partial \bar{r}}\right)^2 - \frac{\xi}{s_f}\frac{\partial^2 s_f}{\partial \bar{r}^2}\right] + \frac{\partial^2 u}{\partial \bar{r}^2}\right\}\left(\frac{1}{r_u-r_l}\right)^2.
\end{aligned}$$

Substituting these expressions into the problem (9.18) yields

$$\begin{cases}
\dfrac{\partial u}{\partial \overline{\tau}} = \mathbf{L}_{\xi,\bar{\mathbf{r}}}u + a_7, & 0 \le \xi \le 1, \quad 0 \le \bar{r} \le 1, \quad 0 \le \tau \le T,\\
u(\xi,\bar{r},0) = 0, & 0 \le \xi \le 1, \quad 0 \le \bar{r} \le 1,\\
u(1,\bar{r},\tau) = s_f(\bar{r},\tau) - v(1,\bar{r},\tau), & 0 \le \bar{r} \le 1, \quad 0 \le \tau \le T,\\
\dfrac{\partial u}{\partial \xi}(1,\bar{r},\tau) = s_f(\bar{r},\tau) - \dfrac{\partial v}{\partial \xi}(1,\bar{r},\tau), & 0 \le \bar{r} \le 1, \quad 0 \le \tau \le T,\\
s_f(\bar{r},0) = \max\left(1,k/D_0\right), & 0 \le \bar{r} \le 1,
\end{cases} \tag{9.21}$$

where

$$\begin{aligned}
\mathbf{L}_{\xi,\bar{\mathbf{r}}} &= a_1\xi^2\frac{\partial^2}{\partial \xi^2} + a_2\xi w\frac{\partial^2}{\partial \xi\partial \bar{r}} + a_3 w^2\frac{\partial^2}{\partial \bar{r}^2} + \left(a_4 + \frac{1}{s_f}\frac{\partial s_f}{\partial \tau}\right)\xi\frac{\partial}{\partial \xi}\\
&\quad + a_5\frac{\partial}{\partial \bar{r}} + a_6,
\end{aligned}$$

$$\begin{aligned}
a_1 &= \frac{1}{2}\sigma^2 - \rho\sigma w \frac{1}{s_f(r_u-r_l)}\frac{\partial s_f}{\partial \bar{r}} + \frac{1}{2}w^2\left[\frac{1}{s_f(r_u-r_l)}\frac{\partial s_f}{\partial \bar{r}}\right]^2, \\
a_2 &= \frac{1}{r_u-r_l}\left[\rho\sigma - \frac{w}{s_f(r_u-r_l)}\frac{\partial s_f}{\partial \bar{r}}\right], \\
a_3 &= \frac{1}{2(r_u-r_l)^2}, \\
a_4 &= r - D_0 - \frac{1}{s_f(r_u-r_l)}\frac{\partial s_f}{\partial \bar{r}}(\rho\sigma w + u - \lambda w) \\
&\quad + \frac{1}{2}w^2\left\{2\left[\frac{1}{s_f(r_u-r_l)}\frac{\partial s_f}{\partial \bar{r}}\right]^2 - \frac{1}{s_f(r_u-r_l)^2}\frac{\partial^2 s_f}{\partial \bar{r}^2}\right\}, \\
a_5 &= \frac{u-\lambda w}{r_u-r_l}, \\
a_6 &= -r, \\
a_7 &= k + \rho\sigma S w \frac{\partial^2 v}{\partial S \partial r} + \frac{1}{2}w^2\frac{\partial^2 v}{\partial r^2} + (u-\lambda w)\frac{\partial v}{\partial r}.
\end{aligned}$$

We will refer to this method as the singularity-separating method for two-factor convertible bonds.

In the problem (9.21), Z and n are not involved. That is, the solution of the problem, $u(\xi, \bar{r}, \tau)$ and $s_f(\bar{r}, \tau)$, does not depend on Z or n. The problem (9.21) is called the problem for a standard convertible bond.

If the asset price S, the asset price volatility σ and the interest rate r are all considered as random variables, then we have American three-factor option problems and three-factor convertible bond problems. It is not difficult to generalize the method here to such three-dimensional problems.

9.2 Implicit Finite-Difference Methods

9.2.1 Solution of One-Dimensional Problems

The problem (9.3) can be solved by different numerical methods, for example, explicit finite-difference methods, implicit finite-difference methods, pseudo-spectral methods, and so forth. In this book, we only discuss the implicit finite-difference methods and the pseudo-spectral methods. In this subsection, we discuss how to use implicit finite-difference methods to solve free-boundary problem (9.3).

As we have pointed out, the problem we are going to solve is defined on $[0,1]\times[0,T]$ on the (ξ,τ)-plane. For simplicity, we assume that we still use the equidistant mesh given by the set of expressions (8.2). Let u_m^n stand for the value of u at the points $\xi = \xi_m \equiv m\Delta\xi$ and $\tau = \tau^n \equiv n\Delta\tau$, and s_f^n represent the value of s_f at $\tau = \tau^n$. At time $t = 0$, the function u and s_f are known, i.e., $u_m^0, m = 0, 1, \cdots, M$ and s_f^0 are known. We need to find $u_m^n, m = 0, 1, \cdots, M$ and s_f^n, $n = 1, 2, \cdots, N$.

The partial differential equation in the problem (9.3) can be discretized by

$$\begin{aligned}&\frac{u_m^{n+1}-u_m^n}{\Delta\tau}\\ &=\frac{1}{2}\left[k_2m^2\left(u_{m+1}^{n+1}-2u_m^{n+1}+u_{m-1}^{n+1}\right)+\frac{k_1m}{2}\left(u_{m+1}^{n+1}-u_{m-1}^{n+1}\right)-k_0u_m^{n+1}\right]\\ &\quad+\frac{1}{2}\left[k_2m^2\left(u_{m+1}^{n}-2u_m^{n}+u_{m-1}^{n}\right)+\frac{k_1m}{2}\left(u_{m+1}^{n}-u_{m-1}^{n}\right)-k_0u_m^{n}\right]\\ &\quad+\frac{s_f^{n+1}-s_f^n}{\left(s_f^{n+1}+s_f^n\right)\Delta\tau}\left[\frac{m}{2}\left(u_{m+1}^{n+1}-u_{m-1}^{n+1}\right)+\frac{m}{2}\left(u_{m+1}^{n}-u_{m-1}^{n}\right)\right]\end{aligned} \tag{9.22}$$

at $m=0,1,2,\cdots,M-1$, for $n=0,1,\cdots,N-1$. Here, in all coefficients, $\xi=m\Delta\xi$ and $\tau=(n+1/2)\Delta\tau$, so from Sect. 6.1, we know that the scheme has a truncation error of $O(\Delta\tau^2,\Delta\xi^2)$. At $m=0$, the equation actually becomes

$$\frac{u_0^{n+1}-u_0^n}{\Delta\tau}=\frac{-k_0}{2}\left(u_0^{n+1}+u_0^n\right),$$

therefore, u_{-1}^n and u_{-1}^{n+1} do not appear in the equations. The boundary conditions at $\xi=1$ in the problem (9.3) can be replaced by

$$u_M^{n+1}=g(s_f^{n+1},\tau^{n+1}), \tag{9.23}$$

and

$$\frac{3u_M^{n+1}-4u_{M-1}^{n+1}+u_{M-2}^{n+1}}{2\Delta\xi}=h\left(s_f^{n+1},\tau^{n+1}\right). \tag{9.24}$$

Here, the condition (9.23) is exact, and the truncation error of the approximate boundary condition (9.24) is $O(\Delta\xi^2)$ because the first derivative is approximated by a one-sided second-order difference scheme. In the system (9.22)–(9.24), if u_m^n, $m=0,1,\cdots,M$ and s_f^n are given, then there are $M+2$ unknowns: $u_m^{n+1}, m=0,1,\cdots,M$ and s_f^{n+1}. The number of equations in the system is also $M+2$. Therefore, we can determine $u_m^{n+1}, m=0,1,\cdots,M$ and s_f^{n+1} from this system. From the initial conditions in problem (9.3), the second and the fifth equations there, we can obtain

$$u_m^0=0,\quad m=0,1,\cdots,M$$

and

$$s_f^0=\max(1,r/D_0).$$

Consequently, starting from $n=0$, we can find the solution at τ^{n+1} from the solution at τ^n successively.

However, the system is a nonlinear one, so we cannot find the solution directly. In order to find the solution of the system, we use iteration methods. For example, Eqs. (9.22)–(9.24) can be written as

$$\begin{aligned}
&\frac{u_m^{(j)} - u_m^n}{\Delta\tau}\\
&= \frac{1}{2}\left[k_2 m^2\left(u_{m+1}^{(j)} - 2u_m^{(j)} + u_{m-1}^{(j)}\right) + \frac{k_1 m}{2}\left(u_{m+1}^{(j)} - u_{m-1}^{(j)}\right) - k_0 u_m^{(j)}\right]\\
&\quad+\frac{1}{2}\left[k_2 m^2\left(u_{m+1}^n - 2u_m^n + u_{m-1}^n\right) + \frac{k_1 m}{2}\left(u_{m+1}^n - u_{m-1}^n\right) - k_0 u_m^n\right]\\
&\quad+\frac{s_f^{(j)} - s_f^n}{\left(s_f^{(j-1)} + s_f^n\right)\Delta\tau}\left[\frac{m}{2}\left(u_{m+1}^{(j-1)} - u_{m-1}^{(j-1)}\right) + \frac{m}{2}\left(u_{m+1}^n - u_{m-1}^n\right)\right],\\
&\qquad\qquad m = 0, 1, \cdots, M-1,
\end{aligned} \tag{9.25}$$

$$u_M^{(j)} = g\left(s_f^{(j)}, \tau^{n+1}\right), \tag{9.26}$$

and

$$\frac{3u_M^{(j)} - 4u_{M-1}^{(j)} + u_{M-2}^{(j)}}{2\Delta\xi} = h\left(s_f^{(j)}, \tau^{n+1}\right), \tag{9.27}$$

where $u_m^{(j)}, s_f^{(j)}$ are the j-th iteration values of u_m^{n+1}, s_f^{n+1} respectively. In order to start an iteration, we set $u_m^{(0)} = u_m^n, m = 0, 1, \cdots, M$ and $s_f^{(0)} = s_f^n$. The system consisting of Eqs. (9.25)–(9.27) is linear for $u_m^{(j)}, m = 0, 1, \cdots, M$, and nonlinear for $s_f^{(j)}$. This system can be solved by a modified LU decomposition method described below.

The system of equations (9.25) can be rewritten as

$$\begin{aligned}
&-\frac{1}{2}\left(k_2 m^2 + \frac{k_1 m}{2}\right)\Delta\tau u_{m+1}^{(j)} + \left[1 + \left(k_2 m^2 + \frac{k_0}{2}\right)\Delta\tau\right] u_m^{(j)}\\
&-\frac{1}{2}\left(k_2 m^2 - \frac{k_1 m}{2}\right)\Delta\tau u_{m-1}^{(j)}\\
&-\frac{1}{\left(s_f^{(j-1)} + s_f^n\right)}\left[\frac{m}{2}\left(u_{m+1}^{(j-1)} - u_{m-1}^{(j-1)}\right) + \frac{m}{2}\left(u_{m+1}^n - u_{m-1}^n\right)\right] s_f^{(j)}\\
&= \frac{1}{2}\left(k_2 m^2 + \frac{k_1 m}{2}\right)\Delta\tau u_{m+1}^n + \left[1 - \left(k_2 m^2 + \frac{k_0}{2}\right)\Delta\tau\right] u_m^n\\
&\quad+\frac{1}{2}\left(k_2 m^2 - \frac{k_1 m}{2}\right)\Delta\tau u_{m-1}^n\\
&\quad-\frac{1}{\left(s_f^{(j-1)} + s_f^n\right)}\left[\frac{m}{2}\left(u_{m+1}^{(j-1)} - u_{m-1}^{(j-1)}\right) + \frac{m}{2}\left(u_{m+1}^n - u_{m-1}^n\right)\right] s_f^n,\\
&\qquad\qquad m = 0, 1, \cdots, M-1.
\end{aligned} \tag{9.28}$$

When $m = 0$, the equation simply becomes:

$$\left(1 + \frac{k_0}{2}\Delta\tau\right) u_0^{(j)} = \left(1 - \frac{k_0}{2}\Delta\tau\right) u_0^n.$$

Thus, no iteration for u_0^{n+1} is needed, and

$$u_0^{n+1} = \frac{1 - \dfrac{k_0}{2}\Delta\tau}{1 + \dfrac{k_0}{2}\Delta\tau} u_0^n.$$

Furthermore, noticing $u_0^0 = 0$, we have $u_0^{n+1} = 0, n = 0, 1, \cdots, N-1$. Therefore, u_0^n can be understood as a given quantity, i.e., for each iteration, there are $M+1$ unknowns: $u_m^{(j)}, m = 1, 2, \cdots, M$, and $s_f^{(j)}$. The $M+1$ unknowns satisfy a system in the following form:

$$\begin{cases} b_1 u_1^{(j)} + c_1 u_2^{(j)} + e_1 s_f^{(j)} = f_1, \\ a_m u_{m-1}^{(j)} + b_m u_m^{(j)} + c_m u_{m+1}^{(j)} + e_m s_f^{(j)} = f_m, \ m{=}2, 3, \cdots, M-1, \\ u_{\scriptscriptstyle M}^{(j)} = g\left(s_f^{(j)}, \tau^{n+1}\right), \\ d_{\scriptscriptstyle M} u_{M-2}^{(j)} + a_{\scriptscriptstyle M} u_{M-1}^{(j)} + b_{\scriptscriptstyle M} u_{\scriptscriptstyle M}^{(j)} = h\left(s_f^{(j)}, \tau^{n+1}\right). \end{cases} \tag{9.29}$$

The top $M-1$ equations of this system are linear equations for $u_m^{(j)}, m = 1, 2, \cdots, M$ and $s_f^{(j)}$. Let us rewrite the first equation as

$$u_1^{(j)} = \alpha_1 u_2^{(j)} + \beta_1 s_f^{(j)} + \gamma_1,$$

where

$$\alpha_1 = -c_1/b_1, \quad \beta_1 = -e_1/b_1, \quad \text{and} \quad \gamma_1 = f_1/b_1.$$

Suppose we have a relation in the form

$$u_{m-1}^{(j)} = \alpha_{m-1} u_m^{(j)} + \beta_{m-1} s_f^{(j)} + \gamma_{m-1}.$$

Substituting this relation into the second equation in the system (9.29) and solving the equation for $u_m^{(j)}$, we have

$$u_m^{(j)} = \alpha_m u_{m+1}^{(j)} + \beta_m s_f^{(j)} + \gamma_m,$$

where

$$\alpha_m = \frac{-c_m}{b_m + a_m \alpha_{m-1}}, \quad \beta_m = -\frac{e_m + a_m \beta_{m-1}}{b_m + a_m \alpha_{m-1}}, \quad \text{and} \quad \gamma_m = \frac{f_m - a_m \gamma_{m-1}}{b_m + a_m \alpha_{m-1}}.$$

This procedure can be done for $m = 2, 3, \cdots, M-1$ successively. Therefore, the first and second equations in the system (9.29) are equivalent to the following relation

$$u_m^{(j)} = \alpha_m u_{m+1}^{(j)} + \beta_m s_f^{(j)} + \gamma_m, \quad m = 1, 2, \cdots, M-1, \tag{9.30}$$

where

$$\begin{cases} \alpha_m = \dfrac{-c_m}{b_m + a_m \alpha_{m-1}}, \\ \beta_m = -\dfrac{e_m + a_m \beta_{m-1}}{b_m + a_m \alpha_{m-1}}, \\ \gamma_m = \dfrac{f_m - a_m \gamma_{m-1}}{b_m + a_m \alpha_{m-1}}. \end{cases} \tag{9.31}$$

Here, we define $a_1 = 0$. Using the two relations in the system (9.30) with $m = M-2$ and $M-1$, we can eliminate $u_{M-2}^{(j)}$ and $u_{M-1}^{(j)}$ in the last equation of the system (9.29) and obtain

$$\begin{aligned} & d_M \left[\alpha_{M-2} \alpha_{M-1} u_M^{(j)} + (\alpha_{M-2} \beta_{M-1} + \beta_{M-2}) s_f^{(j)} + \alpha_{M-2} \gamma_{M-1} + \gamma_{M-2} \right] \\ & + a_M \left(\alpha_{M-1} u_M^{(j)} + \beta_{M-1} s_f^{(j)} + \gamma_{M-1} \right) + b_M u_M^{(j)} \\ = & \, h\left(s_f^{(j)}, \tau^{n+1} \right). \end{aligned}$$

Substituting the third equation in the system (9.29) into this equation yields

$$\begin{aligned} & [(d_M \alpha_{M-2} + a_M) \alpha_{M-1} + b_M] \, g\left(s_f^{(j)}, \tau^{n+1} \right) \\ & + [d_M (\alpha_{M-2} \beta_{M-1} + \beta_{M-2}) + a_M \beta_{M-1}] \, s_f^{(j)} \\ & + d_M (\alpha_{M-2} \gamma_{M-1} + \gamma_{M-2}) + a_M \gamma_{M-1} \\ = & \, h\left(s_f^{(j)}, \tau^{n+1} \right). \end{aligned}$$

This is an equation for $s_f^{(j)}$, and we can use the secant method to get its solution. In order to start the secant method, we need two approximate values of $s_f^{(j)}$. For s_f^1, we can take $s_f^{(0)} = s_f^0$ and $s_f^{(1)} = s_f^0 + \varepsilon$ as the two initial values. Here, ε is a proper positive number because $s_f(t)$ is an increasing function in τ for an American call option. For $s_f^j, j = 2, 3, \cdots, N$, we can take

$$s_f^{(0)} = s_f^{j-1} + 0.75 \cdot \frac{s_f^{j-1} - s_f^{j-2}}{\tau^{j-1} - \tau^{j-2}} (\tau^j - \tau^{j-1})$$

and

$$s_f^{(1)} = s_f^{j-1} + 1.5 \cdot \frac{s_f^{j-1} - s_f^{j-2}}{\tau^{j-1} - \tau^{j-2}}(\tau^j - \tau^{j-1})$$

as the two initial values for s_f^j.

After $s_f^{(j)}$ is found, we can obtain $u_M^{(j)}$ from the third equation in the system (9.29) and $u_m^{(j)}$ from the system (9.30), $m = M-1, M-2, \cdots, 1$, successively. From the system (9.28), we know that a_m, b_m and c_m do not depend on $u_m^{(j-1)}$ and $s_f^{(j-1)}$. Thus, a_m, b_m, and c_m remain unchanged during the iteration. Furthermore, from the expression of α_m in the set of expressions (9.31), we know that α_m and $b_m + a_m\alpha_{m-1}$ also remain unchanged. f_m in the system (9.29) is a sum of two parts:

$$\frac{1}{2}\left(k_2 m^2 + \frac{k_1 m}{2}\right)\Delta\tau u_{m+1}^n + \left[1 - \left(k_2 m^2 + \frac{k_0}{2}\right)\Delta\tau\right]u_m^n$$
$$+\frac{1}{2}\left(k_2 m^2 - \frac{k_1 m}{2}\right)\Delta\tau u_{m-1}^n$$

and

$$\frac{-1}{\left(s_f^{(j-1)} + s_f^n\right)}\left[\frac{m}{2}\left(u_{m+1}^{(j-1)} - u_{m-1}^{(j-1)}\right) + \frac{m}{2}\left(u_{m+1}^n - u_{m-1}^n\right)\right]s_f^n.$$

The first part also does not depend on $u_m^{(j-1)}$ and $s_f^{(j-1)}$. In order to make the computation efficient, all these unchanged quantities during the iteration should be computed once and stored for future use.

The iteration (9.25)–(9.27) will give a second-order accuracy if two iterations are performed. In fact, $u_m^{(1)}$ and $s_f^{(1)}$ are solutions of a first-order scheme, and $u_m^{(2)}$ and $s_f^{(2)}$ are solutions of an improved Euler method in the τ-direction, which gives second-order accuracy in the τ-direction (see any book on numerical methods for ordinary differential equations). This scheme is always second order in the ξ-direction, so the results have an accuracy of $O\left(\Delta\xi^2, \Delta\tau^2\right)$.

The way of solving the system (9.22)–(9.24) is not unique. If s_f^{n+1} is given, then the system consisting of Eqs. (9.22) and (9.23) is a system with $M+1$ linear equations and $M+1$ unknowns $u_m^{(n+1)}$, $m = 0, 1, \cdots, M$. Therefore, this system determines the dependence of u_m^{n+1} on s_f^{n+1}, i.e., the functions $u_m^{n+1}(s_f^{n+1})$, $m = 0, 1, \cdots, M$. Substituting the three functions $u_{M-2}^{n+1}(s_f^{n+1})$, $u_{M-1}^{n+1}(s_f^{n+1})$, $u_M^{n+1}(s_f^{n+1})$ into Eq. (9.24), we have an equation for s_f^{n+1}:

$$f(s_f^{n+1}) \equiv \frac{3u_M^{n+1}(s_f^{n+1}) - 4u_{M-1}^{n+1}(s_f^{n+1}) + u_{M-2}^{n+1}(s_f^{n+1})}{2\Delta\xi} - h\left(s_f^{n+1}, \tau^{n+1}\right) = 0. \tag{9.32}$$

This equation can be solved by the secant method. When using the secant method, we need to evaluate $f(s_f^{n+1})$ for a given s_f^{n+1}. This can be done as follows. Let s_f^{n+1} in Eqs. (9.22) and (9.23) take the given value, then solve the linear system consisting of Eqs. (9.22) and (9.23) by the LU decomposition method described in Sect. 6.2.1. Substituting the value of $u_M^{n+1}, u_{M-1}^{n+1}, u_{M-2}^{n+1}$ into Eq. (9.32) yields the value $f(s_f^{n+1})$. As long as we have $f(s_f^{n+1})$ for two different s_f^{n+1}, we can start the iteration. When $f(s_f^{n+1})$ is very close to zero for some given s_f^{n+1}, we obtain the solution for s_f^{n+1}, and the solution of the linear system corresponding to this s_f^{n+1} gives the values for $u_m^{n+1}, m = 0, 1, \cdots, M$. This is another way to solve the system (9.22)–(9.24).

Wu and Kwok (see [85]) suggested a similar scheme to system (9.22)–(9.24). The main difference is that they computed the option price directly.

9.2.2 Solution of Greeks

In practice, we usually need to know not only the price of the derivative security but also the sensitivities of the price to the parameters, i.e., the derivatives of the price with respect to parameters. As mentioned in Sect. 3.3.4, these derivatives are usually denoted by Greeks on the market. For example, $\dfrac{\partial V}{\partial S}, \dfrac{\partial^2 V}{\partial S^2}, \dfrac{\partial V}{\partial t}, \dfrac{\partial V}{\partial \sigma}, \dfrac{\partial V}{\partial r}$ are usually called Delta (Δ), Gamma (Γ), Theta (Θ), Vega ($\mathcal{V}$), and Rho (ρ), respectively. When we know the price of the derivative security for all S and for all $t \in [0, T]$, it is easy to get Delta, Gamma, and Theta. Here, we discuss how to get the other Greeks.

Let $V(S, t; \sigma, r, D_0)$ be the price of a derivative security. Here, we explicitly indicate that V depends on σ, r, and D_0. Thus, the sensitivities of the option price to them can be described by $\mathcal{V} = \dfrac{\partial V}{\partial \sigma}$, $\rho = \dfrac{\partial V}{\partial r}$, and $\rho_d = \dfrac{\partial V}{\partial D_0}$. In order to get $\dfrac{\partial V}{\partial \sigma}$, we can have $V(S, t; \sigma_1, r, D_0)$ and $V(S, t; \sigma_1 + \Delta\sigma, r, D_0)$, then get $\dfrac{\partial V}{\partial \sigma}$ for a σ near σ_1 by

$$\frac{V(S, t; \sigma_1 + \Delta\sigma, r, D_0) - V(S, t; \sigma_1, r, D_0)}{\Delta\sigma}.$$

We also can solve the problem derived in Sect. 3.3.4 to get $\dfrac{\partial V}{\partial \sigma}$.

Let us take $\dfrac{\partial C}{\partial \sigma}$ as an example to explain how to get such a Greek. Set $\overline{C}(S, t) = C(S, t) - c(S, t)$ and suppose $\overline{C}(S, t)$ and $S_f(t)$ have been obtained. Instead of $\dfrac{\partial C}{\partial \sigma}$, let us discuss how to obtain $\dfrac{\partial \overline{C}}{\partial \sigma}$, which will be denoted by $\overline{C}_\sigma$ in this subsection. As pointed out in Sect. 3.3.4, $\dfrac{\partial C}{\partial \sigma}$ is the solution of problem (3.27). Thus, $\overline{C}_\sigma$ should satisfy

$$\begin{cases} \dfrac{\partial \overline{C}_\sigma}{\partial t}+\dfrac{1}{2}\sigma^2 S^2\dfrac{\partial^2 \overline{C}_\sigma}{\partial S^2}+(r-D_0)S\dfrac{\partial \overline{C}_\sigma}{\partial S}-r\overline{C}_\sigma+\sigma S^2\dfrac{\partial^2 \overline{C}}{\partial S^2}=0, \\ \qquad\qquad 0\le S\le S_f(t),\quad 0\le t\le T, \\ \overline{C}_\sigma(S,T)=0, \qquad 0\le S\le S_f(T), \\ \overline{C}_\sigma(S_f(t),t)=-\dfrac{\partial c\,(S,t)}{\partial\sigma}, \qquad 0\le t\le T. \end{cases}$$

This is a problem with a known moving boundary. By using the transformation

$$\begin{cases} \xi=\dfrac{S}{S_f(t)}, \\ \tau=T-t \end{cases}$$

and letting

$$s_f(\tau)=\frac{1}{E}S_f(T-\tau)$$

and

$$W(\xi,\tau)=\frac{1}{E}\overline{C}_\sigma(S,t)=\frac{1}{E}\overline{C}_\sigma\left(\xi E s_f(\tau),T-\tau\right),$$

the problem above can be written as an initial-boundary value problem on a rectangular domain:

$$\begin{cases} \dfrac{\partial W}{\partial\tau}=k_2\xi^2\dfrac{\partial^2 W}{\partial\xi^2}+\left(k_1+\dfrac{1}{s_f}\dfrac{ds_f}{d\tau}\right)\xi\dfrac{\partial W}{\partial\xi}-k_0W+\sigma\xi^2\dfrac{\partial^2 u}{\partial\xi^2}, \\ \qquad\qquad 0\le\xi\le 1,\quad 0\le\tau\le T, \\ W(\xi,0)=0, \qquad 0\le\xi\le 1, \\ W(1,\tau)=-\dfrac{1}{E}\dfrac{\partial c\,(Es_f(\tau),T-\tau)}{\partial\sigma}, \qquad 0\le\tau\le T, \end{cases} \tag{9.33}$$

where $u(\xi,\tau)$ and $s_f(\tau)$ are the solution of the problem (9.3), and $c(S,t)$ is the price of the European call given in Sect. 2.6.5. The equation in the problem (9.33) can be discretized by

$$\begin{aligned} &\frac{W_m^{n+1}-W_m^n}{\Delta\tau} \\ =&\frac{1}{2}k_2m^2\left(W_{m+1}^{n+1}-2W_m^{n+1}+W_{m-1}^{n+1}+W_{m+1}^n-2W_m^n+W_{m-1}^n\right) \\ &+\frac{1}{2}\left\{\left[\frac{k_1}{2}+\frac{s_f^{n+1}-s_f^n}{\left(s_f^{n+1}+s_f^n\right)\Delta\tau}\right]m\left(W_{m+1}^{n+1}-W_{m-1}^{n+1}\right)-k_0W_m^{n+1}\right. \\ &\left.+\left[\frac{k_1}{2}+\frac{s_f^{n+1}-s_f^n}{\left(s_f^{n+1}+s_f^n\right)\Delta\tau}\right]m\left(W_{m+1}^n-W_{m-1}^n\right)-k_0W_m^n\right\} \\ &+\left(d_m^{n+1}+d_m^n\right)/2,\quad m=0,1,\cdots,M-1, \end{aligned} \tag{9.34}$$

where

$$d = \sigma \xi^2 \frac{\partial^2 u}{\partial \xi^2}.$$

The boundary condition in the problem (9.33) can be written as

$$W_{M}^{n+1} = -\frac{1}{E} \frac{\partial c\left(Es_f(\tau^{n+1}), T - \tau^{n+1}\right)}{\partial \sigma}. \tag{9.35}$$

The system (9.34) and (9.35) is a linear system for W_m^{n+1}, $m = 0, 1, \cdots, M$ and we can get W_m^{n+1} by the LU decomposition method if W_m^n, $m = 0, 1, \cdots,$ M, and s_f^n, s_f^{n+1}, $\dfrac{\partial^2 u_m^n}{\partial \xi^2}$, and $\dfrac{\partial^2 u_m^{n+1}}{\partial \xi^2}$ are given. As soon as we obtain W, $\dfrac{\partial C}{\partial \sigma}$ can be found by

$$\frac{\partial C}{\partial \sigma}(S, t) = EW\left(\frac{S}{Es_f\,(T-t)}, T - t\right) + \frac{\partial c}{\partial \sigma}(S, t).$$

When u and s_f are obtained, we need to solve an initial-boundary value problem in order to get $\dfrac{\partial C}{\partial \sigma}$ if the method above is adopted. If we obtain $\dfrac{\partial C}{\partial \sigma}$ by using

$$\frac{V(S, t; \sigma_1 + \Delta\sigma, r, D_0) - V(S, t; \sigma_1, r, D_0)}{\Delta\sigma},$$

then we need to solve another free-boundary problem in order to have $V(S, t;$ $\sigma_1 + \Delta\sigma, r, D_0)$ when $V(S, t; \sigma_1, r, D_0)$ has been found. The amount of work to solve a free-boundary problem by the method described in Sect. 9.2.1 is more than twice of the amount of the work to solve an initial-boundary value problem by the method given here. This is why we formulate a problem for $\overline{C}_\sigma$ and obtain $\dfrac{\partial C}{\partial \sigma}$ by solving the problem (9.33).

9.2.3 Numerical Results of Vanilla Options and Comparison

In this subsection, we will discuss some issues on the efficiency of the numerical method described in Sect. 9.2.1 and the performance of the method combined with the extrapolation technique. Here, a method combined with the extrapolation technique means that the computation is first done on a mesh by the method, then reduce the mesh sizes in the both directions by a factor of 1/2 (or other numbers) and do the computation on the second mesh again, and finally get the results by the formula (7.30) in Sect. 7.3 (or other similar formulae). The method in Sect. 9.2.1 is an implicit finite-difference version of the SSM and, for simplicity, is referred to as the SSM in this subsection. Here, we also compare the results obtained by the SSM and the combination of the SSM and the extrapolation technique with the results by other methods for two options. Finally, through the shape of the free boundaries, we point out that adopting nonuniform time steps can make the method more accurate.

Table 9.1. Parameters

Interest rates r	$0.05 \sim 0.20$ with $\varDelta r = 0.025$
Volatilities σ	$0.1 \sim 0.5$ with $\varDelta\sigma = 0.1$
Dividend yields D_0	$0.00 \sim 0.15$ with $\varDelta D_0 = 0.025$
Expiries T	3 days, 15 days, 1$\sim$ 12 months with $\varDelta T = 1$ month

Table 9.2. American call options with $r = 0.1$ and $T = 1$ year

$D_0\backslash\sigma$	0.1	0.2	0.3	0.4	0.5
0.000	–	–	–	–	–
0.025	–	–	–	–	–
0.050	–	12×6	12×6	12×6	12×6
0.075	16×8	12×8	12×8	12×8	12×8
0.100	28×14	18×10	16×10	14×8	14×8
0.125	44×16	30×12	24×10	18×8	14×8
0.150	48×18	32×12	26×10	20×8	16×8

The SSM combined with the extrapolation technique has been tested for American vanilla call and put options with various parameters. The parameters tested are given in Table 9.1. Consider the standard American call problem, i.e., the problem with $E = 1$. Suppose $r = 0.1$, $T = 1$, and require the maximum error of C for $S \in [0.9, 1.1]$ to be less than or equal to 10^{-4}. Table 9.2 lists the numbers of mesh intervals needed for different D_0 and σ in order to get such results. There, $M \times N$ means that for the second mesh, M subintervals in the ξ-direction and N time-steps in the τ-direction are taken. In Table 9.2, "–" means that for this set of parameters, and for $S \in [0.9, 1.1]$, the difference between the American call option and the European call option is less than or only a slightly greater than 10^{-4}, so no numerical method is needed. From here, we know that if the method described in Sect. 9.2.1 is used, then a coarse mesh is enough for obtaining a result with error about 10^{-4} for $S \in [0.9, 1.1]$.

Table 9.3. American put option with $r = 0.05$ and $T = 1$ year

$D_0\backslash\sigma$	0.1	0.2	0.3	0.4	0.5
0.000	40×12	24×8	18×6	14×4	12×4
0.025	36×12	26×8	16×4	14×4	12×4
0.050	32×10	22×6	16×4	12×4	12×4
0.075	–	22×6	16×4	12×4	12×4
0.100	–	–	16×4	12×4	12×4
0.125	–	–	–	12×4	12×4
0.150	–	–	–	–	12×4

As pointed out in Chap. 3, using the symmetry relations, we can have the value of an American put option from an American call option with interchanging the interest rate and dividend yield. However, we can also solve the put option problem directly. In Table 9.3, we list the numbers of mesh inter-

Table 9.4. Optimal prices for American call options

($\sigma = 0.2$, $T = 1$ and $E = 100$)

$D_0 \backslash r$	0.050	0.075	0.100	0.125	0.150
0.050	141.540893	170.943495	223.764096	277.831844	331.285054
0.075	128.372144	137.454215	155.027353	186.574326	222.166283
0.100	122.069175	127.037558	134.599182	147.295598	168.445693
0.125	118.119037	121.403431	125.903014	132.417054	142.448401
0.150	115.346132	117.723481	120.800277	124.918028	130.659131

vals needed in order to have an accuracy of about 10^{-4} for $S \in [0.9, 1.1]$ and $r = 0.05$. Thus, for both American call and put options, only a coarse mesh is needed in order to get the accuracy usually needed. From the price of the call option with $r = 0.1$ and $D_0 = 0.05$, we can have the value of the put option with $r = 0.05$ and $D_0 = 0.1$. From Tables 9.2 and 9.3, we know that in order to get the price of the put option with $r = 0.05$, $D_0 = 0.1$, and $\sigma = 0.3$, we can take a 16×4 mesh if we solve a put problem directly or we can take a 12×6 mesh if we solve a corresponding call problem and get the solution using the symmetry relations. For these two meshes, the CPU times needed are very close, so we can choose either way. However, if we already have a code to compute American call option prices, then using the second way would be a better choice since only very little code needs to be added.

With this method, it is not difficult to get results with a high accuracy. In Table 9.4, the optimal price for American call options with various r and D_0 are listed. Analysis shows these results to be exact to at least seven digits (see [98]).

Table 9.5. American call option

($r = 0.1$, $\sigma = 0.2$, $D_0 = 0.05$, $T = 1\,\text{year}$, $S = E = 100$, and the exact value $= 9.94092345\cdots$)

	Without extrapolation			With extrapolation		
Meshes	Results	Errors	CPU(s)	Results	|Errors|	CPU(s)
32×2	9.941663	−0.000739	0.00025	9.940902	0.000021	0.00045
64×4	9.941097	−0.000174	0.00083	9.940908	0.000015	0.0012
128×8	9.940962	−0.000038	0.0027	9.940917	0.000006	0.0038
256×16	9.940932	−0.000009	0.0099	9.9409225	0.000001	0.0125

Now let us discuss the convergence rate of the SSM. Let $r = 0.1, \sigma = 0.2, D_0 = 0.05, T = 1\,\text{year}$, and $S = E = 100$. In order to study the convergence rate, we have to know the exact solution. We do not have the exact solution, but we can get a solution with a very high accuracy and obtain the first few digits of the exact solution. For the parameters given above, our computation shows the exact call option price $C = 9.94092345\cdots$ and the exact put option price $P = 5.92827717\cdots$. As long as we have such a solution, we can find the error of any solution up to the eighth decimal. In Table 9.5,

Table 9.6. American put options

($r = 0.1, \sigma = 0.2, D_0 = 0.05, T = 1\,$year, $S = E = 100$, and the exact value $= 5.92827717\cdots$)

Meshes	Without extrapolation			With extrapolation		
	Results	Errors	CPU(s)	Results	|Errors|	CPU(s)
12×4	5.968338	−0.040060	0.00035	5.925575	0.002702	0.00065
24×8	5.937883	−0.009606	0.00084	5.927732	0.000545	0.0014
48×16	5.930477	−0.002200	0.0025	5.928008	0.000269	0.0035
96×32	5.928819	−0.000542	0.0078	5.928266	0.000011	0.0108
192×64	5.928409	−0.000132	0.0300	5.928272	0.000005	0.0387
384×128	5.928310	−0.000033	0.1200	5.9282767	0.0000005	0.1400

the results without using the extrapolation technique for four meshes and the errors up to the sixth decimal are listed on the second and third columns from the left. When the numbers of intervals in the both directions is doubled, the error is reduced by a factor about 1/4. This means that the error is $O\left(\Delta\xi^2, \Delta\tau^2\right)$. Therefore, it has a second-order convergence rate. In Table 9.6, the results and errors for the put option are given. From there, we see that the convergence rate is also second order for the put option.

A method with a high convergence rate has a better performance if the mesh size is small enough. However, if the mesh size is not small enough, it might not be true. For a fixed mesh, the computational amount of work is different for different methods. Thus, from a practical point of view, a method should be judged by its performance. Therefore, we also list the CPU time needed to perform such a computation on a Space Ultra 10 computer for each mesh in Tables 9.5 and 9.6.

Using these data on errors and CPU times in Tables 9.5 and 9.6, the data given for PEFDII, Binomial, PSOR, and PIFDII in Chap. 8, the graphs of $\log_{10}$(CPU time in second) versus $\log_{10}$(error) for call and put options are plotted in Figs. 9.2 and 9.3, respectively. On these two figures, the lower the point, the better the performance because a lower point means that for a fixed error, it needs less CPU time. From there, we can see that the singularity-separating method (SSM) has the best performance for these two cases if the error required is less than 10^{-2}. Moreover, the higher the accuracy required, the greater the advantage of the SSM.

If the SSM is combined with the extrapolation technique, then the performance is even better. In order to explain this, the results, errors, and CPU times when the SSM is combined with the extrapolation technique are listed in the right three columns of Tables 9.5 and 9.6, and the corresponding graphs of $\log_{10}$(CPU time in second) versus $\log_{10}$(error) are also plotted in Figs. 9.2 and 9.3. There, SSME stands for the singularity-separating method with the extrapolation technique. From here, we can see that the extrapolation technique is very useful. At the beginning of this subsection, we showed that for various parameters, the SSM with the extrapolation technique could give very

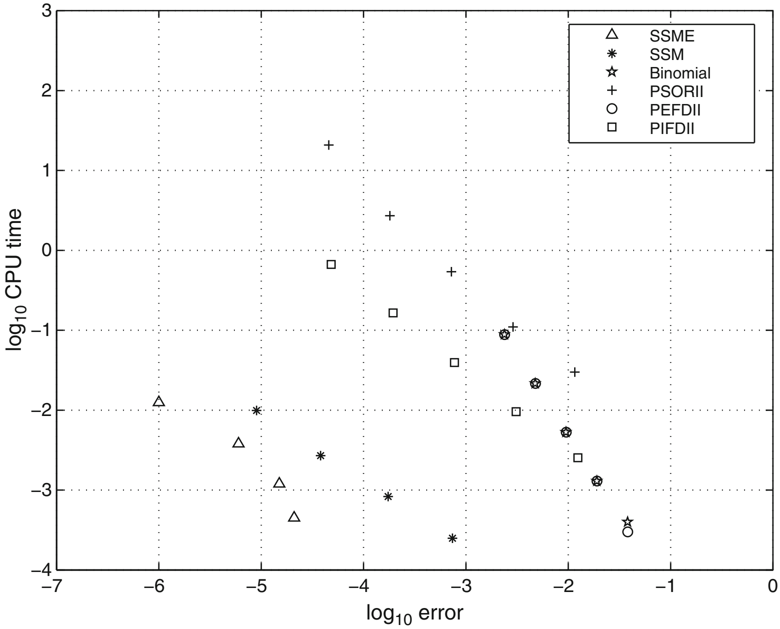

Fig. 9.2. Graphs of CPU time versus error for a call option, $S = 100$

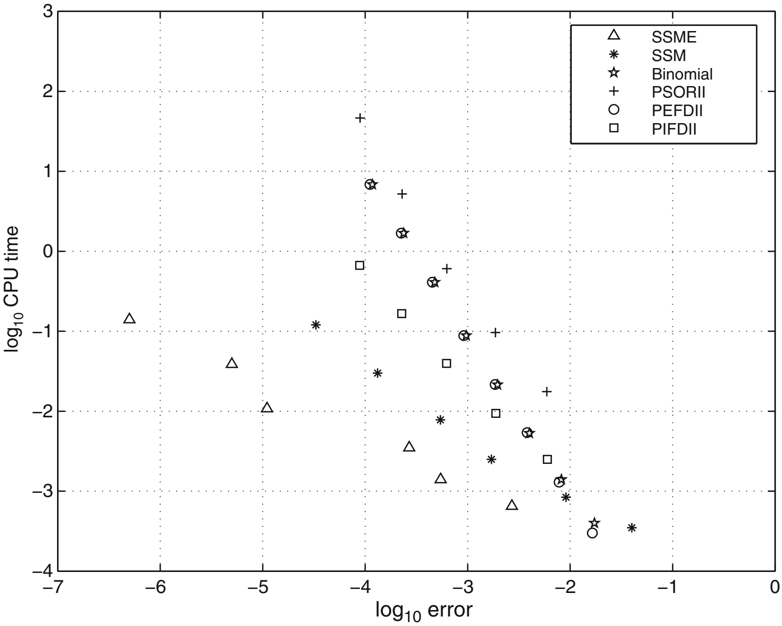

Fig. 9.3. Graphs of CPU time versus error for a put option, $S = 100$

good results on quite coarse meshes. This is because due to the error function being quite smooth, the extrapolation technique is always helpful when combined with SSM.

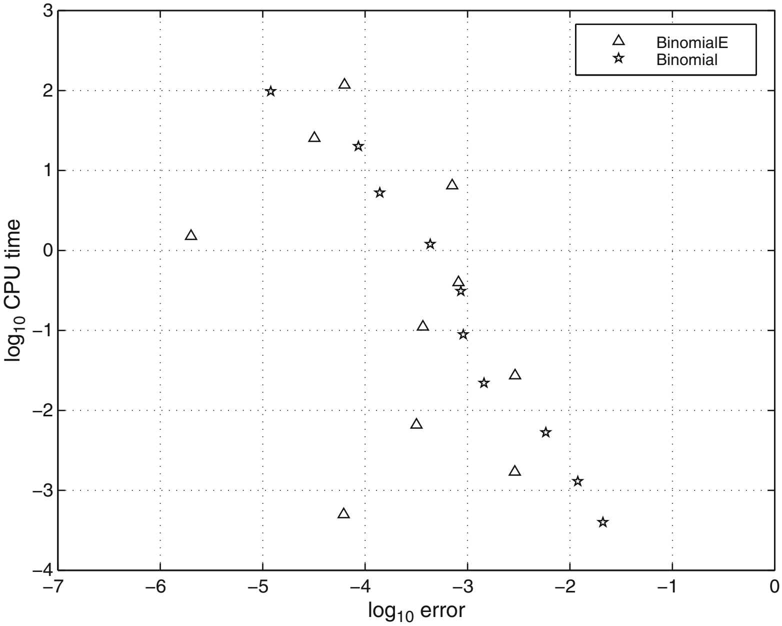

Fig. 9.4. Graphs of CPU time versus error for a call option, $S = 110$

Here, we would like to point out that the extrapolation technique is not always helpful. Let us find out if the performance is improved when the binomial method is combined with the extrapolation technique. Consider a call option with $r = 0.1$, $\sigma = 0.2$, $D_0 = 0.05$, $T = 1$, and $E = 100$. In Fig. 9.4 for $S = 110$ we plot the graphs of $\log_{10}$(CPU time in second) versus $\log_{10}$(error) for the binomial method with and without extrapolation. There, "Binomial" and "BinomialE" mean the binomial method and the binomial method with extrapolation technique. From there, we can see that on some meshes, the extrapolation technique improves the results, but on other meshes, it makes the results worse. In order to have some details about why this happens, the data of the errors and the CPU times are listed for the two cases in Table 9.7. As a first-order method, the error should be reduced by a factor about 1/2 when the number of time steps is doubled. Because the error function is not smooth due to the non-smoothness of the solution, from the table we see that from one mesh size to another, the error before extrapolation does not always show such a property and sometimes the sign of the error even changes. Thus, when the extrapolation technique is used, the error increases for some cases if the sign is unchanged and always increases if the sign changes. This phenomena occurs even if the mesh size is very small. Therefore, the extrapolation technique is not always helpful for the binomial method. However, Broadie and Detemple in [14] suggested an improved binomial method called the binomial Black and Scholes method (BBS). Examples show that the error of BBS decreases and does not change its sign when the mesh size decreases. As long as it is true, the extrapolation technique is helpful for the BBS method.

Table 9.7. American call option (binomial method)

($r = 0.1, \sigma = 0.2, D_0 = 0.05, T = 1$ year, $E = 100$, $S = 110$, and the exact value $= 16.8016638\cdots$)

Numbers of	Without extrapolation			With extrapolation		
time steps	Results	Errors	CPU(s)	Results	|Errors|	CPU(s)
50	16.822670	−0.021006	0.0004	16.801602	0.000062	0.0005
100	16.813618	−0.011954	0.0013	16.804566	0.002902	0.0017
200	16.807482	−0.005818	0.0053	16.801346	0.000318	0.0066
400	16.803114	−0.001450	0.0220	16.798746	0.002918	0.0273
800	16.802573	−0.000909	0.0880	16.802032	0.000370	0.1110
1,600	16.802526	−0.000862	0.3100	16.802479	0.000817	0.3980
3,200	16.802096	−0.000432	1.2000	16.801666	0.000002	1.5100
6,400	16.801525	+0.000139	5.2700	16.800953	0.000710	6.4700
12,800	16.801578	+0.000086	20.100	16.801632	0.000032	25.370
25,600	16.801652	−0.000012	97.600	16.801727	0.000063	117.70

Finally, in this subsection we give two graphs on the location of the free boundaries. In Figs. 9.5 and 9.6, the location of the free boundaries is plotted for three call options and three put options, respectively. There, $E = 100, \sigma = 0.24$, and $t = 0 \sim 10$. The other parameters for the three call options are $(r = 0, D_0 = 0.06)$, $(r = 0.06, D_0 = 0.06)$, and $(r = 0.06, D_0 = 0.03)$, and for the three put options they are $(r = 0.06, D_0 = 0)$, $(r = 0.06, D_0 = 0.06)$, and $(r = 0.03, D_0 = 0.06)$. For all the cases, the location of the free boundary moves quite fast at $t \approx T$. Therefore, the time step at $t \approx T$ should be smaller than the time step at $t << T$. In order to make computation more efficient, the time step used for all the numerical results in this subsection is not constant. When we need to find the solution for $\tau \in [0, T]$ and the total number of time step is N, then τ^n is determined by the formula

$$\tau^n = \frac{n^2}{N^2}T, \quad n = 0, 1, \cdots, N$$

and from τ^n to τ^{n+1}, the time step is $\tau^{n+1} - \tau^n = \dfrac{2n+1}{N^2}T$. When such variable time steps are used, the extrapolation technique can still be used, which is left as an exercise problem for readers to show.

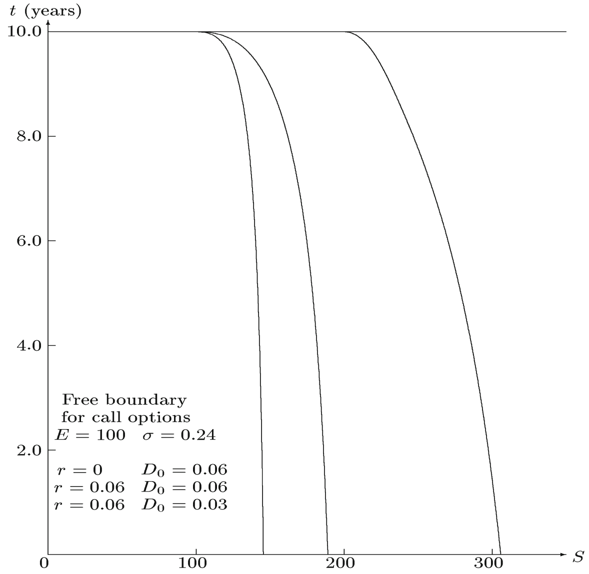

Fig. 9.5. Locations of free boundaries of call options in the (S, t)-plane [The parameters for these curves from the left to the right are (r=0, D_0=0.06), (r=0.06, D_0=0.06), and (r=0.06, D_0=0.03)]

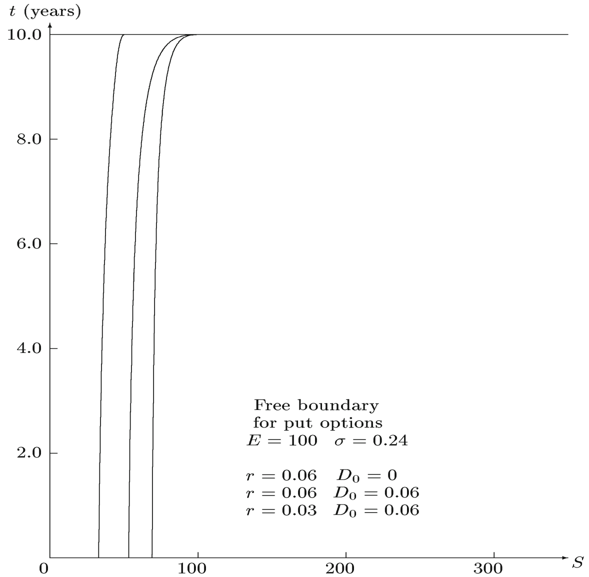

Fig. 9.6. Locations of free boundaries of put options in the (S, t)-plane. [The parameters for these curves from the right to the left are (r=0.06, D_0=0), (r=0.06, D_0=0.06), and (r=0.03, D_0=0.06)]

9.2.4 Solution and Numerical Results of Exotic Options

After making a slight change, the implicit finite-difference method described in Sect. 9.2.1 still can be used for computing free-boundary problems for American-style barrier, Asian, and lookback options. Here, we first show some results for American barrier and lookback options. Then, we discuss some modifications we have used when we compute the prices of American-style Asian options and give some results on Asian options.

For $S = 60 \sim 160$, the prices of American down-and-out call options with $B_l = 80$, 85, 90, 95 and the price of American down-and-out call option with $B_l = 0$—the price of the American vanilla call option—have been shown in Fig. 4.1. There, the parameters are $r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$, $T = 1$ year, and $E = 100$. Here, for $S = 60 \sim 160$ and for the same parameters, the prices of American up-and-out put options with $B_u = 105$, 110, 115, 120 and the price of American up-and-out put option with $B_u = \infty$—the price of the American vanilla put option—are represented in Fig. 9.7. From these curves, we see again that the price of a barrier option is less than a vanilla option. The reason is still that the holder of a barrier option has less rights than a holder of a vanilla option. In Sect. 4.2.3, we have pointed out that for call options, the higher the lower barrier B_l, the less the rights and the cheaper the option. Here, we give some data to show how big the difference between the barrier options and the vanilla options is. In Table 9.8, the prices of the American down-and-out and vanilla call options for S= 80, 85, 90, 95, 100, 105, 110, 115, and 120 are listed. From the data, we can see that the difference is significant for most of the cases.

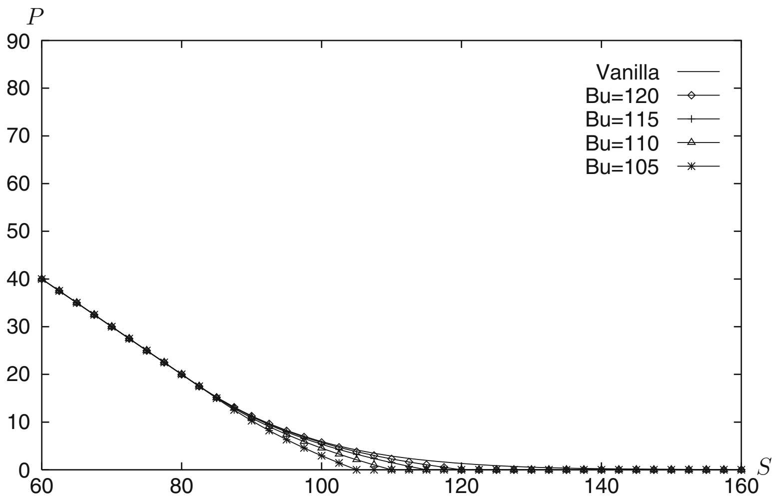

Fig. 9.7. Values of American vanilla put option and American up-and-out put options with $B_u = 105$, 100, 115, 120 ($r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$, $T = 1$ year, and $E = 100$)

Table 9.8. American down-and-out call option

($r = 0.1$, $\sigma = 0.20$, $D_0 = 0.05$, $T = 1$, and $E = 100$)

S	Vanilla	$B_l = 80$	$B_l = 85$	$B_l = 90$	$B_l = 95$
80	1.769	0	0	0	0
85	3.057	2.181	0	0	0
90	4.843	4.418	3.165	0	0
95	7.145	6.943	6.242	4.251	0
100	9.941	9.846	9.464	8.243	5.361
105	13.182	13.138	12.934	12.202	10.292
110	16.802	16.782	16.674	16.244	15.005
115	20.728	20.719	20.663	20.415	19.626
120	24.893	24.889	24.861	24.720	24.226

In Sect. 4.4, for an American lookback strike call option, the values $W(\eta, t)$ as functions of η for $t = 0$, 0.2, 0.4, 0.6, 0.8 are shown in Fig. 4.7. Here, for an American lookback strike put option, similar curves are represented in Fig. 9.8. From this figure, we know that $W(\eta, t) = V(S, H, t)/S$ is an increasing function in $\eta = H/S$. That is, if S is fixed, then $V(S, H, t)$ is an increasing function in H. This is because the payoff $\max(H - S, 0)$ increases for $S \leq H$ as H increases. The highest price up to time t is of course greater than or equal to the price at time t. Thus, $\eta = H/S$ must be greater than or equal to 1. Consequently, $W(\eta, t)$ is defined only for $\eta \geq 1$ and for a fixed t, the price of the option has a minimum at $\eta = 1$. In Fig. 9.8, we can observe this being true and the value of $W(\eta, t)$ at $\eta = 1$ and $t = 0$ being about 0.16. This means that the minimum price at $t = 0$ is about 16 % (the actual value is 16.37 %) of S. From the last subsection, we know that the value of the vanilla put option with $S = E$ is 5.93 % of S. Hence, the price of an American lookback strike put option is much higher than the price of an American vanilla put option. The reason is that the holder of an American lookback strike put option can sell a stock at any time t for the maximum price during the time interval $[0, t]$, whereas a holder of an American vanilla put option can sell a stock at any time t for the price at time t that is always less than or equal to the maximum price during the time interval $[0, t]$.

In Fig. 9.9, the location of the free boundary of the American lookback strike put option is given. In Fig. 4.8, a similar result for a call is represented. In Sect. 3.3.1, it has been shown that the locations of free boundaries for vanilla options are monotone functions in t. In fact, this is also true for American lookback strike options. Figures 4.8 and 9.9 show this fact. In Sect. 3.3.1, we also have pointed out that the monotonicities of the free boundary and of the price with respect to t are related. This reflects that $W(\eta, t)$ should be monotone functions of t for any fixed η. Figures 4.7 and 9.8 show this feature.

For details on how to compute American barrier and lookback options by using SSM, see [18, 99].

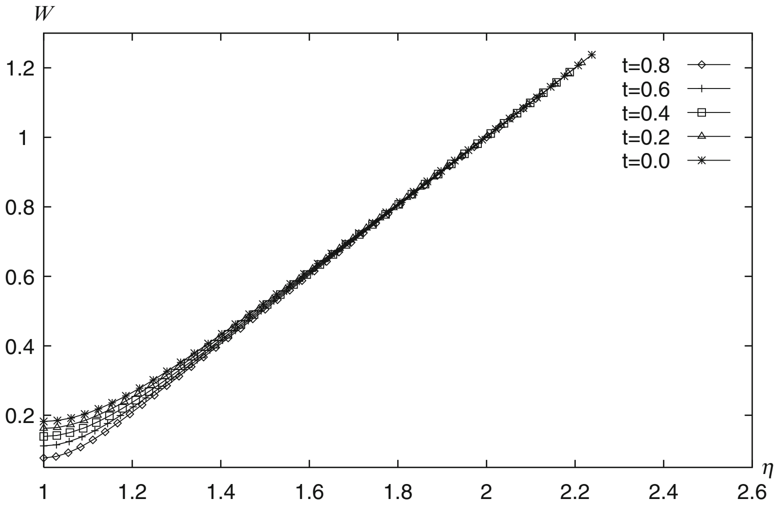

Fig. 9.8. $W(\eta, t)$ of an American lookback strike put option ($r = 0.05$, $D_0 = 0.1$, and $\sigma = 0.2$)

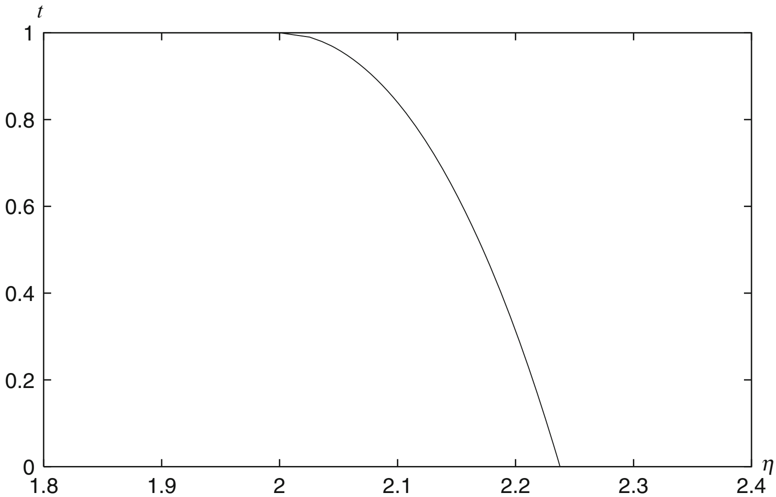

Fig. 9.9. The free boundary of an American lookback strike put option ($r = 0.05$, $D_0 = 0.1$, and $\sigma = 0.2$)

Now let us look at average options. In Fig. 9.10, the line with $*$ gives the solution $W(\eta, 0)$ for an American average strike call option by the singularity-separating method with the implicit finite-difference method (SSMIMP), similar to that described in Sect. 9.2.1. The result of a put option with the same parameters is given in Fig. 9.11 also by a curve with $*$. These two curves are almost horizontal straight lines except near one of the boundaries because

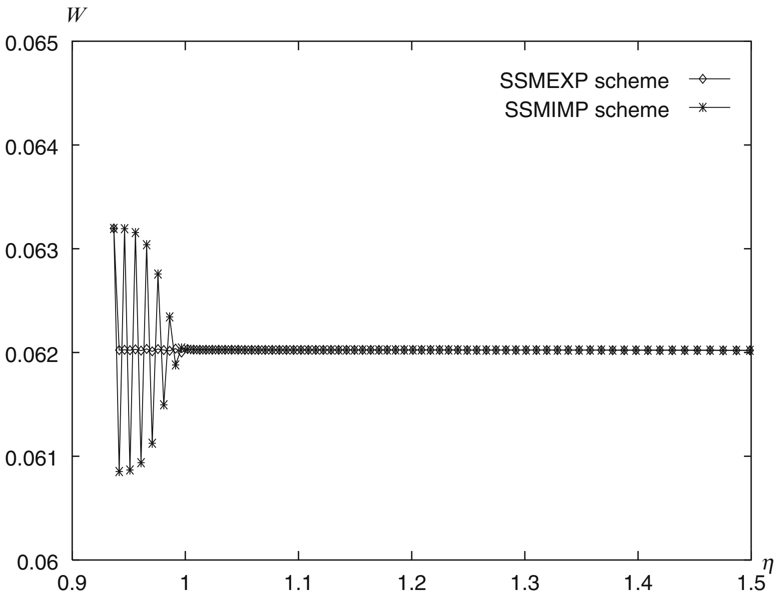

Fig. 9.10. $W(\eta, t)$ of an American average strike call option ($r = 0.1$, $D_0 = 0.1$, $\sigma = 0.2$, and $t = 0$)

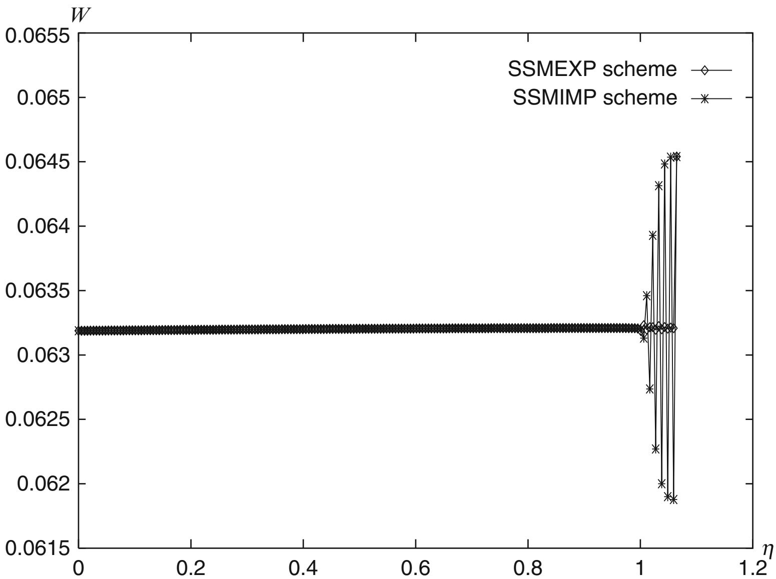

Fig. 9.11. $W(\eta, t)$ of an American average strike put option ($r = 0.1$, $D_0 = 0.1$, $\sigma = 0.2$, and $t = 0$)

there is a term $(1-\eta)/t$ in the partial differential equation of Asian options. Actually, this boundary is the free boundary. Near the free boundary, the exact solution changes very rapidly, and the numerical solution has oscillations. At time $t=0$, the average price of the stock is always equal to the price of the stock, so we actually only need the value of $W(\eta,t)$ at $\eta=A/S=1$, which is the level of the horizontal straight line. Therefore, we can still have a good result for $W(1,t)$ by finding the level of the horizontal straight line. However, in order to get rid of the oscillations and make the entire result nicer, we use the following scheme to approximate the first and second derivatives with respect to ξ in the partial differential equation.

Let us consider the equation:

$$a_i^{n+1/2}\frac{\partial^2 U}{\partial \xi^2}+b_i^{n+1/2}\frac{\partial U}{\partial \xi}+c_i^{n+1/2}U=0.$$

Its characteristic equation is

$$a_i^{n+1/2}\lambda^2+b_i^{n+1/2}\lambda+c_i^{n+1/2}=0. \tag{9.36}$$

When $a_i^{n+1/2}>0$ and $c_i^{n+1/2}<0$, it has two distinct real roots:

$$\begin{cases}\lambda_{1,i}=\dfrac{-b_i^{n+1/2}+\sqrt{\left(b_i^{n+1/2}\right)^2-4a_i^{n+1/2}c_i^{n+1/2}}}{2a_i^{n+1/2}},\\[2ex] \lambda_{2,i}=\dfrac{-b_i^{n+1/2}-\sqrt{\left(b_i^{n+1/2}\right)^2-4a_i^{n+1/2}c_i^{n+1/2}}}{2a_i^{n+1/2}}.\end{cases}$$

Let

$$\varphi(\xi)=\mathrm{e}^{\lambda_{1,i}(\xi-\xi_i)},\quad \psi(\xi)=\mathrm{e}^{\lambda_{2,i}(\xi-\xi_i)} \tag{9.37}$$

be the local basis functions. Then, on a subinterval $[\xi_{i-1},\xi_{i+1}]$ near ξ_i, a function $W(\xi,\tau^{n+1/2})$ can be approximated by

$$\alpha_i\varphi(\xi)+\beta_i\psi(\xi)+\gamma_i, \tag{9.38}$$

where α_i, β_i, and γ_i are determined by the following conditions:

$$\begin{cases}\alpha_i\varphi(\xi_{i-1})+\beta_i\psi(\xi_{i-1})+\gamma_i=W_{i-1}^{n+1/2},\\[1ex] \alpha_i\varphi(\xi_i)+\beta_i\psi(\xi_i)+\gamma_i=W_i^{n+1/2},\\[1ex] \alpha_i\varphi(\xi_{i+1})+\beta_i\psi(\xi_{i+1})+\gamma_i=W_{i+1}^{n+1/2}.\end{cases}$$

From these conditions, we have

$$\begin{cases} \alpha_i = \alpha_{1,i} W_{i-1}^{n+1/2} + \alpha_{2,i} W_i^{n+1/2} + \alpha_{3,i} W_{i+1}^{n+1/2}, \\ \beta_i = \beta_{1,i} W_{i-1}^{n+1/2} + \beta_{2,i} W_i^{n+1/2} + \beta_{3,i} W_{i+1}^{n+1/2}, \\ \gamma_i = W_i^{n+1/2} - \alpha_i - \beta_i, \end{cases} \tag{9.39}$$

where

$$\begin{aligned} \alpha_{1,i} &= \left[\psi(\xi_{i+1}) - \psi(\xi_i)\right]/G_i, \\ \alpha_{2,i} &= \left[\psi(\xi_{i-1}) - \psi(\xi_{i+1})\right]/G_i, \\ \alpha_{3,i} &= \left[\psi(\xi_i) - \psi(\xi_{i-1})\right]/G_i, \\ \beta_{1,i} &= \left[\varphi(\xi_i) - \varphi(\xi_{i+1})\right]/G_i, \\ \beta_{2,i} &= \left[\varphi(\xi_{i+1}) - \varphi(\xi_{i-1})\right]/G_i, \\ \beta_{3,i} &= \left[\varphi(\xi_{i-1}) - \varphi(\xi_i)\right]/G_i, \\ G_i &= \left[\varphi(\xi_{i-1}) - \varphi(\xi_i)\right]\left[\psi(\xi_{i+1}) - \psi(\xi_i)\right] \\ &\quad - \left[\varphi(\xi_{i+1}) - \varphi(\xi_i)\right]\left[\psi(\xi_{i-1}) - \psi(\xi_i)\right]. \end{aligned}$$

If $b_i^{n+1/2}$ is a very large positive number, then $|\lambda_{2,i}|$ is very large and the exponential function $\psi(\xi)$ changes very rapidly. Therefore, even if $W(\xi, \tau^{n+1/2})$ changes very rapidly, as long as its behavior is close to an exponential function, (9.38) can still give a very good approximation not only for the function itself but also for its derivatives. Differentiating function (9.38) with respect to ξ yields

$$\frac{\partial W}{\partial \xi} \approx \alpha_i \lambda_{1,i} \varphi(\xi) + \beta_i \lambda_{2,i} \psi(\xi), \quad \frac{\partial^2 W}{\partial \xi^2} \approx \alpha_i \lambda_{1,i}^2 \varphi(\xi) + \beta_i \lambda_{2,i}^2 \psi(\xi). \tag{9.40}$$

Therefore, we can have the following approximation:

$$\begin{aligned} & a_i^{n+1/2} \frac{\partial^2 W_i^{n+1/2}}{\partial \xi^2} + b_i^{n+1/2} \frac{\partial W_i^{n+1/2}}{\partial \xi} + c_i^{n+1/2} W_i^{n+1/2} \\ = {}& a_i^{n+1/2} \left[\alpha_i \lambda_{1,i}^2 \varphi(\xi_i) + \beta_i \lambda_{2,i}^2 \psi(\xi_i)\right] \\ & + b_i^{n+1/2} \left[\alpha_i \lambda_{1,i} \varphi(\xi_i) + \beta_i \lambda_{2,i} \psi(\xi_i)\right] + c_i^{n+1/2} W_i^{n+1/2} \\ = {}& -c_i^{n+1/2} \left[\alpha_i + \beta_i - W_i^{n+1/2}\right] \\ = {}& -c_i^{n+1/2} \left[(\alpha_{1,i} + \beta_{1,i}) W_{i-1}^{n+1/2} + (\alpha_{2,i} + \beta_{2,i}) W_i^{n+1/2} \right. \\ & \left. + (\alpha_{3,i} + \beta_{3,i}) W_{i+1}^{n+1/2} - W_i^{n+1/2}\right], \end{aligned}$$

where we have used the facts that $\lambda_{1,i}$ and $\lambda_{2,i}$ are roots of Eq. (9.36) and that the expressions of α_i and β_i are given by the set of expressions (9.39).

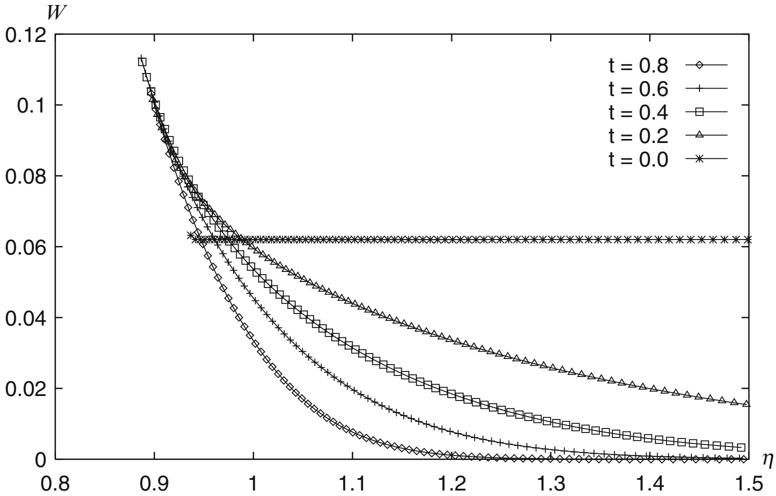

Fig. 9.12. $W(\eta, t)$ of an American average strike call option ($r = 0.1$, $D_0 = 0.1$, $\sigma = 0.2$, and $\alpha = 1$)

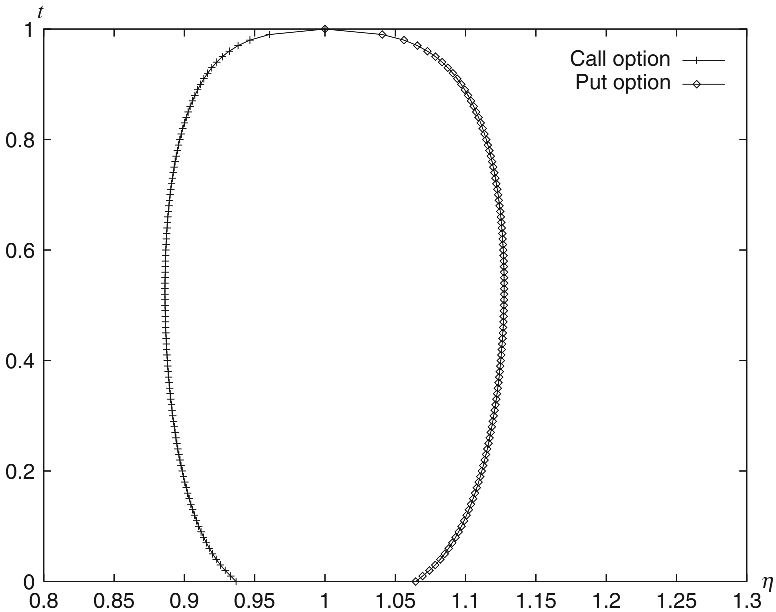

Fig. 9.13. Free boundaries of American average strike options ($r = 0.1$, $D_0 = 0.1$, $\sigma = 0.2$, and $\alpha = 1$)

If the first and second derivatives with respect to ξ in the partial differential equation for an Asian option are not discretized by central schemes but by the set of expressions (9.40), then we have a new scheme, which is called the exponential scheme, and the method is referred to as the singularity-separating method with an exponential scheme (see [17]) and abbreviated as SSMEXP. The results of the exponential scheme are also given in Figs. 9.10 and 9.11 by the curves with $\diamond$, which have no oscillations. From these curves, we see that this scheme improves the results. Therefore, in order to get the price of Asian options, we use this scheme. In Fig. 9.12, for an American average strike call option, the values of $W(\eta, t)$ as functions of η for $t = 0$, 0.2, 0.4, 0.6, 0.8 are given. The price of the option is $V(S, A, t) = AW(S/A, t)$. Because $A = S$ at $t = 0$, in order to find the value of the option at $S = \$100$ and $t = 0$, we need to find $\$100W(1, 0)$. From Fig. 9.12, we see that it is a little higher than $\$100 \times 0.06 = \$6.00$ (from the data we have it is \$6.20). In Fig. 4.3, the values of the American average strike put option with the same parameters are represented. From there, we see that the price for an American average strike put option with the same parameters at $t = 0$ is also a little higher than $\$100 \times 0.06 = \$6.00$ (from the data we have it is \$6.32). Thus, the difference between the call and put prices for the average options is much smaller than for the vanilla options. In Fig. 9.13, the free boundaries of the average strike call and put options are given, which shows that the locations of free boundaries are not monotone functions in t for the average strike options. This indicates that $W(\eta, t)$ is not a monotone function of t for a fixed η, which can be seen in Figs. 4.3 and 9.12.

9.2.5 Solution of Two-Dimensional Problems

In this subsection, we will discuss how to price two-factor vanilla American call options numerically. Here, "two-factor" means that both S and σ are random variables. If D_0 is not equal to zero, then pricing two-factor vanilla American options involves solving two-dimensional free-boundary problems. In what follows, we will give some details on implicit finite-difference methods for two-dimensional free-boundary problems. For the American call, the corresponding free-boundary problem is given by the problem (9.7) or the problem (9.9). Those problems can be converted into a problem on a rectangular domain, for example, the problem (9.9) can be converted into the problem (9.11). Therefore, determining the price on the domain $[0, S_f(\sigma, t)] \times [\sigma_l, \sigma_u] \times [0, T]$ can be reduced to solving the problem (9.11) on a rectangular domain $[0, 1] \times [\sigma_l, \sigma_u] \times [0, T]$ in the (ξ, σ, τ)-space.

We use equidistant grid points on the rectangular domain. Let $\Delta\xi = 1/M$, $\Delta\sigma = (\sigma_u - \sigma_l)/I$, and $\Delta\tau = T/N$ be the mesh sizes in the ξ-, σ-, and τ-directions, respectively, where M, I and N are positive integers. We thus have $M+1$, $I+1$, and $N+1$ nodes in the ξ-, σ-, and τ-directions, respectively. The $M+1$ nodes in the ξ-direction are $\xi_m = m\Delta\xi$, $m = 0, 1, \cdots, M$, the $I+1$ nodes in the σ-direction are $\sigma_i = \sigma_l + i\Delta\sigma$, $i = 0, 1, 2, \cdots, I$, and the $N+1$ nodes in the τ-direction are $\tau^n = n\Delta\tau$, $n = 0, 1, \cdots, N$. In what follows, we also define $\tau^{n+1/2} = (n + 1/2)\Delta\tau$. Let $u^n_{m,i}$ stand for the approximate value of u at $\xi = \xi_m$, $\sigma = \sigma_i$, and $\tau = \tau^n$ and $s^n_{f,i}$ denote the approximate value of s_f at $\sigma = \sigma_i$, and $\tau = \tau^n$.

If $\xi \neq 1$, $\sigma \neq \sigma_l$, and $\sigma \neq \sigma_u$, then at a point $(\xi_m, \sigma_i, \tau^{n+1/2})$, the partial differential equation in the problem (9.11) can be discretized by the following second-order approximation:

$$\begin{aligned}
&\frac{u_{m,i}^{n+1} - u_{m,i}^{n}}{\Delta\tau} \\
&= \frac{a_1 m^2}{2}\left(u_{m+1,i}^{n+1} - 2u_{m,i}^{n+1} + u_{m-1,i}^{n+1} + u_{m+1,i}^{n} - 2u_{m,i}^{n} + u_{m-1,i}^{n}\right) \\
&\quad + \frac{a_2 q m}{8\Delta\sigma}(u_{m+1,i+1}^{n+1} - u_{m+1,i-1}^{n+1} - u_{m-1,i+1}^{n+1} + u_{m-1,i-1}^{n+1} \\
&\qquad\quad + u_{m+1,i+1}^{n} - u_{m+1,i-1}^{n} - u_{m-1,i+1}^{n} + u_{m-1,i-1}^{n}) \\
&\quad + \frac{a_3 q^2}{2\Delta\sigma^2}(u_{m,i+1}^{n+1} - 2u_{m,i}^{n+1} + u_{m,i-1}^{n+1} + u_{m,i+1}^{n} - 2u_{m,i}^{n} + u_{m,i-1}^{n}) \\
&\quad + \frac{a_4 m}{4}(u_{m+1,i}^{n+1} - u_{m-1,i}^{n+1} + u_{m+1,i}^{n} - u_{m-1,i}^{n}) \\
&\quad + \frac{a_5}{4\Delta\sigma}(u_{m,i+1}^{n+1} - u_{m,i-1}^{n+1} + u_{m,i+1}^{n} - u_{m,i-1}^{n}) \\
&\quad + \frac{a_6}{2}(u_{m,i}^{n+1} + u_{m,i}^{n}) + a_7, \\
&\qquad m = 0, 1, \cdots, M-1, \quad i = 1, 2, \cdots, I-1.
\end{aligned} \tag{9.41}$$

Here, q and all the coefficients a_1–a_7 should be evaluated at ξ_m, σ_i and $\tau^{n+1/2}$ in order to guarantee second-order accuracy. For a_1–a_7, the expressions are

$$\begin{aligned}
a_{1,m,i}^{n+1/2} &= \frac{1}{2}(\sigma_l + i\Delta\sigma)^2 \\
&\quad - \frac{\rho_{m,i}^{n+1/2}(\sigma_l + i\Delta\sigma) q_{m,i}^{n+1/2}}{2\Delta\sigma} \frac{(s_{f,i+1}^{n+1} - s_{f,i-1}^{n+1} + s_{f,i+1}^{n} - s_{f,i-1}^{n})}{(s_{f,i}^{n+1} + s_{f,i}^{n})} \\
&\quad + \frac{1}{2}\left[\frac{q_{m,i}^{n+1/2}}{2\Delta\sigma} \frac{(s_{f,i+1}^{n+1} - s_{f,i-1}^{n+1} + s_{f,i+1}^{n} - s_{f,i-1}^{n})}{(s_{f,i}^{n+1} + s_{f,i}^{n})}\right]^2, \\
a_{2,m,i}^{n+1/2} &= \rho_{m,i}^{n+1/2}(\sigma_l + i\Delta\sigma) - \frac{q_{m,i}^{n+1/2}}{2\Delta\sigma} \frac{(s_{f,i+1}^{n+1} - s_{f,i-1}^{n+1} + s_{f,i+1}^{n} - s_{f,i-1}^{n})}{(s_{f,i}^{n+1} + s_{f,i}^{n})}, \\
a_{3,m,i}^{n+1/2} &= \frac{1}{2}, \\
a_{4,m,i}^{n+1/2} &= \frac{2}{(s_{f,i}^{n+1} + s_{f,i}^{n})} \frac{s_{f,i}^{n+1} - s_{f,i}^{n}}{\Delta\tau} + r - D_0 \\
&\quad - \left[\rho_{m,i}^{n+1/2}(\sigma_l + i\Delta\sigma) q_{m,i}^{n+1/2} + p_{m,i}^{n+1/2} - \lambda_{m,i}^{n+1/2} q_{m,i}^{n+1/2}\right] \\
&\qquad \times \frac{s_{f,i+1}^{n+1} - s_{f,i-1}^{n+1} + s_{f,i+1}^{n} - s_{f,i-1}^{n}}{2\Delta\sigma(s_{f,i}^{n+1} + s_{f,i}^{n})}
\end{aligned}$$

$$+\left[\frac{q_{m,i}^{n+1/2}}{2\Delta\sigma}\frac{(s_{f,i+1}^{n+1}-s_{f,i-1}^{n+1}+s_{f,i+1}^{n}-s_{f,i-1}^{n})}{(s_{f,i}^{n+1}+s_{f,i}^{n})}\right]^2$$
$$-\left[\frac{q_{m,i}^{n+1/2}}{\Delta\sigma}\right]^2\frac{s_{f,i+1}^{n+1}-2s_{f,i}^{n+1}+s_{f,i-1}^{n+1}+s_{f,i+1}^{n}-2s_{f,i}^{n}+s_{f,i-1}^{n}}{2\left(s_{f,i}^{n+1}+s_{f,i}^{n}\right)},$$

$$a_{5,m,i}^{n+1/2}=p_{m,i}^{n+1/2}-\lambda_{m,i}^{n+1/2}q_{m,i}^{n+1/2},$$
$$a_{6,m,i}^{n+1/2}=-r,$$
$$a_{7,m,i}^{n+1/2}=-f(m\Delta\xi(s_{f,i}^{n+1}+s_{f,i}^{n})/2,\ \sigma_l+i\Delta\sigma,\ T-(n+1/2)\Delta\tau).$$

At the boundaries $\sigma=\sigma_l$ and $\sigma=\sigma_u$, due to $q=0$, the partial differential equation in the problem (9.11) becomes

$$\frac{\partial u}{\partial\tau}=a_1\xi^2\frac{\partial^2u}{\partial\xi^2}+a_4\xi\frac{\partial u}{\partial\xi}+a_5\frac{\partial u}{\partial\sigma}+a_6u+a_7.$$

Just like the European case, this equation possesses hyperbolic properties in the σ-direction. Hence, we can approximate the partial differential equation in problem (9.11) at the boundary $\sigma=\sigma_l$ by

$$\begin{aligned}&\frac{u_{m,0}^{n+1}-u_{m,0}^{n}}{\Delta\tau}\\
&=\frac{a_1m^2}{2}(u_{m+1,0}^{n+1}-2u_{m,0}^{n+1}+u_{m-1,0}^{n+1}+u_{m+1,0}^{n}-2u_{m,0}^{n}+u_{m-1,0}^{n})\\
&\quad+\frac{a_4m}{4}(u_{m+1,0}^{n+1}-u_{m-1,0}^{n+1}+u_{m+1,0}^{n}-u_{m-1,0}^{n})\\
&\quad+\frac{a_5}{4\Delta\sigma}(-u_{m,2}^{n+1}+4u_{m,1}^{n+1}-3u_{m,0}^{n+1}-u_{m,2}^{n}+4u_{m,1}^{n}-3u_{m,0}^{n})\\
&\quad+\frac{a_6}{2}(u_{m,0}^{n+1}+u_{m,0}^{n})+a_7,\qquad m=0,1,\cdots,M-1\end{aligned}\tag{9.42}$$

and at the boundary $\sigma=\sigma_u$ by

$$\begin{aligned}&\frac{u_{m,I}^{n+1}-u_{m,I}^{n}}{\Delta\tau}\\
&=\frac{a_1m^2}{2}(u_{m+1,I}^{n+1}-2u_{m,I}^{n+1}+u_{m-1,I}^{n+1}+u_{m+1,I}^{n}-2u_{m,I}^{n}+u_{m-1,I}^{n})\\
&\quad+\frac{a_4m}{4}(u_{m+1,I}^{n+1}-u_{m-1,I}^{n+1}+u_{m+1,I}^{n}-u_{m-1,I}^{n})\\
&\quad+\frac{a_5}{4\Delta\sigma}(3u_{m,I}^{n+1}-4u_{m,I-1}^{n+1}+u_{m,I-2}^{n+1}\\
&\qquad+3u_{m,I}^{n}-4u_{m,I-1}^{n}+u_{m,I-2}^{n})\\
&\quad+\frac{a_6}{2}(u_{m,I}^{n+1}+u_{m,I}^{n})+a_7,\qquad m=0,1,\cdots,M-1.\end{aligned}\tag{9.43}$$

Here, $\dfrac{\partial u}{\partial \sigma}$ is discretized by a one-sided second-order scheme in order for all the node points involved to be in the computational domain. Here, a_1 and a_4–a_7 are also evaluated at ξ_m, σ_i and $\tau^{n+1/2}$. The formulae for a_1 and a_4–a_7 are almost the same as those given above, except that the partial derivative $\dfrac{\partial s_f}{\partial \sigma}$ is discretized in the same way as $\dfrac{\partial u}{\partial \sigma}$. That is, $\dfrac{\partial s_f}{\partial \sigma}$ in the difference scheme (9.42) is approximated by

$$\frac{-s_{f,2}^{n+1} + 4s_{f,1}^{n+1} - 3s_{f,0}^{n+1} - s_{f,2}^{n} + 4s_{f,1}^{n} - 3s_{f,0}^{n}}{4\Delta\sigma}$$

and in scheme (9.43) by

$$\frac{3s_{f,I}^{n+1} - 4s_{f,I-1}^{n+1} + s_{f,I-2}^{n+1} + 3s_{f,I}^{n} - 4s_{f,I-1}^{n} + s_{f,I-2}^{n}}{4\Delta\sigma}.$$

From the expression for a_4, we see that because $q = 0$ at $\sigma = \sigma_l$ and $\sigma = \sigma_u$, we do not need one-sided second-order finite-difference schemes for $\dfrac{\partial^2 s_f}{\partial \sigma^2}$.

Noticing that the coefficients of $\dfrac{\partial^2 u}{\partial \xi^2}, \dfrac{\partial u}{\partial \xi}$ in the problem (9.11) at $\xi = 0$ are zero, $u_{-1,i}^n$ does not appear in Eqs. (9.41)–(9.43) with $m = 0$.

At $\xi = 1$, there are two boundary conditions in the problem (9.11). One can be written as

$$u_{M,i}^{n+1} = g(s_{f,i}^{n+1}, \tau^{n+1}), \quad i = 0, 1, 2, \cdots, I, \tag{9.44}$$

where

$$g(s_f, \tau) = s_f\left[1 - \mathrm{e}^{-D_0\tau} N(d_1)\right] - E\left[1 - \mathrm{e}^{-r\tau} N(d_2)\right].$$

The other can be approximated by

$$3u_{M,i}^{n+1} - 4u_{M-1,i}^{n+1} + u_{M-2,i}^{n+1} = 2\Delta\xi h(s_{f,i}^{n+1}, \tau^{n+1}), \quad i = 0, 1, \cdots, I,$$

or

$$\begin{gathered} 3g(s_{f,i}^{n+1}, \tau^{n+1}) - 4u_{M-1,i}^{n+1} + u_{M-2,i}^{n+1} = 2\Delta\xi h(s_{f,i}^{n+1}, \tau^{n+1}), \\ i = 0, 1, \cdots, I, \end{gathered} \tag{9.45}$$

where

$$h(s_f, \tau) = s_f\left[1 - \mathrm{e}^{-D_0\tau} N(d_1)\right].$$

At $\tau = 0$, from

$$u(\xi, \sigma, 0) = 0 \quad \text{ and } \quad s_f(\sigma, 0) = \max(E, rE/D_0),$$

we have

$$\begin{cases} u_{m,i}^0 = 0, \qquad m = 0, 1, \cdots, M, \quad i = 0, 1, \cdots, I, \\ s_{f,i}^0 = \max(E, rE/D_0), \qquad i = 0, 1, \cdots, I. \end{cases} \tag{9.46}$$

For a fixed n, the system (9.41)–(9.45) consists of $(M+2)(I+1)$ equations. If $u_{m,i}^n$, $m = 0, 1, \cdots, M$, $i = 0, 1, \cdots, I$ and $s_{f,i}^n$, $i = 0, 1, \cdots, I$ are known, then in the system there are $(M+2)(I+1)$ unknowns, namely, $u_{m,i}^{n+1}$, $m = 0, 1, \cdots, M$, $i = 0, 1, \cdots, I$ and $s_{f,i}^{n+1}$, $i = 0, 1, \cdots, I$, and these unknowns can be obtained from solving the system. Because the set of initial conditions (9.46) gives $u_{m,i}^0$ for all m and i and $s_{f,i}^0$ for all i, we can have $u_{m,i}^{n+1}$, $i = 0, 1, \cdots, I$, $m = 0, 1, \cdots, M$ and $s_{f,i}^{n+1}$, $i = 0, 1, \cdots, I$ for $n = 0, 1, \cdots, N-1$ successively.

There are many ways to solve the above nonlinear system. If $s_{f,i}^{n+1}$, $i = 0, 1, \cdots, I$ are given, then the system consisting of Eqs. (9.41)–(9.44) is a linear system for $u_{m,i}^{n+1}$, $m = 0, 1, \cdots, M$ and $i = 0, 1, \cdots, I$. One way to solve the system is as follows. Guessing $s_{f,i}^{n+1}$, $i = 0, 1, \cdots, I$ and solving the system (9.41)–(9.44), we get all the approximate $u_{m,i}^{n+1}$, $m = 0, 1, \cdots, M$, and $i = 0, 1, \cdots, I$. Then check if Eq. (9.45) holds. If it does, we get our solution; if not, we determine new $s_{f,i}^{n+1}$, $i = 0, 1, \cdots, I$, in the following way.

For each i, Eq. (9.45) is a nonlinear equation for $s_{f,i}^{n+1}$ when $u_{M-1,i}^{n+1}$ and $u_{M-2,i}^{n+1}$ are given. We take the root of the nonlinear equation as the new value of $s_{f,i}^{n+1}$. This root can be determined by Newton's method based on Eq. (9.45):

$$s_{f,i}^{(k+1)} = s_{f,i}^{(k)} - \frac{\theta(s_{f,i}^{(k)})}{\theta'(s_{f,i}^{(k)})},$$

where $s_{f,i}^{(k)}$ is the k-th iterative value of $s_{f,i}^{n+1}$ and

$$\begin{aligned}
\theta(s_{f,i}, \tau^{n+1}) &= 3g(s_{f,i}, \tau^{n+1}) - 4u_{M-1,i}^{n+1} + u_{M-2,i}^{n+1} - 2\Delta\xi h(s_{f,i}, \tau^{n+1}), \\
\theta'(s_{f,i}, \tau^{n+1}) &= 3\frac{\partial g}{\partial s_{f,i}}(s_{f,i}, \tau^{n+1}) - 2\Delta\xi \frac{\partial h}{\partial s_{f,i}}(s_{f,i}, \tau^{n+1}) \\
&= (3 - 2\Delta\xi)\left[1 - \mathrm{e}^{-D_0\tau^{n+1}} N(d_1)\right] + \frac{2\Delta\xi}{\sigma\sqrt{2\pi\tau^{n+1}}} \mathrm{e}^{-D_0\tau^{n+1} - d_1^2/2}
\end{aligned}$$

with $d_1 = \dfrac{\ln(s_{f,i}/E) + (r - D_0 + \sigma^2/2)\tau^{n+1}}{\sigma\sqrt{\tau^{n+1}}}$. As the starting value $s_{f,i}^{(0)}$ of this procedure, we take the value of $s_{f,i}^{n+1}$ used when the system (9.41)–(9.44) is solved previously.

9.2.6 Numerical Results of Two-Factor Options

Now let us show some results obtained by the numerical method above. We use the following two stochastic volatility models:

$$d\sigma = a(b-\sigma)dt + c\frac{1-\left(1-2\dfrac{\sigma-\sigma_l}{\sigma_u-\sigma_l}\right)^2}{1-0.975\left(1-2\dfrac{\sigma-\sigma_l}{\sigma_u-\sigma_l}\right)^2}\sigma dX_2, \quad \sigma_l \le \sigma \le \sigma_u \quad (9.47)$$

and

$$d\sigma = a(b-\sigma)dt + c\left[\frac{(\sigma-\sigma_l)(\sigma_u-\sigma)}{(\sigma_u-\sigma_l)^2}\right]^{1/2}\sigma dX_2, \quad \sigma_l \le \sigma \le \sigma_u, \qquad (9.48)$$

where a, b, and c are positive parameters. The models (9.47) and (9.48) are referred to as Model I and Model II, respectively, in what follows. Both models are in the form (8.98). There is only a little difference between them. In Model I, $\dfrac{\partial q(\sigma,t)}{\partial \sigma}$ is bounded on $[\sigma_l, \sigma_u]$, and the reversion conditions are reduced to the conditions (8.101) and (8.102). Clearly, $q(\sigma_l) = q(\sigma_u) = 0$, so the equality conditions in the conditions (8.101) and (8.102) hold. In this case, the inequality conditions are $a(b-\sigma_l) \ge 0$ and $a(b-\sigma_u) \le 0$, which can be combined into

$$\sigma_l \le b \le \sigma_u. \qquad (9.49)$$

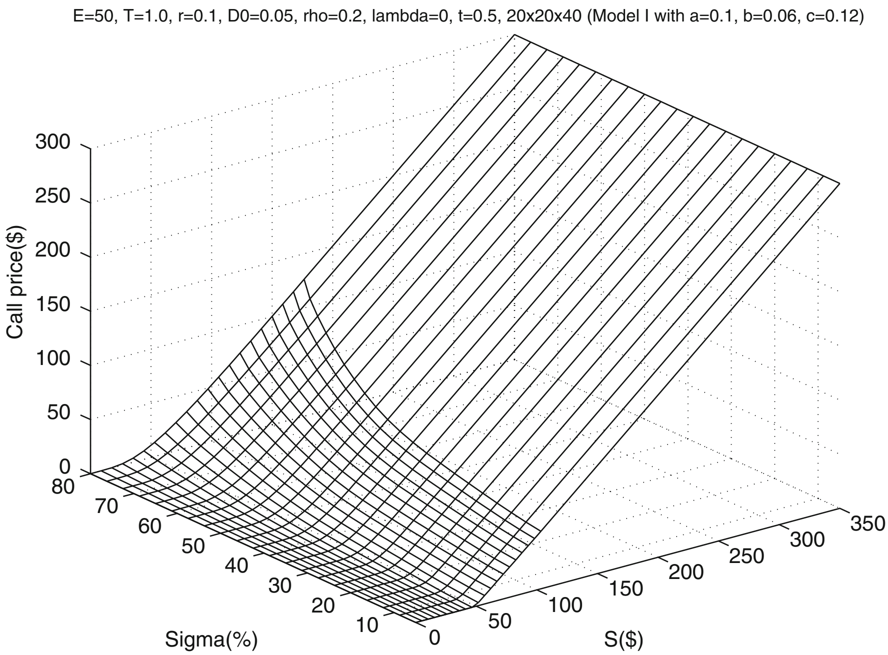

Fig. 9.14. The American call price ($t = 0.5$, $T = 1.0$, $\rho = 0.2$, and $\lambda = 0$)

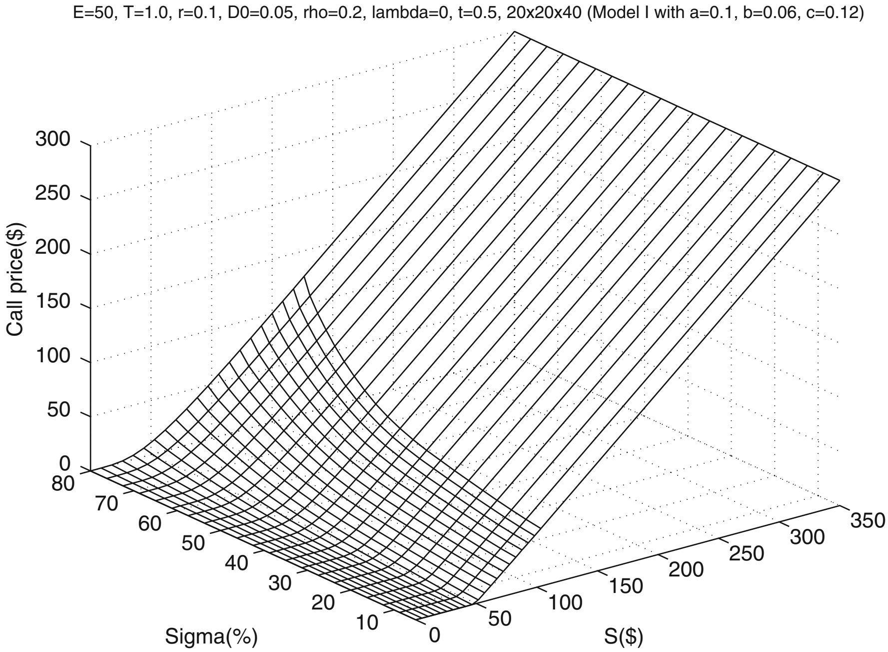

Fig. 9.15. The American call price ($t = 0.5$, $T = 1.0$, $\rho = 0.2$, and $\lambda = 0$)

Consequently, when the relation (9.49) holds, Model I satisfies the reversion conditions. For Model II, the equality conditions of the conditions (8.99) and (8.100) always hold, and the inequality conditions become

$$\begin{cases} p(\sigma_l, t) - q(\sigma_l, t)\dfrac{\partial q(\sigma_l, t)}{\partial \sigma} = a(b - \sigma_l) - 0.5c^2\sigma_l^2/(\sigma_u - \sigma_l) \geq 0, \\ p(\sigma_u, t) - q(\sigma_u, t)\dfrac{\partial q(\sigma_u, t)}{\partial \sigma} = a(b - \sigma_u) + 0.5c^2\sigma_u^2/(\sigma_u - \sigma_l) \leq 0. \end{cases} \tag{9.50}$$

Therefore, in order for Model II to satisfy the reversion conditions (8.99) and (8.100), we require that the set of conditions (9.50) holds. In the following examples, we take $\sigma_l = 0.05$ and $\sigma_u = 0.8$.

Example 1. Here, we calculate a 1-year American call option with Model I. We choose $a = 0.1$, $b = 0.06$, $c = 0.12$, $\rho = 0.2$, and $\lambda = 0$. We take 20 grid points in the ξ-direction and 20 grid points in the σ-direction and 40 time steps in the τ-direction, namely, the mesh is $20 \times 20 \times 40$. The other parameters are $E = 50$, $r = 0.1$, and $D_0 = 0.05$.

Figures 9.14 and 9.15 show the values of the American call option with $T = 1$ at time $t = 0.5$ and $t = 0$. Because those parameters a, b, c, ρ, and λ do not depend on time, Fig. 9.14 also shows the value of an option with $T = 0.5$ at time $t = 0$. Here, the strips represent the plane $C = S - E$, the solution for

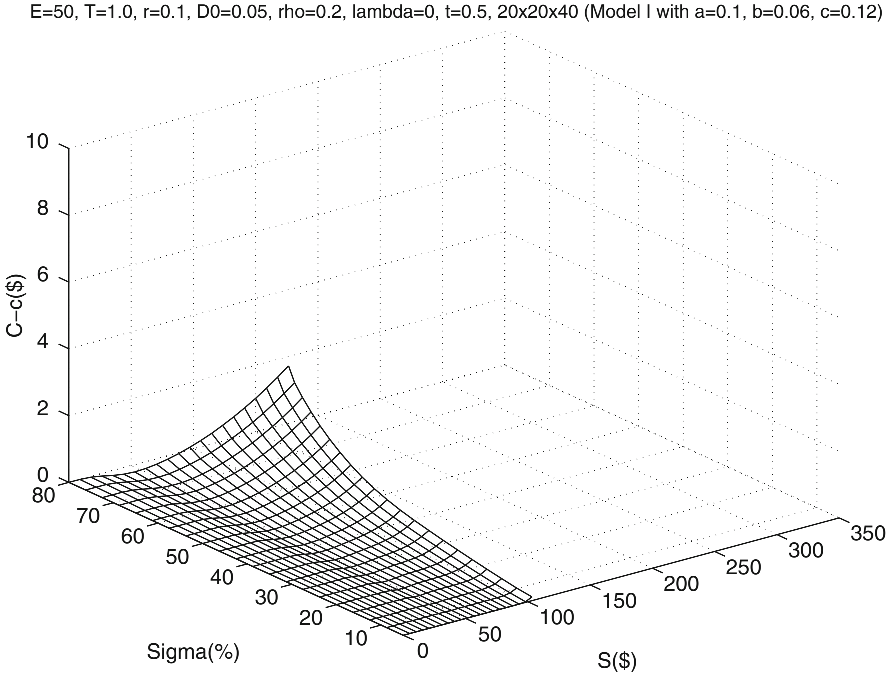

Fig. 9.16. The difference function $\overline{C} = C - c$ ($t = 0.5$, $T = 1.0$, $\rho = 0.2$, and $\lambda = 0$)

Table 9.9. Numerical solutions with extrapolation

($E = 50$, $T = 3.0$, $r = 0.1$, $D_0 = 0.05$, $a = 0.1$, $b = 0.06$, $c = 0.12$, $\rho = 0.2$, and $\lambda = 0$)

σ	S	u_1	u_2	u_3	U_1^*	U_2^*
0.125	45	3.93255	3.93246	3.93148	3.93115	3.93097
0.125	50	7.01873	7.02191	7.02241	7.02257	7.02253
0.125	55	10.7219	10.7224	10.7225	10.7225	10.7226
0.200	45	5.58000	5.57170	5.57137	5.57126	5.57160
0.200	50	8.49808	8.49254	8.49207	8.49191	8.49209
0.200	55	11.8781	11.8756	11.8742	11.8737	11.8736
0.350	45	8.93697	8.92810	8.92576	8.92498	8.92496
0.350	50	11.8615	11.8610	11.8607	11.8606	11.8605
0.350	55	15.1021	15.0953	15.0925	15.0916	15.0914

$S > S_f(\sigma, t)$, and the meshed surface shows the solution for $S \leq S_f(\sigma, t)$. In Figs. 9.16 and 9.17, the difference $\overline{C}$ is shown for $t = 0.5$ and 0, respectively. There, only the solution of the free-boundary problem has been shown. As we know, the derivative of C with respect to S at $t = T$ is discontinuous at $S = E$. Comparing Figs. 9.14 and 9.15, we see that the value of C becomes smoother as t decreases. However, we know from Fig. 9.15 that even at $t = 0$,

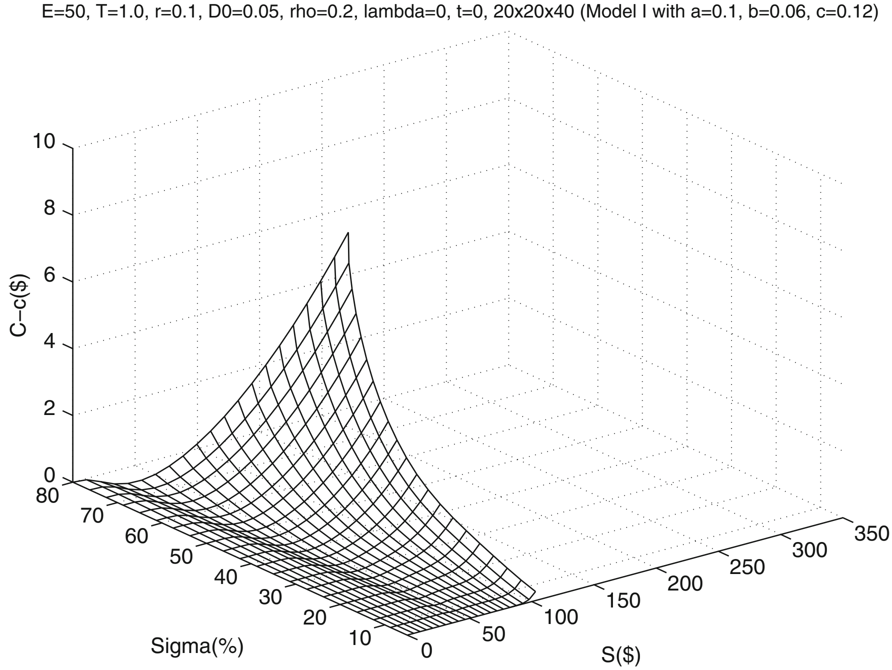

Fig. 9.17. The difference function $\overline{C} = C - c$ ($t = 0.0$, $T = 1.0$, $\rho = 0.2$, and $\lambda = 0$)

for smaller σ, C still changes rapidly with respect to S near $S = E$. The difference $\overline{C}$ at $t = T$ is identically equal to zero and remains smooth as t decreases, which can be seen from Figs. 9.16 and 9.17. Because $\overline{C}$ is much smoother than C, we can have much better numerical results if we use the partial differential equation for $\overline{C}$ instead of C when we do the computation.

Example 2. In this example, we calculate a 3-year American call option for Model II. All the other parameters, except ρ and λ, are the same as those in Example 1. In this case

$$a(b - \sigma_l) - 0.5c^2\sigma_l^2/(\sigma_u - \sigma_l) = 0.000976 > 0$$

and

$$a(b - \sigma_u) + 0.5c^2\sigma_u^2/(\sigma_u - \sigma_l) = -0.067856 < 0.$$

Thus, the set of conditions (9.50) holds, and no boundary condition is needed in order to determine the price.

First, we take $\rho = 0.2$ and $\lambda = 0$ and do the computation on different meshes. In Table 9.9, u_1 is the numerical solution using a mesh of $10 \times 10 \times 20$, u_2 is the value using a mesh of $20 \times 20 \times 40$, and u_3 is the value using a mesh

Table 9.10. Comparison of the results with various λ

($E = 50$, $T = 3.0$, $r = 0.1$, $D_0 = 0.05$, $a = 0.1$, $b = 0.06$, $c = 0.12$, and $\rho = 0.2$)

σ	S	$\lambda = -1.0$	$\lambda = -0.5$	$\lambda = 0$	$\lambda = 0.5$	$\lambda = 1.0$	One-factor
0.125	45	4.1031	4.0170	3.9310	3.8449	3.7589	4.1789
0.125	50	7.1681	7.0953	7.0225	6.9497	6.8770	7.2432
0.125	55	10.825	10.774	10.723	10.672	10.620	10.885
0.200	45	5.9394	5.7555	5.5716	5.3877	5.2037	6.1134
0.200	50	8.8482	8.6701	8.4921	8.3141	8.1360	9.0216
0.200	55	12.193	12.033	11.874	11.714	11.554	12.361
0.350	45	9.6937	9.3085	8.9250	8.5417	8.1576	9.9913
0.350	50	12.647	12.254	11.861	11.467	11.072	12.964
0.350	55	15.876	15.485	15.091	14.701	14.309	16.209

Table 9.11. Comparison of the results with various ρ

($E = 50$, $T = 3.0$, $r = 0.1$, $D_0 = 0.05$, $a = 0.1$, $b = 0.06$, $c = 0.12$, and $\lambda = 0$)

σ	S	$\rho = 0.4$	$\rho = 0.2$	$\rho = 0$	$\rho = -0.2$	$\rho = -0.4$	One-factor
0.125	45	3.9290	3.9310	3.9329	3.9349	3.9369	4.1789
0.125	50	7.0127	7.0225	7.0323	7.0422	7.0520	7.2432
0.125	55	10.711	10.723	10.735	10.747	10.759	10.885
0.200	45	5.5735	5.5716	5.5697	5.5678	5.5659	6.1134
0.200	50	8.4815	8.4921	8.5027	8.5134	8.5240	9.0216
0.200	55	11.854	11.874	11.893	11.913	11.933	12.361
0.350	45	8.9460	8.9250	8.9038	8.8825	8.8616	9.9913
0.350	50	11.866	11.861	11.855	11.849	11.844	12.964
0.350	55	15.083	15.091	15.100	15.109	15.118	16.209

of $40 \times 40 \times 80$. There, we also give results when the extrapolation technique is used. U_1^* is the extrapolation value obtained by

$$U_1^* = \frac{1}{3}(4u_3 - u_2)$$

and U_2^* is the extrapolation value generated by

$$U_2^* = \frac{1}{21}(32u_3 - 12u_2 + u_1).$$

From the table, we see that the errors of u_1 are on the second decimal place, those of u_2 and u_3 are on the third decimal place, and for the extrapolation values U_1^* and U_2^*, they are on the fourth decimal place. This shows that the extrapolation technique increases accuracy.

Then, we take $\rho = 0.2$ and try different λ to see how the results vary. The mesh used is $40 \times 40 \times 80$. In Table 9.10, we compare the values of the options with different parameters λ. The columns with $\lambda = -1, -0.5, 0, 0.5, 1.0$ at the top contain the values of the options when $\lambda = -1, -0.5, 0, 0.5, 1.0$,

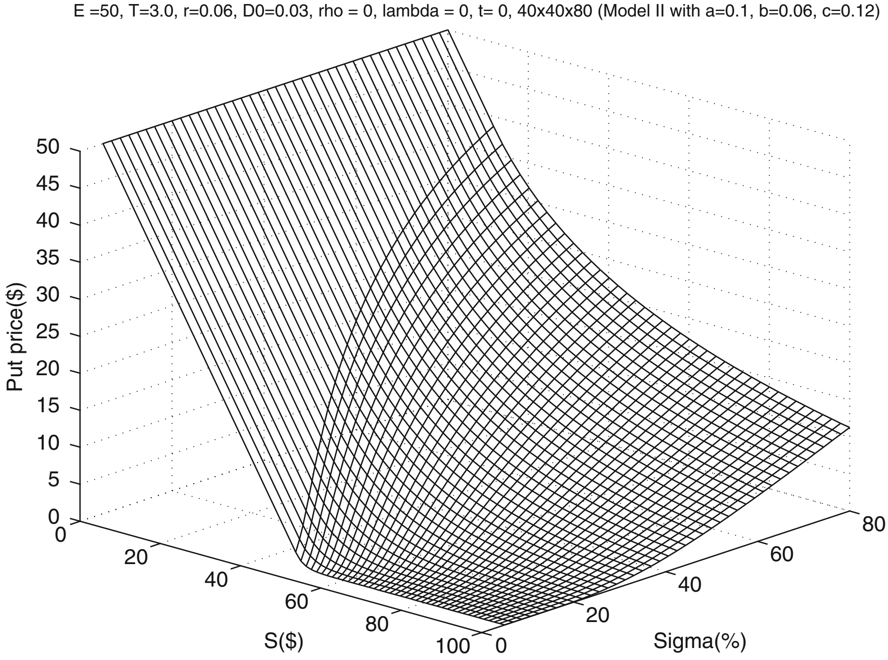

Fig. 9.18. The American put value at $t = 0.0$ ($E = 50$, $T = 3.0$, $\rho = 0$, and $\lambda = 0$)

respectively. For this case, the smaller the λ, the higher the call option price. The difference among the results for $\lambda \in [-1, 1]$ is about 10 %–20 %. This shows that we can calibrate the model to some extent even if we choose a constant λ. We also list the values of the one-factor model with a constant volatility. From Table 9.10, we see that the one-factor model overprices the American call options.

In Table 9.11, we compare the values of the options with a different correlation factor ρ and $\lambda = 0$, while the other parameters are kept unchanged. The notation is similar to Table 9.10. The results show that the option price varies a little when the correlation factor changes. Here, we again see that one factor model overprices the American call option.

An American two-factor put option problem can also be reduced to solving a free-boundary problem. However, the free-boundary problem is defined on an infinite domain. As we have pointed out, a vanilla two-factor put option can be converted into a vanilla two-factor call option with the same parameters except for r, D_0, ρ, and λ. Therefore, as long as we have a code for call options, we can also obtain the price of any put option.

Example 3. We want to have the price of a put option with $r = 0.06$, $D_0 = 0.03$, $\rho = 0$, and $\lambda = 0$ for Model II. The other parameters are the same as those in Example 2. We use a mesh $40 \times 40 \times 80$. In order to do this, we can

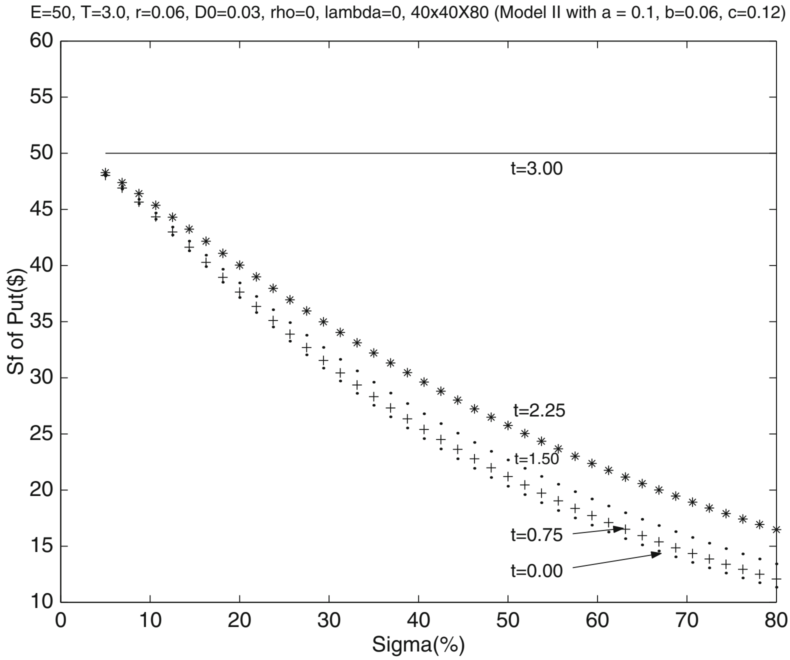

Fig. 9.19. The free boundaries of a put option for different times ($E = 50$, $T = 3.0$, $\rho = 0$, and $\lambda = 0$)

first calculate a call option with $r = 0.03$, $D_0 = 0.06$, $\rho = 0$, and $\lambda = 0$. Then, using the set of relations (9.14), we can have the price and the optimal exercise price of the put option. In Figs. 9.18 and 9.19, the price of the put option at $t = 0$ for $S \in [0, 100]$ and the optimal exercise price at $t = 0, 0.75, 1.5, 2.25$, and 3.0 are shown.

For more results and details on two-factor options, see [56, 93, 94]. Finally, we point out that the models given here are assumptions. In order to use such a computation in practice, the models should be found from the market data.

9.3 Pseudo-Spectral Methods

9.3.1 The Description of the Pseudo-Spectral Methods for Two-Factor Convertible Bonds

A free-boundary problem can also be solved by pseudo-spectral methods. If the solution is smooth, then the pseudo-spectral method as a high-order difference method may be more efficient. Thus, when we compute $\overline{C} = C - c$, the pseudo-spectral method might be another good tool. Also, a parabolic operator always smoothes the solution. Thus, even if the initial value is not smooth, the solution becomes smooth after a while. The life span of a convertible bond is quite long. If the time to the expiry is more than 2 years, the solution is already quite smooth. Thus, if expiry is not soon, then the solution of a convertible bond is quite smooth and for that time, a pseudo-spectral method might be a good choice. In the last section, we already took the American call option as an example to give the details of the implicit difference methods. In this section, we will describe the details of the pseudo-spectral method for the two-factor convertible bond problem.

In Sect. 9.1.2, a two-factor convertible bond problem with $D_0 > 0$ was reduced to the problem (9.15) or the problem (9.21). Suppose that we do not take the difference and want to solve $V(s, r, t)$ directly, that is, we solve the problem (9.15). Using the transformation (9.19) and defining $u(\xi, \bar{r}, \tau)$ and $s_f(\bar{r}, \tau)$ by the set of formulae (9.20), we can rewrite the problem (9.15) as the following problem on $u(\xi, \bar{r}, \tau)$ and $s_f(\bar{r}, \tau)$:

$$\begin{cases} \dfrac{\partial u}{\partial \tau} = \mathbf{L}_{\xi,\bar{\mathbf{r}}} u + a_7, & 0 \le \xi \le 1, \quad 0 \le \bar{r} \le 1, \quad 0 \le \tau \le T, \\ u(\xi, \bar{r}, 0) = \max\left(1, \xi s_f(\bar{r}, 0)\right), & 0 \le \xi \le 1, \quad 0 \le \bar{r} \le 1, \\ u(1, \bar{r}, \tau) = s_f(\bar{r}, \tau), & 0 \le \bar{r} \le 1, \quad 0 \le \tau \le T, \\ \dfrac{\partial u}{\partial \xi}(1, \bar{r}, \tau) = s_f(\bar{r}, \tau), & 0 \le \bar{r} \le 1, \quad 0 \le \tau \le T, \\ s_f(\bar{r}, 0) = \max\left(1, k/D_0\right), & 0 \le \bar{r} \le 1, \end{cases} \tag{9.51}$$

where $\mathbf{L}_{\xi,\bar{\mathbf{r}}}$ is the same as given in the problem (9.21):

$$\mathbf{L}_{\xi,\bar{\mathbf{r}}} = a_1 \xi^2 \frac{\partial^2}{\partial \xi^2} + a_2 \xi w \frac{\partial^2}{\partial \xi \partial \bar{r}} + a_3 w^2 \frac{\partial^2}{\partial \bar{r}^2} + \left(a_4 + \frac{1}{s_f} \frac{\partial s_f}{\partial \tau}\right) \xi \frac{\partial}{\partial \xi} + a_5 \frac{\partial}{\partial \bar{r}} + a_6$$

and

$$a_7 = k.$$

Therefore, finding the value of a convertible bond is now reduced to solving a problem on a rectangular domain. Suppose that we take $M+1$ nodes in the ξ-direction: $\xi_0, \xi_1, \cdots, \xi_M$, $L+1$ nodes in the $\bar{r}$-direction: $\bar{r}_0, \bar{r}_1, \cdots, \bar{r}_L$, and $N+1$ nodes in the τ-direction: $\tau^0, \tau^1, \cdots, \tau^N$, where $\xi_0 = 0$, $\xi_M = 1$, $\bar{r}_0 = 0$, $\bar{r}_L = 1$, $\tau^0 = 0$, and $\tau^N = T$. Furthermore, we assume the nodes in the τ-direction to be equidistant with $\Delta\tau = T/N$. Let $u^n_{m,l}$ denote $u(\xi_m, \bar{r}_l, \tau^n)$ and $s^n_{f,l}$ stand for $s_f(\bar{r}_l, \tau^n)$. For a fixed n, we need to determine $u^n_{m,l}, m = 0, 1, \cdots, M$ and $l = 0, 1, \cdots, L$, and $s^n_{f,l}, l = 0, 1, \cdots, L$. In what follows, let $\left\{u^n_{m,l}\right\}$ and $\left\{s^n_{f,l}\right\}$ denote the sets

$$\left\{u^n_{m,l}, m = 0, 1, \cdots, M \text{ and } l = 0, 1, \cdots, L\right\}$$

and

$$\left\{s_{f,l}^n, l = 0, 1, \cdots, L\right\}$$

respectively. For $n = 0$, $\left\{u_{m,l}^n\right\}$ and $\left\{s_{f,l}^n\right\}$ are determined by the initial conditions of the problem, which gives

$$\begin{aligned} u_{m,l}^0 &= \max\left(1, \xi_m s_{f,l}^0\right), \quad m = 0, 1, \cdots, M, \quad l = 0, 1, \cdots, L, \\ s_{f,l}^0 &= \max\left(1, k/D_0\right), \quad l = 0, 1, \cdots, L. \end{aligned}$$

What we need to do is to find $\left\{u_{m,l}^n\right\}$ and $\left\{s_{f,l}^n\right\}$ for $n = 1, 2, \cdots, N$.

According to Sect. 6.1.2, we may assume the solution on the domain $[0,1] \times [0,1]$ to be polynomials in each direction. Under such an assumption, for a fixed n, $\dfrac{\partial u}{\partial \xi}, \dfrac{\partial u}{\partial \bar{r}}, \dfrac{\partial^2 u}{\partial \xi^2}, \dfrac{\partial^2 u}{\partial \xi \partial \bar{r}}$, and $\dfrac{\partial^2 u}{\partial \bar{r}^2}$ at any point are linear combinations of $u_{m,l}^n$, $m = 0, 1, \cdots, M$ and $l = 0, 1. \cdots, L$, and $\dfrac{\partial s_f}{\partial \bar{r}}$ at any $\bar{r}$ is a linear combination of $s_{f,l}^n$, $l = 0, 1. \cdots, L$. Therefore, the partial differential equation and the boundary conditions in the problem (9.51) can be discretized into algebraic equations, and solving the equations yields $\left\{u_{m,l}^n\right\}$ and $\left\{s_{f,l}^n\right\}$.

Now we describe the details. Suppose that for a fixed pair of n and l, $u_{m,l}^n$, $m = 0, 1, \cdots, M$ are known. According to these values, we can establish a polynomial in ξ with degree M. From this polynomial, we can determine $\dfrac{\partial u}{\partial \xi}, \dfrac{\partial^2 u}{\partial \xi^2}$ at any point for $\bar{r} = \bar{r}_l$ and $\tau = \tau^n$. If ξ_m is defined as follows:

$$\xi_m = \frac{1}{2}\left(1 - \cos\frac{m\pi}{M}\right), \quad m = 0, 1, \cdots, M,$$

then

$$\begin{aligned} \frac{\partial u}{\partial \xi}(\xi_m, \bar{r}_l, \tau^n) &= \sum_{i=0}^{M} D_{\xi,m,i} u(\xi_i, \bar{r}_l, \tau^n), \\ \frac{\partial^2 u}{\partial \xi^2}(\xi_m, \bar{r}_l, \tau^n) &= \sum_{i=0}^{M} D_{\xi\xi,m,i} u(\xi_i, \bar{r}_l, \tau^n) \end{aligned}$$

and according to Sect. 6.1.2,

$$D_{\xi,m,i} = \begin{cases} \dfrac{c_m(-1)^{m+i}}{c_i(\xi_m - \xi_i)}, & m \neq i, \\ -\dfrac{2M^2 + 1}{3}, & m = i = 0, \\ \dfrac{1 - 2\xi_i}{4\xi_i(1 - \xi_i)}, & m = i = 1, 2, \cdots, M - 1, \\ \dfrac{2M^2 + 1}{3}, & m = i = M, \end{cases}$$

where $c_0 = c_M = 2$ and $c_i = 1$, $i = 1, 2, \cdots, M-1$, and

$$D_{\xi\xi,m,i} = \sum_{k=0}^{M} D_{\xi,m,k} D_{\xi,k,i}.$$

For brevity, we define

$$\mathbf{D}_{\xi,\mathbf{m}} u_{m,l}^n = \sum_{i=0}^{M} D_{\xi,m,i} u(\xi_i, \bar{r}_l, \tau^n),$$
$$\mathbf{D}_{\xi\xi,\mathbf{m}} u_{m,l}^n = \sum_{i=0}^{M} D_{\xi\xi,m,i} u(\xi_i, \bar{r}_l, \tau^n)$$

and write the two approximations in difference operator form:

$$\frac{\partial u}{\partial \xi}(\xi_m, \bar{r}_l, \tau^n) = \mathbf{D}_{\xi,\mathbf{m}} u_{m,l}^n,$$
$$\frac{\partial^2 u}{\partial \xi^2}(\xi_m, \bar{r}_l, \tau^n) = \mathbf{D}_{\xi\xi,\mathbf{m}} u_{m,l}^n$$

Similarly, if $\bar{r}_l$ is defined by

$$\bar{r}_l = \frac{1}{2}\left(1 - \cos\frac{l\pi}{L}\right), \quad l = 0, 1, \cdots, L,$$

then

$$\frac{\partial u}{\partial \bar{r}}(\xi_m, \bar{r}_l, \tau^n) = \mathbf{D}_{\bar{\mathbf{r}},\mathbf{l}} u_{m,l}^n,$$
$$\frac{\partial^2 u}{\partial \bar{r}^2}(\xi_m, \bar{r}_l, \tau^n) = \mathbf{D}_{\bar{\mathbf{r}}\bar{\mathbf{r}},\mathbf{l}} u_{m,l}^n,$$

and

$$\frac{\partial^2 u}{\partial \xi \partial \bar{r}}(\xi_m, \bar{r}_l, \tau^n) = \mathbf{D}_{\bar{\mathbf{r}},\mathbf{l}} \mathbf{D}_{\xi,\mathbf{m}} u_{m,l}^n.$$

These difference operators are defined by

$$\mathbf{D}_{\bar{\mathbf{r}},\mathbf{l}} u_{m,l}^n = \sum_{j=0}^{L} D_{\bar{r},l,j} u(\xi_m, \bar{r}_j, \tau^n),$$
$$\mathbf{D}_{\bar{\mathbf{r}}\bar{\mathbf{r}},\mathbf{l}} u_{m,l}^n = \sum_{j=0}^{L} D_{\bar{r}\bar{r},l,j} u(\xi_m, \bar{r}_j, \tau^n),$$
$$\mathbf{D}_{\bar{\mathbf{r}},\mathbf{l}} \mathbf{D}_{\xi,\mathbf{m}} u_{m,l}^n = \sum_{j=0}^{L} D_{\bar{r},l,j} \sum_{i=0}^{M} D_{\xi,m,i} u(\xi_i, \bar{r}_j, \tau^n),$$

where

$$D_{\bar{r},l,j} = \begin{cases} \dfrac{c_l(-1)^{l+j}}{c_j(\bar{r}_l - \bar{r}_j)}, & l \neq j, \\ -\dfrac{2L^2+1}{3}, & l = j = 0, \\ \dfrac{1-2\bar{r}_j}{4\bar{r}_j(1-\bar{r}_j)}, & l = j = 1, 2, \cdots, L-1, \\ \dfrac{2L^2+1}{3}, & l = j = L, \end{cases}$$

with $c_0 = c_L = 2$ and $c_j = 1$, $j = 1, 2, \cdots, L-1$, and

$$D_{\bar{r}\bar{r},l,j} = \sum_{k=0}^{L} D_{\bar{r},l,k} D_{\bar{r},k,j.}$$

In the τ-direction, we can approximate $\dfrac{\partial u}{\partial \tau}$ and $\dfrac{\partial s_f}{\partial \tau}$ by central differences:

$$\begin{aligned} \frac{\partial u}{\partial \tau}(\xi_m, \bar{r}_l, \tau^{n+1/2}) &= \frac{u(\xi_m, \bar{r}_l, \tau^{n+1}) - u(\xi_m, \bar{r}_l, \tau^n)}{\Delta\tau}, \\ \frac{\partial s_f}{\partial \tau}(\bar{r}_l, \tau^{n+1/2}) &= \frac{s_f(\bar{r}_l, \tau^{n+1}) - s_f(\bar{r}_l, \tau^n)}{\Delta\tau}. \end{aligned}$$

Therefore, the first equation in the problem (9.51) can be approximated by

$$\begin{aligned} &\frac{u_{m,l}^{n+1} - u_{m,l}^n}{\Delta\tau} \\ &= \frac{1}{2}\mathbf{L}_{\mathbf{m,l}}^{\mathbf{n+1/2}}\left(u_{m,l}^{n+1} + u_{m,l}^n\right) \\ &\quad + \left(\frac{1}{s_{f,l}^{n+1} + s_{f,l}^n} \frac{s_{f,l}^{n+1} - s_{f,l}^n}{\Delta\tau}\right)\xi_m \mathbf{D}_{\xi,\mathbf{m}}\left(u_{m,l}^{n+1} + u_{m,l}^n\right) + a_{7,m,l}^{n+1/2}, \\ &\quad m = 0, 1, \cdots, M-1, \quad l = 0, 1, \cdots, L. \end{aligned} \tag{9.52}$$

Here, the operator $\mathbf{L}_{\mathbf{m,l}}^{\mathbf{n+1/2}}$ and the scalar $a_{7,m,l}^{n+1/2}$ are defined by

$$\mathbf{L}_{\mathbf{m,l}}^{\mathbf{n+1/2}} = \frac{1}{2}\left(\mathbf{L}_{\mathbf{m,l}}^{\mathbf{n+1}} + \mathbf{L}_{\mathbf{m,l}}^{\mathbf{n}}\right),$$

and

$$a_{7,m,l}^{n+1/2} = \frac{1}{2}\left(a_{7,m,l}^{n+1} + a_{7,m,l}^n\right),$$

where

$$\begin{aligned} \mathbf{L}_{\mathbf{m,l}}^{\mathbf{n}} &= a_{1,m,l}^n \xi_m^2 \mathbf{D}_{\xi\xi,\mathbf{m}} + a_{2,m,l}^n \xi_m w_{m,l}^n \mathbf{D}_{\bar{\mathbf{r}},\mathbf{l}} \mathbf{D}_{\xi,\mathbf{m}} + a_{3,m,l}^n \left(w_{m,l}^n\right)^2 \mathbf{D}_{\bar{\mathbf{r}}\bar{\mathbf{r}},\mathbf{l}} \\ &\quad + a_{4,m,l}^n \xi_m \mathbf{D}_{\xi,\mathbf{m}} + a_{5,m,l}^n \mathbf{D}_{\bar{\mathbf{r}},\mathbf{l}} + a_{6,m,l}^n, \\ a_{i,m,l}^n &= a_i(\xi_m, \bar{r}_l, \tau^n), \quad i = 1, 2, \cdots, 7, \end{aligned}$$

$$w_{m,l}^n = w(\xi_m, \bar{r}_l, \tau^n),$$

and the derivatives $\dfrac{\partial s_f}{\partial \bar{r}}, \dfrac{\partial^2 s_f}{\partial \bar{r}^2}$ appearing in a_1, a_2, and a_4 are approximated by

$$\frac{\partial s_f}{\partial \bar{r}}(\bar{r}_l, \tau^n) = \sum_{j=0}^{L} D_{\bar{r},l,j} s_f(\bar{r}_j, \tau^n),$$

$$\frac{\partial^2 s_f}{\partial \bar{r}^2}(\bar{r}_l, \tau^n) = \sum_{j=0}^{L} D_{\bar{r}\bar{r},l,j} s_f(\bar{r}_j, \tau^n).$$

The boundary conditions, the third and fourth relations in the problem (9.51), can be discretized as follows:

$$u_{M,l}^{n+1} = s_{f,l}^{n+1}, \qquad l = 0, 1, \cdots, L, \tag{9.53}$$

$$\mathbf{D}_{\xi,\mathbf{M}} u_{M,l}^{n+1} = s_{f,l}^{n+1}, \quad l = 0, 1, \cdots, L. \tag{9.54}$$

The system (9.52)–(9.54) has a truncation error of $O(\Delta\tau^2)$ in the τ-direction and is an M-th order scheme in the ξ-direction and an L-th order scheme in the $\bar{r}$-direction.

In the system (9.52)–(9.54), there are $(M+2)(L+1)$ equations. When $\left\{u_{m,l}^n\right\}$ and $\left\{s_{f,l}^n\right\}$ are given, the unknowns are $u_{m,l}^{n+1}$, $m = 0, 1, \cdots, M$, $l = 0, 1, \cdots, L$, $s_{f,l}^{n+1}$, $l = 0, 1, \cdots, L$, the total of which is also $(M+2)(L+1)$. Therefore, it is a closed system. Unfortunately, it is a nonlinear system, and we have to use iteration. Let $u_{m,l}^{(k)}, s_{f,l}^{(k)}$ represent the k-th iteration value of $u_{m,l}^{n+1}, s_{f,l}^{n+1}$, and we rewrite Eq. (9.52) in the form

$$\begin{aligned}
&u_{m,l}^{(k)} - \frac{\Delta\tau}{2}\bar{\mathbf{L}}_{\mathbf{m,l}}^{(\mathbf{k-1})} u_{m,l}^{(k)} - \frac{s_{f,l}^{(k)}}{s_{f,l}^{(k-1)} + s_{f,l}^n}\xi_m \mathbf{D}_{\xi,\mathbf{m}}\left(u_{m,l}^{(k-1)} + u_{m,l}^n\right)\\
&= u_{m,l}^n + \frac{\Delta\tau}{2}\bar{\mathbf{L}}_{\mathbf{m,l}}^{(\mathbf{k-1})} u_{m,l}^n - \frac{s_{f,l}^n}{s_{f,l}^{(k-1)} + s_{f,l}^n}\xi_m \mathbf{D}_{\xi,\mathbf{m}}\left(u_{m,l}^{(k-1)} + u_{m,l}^n\right) + \Delta\tau a_{7,m,l},\\
&\qquad m = 0, 1, \cdots, M-1, \quad l = 0, 1, \cdots, L,
\end{aligned} \tag{9.55}$$

where

$$\bar{\mathbf{L}}_{\mathbf{m,l}}^{(\mathbf{k-1})} = \frac{1}{2}\left(\mathbf{L}_{\mathbf{m,l}}^{(\mathbf{k-1})} + \mathbf{L}_{\mathbf{m,l}}^{\mathbf{n}}\right).$$

Equations (9.53) and (9.54) can be written as

$$u_{M,l}^{(k)} = s_{f,l}^{(k)}, \qquad l = 0, 1, \cdots, L, \tag{9.56}$$

$$\mathbf{D}_{\xi,\mathbf{M}} u_{M,l}^{(k)} = s_{f,l}^{(k)}, \qquad l = 0, 1, \cdots, L. \tag{9.57}$$

The system (9.55)–(9.57) is a linear one for $u_{m,l}^{(k)}$, $m = 0, 1, \cdots, M$, $l = 0, 1, \cdots, L$ and $s_{f,l}^{(k)}$, $l = 0, 1, \cdots, L$. It can be solved by a direct or iteration method. We can let $u_{m,l}^{(0)} = u_{m,l}^{n}$, $m = 0, 1, \cdots, M$, $l = 0, 1, \cdots, L$ and $s_{f,l}^{(0)} = s_{f,l}^{n}$, $l = 0, 1, \cdots, L$. When $\left\{u_{m,l}^{(k-1)}\right\}$ and $\left\{s_{f,l}^{(k-1)}\right\}$ are known, we can find $\left\{u_{m,l}^{(k)}\right\}$ and $\left\{s_{f,l}^{(k)}\right\}$ by solving the system (9.55)–(9.57). When all $u_{m,l}^{(k)} - u_{m,l}^{(k-1)}$ and $s_{f,l}^{(k)} - s_{f,l}^{(k-1)}$ become very small, we can stop the iteration. Just like the case of one-dimensional finite-difference methods, we can stop at $k = 2$, and the result should be second-order accurate in the τ-direction. This is because $\left\{u_{m,l}^{(1)}\right\}$ and $\left\{s_{f,l}^{(1)}\right\}$ can be understood as a result of a first-order scheme in τ. The results $\left\{u_{m,l}^{(2)}\right\}$ and $\left\{s_{f,l}^{(2)}\right\}$ actually are the results of a scheme in which the improved Euler method is used in the τ-direction. Therefore, if $\left\{u_{m,l}^{n}\right\}$ and $\left\{s_{f,l}^{n}\right\}$ are given, we can obtain $\left\{u_{m,l}^{n+1}\right\}$ and $\left\{s_{f,l}^{n+1}\right\}$ by solving the system (9.55)–(9.57). Because $\left\{u_{m,l}^{0}\right\}$ and $\left\{s_{f,l}^{0}\right\}$ are given by the initial conditions, we can repeat the procedure described above for $n = 0, 1, \cdots, N-1$, and finally get $\left\{u_{m,l}^{N}\right\}$ and $\left\{s_{f,l}^{N}\right\}$.

As long as we find $\left\{u_{m,l}^{N}\right\}$ and $\left\{s_{f,l}^{N}\right\}$, for any S, r we can have the price of the convertible bond at $t = 0$ in the following way. If

$$S > Zs_f\left(\frac{r - r_l}{r_u - r_l}, T\right) \Big/ n,$$

then

$$V = \max(Z, nS);$$

while

$$S < Zs_f\left(\frac{r - r_l}{r_u - r_l}, T\right) \Big/ n,$$

then

$$V(S, r, 0) = Zu\left(\frac{nS}{Zs_f\left(\dfrac{r - r_l}{r_u - r_l}, T\right)}, \frac{r - r_l}{r_u - r_l}, T\right).$$

Usually, $\dfrac{r - r_l}{r_u - r_l} \neq \bar{r}_l$ for any l and $\dfrac{nS}{Zs_f\left(\dfrac{r - r_l}{r_u - r_l}, T\right)} \neq \xi_m$ for any m. In order to find $V(S, r, 0)$, we therefore need to use interpolation.

When $t \approx T$ and $S \approx \max\left(\dfrac{Z}{n}, \dfrac{KZ}{D_0 n}\right)$, the solution in the S-direction is not smooth. In order to overcome this problem, we need to solve the problem

(9.21) instead of the problem (9.51). The method for the problem (9.21) is almost the same as the method for the problem (9.51). The only difference is the boundary conditions and a_7. In this case, $a_{7,m,l}$ in the system (9.55) should be replaced by

$$\frac{1}{2}\left(a_{7,m,l}^{(k-1)} + a_{7,m,l}^{n}\right)$$

because a_7 involves the location of the free boundary. Here, $a_{7,m,l}^{(k-1)}$ is the $(k-1)$-th iteration value of $a_{7,m,l}^{n+1}$. If we still want to solve the problem (9.51), then at $t \approx T$, using the finite-difference methods or using the pseudo-spectral methods in the r-direction and using the finite-difference methods in the S-direction might be better. Readers can find the details about how to solve the two-factor convertible bond problems using the implicit finite-difference method in [95] and using the mixture of the pseudo-spectral methods and the finite-difference methods in [77]. In what follows, for brevity, we will refer to the mixture of the pseudo-spectral method and the finite-difference method as the pseudo-spectral method because in the entire computation, the main method is the pseudo-spectral method. It is clear that this problem can also be solved as a linear complementarity problem using an explicit or an implicit finite-difference scheme.

9.3.2 Numerical Results of Two-Factor Convertible Bonds

Here, we show some numerical results of a two-factor convertible bond by the pseudo-spectral method and compare the results by the pseudo-spectral method with the results obtained by the finite-difference method, by the projected explicit and projected implicit finite-difference methods.

The interest rate model we adopted for the example is based on the model used by Brennan and Schwartz (see [12]) and Druskin et al. (see [27]) even though in practice in order to get the interest rate model, we should solve an inverse problem by using the data on the market. Their model is

$$dr = u(r,t)dt + w(r,t)dX_2, \quad 0 \le r,$$

where

$$\begin{cases} u(r,t) = -0.13r + 0.008 + \lambda(r,t)w(r,t), \\ w(r,t) = \sqrt{0.26r}. \end{cases}$$

We made the following modifications. We assume

$$0 \le r \le 0.3$$

and instead of $0.26r$, use

$$0.26r\phi^2(r),$$

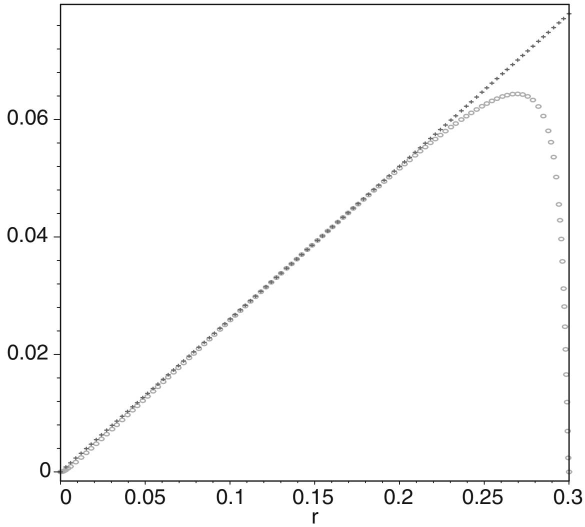

Fig. 9.20. The functions $0.26r$ and $0.26r\phi^2(r)$

where

$$\phi(r) = \frac{1 - (1 - 2r/0.3)^2}{1 - 0.975\,(1 - 2r/0.3)^2}.$$

Thus, our model for the example here is

$$dr = u(r,t)dt + w(r,t)dX_2, \quad 0 \leq r \leq 0.3,$$

where

$$\begin{cases} u(r,t) = -0.13r + 0.008 + \lambda(r,t)w(r,t), \\ w(r,t) = \sqrt{0.26r}\phi(r). \end{cases}$$

The functions $0.26r$ and $0.26r\phi^2(r)$ are shown in Fig. 9.20, and we can see that for $r \in [0, 0.2]$, the difference is very small. Because

$$\phi(0) = \phi(0.3) = 0$$

and $d\phi(r)/dr$ is bounded on $[0, 0.3]$, we have

$$w(0,t)\frac{\partial w(0,t)}{\partial r} = w(0.3,t)\frac{\partial w(0.3,t)}{\partial r} = 0.$$

Therefore, the reversion conditions can be written as:

$$\begin{cases} u(0,t) \geq 0, \\ w(0,t) = 0 \end{cases}$$

and

$$\begin{cases} u(0.3,t) \le 0, \\ w(0.3,t) = 0. \end{cases}$$

Because of

$$u(0,t) = 0.008 > 0$$

and

$$u(0.3,t) = -0.13 \times 0.3 + 0.008 = -0.031 < 0,$$

we do not need any boundary conditions at $r = 0$ and $r = 0.3$.

We still assume the volatility and the dividend yield of the underlying stock to be

$$\sigma(S,t) = 0.20$$

and

$$D_0 = 0.05,$$

respectively, and the correlation of the two random variables dX_1 and dX_2 to be

$$\rho(S,r,t)dt = -0.01dt.$$

Let us consider a standard convertible bond with $k = 0.06$ and $T = 30$. First, we give the result obtained by the pseudo-spectral methods. Concretely, for $\tau \in [0,2]$, in the r-direction the pseudo-spectral method described in Sect. 8.4 is adopted, and in the S-direction the implicit finite-difference method discussed in Sect. 9.2.1 is used, and we take $M = 60$, $L = 10$; for $\tau \in [2,30]$ in both directions, the pseudo-spectral method is used and $M = L = 10$. In the τ-direction, a nonuniform time step is used and $N = 50$. In Fig. 9.21, the values of the two-factor convertible bond at $t = 1$ month, 6 months, 1 year, 5 years, 10 years, and 30 years are plotted. In Fig. 9.22, the location curves of the free boundary at various times are given.

Besides the method mentioned in this section, the implicit finite-difference method similar to the method in Sect. 9.2.5, the projected explicit finite-difference method and the projected implicit finite-difference method have been used to compute the same problem on various meshes. For the implicit finite-difference method, the value of the convertible bond at $r = 0.05, S = 1, t = 30$ years on a very fine mesh is $1.3116835\cdots$ [1] and these eight digits are unchanged as the mesh size further decreases. Therefore, this value is accurate to at least seven digits. After we have a highly accurate result, we can obtain the first few digits of the error of the results on different meshes. For each computation, we also record the CPU time. Thus, for each error, we can have the corresponding CPU time. Figure 9.23 is a $\log_{10}$(error) versus $\log_{10}$(CPU time in second) graph, and each point in the figure represents a performance of the method. Because the ranges of errors and CPU times are very large, we adopt $\log_{10}$(Error) and $\log_{10}$(CPU time in second) as variables. There, a "×"

[1] When this figure was obtained, the function $\phi(r)$ used was $\left[4r(0.3-r)/0.3^2\right]^{1/8}$.

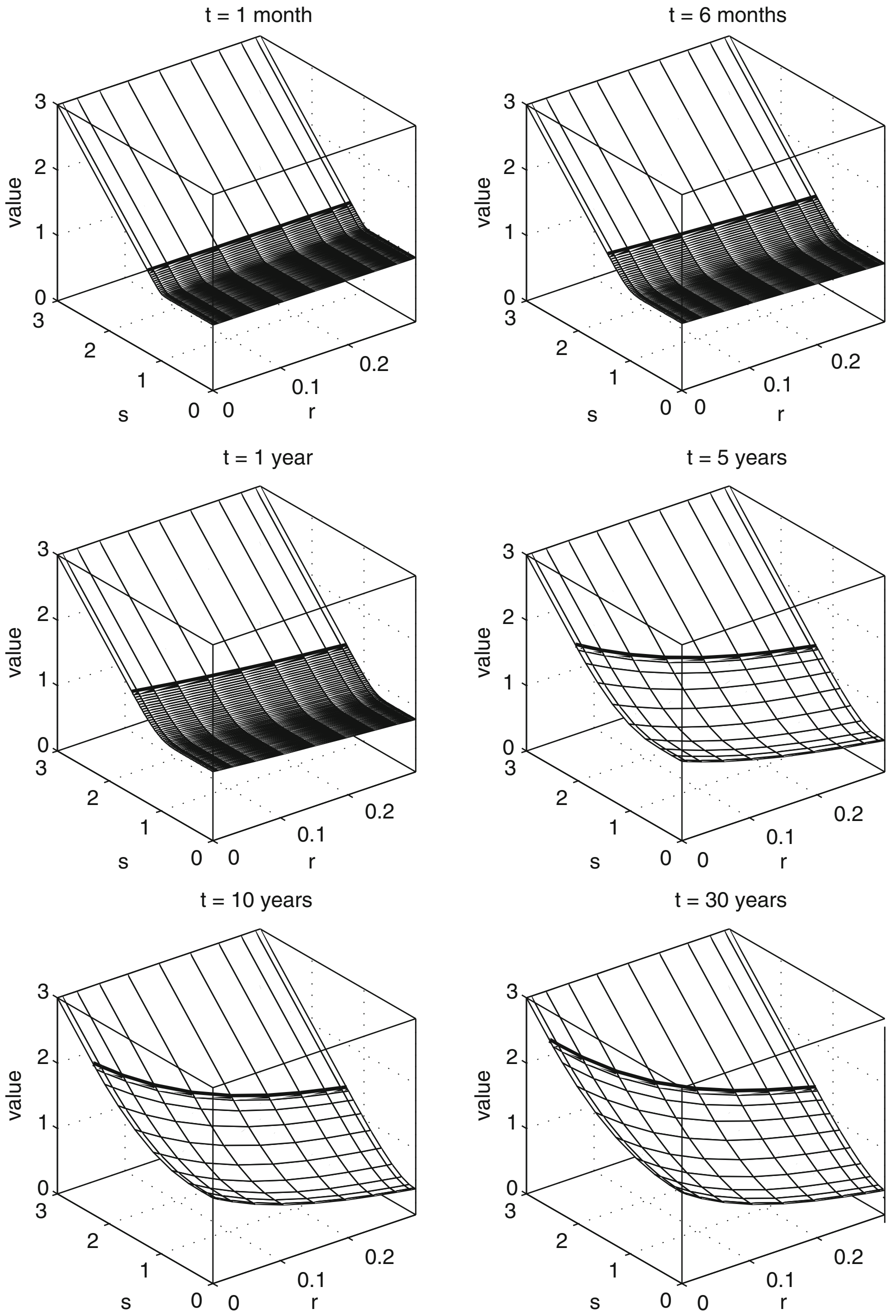

Fig. 9.21. Prices of a two-factor convertible bond at six different times

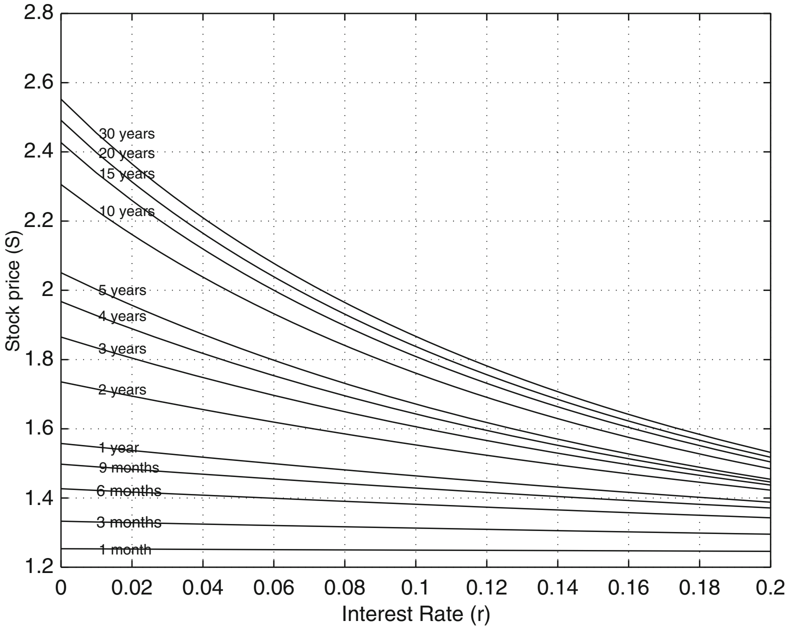

Fig. 9.22. Locations of a free boundary at various times

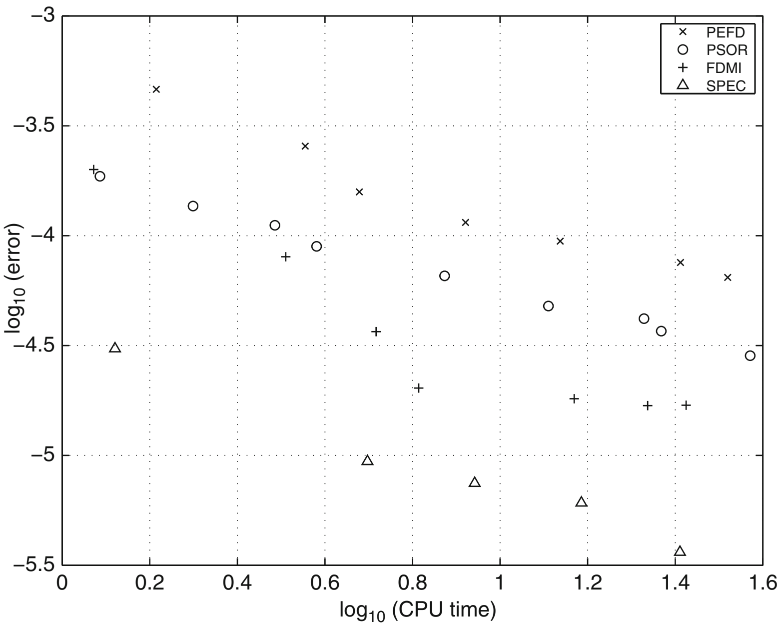

Fig. 9.23. $\log_{10}$(error) versus $\log_{10}$(CPU time in second)

represents the performance of the projected explicit finite-difference method, which is referred to as PEFD in the figure. A "∘" indicates the performance of the projected implicit finite-difference method. The successive over relaxation method is used to get the solution. Therefore, this method is referred to as PSOR in the figure. A "+" stands for the performance of the implicit finite-difference method. In order to get the solution of the nonlinear algebraic equations, the alternating-direction iteration method is used (see [77]). In the figure, it is referred to as FDMI. In the figure, a "Δ" represents the performance of the pseudo-spectral method, which is referred to as SPEC there. Clearly, the lower the point, the better the performance. From Fig. 9.23, we see that the pseudo-spectral method has the best performance for this example.

Problems

Table 9.12. Problems and sections

Problems	Sections	Problems	Sections	Problems	Sections
1–3	9.1	4–10(a, b)	9.2	10(c)–12	9.3

1. Consider the following free-boundary problem that is related to American lookback strike put options:

$$\begin{cases} \dfrac{\partial W}{\partial t}+\dfrac{1}{2}\sigma^2\eta^2\dfrac{\partial^2 W}{\partial \eta^2}+(D_0-r)\,\eta\dfrac{\partial W}{\partial \eta}-D_0W=0, & \\ \qquad\qquad 1\le\eta\le\eta_f(t), \quad 0\le t\le T, & \\ W(\eta,T)=\max(\eta-\beta,0), & 1\le\eta\le\eta_f(T), \\ \dfrac{\partial W}{\partial \eta}(1,t)=0, & 0\le t\le T, \\ W(\eta_f,t)=\eta_f-\beta, & 0\le t\le T, \\ \dfrac{\partial W}{\partial \eta}(\eta_f,t)=1, & 0\le t\le T, \\ \eta_f(T)=\beta\max(1,D_0/r). & \end{cases}$$

By using the closed-form solution of the problem

$$\begin{cases} \dfrac{\partial W_1}{\partial t}+\dfrac{1}{2}\sigma^2\eta^2\dfrac{\partial^2 W_1}{\partial \eta^2}+(D_0-r)\eta\dfrac{\partial W_1}{\partial \eta}-D_0W_1=0, & \eta\ge1,\ \ 0\le t\le T, \\ W_1(\eta,T)=\max(\eta-\beta,0), & \eta\ge1, \\ \dfrac{\partial W_1}{\partial \eta}(1,t)=0, & 0\le t\le T, \end{cases}$$

convert this problem into a problem whose solution has a continuous derivative everywhere. Here we also require that the problem is defined on a rectangular domain: $[0,1] \times [0,T]$, has an initial condition, and the free boundary is the right boundary. (Assume $1 < \beta$).

2. Consider the following free-boundary problem that is related to American average strike call options:

$$\begin{cases} \dfrac{\partial W}{\partial t} + \dfrac{1}{2}\sigma^2\eta^2\dfrac{\partial^2 W}{\partial \eta^2} + \left[(D_0 - r)\eta + \dfrac{1-\eta}{t}\right]\dfrac{\partial W}{\partial \eta} - D_0 W = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad \eta_f(t) \le \eta, \quad t \le T, \\ W(\eta, T) = \max(1-\eta, 0), \qquad \eta_f(T) \le \eta, \\ W(\eta_f(t), t) = 1 - \eta_f(t), \qquad t \le T, \\ \dfrac{\partial W}{\partial \eta}(\eta_f(t), t) = -1, \qquad t \le T, \\ \eta_f(T) = \min\left(1, \dfrac{1 + D_0 T}{1 + rT}\right). \end{cases}$$

Convert this problem into a problem with a singularity weaker than the singularity here for $t > 0$. Also require that the new problem is defined on a rectangular domain, has an initial condition and the right boundary corresponds to the free boundary.

3. *Let $C(S, \sigma, t;\ a, b, c, d)$ and $P(S, \sigma, t;\ a, b, c, d)$ denote the prices of American two-factor call and put options and $S_{cf}(\sigma, t;\ a, b, c, d)$ and $S_{pf}(\sigma, t;\ a, b, c, d)$ be their optimal exercise prices. Here, a, b, c, and d are parameters (or parameter functions) for the risk-free interest rate r, dividend yield rate D_0, correlation coefficient ρ, and market price of volatility risk λ, respectively. Show that between American two-factor put and call options there is the following put–call symmetry relation:

$$\begin{cases} P(S, \sigma, t; a, b, c, d) = \dfrac{S}{E} C\left(\dfrac{E^2}{S}, \sigma, t; b, a, -c, d - c\sigma\right), \\ C(S, \sigma, t; a, b, c, d) = \dfrac{S}{E} P\left(\dfrac{E^2}{S}, \sigma, t; b, a, -c, d - c\sigma\right), \\ S_{pf}(\sigma, t; a, b, c, d) \times S_{cf}(\sigma, t; b, a, -c, d - c\sigma) = E^2. \end{cases}$$

4. Consider the following free-boundary problem:

$$\begin{cases} \dfrac{\partial V}{\partial \tau} = \dfrac{1}{2}\sigma^2\xi^2(1-\xi)^2\dfrac{\partial^2 V}{\partial \xi^2} + (r-D_0)\xi(1-\xi)\dfrac{\partial V}{\partial \xi} \\ \qquad -[r(1-\xi)+D_0\xi]V, & 0 \le \xi < \xi_f(\tau), \quad 0 \le \tau, \\ V(\xi,0) = \max(2\xi-1,0), & 0 \le \xi < \xi_f(0), \\ V\left(\xi_f(\tau),\tau\right) = 2\xi_f(\tau)-1, & 0 \le \tau, \\ \dfrac{\partial V}{\partial \xi}\left(\xi_f(\tau),\tau\right) = 2, & 0 \le \tau, \\ \xi_f(0) = \max\left(\dfrac{1}{2}, \dfrac{r}{r+D_0}\right). \end{cases}$$

It can be easily seen that the free-boundary problem for the American call options under the (S,t)-space can be rewritten as this form if let $\xi = S/(S+E)$ and $\tau = T-t$.

(a) Convert this problem into a problem whose solution has a continuous derivative everywhere. Here we also require that the problem is defined on a rectangular domain and with an initial condition.

(b) Design a second-order implicit method to solve the new problem. (Need to check whether or not the number of equations which can be used is equal to the number of unknowns.)

5. Consider the following the free-boundary problem:

$$\begin{cases} \dfrac{\partial u}{\partial \tau} = \dfrac{1}{2}\sigma^2\left[\xi + \dfrac{1}{\bar{\eta}_f - 1}\right]^2\dfrac{\partial^2 u}{\partial \xi^2} \\ \qquad + \left[(D_0 - r)\left(\xi + \dfrac{1}{\bar{\eta}_f - 1}\right) + \dfrac{\xi}{\bar{\eta}_f - 1}\dfrac{d\bar{\eta}_f}{d\tau}\right]\dfrac{\partial u}{\partial \xi} - D_0 u, & 0 \le \xi \le 1, \\ & 0 \le \tau \le T, \\ u(\xi,0) = 0, & 0 \le \xi \le 1, \\ \dfrac{\partial u}{\partial \xi}(0,\tau) = 0, & 0 \le \tau \le T, \\ u(1,\tau) = \bar{\eta}_f(\tau) - \beta - W_1(\bar{\eta}_f(\tau), T-\tau), & 0 \le \tau \le T, \\ \dfrac{\partial u}{\partial \xi}(1,\tau) = (\bar{\eta}_f(\tau)-1)\left[1 - \dfrac{\partial W_1(\bar{\eta}_f(\tau),T-\tau)}{\partial \eta}\right], & 0 \le \tau \le T, \\ \bar{\eta}_f(0) = \beta\max\left(1, D_0/r\right), \end{cases}$$

where $W_1(\eta, T-\tau)$ is a given function. Design a second-order implicit method to solve this problem which is the new problem obtained in Problem 1. (Need to check whether or not the number of equations which can be used is equal to the number of unknowns.)

6. *Consider the nonlinear system consisting of the following equations

$$\begin{aligned}
&\frac{u_m^{n+1}-u_m^n}{\Delta\tau}\\
&=\frac{1}{2}\left[k_2m^2\left(u_{m+1}^{n+1}-2u_m^{n+1}+u_{m-1}^{n+1}\right)+\frac{k_1m}{2}\left(u_{m+1}^{n+1}-u_{m-1}^{n+1}\right)-k_0u_m^{n+1}\right]\\
&\quad+\frac{1}{2}\left[k_2m^2\left(u_{m+1}^{n}-2u_m^{n}+u_{m-1}^{n}\right)+\frac{k_1m}{2}\left(u_{m+1}^{n}-u_{m-1}^{n}\right)-k_0u_m^{n}\right]\\
&\quad+\frac{s_f^{n+1}-s_f^n}{\left(s_f^{n+1}+s_f^n\right)\Delta\tau}\left[\frac{m}{2}\left(u_{m+1}^{n+1}-u_{m-1}^{n+1}\right)+\frac{m}{2}\left(u_{m+1}^{n}-u_{m-1}^{n}\right)\right],\\
&\qquad m=0,1,2,\cdots,M-1,
\end{aligned}$$

and

$$u_M^{n+1}=g(s_f^{n+1},\tau^{n+1}),$$

$$\frac{3u_M^{n+1}-4u_{M-1}^{n+1}+u_{M-2}^{n+1}}{2\Delta\xi}=h\left(s_f^{n+1},\tau^{n+1}\right),$$

where u_m^n are known, τ^{n+1} is given, k_0, k_1, and k_2 are constants, and $g(s,\tau)$ and $h(s,\tau)$ are given functions. Discuss how to solve this system, provide at least two methods that you think are simple and effective, and give the details for one of the methods.

7. *Is the extrapolation technique always helpful and why?

8. Consider the scheme given in Problem 4. Why the extrapolation technique can still be used when a non-uniform mesh in τ with

$$\tau^n=n^2T/N^2,\quad n=0,1,\cdots,N,$$

is adopted? (Hint: Define $\tau_1=\sqrt{\tau T}$. Solving a problem with a variable step in τ is the same as solving a problem with a constant step in τ_1.)

9. *Design an exponential scheme to approximate

$$a(\xi)\frac{d^2U}{d\xi^2}+b(\xi)\frac{dU}{d\xi}+c(\xi)U,$$

where $a(\xi)>0$ and $c(\xi)<0$.

10. Assume σ to be a random variable satisfying

$$d\sigma=p(\sigma,t)dt+q(\sigma,t)dX,$$

where dX is a Wiener process. In this case, evaluating American call options can be reduced to solving the following free-boundary problem:

$$\begin{cases} \dfrac{\partial C}{\partial t} + \mathbf{L}_{\mathbf{s},\sigma} C = 0, & 0 \le S \le S_f(\sigma, t), \\ & \sigma_l \le \sigma \le \sigma_u, \quad 0 \le t \le T, \\ C(S, \sigma, T) = \max(S - E, 0), & 0 \le S \le S_f(\sigma, T), \\ & \sigma_l \le \sigma \le \sigma_u, \\ C\left(S_f(\sigma, t), \sigma, t\right) = S_f(\sigma, t) - E, & \sigma_l \le \sigma \le \sigma_u, \quad 0 \le t \le T, \\ \dfrac{\partial C\left(S_f(\sigma, t), \sigma, t\right)}{\partial S} = 1, & \sigma_l \le \sigma \le \sigma_u, \quad 0 \le t \le T, \\ S_f(\sigma, T) = \max(E, rE/D_0), & \sigma_l \le \sigma \le \sigma_u, \end{cases}$$

where

$$\begin{aligned} \mathbf{L}_{\mathbf{s},\sigma} = {} & \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + \rho\sigma S q \frac{\partial^2}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2}{\partial \sigma^2} \\ & + (r - D_0) S \frac{\partial}{\partial S} + (p - \lambda q) \frac{\partial}{\partial \sigma} - r. \end{aligned}$$

(a) *Convert this problem into a problem defined on a rectangular domain and whose solution has a singularity weaker than the singularity here.
(b) *Design a second-order implicit method to solve the new problem. (Here and also for part (c), do not require to discuss the solution of the nonlinear system.)
(c) Design a pseudo-spectral method to solve the new problem.

11. Consider the following free-boundary problem related to one-factor convertible bonds:

$$\begin{cases} \dfrac{\partial B_c}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 B_c}{\partial S^2} + (r - D_0) S \dfrac{\partial B_c}{\partial S} - rB_c + kZ = 0, & \\ & 0 \le S \le S_f(t), \quad 0 \le t \le T, \\ B_c(S, T) = \max(Z, nS), & 0 \le S \le S_f(T), \\ B_c\left(S_f(t), t\right) = nS_f(t), & 0 \le t \le T, \\ \dfrac{\partial B_c}{\partial S}\left(S_f(t), t\right) = n, & 0 \le t \le T, \\ S_f(T) = \max\left(\dfrac{Z}{n}, \dfrac{kZ}{D_0 n}\right). & \end{cases}$$

(a) Convert this problem into a problem whose solution has a continuous derivative everywhere, and which is defined on a rectangular domain and has an initial condition.
(b) Design a pseudo-spectral method to solve the new problem. (Do not require to discuss the solution of the nonlinear system.)

12. *Consider the nonlinear system consisting of the following equations:

$$\frac{u_{m,l}^{n+1} - u_{m,l}^{n}}{\Delta\tau}$$
$$= \frac{1}{2}\mathbf{L}_{\mathbf{m,l}}^{\mathbf{n+1/2}}\left(u_{m,l}^{n+1} + u_{m,l}^{n}\right)$$
$$+ \left(\frac{1}{s_{f,l}^{n+1} + s_{f,l}^{n}} \frac{s_{f,l}^{n+1} - s_{f,l}^{n}}{\Delta\tau}\right)\xi_m \mathbf{D}_{\xi,\mathbf{m}}\left(u_{m,l}^{n+1} + u_{m,l}^{n}\right) + a_{7,m,l},$$
$$m = 0, 1, \cdots, M-1, \quad l = 0, 1, \cdots, L,$$
$$u_{M,l}^{n+1} = s_{f,l}^{n+1}, \qquad l = 0, 1, \cdots, L,$$

and

$$\mathbf{D}_{\xi,\mathbf{M}} u_{M,l}^{n+1} = s_{f,l}^{n+1}, \quad l = 0, 1, \cdots, L,$$

where

$$\mathbf{L}_{\mathbf{m,l}}^{\mathbf{n+1/2}} = \frac{1}{2}\left(\mathbf{L}_{\mathbf{m,l}}^{\mathbf{n+1}} + \mathbf{L}_{\mathbf{m,l}}^{\mathbf{n}}\right).$$

Here, $u_{m,l}^{n}, m = 0, 1, \cdots, M,\ l = 0, 1, \cdots, L$ and $s_{f,l}^{n}, l = 0, 1, \cdots, L$ are given and $u_{m,l}^{n+1}, m = 0, 1, \cdots, M,\ l = 0, 1, \cdots, L$ and $s_{f,l}^{n+1}, l = 0, 1, \cdots, L$ are unknown. In the system, $\mathbf{D}_{\xi,\mathbf{m}}$ and $\mathbf{L}_{\mathbf{m,l}}^{\mathbf{n}}$ are difference operators with variable coefficients. $\mathbf{L}_{\mathbf{m,l}}^{\mathbf{n+1}}$ is another difference operator whose coefficients depend on $s_{f,l}^{n+1}, l = 0, 1, \cdots, L$. Discuss how to solve this system and give an outline of a method that you think is simple and effective.

Projects

General Requirements

(A) Submit a code or codes in C or C^{++} that will work on a computer the instructor can get access to. At the beginning of the code, write down the name of the student and indicate on which computer it works and the procedure to make it work.

(B) Each code should use an input file to specify all the problem parameters and the computational parameters for each computation and an output file to store all the results. In an output file, the name of the student, all the problem parameters, and the computational parameters should be given, so that one can know what the results are and how they were obtained. The input file should be submitted with the code.

(C) If not specified, for each case two results are required. For the first result, a 50×10 mesh should be used. For the second result, the accuracy required is 0.001, and the mesh used should be as coarse as possible.

(D) Submit results in form of tables. When a result is given, always provide the problem parameters and the computational parameters.

1. **Implicit Scheme (9.22)–(9.24).** Suppose σ, r, D_0 are constant. Write a code performing implicit singularity-separating method for American calls and puts. In the code, a result of an American call option should be obtained by the implicit scheme (9.22)–(9.24), whereas a result of an American put option should be obtained through solving a corresponding call problem numerically and then using the symmetry relation.
   - For American call and put options, give the results for the case: $S = 100$, $E = 100$, $T = 1$, $r = 0.1$, $D_0 = 0.05$, $\sigma = 0.2$.
   - For American call and put options, give the results for the case: $S = 100$, $E = 100$, $T = 1$, $r = 0.05$, $D_0 = 0.1$, $\sigma = 0.2$.
   - For American call and put options, find the results with an accuracy of 0.00001 under the help of the extrapolation technique. The problem parameters are $S = 90, 100, 110$, $E = 100$, $T = 1.00$, $r = 0.1$, $D_0 = 0.05$, and $\sigma = 0.2$.
2. Using the binomial method (8.28) with the formulae (8.25)–(8.27) try to find the values of American call and put options with an accuracy of 0.00001. The problem parameters are $S = 90, 100, 110$, $E = 100$, $T = 1.00$, $r = 0.10$, $D_0 = 0.05$, and $\sigma = 0.2$.

10

Interest Rate Modeling

As pointed out in Sect. 2.3, when the short-term interest rate is considered as a random variable, there is an unknown function $\lambda(r,t)$, called the market price of risk, in the governing equation. Before using the governing equation for evaluating an interest rate derivative, we have to find this function (or make some assumptions on it). This function cannot be obtained by statistics directly from the market data. In Sect. 5.4, the inverse problem on the market price of risk was formulated. This problem can be solved by numerical methods. However, if the problem is formulated in another way, then the inverse problem may be solved more efficiently. Therefore, in Sect. 10.1, we first discuss another formulation of the inverse problem and then we give numerical methods for both formulations and show some numerical examples. Then, numerical methods for one-factor interest rate derivatives are described, and some numerical results are shown in Sect. 10.2. Because interest rate derivative problems are so complicated, for many cases, use of multi-factor models is necessary. In the last section, we study how to price interest rate derivatives using the three-factor model and the market data.

10.1 Inverse Problems

10.1.1 Another Formulation of the Inverse Problem

As seen in Sect. 5.4, in order to match the bond equation with the market data, we need to find $\lambda(r,t)$ such that the solution $V(r,t;T^*)$ of the problem (5.47) at $r=r^*$ and $t=0$ is equal to today's price of the bond with maturity T^*. There, we also briefly discussed how to solve this inverse problem. Here, we reformulate the inverse problem in Sect. 5.4 (see [89]). This formulation may make the numerical solution easy and efficient. Let us derive this formulation. The problem (5.47) can be rewritten as follows:

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$$\begin{cases} \dfrac{\partial V}{\partial t} = -\mathbf{L}_r V, \qquad r_l \le r \le r_u, \quad 0 \le t \le T^* \le T^*_{max}, \\ V(r, T^*; T^*) = 1, \, r_l \le r \le r_u. \end{cases}$$

Here, we have used the relation

$$\begin{aligned} &\frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} + (u - \lambda w) \frac{\partial V}{\partial r} - rV \\ &= \frac{\partial}{\partial r} \left(\frac{1}{2} w^2 \frac{\partial V}{\partial r} \right) + \left[u - \left(\lambda + \frac{\partial w}{\partial r} \right) w \right] \frac{\partial V}{\partial r} - rV \end{aligned}$$

and adopted the following notation

$$\mathbf{L}_r = \frac{\partial}{\partial r} \left(\frac{1}{2} w^2 \frac{\partial}{\partial r} \right) + \left[u - \left(\lambda + \frac{\partial w}{\partial r} \right) w \right] \frac{\partial}{\partial r} - r. \tag{10.1}$$

Let us define

$$\mathbf{L}_r^* = \frac{\partial}{\partial r} \left(\frac{1}{2} w^2 \frac{\partial}{\partial r} \right) - \frac{\partial}{\partial r} \left\{ \left[u - \left(\lambda + \frac{\partial w}{\partial r} \right) w \right] \right\} - r \tag{10.2}$$

and $U(r,t)$ satisfy the following equation:

$$\frac{\partial U}{\partial t} = \mathbf{L}_r^* U, \qquad r_l \le r \le r_u, \qquad 0 \le t \le T^*_{max}.$$

Because

$$\begin{aligned} &\frac{1}{2} \int_{r_l}^{r_u} \frac{\partial}{\partial r} \left(w^2 \frac{\partial V}{\partial r} \right) U dr \\ &= \frac{1}{2} \left[\left(w^2 \frac{\partial V}{\partial r} U \right) \Bigg|_{r_l}^{r_u} - \int_{r_l}^{r_u} w^2 \frac{\partial V}{\partial r} \frac{\partial U}{\partial r} dr \right] \\ &= \frac{1}{2} \left[\left(w^2 \frac{\partial V}{\partial r} U \right) \Bigg|_{r_l}^{r_u} - \left(w^2 \frac{\partial U}{\partial r} V \right) \Bigg|_{r_l}^{r_u} + \int_{r_l}^{r_u} \frac{\partial}{\partial r} \left(w^2 \frac{\partial U}{\partial r} \right) V dr \right] \end{aligned}$$

and

$$\begin{aligned} &\int_{r_l}^{r_u} \left[u - \left(\lambda + \frac{\partial w}{\partial r} \right) w \right] \frac{\partial V}{\partial r} U dr \\ &= \left\{ \left[u - \left(\lambda + \frac{\partial w}{\partial r} \right) w \right] UV \right\} \Bigg|_{r_l}^{r_u} - \int_{r_l}^{r_u} \frac{\partial}{\partial r} \left\{ \left[u - \left(\lambda + \frac{\partial w}{\partial r} \right) w \right] U \right\} V dr, \end{aligned}$$

we have

$$\begin{aligned} \int_{r_l}^{r_u} \mathbf{L}_r V \, U dr = \int_{r_l}^{r_u} \mathbf{L}_r^* U \, V dr &+ \frac{1}{2} \left[\left(w^2 \frac{\partial V}{\partial r} U \right) \Bigg|_{r_l}^{r_u} - \left(w^2 \frac{\partial U}{\partial r} V \right) \Bigg|_{r_l}^{r_u} \right] \\ &+ \left\{ \left[u - \left(\lambda + \frac{\partial w}{\partial r} \right) w \right] UV \right\} \Bigg|_{r_l}^{r_u}. \end{aligned}$$

Consequently, $\mathbf{L}_r^*$ is called the adjoint operator to $\mathbf{L}_r$. Because $w(r_l,t) = w(r_u,t) = 0$ when the conditions (5.45) and (5.46) hold, we arrive at

$$\int_{r_l}^{r_u} \mathbf{L}_r V\, U dr = \int_{r_l}^{r_u} \mathbf{L}_r^* U\, V dr + \left\{ \left[u - \left(\lambda + \frac{\partial w}{\partial r} \right) w \right] UV \right\} \Bigg|_{r_l}^{r_u}. \tag{10.3}$$

For simplicity, let us consider the case:

$$\begin{cases} u(r_l,t) - w(r_l,t)\dfrac{\partial w(r_l,t)}{\partial r} > 0, \\ u(r_u,t) - w(r_u,t)\dfrac{\partial w(r_u,t)}{\partial r} < 0. \end{cases} \tag{10.4}$$

It is clear that when $u(r_l,t)$ and $u(r_u,t)$ are bounded and the condition (10.4) holds, $u(r_l,t) - w(r_l,t)\dfrac{\partial w(r_l,t)}{\partial r}$ and $u(r_u,t) - w(r_u,t)\dfrac{\partial w(r_u,t)}{\partial r}$ must also be bounded even if $\dfrac{\partial w(r_l,t)}{\partial r}$ or $\dfrac{\partial w(r_u,t)}{\partial r}$ is unbounded because of $\dfrac{\partial w^2(r_l,t)}{\partial r} \geq 0$ and $\dfrac{\partial w^2(r_u,t)}{\partial r} \leq 0$.[1] In this case, in order for $\dfrac{\partial U}{\partial t} = \mathbf{L}_r^* U$ to have a unique solution, two boundary conditions are needed in addition to an initial condition. Therefore, we may add two boundary conditions on $U(r,t)$. Let us choose

$$U(r_l,t) = U(r_u,t) = 0, \qquad 0 \leq t \leq T_{max}^*.$$

Under this choice, equality (10.3) becomes

$$\int_{r_l}^{r_u} \mathbf{L}_r V\, U dr = \int_{r_l}^{r_u} \mathbf{L}_r^* U\, V dr. \tag{10.5}$$

Therefore,

$$\begin{aligned} &\frac{d}{dt}\int_{r_l}^{r_u} VU dr = \int_{r_l}^{r_u} \frac{\partial (VU)}{\partial t} dr = \int_{r_l}^{r_u} \left(\frac{\partial V}{\partial t} U + V \frac{\partial U}{\partial t} \right) dr \\ &= \int_{r_l}^{r_u} (-\mathbf{L}_r V\, U + \mathbf{L}_r^* U\, V) dr = 0, \end{aligned}$$

from which, we further have

$$\int_{r_l}^{r_u} V(r,0;T^*) U(r,0) dr = \int_{r_l}^{r_u} V(r,T^*;T^*) U(r,T^*) dr.$$

Suppose we choose

$$U(r,0) = \delta(r - r^*), \qquad r_l \leq r \leq r_u.$$

Then, noticing $V(r,T^*;T^*) = 1$, we arrive at

[1] This is because $w^2(r,t) \geq 0$ on $[r_l, r_u]$.

$$V(r^*,0;T^*) = \int_{r_l}^{r_u} U(r,T^*)dr. \tag{10.6}$$

Consequently, for any function $\lambda(r,t)$, if $U(r,t)$ is the solution of the problem

$$\begin{cases} \dfrac{\partial U}{\partial t} = \mathbf{L}_r^* U, & r_l \le r \le r_u, \quad 0 \le t \le T^*_{max}, \\ U(r,0) = \delta(r-r^*), & r_l \le r \le r_u, \\ U(r_l,t) = U(r_u,t) = 0, & 0 \le t \le T^*_{max}, \end{cases} \tag{10.7}$$

and $V(r,t;T^*)$ is the solution of the problem (5.47), then condition (10.6) holds.

Let today's time be $t=0$, today's short-term interest rate be r^*, and the prices of zero-coupon bonds with a face value $Z=1$ and with various maturities T^* be $\overline{V}(T^*)$. Assume[2] $\lambda(r,t)=\lambda(t)$ to be such a function that the solution $U(r,T^*)$ of the problem (10.7) satisfies (10.6) with $V(r^*,0;T^*) = \overline{V}(T^*)$. Then, the solution $V(r,t;T^*)$ of the problem (5.47) at $r=r^*$ and $t=0$ gives today's price of the zero-coupon bond with maturity T^* on the market. Consequently, matching $\lambda(r,t)$ with the zero-coupon bond price curve can be reduced to finding $\lambda(t)$ such that $U(r,T^*)$ satisfies Eq. (10.6) with $V(r^*,0;T^*) = \overline{V}(T^*)$.

From Eq. (10.6), we can derive another equivalent relation that can also be used to determine $\lambda(t)$. Differentiating (10.6) with respect to T^* yields

$$\begin{aligned} \frac{\partial V(r^*,0;T^*)}{\partial T^*} &= \int_{r_l}^{r_u} \frac{\partial U(r,T^*)}{\partial T^*} dr \\ &= \int_{r_l}^{r_u} \mathbf{L}_r^* U(r,T^*)dr \\ &= \int_{r_l}^{r_u} \left\{ \frac{\partial}{\partial r}\left(\frac{1}{2}w^2\frac{\partial U}{\partial r}\right) - \frac{\partial}{\partial r}\left\{\left[u - \left(\lambda(t) + \frac{\partial w}{\partial r}\right) w\right] U\right\} - rU \right\} dr \\ &= -\int_{r_l}^{r_u} rU(r,T^*)dr \end{aligned} \tag{10.8}$$

because

$$w(r_l,T^*) = w(r_u,T^*) = U(r_l,T^*) = U(r_u,T^*) = 0.$$

From this relation we can further have

$$\begin{aligned} \frac{\partial^2 V(r^*,0;T^*)}{\partial {T^*}^2} &= -\int_{r_l}^{r_u} r\frac{\partial U(r,T^*)}{\partial T^*} dr \\ &= -\int_{r_l}^{r_u} r\mathbf{L}_r^* U(r,T^*)dr \end{aligned}$$

[2] Or assume $\lambda(r,t) = \bar{\lambda}(t) + u(r,t)/w(r,t)$, which is equivalent to let $u(r,t)=0$ and $\lambda(r,t) = \bar{\lambda}(t)$.

$$
\begin{aligned}
&= -\int_{r_l}^{r_u} \left(r\frac{\partial}{\partial r}\left(\frac{1}{2}w^2\frac{\partial U}{\partial r}\right) - r\frac{\partial}{\partial r}\left\{\left[u - \left(\lambda(t) + \frac{\partial w}{\partial r}\right)w\right]U\right\} - r^2 U\right) dr \\
&= -\left[\frac{r}{2}w^2\frac{\partial U}{\partial r}\right]\Bigg|_{r_l}^{r_u} + \int_{r_l}^{r_u} \frac{1}{2}w^2\frac{\partial U}{\partial r}dr + \left\{r\left[u - \left(\lambda(t) + \frac{\partial w}{\partial r}\right)w\right]U\right\}\Bigg|_{r_l}^{r_u} \\
&\quad - \int_{r_l}^{r_u} \left[u - \left(\lambda(t) + \frac{\partial w}{\partial r}\right)w\right]Udr + \int_{r_l}^{r_u} r^2 Udr \\
&= \int_{r_l}^{r_u} \left(\frac{1}{2}w^2\frac{\partial U}{\partial r} + \frac{\partial w}{\partial r}wU\right) dr + \lambda(t)\int_{r_l}^{r_u} wUdr + \int_{r_l}^{r_u} (r^2 - u)Udr \\
&= \frac{1}{2}\int_{r_l}^{r_u} \frac{\partial(w^2 U)}{\partial r}dr + \lambda(t)\int_{r_l}^{r_u} wUdr + \int_{r_l}^{r_u} (r^2 - u)Udr \\
&= \lambda(t)\int_{r_l}^{r_u} wUdr + \int_{r_l}^{r_u} (r^2 - u)Udr. \qquad (10.9)
\end{aligned}
$$

Consequently, $\lambda(t)$ satisfies the equation:

$$
\lambda(T^*)\int_{r_l}^{r_u} wU(r,T^*)dr + \int_{r_l}^{r_u} (r^2 - u)U(r,T^*)dr = \frac{\partial^2 V(r^*,0;T^*)}{\partial {T^*}^2}. \qquad (10.10)
$$

Here we have shown that from the condition (10.6) we can have the condition (10.10). In order to show that they are equivalent, we also need to show that from the condition (10.10) we can have the condition (10.6). From the derivation procedure of the expressions (10.8) and (10.9), we know that when $U(r,t)$ is the solution of the problem (10.7), the following is true:

$$
\begin{aligned}
\int_{r_l}^{r_u} \frac{\partial U(r,T^*)}{\partial T^*}dr &= -\int_{r_l}^{r_u} rU(r,T^*)dr, \\
-\int_{r_l}^{r_u} r\frac{\partial U(r,T^*)}{\partial T^*}dr &= \lambda(t)\int_{r_l}^{r_u} wUdr + \int_{r_l}^{r_u} (r^2 - u)Udr.
\end{aligned}
$$

When the condition (10.10) holds, we have

$$
-\int_{r_l}^{r_u} r\frac{\partial U(r,T^*)}{\partial T^*}dr = \frac{\partial^2 V(r^*,0;T^*)}{\partial {T^*}^2}.
$$

From this relation we can have

$$
\begin{aligned}
\frac{\partial V(r^*,0;T^*)}{\partial T^*} - \frac{\partial V(r^*,0;0)}{\partial T^*} &= \int_0^{T^*} \frac{\partial^2 V(r^*,0;T^*)}{\partial {T^*}^2}dT^* \\
&= -\int_0^{T^*}\int_{r_l}^{r_u} r\frac{\partial U(r,T^*)}{\partial T^*}drdT^* \\
&= -\int_{r_l}^{r_u} r\left[U(r,T^*) - U(r,0)\right]dr \\
&= \int_{r_l}^{r_u} \frac{\partial U(r,T^*)}{\partial T^*}dr + r^*,
\end{aligned}
$$

which can be reduced into

$$\frac{\partial V(r^*,0;T^*)}{\partial T^*} = \int_{r_l}^{r_u} \frac{\partial U(r,T^*)}{\partial T^*} dr$$

because $\dfrac{\partial V(r^*,0;0)}{\partial T^*} = -r^*$. From the relation just obtained we further have

$$\begin{aligned} V(r^*,0;T^*) - V(r^*,0;0) &= \int_0^{T^*} \frac{\partial V(r^*,0;T^*)}{\partial T^*} dT^* \\ &= \int_0^{T^*} \int_{r_l}^{r_u} \frac{\partial U(r,T^*)}{\partial T^*} dr dT^* \\ &= \int_{r_l}^{r_u} \left[U(r,T^*) - U(r,0)\right] dr \\ &= \int_{r_l}^{r_u} U(r,T^*) dr - 1, \end{aligned}$$

which can be reduced into

$$V(r^*,0;T^*) = \int_{r_l}^{r_u} U(r,T^*) dr$$

because $V(r^*,0;0) = 1$. This completes our proof. Hence, instead of finding $\lambda(t)$ such that the condition (10.6) holds, we may also find $\lambda(t)$ such that the condition (10.10) is satisfied.

Now we discuss how to find $\lambda(t)$ from condition (10.10). From Sect. 5.4,[3] the value of $\lambda(t)$ for $t \in [0,T^*]$ is determined by the portion of the zero-coupon bond price curve on $[0,T^*]$. Suppose we already have the solution of problem (10.7) and the value of $\lambda(t)$ for $t \in [0, T^* - \epsilon]$, ϵ being a small positive number. In order to find the value of $\lambda(t)$ for $t \in (T^* - \epsilon, T^*]$, we need to guess the value of $\lambda(t)$ for $t \in (T^* - \epsilon, T^*]$ and continue to solve the problem (10.7) from $T^* - \epsilon$ to T^* and check the condition (10.10) at any time in $(T^* - \epsilon, T^*]$. As soon as the condition (10.10) holds, we have the value of $\lambda(t)$ on $(T^* - \epsilon, T^*]$. Such a procedure is performed from a very small T^*, gradually increasing, to $T^* = T^*_{max}$, and $\lambda(t)$ can be found for $t \in [0, T^*_{max}]$. This procedure is easy and faster, compared with the procedure of determining $\lambda(t)$ by solving the problem (5.47) if the same mesh sizes are used.

The initial-boundary value problem (10.7) is well-posed because the condition (10.4) holds. If

$$\begin{cases} u(r_l,t) - w(r_l,t)\dfrac{\partial w(r_l,t)}{\partial r} = 0, \\ u(r_u,t) - w(r_u,t)\dfrac{\partial w(r_u,t)}{\partial r} = 0, \end{cases} \tag{10.11}$$

[3]There we assume $\lambda(r,t) = \bar{\lambda}(t) + u(r,t)/w(r,t)$. However the procedures of determining $\bar{\lambda}(t)$ and $\lambda(t)$ from the zero-coupon bond price curve are the same.

then from the relation (10.3), we can still obtain the relation (10.5) without specifying the values for $U(r_l,t)$ and $U(r_u,t)$. In this case, instead of the problem (10.7), $U(r,t)$ is the solution of the following well-posed initial value problem

$$\begin{cases} \dfrac{\partial U}{\partial t} = \mathbf{L}_r^* U, & r_l \le r \le r_u, \quad 0 \le t \le T^*_{max}, \\ U(r,0) = \delta(r-r^*), & r_l \le r \le r_u, \end{cases} \tag{10.12}$$

and we can still derive the conditions (10.6) and (10.10) from the relation (10.5). For more complicated cases, the following treatment can be used. At any point on the lower boundary $r = r_l$, when

$$u(r_l,t) - w(r_l,t)\frac{\partial w(r_l,t)}{\partial r} > 0,$$

we choose $U(r_l,t) = 0$; whereas

$$u(r_l,t) - w(r_l,t)\frac{\partial w(r_l,t)}{\partial r} = 0,$$

we do not specify any value for $U(r_l,t)$ as a boundary condition. For the upper boundary, the situation is similar. Under such a treatment, the conditions (10.6) and (10.10) still hold.

10.1.2 Numerical Methods for the Inverse Problem

Again, let $\overline{V}(T^*)$ denote today's zero-coupon bond curve for bonds with a face value $Z = 1$. Suppose that the values of K zero-coupon bonds with maturities $T_1^*, T_2^*, \cdots, T_K^*$ are $V_{T_1^*}, V_{T_2^*}, \cdots, V_{T_K^*}$, which can be obtained from the market. Assume $T_K^* = T^*_{max}$ and $0 < T_1^* < \cdots < T_K^*$. Let today's time be T_0^* and $T_0^* = 0$. Clearly, $\overline{V}(T_0^*) = 1$ and $\dfrac{\partial \overline{V}(T_0^*)}{\partial T^*} = -r^*$, where r^* is today's short-term interest rate. Based on the data, we can generate a zero-coupon bond price curve $\overline{V}(T^*)$ on $[0, T^*_{max}]$ by the cubic spline interpolation described in Sect. 6.1.1. Because $\dfrac{\partial \overline{V}(T_0^*)}{\partial T^*} = -r^*$, at the left end we require this condition instead of assuming $\dfrac{\partial^2 \overline{V}(T_0^*)}{\partial {T^*}^2} = 0$. At the right end, we assume the function $\overline{V}(T^*)$ to be a polynomial of degree two on $[T^*_{K-1}, T^*_K]$ instead of assuming $\dfrac{\partial^2 \overline{V}(T_M^*)}{\partial {T^*}^2} = 0$. Using the method described in Sect. 6.1.1 for the modified case, we can determine these polynomials on all the subintervals $[T_k^*, T_{k+1}^*]$, $k = 0, 1, \cdots, K-1$. As soon as we have the zero-coupon bond curve, we can determine $\lambda(t)$ by solving inverse problems.

First, let us discuss how to solve the inverse problem (5.47). When $\lambda(t)$ is given on $[0, T^*]$, the partial differential equation can be discretized by the difference scheme (7.12). Hence, for any T^*, as long as $\lambda(t)$ is given on $[0, T^*]$, we can calculate $V(r, 0; T^*)$ from $V(r, T^*; T^*)$. Assume that we have obtained $\lambda(t)$ on $[0, T^* - \Delta t]$ from the value $\overline{V}(t)$ on $[0, T^* - \Delta t]$. We guess $\lambda(T^*)$, assume $\lambda(t)$ to be a linear function on $[T^* - \Delta t, T^*]$, and solve problem (5.47) from $t = T^*$ to $t = 0$. Check if $V(r^*, 0; T^*) = \overline{V}(T^*)$. If it is true, we find $\lambda(t)$ on $[T^* - \Delta t, T^*]$; if not, we adjust $\lambda(T^*)$ until we find a value $\lambda(T^*)$ such that $V(r^*, 0; T^*) = \overline{V}(T^*)$. This procedure can start from $T^* = \Delta t$ and continue successively until $T^* = T^*_{max}$. At $T^* = \Delta t$, if only $\lambda(\Delta t)$ is given, we cannot define a linear function on $[0, \Delta t]$. From the condition (10.10), we see that $\lambda(0)$ can be determined by

$$\lambda(0) = \frac{\dfrac{\partial^2 \overline{V}(0)}{\partial T^{*2}} - r^{*2} + u(r^*, 0)}{w(r^*, 0)}. \tag{10.13}$$

Now let us discuss how to solve problem (10.7). For the domain $[r_l, r_u] \times [0, T^*_{max}]$, we take the following partition: $r_m = r_l + m\Delta r$, $m = 0, 1, \cdots, M$, $t^n = n\Delta t$, $n = 0, 1, \cdots, N$, where $\Delta r = (r_u - r_l)/M$ and $\Delta t = T^*_{max}/N$, M, N being integers. Let U_m^n and $\lambda^{n+1/2}$ be the approximate values of $U(r_m, t^n)$ and $\lambda(t^{n+1/2})$, and $\overline{V}^n$ denote $\overline{V}(t^n)$. We also represent U_m^n, $m = 0, 1, \cdots, M$ by $\{U_m^n\}$. On this partition, the problem (10.7) and the condition (10.10) can be discretized as follows.

Because the initial condition in the problem (10.7) is a Dirac delta function, we discretize the partial differential equation there by the following "conservative" scheme:

$$\begin{aligned}
&\frac{U_m^{n+1} - U_m^n}{\Delta t} \\
&= \frac{1}{4\Delta r}\left[\left(\bar{w}_{m+1/2}^{n+1/2}\right)^2 \left(\frac{U_{m+1}^{n+1} - U_m^{n+1}}{\Delta r} + \frac{U_{m+1}^n - U_m^n}{\Delta r}\right)\right. \\
&\qquad\qquad \left. - \left(\bar{w}_{m-1/2}^{n+1/2}\right)^2 \left(\frac{U_m^{n+1} - U_{m-1}^{n+1}}{\Delta r} + \frac{U_m^n - U_{m-1}^n}{\Delta r}\right)\right] \\
&\quad - \left[\bar{u}_{m+1/2}^{n+1/2} - \left(\lambda^{n+1/2} + \frac{w_{m+1}^{n+1/2} - w_m^{n+1/2}}{\Delta r}\right)\bar{w}_{m+1/2}^{n+1/2}\right] \\
&\qquad\quad \times \frac{U_{m+1}^{n+1} + U_m^{n+1} + U_{m+1}^n + U_m^n}{4\Delta r} \\
&\quad + \left[\bar{u}_{m-1/2}^{n+1/2} - \left(\lambda^{n+1/2} + \frac{w_m^{n+1/2} - w_{m-1}^{n+1/2}}{\Delta r}\right)\bar{w}_{m-1/2}^{n+1/2}\right] \\
&\qquad\quad \times \frac{U_m^{n+1} + U_{m-1}^{n+1} + U_m^n + U_{m-1}^n}{4\Delta r} \\
&\quad - \frac{r_m}{2}(U_m^{n+1} + U_m^n),
\end{aligned}$$

$$m = 1, 2, \cdots, M-1,$$

where $\bar{w}_{m+1/2}^{n+1/2} = (w_{m+1}^{n+1/2} + w_m^{n+1/2})/2$ and $\bar{u}_{m+1/2}^{n+1/2} = (u_{m+1}^{n+1/2} + u_m^{n+1/2})/2$. These equations can be rewritten as

$$a_m U_{m-1}^{n+1} + b_m U_m^{n+1} + c_m U_{m+1}^{n+1} = -a_m U_{m-1}^n + (2 - b_m) U_m^n - c_m U_{m+1}^n,$$
$$m = 1, 2, \cdots, M-1, \tag{10.14}$$

where

$$\begin{aligned}
a_m &= \frac{-\Delta t}{4\Delta r^2}\left(\bar{w}_{m-1/2}^{n+1/2}\right)^2 \\
&\quad - \frac{\Delta t}{4\Delta r}\left[\bar{u}_{m-1/2}^{n+1/2} - \left(\lambda^{n+1/2} + \frac{w_m^{n+1/2} - w_{m-1}^{n+1/2}}{\Delta r}\right)\bar{w}_{m-1/2}^{n+1/2}\right], \\
b_m &= 1 + \frac{\Delta t r_m}{2} + \frac{\Delta t}{4\Delta r^2}\left[\left(\bar{w}_{m+1/2}^{n+1/2}\right)^2 + \left(\bar{w}_{m-1/2}^{n+1/2}\right)^2\right] \\
&\quad + \frac{\Delta t}{4\Delta r}\left[\bar{u}_{m+1/2}^{n+1/2} - \left(\lambda^{n+1/2} + \frac{w_{m+1}^{n+1/2} - w_m^{n+1/2}}{\Delta r}\right)\bar{w}_{m+1/2}^{n+1/2}\right] \\
&\quad - \frac{\Delta t}{4\Delta r}\left[\bar{u}_{m-1/2}^{n+1/2} - \left(\lambda^{n+1/2} + \frac{w_m^{n+1/2} - w_{m-1}^{n+1/2}}{\Delta r}\right)\bar{w}_{m-1/2}^{n+1/2}\right], \\
c_m &= \frac{-\Delta t}{4\Delta r^2}\left(\bar{w}_{m+1/2}^{n+1/2}\right)^2 \\
&\quad + \frac{\Delta t}{4\Delta r}\left[\bar{u}_{m+1/2}^{n+1/2} - \left(\lambda^{n+1/2} + \frac{w_{m+1}^{n+1/2} - w_m^{n+1/2}}{\Delta r}\right)\bar{w}_{m+1/2}^{n+1/2}\right].
\end{aligned}$$

From the boundary conditions in the problem (10.7), we have

$$U_0^{n+1} = U_M^{n+1} = 0. \tag{10.15}$$

When the coefficients in the set of equations (10.14) are known, the sets of equations (10.14) and (10.15) consist of a linear system for $U_0^{n+1}, U_1^{n+1}, \cdots, U_M^{n+1}$.

The initial condition in the problem (10.7) can be approximated by

$$U_m^0 = \begin{cases} \dfrac{1}{\Delta r}\left[1 - \dfrac{r^*}{\Delta r} + \text{int}\left(\dfrac{r^*}{\Delta r}\right)\right], & m = \text{int}\left(\dfrac{r^*}{\Delta r}\right), \\ \dfrac{1}{\Delta r}\left[\dfrac{r^*}{\Delta r} - \text{int}\left(\dfrac{r^*}{\Delta r}\right)\right], & m = \text{int}\left(\dfrac{r^*}{\Delta r}\right) + 1, \\ 0, & \text{otherwise}, \end{cases} \tag{10.16}$$

where $\text{int}(x)$ is the integer part of the number x, and we assume $r^* \in [r_l + \Delta r, r_u - \Delta r]$. As it can be seen, we here approximate the function $\delta(r - r^*)$ in the following way. We let the sum of the two values on the point with

$m = \text{int}\left(\frac{r^*}{\Delta r}\right)$ and the point with $m = \text{int}\left(\frac{r^*}{\Delta r}\right) + 1$ be equal to $\frac{1}{\Delta r}$ and their ratio is inversely proportional to their distances to r^*. On any other point, let the value be equal to zero.

By the trapezoidal rule (see Sect. 6.1.3), the condition (10.10) can be approximated by

$$\begin{aligned}
&\lambda^{n+1/2}\Delta r\left[\frac{1}{4}w_0^{n+\frac{1}{2}}\left(U_0^{n+1}+U_0^n\right)+\tfrac{1}{2}\sum_{m=1}^{M-1}w_m^{n+\frac{1}{2}}\left(U_m^{n+1}+U_m^n\right)\right.\\
&\qquad\qquad\qquad\left.+\frac{1}{4}w_M^{n+\frac{1}{2}}\left(U_M^{n+1}+U_M^n\right)\right]\\
&+\frac{\Delta r}{4}\left[(r_0^{n+\frac{1}{2}})^2-u_0^{n+\frac{1}{2}}\right]\left(U_0^{n+1}+U_0^n\right)\\
&+\frac{\Delta r}{2}\sum_{m=1}^{M-1}\left[(r_m^{n+\frac{1}{2}})^2-u_m^{n+\frac{1}{2}}\right]\left(U_m^{n+1}+U_m^n\right)\\
&+\frac{\Delta r}{4}\left[(r_M^{n+\frac{1}{2}})^2-u_M^{n+\frac{1}{2}}\right]\left(U_M^{n+1}+U_M^n\right)=\frac{\partial^2\overline{V}(t^{n+\frac{1}{2}})}{\partial T^{*2}}.
\end{aligned} \tag{10.17}$$

Here we approximate $U_m^{n+1/2}$ by $\frac{1}{2}(U_m^n + U_m^{n+1})$ for $m = 0, 1, \cdots, M$.

From the expression (10.16), we can have $\{U_m^0\}$. Therefore, we can have the following procedure for $n = 0, 1, \cdots, N-1$ successively. Suppose we already have $\{U_m^n\}$. Guessing[4] $\lambda^{n+1/2}$, we can obtain $\{U_m^{n+1}\}$ by solving the system consisting of Eqs. (10.14) and (10.15). Then, we check if Eq. (10.17) holds. If not, we need to find a new guess by solving $\lambda^{n+1/2}$ from Eq. (10.17) or by other iteration methods, and obtain new $\{U_m^{n+1}\}$ and check again; if it is, we find the value $\lambda^{n+1/2}$. When this procedure is done for $n = 0, 1, \cdots, N-1$ successively, we find the values for $\lambda^{n+1/2}$, $n = 0, 1, \cdots, N-1$. Another condition that can be used to determine $\lambda^{n+1/2}$ is condition (10.6). The advantage of using condition (10.6) is to let the value of the zero-coupon bonds be exactly equal to the data from the market. In this case, we have to design an iteration method to find the next iterative value of $\lambda^{n+1/2}$. It is clear that if the problem (10.7) needs to be replaced by the problem (10.12), the procedure above is almost the same.

For the method based on the problem (5.47), in order to do one iteration to determine $\lambda(t)$, we need to integrate the partial differential equation $n+1$ times from t^{n+1} to t^0. For the method based on the problem (10.7), in order to do the same thing, we need to integrate the partial differential equation only once from t^n to t^{n+1}. Therefore, we pay more attention to the method based on the problem (10.7). The only complication is that the computation based on the problem (10.7) involves the Dirac delta function. This requires us to use more grid points in the r-direction. In order for a function $\lambda(t)$ to be used in practice, we have to check whether or not the computed zero-coupon bond values are matched with the real market data well enough. If

[4] As the first guess, we can let $\lambda^{1/2} = \lambda(0)$ and $\lambda^{n+1/2} = \lambda^{n-1/2}$ for $n \neq 0$.

the formulation (5.47) is adopted, then such a condition is used directly when $\lambda(t)$ is determined. Thus, no further check is needed for this case. However, when the formulation (10.7) is used, theoretically the computed zero-coupon bond values should be consistent with the real market data if the condition (10.6) or the condition (10.10) holds. Because there exists numerical error, this fact will be true only if very large M is used. Thus between these two methods, which has a better performance is not clear.

10.1.3 Numerical Results on Market Prices of Risk

In this subsection, we give two examples on numerical results of inverse problems and the only results obtained by the method of solving the problem (10.7) are given. As an example, we take the following short-term interest rate model:

$$dr = (r^{**} - r)dt + r(0.2 - r)dX, \quad r_l = 0 \le r \le r_u = 0.2,$$

where r^{**} is a constant between r_l and r_u, and $r^{**} = 0.05345$ in these examples given here. This model satisfies conditions (5.45) and (5.46), so these partial differential equation problems we are going to solve are well-posed.

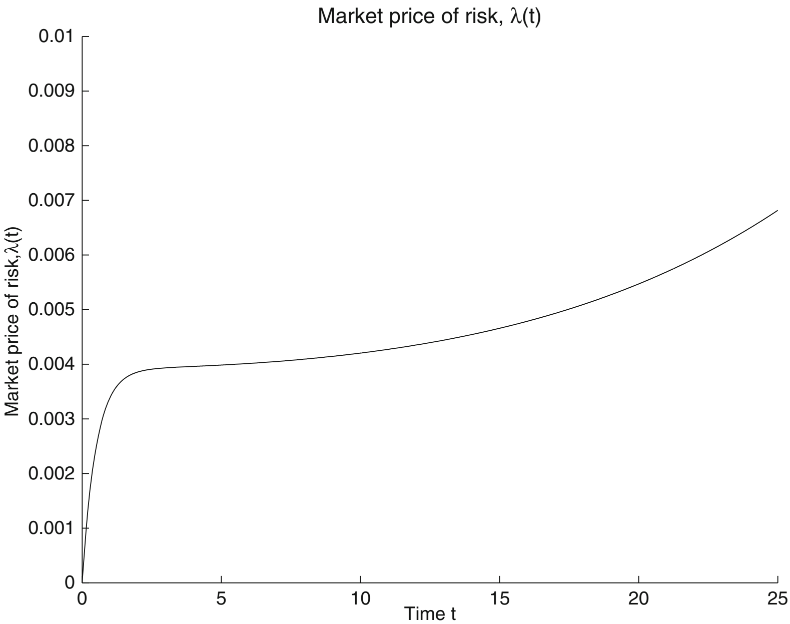

Fig. 10.1. The market price of risk $\lambda(t)$

Example 1. Suppose today's bond prices are given by the exponential function $100e^{-0.05345T^*}$. According to this function, we can use the method

Table 10.1. Comparison between given and computed bond prices

($V_{b,g}$ denotes given bond prices and V_b stands for computed bond prices)

T^*	0.5	1	2	3	5	7	10	15	20	25
$V_{b,g}$	97.36	94.80	89.86	85.18	76.55	68.79	58.60	44.85	34.335	26.283
V_b	97.36	94.80	89.86	85.18	76.55	68.79	58.60	44.85	34.333	26.279

given in the last subsection to find the market price of risk $\lambda(t)$. In Fig. 10.1, the function $\lambda(t)$ is shown. As soon as we have the market price of risk, we can compute the bond price by solving the bond equation. In Table 10.1, we list both the numerical results and the values from the given function. From the table, we see that the difference is on the third decimal place, which means that the inverse problem has been solved quite accurately. In order to do this computation, a $1,000 \times 1,000$ mesh was used.

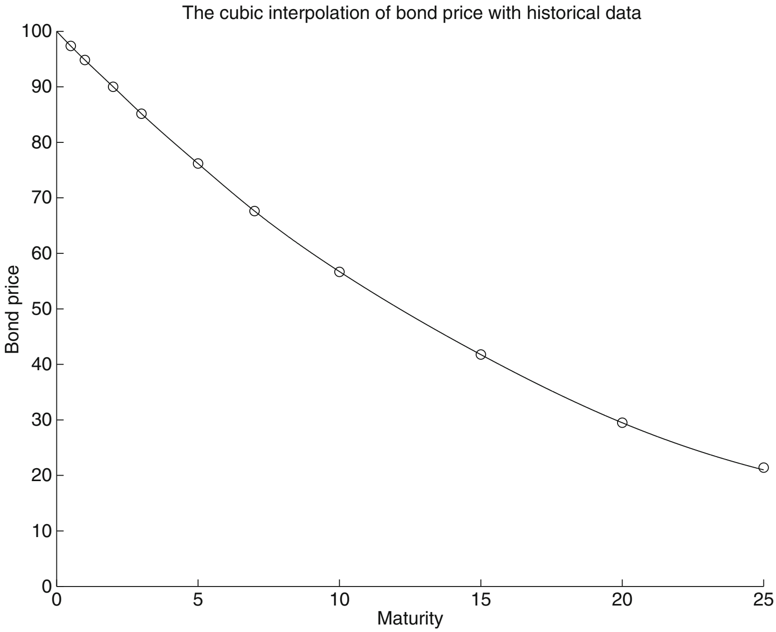

Fig. 10.2. Prices of today's bonds

Table 10.2. Comparison between market and computed bond prices

($V_{b,m}$ represents market bond prices and V_b stands for computed bond prices)

T^*	0.5	1	2	3	5	7	10	15	20	25
$V_{b,m}$	97.35	94.83	90.01	85.16	76.18	67.62	56.72	41.76	29.49	21.00
V_b	97.35	94.83	90.01	85.16	76.18	67.62	56.73	41.77	29.50	21.02

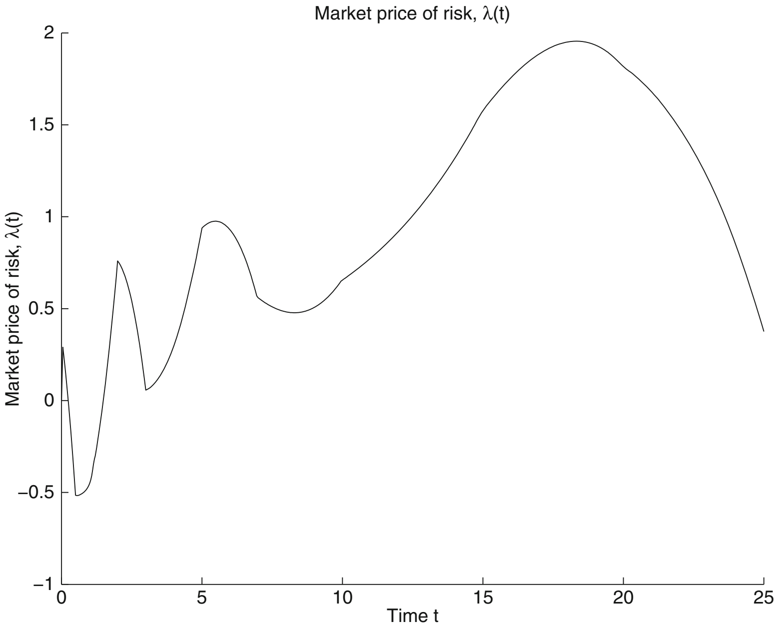

Fig. 10.3. The market price of risk $\lambda(t)$

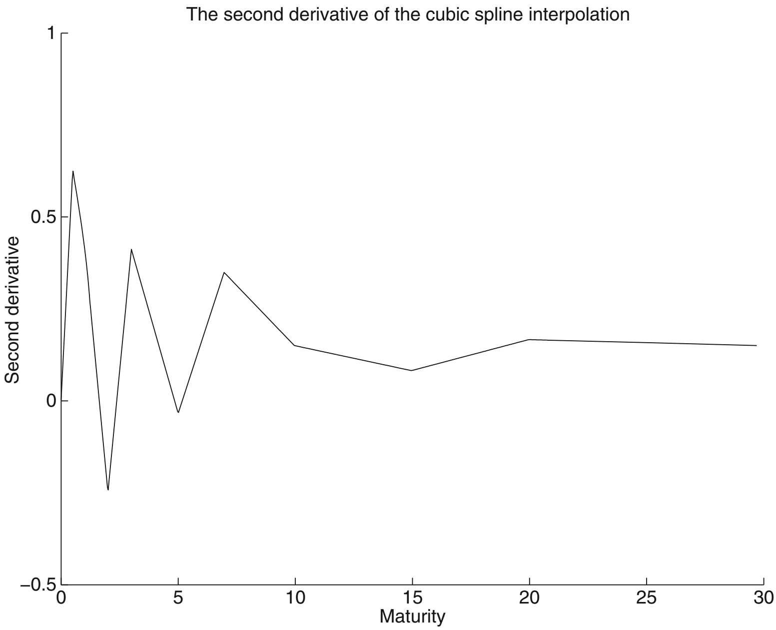

Fig. 10.4. The second derivative of today's bond prices

Example 2. From the market, we obtained the data on the short-term interest rate and the zero-coupon bond prices with maturities 0.5, 1, 2, 3, 5, 7, 10, 15, 20, and 25 years on November 30, 1995. Using the data, we generate a bond price function by the cubic spline interpolation described in Sect. 6.1.1. In Fig. 10.2, the data are given by "∘," and the function is shown by a solid curve. Using the bond price function, we find the market price of risk, which is shown in Fig. 10.3. This function is not as smooth as the market price of risk given in Fig. 10.1. From the condition (10.10), we see that $\lambda(t)$ is closely related to the second derivative of today's bond curve. For this case, the second derivative of bond prices is not smooth (see Fig. 10.4), so the market price of risk has the shape shown in Fig. 10.3. Using the market price of risk, we can compute the bond price by solving the bond equation. In Table 10.2, both the computed bond prices and the bond prices on the market are listed. Their difference is also very small, which means that the inverse problem has been solved successfully even if the market data are used.

10.2 Numerical Results of One-Factor Models

In order to price interest rate derivatives, the market price of risk for the short-term interest rate and today's short-term interest rate r^* must be given. In this section, the market price of risk is given numerically and is based on the data from November 30, 1995. Today's short-term interest rate is assumed to take the value of the short-term interest rate on that day, namely, $r^* = 0.05345$. Also, we suppose today's time t to be zero.

First, let us briefly discuss how to price bond options. Suppose that we need to find today's price of a T-year option with an exercise price E on a N-year bond that has a face value $Z = 1$ and a coupon rate k. Set $T_b = T + N$, and let $V_b(r, t; T_b)$ and $V(r, t)$ be the prices of the bond and the option, respectively. What we need to find is $V(r^*, 0)$. In order to do this, we first need to find $V_b(r, T; T_b)$ for $r \in [r_l, r_u]$ by solving the problem (5.48):

$$\begin{cases} \dfrac{\partial V_b}{\partial t} + \dfrac{1}{2} w^2 \dfrac{\partial^2 V_b}{\partial r^2} + (u - \lambda w) \dfrac{\partial V_b}{\partial r} - r V_b + k = 0, & r_l \le r \le r_u, \quad t \le T_b, \\ V_b(r, T_b; T_b) = 1, & r_l \le r \le r_u \end{cases}$$

from $t = T_b$ to $t = T < T_b$. Based on the function $V_b(r, T; T_b)$, we then obtain $V(r, 0)$ by solving the problem (5.49):

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2} w^2 \dfrac{\partial^2 V}{\partial r^2} + (u - \lambda w) \dfrac{\partial V}{\partial r} - r V = 0, & r_l \le r \le r_u, \quad t \le T, \\ V(r, T) = \max\left(V_b(r, T; T_b) - E, 0\right), & r_l \le r \le r_u \end{cases}$$

from $t = T$ to $t = 0$. When the market price of risk is given numerically, the problems (5.48) and (5.49) have to be solved numerically, and the scheme (7.12) or a modified scheme (7.6) can be adopted. The modified scheme (7.6) is a scheme that is the same as the scheme (7.6) for any

Table 10.3. Prices of bond options with $E = 0.95, 1$ and $k = 0.055$

E	$T\backslash T_b - T$	0.5	1	2	3	5
0.95	0.25	0.0502	0.0516	0.0536	0.0530	0.0515
0.95	0.50	0.0499	0.0514	0.0525	0.0519	0.0498
0.95	0.75	0.0495	0.0509	0.0512	0.0507	0.0478
0.95	1.00	0.0489	0.0500	0.0497	0.0494	0.0457
1.00	0.25	0.0011	0.0024	0.0044	0.0039	0.0026
1.00	0.50	0.0014	0.0029	0.0041	0.0037	0.0022
1.00	0.75	0.0017	0.0031	0.0036	0.0034	0.0016
1.00	1.00	0.0017	0.0029	0.0030	0.0029	0.0010

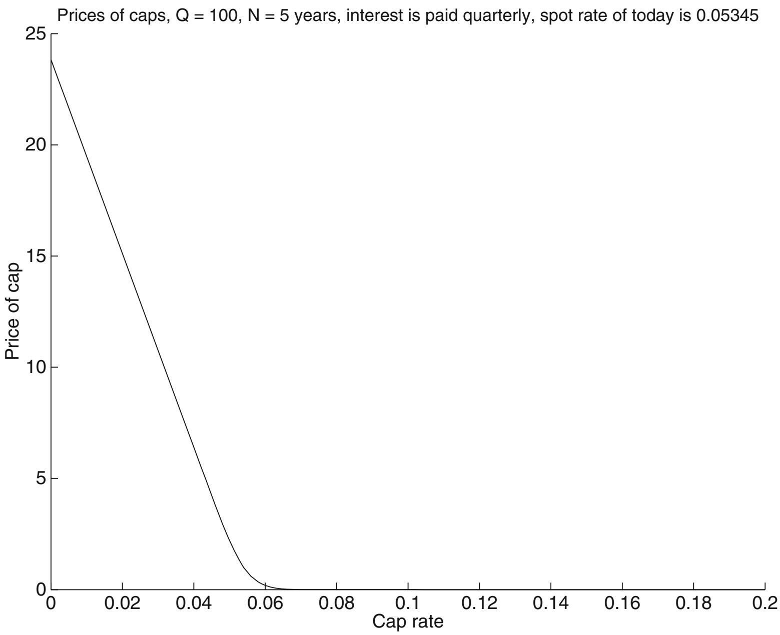

Fig. 10.5. Prices of caps

interior point and the same as the scheme (7.12) for the boundary points. In Table 10.3, the numerical results of the prices on 40 bond options are listed. There, the exercise price E is equal to 0.95 and 1, and the bond pays coupons continuously with a coupon rate $k = 0.055$. The expiries of the options are 0.25, 0.5, 0.75, and 1 year and the life spans of bonds are 0.5, 1, 2, 3, and 5 years.

Pricing a cap is done in the following way. Consider a N-year cap and suppose that money is paid quarterly. As pointed out in Sect. 5.5, the cap is a sum of $4N - 1$ caplets in this case and the maturities of the bonds associated with the $4N - 1$ caplets are $t_k = k/4$, $k = 2, 3, \cdots, 4N$. Let us call the bond

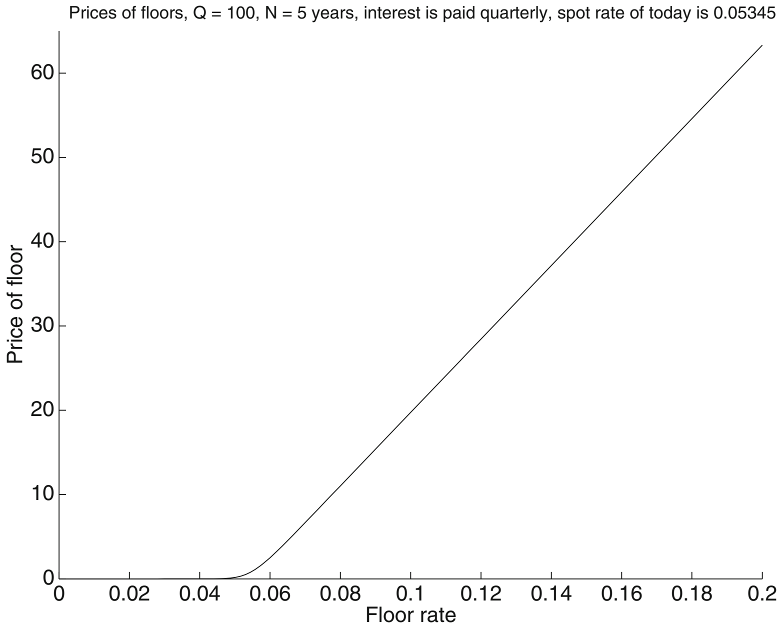

Fig. 10.6. Prices of floors

with maturity t_k the kth bond, and its value is denoted by $V_{bk}(r,t)$. In order to have the value of the kth bond, we solve the problem (5.63):

$$\begin{cases} \dfrac{\partial V_{bk}}{\partial t} + \dfrac{1}{2}w^2 \dfrac{\partial^2 V_{bk}}{\partial r^2} + (u - \lambda w)\dfrac{\partial V_{bk}}{\partial r} - rV_{bk} = 0, & r_l \le r \le r_u, \\ & t_{k-1} \le t \le t_k, \\ V_{bk}(r, t_k) = (1 + r_c/4)\, Q, & r_l \le r \le r_u. \end{cases}$$

After we have all the values of the bonds, we can obtain the total value of the $4N - 1$ caplets by solving the problem (5.64):

$$\begin{cases} \dfrac{\partial V_c}{\partial t} + \dfrac{1}{2}w^2 \dfrac{\partial^2 V_c}{\partial r^2} + (u - \lambda w)\dfrac{\partial V_c}{\partial r} - rV_c \\ \qquad\qquad + \displaystyle\sum_{k=2}^{4N} \max\left(Q - V_{bk}(r, t_{k-1}), 0\right) \delta(t - t_{k-1}) = 0, \\ \qquad\qquad\qquad\qquad r_l \le r \le r_u, \quad t^* \le t \le t_{4N-1}, \\ V_c(r, t_{4N-1}) = 0, \qquad\qquad r_l \le r \le r_u. \end{cases}$$

The value $V_c(r^*, t^*)$ gives the premium of the cap.

The way to solve (5.63) and (5.64) numerically is similar to the way to solve (5.48), namely, by using the scheme (7.12) or modified (7.6). The only difference is that in the problem (5.64) there exist the Dirac delta functions. In this case, the treatment of the Dirac delta function is simple: after $V_c(r, t_{k-1}^+)$ is obtained, we should let $V_c(r, t_{k-1}^-) = V_c(r, t_{k-1}^+) + \max\left(Q - V_{bk}(r, t_{k-1}), 0\right)$ and then continue the computation by using the scheme (7.12) or modified (7.6). We take $Q = 100$, $N = 5$ years, and $r_c = 0, 0.002, 0.004, \cdots, 0.2$ and find these values of caps numerically.[5] The values of caps as a function of r_c are plotted in Fig. 10.5. The curve resembles a price curve of a put option, that is, the price is a decreasing function of r_c and changes rapidly near $r_c = r^* = 0.05345$.

Table 10.4. Pairs of caps and floors with the same values ($Q = 100$, $N = 5$ years, and the interest is paid quarterly)

r_c	r_f	Prices of caps or floors
0.05502	0.05466	0.8
0.05557	0.05422	0.7
0.05618	0.05370	0.6
0.05687	0.05310	0.5
0.05768	0.05242	0.4
0.05868	0.05160	0.3
0.05999	0.05055	0.2
0.06204	0.04899	0.1

The way to price floors is similar to the way to price caps. For the floor rate $r_f \in [0, 0.2]$, the floor prices are shown in Fig. 10.6. Their parameters are the same as the caps. The floor resembles a call option, that is, the price is an increasing function of r_f and changes rapidly near $r_f = r^* = 0.05345$. As soon as we have the prices of a cap with a cap rate r_c and a floor with a floor rate r_f, the difference between them is the price of a collar for the pair of r_c and r_f. If the price of a cap is equal to the price of a floor, then the price of the collar with this pair of r_c and r_f is zero. In Table 10.4, eight such pairs of r_c and r_f are listed. That is, on November 30, 1995, the price of a collar with one of these pairs of r_c and r_f should be zero. In Table 10.4, the corresponding prices of caps and floors are also shown.

Now let us discuss how to price swaps and swaptions, including both European and American swaptions. Let $V_s(r, t; r_s, T)$ be the value of an N-year swap with a swap rate r_s at time t when it is initiated at time T, $t \geq T$. Here, the notation is a little different from the notation used in Chap. 5: the time of the swap being initiated, T, is explicitly given in the notation as a parameter because when American swaptions are priced, many swaps with different initial times are involved. As it is described in Sect. 5.5.2, the procedure of

[5] When these values of caps were computed, a cap was defined as a sum of $4N$ caplets. For those results on floors, the situation is similar.

determining European swaption price is divided into two steps. The first step is to determine the value of swap with r_{se} as the swap rate for all $r \in [r_l, r_u]$, $V_s(r,t;r_{se},T)$, and the second step is to obtain the payoff of swaption and to find the value of swaption. In order to get $V_s(r,t;r_{se},T)$, we need to solve the problem (5.61):

$$\begin{cases} \dfrac{\partial V_s}{\partial t} + \dfrac{1}{2}w^2\dfrac{\partial^2 V_s}{\partial r^2} + (u-\lambda w)\dfrac{\partial V_s}{\partial r} - rV_s \\ \qquad\qquad -\dfrac{Qr_{se}}{2}\displaystyle\sum_{k=1}^{2N}\delta(t-T-k/2) + Q\delta(t-T) = 0, \\ \qquad\qquad r_l \le r \le r_u, \qquad T \le t \le T+N, \\ V_s(r,T+N) = -Q, \quad r_l \le r \le r_u. \end{cases}$$

After we obtain $V_s(r,T;r_{se},T)$, using

$$V_{so}(r,T) = \max\left(V_s(r,T;r_{se},T), 0\right),$$

we can get the payoff of the swaption and then in order to find the value of swaption we need to solve the problem (5.62):

$$\begin{cases} \dfrac{\partial V_{so}}{\partial t} + \dfrac{1}{2}w^2\dfrac{\partial^2 V_{so}}{\partial r^2} + (u-\lambda w)\dfrac{\partial V_{so}}{\partial r} - rV_{so} = 0, \; r_l \le r \le r_u, \qquad t \le T, \\ V_{so}(r,T) = \max\left(V_s(r,T;r_{se},T), 0\right), \qquad\qquad r_l \le r \le r_u \end{cases}$$

from $t = T$ to $t = 0$. $V_{so}(r^*, 0)$ gives today's value of the European swaption.

Table 10.5. Prices of European and American swaptions with $Q = 100$

(The exercise swap rates r_{se} are 0.05335, 0.05423, 0.05506, 0.05712 for $N = 2, 3, 5, 10$, respectively, which are the swap rates computed by using the mode based on the market data of November 30, 1995)

	$T\backslash N$	2	3	5	10
European	0.5	0.167	0.196	0.269	0.278
	1	0.276	0.288	0.499	0.490
	2	0.492	0.548	1.083	1.021
American	0.5	0.213	0.248	0.331	0.342
	1	0.450	0.474	0.731	0.722
	2	0.678	0.753	1.338	1.273

For an American swaption, its value $V_{so}(r,t)$ at any time $t \in [0,T]$ must be greater than or equal to $\max\left(V_s(r,t;r_{se},t), 0\right)$:

$$V_{so}(r,t) \geq \max\left(V_s(r,t;r_{se},t),0\right). \tag{10.18}$$

Therefore, in order to obtain $V_{so}(r,t)$, we need to solve the following linear complementarity problem:

$$\begin{cases} \min\Bigg(-\dfrac{\partial V_{so}}{\partial t} - \dfrac{1}{2}w^2\dfrac{\partial^2 V_{so}}{\partial r^2} - (u-\lambda w)\dfrac{\partial V_{so}}{\partial r} + rV_{so}, \\ \qquad\qquad V_{so}(r,t) - \max\left(V_s(r,t;r_{se},t),0\right)\Bigg) = 0, \\ V_{so}(r,T) = \max\left(V_s(r,T;r_{se},T),0\right), \end{cases} \tag{10.19}$$

where $t \in [0,T]$ and $r \in [r_l, r_u]$. In order to have $V_s(r,t;r_{se},t)$, we need to solve (5.61) with $T = t$ from $t+N$ to t when $V_{so}(r,t)$ for time t needs to be determined. Of course, this LC problem can also be formulated as a free-boundary problem. Readers are asked to write down the free-boundary problem for this case as an exercise.

The problems (5.62) and (10.19) can be solved by the scheme (7.12) or modified (7.6). In Table 10.5, we list some numerical results on European and American swaptions. The exercise swap rates r_{se} are 0.05335, 0.05423, 0.05506, 0.05712 for $N = 2, 3, 5, 10$, respectively. The other parameters are given in the table.

10.3 Pricing Derivatives with Multi-Factor Models

10.3.1 Determining Models from the Market Data

In Sect. 5.6, a three-factor interest rate model was proposed. In this section, we will discuss implicit finite-difference methods for the three-factor interest rate derivative problems and some other related problems. In order to use that model to price an interest rate derivative, we need to know how to find the payoff of the derivative and to determine those coefficients in the partial differential equation (5.83). In this subsection, we will discuss these two problems, and the next subsection is devoted to implicit finite-difference methods.

Suppose we want to price a half-year option on five-year swaps with an exercise swap rate r_{se}. Assume the day we want to price the swaption (the option on swaps) to be denoted as $t = 0$. Thus, according to the notation given in Sect. 5.5.2, $T = 0.5$ and $N = 5$.

First, let us discuss how to determine the final value. On the market, the prices of 3-month, 6-month, 1-year, 2-year, 3-year and 5-year zero-coupon bonds are given every day. Set

$$T_1^* = 0.25,\ T_2^* = 0.5,\ T_3^* = 1,\ T_4^* = 2\ , T_5^* = 3, \ \text{ and }\ T_6^* = 5,$$

let Z_i denote the price of the bond with maturity T_i^*, and define

$$S_i = Z_i/T_i^*, \quad i = 1, 2, \cdots, 6.$$

Suppose we have these values on a period of L days and let $S_{i,l}$ stand for the value of S_i at the lth day, $l = 1, 2, \cdots, L$. By b_i^2 and $b_i b_j \rho_{i,j}$, we denote the variance of S_i and the covariance between S_i and S_j, respectively. From statistics, we know that b_i^2 and $\rho_{i,j}$ can be estimated by

$$\begin{aligned} b_i^2 &= \frac{1}{L-1} \sum_{l=1}^{L} \left(S_{i,l} - \frac{1}{L} \sum_{l=1}^{L} S_{i,l} \right)^2 \\ &= \frac{1}{L-1} \left[\sum_{l=1}^{L} (S_{i,l})^2 - \frac{1}{L} \left(\sum_{l=1}^{L} S_{i,l} \right)^2 \right] \end{aligned}$$

and

$$\begin{aligned} \rho_{ij} &= \frac{\sum\limits_{l=1}^{L} \left(S_{i,l} - \frac{1}{L} \sum\limits_{l=1}^{L} S_{i,l} \right) \left(S_{j,l} - \frac{1}{L} \sum\limits_{l=1}^{L} S_{j,l} \right)}{\sqrt{\left[\sum\limits_{l=1}^{L} \left(S_{i,l} - \frac{1}{L} \sum\limits_{l=1}^{L} S_{i,l} \right)^2 \times \sum\limits_{l=1}^{L} \left(S_{j,l} - \frac{1}{L} \sum\limits_{l=1}^{L} S_{j,l} \right)^2 \right]}} \\ &= \frac{\sum\limits_{l=1}^{L} (S_{i,l} S_{j,l}) - \frac{1}{L} \left(\sum\limits_{l=1}^{L} S_{i,l} \times \sum\limits_{l=1}^{L} S_{j,l} \right)}{\sqrt{\left[\sum\limits_{l=1}^{L} (S_{i,l})^2 - \frac{1}{L} \left(\sum\limits_{l=1}^{L} S_{i,l} \right)^2 \right] \left[\sum\limits_{l=1}^{L} (S_{j,l})^2 - \frac{1}{L} \left(\sum\limits_{l=1}^{L} S_{j,l} \right)^2 \right]}}. \end{aligned}$$

Using the data for the period from January 4, 1982, to February 15, 2002, we obtain

$$\begin{aligned} \mathbf{B} &= \begin{bmatrix} b_1^2 & b_1 b_2 \rho_{1,2} & \cdots & b_1 b_6 \rho_{1,6} \\ b_1 b_2 \rho_{1,2} & b_2^2 & \cdots & b_2 b_6 \rho_{2,6} \\ \vdots & \vdots & \ddots & \vdots \\ b_1 b_6 \rho_{1,6} & b_2 b_6 \rho_{2,6} & \cdots & b_6^2 \end{bmatrix} \\ &= 10^{-3} \begin{bmatrix} 0.4644 & 0.4758 & 0.4637 & 0.4224 & 0.3776 & 0.2993 \\ 0.4758 & 0.4916 & 0.4818 & 0.4413 & 0.3956 & 0.3145 \\ 0.4637 & 0.4818 & 0.4760 & 0.4392 & 0.3952 & 0.3161 \\ 0.4224 & 0.4413 & 0.4392 & 0.4109 & 0.3724 & 0.3014 \\ 0.3776 & 0.3956 & 0.3952 & 0.3724 & 0.3392 & 0.2766 \\ 0.2993 & 0.3145 & 0.3161 & 0.3014 & 0.2766 & 0.2289 \end{bmatrix}. \end{aligned}$$

By the **QR** method given in Sect. 6.2.4 or other methods, we can find the eigenvalues and the unit eigenvectors of **B**. As soon as we have them, **B** can be rewritten as

$$\mathbf{B} = \mathbf{A}^T \mathbf{C} \mathbf{A},$$

where

$$\mathbf{A} = \begin{bmatrix} 0.4366 & 0.4533 & 0.4479 & 0.4151 & 0.3745 & 0.3011 \\ -0.5426 & -0.3546 & -0.0918 & 0.2650 & 0.4190 & 0.5706 \\ -0.5871 & 0.1231 & 0.5461 & 0.2779 & -0.0121 & -0.5143 \\ -0.3980 & 0.6808 & 0.0016 & -0.4305 & -0.1994 & 0.3912 \\ 0.1082 & -0.4337 & 0.7019 & -0.4366 & -0.1869 & 0.2864 \\ -0.0031 & 0.0448 & 0.0113 & -0.5516 & 0.7806 & -0.2902 \end{bmatrix}$$

and

$$\mathbf{C} = 10^{-3} \times \ \text{diag}\,(2.366, 0.04109, 0.003240, \\ 3.953 \times 10^{-4}, 1.996 \times 10^{-4}, 4.498 \times 10^{-5}).$$

Because the last three components of $\mathbf{C}$ are very small compared with the first three components, the six random variables, S_1, S_2, $\cdots$, S_6, almost depend on only three variables. Because

$$\begin{vmatrix} a_{1,1} & a_{1,4} & a_{1,6} \\ a_{2,1} & a_{2,4} & a_{2,6} \\ a_{3,1} & a_{3,4} & a_{3,6} \end{vmatrix} = \begin{vmatrix} 0.4366 & 0.4151 & 0.3011 \\ -0.5426 & 0.2650 & 0.5706 \\ -0.5871 & 0.2779 & -0.5143 \end{vmatrix} \approx -0.3822 \neq 0,$$

we can choose S_1, S_4, and S_6 as the three independent components, which will be denoted by S_{i_1}, S_{i_2}, and S_{i_3} in what follows. From Sect. 5.6.2, we know that the values of S_i, $i \neq i_1, i_2$, and i_3, are uniquely determined by Eq. (5.67) for a given set of S_{i_1}, S_{i_2}, and S_{i_3} when $\mathbf{A}$ is found and $S_i^*, i = 1, 2, \cdots, 6$, are specified.[6] Based on the six values of S_1, S_2, $\cdots$, S_6, a zero-coupon bond curve with a maximum maturity $T^*_{\max} = 5$ can be found by using the cubic spline interpolation. Assume that for the period $t \in [0, T] = [0, 0.5]$, S_i^* are constants, for example, are equal to the values of zero-coupon bonds at $t = 0$. Thus, the possible zero-coupon bond curves for any $t \in [0, T]$ are the same, i.e.,

$$\bar{Z}\left(T^*; Z_{i_1}, Z_{i_2}, Z_{i_3}, t\right) = \bar{Z}\left(T^*; Z_{i_1}, Z_{i_2}, Z_{i_3}, 0\right).$$

Here in order to indicate the dependence of the zero-coupon bond curves on $Z_{i_1}, Z_{i_2}, Z_{i_3}$, instead of $\bar{Z}\left(T^*; t\right)$, we use $\bar{Z}\left(T^*; Z_{i_1}, Z_{i_2}, Z_{i_3}, t\right)$. As soon as we have a zero-coupon bond curve, using the expression (5.55) with $r_s = r_{se}$:

$$Q\left[1 - Z\left(T; T+N\right) - \frac{r_{se}}{2} \sum_{k=1}^{2N} Z\left(T; T+k/2\right)\right],$$

[6]In this way, for any day in the period from January 4, 1982, to February 15, 2002, we can obtain the theoretical values of S_2, S_3, and S_5 by giving the market data of S_1, S_4, and S_6. That is, from the market prices of 3-month, 2-year, and 5-year zero-coupon bonds we can obtain the theoretical prices of 6-month, 1-year, and 3-year zero-coupon bonds for any day. In Fig. 10.7 we compare the theoretical prices of 6-month, 1-year, and 3-year zero-coupon bonds with their market data for any day in the period from January 4, 1982, to February 15, 2002. The figure shows that the theoretical prices and the market data are very close to each other.

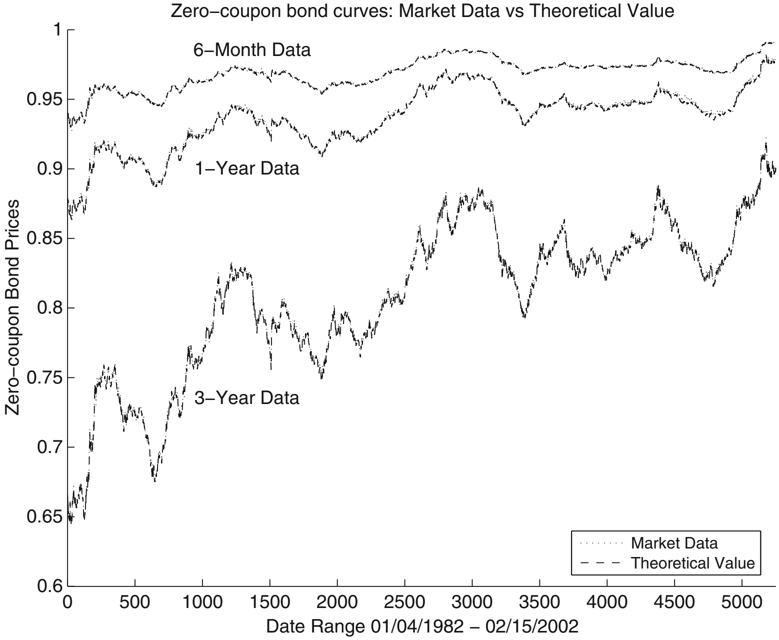

Fig. 10.7. Comparison between the market data and the theoretical values of zero-coupon bonds

we can determine the value of a swap with an exercise rate r_{se}. Here, Q is the notional principal and $Z(T;T+k/2) = \bar{Z}(k/2;Z_{i_1},Z_{i_2},Z_{i_3},T) = \bar{Z}(k/2;Z_{i_1},Z_{i_2},Z_{i_3},0)$. Therefore, the final value of a swaption is

$$Q\max\left(1-\bar{Z}\left(N;Z_{i_1},Z_{i_2},Z_{i_3},0\right)-\frac{r_{se}}{2}\sum_{k=1}^{2N}\bar{Z}\left(k/2;Z_{i_1},Z_{i_2},Z_{i_3},0\right),\ 0\right). \tag{10.20}$$

Before discussing how to determine the coefficients in the partial differential equation, we would like to give some information about how these zero-coupon bond curves generated above are close to the real zero-coupon bond curves. Suppose that one day, the prices of zero-coupon bonds are

$$\begin{cases} Z_1 = 0.9811, \quad Z_2 = 0.9559, \quad Z_3 = 0.9047, \\ Z_4 = 0.7979, \quad Z_5 = 0.7068, \quad \text{and} \quad Z_6 = 0.5475, \end{cases} \tag{10.21}$$

which correspond to the following interest rates:

$$\begin{cases} r_1 = 0.0776, \quad r_2 = 0.0923, \quad r_3 = 0.1027, \\ r_4 = 0.1161, \quad r_5 = 0.1191, \quad \text{and} \quad r_6 = 0.1242. \end{cases}$$

Here, r_i is associated with Z_i by the following expression:

$$Z_i = (1 + r_i/2)^{-2N_i},$$

where N_i is the maturity of the ith zero-coupon bond. From this set of data, we can determine a class of zero-coupon bond curves with $Z_{i_1}, Z_{i_2}, Z_{i_3}$ as parameters. For any day in the period from January 4, 1982, to February 15, 2002, we take the values of $Z_{i_1}, Z_{i_2}, Z_{i_3}$ as input and find a zero-coupon bond curve from the class. From the zero-coupon bond curve, we obtain the values of Z_i, $i \neq i_1, i_2$, and i_3, and the differences between the values determined from the curve and the values from the original market data. We do this for every day. The average value of the differences divided by $(1 - Z_i)$, $i \neq i_1, i_2$, and i_3, is 0.005. The same thing to the swap rate and to the value of the swaption on a 5-year swap with $r_{se} = 0.1225$ is also done. The maximum difference between the swap rates from the market curve and the model curve is 0.0004 (4 basis points), and the average difference is 0.00008 (0.8 basis points). The average error of the swaption value is 0.02 if the notional principal is 100. Therefore, we may conclude that these zero-coupon bond curves reflect the market situation.

Now let us discuss how to determine the coefficients in the partial differential equation. Suppose that derivative securities depend on $Z_{i_1}, Z_{i_2}, Z_{i_3}$, and t. Let

$$\begin{cases} \xi_1 = \dfrac{Z_{i_1} - Z_{i_1,l}}{1 - Z_{i_1,l}}, \\ \xi_2 = \dfrac{Z_{i_2} - Z_{i_2,l}}{Z_{i_1} - Z_{i_2,l}}, \\ \xi_3 = \dfrac{Z_{i_3} - Z_{i_3,l}}{Z_{i_2} - Z_{i_3,l}}, \end{cases} \tag{10.22}$$

where $Z_{i_1,l}, Z_{i_2,l}$, and $Z_{i_3,l}$ are minimums of $Z_{i_1}, Z_{i_2}, Z_{i_3}$ and we set $Z_{i_1,l} = 0.9597$, $Z_{i_2,l} = 0.7209$, and $Z_{i_3,l} = 0.4332$, which are a little less than the observed minimums 0.9634, 0.7463, and 0.4847, respectively. From Sect. 5.6.3, we know that the value of a derivative security, $V(\xi_1, \xi_2, \xi_3, t)$, satisfies the problem (5.83), where coefficients depends on $r, \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3, \tilde{\rho}_{1,2}, \tilde{\rho}_{1,3}, \tilde{\rho}_{2,3}$ besides ξ_1, ξ_2, and ξ_3. Therefore, in order to use that equation, we have to know $r, \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3, \tilde{\rho}_{1,2}, \tilde{\rho}_{1,3}$, and $\tilde{\rho}_{2,3}$. It is clear that r can be determined by the slope of zero-coupon bond curves at the left end, i.e.,

$$r(\xi_1, \xi_2, \xi_3, t) = -\frac{\partial \bar{Z}}{\partial T^*}(0; Z_{i_1}, Z_{i_2}, Z_{i_3}, 0), \tag{10.23}$$

where

$$\begin{cases} Z_{i_1} = Z_{i_1,l} + \xi_1 (1 - Z_{i_1,l}), \\ Z_{i_2} = Z_{i_2,l} + \xi_2 [Z_{i_1,l} + \xi_1 (1 - Z_{i_1,l}) - Z_{i_2,l}], \\ Z_{i_3} = Z_{i_3,l} + \xi_3 \{Z_{i_2,l} + \xi_2 [Z_{i_1,l} + \xi_1 (1 - Z_{i_1,l}) - Z_{i_2,l}] - Z_{i_3,l}\}. \end{cases} \tag{10.24}$$

As we know, for $\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3$ we need to require the condition (5.85):

$$\begin{cases} \tilde{\sigma}_1(0,\xi_2,\xi_3,t) = \tilde{\sigma}_1(1,\xi_2,\xi_3,t) = 0, \\ \tilde{\sigma}_2(\xi_1,0,\xi_3,t) = \tilde{\sigma}_2(\xi_1,1,\xi_3,t) = 0, \\ \tilde{\sigma}_3(\xi_1,\xi_2,0,t) = \tilde{\sigma}_3(\xi_1,\xi_2,1,t) = 0. \end{cases}$$

Let us assume $\tilde{\sigma}_i$ to be in the form

$$\tilde{\sigma}_i(\xi_1,\xi_2,\xi_3,t) = \tilde{\sigma}_i(\xi_i) = \tilde{\sigma}_{i,0}\frac{1-(1-2\xi_i)^2}{1-p_i(1-2\xi_i)^2}, \quad i = 1,2,3, \tag{10.25}$$

where $\tilde{\sigma}_{i,0}$ and p_i are positive constants, and $p_i \in (0,1)$. It is clear that in this case, condition (5.85) is fulfilled. On each day, we have the values of $Z_{i_1}, Z_{i_2}, Z_{i_3}$. Because ξ_1, ξ_2, ξ_3 are defined by the formula (10.22), we can also have the values of ξ_1, ξ_2, ξ_3 every day. Therefore, we can find $\tilde{\sigma}_i(\xi_i)$ from the data on the market using the method described in Sect. 6.3.2 with

$$g(\xi_i) = \frac{1-(1-2\xi_i)^2}{1-p_i(1-2\xi_i)^2} \quad \text{and} \quad N = 0.$$

For $\tilde{\rho}_{1,2}, \tilde{\rho}_{1,3}$, and $\tilde{\rho}_{2,3}$, there is no requirement. We assume that they are constant and that the value can also be obtained using the method described in Sect. 6.3.2.

Taking $p_1 = p_2 = p_3 = 0.8$ and using the data on the market for the period between January 4, 1982, and February 15, 2002, we obtain

$$\tilde{\sigma}_{1,0} = 0.09733, \quad \tilde{\sigma}_{2,0} = 0.08622, \quad \tilde{\sigma}_{3,0} = 0.08148$$

and

$$\tilde{\rho}_{1,2} = 0.5682, \quad \tilde{\rho}_{1,3} = 0.4996, \quad \tilde{\rho}_{2,3} = 0.8585.$$

10.3.2 Numerical Methods and Results

From Sect. 5.6, we know that for a European swaption, $V(\xi_1,\xi_2,\xi_3,t)$ satisfies the problem (5.83):

$$\begin{cases} \dfrac{\partial V}{\partial t} + \mathbf{L}_{3\xi}V = 0 \quad \text{on } \tilde{\Omega} \times [0,T], \\ V(\xi_1,\xi_2,\xi_3,T) = V_T\left(Z_{i_1}(\xi_1), Z_{i_2}(\xi_1,\xi_2), Z_{i_3}(\xi_1,\xi_2,\xi_3)\right) \quad \text{on } \tilde{\Omega}, \end{cases}$$

where $\tilde{\Omega}$ is the domain $[0,1]\times[0,1]\times[0,1]$ in the (ξ_1,ξ_2,ξ_3)-space, $\mathbf{L}_{3\xi}$ is defined by

$$\mathbf{L}_{3\xi} = \frac{1}{2}\tilde{\sigma}_1^2\frac{\partial^2}{\partial\xi_1^2} + \frac{1}{2}\tilde{\sigma}_2^2\frac{\partial^2}{\partial\xi_2^2} + \frac{1}{2}\tilde{\sigma}_3^2\frac{\partial^2}{\partial\xi_3^2}$$

$$+\tilde{\sigma}_1\tilde{\sigma}_2\tilde{\rho}_{1,2}\frac{\partial^2}{\partial\xi_1\partial\xi_2}+\tilde{\sigma}_1\tilde{\sigma}_3\tilde{\rho}_{1,3}\frac{\partial^2}{\partial\xi_1\partial\xi_3}+\tilde{\sigma}_2\tilde{\sigma}_3\tilde{\rho}_{2,3}\frac{\partial^2}{\partial\xi_2\partial\xi_3}$$
$$+b_1\frac{\partial}{\partial\xi_1}+b_2\frac{\partial}{\partial\xi_2}+b_3\frac{\partial}{\partial\xi_3}-r$$

and $Z_{i_1}(\xi_1)$, $Z_{i_2}(\xi_1,\xi_2)$, $Z_{i_3}(\xi_1,\xi_2,\xi_3)$ are given by expression (10.24). For b_1, b_2, and b_3 we have expression (5.86):

$$\begin{cases} b_1 = \dfrac{rZ_{i_1}}{1-Z_{i_1,l}}, \\ b_2 = \dfrac{r\left(Z_{i_2}-Z_{i_1}\xi_2\right)}{Z_{i_1}-Z_{i_2,l}} - \dfrac{\tilde{\sigma}_1\tilde{\sigma}_2\tilde{\rho}_{1,2}\left(1-Z_{i_1,l}\right)}{Z_{i_1}-Z_{i_2,l}}, \\ b_3 = \dfrac{r\left(Z_{i_3}-Z_{i_2}\xi_3\right)}{Z_{i_2}-Z_{i_3,l}} - \dfrac{\tilde{\sigma}_1\tilde{\sigma}_3\tilde{\rho}_{1,3}\xi_2\left(1-Z_{i_1,l}\right)}{Z_{i_2}-Z_{i_3,l}} - \dfrac{\tilde{\sigma}_2\tilde{\sigma}_3\tilde{\rho}_{2,3}\left(Z_{i_1}-Z_{i_2,l}\right)}{Z_{i_2}-Z_{i_3,l}} \end{cases}$$

and r is given by the formula (10.23).

Let

$$\tau = T-t \quad \text{and} \quad \overline{V}(\xi_1,\xi_2,\xi_3,\tau)=V(\xi_1,\xi_2,\xi_3,T-\tau),$$

the above problem becomes

$$\begin{cases} \dfrac{\partial\overline{V}}{\partial\tau} = \mathbf{L}_{3\xi}\overline{V} \quad \text{on } \tilde{\Omega}\times[0,T], \\ \overline{V}(\xi_1,\xi_2,\xi_3,0) = V_T\left(Z_{i_1}(\xi_1), Z_{i_2}(\xi_1,\xi_2), Z_{i_3}(\xi_1,\xi_2,\xi_3)\right) \quad \text{on } \tilde{\Omega}. \end{cases} \tag{10.26}$$

In the last subsection, we discussed how to determine the final value $V_T\left(Z_{i_1}(\xi_1), Z_{i_2}(\xi_1,\xi_2), Z_{i_3}(\xi_1,\xi_2,\xi_3)\right)$ for a swaption, which is given by the expression (10.20), and find r, $\tilde{\sigma}_1$, $\tilde{\sigma}_2$, $\tilde{\sigma}_3$, $\tilde{\rho}_{1,2}$, $\tilde{\rho}_{1,3}$, $\tilde{\rho}_{2,3}$, $Z_{i_1,l}$, $Z_{i_2,l}$, and $Z_{i_3,l}$ from the market. Therefore, we have everything we need in order to solve (10.26) numerically.

Suppose that M, L, I, and N are given integers. Let $\Delta\xi_1 = 1/M$, $\Delta\xi_2 = 1/L$, $\Delta\xi_3 = 1/I$, and $\Delta\tau = T/N$ and $u^n_{m,l,i}$ be an approximate value of $\overline{V}$ at $\xi_1 = m\Delta\xi_1$, $\xi_2 = l\Delta\xi_2$, $\xi_3 = i\Delta\xi_3$, and $\tau = n\Delta\tau$. Here, $m = 0,1,\cdots,M$, $l = 0,1,\cdots,L$, $i = 0,1,\cdots,I$ and $n = 0,1,\cdots,N$.

The partial differential equation in the problem (10.26) is discretized at $\tau = (n+1/2)\Delta\tau$, $n = 0,1,\cdots,N-1$. At any point, the partial derivative with respect to t is discretized by the central difference:

$$\frac{\partial\overline{V}^{n+1/2}_{m,l,i}}{\partial\tau} \approx \frac{u^{n+1}_{m,l,i}-u^n_{m,l,i}}{\Delta\tau}.$$

At any interior point, in $\tilde{\Omega}$, first- and second-order partial derivatives with respect to ξ_i are approximated by central schemes. For example,

$$\frac{\partial\overline{V}^{n+1/2}_{m,l,i}}{\partial\xi_1} \approx \frac{1}{2}\left(\frac{u^{n+1}_{m+1,l,i}-u^{n+1}_{m-1,l,i}}{2\Delta\xi_1}+\frac{u^n_{m+1,l,i}-u^n_{m-1,l,i}}{2\Delta\xi_1}\right)$$

and

$$\frac{\partial^2 \overline{V}_{m,l,i}^{n+1/2}}{\partial \xi_1^2} \approx \frac{1}{2}\left(\frac{u_{m+1,l,i}^{n+1} - 2u_{m,l,i}^{n+1} + u_{m-1,l,i}^{n+1}}{\Delta\xi_1^2} + \frac{u_{m+1,l,i}^{n} - 2u_{m,l,i}^{n} + u_{m-1,l,i}^{n}}{\Delta\xi_1^2}\right).$$

Mixed second-order partial derivatives are discretized by the central finite-difference for mixed partial derivatives. For example,

$$\begin{aligned}\frac{\partial^2 \overline{V}_{m,l,i}^{n+1/2}}{\partial \xi_1 \partial \xi_2} \approx \frac{1}{2}\Biggl(&\frac{u_{m+1,l+1,i}^{n+1} - u_{m+1,l-1,i}^{n+1} - u_{m-1,l+1,i}^{n+1} + u_{m-1,l-1,i}^{n+1}}{4\Delta\xi_1\Delta\xi_2} \\ &+ \frac{u_{m+1,l+1,i}^{n} - u_{m+1,l-1,i}^{n} - u_{m-1,l+1,i}^{n} + u_{m-1,l-1,i}^{n}}{4\Delta\xi_1\Delta\xi_2}\Biggr).\end{aligned}$$

At the boundary $\xi_1 = 0$, because $\tilde{\sigma}_1 = 0$, only $\frac{\partial}{\partial \xi_1}$, $\frac{\partial}{\partial \xi_2}$, $\frac{\partial}{\partial \xi_3}$, $\frac{\partial^2}{\partial \xi_2^2}$, $\frac{\partial^2}{\partial \xi_3^2}$, and $\frac{\partial^2}{\partial \xi_2 \partial \xi_3}$ appear in the partial differential equation. In this case, we can always deal with $\frac{\partial}{\partial \xi_1}$ by the second-order one-sided scheme:

$$\frac{\partial \overline{V}_{0,l,i}^{n+1/2}}{\partial \xi_1} \approx \frac{1}{2}\left(\frac{-u_{2,l,i}^{n+1} + 4u_{1,l,i}^{n+1} - 3u_{0,l,i}^{n+1}}{2\Delta\xi_1} + \frac{-u_{2,l,i}^{n} + 4u_{1,l,i}^{n} - 3u_{0,l,i}^{n}}{2\Delta\xi_1}\right)$$

because of $b_1 \geq 0$ at $\xi_1 = 0$. If $\xi_1 = 0$, $\xi_2 \neq 0$, $\xi_2 \neq 1$, $\xi_3 \neq 0$, and $\xi_3 \neq 1$, then $\frac{\partial}{\partial \xi_2}$, $\frac{\partial}{\partial \xi_3}$, $\frac{\partial^2}{\partial \xi_2^2}$, $\frac{\partial^2}{\partial \xi_3^2}$, and $\frac{\partial^2}{\partial \xi_2 \partial \xi_3}$ can still be discretized by central schemes. If $\xi_1 = 0$, $\xi_3 \neq 0$, $\xi_3 \neq 1$, and $\xi_2 = 0$ or $\xi_2 = 1$, then both $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are equal to zero and only $\frac{\partial}{\partial \xi_1}$, $\frac{\partial}{\partial \xi_2}$, $\frac{\partial}{\partial \xi_3}$, and $\frac{\partial^2}{\partial \xi_3^2}$ are left. The treatment of $\frac{\partial}{\partial \xi_1}$, $\frac{\partial}{\partial \xi_3}$, and $\frac{\partial^2}{\partial \xi_3^2}$ is unchanged, and $\frac{\partial}{\partial \xi_2}$ is approximated by the second-order one-sided differences. For example, at $\xi_2 = 1$, we can use the following approximation:

$$\begin{aligned}\frac{\partial \overline{V}_{m,L,i}^{n+1/2}}{\partial \xi_2} \approx \frac{1}{2}\Biggl(&\frac{3u_{m,L,i}^{n+1} - 4u_{m,L-1,i}^{n+1} + u_{m,L-2,i}^{n+1}}{2\Delta\xi_2} \\ &+ \frac{3u_{m,L,i}^{n} - 4u_{m,L-1,i}^{n} + u_{m,L-2,i}^{n}}{2\Delta\xi_2}\Biggr)\end{aligned}$$

because of $b_2 \leq 0$ at $\xi_2 = 1$. If $\xi_1 = 0$, $\xi_2 = 0$ or 1, and $\xi_3 = 0$ or 1, then $\tilde{\sigma}_1 = \tilde{\sigma}_2 = \tilde{\sigma}_3 = 0$ and only $\frac{\partial}{\partial \xi_1}$, $\frac{\partial}{\partial \xi_2}$, and $\frac{\partial}{\partial \xi_3}$ are left. In this case, all of them need to be dealt with by proper one-sided second-order differences.

For the other boundaries, the situations are similar. All the approximations have a second-order accuracy. In order for the truncation error of the finite-difference equations to have a second-order accuracy, all the coefficients need to take values at the point: $\xi_1 = m\Delta\xi_1$, $\xi_2 = l\Delta\xi_2$, $\xi_3 = i\Delta\xi_3$, and $\tau = (n+1/2)\Delta\tau$, and the term $r\overline{V}$ should be approximated by

$$\frac{1}{2}\left(r_{m,l,i}^{n+1}u_{m,l,i}^{n+1} + r_{m,l,i}^{n}u_{m,l,i}^{n}\right).$$

The number of the finite-difference equations for the time level $\tau = (n+1/2)\Delta\tau$ is $(M+1)\times(L+1)\times(I+1)$. If all the values $u_{m,l,i}^{n}$ are known, the number of unknowns $u_{m,l,i}^{n+1}$ is equal to the number of the equations. Thus, $u_{m,l,i}^{n+1}$ can be determined by the system. It is clear that this system is linear. This system is quite large and usually solved by iteration methods, for example, by successive over relaxation described in Sect. 6.2.2 because iteration methods need less memory space and are usually more efficient than direct methods for this case. The initial condition $\overline{V}(\xi_1,\xi_2,\xi_3,0) = V_T\left(Z_{i_1}(\xi_1), Z_{i_2}(\xi_1,\xi_2), Z_{i_3}(\xi_1,\xi_2,\xi_3)\right)$ gives $u_{m,l,i}^{0}$. Thus, the computation can start with $n=0$ and continue for $n = 1, 2, \cdots, N-1$ successively. Finally, we obtain $u_{m,l,i}^{N}$, the price of the derivative security at time $t=0$.

This problem can also be solved by explicit schemes. If the partial differential equation is discretized at $\tau = n\Delta\tau$ and the time derivative is approximated by the forward finite-difference, then we have an explicit scheme. In this case, $\Delta\tau$ should be small enough so that the stability of computation is guaranteed.

For American swaptions, the value must be greater than or equal to the constraint. In the model here, the value of the constraint does not depend on t and equals to condition (10.20). Therefore, for American swaptions, the method needs to be modified in the following way. At each time step, we should choose the maximum between the computed value by the PDE and the constraint (10.20) as the value of an American swaption.

Consider an American swaption with $r_{se} = 0.1225$, $T = 0.5$, and $N = 5$. We want to have the price of the swaption today. Suppose that the prices of zero-coupon bonds today are given by the expression (10.21), then we can use the numerical methods described here to find the price of the American swaption value. Its value today for $\xi_1 = 0.25, 0.5, 0.75$, $0 \le \xi_2 \le 1$, and $0 \le \xi_3 \le 1$ is shown by the right three graphs in Fig. 10.8. There, the circles are the approximate locations of the free boundary. The final value is also plotted on the left-hand side for comparison. We can see that the derivative of the final value is discontinuous and that for the solution at $t=0$ it is continuous. The result shown in this figure is obtained by the implicit scheme. When for $\tilde{\sigma}_i$, $i = 1,2,3$ and $\tilde{\rho}_{i,j}$, $i,j = 1,2,3$, we use more complicated expressions, the procedure of evaluating the interest rate derivatives is the same. For the details of the procedure and more results, see [96].

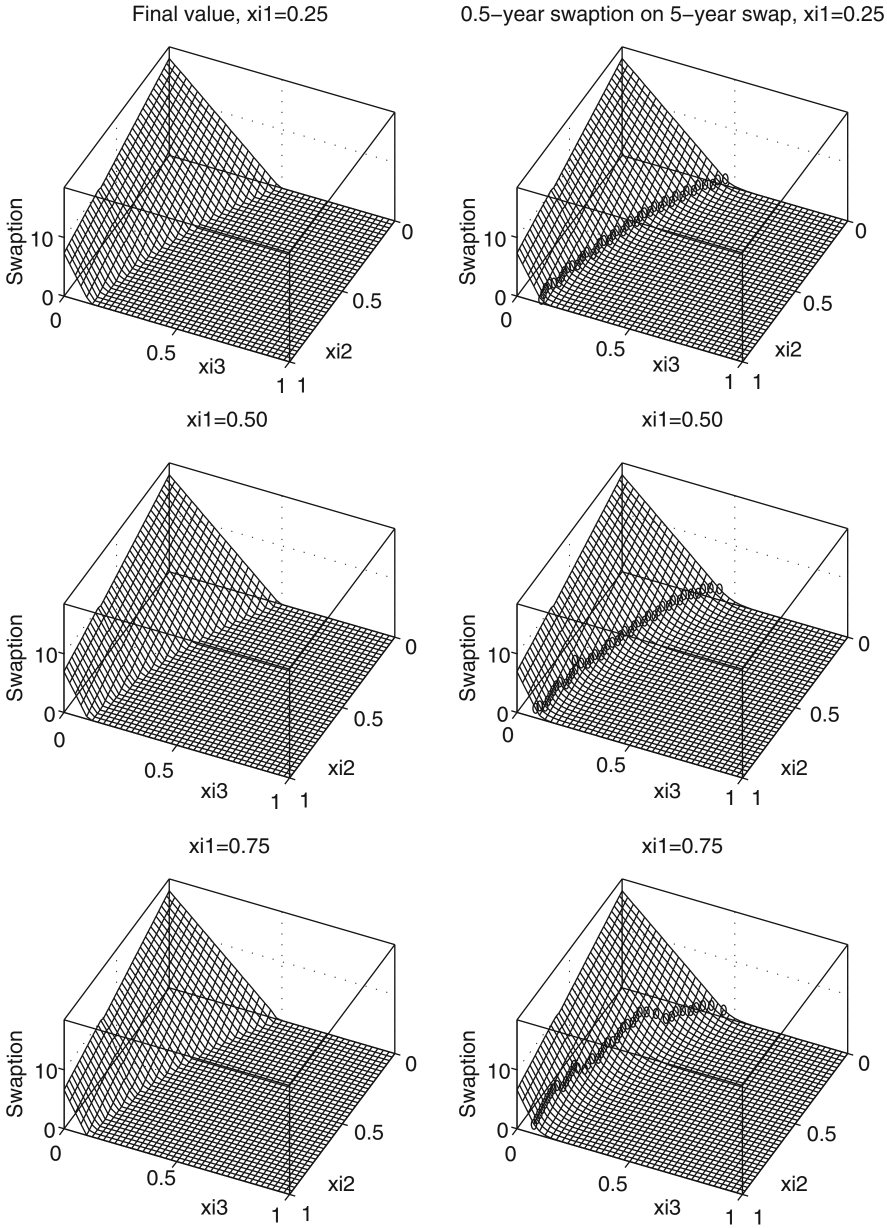

Fig. 10.8. The price of a swaption

Problems

Table 10.6. Problems and sections

Problems	Sections	Problems	Sections	Problems	Sections
1–4	10.1	5–10	10.2	11–13	10.3

1. Define
$$\mathbf{L}_r = \frac{\partial}{\partial r}\left[f_1(r,t)\frac{\partial}{\partial r}\right] - f_2(r,t)\frac{\partial}{\partial r} + f_3(r,t).$$
   (a) Find an operator $\mathbf{L}_r^*$ such that
$$\int_{r_l}^{r_u} \mathbf{L}_r V\, U dr = \int_{r_l}^{r_u} \mathbf{L}_r^* U\, V dr + \left[f_1\left(U\frac{\partial V}{\partial r} - V\frac{\partial U}{\partial r}\right) - f_2 V U\right]\Bigg|_{r_l}^{r_u}.$$
   This operator is called the conjugate operator of $\mathbf{L}_r$.
   (b) Suppose
$$\frac{\partial V}{\partial t} = -\mathbf{L}_r V, \quad \frac{\partial U}{\partial t} = \mathbf{L}_r^* U, \quad f_1(r_l,t) = f_1(r_u,t) = 0,$$
   and
$$U(r_l,t) = U(r_u,t) = 0.$$
   Show
$$\int_{r_l}^{r_u} U(r,t)V(r,t)dr = constant.$$
   (c) Let $U(r,0) = \delta(r - r^*)$ and $V(r,T^*) = 1$. Prove that there is the following relation:
$$V(r^*,0;T^*) = \int_{r_l}^{r_u} U(r,T^*)dr.$$
   Here $V(r,t;T^*)$ stands for the solution $V(r,t)$ with $V(r,T^*) = 1$.
2. Assume that $U(r,t)$ is the solution of the problem
$$\begin{cases} \dfrac{\partial U}{\partial t} = \mathbf{L}_r^* U, & r_l \le r \le r_u, \quad 0 \le t \le T^*_{max}, \\ U(r,0) = \delta(r - r^*), & r_l \le r \le r_u, \\ U(r_l,t) = U(r_u,t) = 0, & 0 \le t \le T^*_{max}, \end{cases}$$
   where
$$\mathbf{L}_r^* = \frac{\partial}{\partial r}\left(f_1(r,t)\frac{\partial}{\partial r}\right) + \frac{\partial}{\partial r}\left(f_2(r,t)\right) - r, \quad f_1(r_u,t) = f_1(r_l,t) = 0.$$

(a) Define $V(r^*, 0; T^*) = \int_{r_l}^{r_u} U(r, T^*)dr$, where $T^* \in [0, T^*_{\max}]$. Show that we can have such an expression:

$$\frac{\partial^2 V(r^*, 0; T^*)}{\partial T^{*2}} = \int_{r_l}^{r_u} F(f_1, f_2, r)\, U dr.$$

and find the concrete expression of $F(f_1, f_2, r)$.

(b) Show that the solution of the problem

$$\begin{cases} \dfrac{\partial^2 V(r^*, 0; T^*)}{\partial T^{*2}} = \displaystyle\int_{r_l}^{r_u} F(f_1, f_2, r)\, U dr, \\ \dfrac{\partial V(r^*, 0; 0)}{\partial T^*} = -r^*, \\ V(r^*, 0; 0) = 1. \end{cases}$$

is $V(r^*, 0; T^*) = \int_{r_l}^{r_u} U(r, T^*)dr$.

3. *Consider the following problem

$$\begin{cases} \dfrac{\partial U}{\partial t} = \dfrac{\partial}{\partial r}\left[f_1(r,t)\dfrac{\partial U}{\partial r}\right] + \dfrac{\partial}{\partial r}\left[f_2\left(r, t, \lambda(t)\sqrt{f_1}\right) U\right] + f_3(r,t)U, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad r_l \le r \le r_u, \quad 0 \le t, \\ U(r, 0) = \delta(r - r^*), \qquad\qquad\qquad\qquad r_l \le r \le r_u, \\ U(r_l, t) = 0, \qquad\qquad\qquad\qquad\qquad 0 \le t, \\ U(r_u, t) = 0, \qquad\qquad\qquad\qquad\qquad 0 \le t, \end{cases}$$

where

$$f_1(r, t) \ge 0 \quad \text{and} \quad f_1(r_l, t) = f_1(r_u, t) = 0,$$

and

$$f_2\left(r_l, t, \lambda(t)\sqrt{f_1(r_l, t)}\right) < 0, \quad f_2\left(r_u, t, \lambda(t)\sqrt{f_1(r_u, t)}\right) > 0.$$

Here, $\lambda(t)$ is a unknown function with a known $\lambda(0)$. We want to find such a function $\lambda(t)$ that

$$\int_{r_l}^{r_u} U(r, t)dr = f(t) \qquad \text{for any } t \in [0, T^*_{max}],$$

where $f(t)$ is a given function with $f(0) = 1$. Design a second-order numerical method for this purpose.

4. *Design a numerical method for finding the market price of risk by using the bond equation directly and taking the prices of today's zero-coupon bonds with various maturities as input.

5. Design an implicit second-order accurate finite-difference method based on the bond equation to solve the European bond option problem.
6. Design an explicit first-order accurate finite-difference method based on the bond equation to solve a cap problem.
7. What is the difference between the numerical methods for a cap problem and for a floor problem if the bond equation is adopted.
8. Design an implicit second-order accurate finite-difference method based on the bond equation to solve the European swaption problem.
9. What is the difference between the numerical methods for the European swaption problem and for the American swaption problem formulated as a linear complementarity problem if the bond equation is adopted.
10. Assume that the prices of American swaptions are the solutions of the following linear complementarity problem:

$$\begin{cases} \min\left(-\dfrac{\partial V_{so}}{\partial t} - \dfrac{1}{2}w^2\dfrac{\partial^2 V_{so}}{\partial r^2} - (u-\lambda w)\dfrac{\partial V_{so}}{\partial r} + rV_{so}, \right. \\ \qquad\qquad \left. V_{so}(r,t) - \max\left(V_s(r,t;r_{se},t),0\right)\right) = 0, \\ V_{so}(r,T) = \max\left(V_s(r,T;r_{se},T),0\right), \end{cases}$$

where $t \in [0,T]$ and $r \in [r_l, r_u]$ and $V_s(r,t;r_{se},t)$ is the price of the swap. Suppose that the price of the swap has been found and assume that there is only one free boundary. Formulate this problem as a free-boundary problem and briefly describe how to solve the free-boundary problem by an implicit second-order finite-difference method.
11. *Briefly describe how to solve a European swaption problem numerically by using the three-factor interest rate model.
12. Briefly describe how to determine the value of a bond option by using the three-factor interest rate model for both European and American cases.
13. Briefly describe how to determine the value of a cap by using the three-factor interest rate model.

Projects

General Requirements

(A) Submit a code or codes in C or C^{++} that will work on a computer the instructor can get access to. At the beginning of the code, write down the name of the student and indicate on which computer it works and the procedure to make it work.

(B) Each code should use an input file to specify all the problem parameters and the computational parameters for each computation and an output file to store all the results. In an output file, the name of the student, all the problem parameters, and the computational parameters should be given, so that one can know what the results are and how they were obtained. The input file should be submitted with the code.

(C) For each case, two results are required. One result is on a 20×24 *mesh, and the accuracy of the other result will be specified individually. (The error of the solution on a* 20×24 *mesh might be quite large.)*

(D) Submit results in form of tables. When a result is given, always provide the problem parameters and the computational parameters.

1. **Implicit Method (7.6) with Modification at the Boundaries for European bond Options and Swaptions**. Suppose

$$dr = (0.05345 - r)dt + r(0.2 - r)dX, \quad r_l = 0 \le r \le r_u = 0.2$$

and $\lambda(t)$ has been found and is given as a function in C. Also, assume that today's short-term interest rate is 0.05345. Write a code to calculate European bond options and a code to calculate European swaptions.

- For European bond options, give results for the case: $E = 0.95, 1$, $k = 0.055$, $T = 0.25, 0.5$, and $T_b - T = 1, 2$. The requirement on the accuracy of the other result is 0.0001, and the mesh used should be as coarse as possible.
- For swaptions, give the results for the cases: $Q = 100$, $N = 5, 10$, $T = 0.5, 1, 2$, $r_{se} = 0.05507$ for $N = 5$, and $r_{se} = 0.05766$ for $N = 10$. The requirement on the accuracy of the other result is 0.001, and the mesh used should be as coarse as possible.

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Y.-l. Zhu et al., *Derivative Securities and Difference Methods*, Springer Finance, DOI 10.1007/978-1-4614-7306-0,

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Index

Y.-l. Zhu et al., *Derivative Securities and Difference Methods*, Springer Finance, DOI 10.1007/978-1-4614-7306-0,