Ronald L. Graham · Jaroslav Nešetřil
Steve Butler *Editors*

The Mathematics of Paul Erdős II

Second Edition

Springer

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Paul Erdős Multi(media)
by Jiří Načeradský and Jarik Nešetřil.

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Preface to the Second Edition

In 2013 the world mathematical community is celebrating the 100th anniversary of Paul Erdős' birth. His personality is remembered by many of his friends, former disciples, and over 500 coauthors, and his mathematics is as alive and well as if he was still among us. In 1995/1996 we were preparing the two volumes of The Mathematics of Paul Erdős not only as a tribute to the achievements of one of the great mathematicians of the twentieth century but also to display the full scope of his œuvre, the scientific activity which transcends individual disciplines and covers a large part of mathematics as we know it today. We did not want to produce just a "festschrift".

In 1995/1996 this was a reasonable thing to do since most people were aware of the (non-decreasing) Erdős activity only in their own particular area of research. For example, we combinatorialists somehow have a tendency to forget that the main activity of Erdős was number theory.

In the busy preparation of the volumes we did not realize that at the end, when published, our volumes could be regarded as a tribute, as one of many obituaries and personal recollections which flooded the scientific (and even mass) media. It had to be so; the old master left.

Why then do we think that the second edition should be published? Well, we believe that the quality of individual contributions in these volumes is unique, interesting and already partly historical (and irreplaceable—particularly in Part I of the first volume). Thus it should be updated and made available especially in this anniversary year. This we feel as our duty not only to our colleagues and authors but also to students and younger scientists who did not have a chance to meet the wandering scholar personally. We decided to prepare a second edition, asked our authors for updates and in a few instances we solicited new contributions in exciting new areas. The result is then a thoroughly edited volume which differs from the first edition in many places.

On this occasion we would like to thank all our authors for their time and work in preparing their articles and, in many cases, modifying and updating them. We are fortunate that we could add three new contributions: one by

Joel Spencer (in the way of personal introduction), one by Larry Guth in Part IV of the first volume devoted to geometry, and one by Alexander Razborov in Part I of the second volume devoted to extremal and Ramsey problems. We also wish to acknowledge the essential contributions of Steve Butler who assisted us during the preparation of this edition. In fact Steve's contributions were so decisive that we decided to add him as co-editor to these volumes. We also thank Kaitlin Leach (Springer) for her efficiency and support. With her presence at the SIAM Discrete Math. conference in Halifax, the whole project became more realistic.

However, we believe that these volumes deserve a little more contemplative introduction in several respects. The nearly 20 years since the first edition was prepared gives us a chance to see the mathematics of Paul Erdős in perspective. It is easy to say that his mathematics is alive; that may sound cliché. But this is in fact an understatement for it seems that Erdős' mathematics is flourishing. How much it changed since 1995 when the first edition was being prepared. How much it changed in the wealth of results, new directions and open problems. Many new important results have been obtained since then. To name just a few: the distinct distances problem, various bounds for Ramsey numbers, various extremal problems, the empty convex 6-gon problem, packing and covering problems, sum-product phenomena, geometric incidence problems, etc. Many of these are covered by articles of this volumes and many of these results relate directly or indirectly to problems, results and conjectures of Erdős. Perhaps it is not as active a business any more to solve a particular Erdős problem. After all, the remaining unsolved problems from his legacy tend to be the harder ones. However, many papers quote his work and in a broader sense can be traced to him.

There may be more than meets the eye here. More and more we see that the Erdős problems are attacked and sometimes solved by means of tools that are not purely combinatorial or elementary, and which originate in the other areas of mathematics. And not only that, these connections and applications merge to new theories which are investigated on their own and some of which belong to very active areas of contemporary mathematics. As if the hard problems inspire the development of new tools which then became a coherent group of results that may be called theories. This phenomenon is known to most professionals and was nicely described by Tim Gowers as two cultures. [W. T. Gowers, *The two cultures of mathematics*, in Mathematics: Frontiers and Perspectives (Amer. Math. Soc., Providence, RI, 2000), 65–78.] On one side, problem solvers, on the other side, theory builders. Erdős' mathematics seems to be on one side. But perhaps this is misleading. As an example, see the article in the first volume *Unexpected applications of polynomials in combinatorics* by Larry Guth and the article in the second volume *Flag algebras: an interim report* by Alexander Razborov for a wealth of theory and structural richness. Perhaps, on the top level of selecting problems and with persistent activity in solving them, the difference between the two sides becomes less clear. (Good) mathematics presents a whole.

Time will tell. Perhaps one day we shall see Paul Erdős not as a theory builder but as a man whose problems inspired a wealth of theories.

People outside of mathematics might think of our field as a collection of old tricks. The second edition of mathematics of Paul Erdős is a good opportunity to see how wrong this popular perception of mathematics is.

La Jolla, USA — R.L. Graham
Prague, Czech Republic — J. Nešetřil

Just a few lines to
remember Szeged restaurant
in Budapest on July 23/96
Paul Erdős

Joachim [illegible]

Laci Lovász

Walter Deuber

[illegible]

Noga Alon

Moshe Rosenfeld

Miki Simonovits

[illegible]

NURIT Alon (NOGA'S BOSS)

[illegible]

Rob Tijdeman

Berrie Tijdeman

Vera T. Sós

[illegible]

[illegible]

[illegible]

Mária [illegible]

IN MEMORIAM

Paul Erdős

26.3.1913–20.9.1996

The week before these volumes were scheduled to go to press, we learned that Paul Erdős died on September 20, 1996. He was 83. Paul died while attending a conference in Warsaw, on his way to another meeting. In this respect, this is the way he wanted to “leave”. In fact, the list of his last month’s activities alone inspires envy in much younger people.

Paul was present when the completion of this project was celebrated by an elegant dinner in Budapest for some of the authors, editors and Springer representatives attending the European Mathematical Congress. He was especially pleased to see the first copies of these volumes and was perhaps surprised (as were the editors) by the actual size and impact of the collection (On the opposite page is the collection of signatures from those present at the dinner, taken from the inside cover of the mock-up for these volumes). We hope that these volumes will provide a source of inspiration as well as a last tribute to one of the great mathematicians of our time. And because of the unique lifestyle of Paul Erdős, a style which did not distinguish between life and mathematics, this is perhaps a unique document of our times as well.

R.L. Graham
J. Nešetřil

Preface to the First Edition

In 1992, when Paul Erdős was awarded a Doctor Honoris Causa by Charles University in Prague, a small conference was held, bringing together a distinguished group of researchers with interests spanning a variety of fields related to Erdős' own work. At that gathering, the idea occurred to several of us that it might be quite appropriate at this point in Erdős' career to solicit a collection of articles illustrating various aspects of Erdős' mathematical life and work. The response to our solicitation was immediate and overwhelming, and these volumes are the result.

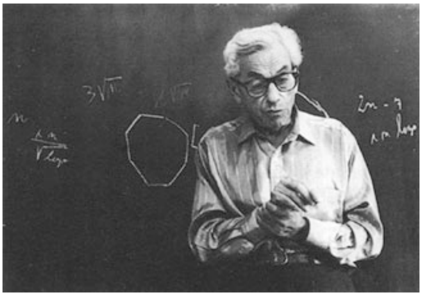
Regarding the organization, we found it convenient to arrange the papers into six chapters, each mirroring Erdős' holistic approach to mathematics. Our goal was not merely a (random) collection of papers but rather a thoroughly edited volume composed in large part by articles explicitly solicited to illustrate interesting aspects of Erdős and his life and work. Each chapter includes an introduction which often presents a sample of related Erdős' problems "in his own words". All these (sometimes lengthy) introductions were written jointly by editors.

We wish to thank the nearly 70 contributors for their outstanding efforts (and their patience). In particular, we are grateful to Béla Bollobás for his extensive documentation of Paul Erdős' early years and mathematical high points; our other authors are acknowledged in their respective chapters. We also want to thank A. Bondy, G. Hahn, I. Ouhel, K. Marx, J. Načeradský and Ché Graham for their help and for the use of their works. At various stages of the project, the book was supported by AT&T Bell Laboratories, GAČR 2167 and GAUK 351. We also are indebted to Dr. Joachim Heinze and Springer Verlag for their encouragement and support. Finally, we would like to record our extreme debt to Susan Pope (at AT&T Bell Laboratories) who somehow (miraculously) managed to convert more than 50 manuscripts of all types into the attractive form they now have.

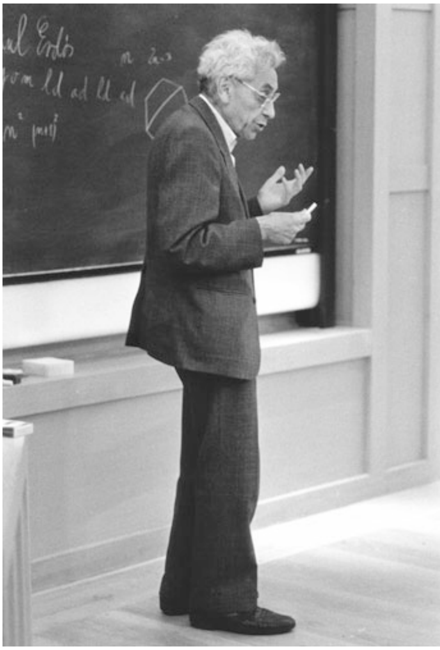
Here then is a unique portrait of a man who has devoted his whole being to "proving and conjecturing" and to the pursuit of mathematical knowledge and understanding. We hope that this will form a lasting tribute to one of the great mathematicians of our time.

Murray Hill, USA R.L. Graham
Praha, Czech Republic J. Nešetřil

Paul Erdős with Fan Chung.
Photo by George Csicsery.



Paul Erdős lecturing.
Photo by Geňa Hahn.



Paul Erdős lecturing.
Photo by George Csicsery.

Paul Erdős with Ron Graham.

Paul Erdős visiting Spelman college in Spring 1989. Photo by Colm Mulcahy.

Paul Erdős with George Szekeres in 1993.

Paul Erdős with an epsilon.

Paul Erdős with epsilon Jakub, Jarik Nešetřil, and Vojtěch Rödl.

Paul Erdős with Wolfgang Haken.

Paul Erdős around 1921.

Paul Erdős with Ralph Faudree.

Paul Erdős with his mother.

Portrait by Fan Chung (watercolor, 2008).

Portrait by Ivan Ouhel (oil, 1992).

Portrait by Karel Marx (oil, 1993).

Contents

I. Combinatorics and Graph Theory

Introduction

Erdős' work in graph theory started early and arose in connection with D. Kőnig, his teacher in prewar Budapest. The classic paper of Erdős' and Szekeres from 1935 also contains a proof in "graphotheoretic terms." The investigation of the Ramsey function led Erdős to probabilistic methods and seminal papers in 1947, 1958 and 1960. It is perhaps interesting to note that three other very early contributions of Erdős' to graph theory (before 1947) were related to infinite graphs: infinite Eulerian graphs (with Gallai and Vászoni) and a paper with Kakutani on nondenumerable graphs (1943). Although the contributions of Erdős to graph theory are manifold, and he proved (and always liked) beautiful structural results such as the Friendship Theorem (jointly with V. T. Sós and Kövári), and compactness results (jointly with N. G. de Bruijn), his main contributions were in asymptotic analysis, probabilistic methods, bounds and estimates. Erdős was the first who brought to graph theory the experience and rigor of number theory (perhaps being preceded by two papers by V. Jarník, one of his early coauthors). Thus he contributed in an essential way to lifting graph theory up from the "slums of topology."

This part contains a "special" problem paper not by Erdős but by his frequent coauthors from Memphis: R. Faudree, C. C. Rousseau and R. Schelp (well, there is actually an Erdős supplement there as well). We encouraged the authors to write this paper and we are happy to include it in this volume. This part also includes two papers coauthored by Béla Bollobás, who is one of Erdős' principal disciples. Bollobás contributed to much of Erdős' combinatorial activities and wrote important books about them (*Extremal Graph Theory*, *Introduction to Graph Theory*, *Random Graphs*). His contributions to this chapter (coauthored with his two former students G. Brightwell and A. Thomason) deal with graphs (and thus are in this chapter) but they by and large employ random graph methods (and thus they could be also be at home in the other volume). The main questions there may be also considered as extremal graph theory questions (and thus they could fit into the following part). Other contributions to this chapter, which are related to some aspect of Erdős' work or simply pay tribute to him are by N. Alon, Z. Füredi, M. Aigner and E. Triesch, S. Bezrukov and K. Engel, A. Gyárfás, S. Brandt, N. Sauer and H. Wang, H. Fleischner and M. Stiebitz, and D. Beaver, S. Haber and P. Winkler.

In 1995/1996, when the contents of these volumes were already crystallizing, we asked Paul Erdős to isolate a few problems, both recent and old, for each of the eight main parts of this book. To this part on Combinatorics and Graph Theory he contributed the following collection of problems and comments.

Erdős in his own words

Many years ago I proved by the probability method that for every k and r there is a graph of girth $\geq r$ and chromatic number $\geq k$. Lovász when he was still in high school found a fairly difficult constructive proof. My proof still had the advantage that not only was the chromatic number of $G(n)$ large but the largest independent set was of size $< \epsilon n$ for every $\epsilon > 0$ if $n > n_0(\epsilon, r, k)$. Nešetřil and V. Rödl later found a simpler constructive proof.

There is a very great difference between a graph of chromatic number $\aleph_0$ and a graph of chromatic number $\geq \aleph_1$. Hajnal and I in fact proved that if G has chromatic number $\aleph_1$ then G must contain a C_4 and more generally G contains the complete bipartite graph $K(n, \aleph_1)$ for every $n < \aleph_0$. Hajnal, Shelah and I proved that every graph G of chromatic number $\aleph_1$ must contain for some k_0 every odd cycle of size $\geq k_0$ (for even cycles this was of course contained in our result with Hajnal), but we observed that for every k and every m there is a graph of chromatic number m which contains no odd cycle of length $< k$. Walter Taylor has the following very beautiful problem: Let G be any graph of chromatic number $\aleph_1$. Is it true that for every $m > \aleph_1$ there is a graph G_m of chromatic number m all finite subgraphs of which are contained in G? Hajnal and Komjáth have some results in this direction but the general conjecture is still open. If it would have been my problem, I certainly would offer 1,000 dollars for a proof or a disproof. (To avoid financial ruin I have to restrict my offers to my problems.)

Let k be fixed and $n \to \infty$. Is it true that there is an $f(k)$ so that if $G(n)$ has the property that for every m every subgraph of m vertices contains an independent set of size $m/2 - k$ then $G(n)$ is the union of a bipartite graph and a graph of $\leq f(k)$ vertices, i.e., the vertex set of $G(n)$ is the union of three disjoint sets S_1, S_2 and S_3 where S_1 and S_2 are independent and $|S_3| \leq f(k)$. Gyárfás pointed out that even the following special case is perhaps difficult. Let m be even and assume that every m vertices of our $G(n)$ induces an independent set of size at least $m/2$. Is it true then that $G(n)$ is the union of a bipartite graph and a bounded set? Perhaps this will be cleared up before this paper appears, or am I too optimistic?

Hajnal, Szemerédi and I proved that for every $\epsilon > 0$ there is a graph of infinite chromatic number for which every subgraph of m vertices contains an independent set of size $(1-\epsilon)m/2$ and in fact perhaps $(1-\epsilon)m/2$ can be replaced by $m/2 - f(m)$ where $f(m)$ tends to infinity arbitrarily slowly. A result of Folkman implies that if G is such that every subgraph of m vertices contains an independent set of size $m/2 - k$ then the chromatic number of G is at most $2k + 2$.

Many years ago Hajnal and I conjectured that if G is an infinite graph whose chromatic number is infinite, then if $a_1 < a_2 < \dots$ are the lengths of the odd cycles of G we have

$$\sum_i \frac{1}{a_i} = \infty$$

and perhaps $a_1 < a_2 < \dots$ has positive upper density. (The lower density can be 0 since there are graphs of arbitrarily large chromatic number and girth.)

We never could get anywhere with this conjecture. About 10 years ago Mihók and I conjectured that G must contain for infinitely many n cycles of length 2^n. More generally it would be of interest to characterize the infinite sequences $A = \{a_1 < a_2 < \dots\}$ for which every graph of infinite chromatic number must contain infinitely many cycles whose length is in A. In particular, assume that the a_i are all odd.

All these problems are unattackable (at least for us). About 3 years ago Gyárfás and I thought that perhaps every graph whose minimum degree is ≥ 3 must contain a cycle of length 2^k for some $k \geq 2$. We became convinced that the answer almost surely will be negative but we could not find a counterexample. We in fact thought that for every r there must be a G_r every vertex of which has degree $\geq r$ and which contains no cycle of length 2^k for any $k \geq 2$. The problem is wide open.

Gyárfás, Komlós and Szemerédi proved that if k is large and $a_1 < a_2 < \dots$ are the lengths of the cycles of a $G(n, kn)$, that is, an n-vertex graph with kn edges, then

$$\sum \frac{1}{a_i} > c \log n.$$

The sum is probably minimal for the complete bipartite graphs.

(Erdős-Hajnal) If G has large chromatic number does it contain two (or k if the chromatic number is large) edge-disjoint cycles having the same vertex set? It surely holds if $G(n)$ has chromatic number $> n^\epsilon$ but nothing seems to be known.

Fajtlowicz, Staton and I considered the following problem (the main idea was due to Fajtlowicz). Let $F(n)$ be the largest integer for which every graph of n vertices contains a regular induced subgraph of $\geq F(n)$ vertices. Ramsey's theorem states that $G(n)$ contains a trivial subgraph, i.e., a complete or empty subgraph of $c \log n$ vertices. (The exact value of c is not known but we know $1/2 \leq c \leq 2$.) We conjectured $F(n)/\log n \to \infty$. This is still open. We observed $F(5) = 3$ (since if $G(5)$ contains no trivial subgraph of 3 vertices then it must be a pentagon) . Kohayakawa and I worked out the $F(7) = 4$ but the proof is by an uninteresting case analysis. (We found that this was done earlier by Fajtlowicz, McColgan, Reid and Staton, see Ars Combinatoria vol 39.) It would be very interesting to find the smallest integer n for which $F(n) = 5$, i.e., the smallest n for which every $G(n)$ contains a regular induced subgraph of ≥ 5 vertices. Probably this will be much more difficult than the proof of $F(7) = 4$ since in the latter we could use properties of perfect graphs. Bollobás observed that $F(n) < c\sqrt{n}$ for some $c > 0$.

Let $G(10n)$ be a graph on $10n$ vertices. Is it true that if every index subgraph of $5n$ vertices of our $G(10n)$ has $\geq 2n^2 + 1$ edges then our $G(10n)$ contains a triangle? It is easy to see that $2n^2$ edges do not suffice. A weaker result has been proved by Faudree, Schelp and myself at the Hakone conference (1992, I believe) see also a paper by Fan Chung and Ron Graham (one of the papers in a volume published by Bollobás dedicated to me).

A related forgotten conjecture of mine states that if our $G(10n)$ has more than $20n^2$ edges and every subgraph of $5n$ vertices has $\geq 2n^2$ edges then our graph must have a triangle. Simonovits noticed that if you replace each vertex of the Petersen graph by n vertices you get a graph of $10n$ vertices, $15n^2$ edges, no triangle and every subgraph of $5n$ vertices contains $\geq 2n^2$ edges.

*****

So much for P. Erdős in 1995. Let us add that since that time some of these problems were solved, some are open and some seem to be dormant. Some were subject of intensive study. The reference to the above Hakone conference is:

P. Erdős, R. J. Faudree, C. C. Rousseau, R. H. Schelp, A local density condition for triangles, Discrete Math. 127, 1–2 (1994), 153–161.
(The conference was The Second Japan Conference on Graph Theory and Combinatorics, Aug 18–22, 1990 in Hakone.)

The mentioned paper by Fan Chung et al. is the following:
F. R. K. Chung and R. L. Graham, On graphs not containing prescribed induced subgraphs, in A Tribute to Paul Erdos, ed. by A. Baker, B. Bollobás and A. Hajnal, Cambridge University Press (1990), 111–120.

One of these problems was quoted by Erdős much earlier. For example the problem of Taylor was mentioned as early as 1975; (W. Taylor: Problem 42. In: Combinatorial Structures and Their Applications, Proc. Calgary Internat. Conf. 1969, Gordon and Breach 1969.)

For more information about Erdős problems on graphs and of their current status see:
F. R. K. Chung, R. L. Graham, *Erdős on Graphs: His Legacy of Unsolved Problems*, A K Peters, Cambridge, MA 1993, xiv+142 pp.

Reconstruction Problems for Digraphs

Martin Aigner and Eberhad Triesch

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Summary. Associate to a finite directed graph $\mathbf{G}(V, E)$ its out-degree resp. in-degree sequences d^+, d^- and the corresponding neighborhood lists N^+, N^- (when $\mathbf{G}$ is a labeled graph). We discuss various problems when sequences resp. lists of sets can be realized as degree sequences resp. neighborhood lists of a directed graph.

1. Introduction

Consider a finite graph $G(V, E)$. Let us associate with G a finite list $P(G)$ of parameters, e.g. the degrees, the list of cliques, the chromatic polynomial, or whatever we like. For any set P of invariants there arise two natural problems:

(R) **Realizability**. Given P, when is $P = P(G)$ for some graph G? We then call P *graphic*, and say that G *realizes* P.

(U) **Uniqueness**. Suppose $P(G) = P(H)$. When does this imply $G \cong H$? In other words, when is P a complete set of invariants?

The best studied questions in this context are probably the reconstruction conjecture for (U), and the degree realization problem for (R). This latter problem was solved in a famous theorem of Erdős-Gallai [4] characterizing graphic sequences. Their theorem reads as follows: Let $d_1 \geq \cdots \geq d_n \geq 0$ be a sequence of integers. Then $(d_1 \geq \cdots \geq d_n)$ can be realized as the degree sequence of a graph if and only if the degree sum is even and

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{j=k+1}^{n} \min(d_j, k) \quad (k = 1, \ldots, n). \tag{1}$$

A variant of this problem concerning neighborhoods was first raised by Sós [13] and studied by Aigner-Triesch [2]. Consider a finite labeled graph $G(V, E)$ and denote by $N(u)$ the neighborhood of $u \in V$. $\mathcal{N}(G) = \{N(u) : u \in V\}$ is called the *neighborhood list* of G. Given a list (multiset) $\mathcal{N} = (N_1, \ldots, N_n)$ of sets. When is $\mathcal{N} = \mathcal{N}(G)$ for some graph G? In contrast to the polynomial

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verification of (1), it was shown in [2] that the neighborhood list problem NL is NP-complete for arbitrary graphs. For bipartite graphs, NL turns out to be polynomially equivalent to the GRAPH ISOMORPHISM problem. A general survey of these questions appears in [3].

In the present paper we consider directed graphs $\mathbf{G}(V,E)$ on n vertices with or without loops and discuss the corresponding realizability problems for the degree resp. neighborhood sequences. We assume throughout that there is at most one directed edge (u,v) for any $u,v \in V$. To every $u \in V$ we associate its *out-neighborhood* $N^+(u) = \{v \in V : (u,v) \in E\}$ and its *in-neighborhood* $N^-(u) = \{v \in V : (v,u) \in E\}$ with $d^+(u) = |N^+(u)|$ and $d^-(u) = |N^-(u)|$ being the *out-degree* resp. *in-degree* of u.

For both the degree realization problem and the neighborhood problem we have three versions in the directed case:

$(\mathcal{D}^+)$ Given a sequence $d^+ = (d_1^+, \dots, d_n^+)$ of non-negative integers. When is d^+ realizable as the out-degree sequence of a directed graph?

Obviously, $(\mathcal{D}^-)$ is the same problem.

$(\mathcal{D}_-^+)$ Given a sequence of pairs $d_-^+ = ((d_1^+, d_1^-), \dots, (d_n^+, d_n^-))$. When is there a graph $\mathbf{G}$ with $d^+(u_i) = d_i^+$, $d^-(u_i) = d_i^-$ for all i?

$(\mathcal{D}^+, \mathcal{D}^-)$ Given two sequences $d^+ = (d_1^+, \dots d_n^+)$, $d^- = (d_1^-, \dots, d_n^-)$. When is there a directed graph such that d^+ is the out-degree sequence (in some order) and d^- the in-degree sequence?

In an analogous way, we may consider the realization problems $(\mathcal{N}^+)$, $(\mathcal{N}_-^+)$, $(\mathcal{N}^+, \mathcal{N}^-)$ for neighborhood lists.

In Sect. 2 we consider the degree problems and in Sect. 3 the neighborhood problems. Section 4 is devoted to simple directed graphs when there is at most one edge between any two vertices and no loops.

2. Degree Sequences

Depending on whether we allow loops or not there are six different reconstruction problems whose solutions are summarized in the following diagram:

Degrees	$(\mathcal{D}^+)$	$(\mathcal{D}_-^+)$	$(\mathcal{D}^+, \mathcal{D}^-)$
With loops	Trivial	Gale-Ryser	Gale-Ryser
Without loops	Trivial	Fulkerson	Fulkerson

The problems $(\mathcal{D}^+)$ have the following trivial solutions: (d^+) is realizable with loops if and only if $d_i^+ \le n$ for all i, and without loops if and only if $d_i^+ \le n-1$ for all i.

Consider $(\mathcal{D}_-^+)$ with loops. We represent $\mathbf{G}(V,E)$ as usual by its adjacency matrix M where the rows and columns are indexed by the vertices $u_1, \dots, u_n$

with $m_{ij} = 1$ if $(u_i, u_j) \in E$ and 0 otherwise. To realize a given sequence (d^{+}_{-}) is therefore equivalent to constructing a $0,1$-matrix with given row-sums d_i^+ and column-sums d_i^- which is precisely the content of the Gale-Ryser Theorem [7, 11]. In fact, the Gale-Ryser Theorem applies to the situation $(\mathcal{D}^+, \mathcal{D}^-)$ as well by permuting the columns. If we do not allow loops, then the realization problem (D^{+}_{-}) is settled by an analogous theorem of Fulkerson [5, 6]. He reduces the problem of constructing a $0,1$-matrix with zero trace and given row and column sums to a network flow problem, an approach which can also be used in the case of the Gale-Ryser theorem thus showing that both problems are polynomially decidable. Finally, we remark that the case $(\mathcal{D}^+, \mathcal{D}^-)$ can be reduced to the case (D^{+}_{-}) in view of the following proposition.

Proposition 1. *Suppose two sequences $d^+ = (d_1^+, \ldots, d_n^+)$, $d^- = (d_1^-, \ldots, d_n^-)$ are given. Denote by $\overline{d}^+$ (resp. $\overline{d}^-$) a non-increasing (resp. non-decreasing) rearrangement of d^+ (resp. d^-). If there exists a $0,1$-matrix $M = (m_{ij})$ with $\sum_{j=1}^n m_{jj} = 0$, $\sum_{j=1}^n m_{ij} = d_i^+$, $\sum_{j=1}^n m_{ji} = d_i^-$, $1 \le i \le n$, then there exists a $0,1$-matrix $\overline{M} = (\overline{m}_{ij})$ satisfying $\sum_j \overline{m}_{jj} = 0$, $\sum_j \overline{m}_{ij} = \overline{d}_i^+$, $\sum_i \overline{m}_{ji} = \overline{d}_i^-$, $1 \le i \le n$.*

Proof. Suppose M is given as above. By permuting the rows and columns of M by the same permutation (which does not change the trace) we may assume that $d^+ = \overline{d}^+$. □

Now suppose that for some indices $i < j$, $d_i^- > d_j^-$. We will show that M can be transformed into a matrix $\hat{M}$ with zero trace, row sum vector d^+ and column sum vector $\hat{d}^-$, where $\hat{d}^-$ arises from d^- by exchanging d_i^- and d_j^-. Since each Permutation is generated by transpositions, this will obviously complete the proof. To keep notation simple, we give the argument only for the case $i = n-1$, $j = n$ but it is immediately clear how the general case works. Suppose

$$M = \left(\begin{array}{c|c|c} & a & b \\ \hline & 0 & y \\ \hline & x & 0 \end{array} \right).$$

(i) If $x \le y$, then we exchange (a_i, b_i) when $a_i = 1$ and $b_i = 0$ for $d_{n-1}^- - d_n^-$ indices $i \le n-2$.

(ii) If $x = 1$, $y = 0$, then since $d_{n-1}^+ \ge d_n^+$ there exists some $\ell < n-1$ such that

$$\begin{pmatrix} m_{n-1,\ell} \\ m_{n,\ell} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Now let

$$M = \left(\begin{array}{c|c|c} & b & a \\ \hline 0 & 0 & 1 \\ \hline 1 & 0 & 0 \end{array}\right)$$

3. Neighborhood Lists

As in the undirected case the neighborhood problems are more involved. Again, we summarize our findings in a diagram and then discuss the proofs.

Neighbors	$(\mathcal{N}^+)$	$(\mathcal{N}^+_-)$	$(\mathcal{N}^+, \mathcal{N}^-)$
With loops	Trivial	$\geq$ GRAPH ISOM.	$\approx$ GRAPH ISOM.
Without loops	Bipartite matching	NP-complete	NP-complete

Let us consider $(\mathcal{N}^+)$ first. Allowing loops, any list (N_i^+) can be realized. In the absence of loops, (N_i^+) can be realized if and only if $(N_1^{+^c}, \ldots, N_n^{+^c})$ has a transversal, where N^c is the complement of N. So, this problem is equivalent to the bipartite matching problem and, in particular, polynomially decidable.

Let us treat next the problem $(\mathcal{N}^+_-)$ without loops. The special case $N_i^+ = N_i^-$ for all i clearly reduces to the (undirected) neighborhood list problem NL which as mentioned is NP-complete. Accordingly, $(\mathcal{N}^+_-)$ is NP-complete as well.

We show next that the decision problem $(\mathcal{N}^+_-)$ with loops is polynomially equivalent to the matrix symmetry problem MS defined as follows:

The input is an $n \times n$-matrix A with $0, 1$-entries, with the question: Does there exist a permutation matrix P such that $(PA)^T = PA$ holds?

Let us represent $\mathbf{G}(V, E)$ again by its adjacency matrix. Then, clearly, MS is the special case of $(\mathcal{N}^+_-)$ where $N_i^+ = N_i^-$ for all i. To see the converse denote by x^i (resp. y^i) the incidence vectors of N_i^+ (resp. N_i^-) as row vectors, and set

$$X = \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}, \quad Y = \begin{pmatrix} y^1 \\ \vdots \\ y^n \end{pmatrix}.$$

The problem $(\mathcal{N}^+_-)$ is thus equivalent to the following decision problem: Does there exist a permutation matrix P such that

$$PX = (PY)^T?$$

Note that $PX = (PY)^T \iff (PX)^T = PY$. Now consider the $4n \times 4n$-matrix

$$\Gamma = \begin{array}{|c|c|c|c|} \hline O & I & X & U \\ \hline I & O & I & W \\ \hline Y & I & O & Z \\ \hline U^T & W^T & Z^T & O \\ \hline \end{array}$$

where O, I are the zero-matrix and identity matrix, respectively, and U, W, Z are matrices with identical rows each which ensure that a permutation matrix R satisfying $(R\Gamma)^T = R\Gamma$ must be of the form

$$R = \begin{array}{|c|c|c|c|} \hline R_1 & O & O & O \\ \hline O & R_2 & O & O \\ \hline O & O & R_3 & O \\ \hline O & O & O & R_4 \\ \hline \end{array} .$$

Clearly, such matrices U, W, Z exist. Now $(R\Gamma)^T = R\Gamma$ if and only if

$$(R_1X)^T = R_3Y, R_2^T = R_1, R_3 = R_2^T = R_1,$$

i.e. if and only if $(R_1X)^T = R_1Y$, and the result follows.

It was mentioned in [2] that MS is at least as hard as GRAPH ISOMORPHISM, but we do not know whether they are polynomially equivalent.

Let us, finally, turn to the problems $(\mathcal{N}^+, N^-)$. We treat $(\mathcal{N}^+, \mathcal{N}^-)$ with loops, the other version is settled by a matrix argument as above. Using again the notation x^i, y^i, X, Y, our problem is equivalent to the following matrix problem: Do there exist permutation matrices P and Q such that

$$PX = Y^TQ, \text{ i.e. } PXQ^T = Y^T?$$

This latter problem is obviously polynomially equivalent to HYPERGRAPH ISOMORPHISM which is known to be equivalent to GRAPH ISOMORPHISM (see [2]).

4. Simple Directed Graphs and Tournaments

Let us now consider simple directed graphs and, in particular, tournaments. In contrast to the non-simple case, where $(\mathcal{D}^+)$ and $(\mathcal{N}^+)$ are trivial resp. polynomially solvable (bipartite matching), the problems now become more involved.

As for $(\mathcal{D}^+)$, a necessary and sufficient condition for (d_i^+) to be realizable as an out-degree sequence of a *tournament* was given by Landau [9]. Assume $d_1^+ \geq \cdots \geq d_n^+$, then (d_i^+) is realizable if and only if

$$\sum_{i=1}^{k} d_i^+ \leq (n-1) + \ldots + (n-k) \quad (K = 1, \ldots, n)$$

with equality for $k = n$.

Deleting the last condition yields the corresponding result for arbitrary simple digraphs.

Theorem 1. *A sequence $(d_1^+ \geq \cdots \geq d_n^+)$ is realizable as out-degree sequence of a simple directed graph if and only if*

$$\sum_{i=1}^{k} d_i^+ \leq \sum_{i=1}^{k} (n-i) \quad (K = 1, \ldots, n). \tag{2}$$

Proof. The condition is obviously necessary. For the converse we make use of the dominance order of sequences $p = (p_1 \geq \cdots \geq p_n)$, $q = (q_1 \geq \cdots \geq q_n)$:

$$p \leq q \Longleftrightarrow \sum_{i=1}^{k} p_i \leq \sum_{i=1}^{k} q_i \quad (k = 1, \ldots, n). \tag{3}$$

Suppose $m = \sum_{i=1}^{n} d_i^+$, and denote by $L(m)$ the lattice of all sequences $p = (p_1 \geq \cdots \geq p_n)$, $\sum_{i=1}^{n} p_i = m$, ordered by (3), see [1] for a survey on the uses of $L(m)$. Let $S(m) \subseteq L(m)$ be the set of sequences which are realizable as out-degree sequences of a simple digraph with m edges.

Claim 1. *$S(m)$ is a down-set, i.e. $p \in S(m), q \leq p \Longrightarrow q \in S(m)$.*

It is well-known that the order $\leq$ in $L(m)$ is transitively generated by successive "pushing down boxes", i.e. it suffices to prove the claim for $q \leq p$ with $q_r = p_r - 1$, $q_s = p_s + 1$ for some $p_r \geq p_s + 2$, and $q_i = p_i$ for $i \neq r, s$. Now suppose $\mathbf{G}$ realizes the sequence p, with $d^+(u_i) = p_i$. Since $d^+(u_r) \geq d^+(u_s) + 2$, there must be a vertex v_t with $(u_r, u_t) \in E, (u_s, u_t) \notin E$. If $(u_t, u_s) \in E$, replace $(u_r, u_t), (u_t, u_s)$ by $(u_t, u_r), (u_s, u_t)$, and if $(u_t, u_s) \notin E$, replace (u_r, u_t) by (u_s, u_t). In either case, we obtain a simple directed graph $\mathbf{G}'$ with q as out-degree sequence, and the claim is proved.

Claim 2. *Suppose $m = (n-1) + \ldots + (n-\ell+1) + r$ with $r \leq n - \ell$, then $\overline{p}_m = (n-1, \ldots, n-\ell+1, r)$ is the only maximal element of $S(m)$ in $L(m)$.*

By the definition (3), the sequence $\overline{p}_m$ clearly dominates any sequence in $S(m)$, and since $\overline{p}_m$ can obviously be realized as an out-degree sequence of a simple digraph, it is the unique maximum of $S(m)$.

Taking Claims 1 and 2 together yields the characterization

$$d \in S(m) \Longleftrightarrow d \leq \overline{p}_m$$

and this latter condition is plainly equivalent to (1). □

As is to be expected the neighborhood list problem ($\mathcal{N}^+$) is considerably more involved. We consider the problem NLSD (neighborhood list of a simple digraph):

Instance: A list $N^+ = (N_1^+, \dots, N_n^+)$.

Question: Does there exist a simple digraph $\mathbf{G}(V, E)$ such that N^+ is the list of its out-neighborhoods?

In terms of the adjacency matrix, the problem reads as follows: We are given an $n \times n$-matrix Γ (whose rows are the incidence vectors of the sets N_i^+). Does there exist a permutation matrix R such that $R\Gamma$ is the adjacency matrix of a simple digraph, i.e. such that

$$R\Gamma + (R\Gamma)^T \le J - I$$

where J is the all-ones matrix, and $\le$ is to be understood coordinate-wise.

Theorem 2. *The decision problem NLSD is NP-complete.*

For the proof we need an auxiliary result. Consider the decision problem k-Aut:

Instance: A simple undirected graph G.

Question: Does G admit an automorphism all of whose orbits have length at least k?

The problem 2-Aut is the FIXED-POINT-FREE AUTOMORPHISM problem which was shown to be NP-complete by Lubiw [10].

Lemma 1. 3-*Aut is NP-complete.*

Proof. We give a transformation from 2-Aut. Note first that the construction given by Lubiw shows that we may assume an instance G of 2-Aut which satisfies the following conditions:

(a) For $u \ne v \in V$, the sets $N(u) = \{w \in V : uw \in E\}$ and $N(v)$ are distinct.
(b) For $u \ne v \in V, N(u) \cup \{u\} \ne N(v) \cup \{v\}$.

For such graphs G, Sabidussi has shown in [12] that the *composition* $G[G]$ the automorphism group $\operatorname{Aut} G \circ \operatorname{Aut} G$ where $\circ$ denotes the *wreath product* and $V(G[G]) = V \times V$,

$$E(G[G]) = \{\{u_1, v_1\}, \{u_2, v_2\} : u_1u_2 \in E \text{ or } u_1 = u_2, v_1v_2 \in E\}.$$

That is, the composition arises from G by replacing each vertex by an isomorphic copy of G and joining either all pairs in different copies or none according to whether the original vertices are adjacent or not. The wreath product $\operatorname{Aut} G \circ \operatorname{Aut} G$ is defined as follows:

$$\operatorname{Aut} G \circ \operatorname{Aut} G = \{(\varphi; \psi_1, \dots, \psi_n) : \varphi, \psi_i \in \operatorname{Aut} G\}$$

where the action on $V \times V$ is given by

$$(\varphi; \psi_1, \dots, \psi_n)(u, v) = (\varphi(u), \psi_u(v)) \text{ with } V = \{1, \dots, n\}.$$

We claim that G has a fixed-point free automorphism if and only if $G[G]$ admits an automorphism with all orbit lengths at least 3. This will then prove our Lemma.

Suppose first that there exists $\varphi \in \operatorname{Aut} G$, $\varphi(u) \neq u$ for all $u \in V$, and let $O_1, \ldots, O_k$ be the orbits of φ on V. Suppose $|O_1| = \ldots = |O_h| = 2$ and $|O_i| \geq 3$ for $h+1 \leq i \leq k$. Define $\rho = (\varphi; \psi_1, \ldots, \psi_n) \in \operatorname{Aut} G \circ \operatorname{Aut} G$ such that $\psi_u \in \{\mathrm{id}, \varphi\}$ for all u and $\psi_u = \varphi$ for exactly one u in each orbit O_i, $1 \leq i \leq h$. Let $(u,v) \in V \times V$ and denote by $\overline{O}$ its ρ-orbit. If the orbit O_i of u under φ has at least three elements, then the same obviously holds for $\overline{O}$. In case $O_i = \{u, w\}$, suppose $\psi_u = \varphi$. Then $\{(u,v), (w, \varphi(v)), (u, \varphi(v))\} \subseteq \overline{O}$ and thus $|\overline{O}| \geq 3$. The case $\psi_w = \varphi$ is analogous.

Now assume, conversely, that each $\varphi \in \operatorname{Aut} G$ has a fixed point $u = u_\varphi$. Then each $\rho = (\varphi; \psi_1, \ldots, \psi_n)$ has a fixed point as well, namely the pair $(u_\varphi, u_{\psi_{u_\varphi}})$. Since the transformation $G \longrightarrow G[G]$ is clearly polynomial, the proof of the Lemma is complete. □

Remark 1. *By iterating the construction of the Lemma, it can be shown that k-*Aut *is NP-complete for every fixed $k \geq 2$.*

Proof of Theorem 2. We provide a transformation from 3-Aut. Suppose G is a simple undirected graph with incidence matrix $A \in \{0,1\}^{n \times m}$, where m is the number of edges. We may assume $n \geq 4$, $m \geq 3$. We denote by $t_{r \times s}$ the $r \times s$-matrix with all entries equal to t, and set $\overline{A} = 1_{n \times m} - A$.

Now consider the $(n+m+2) \times (n+m+2)$-matrix

$$\Gamma = \begin{array}{|c|c|c|c|} \hline I_n & A & 1_{n\times 1} & 1_{n\times 1} \\ \hline \overline{A}^T & 0_{m\times m} & 1_{m\times 1} & 0_{m\times 1} \\ \hline 0_{1\times n} & 0_{1\times m} & 0 & 1 \\ \hline 0_{1\times n} & 1_{1\times m} & 0 & 0 \\ \hline \end{array} \, .$$

Our theorem will be proved by the following claim: There exists a permutation matrix R with $R\Gamma + (R\Gamma)^T \leq J - I$ if and only if G admits an automorphism with all orbit lengths at least 3.

Note first that row $n+m+1$ is the only row with at least $n+m$ zeros. Since the $(n+m+1)$-st column contains $n+m$ ones, R must fix row $n+m+1$. But then row $n+m+2$ must also be fixed, since the entry in position $(n+m+2, n+m+1)$ in $R\Gamma$ must be 0. Furthermore, the rows with numbers $\{1, \ldots, n\}$ and $\{n+1, \ldots, n+m\}$ are permuted among themselves. Hence, R is of the form

$$R = \begin{array}{|c|c|c|c|} \hline P & & & \\ \hline & Q & & \\ \hline & & 1 & \\ \hline & & & 1 \\ \hline \end{array} \, .$$

where P and Q are permutation matrices of order n and m, respectively. Now we have

$$R\Gamma + (R\Gamma)^T \leq J - I \iff \begin{array}{l} P + P^T \leq J - I \ \text{ and} \\ PA + (Q\overline{A}^T)^T \leq 1_{n\times m}. \end{array}$$

It is easily seen, that $P + P^T \leq J - I$ holds if and only if the permutation π corresponding to P has no cycle of length two or less. For the second condition we note

$$PA + \overline{A}Q^T = PA + (1_{n\times m} - A)Q^T \leq 1_{n\times m}$$

if and only if $PA \leq AQ^T$ if and only if $PA = AQ^T$.

Denoting by Perm_n the set of permutation matrices of order n, we have

$$\mathrm{Aut}\, G = \{S \in \mathrm{Perm}_n : \exists U \in \mathrm{Perm}_m \ \text{ with } SA = AU^T\},$$

and the result follows. □

The corresponding decision problem ($\mathcal{N}^+$) for *tournaments* is open, but we surmise that it is also NP-complete.

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Neighborly Families of Boxes and Bipartite Coverings

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Summary. A *bipartite covering of order k* of the complete graph K_n on n vertices is a collection of complete bipartite graphs so that every edge of K_n lies in at least 1 and at most k of them. It is shown that the minimum possible number of subgraphs in such a collection is $\Theta(kn^{1/k})$. This extends a result of Graham and Pollak, answers a question of Felzenbaum and Perles, and has some geometric consequences. The proofs combine combinatorial techniques with some simple linear algebraic tools.

1. Introduction

Paul Erdős taught us that various extremal problems in Combinatorial Geometry are best studied by formulating them as problems in Graph Theory. The celebrated Erdős-de Bruijn theorem [3] that asserts that n non-collinear points in the plane determine at least n distinct lines is one of the early examples of this phenomenon. An even earlier example appears in [4] and many additional ones can be found in the surveys [5] and [12]. In the present note we consider another example of an extremal geometric problem which is closely related to a graph theoretic one. Following the Erdős tradition we study the graph theoretic problem in order to deduce the geometric consequences.

A finite family $\mathcal{C}$ of d-dimensional convex polytopes is called *k-neighborly* if $d - k \leq dim(C \cap C') \leq d - 1$ for every two distinct members C and C' of the family. In particular, a 1-neighborly family is simply called *neighborly*. In this case the dimension of the intersection of each two distinct members of the family is precisely $d - 1$. Neighborly families have been studied by various researchers, see, e.g., [10, 14, 15, 16, 17]. In particular it is known that the maximum possible cardinality of a neighborly family of d-simplices is at least 2^d [16] and at most 2^{d+1} [14]. The maximum possible cardinality of a neighborly family of *standard boxes* in R^d, that is, a neighborly family of d-dimensional boxes with edges parallel to the coordinate axes, is precisely $d + 1$. This has been proved by Zaks [17], by reducing the problem to a theorem of Graham and Pollak [8] about bipartite decompositions of

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complete graphs. In the present note we consider the more general problem of k-neighborly families of standard boxes. The following result determines the asymptotic behaviour of the maximum possible cardinality of such a family.

Theorem 1. *For $1 \leq k \leq d$, let $n(k,d)$ denote the maximum possible cardinality of a k-neighborly family of standard boxes in R^d. Then*

(i) $d+1 = n(1,d) \leq n(2,d) \leq \cdots \leq n(d-1,d) \leq n(d,d) = 2^d.$

(ii) $\left(\frac{d}{k}\right)^k \leq \prod_{i=0}^{k-1} \left(\lfloor \frac{d+i}{k} \rfloor + 1\right) \leq n(k,d) \leq \sum_{i=0}^{k} 2^i \binom{d}{i} < 2\left(\frac{2ed}{k}\right)^k.$

This answers a question of Felzenbaum and Perles [6], who asked if for fixed k, $n(k,d)$ is a nonlinear function of d.

As in the special case $k = 1$, the function $n(k,d)$ can be formulated in terms of bipartite coverings of complete graphs. A *bipartite covering* of a graph G is a family of complete bipartite subgraphs of G so that every edge of G belongs to at least one such subgraph. The covering is of *order* k if every edge lies in at most k such subgraphs. The *size* of the covering is the number of bipartite subgraphs in it. The following simple statement provides an equivalent formulation of the function $n(k,d)$.

Proposition 1. *For $1 \leq k \leq d, n(k,d)$ is precisely the maximum number of vertices of a complete graph that admits a bipartite covering of order k and size d.*

The rest of this note is organized as follows In Sect. 2 we present the simple proof of Proposition 1. The main result, Theorem 1, is proved in Sect. 3. Section 4 contains some possible extensions and open problems.

2. Neighborly Families and Bipartite Coverings

There is a simple one-to-one correspondence between k-neighborly families of n standard boxes in R^d and bipartite coverings of order k and size d of the complete graph K_n. To see this correspondence, consider a k-neighborly family $\mathcal{C} = \{\mathcal{C}_1, \ldots, C_n\}$ of n standard boxes in R^d. Since any two boxes have a nonempty intersection, there is a point in the intersection of all the boxes (by the trivial, one dimensional case of Helly's Theorem, say). By shifting the boxes we may assume that this point is the origin O. If $n \geq 2$, O must lie in the boundary of each box, since it belongs to all boxes, and the dimension of the intersection of each pair of boxes is strictly smaller than d. Put $V = \{1, 2, \ldots, n\}$. For each coordinate x_i, $1 \leq i \leq d$, let H_i be the complete bipartite graph on V whose sets of vertices are $V_i^+ = \{j : C_j$ is contained in the half space $x_i \geq 0\}$, and $V_i^- = \{j : C_j$ is contained in the half space $x_i \leq 0\}$. It is not difficult to see that if the dimension of $C_p \cap C_q$ is $d - r$, then the edge pq of the complete graph on V lies in exactly r of the subgraphs H_i. Therefore, the graphs H_i form a bipartite covering of order k and size d.

Moreover, the above correspondence is invertible; given a bipartite covering of the complete graph on $V = \{1, 2, \ldots, n\}$ by complete bipartite subgraphs $H_1, \ldots, H_d$ one can define a family of n standard boxes as follows. Let V_i^+ and V_i^- denote the two color classes of H_i. For each j, $1 \leq j \leq n$, let C_j be the box defined by the intersection of the unit cube $[-1, 1]^d$ with the half spaces $x_i \geq 0$ for all i for which $j \in V_i^+$ and the half spaces $x_i \leq 0$ for all i for which $j \in V_i^-$. If the given covering is of order k, the family of standard boxes obtained is k-neighborly.

The correspondence above clearly implies the assertion of Proposition 1, and enables us to study, in the next section, bipartite coverings, in order to prove Theorem 1.

3. Economical Bipartite Coverings

In this section we prove Theorem 1. In view of Proposition 1 we prove it for the function $n(k, d)$ that denotes the maximum number of vertices of a complete graph that admits a bipartite covering of order k and size d.

Part (i) of the theorem is essentially known. The fact that $n(1, d) = d + 1$ is a Theorem of Graham and Pollak [8, 9]. See also [7, 11, 18, 13, 1] and [2] for various simple proofs and extensions. The statement that for every fixed d, $n(k, d)$ is a non-decreasing function of k is obvious and the claim that $n(d, d) = 2^d$ is very simple. Indeed, the chromatic number of any graph that can be covered by d bipartite subgraphs is at most 2^d, implying that $n(d, d) \leq 2^d$. To see the lower bound, let V be a set of 2^d vertices denoted by all the binary vectors $\epsilon = (\epsilon_1, \ldots, \epsilon_d)$, and let H_i be the complete bipartite graph whose classes of vertices are all the vertices labelled by vectors with $\epsilon_i = 0$ and all the vertices labelled by vectors with $\epsilon_i = 0$. Trivially $H_1, \ldots, H_d$ form a bipartite covering (of order d and size d) of the complete graph on V, showing that $n(d, d) = 2^d$, as claimed.

The lower bound in part (ii) of the theorem is proved by a construction, as follows. For each i, $0 \leq i \leq k-1$, define $d_i = \lfloor (d+i)/k \rfloor$ and $D_i = \{1, 2, \ldots, d_i, d_i + 1\}$. Observe that $\sum_{r=0}^{k-1} d_r = d$. Let V denote the set of vectors of length k defined as follows

$$V = \{ \ (\epsilon_0, \epsilon_1, \ldots, \epsilon_{k-1}) : \epsilon_i \in \ D_i \ \}.$$

For each r, $0 \leq r \leq k-1$, and each j, $1 \leq j \leq d_r$, let $H_{r,j}$ denote the complete bipartite graph on the classes of vertices

$$A_{r,j} = \{ \ (\epsilon_0, \epsilon_1, \ldots, \epsilon_{k-1}) : \epsilon_r = j \ \}$$

and

$$B_{r,j} = \{ \ (\epsilon_0, \epsilon_1, \ldots, \epsilon_{k-1}) : \epsilon_r \geq j+1 \ \}.$$

Altogether there are $\sum_{r=0}^{k-1} d_r = d$ bipartite subgraphs $H_{r,j}$. It is not too difficult to see that they form a bipartite covering of the complete graph on V. In fact, if $(\epsilon_0, \ldots, \epsilon_{k-1})$ and $(\epsilon'_0, \ldots, \epsilon'_{k-1})$ are two distinct members of V, and they differ in s coordinates, then the edge joining them lies in precisely s of the bipartite graphs. Since $1 \leq s \leq k$ for each such two members, the above covering is of order k, implying the lower bound in part (ii) of the theorem.

The upper bound in part (ii) is proved by a simple algebraic argument. Let $H_1, \ldots, H_d$ be a bipartite covering of order k and size d of the complete graph on the set of vertices $N = \{1, 2, \ldots, n\}$. Let A_i and B_i denote the two vertex classes of H_i. For each $i \in N$, define a polynomial $P_i = P_i(x_1, \ldots, x_d, y_1, \ldots, y_d)$ as follows:

$$P_i = \prod_{j=1}^{k} \left(\sum_{p:i \in A_p} x_p + \sum_{q:i \in B_q} y_q - j \right).$$

For each $i \in N$ let $e_i = (b_{i1}, \ldots, b_{id}, a_{i1}, \ldots, a_{id})$ be the zero-one vector in which $a_{ip} = 1$ if $i \in A_p$ (and $a_{ip} = 0$ otherwise), and, similarly, $b_{iq} = 1$ if $i \in B_q$ (and $b_{iq} = 0$ otherwise). The crucial point is the fact that

$$P_i(e_j) = 0 \text{ for all } i \neq j \text{ and } P_i(e_i) \neq 0. \tag{1}$$

This holds as the value of the sum

$$\sum_{p:i \in A_p} x_p + \sum_{q:i \in B_q} y_q$$

for $x_p = b_{jp}$ and $y_q = a_{jq}$ is precisely the number of bipartite subgraphs in our collection in which i and j lie in distinct color classes. This number is 0 for $i = j$ and is between 1 and k for all $i \neq j$, implying the validity of (1).

Let $\overline{P}_i = \overline{P}_i(x_1, \ldots, x_d, y_1, \ldots, y_d)$ be the multilinear polynomial obtained from the standard representation of P_i as a sum of monomials by replacing each monomial of the form $c \prod_{s \in S} x_s^{\delta_s} \prod_{t \in T} y_t^{\gamma t}$, where all the δ_s and γ_t are positive, by the monomial $c \prod_{s \in S} x_s \prod_{t \in T} y_t$. Observe that when all the variables x_p, y_q attain 0, 1-values, $P_i(x_1, \ldots, y_d) = \overline{P}_i(x_1, \ldots, y_d)$ since for any positive δ, $0^\delta = 0$ and $1^\delta = 1$. Therefore, by (1),

$$\overline{P}_i(e_j) = 0 \text{ for all } i \neq j \text{ and } \overline{P}_i(e_i) \neq 0. \tag{2}$$

By the above equation, the polynomials $\overline{P}_i$ $(i \in N)$ are linearly independent. To see this, suppose this is false, and let

$$\sum_{i \in N} c_i P_i(x_1, \ldots, y_d) = 0,$$

be a nontrivial linear dependence between them. Then there is an $i' \in N$ so that $c_{i'} \neq 0$. By substituting $(x_1, \ldots, y_d) = e_{i'}$ we conclude, by (2) that $c_{i'} = 0$, contradiction. Thus these polynomials are indeed linearly independent. Each polynomial $\overline{P}_i$ is a multilinear polynomial of degree at most k. Moreover, by their definition they do not contain any monomials that contain both x_i and y_i for the same i. It thus follows that all the polynomials $\overline{P}_i$ are in the space generated by all the monomials $\prod_{s \in S} x_s \prod_{t \in T} y_t$, where S and T range over all subsets of N satisfying $|S| + |T| \leq k$ and $S \cap T = \emptyset$. Since there are $m = \sum_{i=0}^{k} 2^i \binom{d}{i}$ such pairs S, T, this is the dimension of the space considered, and as the polynomials $\overline{P}_i$ are n linearly independent members of this space it follows that $n \leq m$. This completes the proof of part (ii) and hence the proof of Theorem 1. □

4. Concluding Remarks and Open Problems

The proof of the upper bound for the function $n(k, d)$ described above can be easily extended to the following more general problem. Let K be an arbitrary subset of cardinality k of the set $\{1, 2, \ldots, d\}$. A bipartite covering $H_1, \ldots, H_d$ of size d of the complete graph K_n on n vertices is called a covering of *type* K if for every edge e of K_n, the number of subgraphs H_i that contain e is a member of K. The proof described above can be easily modified to show that the maximum n for which K_n admits a bipartite covering of type K and size d, where $|K| = k$, is at most $\sum_{i=0}^{k} 2^i \binom{d}{i}$. There are several examples of sets K for which one can give a bigger lower bound than the one given in Theorem 1 for the special case of $K = \{1, \ldots, k\}$. For example, for $K = \{2, 4\}$, there is a bipartite covering $H_1, \ldots, H_d$ of type K of a complete graph on $n = 1 + \binom{d}{2}$ vertices. To see this, denote the vertices by all subsets of cardinality 0 or 2 of a fixed set D of d elements and define, for each $i \in D$, a complete bipartite graph whose classes of vertices are all subsets that contain i and all subsets that do not contain i. Similar examples exist for types K of bigger cardinality.

One can consider bipartite coverings of prescribed type of other graphs besides the complete graph, and the algebraic approach described above can be used to supply lower bounds for the minimum possible number of bipartite subgraphs in such a cover, as a function of the rank of the adjacency matrix of the graph (and the type).

The main problem that remains open is, of course, that of determining precisely the function $n(k, d)$ for all k and d. Even the precise determination of $n(2, d)$ seems difficult.

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On the Isolation of a Common Secret

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Summary. Two parties are said to "share a secret" if there is a question to which only they know the answer. Since possession of a shared secret allows them to communicate a bit between them over an open channel without revealing the value of the bit, shared secrets are fundamental in cryptology.

We consider below the problem of when two parties with shared knowledge can use that knowledge to establish, over an open channel, a shared secret. There are no issues of complexity or probability; the parties are not assumed to be limited in computing power, and secrecy is judged only relative to certainty, not probability. In this context the issues become purely combinatorial and in fact lead to some curious results in graph theory.

Applications are indicated in the game of bridge, and for a problem involving two sheriffs, eight suspects and a lynch mob.

1. Introduction

Suppose two parties—let us call them "Alice" and "Bob"—share a secret, that is, they have common knowledge possessed by no one else; then Alice may use her secret to transmit a bit to Bob in such a way that no eavesdropper can deduce the value of the bit. For example, if Alice and Bob are the only people in the world who know whether the current US President wears a wig, Alice may send Bob the following message (or the same message, with 0 and 1 interchanged):

My bit is 0 if the President wears a wig, 1 otherwise.

While Eve (an eavesdropper) may believe that the President probably does not wear a wig, and therefore that Alice's bit is more likely to be 1 than 0, her inability to determine the value of the bit with certainty is all that concerns us here.

This method of encryption is called a "one-time pad"; Alice and Bob share a bit of information, and they can use it (once) to pass a bit in secret.

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There are, however, situations where even though Alice and Bob appear to possess shared information not available to the public, this information does not take the form of a shared secret. Nonetheless, Alice and Bob may be able to *isolate* a shared secret by communicating with each other, even though their messages are public. Since this is precisely the situation where cryptologic methods are needed (communication lines are available, but not private), Alice and Bob are almost as well off here as if they had begun with a shared secret; they must merely spend a few preliminary rounds of communication in establishing the secret.

Let us give two examples of such situations, before proceeding further.

(1) **The game of bridge:** Here two partners wish to communicate in private, but the rules of the game require that all communication be done by legal bids and plays, about which there may be no prior private understandings. Thus, there are initially no shared secrets. But there is private information: each player knows, by virtue of looking at his own hand, 13 cards that do *not* belong to his partner. Can they make use of this information to communicate in private?

(2) **The 'two sheriffs' problem:** Two sheriffs in neighboring towns are on the track of a killer, in a case involving eight suspects. By virtue of independent, reliable detective work, each has narrowed the list to only two. Now they are engaged in a telephone call; their object is to compare information, and if their pairs overlap in just one suspect, to identify him (the killer) and put out a.p.b.'s so as to catch him in either town.

The difficulty is that their telephone line has been tapped by the local lynch mob, who know the original list of suspects but not which pairs the sheriffs have arrived at. If they are able to identify the killer with certainty as a result of the phone call, he will be lynched before he can be arrested.

Can the sheriffs accomplish their objective without tipping off the mob?

2. The Mathematical Model

One natural model for common knowledge is obtained by imagining that in any situation there is an underlying finite space S of *possibilities* of which any one element may be "the truth." Alice's knowledge concerning S at any point consists of some subset X of S, meaning that X is precisely the set of truths consistent with what Alice knows. As Alice communicates with Bob she obtains more information, and her knowledge set X shrinks accordingly.

At any time the "true point" must lie both in X and in Bob's knowledge set Y, but if there are two or more points in $X \cap Y$ Alice and Bob will never be able to choose among them by communicating with each other. Hence for our purposes if X is a *possible* knowledge set for Alice and Y for Bob, our only concern is whether they intersect.

Consequently we choose to model common knowledge by using a mere vertex to represent a possible knowledge set of Alice's, and similarly for Bob; we connect vertex x of Alice's with vertex y of Bob's when the corresponding knowledge sets intersect, that is, when the two vertices are simultaneously possible. The "truth" is thus represented by some adjacent pair of vertices, i.e. an edge.

Alice and Bob's knowledge at any time thus constitutes a graph, which, in accordance with cryptographic tradition, is assumed to be known to everyone in the world. The interpretation of this graph will, we hope, become clear to the reader after some examples.

It is convenient to formalize our model as follows.

Definition 1. *A bigraph is a finite, non-empty collection H of ordered pairs such that if (x, y) is in H then (y, z) is not.*

Elements of the set $A(H) := \{x : (x, y) \in H$ for some y$\}$ will be termed "Alice's vertices" and are perforce distinct from the symmetrically defined "Bob's vertices" in $B(H)$. Thus the elements of H are edges of a bipartite graph, but note that the vertices come equipped with a labelled left-right (Alice-Bob) partition and that isolated vertices cannot arise.

Our model now consists of a bigraph H, known to all, the edges of which represent possible truths. Alice knows the endpoint in $A(H)$ of the true edge, Bob its endpoint in $B(H)$; in other words, if the true edge is (x, y) then Alice knows x and Bob knows y.

We say that Alice and Bob *share a secret* if there is a question to which they know the answer and Eve does not. In the wig example, the question "Does the President wear a wig?" can be answered only by Bob and Alice, so they indeed share a secret in this case. Here, the bigraph H consists of a pair of disjoint edges, one corresponding to "the President wears a wig" and the other to "the President does not wear a wig." The disconnectivity of H is its crucial property:

Theorem 1. *Two parties share a secret if and only if their bigraph is disconnected.*

Proof. It is immediate that Alice and Bob share a secret whenever their bigraph is disconnected, since if C is one of its connected components, only they can answer the question "Is the true edge in C ?". To see the converse, let Q be the given question and let $a_1, a_2, \ldots$ be its possible answers (from Eve's point of view). Write "$(u, v)\#a_i$" if it is simultaneously possible for a_i to be the answer to Q, and (u, v) to be the true edge of H.

We now note that if $(x, y)\#a_i$ and $(u, v)\#a_j$ for $i \neq j$, then (x, y) and (u, v) can be neither identical nor adjacent; if, for example, $x = u$ then when

Alice's end of the true edge is x she will be unable to decide between answers a_i and a_j.

It follows that the edges consistent with the various answers a_i determine a partition of H, each part of which is a non-empty union of connected components; since the number of possible answers must be at least two, H is disconnected. □

If H is connected, are Alice and Bob doomed never to share a secret? Of course, if they can arrange a private (secure) conversation, they can agree on some string of bits and thus share as many secrets as they wish; this indeed is often done in traditional cryptography, in the name of agreeing on or distributing *key*. Unfortunately this phase is often dangerous and sometimes impossible; else, cryptography would be unnecessary. However, Alice and Bob may be able to use their common knowledge (reflected in the structure of H) to isolate a common secret by means of a *public* conversation; and it is just this process which we wish to investigate.

Consider, for example, the bigraph $H = \{(a_1, b_1), (a_1, b_2), (a_2, b_2), (a_2, b_3), (a_3, b_3), (a_3, b_4), (a_4, b_4), (a_4, b_1)\}$. H is an 8-cycle, thus connected, but Alice and Bob can disconnect it as follows: if Alice's end of the true edge is a_1 or a_3 she says so: "My end of the true edge is either a_1 or a_3." Bob can tell by looking at his end which of the two possibilities is the case, hence they now share a secret; this is reflected in the fact that after Alice's announcement, their bigraph is disconnected (see Fig. 1). Of course, had Alice's end been a_2 or a_4, an announcement to that effect would also have done the job. (We shall see later that two-sided conversations may be necessary to disconnect some bigraphs.)

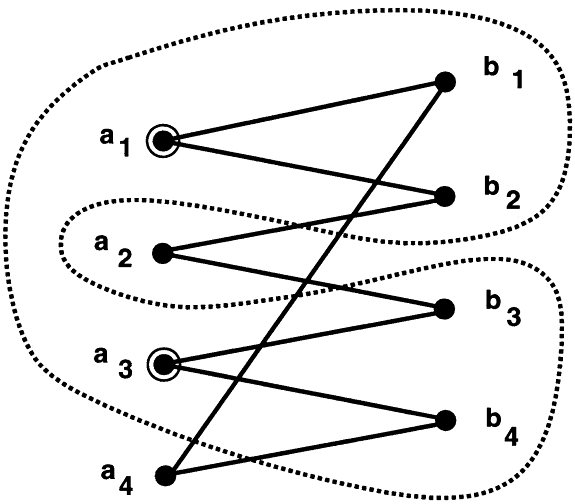

Fig. 1 Alice separates an 8-cycle.

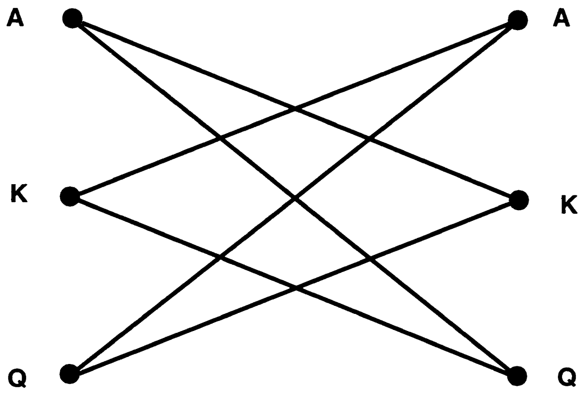

Fig. 2 Bigraph for dealing 2 cards from a 3-card deck.

The next example is inspired by the work of Winkler [4, 5] on cryptologic techniques for the game of bridge, and the recent works of Fischer, Paterson, and Rackoff [2] and Fischer and Wright [3] using random deals of cards for cryptographic key. A card is dealt at random, face down, to each of Alice and Bob, from a three-card deck consisting (say) of an Ace, a King, and a Queen. The remaining card is discarded unseen. Here H is a 6-cycle (see Fig. 2). The six edges correspond to the six possible deals; the set $A(H)$ consists of Alice's three possible holdings ("A", "K" or "Q") and similarly for $B(H)$. The fact that "K" in $A(H)$ and "K" in $B(H)$ are not adjacent corresponds to the fact that Alice and Bob cannot both hold the King. Clearly Alice and Bob share *some* knowledge in this situation, but as we shall see it is not enough for them to be assured of being able to isolate a shared secret.

For a third example, suppose H is a 10-cycle on vertices $v_0, v_1, \ldots, v_9$ with v_i adjacent to v_j just when $|i - j| = 1 \mod 10$. (See Fig. 3.)

Let $A(H)$ consist of the vertices of even index. Then the following protocol allows Alice to disconnect the bigraph: if she holds v_i, she chooses j to be *either* $i + 4$ or $i + 6 \mod 10$, then tells Bob:

$$\text{I hold either } v_i \text{ or } v_j.$$

This protocol is said to be *non-deterministic* because Alice has more than one choice of message for a given holding. Non-determinism as used here is thus quite different from its use in complexity theory, and in fact is more closely related in some respects to randomization (despite the absence of probability in our model). In particular non-deterministic communication protocols are quite practical.

Incidentally, we assume nothing is given away by the order in which objects are named in a semantically symmetric expression such as "v_i or v_j." In a deterministic protocol we can insure this by a naming convention, such as

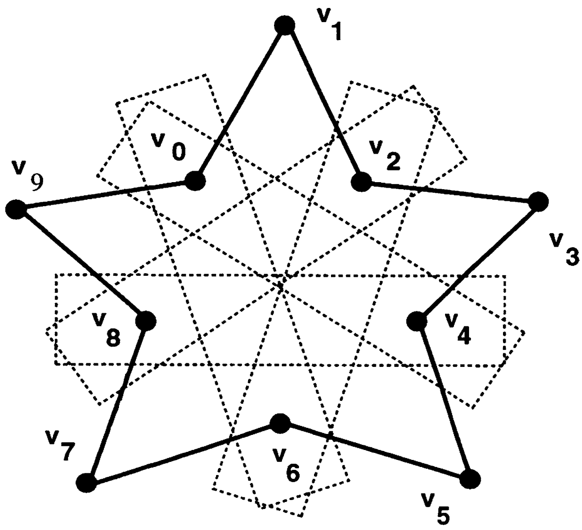

Fig. 3 Non-deterministically separating a 10-cycle.

enforced alphabetical order, but our mathematical model for communication will obviate the problem.

In order to define "communication protocol" rigorously we need to define "conversation", even though the latter definition needs the former for interpretation. Accordingly, a *conversation* will be a finite string $m_1, \ldots, m_t$ of positive integers, communicated alternately by Alice and Bob beginning with Alice.

Although this seems perhaps a limited vocabulary for communication, it is in fact completely general because meanings can be assigned to the numbers via an agreed-upon protocol. The communication protocol specifies *under precisely what circumstances a given number may be uttered.* Thus, we operate in effect in the "political" theory of communication: when someone says something we ask not "what does that mean" but "under what circumstances would he/she have said that." This model is both stronger and simpler than one in which messages are sentences. To see its effect, consider solving the C_{10} bigraph above by using always vertices i and $i+4$. This looks reasonable at first glance but is actually a sham. For example, Alice would not then say "My end is 4 or 8" if she held 8. Hence an eavesdropper can eliminate 8 if she hears this, preventing disconnection of the bigraph and ruining the protocol.

3. Deterministic Separation

Definition 1. A deterministic communication protocol *for the bigraph H is a sequence of functions $f_1, \ldots$ into the positive integers, such that f_1 is a* *function of x (Alice's end of the true edge), and each subsequent f_i depends on the values of $f_1, \ldots, f_{i-1}$ and either on x, if i is odd, or on y (Bob's end of the true edge) if i is even.*

Thus, a deterministic communication protocol, combined with a true edge, produces a unique conversation given by $m_i = f_i(x; m_1, \ldots, m_{i-1})$ for i odd and $m_i = f_i(y; m_1, \ldots, m_{i-1})$ for i even. After step i of the conversation, the situation can be again described by a bigraph H^i consisting of those edges (u, v) such that if (u, v) had been the true edge the conversation would have been as seen. H^i is a sub-bigraph of H which contains the actual true edge; in fact H^i is obtained from H^{i-1} by deleting all edges incident to certain left-hand vertices (for i odd) or certain right-hand vertices (for i even). The vertices which survive on the left (when i is odd) are just $\{u : f_i(u; m_1, \ldots, m_{i-1}) = m_i\}$.

For our purposes a conversation $(m_1, \ldots, m_t)$ will be deemed "successful" if H^t is disconnected, and a communication protocol for H will be said to "separate H" or to be a "separation protocol for H" if it always produces a successful conversation. Finally, H itself will be termed *deterministically separable* if there is a deterministic communication protocol which separates H.

We can also give a recursive characterization of the deterministically separable bigraphs. It will be useful to introduce notation for the *sub-bigraph* $H|A'$ of a bigraph H induced by a subset A' of $A(H)$; namely,

$$H|A' := H \cap (A' \times B(H)).$$

Thus from the standpoint of ordinary graph theory, $H|A'$ is obtained by discarding isolated vertices from the subgraph of H induced by $A' \cup B(H)$. The definition of $H|B'$ for $B' \subset B(H)$ is similar.

Theorem 2. *Let* **DS** *be the smallest symmetric class of bigraphs which contains the disconnected bigraphs and has the following property: for any bigraph H, if there is a partition $A(H) = A_1 \cup A_2$ of Alice's vertices such that $H|A_1$ and $H|A_2$ are both in* **DS***, then H is also in* **DS***. Then* **DS** *is the class of deterministically separable bigraphs.*

Proof. Let us check first that the class of deterministically separable bigraphs is indeed closed under the operation defined in the statement of the theorem. It is certainly symmetrical, since a protocol which separates H can be modified to one which separates the dual of H (i.e. the result of reversing the ordered pairs in H) by switching the roles of Alice and Bob, and adding a meaningless message from Alice to the beginning. If a partition is given along with separation protocols P_1 and P_2 for the two sub-bigraphs of H, we design a separation protocol P for H as follows: first Alice sends "i" if and only if her end of the true edge lies in A_i, then Bob sends back a meaningless "1", then protocol P_i is followed.

It remains to show that any symmetric class **C** containing the disconnected bigraphs and closed under the stated operation contains all deterministically separable bigraphs. This is done by induction on the number of edges.

Let H be connected but deterministically separable via a communication protocol P; then sooner or later P must call for a first meaningful message, say from Alice. We may assume that her options at that point are to send "1", "2", etc. up to "k" for some $k \geq 2$, according to whether her end of the true edge lies in E_1, E_2, etc; there is no dependence on previous conversation here because by assumption said conversation has up to now been predictable. Let us modify P slightly by having Alice send only a 1 or 2 at this point, the former just when her end is in E_1; if she sends a "2" Bob sends a meaningless "1" back, then Alice sends the "2", "3",... or "k" that would have been sent before and the protocol P resumes. We have thus found protocols which separate each of $H|E_1$ and $H|(A(H) - E_1)$. Each of these is thus in $\mathbf{C}$ by the induction assumption, hence $H \in \mathbf{C}$ by the closure condition. □

4. Non-deterministic Separation

We now introduce two ways of weakening the definition of a communication protocol. First, if the functions f_i are permitted to be *multi*-valued, so that at each point Alice or Bob has one *or more* possible messages to send, we say that the protocol is *non-deterministic.* Note that during a non-deterministic communication protocol, the bigraph shrinks as before but this time each party, at his or her turn to speak, is provided with a (message-labelled) *cover* of his or her vertices instead of a *partition.* It is still the case, however, that the knowledge of Alice and Bob is expressed at each point by the state of their bigraph.

Second, if the functions are permitted to depend for odd i on a random number m known only to Alice, and for even i on a random number n known only to Bob, then the protocol is *randomized.* Here the bigraph does *not* any longer describe the situation completely, as Bob and Alice may learn things about each other's random number.

Whether a communication protocol is non-deterministic, randomized or both, however, we continue to insist that the protocol *always* produce a successful conversation in order to qualify for separating H.

Fortunately, the three new categories of separation protocol which arise result in only one new category of bigraph.

Theorem 3. *The following are equivalent, for any bigraph H:*

(a) H is separable by a non-deterministic communication protocol;
(b) H is separable by a randomized communication protocol;
(c) H is separable by a randomized, non-deterministic communication protocol.

Proof. We need to show (c)→(b) and (c)→(a), the reverse implications being trivial. Of these the former is easy: by extending the range of the random numbers, Alice and Bob can use them to decide which message to send when there is more than one choice.

Turning random numbers into non-deterministic choices looks awkward because a random number may be used many times in the protocol, whereas there is no "consistency" built in to nondeterminism. However, this problem is illusory. Suppose, at Alice's first turn to speak, that she is supplied with a randomized separation protocol but no random number; then she chooses a random number and acts accordingly. She cannot "remember" that number at her next turn and use it again, but she can compute which random numbers are consistent with her previous action and choose one of those upon which to base her next message. Bob behaves similarly; at each turn he determines which values of his non-existent random number are consistent with his own previous actions (and his end of the true edge), then picks one such value and acts accordingly. □

A bigraph will be called, simply, *separable* if one (thus all) of the conditions of Theorem 3 obtains. The following recursive characterization is analogous to Theorem 2, although a small additional subtlety arises in the proof.

Theorem 4. *Let* **S** *be the smallest symmetric class of bigraphs which contains the disconnected bigraphs and has the following property: for any bigraph H, if there is a covering $A(H) = A_1 \cup A_2$ of Alice's vertices such that $H|A_1$ and $H|A_2$ are both in* **S***, and both A_1 and A_2 are strictly contained in $A(H)$, then H is also in* **S**. *Then* **S** *is the class of separable bigraphs.*

Proof. The proof that the class of separable bigraphs is symmetrical and closed under the operation defined in the statement of the theorem is as in Theorem 2, except that if Alice's end of the true edge lies in $A_1 \cap A_2$ she may send *either* message "1" or message "2".

It remains to show that any symmetric class **C** containing the disconnected bigraphs and closed under the stated operation contains all separable bigraphs; this is again done by induction on the number of edges.

Let H be connected but separable via a non-deterministic communication protocol P, and suppose that H is the smallest separable bigraph not in **C**. Let us call a vertex u in $A(H)$ (or, dually, in $B(H)$) *weak* if there is no proper subset A' of $A(H)$ containing u for which $H|A'$ is separable.

We claim that there is some weak vertex in $A(H)$. For, if not, define for each $x \in A(H)$ a proper subset A_x which does yield a separable sub-bigraph of H. Since the A_x's cover $A(H)$, we can find $x_1, x_2, \ldots, x_k$ such that the A_{x_i}'s cover $A(H)$ with k minimal (but necessarily greater than 1). Set $A_1 := A_{x_1}$, $A_2 := A_{x_2} \cup \cdots \cup A_{x_k}$. Then A_1 and A_2 are a proper cover of $A(H)$, and $H|A_1$ is separable by assumption. However, $H|A_2$ is also separable, since Alice can reduce it to a separable bigraph by sending some i for which her end of the true edge lies in $A_{x_i}, 2 \leq i \leq k$. These bigraphs are thus both in **C** by the induction assumption, contradicting the fact that H is not in **C**.

Since the class of separable bigraphs is symmetric, the dual of H is also separable but not in **C**; hence the same argument produces a weak vertex in $B(H)$. It may seem to the reader that weak vertices cause trouble only if found on the true edge, and thus that Alice and Bob can't *both* be stymied as long as no two weak vertices are adjacent. However, it turns out that the mere *presence* of weak vertices on both sides is enough to render H inseparable.

To see this, let u be a weak vertex in $A(H)$; there must be some message (say m) which Alice is permitted to send when her end of the true edge is u. Let A' be the set of vertices in $A(H)$ which, like u, allow the message m; then $H|A'$ must be separable, since this is the bigraph which results when m is sent. Thus A' must not be a proper subset of $A(H)$, that is, $A' = A(H)$ and the message m is meaningless.

If m is indeed sent, the protocol turns to Bob who still has all of H before him. By the same reasoning as above, he must also have a meaningless message (i.e. a message he can send regardless of which vertex is his end of the true edge) available to him.

Now we're back to Alice with H still intact. We thus see that Alice and Bob must be allowed by the protocol to pass meaningless messages back and forth ad infinitum, irrespective of which edge of H is the true edge; but then we have a contradiction, since H is required to have a communication protocol which always separates. □

Theorem 4 is often useful in determining separability via case analysis. For example, it is easy to check that no path with fewer than five edges is separable, nor is the 6-cycle (Fig. 2) separable because for any proper subset S of Alice's or Bob's vertices, $H|S$ would be a path of length 2 or 4.

There is one class of bigraphs which is easily seen to be disjoint from the class of separable bigraphs, a fact which helps in obtaining negative results.

Theorem 5. *Suppose that there is an edge of H which is adjacent to all other edges of H. Then H is not separable.*

Proof. Such an edge cannot be contained in any disconnected sub-bigraph of H; thus, if it happens to be the true edge, no protocol can separate H. □

Note that, in particular, no separable bigraph can have a vertex which is adjacent to all the vertices on the other side.

We are now in a position to prove that the class of separable bigraphs is strictly larger than the class of deterministically separable bigraphs.

Let $A(H) = \{a_1, \ldots, a_7\}$ and $B(H) = \{b_1, \ldots, b_7\}$, and put $(a_i, b_j) \in H$ if and only if $j - i = 1$, 2 or 4, where the indices are interpreted always modulo 7. Then H is the incidence graph of a Fano plane (see Fig. 4) and we have:

Theorem 6. *The incidence graph of the Fano plane is separable but not deterministically separable.*

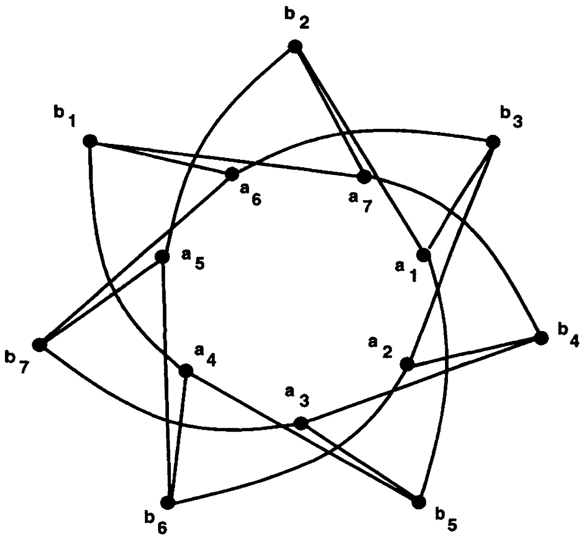

Fig. 4 Incidence graph of the Fano plane.

Proof. We first provide a non-deterministic separation protocol. Alice begins by sending a number k such that her end of the true edge is in the set $\{a_k, a_{k+1}, a_{k+2}\}$. Bob now (deterministically) sends back "1" if his end is b_{k+2} or b_{k+6}; "2" if it is b_{k+4} or b_{k+5}; "3" if it is b_{k+3} or b_{k+1}. (It cannot be b_k.) This separates H into a 2-edge component and a 1-edge component.

If, on the other hand, there were a deterministic separation protocol for H, then one of the parties would eventually have to send a meaningful message, thus effecting a partition of his or her vertices. This may as well be Alice since H is symmetrical. If one of the parts has fewer than 3 of Alice's vertices in it, or has 3 vertices whose neighborhoods intersect, then one of Bob's remaining points will be of full degree, contradicting Theorem 5. Otherwise the partition must be isomorphic to $\{a_1, a_2, a_3\}$ versus $\{a_4, a_5, a_6, a_7\}$. The former part induces a deterministically separable sub-bigraph as we have seen from the above protocol, but the sub-bigraph induced by the latter part contains a vertex (b_1) whose neighborhood intersects the neighborhoods of all other vertices of $B(H)$. Thus, if Bob's end of the true edge is b_1, he cannot separate the bigraph at this time. However, Alice is also stymied because she has only one vertex available (a_5) not adjacent to b_1, thus her vertices can never be partitioned so as to induce disconnected sub-bigraphs. □

The situation changes if we consider bigraphs which are separable *in one round*, that is, by just one message from Alice. (This is not, by the way, a symmetric class; the path on 7 vertices, for example, can be separated only by a message from the party with 4 vertices.) Before proceeding, we need a curious graph-theoretic result.

Theorem 7. *Let G be any graph with no vertex adjacent to all others. Then there is a partition $V_1, \ldots, V_k$ of the vertices of G such that for each $i = 1, \ldots, k$ the subgraph $\langle V_i \rangle$ induced by V_i is disconnected.*

Proof. If not let G be a counterexample with smallest possible number of vertices. For any subset U of the set of vertices V, let $\omega(U)$ be the number of vertices in $V - U$ which are adjacent to all other vertices in $V - U$.

Note first that if $\langle U \rangle$ is disconnected, then $\omega(U)$ must be non-zero; else we may apply the induction hypothesis to get a suitable partition of $V - U$, and appending U itself to this partition yields a partition of V suitable for G.

Choices of U for which $\langle U \rangle$ is disconnected do exist, of course, since U can be taken to be a pair of non-adjacent vertices. Hence we may choose a U for which $\langle U \rangle$ is disconnected and $\omega(U)$ is minimal.

Now let x be any full vertex in $V - U$, that is, any vertex in $V - U$ which is adjacent to all other vertices in $V - U$. By assumption x has *some* non-neighbor, say y, in V; let C be the set of vertices of the component of $\langle U \rangle$ into which y falls.

Suppose first that y is not the only vertex in C, and let $W = U \setminus \{u\}$. Then $\langle W \rangle$ is still disconnected, but x is no longer full in $V - W$ since a non-neighbor y has "moved in." Of course y is not full either, and any other vertex which is full in $V - W$ must already have been full in $V - U$. Hence $\omega(W) < \omega(U)$, a contradiction.

We are reduced to the case where y is an isolated point of $\langle U \rangle$; now we let $W = U \cup \{x\}$. Since x and y are not adjacent y is still isolated in $\langle W \rangle$. Any full vertex of $V - W$ was adjacent to x in $V - U$ and therefore already full in $V - U$; but x itself is now gone from $V - W$ and so we again have $\omega(W) < \omega(U)$, and this contradiction proves the theorem. □

Note that the induction hypothesis, and thus the theorem itself, can be strengthened to read "each $\langle V_i \rangle$ has at least two vertices and contains an isolated point" without changing the proof. However, we will not need the stronger statement.

Theorem 8. *The following are equivalent for a bigraph H.*

(a) H is separable in one round;
(b) H is deterministically separable in one round;
(c) For every vertex u of $A(H)$ there is a vertex v of $A(H)$ such that the neighborhoods of u and v (in $B(H)$) are disjoint.

Proof. It is enough to show (a)→(c)→(b). Suppose that H is separable in one round, let u be any vertex of $A(H)$, and let i be a message that can be sent by Alice when her end of the true edge is u. Since the bigraph that results from sending "i" is disconnected, there is a vertex v on Alice's side of it which is in a different component from u; then u and v must originally have had disjoint neighborhoods.

Now suppose that (c) is satisfied and form a graph G on the vertices in $A(H)$ by defining $\{u, v\}$ to be an edge whenever u and v have *intersecting* neighborhoods. Condition (c) says precisely that G has no vertex of full degree, hence we may apply Theorem 7 to obtain a partition $V_1, \ldots, V_k$ of the vertices of $A(H)$ each part of which induces a disconnected subgraph of G, hence also of H. Sending "i" when Alice's end of the true edge lies in V_i thus yields a deterministic separation protocol. □

We have said that a disconnected bigraph is sufficient to enable Alice and Bob to communicate a bit in secret; we are now in a position to show that separability is in fact *necessary* for such a communication, thus completing the reduction of the original cryptologic problem to a graph-theoretical one.

Let us fix a bigraph H and suppose that Alice (say) has been supplied with a bit ε which she must communicate in secret to Bob, over our usual public channel. The effect of the bit is to double the vertices on Alice's side of H; that is, each vertex $a \in A(H)$ now becomes a pair $a(0), a(1)$ each with the same neighborhood that a had in $B(H)$. The edge $(a(0), b)$ corresponds to the original (a, b) together with the statement "$\varepsilon = 0$".

At the conclusion of a successful, non-randomized communication protocol the question "What is the value of ε?" must be answerable by Bob but not by Eve, hence the bigraph must now be disconnected—and moreover (although we shall not need this fact) the vertices from $A(H)$ in each component must either all correspond to $\varepsilon = 0$ or all to $\varepsilon = 1$.

Theorem 9. *Alice can communicate a bit to Bob in secret, via a randomized and/or non-deterministic communication protocol, if and only if their bigraph is separable.*

Proof. Sufficiency has already in effect been demonstrated; Bob and Alice can cooperatively disconnect the bigraph, ignoring the bit value, then a message of the form "My bit is 0 if the true edge lies in component C, 1 otherwise" does the trick.

For the converse, we double the bigraph as above so that the protocol may be regarded as a special communication protocol, which we denote by P. If P is randomized, then we may replace the random inputs by non-determinism as in Theorem 3; thus we may assume P is merely non-deterministic.

Now we construct from P a randomized (!) communication protocol P' which operates on the original, undoubled bigraph plus a single random bit α for Alice. P' operates by the rule $f'_i(x; \alpha; m_1, \ldots, m_{i-1}) = f_i(x(\alpha); m_1, \ldots, m_{i-1})$ for i odd, and $f'_i = f_i$ for i even.

At each stage, the bigraph associated with P' will be precisely the image of the bigraph associated with P under the collapsing map Φ which sends $x(0)$ and $x(1)$ to x. But the image of a disconnected bigraph under this mapping is again disconnected, so P' is a separation protocol for the original bigraph, completing the proof of the theorem. □

Theorem 9 says, in effect, that if Alice and Bob know that they will need to communicate a bit in secret, then they can disconnect their bigraph *in advance*; when the bit comes in, it can then be communicated (in either direction) by a single message.

5. The 'Two Sheriffs' Problem

Let us now look now at the two sheriffs problem, but generalized as follows: one sheriff (whom we shall call "Lew") has narrowed his list of suspects to p, the other ("Ralph") to q, and the total number of suspects is n. The edges of the bigraph H here represent all possible pairs (L, R) of subsets of the set $N = \{1, 2, \ldots, n\}$ of suspects, with $|L| = p$, $|R| = q$ and $|L \cap R| \neq \emptyset$. Lew's side $A(H)$ of the bigraph will thus contain $\binom{n}{p}$ vertices and Ralph's side $\binom{n}{q}$ vertices, adjacency arising when the corresponding subsets intersect.

If Lew and Ralph succeed in determining the identity of the killer without tipping off the mob, they will share a secret and thus must have disconnected H. Conversely, suppose they manage to disconnect the bigraph; then Lew and Ralph can reduce further to two non-adjacent edges, one of which is the true edge. If that edge represents an overlap of one, the sheriffs will have found the killer.

Theorem 10. *If $n = 2pq$ then there is a deterministic separation protocol for solving the two sheriffs problem.*

Proof. We make use of Baranyai's Theorem [1], which says the following:

If k divides n then there is an array $\{K_{i,j}\}$, $1 \leq i \leq rn/k$, $1 \leq j \leq \binom{n}{k}/(n/k)$ of subsets of $N = \{1, 2, \ldots, n\}$ such that each $|K_{i,j}| = k$, each column $K_{1,j}, \ldots, K_{n/k,j}$ is a partition of N, and each subset of N of size k appears exactly once in the array.

Such an array (known as a 1-factorization of the complete k-uniform hypergraph on n vertices) is fixed by Lew and Ralph (publicly) for $k = p$, and Lew proceeds to tell Ralph on which column his end L of the true edge can be found.

Ralph is thus presented with a partition $L_1, \ldots, L_{n/p}$ of N, one of whose parts is Lew's narrowed-down suspect set. His job will be to split the index set $I = \{1, 2, \ldots, n/p = 2q\}$ into two parts, say I_1 and I_2, so that his suspect set R is contained in $\bigcup_{i \in I_j} L_i$. This will disconnect the bigraph.

To do this the sheriffs employ a fixed (but arbitrary) map Φ from the set of subsets of I of size at most q to the set of subsets of I of size *exactly* q, such that $\Phi(S) \cap S = \emptyset$ for every set S in the domain of Φ. Ralph forms the set $F := \{i : R \cap L_i \neq \emptyset\}$, then puts $F' := \Phi(F)$ and $F'' := \Phi(F')$. F' and F'' are thus complementary subsets of I of cardinality q; let I_1 be the one containing the element "1" of I, and let I_2 be the other. Ralph now identifies

I_1 and I_2 and announces that his set F, defined as above, is contained either in I_1 or I_2.

The resulting bigraph will contain all vertices of $B(H)$ for which the resulting F would have been contained in I_1 or I_2 *and* would have had cardinality q, since in those cases F' and F'' are not dependent on the choice of Φ. Those vertices for which F is contained in I_e will form a connected component, for $e = 1, 2$.

Let k be the index of Lew's end of the true edge, that is, let L_k be Lew's suspect set; suppose $k \in I_e$. Let i be such that k is the ith smallest member of I_e, and let j be the ith smallest element of I_{3-e}. Lew now announces that his set of suspects is in fact either L_j or L_k.

Ralph (but not the mob) will know which of the two is Lew's suspect set: say it is L_j. If $|R \cap L_j| > 1$ then Ralph announces that the killer cannot be identified; otherwise, however, he now knows the killer (say, x). Choosing (again by order of numbers) the corresponding element y of L_k, he announces that the killer is one of x and y. This completes the protocol. □

Let us see how this works in the original case $p = q = 2$, $n = 8$. The following Baranyai array can be used:

$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{1,5\}$	$\{1,6\}$	$\{1,7\}$	$\{1,8\}$
$\{3,4\}$	$\{2,4\}$	$\{2,3\}$	$\{2,6\}$	$\{2,5\}$	$\{2,8\}$	$\{2,7\}$
$\{5,6\}$	$\{5,7\}$	$\{5,8\}$	$\{3,7\}$	$\{3,8\}$	$\{3,5\}$	$\{3,6\}$
$\{7,8\}$	$\{6,8\}$	$\{6,7\}$	$\{4,8\}$	$\{4,7\}$	$\{4,6\}$	$\{4,5\}$

Suppose that the true edge is either $(\{1,2\},\{1,3\})$ or $(\{5,6\},\{5,7\})$. Then Lew will announce that his suspect set belongs to the first column, that is, is one of $\{1,2\}$, $\{3,4\}$, $\{5,6\}$ or $\{7,8\}$. If Ralph himself had one of these sets he would simply announce at this point that the killer cannot be identified; as it is, he splits the index set, telling Lew that his suspect set is either contained in $\{1,2\} \cup \{3,4\}$ or in $\{5,6\} \cup \{7,8\}$. Lew now says “My set is either $\{1,2\}$ or $\{5,6\}$” and Ralph comes back with “The killer is either 1 or 5”.

Non-deterministic versions of the above protocol are more easily described; Lew merely picks some partition of which his suspect set is a part, and Ralph can reduce to two possible suspect sets whose intersections with the partition indices are complementary. Here just one more message, from Lew to Ralph, completes the protocol. Moreover, this can be made to work for any $n > 2pq$ as well.

However, we can do even better when non-determinism is permitted; for example, here is a non-deterministic separation protocol for solving the case where $n = k^2, p = (k-1)^2 + 1$ and $q = 1$, for any $k \geq 2$.

Lew begins by choosing a $k \times k$ array $\{s_{i,j}\}$ of all the suspects, such that for some j', Lew's suspect set consists precisely of $s_{i',1}$ and all $s_{i,j}$ such that $i \neq i'$ and $j \neq 1$. Ralph, who began knowing the identity of the killer, replies as follows: if the killer is $s_{i,j}$ for $j \neq 1$ and any i, he says "The killer is either $s_{1,j}$ or $s_{i,j}$." If the killer is some $s_{i,1}$ then he picks any $j \neq 1$ and makes the same statement.

By first partitioning the suspects into possible vertices (as in the p, q, $2pq$ case) and then making a $k \times k$ array of the sort described above, but where the array elements are members of a partition instead of single suspects, we may combine the techniques for the following result:

Theorem 11. *The two sheriffs problem is solvable non-deterministically whenever* $n \geq q(1+\sqrt{p-1})^2$.

It is perhaps interesting to note that we have separated a very dense bigraph here, regular on each side. In fact, related to these are the following dense bigraphs, which are *deterministically* separable: fix a large k and set H equal to

$$\{((a,b),(c,d)) : 1 \leq a,b,c,d \leq k \text{ and either } a=c \text{ and } b=d, \text{ or } a \neq c \text{ and } b \neq d\}.$$

To separate H, Alice's announces the first coordinate of her pair and Bob the second coordinate of his. Then each will know whether their edge is based on the equalities or the inequalities in the definition above.

6. Multi-Party Generalization

It is evident that many of our definitions and results can be extended to the case where there are more than two conversants. In this case conversation protocols, in order to remain general, allow the identity of the next speaker to depend, at each turn, on the previous conversation.

In [3] Fischer and Wright suggest using random deals to facilitate secret key exchange within a group of persons wishing to communicate privately in yet-to-be-specified subgroups. Among the negative results in [3] is a theorem (Theorem 9, p. 11) which states that no communication protocol for 3 players, each dealt one card of a 3-card deck, can enable them to isolate a secret bit. Fischer and Wright indicate that our methods can be used to generalize the result to $k > 3$ players; we show here how that can be done.

The definition of "bigraph"' extends easily as follows: a *k-graph* is a finite collection H of k-tuples $x = (x_1, \ldots, x_k)$ (which we call "blocks" to avoid confusion) such that $x, y \in H$ implies that x_i and y_j are distinct for $i \neq j$. A k-graph is thus a particular special case of k-uniform hypergraph in which the sets $\{x_i : x \in H\}$ partition the vertex set of H.

The proof of Theorem 1 goes through, as does an appropriate version of Theorem 4; thus we are once more reduced to showing that the players, say X_1 through X_k, cannot cooperatively disconnect their k-graph H_k which in this case consists of a block for each permutation of the cards.

Theorem 12. *The "permutation k-graph"* H_k *is inseparable for* $k \geq 3$.

Proof. H_2 is of course separable, indeed disconnected to begin with. Let us assume that P is a (non-deterministic) separation protocol for H_k, for some $k > 2$, and let $H^0, H^1, \ldots, H^t$ be the state of the k-graph for $X_1, \ldots, X_k$ at each stage of some (successful) conversation using P. Then $H^0 = H_k$ and $H^0, \ldots, H^{t-1}$ are connected k-graphs. We claim first that H^t must consist only of two blocks, which up to permutation of the players and cards, may as well be 1,2,3,..., k and 2,3,..., k, 1. To see this let x and y be two blocks of H^t which lie in different components; then in particular x and y are not adjacent so $x_i \neq y_i$ for $i = 1, 2, \ldots, k$. Let G be the graph on vertices 1,2,..., k with j adjacent to j' when $\{j, j'\} = \{x_i, y_i\}$ for some i; then since x and y are each permutations, G is regular of degree 2. If ϕ is an automorphism of G such that $\phi(j)$ is adjacent to j for all j then $(\phi(x_1), \ldots, \phi(x_k))$ is a block of H^k which is adjacent to x if ϕ has any fixed points and to y if ϕ is not the identity. Since no block of H^t can be adjacent to both x and y, every such ϕ must fix all vertices or none; hence G consists of a single cycle. By relabelling we may assume $x = (1, 2, 3, \ldots, k)$ and $y = (2, 3, \ldots, k, 1)$.

Now if H^t contains any other block some player, say X_k, must have another vertex, say $j \neq k, 1$. But then the block $(1, 2, \ldots, j-2, j-1, j+1, j+2, \ldots, k-1, k, j)$ lies in H^t; and it is adjacent to both x (at player X_1) and y (at player X_k), a contradiction. This proves the claim.

We may now assume that H^t consists exactly of the above blocks x and y, and that the last player to speak was X_1; then the vertices of H^{t-1} are exactly those appearing in x and y, plus some additional vertices held by X_1 which she eliminated in her last message. The fact that H^{t-1} contains those additional vertices means that if one of them had been X_1's "true" vertex, the conversation might have gone exactly as it did until the last message. But H^{t-1} contains no pair of non-adjacent blocks other than x and y, since in every block not equal to x or y, player X_2 holds a 2 and player X_k holds a 1. Hence, the protocol P has failed in this case and this contradiction proves the theorem. □

7. Final Comments

In this work we have only begun to study the combinatorial cryptology of isolating a common secret. In *J. Combin. Theory (B)* **84** (2002) pp. 126–129, Nicole Portmann has shown that for any n, there are bigraphs that are deterministically separable but in no fewer than n steps; and that there are bigraphs that are non-deterministically separable but in no fewer than three steps. We still do not have a proof that the two sheriffs problem cannot be solved deterministically when $n < 2pq$.

We hope that our "bigraphs" may prove to be a useful way of representing common knowledge, even for applications unrelated to the problem of isolating a common secret. Although they carry the same information as do other representations, they may help attract graph-theorists to knowledge problems and thus bring some powerful theorems and sharp combinatorial minds to bear.

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Properties of Graded Posets Preserved by Some Operations

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Summary. We answer the following question: Let P and Q be graded posets having some property and let $\circ$ be some poset operation. Is it true that $P \circ Q$ has also this property? The considered properties are: being Sperner, a symmetric chain order, Peck, LYM, and rank compressed. The studied operations are: direct product, direct sum, ordinal sum, ordinal product, rankwise direct product, and exponentiation.

1. Introduction and Overview

Throughout we will consider finite *graded* partially ordered sets, i.e. finite posets in which every maximal chain has the same length. For such posets P there exists a unique function $r : P \to \mathbb{N}$ (called *rank function*) and a number m (called *rank of* P), such that $r(x) = 0$ ($r(x) = m$) if x is a minimal (resp., maximal) element of P, and $r(y) = r(x)+1$ if y covers x in P (denoted $x \lessdot y$). The set $P_{(i)} := \{x \in P : r(x) = i\}$ is called *i-th level* and its cardinality $|P_{(i)}|$ the *i-th Whitney number*. If S is a subset of P, let $r(S) := \sum_{x \in S} r(x)$, in particular $r(P) := \sum_{x \in P} r(x)$. Let us emphasize, that $r(P)$ is here not the rank of P.

A *symmetric chain* is a chain of the form $C = (x_0 \lessdot x_1 \lessdot \cdots \lessdot x_s)$, where $r(x_0) + r(x_s) = m$. A subset A of P is called a *k-family*, if there are no $k+1$ elements of A lying on one chain in P. Further, $F \subseteq P$ is called a *filter*, if $y \geq x \in F$ implies $y \in F$, and $I \subseteq P$ is said to be an *ideal* if $y \leq x \in I$ implies $y \in I$. Let $d_k(P) := \max\{|A| : A \text{ is a } k\text{-family}\}$ and let $w_k(P)$ denote the largest sum of k Whitney numbers. Obviously, $w_k(P) \leq d_k(P)$, for $k \geq 1$. The poset P is said to be:

(i) *Sperner* (S), if $d_1(P) = w_1(P)$,
(ii) a *symmetric chain order* (SC), if P has a partition into symmetric chains,
(iii) *Peck*, if $d_k(P) = w_k(P)$, for $k \geq 1$, and $|P_{(0)}| = |P_{(m)}| \leq |P_{(1)}| = |P_{(m-1)}| \leq \cdots \leq |P_{(\lfloor m/2 \rfloor)}| = |P_{(\lceil m/2 \rceil)}|$,

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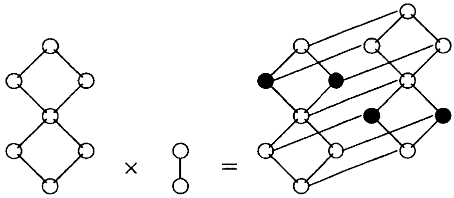

Fig. 1 The property S is not preserved by direct product.

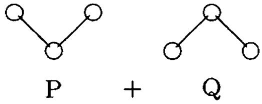

Fig. 2 The properties S, LYM and RC are not preserved by direct sum.

(iv) *LYM*, if $\sum_{x \in A} \frac{1}{|P_{(r(x))}|} \leq 1$ for every antichain A of P,

(v) *rank compressed* (RC), if $\mu_F := \frac{r(F)}{|F|} \geq \frac{r(P)}{|P|} =: \mu_P$ for every filter $F \neq \emptyset$ of P.

Since F is a filter if and only if $P \backslash F$ is an ideal, one can define equivalently:

(v)′ P is rank compressed, if $\mu_I := \frac{r(I)}{|I|} \leq \mu_P$ for every ideal $I \neq \emptyset$ of P.

Extensive information on these properties can be found in [5]. Further related properties are studied in [7].

Already in 1945 Paul Erdös [6] proved that finite Boolean lattices are Peck and thus initiated comprehensive investigations on this subject.

In the following we will study which of these properties are preserved by usual poset operations, i.e. the question is: if P and Q have some property, is it true that $P \circ Q$ has this property as well? Here $\circ$ is some operation.

Throughout let m (resp., n) be the rank of P (resp., Q). If it is not clear from the context whether r is the rank function of P, Q, or $P \circ Q$ we will write r_P, r_Q, $r_{P \circ Q}$, respectively.

A widely studied operation is the *direct product* $P \times Q$, i.e. the poset on the set $\{(x, y) : x \in P \text{ and } y \in Q\}$, such that $(x, y) \leq (x', y')$ in $P \times Q$ if $x \leq_P x'$ and $y \leq_Q y'$. It is well-known, that the direct product preserves the properties SC (de Bruijn et al. [2] and Katona [10]), Peck (Canfield [3]), and RC (Engel [4]), and it does not preserve the properties S and LYM (but with additional condition it does (Harper [8] and Hsieh and Kleitman [9]), see Fig. 1.

A simple operation is the *direct sum* $P + Q$, i.e. the poset on the union $P \cup Q$, such that $x \leq y$ in $P + Q$ if either $x, y \in P$ and $x \leq_P y$, or $x, y \in Q$ and $x \leq_Q y$. In order to obtain again a graded poset we will suppose here $m = n$. Then it is easy to see, that SC and Peck properties are preserved, but S, LYM, and RC not, see Fig. 2.

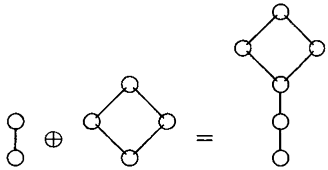

Fig. 3 The properties SC and Peck are not preserved by ordinal sum.

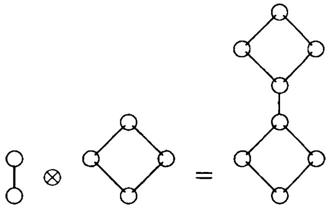

Fig. 4 The properties SC and Peck are not preserved by ordinal product.

Another easy operation is the *ordinal sum* $P \oplus Q$, i.e. the poset on the union $P \cup Q$, such that $x \leq y$ in $P \oplus Q$, if $x, y \in P$ and $x \leq_P y$, or $x, y \in Q$ and $x \leq_Q y$, or $x \in P$ and $y \in Q$. To draw the Hasse diagram of $P \oplus Q$, put Q above P and connect each maximal element of P with each minimal element of Q. Then it is obvious, that properties S and LYM are preserved (note that any antichain in $P \oplus Q$ is either contained completely in P or completely in Q), and also property RC is preserved (see Theorem 1), but properties SC and Peck are not preserved, see Fig. 3.

An interesting operation is the *ordinal product* $P \otimes Q$, i.e. the poset on the set $\{(x, y) : x \in P \text{ and } y \in Q\}$, such that $(x, y) \leq (x', y')$ in $P \otimes Q$, if $x = x'$ and $y \leq_Q y'$, or $x <_P x'$. To draw the Hasse diagram of $P \otimes Q$, replace each element x of P by a copy Q_x of Q, and then connect every maximal element of Q_x with every minimal element of Q_y whenever y covers x in P. In Theorem 2 we will prove, that properties S, LYM, and RC are preserved. Figure 4 shows that properties SC and Peck are not preserved.

Studying posets like square submatrices of a square matrix, Sali [11] introduced the *rankwise direct product* $P \times_r Q$. We will suppose here again $m = n$. Then $P \times_r Q$ is the subposet of $P \times Q$, induced by $\bigcup_{i=0}^{m} P_{(i)} \times Q_{(i)}$. Sali [11] showed, that properties SC, Peck, and LYM are preserved and gave an example that property S is not preserved. Here we present an example, which shows that also property RC is not preserved. Look at the poset P of Fig. 5, which is easily seen to be rank compressed.

The indicated elements form a filter F. Now it is easy to see that the filter $F \times_r F$ in $P \times_r P$ does not verify the filter inequality of (v).

Finally, we will consider also the *exponentiation* Q^P, i.e. the poset on the set of all order-preserving maps $f : P \to Q$ (that is, $x \leq_P y$ implies

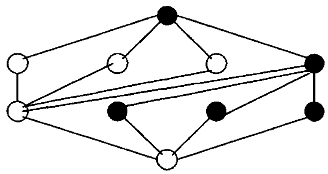

Fig. 5 The property RC is not preserved by rankwise direct product.

$f(x) \leq_Q f(y))$, such that $f \leq g$ if $f(x) \leq_Q g(x)$ for all $x \in P$. In Theorem 3 we will prove that none of the five properties is preserved.

2. Main Results

Theorem 1. *If P and Q are rank compressed, then $P \oplus Q$ is also rank compressed.*

Proof. Obviously, if y has rank i in Q, then it has rank $i+m+1$ in $P \oplus Q$. Hence $\mu_{P\oplus Q} = \frac{r_P(P)+r_Q(Q)+(m+1)|Q|}{|P|+|Q|}$. Let I be an ideal in $P \oplus Q$ and $I \neq \emptyset$.
Case 1. Assume $I \cap Q = \emptyset$. Since $\mu_P \leq \mu_{P\oplus Q}$, and $\mu_I \leq \mu_P$ as P is rank compressed, it follows $\mu_I \leq \mu_{P\oplus Q}$.
Case 2. Now let $I \cap Q \neq \emptyset$. Then $P \subseteq I$ and $\tilde{I} := Q \cap I$ is an ideal in Q. One has $|I| = |P| + |\tilde{I}|$, $r(I) = r_P(P) + r_Q(\tilde{I}) + (m+1)|\tilde{I}|$. Therefore, $\mu_I \leq \mu_{P\oplus Q}$ is equivalent to

$$\big(|\tilde{I}|r_Q(Q) - |Q|r_Q(\tilde{I})\big) + |Q\backslash\tilde{I}|\big((m+1)|P| - r(P)\big) + |P|r(Q\backslash\tilde{I}) \geq 0.$$

This inequality is true, since Q is rank compressed and any element of P has rank at most m. □

Theorem 2. *If P and Q are Sperner or LYM or rank compressed, then $P \otimes Q$ is also resp., Sperner or LYM or rank compressed.*

Proof. Let A be an antichain in $P \otimes Q$. Denote $A_x = \{y \in Q : (x,y) \in A\}$ and $\tilde{A} = \{x \in P : A_x \neq \emptyset\}$. Obviously, $\tilde{A}$ and A_x are antichains in P and Q_x, respectively.

If P and Q are Sperner, then

$$|A| = \sum_{x\in\tilde{A}} |A_x| \leq \sum_{x\in\tilde{A}} w_1(Q) = |\tilde{A}|w_1(Q) \leq w_1(P)w_1(Q) = w_1(P \otimes Q),$$

hence $P \otimes Q$ is Sperner.

Let P and Q be LYM. The level containing (x,y) has $|P_{(r(x))}||Q_{(r(y))}|$ elements. We have

$$\sum_{(x,y)\in A} \frac{1}{|(P\otimes Q)_{(r(x,y))}|} = \sum_{(x,y)\in A} \frac{1}{|P_{(r(x))}||Q_{(r(y))}|}$$
$$= \sum_{x\in\tilde{A}} \frac{1}{|P_{(r(x))}|} \sum_{y\in A_x} \frac{1}{|Q_{(r(y))}|} \le 1.$$

Finally, let P and Q be rank compressed. Let I be an ideal in $P\otimes Q$ and A be the set of maximal elements of I (note that A is an antichain). We use the notation $\tilde{A}$ from above and define further $I_x := I \cap Q_x$, $F_x := Q_x \backslash I_x$, $\tilde{I} := \{x \in\ P : I_x \neq \emptyset\}$. Then I_x (F_x) is an ideal (resp., a filter) in Q_x and $\tilde{I}$ is an ideal in P. It is easy to see that:

$$|I| = |\tilde{I}||Q| - \sum_{x\in\tilde{A}} |F_x|,$$

$$r(I) = |\tilde{I}|r_Q(Q) + (n+1)|Q|r_P(\tilde{I}) - \sum_{x\in\tilde{A}} \big(r_Q(F_x) + (n+1)|F_x|r_P(x)\big),$$

$$|P\otimes Q| = |P||Q|,$$
$$r(P\otimes Q) = |P|r_Q(Q) + (n+1)|Q|r_P(P).$$

Now $\frac{r(I)}{|I|} \le \frac{r(P\otimes Q)}{|P\otimes Q|}$ if and only if

$$|P| \sum_{x\in\tilde{A}} \big(|F_x|r_Q(Q) - |Q|r_Q(F_x)\big)$$
$$\le (n+1)|Q| \left(|Q|\big(|\tilde{I}|r_P(P) - |P|r_P(\tilde{I})\big) + \sum_{x\in\tilde{A}} \big(|P|r_P(x) - r_P(P)\big)|F_x| \right).$$

The LHS is not greater than 0 since Q is rank compressed. So it is sufficient to verify, that the term in the big parentheses on the RHS is not smaller than 0. Denote $\tilde{A}' = \{x \in \tilde{A} : |P|r_P(x) - r_P(P) \le 0\}$. Since one can omit the positive summands in the formula and in view of $|F_x| \le |Q_x|$ it is enough to show that

$$|\tilde{I}|r_P(P) - |P|r_P(\tilde{I}) + \Big(|P|r_P(\tilde{A}') - |\tilde{A}'|r_P(P)\Big) \ge 0,$$

which is equivalent to

$$|\tilde{I}\backslash\tilde{A}'|r_P(P) - |P|r_P(\tilde{I}\backslash\tilde{A}') \ge 0.$$

This inequality is true, since P is rank compressed and $\tilde{I}\backslash\tilde{A}'$ is an ideal in P. □

Let P^l be the direct product of l copies of P. The investigation of rank compressed posets was initiated by the following result of Alekseev [1]:

$$P \text{ is rank compressed iff } d_1(P^l) \sim w_1(P^l) \text{ as } l \to \infty. \quad (1)$$

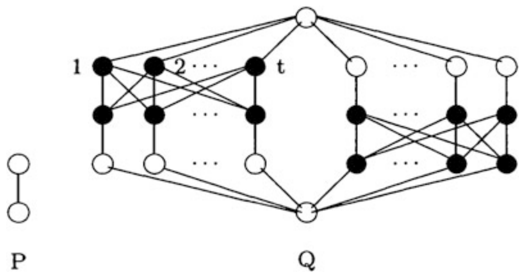

Fig. 6 The properties S, SC, Peck, LYM and RC are not preserved by exponentiation.

Moreover, from the Local Limit Theorem of Gnedenko one can easily derive

$$w_1(P^l) \sim \frac{|P|^l}{\sqrt{2\pi l \sigma_P}} \text{ if } P \text{ is not an antichain,}$$

where $\sigma_P^2 = \frac{1}{|P|} \sum\limits_{x \in P} r^2(x) - \mu_p^2$ (see [5]).

Remark 1. *Straight-forward computations give us the following results:*

$$\sigma^2_{P \times Q} = \sigma^2_P + \sigma^2_Q,$$
$$\sigma^2_{P \otimes Q} = \sigma^2_Q + (n+1)^2 \sigma^2_P,$$
$$\sigma^2_{P \oplus Q} = \frac{|P||Q|}{(|P|+|Q|)^2} \big(\mu_Q + (m+1-\mu_P)\big)^2 + \frac{1}{|P|+|Q|} \big(|P|\sigma_p^2 + |Q|\sigma_Q^2\big).$$

Theorem 3. *The exponentiation does not preserve any of the properties S, SC, Peck, LYM, or RC.*

Proof. First take P and Q from Fig. 6.

Here we mean that the complete bipartite graphs $K_{t,t}$ are formed on the indicated vertices.

Obviously, P and Q have properties S, SC, Peck, LYM, and RC. Now Q^P is isomorphic to the subposet of $Q \times Q$ induced by the set $\{(x, y) : x \le_Q y\}$. Consider the ideal

$$I := \{(x, y) \in Q^P : x \le i \text{ and } y \le i \text{ for some } i = 1, \dots, t\}.$$

Easy calculations give us

$$\mu_I = \frac{9t^2 + 21t}{2t^2 + 7t + 1} \text{ and } \mu_{Q^P} = 4.$$

But $\mu_I > \mu_{Q^P}$ if and only if $t \ge 8$. Hence, Q^P is not rank compressed if $t \ge 8$, and consequently does not have properties SC, Peck, LYM, since these properties imply property RC (see [5]).

Finally let $t \geq 8$ and denote $P' = P + \cdots + P$ (l times). Again, P' has all of the properties above. It is known (see Stanley [12]), that $Q^{P+\cdots+P} \cong Q^P \times \cdots \times Q^P$, hence $Q^{P'} \cong (Q^P)^l$. Since Q^P is not rank compressed, by (1)

$$d_1\big(Q^{P'}\big) = d_1\big((Q^P)^l\big) > w_1\big((Q^P)^l\big) = w_1\big(Q^{P'}\big)$$

if l is sufficiently large. Thus, $Q^{P'}$ is not Sperner. □

Concerning the exponentiation, let us mention that if Q is a distributive lattice, then so is Q^P. Since distributivity implies rank compression (see [5]), in a lot of cases the exponentiation provides a rank compressed poset. In particular, if Q is a two-element chain, Q^P is isomorphic to the lattice of ideals of P, which is consequently rank compressed for any poset P.

3. Summary

In the following table we have summarized which of the considered properties are preserved and which not:

	$P+Q$ $m=n$	$P \oplus Q$	$P \times Q$	$P \otimes Q$	$P \times_r Q$ $m=n$	Q^P
Sperner	No	Yes	No	Yes	No	No
Symm. chain	Yes	No	Yes	No	Yes	No
Peck	Yes	No	Yes	No	Yes	No
LYM	No	Yes	No	Yes	Yes	No
Rank compr.	No	Yes	Yes	Yes	No	No

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The Dimension of Random Graph Orders

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Summary. The random graph order $P_{n,p}$ is obtained from a random graph $G_{n,p}$ on $[n]$ by treating an edge between vertices i and j, with $i \prec j$ in $[n]$, as a relation $i < j$, and taking the transitive closure. This paper forms part of a project to investigate the structure of the random graph order $P_{n,p}$ throughout the range of $p = p(n)$. We give bounds on the dimension of $P_{n,p}$ for various ranges. We prove that, if $p \log\log n \to \infty$ and $\epsilon > 0$ then, almost surely,

$$(1+\epsilon)\sqrt{\frac{\log n}{\log(1/q)}} \leq \dim P_{n,p} \leq (1+\epsilon)\sqrt{\frac{4\log n}{3\log(1/q)}}.$$

We also prove that there are constants c_1, c_2 such that, if $p \log n \to 0$ and $p \geq \log n/n$, then

$$c_1 p^{-1} \leq \dim P_{n,p} \leq c_2 p^{-1}.$$

We give some bounds for various other ranges of $p(n)$, but several questions are left open.

1. Introduction

The *random graph order* $P_{n,p}$ is defined as follows. The vertex set is $[n] \equiv \{1, \ldots, n\}$: we use the symbol $\prec$ to denote the standard linear order on this set. We take a random graph $G_{n,p}$ on $[n]$, and interpret an edge between vertices i and j with $i \prec j$ as a relation $i < j$ of the order. (For the theory of random graphs, founded by Erdős and Rényi [10, 11], see Bollobás [4].) The full order $<$ is then defined by taking the transitive closure. Put another way, if $i \prec j$, then we have $i < j$ in $P_{n,p}$ if and only if there is a $\prec$-increasing sequence $i = i_1, i_2, \ldots, i_m = j$ of vertices such that each of $i_1i_2, i_2i_3, \ldots, i_{m-1}i_m$ is an edge of the underlying random graph.

In general, the probability p will be a function of n. We say that $P_{n,p}$ has a property *almost surely* if, as $n \to \infty$, the probability that $P_{n,p(n)}$ has

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the property tends to 1. Throughout the paper, any inequalities we state are only claimed to hold for n sufficiently large. Also throughout the paper, we set $q = 1 - p$.

Random graph orders have been studied by Barak and Erdős [3], Albert and Frieze [1], Alon, Bollobás, Brightwell and Janson [2], Newman and Cohen [17], Newman [16] and Simon, Crippa and Collenberg [18]. The first three of these papers deal exclusively with the case where the probability p is constant; the next two are concerned with the height of the random order, and the final one is essentially concerned with the number of incomparable pairs of elements. See also the survey by Brightwell [8]. This paper is part of a project to investigate the structure of the random graph order more fully throughout the range of $p = p(n)$. The authors have produced a paper [5] dealing with the width of $P_{n,p}$ throughout the range, and a further paper [6] on the general structure of random graph orders.

This paper is concerned with the dimension of $P_{n,p}$. Recall that a *realiser* of a partial order $(X, <)$ is a set of linear orders on X whose intersection is exactly $(X, <)$. The *dimension* of $(X, <)$ is the minimum cardinality of a realiser. Alternatively, the dimension is the minimum d such that $(X, <)$ can be embedded in $\mathbb{R}^d$ with the co-ordinate order. Trotter's book [21] is a thorough treatment of dimension theory for partial orders. The dimension has already been studied to some effect for another model of random orders by Erdő's, Kierstead and Trotter [9]. We discuss there result here, partly because we shall use it later, and also because it motivates our approach to dimension.

Define a *random bipartite order* $B_{n,p}$ by taking a random bipartite graph on the union of two disjoint sets V and W of vertices, with $|V| = |W| = n$, and joining each pair of vertices $(v, w) \in V \times W$ with probability $p = p(n)$, independently. Then the relations of $B_{n,p}$ are exactly the relations $v < w$, for a vertex $v \in V$ adjacent to a vertex $w \in W$.

The dimension of $B_{n,p}$ is certainly at most n, since the dimension of any partial order is bounded above by its width (see, for instance, Trotter [21]). The result of Erdős, Kierstead and Trotter [9] implies that, even if p is fairly small, the dimension of $B_{n,p}$ is almost surely almost as large as n.

Theorem 1. *For every $\epsilon > 0$, there is a $\delta > 0$ such that, if $\log^{1+\epsilon} n/n < p < 1 - n^{-1+\epsilon}$, then, almost surely,*

$$\dim B_{n,p} > \frac{\delta pn \log pn}{1 + \delta p \log pn}.$$

In particular, their is a constant c such that, if $1/\log n < p < 1 - n^{-1/2}$, then, almost surely,

$$\dim B_{n,p} \geq n(1 - c/p \log n).$$

The proof of Theorem 1 can be found in the original paper [9], or in Trotter's book [21, Chap. 7].

Theorem 1 suggests a method of proving that a partial order has large dimension; namely that one should try to find a large "random" bipartite order within it. We shall see this idea applied in due course.

We also note here a powerful upper bound for the dimension, due to Füredi and Kahn [12] (see also Trotter [21, Chap. 7]). The *maximum degree* of a partial order $(X, <)$ is the maximum, over all vertices $x \in X$, of the number of vertices of X comparable with x.

Theorem 2. *Let* $(X, <)$ *be a partial order with maximum degree* Δ. *Then*

$$\dim(X, <) \le 50\Delta \log^2 \Delta.$$

A random bipartite order with $p = 2c/\log n$, where c is the constant of Theorem 1, almost surely has dimension at least $\epsilon\Delta \log \Delta$, for some fixed $\epsilon > 0$ where Δ is the maximum degree.

Returning now to random graph orders, the only previous result concerning the dimension is due to Albert and Frieze [1], who showed the following,

Theorem 3. *Let* p *be a fixed constant with* $0 < p < 1$, *and set* $q = 1 - p$. *Then, for any* $\epsilon > 0$, *almost surely,*

$$(1-\epsilon)\sqrt{\frac{\log n}{\log(1/pq)}} \le \dim P_{n,p} \le \sqrt{\frac{2\log n}{\log(1/q)}} + \frac{1}{2} + \epsilon.$$

Thus if $p = q = 1/2$, the upper bound is essentially twice the lower bound; and the ratio of upper to lower bounds increases as p becomes smaller. The upper bound in Theorem 3 is an immediate consequence of the same upper bound for the width of $P_{n,p}$, as proved by Barak and Erdós [3] (and subsequently sharpened by Bollobás and Brightwell [5] w). The lower bound is obtained by showing that a specific partial order of the given dimension almost surely exists as a suborder of $P_{n,p}$. Albert and Frieze conjectured that this lower bound was correct. One of our principal aims here is to improve both the upper and lower bounds in Theorem 3, thus disproving that conjecture. We prove the following result.

Theorem 4. *Suppose* $p(n)$ *is a function of* n *such that* $p \log\log n \to \infty$ *and* $p \le 1 - 1/\sqrt{\log n}$. *For any* $\epsilon > 0$ *we have, almost surely,*

$$(1-\epsilon)\sqrt{\frac{\log n}{\log(1/q)}} \le \dim P_{n,p} \le (1+\epsilon)\sqrt{\frac{4\log n}{3\log(1/q)}}.$$

The gap between the upper and lower bounds for the dimension of $P_{n,p}$ is thus reduced to a multiplicative factor of $(1+\epsilon)2/\sqrt{3}$, for each constant value of p, and indeed whenever $p(n)$ tends to 0 more slowly than $1/\log\log n$. We strongly suspect that the lower bound here is essentially correct, at least for constant p, but have been unable to prove it. Section 2 is devoted to a proof of Theorem 4: in fact, we prove slightly more general results.

Our other main aim is to estimate the dimension for rather lower ranges of $p = p(n)$. In [5], we showed that there is a type of "phase transition" as $p(n)$ passes through the range $p = c/\log n$; this is investigated further, and to some extent explained, in [6]. There is a second, equally radical, change of behaviour near $p = \log n/n$, the nature of which will hopefully become clear in the course of our proofs. A third transition phase, as p decreases through c/n, is readily understood, since the underlying random graph changes from being almost all in one component, to being a collection of small trees. The following result of Bollobás and Brightwell [5] gets close to pinning down the behaviour of the width of $P_{n,p}$ between the first and last of these three transitions.

Theorem 5. *If $p \log n \to 0$ and $pn \to \infty$, then the width of $P_{n,p}$ almost surely lies between $1.455p^{-1}$ and $2.428p^{-1}$.*

We thus have an upper bound for $\dim P_{n,p}$ of $2.428p^{-1}$ in this range. It is natural to ask whether this is roughly the right order of magnitude: we shall prove that it is, provided we are above the second phase transition point. The main result of Sect. 3 is as follows.

Theorem 6. *There is an $\epsilon > 0$ such that, if $p \geq \log n/n$ and $p \log n \to 0$, then, almost surely,*

$$\epsilon p^{-1} \leq \dim P_{n,p} \leq 2.428p^{-1}.$$

Below $p = \log n/n$, the maximum degree of the random partial order is smaller than p^{-1}, so the Füredi-Kahn bound, Theorem 2, gives us a better upper bound on the dimension. We show that, at least in some range below $p = \log n/n$, this upper bound is not too far off being correct.

Theorem 7. *There are positive constants c_1, c_2 such that, if $\frac{4}{5}\log n/n \leq p \leq (\log n - \log\log n)/n$, then, almost surely,*

$$c_1 e^{pn} \leq \dim P_{n,p} \leq c_2 e^{pn} \log^3 n.$$

In particular, there is a genuine change of behaviour near $p = \log n/n$, which is associated with a sudden drop in the value of the maximum degree from ϵn to $o(n)$. A later result, Lemma 1, gives some more information on the distribution of the number of vertices above a given vertex in $P_{n,p}$.

The proofs of Theorem 6 and the lower bound in Theorem 7 are practically identical, and will be dealt with together in Sect. 3. The constant $4/5$ appearing in Theorem 7 can be improved, and indeed it appears that, with some difficulty, it can be replaced by any $\epsilon > 0$. The upper bound in Theorem 7 remains valid for much smaller values of p, and it would be of some interest to determine if a lower bound of the form given in Theorem 7 holds further down. The following easy result deals with very small values of p.

Theorem 8. *(i) If $pn \to 0$, but $pn^{7/6} \to \infty$, then, almost surely,* $\dim P_{n,p} = 3$.
(ii) If $pn^{7/6} \to 0$, then, almost surely, $\dim P_{n,p} = 2$.

Proof. This result is an immediate consequence of well-known structural results for random graphs $G_{n,p}$ with $pn \to 0$ (see, for instance, Bollobás [8]).

If $pn \to 0$, then all the components of the underlying random graph are trees, so the dimension of the random order is at most 3 (see, for instance, Trotter [21, Chap. 5, (2.6)]). If $pn^{7/6} \to \infty$, then some of the components are seven-vertex trees of dimension 3, whereas if $pn^{7/6} \to 0$, all components are trees of at most six vertices, so $\dim P_{n,p} = 2$. □

If $p = c/n$ with $c < 1$, then all components are either trees or unicyclic graphs The following conjecture would imply that Theorem 8(i) can be extended to this range of $p(n)$.

Conjecture 1. *If P is a partial order whose covering graph is unicyclic, then* $\dim P \leq 3$.

As pointed out to us by Tom Trotter, Theorem (4.4) of Chap. 4 of [21] implies that, if P is as above, then $\dim P \leq 5$. What happens if $p = c/n$ with $c > 1$?

The reader will perhaps have noticed that there is one intermediate range of $p(n)$ that we have not yet discussed, namely the range where $p \log n \to \infty$ but $p \log\log n \to 0$. In this range, the width of $P_{n,p}$ is still almost surely about $\sqrt{2p^{-1}\log n}$ and indeed the upper bound for $\dim P_{n,p}$ in Theorem 4 still holds. Our methods can be adapted to give apparently rather weak lower bounds for the dimension, but we have been unable to answer the following question.

Question 1. *Is there an $\epsilon > 0$ such that, if $p \geq \log n/n$, then, almost surely,*

$$\dim P_{n,p} \geq \epsilon\,\mathrm{Width}(P_{n,p})?$$

Note that we have an affirmative answer if either $p \log n \to 0$ or $p \log\log n \to \infty$. However, we suspect that the answer is “no”.

2. Large $p(n)$

This section is devoted to the proof of Theorem 4. In fact, we shall prove two slightly sharper and more general results, which together imply Theorem 4. We first consider the lower bound.

Theorem 9. *There is a constant k such that, if $2/\log\log n \leq p \leq 1 - 1/\sqrt{\log n}$, then, almost surely,*

$$\dim P_{n,p} \geq \sqrt{\frac{logn}{\log(1/q)}} \left(1 - \frac{k}{p \log\log n}\right).$$

Proof. We shall use Theorem 1, due to Erdős, Kierstead and Trotter [9]. Set

$$m = \left\lfloor \sqrt{\frac{\log n}{\log(1/q)}} \left(1 - \frac{1}{\log\log n}\right) \right\rfloor,$$

and note that $m \leq \sqrt{\log n \log\log n}$. Our aim is to find a "random $B_{m,p}$" inside our random graph order $P_{n,p}$, and to apply Theorem 1.

We break our vertex set $[n]$ into $l = \lfloor n/2m \rfloor$ consecutive pieces $A_1, \ldots, A_l$, so that A_{j+1} consists of the vertices between $2mj+1$ and $2m(j+1)$ inclusive. Each set A_i thus consists of $2m$ vertices; we subdivide it further into two consecutive sets B_i and C_i of m vertices each, so that B_{j+1} consists of the vertices between $2mj+1$ and $2mj+m$ inclusive, and C_{j+1} contains the remainder.

Note that, if there are no edges of the underlying random graph inside a set B_i or C_i, then that set forms an antichain of size m in $P_{n,p}$. We call a set A_i *good* if both B_i and C_i form antichains. Note that, conditional on A_i being good, the random order $P_{n,p}$ restricted to A_i is distributed as a random bipartite order $B_{m,p}$ (with vertex classes B_i and C_i).

The probability that A_i is good is equal to $q^{2\binom{m}{2}} \geq q^{m^2}$. Also, the events that the various A_i are good are independent. Thus the probability P that there is no good A_i is at most

$$\left(1 - q^{m^2}\right)^l.$$

Substituting our chosen value for m, and using the inequality $l \geq n/3m$, we obtain:

$$\begin{aligned} P &\leq \exp\left(-\frac{n}{3m} q^{\frac{\log n}{\log(1/q)}(1-1/\log\log n)^2}\right) \\ &\leq \exp\left(-\frac{1}{3\sqrt{\log n \log\log n}} n^{1/\log\log n}\right) \\ &= o(1). \end{aligned}$$

Therefore there is almost surely some good A_i. Thus we have, almost surely, $\dim P_{n,p} \geq \dim B_{m,p}$.

Note that $1/\log m < p < 1 - m^{-1/2}$, so, by Theorem 1, we almost surely have

$$\begin{aligned} \dim P_{n,p} \geq \dim B_{m,p} &\geq m \left(1 - \frac{c}{p \log m}\right) \\ &\geq \sqrt{\frac{\log n}{\log(1/q)}} \left(1 - \frac{2}{\log\log n}\right) \left(1 - \frac{3c}{p \log\log n}\right), \end{aligned}$$

where c is the constant of Theorem 1. The result now follows on choosing an appropriate value of k. □

We now turn our attention to the upper bound. The main point here is that the dimension is bounded away from the width. Informally, very large antichains are rare, and one needs more than one large antichain to have large dimension.

Furthermore, for p above our first transition phase around $1/\log n$. it is shown in [6] that the random graph order has the structure of many small orders arranged one above another. Thus we shall make use of the following two results.

Theorem 10. *Let A be a maximal antichain in $(X, <)$, with $A \neq X$. Set $U = \{x \in X : x > a$ for some $a \in A\}$ and $D = \{x \in X : x < a$ for some $a \in A\}$, so that X is the disjoint union of A, U and D. Let W_U and W_D denote the widths of the partial order $(X, <)$ restricted to U and D respectively. Then*

$$\dim(X, <) \leq 1 + W_U + W_D.$$

Trotter [19] proves Theorem 10 in the cases where (i) $W_D = 0$ or (ii) $W_D = W_U$, and in fact the proof of the latter result generalises immediately to give Theorem 10. This proof can also be found in Trotter's book [21, Chap. 1 (11.3)].

A *post* in a partial order is an element comparable with all others. Posts in $P_{n,p}$ are considered in Alon, Bollobás Brightwell and Janson [2], and in more detail in Bollobás and Brightwell [6]. Assume for the moment that $P_{n,p}$ has at least one post. If i and j are posts in $P_{n,p}$ such that there are no posts k with $i \prec k \prec j$, then the partial order restricted to the interval $[i, j)$ of $[n]$ is called a *factor* of $P_{n,p}$. If i is the first and j the last post, then the partial order restricted to either of $[1, i)$ or $[j, n]$ is also regarded as a factor. If the factors of $P_{n,p}$ are $F_1, \ldots, F_m$, with $F_1 \prec \ldots \prec F_m$ in $[n]$, then $P_{n,p}$ is the *linear sum* of $F_1, \ldots, F_m$, i.e., the partial order defined by taking the union of the F_i, and putting in each relation of the form $x < y$, where x is an a strictly lower factor than y. Note that the dimension of a linear sum is equal to the maximum of the dimensions of its constituent factors.

Set $\eta(p) = \prod_{i=1}^{\infty}(1 - q^i)$. We note that there is a constant c such that, for $p \leq 1/2$, $\eta(p) \geq c\exp(-\pi^2/6\log(1/q))$. This can be read out of Hall [13, equation 4.2.11]. We quote the following result from [6].

Theorem 11. *Suppose $p\log n \to \infty$. Then there is almost surely no set of $\eta(p)^{-1}\log^3 n$ consecutive vertices of $[n]$ that does not contain a post in $P_{n,p}$.*

We shall apply this $p\log n \to \infty$, so that $\eta(p)^{-1}\log^3 n = o(n^\epsilon)$ for every $\epsilon > 0$. The significance of Theorem 11 is then that the partial order $P_{n,p}$ is almost surely the linear sum of factors, all of which have at most $\eta(p)^{-1}\log^3 n$ elements. In view of Theorem 10, this implies that, if $P_{n,p}$ does have dimension at least d, then one of the factors must contain both an antichain A of size

d, and a pair of disjoint antichains, also disjoint from A, of sizes summing to at least $d-1$.

Our plan is to estimate the expected number of occurrences of such structures.

Theorem 12. *Let $\epsilon > 0$ be any fixed constant. Suppose $p \log n \to \infty$. Then, almost surely,*

$$\dim P_{n,p} \leq (1+\epsilon)\sqrt{\frac{4 \log n}{3 \log(1/q)}}.$$

Proof. In [5], it is proved that the expected number of antichains of size w whose $\prec$-first vertex is x is at most $\eta(p)^{-2} q^{\binom{w}{2}}$. One can readily adapt the argument there to show that, for any triple (x, y, z) of vertices, the expected number of triples of antichains of size w_1, w_2, w_3 with $\prec$-first vertices x, y, z respectively is at most

$$\eta(p)^{-6} q^{\binom{w_1}{2}+\binom{w_2}{2}+\binom{w_3}{2}}.$$

The number of triples of vertices (x, y, z) in the same factor of $P_{n,p}$ is almost surely at most $n(\eta(p)^{-1} \log^3 n)^2$ from Theorem 11.

Hence, by Theorem 10, the expected number of factors of dimension at least d, conditional on the conclusion of Theorem 11 holding, is at most

$$n\eta(p)^{-8} (\log n)^6 q^{\binom{d}{2}} \sum_{i=0}^{d-1} q^{\binom{i}{2}+\binom{d-i-1}{2}}.$$

The largest term in the sum is the term with $i = \lfloor (d-1)/2 \rfloor$, so this expectation is at most

$$n\eta(p)^{-8} (\log n)^6 d q^{d^2/2+2(d/2)^2/2-2d} = n\eta(p)^{-8} (\log n)^6 d q^{3d^2/4-2d}.$$

Setting $d = (1+\epsilon)\sqrt{4 \log n / 3 \log(1/q)}$, for any $\epsilon > 0$, we see that this is at most

$$n^{1-(1+\epsilon)^2} \eta(p)^{-8} (\log n)^7 e^{-3\sqrt{p^{-1} \log n}} = o(1).$$

Since the dimension of $P_{n,p}$ is at most the maximum dimension of its factors, this implies that, almost surely, $\dim P_{n,p} \leq d$, as required. □

As mentioned earlier, we do not expect that the constant 4/3 in Theorem 12 is correct. One possible way to decrease it would be to try to improve on Theorem 10, under suitable extra hypotheses. Although Trotter [20, 21] has shown that Theorem 10 is sharp, even when $W_D = W_U$, the example he constructs looks completely unlike anything that is likely to arise as a suborder of $P_{n,p}$: in particular the size of the antichain A is exponential in W_D. On the other hand, one can almost surely find in $P_{n,p}$ three consecutive sets of vertices of sizes $w/2$, w and $w/2$ successively, where $w = (1-\epsilon)\sqrt{4 \log n / 3 \log(1/q)}$; however the (random) order induced by these sets almost surely has dimension at most $w/2$—see Erdős, Kierstead and Trotter [9].

We suspect that the lower bound in Theorem 9 gives the correct constant, at least for constant values of p, so we make the following conjecture.

Conjecture 2. *Let p be a fixed constant with $0 < p < 1$. For every $\epsilon > 0$, we have, almost surely,*

$$\dim P_{n,p} \leq (1+\epsilon)\sqrt{\frac{\log n}{\log(1/q)}}.$$

3. Small $p(n)$

Our principal aim in this section is to prove Theorem 6, giving a lower bound of ϵp^{-1} on the dimension for the major range of $p(n)$. As mentioned earlier, this bound is best possible up to the value of ϵ. We shall work in slightly greater generality so as to prove Theorem 7 as well.

We start with an overview of the proof, assuming for simplicity that $p \geq \log n/n + \omega(n) \log\log n/n$, where $\omega(n) \to \infty$. Here and later, we treat integer and real quantities interchangeably, for the sake of clarity: the proof is not materially affected.

Throughout this section, we set $m = p^{-1}$. We restrict attention to the set $[N]$ consisting of the $\prec$-first $N = m \log m + 6m$ vertices of $[n]$, so we are dealing with a random graph order $P_{N,p}$. We partition $[N]$ into sets $W \prec U \prec U' \prec W'$, with $|U| = |U'| = \frac{1}{2} m \log m$, and $|W| = |W'| = 3m$. Let A be the set of maximal elements of the partial order restricted to W, and let A' be the set of minimal elements in the partial order restricted to W'. It was noted in our earlier paper [5] that, almost surely, $|A|, |A'| \geq 9m/10$. Our aim is to prove that the random order $P_{N,p}$ restricted to $A \cup A'$ almost surely has dimension at least ϵm.

Intuitively, the size of $U \cup U', m \log m$, has been fixed so that, for each pair $(a, a') \in A \times A'$, the probability that $a < a'$ in $P_{N,p}$ is a constant. If these relations were independent, then the partial order we consider would be a random bipartite order, and, by the result of Erdős, Kierstead and Trotter [9], Theorem 1, would almost surely have dimension at least $(1-\epsilon)9m/10$. However of course the relations are far from independent. We overcome this, at the expense of finishing with a rather small constant, in the following manner. We prove that, almost surely, there are at least γm^2 pairs $(a, a') \in A \times A'$ which are not related in $P_{N,p}$. If we have a realiser $L_1, \ldots, L_d$ of $P_{N,p}$, then, for each such unrelated pair, there must be an $i \in \{1, \ldots, d\}$ such that a is above a' in L_i—we say L_i reverses (a, a'). On the other hand, we prove that, for any subset T of A of (small) size t, the number of elements of A' incomparable with all elements of T is at most $(1-\delta)^t m$. This will imply that the total number of pairs (a, a') reversed by a linear extension L is at most $2m/\delta$, and so we must have $\dim P_{N,p} \geq d \geq \gamma\delta m/2$.

For X a subset of $[N]$ and B a subset of A, let $\Gamma x(B)$ be the set of vertices $x \in X$ such that x is above some vertex of B in $P_{N,p}$, i.e., there is a $\prec$-increasing path in the underlying random graph from some vertex of B to x. Our strategy will be to prove that, for every (small) subset B of A, $\Gamma_U(B)$ is about the "right" size, namely $n^{1/2}|B|$. Unfortunately, this will fail as it stands because a few vertices a in A are likely to be below far too few or too many vertices in U. So our first step will be to exclude such vertices, and certain others, from consideration.

Formally, we shall begin by looking just at the set $A \cup U$. We shall prove that the partial order restricted to this set almost surely has certain properties, and by symmetry the partial order restricted to $A' \cup U'$ has analogous properties. These properties will of course be independent of edges between U and U'. Then we prove that, conditional on these properties, the partial order $P_{N,p}$ almost surely has large dimension.

For the sake of convenience, we begin by working in a slightly altered model of random orders. Define the model $P'(s,p)$ by taking two disjoint vertex sets R and $S = [s]$, with $|R| = 9p^{-1}/10$. Then we put in edges independently, with probability p, between each pair of vertices, not both in R. Thus the model restricted to S is just a copy of $P_{s,p}$. All vertices in R are to be thought of as "below" S, so a vertex $x \in R$ is below $y \in S$ if there is a vertex $z \in S$ such that xz is an edge of the graph, and $z < y$ in $P_{s,p}$. We shall apply this with $R = A$ and $S = U$. For the greater generality required to prove Theorem 7, we need to consider cases where s is a little smaller than $\frac{1}{2}m \log m$ where again $m = p^{-1}$.

For the next few steps, we work in the model $P'(s,p)$ assuming that $\frac{2}{5}m \log m \leq s \leq \frac{1}{2}m \log m$. As mentioned above, we wish to show that there is almost surely a large subset R_G of R such that, for every (small) subset T of R_G, $R_G, \Gamma_S(T)$ is not too far from its expected size. There are essentially two steps here, first to ensure that this holds for every single-element subset of R_G, and then to extend the argument to larger subsets.

We start with a lemma which is perhaps of independent interest. Let the random variable $X(s,p)$ be the number of vertices of S above a given vertex $x \in R$ in $P'(s,p)$. Of course, this, is distributed as the number of vertices above vertex 1 in $P_{s+1,p}$. The lemma below gives the exact distribution of $X(s,p)$, and also some convenient estimates. Note that our estimates are only good in the case where $pq^{-s} = o(1)$, which is when s is rather less than $m \log m$. The identity given in part (i) has been obtained independently by Simon, Crippa and Collenberg [18], who study in more detail the mean and variance of $X(s,p)$ and the behaviour in the limit as $s \to \infty$ with p fixed.

In the lemma below, $\mathrm{Geom}(a^{-1})$ denotes a geometric random variable with mean a^{-1}, i.e., with $\Pr(\mathrm{Geom}(a^{-1}) = t) = a(1-a)^t$ for t a non-negative integer. The *total variation distance* $d_{\mathrm{TV}}(X,Y)$ between two real-valued random variables X and Y is the supremum over $x \in \mathbb{R}$ of $|\Pr(X \leq x) - \Pr(Y \leq x)|$.

Lemma 1. *(i) For* $0 \leq t \leq s$*,*

$$\Pr(X(s,p) = t) = q^{s-t} \prod_{i=s-t+1}^{s} (1 - q^i).$$

(ii) Suppose $p \leq 1/2$*. Then*

$$d_{\mathrm{TV}}(X(s,p), \mathrm{Geom}(q^{-s})) \leq 3pq^{-s}.$$

(iii) For every t*,*

$$\Pr(X(s,p) > t) \leq (1 - q^s)^t.$$

(iv) If $p \leq 1/2$*, then, for every* t*,*

$$\Pr(X(s,p) \leq t) \leq \exp(2pq^{-s})(1 - (1 - q^s)^t).$$

Proof. Of course, the formula in (i) can be proved by induction on s, using the recurrence

$$r(t,s) = r(t, s-1)q^{t+1} + r(t-1, s-1)(1 - q^t),$$

where $r(t,s) = \Pr(X(s,p) = t)$: see Simon, Crippa and Collenberg [18] for details. We prefer to give a combinatorial proof, which is hopefully slightly more informative.

The probability that the vertices of $S = [s]$ above x are exactly $k_1 + 1, k_1 + k_2 + 2, \ldots, k_1 + \ldots + k_t + t$ is

$$\left(\prod_{i=1}^{t}(1 - q^i)\right)\left(\prod_{i=1}^{t+1} q^{ik_i}\right),$$

where $k_{t+1} = s - \sum_{i=t}^{t} k_i - t$. As the possible vectors $(k_1, \ldots, k_{t+1})$ are exactly the ordered partitions of $s - t$ into $t + 1$ parts, we have

$$\Pr(X(s,p) = t) = \left(\prod_{i=1}^{t}(1 - q^i)\right) [X^{s-t}] \prod_{i=1}^{t+1} \frac{1}{1 - q^i X}.$$

By Theorem 349 of Hardy and Wright [14], the coefficient of X^{s-t} in the product is

$$q^{s-t} \prod_{i=1}^{s-t} \frac{\mathbf{1} - q^{t+i}}{1 - q^i},$$

which implies our formula for $\Pr(X(s,p) = t)$.

For the estimates (ii)–(iv), we have that

$$a_t \equiv \Pr(X(s,p) = t) = q^{s-t} \prod_{j=0}^{t-1} (1 - q^{s-j}) \quad (0 \le t \le s)$$

$$b_t \equiv \Pr(\mathrm{Geom}(q^{-s}) = t) = q^s (1 - q^s)^t \quad (0 \le t).$$

We claim that:

(1) $a_t \ge b_t$ for all t at most some t_0, and $a_t < b_t$ thereafter;
(2) If $p \le 1/2$, then $a_t/b_t \le \exp(2pq^{-s})$ for all t.

Claim (1) will imply (iii) immediately, since the right hand side in (iii) is the probability that $\mathrm{Geom}(q^{-s})$ is greater than t. Similarly Claim (2) implies (iv) immediately. For (ii), the two claims together give that

$$\begin{aligned} d_{\mathrm{TV}}(X(s,p), \mathrm{Geom}(q^{-s})) &= \sum_{t=0}^{t_0} (a_t - b_t) \\ &\le \sum_{t=0}^{t_0} b_t (\exp(2pq^{-s}) - 1) \\ &\le \exp(2pq^{-s}) - 1. \end{aligned}$$

This final expression is at most $3pq^{-s}$ whenever $3pq^{-s} < 1$. Since the total variation distance is certainly at most 1, we thus have the result as stated.

For $j = 1, \dots, s$, set $\alpha_j = q^{-1}(1 - q^{s-j})/(1 - q^s)$. Note that $a_t/b_t = \prod_{j=0}^{t-1} \alpha_j$. Observe that the sequence α_j starts greater than 1 and decreases to 0 by $j = s$. This establish Claim (1). (Note that the claim is certainly true if $t > s$, when $a_t = 0 < b_t$.)

For Claim (2), we have

$$\begin{aligned} \log \alpha_j &= -\log(1-p) + \log\left(1 - \frac{q^s}{1-q^s}(q^{-j} - 1)\right) \\ &\le p + p^2 - q^s(q^{-j} - 1) \\ &\le p + p^2 - jpq^s, \end{aligned}$$

and hence

$$\log(a_t/b_t) \le p\left(t(1+p) - q^s \sum_{j=0}^{t-1} j\right) = \frac{tp}{2}(2 + 2p - q^s(t-1)). \qquad (*)$$

The expression on the right hand side of $(*)$ is maximised for $2t - 1 = 2(1+p)q^{-s}$ when it is equal to

$$\frac{pq^{-s}}{2}(1 + 2p + p^2 + q^s + pq^s + q^{2s}/4) \le 2pq^{-s},$$

as required to establish Claim (2) and complete the proof of the lemma. (Here, we have used the weak bounds $p \le 1/2$ and $q^s \le 1$.) □

Returning to our main thread, we wish to prove that "many" vertices of R have about the expected number, namely about q^{-s}, of vertices of S above them. Lemma 1 will suffice for that, and we shall simply ignore vertices of R below too few or too many vertices. We also wish to exclude certain other vertices $a \in R$, namely those which only have sufficiently many vertices of S above them because of the large number of directed paths through one particular neighbour a^*: the problem with such vertices is that if another vertex b of R is also adjacent to a^*, the combined neighbour-set $\Gamma_S(\{a,b\})$ may be too small.

It turns out to be sufficient for our purposes to ensure that vertices of R are below about the right number of vertices from the bottom portion of S. Define then C to be the set of the bottom m vertices of S, and W to be the set consisting of the next $w \equiv 6m \log\log m$, with $Y = C \cup W$.

We begin by choosing a large matching among those edges of the random graph between R and C. A result in our earlier paper [5, Theorem 14], implies that there is almost surely such a matching of size at least $m/3$. Given such a matching M, let R_M be the set of vertices of R incident with an edge of M. For $a \in R_M$, let a^* be the vertex of C with $aa^* \in M$. For $a \in R_M$, let $N(a)$ be the set of vertices of W sending a path down to a avoiding a^*, and $N(a^*)$ be the set of vertices of W sending a path down to a^*; i.e., $N(a^*) = \Gamma_W(a^*)$. The idea is that, if both $N(a)$ and $N(a^*)$ are large, then any subset of R including a will receive its due "contribution" from a.

We shall call a vertex $a \in R_M$ *good* if it satisfies

(i) $5\log^6 m \geq |\Gamma_Y(a)|$,
(ii) Both $|N(a)|$ and $|N(a^*)|$ are at least $\frac{1}{8}\log^6 m$.

Lemma 2. *There are almost surely at least $m/12$ good vertices in R_M.*

Proof. Let D be the set of vertices incident with M. Note that the choice of D depends only on the edges of the random graph between R and C. For $x \in D$, let $Z(x)$ be the number of vertices of W having a directed path to x whose penultimate vertex is in W. Thus $Z(x) \subseteq N(x)$ for $x \in D$. The random variable $Z(x)$ is distributed as $X(w,p)$, so by Lemma 1 the probability that $Z(x)$ is at most $\frac{1}{8}\log^6 m$ is at most

$$e^{2pq^{-w}}\left(1-(1-q^{6m\log\log m})^{\log^6 m/8}\right) = 1 - e^{-1/8} + o(1).$$

For $x, y \in D$, the probabilities that $Z(x)$ and $Z(y)$ are too small are not independent, so we cannot immediately deduce that, almost surely, many vertices $a \in R$ have $Z(a)$ and $Z(a^*)$ large enough. However, the problem is easily overcome. For any subset V of W, and any $x \in D$, the event that $\{v \in W : v > x\} = V$ depends only on the set of edges of the random graph upwards from $V \cup \{x\}$, so is independent of the random graph restricted to $(W \setminus V) \cup \{y\}$, for any other element $y \in D$. Thus the random variable $Z(y)$, conditioned on the event that $Z(x) < \frac{1}{8}\log^6 m$, dominates a random variable distributed as $X(W', p)$ where $w' = w - \frac{1}{8}\log^6 m$. The probability that such a random variable is at most $\frac{1}{8}\log^6 m$ is again at most $1 - e^{-1/8} + o(1)$. Thus the variance of the number of vertices of D below too few vertices in W is $o(m^2)$, so the number of such vertices is almost surely at most $\frac{2m}{3}(1 - e^{-1/8} + \epsilon) < m/12$. Therefore, almost surely, the number of vertices $a \in R_M$ such that a and a^* both have sufficient neighbours in W is at least $m/3 - 2m/12 > m/6$.

Finally, the expected number of vertices $a \in R_M$ below more than $5\log^6 m$ vertices of $C \cup W$ can also be estimated from Lemma 1 to be at most

$$\frac{1}{3}m\exp(-5\log^6 m q^{m+6m\log\log m}) = \frac{1}{3}m\exp(-5e^{-1} + o(1)) < m/13.$$

As before, the variance of this number is $o(m^2)$, so there are almost surely at most $m/12$ vertices in R_M with too many neighbours. Hence the number of good vertices in R is almost surely at least $m/6 - m/12$, as desired. □

Let R_G be a set of $m/12$ good vertices in R_M. We have that each vertex of R_G is below about $\log^6 m$ vertices of W. We need a little more, namely that each smallish set T of good vertices has about $|T|\log^6 m$ neighbours in W. This follows from the principle that random graphs do not possess small dense subgraphs.

Lemma 3. *Almost surely, for every subset T of R_G with $|T| = t \le m^{1/4}$, $|T \cup \Gamma_Y(T)| \ge t\log^6 m/16$.*

Proof. For any subset T of the whole set R, define $G(T) = T \cup \Gamma_Y(T)$ and $g(T) = |G(T)|$. Define a spanning forest $F(T)$ in the graph restricted to $G(T)$ by choosing, for each element of $\Gamma_Y(T)$, one edge down to an element of $G(T)$.

We first prove that, almost surely, for each subset T of R, either $g(T) > g(t) \equiv t\log^6 m/16$, or the number of edges induced by the random graph on $G(T)$ is at most $|F(T)| + 3t/2$. Indeed, the probability that a given set T of size t fails is at most the probability that (a) $g(T) \le g(t)$ and (b) of a given set of at most $\binom{g(t)}{2} - |F(T)|$ pairs of vertices, at least $3t/2$ of them are in the random graph. The probability of (b) is at most

$$\binom{g(t)^2}{3t/2} p^{3t/2} \le \left(\frac{2eg(t)^2}{3tm}\right)^{3t/2} \le \left(\frac{t\log^{12} m}{m}\right)^{3t/2} \le \left(\frac{\log^{12} m}{m^{3/4}}\right)^{3t/2},$$

for $t \le m^{1/4}$. The expected number of failing t-sets is thus at most

$$m^t\left(\frac{\log^{12} m}{m^{3/4}}\right)^{3t/2} = ((\log m)^{18} m^{-1/8})^t \le (\log m)^{-t}.$$

Hence, almost surely, there are no over-dense sets $G(T)$. We assume from now on that this is the case.

Now let T be a subset of R_G of size t, with $g(T) \le t\log^6 m/10$. Set $T^* = \{a^* : a \in T\}$, and $D(T) = T \cup T^*$. By the above, $G(T)$ spans at most $3t/2$ edges other than those in $F(T)$. For each such edge, take the lower endpoint, and then follow down the edges of $F(T)$ until a vertex of $D(T)$ is reached. Thus there are at least $t/2$ vertices of $D(T)$ that are not hit. Now consider the collection of sets $N(a)$, for a a vertex of $D(T)$ that is not hit. Each such set has size at least $\log^6 m/8$, by our choice of R_G. We claim that these sets are mutually disjoint. If not, there is a minimal vertex w in some pair of them, so w sends edges down to both sets, and one of these edges is not in $F(T)$: the path thus generated hits the vertex a defining one of the two sets, leading to a contradiction. Thus the union of these sets has size at least $t\log^6 m/16$, and this union is certainly contained in $G(T)$, which is a contradiction. □

Once we have all our small sets "spreading" into the bottom $6m\log\log m + m$ elements of S, it is not hard to show that they continue to spread at a very steady rate through S.

Lemma 4. *Almost surely, for every subset T of R_G with $|T| = t \le m^{1/4}$,*

$$2te^{s/m} \ge |T \cup \Gamma_S(T)| \ge te^{s/m}/80.$$

Proof. We break $S \setminus Y$ into consecutive sets of size $m/\log m$, say $A_{k+1} < A_{k+2} < \ldots < A_l$, where $l = s\log m/m \le \frac{1}{2}\log^2 m$ and $k = 6\log m \log\log m + \log m$. For $j = k, \ldots, l$, set $C_j = Y \cup \bigcup_{i=k+1}^{j} A_i$. For a set T of size t, and $k \le j < l$, we estimate the probability that

$$\big||\Gamma_{C_{j+1}}(T)| - |\Gamma_{C_j}(T)|(1 + 1/\log m)\big| \ge |\Gamma_{C_j}(T)| \log^{-5/2} m,$$

given that $m^{3/4} \ge \gamma \equiv |\Gamma_{C_j}(T)| \ge t\log^6 m/16$.

The number $N_j(T)$ of vertices of A_{j+1} sending an edge to $\Gamma_{C_j}(T)$ is a binomial random variable with parameters $m/\log m$ and

$$1 - (1-p)^{\gamma} = \frac{\gamma}{m}(1 + O(m^{-1/4})).$$

Hence we have

$$\Pr\left(\left|N_j(T) - \frac{\gamma}{\log m}\right| \ge \frac{\gamma}{\log^{5/2} m}\right) \le 2e^{-\gamma/3\log^4 m} \le 2e^{-t\log^2 m/48},$$

where we used the Chernoff bounds on the tail of the Binomial distribution, the lower bound on γ, and implicitly the fact that the variation we tolerate is much greater than the error in our estimate of $\gamma/\log m$ for the mean of $N_j(T)$.

Hence, for each fixed t, the probability that, for some T of size t, and some j, the set $\Gamma_{C_j}(T)$ fails to spread within the prescribed bounds is at most

$$2m^t l e^{-t\log^2 m/48} = o(m^{-1/4}).$$

Thus, almost surely, every set $\Gamma_{C_j}(T)$ spreads at the required rate.

Hence, almost surely, for every set T of size at most $m^{1/4}$,

$$
\begin{aligned}
|\Gamma_S(T)| &\geq \frac{1}{16} t \log^6 m \left(1 + \frac{1}{\log m} - \frac{1}{\log^{5/2} m}\right)^{l-k} \\
&\geq \frac{t}{16} \exp\left(6 \log\log m + \left(\frac{s \log m}{m} - 6 \log m \log\log m - \log m\right)\left(\frac{1}{\log m} - \frac{1}{\log^2 m}\right)\right) \\
&\geq \frac{t}{16} \exp\left(\frac{s}{m} - 1 - \frac{s}{m \log m}\right) \\
&\geq \frac{t}{16} e^{s/m - 3/2} \geq \frac{t}{80} e^{s/m}.
\end{aligned}
$$

Similarly, almost surely, for every set T,

$$
\begin{aligned}
|\Gamma_S(T)| &\leq 5t \log^6 m (1 + 1/\log m + 10^5/\log^{5/2} m)^{l-k} \\
&\leq 5t \exp\left(6 \log\log m + \left(\frac{s \log m}{m} - 6 \log m \log\log m - \log m\right) \frac{1}{\log m}\right) \\
&= 5t e^{s/m-1} \leq 2t e^{s/m}.
\end{aligned}
$$

This completes the proof. □

There is one more property we want from our model $P'(s,p)$, namely that no single element of S is above too many members of R. A reasonably sharp bound follows readily from Lemma 1 (iii): the number $J(x)$ of elements of $R \cup S$ below any member x of S is dominated by a random variable distributed as $X(s+r,p)$, so the probability that $J(x)$ is greater than $3e^{s/m} \log m$ is at most

$$
\begin{aligned}
(1 - q^{s+r})^{3e^{s/m} \log m} &\leq (1 - (1 - 1/m)^{s+m})^{3e^{s/m}} \log m \\
&\leq \exp\left(\frac{-3e^{s/m} \log m}{e^{s/m+1}(1 + o(1))}\right) = o(m^{-11/10}) = o(s^{-1}),
\end{aligned}
$$

so there is almost surely no vertex $x \in S$ above more than $3e^{s/m} \log m$ element of R.

Collecting together our results so far, we have the following.

Lemma 5. *Suppose $p \to 0$, and $s \leq \frac{1}{2} m \log m$. Then, almost surely, the random order $P'(s,p)$ has the following properties.*

(1) There is a subset R_G of R of size at least $m/12$ such that, for every subset T of R_G of size $t \leq m^{1/4}$,

$$2te^{s/m} \geq |T \cup \Gamma_S(T)| \geq te^{s/m}/80.$$

(2) No element of S is above more than $3e^{s/m} \log m$ elements of R.

Our strategy for the next few steps is as follows. We take four sets A, A', S, S' with $|A| = |A'| = m/12$ and $|S| = |S'| = s$. (One should think of $R \prec S \prec S' \prec R'$. with a $P'(s,p)$ random partial order on $R \cup S$ and another independent copy on the dual of $R' \cup S'$, then $A = R_G$ and $A' = R'_G$.) We then fix *any* partial orders $<$ and $<'$ on $A \cup S$ and $A' \cup S'$ respectively, satisfying the conclusions of Lemma 5. To be precise, we state the following three properties.

(P1) Every element of A is minimal in $<$, and every element of A' is maximal in $<'$.

(P2) For every subset T of A of size $t \leq m^{1/4}$,

$$2te^{s/m} \geq |T \cup \Gamma_S(T)| \geq te^{s/m}/80;$$

and for every subset T of A' of size $t \leq m^{1/4}$,

$$2te^{s/m} \geq |T \cup \Gamma'_{S'}(T)| \geq te^{s/m}/80,$$

where $\Gamma'_S(T)$ is the set of elements of S' below some element of T in $<'$.

(P3) No element of S is above more than $3e^{s/m} \log m$ elements of A in $<$, and no element of S' is below more than $3e^{s/m} \log m$ elements of A' in $<'$.

Given orders $<$, $<'$, $<_B$ with: (i) $<$ and $<'$ orders on $S \cup A$ and $S' \cup A'$ respectively, satisfying (P1)–(P3), (ii) $<_B$ a bipartite order, in which $x <_B y$ implies that $x \in A \cup S$ and $y \in A' \cup S'$, we define the order $Q(<, <', <_B)$ to be the transitive closure of the union of the three orders. Thus, for $a \in A, a' \in A', a$ is below a' in $Q(<, <', <_B)$ if and only if $x <_B y$ for some $x \in \Gamma_S(a)$ and $y \in \Gamma'_{S'}(a')$.

Given $<$ and $<'$ as a bove, we define a random partial order $P'' = P''(<, <', p)$ on $A \cup S \cup S' \cup A'$ by taking a random bipartite order $<_B = <_B (s, p)$, with edge probability p, and vertex sets $A \cup S, S' \cup A'$, and forming $Q(<, <', <_B)$.

(Note that, if $<$ and the dual of $<'$ were chosen as random $P'(s, p)$ orders, with S immediately below S' in $\prec$, and then A and A' selected as sets of good vertices, then the random partial order just defined is distributed as the restriction of the random graph order to the chosen vertex set, conditional on $<$ and $<'$.)

Our aim is to prove that, almost surely, the random order P'' has large dimension.

We first show that, almost surely, many pairs $(a, a') \in A \times A'$ are unrelated in P''. For this, we may as well assume that $s = \frac{1}{2} m \log m$.

A pair $(a, a') \in A \times A'$ is related exactly when there is some relation of $<_*$ between an element of $\Gamma_S(a)$ and an element of $\Gamma'_{S'}(a')$. These sets both have sizes at most $2e^{s/m} = 2m^{1/2}$, by property (P2), so the probability that the pair is unrelated is at least $q^{4m} = e^{-4}(1 + o(1))$. Therefore the expected number of unrelated pairs is at least $\left(\frac{m}{12}\right)^2 e^{-4}(1 + o(1)) \geq m^2/8{,}000$.

Again, we have to confront the problem that the relations between these pairs are far from independent. We deal with this by using the following "isoperimetric inequality" of Bollobás and Leader [7] (see also Leader [15]). Let $\mathcal{Q}_k(p)$ denote the "weighted" k-dimensional cube, i.e., the set of subsets of $\{1,\ldots,k\}$ with a set V given weight $p^{|A|}q^{k-|A|}$. This weight is a probability measure, so $\mathcal{Q}_k(p)$ is thus a probability space. Our intention here is to identify $\{1,\ldots,k\}$ with the elements of $(S\cup A)\times(S'\cup A')$, when the probability measure $\Pr_p$ on $\mathcal{Q}_k(p)$ coincides with our probability measure on bipartite orders $<_B$. For $\mathcal{A}$ a subset of $\mathcal{Q}_k(p)$, and $l\in\mathbb{N}$, we define

$$\mathcal{A}_{(l)}=\{B\in\mathcal{Q}_k(p):|B\triangle A|\le l \text{ for some } A\in\mathcal{A}\}.$$

Finally, for $r\in\mathbb{N}$, we define $\mathcal{B}^{(r)}$ to be the set of subsets of $\{1,\ldots,k\}$ with at most r elements. The inequality of Bollobás and Leader [7] is as follows.

Lemma 6. *(i) Let $\mathcal{A}\subseteq\mathcal{Q}_k(p)$ be a down-set with $\Pr_p(\mathcal{A})\ge\Pr_p(\mathcal{B}^{(r)})$. Then, for every $l\in\mathbb{N}$, we have $\Pr_p(\mathcal{A}_{(l)})\ge\Pr_p(\mathcal{B}^{(r+l)}$.*
(ii) For $z\ge 0$, let $\mathcal{A}\subseteq\mathcal{Q}_k(p)$ be either a down-set or an up-set, such that $\Pr_p(\mathcal{A})\ge e^{-z}$. Then $\Pr_p(\mathcal{A}_{(l)})\ge 1-e^{-z}$, where $l=\sqrt{12zpk}$.

Part (ii) of Lemma 6 follows immediately from part (i) on applying the Chernoff bounds for the binomial distribution.

As indicated, we apply Lemma 6 with $z=\log m, k=(s+m/12)^2\le \frac{1}{3}m^2\log^2 m$, and the set $\{1,\ldots,k\}$ identified with the pairs $(x,y)\in(S\cup A)\times(S'\cup A')$. Let $\mathcal{A}$ be the set of those bipartite orders $<_B$ giving rise to a partial order $Q(<,<',<_B)$ in which at least $m^2/10{,}000$ pairs $(a,a')\in A\times A'$ are unrelated. Note that $\mathcal{A}$ is a down-set. Since the number of unrelated pairs has expectation at least $m^2/8{,}000$ and is bounded above by m^2, the set $\mathcal{A}$ has probability at least $1/4{,}000\ge 1/m=e^{-z}$. Therefore, by Lemma 6, $\Pr_p(\mathcal{A}_{(l)})$ is at least $1-o(1)$, i.e., a random order $<_B$ is almost surely within $l\le 2m^{1/2}\log^{3/2}m$ edges of a bipartite order in $\mathcal{A}$.

By property (P3) of the orders $<$ and $<'$, the addition or removal of any edge (x,y) between $S\cup A$ and $S'\cup A'$ can only change the number of unrelated $A-A'$ pairs by at most $(3e^{s/m}\log m)^2\le 9m\log^2 m$. Thus the number of unrelated pairs in P'' is almost surely at least $m^2/10{,}000-(2m^{1/2}\log^{3/2}m)(9m\log^2 m)\ge m^2/20{,}000$. We say that a partial order P'' with at least $m^2/20{,}000$ unrelated $A-A'$ pairs satisfies property (Q1).

To complete our project, we now have to show that, almost surely, every small subset of A has fairly many elements of A' above it, and vice versa. The proof of this is very similar to the previous part.

Consider any subset T of A of size $t\le m^{1/4}$, and any element x of A'. By property (P2), $|T\cup\Gamma_S(T)|\ge te^{s/m}/80$, and $|\Gamma'_{S'}(x)|\ge e^{s/m}/80$. Thus the probability that x is above some element of T in P'' is at least $1-(1-1/m)^{te^{2s/m}/6{,}400}\ge 1-\exp(-te^{2s/m}/6{,}500m)$. Hence the number $N(T)$ of elements of A' above some element of T has mean at least $(m/12)(1-x^t)$, where $x=\exp(-e^{2s/m}/6{,}500m)$. Note that $1-x\ge e^{2s/m}/7{,}000m$.

We again use the result of Bollobás and Leader [7], Lemma 6, to show that $N(T)$ is almost surely never far from its mean. This time, we identify

$\{1, \ldots, k\}$ with the set of pairs $(x, y) \in (T \cup \Gamma_S(T)) \times (A' \cup S')$, so $k \le 3ste^{s/m}$: clearly $N(T)$ depends only on the relations of $<_B$ in this set.

Let $\mathcal{A} = \mathcal{A}(T)$ be the set of bipartite orders $<_B$ such that $N(T) \ge (m/12)(1 - x^t) - 1$. Note that $\mathcal{A}$ is an up-set. Also, $\mathbb{E}N(T) < (m/12)(1 - x^t) - 1 + m \Pr_p(\mathcal{A})$, so $\Pr_p(\mathcal{A}) \ge 1/m$.

We apply Lemma 6 with $z = t \log m$, and deduce that

$$\Pr_p(\mathcal{A}_{(l)}) \ge 1 - e^{-t \log m} = 1 - o\binom{m}{t}\left(\binom{m}{t}^{-1} m^{1/4}\right).$$

Therefore, with at least this probability, the addition of at most $l = \sqrt{12zpk}$ relations to $<_B$ will ensure that $N(T) \ge (m/12)(1 - x^t) - 1$. Note that

$$l = \sqrt{12zpk} \le \sqrt{36t \log m m^{-1} ste^{s/m}} \le 6te^{s/2m} \log m.$$

By property (P3) of $<$, the addition of any one relation only increases $N(T)$ by at most $3e^{s/m} \log m$.

Therefore, almost surely, every set T of size $t \le m^{1/4}$ has

$$N(T) \ge \frac{m}{12}(1 - x^t) - 1 - 18te^{3s/2m} \log^2 m.$$

Hence, for each T, the number of elements of A' *not* above some element of T is almost surely at most

$$\frac{m}{12}x^t + 20te^{3s/2m} \log^2 m.$$

We say that a partial order P'' satisfying this condition has property (Q2). Of course, the analogous property (Q2′) also holds almost surely for all small subsets T' of A'.

Let (Q3) be the property that, for every pair of sets $T \subset A$, $T' \subset A'$, of sizes

$$t_0 \equiv 10^6 m \log m e^{-2s/m},$$

we have x below y in P'' for some $x \in T, y \in T'$. We prove that P'' almost surely has property (Q3). Recall that $s \ge \frac{2}{5} m \log m$, so that $t_0 \le m^{1/4}$ and therefore, by property (P2), $|T \cup \Gamma_S(T)|$ and $|T' \cup \Gamma_{S'}(T')|$ are both at least $t_0 e^{s/m}/80$. The probability that a particular pair (T, T') fails is at most

$$(1 - 1/m)^{t_0^2 e^{2s/m}/6{,}400} \le e^{-10t_0 \log m} = o\left(\binom{m}{t_0}^{-2}\right).$$

Thus the expected number of failing pairs is $o(1)$.

Now we assume that P'' does have properties (Q1), (Q2), (Q2′) and (Q3). Thus there is a relation between every pair of subsets of size $t_0 = 10^6 m \log m e^{-2s/m}$, every subset T of A of size $t \le m^{1/4}$ has $N(T) \ge (m/12)(1 - x^t) - 20te^{3s/2m} \log^2 m$, and the analogous inequality holds for subsets of A', and there are at least $m^2/20{,}000$ unrelated pairs in $A \times A'$.

Recall here that $x \le 1 - e^{2s/m}/7{,}000m$. We now show that these assumptions imply that the dimension of P'' is large.

We bound the number of pairs $(a, a') \in A \times A'$ reversed by a linear extension L of P''. Consider the top t_0 elements of A in L, say $a_1 >_L a_2 >_L \cdots >_L a_{t_0}$, and similarly the bottom t_0 elements of A', say $a'_1 <_L \cdots <_L a'_{t_0}$. By property (Q3), there is some pair $i, j (1 \le i, j \le t_0)$ such that $a_i < a'_j$ in P'', so every pair reversed by L involves one of these $2t_0$ elements.

For $1 \le j \le t_0$, the number of reversed pairs involving a_j is at most the number $N(\{a_1, \ldots, a_j\})$ of elements of A' incomparable with all of $a_1, \ldots, a_j$, and so by (Q2) is at most $\frac{m}{12}x^j + 20je^{3s/2m}\log^2 m$. Hence the total number of reversed pairs is at most

$$\begin{aligned} 2\sum_{j=1}^{t_0} &\left(\frac{m}{12}x^j + 20je^{3s/2m}\log^2 m\right) \\ &\le \frac{m}{6}\sum_{j=0}^{\infty}\left(1 - \frac{e^{2s/m}}{7{,}000m}\right)^t + 20t_0^2 e^{3s/2m}\log^2 m \\ &= \frac{7{,}000m^2}{6e^{2s/m}} + 2 \times 10^{13} m^2 e^{-5s/2m}\log^4 m \\ &\le 1{,}200m^2 e^{-2s/m}. \end{aligned}$$

Therefore, under our assumptions, the dimension of P'' is at least

$$\frac{m^2}{20{,}000}\bigg/\frac{1{,}200m^2}{e^{2s/m}} \geqslant 10^{-8}e^{2s/m}.$$

As we have been indicating throughout, this suffices to prove Theorem 6, and indeed also Theorem 7. We fill in the details below.

Proof of Theorem 6. The upper bound follows immediately from the upper bound on the width given in Theorem 5.

For the lower bound, we set $m = p^{-1}$ as usual, and take $s = \frac{1}{2}m\log m$. We are given $p \ge \log n/n$, so $n \ge 2s + 6m$. We define four sets as follows: B consists of the $\prec$-first $3m$ elements of $[n]$, U consists of the $\prec$-next s elements, U' of the $\prec$-next s elements, and B' of the $\prec$-next $3m$. There are almost surely at least $9m/10$ maximal elements in the random order $P_{n,p}$ restricted to B, and $9m/10$ minimals in the order restricted to B', as noted in [5]. If this is the case, then, to be definite, take A to be the set of the $9m/10$ $\prec$-largest maximals from B, and A' to consist of the $9m/1O$ $\prec$-smallest minimals from B'. Note that the choice of A and A' depends only on the order restricted to B and to B'.

We next consider the random order restricted to the sets $A \cup U$ and $A' \cup U'$. These are distributed as a random order $P'(s, p)$ and its dual, so the conclusions of Lemma 5 hold almost surely for $R = A$ and $S = U$, and dually for $R = A'$ and $S = U'$. In other words, the restrictions $<$ and $<'$ of the random order $P_{n,p}$ to $A \cap U$ and to $A' \cap U'$ almost surely have properties (P1)–(P3).

Conditioned on the restrictions of $P_{n,p}$ being equal to any particular pair $<, <'$ satisfying (P1)–(P3), we have seen that the dimension of the random order restricted to $A \cup A'$, and hence $\dim P_{n,p}$, is almost surely at least $10^{-8}e^{2s/m} = 10^{-8}m$.

Combining all the above, we see that the dimension of $P_{n,p}$ is almost surely at least $10^{-8}p^{-1}$, as claimed. □

Proof of Theorem 7. Set $m = p^{-1}$ and $s = \frac{1}{2}(n - 3m)$. Note that $s \geq \frac{2}{5}m \log m$. The proof of Theorem 6 goes through without alteration, and we deduce that

$$\dim P_{n,p} \geq 10^{-8}e^{2s/m} \geq 10^{-10}e^{n/m},$$

almost surely, as required.

For the upper bound, we apply Theorem 2, the upper bound of Füredi and Kahn [12]. Thus we need an estimate on the maximum degree Δ of an element of $P_{n,p}$. A bound on Δ follows from Lemma 1(iii), just as in the proof of Lemma 5(2), namely, almost surely, $\Delta \leq 3e^{pn} \log n$. Theorem 2 now tells us that

$$\dim P_{n,p} \leq 50\Delta \log^2 \Delta \leq 150e^{pn} \log n(pn + \log\log n + 2)^2 \leq 40e^{pn} \log^3 n,$$

as required. □

We have proved Theorem 6 with 10^{-8} for the value of ϵ; clearly this can be improved substantially, but tinkering with the method is unlikely to produce a reasonable constant. We suspect the result is true with $\epsilon = 1$, at least, and maybe it is even true that the dimension is almost surely $(1 - o(1))$ times the width.

On a slightly more abstract note, it is likely that there is a constant c such that, if $pn/\log n \to \infty, p \log n \to 0$ and $\epsilon > 0$, then, almost surely,

$$(c - \epsilon)p^{-1} \leq \dim P_{n,p} \leq (c + \epsilon)p^{-1},$$

but this may be rather hard to prove.

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Hereditary and Monotone Properties of Graphs

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Summary. Given a hereditary graph property $\mathcal{P}$ let $\mathcal{P}^n$ be the set of those graphs in $\mathcal{P}$ on the vertex set $\{1,\dots,n\}$. Define the constant c_n by $|\mathcal{P}^n| = 2^{c_n\binom{n}{2}}$. We show that the limit $\lim_{n\to\infty} c_n$ always exists and equals $1 - 1/r$, where r is a positive integer which can be described explicitly in terms of $\mathcal{P}$. This result, obtained independently by Alekseev, extends considerably one of Erdős, Frankl and Rödl concerning principal monotone properties and one of Prömel and Steger concerning principal hereditary properties.

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1. Introduction

A *property* $\mathcal{P}$ of graphs is an infinite class of graphs which is closed under isomorphism. A property $\mathcal{P}$ is *hereditary* if every *induced subgraph* of every member of $\mathcal{P}$ is also in $\mathcal{P}$, and is *monotone* if every *subgraph* of every member of $\mathcal{P}$ is also in $\mathcal{P}$; a monotone property is therefore also hereditary. Let $\mathcal{P}^n$ be the set of graphs in $\mathcal{P}$ with vertex set $[n] = \{1,\dots,n\}$. In this paper we are interested in the rate of growth of $\mathcal{P}^n$ with n, so it is convenient to define the constant $c_n = c_n(\mathcal{P})$ by $|\mathcal{P}^n| = 2^{c_n\binom{n}{2}}$. Note that eventually $c_n < 1$ unless $\mathcal{P}$ is the *trivial* property consisting of all graphs.

Scheinermann and Zito [26] asked if, for a hereditary property, $\lim_{n\to\infty} c_n$ always exists, and if so what the possible values are. The limit had been evaluated earlier for the monotone property of K_n-free graphs by Erdős, Kleitman and Rothschild [11], by using the method of Kleitman and Rothschild [13]. Their result was generalized by Erdős, Frankl and Rödl [10], who considered the monotone property of graphs not containing a given graph F as a subgraph. The structure of K_n-free graphs was investigated by Kolaitis, Prömel and Rothschild [14].

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It is considerably more difficult to determine the asymptotic size of the hereditary, but non-monotone, property $\mathcal{P}$ of not containing a given graph F as an induced subgraph; this problem has been studied by Prömel and Steger in a series of papers [17, 18, 19, 20, 21]. In particular in [18] they gave sharp estimates for the property of not containing an induced quadrilateral, and in [19] they evaluated $\lim_{n\to\infty} c_n$ for every given graph F.

Our purpose in this paper is to evaluate $\lim_{n\to\infty} c_n$ for *every* hereditary property $\mathcal{P}$, not only for *principal* properties, namely those defined by a single forbidden subgraph (induced or otherwise). Our main result (Theorem 4) claims that this limit equals $1-1/r$, where $r = r(\mathcal{P})$ is an integer which we call the *colouring number* of $\mathcal{P}$, to be defined below. Since writing this paper we have discovered that we were anticipated in the result by Alekseev [2], but the present proof appears to be simpler and more natural. In particular, it is reasonable to suspect a relationship between the main theorem and the Erdős-Stone theorem. In our proof this relationship is established and is made transparent.

The proof of Theorem 4 contains an implicit demonstration of the existence of $\lim_{n\to\infty} c_n$. In fact, Alekseev [1] already showed that the limit exists. Moreover it was shown in [7] that the sequence (c_n) is monotone decreasing (and so the limit must therefore exist). The analogous sequence monotonicity property was shown to pertain to uniform hypergraph properties, and so the corresponding limits again exist. However, for hypergraphs we are unable to say anything about the value of the limits, or even whether every real number between zero and one is the limit for some property.

The colouring number $r(\mathcal{P})$ of a property $\mathcal{P}$ is defined as follows. Let $0 \le s \le r$ be integers. An (r,s)-*colouring* of a graph H is a map $\psi : V(H) \to [r]$ such that $H[\psi^{-1}(i)]$ is complete for $1 \le i \le s$ and is empty otherwise. Note that H is $(r,0)$-colourable if and only if $\chi(H) \le r$, since an $(r,0)$-colouring is just an r-colouring in the usual sense. Note too that a graph is (r,r)-colourable if and only if $\chi(\overline{H}) \le r$ (all notation used but not defined in this paper is described in [3]). Now let

$$\mathcal{C}_k(r,s) = \{H \ : \ |H| = k \text{ and } H \text{ is } (r,s)\text{-colourable}\}.$$

Then the *colouring number* $r(\mathcal{P})$ of a hereditary property $\mathcal{P}$ is defined by

$$r(\mathcal{P}) = \max\{r \ : \ \text{there exists } 0 \le s \le r \text{ such that } \mathcal{P} \supset \bigcup_{k\ge 1} \mathcal{C}_k(r,s)\},$$

that is, $r(\mathcal{P})$ is the largest integer r such that, for some s, $\mathcal{P}$ contains every (r,s)-colourable graph. Since a graph of order r is (r,s)-colourable for every s, $0 \le s \le r$, it follows that $r(\mathcal{P})$ is finite if $\mathcal{P}$ is non-trivial. Note also that the only $(1,0)$-colourable graphs are empty, and the only $(1,1)$-colourable graphs are complete, so by Ramsey's theorem $r(\mathcal{P}) \ge 1$.

Consideration of the property complementary to $\mathcal{P}$ gives us another way to view the colouring number of $\mathcal{P}$. A property $\mathcal{P}$ is hereditary if, and only if, for some sequence $F_1, F_2, \dots$ of graphs, $\mathcal{P}$ is the collection of graphs having no induced subgraph isomorphic to an F_i. Then

$$r(\mathcal{P}) = \max\{r : \text{ for some } 0 \le s \le r \text{ no } F_i \text{ is } (r,s)\text{-colourable}\}.$$

Although monotone properties $\mathcal{P}$ are also hereditary, and so the above serves to define $r(\mathcal{P})$, it is worth giving the definition in a third and simpler form for these properties. A property $\mathcal{P}$ is monotone if, and only if, for some sequence $F_1, F_2, \ldots$ of graphs, $\mathcal{P}$ is the collection of graphs having no subgraph isomorphic to an F_i. Since any (r,s)-colourable graph contains an $(r,0)$-colourable subgraph, we see that for a *monotone* property $\mathcal{P}$ the colouring number is

$$r(\mathcal{P}) = \max\{r : \text{ no } F_i \text{ is } r\text{-colourable}\}.$$

Therefore for monotone properties our result is that $\lim_{n\to\infty} c_n = 1 - 1/r$, where $r = \min\{\chi(F_i)\} - 1$. This is in contrast to the case of hereditary properties in general, for which the colouring number is not merely the minimum of those colouring numbers of the properties defined by excluding a single F_i.

It is immediate that $\liminf_{n\to\infty} c_n \ge 1 - 1/r(\mathcal{P})$ for any hereditary property $\mathcal{P}$. For let s be an integer for which $\mathcal{P}$ contains every (r,s)-colourable graph. Partition the set $[n]$ into r disjoint classes $V_1, \ldots, V_r$, where $\lfloor n/r \rfloor \le |V_i| \le \lceil n/r \rceil$. Every graph with vertex set $[n]$ in which the subgraph spanned by V_i is complete for $1 \le i \le s$, and in which the subgraph spanned by V_i is empty for $s < i \le r$, is (r,s)-colourable and is therefore in $\mathcal{P}^n$. Hence $|\mathcal{P}^n| \ge 2^{(1-1/r+O(1/n))\binom{n}{2}}$, which shows that $\liminf_{n\to\infty} c_n \ge 1 - 1/r$.

Let us denote by $\mathrm{ex}_{\mathrm{ind}}(n,\mathcal{P})$ the maximal number of edges in a graph G_0 of order n, for which there is an edge-disjoint graph G_1 on the same vertex set, such that every graph G with $G_1 \subseteq G \subseteq G_0 \cup G_1$ is in the hereditary property $\mathcal{P}$. This invariant was introduced (for principal properties) by Prömel and Steger [19]. Trivially $|\mathcal{P}^n| \ge 2^{\mathrm{ex}_{\mathrm{ind}}(n,\mathcal{P})}$, and it is clear from the construction in the paragraph above that $\mathrm{ex}_{\mathrm{ind}}(n,\mathcal{P}) \ge (1 - \frac{1}{r(\mathcal{P})} + o(1))\binom{n}{2}$; it is, therefore, a consequence of our result that this last inequality is, in fact, an equality.

The proof of Theorem 4 is based on three well-known results, namely those of Ramsey [22], of Szemerédi [27] and of Erdős and Stone [12]. Ramsey's theorem states that for each positive integer k there exists an integer $R(k)$, such that if $n \ge R(k)$ and the edges of the complete graph K_n are coloured with two colours then there will be a monochromatic complete subgraph of order k. Szemerédi's lemma will be stated and discussed in Sect. 3. The Erdős-Stone theorem will be discussed in Sect. 2; in fact, for the present purpose we have to prove a slight extension of that theorem.

Each of the three cited fundamental results asserts the existence of certain constants. For the achievement of our aim, which is to describe $\lim_{n\to\infty} c_n$, it is sufficient that these constants exist. To investigate the rate of convergence of the sequence (c_n) we would need effective versions of these theorems, but we make no attempt whatsoever to carry out this investigation.

2. An Extension of the Erdős-Stone Theorem

Let $r, t \geq 1$ and $\varepsilon > 0$. The well-known theorem of Erdős and Stone [12] states that every graph of order n and size at least $(1 - 1/r + \epsilon)\binom{n}{2}$ contains $K_{r+1}(t)$, the complete $(r+1)$-partite graph with t vertices in each class, provided n is large enough. Exactly how large n needs to be, as a function of r, t and ϵ, was investigated by Bollobás and Erdős [5], Bollobás, Erdős and Simonovits [6], and Chvátal and Szemerédi [9].

We shall prove here an extension of the Erdős-Stone theorem, in which a certain number of 'forbidden' edges are added to the graph, and it is required that the $K_{r+1}(t)$ span no forbidden edge. To be precise, let G and F be two graphs on the same vertex set. We say that a subgraph H of G is *F-avoiding* if $V(H)$ spans no edge of F. Our theorem shows that if G satisfies the conditions of the Erdős-Stone theorem, and if $e(F)$ is sufficiently small, then G will contain an F-avoiding $K_{r+1}(t)$.

An alternative proof of the theorem has been pointed out by Rödl [23]; his proof depends on a 'supersaturated' version of the Erdős-Stone theorem, stating that every graph of order n (sufficiently large) and size at least $(1 - 1/r + \epsilon)\binom{n}{2}$ contains not just one but $cn^{(r+1)t}$ copies of $K_{r+1}(t)$. Given this theorem, our theorem follows at once since if $e(F)$ is small it cannot meet all of the copies of $K_{r+1}(t)$. However, our purpose here is to give a short self-contained proof. For the proof it will be convenient to have the following weak form of Turán's theorem [29], wherein $\overline{G}$ denotes the complement of G.

Lemma 1. *If G is a graph of order $m \geq t^2$ and $e(\overline{G}) \leq m^2/2t$ then $G \supset K_t$.*

Proof. By Turán's theorem, if $G \not\supset K_t$ then

$$e(\overline{G}) \geq \sum_{i=0}^{t-2} \binom{\lfloor (m+i)/(t-1) \rfloor}{2} \geq (t-1)\binom{m/(t-1)}{2} = \frac{m(m-t+1)}{2(t-1)} > \frac{m^2}{2t},$$

as claimed. □

Theorem 1. *Given $r \geq 0$, $t \geq 1$ and $\epsilon > 0$, there exist $\delta = \delta(r, t, \epsilon)$ and $n_0 = n_0(r, t, \epsilon)$ such that the following holds. Let F and G be graphs on the same vertex set of order $n \geq n_0$, with $e(F) \leq \delta n^2$ and, if $r \geq 1$, with*

$$e(G) \geq (1 - \frac{1}{r} + \epsilon)\binom{n}{2}.$$

Then G contains an F-avoiding $K_{r+1}(t)$ subgraph.

Proof. Note first that we may assume that F and G share no edges. We shall apply induction on r. If $r = 0$ then Lemma 1 applied to $\overline{G}$ shows that $\delta = 1/(2t)$ and $n_0 = t^2$ will do.

Suppose then that $r \geq 1$ and that the assertion holds for smaller values of r. Let us first make the customary observation that for some m, $(\epsilon/2)^{1/2} n \leq m \leq n$, G has an induced subgraph G_m of order m and minimal degree at least

$(1-1/r+\epsilon/2)m$. Indeed, if this were not the case then, with $s = \lfloor(\epsilon/2)^{1/2}n\rfloor$, we could find subgraphs $G_n = G \supset G_{n-1} \supset \ldots \supset G_s$ such that, for $n > i \geq s$, G_i is a subgraph induced by i vertices and the only vertex of G_{i+1} not in G_i has degree less than $(1-1/r+\epsilon/2)(i+1)$ in G_{i+1}. But then

$$e(G_s) > (1 - \frac{1}{r} + \epsilon)\binom{n}{2} - (1 - \frac{1}{r} + \frac{\epsilon}{2})\sum_{s}^{n-1}(i+1) \geq \epsilon n^2/4 \geq s^2/2$$

if n is large, which is a contradiction.

Let F_m be the subgraph of F induced by $V(G_m)$. Note that $e(F_m) \leq \delta n^2 \leq (2\delta/\epsilon)m^2$. Let $T = \lceil 4t/\epsilon r\rceil$. By the induction hypothesis, if n is large enough and δ is small enough, G_m contains an F-avoiding $K_r(T)$, say K. The bound on the minimal degree of G_m implies that each vertex of K sends at least $(1-1/r+\epsilon/2)m - rT$ edges to $G_m - K$. We claim that the set U of vertices of $G_m - K$ sending at least $(r-1+\epsilon r/4)T$ edges to K has at least $\epsilon rm/5$ members. Indeed, if this were not the case then the number of edges f between K and $G_m - K$ would satisfy

$$\frac{\epsilon rm}{5}rT + \left(1 - \frac{\epsilon r}{5}\right)m\left(r - 1 + \frac{\epsilon r}{4}\right)T > f \geq \left\{\left(1 - \frac{1}{r} + \frac{\epsilon}{2}\right)m - rT\right\}rT,$$

which is false if n is large.

Each vertex of U is joined to a $K_r(t)$ subgraph of K. There are only $\binom{T}{t}^r$ such subgraphs, so for some $K_r(t)$ subgraph K' the set W of vertices of U joined to every vertex of K' has at least $\frac{\epsilon rm}{5}\binom{T}{t}^{-r}$ members. Now W spans at most $(2\delta/\epsilon)m^2$ edges of F, so again by Lemma 1 if n, and hence m, is large enough and δ is small enough, the set W contains t vertices spanning an independent set in F, and hence forming with K' an F-avoiding $K_{r+1}(t)$ subgraph of G. □

3. Universal Graphs

Given a finite class $\mathcal{G}$ of graphs, we say that a graph G is *$\mathcal{G}$-universal* if every member of $\mathcal{G}$ is an induced subgraph of G. The classes of graphs of interest to us are the classes $\mathcal{C}_k(r,s)$ of (r,s)-colourable graphs defined earlier. Our aim in this section is to show that every suitably large collection of graphs has a $\mathcal{C}_k(r,s)$-universal member, for some s.

For the proof of our theorem we will need a lemma of Szemerédi [27]. In order to state this lemma the notion of uniformity must be defined: a pair of subsets U and W of the vertex set of a graph G is said to be *η-uniform* if $|d(U,W) - d(U',W')| < \eta$ whenever $U' \subseteq U$, $|U'| > \eta|U|$ and $W' \subseteq W$, $|W'| > \eta|W|$, where $d(U,W) = e(U,W)/|U||W|$. Szemerédi's Uniformity Lemma is (equivalent to) the statement that, given $\eta > 0$ and an integer l, there is an integer $L = L(l,\eta)$ such that the vertices of every graph of order n can be partitioned into m classes $V_1, \ldots, V_m$, for some $l \leq m < L$, so that

$\lfloor n/m \rfloor \le |V_i| \le \lceil n/m \rceil$ and all but at most $\eta\binom{m}{2}$ of the pairs (V_i, V_j) are η-uniform, $1 \le i < j \le m$.

The following lemma is a standard application of the notion of uniformity (see for example [24]); once again we shall give a self-contained proof in a form convenient for our needs.

Lemma 2. *Let H be a graph with vertex set $\{x_1, \dots, x_k\}$. Let $0 < \lambda, \eta < 1$ satisfy $k\eta \le \lambda^{k-1}$. Let G be a graph with vertex set $\bigcup_{i=1}^k V_i$ where the V_i are disjoint sets each of order $u \ge 1$. Suppose that each pair (V_i, V_j), $1 \le i < j \le k$, is η-uniform, that $d(V_i, V_j) \le 1 - \lambda$ if $x_i x_j \notin E(H)$ and that $d(V_i, V_j) \ge \lambda$ if $x_i x_j \in E(H)$. Then there exist vertices $v_i \in V_i$, $1 \le i \le k$, such that the map $x_i \mapsto v_i$ gives an isomorphism between H and the subgraph of G spanned by $\{v_1, \dots, v_k\}$.*

Proof. Observe that by replacing, if necessary, the set of V_i–V_j edges of G by the complementary set of V_i–V_j edges, we may assume that H is the complete graph and that $d(V_i, V_j) \ge \lambda$ for all $1 \le i < j \le k$. We shall select, one by one, the vertices $v_1, \dots, v_k$ so that after $v_1, \dots, v_l$ have been chosen there are sets $U_j^l \subset V_j$, $l < j \le k$, such that each of $v_1, \dots, v_l$ is joined to every vertex of U_j^l and $|U_j^l| \ge (\lambda - \eta)^l u$. Clearly we can begin (with $l = 0$) by taking $U_j^0 = V_j$.

In order to find v_{l+1}, having found $v_1, \dots, v_l$, let

$$W_j = \{v \in U_{l+1}^l : |\Gamma(v) \cap U_j^l| < (\lambda - \eta)|U_j^l|\}$$

for $l + 1 < j \le k$. Since $d(W_j, U_j^l) < \lambda - \eta$, the definition of η-uniformity applied to the pairs (W_j, U_j^l) implies that either $|W_j| < \eta u$ or $|U_j^l| < \eta u$. However the latter is ruled out because $l \le k - 1$ and hence $|U_j^l| \ge (\lambda - \eta)^{k-1} u > (\lambda^{k-1} - (k-1)\eta)u \ge \eta u$. Hence

$$|U_{l+1}^l \setminus \bigcup_{j > l+1} W_j| > (\lambda - \eta)^l u - (k - l - 1)\eta u \ge (\lambda^l - l\eta - (k - l - 1)\eta)u > 0.$$

We may therefore choose a vertex $v \in U_{l+1}^l$ so that if we let $U_j^{l+1} = \Gamma(v) \cap U_j^l$ then $|U_j^{l+1}| \ge (\lambda - \eta)^{l+1} u$ for $l + 1 < j \le k$. In this way we may proceed to find each of the vertices $v_1, \dots, v_k$. □

We define a *coloured partition* π to be an edge colouring of the complete graph on the vertex set $[m]$ which uses four colours, namely grey, red, green and blue. The number m is called the *order* of π and is denoted by $|\pi|$. We associate with π two graphs, F_π and G_π, both on the vertex set $[m]$; the edge set of F_π is $\{ij : ij \text{ is grey}\}$ and the edge set of G_π is $\{ij : ij \text{ is green}\}$.

Given a graph G and constants $0 < \lambda, \eta < 1$, we say that G *satisfies* the coloured partition π with respect to λ and η, if there is a partition of the vertices of G into $|\pi|$ classes $V_1, \dots, V_{|\pi|}$, with $|V_1| \le |V_2| \le \dots \le |V_{|\pi|}| \le |V_1| + 1$, such that the pair (V_i, V_j) is not η-uniform only if ij is grey, and otherwise $0 \le d(V_i, V_j) \le \lambda$, $\lambda < d(V_i, V_j) < 1 - \lambda$ or $1 - \lambda \le d(V_i, V_j) \le 1$ according as ij is red, green or blue. Szemerédi's Uniformity Lemma asserts that, given λ, η and some integer l, there exists an integer $L = L(l, \eta)$ such that any graph G satisfies some coloured partitition π with respect to λ and η, with $l \le |\pi| < L$ and $e(F_\pi) \le \eta\binom{|\pi|}{2}$.

The following theorem shows that a graph satisfying π will be $\mathcal{C}_k(r, s)$-universal if the size of G_π is large. We shall use this result only to prove Theorem 3, but we state it because it may be useful in further investigations of hereditary graph properties. The dependencies of some of the parameters on other ones gives the theorem a technical appearance, but these dependencies are likely to be crucial in applications. In this theorem and the next one, we will use $r - 1$ where in Theorem 4 we used r. Despite the potential confusion, we adopt this usage because it seems more natural here.

Theorem 2. *Let $r, k \in \mathbb{N}$, $r \ge 2$, $\epsilon > 0$ and $0 < \lambda < 1$ be given. Then there exist positive constants $l_1 = l_1(r, k, \epsilon)$ and $\eta_1 = \eta_1(r, k, \epsilon, \lambda)$ with the following property. Let $0 < \eta \le \eta_1$ and let π be a coloured partition with $|\pi| \ge l_1$, $e(F_\pi) \le \eta\binom{|\pi|}{2}$ and*

$$e(G_\pi) \ge \left(1 - \frac{1}{r-1} + \epsilon\right)\binom{|\pi|}{2}.$$

Then there is an integer $s = s(\pi)$, $0 \le s \le r$, such that every graph of order $n \ge |\pi|$ that satisfies π with respect to λ and η is $C_k(r, s)$-universal.

Proof. Let t be the Ramsey number $R(k)$, let $l_1 = n_0(r-1, t, \epsilon)$ and let $\eta_1 = \min\{\delta(r-1, t, \epsilon), k^{-1}\lambda^{k-1}\}$, where n_0 and δ are the functions appearing in Theorem 1. We shall show that these functions have the required properties.

By Theorem 1 there is an F_π-avoiding copy of $K_r(t)$ in G_π. Consider one of the r vertex classes of this $K_r(t)$, say T where $|T| = t$. The edges ij, where $\{i, j\} \subset T$, are coloured either red, green or blue. Let us recolour *orange* the edges coloured red or green. By the definition of t, there is a subset $T' \subset T$, $|T'| = k$, such that the edges ij where $\{i, j\} \subset T'$, are either all orange or all blue. This argument can be applied to each class of the $K_r(t)$. Therefore our $K_r(t)$ in G_π contains a subgraph $K = K_r(k)$ such that the edge ij is green if i and j are in different classes of K, the edge ij is blue if i and j are both in one of the first s classes of K, and the edge ij is orange if i and j are both in one of the remaining $r - s$ classes of K, where $s = s(\pi)$ lies in the range $0 \le s \le r$.

Let $H \in \mathcal{C}_k(r, s)$, so that $|H| = k$ and H is (r, s)-colourable. Choose an (r, s)-colouring of H and label the vertex set of H by $x_1, \ldots, x_k$ so that the first k_1 vertices have colour 1, the next k_2 vertices have colour 2 and so on. Now construct a subgraph K' of K by choosing k_1 vertices from the first class of K, k_2 vertices from the second class, and so on. Thus $|K'| = k$. Let G be a graph of order $n \ge |\pi|$ that satisfies π with respect to λ and η, and let G' be the subgraph of G induced by $\bigcup\{V_i \ : \ i \in V(K')\}$. The definition of the colouring and the choice of K' mean that Lemma 2 can be applied to H and G'. This shows that H is an induced subgraph of G' and so also of G. □

We are now able to prove the main theorem of this section, which shows that any suitably large collection of graphs will have a $\mathcal{C}_k(r,s)$-universal member.

Theorem 3. *Let $r,k \in \mathbb{N}$, $r \geq 2$ and $\epsilon > 0$ be given. Then there exists $n_1 = n_1(r,k,\epsilon)$ such that if $n > n_1$ and $\mathcal{P}^n$ is a collection of at least $2^{(1-1/(r-1)+\epsilon)\binom{n}{2}}$ labelled graphs with vertex set $[n] = \{1,\ldots,n\}$, then $\mathcal{P}^n$ contains a $\mathcal{C}_k(r,s)$-universal graph for some s, $0 \leq s \leq r$.*

Proof. We begin by choosing constants λ, l and η as follows. Choose $0 < \lambda < 1/4$ small enough so that $(e/\lambda)^\lambda < 2^{\epsilon/10}$. Now choose $0 < \eta < \min\{\epsilon/8, \eta_1(r,k,\epsilon/8,\lambda)\}$ and an integer l larger than $\max\{20/\epsilon, l_1(r,k,\epsilon/8)\}$, where the functions η_1 and l_1 are those appearing in Theorem 2.

Apply Szemerédi's Uniformity Lemma, with the usual parameters l and η, to each graph G in $\mathcal{P}^n$, thus obtaining, for each such graph, a coloured partition $\sigma(G)$ which G satisfies with respect to λ and η, with $l \leq |\sigma(G)| < L = L(l,\eta)$ and $e(F_{\sigma(G)}) \leq \eta\binom{|\sigma(G)|}{2}$. Since there are at most $4^{\binom{L}{2}}$ coloured partitions of order less than L and at most n^L ways to split the set $[n]$ into fewer than L parts, there is some coloured partition π satisfied by at least $n^{-L}4^{-\binom{L}{2}}2^{(1-1/(r-1)+\epsilon)\binom{n}{2}}$ graphs of $\mathcal{P}^n$, which number being at least $2^{(1-1/(r-1)+\epsilon/2)\binom{n}{2}}$ if n is large. From now on we consider only this particular coloured partition π which is satisfied by many graphs. We shall show that any graph in $\mathcal{P}^n$ satisfying π is $\mathcal{C}_k(r,s)$-universal, where $s = s(\pi)$, provided n is both larger than L and is large enough to ensure the validity of the estimates below. This assumption on the size of n will be made without further mention throughout the remainder of the proof.

Our aim is now to show that the coloured partition π must have many green edges. This can be done by estimating the number of graphs which can satisfy the coloured partition π. Within such a graph there are at most $2^{\binom{N}{2}}$ possible distributions of edges inside a vertex class V_i, where $N = \lceil n/|\pi| \rceil$. Between two classes V_i and V_j the possible distributions of edges of the graph number at most 2^{N^2} for grey and green edges ij of π, and at most $\sum_{i=0}^{\lambda N^2}\binom{N^2}{i}$ for red and blue edges. Let f be the number of grey and green edges in the coloured partition π. Then the number of graphs which may satisfy π with respect to λ and η is at most

$$2^{|\pi|\binom{N}{2}} \times \left[\sum_{i=0}^{\lambda N^2}\binom{N^2}{i}\right]^{\binom{|\pi|}{2}} \times 2^{fN^2}.$$

Each of the first two factors here is bounded above by $2^{\frac{\epsilon}{10}\binom{n}{2}}$; the first because

$$|\pi|\binom{N}{2} < \frac{2}{|\pi|}\binom{n}{2} < \frac{\epsilon}{10}\binom{n}{2}$$

and the second because

$$\left[\sum_{i=0}^{\lambda N^2}\binom{N^2}{i}\right]^{\binom{|\pi|}{2}} < \left[2\binom{N^2}{\lambda N^2}\right]^{\binom{|\pi|}{2}} < \left(\frac{e}{\lambda}\right)^{\lambda N^2\binom{|\pi|}{2}} \le \left(\frac{e}{\lambda}\right)^{\lambda n^2/2} < 2^{\frac{\epsilon}{10}\binom{n}{2}}.$$

It follows that the number of graphs satisfying the coloured partition π is at most $2^{\frac{\epsilon}{5}\binom{n}{2}+fN^2}$, and since we have at least $2^{(1-1/(r-1)+\epsilon/2)\binom{n}{2}}$ such graphs we see that

$$f \ge \left(1 - \frac{1}{r-1} + \frac{3\epsilon}{10}\right)\binom{n}{2}N^{-2} > \left(1 - \frac{1}{r-1} + \frac{\epsilon}{4}\right)\binom{|\pi|}{2}.$$

Now $f = e(F_\pi) + e(G_\pi)$, and Szemerédi's lemma asserts that $e(F_\pi) \le \eta\binom{|\pi|}{2}$. Since $\eta < \epsilon/8$, we see that $e(G_\pi) \ge (1-1/(r-1)+\epsilon/8)\binom{|\pi|}{2}$. Theorem 2 now implies that any graph of order n which satisfies π is $\mathcal{C}_k(r, s(\pi))$-universal, as claimed. □

4. Hereditary Properties

Recall that for a non-trivial hereditary property $\mathcal{P}$ of graphs the colouring number $r(\mathcal{P})$ is defined by

$$r(\mathcal{P}) = \max\{r \ : \ \text{there exists } 0 \le s \le r \text{ such that } \mathcal{P} \supset \bigcup_{k\ge 1}\mathcal{C}_k(r,s)\}.$$

We can now state and prove the main result of this paper.

Theorem 4. *Let $\mathcal{P}$ be a non-trivial hereditary property of graphs and let $\mathcal{P}^n$ be the set of graphs in $\mathcal{P}$ with vertex set $[n] = \{1, \ldots, n\}$. Set $|\mathcal{P}^n| = 2^{c_n\binom{n}{2}}$. Then*

$$\lim_{n\to\infty} c_n = 1 - 1/r(\mathcal{P}),$$

where $r(\mathcal{P})$ is the colouring number of $\mathcal{P}$.

Proof. Let $r = r(\mathcal{P})$. We saw in the introduction that $\liminf_{n\to\infty} c_n \ge 1 - 1/r$. Suppose that the assertion of the theorem is false, so that $\limsup_{n\to\infty} c_n > 1 - 1/r$, which is to say there exists $\epsilon > 0$ such that $|\mathcal{P}^n| > 2^{(1-1/r+\epsilon)\binom{n}{2}}$ for infinitely many values of n. It follows from Theorem 3 that for each integer k there is an integer s, $0 \le s \le r+1$, such that for some n the set $\mathcal{C}_k(r+1, s)$ is contained in $\mathcal{P}^n$ and therefore in $\mathcal{P}$. Consequently, for some value of s, we have $\mathcal{C}_k(r+1,s) \subset \mathcal{P}$ for infinitely many k, and hence for all k. But this contradicts the definition of $r(\mathcal{P})$, so our theorem is proved. □

As an example, let $\mathcal{P}$ be the property consisting of planar graphs and let $\mathcal{P}'$ be those graphs containing no induced subgraph isomorphic to either K_5 or $K_{3,3}$. Clearly $\mathcal{P} \subset \mathcal{P}'$. Since K_5 is $(2,s)$-colourable for $s = 1, 2$ and $K_{3,3}$ is $(2,0)$-colourable, it follows that $r(\mathcal{P}) = r(\mathcal{P}') = 1$, so $|\mathcal{P}'| = 2^{o(n^2)}$.

For a slightly less simple example, let $\mathcal{P}_1$ be the (monotone) property of containing no complete graph K_4, and let $\mathcal{P}_2$ be the property of containing no induced subgraph isomorphic to the 7-cycle C_7. Now let $\mathcal{P}$ be $\mathcal{P}_1 \cap \mathcal{P}_2$; that is, let $\mathcal{P}$ be the property of having no induced subgraph isomorphic to either K_4 or C_7. Each of K_4 and C_7 is $(4,s)$-colourable for every s, $0 \le s \le 4$. Now K_4 is $(3,s)$-colourable for $1 \le s \le 3$ but is not $(3,0)$-colourable, so $r(\mathcal{P}_1) = 3$. On the other hand, C_7 is $(3,0)$-colourable but has no $(3,s)$-colouring if $s = 2, 3$, so $r(\mathcal{P}_2) = 3$. Finally, neither K_4 nor C_7 has a $(2,0)$-colouring so $r(\mathcal{P}) = 2$. Consequently $|\mathcal{P}_1^n| \approx 2^{n^2/3}$ and $|\mathcal{P}_2^n| \approx 2^{n^2/3}$, whereas $|\mathcal{P}^n| \approx 2^{n^2/4}$.

5. Additional Remarks for the Second Edition

The preceding text is (apart from one or two updated references) the same as that in the original edition of this book. But the subject has moved on since the paper first appeared: we give here just some pointers to further developments.

For smaller hereditary graph properties, namely those for which $r(\mathcal{P}) = 1$, the theorem here gives little information. The sizes of such properties, as well as the sizes of hereditary properties for numerous other structures (two examples being oriented graphs and permutations), have been extensively investigated by many people, including Balogh, Morris and the first author. There is by now an enormous literature: for more, the reader is advised to begin with the survey [4].

The size of the hereditary property $\mathcal{P}$ is determined by the probability that a random graph $\mathcal{G}(n, 1/2)$ has property $\mathcal{P}$, and the corresponding probability in the model $\mathcal{G}(n,p)$ has been studied by us in [8]. In [16] Marchant and the second author show how these probabilities can be calculated more readily because of their relationship to extremal functions for weighted multigraphs, as developed in [15] and surveyed in [28].

Finally, we remark that a different approach to several of these results, avoiding the use of Szemerédi's Lemma, was found by Saxton and the second author [25].

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Cycles and Paths in Triangle-Free Graphs

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Summary. Let G be a triangle-free graph of order n and minimum degree $\delta > n/3$. We will determine all lengths of cycles occurring in G. In particular, the length of a longest cycle or path in G is exactly the value admitted by the independence number of G. This value can be computed in time $O(n^{2.5})$ using the matching algorithm of Micali and Vazirani. An easy consequence is the observation that triangle-free non-bipartite graphs with $\delta \geq \frac{3}{8}n$ are hamiltonian.

1. Introduction

In recent years a lot of research was performed on sufficient conditions for cycles in graphs which do not contain certain graphs as induced subgraphs. Mostly one of the forbidden subgraphs is the claw $K_{1,3}$.

A completely different class of graphs are triangle-free graphs, i.e. graphs containing no K_3 (which is always an induced subgraph). Note that no graph with maximum degree more than 2 is claw-free and triangle-free at the same time. Most attention focused on a very special class of triangle-free graphs, namely bipartite graphs. Our main objective are cycle lengths in triangle-free non-bipartite graphs.

Häggkvist [14] proved that computing the circumference (i.e. the length of a longest cycle) in general graphs of order n with minimum degree $\delta > \left(\frac{1}{2} - \varepsilon\right) n$ is $\mathcal{NP}$-hard for every $\varepsilon > 0$. Note that for $\varepsilon = 0$ the circumference is n by Dirac's celebrated result [12]. It seems even difficult to determine precisely the circumference or the length of a longest path in terms of other invariants of the graph (whose determination might be $\mathcal{NP}$-hard themselves).

The situation changes if we consider triangle-free graphs. The main observation in this paper is that in triangle-free graphs with minimum degree $\delta > n/3$ the circumference is exactly $\min\{n, 2(n - \alpha)\}$ and the length of a longest path is $\min\{n - 1, 2(n - \alpha)\}$ where α denotes the independence number of the graph, and both values can be computed in polynomial time. A simple derivation of the circumference result is that

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triangle-free non-bipartite graphs with minimum degree $\delta \geq 3n/8$ are hamiltonian.

Moreover, we obtain that triangle-free non-bipartite graphs with $\delta > n/3$ are **weakly pancyclic**, i.e. they contain all cycles between their girth (length of a shortest cycle) and circumference. This settles the triangle-free part of a result of Brandt, Faudree and Goddard (see [7]) who prove that every graph with minimum degree $\delta \geq (n+2)/3$ is weakly pancyclic or bipartite.

For 2-connected graphs with an odd cycle of length at most 5, Brandt, Faudree and Goddard [8] proved that even the degree condition $\delta > n/4 + c$ for a moderately large constant c suffices to ensure that the graph is weakly pancyclic.

Very recently, substantial progress was obtained concerning cycles in triangle-free non-bipartite graphs. Dingjun Lou [15] proved that triangle-free non-bipartite graphs $\neq C_5$ satisfying the famous Chvátal-Erdős condition for hamiltonian graphs [11] are weakly pancyclic with girth 4 and circumference n, thereby answering a conjecture of Amar et al. [3] in the affirmative. Bauer, van den Heuvel and Schmeichel [5] proved that there are triangle-free graphs with arbitrary large toughness which contradicts a conjecture of Chvátal [10] saying that there is a constant t_0 such that t_0-tough graphs are pancyclic. Subsequently Alon [1] determined graphs with arbitrary large girth and toughness.

Since there are still extensive variations in the notation of standard invariants of a graph $G = (V(G), E(G))$ in graph-theoretical literature we give a brief collection of the notation used in this paper:

$\alpha(G)$ (vertex-)independence number of G,
$\kappa(G)$ (vertex-)connectivity of G,
$\nu(G)$ edge-independence number (or matching number) of G,
$\omega(G)$ number of components of G,
$\delta(G)$ minimum degree of G,
$\Delta(G)$ maximum degree of G,
$c(G)$ circumference of G, i.e. the length of a longest cycle,
$p(G)$ length (i.e. number of edges) of a longest path of G,

If there are no ambiguities we frequently omit the explicit reference to the graph, by simply writing α, κ, ν, etc. For the meaning of these parameters we refer the reader to introductory graph theory literature, e.g. [9]. As usual $|G|$ denotes the order of G. Moreover, for a subset $S \subseteq V(G)$ we denote the induced subgraph of $V(G) \setminus S$ by $G - S$, and for $v \in V(G)$ we denote the number of neighbors of v in S by $d(v, S)$. If H is a fixed subgraph of G we briefly write $G - H$ and $d(v, H)$ instead of $G - V(H)$ and $d(v, V(H))$, respectively.

The **toughness** of $G \neq K_n$ is

$$\tau(G) := \min_{s} |S|/\omega(G - S)$$

where the minimum is taken over all separating vertex sets, i.e. sets $S \subseteq V(G)$ with $\omega(G - S) > 1$. A graph is called t-tough if $\tau \geq t$. Cycles and paths on k vertices are denoted by C_k (P_k, resp.).

Finally, for a graph G and a fixed subgraph H we say that H is **matching compatible**, if $\nu(G - H) \geq \nu(G) - \lfloor \frac{1}{2}|H| \rfloor$. Note that for graphs G with a perfect matching there are only even order subgraphs which are matching compatible. Matching compatible paths and cycles in bipartite graphs and digraphs were considered by Amar and Manoussakis in [4].

2. Main Results

Theorem 1. *Let G be a triangle-free non-bipartite graph of order n and independence number α. If the minimum degree $\delta > n/3$ then G is weakly pancyclic with $c(G) = \min\{n, 2(n - \alpha)\}$ and girth 4 unless $G = C_5$.*

Theorem 2. *Let G be a triangle-free graph of order n and independence number α. If $\delta \geq n/3$ then $p(G) = \min\{n - 1,\ 2(n - \alpha)\}$.*

Consider the graphs $G_1(r)$ and $G_2(r)$ obtained from C_5 by replacing four vertices by a set of r independent vertices and the fifth vertex by $2r$ ($2r + 1$, resp.) independent vertices and by joining two sets by a complete bipartite graph whenever the original vertices in C_5 were adjacent (see Fig. 1).

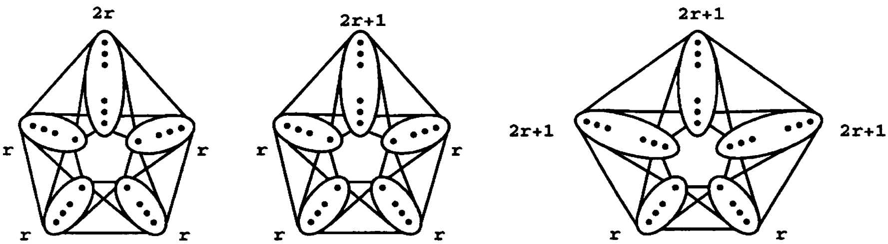

Fig. 1 The graphs $G_1(r)$, $G_2(r)$ and $G_3(r)$.

The graph $G_1(r)$ has minimum degree $\delta = n/3$ and independence number $\alpha = n/2$ but no hamiltonian cycle, thus Theorem 1 is best possible. Moreover the Petersen graph has $\delta = (n - 1)/3$ and $\alpha = 4$ but it contains only cycles of lengths 5, 6, 8, 9, so it is not weakly pancyclic. The graph $G_2(r)$ shows that Theorem 2 is best possible, since it has $\delta = (n - 1)/3$ and $\alpha = (n + 1)/2$ but no hamiltonian path.

As an easy consequence of Theorem 1 we obtain a degree bound for triangle-free non-bipartite graphs to be hamiltonian.

Theorem 3. *Let G be a triangle-free non-bipartite graph of order n. If $\delta \geq 3n/8$ then G is hamiltonian.*

Consider the graph $G_3(r)$ obtained from C_5 by replacing two consecutive vertices of C_5 by r independent vertices and the remaining three vertices by $2r+1$ vertices each and by joining sets in the same way as for $G_1(r)$ and $G_2(r)$ (see Fig. 1). This graph has $\delta = (3n-1)/8$ but is non-hamiltonian, thus Theorem 3 is best possible.

There is a related result for bipartite graphs involving smaller degree constraints, which is easily obtained by combining results from [2] and [4]. We will denote the graph consisting of two complete balanced bipartite graphs $K_{r,r}$ intersecting in one vertex by $H(r)$.

Theorem 4. *Let G be a bipartite graph of order $n \geq 3$. If $\delta > n/4$ then $c(G) = 2(n-\alpha)$ and G contains all even length cycles between 4 and $c(G)$ unless $G = C_6$, $H(r)$.*

Now, by combining Theorem 1 with Theorem 4 we immediately obtain the following Corollary.

Corollary 1. *Let G be a triangle-free graph of order $n \geq 3$. If $\delta > n/3$ then $c(G) = \min\{n,\ 2(n-\alpha)\}$ and G contains all even length cycles between 4 and $c(G)$ unless $G = C_5$.*

As a consequence of this observation we will show that the circumference of such graphs can be computed in polynomial time.

Theorem 5. *The circumference of every triangle-free graph of order n with $\delta > n/3$ can be computed in time $\mathcal{O}(n^{2.5})$.*

This result is based on the matching algorithm of Micali and Vazirani [16]. H. J. Veldman [19] proved that computing the circumference in the class of bipartite graphs with $\delta > \left(\frac{1}{4} - \varepsilon\right) n$ is $\mathcal{NP}$-hard for every $\varepsilon > 0$ by a variation of Häggkvist's simple construction for general graphs [14]. It would be interesting to know whether there is a constant $c < 1/3$ such that the determination of the circumference in triangle-free graphs with $\delta > cn$ is still polynomial.

3. Matchings and Independence Number

The number of edges in a maximum matching $\nu(G)$ of any graph G of order n is bounded by t

$$\nu(G) \leq \min\{\lfloor \tfrac{1}{2}n \rfloor,\ n - \alpha(G)\}. \tag{3.1}$$

Our first step will be to show that for triangle-free graphs with $\delta(G) \geq n/3$ equality holds in (3.1).

For the further results in this paragraph it is helpful to observe the following sequence of inequalities for a triangle-free graph G:

$$\alpha(G) \geq \Delta(G) \geq \delta(G) \geq \kappa(G).$$

The second and third inequality hold in arbitrary graphs. The following result ensures that for triangle-free Paths with $\delta \geq n/3$ the third inequality is an equality.

Lemma 1. *Let G be a triangle-free graph. Then $\kappa(G) \geq \min\{\delta(G), 4\delta(G) - n\}$. In particular, if $\delta(G) \geq n/3$ then $\kappa(G) = \delta(G)$.*

Proof. If $\kappa(G) = \delta(G)$ there is nothing to prove. So assume $\kappa(G) \leq \delta(G) - 1$ and let S be a disconnecting set of cardinality $\kappa(G)$ and G_1 and G_2 component of $G - S$. Let $v_i \in V(G_i)$, $i = 1, 2$. Since $\kappa(G) < \delta(G), v_i$ has a neighbor $\omega_i \in V(G_i)$. Since G is triangle-free v_i and w_i have no common neighbor. So from

$$4\delta(G) \leq \sum_{i=1,2}(d(v_i) + d(w_i)) \leq |G_1| + |G_2| + 2|S| \leq n + \kappa(G)$$

we derive $\kappa(G) \geq 4\delta(G) - n$. □

In the next result it will be shown that for triangle-free graphs with $\delta > n/3$ which are not 1-tough the complements of maximum independent sets determine the toughness.

Theorem 6. *Let G be a triangle-free graph with $\delta \geq n/3$ and toughness $\tau < \delta/(n - 2\delta + 1)$. Then $\tau = (n - \alpha)/\alpha$.*

Proof. Obviously $\tau \leq (n - \alpha)/\alpha$ so it remains to prove "$\geq$". Suppose S is a separating set with $\tau = |S|/\omega(G - S) < (n - \alpha)/\alpha$. Then $G - S$ is not the empty graph, so there is an edge vw in $G - S$. By Lemma 1 we have

$$|S| \geq \kappa(G) = \delta(G),$$

and since all neighbors of v and w are distinct, and lie in the same component of $G - S$ or in S we get

$$\omega(G - S) \leq n - 2\delta + 1$$

thus $\tau = |S|/\omega(G - S) \geq \delta/(n - 2\delta \ + 1)$, a contradiction. □

Note that $G_1(r)$ is a graph with $\delta = n/3$ and $\alpha = n/2$ which is not 1-tough, So its toughness is not determined by the complements of maximum independent sets. While 1-toughness is a necessary condition for being hamiltonian it is a sufficient condition for even order graphs to contain a perfect matching. This follows from a famous result of Tutte [18] saying that a graph contains a perfect matching if and only if for every subset S of the vertex set $o(G - S) \leq |S|$ holds, where $o(G - S)$ denotes the number of odd order components of $G - S$. We will need the defect version of this result, which was first discovered by Berge [6].

Theorem 7. *The maximum number of independent edges of a graph G is*

$$\nu(G) = \min_{S \subseteq V(G)} \frac{1}{2}(n - o(G - S) + |S|).$$

Now the main result of this paragraph is an easy consequence of the two previous results:

Theorem 8. *If G is triangle-free with $\delta \geq n/3$ then*

$$\nu(G) = \min\{\lfloor \tfrac{1}{2}n \rfloor,\, n-\alpha\}.$$

Proof. By (3.1) it suffices to show $\nu(G) \geq \min\{\lfloor \frac{1}{2}n \rfloor,\, n-\alpha\}$. If the toughness $\tau \geq \delta/(n-2\delta+1)$ then for every S we have

$$\omega(G-S) \leq |S| \left(\tfrac{1}{3}n+1\right)/\tfrac{1}{3}n = |S| + 3|S|/n < |S|+2$$

since $|S| < 2n/3$ (otherwise $\omega(G-S) \leq n/3 < |S|$). So $o(G-S) \leq \omega(G-S) \leq |S|+1$ and using Theorem 7 we get $\nu(G) \geq (n-1)/2$, thus $\nu(G) = \lfloor \frac{1}{2}n \rfloor$.

Otherwise using Theorem 6 we have for every S

$$o(G-S) - |S| \leq \omega(G-S) - |S| \leq \alpha - (n-\alpha) = 2\alpha - n$$

implying $\nu(G) \geq n-\alpha$ again using Theorem 7. □

Paths

We are now going to prove Theorem 2.

Proof of Theorem 2. Clearly a longest path has at most $s = 1 + \min\{n-1, 2(n-\alpha)\}$ vertices. Moreover, by Theorem 8, $\nu(G) = \min\{\lfloor \frac{1}{2}n \rfloor, (n-\alpha)\}$. First observe that G contains a path which is matching compatible, having odd order if $\nu(G) < n/2$. Let $P = P_\ell$ be a longest such path and suppose $\ell < s$. Let v and w be the end vertices of P. Note that the subgraph induced by P cannot be hamiltonian, since by Lemma 1 the graph G is connected, yielding a longer matching compatible path. Thus, by Ore's Lemma [17], $\delta \leq \min\{d(v,P), d(w,P)\} \leq (\ell-1)/2$, implying $\ell \geq \frac{2}{3}n+1$. Since $\ell < s$ we have $\nu(G-P) \geq 1$ so there is an edge xy of a maximum matching in $G-P$. Again we calculate $d(x, G-P) + d(y, G-P) \leq n-\ell$ which implies

$$d(x,P) + d(y,P) \geq 2\delta - (n-\ell) \geq \ell - n/3 > (\ell-1)/2$$

since $\ell > \frac{2}{3}n-1$. By the maximality of ℓ neither v nor w can have a neighbor in $\{x,\ y\}$. So there must be two consecutive vertices in $P-v-w$ both having a neighbor in $\{x,\ y\}$. Since G is triangle-free this contradicts the maximality of ℓ. Thus $\ell = s$ so, indeed, $p(G) = s-1$. □

4. Lassos

An ℓ-lasso $L_{k,\ell}$, $k \geq \ell \geq 3$, consists of a path $x_1x_2\ldots x_{k-1}x_k$ with the additional edge x_1x_ℓ and we refer to k as the **length** of $L_{k,\ell}$. A lasso is **even** or **odd** according to ℓ being even or odd. In our proofs 5-lassos $L_{k,5}$ are very important, and we will always assume that the vertices of a 5-lasso are labeled as in Fig. 2. Note that in graph theoretical literature the lassos are also called cups or lollipops.

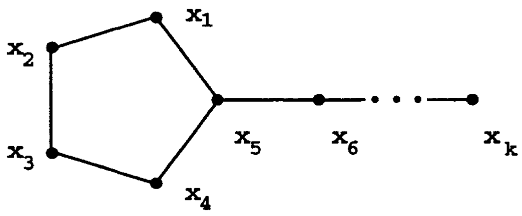

Fig. 2 The 5-lasso.

Lassos and Cycles

The only aim of this part is to discover a surprising property of triangle-free graphs. The existence of a subgraph $L_{k,5}$, in a triangle-free graph with $\delta \geq n/3$ implies the existence of cycles of all lengths between 5 and $k-1$. Here $n/3$ is best possible since the Petersen graph has $\delta = (n-1)/3$ and contains $L_{10,5}$ but does not contain C_7. For the proof it is helpful to define 5-lassos with the additional edge x_3x_6 which we will denote by $L^+_{k,5}$ (see Fig. 3).

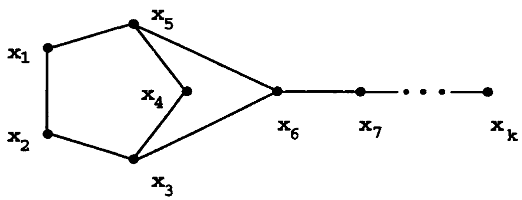

Fig. 3 The lasso $L^+_{k,5}$.

Theorem 9. *Let G be a triangle-free graph containing a lasso $L = L_{k,5}$ for $k \geq 5$. If $\delta \geq n/3$ then G contains C_ℓ for all ℓ, $5 \leq \ell \leq k-1$. Moreover, if L is a longest 5-lasso then G contains $C = C_{k-2}$ such that $G - C$ contains an edge.*

Proof. We will show that the following two claims are true whenever G contains $L_{k,5}$:

(1) G contains C_{k-1},
(2) G contains $L_{k+1,5}$ or G contains a C_{k-2} where $G - C_{k-2}$ contains an edge.

Since with $L_{k,5}$ all shorter 5-lassos are contained in G the claims (1) and (2) prove our statement.

For a lasso $L = L_{k,5}$ define $X_k = \{i | x_k x_{i-1} \in E(G)\}$, and $X_j = \{i | x_j x_i \in E(G)\}$ for $1 \le j < k$.

First suppose that G contains a lasso $L = L_{k,5}$ but no subgraph isomorphic to $L^+_{\kappa,5}$. So x_k has at most one neighbor in $\{x_1, x_2, \ldots, x_5\}$ and by symmetry we may assume $x_k x_1 \notin E(G)$. Moreover, x_6 is adjacent to neither x_2 nor x_3. If a vertex $v \in V(G - L)$ is adjacent to two vertices of x_2, x_3 and x_k then it is easily seen that G contains a longer 5-lasso and a C_{k-1} because G is triangle-free. So $d(x_2, G - L) + d(x_3, G - L) + d(x_k, G - L) \le n - k$.

Consider $X = X_2 \cup X_3 \cup X_k$. By the above reasoning on the neighbors of x_6 and x_k and by the observation that x_5 is adjacent to neither x_2 nor x_3 we get that either $5 \notin X$ or $6 \notin X$ thus $|X| \le k - 1$. On the other hand we have

$$d(x_2, L) + d(x_3, L) + d(x_k, L) \ge 3\delta - (n - k) \ge k > |X|.$$

Thus there is an index i such that $i \in X_k$ and $i \in X_j$ for $j \in \{2, 3\}$ since G is triangle-free. So, again, it is easily verified that G contains C_{k-1} and a C_{k-2} with an edge in $G - C_{k-2}$.

Now assume $L^+ = L^+_{k,5} \subseteq G$. For $k = 6$ the lasso $L^+_{6,5}$ contains C_5 and a C_4 with an edge in $G - C_4$ so assume $k \ge 7$. Consider this time x_1, x_2 and x_k. By the same reasoning as above we are done if there is a vertex in $G - L^+$ adjacent to two of them. Since G is triangle-free $X_1 \cap X_2 = \emptyset$ and $X_1 \cup X_2 \subseteq \{1, 2, 3, 5, 7, 8, \ldots, k\}$. Moreover, we are done if x_k is adjacent to one vertex of x_1, x_2, x_3 or x_5, so we may assume $X_k \subseteq \{5, 7, 8, \ldots, k\}$. Thus we have $X = X_1 \cup X_2 \cup X_k \subseteq \{1, 2, 3, 5, 7, 8, \ldots, k\}$, in particular $|X| \le k - 2$. Again we calculate

$$d(x_1,\ L^+) + d(x_2,\ L^+) + d(x_k,\ L^+) \ge 3\delta - (n - k) \ge |X| + 2$$

so X_k and $X_1 \cup X_2$ must intersect in an index $i \ne 5$, yielding a C_{k-1} and a C_{k-2} with an edge in $G - C_{k-2}$. □

Lassos and Matchings

The previous observation suggests looking for long 5-lassos in triangle-free graphs in order to obtain many cycles at once. We will now examine how to determine the length of a longest 5-lasso in such a graph. The starting point will be a powerful result relating the length of a longest path to the length of a longest cycle.

Theorem 10 (**Enomoto, van den Heuvel, Kaneko, Saito** [13]). *Let G be a graph of order n. If every set of three independent vertices u, v, w satisfies $d(u)+d(v)+d(w) \ge n$ then $c(G) \ge p(G)$ or G is a spanning subgraph of one of six exceptional families of graphs each of which has connectivity $\kappa < n/3$.*

For our purposes it suffices to replace the degree sum condition by $\delta \geq n/3$. Note that the condition $c(G) \geq p(G)$ for connected graphs implies equality unless G is hamiltonian where $c(G) = p(G) + 1$.

Lemma 2. *If G is a triangle-free non-bipartite graph of order $n \geq 6$ with $\delta \geq n/3$ then G contains a lasso $L = L_{k,5}$ which is matching compatible and where k is odd if n is odd and $\alpha < n/2$.*

Proof. First assume $\alpha \geq n/2$. Take an independent set A, $|A| = \alpha$, and a maximum matching M, $|M| = n - \alpha$, which exists by Theorem 8. Note that M saturates every vertex of $G - A$. Since G is not bipartite $G - A$ contains an edge vw. Moreover, because $2\delta > n - \alpha$ the matching neighbors v' and w' of v and w, respectively, must have a common neighbor $u \in G - A$. Now the graph spanned by the set $\{u, u', v, v', w, w'\}$ where u' is the matching neighbor of u contains a lasso $L = L_{6,5}$ and $G - L$ contains a matching on $\nu - 3$ edges.

For $\alpha < n/2$ we will first show that G contains an odd lasso $L_{k,\ell}$ where $G - L_{k,\ell}$ has a perfect matching. From Theorem 2 we obtain that $p(G) \geq n-1$. Since G satisfies the hypothesis of Theorem 10 and using Lemma 1 we infer that G cannot be an exceptional graph, so $c(G) \geq n - 1$. If $c(G) = n$ then every hamiltonian cycle has an even length chord since G is non-bipartite. This yields a spanning odd lasso $L_{n,\ell}$. If $c(G) = n - 1$ then we have $L_{n,n-1}$ and we are done if $n - 1$ is odd. Otherwise recall the standard labeling of the lasso vertices. If x_n is adjacent to an odd labeled vertex x_ℓ then we obtain $L_{n,\ell+2}$. If x_n is adjacent only to even labeled vertices then since $\alpha < n/2$ two even labeled vertices must be adjacent. So we get an even length chord splitting the cycle into an odd cycle where x_n has a neighbor and an odd length path. So we get a perfect matching from the path and an odd lasso.

Take a smallest order odd lasso $L = L_{k,\ell}$ where $G - L$ has a perfect matching. Such a lasso exists by the above reasoning. Clearly we may assume $k \leq \ell + 1$ since otherwise we add the edge $x_k x_{k-1}$ to the matching. Note that the cycle of the lasso is an induced cycle in G since every chord yields a shorter odd lasso and a perfect matching. If $\ell = 5$ we are done so suppose $\ell \geq 7$. Now neither x_3 nor x_4 can be adjacent to $x_{\ell+1}$ so for $X = \{x_3, x_4, x_\ell\}$ we calculate

$$\sum_{x \in X} d(x, G - L) \geq 3\delta - 6 - (k - \ell) > n - k.$$

So one of the $(n - k)/2$ matching edges in $G - L$ has at least 3 neighbors in X. It is now easily checked that in all cases we obtain a shorter odd lasso L' where $G - L'$ has a perfect matching. □

It should be mentioned that there is a direct way of proving Lemma 2 without using the powerful Theorem 10, at the expense of some tedious case analysis. However this might be of interest for giving an algorithm which actually finds cycles of given length in triangle-free graphs with $\delta > n/3$.

Proposition 1. *Every triangle-free non-bipartite graph G with $\delta > n/3$ contains a lasso $L_{s,5}$ where $s = \min\{n, 2(n-\alpha)\}$.*

Proof. Clearly every 5-lasso in G can have at most s vertices since at most half of its vertices form an independent set. Let r be the length of a longest lasso $L = L_{r,5}$ contained in G which is matching compatible and which has odd order if $\alpha < n/2$ and n is odd. By Lemma 2 such a lasso exists and it is easily observed that L has even order if $\alpha \geq n/2$ and order of the same parity as the order of the graph otherwise. Consider the end vertex x_r of the lasso. If x_r has a neighbor in $G - L$ then the maximality requirement for r is obviously violated. So from $(n+1)/3 \leq d(x_r, L) \leq (r-1)/2$ we obtain $r \geq (2n+5)/3$.

Now consider a matching edge vw in $G - L$. First observe that v and w have together at most 4 neighbors in $\{x_1, \ldots, x_6\}$, since if one vertex, say v, is adjacent to x_6 the other vertex w cannot be adjacent to a vertex with a smaller index without enlarging the length of the lasso by two. Since $d(v, G-L) + d(w, G-L) \leq n - r$ we obtain

$$d(v,L) + d(w,L) \geq (2n+2)/3 - n + r = r - (n-2)/3 \geq (r+3)/2.$$

Thus

$$d(v, \{x_7, \ldots, x_r\}) + d(w, \{x_7, \ldots, x_r\}) \geq (r-5)/2.$$

Since G is triangle-free, and v and w are both non-adjacent to x_r they must be adjacent to two consecutive vertices on the path $x_6 x_7 \ldots x_{r-1}$, contradicting the maximality of the lasso. So we conclude that the lasso must have at least $2v(G)$ vertices and one vertex more if $\alpha < n/2$ and n is odd. Now Theorem 8 completes the proof. □

5. Proofs of the Main Results

Non-bipartite Graphs

In order to carry out the proofs of Theorems 1 and 3 we need a better bound for the independence number than the obvious bound $\alpha \leq n - \delta$ holding for arbitrary graphs.

Lemma 3. *For any triangle-free non-bipartite graph the inequality* $\alpha \leq \min\{2(n-2\delta), n - \delta \ - 1\}$ *holds.*

Proof. Let A be an independent set on α vertices. Since G is non-bipartite $G - A$ contains an edge xy. Since no vertex from A is adjacent to both x and y we get $\alpha + \delta < n$. implying $\alpha \leq n - \delta - 1$. Now assume $d(x, G-A) \geq d(y, G-A)$. Since every neighbor of x in A has all its neighbor in $G - A$ we have $d(x, G-A) \leq (n-\alpha) - \delta$. Furthermore x and y have no common neighbor in A. Thus

$$2\delta \leq d(x) + d(y) \leq 2(n - \alpha - \delta) + |A|,$$

implying $\alpha \leq 2(n - 2\delta)$. □

We are now prepared to prove the main result.

Proof of Theorem 1. First note that there is no graph of order $n < 8$ satisfying the hypothesis. By Proposition 1 the graph G contains a lasso $L = L_{r,5}$ where $r = \min\{n,\ 2(n-\alpha)\}$. In particular G contains C_5. A simple counting argument proves that there must be a vertex in $G - C_5$ having two neighbors in C_5, hence G contains C_4.

Now consider L. Theorem 9 implies that G contains all cycles of lengths $5 \leq \ell \leq r-1$ and a cycle $C = C_{r-2}$ where $G - C$ contains an edge vw. Using Lemma 3 we get $\alpha \leq (2n-4)/3$, thus $r \geq (2n+8)/3$. Since $d(v, G - C) + d(w, G - C) \leq n - r + 2$ we get

$$d(v, C) + d(w, C) \geq 2\delta - (n - r + 2) > |C|/2.$$

Thus there must be two consecutive vertices on the cycle each having a neighbor in vw. Since G is triangle-free this provides the missing C_r. □

Proof of Theorem 3. This is immediate from Theorem 1 since by Lemma 3 we obtain $\alpha(G) \leq n/2$. □

Bipartite Graphs

The following result of Amar and Manoussakis [4, Corollary 4] settles the circumference part of Theorem 4.

Theorem 11. *Let G be bipartite with bipartition $A \cup B$, $|A| \leq |B|$. If $\delta > |A|/2$ then $c(G) = 2|A|$ or $G = H(r)$.*

Now the following local condition of Amar [2] provides all the even length cycles but the 4-cycle (actually Amar's result gives more information than we mention here).

Theorem 12. *Suppose C is a hamiltonian cycle in a bipartite graph G and x and x^{++} are vertices at distance 2 in C such that the neighborhoods of x and x^{++} are not identical. If $d(x) + d(x^{++}) \geq (n+1)/2$ then G contains all even length cycles between 6 and n.*

Proof of Theorem 4. For a bipartition $A \cup B$, $|A| \leq |B|$, fix a cycle C of length $2|A|$ which is guaranteed by Theorem 11 and consider the subgraph of G induced by C. This graph contains every vertex of A. If $|A| = 2$ the conclusion is certainly true so assume $|A| \geq 3$. Thus $|B \cap C| \geq 3$ and it is easily calculated that among any three vertices of $B \cap C$ there is a pair whose neighborhoods intersect in at least two vertices (unless $G = C_6$), so G contains C_4. If the neighborhoods of all vertices in $B \cap C$, are identical then the subgraph induced by C is complete balanced bipartite otherwise there are two vertices in $B \cap C$ at distance 2 in C satisfying the hypothesis of Theorem 12. So in any case G contains cycles from 4 through $2|A|$. Since B is an independent set $2|A| \geq 2(n-\alpha)$. □

Computing the Circumference

Proof of Theorem 5. Determine a maximum matching M in $\mathcal{O}(n^{2.5})$ time using the algorithm of Micali and Vazirani [16]. By Corollary 1 and Theorem 8 we have $c(G) = 2\nu(G)$ unless $\nu(G) = (n-1)/2$ and $\alpha(G) \neq (n+1)/2$, in which case $c(G) = n$. So if $\nu = (n-1)/2$ select a set S' of vertices as follows: take the vertex outside the matching and the matching neighbors of its neighbors. Define S as the set of vertices having no neighbor in S'. Since $|S'| \geq \delta + 1$ and $|N(v) \cup N(w)| \geq 2\delta$ for every edge vw, at most one vertex of every matching edge is in S. It is easily checked that $\alpha = (n+1)/2$ if and only if S is an independent set of cardinality $(n+1)/2$. Finally observe that computing S and checking its independence can be done in a straightforward way in $\mathcal{O}(n^2)$ time, once the matching is given. □

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Problems in Graph Theory from Memphis

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Summary. This is a summary of problems and results coming out of the 20 year collaboration between Paul Erdős and authors.

1. Introduction (by Paul Erdős)

Over a 20 year period, my friends at Memphis State University and I have done a great deal of work in various branches of combinatorics and graph theory. We have published more than 40 papers and posed many problems. Here I only mention two examples. We wrote the first paper on the size Ramsey number [35] and stated many problems and conjectures in this subject. Some of our questions have been answered by Beck [3] but many of them are still open, and the subject is very much alive. We also proved the conjecture of Hajnal and myself concerning monochromatic domination in edge colored complete graphs.

During my present visit, many new problems were raised. I state only one. Let $f(n,k)$ be the smallest integer for which there is a graph with k vertices and $f(n,k)$ edges in which every set of $n+2$ vertices induces a subgraph with maximum degree at least n. This problem was raised in trying to settle an old conjecture of ours: is it true that for every $m \geq 2n$ every graph with m vertices and $\binom{2n+1}{2} - \binom{n}{2} - 1$ edges is the union of a bipartite graph and a graph every vertex of which has degree less than n. Faudree has a very nice proof for $m = 2n+1$. We tried to prove it for $m = 2n+2$ and this led us to our problem.

I hope that if I live we will have many more new problems and results.

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2. Generalized Ramsey Theory

2.1. Terminology and Notation

Let $G_1, G_2, \ldots, G_k$ be a sequence of graphs, each having no isolated vertices. The *Ramsey number* $r(G_1, G_2, \ldots, G_k)$ is the smallest integer r so that in every k-coloring of the edges of K_r, there is a monochromatic copy of G_i in the ith color for at least one i. Most of the known results concern the two-color case. In case $k = 2$ and $G_1 = G_2 = G$, we denote the Ramsey number by $r(G)$. Throughout this paper, the notation will generally follow that of the well-known text by Chartrand and Lesniak [21]. In particular, the graph theoretic functions $\delta(G)$, $\Delta(G)$, $\beta(G)$ and $\chi(G)$ are used to represent the minimum degree, maximum degree, independence number, and chromatic number, respectively, of the graph G.

2.2. Cycle Versus Complete Graph

The cycle-complete graph Ramsey number $r(C_m, K_n)$ was first studied by Bondy and Erdős in [5]. They obtained the exact value in case m is large in comparison with n.

Theorem 1 ([**5**]). *For $m \geq n^2 - 2$,*

$$r(C_m, K_n) = (m-1)(n-1) + 1.$$

Question 1 ([**34**]). *What is the smallest value of m such that*

$$r(C_m, K_n) = (m-1)(n-1) + 1?$$

It is possible that $r(C_m, K_n) = (m-1)(n-1) + 1$ for all $m \geq n$.

Additional results concerning cycle-complete graph Ramsey numbers were obtained in [34], among them the following upper bound.

Theorem 2 ([**34**]). *For all $m \geq 3$ and $n \geq 2$,*

$$r(C_m, K_n) \leq \{(m-2)(n^{1/k} + 2) + 1\}(n-1),$$

where $k = \lfloor (m-1)/2 \rfloor$.

For m fixed and $n \to \infty$, the probabilistic method [57] yields

$$r(C_m, K_n) > c\left(\frac{n}{\log n}\right)^{(m-1)/(m-2)} \qquad (n \to \infty). \tag{1}$$

The case of $m = 3$ is well studied. By the probability result, $r(C_3, K_n) > c(n/\log n)^2$, and in [56] Shearer improves the technique of Ajtai, Komlós and Szemerédi [1]

$$r(C_3, K_n) < \left\lceil \frac{n^2}{\log(n/e)} \right\rceil.$$

The case of $m = 4$ is comparatively open. As above, the probability method yields

$$r(C_4, K_n) > c\left(\frac{n}{\log n}\right)^{3/2} \qquad (n \to \infty).$$

In [34] we proved that $r(C_4, K_n) = o(n^2)$, and an unpublished observation of Szemerédi yields

$$r(C_4, K_n) < c\left(\frac{n}{\log n}\right)^2. \tag{2}$$

It is expected that

$$\lim_{n\to\infty} \frac{r(C_4, K_n)}{r(C_3, K_n)} = 0,$$

but the known bounds [(1) with $m = 3$ and (2)] just fail to produce such a result.

Conjecture 1. *There exist an $\varepsilon > 0$ such that $r(C_4, K_n) < n^{2-\varepsilon}$ for all sufficiently large n.*

From Theorem 2 and (1) it follows that if m is fixed,

$$r(C_m, K_n) > r(C_{2m-1}, K_n) \quad \text{and} \quad r(C_m, K_n) > r(C_{2m}, K_n),$$

for all sufficiently large n. In view of this, it may be that for n appropriately large, $r(C_m, K_n)$ first decreases as m increases. In view of Theorem 1, we know that $r(C_m, K_n)$ ultimately increases with m. It may be that there is a unique value of m at which the minimum is attained.

Question 2 ([**34**]). *For n fixed, where is the minimum value of $r(C_m, K_n)$ attained?*

Notable progress has been made on some of the cycle—complete graph problems.

Answer 1 ([**51**]). *J. H. Kim has proved that*

$$r(K_3, K_n) = r(C_3, K_n) = \theta(n^2/\log n), \qquad n \to \infty.$$

Answer 2 ([**53**]). *V. Nikiforov has proved that if $m \geq 4n + 2$ then*

$$r(C_m, K_n) = (m-1)(n-1) + 1.$$

2.3. "Goodness" in Generalizes Ramsey Theory

Let F be a graph with chromatic number $\chi(F)$. The *chromatic surplus* of F, denoted $s(F)$, is the least number of vertices in a color class under any proper $\chi(F)$-coloring of the vertices of F. If G is any connected graph of order $n \geq s(F)$, then

$$r(F,G) \geq (\chi(F)-1)(n-1)+s(F).$$

The example which establishes this bound is simple.

Example 1. *With* $p = (\chi(F)-1)(n-1)+s(F)-1$, *two-color the edges of* K_p *so that the blue graph consists of* $\chi(F)$ *disjoint complete graphs,* $\chi(F)-1$ *of order* $n-1$ *and one of order* $s(F)-1$. *In this two-coloring, there is no connected blue subgraph of order* n *and there is no red copy of* F.

In case $r(F,G) = (\chi(F)-1)(n-1)+s(F)$, we shall say that G is F-*good*. Chvátal observed that for each m and each tree T of order n,

$$r(K_m,T) = (m-1)(n-1)+1.$$

In other words, every tree is K_m-good. In large part, the study of "goodness" in Ramsey theory is an attempt to see how far this simple result of Chvátal can be generalized. Known cases where G is F-good involve some restriction on the number of edges and/or maximum degree of G [15, 16, 37].

Theorem 3 ([**38**]). *Given* k, Δ *and* $s = p_1 \leq \cdots \leq p_m$, *there exists a corresponding number* n_0 *such that every connected graph* G *with* $n \geq n_0$ *vertices,* $q \leq n+k$ *edges and maximum degree* $\leq \Delta$ *is* $K(p_1,p_2,\ldots,p_m)$-*good.*

Corollary 1 ([**38**]). *Let* F *be a fixed graph with chromatic number* χ *and chromatic surplus* s. *Set* $\alpha = 1/(2|V(F)|-1)$. *Then there are constants* C_1 *and* C_2 *such that for all sufficiently large* n, *every connected graph* G *of order* n *satisfying* $|E(G)| \leq n + C_1 n^{\alpha}$ *and* $\Delta(G) \leq C_2 n^{\alpha}$ *is* F-*good.*

The *edge density* of a graph G is defined to be $\max\{q(H)/p(H)\}$, where $p(H)$ and $q(H)$ denote the number of vertices and edges, respectively, of H, and the maximum is taken over all subgraphs $H \subset G$. As usual, $\Delta(G)$ denotes the maximum degree in G.

Conjecture 2 ([**38**]). *Let* F *be an arbitrary graph and let* (G_n) *be a sequence of connected graphs such that (i)* G_n *is of order* n, *(ii)* $\Delta(G_n) = o(n)$ *and (iii) each graph in* (G_n) *has edge density at most* ρ *(a constant). Then* G_n *is* F-*good for all sufficiently large* n.

Conjecture 3 ([**38**]). *If* F *is an arbitrary graph and* (G_n) *is a sequence of connected graphs such that* $\Delta(G_n)$ *is bounded, then* G_n *is* F-*good for all sufficiently large* n.

If all trees are F good, it is natural to ask for the largest integer $q = f(F,n)$ so that every connected graph with n vertices and at most q edges is F-good. For $F = K_3$ we have the following result.

Theorem 4 ([**10**]). *Let* $f(K_3,n)$ *be the largest integer* q *so that every connected graph with* n *vertices and at most* q *edges is* K_3-*good. Then* $f(K_3,n) \geq (17n+1)/15$ *for all* $n \geq 4$ *and* $f(K_3,n) < (27/4+\varepsilon)n(\log n)^2$ *for all sufficiently large* n.

Question 3 ([10]). *Does $f(K_3, n)/n$ tend to infinity with n?*

Also, it is appropriate to ask for the largest integer $q = g(F, n)$ so that there exists a connected graph with n vertices and q edges that is F-good. The following result is known concerning $f(K_m, n)$ and $g(K_m, n)$ where $m \geq 3$.

Theorem 5 ([10]). *For each $m \geq 3$, there exist positive constants A, B, C, D such that*

$$n + An^{\frac{2}{m-1}} < f(K_m, n) < n + Bn^{\frac{4}{m+1}}(\log n)^2,$$

and

$$Cn^{\frac{m}{m-1}} < g(K_m, n) < Dn^{\frac{m+2}{m}}(\log n)^{\frac{(m+1)(m-2)}{m(m-1)}},$$

for all sufficiently large n.

The example that provides the lower bound for $g(K_3, n)$ is easily described. (A similar example works in general to provide a lower bound for $g(K_m, n)$.)

Example 2 ([10]). *Choose t and s so that $r(K_3, tK_s) \leq n - 1$. Then set $G = K_1 + H$ where H is the graph consisting of t disjoint copies of K_s together with $n - 1 - ts$ isolated vertices. Then G is a connected graph with n vertices and $t\binom{s}{2} + n - 1$ edges that is K_3-good.*

Are there cases where this example is best possible?

Question 4. *For what values of n (if any) is $g(K_3, n)$ the number of edges in the above example where s and t have been chosen so that $t\binom{s}{2}$ is maximized (or possibly a generalized version in which there are t complete graphs $K_{s_1}, K_{s_2}, \ldots, K_{s_t}$, not all of the same order)?*

In an influential paper [6], Burr and Erdős gave six questions, the resolutions of which would in all likelihood require new and widely applicable techniques. One of these six asks "is the n-dimensional cube K_3-good for all large n?" This question is still open. However, the remaining five are proven to be true in a paper by Nikiforov and Rousseau [54]. The methods of proof are indeed modern—a mix of the regularity lemma, embedding of sparse graphs, Turán type stability, and other structural techniques. They define a class of graphs "degenerate and crumbling" and prove that all sufficiently large graphs from this class are K_p-good. This large class of "degenerate and crumbling" graphs contains well know classes of graphs referenced in [6]. Two examples of a specific results from this paper is the following.

Answer 3 ([54]). *If each edge of K_n is subdivided by one vertex, then the resulting graph is K_p-good for p fixed and n large?*

Answer 4 ([54]). *Suppose $K \geq 1$, $p \geq 3$ and T_n is a tree of order n where $\mathbf{k} = (k_1, \ldots, k_n)$. If $0 < k_i \leq K$ for all $i \in [n]$ then $T_n^{\mathbf{k}}$ is K_p-good for n large, where $T_n^{\mathbf{k}}$ is the graph obtained from T_n by replacing vertex i with a clique K_{k_i} and replacing and an edge ij with a complete bipartite graph K_{k_i,k_j}.*

2.4. Graph Versus Tree

If T is any tree, $r(C_4, T)$ is determined by $r(C_4, K_{1,\Delta(T)})$, where $\Delta(T)$ denotes the maximum degree in T.

Theorem 6 ([17]). *Let T be any tree of order n. Then*

$$r(C_4, T) = \max\{4, n+1, r(C_4, K_{1,\Delta(T)})\}.$$

In view of the importance of $r(C_4, K_{1,n})$, it would be helpful to have more precise information concerning this Ramsey number. It is easy to prove that

$$r(C_4, K_{1,n}) \le n + \lceil \sqrt{n} \rceil + 1,$$

and in [17] we obtain

$$r(C_4, K_{1,n}) > n + \sqrt{n} - 6n^{11/40}, \tag{3}$$

for all sufficiently large n. The proof of the later result uses the fact that for any prime p there is a C_4-free graph $\mathcal{G}_p$ of order p^2+p+1 in which each vertex has degree p or $p+1$. By letting p be the smallest prime greater than $\sqrt{n}$ and randomly deleting a suitable number of vertices from $\mathcal{G}_p$, we obtain a graph that yields (3). The bound is based on information about the distribution of primes. Let p_k denote the k-th prime. If has been conjectured that $p_{k+1} - p_k < (\log p_k)^\alpha$ for some constant α. If this conjecture is true, the lower bound in (3) can be improved to $n + \sqrt{n} - 6\left(\frac{1}{2}\log n\right)^\alpha$.

Question 5 ([17]). *Is it true that $r(C_4, K_{1,n}) < n + \sqrt{n} - c$ holds infinitely often, where c is an arbitrary constant?*

Paul Erdős offers \$100 for a proof (one way or the other).

Question 6 ([17]). *Is it true that $r(C_4, K_{1,n+1}) \le r(C_4, K_{1,n}) + 2$ for all n?*

In [41] it is proved that if $n \ge 3m-3$ then $r(K(1,1,m), T) = 2n-1$ for every tree T of order n. This was shown to be the case when T is a star ($K_{1,n-1}$) by Rousseau and Sheehan [55] who also proved that the condition $n \ge 3m-3$ cannot be weakened. This is one of many examples in Ramsey problems involving trees where the star turns out to be the worst, i.e. largest Ramsey number, case. How general is this phenomenon?

Conjecture 4. *If G is fixed and n is sufficiently large, $r(G, T) \le r(G, K_{1,n-1})$ for every tree T of order n.*

What are the graphs G relative to which all sufficiently large trees are good? An early result showed that this class of graphs includes all those in which some $\chi(G)$-coloring has two color classes consisting of a single vertex.

Theorem 7 ([7]). *Let G be a graph with $\chi(G) \ge 2$ that has a vertex $\chi(G)$-coloring with at least two color classes consisting of a single vertex. Then*

$$r(G, T) = (\chi(G) - 1)(n-1) + 1$$

for every tree T of order $n \ge n_0(G)$.

A complete characterization for the case of $s(G) = 1$ has been obtained by Burr and Faudree [18].

Theorem 8 ([**18**]). *Let G be a graph with chromatic number k. Then for all sufficiently large n, every tree T of order n satisfies $r(G,T) = (k-1)(n-1)+1$ iff there is an integer m, such that G is a subgraph of both the chromatic number k graphs $K_{1,m,m,\ldots,m}$ and $mK_2 + K_{m,m,\ldots,m}$.*

If $G \cong K_{m_1,m_2,\ldots,m_k}$ with $m_1 = 1$, the fact that G has chromatic number k and chromatic surplus 1, gives the lower bound $(k-1)(n-1)+1$ and there is an upper bound that depends only on $r(K_{1,m_2}, T)$.

Theorem 9 ([**42**]). *For all sufficiently large n, every tree T of order n satisfies*

$$(k-1)(n-1)+1 \le r(K_{1,m_2,\ldots,m_k}, T) \le (k-1)(r(K_{1,m_2}, T) - 1) + 1.$$

For $m_1 \neq 1$ it is natural to suspect the following generalization.

Conjecture 5 ([**42**]). *If $m_1 \le m_2 \le \cdots \le m_k$ and T is any sufficiently large tree,*

$$r(K_{m_1,m_2,\ldots,m_k}, T) \le (k-1)(r(K_{m_1,m_2}, T) - 1) + m_1.$$

The *decomposition class* of a graph G, denoted $\mathcal{B}(G)$, is defined as follows: a given bipartite graph B belongs to $\mathcal{B}(G)$ whenever there exists a $\chi(G)$ vertex coloring such that B is the bipartite graph induced by some pair of color classes. Let $r(\mathcal{B}(G), T)$ denote the smallest integer r so that in every two-coloring of $E(K_r)$ there is at least one member of $\mathcal{B}(G)$ in the first color or a copy of T in the second color. In [40] it is shown that if B is a fixed bipartite graph and $T_1, T_2, T_3, \ldots$ is any sequence of trees where T_n is of order n, then $r(B, T_n) = n + o(n)$ as $n \to \infty$. Thus $r(\mathcal{B}(G), T_n) = n + o(n)$. The result of Faudree and Burr can be written as follows: if $\mathcal{B}(G)$ contains K_2, then for all sufficiently large n, every tree T of order n satisfies $r(G,T) = (\chi(G)-2)(n-1) + r(\mathcal{B}(G), T)$. It would be interesting to know how generally this formula holds.

Question 7. *What graphs G satisfy*

$$r(G,T) = (\chi(G) - 2)(n-1) + r(\mathcal{B}(G), T)$$

for every tree T of order $n \ge n_0(G)$?

2.5. Trees

Some very interesting open problems concern the diagonal Ramsey number $r(T)$ where T is a tree. It has been conjectured by Erdős and Sós that each graph with m vertices and at least $(n-2)m/2$ edges contains every tree T of order n; if true, this yields $r(T) \le 2n-2$. A general lower bound for $r(T)$ is given by the following example.

Example 3. *Assume $a \leq b$. Construction (1): Two-color the edges of K_{2a+b-2} so that the red graph is $K_{a-1} \cup K_{a+b-1}$. Construction (2): Two-color the edges of K_{2b-2} so that the red graph is $2K_{b-1}$. In each of these two constructions, there is no monochromatic connected bipartite graph with a vertices in one class and b vertices in the other.*

In view of this example, we see that any tree T with a vertices in one color class and b in the other satisfies

$$r(T) \geq \max\{2a+b-1, 2b-1\}.$$

The *broom* $B_{k,l}$ is the tree on $k+l$ vertices obtained by identifying an endvertex of a path P_l on l vertices with the central vertex of a star $K_{1,k}$ on k edges.

Theorem 10 ([36]).

(1) $r(B_{k,l}) = k + \lceil 3l/2 \rceil - 1$ *for* $k \geq 1$ *and* $l \geq 2k$,
(2) $r(B_{k,l}) \leq 2k + l$ *for* $5 \leq l \leq 2k$.

From the lower bound given by Example 3,

$$r(B_{k,l}) \geq \begin{cases} 2k + 2\lfloor l/2 \rfloor - 1 & \text{when } l < 2k-1, \\ 2k + 2\lfloor l/2 \rfloor & \text{when } l = 2k-1. \end{cases}$$

Thus in case (2) of Theorem 10, $r(B_{k,l})$ differs from the upper bound by at most 2.

The smallest value of $\max\{2a+b-1, 2b-1\}$ occurs when $2a = b$, so Example 3 shows that $r(T) \geq \lceil 4n/3 \rceil$ for every tree of order n. Moreover, this bound cannot be improved since equality holds for certain booms.

Question 8 ([36]). *Is $r(T) = 4a$ for every tree with a vertices in one color class and $2a$ in the other?*

A special case of the previous question is when T is the tree obtained from subdividing $a-1$ edges of the star $K_{1,2a}$ [24].

2.6. Multicolor Results

When the list of graphs $(G_1, G_2, \ldots, G_c)$ is restricted to complete bipartite graphs, odd cycles, and exactly one large cycle (even or odd), there is an exact formula for $r(G_1, G_2, \ldots, G_c)$.

Theorem 11 ([33]). *Let $[B]$ and $[C]$ denote the following fixed sequences of graphs:*

$$[B] = (K_{b_1,c_1}, \cdots, K_{b_s,c_s}) \quad b_i \leq c_i \quad (i = 1, 2, \ldots, s),$$

$$[C] = (C_{2d_1+1}, \ldots, C_{2d_t+1}).$$

Further, let $\ell = \sum_{i=1}^{s}(b_i - 1)$, *and require that* $d_i \geq 2^{t-1}$ *for* $i = 1, \ldots, t$. *Then, if* n *is sufficiently large,* $r(C_n, [B], [C]) = 2^t(n + \ell - 1)$.

All of the results in [33] are for the case where exactly one cycle length is large in comparison with the orders of the remaining graphs $([B], [C])$. It would be quite helpful to have corresponding results when there are two or more large cycles.

Question 9 ([**33**]). *What is* $r(C_n, C_m, [B], [C])$ *when both* n *and* m *are large in comparison with the orders of the remaining graphs?*

Let $[C_{odd}] = (C_{2d_1+1}, \ldots, C_{2d_t+1})$. Then $r(\leq [C_{odd}])$ denotes the smallest number r such that in every t-coloring of the edges of K_r, there is an odd cycle of length at most $2d_i + 1$ in the ith color for some i. Using the proof technique of Theorem 11, we have the following multicolor Ramsey result for cycles.

Corollary 2 ([**33**]). *Let* $[C_{even}] = (C_{2b_1}, \ldots, C_{2b_s})$, $[C_{odd}] = (C_{2d_1+1}, \ldots, C_{2d_t+1})$, *and let* $\ell = \sum_{i=1}^{s}(b_i - 1)$. *If* n *is sufficiently large,*

$$r(C_n, [C_{even}], [C_{odd}]) = (r(\leq [C_{odd}]) - 1)(n + \ell - 1) + 1.$$

For the case of three colors, the following results were found for $r(\leq [C_{odd}])$.

Theorem 12 ([**33**]). *Let* $\ell \geq k \geq m$. *Then*

$$r(\leq (C_{2k+1}, C_{2l+1}, C_{2m+1})) = \begin{cases} 9 & k \geq 4,\ l \geq 2,\ m \geq 1, \\ 11 & k \geq 5,\ l = m = 1. \end{cases}$$

In the remaining cases,

$$r(\leq (C_{2k+1}, C_{2l+1}, C_{2m+1})) = \begin{cases} 9 \\ 10 \\ 12 \\ 17 \end{cases}$$

for (respectively)

$$(k, l, m) = \begin{cases} (3,3,3), (3,3,2), (3,3,1), (3,2,2), (2,2,2), \\ (3,2,1), (2,2,1), \\ (4,1,1), (3,1,1), (2,1,1), \\ (1,1,1). \end{cases}$$

By considering a bipartite decomposition of a complete graph, it is clear that, in general,

$$r(\leq (C_{2m_1+1}, \cdots, C_{2m_t+1})) \geq 2^t + 1.$$

Note that $r(\leq (C_{2k+1}, C_{2l+1}, C_{2m+1})) = 2^3 + 1 = 9$ except for $(k \geq 5)$, $l = m = 1$ and six special cases.

Question 10. *When does $r(\leq (C_{2m_1+1}, \cdots, C_{2m_t+1})) = 2^t + 1$ hold? In particular, what is true for $t = 4$?*

2.7. Ramsey Size Linear Graphs

A graph G is *Ramsey size linear* if there is a constant C such that

$$r(G, H) \leq Cn$$

for every graph H with n edges. The following result is easily proved using the Erdős-Lovász local lemma and the techniques of Spencer [57].

Theorem 13 ([**39, 44**]). *Given a graph G with p vertices and q edges, there exists a constant C such that for n sufficiently large,*

$$r(G, K_n) > \left(\frac{n}{\log n}\right)^{\frac{q-1}{p-2}}.$$

Corollary 3. *If G is a graph with p vertices and $2p - 2$ edges, then G is not Ramsey size linear.*

In the other direction, we have the following result.

Theorem 14 ([44]). *If G is any graph with p vertices and at most $p + 1$ edges, then G is Ramsey size linear.*

There are various small graphs for which the question of Ramsey size linearity is open.

Question 11. *Are the following graphs Ramsey size linear: $K_{3,3}$, H_5 (the graph obtained from C_5 by adding two vertex disjoint chords), Q_3 (the 3-dimensional cube)?*

Question 12 ([**44**]). *If G is a graph such that each subgraph on p vertices has at most $2p - 3$ edges, then is G Ramsey size linear?*

The graph K_4 is not Ramsey size linear, but the deletion of any edge gives a Ramsey size linear graph. It would be of interest to find a specific infinite family of graphs with this property.

2.8. Repeated Degrees and Degree Spread

If $n \geq r(G)$, and the edges of K_n are two-colored, there must be at least one monochromatic copy of G. If there must be such a monochromatic copy of G in which two vertices have the same degree in the two-colored K_n, at least for all sufficiently large n, then we say that G has the *Ramsey repeated* *degree property*. In [2] it was shown that K_m where $m \geq 4$ *does not* have this property.

Theorem 15 ([**20**]). *For each $m \geq 1$, the graphs $K_{m,m}$ and C_{2m+1} have the Ramsey repeated degree property.*

Question 13. *Are there graphs other than K_m for $m \geq 4$ that fail to have the Ramsey repeated degree property?*

The *degree spread* of $X \subseteq V(\mathcal{G})$ is the difference between the largest and smallest degree (in $\mathcal{G}$) over all vertices in X. Given graphs G and H and a two-coloring of the edges of K_n with $n \geq r(G, H)$, the degree spread of the coloring is the minimum degree spread over all vertex sets of copies of G and/or H that are appropriately monochromatic. Then $\Phi_n(G, H)$ is the maximum degree spread over all two-colorings of $E(K_n)$.

Theorem 16 ([**20**]). *If $n \geq 4(r-1)(r-2)$ where $r = r(G, H)$, then $\Phi_n(G, H) = r(G, H) - 2$.*

Question 14 ([**20**]). *What is the smallest n for which $\Phi_n(G, H) = r(G, H) - 2$? Does this relation hold for all $n \geq r(G, H)$?*

3. Size Ramsey Numbers

3.1. Definitions and Notation

Write $F \to (G, H)$ to mean that in every two-coloring of the edges of F, there is a copy of G in the first color or a copy of H in the second. The *size Ramsey number* of the pair (G, H), denoted $\hat{r}(G, H)$, is the smallest number $q = \hat{r}(G)$ for which there exists a graph F with q edges satisfying $F \to (G, H)$. In case $G = H$ we write $F \to G$ for the "arrow" relation and $\hat{r}(G)$ for the size Ramsey number. The size Ramsey number was introduced in [35]. Interestingly enough, the size Ramsey number of K_n is simply expressed in terms of its ordinary Ramsey number:

$$\hat{r}(K_n) = \binom{r(K_n)}{2}.$$

(See [35].) Questions concerning $\hat{r}(P_n)$ were raised in [35]. In [3] Beck proved the rather surprising fact that $\hat{r}(P_n) \leq Cn$ for some constant C

3.2. Complete Bipartite Graphs

The asymptotic behavior of the size Ramsey number of $K_{n,n}$ is studied in [48].

Theorem 17 ([**48**]). *For all $n \geq 6$,*

$$\frac{1}{60} n^2 2^n < \hat{r}(K_{n,n}) < \frac{3}{2} n^3 2^n.$$

The upper bound comes from the fact that $K_{a,b}$ with $a = \lfloor n^2/2 \rfloor$ and $b = 3n2^n$ arrows $K_{n,n}$ for $n \geq 6$. The lower bound comes from a simple application of the probabilistic method using a bound for the number of copies of $K_{n,n}$ in any graph with q edges. Of course, the immediate goal is to bring the bounds for $\hat{r}(K_{n,n})$ to within a constant factor.

Question 15. *Is there a choice of $a = a(n)$ and $b = b(n)$ so that $ab = O(n^2 2^n)$ and $K_{a,b} \to K_{n,n}$?*

3.3. Stars and Star-Forests

Theorem 18 ([8]). *For positive integers k, l, m, and n,*

$$\hat{r}(mK_{1,k}, nK_{1,\ell}) = (m+n-1)(k+l-1).$$

Moreover if $F \to (mK_{1,k}, nK_{1,\ell})$ and F has $(n+m-1)(k+l-1)$ edges, then $F = (m+n-1)K_{1,k+\ell-1}$ or $k = l = 2$ and $F = tK_3 \cup (m+n+t-1)K_{1,3}$ for some $1 \leq t \leq m+n-1$.

This leaves open the size Ramsey number for a pair of star forests. Assume that

$$F_1 = \bigcup_{i=1}^{s} K_{1,n_i} \quad (n_1 \geq n_2 \geq \cdots \geq n_s) \quad \text{and}$$

$$F_2 = \bigcup_{i=1}^{t} K_{1,m_i}, \quad (m_1 \geq m_2 \geq \cdots \geq m_t).$$

For $2 \leq k \leq s+t$ set $l_k = \max\{n_i + m_j - 1 : i + j = k\}$. It is not difficult to prove that $\cup_{k=2}^{s+t} K_{1,\ell_k} \to (F_1, F_2)$ so $\hat{r}(F_1, F_2) \leq \sum_{k=2}^{s+t} \ell_k$.

Conjecture 6 ([8]). $\hat{r}(F_1, F_2) = \displaystyle\sum_{k=2}^{s+t} l_k$.

Note that if $n_i = n$ for all i and $m_j = m$ for all j, then the conjectured value agrees with the number $\hat{r}(sK_{1,n}, tK_{1,m}) = (m+n-1)(s+t-1)$ given in Theorem 18.

3.4. Star Versus Complete Graph

The following result is easily established.

Theorem 19. *The arrow relation $K_{2n+1} - K_n \to (K_{1,n}, K_3)$ holds for each $n \geq 2$.*

Possibly $K_{2n+1} - K_n$ is an arrowing graph with the minimum possible number of edges, and thus $\hat{r}(K_{1,n}, K_3) = \binom{2n+1}{2} - \binom{n}{2} = 3\binom{n+1}{2}$. To verify this, it would be sufficient to prove the following conjecture.

Conjecture 7 ([23]). *For $n \geq 3$, any graph with $\binom{2n+1}{2} - \binom{n}{2} - 1$ edges is the union of a bipartite graph and a graph with maximum degree less than n.*

It is not difficult to verify this conjecture if the graph has at most $2n+1$ vertices.

3.5. Graph Versus Matching

No general result is known for $\hat{r}(G, tK_2)$, but some information about this number can be obtained by considering

$$\hat{r}_\infty(G) = \lim_{t\to\infty} \frac{\hat{r}(G, tK_2)}{t\hat{r}(G, K_2)} = \lim_{t\to\infty} \frac{\hat{r}(G, tK_2)}{t|E(G)|}.$$

Clearly, $0 < \hat{r}_\infty(G) \leq 1$. The following results were obtained in [25].

Theorem 20 ([25]). *The set $\{\hat{r}_\infty(G) : G \text{ is a connected graph}\}$ is dense in $[0, 1]$.*

Theorem 21 ([25]). *For each $n \geq 2$, there is a corresponding positive constant a such that*

$$\frac{1}{n} \leq \hat{r}_\infty(P_n), \hat{r}_\infty(C_n) \leq \frac{a}{n}.$$

Also

$$\frac{1}{n} \leq \hat{r}_\infty(K_{n,n}) \leq \frac{2+\sqrt{2}}{n} \qquad \text{and} \qquad \frac{2}{n} \leq \hat{r}_\infty(K_n) \leq \frac{8}{n}.$$

Thus $\lim_{n\to\infty} \hat{r}_\infty(H_n) = 0$ for $H_n = P_n, C_n, K_{n,n}$ or K_n. On the other hand, we can show that $\lim_{n\to\infty} \hat{r}_\infty(K_{1,n}) = 1$. This suggests the following question.

Question 16 ([25]). *Does $\lim_{n\to\infty} \hat{r}_\infty(G_n) = 0$ hold for every sequence of graphs (G_n) such that $|V(G_n)| \to \infty$ and $\Delta(G_n)$ is bounded as $n \to \infty$? What sequences (G_n) yield $\lim_{n\to\infty} \hat{r}_\infty(G_n) = 1$?*

4. Other Ramsey Problems

4.1. Multiple Copies

Assume $F \to (mG, H)$ where mG denotes m vertex disjoint copies of G. How many copies of G must F contain?

Theorem 22 ([8]). *If $F \to (mG, H)$, then $tG \subseteq F$ where*

$$t = \left\lfloor \frac{m|V(G)| + |V(H)| - \beta(H) - 1}{|V(G)|} \right\rfloor.$$

Question 17 ([8]). *If $F \to (nG)$, then must F contain at least $\left\lfloor \frac{r(nG)}{|V(G)|} \right\rfloor$ copies of G?*

We have shown that if G and $\varepsilon > 0$ are fixed then for all sufficiently large n, every graph F satisfying $F \to (nG)$ contains at least $r(nG)(1-\varepsilon)/|V(G)|$ copies of G.

4.2. Ramsey Minimal Problems

Let F, G, H be graphs without isolates. The graph F is (G,H)-*minimal* if $F \to (G,H)$ but $F - e \not\to (G,H)$ for any edge e of F. The pair (G,H) is called *Ramsey-finite* or *Ramsey-infinite* according to whether the class of all (G,H)-minimal graphs is a finite or infinite set. It has been shown by Nešetřil and Rödl [52] that (G,H) is Ramsey-infinite when at least one of the following hold:

(i) G and H are both 3-connected.
(ii) $\chi(G)$ and $\chi(H)$ are both ≥ 3.
(iii) G and H are both forests, neither of which is a union of stars.

Statement (iii) of the above result has been strengthened in the following way.

Theorem 23. **(i)** [14] *The pair (G,H) is Ramsey-infinite when both G and H are forests, at least one having a non-star component.*
(ii) [11] *If G and H are star forests with no single edge star (K_2), then (G,H) is Ramsey-finite if and only if both G and H are single stars, each with an odd number of edges.*
(iii) [12] *The pair $(K_{1,m} \cup kK_2, K_{1,n} \cup \ell K_2)$ is Ramsey-finite when both m and n are odd.*

Subsequently, Faudree characterized Ramsey-finite pairs of forests.

Theorem 24 ([49]). *If G and H are forests, then (G,H) is Ramsey-finite if and only if*

$$G = \left(\bigcup_{i=1}^{s} K_{1,m_i}\right) \bigcup mK_2, \quad m_1 \geq m_2 \geq \cdots \geq m_s \geq 2,\ m \geq 0,$$

$$H = \left(\bigcup_{i=1}^{t} mK_{1,n_i}\right) \bigcup nK_{1,1}, \quad n_1 \geq n_2 \geq \cdots \geq n_t \geq 2,\ s \geq t \geq 0,$$

and one of the following hold:

(1) $t = 0$ *and* $n > 0$,
(2) $s = t = 1$ *and* n_1, n_1 *are odd,*
(3) $s \geq 2, t = 1,\ m_1, n_1$ *are odd,* $m_1 \geq n_1 + m_2 - 1$ *and* $n \geq n_0 = n_0(G,H)$.

It should be noted that the precise value of $n_0(G,H)$ in (3) is not known. This is the only place where the Ramsey-finiteness question for a pair of star forests is not explicitly settled.

The above results leave a considerable gap when either G or H has connectivity two or less. It is likely that (G,H) is Ramsey-infinite for all pairs not identified above as *Ramsey-finite*.

Conjecture 8 ([**13**]). *The pair (G,H) is Ramsey-infinite unless both G and H are stars with an odd number of edges or at least one of G and H contains a single edge component.*

An interesting case of the above conjecture is when G is a cycle and H is two-connected. No technique is presently known for showing such a pair is Ramsey-infinite.

Other results concerning Ramsey finiteness were obtained in [9] and [11].

Theorem 25. (**i**) [9] *If G is a matching and H is an arbitrary graph, then (G,H) is Ramsey-finite.*

(**ii**) [11] *The pair $(K_{1,k},G)$ is Ramsey-infinite if $k \geq 2$ and G is a two-connected graph. Also $(K_{1,2},H)$ is Ramsey-infinite if H is a bridgeless connected graph.*

Conjecture 9 ([**10**]). *The pair $(K_{1,2},G)$ is Ramsey-finite only if G is a matching.*

Conjecture 10 ([**10**]). *If (G,H) is Ramsey-finite for each graph H, then G must be a matching.*

There is no known case where adding independent edges to G (or H) changes a Ramsey-finite pair (G,H) to a Ramsey-infinite pair.

Conjecture 11 ([**13**]). *If (G,H) is Ramsey-finite then $(G \cup K_2, H)$ is Ramsey-finite.*

5. Extremal Problems

5.1. Edge Density and Triangles

The classical result of Turán implies that any graph with n vertices and at least $\lfloor n^2/4\rfloor+1$ edges contains a triangle. A natural generalization is obtained by fixing α between 0 and 1 and asking for the minimum number $f_\alpha(n)$ so that a graph of order n must contain a triangle if each set of $\lfloor \alpha n\rfloor$ vertices induces a subgraph with at least $f_\alpha(n)$ edges.

Example 4 ([**45**]). *Let $M_1 = K_2$, $M_2 = C_5$, and let M_3, be the graph obtained from C_8 by adding all of the long chords. From M_i obtain a graph H_i of order $n = k|V(M_i)|$ by replacing each vertex of M_i by an independent set of k vertices, and making each pair of vertices of H_i in different expanded* *sets adjacent if and only if the replaced vertices are adjacent in M_i. Each graph H_i is extremal triangle-free in that the addition of any edge creates a triangle.*

Note that the minimum number of edges induced by a set of $n/2$ vertices in H_2 is $n^2/50$. This leads to the following conjecture of Erdős.

Conjecture 12 ([45]). *If each set of $\lfloor n/2 \rfloor$ vertices in a graph G of order n spans more than $n^2/50$ edges, then G contains a triangle.*

An analysis of the graphs H_1, H_2, and H_3 leads to the following more general conjecture.

Conjecture 13 ([45]). *Suppose $17/30 \le \alpha \le 1$ and $\beta > (2\alpha - 1)/4$ or $43/120 \le \alpha < 17/30$ and $\beta > (5\alpha - 2)/25$. For all sufficiently large n, a graph of order n in which each set of $\lfloor \alpha n \rfloor$ vertices spans at least βn^2 edges must contain a triangle.*

In [45] this conjecture is proved for $.648 \ldots \le \alpha \le 1$; otherwise it is open. The case of $\alpha = 1/2$ is of particular interest.

One direction would be to extend these results from triangles to arbitrary complete graphs. For $\alpha = 1/2$, the following conjecture has been made by Chung and Graham.

Conjecture 14 ([22]). *Let $b_t(n)$ denote the minimum number of edges induced by any set of $n/2$ vertices in the Turán graph on n vertices for K_t. If each set of $\lfloor n/2 \rfloor$ vertices in a graph G of order n spans more than $b_t(n)$ edges, then G contains a K_t.*

5.2. Minimal Degree

The wheel $W_n = K_1 + C_{n-1}$ has n vertices, $2n - 2$ edges, and minimum degree 3, but any proper subgraph of W_n has a vertex of degree less than 3. On the other hand, it is easy to prove that any graph with n vertices and $2n - 1$ edges has a proper subgraph of minimum degree at least 3 [27].

Theorem 26 ([43]). *If G is a graph with n vertices and $2n - 1$ edges, then G contains a subgraph with less than $n - \sqrt{n/48}$ vertices that has minimum degree at least 3.*

A stronger result has been conjectured.

Conjecture 15 ([43]). *There is an $\varepsilon > 0$ such that every graph with n vertices and $2n - 1$ edges has a subgraph with $(1 - \varepsilon)n$ or fewer vertices that has minimum degree at least 3.*

A more general version of Theorem 26 was proved in [43] and independently by other authors.

Theorem 27 ([**43**]). *For $k \geq 2$, any graph with n vertices and $(k-1)(n-k+2)+\binom{k-2}{2}+1$ edges contains a subgraph with at most $n - \lfloor\sqrt{n/(6k^3)}\rfloor$ vertices that has minimum degree at least k.*

The generalized wheel $W(k-2,n) = K_{k-2} + C_{n-k+2}$ shows that the number of edges in the last result cannot be reduced. Conjecture 15 has the following generalization for subgraphs of minimum degree at least $k \geq 2$.

Conjecture 16 ([**43**]). *For each $k \geq 2$, there is a corresponding $\varepsilon > 0$ such that every graph with n vertices and $(k-1)(n-k+2)+\binom{k-2}{2}+1$ edges has a subgraph with $(1-\varepsilon)n$ or fewer vertices that has minimum degree at least k.*

5.3. Odd Cycles in Graphs of Given Minimum Degree

Example 5 ([**29**]). *Let $k \geq 3$ be an odd integer and let H be the two-connected nonbipartite graph obtained from C_{k+2} by replacing each of its $k+2$ vertices by an independent set of size s. Then H is a graph of order $n = (k+2)s$ that contains no odd cycles of length $\leq k$ and contains all possible odd cycles of larger length. Also H is regular of degree $2s$.*

Theorem 28 ([29]). *Let $k \geq 3$ be an odd positive integer. There exists $f(k)$ such that if $n \geq f(k)$ and G is a two-connected nonbipartite graph of order n with minimum degree $\geq 2n/(k+2)$, then either G contains a C_k or $G \cong H$.*

If G is a two-connected nonbipartite graph of appropriately large order n and $\delta(G) \geq 2n/(k+2)$, what range of odd cycles must G contain?

Example 6 ([**29**]). *Set $a = \lceil 2n/(k+2)\rceil$ and $b = n-a$. The graph G obtained by adding an edge to the smaller part of $K_{a,b}$ contains C_{2t+1} for $1 \leq t \leq a-1$, but no odd cycle of length greater than $2a-1$.*

Thus the best possible result would show that, aside from H, any two-connected nonbipartite graph with large order n and minimum degree at least $2n/(k+2)$ contain odd cycles of all lengths in an a range up to about $4n/(k+2)$. The next result shows that such a result is true (asymptotically) if the factor 4 in the last statement is replaced by $8/3 - o(n)$.

Theorem 29 ([**29**]). *Given $0 < c < \frac{1}{3}$ and $0 < \varepsilon < 1$, there exist $h_1(c,\varepsilon)$ and $h_2(c,\varepsilon)$ so that for all $n \geq h_2(c,\varepsilon)$ every two-connected nonbipartite graph G of order n and minimum degree $\delta(G) \geq cn$ contains C_{2t+1} for $h_1(c,\varepsilon) \leq 2t+1 \leq 4(1-\varepsilon)cn/3$.*

Is Example 6 really the truth?

Question 18 ([**29**]). *If n is appropriately large and $G \ncong H$ is a two-connected nonbipartite graph of order n satisfying $\delta(G) \geq 2n/(k+2)$, must G contain C_{2t+1} for $k \leq 2t+1 \leq 2\lceil 2n/(k+2)\rceil - 1$?*

5.4. Vertices and Edges on Odd Cycles

Every graph with $n \geq 3$ vertices and at least $\lfloor n^2/4 \rfloor + 1$ edges contains a triangle, and in fact a cycle of length $2k+1$ for every $k \leq \lfloor (n+1)/4 \rfloor$. For such a graph, how many vertices (edges) must lie on triangles and, more generally, on cycles of length $2k+1$?

Theorem 30 ([32]). *Let G be a graph with n vertices and at least $\lfloor n^2/4 \rfloor + 1$ edges. (a) At least $\lfloor n/2 \rfloor + 2$ vertices of G are on triangles, and this result is sharp. (b) At least $2\lfloor n/2 \rfloor + 1$ edges of G are on triangles, and this result is sharp. (c) If $k \geq 2$ and $n \geq \max\{3k(3k+1), 216(3k-2)\}$, then at least $2(n-k)/3$ vertices of G are on cycles of length $2k+1$, and this result is asymptotically best possible. (d) If $k \geq 2$ is fixed, then at least $11n^2/144 - O(n)$ edges of G are on cycles of length $2k+1$.*

The example used to show that the result in (c) is asymptotically best possible suggests what the sharp result should be in place of (d).

Example 7. *Consider the graph $K_s \cup T_2(n-s)$ where $s = \lceil 2n/3 \rceil + 1$ and $T_2(m)$ denotes the complete bipartite graph with parts $\lfloor m/2 \rfloor$ and $\lceil m/2 \rceil$ (the Turán graph). This graph has $\lceil 2n/3 \rceil + 1$ vertices on cycles of length $2k+1$ and $\sim 2n^2/9$ edges on cycles of length $2k+1$.*

Conjecture 17. *If $k \geq 2$ is fixed, then as $n \to \infty$ every graph with n vertices and $\lfloor n^2/4 \rfloor + 1$ or more edges has at least $2n^2/9 - O(n)$ edges on cycles of length $2k+1$?*

5.5. Extremal Paths

Let m, n and k be fixed positive integers with $m > n \geq k$. We wish to find the minimum value of l such that each m vertex graph with at least l vertices of degree $\geq n$ contains a P_{k+1}. A plausible minimum value for l is suggested by the following construction.

Example 8. *Let $m = t(n+1) + r$ where $0 \leq r < n+1$ and assume $k < 2n+1$. Let $s = \lfloor (k-1)/2 \rfloor$. The graph consisting of t vertex disjoint copies of $H = \bar{K}_{n+1-s} + K_s$ and r isolated vertices contains ts vertices of degree n and no P_{k+1}. When k is even and $r + s \geq n$, the number of vertices of degree $\geq n$ in this graph can be increased by 1 to $ts + 1$ without forcing the graph to contain a P_{k+1}. Simply take one of the vertices of degree s and make it adjacent to each of the r isolated vertices.*

Theorem 31 ([46]). *Let m and n be positive integers such that $n+1 \leq m \leq 2n+1$. If G is of order m and contains at least k vertices of degree $\geq n$, then G contains (i) a P_{2k-5} if $k \leq n/2 + 3$, (ii) a P_{n+1} if $(n+1)/2 + 3 \leq k \leq n$, and (iii) a P_k if $n + 1 \leq k$.*

Conjecture 18 ([46]). *Let m, n and k be positive integers with $m > n \geq k$ and set $\delta = 2$ if k is even and $\delta = 1$ if k is odd. If G is a graph of order m*

and at least $l = \lfloor (k-1)/2 \rfloor \lfloor m/(m+1) \rfloor + \delta$ *vertices of degree* $\geq n$, *then* G *contains a* P_{k+1}.

5.6. Degree Sequence and Independence in K_4-Free Graphs

For a graph G with $\delta(G) \geq 1$ let $f(G)$ denote the largest number of times any entry in the degree sequence of G is repeated. If G is K_r-free graph and $f(G) \leq k$, what is the smallest possible value of the independence number $\beta(G)$?

Theorem 32 ([31]).

- **(i)** *If* G *is a* K_3*-free graph of order* n *and* $f(G) \leq k$, *then* $\beta(G) \geq n/k$.
- **(ii)** *If* $r \geq 5$ *and* $k \geq 2$, *then there exists a sequence* (G_n) *of* K_r*-free graphs such that* $|V(G_n)| = n$ *and* $f(G_n) \leq k$ *for each* n *and* $\beta(G_n) = o(n)$ *as* $n \to \infty$.
- **(iii)** *If* G *is a* K_4*-free graph of order* n *and* $f(G) = 2$, *then* $\beta(G) \geq n/12$.
- **(iv)** *There exists no sequence* (G_n) *of* K_4*-free graphs such that* $|V(G_n)| = n$ *and* $f(G_n) = 3$ *for each* n *and* $\beta(G_n) = o(n)$ *as* $n \to \infty$.

Bollobás has shown that there exists a sequence (G_n) of K_4-free graphs such that $|V(G_n)| = n$ and $f(G_n) \geq 5$ for each n and $\beta(G_n) = o(n)$ as $n \to \infty$ [4]. The following questions are open.

Question 19 ([31]). *Does there exist a sequence* (G_n) *of* K_4*-free graphs such that* $|G_n)| = n$ *and* $f(G_n) = 4$ *for each* n *and* $\beta(G_n) = o(n)$ *as* $n \to \infty$?

Question 20 ([31]). *If* G *is a* K_3*-free graph of order* n *with* $f(G) \leq k$, *is the lower bound* $\beta(G) \geq n/k$ *best possible?*

6. Other Problems

6.1. Monochromatic Coverings

If A and B are sets of vertices in a given graph G, we say that A *dominates* or *covers* B if for every $y \in B \setminus A$ there is an $x \in A$ such that $xy \in E(G)$. The following result was conjectured by Erdős and Hajnal.

Theorem 33 ([28]). *For any fixed* t, *in every two-coloring of the edges of* K_n *there exists a set of* t *or fewer vertices that monochromatically dominates at least* $n(1 - 2^{-t})$ *vertices. This result is essentially sharp.*

In [26] we studied monochromatic domination when more than two colors are involved. For three colors, we have the following result.

Theorem 34 ([26]). *Three-color the edges of* K_n. *Then in at least one color, there is a set consisting of 22 or fewer vertices that dominates a set of at least* $2n/3$ *vertices.*

The fact that no small set will, in general, monochromatically dominate more than two-thirds of the vertices comes from the following example due to Kierstead.

Example 9 (**Kierstead**). *Three-color the edges of K_{3m} as follows. Form the partition $V(K_{3m}) = (A_1, A_2, A_3)$ and for $i \equiv 1, 2, 3$ $(mod\ 3)$ color the internal edges of A_i and all the edges between A_i and A_{i+1} with color i. Then each monochromatic subgraph is isomorphic to $K_m + \overline{K}_m$ and there is no set that monochromatically dominates more than $2m$ vertices.*

It is possible that Theorem 34 holds with "22" replaced by "3."

Question 21. *Is it true that in every three-coloring of the edges of K_n there is a set of three vertices that monochomatically dominates two-thirds or more of all vertices?*

6.2. Spectra

A nonnegative integer q belongs to the *k-spectrum* of a graph G if there is some set of k vertices in G spanning q edges. Denote the k-spectrum of G by $s_k(G)$. Then $s_k(G) \subseteq \{0, 1, \cdots \binom{k}{2}\}$, but not every such subset is an example of a k-spectrum. Let n_k denote the number of distinct subsets of $\{0, 1 \cdots, \binom{k}{2}\}$ that are realizable as k-spectra. The k-spectra of all large trees were characterized in [47].

Theorem 35 ([**47**]). *If $n \geq \max\{2k-1, 3k-5\}$, then for each tree T of order n there exist corresponding integers r and s with $0 \leq \lceil r/2 \rceil \leq s \leq r \leq k-1$ such that $s_k(T_n) = [0, r] \cup [s, k-1]$.*

The following bounds have been obtained for number of realizable k-spectra.

Theorem 36 ([47]). *For any integer $k \geq 2$, the number of distict realizable k-spectra satisfies $\frac{1}{16}\left(\frac{5}{2}\right)^{k-1} < n_k < 2^{\binom{k}{2}+1}$.*

For $k = 2, 3$ and 4, the realizable k-spectra have been determined [50].

Question 22. *For $k \geq 5$, what is n_k, the number of distinct realizable k-spectra?*

A similar problem was asked by Erdős and Faudree concerning the *cycle spectrum* of a graph, namely the set of distinct cycle lengths of a graph. Let c_n be the number of distinct cycle spectra for graphs of order n. The following family of graphs gives a lower bound for the c_n.

Example 10. *Let v be a vertex in C_n and let $S \subseteq \{2, 3, \ldots, \lfloor n/2 \rfloor\}$. Add to C_n all chords extending from v whose lengths belong to S. Distinct sets S will give distinct cycle spectra, since the cycles of length at least $\lceil n/2 + 1 \rceil$ will be different.*

Using this example, we see that $2^{n/2} - 1 \leq c_n < 2^n - 2$.

Question 23. *What is the number of distinct cycle spectra of a graph of order n?*

6.3. Clique Coverings and Partitions

A *clique covering* of a graph G is a set of cliques that together contain all of the edges of G; if each edge is contained in precisely one of these cliques, we have a *clique partition*. The *clique covering number* $cc(G)$ of G is the smallest cardinality of any clique covering, and the *clique partition number* $cp(G)$ is the smallest cardinality of any clique partition. The relationship between $cc(G)$ and $cp(G)$ was investigated in [30], where the following example was described.

Example 11. *For any integer n divisible by 8, let $G_n = K_{n/2} + 4K_{n/8}$. Then, clearly $cc(G_n) = 4$, and it was shown in [30] that $cp(G_n) = n^2/16 + 3n/4$. Thus,*

$$\frac{cp(G_n)}{cc(G_n)} > n^2/64.$$

Question 24. *What is the largest C for which there is a sequence of graphs (G_n) such that $|V(G_n)| = n$ and*

$$\frac{cp(G_n)}{cc(G_n)} > Cn^2?$$

In [19] the authors exhibit a sequence of graphs (G_n) such that $|V(G_n)| = n$

$$cp(G_n) - cc(G_n) = \frac{n^2}{4} - \frac{n^{3/2}}{2} + \frac{n}{4} + O(1), \qquad (n \to \infty).$$

Paul Erdős has asked whether the $n^{3/2}$ term is really necessary.

Question 25. *Is there a sequence of graphs (G_n) such that $|V(G_n)| = n$ and*

$$cp(G_n) - cc(G_n) = n^2/4 + O(n), \qquad (n \to \infty)?$$

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Some Remarks on the Cycle Plus Triangles Problem

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1. Introduction and Main Results

All (undirected) graphs and digraphs considered are assumed to be finite (if not otherwise stated) and loopless. Multiple edges (arcs) are permitted. For a graph G, let $V(G)$, $E(G)$, and $\chi(G)$ denote the vertex set, the edge set, and the chromatic number of G, respectively. If $X \subseteq V(G)$ and $F \subseteq E(G)$, then $G - X - F$ denotes the subgraph H of G satisfying $V(H) = V(G) - X$ and $E(H) = \{xy \mid xy \in E(G) - F \text{ and } x, y \notin X\}$.

A system of non-empty subgraphs $\{G_1, \ldots, G_m\}$, $m \geq 1$, in a graph G is called a *decomposition* of G if and only if $E(G) = \bigcup_{i=1}^{m} E(G_i)$ and $E(G_i) \cap E(G_j) = \emptyset$, $1 \leq i < j \leq m$. For an integer $k \geq 2$, let $\mathcal{G}_k$ denote the family of all graphs G which have a decomposition into a Hamiltonian cycle and $m \geq 0$ pairwise vertex disjoint complete subgraphs each on k vertices. Note that every vertex of a graph $G \in \mathcal{G}_k$, $k \geq 2$, has degree either $k+1$ or 2. In particular, $\mathcal{G}_k$ contains every cycle.

The authors proved the following result providing an affirmative solution to a colouring problem of P. Erdős, which became known as the "cycle-plus-triangles problem" (see e.g. [5] or [7]).

Theorem 1 (**[9]**). *If $G \in \mathcal{G}_3$ then $\chi(G) \leq 3$.*

Actually, Theorem 1 was proved only for 4-regular graphs. But every non-regular graph $G' \in \mathcal{G}_3$ can be obtained from a 4-regular graph $G \in \mathcal{G}_3$ by subdividing some edges of the Hamiltonian cycle of G, and hence every legal 3-colouring of G can be extended to a legal 3-colouring of G'. Recently, H. Sachs [12] found a purely elementary proof of Theorem 1.

Theorem 2. *Let $k \geq 4$ be an integer and let G be a graph. Assume that G has a decomposition $\mathcal{D}$ into a Hamiltonian cycle and $m \geq 0$ pairwise vertex disjoint complete subgraphs each on at most k vertices. Then $\chi(G) \leq k$.*

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Proof (by induction on k). Let $X = \bigcup V(K)$ where the union is taken over all complete subgraphs $K \in \mathcal{D}$ of size $\leq k-2$. Then every vertex of X has degree $\leq k-1$ in G implying that every legal k-colouring of $G - X$ can be extended to a legal k-colouring of G. Therefore, we need only to show that $\chi(G - X) \leq k$. Clearly, $G - X$ is a subgraph of some graph G' which has a decomposition $\mathcal{D}'$ into a Hamiltonian cycle and a (possibly empty) set of pairwise vertex disjoint complete subgraphs on $k-1$ or k vertices each. We claim that $\chi(G') \leq k$. Let $\mathcal{D}_1$ be the set of all complete subgraphs from $\mathcal{D}'$ with exactly k vertices. If $\mathcal{D}_1 = \emptyset$ then from the induction hypothesis or, in case $k = 4$, from Theorem 1 we conclude that $\chi(G') \leq k - 1 < k$. If $\mathcal{D}_1 \neq \emptyset$ then we argue as follows. First, choose from every complete subgraph $K \in \mathcal{D}_1$ an arbitrary vertex $y = y(K)$ and let Y be the set of all these vertices. Then $G' - Y$ is a subgraph of some graph in $\mathcal{G}_{k-1}$ and therefore, the induction hypothesis or, in case $k = 4$, Theorem 1 implies that $G' - Y$ has a legal colouring with $k-1$ colours, say $1, 2, \ldots, k-1$. Obviously, each complete subgraph $K - y(K)$ of $G' - Y$, $K \in \mathcal{D}_1$, contains precisely one vertex coloured 1 (note that $K - y(K)$ has $k-1$ vertices). Now, let Z be the set of all such vertices. Then Z is an independent set in $G' - Y$, and hence also in G'. By the same argument as before we conclude that $G' - Z$ has a legal $(k-1)$-colouring implying that G' has a legal k-colouring, i.e. $\chi(G') \leq k$. This proves our claim, and hence the desired result. □

Theorem 2 is not true for $k = 3$. A counterexample is the complete graph on 4 vertices, a further one due to H. Sachs is shown in Fig. 1.

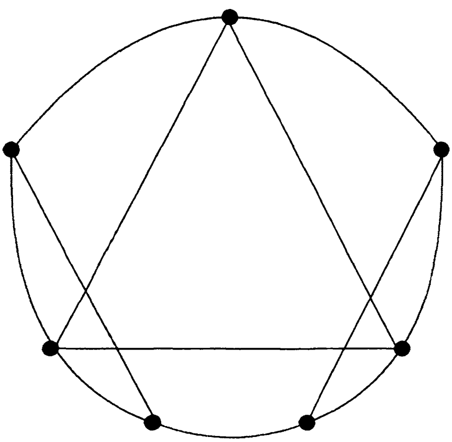

Fig. 1

As a consequence of Theorems 1 and 2 we obtain the following proposition.

Corollary 1. *If $G \in \mathcal{G}_k$ for some $k \leq 3$, then $\chi(G) \leq k$.*

The proof of Theorem 1 given in [9] uses a recent result of N. Alon and M. Tarsi [2] which provides a colouring criterion for a graph G based on orientations of G. On the one hand, this yields a somewhat stronger result than Theorem 1, namely that every graph $G \in \mathcal{G}_3$ has list chromatic number at most 3. On the other hand, this implies that the proof of Theorem 1 is not constructive in the sense that it does not yield any algorithm which, given $G \in \mathcal{G}_3$, allows a legal 3-colouring of G to be constructed. For a graph $G \in \mathcal{G}_k$, $k \geq 4$, a legal k-colouring of G can be easily obtained using the induction step of Theorem 2 and the following colouring procedure for regular graphs from $\mathcal{G}_4$.

Let $G \in \mathcal{G}_4$ be any 5-regular graph, and let $\mathcal{D}$ be a decomposition of G into a Hamiltonian cycle C and $m \geq 1$ pairwise vertex disjoint complete subgraphs $K^1, \ldots, K^m$ each on 4 vertices. For $E' \subseteq E(G)$, let $G(E')$ denote the spanning subgraph of G with vertex set $V(G)$ and edge set E'. By a linear factor of G we mean a set $L \subseteq E(G)$ such that $G(L)$ is 1-regular.

To obtain a legal 4-colouring of G, we first choose an arbitrary linear factor L of C (note that the cycle C has length $0 \mod 4$) and, for each $i \in \{1, 2, \ldots, m\}$, an arbitrary linear factor L_1^i of K^i. Obviously, the subgraph $G(F)$, where $F = L \cup L_1^1 \cup \ldots \cup L_1^m$, is the disjoint union of even cycles, and hence we easily find a legal 2-colouring c_1 of $G(F)$. Now it is easy to check that we can choose a linear factor L_2^i of $K^i - L_1^i$ $(i = 1, \ldots, m)$ such that c_1 remains a legal 2-colouring of $G(F_1)$, where $F_1 = F \cup L_2^1 \cup \ldots \cup L_2^m$. Then $F_2 = E(G) - F_1$ consists of the linear factor $L' = E(C) - L$ and the linear factors $L_3^i = E(K^i) - L_1^i - L_2^i$ $(i = 1, \ldots, m)$. Therefore, $G(F_2)$ is the disjoint union of even cycles and has a legal 2-colouring c_2. Eventually, because of $G = G(F_1 \cup F_2)$, the mapping c defined by $c(x) = (c_1(x), c_2(x))$, $x \in V(G)$, is a legal 4-colouring of G.

Noga Alon [1] proposed a more general situation. Let H be a graph on n vertices. If k divides n, then H is said to be *strongly k-colourable* if, for any partition of $V(H)$ into pairwise disjoint sets V_i each having cardinality k, there is a legal k-colouring of H in which each colour class intersects each V_i by exactly one vertex. Notice that H is strongly k-colourable if and only if the chromatic number of any graph G obtained from H by adding to it a union of vertex disjoint complete subgraphs (on the set $V(H)$) each having k vertices is k. If k does not divide n, then H is said to be strongly k-colourable if the graph obtained from H by adding to it $k\lceil n/k \rceil - n$ isolated vertices is strongly k-colourable. Of course, the *strong chromatic number* of a graph H, denoted by $s\chi(H)$, is the minimum k such that H is strongly k-colourable. In this terminology, Theorem 1 says that, for every cycle C_{3m} on $3m$ vertices, $s\chi(C_{3m}) \leq 3$. Moreover, Theorem 2 implies that, for every cycle C, $s\chi(C) \leq 4$. For cycles on $4m$ vertices this statement was established by several researchers including F. de la Vega, M. Fellows and N. Alon (see [7]). The 4-chromatic graph depicted in Fig. 1 shows that $s\chi(C_7) = 4$.

Noga Alon [1] investigated the function $s\chi(d) = \max(s\chi(H))$, where H ranges over all graphs with maximum degree at most d. On the one hand, using similar arguments as above he gives a short proof that $s\chi(H) \leq 2^{\chi'(H)}$, where $\chi'(H)$ denotes the chromatic index of H. Because of Vizing's Theorem, this yields $s\chi(d) \leq 2^{d+1}$. On the other hand, using probabilistic arguments, he proved that there is a (very large) constant c such that $s\chi(d) \leq cd$ for every d. This leaves the following problem. What is the minimum value k such that every graph with maximum degree d is strongly k-colourable? Even for $d = 2$, the answer is not known. Clearly, for $d = 1$ we have $s\chi(1) = 2$. To obtain a lower bound for $s\chi(d)$, we construct a graph H_d for every $d \geq 1$ in the following way. Let X_1, X_2, X_3, X_4 be disjoint sets on r vertices each, $r \geq 1$. Join X_i to X_{i+1} ($i = 1, 2, 3$) and X_4 to X_1 by all possible edges. The resulting graph is H_{2r}. The graph H_{2r-1} is obtained from H_{2r} by removing two adjacent vertices. Then the graph H_d, $d \geq 1$, is regular of degree d and it is easy to check that H_d is not strongly k-colourable for $k \leq 2d - 1$, i.e. $s\chi(H_d) \geq 2d$. This yields $s\chi(d) \geq 2d$ for every $d \geq 1$.

In [5] P. Erdős asked whether the statement of Theorem 1 remains true if the Hamiltonian cycle of the decomposition is replaced by a 2-regular graph not containing a cycle of length 4. The graph G depicted in Fig. 2 provides a negative answer. This graph belongs to an infinite family of 4-regular 4-chromatic (in fact, 4-critical) graphs due to Gallai [6] all of which (except one) are counterexamples with respect to Erdős's question.

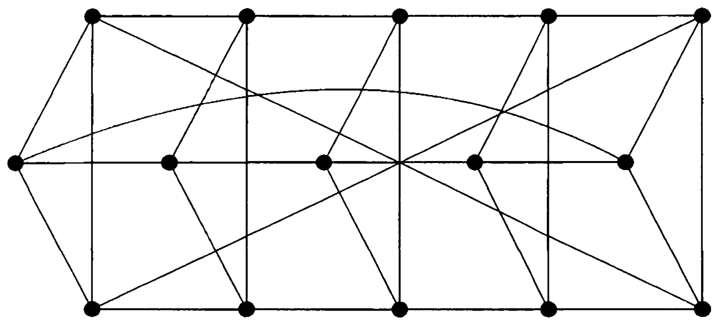

Fig. 2

We define a (not necessarily connected) graph to be *Eulerian* if and only if each vertex has even degree. Analogously, a digraph is said to be *Eulerian* if and only if each vertex has equal out-degree and in-degree. Let G be an Eulerian graph. Clearly, G has an orientation D (this means, that D is a digraph whose underlying graph is G) which is Eulerian. Such an orientation is briefly called an *Eulerian orientation* of G. We denote the number of all Eulerian orientations of G by $e(G)$. If D is an arbitrary Eulerian orientation of the Eulerian graph G, then (see [8]) $e(G)$ is equal to the number of all spanning Eulerian subdigraphs of D. Therefore, Theorem 2.1 in [9] implies the following.

Theorem 3 ([**9**]). *If* $G \in \mathcal{G}_3$ *then* $e(G) \equiv 2 \mod 4$.

Theorem 3 may be considered as the main result of [9]. To deduce Theorem 1 from Theorem 3, the colouring criterion of Alon and Tarsi (see [2] or [9]) is needed. As an immediate consequence of Theorem 3 we obtain the following result. This was first noted by Alon, Gutner and Tarsi.

Corollary 2. *Let G be a graph, which has a decomposition into a Hamiltonian cycle C and $m \geq 1$ pairwise vertex disjoint triangles $T_1, \ldots, T_m$, i.e. $G \in \mathcal{G}_3$. Then G has an Eulerian orientation D such that no triangle $T_i (i = 1, \ldots, m)$ is cyclically oriented in D. Moreover, if $V_1 (V_2)$ is the set of all those vertices from G which are sinks (sources) of some triangle T_i with respect to D, then both V_1 and V_2 are independent sets in G each of size m.*

Proof. Let $\mathcal{O}$ denote the set of all Eulerian orientations of G, and let $\mathcal{O}_i$ $(i = 1, \ldots, m)$ denote the set of all Eulerian orientations of G in which the triangle T_i is cyclically oriented. To prove that a desired orientation D of G exists, suppose the contrary. Then $\mathcal{O} = \bigcup_{i=1}^{m} \mathcal{O}_i$, and hence

$$e(G) = |\mathcal{O}| = \sum\nolimits_{I \subseteq \{1,\ldots,m\}} (-1)^{|I|+1} o(I),$$

where $o(I) = |\bigcap_{i \in I} \mathcal{O}_i|$. It is easy to check that, for $I \neq \emptyset$, we have $o(I) = 2^{|I|} e(G')$, where $G' = G - \bigcup_{i \in I} E(T_i)$. Since $G' \in \mathcal{G}_3$, from Theorem 3 it then follows that $o(I) \equiv 0 \mod 4$, and hence $e(G) \equiv 0 \mod 4$, a contradiction. Therefore, the desired orientation D of G exists. Obviously, the sets V_1 and V_2 satisfying the hypothesis of Corollary 2 are both independent sets of size m in G. This proves Corollary 2. □

In particular, from Theorem 3 we obtain immediately that every 4-regular graph $G \in \mathcal{G}_3$ with $3m$ vertices has independence number m, as conjectured by D. Z. Du and D. F. Hsu in 1986 (see [3, 7]).

In connection with the Cycle Double Cover Conjecture R. Goddyn (private communication) suspects that there is no snark H with a chordless dominating cycle C (this means, that $V(H) - V(C)$ is an independent set in H). The first author and M. Tarsi observed that the following result can be deduced from Corollary 2.

Corollary 3. *Assume that H is a 3-regular graph which has a dominating cycle C. Then there is a matching M of $H - E(C)$ such that $H - M$ is 2-regular or isomorphic to a subdivision of a 3-regular bipartite graph.*

Proof. Put $X = V(H) - V(C)$, and let M_1 be the set of all chords of C in H. From the cycle C we construct a new graph in the following way. For every vertex $x \in X$, join the three neighbours of x on C by three new edges forming a triangle T_x. Clearly, the resulting graph G belongs to $\mathcal{G}_3$. From Corollary 2 it then follows that there is an Eulerian orientation D of G such that no triangle T_x is cyclically oriented. Now, let V_1 (V_2) be the set of all those vertices from G which are sinks (sources) of some triangle T_x with respect to D. Then, for every vertex $x \in X$, there is exactly one edge in H joining x to some vertex of $V(C) - V_1 - V_2$. Denote the set of all such edges by M_2, and put $M = M_1 \cup M_2$. Clearly, M is a matching of $H - E(C)$. Since both V_1 and V_2 are independent sets in G (see Corollary 2), it is easy to check that $H - M$ is 2-regular (in case of $X = \emptyset$) or isomorphic to a subdivision of a 3-regular bipartite graph (in case of $X \neq \emptyset$). This proves Corollary 3. □

One can easily show that if a 3-regular graph H with a dominating cycle contains two disjoint matchings as described in Corollary 3, then H has a nowhere-zero 5-flow.

Remark (added in 2013). Moreover, if H as above has two such disjoint matchings, then H also has a 5-cycle double cover containing the dominating cycle, [10]. It was therefore conjectured by the first author of this article that if H is cyclically 4-edge-connected and has a dominating cycle C, then it has two such disjoint matchings with respect to C. This was disproved by his PhD student Arthur Hoffmann-Ostenhof, [11]. However, together with the Dominating Cycle Conjecture (DCC) it would suffice to prove the existence of two such disjoint matchings with respect to some dominating cycle, to ensure the validity of the cycle double cover conjecture (CDCC) and the nowhere-zero 5-flow conjecture (NZ5FC). In fact, to prove the CDCC it would suffice to prove the DCC and the existence of two such disjoint matchings with respect to a stable dominating cycle; however, this would not be enough to prove the NZ5FC.

Finally, let us mention the following infinite version of Corollary 1. A set of cardinality k is briefly called a k-set.

Theorem 4. *Let $k \geq 3$ be an integer. For any partition $\mathcal{D}$ of the integers Z into k-sets, there is another partition $\{X_1, \dots, X_k\}$ of Z such that X_i $(i = 1, \dots, k)$ contains a member of each k-set from $\mathcal{D}$ but no consecutive pair of integers.*

Proof. Define G to be the infinite graph with vertex set Z, where $xy \in E(G)$ if and only if $|x-y| = 1$ or $x, y \in K \in \mathcal{D}$. Consider an arbitrary finite subgraph H of G. Then it is easy to check that H is a subgraph of some graph from $\mathcal{G}_k$ and hence, using Corollary 1, we conclude that H is k-colourable. From a well-known result of Erdős and de Bruijn [4] we then obtain that G is itself k-colourable. This immediately implies that $\chi(G) = k$. Obviously, any legal k-colouring of G provides a partition $\{X_1, X_2, \dots, X_k\}$ of Z having the desired properties. □

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Intersection Representations of the Complete Bipartite Graph

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Summary. A p-representation of the complete graph $\mathcal{K}_{n,n}$ is a collection of sets $\{S_1, S_2, \ldots, S_{2n}\}$ such that $|S_i \cap S_j| \geq p$ if and only if $i \leq n < j$. Let $\vartheta_p(\mathcal{K}_{n,n})$ be the smallest cardinality of $\cup S_i$. Using the Frankl-Rödl theorem about almost perfect matchings in uncrowded hypergraphs we prove the following conjecture of Chung and West. For fixed p while $n \to \infty$ we have $\vartheta_p(\mathcal{K}_{n,n}) = (1 + o(1))n^2/p$. Several problems remain open.

1. The p-Intersection Number of $\mathcal{K}(n, n)$

One of the important topics of graph theory is to represent graphs, or an interesting class of graphs, using other simple structures. One approach is to represent the vertices by sets so that vertices are adjacent if and only if the corresponding sets intersect (line graphs). More generally, the p-intersection number of a graph is the minimum t such that each vertex can be assigned a subset of $\{1, \ldots, t\}$ in such a way that vertices are adjacent if and only if the corresponding sets have at least p common elements. Such a system is called a *p-representation* (or p-intersection representation) of the graph $\mathcal{G}$, and the minimum t is denoted by $\vartheta_p(\mathcal{G})$.

Problem 1. *Given n and p determine* $\max \vartheta_p(G)$ *among the n-vertex graphs.*

For any graph of v vertices Erdős, Goodman, and Pósa [10] showed that $\vartheta_1 \leq \lfloor v^2/4 \rfloor$, and here equality holds for $\mathcal{K}_{\lfloor v/2 \rfloor, \lceil v/2 \rceil}$. This settles the case $p = 1$ in Problem 1. Myung S. Chung and D. B. West [5] conjectured that the complete bipartite graph also maximizes ϑ_p. Their lower bound for $p > 1$ is

$$\vartheta_p(\mathcal{K}_{n,n}) \geq (n^2 - n)/p + 2n. \quad (1)$$

In this note we determine ϑ_p for these and a few more graphs. Especially, we show that in (1) equality holds asymptotically (p is fixed $n \to \infty$) with a smaller order error term. One of our main construction (Theorem 1) along with the lower bound (7) was independently discovered by Eaton, Gould and Rödl [6, 9]. They also considered 2-representations of bounded degree trees.

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More interestingly, considering the the unbalanced complete bipartite graph Eaton [7] disproved the Chung-West conjecture by showing that

$$\max_{a+b=2n} \vartheta_p(\mathcal{K}_{a,b}) \geq (1+o(1))\frac{16p^2-12p+1}{16p^2-16p+4}\,\frac{n^2}{p}. \tag{2}$$

Her method was further developed in [8] when $\vartheta_p(\mathcal{K}_{a,b})$ was exactly determined by infinitely many values using certain designs and Steiner systems.

The *complete k-partite graph*, $\mathcal{K}^{(k)}_{n,\dots,n}$, has kn vertices, k disjoint independent sets of sizes n and all the $\binom{k}{2}n^2$ edges between different classes. $\mathcal{K}(n\times k)$ denotes a graph with vertex set $V^1 \cup \ldots \cup V^k$, $V^\ell = \{v^\ell_1, \dots, v^\ell_n\}$ and v^ℓ_i is joined to v^m_j if and only if $i \neq j$ and $\ell \neq m$. So $\mathcal{K}(n\times 2)$ is obtained from $\mathcal{K}_{n,n}$ by deleting a one factor.

Theorem 1. *For fixed p and k, the p-intersection number of the complete k-partite graph $\mathcal{K}^{(k)}_{n,\dots,n}$ is $(1+o(1))n^2/p$.*

Note that the asymptotic is independent from the fixed value of k. In Sect. 4 we will give a partial proof for Theorem 1 using classical design theory and obtain a better error term. A *Hadamard matrix* of order n is a square matrix M with ± 1 entries such that $MM^t = nI_n$. It is conjectured that it exists for all $n \equiv 0 \pmod 4$. The smallest undecided case is larger than 184. An $S_\lambda(v,l,t)$ *block design* is a l-uniform (multi)hypergraph on v vertices such that each t-subset is contained in exactly λ hyperedges (blocks). Block designs with $\lambda = 1$ are called *Steiner systems*. All notions we use about designs can be found, e.g., in Hall's book [15]. Wilson [22] proved that for any l there exists a bound $v_0(l)$ such that for all $v \geq v_0$ there exists a Steiner system $S(v,l,2)$ if $\binom{v}{2}/\binom{l}{2}$ and $(v-1)/(l-1)$ are integers.

Theorem 2. *(a) If there exists a Hadamard matrix of size $4p$, and a Steiner system $S(n,2p,2)$, then $\vartheta_p(\mathcal{K}(n\times 2)) = (n^2-n)/p$.*

(b) If $p = q^d$ where q is a prime power, d a positive integer, $k \leq q$, and there exists a Steiner system $S(n,q^{d+1},2)$, then $\vartheta_p(\mathcal{K}(n\times k)) = (n^2-n)/p$.

Corollary 1. *For all p in Theorem 2, and $n > n_0(p)$*

$$n^2/p < \vartheta_p(\mathcal{K}_{n,n}) \leq \vartheta_p(\mathcal{K}(n\times 2)) + pn \leq (n^2/p) + 4pn.$$

This covers all cases $p \leq 46$. A construction from the finite projective space is given in Sect. 3. In Sect. 5 we list a few open problems.

2. A Random Construction

A hypergraph $\mathcal{H}$ with edge set $\mathcal{E}(\mathcal{H})$ and vertex set $V(\mathcal{H})$ is called *r-uniform* (or an r-graph) if $|E| = r$ holds for every edge $E \in \mathcal{E}(\mathcal{H})$. The *degree*, $\deg_{\mathcal{H}}(x)$, of the vertex $x \in V$ is the number of edges containing it. The degree of a pair, $\deg_{\mathcal{H}}(x,y)$, is the number of edges containing both vertices x and y. The *dual*,

$\mathcal{H}^*$, of $\mathcal{H}$ is the hypergraph obtained by reversing the roles of vertices and edges keeping the incidences, i.e., $V(\mathcal{H}^*) = \mathcal{E}(\mathcal{H})$. A *matching* $\mathcal{M} \subseteq \mathcal{E}(\mathcal{H})$ is a set of mutually disjoint edges, $\nu(\mathcal{H})$ denotes the largest cardinality of a matching in $\mathcal{H}$.

We are going to use a theorem of Frankl and Rödl [11]. The following slightly stronger form is due to Pippenger and Spencer [20]: For all integers $r \geq 2$ and real $\varepsilon > 0$ there exists a $\delta > 0$ so that: If the r-uniform hypergraph $\mathcal{H}$ on z vertices has the following two properties (i) $(1-\delta)d < \deg_{\mathcal{H}}(x) < (1+\delta)d$ holds for all vertices, (ii) $\deg_{\mathcal{H}}(x,y) < \delta d$ for all distinct x and y, then there is a matching in $\mathcal{H}$ almost as large as possible, more precisely

$$\nu(\mathcal{H}) \geq (1-\varepsilon)(z/r). \tag{3}$$

Far reaching generalizations of (3) have been proved by Kahn [18].

Suppose $\mathcal{G}$ is a graph and $\mathcal{F} = \{F_1, \dots, F_t\}$ is a family of subsets of the vertex set $V(\mathcal{G})$, repetition allowed. Such a system $\mathcal{F}$ is called a *p-edge clique cover* if every edge of $\mathcal{G}$ is contained in at least p members of $\mathcal{F}$ and the non-edge pairs are covered by at most $p-1$ of the F_i's. A p-edge clique cover is the dual of a p-representation (and vice versa), so the smallest t for which there is a p-edge clique cover is $\vartheta_p(\mathcal{G})$. This was the way Kim, McKee, McMorris and Roberts [19] first defined and investigated $\vartheta_p(\mathcal{G})$.

Proof of Theorem 1. To construct a p-edge clique cover of the complete k-partite graph with n-element classes $V^1, \dots, V^k$ consider the following multigraph $\mathcal{M}$. Every edge contained in a class V^i has multiplicity $p-1$, and all edges joining distinct classes (crossing edges) have multiplicity p. The total number of edges is

$$\left|\mathcal{E}(\mathcal{M})\right| = (p-1)k\binom{n}{2} + p\binom{k}{2}n^2 = (1+o(1))\frac{n^2}{p}\left(k\binom{p}{2} + \binom{k}{2}p^2\right) \tag{4}$$

Let $r = |\mathcal{E}(\mathcal{K}^{(k)}_{p,\dots,p})| = k\binom{p}{2} + \binom{k}{2}p^2$. Define the r-uniform hypergraph $\mathcal{H}$ with vertex set $\mathcal{E}(\mathcal{M})$ as follows. The hyperedges of $\mathcal{H}$ are those r-subsets of $\mathcal{E}(\mathcal{M})$ which form a complete k-partite subgraph with p vertices in each V^i. The number of such subgraphs is

$$\left|\mathcal{E}(\mathcal{H})\right| = \binom{n}{p}^k (p-1)^{k\binom{p}{2}} p\binom{k}{2}p^2.$$

Let $e \in \mathcal{E}(\mathcal{M})$ be an edge contained in a class V^i. The number of $\mathcal{K}^{(k)}_{p,\dots,p}$'s, i.e., the number of hyperedges of $\mathcal{H}$ containing e is exactly

$$\deg_{\mathcal{H}}(e) = \binom{n-2}{p-2}\binom{n}{p}^{k-1}(p-1)^{k\binom{p}{2}-1}p^{\binom{k}{2}}p^2. \tag{5}$$

For any crossing edge $f \in \mathcal{E}(\mathcal{M})$ connecting two distinct classes we have

$$\deg_{\mathcal{H}}(f) = \binom{n-1}{p-1}^2\binom{n}{p}^{k-2}(p-1)^{k\binom{p}{2}}p^{\binom{k}{2}p^2-1}. \tag{6}$$

The ratio of the right hand sides of (5) and (6) is $n/(n-1)$, so the hypergraph $\mathcal{H}$ is nearly regular, it satisfies the first condition in the Frankl-Rödl theorem for any $\delta > 0$ if n is sufficiently large. For two distinct edges, $e_1, e_2 \in \mathcal{E}(\mathcal{M})$, obviously $\deg_{\mathcal{H}}(e_1, e_2) = O(n^{kp-3})$, so condition (ii) is fulfilled, too. Apply (3) to $\mathcal{H}$. We get a system $\mathcal{F} = \{F_1, \dots, F_\nu\}$ of kp-element subsets of $\cup V^i$ such that every pair e contained in a class V^i is covered at most $p-1$ times, every pair f joining two distinct classes is covered at most p times. Moreover, $\nu = (1 - o(1))n^2/p$, by (4). It follows that almost all edges of $\mathcal{K}^{(k)}_{n,\dots,n}$ are covered exactly p times, so the system $\mathcal{F}$ can be extended to a p-edge clique cover by adding sufficiently many (but only $o(n^2)$) edges. □

3. Exact Results from Finite Projective Spaces

Proposition 1. *For all $p \geq 1$, $\vartheta_p(\mathcal{K}(n \times 2)) \geq (n^2 - n)/p$.*

Proof. Let V^1 and V^2 be the two parts of the vertex set of the graph, $|V^1| = |V^2| = n$, let $\{A_i \cup B_i : 1 \leq i \leq t\}$ be a p-edge clique cover, their average size on one side is $\ell := \sum_i (|A_i| + |B_i|)/(2t)$. Using the inequalities

(1) $\sum_i \binom{|A_i|}{2} \leq (p-1)\binom{n}{2}$, and
(2) $\sum_i \binom{|B_i|}{2} \leq (p-1)\binom{n}{2}$, and the fact that
(3) all the $n^2 - n$ crossing edges are covered at least p times we have

$$p(n^2 - n) \leq \sum_i |A_i||B_i| \leq \sum_i \left(\binom{|A_i|}{2} + \binom{|B_i|}{2}\right)$$
$$+\ell t \leq (p-1)(n^2 - n) + \ell t.$$

This gives $(n^2 - n) \leq \ell t$. On the other hand, (1) and (2) give $2t\binom{\ell}{2} \leq 2(p-1)\binom{n}{2}$. Hence $\ell \leq p$ and $t \geq (n^2 - n)/p$ follows. □

Replacing (3) by $pn^2 \leq \sum_i |A_i||B_i|$ the above proof gives

$$\vartheta_p(\mathcal{K}_{n,n}) \geq (n + p - 1)^2/p, \tag{7}$$

which is better than (1) for $n < (p-1)^2$.

Consider a $\mathcal{K}(n \times k)$ with classes $V^1, \dots, V^k, V^\ell = \{v_1^\ell, \dots, v_n^\ell\}$. We call the p-edge clique cover $\mathcal{F} = \{F_1, \dots, F_t\}$ *perfect* if the sets $\{F_i \cap V^\ell : 1 \leq i \leq n\}$ form an $S_{p-1}(n, p, 2)$ design for all ℓ. It follows, that $|F_i| = kp$ for all F_i, every edge of $\mathcal{K}(n \times k)$ is contained in exactly p sets, every pair from V^ℓ is covered $p-1$ times, every pair of the form $\{v_\alpha^\ell, v_\alpha^m\}$ is uncovered, and $t = (n^2 - n)/p$.

Proposition 2. *If $p = q^d$, where q is a prime power, d a positive integer and $k \leq q$, then there exists a perfect p-edge clique cover of $\mathcal{K}(q^{d+1} \times k)$. Hence, in this case, $\vartheta_p(\mathcal{K}(q^{d+1} \times k)) = q^{d+2} - q$.*

Proposition 3. *If $p = q^d + q^{d-1} + \cdots + 1$, where q is a prime power, d a positive integer and $k \le q+1$, then $\vartheta_p(\mathcal{K}^{(k)}_{q^{d+1},\ldots,q^{d+1}}) = q^2(q^d+q^{d-1}+\cdots+1)$.*

Proof. The lower bounds for ϑ_p are implied by Proposition 1 and (2), respectively. The upper bounds are given by the following construction. Let X be the point set of a $(d+2)$-dimensional projective space of order q, $PG(d+2,q)$, let $Z \subset X$ be a subspace of dimension d, and let $Y^1,\ldots,Y^{q+1}$ be the hyperplanes containing $Z, V^\ell = Y^\ell \setminus Z$. The sets V^ℓ partition $X \setminus Z$ into q^{d+1}-element classes. Choose a point $c \in V^{q+1}$ and label the vertices of $V^\ell = \{v_i^\ell : 1 \le i \le q^{d+1}\}$ in such a way that $\{v_i^\ell : 1 \le \ell \le q\} \cup \{c\}$ form a line for all i. □

The hyperplanes not containing Z and avoiding c induce a perfect p-edge clique cover of $\mathcal{K}(p^{d+1} \times k)$ with classes $V^1,\ldots,V^k (k \le q)$. Indeed, $PG(d+2,q)$ contains $q^{d+2}+q^{d+1}+\cdots+q+1$ hyperplanes and they cover each pair of points exactly p times. The point c is contained by exactly $q^{d+1}+\cdots+q+1$ of the hyperplanes, Z is contained in $q+1$ of them, and $Z \cup \{c\}$ is contained in a unique one. So the above defined cover consists of $q^{d+2}-q$ sets. These sets still cover each pair of the form $\{v_i^\ell, v_j^m\}, i \ne j$ exactly p times. However, the pairs of the form $\{v_\alpha^\ell, v_\alpha^m\}$ are uncovered, because any subspace containing these tow points must contain the line through them, so it must contain the element c.

Similarly, considering all the hyperplanes not containing Z, we get a p-edge clique cover of $\mathcal{K}^{(k)}_{n,\ldots,n}$ with classes $V^1,\ldots,V^k$ where $n = q^{d+1}$ and $k \le q+1$.

The dual (perfect) p-representation of $\mathcal{K}(q^{d+1} \times k)$ can be obtained by considering a line, L, in the affine space of dimension $d+2$, and assigning all sets of the form $Y \setminus \{v^\ell\}$ to the vertices of the ℓth color class, where $v^\ell \in L$, and Y is a hyperplane with $Y \cap L = \{v^\ell\}$. Similarly, the dual p-representation of $\mathcal{K}^{(k)}_{n,\ldots,n}$ on the underlying set $X \setminus L$ can be obtained by assigning the sets $Y \setminus \{v^\ell\}$ to the ℓth color class. There might be more optimal constructions using higher dimensional spaces.

4. Constructions from Steiner Systems

Proposition 4. *If there exists a Hadamard matrix of size $4p$, then there exists a perfect p-edge clique cover of $\mathcal{K}(2p \times 2)$, so its $\vartheta_p = 4p-2$.*

Proof. We are going to give a perfect p-intersection representation with underlying set $\{1,\ldots,4p-2\}$. Its dual is a perfect p-edge clique cover. Let M be a Hadamard matrix of order $4p$. We may suppose that the last row contains only $+1$'s. The ±1's in any other row define a partition of $\{1,\ldots,4p\}$ into two $2p$-element sets $P_i^+ \cup P_i^-$. We may also suppose that the last two entries are $M_{i,4p-1} = 1$, $M_{i,4p} = -1$ for $1 \le i \le 2p$. Finally, assign the set $P_i^+ \setminus \{4p-1\}$ to the vertex v_i^1, and $P_i^- \setminus \{4p\}$ to v_i^2. □

Note that both in Proposition 2 and here we have got the perfect p-edge clique cover from a resolvable $S_\lambda(sp, p, 2)$ design, where s is an integer $\lambda = (p-1)/(s-1)$.

Proof of Theorem 2(b). First, we consider a perfect p-edge clique cover, $\mathcal{F}$, in the case $n = q^{d+1}$ given by Proposition 2. Consider k identical copies of a Steiner system $S(n, q^{d+1}, 2)$ over the n-element sets V^ℓ. Replace each block and its corresponding pairs by a copy of $\mathcal{F}$. Then we obtain a system is a perfect p-edge clique cover.

The proof of case (a) is similar, we put together a perfect p-edge clique cover using a building block of size $2p$ supplied Proposition 4 and a Steiner system $S(n, 2p, 2)$. Taking the sets $\{v_i^1, v_i^2, \dots, v_i^k\}$ p times, we get that $\vartheta_p(K_{n,\dots,n}^{(k)}) \le \vartheta_p(\mathcal{K}(n \times k)) + pn$. As $\vartheta_p(K_{n,\dots,n}^{(k)})$ is a monotone function of n we got Corollary 1 . □

Conjecture 1. *If $(n^2-n)/p$ is an integer and $n > n_0(p,k)$, then there exists a perfect p-edge clique cover of $\mathcal{K}(n \times k)$, hence its p-intersection number $\vartheta_p = (n^2-n)/p$.*

The case $p = 1$ corresponds to the fact that there are transversal designs $T(n,k)$ (i.e., mutually orthogonal Latin squares of sizes n) for $n > n_0(k)$ (Chowla, Erdős, Straus [4], also see Wilson [23]).

Chung and West [5] proved the case $k = p = 2$. They showed $\vartheta_2(\mathcal{K}(n \times 2)) = (n^2-n)/2$ by constructing a perfect 2-edge cover (they call it a perfect 2-generator) for the cases $n \equiv 1, 2, 5, 7, 10$, or $11 \pmod{12}$. This and (1) imply that

$$\vartheta_2(\mathcal{K}_{n,n}) = (n^2+3n)/2 \tag{8}$$

holds for these cases. Their conjecture about the so-called orthogonal double covers (a conjecture equivalent to the existence of a perfect 2-edge cover of $\mathcal{K}(n \times 2)$) which conjecture had appeared in [6], too, is true for all $n > 8$. This was proved by Ganter and Gronau [14] and independently by Bennett and Wu [1]. So $n_0(2,2) = 8$ and (8) holds for all $n > 8$.

There are two more values proved in [5], namely the special cases $q = 2$ and $q = 3$ of the following conjecture. $\mathcal{K}(q^2+q+1 \times 2)$ has a perfect q-edge cover whenever a projective plane of order q exists. This would imply that equality holds in (1) for $(p,n) = (q+1, q^2+q+1)$.

5. Further Problems, Conjectures

The first nontrivial lower bound for $\vartheta_p(\mathcal{K}_{n,n})$ was proved by Jacobson [16]. He and Kézdy and West [17] also investigated $\vartheta_2(\mathcal{G})$ for other classes of graphs, like paths and trees.

How large $\vartheta_p(\mathcal{K}(n \times k))$ and $\vartheta_p(\mathcal{K}_{n,\dots,n}^{(k)})$ if n is fixed and $k \to \infty$?

Estimate ϑ_p for complete bipartite graph with parts of sizes a and b when $a \to \infty, p$ is fixed and a/b goes to a finite limit.

Another interesting graph where one can expect exact results is a cartesian product, its vertex set is $I_1 \times \cdots \times I_\ell$, and $(i_1, \ldots, i_\ell)$ is joined to $(i'_1, \ldots, i'_\ell)$ if and only if $i_\alpha \neq i'_\alpha$ for all $1 \leq \alpha \leq \ell$.

For a matching, $\mathcal{M}$, of size n it easily follows that $\vartheta_p(\mathcal{M}) = \min\{t : \binom{t}{p} \geq n\}$.

One can ask the typical value of $\vartheta_p(\mathcal{G})$, i.e., the expected value of ϑ_p for the random graph of n vertices. The case of $p = 1$ was proposed in [13], and the best bounds are due to Frieze and Reed [12] (for an intermediate result see Bollobás, Erdős, Spencer, and West [3]): For almost all graphs its edge set can be covered by $O(n^2/(\log n)^2)$ cliques and this magnitude is the best possible. Obviously,

$$\vartheta_p(\mathcal{G}) \leq \vartheta_{p-1}(\mathcal{G}) + 1 \leq \vartheta_1(\mathcal{G}) + p - 1,$$

so the order of magnitude of $E(\vartheta_p)$ is at most that of $E(\vartheta_1)$. Can we extend the method of [12] to estimate for $E(\vartheta_p)$?

The notion of ϑ_p was generalized from the study of the p-*competition* graphs. Another generalization, also having several unsolved questions, is the clique coverings by p *rounds*. Let $\mathcal{G}$ be a simple graph and let $\varphi_p(\mathcal{G})$ be the minimum of $\sum_{1 \leq i \leq p} n_i$ such that there are families $\mathcal{A}_1, \ldots, \mathcal{A}_p$ with $|\mathcal{A}_i| = n_i$, such that each edge $e \in \mathcal{E}(\mathcal{G})$ is covered by each family (i.e., there exists an $A \in \mathcal{A}_i$ with $e \subset A$), but this does not hold for the non-edges. It is known [13], that for all graphs on n vertices $\varphi_2 \leq 3n^{5/3}$ and for almost all graphs $\varphi_2 > 0.1n^{4/3}/(\log n)^{4/3}$. For further problems and questions, see [13].

Bollobás [2] generalized the Erdős-Goodman-Pósa result as follows. The edge set of every graph on n vertices can be *decomposed* into $t(k-1, n)$ parts using only $\mathcal{K}_k$'s and edges, where $t(k-1, n)$ is the maximum number of edges in a $(k-1)$-colored graph on n vertices, e.g., $t(2, n) = \lfloor n^2/4 \rfloor$. There are many beautiful results and problems of this type, the interested reader can see the excellent survey by Pyber [21]. Most of the problems can be posed to multigraphs, obtaining new, interesting, non-trivial problems.

Let $\vartheta^*_p(\mathcal{G})$ the minimum t such that each vertex can be assigned a subset of $\{1, \ldots, t\}$ in such a way that the intersection of any two of these sets is at most p, and vertices of $\mathcal{G}$ are adjacent if and only if the corresponding sets have *exactly* p common elements. Note that in all of the results in this paper were proved an upper bound for ϑ^*_p. If $\mathcal{G}^n$ is a graph on n vertices with with $2n - 3$ edges such that two vertices are connected to all others, then one can show that $\lim_{n\to\infty} \vartheta^*_p(\mathcal{G}^n) - \vartheta_p(\mathcal{G}^n) = \infty$ for any fixed p. What is $\max(\vartheta^*_p(\mathcal{G}) - \vartheta_p(\mathcal{G}))$ and $\max(\vartheta^*_p/\vartheta_p)$ for different classes of graphs?

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Reflections on a Problem of Erdős and Hajnal

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Summary. We consider some problems suggested by special cases of a conjecture of Erdős and Hajnal.

1. Epsilons

The problem I am going to comment on reached me in 1987 at Memphis in a letter of Uncle Paul. He wrote: 'We have the following problem with Hajnal. If $G(n)$ has n points and does not contain induced C_4, is it true that it has either a clique or an independent set with n^ε points? Kind regards to your boss + colleagues, kisses to the ε-s. E.P.' After noting that ε has been used in different contexts I realized soon that $\frac{1}{3}$ is a good ε (in both senses since I have three daughters). About a month later Paul arrived and said he meant C_5 for C_4. And this minor change of subscript gave a problem still unsolved. And this is just a special case of the general problem formulated in the next paragraph.

2. The Erdős-Hajnal Problem (from [7])

Call a graph H-free if it does not contain induced subgraphs isomorphic to H. Complete graphs and their complements are called *homogeneous sets*. As usual, $\omega(G)$ and $\alpha(G)$ denotes the order of a maximum clique and the order of a maximum independent set of G. It will be convenient to define $\hom(G)$ as the size of the largest homogeneous set of G, i.e. $\hom(G) = \max\{\alpha(G), \omega(G)\}$ and

$$\hom(n, H_1, H_2, \ldots) = \min\{\hom(G) : |V(G)| = n, G \text{ is } H_i\text{-free}\}$$

A well-known result of Paul Erdős [5] says that there are graphs of n vertices with $\hom(G) \le 2\log n$ (log is of base 2 here). The following problem of Erdős and Hajnal suggests that in the case of forbidden subgraphs $\hom(G)$ is much larger: Is it true, that for every graph H there exists a positive ε and

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n_0 such that every H-free graph on $n \geq n_0$ vertices contains a homogeneous set of n^ε vertices? If such ε exists for a particular H, one can define the 'best' exponent, $\varepsilon(H)$ for H as

$$\varepsilon(H) = \sup\{\varepsilon > 0 : \hom(n, H) \geq n^\varepsilon \text{ for } n \geq n_0\}.$$

The existence of $\varepsilon(H)$ is proved in [7] for P_4-free graphs (usually called *cographs* but in [7] the term *very simple graphs* has been used). In fact, a stronger statement is proved in [7]: if $\varepsilon(H_i)$ exists for $i = 1, 2$ and H is a graph formed by putting all or no edges between vertex disjoint copies of H_1 and H_2 then $\varepsilon(H)$ also exists. Combining this with the well known fact that P_4-free graphs are perfect [14], it follows that $\varepsilon(H)$ exists for those graphs H which can be generated from the one-vertex graph and from P_4, using the above operations. In the spirit of [7], call this class SVS (still very simple). In terms of graph replacements (see Sect. 4 below), SVS is generated by replacements into two-vertex graphs starting from K_1 and P_4. As far as I know, the existence of $\varepsilon(H)$ is not known for any graph outside SVS.

3. Large Perfect Subgraphs

A possible approach to finding a large homogeneous set in a graph is to find a large perfect subgraph. It was shown in [7] that any H-free graph of n vertices has an induced cograph of at least $e^{c(H)\sqrt{\log n}}$ vertices for sufficiently large n. This shows that the size of the largest homogeneous set makes a huge jump in the case of any forbidden subgraph. In a certain sense it is not so far from $n^{c(H)}$. What happens if cographs are replaced by other perfect graphs? A deep result of Prömel and Steger [13] says that almost all C_5-free graphs are perfect (generalized split graphs) This suggest the possibility to find a large (n^ε) generalized split graph in a C_5-free graph of n vertices and prove the existence of $\varepsilon(C_5)$ this way.

4. Replacements

A well-known important concept in the theory of perfect graphs is the replacement of a vertex by a graph. The replacement of vertex x of a graph G by a graph H is the graph obtained from G by replacing x with a copy of H and joining all vertices of this copy to all neighbors of x in G. According to a key lemma (Replacement Lemma) of Lovász (see for example in [12]), perfectness is preserved by replacements. The property of being H-free is obviously preserved by replacements if (and in some sense only if) H can not be obtained from a smaller graph by a nontrivial (at least two-vertex) replacement. For such an H, replacements can be applied to get an upper bound on $\varepsilon(H)$. Analogues of the Replacement Lemma can be also useful to find large homogeneous sets (an example is Lemma 1 below)

5. Partitions into Homogeneous Sets

For certain graphs H, the existence of $\varepsilon(H)$ follows from stronger properties. An H-free graph G may satisfy $\chi(G) \leq p(\omega(G))$ or $\theta(G) \leq p(\alpha(G))$ or more generally $cc(G) \leq p(\alpha(G), \omega(G))$ where p is a polynomial of constant degree and $\chi, \theta, \alpha, \omega, cc$ denote the chromatic number, clique cover number, independence number, clique number and cochromatic number of graphs. Using terminology from [10], p is a *polynomial binding function* (for χ, θ, cc, respectively). It is clear that if H-free graphs have a polynomial binding function of degree k then $\varepsilon(H) \geq \frac{1}{k+1}$. Binding functions for χ (for θ) in H-free graphs may exist only if $H(\overline{H})$ is acyclic. However, the existence of a polynomial binding function for cc in H-free graphs is equivalent with the existence of $\varepsilon(H)$.

6. Small Forbidden Subgraphs

The existence of $\varepsilon(H)$ follows if H has at most four vertices since these graphs are all in SVS. But, as will be shown below, to find $\varepsilon(H)$ for these small graphs is not always that simple...

Since $\hom(n, H) = \hom(n, \overline{H})$ from the definition, it is enough to consider one graph from each complementary pair. For $H = K_m$, finding $\hom(n, H)$ is the classical Ramsey problem. In case of $m = 2, 3, 4, \varepsilon(K_2) = 1$ (trivial), $\varepsilon(K_3) = \frac{1}{2}$ (from Uncle Paul's lower bound on $R(3, m)$ in [4]), $\frac{1}{3} \leq \varepsilon(K_4) \leq 0.4$ (the upper bound is due to Spencer [15]). If $H = P_3$ or $H = P_4$ then an H-free graph is perfect and thus $\varepsilon(H) = \frac{1}{2}$.

There are four more graphs with four vertices to look at. Let H_1 be $K_{1,3}$, the *claw*, and let H_2 be K_3 with a pendant edge. It is not difficult to see that $\varepsilon(H_i) = \frac{1}{3}$ in this case. The construction is simple: let G be a graph on m vertices with no independent set of three vertices and with no complete subgraph of much more than $\sqrt{m}$ vertices [4]. Take about $\frac{\sqrt{m}}{2}$ disjoint copies of G. This graph is H_1-free, has $m^{\frac{3}{2}}/2$ vertices and has no homogeneous subset with much more than $m^{\frac{1}{2}}$ vertices. The complement of this graph is good for H_2. On the the other hand, let G be an H_i-free graph with n vertices (i is 1 or 2). If the degree of a vertex v is at least $n^{\frac{2}{3}}$ then $\Gamma(v)$ (the set of vertices adjacent to v) contains a homogeneous set of at least $n^{\frac{1}{3}}$ vertices (in the case of H_1 by Ramsey's theorem, in the case of H_2 by perfectness). Otherwise G has an independent set of at least $n^{\frac{1}{3}}$ vertices. This gives

Proposition 1. *If H is the claw or K_3 with a pendant edge then $\varepsilon(H) = \frac{1}{3}$.*

The remaining two H-s are the C_4 and K_4 minus an edge (the *diamond*). The following argument is clearly discovered by many of us, could be heard from Uncle Paul too. It was used for example in [9, 16]. Let $S = \{v_1, \ldots, v_\alpha\}$

be a maximum independent set of a C_4-free or diamond-free graph G. Then $V(G)$ is covered by the following $\binom{\alpha+1}{2}$ sets: $A(i)$, $B(i,j)$, $1 \le i < j \le \alpha$ where

$$A(i) = \{v \in V(G) - S : \Gamma(v) \cap S = \{v_i\}\} \cup \{v_i\}$$

and

$$B(i,j) = \{v \in V(G) - S : \Gamma(v) \cap S \supseteq \{v_i, v_j\}\}.$$

The sets $A(i)$ induce complete subgraphs by the maximality of S and the sets $B(i,j)$ induce homogeneous sets (complete if G is C_4-free, independent if G is diamond-free). This gives

Proposition 2. *If G is C_4-free or diamond-free then $cc \le \binom{\alpha+1}{2}$.*

Corollary 1. *If H is either C_4 or the diamond then* $\hom(n,H) \ge (2n)^{\frac{1}{3}}$. *Therefore $\varepsilon(H) \ge \frac{1}{3}$.*

Vertex disjoint unions of complete graphs shows that $\varepsilon(H) \le \frac{1}{2}$ for any connected graph H. In the case of $H = C_4$ this upper bound can be improved as follows. Let $R(C_4, m)$ be the smallest integer k such that any graph on k vertices either contains a C_4 (not necessarily induced C_4!) or contains an independent set on m vertices. F. K. Chung gives a graph G_m in [3] which shows that $R(C_4, m) \ge m^{\frac{4}{3}}$ for infinitely many m. Replacing each vertex of G_m by a clique of size $\frac{m}{3}$ we have a graph with no induced C_4 and with no homogeneous subset larger than m. This gives

Proposition 3. $\varepsilon(C_4) \le \frac{3}{7}$.

Notice that if $R(C_4, m) \ge m^{2-\varepsilon}$ with every $\varepsilon > 0$ as asked by Uncle Paul then the replacement described above would show that $\varepsilon(C_4) = \frac{1}{3}$.

Perhaps the next construction has a chance to improve the upper bound on $\varepsilon(H)$ if H is the diamond. The vertices of G_q are the points of a *linear complex* [11] of a 3-dimensional projective space of order q. Two points are adjacent if and only if they are on a line of the linear complex. The graph G_q has $q^3 + q^2 + q + 1$ vertices, $\omega(G_q) = q + 1$ and G_q is diamond-free. But is it true that $\alpha(G_q) < q^{\frac{3}{2}-\varepsilon}$ for some positive ε and for infinitely many q? Thanks for the conversations to T. Szőnyi who thinks this is not known.

Problem 1. *Improve the exponents in the above estimates of* $\hom(n,H)$ *if H is either C_4 or the diamond.*

What happens if G is C_4-free *and* diamond-free? In these graphs each four cycle induces a K_4, The sets $B(i,j)$ collapse implying

Corollary 2. $\hom(n, C_4, \mathit{Diamond}) \ge \sqrt{\frac{2}{3}n} - 1$.

Problem 2. *Is it true that* $\hom(n, C_4, \mathit{Diamond}) = \sqrt{n} + o(\sqrt{n})$*?*

During the years between the submission and publication of this paper, Problem 2 had been answered affirmatively, in fact $\hom(n, C_4, \text{Diamond}) = \lceil\sqrt{n}\rceil$ [6].

There are eight five-vertex graphs outside SVS. Keeping one from each complementary pair reduces the eight to five: $K_{1,3}$ with a subdivided edge, the *bull* (the self-complementary graph different from C_5), C_4 with a pendant edge, P_5, and C_5. The existence of $\varepsilon(H)$ is open for all of them, perhaps the list is about in the order of increasing difficulty. The construction in Proposition 1 shows that $\varepsilon(H)$ is at most $\frac{1}{3}$ for all but C_5. In the case of C_5 repeated replacements of $\overline{C_7} \cup K_3$ into itself shows $\varepsilon(C_5) \le \frac{\log 3}{\log 10}$. (Any C_5-free graph G with $\hom(G) = 3$ and with at least 11 vertices would improve this.)

7. Forbidden Complementary Pairs

Perhaps an interesting subproblem is to find bounds on $\hom(n, H, \overline{H})$. In the case of four-vertex H, the structure of graphs which are both H-free and $\overline{H}$-free is well understood and values of $\hom(n, H, \overline{H})$ can be determined as follows: $n^{\frac{1}{2}}$ if $H = P_4$ (from perfectness); $n^{\frac{1}{2}} - 1$ if $H = P_3 + K_1$ (from structure, [10]); $\frac{n-1}{2}$ if $H = C_4$ (from structure, [2]); $n - 4$ if H is a diamond (from structure, [10]); $\frac{2n}{5}$ if H is a claw (from structure, [10]).

The rest of this section is devoted to the case $H = P_5$. The upper bound $\hom(n, P_5, \overline{P_5}) \le n^{\frac{1}{\log 5}}$ is shown by replacing repeatedly C_5 into itself. The lower bound $n^{\frac{1}{3}}$ will follow from Corollary 3 which is the consequence of the following result.

Theorem 1. *If G is P_5-free and $\overline{P_5}$-free then G satisfies the following properly SP^*: there is an induced perfect subgraph of G whose vertices intersect all maximal cliques of G.*

Notice that property SP^* is a generalization of *strong perfectness* introduced by Berge and Duchet in [1]. (Maximal clique is a clique which is not properly contained in any other clique.) By Theorem 1, if G is both P_5-free and $\overline{P_5}$-free then G can be partitioned into at most $\omega(G)$ vertex disjoint perfect subgraphs. Each of these perfect graphs has clique number at most $\omega(G)$ thus each has chromatic number at most $\omega(G)$. This gives the next corollary.

Corollary 3. *If a graph is both P_5-free and $\overline{P_5}$ free then $\chi \le \omega^2$.*

The proof of Theorem 1 is combining a result of Fouquet [8] with the following analogue of the Lovász replacement lemma.

Lemma 1. *Property SP^* is preserved by replacements.*

Proof. The proof of the Lemma is along the same line as the replacement Lemma of Lovász. Assume that G and H have property SP^* and R is the graph obtained by replacing $v \in V(G)$ by H. Let G_1 and H_1 be perfect subgraphs of G and H such that $V(G_1)$ intersects all maximal cliques of G and $V(H_1)$ intersects all maximal cliques of H.

Case 1. $v \notin V(G_1)$. We claim that $V(G_1)$ intersects all maximal cliques of R. Let K be a maximal clique of R. If $V(K) \cap V(H)$ is empty then the claim follows from the definition of G_1. Otherwise $\{v\} \cup (K \cap V(G))$ is a clique of G which can be extended in G to a maximal clique K' intersecting $V(G_1)$. Since K is obtained by replacing $v \in K'$ by $K \cap V(H)$, K intersects $V(G_1)$.

Case 2. $v \in V(G_1)$. By the Lovász replacement lemma, the subgraph Z of R induced by $(V(G_1) \cup V(H_1)) - \{v\}$ is perfect. If a maximal clique K of R intersects $V(H)$, it intersects it in a maximal clique of H which (by the definition of H_1) intersects $V(H_1)$. If K does not intersect H then it does not contain v so it intersects $V(G_1) - \{v\}$ by the definition of G_1. Therefore K intersects Z. □

Theorem 2 (**Fouquet** [**8**]). *Each graph from the family of P_5-free and $\overline{P_5}$-free graphs is either perfect or isomorphic to C_5 or can be obtained by a nontrivial replacement from the family.*

Now Theorem 1 follows by induction from Lemma 1 and Theorem 2.

8. Berge Graphs

These are graphs which do not contain induced subgraphs isomorphic to C_{2k+1} or to $\overline{C_{2k+1}}$ for $k \geq 2$. According to the Strong Perfect Graph Conjecture (of Berge), Berge graphs are perfect. The following weaker form of this conjecture is attributed to Lovász in [7] (illustrating the difficulty of proving the existence of $\varepsilon(C_5)$).

Problem 3. *There exists a positive constant ε such that Berge graphs with n vertices contain homogeneous subsets of n^{ε} vertices.*

Similar problems can be asked for subfamilies of Berge graphs for which the validity of SPGC is not known. One of them is the following.

Problem 4. *Show that C_4-free Berge graphs with n vertices contain homogeneous subsets of $n^{\frac{1}{2}}$ (or at least $cn^{\frac{1}{2}}$) vertices.*

9. Notes Added in 2013

Roughly 15 years went by... Replacements suggested in Sect. 4 have been developed in [17], implying the existence of $\varepsilon(H)$ for many new graphs H. Two of them, together with the bull resolved separately in [19], leave only two five-vertex graphs (from the five in Sect. 6) for which the Erdős-Hajnal conjecture is open: P_5 (or its complement) and C_5. The theme of Sect. 7

is recently revitalized, see [20, 21]. Finally, with the Strong Perfect Graph Theorem [18], the problems of Sect. 8 are resolved.

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The Chromatic Number of the Two-Packing of a Forest

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Summary. A two-packing of a graph G is a bijection $\sigma : V(G) \mapsto V(G)$ such that for every two adjacent vertices $a, b \in V(G)$ the vertices $\sigma(a)$ and $\sigma(b)$ are not adjacent. It is known [2, 6] that every forest G which is not a star has a two-packing σ. If F_σ is the graph whose vertices are the vertices of G and in which two vertices a, b are adjacent if and only if a, b or $\sigma^{-1}(a)$, $\sigma^{-1}(b)$ are adjacent in G then it is easy to see that the chromatic number of F_σ is either 1, 2, 3 or 4. We characterize, for each number n between one and four all forests F which have a two-packing σ such that F_σ has chromatic number n.

Keywords Packing, Placement, Factorization, Tree, Forest, Chromatic number

AMS Subject Classification: 05C70

1. Introduction

We only discuss finite simple graphs and use standard terminology and notation from [1] except as indicated. For any graph G, we use $V(G)$ and $E(G)$ to denote the set of vertices and the set of edges of G, respectively. A forest is a graph without cycles. A tree is a connected forest. The *length* of a path or cycle is the number of its edges. A path of length n is denoted by P_n and a cycle of length n is denoted by C_n. The complete graph on n vertices is denoted by K_n. A vertex of a graph G is an *isolated point* if its degree is zero and it is an *endpoint of* G if its degree is one. A vertex of a forest F adjacent to an endpoint of F is called a *node of* F. The set of vertices

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of a forest F which are adjacent to at least one of the endpoints of a set A of endpoints is called the set of *nodes belonging* to the set A of endpoints.

The *distance* between two vertices x and y of a graph G is the length of the shortest path in G from x to y. The *diameter* of a graph G is the largest distance between any two vertices of G. Observe that if T is a tree and the distance between the vertices x and y of T is the diameter of T, then the vertices x and y are endpoints of T. A vertex x of a tree T is in the *center* of T, or is a *central point* of T, if the maximal distance from x to any other vertex of T is minimal for the vertex x. Observe that if the diameter of T is odd then T has exactly two central points and if the diameter of T is even, then T has exactly one central point.

A tree of order at least two has at least two endpoints and if it has exactly two endpoints it is a path. A tree with exactly two nodes is called a *dragon.* Note that every tree of diameter three is a dragon. A tree of diameter two or one with $n \geq 1$ edges is called a *star* S_n. Hence an isolated point is not a star. If A is a set of vertices of the graph G then $G-A$ is the graph obtained from G by removing the vertices in A together with all of the incident edges from G. We will write $G - a$ for $G - \{a\}$. If F is a forest and A a set of endpoints of F then the forest $F - A$ is called a *derived* forest of F. If A is the set of all endpoints of F then $F - A$ is the *completely derived* forest of F. A tree T is a *crested dragon* if T has a path $P = a_1, a_2, a_3, a_4$ of length three as a derived tree, the vertices a_1, a_4 and possibly a_3 are nodes of T, every node of T is in the set $\{a_1, a_3, a_4\}$ and the number of endpoints adjacent to a_1 is strictly larger than the number of endpoints adjacent to a_3. Observe that the crested dragon T is a dragon of diameter five if and only if a_3 is not a node of T. See Fig. 1.

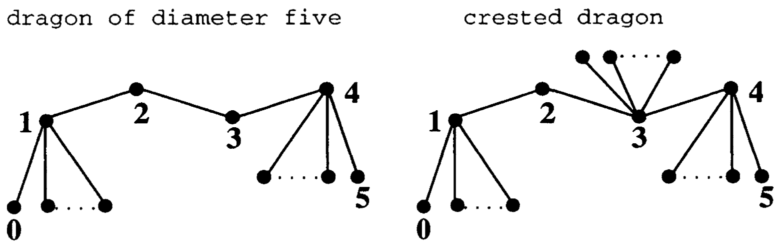

Fig. 1

Let G be a graph and σ a bijection from $V(G)$ to $V(G)$. If σ has the property that for every pair a and b of adjacent vertices of G the vertices $\sigma(a)$ and $\sigma(b)$ are not adjacent then σ is called a *two-packing of the graph* G. If σ is a two-packing of G then the graph $\sigma(G)$ has the same vertices as the graph G and two vertices a and b of $\sigma(G)$ are adjacent in $\sigma(G)$ if and only if the vertices $\sigma^{-1}(a)$ and $\sigma^{-1}(b)$ are adjacent in the graph G. If E is the set of edges of G then $\sigma(E)$ denotes the set of edges of $\sigma(G)$. Note that $E \cap \sigma(E) = \emptyset$. The graph G_σ has the same vertices as the graph G and the edges of G_σ are given by $E(G_\sigma) = E(G) \cup E(\sigma(G))$. The graph G_σ can be *factorized* into the graphs G and $\sigma(G)$. Hence results about two-packings of graphs can also be understood as results about factorizations of graphs.

Burns and Schuster proved that if a graph G of order n does not contain a vertex of degree $n-1$ and contains no cycles of length 3 or 4 and $|E(G)| = n-1$ then there is a two-packing of G, [2]. Faudre, Rousseau, Schelp and Schuster proved that if a graph G of order n does not contain a vertex of degree $n-1$ and contains no cycles of length 3 or 4 and $|E(G)| \le 6n/5 - 2$, then there is a two packing of G, [3]. St. Brandt proved that if G is a non-star graph of girth larger than or equal to seven then there is a packing of two copies of G, [4]. It follows from [2] that every tree which is not a star has a two-packing and we recently [6] characterised those trees which have a "three-packing".

The results mentioned above establish the existence of two-packings of a graph G but are not concerned with any properties of the graph G_σ. From the point of factorizations of graphs this would be a very interesting question. Given the difficulties in establishing the existence of two-packings we looked for a simple case in which we might be able to determine the possible chromatic numbers of the two-packings of a graph. Note first that for any two-packing σ of G the chromatic number of G_σ satisfies the inequality $\chi(G) \le \chi(G_\sigma) \le \chi(G)^2$. This means that if F is a forest which has at least one edge and σ is a two-packing of F then $2 \le \chi(F_\sigma) \le 4$. If F_σ has chromatic number less than or equal to two we will call the two-packing σ a bipartite two-packing.

If G is a connected bipartite graph then the partition of G into the two color classes is unique. We will call the color class of a vertex x the *parity* of x and will often call the vertices in one class the vertices of even parity or even vertices and the vertices in the other class the vertices of odd parity or the odd vertices. We will then take care to label the even vertices with even and the odd vertices with odd numbers. A consequence of the uniqueness of the color-classes of a connected bipartite graph G is, that if σ is a bipartite two-packing of G then either for every vertex x of G, $\sigma(x)$ has the same parity as x or for every vertex x of G, $\sigma(x)$ has a different parity than x. In the first case we will say that the bipartite two-packing σ is an equal parity packing and in the second case we will say that the bipartite two-packing σ is an unequal parity packing. If F is a forest and σ a two-packing of F then σ is an *equal parity two-packing* of F if for every vertex a of F, whenever $\sigma(a)$ and a are in the same connected component of F then, in this connected component of F, the parity of $\sigma(a)$ is the same as the parity of a.

It will be often necessary to exhibit a two-packing σ for some small forest F. We usually will do this in a figure in which the edges of F are solid lines and the edges of $\sigma(F)$ are interrupted lines. In order to check the claim that the exhibited map σ is indeed a two-packing one has to check that the graph with the interrupted edges is indeed isomorphic to the graph having solid edges and that the graph with the solid edges is the intended graph. It is then visually clear that the two edge-sets have no edge in common. In the case of a bipartite two-packing σ, if in the text a vertex is described by a labeled letter then in the figure the label alone will be used to name the vertex. The labels of the vertices of F will be in bold times italic typeset and the vertices of $\sigma(F)$ in gray courier oblique. All vertices of one parity will have even and all the other vertices will have odd labels. In order to check that σ is an equal parity two-packing it suffices to check that the parity of the two labels of each vertex is the same.

We will, for each number n between one and four, completely characterize all forests F which have a two-packing a such that F_σ has chromatic number n. More exactly we will prove the following theorems:

Theorem 1. *A forest has a bipartite two-packing if and only if it is either a singleton vertex or disconnected or a tree of diameter at least five which is not a crested dragon. Every forest which has a bipartite two-packing has an equal parity two-packing.*

Theorem 2. *A forest F has a two-packing σ such that F_σ has chromatic number three if and only if either F is a tree with diameter at least four or F is disconnected and contains a path of length two.*

Theorem 3. *A forest F has a two-packing a such that F_σ has chromatic number four if and only if F contains a path of length three and F does not consist of exactly two connected components where one of the two components is a path of length three and the other is a star.*

2. Preliminary Results

If F is a forest, B a set of vertices of F and σ a two-packing with the property that for all $b \in B$, $\sigma(b) \neq b$ holds, then we say that σ *splits* the vertices of B. If B consists of a single vertex b we say that the two-packing σ splits the vertex b.

If σ is an equal parity two-packing which splits all vertices of $V(F)$ then we say that σ is a *perfect packing*. Let A be a set of endpoints of a forest G and B the set of nodes which belong to A. Assume that σ is a two-packing of $G - A$ which splits the vertices of B and λ the extension of σ from $G - A$ to F such that for all $a \in A$, $\lambda(a) = a$ holds. Obviously λ is a two-packing of G. We will call λ a *trivial extension* of σ to the forest G. The following *First Preservation Lemma* holds for trivial extensions of two-packings.

Lemma 1 (**First Preservation Lemma**). *Let σ be a two-packing of a forest F and λ a trivial extension of σ to the forest G. If F_σ has chromatic number four then G_λ has chromatic number four. If F_σ has chromatic number three then G_λ has chromatic number three. If σ is an equal parity two-packing then λ is an equal parity two-packing.*

Proof. Assume that A is the set of endpoints of G such that $G - A = F$ and that B is the set of nodes of G which belong to the set A of endpoints. Then by the definition of trivial extension, the two-packing σ splits the nodes in B. The chromatic number of G_λ is larger than or equal to the chromatic number of F_σ, and because the chromatic number of any two-packing of a forest is at most four, the first of the assertions follows. The endpoints in A will have degree two in G_λ, which implies the second assertion. Let us now assume that σ is an equal parity two-packing. Then, for every node $b \in B$ and every two coloring of F_σ, the vertices b and $\sigma(b)$ will be colored with the same color. This of course implies the third assertion. □

Corollary 1. *If some derived forest of a forest F has a perfect packing then F itself has an equal parity two-packing.*

Let F be a forest, a and b two endpoints of F, a_1 the node adjacent to a and b_1 the node adjacent to b. If a_1 and b_1, are different vertices then the pair of endpoints a and b is called a *reducing pair of endpoints* of the forest F. If the nodes a_1 and b_1 are in the same connected component of F and have the same parity then clearly also the endpoints a and b have the same parity and then a and b is an equal parity *reducing pair* of endpoints of the forest F. Let a, b be a reducing pair of endpoints of the forest F, a_1 and a_2 the nodes adjacent to a and b respectively and σ a two-packing of the forest $F - \{a, b\}$. Let λ be the extension of σ from $F - \{a, b\}$ to F such that if $\sigma(a_1) = a_1$ or $\sigma(b_1) = b_1$ then $\lambda(a) = b$ and $\lambda(b) = a$. In all other cases we put $\lambda(a) = a$ and $\lambda(b) = b$. Clearly λ is a two-packing of the forest F. We will call λ the extension of σ to the reducing pair a, b of endpoints. The following *Second Preservation Lemma* holds for extensions of two-packings to reducing pairs of endpoints.

Lemma 2 (**Second Preservation Lemma**). *Let F be a forest, a, b a reducing pair of endpoints of F, σ a two-packing of the forest $F - \{a, b\}$ and λ an extension of σ to the reducing pair a, b of endpoints of F. If F_σ has chromatic number four then F_λ has chromatic number four. If F_σ has chromatic number three then F_λ has chromatic number three. If σ is an equal parity two-packing and a_1 and b_1 have the same parity then λ is an equal parity two-packing. If σ splits a set B of vertices then λ splits the set B of vertices.*

Proof. The chromatic number of F_λ is larger than or equal to the chromatic number of F_σ, and because the chromatic number of any two-packing of a forest is at most four, the first of the assertions follows. The endpoints a and b will have degree two in F_λ which implies the second assertion. Let us now assume that σ is an equal parity two-packing and that the nodes a_1 and b_1 have the same parity. Then, for every proper two coloring of F_σ, the vertices a_1, b_1, $\sigma(a_1)$ and $\sigma(b_1)$ will be colored with the same color. This of course implies the third assertion. Because σ and λ agree on $F - \{a, b\}$ the two-packing λ splits all vertices which are split by σ. □

Observe the following fact:

Lemma 3. *If T is a tree with diameter one or two then T does not have a two-packing.*

Lemma 4. *For every two-packing σ of a tree T with diameter three the graph T_σ has chromatic number four.*

Proof. Denote the two central points of T by a and b. If $\sigma(a) = a$ then $\sigma(b)$ is an endpoint not adjacent to a and hence adjacent to b. We will show that $\sigma(b)$ and b are also adjacent in $\sigma(T)$, a contradiction. Because σ is an onto map and σ maps all vertices adjacent to a to endpoints adjacent to b there is some endpoint x adjacent to b such that $\sigma(x) = b$. But then $\sigma(x) = b$ is adjacent to $\sigma(b)$ in $\sigma(T)$. If $\sigma(a) = b$ then $\sigma(b)$ is an endpoint not adjacent to b and hence adjacent to a. We will show that $\sigma(b)$ and a are also adjacent in $\sigma(T)$, a contradiction. Because σ is an onto map and σ maps all vertices adjacent to a to endpoints adjacent to a there is some endpoint x adjacent to b such that $\sigma(x) = a$. But then $a(x) = a$ is adjacent to $\sigma(b)$ in $\sigma(T)$. We conclude that $\{\sigma(a), \sigma(b)\} \cap \{a, b\} = \emptyset$.

We will now prove that $T \cup \sigma(T)$ induces the complete graph on the set $\{\sigma(a), \sigma(b), a, b\}$ of four vertices. It follows from the result of the previous paragraph that there are two endpoints x and y of T such that $\sigma(a) = x$ and $\sigma(b) = y$. Again because σ is onto there are two endpoints u and v of T such that $\sigma(u) = a$ and $\sigma(v) = b$. The graph $T \cup \sigma(T)$ induces then six different edges in the set $\{\sigma(a), \sigma(b), a, b\}$ of vertices. Those are first the two edges from a to b and $\sigma(a)$ to $\sigma(b)$. Then each of the four different points x, y which are endpoints of T and $\sigma(u)$, $\sigma(v)$ which are endpoints of $\sigma(T)$ is in the graph $T \cup \sigma(T)$ adjacent to at least one of the vertices $\{\sigma(a), \sigma(b), a, b\}$. □

Lemma 5. *If T is a tree with diameter at least two and at most four then T does not have a bipartite two-packing.*

Proof. Observe that every tree T of diameter at most four contains a vertex a which is adjacent to every vertex of T whose parity is different than the parity of the vertex a. If T has a bipartite two-packing σ then because σ is a bijection, there is some vertex b of T such that $\sigma(b) = a$. This means that the parity of a is different from the parity of b, otherwise b would have to be an isolated point (and if the diameter of a tree is larger than one it does not contain an isolated point). Because the vertex a is adjacent to every vertex whose parity is different from the parity of a, the vertices a and b are adjacent in T and therefore $\sigma(a)$ and $\sigma(b)$ are adjacent in $\sigma(T)$. Remember that every bipartite two-packing of T either changes the parity of all vertices or maps each parity class into itself. Hence $\sigma(a)$ is a vertex c adjacent to a. This now leads to a contradiction. The vertices a and c are adjacent in T, and because $a = \sigma(b)$ and $c = \sigma(a)$ they are adjacent in $\sigma(T)$. □

3. Bipartite Two-Packings

Lemma 6. *Every path of length at least seven has a perfect packing.*

Proof. If the length r of the path is seven, eight or nine Lemma 6 follows from Figs. 2, 3, and 4. So assume that $P_{r+1} = a_0, a_1, \ldots, a_{r+1}$ is a path with $r \geq 9$ and σ is a perfect packing of the path $P_r = a_0, a_1, \ldots, a_r$. Assume that for x

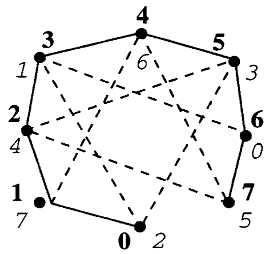

Fig. 2

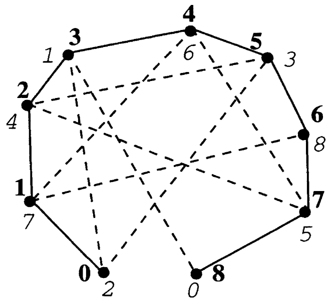

Fig. 3

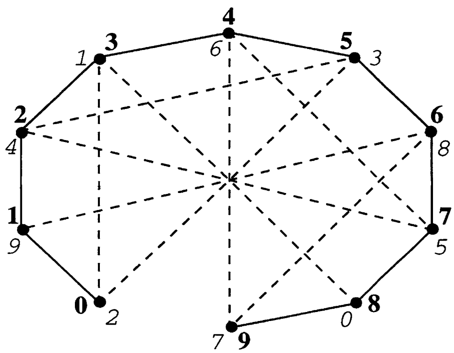

Fig. 4

in P_r, $\sigma(x) = a_r$. There are at least five vertices in the path P_r which have the same parity as the vertex a_{r+1}. This means that at least one of them, say a_i with $\sigma(y) = a_i$ has the following properties: $a_{i+1} \neq \sigma(a_r) \neq a_{i-l}$ and y is not adjacent to x. The following bijection λ from (P_{r+1}) to (P_{r+1}) is then a perfect packing of P_{r+1}. First of all λ agrees with σ on (P_{r-y}) and then $\lambda(y) = a_{r+1}$ and $\lambda(a_{r+1}) = a_i$. It is easy to check that λ is a perfect packing of P_{r+1}. □

Lemma 7. *If a forest G has an equal parity two-packing which splits the vertices of some set B, then the forest F which consists of G together with an additional isolated point e has an equal parity two-packing which splits every vertex in B and which also splits the isolated point e.*

Proof. Let σ be an equal parity two-packing of G which splits every vertex in B. Let a be any vertex of G and λ a function from $V(G) \cup \{e\}$ to $V(G) \cup \{e\}$ which agrees with σ on all vertices of $V(G) - a$ and for which $\lambda(e) = \sigma(a)$ and $\lambda(a) = e$. Clearly λ is an equal parity two-packing of G which splits the vertices of $B \cup \{e\}$. □

Lemma 8. *Every forest F with two components where one of the components is an isolated point e and the other component T is a dragon, a star, or an isolated point, has an equal parity two-packing which splits the isolated point e.*

Proof. If T is an isolated point then F clearly has an equal parity two-packing which splits e. If T is a star then a derived forest of F consists of two isolated vertices and hence there is a derived forest of F which has a perfect packing. Lemma 8 follows from the Corollary to Lemma 1. Let the path $P_r = a_1, a_2, \ldots, a_r$ be the completely derived tree of T and denote the nonempty set of endpoints adjacent to a_r by A and the nonempty set of endpoints adjacent to a_1 by B. Let $a \in A$ and $b \in B$ be two endpoints of T. The path P_r together with a and b is then a path P of length $r + 1$. If the forest whose connected components are P_r and e has an equal parity two-packing in which the vertices a_1, a_r and e are split, then according to the First Preservation Lemma, the forest F has an equal parity two-packing which splits e. It follows from Lemmas 6 and 7 that the forest whose connected components are P and e has an equal parity two-packing in which the vertices a_1, a_r and e are split if $r \geq 6$. If $r = 3$ or $r = 5$ then P_r is a star S_2 or has an equal parity reducing pair of endpoints a and b such that $P_r - \{a, b\}$ is a star S_2. Using the Second Preservation Lemma we see that Lemma 8 holds in this case. This leaves the cases $r = 2$ and $r = 4$. If $r = 2$ Fig. 5 exhibits an equal parity two-packing of P together with an isolated point which splits the two nodes of P and the isolated point.

Consider $r = 4$. Let $a_0 \in A$, $a_5 \in B$. See Fig. 6 in which 1, 4 and e are split. Then by the First Preservation Lemma, Lemma 8 holds for this case as well. □

Lemma 9. *Every forest F with two components, where one of the components is an isolated vertex e and the other a tree T, has an equal parity two-packing which splits e.*

Proof. By the Second Preservation Lemma we may assume that T does not contain two different nodes which have the same parity. This means that T can not have three different nodes, hence T is a dragon, a star or an isolated point. Lemma 9 follows then from Lemma 8. □

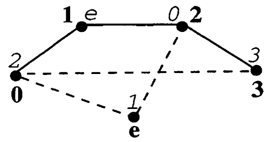

Fig. 5

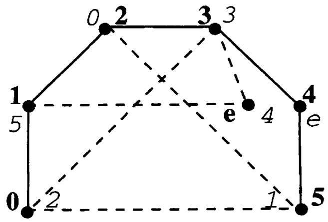

Fig. 6

Lemma 10. *Every forest F with at least two connected components has an equal parity two-packing.*

Proof. Observe first that if we obtain the forest G from F by adding an edge which is adjacent to vertices in different connected components of F, and if G has an equal parity two-packing, then F has an equal parity two-packing. We may therefore assume without loss of generality that the forest F has exactly two components G and H. Let G_1 be the forest which consists of G together with an additional vertex e_1 and H_1 the forest which consists of H together with an additional vertex e_2. Let γ be an equal parity two-packing of G_1 which splits e_1 and λ be an equal parity two-packing of H_1 which splits e_2. Identify the vertex $\gamma(e_1)$ with the vertex e_2 and the vertex $\lambda(e_2)$ with the vertex e_1. It is not difficult to check that $\gamma \cup \lambda = \sigma$ is an equal parity two-packing of the forest F. □

Lemma 11. *A crested dragon T does not have a bipartite two-packing.*

Proof. Let $P = a_1, a_2, a_3, a_4$ be the completely derived tree of the crested dragon T. Assume that the nodes of T are a_1, a_4 and possibly a_3. Let A be the set of endpoints adjacent to a_1, B the set of endpoints adjacent to a_4 and C the set of endpoints adjacent to a_3. Then $|A| > |C|$ and C might be empty if T is a dragon of diameter five. Assume for a contradiction that σ is a bipartite two packing of T.

Case 1. *The two-packing σ is an equal parity two-packing.*

Every even vertex of T is adjacent to at least one of the odd vertices a_1 and a_3. If the distance in T between $\sigma(a_1)$ and $\sigma(a_3)$ is two, then for some even vertex x of T, $\sigma(x)$ would be in T adjacent to both vertices $\sigma(a_1)$ and $\sigma(a_3)$ and in $\sigma(T)$ to at least one of the vertices $\sigma(a_1)$ and $\sigma(a_3)$. This is not possible because T and $\sigma(T)$ would then have an edge in common. Hence the distance between $\sigma(a_1)$ and $\sigma(a_3)$ is larger than two and because this distance must be an even number it is at least four. Also both vertices $\sigma(a_1)$ and $\sigma(a_3)$ are odd vertices of T. Hence one of the vertices $\sigma(a_1)$ or $\sigma(a_3)$ must be equal to a_1. If $\sigma(a_1) = a_1$ then the $|A| + 1$ even vertices adjacent to a_1 must be mapped by σ to even vertices not adjacent to a_1. But there are only $|C| + 1 < |A| + 1$ even vertices not adjacent to a_1. If $\sigma(a_3) = a_1$ then the $|C| + 2$ even vertices adjacent to a_3 must be mapped by σ to even vertices not adjacent to a_1. But there are only $|C| + 1 < |C| + 2$ even vertices not adjacent to a_1.

Case 2. *The two packing σ is an unequal parity two-packing.*

The vertex a_4 is adjacent to all odd vertices except a_1. Hence $\sigma(a_4)$ must be an odd endpoint of T, that is $\sigma(a_4)$ is some endpoint adjacent to a_4 and $\sigma(a_1) = a_4$. But this implies that $\sigma(a_1)$ can only be adjacent to a_1 in $\sigma(T)$. Hence a_1 must be an endpoint of T, a contradiction. □

Lemma 12. *If T is a tree of diameter five which is not a crested dragon but every equal parity reduction of T is a crested dragon or a tree of diameter less than five then T has an equal parity two-packing.*

Proof. We assume first that T contains two equal parity reducing endpoints x and y such that $T - \{x, y\}$ is a crested dragon D. Let m be the node adjacent to x and n the node adjacent to y. Choose a path $P = a_0, a_1, a_2, a_3, a_4, a_5$ in the crested dragon D such that the vertices a_1, a_4 and possibly a_3 are nodes of D but no other vertices are nodes of D. Note that P is a derived tree of T. Denote by A the set of endpoints adjacent to a_1, by B the set of endpoints adjacent to a_4 and by C the set of endpoints adjacent to a_3. Then $|A| > |C|$ and C might be empty. Observe that the nodes m and n are vertices of D. We are going to discuss several cases depending on which vertices of D are the nodes m and n of T. Clearly $\{m, n\} \cap (A \cup B) = \emptyset$ otherwise the diameter of T would be larger than five.

We assume first that both nodes m and n are elements of C. Let then R be the subtree of T which is spanned by the vertices of P together with the vertices m, n, x and y. Clearly R is a derived tree of T. Figure 7 exhibits a perfect packing of R. It follows then from the First Preservation Lemma that T has an equal parity two-packing. (The two-packing σ maps a bold labelled vertex i to the gray labeled vertex i. The reason for using numbers and not letters in Fig. 7 is to make it obvious that the graph F_σ is bipartite. Indexed letters take too much space in the figure.)

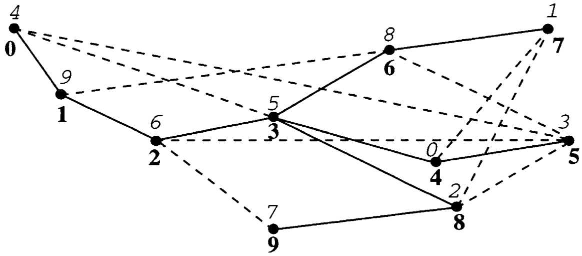

Fig. 7

If only one of the nodes m and n, say m, is an element of C, then the other node n must be equal to some vertex of even parity in D. Hence n is equal to a_2 or a_4. If $n = a_2$ let R then be the subtree of T which is spanned by the vertices of P together with the vertices m and y. Clearly R is a derived tree of T. Figure 8 exhibits a perfect packing of R. It follows from the First Preservation Lemma that T has an equal parity two-packing. If m is an element of C and $n = a_4$ we consider R to be the subtree of T which is spanned by the vertices of P and the vertices m and x. Note that a_2 is now not a node of T. Clearly R is a derived tree of T. Figure 9 exhibits a bipartite two-packing of R in which every vertex splits except the vertex a_2, which corresponds to 2 in the figure. It follows then from the First Preservation Lemma that T has an equal parity two-packing. We arrived now at the situation that neither m nor n is an element of C. Because m and n have the same parity it follows that either $\{m, n\} = \{a_1, a_3\}$ or that $\{m, n\} = \{a_2, a_4\}$ holds. If $\{m, n\} = \{a_1, a_3\}$ then T is a crested dragon, contrary to our assumptions on T. (Because $|A| > |C|$ clearly $|A| + 1 > |C| + 1$). If $\{m, n\} = \{a_2, a_4\}$ and C is not empty then the same graph R as used for Fig. 8 is a derived tree of T. If C is empty then T is a crested dragon, (label the path P beginning at a_5 with a_0), contrary to the assumption on T.

We can now assume that whenever x and y is an equal parity reducing pair of endpoints of T, then the diameter of the tree $K = T - \{x, y\}$ is smaller than five. Because T has diameter five it contains a path $P = a_0, a_1, a_2, a_3, a_4, a_5$ of length five such that the vertices a_0 and a_5 are endpoints of T and a_1 and a_4 are nodes of T. Since T is not a crested dragon, the tree T has at least one node different from a_1 and a_4. The tree T has at most two more nodes besides a_1 and a_4, otherwise two of those nodes would have the same parity and T could be further equal parity reduced to a tree which has diameter five. If T has two nodes besides a_1 and a_4 we will denote them by n and m and if T has

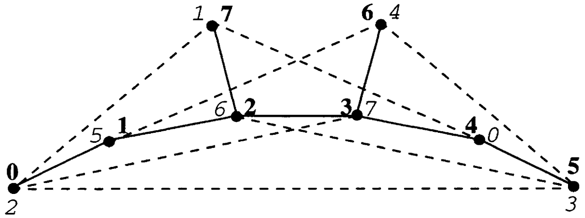

Fig. 8

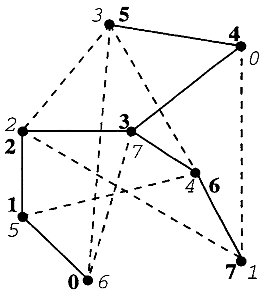

Fig. 9

only one such a node we will denote it by m. Note that the two nodes m and n have different parity otherwise T could be equal parity reduced to a tree which has diameter five. For a vertex a in T the distance of a from P is the length of the shortest path from a to any vertex of P. The vertex of P which has shortest distance from a is called the point of attachment of a. Because T has diameter five there is no vertex in T which has distance greater than or equal to three from P. This means that the nodes m and n have distance at most one from P.

If both nodes m and n have distance one from P the points of attachment of m and n are in the set $\{a_2, a_3\}$. Otherwise T would have diameter at least six. If the points of attachment of m and n are different we may assume without loss of generality that the point of attachment of m is a_2 and the point of attachment of n is a_3. If we remove all endpoints from T except the endpoints a_0 and a_5 we arrive at the subtree of T spanned by the vertices $a_0, a_1, a_2, a_3, a_4, a_5, m, n$. We see from Fig. 8 that this tree has a perfect packing. Hence by the First Preservation Lemma, T has an equal parity two-packing. If m and n have the same vertex as point of attachment then m and n would have the same parity.

Hence at most one node, say m, has distance one from P while the other node is a vertex in P. As above the point of attachment of m is one of the points a_2 or a_3 and we may assume without loss of generality that a_3 is the point of attachment of the node m. The node n can not be one of the

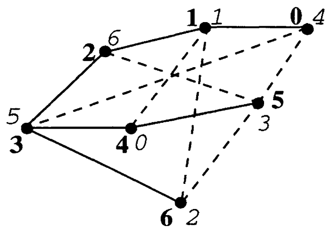

Fig. 10

endpoints of T adjacent to a_1 or a_4 because the diameter of T is five. By assumption n is not equal to a_1 or a_4. If n is equal to a_2 then m and n would have the same parity. We conclude that $n = a_3$ holds. The nodes $a_3 = n$ and a_1 of T have the same parity. If a_1 would be adjacent to more than the one endpoint a_0 then T could be further equal parity reduced while still containing a path of length five. Hence a_1 is only adjacent to the one endpoint a_0. If we remove all endpoints from T except the endpoints a_0 and a_5 we arrive at the subtree of T spanned by the vertices $a_0, a_1, a_2, a_3 = n, a_4, a_5, m$. We see from Fig. 10 that this tree has an equal parity two-packing in which every vertex, except the vertex a_1, splits. Hence by the First Preservation Lemma the tree T has an equal parity two-packing.

The next case to investigate is the one in which both of the nodes m and n of T are vertices of the path P. Because the diameter of T is five the nodes m and n can not be endpoints of P and by assumption m and n are different from a_1 and a_4. Also, m and n are different from each other. Hence without loss, $m = a_2$ and $n = a_3$. There is then at least one endpoint of T adjacent to a_2 and at least one endpoint of T adjacent to a_3. We can now use Fig. 8 and the First Preservation Lemma to conclude that T has a bipartite packing.

We assume from now on that there is only the one node m in T which is not equal to a_1 or a_4. If m has distance one from P we may assume without loss that the point of attachment of m is a_3. Because m is a node there exists at least one endpoint a_7 of T adjacent to m. If we remove all endpoints from T except the endpoints a_0, a_5 and a_7 we arrive at the subtree of T spanned by the vertices $a_0, a_1, a_2, a_3, a_4, a_5, a_7, m$. We see from Fig. 9 that this tree has an equal parity bipartite packing in which every vertex splits, except for the vertex a_2 which is not a node of T. Hence by the First Preservation Lemma, T has an equal parity two-packing.

The last case is when m is a vertex in P. As before we may assume without loss that $m = a_3$ holds. The node $m = a_3$ as adjacent to at least one endpoint, say a_6. The parity of $m = a_3$ is the same as the parity of the node a_1. Hence the node a_1 is adjacent to at most one endpoint of T, otherwise T could be further equal parity reduced while still containing a path of length five. If we remove all endpoints from T except the endpoints c_0, a_5 and a_6 we arrive at the subtree of T spanned by the vertices $a_0, a_1, a_2, a_3 =$

m, a_4, a_5, a_6. We see from Fig. 10 that this tree has an equal parity bipartite packing in which every vertex, except the vertex a_2, splits. Hence by the First Preservation Lemma, T has an equal parity two-packing. □

Lemma 13. *A tree T of diameter five has an equal parity two-packing if and only if T is not a crested dragon.*

Proof. If T is a crested dragon it follows from Lemma 11 that T does not have a bipartite two-packing. If T has an equal parity reducing pair a and b of endpoints such that $T - \{a, b\}$ is not a crested dragon and has diameter at least five then, using the Second Preservation Lemma, we have to consider the tree $T - \{a, b\}$ instead. Because T is finite we can assume that for every equal parity reducing pair a and b of endpoints of T, $T - \{a, b\}$ is either a crested dragon or has diameter smaller than five. By Lemma 12, the tree T has then an equal parity two-packing. □

Lemma 14. *Every tree T of diameter larger than or equal to five which is not a crested dragon has an equal parity two-packing.*

Proof. By the Second Preservation Lemma we may assume that any equal parity reduction of T either yields a tree of diameter less than five or a crested dragon. Because of Lemma 13 we may also assume that the diameter of T is larger than five. If T does not contain an equal parity reducing pair of endpoints then because the diameter of T is larger than five, T would have to be a dragon of diameter at least seven. Because every path of length at least seven has a perfect packing (Lemma 6), every dragon of length at least seven has an equal parity two-packing. We may therefore further assume that the tree T contains a pair of endpoints a, b of equal parity which are adjacent to two different nodes.

Case 3. *The tree T contains endpoints a, b of equal parity which are adjacent to two different nodes such that the tree $T - \{a, b\}$ is a crested dragon D.*

We assume that the path a_1, a_2, a_3, a_4 is a derived tree of the crested dragon D such that a_1, a_4 and possibly a_3 are the nodes of D. Let A be the set of endpoints of D adjacent to a_1, B the set of endpoints of D adjacent to a_4, C the set of endpoints of D adjacent to a_3 and assume that a_0 is an endpoint in A and that a_5 is an endpoint in B. Because the diameter of T is larger than five it follows that at least one of the endpoints a or b of the equal parity reducing pair of endpoints of T is adjacent to one of the endpoints in the set $A \cup B$.

If both vertices a and b are adjacent to vertices in the set $A \cup B$ then, because a and b have the same parity, they are either both adjacent to vertices in A or they are both adjacent to vertices in B. In either case the tree R, for which Fig. 10 exhibits an equal parity two-packing σ, is a derived tree of T. The two-packing σ splits all of the vertices of R except the vertex labelled 2. Using the First Preservation Lemma we conclude that T has an equal parity two-packing except in the case where both endpoints a and b are adjacent to vertices in B and the set C is not empty. In this case we use the tree for which Fig. 11 exhibits a perfect packing and which is then a derived tree of T. We are therefore now in the situation that exactly one of the endpoints a and b, say a, is in the set $A \cup B$. Assume first that $a \in B$. Because a and b have the same parity the endpoint b is not adjacent to a vertex in C. The tree T then contains the path $P = a_0, a_1, a_2, a_3, a_4, a_5, a_6 = a$ of length six and every endpoint of T is adjacent to some vertex in this path. If the center a_3 of P is not a node of T we are done by the First Preservation Lemma and the following Fig. 12 which exhibits a bipartite two-packing of P_6 in which every vertex except the center splits. If a_3 is a node of T we may assume that a_3 is adjacent to a_8. The vertices of the path P together with a_8 span a subtree R of T which is a derived tree of T. If the node a_5 of T is adjacent to more than one endpoint then T would have an equal parity reducing pair of endpoints x and y such that $T - \{x, y\}$ has diameter six. Hence we assume without loss of generality that a_5 is adjacent to only one endpoint. Figure 13 exhibits an equal parity two-packing of R in which all of the nodes which belong to endpoints of T which are not in R are split. Hence T has an equal parity two-packing.

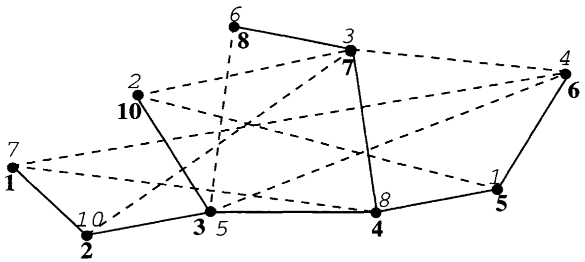

Fig. 11

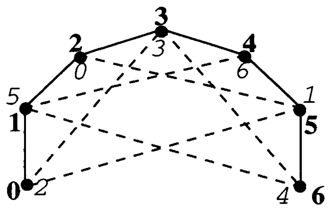

Fig. 12

Assume next that a is adjacent to some endpoint, say a_0 in A. The tree T contains then the path $P = a, a_0, a_1, a_2, a_3, a_4, a_5$ of length six. If b is adjacent to an element of C then a_2 is not a node of T. Hence we can use the tree exhibited in Fig. 10 together with the First Preservation Lemma to construct

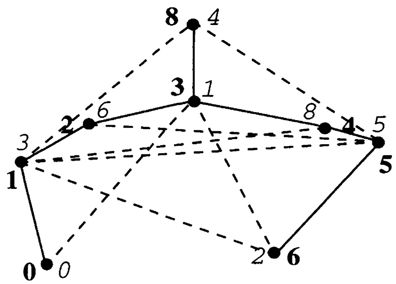

Fig. 13

an equal parity two-packing of T. If b is not adjacent to a vertex in C but to the center a_2 of P we use Fig. 14 and if b is neither adjacent to a_2 nor to a vertex in C we use Fig. 13 together with the First Preservation Lemma to construct an equal parity two-packing of T.

Case 4. *For any two endpoints a, b of equal parity which are adjacent to two different nodes, the tree $T - \{a, b\}$ has diameter four.*

Let P be the path $a = a_0, a_1, a_2, a_3, a_4, a_5, a_6 = b$ of length six which joins the endpoints a and b in T. Observe that T does not contain a node, say m, different from the nodes a_1 and a_5 which has parity 1. Otherwise some endpoint x adjacent to m would form an equal parity reducing pair with a_0 and $T - \{x, a_0\}$ would have diameter five. If T would contain two nodes, say m and n, besides the nodes a_1 and a_5 they would both have to have even parity. But then T could be equal parity reduced by some two endpoints adjacent to m and n respectively and still have diameter six. If T contains no other node besides the nodes a_1 and a_5 then T is a dragon of diameter six which has an equal parity two-packing by Fig. 12 and the First Preservation Lemma.

It follows that the tree T has exactly three nodes, namely a_1, a_5 and a third one, say m which has even parity. The distance from m to either one of a_1 and a_5 is smaller than four. Otherwise we could reduce T by the pair a_0, a_6 of endpoints and obtain a tree of diameter at least five. Hence m is either a vertex in P or has distance one from a_3. If m is a vertex in P different from the center a_3 we are done by Fig. 12 and if m is adjacent to a_3 or equal to a_3 we are done by Fig. 13 and the First Preservation Lemma. □

Theorem 4. *A forest has a bipartite packing if and only if it is either a singleton vertex or disconnected or a tree of diameter at least five which is not a crested dragon. Every forest which has a bipartite packing has an equal parity packing.*

Proof. Follows immediately from Lemmas 5, 10, 13, and 14. □

4. Two-Packings Which Yield a Graph of Chromatic Number Three

Lemma 15. *Every tree with diameter at least four has a three chromatic two-packing.*

Proof. Using the Second Preservation Lemma we may assume that if a, b is any reducing pair of endpoints of T then $T - \{a, b\}$ does not have diameter at least four. This implies that T has at most three different nodes.

Case 5. *The tree T has exactly three nodes.*

Assume that the nodes are the vertices a_1, b_1 and c_1 and that a is an endpoint adjacent to a_1, b is an endpoint adjacent to b_1 and that c is an endpoint adjacent to c_1. Assume also that the distance from a to b is the diameter of T. The diameter of T can not be larger than or equal to five because then the tree $T - \{b, c\}$ would still have diameter at least four. Hence there is a path $P = a, a_1, d, b_1, b$ of length four from a to b. Let the point of attachment of c_1 be the vertex in the path P which has shortest distance to c_1. The point of attachment of c_1 can not be a or b because a and b are endpoints of T. If the point of attachment of c_1 is a_1 then c_1 is not a vertex in P, otherwise $c_1 = a_1$ contrary to the assumption that the three nodes a_1, b_1 and c_1 are pairwise different. But c_1 has then at least distance one from a_1 and the distance from c to b would be at least five. Similarly the point of attachment of c_1 is not equal to b_1. Hence the point of attachment of c_1 must be the central point d of the path P. The distance from c_1 to d is at most one otherwise the distance from c to a would be at least five. Hence the distance from c_1 to d is either one or zero. In either case the tree R which consists of the path P of length four together with one endpoint x adjacent to the central point d will be a derived tree of the tree T. Figure 14 shows a two-packing of the tree R of chromatic number three in which every vertex splits. Using the First Preservation Lemma we conclude that T has a three chromatic two-packing.

Case 6. *The tree T has exactly two nodes.*

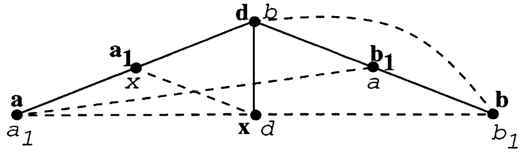

Fig. 14

This means that T is a dragon. Assume that the nodes are the vertices a_1 and b_1 and that a is an endpoint adjacent to a_1 and b is an endpoint adjacent to b_1. Observe that either a_1 or b_1 is adjacent to exactly one endpoint. Otherwise T can be further reduced to a tree which has diameter at least four. For the same reason the dragon T has diameter four or five. In one case a path of length four and in the other case a path of length five is a derived tree of T. Because at least one of the two nodes is adjacent to only one endpoint it is sufficient to find a two-packing of a path of length four and a two-packing of a path of length five in which at least one of the nodes splits and which has chromatic number three. Such two-packings are exhibited in Fig. 15. □

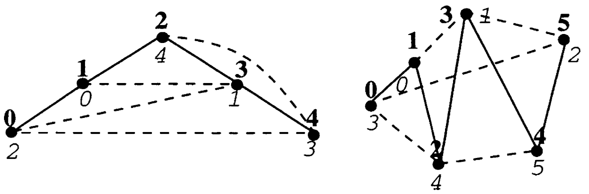

Fig. 15

Lemma 16. *A forest F which is not connected has a three chromatic two-packing if and only if F contains a path of length two.*

Proof. If F does not contain a path of length two then the degree of every vertex of F is at most one and F consists of a partial matching (that is a set of pairwise not adjacent edges), together with some isolated vertices. Hence for any two-packing σ of F the maximal degree of F_σ is at most two and the edges of F_σ can be factorized into two matchings. This means that every connected component of F_σ is either a path or a circuit. Since an odd circuit can not be factorized into two matchings each of the circuits has even length which implies that the chromatic number of F_σ is at most two.

Let us now assume that F contains a path of length two. Using the Second Preservation Lemma we may assume that F can not be reduced without yielding a forest which has no path of length two. That implies that F has at most two connected components which contain a path of length one or two and all other components of F are isolated points. Otherwise F could be further reduced. Observe that every isolated point has a bipartite two-packing. This means that we can assume that F has at most two connected components and that if F contains an isolated point then F has exactly two components, one the isolated point and the other a tree of diameter two or three. There are three more cases to be discussed. In each case, a two-packing σ of F is exhibited such that every vertex of F is split and F_σ has chromatic number three.

Case 7. *The forest F has two connected components A and B which contain a path of length two.*

Since the forest F can not be further reduced, each of the two connected components A and B is a path of length two. Figure 16 exhibits a two-packing σ such that F_σ has chromatic number three.

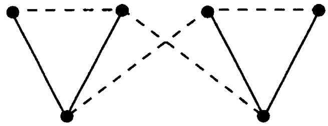

Fig. 16

Case 8. *The forest F has only one component A which contains a path of length two and an isolated point e.*

The forest F consists then of two components, A and the isolated point e. The diameter of A is at most three, otherwise F could be further reduced. We conclude that A is either a star or a dragon of diameter three. If A is a dragon of diameter three we can assume that at most one of the two nodes has degree larger than or equal to three, otherwise F could be further reduced. Figure 17 exhibits for both cases a two-packing σ such that F_σ has chromatic number three.

Case 9. *The forest F has only one component A which contains a path of length two and no isolated point.*

Then the other component of F contains exactly one edge. Since F can not be reduced further, F must consist of a path of length two and a second component of diameter one. Figure 18 exhibits a two-packing σ such that F_σ has chromatic number three. □

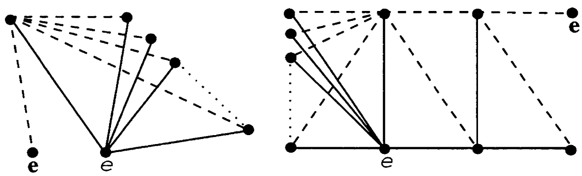

Fig. 17

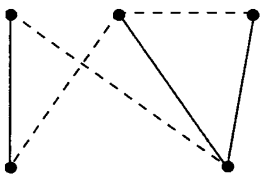

Fig. 18

Theorem 5. *A forest F has a two-packing σ such that F_σ has chromatic number three if and only if F is a tree with diameter at least four or F is disconnected and contains a path of length two.*

Proof. If F is disconnected then the theorem follows from Lemma 16. If F is connected and contains a path of length four the theorem follows from Lemma 15. If F is connected and has diameter zero, that is F is an isolated point, then for every two-packing σ of F, the graph F_σ has chromatic number one. If F is connected and has diameter one or diameter two, then F does not have a two-packing, by Lemma 3. If F is connected and has diameter three then it follows from Lemma 4 that for every two packing σ of F, the graph F_σ has chromatic number four. □

5. Two-Packings Which Yield a Graph of Chromatic Number Four

For $n \geq 3$, the wheel W_n is a graph which consists of a cycle with n vertices and an additional vertex which is adjacent to all of the vertices in the cycle. The complete graph K_4 for example is the wheel W_3.

Lemma 17. *A graph G with at most six vertices and chromatic number four contains the complete graph K_4 or the wheel W_5 as a subgraph.*

Proof. We only need consider the cases that G has five vertices and that G has six vertices. If G has six vertices and does not contain a triangle, then because G has chromatic number four it must contain a circuit C of length five. Because G does not contain a triangle C has no chords. If the sixth vertex is not adjacent to all of the vertices of C, G would have chromatic number three and if the sixth vertex is adjacent to all vertices of C then G is the wheel W_5. If G contains a triangle with vertices a, b, c then color the vertex a with color 0 the vertex b with color 1 and the vertex c with color 2. If G does not contain the complete graph K_4, then each of the other three vertices x, y, z is adjacent to at most two of the vertices in the triangle spanned by a, b, c. If two of the vertices x, y and z are adjacent then they are not adjacent to the same two vertices of the triangle spanned by a, b, c. We color the vertices x, y and z with the colors 0, 1 and 2 in such a way that none of the vertices x, y, z is adjacent to a vertex in the set $\{a, b, c\}$ of vertices which has the same color. This can be done because each of the vertices x, y, z is adjacent to at most two of the vertices in the triangle spanned by a, b, c. We also assume that the number of adjacent pairs of vertices which received the same color is as small as possible under the coloring condition given above. Observe that if all three colors are used to color the vertices x, y and z then G would have a good three coloring. Hence we can assume that at least two of the three vertices x, y and z have the same color.

Assume first that all three vertices x, y and z have the same color, say 2. If no two of them are adjacent then we arrived at a three coloring of the graph G. Hence assume that the vertices x and y are adjacent. At least one of the two vertices x and y, say y is not adjacent to one of the two vertices, say b. We can then color y with the color 1 instead of with the color 2 and obtain a coloring in which there are fewer adjacent pairs of vertices which have the same color.

Assume next that the vertices x and y have the same color, say 2, and that the vertex z has a different color, say 1. The vertices x and y are adjacent, otherwise the given coloring would be a good three coloring of the graph G. If one of the two vertices x and y, say y, is not adjacent to the vertex a, then y could be colored with the color 0 to obtain a good three coloring of G. Hence both vertices x and y are adjacent to the vertex a. This means that one of the two vertices x and y, say y is not adjacent to the vertex b. If the vertices z and y would not be adjacent then coloring y with the color 1 instead of 2 would give a good three coloring of G. Hence y is adjacent to z. If the vertex z would not be adjacent to the vertex a we would obtain a good three coloring of G by changing the color or z from 1 to 0 and the color of y from 2 to 1. Hence z is adjacent to a. The vertices z and x are not adjacent otherwise the vertices x, y, z and a would form a K_4. This implies that x is adjacent to b otherwise x could be colored with color 1 to give a good three coloring of G. Also the vertices z and c are adjacent, otherwise we could change the color of z from 1 to 2 and the color of y from 2 to 1 to obtain a good three coloring of G. This now finally implies that G contains the wheel W_5 because x, y, z, c, b, x form a cycle of length five and the vertex a is adjacent to the other five vertices.

If G has five vertices let G_1 be the graph obtained from G by adding an additional isolated point. We can then apply the result for six vertices to G_1 and obtain the lemma in the case of five vertices as well. □

Lemma 18. *If A and B is a factorization of the complete graph K_4 into two forests, then both of the factors A and B are isomorphic to the path P_3 of length three.*

Proof. If one of the factors A or B contains less than three edges the other factor must contain at least four edges and hence a circuit. This means that both of the factors A and B contain exactly three edges. Hence each of the factors A and B is either the path P_3 or the star S_3. The complement of S_3 in K_4 is a triangle. Hence both factors A and B are isomorphic to the path P_3.□

Lemma 19. *A forest F with exactly two connected components P and S, where P is a path of length three and S is a star S_n with $n \geq 1$, does not have a two-packing σ such that F_σ has chromatic number four.*

Proof. Assume for a contradiction that F_σ has chromatic number four. Let G be the graph obtained from F_σ by removing all vertices from F_σ which have degree at most two. Clearly, F_σ has chromatic number four if and only if G has chromatic number four. Because F contains at most three vertices which have degree larger than one, the graph G contains at most six vertices. Hence by Lemma 17, G and therefore F_σ contains the complete graph K_4 or the wheel W_5. According to Lemma 18, if A, B is a two-factorization of K_4 into two forests then both factors A and B are isomorphic to P_3. Because F contains only one path of length three, the two factors A and B are a two-packing of P_3. But this is not possible because the star S does not have a two-packing.

Let us now assume that G contains the wheel W_5. The wheel W_5 contains six vertices each of which has degree at least three. This means that the vertices of this wheel consist of the three vertices of F which are not endpoints of F together with their σ images. Notice that if y is a vertex in W_5 which is not an endpoint of F then both $\sigma^{-1}(y)$ and $\sigma(y)$ is an endpoint of F. Let a be the center of the star S. The vertex x of W_5 which is adjacent to all other vertices of W_5 has degree five. We may assume without loss of generality that three of the edges adjacent in G to x are edges of F and hence that $x = a$. The two central points c, d of P are then in W_5 adjacent to $x = a$. The two edges of W_5 from c, d to $x = a$ must be edges of $\sigma(F)$. But then $\sigma^{-1}(a)$ could not be an endpoint of F. □

Lemma 20. *A forest F which does not contain a path of length three does not have a two-packing σ such that F_σ has chromatic number four.*

Proof. Let us direct the edges of F in such a way that every directed edge points towards an endpoint of F. Then we direct the edges of $\sigma(F)$ such that σ preserves the directions. This makes F_σ into a directed graph. Observe that a vertex in F_σ has degree larger than or equal to three if and only if its indegree is at most one. If F_σ has chromatic number four then the graphs which we obtain from F_σ by successively removing vertices of degree at most two will also have chromatic number four. Hence this process of removing vertices of degree at most two ends in a graph G of chromatic number four in which every vertex has degree at least three. This is a contradiction because in G every vertex has indegree at most one and hence outdegree at least two. □

Lemma 21. *Every tree T of diameter at least three has a two packing σ such that F_σ has chromatic number four.*

Proof. By the Second Preservation Lemma we may assume that T is reduced, that means that whenever a, b is a reducing pair of endpoints then $T - \{a, b\}$ has diameter smaller than three. This implies that T has diameter either three or four and if T has diameter four then T has exactly two endpoints, that is T is a path of length four, and if T has diameter three then T is a dragon of diameter three because every tree of diameter three is a dragon. Observe that in both cases the path of length three is a derived tree. Any factorization of the complete graph K_4 into two paths of length three results in a two packing σ of the path of length three, in which every vertex splits. □

Lemma 22. *If F is a forest which contains at least one path of length three and if F does not have exactly two components one of which is a star and the other a path of length three then F has a two-packing a such that F_σ has chromatic number four.*

Proof. By the Second Preservation Lemma we may assume that F is reduced, that is if a and b is a reducing pair of vertices then $F-\{a,b\}$ is a forest which either does not contain a path of length three or $F-\{a,b\}$ has exactly two components one of which is a star and the other a path of length three. If F has more than two components $A, B_1, B_2, \ldots, B_n$ with $n \geq 2$, where A contains a path of length three then let σ be a two-packing of A such that F_σ has chromatic number four, (Lemma 21), and λ a two-packing of the forest which consists of the other components. Clearly $\sigma \cup \lambda$ has then the required properties. We may therefore assume that F has exactly two components A and B where A has diameter at least three. If B is not a star then again the union of a four chromatic two-packing of A and a two-packing of B will be a four chromatic two-packing of F. We assume therefore that B is a star. If A has diameter larger than or equal to five then F could be reduced further by a pair of endpoints, one from B and the other from A. If A has diameter four then A has exactly two endpoints or else we could reduce F by a pair of endpoints one from A and the other from B. We arrived then at the case where A is a path of length four and B a star with at least two vertices. Figure 19 exhibits a two packing σ for this case such that F_σ has chromatic number four. If A has diameter three then A is a dragon of diameter three. If A would have more than three endpoints then F could be further reduced. Hence A has exactly three endpoints, otherwise A would be a path of length three. Figure 20 exhibits a two packing λ such that F_λ has chromatic number four. □

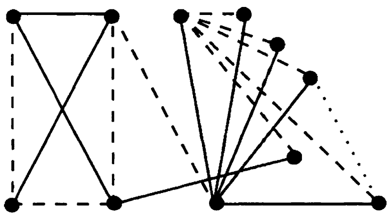

Fig. 19

Theorem 6. *A forest F has a two-packing a such that F_σ has chromatic number four if and only if F contains a path of length three and F does not consist of exactly two connected components where one of the two components is a path of length three and the other is a star.*

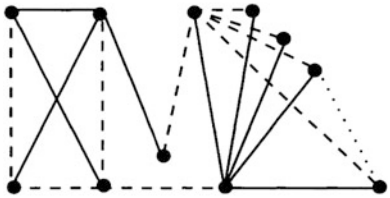

Fig. 20

Proof. Theorem 6 follows immediately from Lemmas 19, 20, 21, and 22. □

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II. Ramsey and Extremal Theory

Introduction

It is perhaps evident from several places of this volume that Ramsey theorem played a decisive role in Erdős' combinatorial activity. And perhaps no other part of combinatorial mathematics is so dear to him as Ramsey theory and extremal problems. He was not creating or even aiming for a theory. However, a complex web of results and conjectures did, in fact, give rise to several theories. They all started with modest short papers by Erdős, Szekeres and Turán in the thirties. How striking it is to compare these initial papers with the richness of the later development, described, e.g., by the survey articles of Miki Simonovits (extremal graph theory) and Jeff Kahn (extremal set theory). In addition, the editors of these volumes tried to cover in greater detail the development of Ramsey theory mirrored and motivated by Erdős' papers. In a way (and this certainly is one of the leitmotivs of Erdős' work), there is little difference between, say, density Ramsey type results and extremal problems.

One can only speculate on the origins of density questions. It is clear that in the late 1930s and 1940s, the time was ripe in ideas which later developed into extremal and density questions: we have not only the Erdős-Turán 1941 paper but also, Erdős and Tomsk 1938 paper on number theory which anticipated extremal theory by determining $n^{3/2}$ as upper bound for C_4-free graphs, the Sperner paper and also the Erdős–Ko–Rado work (which took several decades to get into print). All these ideas, together with Turán's extremal results provided a fruitful cross-interaction of ideas from various fields which, some 30 years later, developed into density Ramsey theorems and extremal theory. We are happy to include in this chapter papers by Gyula Katona (which gives a rare account of Erdős method of encouraging and educating young talented students). The article by Vojtěch Rödl and Robin Thomas related to another aspect of Erdős' work was the start of many further refinements of arrangeability for special classes of sparse graphs (see J. Nešetřil, P. Ossona de Mendez: Sparsity, Springer 2012 for a recent discussion of the problem). We also include an important paper by Alexander Kostochka which gives a major breakthrough to the Erdős–Rado Δ-system problem (for triples; meanwhile Kostochka succeeded in generalizing the result to k-tuples).

Finally, we have the paper of Saharon Shelah, solving a model theoretic Ramsey question of Väänänen, illustrating (once again) that Ramsey theory is alive and well.

In 1995/1996, when the content of these volumes was already crystallising, we asked Paul Erdös to isolate a few problems, both recent and old, for each of the eight main parts of this book. To this part on Ramsey theory he contributed the following collection of problems and comments.

Erdős in his own words

Hajnal, Rado and I proved

$$2^{ck^2} < r_3(k,k) < 2^{2^k},$$

we believe that the upper bound is correct or at least is closer to the truth, but Hajnal and I have a curious result: If one colors the triples of a set of n elements by two colors, there always is a set of size $(\log n)^{1/2}$ where the distribution is not just, i.e., one of the colors has more than $0.6\binom{t}{3}$ of the triples for $t = (\log n)^{1/2}$. Nevertheless we believe that the upper bound is correct. It would perhaps change our minds if we could replace 0.6 by $1-\epsilon$ for some t, $t > (\log n)^{\epsilon}$. We never had any method of doing this.

Let H be a fixed graph and let n be large. Hajnal and I conjectured that if $G(n)$ does not contain an induced copy of H then $G(n)$ must contain a trivial subgraph of size $> n^{\epsilon}$, $\epsilon = \epsilon(H)$. We proved this for many special cases but many problems remain. We could only prove $\exp(c\sqrt{\log n})$.

Extremal Graph Theory

Denote by $T(n;G)$ the smallest integer for which every graph of n vertices and $T(n;G)$ edges contains G as a subgraph. Turán determined $T(n;G)$ if G is $K(r)$ for all r, $T(n,C_4) = \left(\frac{1}{2}+o(1)\right)n^{3/2}$ was proved by V. T. Sós, Rényi and myself. The exact formula for $T(n,C_4)$ is known if $n = p^2+p+1$ (Füredi).

I have many asymptotic and exact results for $T(n,C_4)$ and many results and conjectures with Simonovits. I have to refer to the excellent book of Bollobás and the excellent survey of Simonovits and a very good recent survey article of Füredi. Here I only state two results of Simonovits and myself: $T(n,G) < cn^{8/5}$ where G is the edge graph of the three-dimensional cube. Also we have fairly exact results for $T(n,K(2,2,2))$.

Ramsey–Turán Theorems

The first papers are joint with V. T. Sós and then there are comprehensive papers with Hajnal, Simonovits, V. T. Sós and Szemerédi. Here I only state a result with Bollobás, Szemerédi and myself: For every $\epsilon > 0$ and $n > n_0(\epsilon)$

there is a graph of n vertices $\frac{n^2}{8}(1-\epsilon)$ edges with no $K(4)$ and having largest independent set $o(n)$, but such a graph does not exist if the number of edges is $\frac{n^2}{8}(1+\epsilon)$. We have no idea what happens if the number of edges is $\frac{n^2}{8}$.

One last Ramsey type problem: Let n_k be the smallest integer (if it exists) for which if we color the proper divisors of n_k by k colors then n_k will be a monochromatic sum of distinct divisors, namely a sum of distinct divisors in a color class. I am sure that n_k exists for every k but I think it is not even known if n_2 exists. It would be of some interest to determine at least n_2. An old problem of R. L. Graham and myself states: Is it true that if m_k is sufficiently large and we color the integers $2 \leq t \leq m_k$ by k colors, then

$$1 = \sum \frac{1}{t_i}$$

is always solvable monochromatically? I would like to see a proof that m_2 exists. (Clearly $m_k \geq n_k$.) Perhaps this is really a Turán type problem and not a Ramsey problem. In other words, if m is sufficiently large and $1 < a_1 < a_2 < \cdots < a_\ell \leq m$ is a sequence of integers for which $\sum_{\ell 1} /a_\ell > \delta \log m$, then

$$1 = \sum \frac{\epsilon_i}{a_i} \quad (\epsilon_i = 0 \text{ or } 1)$$

is always solvable. I offer 100 dollars for a proof or disproof. Perhaps it suffices to assume that

$$\sum_{a_i < m} \frac{1}{a_i} > C(\log \log m)^2$$

for some large enough C. For further problems of this kind as well as for related results see my book with R. L. Graham. I hope before the year 2000 a second edition will appear.

*****

So much for Paul Erdős. Of course progress since 1995/1996 has been very rapid, particularly so in extremal and Ramsey theory. Most of the papers were updated for this second edition. Note that the main result of the paper by Shelah in this chapter was improved very recently by D. Conlon, J. Fox and B. Sudakov (Two extensions of Ramsey theorem, to appear in Duke Math. J.): the order of the modified Ramsey function is single exponential.

We are fortunate that Sasha Razborov contributed to our volume with a new survey on an exciting new tool for extremal theory problems—the flag algebra calculus. (See also the recent book L. Lovász: Graph Limits, AMS 2012.)

At this place let us comment on the last part of Erdős' text: The problem of Erdős and Graham has been solved by Ernie Croot (then only a student) in

E. Croot, On a coloring conjecture about unit fractions, Ann. Math. 157 (2003), 545–556.

On the other hand the second edition of
P. Erdős, R. L. Graham, *Old and new problems and results in combinatorial number theory*, Monographie L'Enseignements Math. 28,(1980) definitively did not appear before 2000 (in fact it has not even appeared up to now (2013)). Well, even Paul Erdős was sometimes wrong!

Ramsey Theory in the Work of Paul Erdős

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Summary. Ramsey's theorem was not discovered by P. Erdős. But perhaps one could say that Ramsey theory was created largely by him. This paper will attempt to demonstrate this claim.

1. Introduction

Ramsey's theorem was not discovered by Paul Erdős. This was barely technically possible: Ramsey proved his theorem in 1928 (or 1930, depending on the quoted source) and this is well before the earliest Erdős publication in 1932. He was then 19. At such an early age 4 years makes a big difference. And also at this time Erdős was not even predominantly active in combinatorics. The absolute majority of the earliest publications of Erdős is devoted to number theory, as can be seen from the following table:

	1932	1933	1934	1935	1936	1937	1938	1939
All papers	2	0	5	10	11	10	13	13
Number theory	2	0	5	9	10	10	12	13

The three combinatorial exceptions among his first 8^2 papers published in 8 years are 2 papers on infinite Eulerian graphs and the paper [46] by Erdős and G. Szekeres. Thus, the very young P. Erdős could not have been a driving force for the development of Ramsey theory or Ramsey-type theorems in the 30s. That position should be perhaps reserved for Issac Schur who not only proved his sum theorem [114] in 1916 but, as it appears now [115], also conjectured van der Waerden's theorem [124], proved an important extension, and thus put it into a context which inspired his student R. Rado to completely settle (in 1933) the question of monochromatic solutions of linear equations [102]. This result stands apart even after 60 years.

Yet, in retrospect, it is fair to say that P. Erdős was responsible for the continuously growing popularity of the field. Ever since his pioneering work in the 30s he proved, conjectured and asked seminal questions which together,

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DOI 10.1007/978-1-4614-7254-4_13,

some 40–50 years later, formed Ramsey theory. And for Erdős, Ramsey theory was a constant source of problems which motivated some of the key pieces of his combinatorial research.

It is the purpose of this article to partially justify these claims, using a few examples of Erdős' activity in Ramsey theory which we will discuss from a contemporary point of view.

In the first section we cover paper [46] and later development in great detail. In Sect. 2, we consider the development based on Erdős' work related to bounds on various Ramsey functions. Finally, in Sect. 3 we consider his work related to structural extensions of Ramsey's theorem.

No mention will be made of his work on infinite extensions of Ramsey's theorem. This is covered in this volume by the comprehensive paper of A. Hajnal.

2. The Erdős-Szekeres Theorem

F. P. Ramsey discovered his theorem [104] in a sound mathematical context (of the decision problem for a class of first-order formulas; at the time, the undecidability of the problem was not known). But since the time of Dirichlet the "Schubfach principle" and its extensions and variations played a distinguished role in mathematics. The same holds for the other early contributions of Hilbert [67], Schur [114] and van der Waerden [124].

Perhaps because of this context Ramsey's theorem was never regarded as a puzzle and/or a combinatorial curiosity only. Thanks to Erdős and Szekeres [46] the theorem found an early application in a quite different context, namely, plane geometry:

Theorem 1 (**[46]**). *Let n be a positive integer. Then there exists a least integer $N(n)$ with the following property: If X is a set of $N(n)$ points in the plane in general position (i.e. no three of which are collinear) then X contains an n-tuple which forms the vertices of a convex n-gon.*

One should note that (like in Ramsey's original application in logic) this statement does not involve any coloring (or partition) and thus, by itself, fails to be of "Ramsey type". Rather it fits to a more philosophical description of Ramsey type statements as formulated by Mirsky:

> There are numerous theorems in mathematics which assert, crudely speaking, that every system of a certain class possesses a large subsystem with a higher degree of organization than the original system.

It is perhaps noteworthy to list the main features of the paper. What a wealth of ideas it contains! We can list at least 6 main aspects of this paper (numbered **I**–**VI**):

I. It is proved that $N(4) = 5$ and this is attributed to Mrs. E. Klein. This is tied to the social and intellectual climate in Budapest in the 30s which has been described both by Paul Erdős and Szekeres on several occasions (see e.g. [30]), and with names like the Happy End Theorem.

II. The following two questions related to statement of Theorem 1 are explicitly formulated:

(a) Does the number $N(n)$ exist for every n?
(b) If so, estimate the value of $N(n)$.

It is clear that the estimates were considered by Erdős from the very beginning. This is evident at several places in the article.

III. The first proof proves the existence of $N(n)$ by applying Ramsey's theorem for partitions of quadruples. It is proved that $N(n) \leq r(2, 4, \{5, n\})$. This is still a textbook argument. Another proof based on Ramsey's theorem for partitions of triples was found by A. Tarsi (see [63]). So far no proof has emerged which is based on the graph Ramsey theorem only.

IV. The authors give "a new proof of Ramsey's theorem which differs entirely from the previous ones and gives for $m_i(k, \ell)$ slightly smaller limits". Here $m_i(k, \ell)$ denotes the minimal value of $|X|$ such that every partition of i-element subsets of X into two classes, say α and β, each k-element contains an i-element subset of class α or each i-element subset contains an i-element subset of class β.

Thus, $m_i(k, \ell)$ is the Ramsey number for 2-partitions of i-element subsets. These numbers are denoted today by $r(2, i, \{k, l\})$ or $r_i(k, l)$. The proof is close to the standard textbook proofs of Ramsey's theorem. Several times P. Erdős attributed it to G. Szekeres.

Erdős and Szekeres explicitly state that $r_2(k + 1, \ell + 1) = m_2(k + 1, \ell + 1) \leq \binom{k+\ell}{2}$ and this value remained for 50 years essentially the best available upper bound for graph Ramsey numbers until the recent improvements by Rödl, Thomason [122] and finally by Conlon [17]. The current best upper bound (for $k = \ell$) is

$$\binom{2k}{k} k^{-C\frac{\log k}{\log\log k}}.$$

V. It is not as well known that [46] contains yet another proof of the graph theoretic formulation of Ramsey's theorem (in the above notation, $i = 2$) which is stated for its particular simplicity. We reproduce its formulation here.

Theorem 2. *In an arbitrary graph let the maximum number of independent points be k; if the number of points is $N \geqq m(k, \ell)$ then there exists in our graph a complete graph of order ℓ.*

Proof. For $\ell = 1, 2$, the theorem is trivial for any k, since the maximum number of independent points is k and if the number of points is $(k+1)$, there must be an edge (complete graph of order 2).

Now suppose the theorem proved for $(\ell - 1)$ with any k. Then at least $\frac{N-k}{k}$ edges start from one of the independent points. Hence if

$$\frac{N-k}{k} \geqq m(k, \ell - 1),$$

i.e.,

$$N \geqq k \cdot m(k, \ell - 1) + k,$$

then, out of the end points of these edges we may select, in virtue of our induction hypothesis, a complete graph whose order is at least $(\ell - 1)$. As the points of this graph are connected with the same point, they form together a complete graph of order ℓ. □

This indicates that Erdős and Szekeres were well aware of the novelty of the approach to Ramsey's theorem. Also this is the formulation of Ramsey's problem which motivated some of the key pieces of Erdős' research. First an early use of the averaging argument and then the formulation of Ramsey's theorem in a "high off-diagonal" form: If a graph G has a bounded clique number (for example, if it is triangle-free) then its independence number has to be large. The study of this phenomenon led Erdős to key papers [25, 27, 28] which will be discussed in the next section in greater detail.

VI. The paper [46] contains a second proof of Theorem 1. This is a more geometrical proof which yields a better bound

$$N(n) \leq \binom{2n-4}{n-2} + 1$$

and it is conjectured (based on the exact values of $N(n)$ for $n = 3, 4, 5$) that $N(n) = 2^{n-2} + 1$. This is still an unsolved problem. The second proof (which 50 years later very nicely fits to a computational geometry context) is based on yet another Ramsey-type result.

Theorem 3 (**Ordered pigeon hole principle; Monotonicity lemma).** *Let m, n be positive integers. Then every set of $(m-1)(n-1)+1$ distinct integers contains either a monotone increasing n-set or monotone decreasing m-set.*

The authors of [46] note that the same problem was considered by R. Rado. The stage has been set.

The ordered pigeon-hole principle has been generalized in many different directions (see e.g., [14, 90] and more recently [10]).

All this is contained in this truly seminal paper. Viewed from a contemporary perspective, the Erdős-Szekeres paper did not solve any well-known problem at the time and did not contribute to Erdős' instant mathematical fame (as a number theorist). But the importance of the paper [46] for the later development of combinatorial mathematics cannot be overestimated. To illustrate this development is one of the aims of this paper.

Apart from the problem of a good estimation of the value of N there is a peculiar structural problem related to [46]:

Call a set $Y \subseteq X$ an n-hole in X if Y is the set of vertices of a convex n-gon which does not contain other points in X. Does there always exist $N^*(n)$ such that if X is any set of at least $N^*(n)$ points in the plane in general position then X contains an n-hole?

It is easy to prove that $N^*(n)$ exists for $n \leq 5$ (see Harborth (1978) where these numbers are determined). Horton (1983) showed that $N^*(7)$ does not exist. The fact that $N^*(6)$ exists was established only very recently. The best bounds currently available are $30 \leq N^*(6) \leq 463$ (see [57, 96, 73, 97]).

3. Estimating Ramsey Numbers

Today it seems that the first question in this area which one might be tempted to consider is the problem of determining the actual sizes of the sets which are guaranteed by Ramsey's theorem (and other Ramsey-type theorems). But one should try to resist this temptation since it is "well-known" that Ramsey numbers (of all sorts) are difficult to determine and even good asymptotic estimates are difficult to find.

It seems that these difficulties were known to both Erdős and Ramsey. But Erdős considered them very challenging and addressed this question in several of his key articles. In many cases his estimations obtained decades ago are still the best available. Not only that, his innovative techniques became standard and whole theories evolved from his key papers.

Here is a side comment which may partly explain this success: Erdős was certainly one of the first number theorists who took an interest in combinatorics in the contemporary sense (being preceded by isolated events, for example, by V. Jarník's work on the minimum spanning tree problem and the Steiner problem see [69] and e.g. [66] and more recent [89] for the history of the problem. Incidentally, Jarník was one of the first coauthors of Erdős.) Together with Turán, Erdős brought to the "slums of topology" not only his brilliance but also his expertise and "good taste". It is our opinion that these facts profoundly influenced further development of the whole field. Thus it is not perhaps surprising that if one would isolate a single feature of Erdős' contribution to Ramsey theory then it is perhaps his continuing emphasis on estimates of various Ramsey-related questions. From the large number of his results and papers we decided to cover several key articles and comment on them from a contemporary point of view.

I. The 1947 paper [25]. In a classically clear way, Erdős proved

$$2^{k/2} \leq r(k) < 4^k \tag{1}$$

for every $k \geq 3$.

His proof became one of the standard textbook examples of the power of the probabilistic method. (Another example perhaps being the strikingly simple proof of Shannon of the existence of exponentially complex Boolean functions.)

The paper [25] proceeds by stating (1) in an inverse form: Define $A(n)$ as the greatest integer such that given any graph G of n vertices, either it or its complementary graph contains a complete subgraph of order $A(n)$. Then for $A(n) \geq 3$,

$$\frac{\log n}{2\log 2} < A(n) < \frac{2\log n}{\log 2}.$$

Despite considerable efforts over many years, these bounds have been improved only slightly (see [121, 117]). We commented on the upper bound improvements above. The best current lower bound is

$$r(n) \geq (1 + O(1))\frac{\sqrt{2}n}{e}2^{n/2}$$

which is twice the Erdős bound (when computed from his proof).

The paper [25] was one of 23 papers which Erdős published within 3 years in the *Bull. Amer. Math. Soc.* and already here it is mentioned that although the upper bound for $r(3,n)$ is quadratic, the present proof does not yield a nonlinear lower bound. That had to wait for another 10 years.

II. The 1958 paper [27]—Graph theory and probability. The main result of this paper deals with graphs, circuits, and chromatic number and as such does not seem to have much to do with Ramsey theory. Yet the paper starts with the review of bounds for $r(k,k)$ and $r(3,k)$ (all due to Erdős and Szekeres). Ramsey numbers are denoted as in most older Erdős papers by symbols of $f(k)$, $f(3,k)$, $g(k)$. He then defines analogously the function $h(k,\ell)$ as "the least integer so that every graph of $h(k,\ell)$ vertices contains either a closed circuit of k or fewer lines or the graph contains a set of independent points. Clearly $h(3,\ell) = f(3,\ell)$".

The main result of [27] is that $h(k,\ell) > \ell^{1+1/2k}$ for any fixed $k \geq 3$ and ℓ sufficiently large. The proof is one of the most striking early uses of the probabilistic method. Erdős was probably aware of it and this may explain (and justify) the title of the paper. It is also proved that $h(2k+1,\ell) < c\ell^{1+1/k}$ and this is proved by a variant of the greedy algorithm by induction on ℓ. Now after this is claimed, it is remarked that the above estimation (1) leads to the fact that there exists a graph G with n vertices which contain no closed circuit of fewer than k edges and such that its chromatic number is $> n^{\epsilon}$.

This side remark is in fact perhaps the most well known formulation of the main result of [27]:

Theorem 4. *For every choice of positive integers k, t and ℓ there exists a k-graph G with the following properties:*

(1) The chromatic number of $G > t$.
(2) The girth of $G > \ell$.

This is one of the few true combinatorial classics. It started in the 40s with Tutte [20] and Zykov [126] for the case $k = 2$ and $\ell = 2$ (i.e., for triangle-free graphs). Later, this particular case was rediscovered and also conjectured several times [22, 70]. Kelly and Kelly [70] proved the case $k = 2$, $\ell \leqq 5$, and conjectured the general statement for graphs. This was settled by Erdős in [27] and the same probabilistic method has been applied by Erdős and Hajnal [35] to yield the general result for hypergraphs.

Erdős and Rado [41] proved the extension of $k = 2$, $\ell = 2$ to transfinite chromatic numbers while Erdős and Hajnal [36] gave a particularly simple construction of triangle-free graphs, so called shift graphs $G = (V, E) : V = \{(i, j); 1 \leq i < j \leq n\}$ and $E = \{(i, j), (i, j); i < j = i < j\}$. G_n is triangle-free and $\chi(G_n) = [\log n]$.

For many reasons it is desirable to have a constructive proof of Theorem 4. This has been stressed by Erdős on many occasions (and already in [27]). This appeared to be a difficult problem and a construction in full generality was finally given by Lovász [81]. A simplified construction has been found in the context of Ramsey theory by Nešetřil and Rödl [91]. The graphs and hypergraphs with the above properties (i), (ii) are called *highly chromatic (locally) sparse graphs*, for short. Their existence could be regarded as one of the true paradoxes of finite set theory (see [35]) and it has always been felt that this result is one of the central results in combinatorics.

Recently it has been realized that sparse and complex graphs may be used in theoretical computer science for the design of fast algorithms. However, what is needed there is not only a construction of these “paradoxical” structures but also their reasonable size. In one of the most striking recent developments, a program for constructing complex sparse graphs has been successfully carried out. Using several highly ingenious constructions which combine algebraic and topological methods it has been shown that there are complex sparse graphs, the size of which in several instances improves the size of random objects. See Margulis [84], Alon [2] and Lubotzky et al. [83].

Particularly, it follows from Lubotzky et al. [83] that there are examples of graphs with girth ℓ, chromatic number t and the size at most $t^{3\ell}$. A bit surprisingly, the following is still open:

Find a primitive recursive construction of highly chromatic locally sparse k-uniform hypergraphs. Indeed, even triple systems (i.e., $k = 3$) present a problem. The best construction seems to be given in [75].

III. $r(3,n)$ [**28**]. The paper [28] provides the lower bound estimate on the Ramsey number $r(3,n)$. Using probabilistic methods Erdős proved

$$r(3,n) \geq \frac{n^2}{\log^2 n} \tag{2}$$

(while the upper bound $r(3,n) \leq \binom{n+1}{2}$ follows from [46]). The estimation of Ramsey numbers $r(3,n)$ was Erdős' favorite problem for many years. We find it already in his 1947 paper [25] where he mentioned that he cannot prove the nonlinearity of $r(3,n)$. Later he stressed this problem (of estimating $r(3,n)$) on many occasions and conjectured various forms of it. He certainly felt the importance of this special case. How right he was is clear from the later developments, which read as a saga of modern combinatorics. And as isolated as this may seems, the problem of estimating $r(3,n)$ became a cradle for many methods and results, far exceeding the original motivation.

In 1981 Ajtai, Komlós and Szemerédi in their important paper [1] proved by a novel method

$$r(3,n) \leq c\frac{n^2}{\log n}. \tag{3}$$

This bound and their method of proof has found many applications. The Ajtai, Komlós and Szemerédi proof was motivated by yet another Erdős problem from combinatorial number theory. In 1941 Erdős and Turán [48] considered problem of dense Sidon sequences (or B_2-sequences). An infinite sequence $S = \{a_1 < a_2 < \cdots\}$ of natural numbers is called *Sidon sequence* if all pairwise sums $a_i + a_j$ are distinct. Define

$$f_S(n) = \max\{x : a_x \leq n\}$$

and for a given n, let $f(n)$ denote the maximal possible value of $f_s(n)$. In [48], Erdős and Turán prove that for finite Sidon sequences $f(n) \sim n^{1/2}$ (improving Sidon's bound of $n^{1/4}$; Sidon's motivation came from Fourier analysis [116]). However for every infinite Sidon sequence S growth of $f_s(n)$ is a more difficult problem and as noted by Erdős and Turán,

$$\liminf f_s(n)/n^{1/2} = 0.$$

By using a greedy argument it was shown by Erdős [26] that $f_s(n) > n^{1/3}$. (Indeed, given k numbers $x_1 < \ldots < x_k$ up to n, each triple $x_i < x_j < x_k$ kills at most 3 other numbers x, $x_i + x_j = x_k + x$, $x_i + x_k = x_j + x$ and $x_j + x_k = x_i + x$ and thus if $k + 3\binom{k}{3} < ck^2 < n$ we can always find a number $x < n$ which can be added to S.) Ajtai, Komlós and Szemerédi proved [1] using a novel "random construction" the existence of an infinite Sidon sequence S such that

$$f_s(n) > c \cdot (n \log n)^{1/3}.$$

An analysis of independent sets in triangle-free graphs is the basis of their approach and this yields as a corollary the above mentioned upper bound on $r(3, n)$. (The best upper bound for $f_s(n)$ is of order $c \cdot (n \log n)^{1/2}$.)

It should be noted that the above Erdős-Turán paper [48] contains the following still unsolved problem: Let $a_1 < a_2 < \cdots$ be an arbitrary sequence. Denote by $f(n)$ the number of representations of n as $a_i + a_j$. Erdős and Turán prove that $f(n)$ cannot be a constant for all sufficiently large n and conjectured that if $f(n) > 0$ for all sufficiently large n then $\limsup f(n) = \infty$. This is still open. Erdős provided a multiplicative analogue of this conjecture (i.e., for the function $g(n)$, the number of representation of n as $a_i a_j$); this is noted already in [48]. One can ask what this has to do with Ramsey theory. Well, not only was this the motivation for [1] but a simple proof of the fact that $\limsup g(n) = \infty$ was given by Nešetřil and Rödl in [93] just using Ramsey's theorem.

We started this paper by listing the predominance of Erdős's first works in number theory. But in a way this is misleading since the early papers of Erdős stressed elementary methods and often used combinatorial or graph-theoretical methods. The Erdős-Turán paper [48] is such an example and the paper [24] even more so.

The innovative Ajtai-Komlós-Szemerédi paper [1] was the basis for a further development (see, e.g., [6]) and this in turn led somewhat surprisingly to the remarkable solution of Kim [72], who proved that the Ajtai-Komlós-Szemerédi bound is up to a constant factor, the best possible, i.e.,

$$r(n, 3) > c\frac{n^2}{\log n}.$$

Thus $r(n, 3)$ is a nontrivial infinite family of (classical) Ramsey numbers with known asymptotics. Recently, there are more such examples, see [3, 4, 5].

IV. Constructions. It was realized early by Erdős the importance of finding explicit constructions of various combinatorial objects whose existence he justified by probabilistic methods (e.g., by counting). In most cases such constructions have not yet been found but even constructions producing weaker results (or bounds) formed an important line of research. For example, the search for an explicit graph of size (say) $2^{n/2}$ which would demonstrate this Ramsey lower bound has been so far unsuccessful. This is not an entirely satisfactory situation since it is believed that such graphs share many properties with random graphs and thus they could be good candidates for various lower bounds, for example, in theoretical computer science for lower bounds for various measures of complexity. (See the papers [13] and [122] which discuss properties of pseudo- and quasirandom graphs.)

The best constructive lower bound for Ramsey numbers $r(n)$ is due to Frankl and Wilson. This improves on an earlier construction of Frankl [51] who found the first constructive superpolynomial lower bound.

The construction of Frankl-Wilson graphs is simple:

Let p be a prime number, and put $q = p^3$. Define the graph $G_p = (V, E)$ as follows:

$$V = \binom{[q]}{p^2 - 1} = \{F \subseteq \{1, \dots, p^3\} : |F| = p^2 - 1\},$$

$$\{F, F\} \in E \quad \text{iff} \quad |F \cap F| \equiv -1 \pmod{q}.$$

The graph G_p has $\binom{p^3}{p^2-1}$ vertices. However, the Ramsey properties of the graph G_p are not trivial to prove: It follows only from deep extremal set theory results due to Frankl and Wilson [53] that neither G_p nor its complement contain K_n for $n \geq \binom{p^3}{p-1}$. This construction itself was motivated by several extremal problems of Erdős and in a way (again!) the Frankl-Wilson construction was a byproduct of these efforts.

We already mentioned earlier the developments related to Erdős paper [27]. The constructive version of bounds for $r(3, n)$ led Erdős to geometrically defined graphs. An early example is Erdős-Rogers paper [45] where they prove that there exists a graph G with ℓ^{1+c_k} vertices, which contains no complete k-gon, but such that each subgraph with ℓ vertices contains a complete $(k-1)$-gon.

If we denote by $h(k, \ell)$ the minimum integer such that every graph of $h(k, \ell)$ vertices contains either a complete graph of k vertices or a set of ℓ points not containing a complete graph with $k - 1$ vertices, then

$$h(k, \ell) \leq r(k, \ell).$$

However, for every $k \geq 3$ we still have $h(k, \ell) > \ell^{1+c_k}$.

This variant of the Ramsey problem is due to A. Hajnal. The construction of the graph G is geometrical: the vertices of G are points on an n-dimensional sphere with unit radius, and two points are joined if their Euclidean distance exceeds $\sqrt{2k/(k-1)}$.

Graphs defined by distances have been studied by many people (e.g., see [101]). The best constructive lower bound on $r(3, n)$ is due to Alon [3] and gives $r(3, n) \geq cn^{3/2}$. See also a remarkable elementary construction [12] (and also [16] which gives a weaker result).

4. Ramsey Theory

It seems that the building of a theory per se was never Erdős's preference. He was a life-long problem solver, problem poser, admirer of mathematical miniatures and beauties. THE BOOK is an ideal. Instead of developing the whole field he seemed always to prefer consideration of particular cases. However, many of these cases turned out to be key cases and somehow theories emerged.

Nevertheless, one can say that Erdős and Rado systematically investigated problems related to Ramsey's theorem with a clear vision that here was a new basis for a theory. In their early papers [42, 43] they investigated possibilities of various extensions of Ramsey's theorem. It is clear that these papers are a result of a longer research and understanding of Ramsey's theorem. As if these two papers summarized what was known, before Erdős and Rado went on with their partition calculus projects reflected by the grand papers [44] and [37]. But this is beyond the scope of this paper. Erdős and Rado [42] contains an extension of Ramsey's theorem for colorings by an infinite number of colors. This is the celebrated Erdős-Rado canonization lemma:

Theorem 5 (**[42]**). *For every choice of positive integers p and n there exists $N = N(p,n)$ such that for every set X, $|X| \geq N$, and for every coloring $c : \binom{x}{p} \to \mathbb{N}$ (i.e., a coloring by arbitrarily many colors) there exists an n-element subset Y of X such that the coloring c restricted to the set $\binom{Y}{p}$ is "canonical".*

Here a coloring of $\binom{Y}{p}$ is said to be canonical if there exists an ordering $Y = y_1 < \ldots < y_n$ and a subset $w \subseteq \{1, \ldots, p\}$ such that two n-sets $\{z_l < \ldots < z_p\}$ and $\{z'_1 < \cdots < z'_p\}$ get the same color if and only if $z_i = z'_i$ for exactly $i \in w$. Thus there are exactly 2^p canonical colorings of p-tuples. The case $w = \phi$ corresponds to a monochromatic set while $w = \{1, \ldots, p\}$ to a coloring where each p-tuple gets a different color (such a coloring is sometimes called a "rainbow" or totally multicoloring).

Erdős and Rado deduced Theorem 5 from Ramsey's theorem. For example, the bound $N(p,n) \leq r(2p, 2^{2p}, n)$ gives a hint as to how to prove it. One of the most elegant forms of this argument was published by Rado [103] in one of his last papers.

The problem of estimating $N(p,n)$ was recently attacked by Lefman and Rödl [80] and Shelah [113]. One can see easily that Theorem 5 implies Ramsey's theorem (e.g., $N(p,n) \geq r(p, n-2, n)$) and the natural question arises as to how many exponentiations one needs. In [80] this was solved for graphs ($p = 2$) and Shelah [113] solved recently this problem in full generality: $N(p,n)$ is the lower function of the same height $r(p,4,n)$ i.e., $(p-1)$ exponentiations.

The Canonization Lemma found many interesting applications (see, e.g., [98]) and it was extended to other structures. For example, the canonical van der Waerden theorem was proved by Erdős and Graham [31].

Theorem 6 (**[31]**). *For every coloring of positive integers one can find either a monochromatic or a rainbow arithmetic progression of every length. (Recall: a rainbow set is a set with all its elements colored differently.)*

This result was extended by Lefman [79] to all regular systems of linear equations (see also [21]) and in an extremal setting by Erdős et al. [38].

One of the essential parts of the development of the "new Ramsey theory" age was the stress on various structural extensions and structure analogies of the original results. A key role was played by the Hales-Jewett theorem (viewed as a combinatorial axiomatization of van der Waerden's theorem), Rota's conjecture (the vector-space analogue of Ramsey's theorem), Graham-Rothschild parameter sets, all dealing with new structures. These questions and results displayed the richness of the field and attracted so much attention to it.

It seems that one of the significant turns appeared in the late 60s when Erdős, Hajnal and Galvin started to ask questions such as "which graphs contain a monochromatic triangle in any 2-coloring of its edges". Perhaps the essential parts of this development can be illustrated with this particular example.

We say that a graph $G = (V, E)$ is t-Ramsey for the triangle (i.e., K_3) if for every coloring of E by t-colors, one of the colors contains a triangle. Symbolically we denote this by $G \to (K_3)^2_t$. This is a variant of the Erdős-Rado partition arrow. Ramsey's theorem gives us $K_6 \to (K_3)^2_2$ (and $K_{r(2,t,3)} \to (K_3)^2_t$). But there are other essentially different examples. For example, a 2-Ramsey graph for K_3 need not contain K_6. Graham [60] constructed the unique minimal graph with this property: The graph $K_3 + C_5$ (triangle and pentagon completely joined) is the smallest graph G with $G \to (K_3)^2_2$ which does not contain a K_6. Yet $K_3 + C_5$ contains K_5 and subsequently van Lint, Graham and Spencer constructed a graph G not containing even a K_5, with $G \to (K_3)^2_2$. Until recently, the smallest example was due to Irving [68] and had 18 vertices. Very recently, two more constructions appeared by Erickson [49] and Bukor [11] who found examples with 17 and 16 vertices (both of them use properties of Graham's graph).

Of course, the next question which was asked is whether there exists a K_4-free graph G with $G \to (K_3)^2_2$. This question proved to be considerably harder and it is possible to say that it has not yet been solved completely satisfactorily.

The existence of a K_4-free graph G which is t-Ramsey for K_3 was settled by Folkman [50] ($t = 2$) and Nešetřil and Rödl [94]. The proofs are complicated and the graphs constructed are very large. Perhaps just to be explicit Erdős [29] asked whether there exists a K_4-free graph G which arrows triangle with fewer than 10^{10} vertices. This question proved to be very motivating and it was later shown by Spencer [118] that there exists such a graph with 3×10^9 vertices. More recently, it was shown by Lu [82] with the help of computers that such a graph exists with 9,697 vertices, and subsequently Dudek and Rödl reduced this number to 941. The record is currently held by Lange, Radziszowski and Xu [76] who found such a graph on just 786 vertices. Of course, it is possible that such a graph exists with fewer than 100 vertices! (In fact, one of the authors offers US$100 for the first person to find such a graph). However, for more than 2 colors the known K_4-free Ramsey graphs are still astronomical.

Probabilistic methods were not only applied to get various bounds for Ramsey numbers. Recently, the Ramsey properties of the Random Graph $K(n,p)$ were analyzed by Rödl and Ruciński and the threshold probability for p needed to guarantee $K(n,p) \to (K_3)^2_t$ with probability tending to 1 as $n \to \infty$, was determined (see [107]).

Structural properties of Ramsey's theorem have also been investigated. For example, the Erdős problems involving $\sum \frac{1}{\log i}$ where the sum is over homogeneous subsets of $\{1, 2, \ldots, n\}$ and problems concerning the relative order of gaps of homogeneous sets were treated for graphs in [106, 18] (see also similar problems for ordering pigeonhole [10]).

Many of these questions were answered in a much greater generality and this seems to be a typical feature for the whole area. On the other side these more general statements explain the unique role of the original Erdős problem. Let us be more specific. We need a few definitions: An ordered graph is a graph with a linearly ordered set of vertices (we speak about "admissible" orderings). Isomorphism of ordered graphs means isomorphism preserving admissible orderings. If A, B are ordered graphs (for now we will find it convenient to denote graphs by $A, B, C, \ldots$) then $\binom{B}{A}$ will denote the set of all induced subgraphs of B which are isomorphic to A. We say that a class $\mathcal{K}$ of graphs is *Ramsey* if for every choice of ordered graphs A, B from $\mathcal{K}$ there exists $C \in K$ such that $C \to (B)^A_2$. Here, the notation $C \to (B)^A_2$ means: for every coloring $c : \binom{C}{A} \to \{1, 2\}$ there exists $B \in \binom{C}{B}$ such that the set $\binom{B'}{A}$ is monochromatic (see, e.g., [88].) Similarly we say that a class $\mathcal{K}$ of graphs is *canonical* if for every choice of ordered graphs A, B from $\mathcal{K}$ there exists $C \in \mathcal{K}$ with the following property: For every coloring $c : \binom{C}{A} \to \mathbb{N}$ there exists $B \in \binom{C}{B}$ such that the set $\binom{B}{A}$ has a canonical coloring.

Denote by $Forb(K_k)$ the class of all K_k-free graphs. Now we have the following

Theorem 7. *For a hereditary class $\mathcal{K}$ of graphs the following statements are equivalent:*

1. *$\mathcal{K}$ (with some admissible orderings) is Ramsey;*
2. *$\mathcal{K}$ (with some admissible orderings) is canonical;*
3. *$\mathcal{K}$ is a union of the following 4 types of classes: the class $Forb(K_k)$, the class of complements of graphs from $Forb(K_k)$, the class of Turán graphs (i.e., complete multipartite graphs) and the class of equivalences (i.e., complements of Turán graphs).*

(1. $\Leftrightarrow$ 3. is proved in [87] establishing important connection of Ramsey classes and ultra homogeneous structures. 2. $\Rightarrow$ 1. is easy, and one can prove 1. $\Rightarrow$ 2. directly along the lines of Erdős-Rado proof of the canonization lemma.) Thus, as often in Erdős' case, the triangle-free graphs was not just any case but rather the typical case.

From today's perspective it seems to be just a natural step to consider. Ramsey properties of geometrical graphs. This was initiated in a series of papers by Erdős, Graham, Montgomery, Rothschild, Spencer and Straus, [32, 33, 34]. Let us call a finite configuration C of points in $\mathbb{E}^n$ *Ramsey* if for every r there is an $N = N(r)$ so that in every r-coloring of the points of $\mathbb{E}^n$, a monochromatic congruent copy of C is always formed. For example, the vertices of a unit simplex in $\mathbb{E}^n$ is Ramsey (with $N(r) = n(r-1)+n$), and it is not hard to show that the Cartesian product of two Ramsey configurations is also Ramsey. More recently, Frankl and Rödl [52] showed that any simplex in $\mathbb{E}^n$ is Ramsey (a simplex is a set of $n+1$ points having a positive n-volume).

In the other direction, it is known [32] that any Ramsey configuration must lie on the surface of a sphere (i.e., be "spherical"). Hence, 3-collinear points do not form a Ramsey configuration, and in fact, for any such set C_3, $\mathbb{E}^N$ can always be 16-colored so as to avoid a monochromatic congruent copy of C_3. It is not known if the value 16 can be reduced (almost certainly it can). The major open question is to characterize the Ramsey configurations. It is natural to conjecture that they are exactly the class of spherical sets. Additional evidence of this was found by Kříž [74] who showed for example, that the set of vertices of any regular polygon is Ramsey (see [85] for a positive answer to a weaker version). However, Leader, Russell and Walters [77] have a different conjecture as to which sets are Euclidean Ramsey sets. Let us call a finite set in Euclidean space *subtransitive* if it is a subset of a set which has a transitive automorphism group. They conjecture that the Euclidean Ramsey sets are exactly the subtransitive sets. These two conjectures are not compatible since they also show [78] that almost all 4-points subsets of a (unit) circle are not subtransitive. A fuller discussion of this interesting topic can be found in [61] and [62].

5. Adventures in Arithmetic Progressions

Besides Ramsey's theorem itself the following result provided constant motivation for Ramsey Theory:

Theorem 8 (van der Waerden [124]). *For every choice of positive integers k and n, there exists a least $N(k,n) = N$ such that for every partition of the set $\{1, 2, \ldots, N\}$ into k classes, one of the classes always contains an arithmetic progression with n terms.*

The original proof of van der Waerden (which developed through discussions with Artin and Schreier—see [125] for an account of the discovery) and which is included in an enchanting and moving book of Khinchine [71] was until recently essentially the only known proof. However, interesting modifications of the proof were also found, the most important of which is perhaps the combinatorial formulation of van der Waerden's result by Hales and Jewett [65].

The distinctive feature of van der Waerden's proof (and also of Hales-Jewett's proof) is that one proves a more general statement and then uses double induction. Consequently, this procedure does not provide a primitive recursive upper bound for the size of N (in van der Waerden's theorem). On the other hand, the best bound (for n prime) is (only!) $W(n+1) \geq n2^n$, n prime (due to Berlekamp [9]). Thus, the question of whether such a huge upper bound was also necessary, was and remains to be one of the main research problems in the area. In 1988, Shelah [112] gave a new proof of both van der Waerden's and the Hales-Jewett's theorem which provided a primitive recursive upper bound for $N(k,n)$. However the bound was still very large, being of the order of fifth function in the Ackermann hierarchy—"tower of tower functions".

Even for a proof of the modest looking conjecture $N(2,n) \leq 2^{2^{2^{2^{\cdots}}}}$ where the tower of 2's has height n, the first author of this paper offered \$1,000. (He subsequently happily paid this reward to Tim Gowers for his striking improvement for upper bounds on the related function $r_k(n)$ which we define in the next section). The first author currently (foolishly?) offers \$1,000 for a proof (or disproof) that $N(2,n) \leq 2^{n^2}$ for every n.

The discrepancy between the general upper bound for van der Waerden numbers and the known values is the best illustrated for the first nontrivial value: while $N(2,3) = 9$, Gowers' proof gives the bound

$$N(2,3) \leq 2^{2^{2^{2^{4,096}}}}\,!$$

These observations are not new and were considered already in the Erdős and Turán 1936 paper [47]. For the purpose of improving the estimates for the van der Waerden numbers, they had the idea of proving a stronger—now called a *density*—statement. They considered (how typical!) the particular case of 3-term arithmetic progressions and for a given positive integer N, defined $r(N)$ (their notation) to denote the maximum number elements of a sequence of numbers $\leq N$ which does not contain a 3-term arithmetic progression. They observed the subadditivity of function $r(N)$ (which implies the existence of a limiting value of $r(N)/N$ and proved $r(N) \leq (\frac{3}{8} + \epsilon)N$ for all $N \geq N(E)$.

After that they remarked that probably $r(N) = o(N)$. And in the last few lines of their paper, they define numbers $r_k(N)$ to denote the maximum number of integers less than or equal to N such that no k of them form an arithmetic progression. Although they do not ask explicitly whether $r_k(N) = o(N)$ (as Erdős did many times since), this is clearly in their mind as they list consequences of a good upper bound for $r_k(N)$: long arithmetic progressions formed by primes (yes, already there!) and a better bound for the van der Waerden numbers.

As with the Erdős-Szekeres paper [46], the impact of the modest Erdős-Turán note [47] is hard to overestimate. Thanks to its originality, both in combinatorial and number theoretic contexts, and to Paul Erdős' persistence, this led eventually to beautiful and difficult research, and probably beyond Erdős' expectations, to a rich general theory. We wish to briefly mention some key points of this development where the progress has been remarkably rapid, so that van der Waerden's theorem with it many variations and related problems has become one of the fastest growing (and successful) areas in mathematics. It cannot be the purpose of this article (which concentrates narrowly on the work of Erdős) to survey this body of work (for a good start, see [119]). In particular, this development has lead to 2 Fields Medals (Gowers 2002, Tao 2006) and more recently, to an Abel Prize (Szemerédi 2012). In particular, Gowers [58] gave a new bound for $r_k(n)$ which as a consequence gave the strongest current upper bound for the van der Waerden function $W(2,n)$ of the form

$$W(2,n) < 2^{2^{2^{2^{2^{n+9}}}}},$$

thereby earning the above-mentioned \$1,000 prize. (Strictly speaking, Gowers' bound for $W(2,n)$ is larger then required conjectured bound given by the tower of n 2's for the values of $n = 7$ and 8 but it was judged to be close enough to deserve the full prize!) In addition, Green and Tao [64] proved the existence of arbitrarily long arithmetic progression of primes in any set of integers of positive upper density (thus solving a problem attributed to Legendre). Most of these advances were motivated by and more or less directly related to the Erdős-Turán function $r_k(n)$. Soon after [47] good lower estimates for $r(N)$ were obtained by Salem and Spencer [110] and Behrend [8] which still gives the best bounds. These bounds recently found a surprising application in a least expected area, namely in the fast multiplication of matrices (Coppersmith and Winograd [19]).

The upper bounds and $r_k(N) = o(N)$ appeared to be much harder. In 1953 K. Roth [109] proved $r_3(N) = o(N)$ and after several years of partial results, E. Szemerédi in 1975 [92] proved the general case

$$r_k(N) = o(N) \text{ for every } k.$$

This is generally recognized as the single most important solution of an Erdős problem, the problem for which he has paid the largest reward. By now there are more expensive problems (see Erdős' article in these volumes) but they have not yet been solved. And taking inflation into account, possibly none of them will ever have as an expensive solution. Szemerédi's proof changed Ramsey theory in at least two aspects. First, several of its pieces, most notably the so-called Regularity Lemma, proved to be very useful in many other combinatorial situations (see e.g., [15, 92, 107]). Secondly, perhaps due to the complexity of Szemerédi's combinatorial argument, and the beauty of the result itself, an alternative approach was called for. Such an approach was found by Hillel Furstenberg [54, 55] and developed further in many aspects in his joint work with B. Weiss, Y. Katznelson and others. Let us just mention two results which in our opinion best characterize the power of this approach: In [56] Furstenberg and Katznelson proved the density version of Hales-Jewett theorem, and Bergelson and Leibman [7] proved the following striking result (conjectured by Furstenberg):

Theorem 9 ([7]). *Let $p_1, \dots, p_k$ be polynomials with rational coefficients taking integer values on integers and satisfying $p_i(0) = 0$ for $i = 1, \dots, k$. Then every set X of integers of positive density contains for every choice of numbers $v_1, \dots, v_k$, a subset*

$$\mu + p_1(d)v_1, \mu + p_2(d)v_2, \dots, \mu + p_k(d)v_k$$

for some μ and $d > 0$.

Choosing $p_i(x) = x$ and $v_i = i$ we get the van der Waerden theorem. Already, the case $p_i(x) = x^2$ and $v_i = i$ was open for several years [111] (this gives long arithmetic progressions in sets of positive density with their differences being a square).

Originally, none of these results was proved by combinatorial methods. Instead, they were all proved by a blend of topological dynamics and ergodic theory methods, proving countable extensions of these results. For this part of Ramsey theory this setting seems to be most appropriate. In some sense, this is a long way from the original Erdős-Turán paper. However, this emphasis as been changing recently with combinatorial proofs of many of the results in the area, most notably of the density version of the Hales-Jewett theorem (see [100]).

And even more recently, the situation reversed as Rödl's project of a combinatorial approach to Szemerédi's theorem [105] using a hypergraph generalization of the regularity lemma was successful, see e.g., [59, 108]. This generalization in turn was related to model theory, probability and analysis, see e.g., recent papers [119, 120]. This development probably far exceeded even Erdős' expectations.

Let us close this section with a very concrete and still unsolved example. In 1983, G. Pisier [99] formulated (in a harmonic analysis context) the following problem: A set of integers $x_1 < x_2 < \dots$ is said to be *independent* if all finite subsums of distinct elements are distinct. Now let X be an infinite set and suppose for some $\epsilon > 0$ that every finite subset $Y \subseteq X$ contains a subsubset Z of size $\geq \epsilon|Z|$ which is independent. Is it then true that X is a finite union of independent sets?

Despite much effort and partial solutions, the problem is still open. It was again Paul Erdős who quickly realized the importance of the Pisier problem and as a result, Erdős, Nešetřil and Rödl [39, 40] studied "Pisier type problems". For various notions of an independence relation, the following question was considered: Assume that an infinite set X satisfies for some $\epsilon > 0$, some hereditary density condition (i.e., we assume that every finite set Y contains an independent subsubset of size $\geq \epsilon|Y|$). Is it then true that X can be partitioned into finitely many independent sets?

Positive instances (such as collinearity, and linear independence) as well as negative instances (such as Sidon sets) were given in [39, 40]. Also various "finitization versions" and analogues of the Pisier problem were answered in the negative. But at present the original Pisier problem is still open. In a way one can consider Pisier type problems as dual to the density results in Ramsey theory: One attempts to prove a positive Ramsey type statement under a strong (hereditary) density condition. This is exemplified in [40] by the following problem which is perhaps a fitting conclusion to this paper surveying 60 years of Paul Erdős' service to Ramsey theory.

The Anti-Szemerédi Problem [40]

Does there exist a set X of positive integers such that for some $\epsilon > 0$ the following two conditions hold simultaneously:

(1) For every finite $Y \subseteq X$ there exists a subset $Z \subseteq X$, $|Z| \geq \epsilon|Y|$, which does not contain a 3-term arithmetic progression;
(2) Every finite partition of X contains a 3-term arithmetic progression in one of its classes.

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Shadows of Memories (Ramsey Theory, 1984)

From left to right: B. L. Rothschild, W. Deuber, P. Erdős,
B. Voigt, H.-J. Promel, R. L. Graham, J. Nešetřil, V. Rödl.

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Memories on Shadows and Shadows of Memories

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I am one of the very few mathematicians who knew Paul's aunt Irma before I knew him. She and my grandmother were neighbors during World War II. Aunt Irma was one of the few Jews in Budapest who survived the holocaust. This is how I met her since I was raised from this time by my grandmother and my aunt. Aunt Irma must have had good memories about my grandmother since they kept a good relationship, she regularly visited my grandmother even after her move to another place. She learned about my interest in mathematics and suggested I meet her nephew *Pali* who happened to be a mathematician. "Of course" I had never heard of him, but I was very glad to meet an old "real mathematician". He was a very respectful old man (46!). I immediately understood that I was seeing an extraordinary personality.

He gave me three problems to solve. I remember one of them.

> *How many numbers can we choose from the set* $\{1, 2, \ldots, 2n\}$ *without having a number and its proper divisor in the set?*

He probably had in his mind that I should call him the next day with the solutions. I took the problems very seriously. But only with my pace. It was summer. The summer between high school and university studies. I had to do so much. But I regularly returned to the problems, I solved all of them by September and reported the solutions to him.

Next time when Aunt Irma visited us forwarded the following message:

> *He is probably very talented in other fields but not so much in mathematics.*

It did not touch me deeply. I had a lot of self-confidence. I won the National Olympiad in mathematics for high school students and was a member of the Hungarian team at the first International Olympiad. But I should have taken his opinion more seriously! At least concerning my pace and assertiveness. Is it too late?

I started my studies. Beside my regular classes, I attended several special lectures and seminars. Turán's seminar proved to be the most important for me. In spite of his negative (perhaps non-positive) opinion on my abilities, Uncle Paul (in Hungarian: Pali bácsi (=Pauli baachy)) did not forget about me. He handed me a reprint of the famous Erdős-Ko-Rado paper [1], saying that there are some open problems in it. I was in my second year of studies,

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I had a joint result with my friend Domokos Szász, but no one was too much interested in it because the problem was posed by ourselves. So I started to read the paper which was not without difficulties: I did not know English. The influence of this paper on my mathematical life was decisive.

Let me remind the reader what the main theorem of this paper was.

> *If we have a family of k element subsets of an n-element set, $2k \geq n$ and any two sets meet then the size of the family is at most $\binom{n-1}{k-1}$.*

This theorem became one of the centers of my interest. Much later, in 1971, I found an elegant proof of it [5]. The open problem I started to work on was the following.

> *Determine the largest family of subsets of an n-element set if any two of the sets meet in at least l elements.*

I spent all my free time in a period of 3 or 4 months thinking on this problem. Knowing my pace, this is not so much! Let me show you what I observed.

For sake of simplicity consider only the case when $l = 2$ and n is even. The conjectured optimal family for this case was the family of all subsets having at least $\frac{n+2}{2}$ elements. A family satisfying the conditions cannot contain the empty set or a one-element set. The total number of the 2-element and $n-1$-element sets is maximum when all $n-1$-element ones are chosen. The total number of the 3-element and $n-2$-element sets is maximum when all $n-2$-element ones are chosen. In general, it seemed that the total number of the i-element and $n-i+1$-element sets was maximum when all $n-i+1$-element ones are chosen. That is, if we have m sets of size i in the family then they push out at least m sets of size $n-i+1$. A set of size i pushes out exactly the complements of its $i-1$-element subsets. So, it would be enough to prove that the number of $i-1$-element subsets of i-element members of the family is at least m.

Let us repeat this more formally. Let $\mathcal{A}$ be the family satisfying the condition of the problem and let $\mathcal{A}_i$ be the subfamily of its i-element members.

The *shadow* (this name was introduced later by someone else) of $\mathcal{A}_i$ is

$$\sigma(\mathcal{A}_i) = \{B : |B| = i-1, B \subset A \in \mathcal{A}\}.$$

We need $|\mathcal{A}_i| \leq |\sigma(\mathcal{A}_i)|$. After several months of work I managed to prove a somewhat stronger statement:

> *If any two members of the family $\mathcal{A}_i$ meet in at least 2 elements then*

$$\frac{\binom{2i-2}{i-1}}{\binom{2i-2}{i}} \leq \frac{|\sigma(\mathcal{A}_i)|}{|\mathcal{A}_i|} \tag{1}$$

The left hand side is $\frac{i}{i-1}$, therefore (1) implies that the size of the shadow is larger than the size of the original family. This gives the solution of the problem. The case of general l is much the same [3].

I was extremely happy and reported the result to Uncle Paul (that time Professor). He seemed to be satisfied and mentioned some consequences of "my theorem". Moreover, he invented an unusual reward for me. He invited me for lunch in a very nice hotel. The name of the hotel was Red Star (Vörös Csillag). It was on the top of a larger hill. Its restaurant was partially open air and had a fantastic view on Budapest. Of course, his mother (Anyuka) was with us. I was on one hand very proud on the other hand embarrassed. This was the first time I ate in a restaurant. Uncle Paul probably did understand my embarrassment and helped me to choose the food.

Equation (1) determined the minimum of the ratio of the size of the shadow and the original family. It is easy to see that this estimate is sharp when the size of the family is $\binom{2i-2}{i}$. However it is not sharp if the number of sets is different. It was disturbing that I could not determine the minimum of $|\sigma(\mathcal{A}_i)|$ when $|\mathcal{A}_i|$ was given. However I did not see any nice construction that could be conjectured to be the optimum. Then I realized that the problem makes sense without the intersection-property, too.

From this time (about 1962) I concentrated on this problem and after 2 or 3 years I found a complicated inductional solution. I presented my theorem in 1965 in a lecture organized by the János Bolyai Mathematical Society. This became my best known result [4]. Let me formulate it in a special case, only.

The minimum of $|\sigma(\mathcal{A}_i)|$ under the condition that $|\mathcal{A}_i| = \binom{a}{i}$ for some fixed integer a is $\binom{a}{i-1}$. The optimal construction is the family of all i-element subsets of an a-element set.

Although it was not asked or conjectured by Erdős, it was an indirect consequence of Uncle Paul giving me his reprint. Later Branko Grünbaum called my attention to the fact that Kruskal [6] proved the same theorem earlier. His motivation was coding, therefore the combinatorial world was not aware of his result. Both (very different) proofs were quite lengthy. Many authors tried to find simpler proofs. Probably the shortest one is due to Frankl [2].

A young student from Greifswald (that time German Democratic Republic) spent a few months in Budapest in 1990 as an exchange student. I suggested to him to think about the following problem. What is the minimum size of the shadow if $|\mathcal{A}_i|$ is fixed, like before, and the family $\mathcal{A}_i$ has a system of distinct representatives, that is, one can find elements in each member of $\mathcal{A}_i$ in such a way that these elements are distinct. (More precisely, I asked a somewhat different problem and he found this nicer variant.) After 3 years of working on it he found the solution what I would formulate here only in a special case.

Theorem 1 (**Leck** [7]). *Suppose that $|\mathcal{A}_i|$ is fixed, is of the form $m(i+1)$ and $\mathcal{A}_i$ has a system of distinct representatives. The minimum of $|\sigma(\mathcal{A}_i)|$ is $m\binom{i+1}{2}$ under these conditions. The optimal construction is a disjoint union of $m(i+1)$-element sets containing all i-element subsets.*

The underground stream of problems started by Uncle Paul has reached Uwe Leck. But it is my responsibility to find a reward for him. Should I take him to Hotel Red Star? Its name is Golf now. And Uwe Leck has since become a "rich (West-)German". I do not think lunch in a restaurant would be a great experience for him. Any suggestions?

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A Bound of the Cardinality of Families Not Containing Δ-Systems

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Dedicated to Professor Paul Erdős on the occasion of his 80th birthday

Summary. P. Erdős and R. Rado defined a Δ-system as a family in which every two members have the same intersection. Here we obtain a new upper bound of the maximum cardinality $\varphi(n)$ of an n-uniform family not containing any Δ-system of cardinality 3. Namely, we prove that for any $\alpha > 1$, there exists $C = C(\alpha)$ such that for any n,

$$\varphi(n) \leq Cn!\alpha^{-n}.$$

1. Introduction

P. Erdős and R. Rado [2] introduced the notion of a Δ-system. They called a family $\mathcal{H}$ of finite sets a Δ-*system* if every two members of $\mathcal{H}$ have the same intersection.

Let $\varphi(n)$ (respectively, $\gamma(n)$) denote the the maximum cardinality of an n-uniform family (respectively, intersecting n-uniform family) not containing any Δ-system of cardinality 3.

P. Erdős and R. Rado [2] proved that

$$2^n n! > \varphi(n) \geq 2^n$$

and conjectured that

$$\varphi(n) < c^n \quad \text{for some absolute constant } c.$$

The best published upper bound for $\varphi(n)$ is due to J. Spencer [3]:

$$\varphi(n) < e^{cn^{3/4}} n!.$$

Z. Füredi and J. Kahn (see [1]) proved that

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$$\varphi(n) < e^{c\sqrt{n}} n!.$$

The aim of the present paper is to prove

Theorem 1. *For any integer* $\alpha > 1$*, there exists* $C = C(\alpha)$ *such that for any* n*,*

$$\varphi(n) \leq Cn!\alpha^{-n}.$$

We follow here the ideas of J. Spencer [3]. In particular, in the course of proofs some inequalities are true if n is large in comparison with α. And we choose C so that the statement of the theorem holds for smaller n.

Remark 1. *Certainly, since the statement is true for any constant* α*, it is also true for some function on* n *tending to infinity. In fact, along the lines of the proof of the theorem one can prove that there exists a positive constant* C *such that*

$$\varphi(n) \leq Cn! \left(\frac{30 \log\log\log n}{\log\log n} \right)^n.$$

(All the logarithms throughout the paper are taken to the base e.) It is enough to change slightly Lemma 2 and to take $k = \lceil \frac{n}{\log n} (\log\log n)^3 \rceil$ *in Sect. 3.*

2. Preliminary Lemmas

Call a family $\mathcal{F}$ of sets a $(3, n, k)$-*family* if it is an n-uniform family not containing any Δ-system of cardinality 3 such that the cardinality of the intersection of each two members of $\mathcal{F}$ is at most $n - k$.

Lemma 1. *For any* $(3, n, k)$*-family* $\mathcal{F}$*,*

$$|\mathcal{F}| \leq 2^{n-k+1} \frac{n!}{k!}.$$

Proof. We use induction on $n-k$. Obviously, any $(3, k, k)$-family has at most two members. Hence the lemma is true for $n - k = 0$.

Let the lemma be valid for $n - k \leq m - 1$ and $\mathcal{F}$ be a $(3, m+k, k)$-family. Choose in $\mathcal{F}$ two edges A_1 and A_2 with minimum cardinality of intersection and denote $Z = A_1 \cup A_2$. Then each $A \in \mathcal{F}$ has a non-empty intersection with Z.

For any $x \in Z$, let $\mathcal{F}(x) = \{A \in \mathcal{F} | x \in A\}$, $\tilde{\mathcal{F}}(x) = \{A \setminus \{x\} | A \in \mathcal{F}(x)\}$. Then for any $x \in Z$, $\tilde{\mathcal{F}}(x)$ is a $(3, m+k-1, k)$-family. Thus,

$$|\mathcal{F}| \leq \sum_{x \in Z} |\tilde{\mathcal{F}}(x)| \leq |Z| 2^m \frac{(m-1+k)!}{k!} \leq 2^{m+1} \frac{(m+k)!}{k!}.$$

□

Lemma 2. *If* $\gamma(k) \leq Ck!\alpha^{-k}e^{-\alpha}$ *for any* $k \leq n$*, then* $\varphi(n) \leq Cn!\alpha^{-n}$*.*

Proof. Let $\mathcal{F}$ be a $(3, n, 1)$-family with $|F| = \varphi(n)$ and $A \in \mathcal{F}$. For any $X \subset A$, the family $\mathcal{F}(A, X) = \{B \setminus X | B \in \mathcal{F} \& B \cap A = X\}$ is a $(3, n-|X|, 1)$-family. Moreover, if for some $X \subset A$, there exist two disjoint sets $B_1 \setminus X$, $B_2 \setminus X \in \mathcal{F}(A, X)$ then A, B_1 and B_2 form a Δ-system. Hence $\mathcal{F}(A, X)$ is an intersecting family and

$$\varphi(n) \leq \sum_{X \subset A} |\mathcal{F}(A, X)| \leq \sum_{i=0}^{n} \binom{n}{i} \gamma(n-i) \leq Cn! \alpha^{-n} e^{-\alpha} \sum_{i=0}^{n} \frac{\alpha^i}{i!} < Cn! \alpha^{-n}.$$

From now on, we suppose that for each $m \leq n-1$,

$$\gamma(m) \leq Cm! \alpha^{-m} e^{-\alpha}, \tag{1}$$

$$\varphi(m) \leq Cm! \alpha^{-m}. \tag{2}$$

In view of Lemma 2 it is enough to show that (1) holds for $m = n$.

The following observation from [3] will be used throughout the paper. Let $B_1, \ldots, B_t$ be disjoint finite sets and $\mathcal{F}$ be a $(3, n, 1)$-family such that $|A \cap B_i| \geq b_i$ for each $A \in \mathcal{F}$. Then

$$|\mathcal{F}| \leq \binom{|B_1|}{b_1} \cdot \ldots \cdot \binom{|B_t|}{b_t} \varphi(n - b_1 - \cdots - b_t). \tag{3}$$

Lemma 3. *Let $0 < r \leq k \leq n/2$ and for any members $A_1, \ldots, A_r$ of a $(3, n, 1)$-family $\mathcal{F}$,*

$$|A_1 \cup \ldots \cup A_r| \leq rn - kr^2/2. \tag{4}$$

Then

$$|\mathcal{F}| \leq C \frac{n!}{k!}.$$

Proof. For $r = 1$ the lemma is valid since (4) is impossible as $r = 1$. Suppose that the lemma is true for $r \leq s-1$ and $|\mathcal{F}| > Cn!/k!$. By the induction hypothesis there exist $A_1, \ldots, A_{s-1} \in \mathcal{F}$ such that for the set $B = A_1 \cup \ldots \cup A_{s-1}$ we have $|B| > (s-1)n - k(s-1)^2/2$. If the lemma does not hold for $\mathcal{F}$ then for any $A \in \mathcal{F}$,

$$|A \cap B| > n + ((s-1)n - k(s-1)^2/2) - (sn - ks^2/2) = k(s - 1/2),$$

and there is an i, $1 \leq i \leq s-1$ such that $|A \cap A_i| > k$. Thus by (3),

$$|\mathcal{F}| \leq (s-1)\binom{n}{k+1}\varphi(n-k-1) \leq$$

$$(s-1)\binom{n}{k+1} C(n-k-1)! \alpha^{-n+k+1} = Cn! \alpha^{-n+k+1} \frac{s-1}{(k+1)!} < C\frac{n!}{k!}.$$

□

Lemma 4. *Let $\xi \geq 2$, $1 \leq t < s \leq n$ and $\mathcal{F}$ be a $(3, s, 1)$-family with $|\mathcal{F}| \geq Cs! \xi^{-s}$. Then there exist $\mathcal{F}' \subset F$ and X such that*

(1) $|X| = s - t$;
(2) *For any* $A \in \mathcal{F}'$, $A \supset X$;
(3) $|\mathcal{F}'| \geq Ct!\beta^{-t}$, *where* $\beta = (4\xi)^{s/t}$.

Proof. Case 1. *For any* $A \in \mathcal{F}$, $|\{B \in \mathcal{F}|\ |B \cap A| \geq s - t\}| \leq Ct!2^s\beta^{-t} - 1$. Then a simple greedy algorithm gives us a $(3, s, t+1)$-subfamily of $\mathcal{F}$ with cardinality at least

$$\frac{|\mathcal{F}|}{Ct!2^s\beta^{-t}} = \frac{s!(4\xi)^s}{t!(2\xi)^s} = 2^s\frac{s!}{t!}.$$

But the existence of such a big $(3, s, t+1)$-family contradicts Lemma 1.

Case 2. *There exists* $A \in \mathcal{F}$ *such that* $|\{B \in \mathcal{F}|\ |B \cap A| \geq s - t\}| \geq \lfloor Ct!2^s\beta^{-t}\rfloor$.

Then for some $X \subset A$ with $|X| = s - t$ we have

$$|\{B \in \mathcal{F}| B \cap A \supset X\}| \geq \lfloor Ct!2^s\beta^{-1}\rfloor \binom{s}{s-t}^{-1} > Ct!\beta^{-t}.$$

This is the family we need. □

3. Main Construction

Let $\mathcal{F}$ be an intersecting $(3, n, 1)$-family with $|\mathcal{F}| = \gamma(n)$. The idea is to find a (not too large) family of collections of disjoint (and considerably small) sets such that each member of $\mathcal{F}$ intersects each set from some collection and then apply (3). We put

$$y = \lfloor n/3\alpha\rfloor,\ m = 3\alpha - 1, k = \left\lceil \frac{n}{\log n}\log\log n\right\rceil,\ r = \lfloor \log\log n\rfloor.$$

Lemma 5. *For all* $s = 0, 1, \ldots, m$ *and for* $i_0 = 1$ *and any* $i_1, \ldots, i_s \in \{1, \ldots, r\}$ *there are subfamilies* $\mathcal{F}(1, i_1, \ldots, i_s)$ *of the family* $\mathcal{F}$ *and sets* $X(i_1, \ldots, i_s)$ *and* $Z(1, i_1, \ldots, i_{s-1})$ *such that for any* $s = 1, \ldots, m$ *and for any* $i_1, \ldots, i_s, i'_s \in \{1, \ldots, r\}$,

(1) $\mathcal{F}(1, i_1, \ldots, i_s) \subset \mathcal{F}(1, i_1, \ldots, i_{s-1})$;
(2) *For all* $A \in \mathcal{F}(1, i_1, \ldots, i_s)$,

$$A \supset X(i_1) \cup X(i_1, i_2) \cup \ldots \cup X(i_1, i_2, \ldots, i_s);$$

(3) *The sets* $X(i_1)$, $X(i_1, i_2), \ldots, X(i_1, i_2, \ldots, i_s)$ *are pairwise disjoint;*
(4) $|X(i_1, i_2, \ldots, i_s)| = y$;
(5) $|Z(1, i_1, i_2, \ldots, i_{s-1})| \leq kr(r+1)/2$;
(6) $X(i_1, i_2, \ldots, i_{s-1}, i_s) \cap X(i_1, i_2, \ldots, i_{s-1}, i'_s) \subset Z(1, i_1, i_2, \ldots, i_{s-1})$;
(7) $|\mathcal{F}(1, i_1, \ldots, i_s)| \geq C(n - sy)!\xi_s^{sy-n}$, *where*

$$\xi_s = \left((2\alpha)^{\frac{n}{n-sy}} 8^{\frac{n+(n-y)+\ldots+(n-(s-1)y)}{n-sy}}\right) = \left((2\alpha)^n 8^{ns-s(s-1)y/2}\right)^{\frac{1}{n-sy}}.$$

Proof. We use induction on s. Put $\mathcal{F}(1) := \mathcal{F}$, $\xi_0 := 2\alpha$.

Step s $(0 \le s < m)$. We have at hand $\mathcal{F}(1, i_1, \ldots, i_s)$ for any $i_1, \ldots, i_s \in \{1, \ldots, r\}$ and if $s > 0$ we also have sets $X(i_1, \ldots, i_s)$ and $Z(1, i_1, \ldots, i_{s-1})$ as needed. Consider

$$\begin{aligned}\tilde{\mathcal{F}} &= \tilde{\mathcal{F}}(1, i_1, \ldots, i_s)\\ &= \{A \setminus (X(i_1) \cup X(i_1, i_2) \cup \ldots \cup X(i_1, i_2, \ldots, i_s)) | A \in \mathcal{F}(1, i_1, \ldots, i_s)\}.\end{aligned}$$

According to the statements of the lemma, $\tilde{\mathcal{F}}$ is a $(3, n - sy, 1)$-family. Note that

$$(ns - s(s-1)y/2)/(n - sy) \le ns/(n - (m-1)y) \le \frac{mn}{n - (n - 2y)} < (3\alpha)^2.$$

Hence $\xi_s \le (2\alpha)^{3\alpha} 8^{9\alpha^2}$ and due to Statement 7 of the lemma, we can use Lemma 4. This Lemma 4 provides that there exists X_1 of cardinality y and $\mathcal{H}_1 \subset \tilde{\mathcal{F}}$ with $|\mathcal{H}_1| \ge C(n-(s+1)y)!\beta^{(s+1)y-n}$ (where $\beta = (4\xi_s)^{\frac{n-sy}{n-(s+1)y}}$) such that any $A \in \mathcal{H}_1$ contains X_1. We put $Z_1 := \emptyset$. Suppose that $(3, n - sy, 1)$-families $\mathcal{H}_1, \ldots, \mathcal{H}_l$ and sets $X_1, \ldots, X_l, Z_1, \ldots, Z_l$ are constructed and that for each $1 \le j \le l$, $1 \le j' \le l$, $j \ne j'$,

(i) $|X_j| = y$;
(ii) $|Z_l| \le kl(l-1)/2$;
(iii) $X_j \cap X'_j \subset Z_l$;
(iv) For any $A \in \mathcal{H}_j$, $X_j \subset A$;
(v) $|\mathcal{H}_j| \ge C(n - (s+1)y)!\xi_{s+1}^{(s+1)y-n}$.

If $l < r$ then we construct $\mathcal{H}_{l+1}$, X_{l+1} and Z_{l+1} as follows. Remark that for each $A \in \tilde{\mathcal{F}}$, we have $|A| = n - sy > lk$ and the number of $A \in \tilde{\mathcal{F}}$ with $|A \cap (X_1 \cup \ldots \cup X_l)| \ge lk$ does not exceed (by (3))

$$\begin{aligned}\binom{|X_1 \cup \ldots \cup X_l|}{lk}\varphi(n - sy - lk) &\le \binom{ly}{lk} C(n - sy - lk)!\alpha^{sy+lk-n}\\ &\le \left(\frac{eln}{3\alpha lk}\right)^{lk} C(n - sy - lk)!\alpha^{sy+lk-n} \le \frac{C(n-sy)!}{\alpha^{n-sy}k^{lk}}.\end{aligned}$$

But for large n we have

$$k^k \ge \left(\frac{n}{\log n}\right)^{\frac{n \log\log n}{\log n}} \ge e^{0.5n \log\log n} > (2\xi_m)^n.$$

Hence for the family $\mathcal{H}' := \{A \in \tilde{\mathcal{F}} \mid |A \cap (X_1 \cup \ldots \cup X_l)| < lk\}$ we have $|\mathcal{H}'| \ge |\tilde{\mathcal{F}}| - C(n - sy)!(2\xi_s)^{sy-n} \ge C(n - sy)!(2\xi_s)^{sy-n}$. Then by Lemma 4 there exist $\mathcal{H}_{l+1} \subset \mathcal{H}'$ and X_{l+1} with $|X_{l+1}| = y$ such that each $A \in \mathcal{H}_{l+1}$ contains X_{l+1} and $|\mathcal{H}_{l+1}| \ge C(n - (s+1)y)!(\beta)^{(s+1)y-n}$, where

$$\beta = (4 \times 2\xi_s)^{\frac{n-sy}{n-(s+1)y}} = \left(8 \times \left((2\alpha)^n 8^{ns-s(s-1)y/2}\right)^{\frac{1}{n-sy}}\right)^{\frac{n-sy}{n-(s+1)y}}$$
$$= \left((2\alpha)^n 8^{n(s+1)-s(s+1)y/2}\right)^{\frac{1}{n-(s+1)y}} = \xi_{s+1}.$$

By definition of $\mathcal{H}'$,

$$|X_{l+1} \cap (X_1 \cup \ldots \cup X_l)| < lk.$$

Putting $Z_{l+1} := Z_l \cup (X_{l+1} \cap (X_1 \cup \ldots \cup X_l))$, we have $|Z_{l+1}| \leq |Z_l| + lk \leq kl(l+1)/2$ and conditions (i)–(v) are fulfilled for $l+1$. Thus we can proceed till $l = r$.

After constructing $\mathcal{H}_r$, X_r and Z_r we put for j=$1, \ldots, r$, $X(i_1, i_2, \ldots, i_s, j) :=X_j$, and

$$\mathcal{F}(1, i_1, \ldots, i_s, j) := \{A \cup X(i_1) \cup X(i_1, i_2) \cup \ldots \cup X(i_1, i_2, \ldots, i_s, j) | A \in \mathcal{H}_j\},$$

and

$$Z(1, i_1, \ldots, is) := Z_r.$$

By construction, the statements 1–7 of the lemma will be fulfilled for $s+1$. □

Lemma 6. *For all $s = 0, 1, \ldots, m+1$ and for any $i_1, \ldots, i_s \in \{1, \ldots, r\}$ there are sets $X(i_1, \ldots, i_s)$ and $Z(1, i_1, \ldots, i_{s-1})$ and for any $i_1, \ldots, i_{m+1} \in \{1, \ldots, r\}$ there are sets $A(i_1, \ldots, i_{m+1}) \in \mathcal{F}$ such that*

(1) The sets $X(i_1), X(i_1, i_2), \ldots, X(i_1, i_2, \ldots, i_{m+1})$ are pairwise disjoint;
(2) $|X(i_1, i_2, \ldots, i_s)| = y$ if $1 \leq s \leq m$;
(3) $|X(i_1, i_2, \ldots, i_{m+1})| = n - my$;
(4) $A(i_1, \ldots, i_{m+1}) = X(i_1) \cup X(i_1, i_2) \cup \ldots \cup X(i_1, i_2, \ldots, i_{m+1})$; for any $s = 1, \ldots, m$ and for any $i_1, \ldots, i_s, i'_s \in \{1, \ldots, r\}$
(5) $X(i_1, i_2, \ldots, i_{s-1}, i_s) \cap X(i_1, i_2, \ldots, i_{s-1}, i'_s) \subset Z(1, i_1, i_2, \ldots, i_{s-1})$;
(6) $|Z(1, i_1, i_2, \ldots, i_{s-1})| \leq kr(r+1)/2$.

Proof. For $s = 0, 1, \ldots, m$ and for any $i_1, \ldots, i_s \in \{1, \ldots, r\}$, consider $\mathcal{F}(1, i_1, \ldots, i_s)$, $X(i_1, \ldots, i_s)$ and $Z(1, i_1, \ldots, i_{s-1})$ from Lemma 5.

Now, for an arbitrary $(m+1)$-tuple $(1, i_1, \ldots, i_m)$, consider

$$\mathcal{H} = \mathcal{H}(1, i_1, \ldots, i_m)$$
$$:= \{A \setminus (X(i_1) \cup X(i_1, i_2) \cup \ldots \cup X(i_1, i_2, \ldots, i_m)) | A \in \mathcal{F}(1, i_1, \ldots, i_m)\}.$$

By construction, $\mathcal{H}$ is a $(3, n-my, 1)$-family and by Lemma 5, $|\mathcal{H}| \geq C(n-my)!\xi_m^{my-n}$.

Recall that $m = 3\alpha - 1$, and $n - my \geq y \geq n/(3\alpha) - 1$. For large n,

$$\xi_m^{n-my} = (2\alpha)^n 8^{nm-m(m-1)y/2} \leq (2\alpha)^n 8^{nm} \leq (2\alpha 8^{3\alpha-1})^n < k!\,.$$

Hence $|\mathcal{H}| > C(n-my)!/k!$ and by Lemma 3 (note that $0 < r < k < (n/3\alpha - 1)/2 \leq (n-my)/2$) there exist $A_1, \ldots, A_r \in \mathcal{H}$ such that

$$|A_1 \cup \ldots \cup A_r| > r(n-my) - kr^2/2. \tag{5}$$

Let us denote

$$Z(1, i_1, \ldots, i_m) := \bigcup_{1 \leq j < h \leq r} A_j \cap A_h,$$

and for $j = 1, \ldots, r, X(i_1, i_2, \ldots, i_s, j) := A_j$ and

$$A(i_1, \ldots, i_m, j) = X(i_1) \cup X(i_1, i_2) \cup \ldots \cup X(i_1, i_2, \ldots, i_m) \cup A_j.$$

In view of (5), $|Z(1, i_1, \ldots, i_m)| \leq kr^2/2$. Now, by Lemma 5 and the construction, all the statements of the lemma are fulfilled. □

Lemma 7. *For each $A \in \mathcal{F}$, there exist s, $0 \leq s \leq m$ and $i_1, \ldots, i_s \in \{1, \ldots, r\}$ such that*

$$A \bigcap X(i_1, i_2, \ldots, i_s, j) \neq \emptyset \quad \forall j \in \{1, \ldots, r\}. \tag{6}$$

Proof. Assume that for some $B \in \mathcal{F}$ for each s, $0 \leq s \leq m$ and each $i_1, \ldots, i_s \in \{1, \ldots, r\}$, there exists $j^*(1, \ldots, i_s)$ such that

$$B \bigcap X(i_1, i_2, \ldots, i_s, j^*(1, \ldots, i_s)) = \emptyset.$$

Let further $q_0 = 1$ and for $s = 1, \ldots, m+1$,

$$q_s = j^*(q_0, \ldots, q_{s-1}).$$

Then B has empty intersection with every member of the sequence $X(q_1)$, $X(q_1, q_2)$, $\ldots$, $X(q_1, q_2, \ldots, q_{m+1})$. But this means that B is disjoint from $A(q_1, q_2, \ldots, q_{m+1})$, a contradiction to the definition of $\mathcal{F}$. □

Completion of the proof of the theorem. Consider

$$Z := \bigcup_{s=1}^{m+1} \bigcup_{(1, i_1, \ldots, i_{s-1})} Z(1, i_1, \ldots, i_{s-1}).$$

Clearly, $|Z| \leq (1 + r + r^2 + \ldots + r^m)kr(r+1)/2 \leq kr^{m+2} = kr^{3\alpha+1}$.

Let $\mathcal{E} = \{A \in \mathcal{F} \mid A \cap Z \neq \emptyset\}$ and

$$\mathcal{H}(1, i_1, \ldots, i_s) = \{A \in \mathcal{F} \setminus \mathcal{E} \mid A \bigcap X(i_1, i_2, \ldots, i_s, j) \neq \emptyset \quad \forall j \in \{1, \ldots, r\}\}.$$

By Lemma 6, for each s, $0 \leq s \leq m$ and $i_1, \ldots, i_s \in \{1, \ldots, r\}$ the sets $X(i_1, i_2, \ldots, i_s, j) \setminus Z$ for distinct j are disjoint. Hence by Lemma 7, we can write $\mathcal{F}$ in the form

$$\mathcal{F} = \mathcal{E} \bigcup \left(\bigcup_{s=0}^{m} \bigcup_{(1, i_1, \ldots, i_s)} \mathcal{H}(1, i_1, \ldots, i_s) \right).$$

Let us estimate

$$|\mathcal{E}| \leq |Z|\varphi(n-1) \leq kr^{3\alpha+1}C(n-1)!\alpha^{1-n}.$$

Note that each $A \in \mathcal{H}(1, i_1, \dots, i_s)$ should intersect each of r disjoint sets $X(i_1, i_2, \dots, i_s, 1) \setminus Z, X(i_1, i_2, \dots, i_s, 2) \setminus Z, \dots, X(i_1, i_2, \dots, i_s, r) \setminus Z$. The cardinalities of these sets for $s < m$ are at most y and for $s = m$ are less than $2y$. Consequently by (3),

$$\begin{aligned} |\mathcal{H}(1, i_1, \dots, i_s)| &\leq (2y)^r \varphi(n-r) \\ &\leq \left(\frac{2n}{3\alpha}\right)^r C(n-r)!\alpha^{r-n} \leq \left(\frac{2+o(1)}{3}\right)^r Cn!\alpha^{-n}. \end{aligned}$$

Thus,

$$\begin{aligned} |\mathcal{F}| &\leq kr^{3\alpha+1}C(n-1)!\alpha^{1-n} + \frac{r^{m+1}-1}{r-1}\left(\frac{2+o(1)}{3}\right)^r Cn!\alpha^{-n} \\ &< Cn!\alpha^{-n}e^{-\alpha}\left(\frac{1.5(\log\log n)^{3\alpha+2}e^{\alpha}\alpha}{\log n} + \frac{(\log\log n)^{3\alpha}e^{\alpha}}{(1.5-o(1))^{\lfloor \log\log n \rfloor}}\right). \end{aligned}$$

But for large n the expression in big parentheses is less than one. So, the theorem is proved.

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Flag Algebras: An Interim Report

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To the memory of my mother,
Ludmila Alexeevna Razborova

Summary. For the most part, this article is a survey of concrete results in extremal combinatorics obtained with the method of flag algebras. But our survey is also preceded, interleaved and concluded with a few general digressions about the method itself. Also, instead of giving a plain and unannotated list of results, we try to divide our account into several connected stories that often include historical background, motivations and results obtained with the help of methods other than flag algebras.

A Foreword

When I was asked by the organizers to contribute something on flag algebras, I was a bit uncertain at first. The reasons will become clear from the text below, but a two-sentence summary is this. In just a few recent years we have witnessed a tremendous explosion of activity in this area, and the explosion is still ongoing. It does not look (at least to me) quite consistent with the inevitable stamp of finality a full-fledged survey is supposed to convey.

As a consequence, this contribution has a very clear flavor of an accounting book. I will try my best to summarize in Sect. 3, in a categorized and annotated form, *concrete* results in extremal combinatorics obtained with the method of flag algebras so far. Or, in other words, where do we stand now, in February of 2013.

That said, I still feel obliged to say at least a few general words about the method itself, and this is where we begin. This introductory part is rather loose and informal, and a disinterested reader may proceed directly to Sect. 2.

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1. The Method

The theory of flag algebras is supposed to treat in an entirely uniform way all classes of combinatorial structures $\mathcal{C}$ that possess the *hereditary property*: any subset of vertices of a structure from $\mathcal{C}$ gives rise to another ("induced") structure in $\mathcal{C}$. A precise definition at the appropriate level of generality is best given in logical terms [63, §2], but for the purposes of this text we can safely assume that $\mathcal{C}$ is the class of either ordinary simple graphs or r-uniform hypergraphs (*r-graphs*) or oriented graphs (*orgraphs*). In this section the specific choice of the class $\mathcal{C}$ is almost never important, and for simplicity we will use the word "graph" cumulatively.

The main quantity studied in the part of extremal combinatorics that is amenable to the method of flag algebras is the number $i(H,G)$ of induced copies (up to automorphisms of H) of a graph H in a larger graph G. One of the most basic paradigms underlying the theory of flag algebras tells us to normalize whenever possible so we immediately replace these numbers with the corresponding densities and let

$$p(H,G) \stackrel{\text{def}}{=} \binom{L}{\ell}^{-1} i(H,G) \ (L \stackrel{\text{def}}{=} |V(G)|,\ \ell \stackrel{\text{def}}{=} |V(H)|).$$

One useful interpretation is that $p(H,G)$ is the probability that a randomly chosen ℓ-subset of $V(G)$ induces a subgraph isomorphic to H [63, §2.1].

In many contexts, notably in the theory of graph limits, researchers are often interested in the number of all copies, not necessarily induced, and sometimes also other variants. It turns out, however, that all these variants are essentially equivalent; let us review some simple formulas connecting different versions (see [50, Chap. 5.2.2]) as we will occasionally need them below.

Let $\mathsf{ind}(H,G)$ be the number of induced embeddings $\alpha : V(H) \longrightarrow V(G)$, that is embeddings preserving both adjacency and non-adjacency. Denoting by

$$t_{\mathsf{ind}}(H,G) \stackrel{\text{def}}{=} \frac{\mathsf{ind}(H,G)}{L(L-1)\cdots(L-\ell+1)}$$

the corresponding density, we see that $\mathsf{ind}(H,G) = |\mathrm{Aut}(H)| \cdot i(H,G)$ and, hence,

$$t_{\mathsf{ind}}(H,G) = \frac{|\mathrm{Aut}(H)|}{\ell!} p(H,G).$$

$t_{\mathsf{inj}}(H,G)$ is defined similarly to $t_{\mathsf{ind}}(H,G)$ with the difference that now the embedding α need not necessarily be induced, i.e. it must respect adjacencies only. Clearly,

$$t_{\mathsf{inj}}(H,G) = \sum_{H'\supseteq H} t_{\mathsf{ind}}(H',G) = \frac{1}{\ell!}\sum_{H'\supseteq H} |\mathrm{Aut}(H')| p(H',G), \tag{1}$$

and the inverse formula is given by the Möbius transform (see [50, (5.20)]):

$$t_{\mathsf{ind}}(H,G) = \sum_{H'\supseteq H} (-1)^{|E(H')|-|E(H)|} t_{\mathsf{inj}}(H',G). \tag{2}$$

Two more variants, homomorphism density $t(H,G)$ [50] and strong homomorphism density [40, Sect. 2.3] are obtained from $t_{\mathsf{inj}}(H,G)$, $t_{\mathsf{ind}}(H,G)$, respectively, by dropping the requirement that the mapping α must be injective, followed by an obvious re-normalization. They are related to each other via formulas completely analogous to (1), (2). There is no neat formula, however, relating "injective" densities $p(H,G), t_{\mathsf{ind}}(H,G), t_{\mathsf{inj}}(H,G)$ with their non-injective versions: any such formula must necessarily involve the number of vertices L, which is grossly inconsistent with the philosophy of flag algebras. What is important, however, is that as $L \to \infty$, the difference between these two classes of measures becomes negligible (see e.g. [50, (5.21)]).

A significant part of extremal combinatorics studies arithmetic and Boolean relations existing between the densities $p(H_1,G),\ldots,p(H_h,G)$ (or sometimes their equivalent versions $t_{\mathsf{ind}}(H_i,G), t_{\mathsf{inj}}(H_i,G)$) where $H_1,\ldots,$ H_h are small fixed templates, and G is an unknown graph. Sometimes problems of interest (like the Caccetta–Häggkvist conjecture that we will discuss in Sect. 3.3) also involve concepts like minimal/maximal degree; these fit into our framework with very minimal changes.

And the *asymptotic* extremal combinatorics additionally assumes that the size of G is very large, and thus these relations are to be satisfied only in the limit. More precisely, in every increasing sequence $G_1, G_2, \ldots G_n, \ldots$ of graphs, we can by compactness choose a subsequence $G'_1,\ldots,G'_n,\ldots$ such that all h limits $\lim_{n\to\infty} p(H_\nu, G'_n)$ $(\nu \in [h])$ exist; denote them by $\phi(H_1),\ldots,\phi(H_h)$. The question is then re-phrased as follows: which properties should the tuple $(\phi(H_1),\ldots,\phi(H_h))$ satisfy?

The next observation is that by going to an infinite subsequence we can ensure that the limits $\phi(H) \stackrel{\text{def}}{=} \lim_{n\to\infty} p(H,G'_n)$ exist for *all* (countably many) graphs H, not only those we are actually interested in. This follows from Tychonoff's theorem on the compactness of products of compact sets (that in our particular case can be replaced by a simple diagonal argument). Such sequences are called *convergent*, and the function ϕ that maps isomorphism classes of finite graphs[1] is a paradigmical example of what in the theory of graph limits is called a simple graph parameter [50, Chap. 4.1].

[1] It is perhaps a good time to remind that in this section we use the word "graph" in a broader sense that also includes hypergraphs, orgraphs, etc.

Convergent sequences of graphs $\{G_i\}$ and associated graph parameters ϕ make the main object of study in both theories: graph limits and flag algebras. From this point, however, they diverge significantly: a logician might have said that the theory of graph limits is semantical in its nature while flag algebras strongly focus on syntax. Indeed, a very substantial part of the theory of graph limits deals with the question of what is the *actual* limit object for a converging sequence of graphs and with studying its properties. This limit object was successfully described by Lovász and B. Szegedy for ordinary graphs (graphons, see [50, Chap. 7]), by Elek and B. Szegedy for hypergraphs [50, Chap. 23.3], and it looks as if a sort of a description is possible even for directed graphs [50, Chap. 23.5].

The approach taken by flag algebras is on the contrary manifestly minimalistic, which is dictated by the utilitarian purpose of the theory. Semantics is substantially demoted as not being very useful for proving new concrete results; one immediate advantage of this is that the theory can be applied to arbitrary combinatorial structures without any changes at all. Instead, it focuses on developing *syntactic* tools for proving universal statements about the quantities $\phi(H_1),\ldots,\phi(H_h)$ using more or less formal manipulations. Careful attention is paid to notational uniformity, simplicity and transparency: this is particularly important since, as the experience shows, the method begins to bring real fruit dangerously close to the region where it becomes unfeasible for purely computational reasons, see the discussion in [34, Sect. 4.1]. Another characteristic feature of the method is its strong tendency to expose and exploit (usually simple) mathematical structure in an uniform way wherever it can be found. Besides obvious mathematical connections, this paradigm, somewhat surprisingly, has its own non-negligible utilitarian value. For example, it adds versatility to some existing packages for working with flag algebras that can be easily re-programmed to work with different types of combinatorial objects.

As we indicated at the beginning, this text is not intended to be an exposition of the method itself. Almost all necessary formalism can be found in the original paper [63, 67, Sect. 2.1.1] adds half a page of notation and definitions that are particularly useful when one has to work with several different types of combinatorial structures at once. An informal account can be found in [45, Sect. 7], and almost every paper with concrete results surveyed below also strives to explain its own version of the formalism in its own way. But in the next Sect. 3 we will use the distinction between "plain" (Cauchy–Schwarz) applications and those using more advanced concepts. So we conclude with a somewhat informal account of the fragment of the general theory that is necessary to understand this distinction. This part is similar to quantum graphs, graph algebras, reflection positivity, etc., studied in the context of graph limits [50, Part 2], but there are also important differences dictated, as almost everything else in flag algebras, by pragmatic purposes.

Let $\mathcal{M}$ be the set of all finite graphs up to an isomorphism, and $\mathbb{R}\mathcal{M}$ be the set of all their finite formal linear combinations with real coefficients. Then any graph parameter ϕ can be extended by linearity to a linear mapping $\mathbb{R}\mathcal{M} \to \mathbb{R}$ that we will also denote by the same letter ϕ. Graph parameters ϕ resulting from convergent sequences of graphs turn out to satisfy $\phi(f) = 0$ for certain elements $f \in \mathbb{R}\mathcal{M}$ expressing the most basic *chain rule* [63, Lemma 2.2]. Factoring out by these relations, we obtain a linear space that is denoted by $\mathcal{A}^0$ (the meaning of the superscript 0 will become clear soon). It turns out that for every pair H_1, H_2 of graphs, $\phi(H_1)\phi(H_2)$ can be always expressed as $\phi(f)$ for an easily computable element $f \in \mathcal{A}^0$ not depending on ϕ. This allows us to endow $\mathcal{A}^0$ with the structure of an associative commutative algebra [63, Lemma 2.4], and thus ϕ defines an algebra homomorphism from this algebra to the reals. It clearly satisfies $\phi(H) \geq 0$ for any graph H, and we let $\mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$ denote the set of all algebra homomorphisms with the latter property.

One extremely important fact is that at this point our search for "generic", "logical" relations satisfied by all graph parameters resulting from convergent sequences is over. Namely, the "completeness theorem" [50, Theorem 11.52], [63, Theorem 3.3] states that *every* element $\phi \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$ can be realized as a convergent sequence of graphs, and this allows us to focus on $\mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$ as an *axiomatic* description of our main object of study. Of course, even under this view, the intended semantical interpretation is still indispensable for intuition and is occasionally used in arguments (see e.g. [63, Theorem 4.3]).

The backbone of the theory is made by the real cone $\mathcal{C}^0_{\mathrm{sem}}$ consisting of all those f for which $\forall \phi \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})(\phi(f) \geq 0)$, and what we refer to as "plain" Cauchy–Schwarz applications is just a systematic way of finding "interesting" elements in this cone by semi-definite programming. More specifically, all notions reviewed so far readily generalize to the relative framework in which a prescribed number of vertices k spanning a prescribed graph σ are labeled in all objects under consideration and are always required to be preserved [63, §2.1]. σ itself is called a *type* [of size k], relativized graphs become *flags* [of type σ], and the relativized version $\mathcal{A}^\sigma$ is (finally!) called the *flag algebra*. For every $f \in \mathcal{A}^\sigma$ we clearly have $f^2 \in \mathcal{C}^\sigma_{\mathrm{sem}}$, and we also have a naturally defined *averaging* (or *label-erasing*) linear operator $[\![\cdot]\!]_\sigma : \mathcal{A}^\sigma \longrightarrow \mathcal{A}^0$ preserving the set of positive elements: $[\![\mathcal{C}^\sigma_{\mathrm{sem}}]\!]_\sigma \subseteq \mathcal{C}^0_{\mathrm{sem}}$ [63, Theorem 3.1]. This already provides us with a supply of non-trivial elements in $\mathcal{C}^0_{\mathrm{sem}}$ of the form $[\![f^2]\!]_\sigma$ ($f \in \mathcal{A}^\sigma$), and we can also take their linear combinations with non-negative coefficients. The resulting set is a quadratic sub-cone $\mathcal{C}^0 \subseteq \mathcal{C}^0_{\mathrm{sem}}$ defined by positive semi-definite constraints. And when the size of all flags involved is bounded by a constant ℓ (in a "typical" plain application of the method ℓ varies between 4 and 6), the corresponding SDPs become finitely defined, and, what is even more important can be handled by the existing solvers[2] sufficiently well to actually solve problems. This is what we will refer to as the "plain" method, and in what follows we will use the word "plain" in this rather technical and exact sense.

The structure that can be extracted from the objects $\mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$ is, however, much richer than that and includes other things like various algebra homomorphisms allowing us to move true statements around [63, §2.3], ensembles of random homomorphisms extending a given one [63, §3.2] or variational principles [63, §4.3]. We can not go into details here, but in Sect. 3 we will sometimes mention these structures by name whenever they are used in the arguments.

What are the relations between the cone $\mathcal{C}^0_{\mathrm{sem}}$ we are interested in and its approximation $\mathcal{C}^0$ corresponding to what we can *prove* using Cauchy–Schwarz arguments? Topologically, $\mathcal{C}^0$ is dense in $\mathcal{C}^0_{\mathrm{sem}}$, and one does not even have to use quadratic relations for that. Namely, it is a simple consequence of the completeness result [50, Theorem 11.52], [63, Theorem 3.3] that the linear subcone in $\mathcal{C}^0$ consisting of non-negative linear combinations of flags is already dense in $\mathcal{C}^0_{\mathrm{sem}}$.

In terms of *logical* complexity, however, the difference is huge. If we, for simplicity, focus on rational points in these cones, then the sub-cone $\mathcal{C}^0_\ell \subseteq \mathcal{C}^0$ consisting of all inequalities provable by using only ℓ-sized flags is decidable and hence $\mathcal{C}^0 = \bigcup_\ell \mathcal{C}^0_\ell$ is recursively enumerable. The fundamental result by H. Hatami and Norin [44] states that $\mathcal{C}^0_{\mathrm{sem}}$ is not r.e. already for ordinary simple graphs. Informally, this means that *every* proof system that will try to generate true statements in the asymptotic extremal combinatorics will necessarily be incomplete. Very recently, Lovett, H. Hatami, P. Hatami and Norin have extended this result to the theory of 2-colored graphs with distinguishable parts.

Finally, the theory of flag algebras has not appeared overnight out of nowhere, it had many predecessors. First of all, most constructions and arguments are modeled after their discrete counterparts that have been used in extremal combinatorics for many decades. Next, one should definitely mention the method of *Lagrangians* [55] that was perhaps the first successful usage of analytical methods in the area. Quasi-random graphs [16] are, in our language, devoted to the study of one specific and, arguably, the most natural element of $\mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$, and many central results and proofs there have a distinct syntactic flavor. Bondy [9] used what we would now call "Cauchy–Schwarz calculus" in the specific context of the Caccetta–Häggkvist problem.

[2] In my own work, I interchangeably use CSDP [10] and SDPA http://sdpa.sourceforge.net/, and my special thanks go to their developers.

2. Notation

We review the main definitions for the case of simple r-uniform hypergraphs (r-*graphs* in what follows), where $r \geq 2$ is a fixed number. We will also be occasionally considering oriented graphs,[3] but it is very straightforward how to adopt our definitions to that case.

2.1. Turán Densities

For two r-graphs F and G, G is F-*free* if it does not contain (not necessarily induced) subgraphs isomorphic to F. Given a family $\mathcal{F}$ of r-graphs, G is $\mathcal{F}$-*free* if it is F-free for every $F \in \mathcal{F}$. Let $\mathrm{ex}_H(n;\mathcal{F})$ be the maximal possible number of induced copies of an r-graph H in an $\mathcal{F}$-free r-graph on n vertices and

$$\pi_H(\mathcal{F}) \stackrel{\mathrm{def}}{=} \lim_{n\to\infty} \frac{\mathrm{ex}_H(n;\mathcal{F})}{\binom{n}{|V(H)|}}.$$

In the language of flag algebras, $\pi_H(\mathcal{F})$ is the maximal possible value of $\phi(H)$, where the maximum is taken over all $\phi \in \mathrm{Hom}^+(\mathcal{A}^0,\mathbb{R})$ for which $\phi(\hat{F}) = 0$ whenever $\hat{F}$ contains a spanning subgraph isomorphic to some $F \in \mathcal{F}$. We let

$$\pi(\mathcal{F}) \stackrel{\mathrm{def}}{=} \pi_{\{e\}}(\mathcal{F}),$$

where e is a single (hyper)edge. For better understanding the context of this survey, it is useful to recall that in the case of ordinary simple graphs the quantities $\pi(\mathcal{F})$ are completely described by the Erdős–Stone–Simonovits theorem [26, 27]:

$$\pi(\mathcal{F}) = 1 - \frac{1}{r-1}, \tag{3}$$

where $r \stackrel{\mathrm{def}}{=} \min\{\,\chi(G) \mid G \in \mathcal{F}\,\}$.

In order to cover more situations of interest, we define $\mathrm{ex}_{\min,H}(n;\mathcal{F})$, $\pi_{\min,H}(\mathcal{F})$ analogously to $\pi(\mathcal{F})$, but with the following two differences:

1. We are interested in the *minimal* possible number of induced copies of H;
2. r-graphs from $\mathcal{F}$ are forbidden only as *induced* subgraphs.

Again, when H is a single (hyper)edge, $\mathrm{ex}_{\min,H}(n;\mathcal{F}), \pi_{\min,H}(\mathcal{F})$ are abbreviated to $\mathrm{ex}_{\min}(n;\mathcal{F}), \pi_{\min}(\mathcal{F})$. Very recently, Norin (personal communication) was able to give a nice and complete description of $\pi_{\min}(\mathcal{F})$ for the case

[3] That is, directed graphs without loops, parallel or anti-parallel edges. By analogy with the abbreviation "digraph", in this survey oriented graphs will be often called *orgraphs*.

of ordinary graphs. More generally, given a finite set $\mathcal{F}$ of graphs he fully describes the set $D(\mathcal{F}) \subseteq [0,1]$ consisting of those $x \in [0,1]$ for which there exists $\phi \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$ with $\phi(F) = 0$ $(F \in \mathcal{F})$ and $\phi(e) = x$. The situation for 3-graphs is very different, and some related results will be thoroughly discussed in Sect. 3.4.

When $\mathcal{F} = \{F\}$ consists of a single graph, the quantities $\pi_H(F)$ and $\pi_{\min,H}(F)$ can be readily generalized to their relaxed versions when instead of forbidding copies of F entirely, we are interested in minimizing their number. For example, given $x \in [0,1]$, we let

$$g_H^F(x) \stackrel{\mathrm{def}}{=} \lim_{n\to\infty} \frac{\mathrm{ex}_{H,x}(n;\mathcal{F})}{\binom{n}{|V(F)|}}, \qquad (4)$$

where $\mathrm{ex}_{H,x}(n;\mathcal{F})$ is the minimal possible density of copies of F in an r-graph on n vertices, in which the density of (induced) copies of H is at least x. Thus, g_H^F is a non-decreasing function and $\pi_H(F)$ is the maximal x for which $g_H^F(x) = 0$.

2.2. Frequently Used [or] Graphs

K_ℓ is a clique on ℓ vertices, I_ℓ is an independent set on ℓ vertices, C_ℓ $[\mathbf{C}_\ell]$ is a non-oriented [oriented, respectively] cycle of length ℓ, and P_ℓ $[\mathbf{P}_\ell]$ is a non-oriented [oriented] path on ℓ vertices, i.e., of length $(\ell-1)$. $\mathbf{K}_{1,\ell}$ is the oriented star on $(\ell+1)$ vertices in which all edges are oriented from the center.

2.3. Frequently Used Hypergraphs

K_ℓ^r is a complete r-graph on ℓ vertices, and I_ℓ^r is an empty r-graph on ℓ vertices (thus, $K_\ell = K_\ell^2$ and $I_\ell = I_\ell^2$). J_k is the 3-graph on $(k+1)$ vertices consisting of all $\binom{k}{2}$ edges that contain a distinguished vertex v. G_ℓ is the uniquely defined 3-graph on 4 vertices with ℓ edges; thus, $G_4 = K_4^3$, and G_3 is often denoted by K_4^-. $\mathcal{C}_5$ is the 3-graph on 5 vertices with the edge set $\{(123), (234), (345), (451), (512)\}$. $F_{3,2}$ is the 3-graph, also on 5 vertices, with the edge set $\{(123), (145), (245), (345)\}$.

$\bar{F}$ is the edge-complement of a (hyper)graph F (on the same set of vertices). For a (hyper)graph F, $\lambda(F)$ is its *Lagrangian* defined as the maximal possible edge density of all weighted hypergraphs resulted from placing probability distributions on the vertices of F.

3. Results

In our survey of existing results, we are trying to group them into a few large groups centered either around a "big" problem or a reasonably broad topic. In all these cases the contribution made by flag algebras has been very substantial, but seldom it was exclusive. Therefore, we feel that our purpose will be served better if we give more coherent account by including, whenever appropriate, historical context, motivations, results proved by other methods, etc.

3.1. Clique Densities

In this section we consider only simple ordinary graphs, and we are interested in the functions $g_{K_p}^{K_r}$ (see (4)). The case $p = 2$ has received most attention, and we abbreviate

$$g_r(x) \stackrel{\text{def}}{=} g_{K_p}^{K_r}(x).$$

In words, $g_r(x)$ is the (asymptotically) minimal possible density of K_r in graphs with edge density $\geq x$.

The first general bound on $g_r(x)$ was proved by Goodman [35]:

$$g_3(x) \geq x(2x-1); \tag{5}$$

in the framework of flag algebras his proof amounts to a one-line calculation [63, Example 11]. This result was later rediscovered by Nordhaus and Stewart [58] who also conjectured that

$$g_3(x) \geq \frac{2}{3}(2x-1). \tag{6}$$

Goodman's bound (5) was extended to the case $r = 4$ by Moon and Moser [52] as follows:

$$g_4(x) \geq x(2x-1)(3x-2) \ (x \geq 2/3).$$

Following the pattern, they also stated without proof the natural generalization

$$g_r(x) \geq \prod_{i=1}^{r-1}(ix-(i-1)) \ \left(x \geq 1 - \frac{1}{r-1}\right) \tag{7}$$

for an arbitrary r; a complete proof was later provided in [48, 51].

Values of the form $x = 1 - \frac{1}{t}$ are called *critical*. These are precisely edge densities of complete balanced t-partite graphs, and at critical values the right-hand side of (7) computes the densities of K_r in these graphs. Thus, the bound (7) (and its partial case (5)) is tight at the critical points $1-\frac{1}{t}$; the question is what is happening between them.

The bound (7) is convex. Let $\psi_r(x)$ be the piecewise linear function that is linear in every interval $\left[1-\frac{1}{t}, 1-\frac{1}{t+1}\right]$ and coincides with g_r at its ends. Then, by convexity, $\psi_r(x) \geq g_r(x)$ (note that in the interval $[1/2, 2/3]$ the bound conjectured in (6) is precisely $\psi_3(x)$). More generally, let $\psi_r^p(x)$ be the piecewise linear function that is linear in every interval $\left[g_p\left(1-\frac{1}{t}\right), g_p\left(1-\frac{1}{t+1}\right)\right]$ and coincides with $g_r\left(1-\frac{1}{t}\right), g_r\left(1-\frac{1}{t+1}\right)$ at its ends.

In the beautiful paper [8], Bollobás proved that $\psi_r^p(x)$ still provides a lower bound on the function $g_{K_p}^{K_r}$:

$$g_{K_p}^{K_r}(x) \geq \psi_r^p(x). \tag{8}$$

A brief survey of these and related early developments can be found in [7].

We are not aware of any improvements on Bollobás's bound (8) for $p > 2$, which, in our opinion, makes an interesting open problem. The follow-up research concentrated on computing the functions $g_r(x)$.

As for upper bounds, let us consider a complete $(t+1)$-partite graph in which t parts are of the same size while the remaining part is smaller. Given $x \in \left[1-\frac{1}{t}, 1-\frac{1}{t+1}\right]$, there exists an asymptotically unique graph in this class with the edge density x. Computing the density of K_r in it leads to the following (somewhat ugly) upper bound on $g_r(x)$:

$$\left.\begin{array}{r} g_r(x) \leq \frac{(t-1)!}{(t-r+1)!(t(t+1))^{r-1}} \cdot \left(t-(r-1)\sqrt{t(t-x(t+1))}\right) \\ \cdot \left(t+\sqrt{t(t-x(t+1))}\right)^{r-1} \quad \left(x \in \left[1-\frac{1}{t}, 1-\frac{1}{t+1}\right]\right). \end{array}\right\} \tag{9}$$

This bound is *concave* in the interval $\left[1-\frac{1}{t}, 1-\frac{1}{t+1}\right]$.

I was not able to trace the origin of the conjecture that the bound (9) is actually tight, but in explicit form it appears already in the paper [51] by Lovász and Simonovits. The same paper proved the conjecture in some sub-intervals of the form $\left[1-\frac{1}{t}, 1-\frac{1}{t}+\epsilon_{r,t}\right]$, where $\epsilon_{r,t}$ is a (very small) constant. The next development occurred in 1989 when Fisher [30] proved[4] that (9) is tight for $r=3, t=2$.

And this is where flag algebras entered the stage. Firstly, Razborov [63, §5] independently re-discovered Fisher's result. More generally, a relatively

[4] Fisher's proof was incomplete as it implicitly used a fact about clique polynomials unknown at the time. This missing statement, however, was verified in 2000.

simple calculation [64, (3.6)] shows that if the bound (9) is tight for some t and $r = 4$, then it is also tight for the same value of t and $r = 3$. Fisher's result follows immediately since (9) is tight when $t = 2, r = 4$ (both sides are zero).

Then, using much more involved flag-algebraic constructs and calculations, Razborov [64] proved that the bound (9) is tight for $r = 3$ and an arbitrary t. In the classification scheme outlined in Sect. 1, this proof is certainly not plain, and in fact it barely uses Cauchy–Schwarz at all. Instead, it significantly employs more elaborated parts of the theory like ensembles of random homomorphisms or variational principles; we can not go into further details here.

While the next two papers do not directly use the language of flag algebras (see, however, the discussion at the conclusion of Sect. 1 in [69]), the proofs are still highly analytical. Nikiforov [57] proved that (9) is tight for $r = 4$ (and any t). And, finally, Reiher [69] established the same for arbitrary r, t thus completing the quest for computing the function g_r itself. Let me, however, remind here again that no progress on the relative values $g_{K_p}^{K_r}$ for $p > 2$ has apparently been made since Bollobás's seminal paper [8].

As for exact bounds, infinite blow-ups in general provide a powerful tool for converting asymptotic results into exact ones. In our context (we will discuss one more case in Sect. 3.5) this simple idea immediately implies the bound

$$\mathrm{ex}_{e,x}(n; K_r) \geq \frac{n^r}{r!} g_r\left(\frac{2m}{n^2}\right) \tag{10}$$

[64, Theorem 4.1]. Nikiforov [57, Theorem 1.3] showed that it is rather close to optimal.

Lovász and Simonovits [51] made several quite precise conjectures about the behavior of $\mathrm{ex}_{e,x}(n; K_r)$ and the corresponding extremal configurations, but these conjectures still remain unanswered. A partial progress toward them was made by Pikhurko and Razborov in [62]. Firstly, using a genuine flag-algebraic argument, they completely described the set $\Phi \subseteq \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$ of all asymptotically extremal configurations, i.e., those ϕ for which $\phi(K_3) = g_3(\phi(K_2))$. Then, by standard techniques, [62] proved stability, i.e., that actual extremal configurations are $o(n^2)$-close to the conjectured ones in the *edit* distance. These are precisely the first two steps of the program that we will discuss in the next Sect. 3.2. The third step, however (extracting an exact result from the stability version) is still missing. And for $r > 3$ nothing along these lines seems to be known at all.

In conclusion, let me note again that since [8] and [51] all improvements have been very analytical in their nature. Proving comparable results with entirely combinatorial techniques remains an unanswered challenge.

3.2. Turán's Tetrahedron Problem

In this section we switch gears and work with 3-graphs. The value $\pi(K_\ell^r)$ is unknown for *any* pair $\ell > r > 2$, this is the famous Turán problem. More information on its history and state of the art can be found in the recent comprehensive survey [45] (see also much older but still useful text [71]). In this section we concentrate on the simplest case $r = 3,\ \ell = 4$, with a brief digression to the next one, $r = 3,\ \ell = 5$. $\pi(K_4^3) = 1 - \pi_{\min}(I_4^3)$, and it will be convenient to us (partially for historical reasons) to switch to this dual notation. Turán [74] conjectured that $\pi_{\min}(I_4^3) = 4/9$, and this conjecture is sometimes called *Turán's (3,4)-problem* or *tetrahedron problem*. De Caen [14], Giraud (unpublished) and Chung and Lu [18] proved increasingly stronger lower bounds on $\pi_{\min}(I_4^3)$, the latter being of the form $\pi_{\min}(I_4^3) \geq \frac{9-\sqrt{17}}{12} \geq 0.406407$.

A plain (remember that we use this word in a technical sense) flag-algebraic calculation leads to the numerical bound

$$\pi_{\min}(I_4^3) \geq 0.438334 \tag{11}$$

[65] that was verified in [12] and later in [34] using the *flagmatic software* (we will discuss the latter in Sect. 4.1). The scale of this improvement reflects a general phenomenon: let me cautiously suggest that I am not aware of a *single* example of a *non-exact* bound in asymptotic extremal combinatorics that could not be improved by a plain application of flag algebras.

The remaining results in this section were distinctly motivated by the structure of known extremal configurations elaborated in a series of early papers by Turán [74], Brown [11], Kostochka [49] and Fon-der-Flaass [28], and we review it first. Our description (borrowed from [67]) has a rather distinct analytical flavor; for more combinatorial treatment see, e.g., [45, Sect. 7].

Let Γ be a (possibly infinite) orgraph without induced copies of $\mathbf{C}_4$. Let $FDF(\Gamma)$ be the 3-graph on $V(\Gamma)$ in which (u, v, w) spans an edge if and only if $\Gamma|_{\{u,v,w\}}$ contains either an isolated vertex (i.e., a vertex of both in-degree and out-degree 0) or contains a vertex of out-degree 2. Then $FDF(\Gamma)$ does not contain induced copies of I_4^3 [28].

Next, let $\Omega \stackrel{\text{def}}{=} \mathbb{Z}_3 \times \mathbb{R}$, and consider the (infinite) orgraph $\Gamma_K = (\Omega, E_K)$ given by

$$E_K \stackrel{\text{def}}{=} \{\, \langle (a, x), (b, y) \rangle \mid (x + y < 0 \wedge b = a + 1) \vee (x + y > 0 \wedge b = a - 1) \,\}\,.$$

Γ_K does not have induced copies of $\mathbf{C}_4$ and hence $FDF(\Gamma)$ does not contain induced copies of I_4^3. The set of known extremal configurations when the number of vertices is divisible by three is precisely the set of all induced subgraphs of this 3-graph that are of the form $FDF(\Gamma_K|_{\mathbb{Z}_3 \times S})$, where S is an arbitrary finite set of reals.

Turán's original configuration [74] corresponds to the case when $S \subseteq \mathbb{R}^+$. Brown's examples are obtained when negative entries in S are allowed, but are always smaller in absolute values than positive entries. Kostochka's examples [49] correspond to arbitrary finite S. And if we replace [the uniform measure on] S by a non-atomic measure on the real line, we will get a full description of all known $\phi \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$ with $\phi(I_4^3) = 0$ and $\phi(e) = 4/9$.

Turán's original example does not contain induced copies of G_3 which implies $\pi_{\min}(I_4^3, G_3) \leq \frac{4}{9}$. Razborov [65] proved that in fact

$$\pi_{\min}(I_4^3, G_3) = \frac{4}{9} \tag{12}$$

which also was the first application of the method in its genuinely plain form. Baber and Talbot [13, Theorem 25] gave a list of ten 3-graphs $\{H_1, \ldots, H_{10}\}$ on six vertices for which non-induced results of the same nature hold: $\pi(K_4^3, H_i) = \frac{5}{9}$ $(1 \leq i \leq 10)$; their proof method is also plain.

Pikhurko [59] proved that for a sufficiently large n, Turán's example is the *only* 3-graph on which $\mathrm{ex}_{\min}(n; I_4^3, G_3)$ is attained. This was also one of the first papers to demonstrate the three-step program for converting asymptotic flag-algebraic results into exact ones:

1. Describe the set of all extremal elements in $\mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$ (which in this particular case consists of a single element).
2. Prove stability, that is that the convergence in the pointwise topology described in Sect. 1 can be strengthened to convergence in the edit distance.
3. Move from stability to exact results using combinatorial techniques.

Let us now take a brief de-tour and discuss a couple results of similar nature inspired by the next case $r = 3$, $\ell = 5$ in Turán's problem. The situation with computing $\pi_{\min}(I_5^3)$ itself is very similar to $\pi_{\min}(I_4^3)$: Turán's conjecture says that $\pi_{\min}(I_5^3) = 1/4$, and there are many non-isomorphic configurations realizing this bound. The simplest of them given by Turán himself is the disjoint union $K_{n/2}^3 \dot\cup K_{n/2}^3$ of two cliques of the same size. Let H_1, H_2 be the two non-isomorphic 3-graphs on 5 vertices with precisely two edges. Then they are missing in Turán's example above, and Falgas–Ravry and Vaughan proved in [34] that

$$\pi_{\min}(I_5^3, H_1, H_2) = 1/4$$

which is analogous to (12). Their proof method is plain.

To review another remarkable result, it is convenient to switch to the dual notation. Turán's construction from the previous paragraph implies that $\pi(K_5^3) \geq 3/4$ and, more generally, $\pi(G) \geq 3/4$ for any 3-graph G that is not 2-colorable. In particular, this applies to critical (that is, edge minimal) 3-graphs on six vertices with chromatic number 3. There are precisely six such graphs; one of them being K_5^3 plus an isolated vertex (in other words, $\bar{J}_5$) and, obviously, $\pi(K_5^3) = \pi(\bar{J}_5)$. Quite remarkably, using flag algebras, Baber [1, Theorem 2.4.1] proved that $\pi(G) = 3/4$ for every one of the remaining five graphs on the list; his proof is plain.

We now return to the tetrahedron problem. Clearly, not all graphs without copies of I_4^3 can be realized in the form $FDF(\Gamma)$, and Fon-der-Flaass [28] asked whether Turán's conjecture can at least be proved for 3-graphs of his special form. He himself showed a lower bound of 3/7 (superseded by (11)). While the Fon-der-Flaass conjecture is still open, Razborov [67] verified it under either one of the two following assumptions:

1. Γ is an orientation of a complete t-partite graph (not necessarily balanced) for some t;
2. The edge density of Γ is $\geq 2/3 - \epsilon$ for some absolute constant ϵ.

Note that (2) settles a local version of the Fon-der-Flaass conjecture, that is proves it in an open neighborhood of the set $\Phi \subseteq \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$ of known extremal configurations. The proof method is a combination of plain and more sophisticated techniques heavily based upon working with several different kinds of combinatorial structures at once and frequently transferring auxiliary results from one context to another. The author expresses his hope that this kind of interaction (mostly human reasoning aided in appropriate places by the hammer-like power of plain flag-algebraic arguments) will become increasingly more popular in the area.

The result (12) is relevant only to the original extremal example given by Turán as all others contain plenty of induced copies of G_3. Razborov [68] identified three 3-graphs on 5 vertices given by their set of edges as follows:

$$E(H_1) \stackrel{\text{def}}{=} \{(123)(124)(134)(234)(125)(345)\}$$

$$E(H_2) \stackrel{\text{def}}{=} \{(123)(124)(134)(234)(135)(145)(235)(245)\}$$

$$E(H_3) \stackrel{\text{def}}{=} \{(123)(124)(134)(234)(125)(135)(145)(235)(245)\}$$

and proved that

$$\pi_{\min}(I_4^3, H_1, H_2, H_3) = 4/9. \qquad (13)$$

The motivation behind this result is that, as induced subgraphs, H_1, H_2, H_3 are missing in $FDF(\Gamma_K)$ and, thus, in all known extremal configurations. Flag algebras are used in this proof only "behind the scene", but the proof method itself deserves a few words here. Let us call a 3-graph H *singular* if its edge set is *not* a superset of $E(FDF(\Gamma))$ for any orgraph Γ which is an orientation of a complete t-partite graph (cf. the first result from [67] cited in item (1) on page 220) and does not contain induced copies of $\mathbf{C}_4$. Then [68] proved that

$$\hat{\pi}_H(H_1, H_2, H_3) = 0,$$

where H is an *arbitrary* singular 3-graph and $\hat{\pi}_H(\mathcal{F})$ is defined similarly to $\pi_H(\mathcal{F})$, with the difference that only induced copies of elements in $\mathcal{F}$ are forbidden. The proof uses Ramsey theory, and the main result (13) follows almost immediately from this and the first result in [67]. We are not aware of a similar "zero-inducibility" phenomenon that would not have held for some trivial reasons.

In conclusion, [67, 68] provide several results that verify Turán's conjecture for several natural classes containing the set Φ_{Turan} of all known extremal examples. None of them, however, covers an *open* neighborhood of Φ_{Turan}, and we believe that obtaining such a local result would have been a major step toward resolving the unrestricted version of the tetrahedron problem.

3.3. Caccetta–Häggkvist Conjecture

In this section we work with oriented graphs.

In 1970, Behzad, Chartrand and Wall [4] asked the following question: if G is a bi-regular orgraph on n vertices of girth $(\ell+1)$ (i.e., $\mathbf{C}_k$-free for any $k \leq \ell$), how large can be its degree? They conjectured that the answer is $\lfloor \frac{n-1}{\ell} \rfloor$ and presented a simple construction attaining this bound. Eight years later, Caccetta and Häggkvist [17] proposed to lift in this conjecture the restriction of bi-regularity and, moreover, restrict attention to minimal *outdegree* only. In other words, they asked if every orgraph without oriented cycles of length $\leq \ell$ must contain a vertex of out-degree $\leq \frac{n-1}{\ell}$, and it is this question that became known as the *Caccetta–Häggkvist conjecture*. Like in the previous Sect. 3.2 we concentrate on the case $\ell = 3$ even if some prominent work has been done for higher values of ℓ.

Let c be the minimal x for which every $\mathbf{C}_3$-free orgraph on n vertices contains a vertex of outdegree $\leq (c + o(1))n$; the Caccetta–Häggkvist conjecture then says[5] that $c = 1/3$. Caccetta and Häggkvist themselves proved the bound $c \leq \frac{3-\sqrt{5}}{2} \approx 0.382$ [17]. In the paper [9] that, as we acknowledged in Sect. 1, was one of the predecessors of flag algebras, Bondy proved that $c \leq \frac{2\sqrt{6}-3}{5} \leq 0.379$. His proof is essentially what we would call here a plain application of the method using $\mathbf{C}_3$-free orgraphs on 4 vertices (there are 32 of them). However, instead of actually solving the resulting SDP, Bondy gives a hand-manufactured (non-optimal) solution to it. Shen [70] improved this to $c \leq 0.3543$, and Hamburger, Haxell and Kostochka [39] proved a bound of $c \leq 0.3532$.

The current record of $c \leq 0.3465$ was established by Hladký, Král' and Norin in [43] using flag algebras. After incorporating an inductive argument previously used by Shen in [70], their proof method is mostly plain, but it also introduces one more novel and important feature. Namely, [43] utilizes a result by Chudnovsky, Seymour and Sullivan [19] on eliminating cycles in triangle-free digraphs that is only *somewhat* related to the Caccetta–Häggkvist conjecture, and adding that auxiliary result to the computational

[5] It is well-known that its asymptotic and exact versions are equivalent.

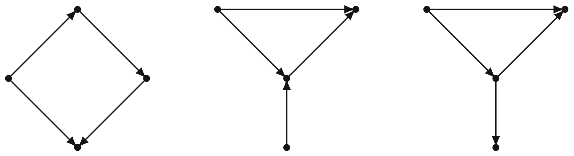

Fig. 1 Forbidden orgraphs.

brew is paramount for the improvement. Again, I would like to express my hope that in the future we will see more examples of interaction of this sort between different problems.

As for partial but exact results, Razborov [66] proved the Caccetta–Häggkvist conjecture under the additional assumption that the three orgraphs on Fig. 1 are missing as induced subgraphs. Like in the previous section, the point here is that these orgraphs are missing in all known extremal configurations; for the description of the latter see [9, Sect. 3] and [66, Sect. 2]. The proof is not plain and in fact does not use Cauchy–Schwarz at all. Moreover, all concrete calculations are so simple that the proof was presented in an entirely finite setting but using flag-algebraic notation.

3.4. Topics in Hypergraphs Motivated by the Erdős–Stone–Simonovits Theorem

From this point on, all flag-algebraic proofs we review are plain. Therefore, we will normally omit this qualification.

As we already noted in Sect. 2, the Erdős–Stone–Simonovits theorem (3) completely settles the question of computing $\pi(\mathcal{F})$ for finite families of *ordinary* graphs. But it also implies several interesting *structural* consequences pertaining to the behavior of this function. In this section we survey a few contributions to the hypergraph theory bound together by the general intention to understand how precisely badly this theorem fails for hypergraphs.

To start with, (3) implies that $\pi(\mathcal{F}) = \min_{F\in\mathcal{F}} \pi(F)$ (one direction is obvious), and by analogy with objects like principal ideals, etc. it is natural to say that for ordinary graphs the function $\pi(\mathcal{F})$ displays *principle behavior.* It is also natural to ask if this is true for hypergraphs, and, indeed, Mubayi and Rödl [54] conjectured that *non-principal families* $\mathcal{F}$ (i.e., those for which $\pi(\mathcal{F}) < \min_{F\in\mathcal{F}} \pi(F)$) exist already for 3-graphs. They further conjectured that they exist even with $|\mathcal{F}| = 2$.

The first question was answered in affirmative by Balogh [2], but, in his own words, "the cardinality of the set $\mathcal{F}$ is not immediately obvious". The second question was answered by Mubayi and Pikhurko [53] who showed that the pair (K_4^3, J_5) is not principal.

From the discussion in [54] it is sort of clear that the authors expect the pair $(G_3, \mathcal{C}_5)$ to be non-principal, and to that end they note the known inequality [29]

$$\pi(G_3) \geq \frac{2}{7}, \tag{14}$$

as well as prove new results $\pi(\mathcal{C}_5) \geq 0.464$ and $\pi(G_3, \mathcal{C}_5) \leq \frac{10}{31}$.

Using flag algebras, Razborov [65] improved the latter bound to

$$\pi(G_3, \mathcal{C}_5) \leq 0.2546 < \frac{2}{7}$$

thus proving that $(G_3, \mathcal{C}_5)$ is indeed a non-principal pair. Then Falgas–Ravry and Vaughan [34], also using flag algebras, proved that the pairs $(G_3, F_{3,2})$ and (K_4^3, J_4) are non-principal. The former example is remarkable since they were also able to compute

$$\pi(G_3, F_{3,2}) = \frac{5}{18},$$

and $\pi(F_{3,2}) = \frac{4}{9}$ had been known before [31] (for a several-line flag-algebraic proof of this result see [65, Theorem 5]). Nonetheless, $\pi(G_3)$ is still unknown, and it is interesting to note in this respect that we still do not know of any example of a non-principal family $\mathcal{F}$ for which we can *actually* compute all involved quantities $\pi(\mathcal{F})$ and $\pi(F)$ $(F \in \mathcal{F})$.

Another obvious consequence of the Erdős–Stone–Simonovits theorem is that for ordinary graphs, $\pi(\mathcal{F})$ is always rational. The book [15] mentions the conjecture, believed to be due to Erdős, that this will also be the case for r-graphs. This conjecture was disproved using flag algebras by Baber and Talbot [13] who gave a family of three 3-graphs $\mathcal{F}$ such that $\pi(\mathcal{F}) = \lambda(F_{3,2}) = \frac{189+15\sqrt{15}}{961}$. It was also independently disproved by Pikhurko using different methods [61], but his family $\mathcal{F}$ is huge.

Yet another consequence of the Erdős–Stone–Simonovits theorem is that in case of ordinary graphs, for any $\alpha \in [0,1)$ the density bound $\pi(\mathcal{F}) \leq \alpha$ can be forced by a finite family $\mathcal{F}$ such that all graphs $G \in \mathcal{F}$ have larger density $\geq \beta$ for some fixed $\beta > \alpha$. Moreover, the graphs $G \in \mathcal{F}$ can be made arbitrarily large, and (this is important!) β does not depend on $\min_{G\in\mathcal{F}} |V(G)|$. For example, if $\alpha \in [1/2, 2/3)$, then this property is witnessed by taking $\beta = 2/3$ and letting $\mathcal{F}$ consist of a single balanced complete tripartite graph. α is said to be a *jump* for an integer $r \geq 2$ if the analogous property holds for r-graphs.

Erdős [24] showed that for all r, every $\alpha \in \left[0, \frac{r!}{r^r}\right)$ is a jump and conjectured that, like in the case of ordinary graphs, every $\alpha \in [0,1)$ is a jump. And perhaps the most surprising fact about jumps is that this "jumping constant conjecture" is not true. The first examples of non-jumps were given by Frankl and Rödl in [32], and a number of other examples followed. All of them, however, live in the interval $\left[\frac{5r!}{2r^r}, 1\right)$, and what happens in between (i.e., for $\alpha \in \left[\frac{r!}{r^r}, \frac{5r!}{2r^r}\right)$) was a totally grey area.

Using flag algebras, Baber and Talbot [12] gave the first example of jumps in this intermediate interval by showing that all $\alpha \in [0.2299, 0.2316)$ are jumps for $r = 3$. Their proof also uses a previous characterization from [32] that allows to get rid of the condition that $G \in \mathcal{F}$ must be arbitrarily large by considering their Lagrangians instead. Given this reduction, Baber and Talbot produced a set $\mathcal{F}$ consisting of five 3-graphs on 6 vertices such that $\lambda(F) \geq 0.2316$ ($F \in \mathcal{F}$) while $\pi(\mathcal{F}) \leq 0.2299$. It is worth noting that whether $\alpha = 2/9$ is a jump for $r = 3$ (which was one of the questions asked in the original paper by Erdős) still remains open.

3.5. Induced *H*-Densities

So far we predominantly dealt with "normal" Turán densities $\pi(\mathcal{F})$, $\pi_{\min}(\mathcal{F})$, i.e. special cases of $\pi_H(\mathcal{F})$, $\pi_{\min,H}(\mathcal{F})$ when H is a (hyper)edge. In this section, on the contrary, we review a few results proven with the help of flag algebras in which the graph H is more complicated.

Triangle-free graphs need not be bipartite. But how exactly far from being bipartite can they be? In 1984, Erdős [25, Questions 1 and 2] considered three quantitative refinements of this question, and one of them was to determine $\mathrm{ex}_{C_5}(n, K_3)$. Györi [38] had a partial result in that direction.

The asymptotic version of Erdős's question was solved using flag algebras by H. Hatami et al. [40] and Grzesik [37] who independently proved that

$$\pi_{C_5}(K_3) = \frac{5!}{5^5}. \tag{15}$$

The standard trick with blow-ups (cf. (10)) immediately implies that

$$\mathrm{ex}_{C_5}(5\ell, K_3) = \ell^5. \tag{16}$$

Hatami et al. [40] also proved that the infinite blow-up of C_5 is the only element $\phi \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$ realizing equality in (15), and that the finite balanced blow-up of C_5 is the only graph realizing equality in (16). Remarkably, the proof of the latter result *bypasses the stability approach* outlined in Sect. 3.2. Namely, given a finite K_3-free graph G, instead of viewing this graph as a member of a converging sequence, [40] simply considers its infinite blow-up $\phi_G \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$ and *directly* applies to it the asymptotic uniqueness result.

When $n = 5\ell + a$ $(1 \le a \le 4)$, [40] also proved that $\mathrm{ex}_{C_5}(n, K_3) = \ell^{5-a}(\ell+1)^a$, the equality being attained at almost balanced blow-ups of C_5. But this proof already uses the traditional stability approach, and, as a consequence, works only for sufficiently large n.

Somewhat similar in spirit is another question asked by Erdős in [23]. The Ramsey theorem is equivalent to the statement that for any fixed $k, \ell > 0$, for all sufficiently large n we have $\mathrm{ex}_{\min, I_k}(n, K_\ell) > 0$. Erdős asked about the *quantitative* behavior of this function and conjectured that the minimum is attained for the balanced $(\ell-1)$-partite graph. Asymptotically, if we let

$$c_{k,\ell} \stackrel{\mathrm{def}}{=} \pi_{\min, I_k}(K_\ell),$$

Erdős's conjecture says that

$$c_{k,\ell} = (\ell - 1)^{1-k}. \tag{17}$$

This was disproved by Nikiforov [56] who observed that the blow-up of C_5 we just discussed in a different context actually implies that $c_{4,3} \le \frac{3}{25}$. Moreover, Nikiforov showed that Erdős's conjecture (17) can be true only for finitely many pairs (k, ℓ).

Using flag algebras, Das et al. [20] and Pikhurko [60] independently proved that $c_{3,4} = 1/9$ (thus confirming Erdős's conjecture in this case) and that $c_{4,3} = 3/25$ (thus showing that Nikiforov's counterexample is the worst possible). Both papers use the stability approach to get exact results for sufficiently large n. Das et al. [20, Sect. 6] states that their unverified calculations confirm Erdős's conjecture in two more cases: $c_{3,5} = 1/16$ and $c_{3,6} = 1/25$. Both these calculations were verified by Vaughan (referred to in [60]) who also confirmed Erdős's conjecture in one more case: $c_{3,7} = 1/36$. Along the other axis, Pikhurko [60] calculated $c_{5,3}, c_{6,3}$ and $c_{7,3}$.

Let us now discuss "pure" *inducibility* $i(H)$ of a graph/orgraph/hypergraph H that in our notation is simply equal to $\pi_H(\emptyset)$.

There is one self-complementary graph on 4 vertices, P_4 and five complementary pairs, which (since $\pi_H(\emptyset) = \pi_{\bar{H}}(\emptyset)$) give rise to six different problems of determining $\pi_H(\emptyset)$. One of them (K_4/I_4) is trivial, and two problems had been solved before with other methods.

Using flag algebras, Hirst [42] solved two more cases: he showed that

$$\pi_{K_4 - K_2}(\emptyset) = \frac{72}{125}$$

and that

$$\pi_{K_4 - P_3}(\emptyset) = \frac{3}{8}.$$

Thus, now P_4 is the only remaining graph on 4 vertices whose inducibility is still unknown.

Sperfeld [72] studied inducibility for *oriented* graphs. Using flag algebras, he showed that $\pi_{\mathbf{C}_3}(\emptyset) = \frac{1}{4}$ and obtained a few non-exact results improving on previous bounds: $\pi_{\mathbf{P}_3}(\emptyset) \leq 0.4446$ (the conjectured value is 2/5), $\pi_{\mathbf{C}_4}(\emptyset) \leq 0.1104$ and $\pi_{\mathbf{K}_{1,2}}(\emptyset) \leq 0.4644$. Then Falgas–Ravry and Vaughan [33] were able to actually compute the latter quantity:

$$\pi_{\mathbf{K}_{1,2}}(\emptyset) = 2\sqrt{3} - 3.$$

They also computed $\pi_{\mathbf{K}_{1,3}}(\emptyset)$ that turned out to be a rational function in a root of a cubic polynomial.

In the department of 3-graphs, the same paper [33] calculated the inducibility of G_2:

$$\pi_{G_2}(\emptyset) = \frac{3}{4}.$$

Slightly stretching our notation, let $\pi_{m.k}(\emptyset)$ be the minimal induced density of the *collection* of all 3-graphs on m vertices with exactly k edges (thus e.g. $\pi_{G_2}(\emptyset) = \pi_{4.2}(\emptyset)$). Falgas–Ravry and Vaughan also proved in [33] that

$$\pi_{5.1}(\emptyset) = \pi_{5.9}(\emptyset) = \frac{5}{8}$$

and

$$\pi_{5.k}(\emptyset) = \frac{20}{27}\ (3 \leq k \leq 7).$$

3.6. Miscellaneous Results

A graph H is *common* if the sum of the number of its copies (not necessarily induced) in a graph G and the number of such copies in the complement of G is asymptotically minimized by taking G to be a random graph. Erdős [23] conjectured that all complete graphs are common, and this conjecture was disproved by Thomason [73] who showed that for $p \geq 4$, K_p is not common. It is now known that common graphs are very rare, and several authors specifically asked if the wheel W_5 shown on Fig. 2 is common.

This question becomes amenable to the (manifestly induced) framework of flag algebras by using the transformation (1). And, indeed, H. Hatami et al. [41] proved that W_5 is common. This is the only result in our survey where optimization takes place over a rather complicated linear combination of "primary" induced densities rather than individual densities.

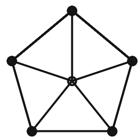

Fig. 2 The 5-wheel.

Erdős et al. asked in [22] a question that later became known as the *(2/3)-conjecture*. Given a 3-coloring of the edges of K_n, what is the smallest t such that there exists a color c and a set A of t vertices whose c-neighborhood has density at least 2/3? The conjecture says that $t = 3$, the previously known bound was $t = 22$, and using flag algebras, Král' et al. proved in [46] that one can actually take $t = 4$.

In [47], Král', Mach and Sereni looked at the following geometric problem resulted from the work by Boros and Füredi [5] and Bárány [3]. What is the minimal constant c_d such that for every set P of n points in $\mathbb{R}^d$ in general position there exists a point of $\mathbb{R}^d$ contained in at least $c_d\binom{n}{d+1}$ d-simplices with vertices at the points of P. As stated, it is not amenable to the approach of flag algebras, but Gromov [36] was able to find a topological approach to it, and its later expositions (see [47] for details) brought it rather close to that realm. One remaining concepts that still can not be handled by flag algebras in full generality is that of *Seidel minimality* as it quantifies over arbitrary sets of vertices. Král', Mach and Sereni, however, showed that by applying this property only to certain sets definable in this language they can improve known bounds on c_d.

In the rest of this section we review a few more results about 3-graphs obtained with the method of flag algebras that were not addressed in our previous account. This activity started with the *Mubayi challenge* when all exact results presented to the author by Dhruv Mubayi found their new flag-algebraic proofs in [65, Sect. 6.2] of varying and surprisingly unpredictable computational difficulty. Razborov [65] also gave a few non-exact results, of which we would like to mention here only $\pi(G_3) \leq 0.2978$ later improved by Baber and Talbot [12] to $\pi(G_3) \leq 0.2871$, which is already quite close to the conjectured value 2/7 (see (14)).

Baber and Talbot [13] go over "critical" densities 2/9, 4/9, 5/9, 3/4 (recall that $\pi(F_{3,2}) = 4/9$ and 5/9, 3/4 are conjectured values for $\pi(K_4^3), \pi(K_5^3)$, respectively). For every α from this set they were able to construct one (for $\alpha = 2/9$) or more (for $\alpha \in \{4/9, 5/9, 3/4\}$) 3-graphs F with $\pi(F) = \alpha$.

Falgas–Ravry and Vaughan [34] proved, besides the results we already cited above in various contexts, several more exact results:

$$\pi(G_3, \mathcal{C}_5, F_{3,2}) = \frac{12}{49},$$

$$\pi(G_3, F_{3,2}) = \frac{5}{18},$$

$$\pi(J_4, F_{3,2}) = \frac{3}{8}.$$

In the second paper [33] of the same authors they prove (again, in addition to what we already surveyed before) several more inducibility results:

$$\pi_{G_3}(K_4^3) = \frac{16}{27}, \quad \pi_{G_3}(F_{3,2}) = \frac{27}{64}, \quad \pi_{K_4^3}(F_{3,2}) = \frac{3}{32}$$

$$\pi_{G_2}(\mathcal{C}_5, F_{3,2}) = \frac{9}{16}, \quad \pi_{G_2}(G_3, F_{3,2}) = \frac{5}{9}, \quad \pi_{G_2}(G_3, \mathcal{C}_5, F_{3,2}) = \frac{4}{9}.$$

Two forthcoming papers study *codegree density* $\pi_2(F)$ for 3-graphs (see [45, Sect. 13.2] for definitions). Falgas–Ravry, Marchant, Pikhurko and Vaughan give a new flag-algebraic proof of the result $\pi_2(F_{3,2}) = 2/3$ from Marchant's thesis. In the second paper, Falgas–Ravry, Pikhurko and Vaughan prove that $\pi_2(G_3) = 1/4$.

4. Concluding Remarks

4.1. Flagmatic Software

In the first few years since the inception of the method, researchers who needed it for their work had to write the code on their own, the only thing that was available from the shelf were SDP-solvers like CSDP [10] or SDPA. It appears as if these homemade pieces of software greatly differ in the level of their public availability, user-friendliness and, most important, versatility. Like in many other similar scenarios, it can be expected that this period of anarchy will eventually be over, and the separation between users and developers (with the clear understanding that these two groups are likely to overlap significantly) will be defined more clearly. This will also likely imply that the many ad hoc programs around will give way to one or a few "standard" packages, and researchers that are new to the method will largely lose the initiative to program on their own.

One very serious bid to become such a "golden standard" has been made by the *Flagmatic software* developed by Vaughan and, in fact, many results that we surveyed above were obtained using this program. It is publicly available from http://www.flagmatic.org, and (from everything I know) it is user-friendly. Versatility is also improving: while this project started with

3-graphs, the last version 2 also has support for ordinary graphs and oriented graphs. Time will tell if Flagmatic gets a serious competitor, but at the moment this seems to be the only option for a researcher who needs to use the method on a reasonably recurrent basis but does not want to invest time into writing his/her own code.

4.2. Beyond Turán Densities?

Turán densities for dense graphs is by far not the only area in discrete mathematics and beyond where Cauchy–Schwarz and positive semi-definite programming are used extensively. Thus, it is natural to wonder if formal methods similar to flag algebras can be applied elsewhere. In cases we potentially have in mind it is more or less clear how to come up with a mathematically beautiful calculus that "works in theory". But our question is more pragmatic: can it be done in such a way that it will *actually* allow to prove *new concrete* results in the area in question. See [34, Sect. 4.1] for a very relevant discussion of the complexity barrier that (as we believe) prevents us from getting many more, and possibly very great, results with this method even on its home field, asymptotic extremal combinatorics.

We are aware of two moderate but concrete and successful steps in that direction. Baber [1, Chap. 2.5] (some of these results were later independently rediscovered by Balogh et al. in [6]) extends the method to Turán densities for subgraphs of the hypercube Q_n. The latter is a rather sparse graph, so significant modifications are necessary. And Norin and Zwols (personal communication) started considering applications of the flag algebra framework to the study of crossing numbers, particularly of the complete bipartite graph $K_{n,n}$. They already were able to get a numerical improvement on the previously best known bound from [21].

One more paper that might be mentioned here is the work by Král', Mach and Sereni [47] on the Boros–Füredi–Bárány problem that we already discussed in Sect. 3.6. But their approach is sort of opposite: they "massage" the problem they are interested in until it fits nicely the framework of flag algebras as originally defined in [63].

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Arrangeability and Clique Subdivisions

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Summary. Let k be an integer. A graph G is k*-arrangeable* (concept introduced by Chen and Schelp) if the vertices of G can be numbered $v_1, v_2, \ldots, v_n$ in such a way that for every integer i with $1 \le i \le n$, at most k vertices among $\{v_1, v_2, \ldots, v_i\}$ have a neighbor $v \in \{v_{i+1}, v_{i+2}, \ldots, v_n\}$ that is adjacent to v_i. We prove that for every integer $p \ge 1$, if a graph G is not $2500(p+1)^8$-arrangeable, then it contains a K_p-subdivision. By a result of Chen and Schelp this implies that graphs with no K_p-subdivision have "linearly bounded Ramsey numbers," and by a result of Kierstead and Trotter it implies that such graphs have bounded "game chromatic number."

The Theorem

In this paper graphs are finite, may have parallel edges, but may not have loops. We begin by defining the concept of admissibility, introduced by Kierstead and Trotter [8].

Let G be a graph, let $M \subseteq V(G)$, and let $v \in M$. A set $A \subseteq V(G)$ is called an M*-blade* with center v if either

(i) $A = \{a\}$ and $a \in M$ is adjacent to v, or
(ii) $A = \{a, b\}$, $a \in M$, $b \in V(G) - M$, and b is adjacent to both v and a.

An M*-fan with center* v is a set of pairwise disjoint M-blades with center v. Let k be an integer. A graph G is k*-admissible* if the vertices of G can be numbered $v_1, v_2, \ldots, v_n$ in such a way that for every $i = 1, 2, \ldots, n$, G has no $\{v_1, v_2, \ldots, v_i\}$-fan with center v_i of size $k+1$.

As pointed out in [8] the concepts of arrangeability and admissibility are asymptotically equivalent in the sense that if a graph is k-arrangeable, then it is $2k$-admissible, and if it is k-admissible, then it is (k^2-k+1)-arrangeable.

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Let p be an integer. A graph G has a K_p-subdivision if G contains p distinct vertices $v_1, v_2, \ldots, v_p$ and $\binom{p}{2}$ paths P_{ij} $(i, j = 1, 2, \ldots, p,\ i < j)$ such that P_{ij} has ends v_i and v_j, and if a vertex of G belongs to both P_{ij} and $P_{i'j'}$ for $(i, j) \neq (i', j')$, then it is an end of both. The following is our main result.

Theorem 1. *Let $p \geq 1$ be an integer. If a graph G is not $50p^2(p^2+1)$-admissible, then it has a K_p-subdivision.*

We first prove Theorem 1, and then discuss its applications. For the proof we need the following result, originally proved with a larger constant by Bollobás and Thomason [3], and independently by Komlós and Szemerédi [9], who proved it with a smaller constant, but only for sufficiently large p. Our version follows from a result of Thomas and Wollan [11] and can be found in [7, Theorem 7.2.1].

Theorem 2. *Let $p \geq 1$ be an integer. If a simple graph on n vertices has at least $5p^2n$ edges, then it has a K_p-subdivision.*

We first prove a lemma.

Lemma 1. *Let $p \geq 1$ be an integer, let G be a graph, and let M be a non-empty subset of $V(G)$. If for every $v \in M$ there is an M-fan in G with center v of size $50p^2(p^2+1)$, then G has a K_p-subdivision.*

Proof. Let p, G, and M be as stated in the lemma, and for $v \in M$ let F_v be a fan in G with center v of size $50p^2(p^2+1)$. We may assume that G is minimal subject to $M \subseteq V(G)$ and the existence of all $F_v (v \in M)$. Let $|M| = m$, let e_1 be the number of edges of G with both ends in M, and let e_2 be the number of edges of G with one end in M and the other in $V(G) - M$. Then from the existence of the fans F_v for $v \in M$ we deduce that $2e_1 + e_2 \geq 50p^2(p^2+1)m$.

We claim that if $|V(G) - M| \geq 10p^2m - e_1$, then G has a K_p-subdivision. Indeed, by our minimality assumption for every $w \in V(G) - M$ there exist vertices u, $v \in M$ such that $\{u, w\} \in F_v$. For $w \in V(G) - M$ let us denote by $e(w)$ some such pair of vertices. Let J be the graph obtained from G by deleting $V(G) - M$ and for every $w \in V(G) - M$ adding an edge between the vertices in $e(w)$. Then $|E(J)| \geq 10p^2m$, and since every pair of vertices is joined by at most two (parallel) edges, J has a simple subgraph J' on the same vertex-set with at least $5p^2m$ edges. By Theorem 2 J' has a K_p-subdivision L. Every edge of L that does not belong to G joins two vertices u, v with $\{u, v\} = e(w)$ for some $w \in V(G) - M$. By replacing each such edge by the edges uw, vw we obtain a K_p-subdivision in G. This proves our claim, and so we may assume that $|V(G) - M| \leq 10p^2m - e_1$.

Now $|V(G)| \leq (10p^2+1)m - e_1$, and

$$|E(G)| \geq e_1 + e_2 = 2e_1 + e_2 - e_1 \geq 50p^2(p^2+1)m - e_1$$
$$\geq 5p^2((10p^2+1)m - e_1) \geq 5p^2|V(G)|,$$

and hence G has a K_p-subdivision by Theorem 2, as required. □

Proof of Theorem 1. Let p be an integer, and let G be a graph on n vertices with no K_p-subdivision. We are going to show that G is $50p^2(p^2+1)$-admissible by exhibiting a suitable ordering of $V(G)$. Let $i \in \{0, 1, \dots, n\}$ be the least integer such that there exist vertices $v_{i+1}, v_{i+2}, \dots, v_n$ with the property that for all $j = i, i+1, \dots, n$, G has no $(V(G) - \{v_{j+1}, v_{j+2}, \dots, v_n\})$-fan with center v_j of size $50p^2(p^2+1)$. We claim that $i = 0$. Indeed, otherwise by Lemma 1 applied to $M = V(G) - \{v_{i+1}, v_{i+2}, \dots, v_n\}$ there exists a vertex v_i with no M-fan with center v_i of size $50p^2(p^2+1)$, and so the sequence v_i, $v_{i+1}, \dots, v_n$ contradicts the choice of i. Hence $i = 0$, and $v_1, v_2, \dots, v_n$ is the desired enumeration of the vertices of G. □

Applications

We now mention two applications of Theorem 1. Let $\mathcal{G}$ be a class of graphs. We say that $\mathcal{G}$ has *linearly bounded Ramsey numbers* if there exists a constant c such that if $G \in \mathcal{G}$ has n vertices, then for every graph H on at least cn vertices, either H or its complement contain a subgraph isomorphic to G. The class of all graphs does not have linearly bounded Ramsey numbers, but some classes do. Burr and Erdős [4] conjectured the following.

Conjecture 1. *Let t be an integer, and let $\mathcal{G}$ be the class of all graphs whose edge-sets can be partitioned into t forests. Then $\mathcal{G}$ has linearly bounded Ramsey numbers.*

Chvátal, Rödl, Szemerédi and Trotter [6] proved that for every integer d, the class of graphs of maximum degree at most d has linearly bounded Ramsey numbers, and Chen and Schelp [5] extended that to the class of k-arrangeable graphs. Chen and Schelp also showed that every planar graph has arrangeability at most 761, a bound that has been subsequently lowered to 10 by Kierstead and Trotter [8]. From Chen and Schelp's result and Theorem 1 we deduce

Corollary 1. *For every integer $p \geq 1$, the class of graphs with no K_p-subdivision has linearly bounded Ramsey numbers.*

For the second application we need to introduce the following two-person game, first considered by Bodlaender [2]. Let G be a graph, and let t be an integer, both fixed in advance. The game is played by two players Alice and Bob. Alice is trying to color the graph, and Bob is trying to prevent that from happening. They alternate turns with Alice having the first move. A move consists of selecting a previously uncolored vertex v and assigning it a color from $\{1, 2, \dots, t\}$ distinct from the colors assigned previously (by either player) to neighbors of v. If after $|V(G)|$ moves the graph is (properly) colored, Alice wins, otherwise Bob wins. More precisely, Bob wins if after less than $|V(G)|$ steps either player cannot make his or her next move. The *game chromatic number* of a graph G is the least integer t such that Alice has a winning strategy in the above game. Kierstead and Trotter [8] have shown the following.

Theorem 3. *Let k and t be positive integers. If a k-admissible graph has chromatic number t, then its game chromatic number is at most $kt + 1$.*

They have also shown that planar graphs have admissibility at most 8, and hence planar graphs have game chromatic number at most 33 by Theorem 3 and the Four Color Theorem [1]. From Theorems 1 to 3 we deduce

Corollary 2. *Let p be a positive integer. Then every graph with no K_p-subdivision has game chromatic number at most $500p^4(p^2 + 1) + 1$.*

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A Finite Partition Theorem with Double Exponential Bound

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Dedicated to Paul Erdős

Summary. We prove that double exponentiation is an upper bound to Ramsey's theorem for colouring of pairs when we want to predetermine the order of the differences of successive members of the homogeneous set.

The following problem was raised by Jouko Vaananen for model theoretic reasons (having a natural example of the difference between two kinds of quantifiers, actually his question was a specific case), and propagated by Joel Spencer: Is there for any n, c an m such that

($*$) For every colouring f of the pairs from $\{0, 1, \ldots, m-1\}$ by 2 (or even c) colours, there is a monochromatic subset $\{a_0, \ldots, a_{n-1}\}$, $a_0 < a_1 < \cdots$ such that the sequence $\langle a_{i+1} - a_i : i < n-1 \rangle$ is with no repetition and is with any pregiven order.

Noga Alon [1] and independently Janos Pach proved that for every n, c there is such an m as in ($*$); Alon used van der Waerden numbers (see [2]) (so obtained weak bounds).

Later Alon improved it to iterated exponential (Alan Stacey and also the author have later and independently obtained a similar improvement). We get a double exponential bound. The proof continues [4]. Within the "realm" of double exponential in c, n we do not try to save.

We thank Joel Spencer for telling us the problem, and Martin Goldstern for help in proofreading.

Notation. Let ℓ, k, m, n, c, d belong to the set $\mathbb{N}$ of natural numbers (which include zero). A sequence η is $\langle \eta(0), \ldots, \eta(\ell g n - 1) \rangle$, also ρ, ν are sequences. $\eta \lhd v$ means that η is a proper initial subsequence of v. We consider sequences as graphs of functions (with domain of the form $n = \{0, \ldots, n-1\}$) but $\eta \cap v$ means the largest initial segment common to η and v. $\eta^\frown \langle s \rangle$ is the sequence $\langle \eta(0), \ldots, \eta(\ell g \eta - 1), s \rangle$ (of length $\ell g(\eta) + 1$).

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Let ${}^{\ell}m := \{\eta : \ell g(\eta) = \ell, \text{ and } \mathrm{Rang}(\eta) \subseteq \{0,\ldots,m-1\}\}$, ${}^{\ell>}m = \bigcup_{k<\ell} {}^{k}m$.
For $v \in {}^{\ell>}m$ we write $[v]_{\ell_m}$ (or $[v]$ if ℓ and m are clear from the context) for the set $\{\eta \in {}^{\ell}m : v \lhd \eta\}$.

$[A]^n = \{w \subseteq A : |w| = n\}$. Intervals $[a,b)$, (a,b), $[a,b)$ are the usual intervals of integers. The proof is similar to [4] but sets are replaced by trees.

1. Definition

$r = r(n,c)$ is the first number m such that:

$(*)^{n,c}_{m}$ for every $f : [\{0,\ldots,m-1\}]^2 \to \{0,\ldots,c-1\}$ and linear order $<^*$ on $\{0,\ldots,n-2\}$ we can find $a_0 < \ldots < a_{n-1} \in [0,m)$ such that:

(a) $f|\{a_i, a_j\} : 0 \le i < j < n\}$ is constant.
(b) The numbers $b_\ell := a_{\ell+1} - a_\ell$ (for $\ell < n-1$) are with no repetitions and are ordered by $<^*$, i.e. $i <^* j \Rightarrow b_i < b_j$.

We will find a double exponential bound for $r = r(n,c)$, specifically, $r = r(n,c) \le 2^{(c(n+1)^3)^{nc}}$ (so our bound is double exponential in n and in c).

This is done in Sect. 6. Alon conjectures that the true order of magnitude of $r(n,c)$ is single exponential, and Alon and Spencer have proved this for the case where the sequence $\langle a_{i+1} - a_i : i < n-1\rangle$ is monotone.

2. Definition

We say S is an (ℓ, m^*, m, u)-tree if:

(a) $u \subseteq \{0,1,\ldots,\ell-1\}$ and $m^* \ge m$
(b) $S \subseteq^{\ell\geq} (m^*)$
(c) S is closed under initial segments
(d) If $v \in S$ is $\lhd$-maximal then $\ell g(v) = \ell$
(e) If $v \in S$ and $\ell g(v) \in u$ then for m numbers $j < m^*$ we have $v^\frown\langle j\rangle \in S$

3. Claim

Suppose $k, \ell, m, p, m^* \in \mathbb{N}$ satisfy

$$(*)_{k,\ell,m,p,m^*} \qquad m^* \ge p^k m^{\ell k+1},$$

then for every $i(*) < \ell$, and $u \subseteq [0,\ell-1)$, and $|u| \le p$ and $f : [{}^{\ell}(m^*)]^2 \to \{0,1\}$ there is $T \subseteq^{\ell\geq} (m^*)$ closed under initial segments, $(T,\lhd) \cong ({}^{\ell\geq}m, \lhd)$ satisfying

$\bigoplus$ for every $\eta_1, \dots, \eta_k \in T \cap^{\ell} (m^*)$ with $\langle \eta_1 \restriction i(*), \dots, \eta_k \restriction i(*) \rangle$ pairwise distinct we can find a set S such that:

(a) S is an $(\ell, m^*, m, u\ i(*))$-tree
(b) If $j \in [1, k)$ and $\nu \in S \cap^{\ell} (m^*)$ then $f(\{\eta_j, \eta_k\}) = f(\{\eta_j, \nu\})$
(c) $\nu \in S \cap^{\ell} (m^*) \Rightarrow \eta_k \restriction i(*) \triangleleft \nu$

Remark 1. *(1) With minor change we can demand in $\bigoplus$ "for any $i(*) < \ell$".*
(2) We could use here f with range $\{0, \dots, c-1\}$, and in claim 3 get a longer sequence $\langle \nu_\ell : \ell < n^* \rangle$ such that $f(\{v_{\ell_1}, v_{\ell_2}\})$ depends just on $v_{\ell_1} \cap v_{\ell_2}$, then use a partition theorem on such colouring.

Proof. For each $\eta \in^{\ell >} (m^*)$ choose randomly a set $A_\eta \subseteq [0, m^*), |A_\eta| = m, A_\eta = \{x_1^\eta, \dots, x_m^n\}$ (pairwise distinct, chosen by order) (not all are relevant, some can be fixed).

We define $T = \{\eta : \eta \in^{\ell \geq} (m^*)$, and $i < lg(\eta) \Rightarrow \eta(i) \in A_{n \restriction i}\}$.

We have a natural isomorphism h from ${}^{\ell \geq} m$ onto T:

$$h(\nu) = \eta \Leftrightarrow \bigwedge_{i < \ell g \nu} \eta(i) = x_{\nu(i)}^{\eta | i}.$$

Our problem is to verify $\bigoplus$, we prove that the probability that it fails is < 1, this suffices.

We can represent it as:

$(*)_1$ if $\nu_1, \dots, \nu_k \in {}^{\ell} m$ and $\nu_1 \restriction i(*), \dots, \nu_k \restriction i(*)$ distinct, then for $h(\nu_1), \dots, h(\nu_k)$ there is S as required there.

So it suffices to prove that for any given such $i(*) < \ell$ and $\nu_1, \dots, \nu_k$ the probability of failure is $< \frac{1}{\binom{|^{\ell} m|}{k}} = \frac{1}{\binom{m^\ell}{k}}$ as it suffices the demand in $(*)$, to hold for the minimal suitable $i(*)$. Without loss of generality we may assume $u = u \setminus i(*)$. For this we can assume x_j^ρ are fixed whenever $\neg[h(\nu_k) \restriction i(*) \trianglelefteq \rho]$ or $\ell g(\rho) \notin u$.

Let $Y = \{\eta \in {}^{\ell}(m^*) : \text{Prob}[h(\nu_k) = \eta] \neq 0\}$, so $|Y| = m^{|u|}$.

So $h(\nu_1), \dots, h(\nu_{k-1})$ are determined. Now $h(\nu_1), \dots, h(\nu_{k-1})$ and f induces an equivalence relation E on Y:

$$\eta' E \eta'' \text{ iff } \bigwedge_{j=1}^{k-1} f(\{h(\nu_j), \eta'\}) = f(\{h(\nu_j), \eta''\}).$$

The number of classes is $\leq 2^{k-1}$, let them be $A_1, \dots, A_{2k-1}$ (they are pairwise disjoint, some may be empty).

We call A_j large if there is S as required in clauses (a) and (c) of $\bigoplus$ such that $(\forall \rho)[\rho \in S \cap {}^{\ell}(m^*) \Rightarrow \rho \in A_j]$.

It is enough to show that the probability of $h(v_k)$ belonging to a non-large equivalence class is $< \frac{1}{\binom{m^\ell}{k}}$, hence it is enough to prove:

$(*)_2$ A_j not large $\Rightarrow \text{Prob}(h(\nu_k) \in A_j) < \frac{1}{\binom{m^\ell}{k} \times 2^k}$.

So assume A_j is not large. Let $Y^* := \{\eta \restriction i : \eta \in Y \text{ and } i \leq \ell\}$.

$$\text{Let } Z_j := \left\{ \begin{array}{ll} \eta \in Y^* : \text{there is } S \subseteq Y^* \text{ satisfying} & \\ \qquad (a)' & S \text{ is an } (\ell, m^*, m, u \setminus \ell g(\eta)) - \text{tree} \\ \qquad (b)' & \nu \in S \cap {}^\ell(m^*) \Rightarrow \eta \trianglelefteq \nu \\ \qquad (c)' & \text{for every } \nu \in S \cap {}^\ell(m^*) \text{ we have } \nu \in A_j \end{array} \right\}.$$

Let $Z_j^* := \{\eta \in Z_j : \text{ there is no } \eta' \triangleleft \eta, \eta' \in Z_j\}$.

Clearly $h(\nu_k \restriction i(*)) \notin Z_j$ (as A_j is not large) hence $h(\nu_k \restriction i(*)) \notin Z_j^*$.

Clearly

$(*)_3$ for $\eta \in Y^* \setminus Z_j$ such that $\ell g(\eta) \in u$ we have

$$|\{i : \eta^\frown\langle i\rangle \in Z_j^* \text{ (or even } \in Z_j)\}| < m.$$

But if $\nu \triangleleft \eta$, $\nu \in Z_j$ then $\eta \notin Z_j^*$. Hence

$(*)_4$ for $\eta \in Y^*$ such that $\ell g(\eta) \in u$ we have

$$|\{i : \eta^\frown\langle i\rangle \in Z_j^*\}| < m.$$

Now

$(*)_5$ if $\eta \in A_j$ (hence $\eta \in Y$; remember that A_j is not large) <u>then</u>, $\eta \in Z_j$, hence $\bigvee_{j \in u} \eta \restriction (j+1) \in Z_j^*$.

So

$$\begin{aligned} \text{Prob}(h(\nu_k) \in A_j) &\leq \text{Prob}\Big(\bigvee_{j \in u} [h(\nu_k \restriction (j+1)) \in Z_j^*]\Big) \\ &\leq \sum_{j \in u} \text{Prob}(h(\nu_k \restriction (j+1) \in Z_j^*)) \\ &< |u| \times \frac{m}{m^*} \end{aligned}$$

(first inequality by $(*)_5$, second inequality trivial, last inequality by $(*)_4$ above). So it suffices to show:

$$|u| \times \frac{m}{m^*} \leq \frac{1}{\binom{m^\ell}{k} \times 2^k}$$

equivalently

$$m^* \geq |u| \times m \times \binom{m^\ell}{k} \times 2^k$$

as $\binom{m^\ell}{k} \leq m^{\ell k}/k!$, and $|u| \leq p$ by the hypothesis $(*)_{k,\ell,m,p,m^*}$ we finish. $\square$

4. Lemma

Assume

(a) $\rho_1, \ldots, \rho_n \in^{n-1} 2$ *are distinct, for* $\ell \in \{2, \ldots, n\}$*, we have* $r_\ell \in \{1, \ldots, \ell - 1\}$ *such that* $\ell g(\rho_\ell \cap \rho_{r_\ell}) = \ell - 1$ *and* $r \in \{1, \ldots, \ell-1\} \setminus \{r_\ell\} \Rightarrow \ell g(\rho_\ell \cap \rho_r) < \ell - 1$
(b) $f : [{}^\ell m]^2 \to [0, c) = \{0, \ldots, c - l\}$
(c) $m = 2^{(n+1)^{(c+1)n}}$.

<u>Then</u> we can find $\eta_1, \ldots, \eta_n, \in {}^\ell m$ such that:

(α) $f \restriction [\{\eta_1, \ldots, \eta_n\}]^2$ is a constant function
(β) $\langle \ell g(\eta_{i+1} \cap \eta_{r_{i+1}}) : i = 1, \ldots, n-1\} \rangle$ is a sequence with no repetitions ordered just like $\langle \ell g(\rho_{i+1} \cap \rho_{r_{i+1}}) : i = 1, n-1\} \rangle$; also:

$$\eta_{i+1}(\ell g(\eta_{i+1} \cap \eta_{r_{i+1}})) < \eta_{r_{i+1}}(\ell g(\eta_i \cap \eta_{r_{i+1}}))$$
$$\Leftrightarrow \rho_{i+1}(\ell g(\rho_{i+1} \cap \rho_{r_{i+1}})) < \rho_{r_{i+1}}(\ell g(\rho g(\rho_{i+1} \cap \rho_{r_{i+1}})).$$

Remark 1. *(1) Note that if* $\Gamma \subseteq^{n-1} 2$*,* $|\Gamma| = n$ *and the set* $\{\rho_1 \cap \rho_2 : \rho_1 \neq \rho_2$ *are from* $\Gamma\}$ *has no two distinct members with the same length then we can list* Γ *as* $\langle \rho_1, \ldots, \rho_n \rangle$ *as required in clause (a) of Lemma 4.*
(2) So if $<^*$ *is a linear order on* $\{1, \ldots, n-1\}$ *then we can find distinct* $\rho_1, \ldots, \rho_n \in^{n-1} 2$ *as in clause (a) of Lemma 4 and a permutation* σ *of* $\{1, \ldots, n\}$ *such that: for* $i \neq j \in \{1, \ldots, n-2\}$ *we have*

$$i <^* j \quad \text{iff } \ell g(\rho_{\sigma(i)} \cap \rho_{\sigma(i+1)}) > \ell g(\rho_{\sigma(j)} \cap \rho_{\sigma(j+1)}).$$

(E.g. use induction on n.)

Proof. Let us define $(m_j : 2 \leq j \leq cn)$ by induction on j:

$$m_2 = n^c, \qquad m_{j+1} = n^c m_j^{n^{c(n+1)+1}}.$$

Check that $m_j \leq 2^{(n+1)^{(c+1)j}}$, so in particular $m_{(cn)} \leq m$. Now we claim that for any number $d \in [1, c]$ the following holds:

$\bigotimes_d$ Assume $q \in [0, n^c)$ such that q divisible by n^d and $q + n^d \leq cn$, $u = [q, q + n^d)$, T^* is an $(\ell, m, m_{(q+dn)}, u)$-tree and f is a function from $[T^* \cap {}^\ell m]^2$ with range of cardinality d. <u>Then</u> we can find $\eta_1, \ldots, \eta_n \in T^* \cap ({}^\ell m)$ such that clauses (α) and (β) of the conclusion of Lemma 4 hold.

This suffices: use $q = 0$, $d = c$. We prove this by induction on d. If $d = 1$, trivial as only one colour occurs. For $d + 1 > 1$, without loss of generality $\text{Rang}(f) = [0, d]$, let $f' : [{}^\ell m]^2 \to \{0, 1\}$ be $f'(\{\eta', \eta''\}) = \text{Min}\{f(\{\eta', \eta''\}), 1\}$. Let for $j < n$, $u_j := [q + n^d j, q + n^d j + n^d)$. By downward induction on $j \in [1, n]$ we try to define T_j such that:

(i) T_j is an $(\ell, m, m_{(q+nd+j)}, \bigcup_{i<j} u_i)$-tree

(ii) $T_j \subseteq T_{j+1}$

(iii) For every $j \in [1, n-1]$ and $\eta_1, \ldots, \eta_j \in T_j \cap {}^{\ell}m$ with $\langle \eta_1 \restriction (q + n^d j), \ldots, \eta_j \restriction (q+n^d j)\rangle$ pairwise distinct, we can find $\eta' \neq \eta'' \in T_{j+1} \cap {}^{\ell}m$ such that:

$$\begin{gathered}\eta_j \restriction (q + n^d j) \triangleleft \eta', \eta'' \\ \ell g(\eta' \cap \eta'') \in u_j \\ f'(\{\eta', \eta''\}) = 0\end{gathered}$$

$$\bigwedge_{t \in [1,j)} \big[0 = f'(\{\eta_t, \eta_j\}) \Rightarrow 0 = f'(\{\eta_t, \eta'\}) = f'(\{\eta_t, \eta''\})\big].$$

<u>This suffices</u> as then we can choose by induction on $j = 1, \ldots, n$ a sequence $v_i \in T_j \cap ({}^{\ell}m)$ such that (after reordering) the set $\{v_1, \ldots, v_n\}$ will serve as $\{\eta_1, \ldots, \eta_n\}$ of $\bigotimes_d$ (with the constant colour being zero). Let us do it in detail.

By induction on $j = 1, \ldots, n$ we choose $v_1^j, \ldots, v_{j-1}^j$ such that:

(a) $v_1^j, \ldots, v_j^j$ are distinct members of $T_j \cap ({}^{\ell}m)$

(b) $f \restriction [\{v_1^j, \ldots, v_j^j\}]^2$ is constantly zero

(c) For $\ell = \{2, \ldots, j\}$ we have $\ell g(v_\ell^j \cap v_{r_\ell}^j) \in u_{r\ell} - 1$

For $j = 1$ no problem. In the induction step, i.e. for $j+1$, we apply the condition (iii) above with $(v_\ell^j : \ell \in [1,j], \ell \neq q_{j+1}\rangle^\frown\langle v_{q_{j+1}}^j\rangle$ here standing for $\eta_1, \ldots, \eta_j$ there (we want $v_{q_{j+1}}^j$ be the last), the condition $\langle \eta_\ell \restriction (q + n^d j) : \ell = 1, \ldots, j\rangle$ with no repetition follows by clause (c), so we get η', $\eta'' \in T_{j+1} \cap ({}^{\ell}m)$ as there. W.l.o.g. $\eta'(\ell g(\eta' \cap \eta'')) < \eta''(\ell g(\eta' \cap \eta''))$.

We now define v_ℓ^{j+1} for $\ell = 1, \ldots, j+1$:

If $\ell \in \{1, \ldots, j+1\} \setminus \{j+1, r_{j+1}\}$ then $v_\ell^{j+1} = v_\ell^j$ (remember $T_j \subseteq T_{j+1}$).

If $\rho_{j+1}(\ell g(\rho_{j+1} \cap \rho_{r_{j+1}})) < \rho_{r_{j+1}}(\ell g(\rho_{j+1} \cap \rho_{r_{j+1}}))$ then $v_{j+1}^{j+1} = \eta'$ and $v_{r_{j+1}}^{j+1} = \eta''$.

If $\rho_{r_{j+1}}(\ell g(\rho_{j+1} \cap \rho_{r_{j+1}})) < \rho_{j+1}(\ell g(\rho_{j+1} \cap \rho_{r_{j+1}}))$ then $v_{r_{j+1}}^{j+1} = \eta''$ and $v_{r_{j+1}}^{j+1} = \eta''$.

Now check. (Note T_0, u_0 could be omitted above.)

Carrying the Inductive Definition. For $j = n$, trivial : let $T_n = T^*$ (given in $\bigotimes_d$).

For $j \in [2, n)$, where $T_{j+1}, \ldots, T_n$ are already defined, we apply Claim 2 with

$$m_{q+nd+j+1},\ \ m_{q+nd+j},\ \ n^d(j+1),\ \ j,\ \ q+n^d j,\ \ [q+n^d j, q+n^d j+n^d),\ \ n^d j$$

here standing for

$$m^*, \qquad m, \qquad \ell, \qquad k, \qquad i(*), \qquad u, \qquad p$$

there. (I.e. the tree in Claim 2 is replaced by one isomorphic to it, levels outside $\bigcup_{i<j} u_i$ can be ignored.)

So we need to check $m_{q+nd+j+1} \geq n^d j (m_{q+nd+j})^{n^d(j+1)j+1}$, which holds by the definition of the m_i's (as $d \leq c-1$). But we require above more than in Claim 2 (preferring the colour 0). But if it fails for T_j then for some $\eta_1, \ldots, \eta_j$ in T_j we have S as in $\oplus$ of Claim 2, with no $\eta^1 \neq \eta^2 \in S \cap {}^{\ell}m$ such that $f'(\{\eta', \eta''\}) = 0$. On S we can apply our induction hypothesis on d—allowed as the original f misses a colour (the colour zero) when restricted to S. $\square_3$

5. Fact

Let $\langle \eta_i : i < (2m-1)^{\ell} \rangle$ enumerate ${}^{\ell}(2m-1)$ in lexicographic order. Let

$$A = \{0, 2, \ldots, 2m-2\} \subseteq [0, 2m-1), \text{so } |A| = m,$$
$$B := \{i < (2m-1)^{\ell} : \eta_i \in A\}.$$

Let

$$low_{m,\ell}(k) = (2m-1)^{\ell-k-1}, \text{ and}$$
$$high_{m,\ell}(k) = (2m-1)^{\ell-k} - 1.$$

Then:

(0) If $i \neq j$ are in B, $|\eta_i \cap \eta_j| = k$ then:

$$i < j \text{ iff } \eta_i(k) < \eta_j(k).$$

(1) If $i < j$ are in B, $|\eta_i \cap \eta_j| = k$, then:

$$low_{m,\ell}(k) \leq j - i \leq high_{m,\ell}(k)$$

(2) If $i_1 < j_1$, $i_2 < j_2$ are all in B, and $|\eta_{i_1} \cap \eta_{j_1}| \neq |\eta_{i_2} \cap \eta_{j_2}|$, then:

$$j_1 - i_1 < j_2 - i_2 \quad \text{iff} \quad |\eta_{i_1} \cap \eta_{j_1}| > |\eta_{i_2} \cap \eta_{j_2}|.$$

Proof. (0) Check.

(1) Let $\nu := \eta_i \cap \eta_j$ and $k := |\nu|$. We are looking for upper and lower bounds of the cardinality of the set (the order is lexicographic)

$$C := \{\eta \in {}^{\ell}(2m-1) : \eta_i \leq \eta < \eta_j\}.$$

Clearly each $\eta \in C$ satisfies $\nu \lhd \eta$. Moreover, since $\eta_j \in {}^{\ell}m$, each element $\eta \in C$ must satisfy $\eta(k) \leq \eta_j(k) < 2m-1$. Hence

$$C \subseteq \bigcup_{s<2m-1} [\nu^\frown\langle s \rangle]_{({}^{\ell}(2m-1))} \setminus \{\eta_j\}$$

so we get $|C| \leq (2m-1) \cdot (2m-1)^{\ell-k-1} - 1 = (2m-1)^{\ell-k} - 1$.

For $k = \ell-1$ the lower bound claimed in (1) is trivial, so assume $k \leq \ell-2$. Let $\nu' := (\eta_i \restriction k)^\frown\langle \eta_i(k)+1 \rangle$ (note: as $\eta_i(k) < \eta_j(k)$ are both in A, necessarily $\nu'(k) < \eta_j(k)$). Then we have

$$C \supseteq \bigcup_{s<2m-1} [\nu'{}^\frown\langle s\rangle]_{({}^{\ell}(2m-1))} \cup \{\eta_i\},$$

so $|C| \geq (2m-1)^{\ell-k-1}+1$.

Proof of (2): Check that $high_{m,\ell}(k+1) < low_{m,\ell}(k)$, and use (1). $\square_4$

Remark. Also $low_{m,\ell}(k) = (2m-1)^{\ell-k-1}+1$ is O.K. but with the present bound we can use only η_i with $\eta_i(\ell-1) < m$, $B = \{i : \eta_i \restriction (\ell-1) \in^{\ell} A\}$. So $(2m-1)^{\ell}$ can be replaced by $m(2m-1)^{\ell-1}$.

6. Conclusion

If $f : [0,(2m-1)^{\ell})^{[2]} \to [0,c)$, $\ell := nc$, $m := 2^{(n+1)^{(c+1)n}}$, then we can find $a_0 < \ldots < a_{n-1} < m$ such that $f \restriction [\{a_1,\ldots,a_{n-1}\}]^2$ constant and $\langle a_{i+1}-a_i : i < n-1\rangle$ in any pregiven order

Proof. As in Fact 5, let $\langle \eta_i : i < (2m-1)^{\ell}\rangle$ enumerate ${}^{\ell}(2m-1)$ in lexicographic order, and let $\bar{B} := \{i < (2m-1)^{\ell} : \eta_i \in {}^{\ell}m\}$ and for $i \in \bar{B}$ define η'_i by $\eta'_i(j) = 2\cdot\eta_i(j)$ for $j < \ell$. So η'_i is in the set B from Fact 5. Define a function $f' : [{}^{\ell}m]^2 \to c$ by requiring $f'(\{\eta'_i,\eta'_j\}) = f(\{i,j\})$ for all i, $j \in B$. Now the conclusion follows from Lemma 4., Remark 1(2) and Fact 5., particularly clause (2). $\square_5$

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Paul Erdős' Influence on Extremal Graph Theory

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Dedicated to Paul Erdős on the occasion of his 80th birthday,
rewritten in 2013, on the occasion of 100th aniversary of Paul Erdős birth

Preface

This paper is a somewhat revised version of an old survey paper of mine [197]. Several minor corrections, new remarks, and a new section of REFERENCES are added. The later one includes references to some newer survey papers and some newer results. ... Yet, this paper does not try to cover the new developments: that would be too much.[1]

Wherever I could I tried to keep the original text. Thus, e.g., mostly I ignored that Erdős died 3 years after the original paper had been finished: I kept the "Present Time" even in places where today this may seem strange.

Remark [N] 1. Mostly, but not always, the added parts have the form of distinguished remarks, like this one. Further, I have added many references at the end of most sections.

As to the references, I have to mention here that the Erdős papers up to 1989 are scanned in and are available at [45]
http://www.renyi.hu/~p_erdos

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[1] Including or leaving out some newer results does not necessarily reflect a value judgement!

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Summary. Paul Erdős is 80[2] and the mathematical community is celebrating him in various ways. Jarik Nešetřil also organized a small conference in Prague in his honour, where we, combinatorists and number theorists attempted to describe in a limited time the enourmous influence Paul Erdős made on the mathematics of our surrounding (including our mathematics as well). Based on my lecture given there, I shall survey those parts of Extremal Graph Theory that are connected most directly with Paul Erdős's work.

In *Turán type extremal problems* we usually have some sample graphs $L_1, \dots, L_r$, and consider a graph G_n on n vertices not containing any L_i. We ask for the maximum number of edges such a G_n can have. We may ask similar questions for hypergraphs, multigraphs and digraphs.

We may also ask, how many copies of forbidden subgraphs L_i must a graph G_n contain with a given number of edges superseding the maximum in the corresponding extremal graph problems. These are the problems on *Supersaturated Graphs*.

We can mix these questions with Ramsey type problems, (Ramsey-Turán Theory). This topic is the subject of a survey by Simonovits and V. T. Sós (Discrete Math 229:293–340, 2001).

These topics are definitely among the favourite areas in Paul Erdős's graph theory.

Keywords Graphs, Extremal graphs, Graph theory

1. Introduction

Extremal graph theory is a wide and fast developing area of graph theory. Having many ramifications, this area can be defined in a broader and in a more restricted sense. In this survey we shall restrict our considerations primarily to **"Turán Type Extremal Graph Problems"** and some closely related areas.

Extremal graph theory is one of the wider theories of graph theory and – in some sense – one of those where Paul Erdős's profound influence can really be seen and appreciated.

Remark [N] 2. Recently, several new schools emerged in this topic, and also, mostly in the last 10–15 years, a completely new approach emerged, which,

(a) On the one hand, connects discrete methods with continuous ones,
(b) On the other hand, generalizes the "classical notion" of Extremal Graph Problems and often tries to use a "Calculus"; further the new schools occasionally use computers in solving extremal problems.

Here the reader is refered to Lovász' book [193] or to [295], or to Lovász' homepage, and also to Razborov's works [311], or [271, 272].

[2] This refers to the time of the conference, not to when this volume appears.

Actually, there are two distinct theories here: the theory of dense and the theory of sparse graphs (see e.g. [193, 248, 249]) but we skip the sparse case.

What Is a Turán Type Extremal Problem?

We shall call the *Theory of Turán type extremal problems* the area which – though being much wider – still is originated from problems of the following type:

> Given a family $\mathcal{L}$ of sample graphs, what is the maximum number of edges a graph G_n can have without containing subgraphs from $\mathcal{L}$.

Here "subgraph" means "not necessarily induced". In Sect. 11 we shall also deal with the case of "excluded induced subgraphs", as described by Prömel and Steger.

Below K_t, C_t and P_t will denote the complete graph, the cycle and the path of t vertices and $e(G)$ will be the number of edges of a graph G. G_n will be a graph of n vertices, $G(X,Y)$ a bipartite graph with colour classes X and Y.

The first result in our field may be that of Mantel [132] back in 1907, asserting that if a graph G_n contains no K_3, then

$$e(G_n) \leq \left[\frac{n^2}{4}\right].$$

Mantel's result soon became forgotten. The next extremal problem was the problem of C_4 (Erdős [46]).

The C_4–Theorem and Number Theory

In 1938 Erdős published a paper [46]

P. Erdős: On sequences of integers no one of which divides the product of two others and related problems, Mitt. Forsch. Institut Mat. und Mech. Tomsk 2 (1938) 74–82.

In this paper Erdős investigated two problems:

> (A) Assume that $n_1 < \cdots < n_k$ are positive integers such that n_i does not divide $n_h n_\ell$, except if either $i = h$ or $i = \ell$. What is the maximum number of such integers in $[2, n]$? Denote this maximum by $A(n)$.

Let $\pi(n)$ denote the number of primes in $[2, n]$. Clearly, the primes in $[2, n]$ satisfy our condition, therefore $A(n) \geq \pi(n)$. One could think that one can find much larger sets of numbers satisfying this condition. Surprisingly enough, the contrary is true: Erdős has proved that $A(n) \approx \pi(n)$. More precisely,

$$\pi(n) + \frac{n^{2/3}}{80\,\log^2 n} \le A(n) \le \pi(n) + O\left(\frac{n^{2/3}}{\log^2 n}\right).$$

For us the other problem of [46] is more important:

> (B) Assume that $n_1 < \cdots < n_k$ are positive integers such that $n_i n_j \neq n_h n_\ell$ unless $\{i,j\} = \{h,\ell\}$. What is the maximum number of such integers in $[1,n]$? Denote this maximum by $B(n)$.

Again, the primes of $[2,n]$ satify this condition. Here Erdős proved that

$$\pi(n) + \frac{cn^{3/4}}{(\log\, n)^{3/2}} \le B(n) \le \pi(n) + O\left(n^{3/4}\right).$$

Later Erdős improved the upper bound to

$$B(n) \le \pi(n) + O\left(\frac{n^{3/4}}{\log^{3/2} n}\right),$$

see [58]. It is still open if, for some $c \neq 0$,

$$B(n) = \pi(n) + (1+o(1))\frac{cn^{3/4}}{(\log\, n)^{3/2}}$$

or not.

Solving this unusual type of number theoretical problem, Erdős (probably first) applied Graph Theory to Number Theory. He did the following: Let $\mathcal{D}$ be the set of integers in $[1, n^{2/3}]$, $\mathbb{P}$ be the set of primes in $(n^{2/3}, n]$ and $\mathcal{B} = \mathcal{D} \cup \mathbb{P}$.

Lemma (Erdős). *Every integer $a \in [1,n]$ can be written as*

$$a = bd, \qquad \text{where} \qquad b \in \mathcal{B},\ d \in \mathcal{D}.$$

Let $\mathcal{A}$ be a set satisfying the condition in (B). Let us represent each $a \in \mathcal{A}$ as described in the Lemma: $a_i = b_{j(i)} d_{h(i)}$. We may assume that $b_{j(i)} > d_{h(i)}$. Build a bipartite graph $G(\mathcal{B},\mathcal{D})$ by joining b_j to d_h if $a = b_j d_h \in \mathcal{A}$. Thus we represent each $a \in \mathcal{A}$ by an edge of a bipartite graph $G(\mathcal{B},\mathcal{D})$. Erdős observed that

> the number theoretic condition in (B) implies that $C_4 \not\subseteq G(\mathcal{B},\mathcal{D})$.

Indeed, if we had a 4–cycle $(b_1 d_1 b_2 d_2)$ in $G(\mathcal{B},\mathcal{D})$, then $a_1 = b_1 d_1$, $a_2 = d_1 b_2$, $a_3 = b_2 d_2$ and $a_4 = d_2 b_1$ all would belong to $\mathcal{A}$ and $a_1 a_3 = a_2 a_4$ would hold, contradicting our assumption. So the graph problem Erdős formulated was the following:

> Given a bipartite graph $G(X,Y)$ with m and n vertices in its colour classes. What is the maximum number of edges such a graph can have without containing a C_4?

Erdős proved the following theorem:

Theorem 1.1. *If $C_4 \not\subseteq G(X,Y)$, $|X| = |Y| = k$, then*

$$e(G(X,Y)) \le 3k^{3/2}.$$

Here the constant 3 is not sharp (see Sect. 4). Basically this theorem implied the upper bound on $B(n)$. To get the lower bound Erdős used finite geometries. Erdős writes:

> ... Now we prove that the error term cannot be better than $O\left(\frac{cn^{3/4}}{(\log n)^{3/2}}\right)$. First I prove the following lemma communicated to me by Miss E. Klein.[3]
>
> **Lemma (Eszter Klein).** *Given $p(p+1)+1$ elements, (for some prime p) we can construct $p(p+1)+1$ combinations, taken $(p+1)$ at a time*[4] *having no two elements in common.*

Clearly, this is a finite geometry, and this seems to be the first application of Finite Geometric Constructions in proving lower bounds in Extremal Graph Theory. Yet, Erdős does not speak here of finite geometries, neither of lower bounds for the maximum of $e(G(X,Y))$ in Theorem 1.1.

In the last years[5] Erdős, András Sárközy and V. T. Sós started applying similar methods in similar number theoretic problems, which, again, led to new extremal graph problems, [83]. I mention just one of them:

(C) Let $F_k(N)$ be the maximum number of integers $a_1 < a_2 < \cdots < a_t$ in $[1,N]$ with the property that the product of k different ones is never a square.

Theorem 1.2 (Erdős–A. Sárközy–T. Sós [83]). There exist a positive absolute constant $c > 0$ and for every $\varepsilon > 0$ an $N_0(\varepsilon)$ such that for $N > N_0(\varepsilon)$ we have

$$\frac{(\sqrt{2}-\varepsilon)N^{2/3}}{\log^{4/3} N} < F_6(N) - \pi(N) - \pi(N/2) < cN^{7/9}\log N. \qquad (1)$$

Taking all the primes of $[2,N]$ and all the numbers $2p$ where p is a prime in $[1,N/2]$ we get $\pi(N)+\pi(N/2)$ such numbers (satisfying (C)) and the above theorem suggests that this construction is almost the best.

The solution of this problem depends on extremal graph theorems connected to excluding C_6. Theorems analogous to Theorem 1.2 hold for the even values of k, and somewhat different ones for odd values of k. The question which was asked is:

---

[3] Eszter Klein, later Mrs. Szekeres.

[4] Meaning: $p+1$-tuples.... The text here does not tell us if the role of Mrs. Szekeres was important here or not, but somewhere else Erdős writes: "Mrs. Szekeres and I proved ..."

[5] Watch out: I wrote this many years ago!

> What is the maximum number of edges a bipartite graph $G(U, V)$ with u RED and v BLUE vertices can have if $G(U, V)$ contains no C_6 and $uv \le N$?

In [83] the following conjecture was formulated:

Conjecture 1.3. If $G(U, V)$ is a bipartite graph with $u = |U|$ RED vertices and $v = |V|$ BLUE ones, and $G(U, V)$ contains no C_6, and $v \le u \le v^2$, then $e(G(U, V)) \le c(uv)^{2/3}$.

De Caen and Székely [34] and Faudree and Simonovits [253] had earlier some related estimates. The upper bound (1) (of [83]) has been improved first by Gábor Sárközy [148]. Then E. Győri [268, 269] proved the above conjecture, which in turn brought down the upper bound of (1) to the lower bound, apart from some log-powers.

Theorem 1.4 (Győri). If $C_{2k} \not\subseteq G(m, n)$, then $e(G(m, n)) \le (k-1)n + O(m^2)$.

Conjecture 1.5 (Győri). There exists a $c > 0$ for which, if $C_{2k} \not\subseteq G(m, n)$, then $e(G(m, n)) \le (k-1)n + O(m^{2-c})$.

Perhaps even $e(G(m, n)) \le (k-1)n + O(m^{3/2})$ could be proved for C_6.

Further sources to read: Works of Lazebnik, Ustimenko, and Woldar should be mentioned here, e.g., [124, 125]. See also Füredi [101]

There is also a similar, related but slightly different branch of combinatorial number theory, where the extremal numbers of C_{2k} play important role see e.g., Dietmann, Elsholz, Gyarmati and Simonovits [246].

"How Did Crookes Miss to Invent the X–Ray?"

Erdős feels that he "should have invented" Extremal Graph Theory, back in 1938. He has failed to notice that his theorem was the root of an important and beautiful theory. Two to three years later Turán proved his famous theorem and right after that he posed a few relevant questions, thus initiating a whole new branch of graph theory. Erdős often explains his "blunder" by telling the following story.

> Crookes observed that leaving a photosensitive film near the cathod-ray-tube causes damage to the film: it becomes exposed. He concluded that "Nobody should leave films near the cathod-ray-tube." Röntgen observed the same phenomenon a few years later and concluded that this can be used for filming the inside of various objects. His conclusion changed the

whole Physics.[6] "It is *not enough* to be in the right place at the right time. You should also have an open mind at the right time," Paul concludes his story.

Erdős's influence on the field is so thorough that we do not even attempt to describe it in its full depth and width. We shall neither try to give a very balanced description of the whole, extremely wide area. Instead, we pick a few topics to illustrate Erdős's role in developing this subject, and his vast influence on others.

Also, I shall concentrate more on the new results, since the book of Bollobás [12] (see also [183]), or the surveys of myself, [156, 158, 159], or the surveys of Füredi [102] and Sidorenko [151] provide a lot of information on the topic and some problem-papers of Erdős, e.g., [54, 62, 64, 65] are also highly recommended for the reader wishing to learn about the topic. Also, wherever it was possible, I selected newer results, or older but less known theorems (partly to avoid *unnecessary* repetition compared to the earlier surveys).

I had to leave out quite a few very interesting topics. Practically,

(a) I skipped all the hypergraph theorems, [102, 151]
(b) the covering problems connected to the Erdős–Goodman-Pósa theorem [251],
(c) applications of finite geometrical methods in extremal graph theory, see, e.g. [163, 158], ... application of Lazebnik–Ustimenko type constructions, [124] and many more... Actully, I shall return to some of them in [188].
(d) Among others, I had to leave out that part of Ramsey Theory, which is extremely near to Extremal Graph Theory, (see [97]) ... and for many other things see Bollobás [12, 15, 16] ...

"The Complete List of Theorems"

If one watches Erdős in work, beside of his great proving power and elegance, one surprising feature is, how he poses his conjectures. This itself would deserve a separate note. Sometimes one does not immediately understand the importance of his questions. Slightly mockingly, once his friend, András Hajnal told to him: "You would like to have a Complete List of Theorems". I think there is some truth in this remark, still one modification should be made.

[6] When I asked Paul, why did he think that Röntgen's discovery changed the whole Physics, he answered that Röntgen's findings had led to certain results of Curie and from that point it was only a short step to the A-bomb.

Erdős does not like to state his conjectures immediately in their most general forms. Instead, he picks very special cases and attacks first these ones. Mostly he picks his examples “very fortunately”. Therefore, having solved these special cases he very often discovers whole new areas, and it is difficult for the surrounding to understand how can he be so “fortunate”. So, the reader of Erdős and the reader of this survey should keep in mind that Erdős's method is always to attack *important special cases.*

Notation

We shall primarily consider simple graphs: graphs without loops and multiple edges. Later there will be paragraphs where we shall also consider digraph- and hypergraph extremal problems.

Given a family $\mathcal{L}$ of – so called – *excluded* or *forbidden* subgraphs, $\mathrm{ex}(n,\mathcal{L})$ will denote the maximum number of edges a graph G_n can have without containing forbidden subgraphs. (Containment does not assume “induced subgraph” of the given type.) The family of graphs attaining the maximum will be denoted by $\mathrm{EX}(n,\mathcal{L})$. If $\mathcal{L}$ consists of a single L, we shall use the notation $\mathrm{ex}(n,L)$ and $\mathrm{EX}(n,L)$ instead of $\mathrm{ex}(n,\{L\})$ and $\mathrm{EX}(n,\{L\})$.

For a set Q, $|Q|$ will denote its cardinality.[7] Given a graph G, $e(G)$ will denote the number of its edges, $v(G)$ the number of its vertices, $\chi(G)$ and $\alpha(G)$ its chromatic and independence numbers, respectively. For graphs the (first) subscript will mostly denote the number of vertices: G_n, S_n, $T_{n,p}$, …denote graphs on n vertices. There will be one exception: speaking of excluded graphs $L_1,\dots,L_r$ we use superscripts just to enumerate these graphs. Given two disjoint vertex sets, X and Y, in a graph G_n, $e(X,Y)$ denotes the number of edges joining X and Y. Given a graph G and a set X of vertices of G, the number of edges in a subgraph spanned by a set X of vertices will be denoted by $e(X)$, the subgraph of G spanned by X is $G(X)$.

Special graphs. K_p will denote the complete graph on p vertices, $T_{n,p}$ is the so called Turán graph on n vertices and p classes: n vertices are partitioned into p classes as uniformly as possible and two vertices are joined iff they belong to different classes. This graph is the (unique) p–chromatic graph on n vertices with the maximum number of edges among such graphs. $K_p(n_1,\dots,n_p)$ (often abbreviated to $K(n_1,\dots,n_p)$) denotes the complete p–partite graph with n_i vertices in its ith class, $i=1,2,\dots,p$.

We shall say that X is *completely joined* to Y if every vertex of X is joined to every vertex of Y. Given two vertex-disjoint graphs, G and H, their *product* $G\otimes H$ is the graph obtained by joining each vertex of G to each one of H.

[7] Our notation is mostly standard. Below some of the notations are just the repetitions of what we wrote rarlier.

Quoting. Below sometimes I quote some paragraphs from other papers, but the references and occasionally the notations too are changed to comply with mines.

2. Turán's Theorem

Perhaps Turán was the third to arrive at this field. In 1940 he proved the following theorem, [174] (see also [175, 173]):

Theorem 2.1 (Turán). (a) If G_n contains no K_p, then $e(G_n) \le e(T_{n,p-1})$. In case of equality $G_n = T_{n,p-1}$.

Turán's original paper contains much more than just this theorem. Still, the main impact coming from Turán was that he asked the general question:

> *What happens if we replace K_p with some other forbidden graphs, e.g., with the graphs coming from the Platonic polyhedra, or with a path of length ℓ, etc.*

Turán's theorem also could have sunk into oblivion. However, this time Erdős was more open-minded. He started proving theorems, talked to people about this topic and people started realizing the importance of the field.

Turán died in 1976. The first issue of Journal of Graph Theory came out around that time. Both Paul [63] and I were asked to write about Turán's graph theory [156]. (In the introductory issue of the journal Turán himself wrote a Note of Welcome, also mentioning some historical facts about his getting involved in graph theory [177].) Let me quote here some parts of Paul Erdős's paper [63].

> In this short note I will restrict myself to Turán's work in graph theory, even though his main work was in analytic number theory and various other branches of real and complex analysis. Turán had the remarkable ability to write perhaps only one paper in various fields distant from his own; later others would pursue his idea and new subjects would be born.
>
> In this way Turán initiated the field of extremal graph theory. He started this subject in 1941,[8] (see [174, 175]) He posed and completely solved the following problem . . .

Here Erdős describes Turán Theorem and Turán's hypergraph conjecture, and a result of his own to which we shall return later. Then he continues:

[8] To be precise, Turán proved his theorem in 1940, in a labour camp, and published it in 1941.

> Turán also formulated several other problems on graphs, some of which were solved by Gallai and myself [69]. *I began a systematic study of extremal graph theory in 1958 on the boat from Athens to Haifa and have worked on it since then.* The subject has a very large literature; Bollobás has written a comprehensive book on extremal problems in graph theory which will appear soon. (Paul meant [12].)

One final remark should be made here. As I stated in other places, Paul Turán's role was crucial in the development of Extremal Graph Theory. Still, even here there is a point, where Erdős's influence should be mentioned again. More precisely, the influence of an Erdős–Szekeres paper [94]. As today it is already well known, Erdős and Szekeres tried to solve a problem in convex geometry [94] and rediscovered Ramsey's Theorem [145]. They informed Turán about their theorem, according to which either the graph or the complementary graph contains a large complete graph. Turán regarded this result as a theorem where one ensures the existence of a large complete subgraph in G_n by assuming something about the complementary graph. So Turán wanted to change the condition and still arrive at the same conclusion. This is why he supposed that a lower bound is given on the number of edges and deduced the existence of a large complete subgraph of G_n. Turán writes in [174]:

> Theorem I gives a condition to guarantee the existence of a complete subgraph on k vertices in a graph on a finite number of vertices. The only related theorem – as far as I know – can be found in a joint paper of Pál Erdős and György Szekeres [94] and essentially states that if a graph A on n vertices is such that its complement $\overline{A}$ contains only complete subgraphs having "few" vertices, then the graph A contains a complete subgraph on "many" vertices. Their theorem contains only bounds in the place of the expressions "few" and "many", in fact it gives almost only the existence; the exact solution seems to be very interesting but difficult ...

Some (Further) Historical Remarks

(a) Turán's paper contains an infinite Ramsey theorem. I quote:

Theorem II. *Let us suppose that for the infinite graph A containing countably many vertices $P_1, P_2, \ldots$ there is an integer $d > 1$ such that if we choose arbitrary d different vertices of A, there will be at least two among these vertices joined by an edge in A. Then A has at least one complete subgraph of infinitely many vertices.*

This theorem is weaker than the one we usually teach in our Combinatorics courses, nevertheless, historically it is interesting to see it in Turán's paper.

(b) Turán's theorem is connected to the Second World War in two ways. On the one hand, Turán, sent to forced labour service and deprived of paper and pencil, started working on problems that were possible to follow without writing them down. Also he made his famous hypergraph conjecture, thinking that would he have paper and pencil, he could have easily proved it.

On the other hand, it is worth mentioning that Turán's Theorem was later rediscovered by A. A. Zykov [182, 1949] who (because of the war) learned too late that it had already been published.

(c) As to Mantel's result, I quote the last 4 lines of Turán [174]:

> **Added in proof.** ... Further on, I learned from the kind communication of Mr. József Krausz that the value of $d_k(n)$ ($= \mathrm{ex}(n, K_3)$) is given on p. 438, for $k = 3$ was found already in 1907 by W. Mantel, (Wiskundige Opgaven, vol. **10**, p. 60–61). I know his paper only from the reference of Fortschritte d. Math., vol **38** p. 270.

(d) During the war Turán was trying to prove that either a graph G_n or its complementary graph contains a complete graph of order $[\sqrt{n}]$. He writes in [177]:

> I still have the copybook in which I wrote down various approaches by induction, all they started promisingly, but broke down at various points. I had no other support for the truth of this conjecture, than the symmetry and some dim feeling of beauty: ... In one of my first letters to Erdős after the war I wrote of this conjecture to him. In his answer he proved that my conjecture was utterly false ...

Of course, all we know today, what Erdős wrote to Turán: the truth is around $c \log n$. This was perhaps the first application of probability to Graph Theory, though many would deny that Erdős's elegant answer uses more than crude counting. Probably this is where the Theory of Random Graphs started. (To be quite precise, one should mention, that T. Szele had a similar proof for Rédei's theorem on directed Hamiltonian cycles in tournaments, [167], already in 1943, however, Erdős's proof was perhaps of more impact and it was the first where no other approach could replace the counting argument. Another early breakthrough of the Random Graph Method was when Erdős easily answered the following problem of Schütte [52]: Is there a tournament where for every k players there is a player which beats all of them?)

I would suggest the reader to read also the beautiful paper of Turán [177], providing a lot of information on what I have described above shortly.

(e) For a longer account on the birth of the Erdős–Szekeres version of Ramsey theorem see the account of Gy. Szekeres in the introduction of the Art of Counting, [60]. For geometric aspects see also Pach–Agarwal [139].

3. Erdős–Stone Theorem

Setting out from a problem in topology, Erdős and A. H. Stone proved the following theorem in 1946 [93]:

Theorem 3.1 (Erdős–Stone). For every fixed p and m

$$\operatorname{ex}(n, K_{p+1}(m, \dots, m)) = \left(1 - \frac{1}{p}\right)\binom{n}{2} + o(n^2). \tag{2}$$

Moreover, if p is fixed and $m := \sqrt{\ell_p(n)}$ where $\ell_p(x)$ denotes the p times iterated logarithm of x, (2) still holds.

Here $m := \sqrt{\ell_p(n)}$ is far from being the best possible. The sharp order of magnitude is $c \log n$. Let $m = m(n, \varepsilon)$ be the largest integer such that, if $e(G_n) > e(T_{n,p}) + \varepsilon n^2$, then G_n contains the regular $(p+1)$–partite graph $K_{p+1}(m, m, \dots, m)$. One can ask how large is $m = m(n, \varepsilon)$, defined above? This was determined by Bollobás, Erdős, Simonovits [17, 19] and Chvátal and Szemerédi [39].

The first breakthrough was that the p–times iterated log was replaced by $K \log n$, where K is a constant [17]. In the next two steps the dependence of this constant on p and ε were determined.

Theorem 3.2 (Bollobás–Erdős–Simonovits [19]). There exists an absolute constant $c > 0$ such that every G_n with

$$e(G_n) \ge \left(1 - \frac{1}{p} + \varepsilon\right)\binom{n}{2}$$

contains a $K_{p+1}(m, m, \dots, m)$ with

$$m > \frac{c \log n}{p \log(1/\varepsilon)}.$$

The next improvement, essentially settling the problem completely is the result of Chvátal and Szemerédi, providing the exact dependence on all the parameters, up to an absolute constant.

Theorem 3.3 (Chvátal–Szemerédi [39]).

$$\frac{\log n}{500 \log(1/\varepsilon)} < m(n, \varepsilon) < \frac{c \log n}{\log(1/\varepsilon)}.$$

One could have thought that the problem is settled but here is a nice result of Bollobás and Kohayakawa, improving Theorem 3.3.

Conjecture 3.4 (Bollobás–Kohayakawa [20]). There exists an absolute constant $\alpha > 0$ such that for all $r \ge 1$ and $0 \le \varepsilon \le 1/r$ every G_n of sufficiently large order satisfying

$$e(G_n) \ge \left(1 - \frac{1}{r} + \varepsilon\right)\binom{n}{2}$$

contains a $K_{r+1}(s_0, m_0, \ldots, m_0)$, where

$$s_0 = s_0(n) = \left\lfloor \frac{\alpha \log n}{\log(1/\varepsilon)} \right\rfloor \quad \text{and} \quad m_0 = m_0(n) = \left\lfloor \frac{\alpha \log n}{\log r} \right\rfloor.$$

Bollobás and Kohayakawa [20] succeded in proving that under the above conditions, if $0 < \gamma < 1$, then G_n contains a $K_{r+1}(s_1, m_1, \ldots, m_1, \ell)$, where

$$s_1 = \left\lfloor \frac{\alpha(1-\gamma)\log n}{r \log(1/\varepsilon)} \right\rfloor, \qquad m_1 = \left\lfloor \frac{\alpha(1-\gamma)\log n}{\log r} \right\rfloor, \quad \text{and } \ell = \lfloor \alpha\varepsilon^{1+\gamma/2} n^{\gamma} \rfloor.$$

Observe that this result is fairly close to proving Conjecture 3.4: the first class is slightly smaller and the last class much larger than in the conjecture.

Remark [N] 3. In some sense the most important consequence of the Chvátal–Szemerédi theorem was that this was the point where the Szemerédi Regularity Lemma [169] was really formulated. In the earlier cases, e.g., in [168], only weaker, more involved versions of it were used.[9]

The Kővári–V. T. Sós–Turán Theorem

The Kővári–T. Sós–Turán theorem [123] solves the extremal graph problem of $K_2(p, q)$, at least, provides an upper bound, which in some cases proved to be sharp.[10] This theorem is on the one hand a generalization of the C_4–theorem, since $C_4 = K(2, 2)$, and, on the other hand, is a special case of the Erdős–Stone theorem, apart from the fact that here we get sharper estimates.[11]

Theorem 3.5 (Kővári–T. Sós–Turán). Let $2 \le p \le q$ be fixed integers. Then

$$\mathrm{ex}(n, K(p, q)) \le \frac{1}{2} \sqrt[p]{q-1}\, n^{2-1/p} + O(n).$$

Conjecture 3.6. The exponent $2 - (1/p)$ is sharp: for every $p, q \ge 2$,

$$\mathrm{ex}(n, K(p, q)) > c_{p,q} n^{1-(1/p)}.$$

[9] Szemerédi in [168] used what I call either bipartite or "ugly" regularity lemma. In some places, like [120], I stated that this bipartite version was used in [170], however, that is not quite correct: a simpler version was enough there.

[10] A footnote of [123] tells us that the authors have received a letter from Erdős in which Erdős informed them that he also had proved most of the results of [123].

[11] Estimates, sharper than in the original Erdős–Stone.

Unfortunately, this is known only for $p = 2$ and $p = 3$, (see Erdős, Rényi, V. T. Sós [81] and independently W. G. Brown [27]). Random graph methods [92] show that

$$\mathrm{ex}(n, K(p,p)) > c_p n^{2-\frac{2}{p+1}}.$$

Recently Füredi [104] improved the constant in the upper bound, showing that

$$\mathrm{ex}(n, K(2,t+1)) = \frac{1}{2}\sqrt{t}n^{3/2} + O(n^{4/3}),$$

and that the constant provided by Brown's construction is sharp. While one conjectures that $\mathrm{ex}(n, K(4,4))/n^{7/4}$ converges to a positive limit, we know only, by the Brown construction, that $\mathrm{ex}(n, K(4,4)) > \mathrm{ex}(n, K(3,3)) > cn^{5/3}$. It is unknown if

$$\frac{\mathrm{ex}(n, K(4,4))}{n^{5/3}} \to \infty.$$

Remark [N] 4 (New constructions). It was a surprising beakthrough, when Kollár, Rónyai and Tibor Szabó – using Commutative Algebra – showed [219] that

$$\mathrm{ex}(n, K(p,q)) > c_{p,q} n^{1-(1/p)}, \qquad \text{if} \qquad q > p!,$$

proving Conjecture 3.6 for infinitely many cases. This was improved to

$$\mathrm{ex}(n, K(p,q)) > c_{p,q} n^{1-(1/p)}, \qquad \text{if} \qquad q > (p-1)!,$$

by Alon, Rónyai and Tibor Szabó, see [294]; an even lower bound is given by Ball and Pepe, see [8].

Further sources to read: the surveys of Alon [2], Simonovits [159], Füredi and Simonovits [188].

The Matrix Form

The problem of Zarankiewicz [181] is to determine the maximum integer $k_p(n)$ such that if A_n is a matrix with n rows and n columns consisting exclusively of 0's and 1's, and the number of 1's is at least $k_p(n)$, then A_n contains a minor B_p of p rows and columns so that all the entries of B_p are 1's.

One can easily see that this problem is equivalent with determining the maximum number of edges a bipartite graph $G(n,n)$ can have without containing $K(p,p)$.

In [123] the authors remark that the problem can be generalized to the case of general matrices: when A has m rows and n columns and B has p rows and q columns. Denote the maximum by $k(m,n,p,q)$. There are many results on estimating this function but we shall not go here into details. Rather, we explain the notion of symmetric and asymmetric bipartite graph problems.

As Erdős pointed out,

Theorem 3.7. Every graph G_n has a bipartite subgraph $H(U,V)$ in which each vertex has at least half of its original degree: $d_H(x) \geq \frac{1}{2}d_G(x)$, and therefore $e(H(U,V)) \geq \frac{1}{2}e(G_n)$.

One important consequence of this almost trivial fact (proved later) is that (as to the order of magnitude), it does not matter if we optimize $e(G_n)$ over *all* graphs or only over the *bipartite* graphs. Another important consequence is that some matrix extremal problems are equivalent to graph extremal problems. Conversely, many extremal graph problems with bipartite excluded subgraphs have equivalent matrix forms as well:

As usually, having a bipartite graph, $G(U,V)$ we shall associate with it a matrix A, where the rows correspond to U, the columns to V and $a_{ij} = 1$ if the ith element of U is joined to the jth element of V, otherwise $a_{ij} = 0$.

Given a bipartite graph $L = L(X,Y)$ and another bipartite graph $G(U,V)$, $|U| = m$ and $|V| = n$, take the $m \times n$ adjacency matrix A of G and the adjacency-matrix B of L. Assume for a second that the colour-classes of L are symmetric (in the sense that there is an automorphism of L exchanging the two colour-classes). Then the condition that $L \not\subseteq G(U,V)$ can be formulated by saying that the matrix A has no submatrix equivalent to B, where equivalency means that they are the same apart from some permutation of the rows and columns. So Turán type problems lead to problems of the following forms:

> Given an $m \times n$ $0-1$ matrix, how many 1's ensure a submatrix equivalent to B?

If, on the other hand, the forbidden graph $L = L(X,Y)$ has no automorphism exchanging X and Y, then the matrix-problem and the graph-problem may slightly differ. Excluding the submatrices equivalent to B means that we exclude that $G(U,V)$ contains an L with $X \subseteq U$ and $Y \subseteq V$, but we do not exclude $L \subseteq G(U,V)$ in the opposite position. Denote by $\mathrm{ex}^*(n,L)$ the maximum in this asymmetric case. Clearly, $\mathrm{ex}^*(n,L) \geq \mathrm{ex}(n,L)$, and they are equal if L has a colour-swapping automorphism.

Conjecture 3.8 (Simonovits). If L is bipartite, then $\mathrm{ex}^*(n,L) = O(\mathrm{ex}(n,L))$.

We do not know this even for $K(4,5)$. The difficulty in disproving such a conjecture is partly that in all the proofs of upper bounds on degenerate extremal graph problems, we use only "one-sided" exclusion. Therefore the upper bounds we know are always upper bounds on $\mathrm{ex}^*(n,L)$.

Conjecture 3.9 (Erdős–Simonovits). For every $\mathcal{L}$ with a bipartite $L \in \mathcal{L}$ there is a bipartite $L^* \in \mathcal{L}$ for which $\mathrm{ex}(n,\mathcal{L}) = O(\mathrm{ex}(n,L^*))$.

Remark [N] 5. Certain unpublished result of Faudree and Simonovits [253] indicate that perhaps this conjecture does not always hold.

We close this part with a beautiful but probably very difficult problem of Erdős.

Conjecture 3.10. $\mathrm{ex}(n, \{C_3, C_4\}) = \frac{1}{2\sqrt{2}} n^{3/2} + o(n^{3/2})$.

The meaning of this conjecture is that excluding C_3 beside C_4 has the same effect as if we excluded all the odd cycles. If we replace C_3 by C_5, then this is true, see [87]. Erdős risks the even sharper conjecture that the exact equality may hold:

$$\mathrm{ex}(n, \{C_3, C_4\}) = \mathrm{ex}(n, \{C_3, C_4, C_5, C_7, C_9, C_{11} \dots\}).$$

Remark [N] 6. Some related results can be found in the paper of Keevash, Sudakov, and Verstraete [279]: they prove the above conjecture for several cases. (See also [209].)

Further sources to read: For some further information, see a survey paper of Richard Guy [106] and also a paper of Guy and Znam [107] on $K(p, q)$ and the results of Lazebnik, Ustimenko and Woldar on cycles [124, 125].

Also, the problems, posing the problems was a very characteristic feature of Paul's mathematics. Here we neglect this aspect a little. The reader is referred to the many problem-posing papers of Erdős, and also to the book of Chung and Graham [185]. See also [188].

Applications of Kővári–T. Sós–Turán Theorem

It is interesting to observe that the C_4–theorem and its immediate generalizations (e.g. the Kővári–T. Sós–Turán theorem) have quite a few applications. Some of them are in geometry. For example, as Erdős observed, if we have n points in the plane, and join two of them if their distance is exactly 1, then the resulting graph contains no $K(2, 3)$. So the number of unit distances among n points of the plane is $O(n^{3/2})$. Similarly, the unit-distance-graph of the 3-dimensional space contains no $K(3, 3)$, therefore the number of unit distances in the 3-space is $O(n^{5/3})$. There are deeper and sharper estimates on this subject, see Spencer, Szemerédi and Trotter [164] or Clarkson, Edelsbrunner, Guibas, Sharir and Welzl [40].

Conjecture 3.11 (Erdős). For every $\varepsilon > 0$, the number of unit distances among n points of the plain is $O(n^{1+\varepsilon})$.

We mention one further application: the *chromatic number of the product* hypergraph. Claude Berge was interested in calculating the chromatic number of the product of two graphs, $\mathcal{H}$ and $\mathcal{H}'$. Generally there are various ways to define the product of r–uniform hypergraphs. *This* product is defined as the r^2–uniform hypergraph whose vertex-set is the Cartesian product $V(\mathcal{H}) \times V(\mathcal{H}')$ and the edge-set is

$$\{H \times H' \; : \; H \in E(\mathcal{H}) \quad \text{and} \quad H' \in E(\mathcal{H}')\} \, .$$

The chromatic number of the hypergraph is at most k if the vertices can be coloured in k colours without having monochromatic r^2–tuples. Berge and I [10] estimated the chromatic number of products of graphs (hypergraphs) using Kővári–T. Sós–Turán theorem. Also we found a matching lower bound by using basically the Brown and Erdős–Rényi–T. Sós constructions [27, 81]. The same time a student of Berge, F. Sterboul [165, 166] have proved the same theorem and an *earlier* paper of V. Chvátal [38] used the same technique to prove some assertions roughly equivalent with this part of our paper [10].

Remark [N] 7. It is an interesting feature of the Berge-Simonovits paper that – connected to the above topics – the paper also defines the fractional chromatic number.

4. Graph Theory and Probability

Erdős wrote two papers with the above title, one in 1959, [48], and the other in 1961, [49]. These papers were of great importance. In the first one Erdős proved the following theorem.

Theorem 4.1. For fixed k and sufficiently large ℓ, if $n > \ell^{1+1/(2k)}$, then there exist (many) graphs G_n of girth k and independence number $\alpha(G_n) < \ell$.

Clearly, $\chi(G_n) \geq \frac{v(G_n)}{\alpha(G_n)}$. So, as Erdős points out, a corollary of this is

Corollary 4.2. For every integer k for $n > n_0(k)$ there exist graphs G_n of girth $\geq k$ and chromatic number $\geq n^{\frac{1}{2k+1}}$.

This theorem seems to be a purely Ramsey theoretical result, fairly surprising in those days, but, it has many important consequences in Extremal Graph Theory as well. The same holds for the next theorem, too:

Theorem 4.3 ([49]). Assume that $n > n_0$. Then there exist graphs G_n with $K_3 \not\subseteq G_n$ and $\alpha(G_n) = O(\sqrt{n} \log n)$.

One important corollary of Theorem 4.1, more precisely, of its proof is that

Theorem 4.4. If $\mathcal{L}$ is finite and contains no trees, then $\mathrm{ex}(n, \mathcal{L}) > c^*_{\mathcal{L}} n^{1+c_{\mathcal{L}}}$, for some constants $c^*_{\mathcal{L}}, c_{\mathcal{L}} > 0$.

On the other hand, it is easy to see that if $L \in \mathcal{L}$ is a tree (or a forest), then $\mathrm{ex}(n, \mathcal{L}) = O(n)$.

These theorems use random graph methods. They and some of their generalizations play also important role, in obtaining lower bounds in Turán–Ramsey Theorems. (See also Füredi-Seress [105].) For general applications of the probabilistic methods in graph theory see, e.g. Erdős–Spencer [92], Bollobás [15], Alon–Spencer [7].

Of course, speaking of Graph Theory and Probability, one should also mention the papers of Erdős and Rényi, perhaps above all, [80].[12]

Further sources to read: Janson, Łuczak, Ruciński [190], Molloy and Reed [195].

5. The General Theory

In this section, we present the asymptotic solution to the general extremal problem.

> *General Extremal Problem.* Given a family $\mathcal{L}$ of forbidden subgraphs, find those graphs G_n that contain no subgraph from $\mathcal{L}$ and have the maximum number of edges.

The problem is considered to be "completely solved" if all the extremal graphs have been found, at least for $n > n_0(\mathcal{L})$. Quite often this is too difficult, and we must be content with finding $\mathrm{ex}(n, \mathcal{L})$, or at least good bounds for it.

It turns out that a parameter related to the chromatic number plays a decisive role in many extremal graph theorems. The *subchromatic number* $p(\mathcal{L})$ of $\mathcal{L}$ is defined by

$$p(\mathcal{L}) = \min\{\chi(L) : L \in \mathcal{L}\} - 1.$$

The following result is an easy consequence of the Erdős–Stone theorem [93]:

Theorem 5.1 (The Erdős–Simonovits Theorem [84]). *If $\mathcal{L}$ is a family of graphs with subchromatic number p, then*

$$\mathrm{ex}(n, \mathcal{L}) = \left(1 - \frac{1}{p}\right)\binom{n}{2} + o(n^2).$$

The meaning of this theorem is that $\mathrm{ex}(n, \mathcal{L})$ depends only very loosely on $\mathcal{L}$; up to an error term of order $o(n^2)$, it is already determined by the minimum chromatic number of the graphs in $\mathcal{L}$.

[12] They are reprinted in [60, 196]

Classification of Extremal Graph Problems

By the above theorem,

$$\mathrm{ex}(n, \mathcal{L}) = o(n^2)$$

if and only if $p(\mathcal{L}) = 1$, i.e. there exist bipartite graphs in $\mathcal{L}$. From the Kővári-T. Sós-Turán Theorem we get that here $\mathrm{ex}(n, \mathcal{L}) = O(n^{2-c})$ for some $c = c(\mathcal{L})$. We shall call these cases *degenerate extremal graph problems* and find them among the most interesting problems in extremal graph theory. One special case is when $\mathcal{L}$ contains a tree (or a forest). These cases could be called *very degenerate*. Observe, that if a problem is *non-degenerate*, then $T_{n,2}$ contains no excluded subgraphs. Therefore $\mathrm{ex}(n, \mathcal{L}) \geq \left[\frac{n^2}{4}\right]$.

Structural Results

The structure of the extremal graphs is also almost determined by $p(\mathcal{L})$, and is very similar to that of $T_{n,p}$ This is expressed by the following results of Erdős and Simonovits [55, 57, 152]:

Theorem 5.2 (The Asymptotic Structure Theorem). Let $\mathcal{L}$ be a family of forbidden graphs with subchromatic number p. If S_n is any graph in $\mathrm{EX}(n, \mathcal{L})$, then it can be obtained from $T_{n,p}$ by deleting and adding $o(n^2)$ edges. Furthermore, if $\mathcal{L}$ is finite, then the minimum degree

$$d_{min}(S_n) = \left(1 - \frac{1}{p}\right) n + o(n).$$

The structure of extremal graphs is fairly stable, in the sense that the almost-extremal graphs have almost the same structure as the extremal graphs (for $\mathcal{L}$ or for K_{p+1}). This is expressed in our next result:

Theorem 5.3 (The First Stability Theorem). Let $\mathcal{L}$ be a family of forbidden graphs with subchromatic number $p \geq 2$. For every $\varepsilon > 0$, there exist a $\delta > 0$ and n_ε such that, if G_n contains no $L \in \mathcal{L}$, and if, for $n > n_\varepsilon$,

$$e(G_n) > \mathrm{ex}(n, \mathcal{L}) - \delta n^2,$$

then G_n can be obtained from $T_{n,p}$ by changing at most εn^2 edges.

These theorems are interesting on their own and also widely applicable.

In the remainder of this section we formulate a sharper variant of the stability theorem. One can ask whether further information on the structure of forbidden subgraphs yields better bounds on $\mathrm{ex}(n, \mathcal{L})$ and further information on the structure of extremal graphs. At this point, we need a definition.

Definition 5.4. Let $\mathcal{L}$ be a family of forbidden subgraphs, and let $p = p(\mathcal{L})$ be its subchromatic number. The *decomposition* $\mathcal{M}$ of $\mathcal{L}$ is the family of graphs M with the property that, for some $L \in \mathcal{L}$, L contains M as an induced subgraph and $L - V(M)$ is $(p-1)$–colorable.

In other words, for $r = v(L)$, $L \subseteq M \times K_{p-1}(r, \dots, r)$, and M is minimal with this property. In case of $\mathcal{L} = \{K_p\}$ the family $\mathcal{M}$ consists of one graph K_2. The following result is due to Simonovits [152], (see also [57]).

Theorem 5.5 (**The Decomposition Theorem, [152]**). Let $\mathcal{L}$ be a forbidden family of graphs with $p(\mathcal{L}) = p$ and decomposition $\mathcal{M}$. Then every extremal graph $S_n \in \mathrm{EX}(n, \mathcal{L})$ can be obtained from a suitable $K_p(n_1, \dots, n_p)$ by changing $O(\mathrm{ex}(n, \mathcal{M}) + n)$ edges. Furthermore, $n_j = (n/p) + O(\mathrm{ex}(n, \mathcal{M})/n) + O(1)$, and

$$d_{\min}(S_n) = \left(1 - \frac{1}{p}\right) n + O(\mathrm{ex}(n, \mathcal{M})/n) + O(1).$$

It follows from this theorem that, with $m = \lceil n/p \rceil$, $\mathrm{ex}(n, \mathcal{L}) = e(T_{n,p}) + O(\mathrm{ex}(m, \mathcal{M}) + n)$. If $\mathrm{ex}(n, \mathcal{M}) > cn$, then $O(\mathrm{ex}(m, \mathcal{M}))$ is sharp: put edges into the first class of a $T_{n,p}$ so that they form a $G_m \in \mathrm{EX}(m, \mathcal{M})$; the resulting graph contains no L, and has $e(T_{n,p}) + \mathrm{ex}(m, \mathcal{M})$ edges.

A second stability theorem can be established using the methods of [152]. To formulate it, we introduce some new terms. Consider a partition $U_1, U_2, \dots, U_p$ of the vertex-set of G_n, and the complete p–partite graph $H_n = K(u_1, \dots, u_p)$ corresponding to this partition of $V(G_n)$, where $u_i = |U_i|$. A pair of vertices is called an *extra edge* if it is in G_n but not in H_n, and is a *missing edge* if it is in H_n but not in G_n. For given p and G_n, the partition $U_1, U_2, \dots, U_p$ is called *optimal* if the number of missing edges is minimum. Finally, for a given vertex v, let $a(v)$ and $b(v)$ denote the numbers of missing and extra edges at v, respectively.

Theorem 5.6 (The Second Stability Theorem). Let $\mathcal{L}$ be a forbidden family of graphs with $p(\mathcal{L}) = p$ and decomposition $\mathcal{M}$, and let $k > 0$. Suppose that G_n contains no $L \in \mathcal{L}$,

$$e(G_n) \geq \mathrm{ex}(n, \mathcal{L}) - k \cdot \mathrm{ex}(n, \mathcal{M}),$$

and let $U_1, \dots, U_p$ be the optimal partition of G_n, $G_i := G(U_i)$. Then

(i) G_n can be obtained from $\times G_i$ by deleting $O(\mathrm{ex}(n, \mathcal{M}) + n)$ edges;
(ii) $e(G_j) = O(\mathrm{ex}(n, \mathcal{M})) + O(n)$, and $|U_j| = (n/p) + O(\sqrt{\mathrm{ex}(n, \mathcal{M})} + \sqrt{n})$;
(iii) For any constant $c > 0$, the number of vertices v in G_i with $a(v) > cn$ is only $O(1)$, and the number of vertices with $b(v) > cn$ is only $O(\mathrm{ex}(n, \mathcal{M})/n) + O(1)$;
(iv) Let $L \in \mathcal{L}$, with $v(L) = r$, and let A_i be the set of vertices v in U_i for which $b(v) < (n/2pr)$; if $M \times K_{p-1}(r_1, \dots, r) \supseteq L$, then the graph $G(A_i)$ contains no M.

The constant k of the condition cannot be seen in (i)–(iv): it is hidden in the constants of the $O(.)$'s This theorem is useful also in applications. The deepest part is the first part of (iii). This implies (iv), which in turn implies all the other statements. A proof is sketched in [155], where the theorem was needed.

We conclude this section with the theorem characterizing those cases where $T_{n,p}$ is the extremal graph.

Theorem 5.7 (Simonovits [155]). The following statements are equivalent:

(a) The minimum chromatic number in $\mathcal{L}$ is $p+1$ but there exists (at least one) $L \in \mathcal{L}$ with an edge e such that $\chi(L-e) = p$. (Colour critical edge.)
(b) There exists an n_0 such that for $n > n_0(\mathcal{L})$, $T_{n,p}$ is extremal.
(c) There exists an n_1 such that for $n > n_1(\mathcal{L})$, $T_{n,p}$ is the *only* extremal graph.

Remark [N] 8. Here we can observe an important phenomenon, namely, that (mostly) if we can prove some theorems for complete graphs K_{p+1}, then we can also prove it for cases, where a $p+1$-chromatic $L \in \mathcal{L}$ has a critical edge.

We shall see this phenomenon in the Kolaitis–Prömel–Rothschild theorem, and in many other cases.

The Product Conjecture

When I started working in extremal graph theory, I formulated (and later slightly modified) a conjecture on the structure of extremal graphs in non-degenerate cases (i.e., when no excluded graph L is bipartite). The meaning of this conjecture is that all the non-degenerate extremal graph problems can be reduced to degenerate extremal graph problems.

Conjecture 5.8 (Product structure). Let $\mathcal{L}$ be a family of forbidden graphs and $\mathcal{M}$ be the decomposition family of $\mathcal{L}$. If no trees and forests occur in $\mathcal{M}$, then all the extremal graphs S_n for $\mathcal{L}$ have the following structure: $V(S_n)$ can be partitioned into $p = p(\mathcal{L})$ subsets $V_1, \dots, V_p$ so that V_i is *completely joined* to V_j for every $1 \le i < j \le p$.

This implies that each S_n is the product of p graphs G_i, where each G_i is extremal for some degenerate family $\mathcal{L}_{i,n} \supseteq \mathcal{M}$. The meaning of this conjecture is that (almost) all the non-degenerate extremal graph problems can be reduced to degenerate extremal graph problems.

One non-trivial illustration of this conjecture is the Octahedron theorem:

Theorem 5.9 (Erdős–Simonovits [85]). Let $O_6 = K_3(2,2,2)$ (i.e. O_6 is the graph defined by the vertices and edges of the octahedron.) If S_n is an extremal graph for O_6 for $n > n_0(O_6)$, then $S_n = H_m \otimes H_{n-m}$ for some

$m = \frac{1}{2}n + o(n)$. Further, H_m is an extremal graph for C_4 and H_{n-m} is extremal for P_3.

Remark 5.10. The last sentence of this theorem is an easy consequence of that S_n is the product of two other graphs of roughly the same size.

Remark 5.11. In [85] some generalizations of the above theorem can also be found. Thus e.g., the analogous product result holds for all the forbidden graphs $L = K_{p+1}(2, t_2, \dots, t_p)$ and $L = K_{p+1}(3, t_2, \dots, t_p)$. Instead of formulating the general result we just give an illustration. If e.g., we try to apply this to $L = K(3, a, b, c)$, where $3 \le a \le b \le c$, then any $S_n \in \mathrm{EX}(n, L)$ will be the product of three graphs, where the first one is $K(3, a)$-extremal, the other two are $K(1, b)$-extremal (i.e. almost $(b-1)$-regular).

Probably the octahedron theorem can be extended to all graphs $L = K_{p+1}(t_1, t_2, \dots, t_p)$ and even to more general cases, see [330]. On the other hand, in [157] counterexamples are constructed to the product-conjecture if we allow trees or forests in the decomposition family. In this case, when the decomposition contains trees, both cases can occur: the extremal graphs may be non-products and also they may be products. Turán's theorem itself is a product-case, where the decomposition family contains $K_2 = P_2$.

Szemerédi Lemma on Regular Partitions of Graphs

There are many important tools in Extremal Graph Theory that became quite standard to use over the last 20 years. One of them is the Szemerédi Regularity Lemma [169].

Let G be an arbitrary graph, $X, Y \subset V(G)$ be two disjoint vertex-sets and let $d(X, Y)$ denote the edge-density between them:

$$d(X, Y) = \frac{e(X, Y)}{|X| \cdot |Y|}.$$

Regularity lemma ([169]). *For every $\varepsilon > 0$, and every integer κ there exists a $k_0(\varepsilon, \kappa)$ such that for every G_n, we can partition $V(G_n)$ into sets $V_0, V_1, \dots, V_k$ – for some $\kappa < k < k_0(\varepsilon, \kappa)$ – so that $|V_0| < \varepsilon n$, each $|V_i| = m$ for $i > 0$ and for all but at most $\varepsilon \cdot \binom{k}{2}$ pairs (i, j), for every $X \subseteq V_i$ and $Y \subseteq V_j$, satisfying $|X|, |Y| > \varepsilon m$, we have*

$$|d(X, Y) - d(V_i, V_j)| < \varepsilon.$$

The applications of Szemerédi's Regularity Lemma are plentiful and are explained in details in [120], so here we shall describe it only very briefly.

One feature of the Regularity Lemma is that – in some sense – it allows us to handle a deterministic graph as if it were a (generalized) random one. One can easily prove for random graphs the existence of various subgraphs and the Regularity Lemma often helps us to ensure the existence of the same subgraphs when otherwise that would be far from trivial.

One example of this is the Erdős–Stone Theorem. Knowing Turán's theorem, the Szemerédi Lemma immediately implies the Erdős–Stone theorem. In the previous section we have mentioned a few improvements of the original Erdős–Stone theorem. The proof of the Chvátal–Szemerédi version [39] also uses the Regularity Lemma as its main tool. Joining to the work of Thomason [171, 172], Fan Chung, Graham, Wilson [37, 36] and others, V. T. Sós and I used the Regularity Lemma to give a transparent description of the so-called *quasi-random* graph sequences [160], that was generalized by Fan K. Chung to hypergraphs [35].

Remark [N] 9. Actually, the Regularity Lemma described in [35] was a "weak hypergraph regularity lemma". Weak, because it could not be connected to an appropriate Counting Lemma. This lemma was obtained slightly earlier by Frankl and Rödl [99], but published slightly after [35]. Later a Strong Hypergraph Lemma was established by Rödl and his school, among others, Nagle [303], Skokan [323, 324], Schacht [319, 320], directly connected to a Counting Lemma and also to the Removal Lemma. Tim Gowers also needed and established (a) strong hypergraph lemma [263]. See also the results of Tao on this subject [334, 335, 336, 337].

These results are much more complicated than the weak regularity lemma. Just to formulate them takes a lot of time and energy. Here we skip the details.

The main advantage of Chung's Hypergraph Regularity Lemma [35] was that it helped to extend the Simonovits-Sós results of [160] to hypergraphs.

Here I mention only that it was extended to hypergraphs by Frankl and Rödl [99], see also Chung [35]. Prömel and Steger also use a hypergraph version of the Regularity lemma in "induced extremal graph problems" [143].

Further sources to read: Diestel's book [187] also contains a classical proof of the regularity lemma.

Frieze and Kannan produced a Weak Regularity Lemma [257, 258, 259], that was very applicable in algorithms, and iterating it we got an other proof of the Szemerédi Regularity Lemma.

Lovász and Szegedy [296] followed a similar approach, however, used limits of graphs and analysis.

See also the Theorem and the Removal Lemma of Ruzsa and Szemerédi, answering a question of [239], [147], Solymosi [333], Conlon and Fox [242], G. Elek and B. Szegedy [247] and many others.

Rödl and Schacht, setting out from some results of Alon and Shapira, extend results on the connection of Property Testing and Regularity Lemma to hypergraphs [322].

Back to Ordinary Graphs?

The (ordinary graph) Regularity Lemma can be generalized in various ways. One of these generalizations states that if the edges of G_n are r–coloured for some fixed r, then we can partition the vertices of the graph so that the above Regularity Lemma remains true in all the colours simultaneously. This is what we used among others in proving some Turán–Ramsey type theorems [72, 73, 74] but it has also many other applications.

Generalized Regularity lemma. *For every $\varepsilon > 0$, and integers r, κ there exists a $k_0(\varepsilon, \kappa, r)$ such that for every graph G_n the edges of which are r–coloured, the vertex set $V(G_n)$ can be partitioned into sets $V_0, V_1, \dots, V_k$ – for some $\kappa < k < k_0(\varepsilon, \kappa, r)$ – so that $|V_0| < \varepsilon n$, $|V_i| = m$ (is the same) for every $i > 0$, and for all but at most $\varepsilon\binom{k}{2}$ pairs (i, j), for every $X \subseteq V_i$ and $Y \subseteq V_j$ satisfying $|X|, |Y| > \varepsilon m$, we have*

$$|d_\nu(X, Y) - d_\nu(V_i, V_j)| < \varepsilon \quad \text{simultaneously for} \quad \nu = 1, \dots, r,$$

where $d_\nu(X, Y)$ is the edge-density in colour ν.

As I mentioned, we describe the various applications of Szemerédi Lemma in more details in some other places [120, 292].

Algorithmic versions were found by Alon, Duke, Lefmann, Rödl, and Yuster [5], and in some sense it was extended to sparse graphs by Kohayakawa [119], and Kohayakawa–Rödl [192].

I should also mention some new variants due to Komlós, see, for example, [120].

Remark [N] 10 (Historical remarks). Actually, most of the mathematicians may think that Szemerédi invented the Regularity Lemma to prove his famous result (the Erdős-Turán conjecture) on the existence of arithmetic progressions in sequences of integers of positive density. The truth is that the first applications used a less appealing version of the Regularity Lemma, or some simpler version. The first occasion where the Szemerédi Regularity Lemma was used is the Quantitative Erdős-Stone Theorem of Chvátal and Szemerédi [39], (see above as Theorem 3.3).

Remark [N] 11 (Regularity Lemma, Old and New Areas). Let us return to the field of ordinary regularity lemma. If one is interested in it, there is a long list of papers that are reasonably easy to read and introduce to various aspects of the Regularity Lemma.

1. Blow Up Lemmas: There are cases, when we wish to embed a spanning subgraph into G_n, and it would be easy to embed $n - o(n)$ vertices, but we may have trouble with embedding the last few vertices. The Blow-Up Lemma was “invented” for this purpose, by Komlós, Sárközy and Szemerédi, see [289, 290], see also Rödl and Ruciński [318],
2. Algorithmic Aspects of Regularity Lemmas [5], ...

3. Sparse Regularity Lemmas and their applications [192, 284, 285, 286, 287, 288],
4. Regularity Lemma and Property Testing: Alon and Shapira [221, 220], Alon, Fischer, Newman, and Shapira [216], Lovász and Szegedy [297, 298]. (An early result on this topic, slightly disguised, can be found in Bollobás–Erdős–Simonovits–Szemerédi [234].) For a "Strong" Regularity Lemma see Alon, Fischer, Krivelevich and M. Szegedy [4].

6. Turán–Ramsey Problems

Simonovits and V. T. Sós has a survey [331] on Turán–Ramsey type theorems initiated by V. T. Sós, and the connection of this to other fields. These fields belong to Extremal Graph Theory and are strongly influenced by Paul Erdős. I will touch on these fields only very briefly.

These problems were partly motivated by applications of graph theory to distance distribution. Turán theorem combined with some geometrical facts can provide us with estimates on the number of short distances in various geometrical situations. Thus they can be applied in some estimates in analysis, probability theory, and so on. It was Erdős who first pointed out this possibility of applying Graph Theory to distance distribution theorems [47] and later Turán in [176] initiated investigating these problems more systematically. This work culminated in three joint papers of Erdős, Meir, Sós and Turán [76, 77, 78].[13]

The structure of the extremal graphs in Turán type theorems seems to be too regular. So we arrive at the question: How do the upper bounds in extremal graph theorems improve if we exclude graphs very similar to the Turán graphs? Basically this was what motivated V. T. Sós [162] in initiating a new field of investigation. Erdős joined her and they have proved quite a few nice results, see e.g., [89, 90, 91].

Let $\alpha_p(G)$ denote the maximum cardinality of vertices in G such that the subgraph spanned by these vertices contains no K_p.

> *General Problem.* Assume that $L_1, \dots, L_r$ are given graphs, and G_n is a graph on n vertices the edges of which are coloured by r colours $\chi_1, \dots, \chi_r$, and
>
> $$\begin{cases} \text{for } \nu = 1, \dots, r \text{ the subgraph of colour } \chi_\nu \text{ contains no } L_\nu \\ \text{and } \alpha_p(G_n) \le m. \end{cases}$$
>
> What is the maximum of $e(G_n)$ under these conditions?

Originally the general problem was investigated only for $p = 2$,[14] and one breakthrough was the Szemerédi–Bollobás–Erdős theorem:

[13] See also Corrigendum to [76].

[14] This means that in all the papers quoted but [74] we restrict ourselves to ordinary independent sets.

Theorem 6.1 (Szemerédi [170]). If (G_n) is a sequence of graphs not containing K_4 and the stability number $\alpha(G_n) = o(n)$, then

$$e(G_n) \le \frac{1}{8}n^2 + o(n^2). \tag{3}$$

Erdős asks if the $o(n^2)$ error term is necessary: Is it true that in Theorem 6.1 the stronger

$$e(G_n) \le \frac{n^2}{8}$$

also holds?

Theorem 6.2 (Bollobás–Erdős [18]). *Equation (3) is sharp.*

Many estimates concerning general and various special cases of this field are proved in [72, 73, 74]. Here we mention just one result:

Theorem 6.3 (Erdős–Hajnal–Simonovits–Sós–Szemerédi [74]).

(a) For any integers $p > 1$ and $q > p$ if $\alpha_p(G_n) = o(n)$ and $K_q \not\subseteq G_n$, then

$$e(G_n) \le \frac{1}{2}\left(1 - \frac{p}{q-1}\right)\binom{n}{2} + o(n^2).$$

(b) For $q = pk + 1$ this upper bound is sharp.

One of the most intriguing open problems of the field is (among many other very interesting questions)

Problem 6.4. Assume that (G_n) is a sequence of graphs not containing $K(2,2,2)$. If $\alpha(G_n) = o(n)$, does it follow that $e(G_n) = o(n^2)$?

I conclude this section with a slightly different result of Ajtai, Erdős, Komlós and Szemerédi. Let t be the average degree of G_n. Turán's theorem guarantees an independent set of size $\frac{n}{t+1}$.

Theorem 6.5 ([1]). There exists a contant $c > 0$ for which, if the number of $K_3 \subseteq G_n$ is $o(n^3)$, then

$$\alpha(G_n) > c\frac{n}{t}\log t \quad \text{for} \quad t = \frac{2e(G_n)}{n}.$$

The nice feature of the above theorem is that it says: if the number of triangles in G_n is $o(n^3)$, then the size of the maximum independent set jumps by a log-factor. This is sharp: $\frac{n}{t}\log t$ is achieved for random graphs.

Theorem 6.5 can also be interpreted as follows: excluding the triangles (or assuming that there are only few triangles in our graph) leads to randomlike behaviour. (See also [206, 207].)

Remark [N] 12 (Some newer results.). Sudakov proved that if $\alpha(G_n)$ is "slightly smaller than" $e^{c\sqrt{\log n}}$, and $K_4 \not\subseteq G_n$, then $e(G_n) = o(n^2)$ [332].

Jacob Fox, Po-Shen Loh and Y. Zhao [254] described very precisely the phase-transition when G_n does not contain K_4 and $e(G_n)$ is nearly $\frac{1}{8}n^2$.

József Balogh and Lenz found very nice new constructions to obtain good lower bounds for several cases, [227, 228]

Balogh-Ping-Simonovits [229] investigated, when does $e(G_n)$ drastically drops, as we decrease $\alpha(G_n)$ "continuously" from $\binom{n}{2}$ to 2.

Remark [N] 13 (A minor correction). In one of the proofs of [74] we (Erdős, Hajnal, Simonovits, Sós, and Szemerédi) used an isoperimetric theorem from [232], which turned out to be false. The original proof could not be saved. However, Balogh and Lenz [227] saved the essential part of our result and extended it to several other cases. For details, see [227].

7. Cycles in Graphs

Cycles play central role in graph theory. Many results provide conditions to ensure the existence of some cycles in graphs. Among others, the theory of Hamiltonian cycles (and paths) constitute an important part of graph theory. The Handbook of Combinatorics contains a chapter by A. Bondy [23] giving a lot of information on ensuring cycles via various types of conditions. Also, the book of Walther and Voss [179] and the book of Voss [178] contain many relevant results. Below we shall approach the theory of degenerate extremal graph problems (see Sect. 8) through extremal graph problems with forbidden cycles. Of course, one of the simplest extremal graph problems is when $\mathcal{L}$ is the family of all cycles. If we exclude them, the considered graphs will be all the trees and forests; the extremal graphs are the trees.

Remark 7.1. Describing walks and cycles in graphs is perhaps one of those parts of extremal graph theory, where algebraic methods may come in more often than in other extremal problems. So here occasionally, and very superficially, I will speak of Margulis graphs, Ramanujan graphs and Cayley graphs. I feel, these topics are very important, not only because of expander graphs but also because they provide new methods to construct nice graphs in extremal graph theory. I warmly recommend Noga Alons' chapter from the Handbook of Combinatorics: Tools from Higher Algebra [2], which provides a lot of interesting and useful information – among others – on topics I had to describe very shortly.

Excluding Long Cycles

One problem posed by Turán was the extremal problem of cycles of length m. If we exclude all the odd cycles, the extremal graph will be the Turán graph $T_{n,2}$. What are the extremal graphs if the family of excluded graphs is the family $\mathcal{L}_m$ of cycles of length at least m. The answer is given by the Erdős–Gallai theorem:

Theorem 7.2 (**Erdős and Gallai** [69]). Let $\mathcal{L}_m = \{C_k : k \geq m\}$. Then

(i) $\frac{m-1}{2}n - \frac{1}{2}m^2 < \mathrm{ex}(n, \mathcal{L}_m) \leq \frac{m-1}{2}n$ and
(ii) The connected graphs G_n whose 2-connected blocks are K_{m-1}'s are extremal.

Graphs described in (ii) do not exist for all n, but we get asymptotically extremal graphs for all n, by taking those graphs in which one 2-connected component has size at most $m-1$ and all the other blocks are complete $m-1$–graphs.

The following theorem is the twin of the previous one's.

Theorem 7.3 (**Erdős and Gallai** [69]).

$$\mathrm{ex}(n, P_m) \leq \frac{m-2}{2}n.$$

The union of $\lfloor \frac{n}{m-1} \rfloor$ vertex disjoint K_{m-1} (and one smaller K_q) shows that this is sharp: $\mathrm{ex}(n, P_m) = \frac{m-2}{2}n + O(m^2)$.

This theorem has a sharper form, proved by Faudree and Schelp [95]. They needed the sharper form to prove some Ramsey theorems on paths. (See also Kopylov [293].)

These theorems can also be used to deduce the existence of Hamilton paths and cycles. Thus, for example, Theorem 7.2 implies Dirac's famous result:

Theorem 7.4 (**Dirac**). If the minimum degree of G_{2k} is at least k, then G_{2k} is Hamiltonian.

7.1. Excluding Fixed Trees

Erdős and T. Sós observed that the same estimates hold both for the path P_m and the star $K_2(1, m-1)$ and these being two extremes among the trees of m vertices, they conjectured that [54]:

Conjecture 7.5 (**Erdős–T. Sós**). *For any tree T_m,*

$$\mathrm{ex}(n, T_m) \leq \frac{m-2}{2}n.$$

Of course, this implies $\mathrm{ex}(n, T_m) \leq \frac{m-2}{2}n + O(1)$. Some asymptotical approximations of this conjecture were proved by Ajtai, Komlós and Szemerédi, (unpublished), also, the conjecture is proved in its sharp form for some special families of trees, like caterpillars, large girth graphs [238], graphs with many leaves [327], graphs of small diameters [301], . . .

Remark [N] 14 (**Embedding trees into graphs**). The above Conjecture 7.5 is now solved for all suffficiently large trees, by Ajtai, Komlós, Simonovits and Szemerédi, though the publishing of this theorem is not yet finished [202, 204, 203].

Theorem 7.6 (Ajtai-Komlós-Simonovits-Szemerédi). *There exists a k_0 such that for all $k > k_0$ the Erdős-Sós conjecture holds.*

There is another conjecture, strongly related to the Erdős-Sós Conjecture, namely, the Loebl-Komlós-Sós conjecture. This question has a special form, (Loebl) and a general form (Komlós, Sós) and the Loebl Conjecture comes from a problem of Erdős, Füredi, Loebl and Sós [252].

Conjecture 7.7 (Komlós-Sós). If G_n has at least $n/2$ vertices of degree at least k then it contains all the trees of $k+1$ vertices.

The original problem of Loebl was this Conjecture for $n = k$.

This was obtained through a series of partial results by Ajtai-Komlós-Szemerédi [208], Yi Zhao [341], Piguet-Stein [309], Cooley–Hladký–Piguet [244]... and finally

Theorem 7.8 (Hladký-Komlós-Piguet-Simonovits-Stein-Szemerédi). *There exists a k_0 such that for all $k > k_0$ the Komlós–Sós conjecture holds.*

Further sources to read:

(a) A much more detailed description of this can be found in Simonovits–Füredi [188], see also Sidorenko [327], Brandt and Dobson [238], and Andrew McLennan [301], Saclé and Wozniak [325, 339].
(b) As to the Komlós–Sós conjecture, see [273] for the "approximative solution" and a longer description of the situation, or [188].
(c) We have to remark here that Yi Zhao (a Student of Szemerédi) was the first to prove a result in this field, superseding [208], see his PhD thesis [340], or his preprint, [341], however, this paper was very difficult to read and finally it was superseded by Piguet and Stein [309, 308] and Cooley [243]. (See also [274].)

Excluding C_{2k}

Since the odd cycles are 3-chromatic colour-critical, one can apply Theorem 5.7 to them to get

$$\mathrm{ex}(n, C_{2k+1}) = \left[\frac{n^2}{4}\right] \quad \text{if} \quad n > n_0(k).$$

The case of even cycles is much more fascinating. The upper bound would become trivial if we assumed that G_n is (almost) regular and contains no cycles of length $\le 2k$. The difficulty comes from that we exclude *only* C_{2k}.

Theorem 7.9 (Erdős, Bondy–Simonovits [25]).

$$\mathrm{ex}(n, C_{2k}) < ckn^{1+1/k} + o(n^{1+1/k}).$$

Theorem 7.10 (Bondy–Simonovits [25]). If $e(G_n) > 100kn^{1+1/k}$, then

$$C_{2\ell} \subseteq G_n \quad \text{for every integer} \quad \ell \in [k, kn^{1/k}].$$

Erdős stated Theorem 7.9 in [54] without proof and conjectured Theorem 7.10, which we proved. The upper bound on the cycle-length is sharp: take a G_n which is the union of complete graphs.

Let us return to Theorem 7.9. Is it sharp? Finite geometrical (and other) constructions show that for $k = 2, 3, 5$ YES. (Singleton [161], Benson [9], Wenger [180] ....) Unfortunately, nobody knows if this is sharp for C_8, or for other C_{2k}'s.

Faudree and I sharpened Theorem 7.9 in another direction:

Definition 7.11 (Theta-graph). $\Theta(k, p)$ is the graph consisting of p vertex-independent paths of length k joining two vertices x and y.

Clearly, $\Theta(k, p)$ is a generalization of C_{2k}. We have proved

Theorem 7.12 (Faudree–Simonovits [96]). $\mathrm{ex}(m, \Theta(k, p)) < c_{k,p}n^{1+1/k}$.

The Erdős–Rényi Theorem [80] shows that Theorem 7.12 is sharp in the sense that

$$\mathrm{ex}(n, \Theta(k, p)) > c^*_{k,p}n^{1+\frac{1}{k}+\frac{1}{kp}} \quad \text{as} \quad n \to \infty.$$

One could ask if there are other global ways to state that if a graph has many edges then it has many cycles of different length. Erdős and Hajnal formulated such a conjecture, which was proved by A. Gyárfás, J. Komlós and E. Szemerédi. Among others, they proved

Theorem 7.13 (Gyárfás–Komlós–Szemerédi [108]). If $d_{min}(G) \geq \delta$ and $\ell_1, \ldots, \ell_m$ are the cycle-lengths of G, then

$$\sum \frac{1}{\ell_i} \geq c_1 \log \delta.$$

The meaning of this is as follows: If we regard all the graphs with minimum degree δ and try to minimize the sum of the reciprocals of the cycle-lengths, two candidates should first be checked. One is the union of disjoint $K_{\delta+1}$'s, the other is the union of disjoint complete bipartite graphs $K(\delta, \delta)$'s. In the first case we get $\log \delta + O(1)$, in the second one $\frac{1}{2} \log \delta + O(1)$. The above theorem asserts that these cases minimize $\sum \frac{1}{\ell_i}$.

Some graph theorists could be surprised by measuring the density of cycle lengths this way. Yet, whenever we want to express that something is nearly linear, then in number theory we tend to use this measure. Thus, e.g. the famous \$3,000 problem of Erdős asks for the following sharpening of Szemerédi's theorem on Arithmetic Progressions [168]:

Conjecture 7.14 (Erdős). Prove that if $A = \{a_1 < a_2 < \cdots\}$ is an infinite sequence of positive integers and

$$\sum \frac{1}{a_i} = \infty,$$

then for every k, A contains a k–term arithmetic progression.

Remark [N] 15. When Erdős posed the above conjecture, he was interested among other if

Q: the set of primes contain arbitrary long arithmetic progressions.

In the paper of Erdős [250] that appeared in the Birthday Volume "Paul Erdős is 80" he mentioned that the longest arithmetic progression of primes was of length 17 those days. It was a fantastic breakthrough when Ben Green and Terence Tao proved [266] that YES, Q is true.

Remark [N] 16 (Θ-click-in). One could weaken Erdős Conjecture by asking: Is it true at least, that for every k there is a $p = p_k$ for which, for some $c_k > 0$

$$\text{ex}(n, \Theta(k, p)) \;>\; c_k \cdot n^{1+1/k}?$$

One feels that similarly to the Kollar-Rónyai-Szabó construction [294] (see also Alon-Rónyai-Szabó [219]) one should be able to find a construction proving this. J. Verstraëte has some results into this direction [338].

Very Long Cycles

We know that a graph with minimum degree 3 contains a cycle of length at most $2\log_2 n$. The other extreme is when (instead of short cycles) we wish to ensure very long cycles. We may go much beyond the Erdős–Gallai theorem if we increase the connectivity and put an upper bound on the maximum degree.

Theorem 7.15 (Bondy–Entringer [24]). Let $f(n, d)$ be the largest integer k such that every 2–connected G_n with maximum degree d contains a cycle of length at least k. Then

$$4\log_{d-1} n - 4\log_{d-1}\log_{d-1} n - 20 < f(n, d) < 4\log_{d-1} n + 4.$$

Clearly, the connectivity is needed, otherwise – as we have seen – the Erdős–Gallai theorem is sharp. They also considered the case of d-regular graphs, proving that

Theorem 7.16 (Bondy–Entringer [24]). If G_n is 2–connected and d-regular, then G_n contains a cycle of length at least

$$\max\{2d, 4\log_{d-1} n - 4\log_{d-1}\log_{d-1} n - 20\}.$$

Bondy and I proved that increasing the connectivity leads to a steap jump:

Theorem 7.17 (Bondy–Simonovits [26]). If G_n is 3–connected and the minimum degree of G_n is d, the maximum degree is D, then G_n contains a cycle of length at least $e^{c\sqrt{\log n}}$ for some $c = c(d, D)$.

We conjectured that $e^{c\sqrt{\log n}}$ can be improved to n^c. Bill Jackson, Jackson and Wormald succeded in proving this:

Theorem 7.18 (B. Jackson [112, 115]). If G_n is 3–connected and the minimum degree of G_n is d, the maximum degree is D, then G_n contains a cycle of length at least n^c for some $c = c(d, D)$.

Increasing the connectivity higher does not help in getting longer cycles:

Theorem 7.19 (Jackson, Parson [113]). For every $d > 0$ there are infinitely many $d + 2$–regular d–connected graphs without cycles longer than n^γ for some $\gamma = \gamma_d < 1$.

See also [114]. We close this topic with an open problem:

Conjecture 7.20 (J. A. Bondy). *There exists a constant $c > 0$, such that every cyclically 4-connected 3-regular graph G_n contains a cycle of length at least cn.*

Erdős–Pósa Theorem

The following question of Gallai is motivated partly by Menger Theorem. If G is a graph

(∗) not containing two independent cycles,

how many vertices are needed to represent all the cycles of G?

K_5 satisfies (∗) and we need at least 3 vertices to represent all its cycles. Bollobás [11] proved that in all the graphs satisfying (∗) there exist 3 vertices the deletion of which results in a tree (or forest). More generally,

> Let $RC(k)$ denote the minimum t such that if a graph G contains no $k+1$ independent cycles, then one can delete t vertices of G ruining all the cycles of the graph. Determine $RC(k)$!

Erdős and Pósa [79] proved the existence of two positive constants, c_1 and c_2 such that

$$c_1 k \log k \le RC(k) \le c_2 k \log k. \tag{4}$$

This theorem is strongly connected to the following extremal graph theoretical question:

> Assume that G_n is a graph in which the minimum degree is D. Find an upper bound on the girth of the graph.

Here the usual upper bound is

$$\approx \frac{2\,\log n}{\log(D-1)}. \tag{5}$$

The proof is easy: Assume that the girth is g and let $k = \lfloor \frac{g-1}{2} \rfloor$. Take a vertex x and denote the set of vertices having distance t from x by X_t. Then for $t \le k$ we have $|X_t| \ge (D-1)|X_{t-1}|$. Therefore

$$n = v(G_n) \ge 1 + D + D(D-1) + D(D-1)^2 + \cdots + D(D-1)^k.$$

This implies (5).

These things are connected to many other parts of Graph Theory, in some sense even to the Robertson–Seymour theory. Below I shall try to convince the reader that the Gallai problem is strongly connected to the girth problem.

In [153] I gave a short proof of the upper bound of (4). My proof goes as follows (sketch!):

Let G_n be an arbitrary graph not containing $k+1$ independent circuits. Let H_m be a maximal subgraph of G_n all whose degrees are 2, 3 or 4. Then one can immediately see that the ramification vertices of H_m, i.e. the vertices of degree 3 or 4 represent all the cycles of G_n.[15] Let μ be the number of these vertices. Replacing the hanging chains[16] by single edges, we get an H_μ each degree of which is 3 or 4. So one can easily find a cycle $C^{(1)}$ of length $\le c_3\,\log\mu$ in H_μ. Applying this to $H_\mu - C^{(1)}$ (but first cleaning up the resulting low degrees) we get another short cycle $C^{(2)}$. This cleaning up is where we have to use that the degrees are bounded from above. Iterating this (and using that the degrees are bounded from above) one can find $c_4\mu/\log\mu$ vertex–independent cycles in G_n. Since $k \le c_4\mu/\log\mu$, therefore $\mu \le c_5 k\,\log k$. ∎

The Erdős–Pósa theorem is strongly connected with the girth problem. If, e.g. we had shorter circuits in graphs with degrees 3 and 4 then the above proof would give better upper bound on $RC(k)$ – but that is ruled out.

Remark [N] 17 (Directed graph version). It was Gallai [262] who first asked for extending the Erdős–Pósa theorem to digraphs. The problem is highly non-trivial even in its simplest case, when any pair of directed cycles have a vertex in common. Perhaps this is why in its published form Gallai asked only to prove this simplest case, for me it is obvious that he also meant the general case. The general case formally was formulated only 5 years later, by D. Younger [342].

McCuaig [300], proved that if in a digraph any two directed cycles have a common vertex, then one can delete 3 vertices to turn the graph into an acyclic one. Reed, Robertson, Seymour, and Thomas settled the general question [314], proving that for every integer $k \ge 0$ there exists an integer

[15] The vertices of degree 4 are not really needed ...

[16] paths all whose inner vertices have degree 2

$t > 0$ such that for every digraph G, either G has k vertex-disjoint directed circuits, or G can be made acyclic by deleting at most t vertices.

The Margulis Graphs and the Lubotzky–Phillips–Sarnak Graphs

Sometimes we insist on finding *constructions* in certain cases when the probabilistic methods work easily. Often finding explicit constructions is very difficult. A good example of this is the famous case of the Ramsey 2–colouring, where Erdős offered a larger sum (money) for finding a construction of a graph of n vertices not containing complete graphs or independent sets of at least $c \log n$ vertices, for some large constant c. (See Frankl and Wilson [100].)

Another similar case is the *girth* problem discussed above, with one exception. Namely, in the girth problem Margulis [133] and in constructing Ramanujan graphs Margulis [134, 135, 136] and Lubotzky–Phillips-Sarnak [128, 129], succeeded in constructing regular graphs G_n of (arbitrary high) but fixed degree d and girth at least $c_d \log n$. (The original existence proof is due to Erdős and Sachs [82] and uses induction.)

The graphs we consider here are Cayley graphs. Below (skipping many details)

(a) First we explain, why should one try Cayley graphs of *non-commutative groups*,
(b) Then we give a sketch of the description of the first, simpler Margulis graph [133].
(c) Finally we list the main features of the Lubotzky–Phillips–Sarnak graph [133].

  (a) Often cyclic graphs are used in the constructions. Cyclic graphs are the graphs where a set $A_n \subseteq [1, n]$ is given, the vertices of our graphs are the residue classes $Z_i \pmod n$, and Z_i is joined to Z_j if $|i-j| \in A_n$ (or $|n+i-j| \in A_n$). One such well known graph is Q_p (the Paley graph) obtained by joining Z_i to Z_j if their difference is a quadratic non-residue. The advantage of such graphs is that they have great deal of fuzzy (randomlike) structure. From the point of view of the short cycles they are not the best: they have many short even cycles.
  (b) Given an arbitrary group $\mathcal{G}$ and some elements $g_1, \ldots, g_t \in \mathcal{G}$, these elements generate a Cayley graph on $\mathcal{G}$: we join each $a \in \mathcal{G}$ to the elements $ag_1, \ldots, ag_t$. This is a digraph. If we are interested in ordinary graphs, we choose $g_1, \ldots, g_t$ so that whenever g is one of them, then also $g^{-1} \in \{g_1, \ldots, g_t\}$. Thus we get an undirected graph. Still, if $\mathcal{G}$ is commutative, then this Cayley graph will have many even cycles. For example, $a, ag_1, ag_1g_2, ag_1g_2g_1^{-1}, ag_1g_2g_1^{-1}g_2^{-1}$ is (mostly) a C_4 for commutative groups and a P_5 for our non-commutative

groups. So, if we wanted to obtain Cayley graphs with large girth, we have better to start with non-Abelian groups. This is what Margulis did in [133]:

Let X denote the set of all 2×2 matrices with integer entries and with determinant 1. Pick the following two matrices:

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

It is known that they are independent in the sense that there is no non-trivial multiplicative relation between them.[17] So, if we take the 4 matrices A, B, A^{-1} and B^{-1}, they generate an infinite Cayley graph which is a 4-regular tree. If we take everything mod p, then it is easy to see that the tree collapses into a graph of $n \approx p^3$ vertices, in which the shortest cycle has length at least $c \log p$ for some constant $c > 0$. This yields a sequence of 4–regular graphs X_n with girth $\approx c^* \log n$ for some $c^* \approx 0.91\ldots$ Margulis also explains, how the above graphs can be used in constructing certain (explicit) error-correcting codes. Margulis has also generalized this construction (in the same paper) to arbitrary even degrees.

Theorem 7.21 (Margulis [133]). For every $\varepsilon > 0$ we have infinitely many values of r, and for each of them an infinity of regular graphs X_j of degree $2r$ with girth

$$g(X_j) > \left(\frac{4}{9} - \varepsilon\right) \frac{\log v(X_j)}{\log r}.$$

(c) The next breakthrough was due to Margulis [134] and to Lubotzky, Phillips and Sarnak [128]. The graph of Lubotzky, Phillips and Sarnak was obtained *not* for extremal graph purposes. They were interested in the extremal spectral gap of d-regular graphs: they constructed graphs where the difference between the first and second eigenvalues is as large as possible. Graphs with large spectral gaps are good expanders, and this was perhaps the primary interest in [128] or in [134]. As the authors of [128] remarked, Noga Alon turned their attention to the fact that their graphs can be "used" also for many other, "classical" purposes.

Definition 7.22. Let X be a connected k–regular graph. Denote by $\lambda(X)$ the second largest eigenvalue (in absolute value) of the adjacency matrix of X.

Definition 7.23. A k–regular graph on n vertices, $X = X_{n,k}$, will be called a *Ramanujan graph*, if $\lambda(X_{n,k}) \le 2\sqrt{k-1}$.

[17] This is far from being trivial but can be proved, or found e.g. in [194].

I do not have the place here to go into details, but the basic idea is that random graphs have roughly the spectral gap[18] required above and vice versa: if the graph has a large spectral gap, then it may be regarded in some sense, as if it were a random graph. So the Ramanujan graphs provide near-extremum in some problems, where random graphs are near-extremal. (See also Alon [3], Chung, Graham and Wilson [37], Füredi and Komlós [260], and Alon–Boppana [214].)

Let p, q be distinct primes congruent to 1 mod 4. The Ramanujan graph $X^{p,q}$ of [128] is a $p+1$–regular Cayley graph of $PSL(2, \mathbb{Z}_q)$ if the Legendre symbol $(\frac{p}{q}) = 1$ and of $PGL(2, \mathbb{Z}_q)$ if $(\frac{p}{q}) = -1$. (Here $\mathbb{Z}_q$ is the field of integers mod q.)

Theorem 7.24 (**Alon, quoted in [128]**). Let $X_{n,k} = X^{p,q}$ be a non-bipartite Ramanujan graph; $(\frac{p}{q}) = 1$, $k = p+1$, $n = q(q^2-1)/2$. Then the independence number

$$\alpha(X_{p,q}) \le \frac{2\sqrt{k-1}}{k} n.$$

Corollary 7.25 (**[128]**). If $X_{n,k}$ is a non-bipartite Ramanujan graph, then

$$\chi(X_{n,k}) \ge \frac{k}{2\sqrt{k-1}}.$$

Margulis, Lubotzky, Phillips and Sarnak have constructed Ramanujan graphs which are $p+1$–regular, and

(a) *bipartite* with $n = q(q^2-1)$ vertices, satisfying

$$\text{girth}(X_{n,p+1}) \ge \frac{4}{3}\frac{\log n}{\log p} - O(1) \quad \text{and} \quad \text{diam}(X_{n,p+1}) \le \frac{2}{3}\frac{\log n}{\log p} + 3.$$

Further, they constructed *non-bipartite Ramanujan graphs* with $n = q(q^2-1)/2$ vertices, and with the same diameter estimate and with

$$\text{girth}(X_{n,p+1}) \ge \frac{2}{3}\frac{\log n}{\log p} + O(1), \quad \alpha(X_{n,p+1}) \le \frac{2\sqrt{p}}{p+1} n, \quad \chi(X_{n,p+1}) \ge \frac{p+1}{2\sqrt{p}}.$$

Putting $p = const$ or $p \approx n^c$ we get constructions of graphs the existence of which were known earlier only via random graph methods. As a matter of fact, they are better than the known “random constructions”, showing that

$$\text{ex}(n, C_{2k}) > c_k n^{1+\frac{4}{3k+25}}.$$

---

[18] = difference between the largest and second largest eigenvalues

Further sources to read:

(a) One may be interested in expander graphs and their constructions without really using this deep number theory. An extremely good start is to look for results of Avi Wigderson and his surrounding. Here I mention just a few papers: [215, 275, 316, 317].
(b) Another direction is to read about constructions connected to the constructions above, see the book of Lubotzky, or [194, 299], Sarnak [245, 326].
(c) There is also a fascinating area to be mentioned here, on graphs and their eigenvalues, Cvetkovic–Doob–Sachs [186], F. Chung [184] Alon–Boppana [214], Alon, Alon–Milman [218].
(d) One fascinating area is the use of Expanders in Theoretical Computer Science [205, 215]. Perhaps the Ajtai-Komós-Szemerédi Sorting Network [205] was the first example where a bounded degree expander graph was used to derandomize the algorithm.

8. Further Degenerate Extremal Graph Problems

We have already seen the most important degenerate extremal graph problems. Unfortunately we do not have as many results in this field as we would like to. Here we mention just a few of them.

Topological Subgraphs

Given a graph L, we may associate with it all its topologically equivalent forms. Slightly more generally, let $\mathcal{T}(L)$ be the set of graphs obtained by replacing some edges of L by "hanging chains", i.e., paths, all inner vertices of which are of degree 2.

Problem 8.1. Find the maximum number of edges a graph G_n can have without containing subgraphs from $\mathcal{T}(L)$.

Denote the topological complete p-graphs by $< K_p >$. G. Dirac [41] have proved that every G_n of $2n-2$ edges contains a $< K_4 >$. This is sharp: Dirac gave a graph G_n of $2n-3$ edges and not containing $< K_4 >$. Erdős and Hajnal pointed out that there exist graphs G_n of cp^2n edges and not containing $< K_p >$. (This can be seen, e.g. by taking $[n/q]$ vertex-disjoint union $K(q,q)$'s for $q = \binom{p/2}{2}$.) It was a breakthrough when Mader [130] (also see [131]) showed

Theorem 8.2. For every integer $p > 0$ there exists a $D = D(p)$ such that if the minimum degree of G is at least $D(p)$, then G contains a $< K_p >$.

More precisely,

Theorem 8.3. There exists a constant $c > 0$ such that if $e(G) > tn$, then G contains a $< K_p >$ for $p = [c\sqrt{\log t}]$.

Corollary 8.4. For every L, $\mathrm{ex}(n, \mathcal{T}(L)) = O(n)$.

Conjecture 8.5 (Erdős–Hajnal–Mader [71, 130]). *If $e(G_n) > tn$, then G_n contains a $< K_p >$ with $p \geq c\sqrt{t}$.*

Mader's result was improved by Komlós and Szemerédi to almost the best:

Theorem 8.6 ([121]). There is a positive c_1 such that if $e(G_n) > tn$, then G_n contains a $< K_p >$ with

$$p > c_1 \frac{\sqrt{t}}{(\log\ t)^6}.$$

Very recently, improving some arguments of Alon and Seymour, Bollobás and Thomason completely settled Mader's problem:

Theorem 8.7 (Bollobás and Thomason [22]). If $e(G_n) > 256p^2n$, then G_n contains a $< K_p >$

This means that they have got rid of the log t-power in Theorem 8.6. Their proof-method was completely different from that of Komlós and Szemerédi. Komlós and Szemerédi slightly later also obtained a proof of Theorem 8.7 (with some other constants) along their original lines [122].

Recursion Theorems

Recursion theorems could be defined for ordinary graphs and hypergraphs, for ordinary degenerate extremal problems and non-degenerate extremal graph problems, for supersaturated graph problems,... However, here we shall restrict our considerations to ordinary degenerate extremal graph problems. In this case we have a bipartite L and a procedure assigning an L' to L. Then we wish to deduce upper bounds on $\mathrm{ex}(n, L')$, using upper bounds on $\mathrm{ex}(n, L)$. To illustrate this, we start with two trivial statements.

Claim 8.8. Let L be a bipartite graph and L' be a graph obtained from L by attaching a rooted tree T to L at one of its vertices.[19] Then

$$\mathrm{ex}(n, L') = \mathrm{ex}(n, L) + O(n).$$

Claim 8.9. Let L be a bipartite graph and L' be a graph obtained by taking two vertex-disjoint copies of L. Then (again)

[19] This means that we take vertex-disjoint copies of L and T, a vertex $x \in V(T)$ and a vertex $y \in V(L)$ and identify x and y.

$$\text{ex}(n, L') = \text{ex}(n, L) + O(n).$$

The proofs are trivial.

One of the problems Turán asked in connection with his graph theorem was to find the extremal numbers for the graphs of the regular (Platonic) polytopes. For the tetrahedron the answer is given by Turán Theorem (applied to K_4). The question of the Octahedron graph is solved by Theorem 5.9, the problems of the Icosahedron and Dodecahedron can be found in Sect. 12, [154, 155]. On the cube-graph we have

Theorem 8.10 (Cube Theorem, Erdős–Simonovits [86]).

$$\text{ex}(n, Q_8) = O(n^{8/5}).$$

We conjecture that the exponent 8/5 is sharp. Unfortunately we do not have any "reasonable" lower bound.

The above theorem and many others follow from a recursion theorem:

Theorem 8.11 (Recursion Theorem, [86]). Let L be a bipartite graph, coloured in BLUE and RED and $K(t,t)$ be also coloured in BLUE and RED. Let L^* be the graph obtained from these two (vertex-disjoint) graphs by joining each vertex of L to all the vertices of $K(t,t)$ of the other colour. If $\text{ex}(n, L) = O(n^{2-\alpha})$ and

$$\frac{1}{\beta} - \frac{1}{\alpha} = t,$$

then $\text{ex}(n, L^*) = O(n^{2-\beta})$.

Applying this recursion theorem with $t = 1$ and $L = C_6$ we obtain the Cube-theorem.[20] Another type of recursion theorem was proved by Faudree and me in [96].

Regular Subgraphs

Let $\mathcal{L}_{r-reg}$ denote the family of r–regular graphs. Erdős and Sauer posed the following problem [64]:

> What is the maximum number of edges in a graph G_n not containing any k–regular subgraph?

Since $K(3,3)$ is 3-regular, one immediately sees that $\text{ex}(n, \mathcal{L}_{3-reg}) = O(n^{5/3})$. Using the Cube Theorem one gets a better upper bound, $\text{ex}(n, \mathcal{L}_{3-reg}) = O(n^{8/5})$. Erdős and Sauer conjectured that for every $\varepsilon > 0$ there exists an

[20] This approach proved the upper bound for most of the bypartite extremal results (at least, up to a contant) known those days. It did not cover Füredi's Theorem 8.13.

$n_0(k,\varepsilon)$ such that for $n > n_0(k,\varepsilon)$ $\mathrm{ex}(n, \mathcal{L}_{k-reg}) \le n^{1+\varepsilon}$. Pyber proved the following stronger theorem.

Theorem 8.12 (Pyber [144]). For every k, $\mathrm{ex}(n, \mathcal{L}_{k-reg}) = 50k^2 n \log n$.

The proof is based on a somewhat similar but much less general theorem of Alon, Friedland and Kalai [6]. For further information, see e.g. Noga Alon [2]

One More Theorem

We close this section with an old problem of Erdős solved not so long ago by Füredi. Let $F(k,t)$ be the bipartite graph with k vertices $x_1,\dots,x_k$ and $\binom{k}{2}t$ further vertices in groups U_{ij} of size t, where all the vertices of $\cup U_{ij}$ are independent and the t vertices of U_{ij} are joined to x_i and x_j $(1 \le i < j \le k)$. Erdős asked for the determination of $\mathrm{ex}(n, F(k,t))$ for $t = 1$. For $t = 1$ and $k = 2$ this is just C_4, so the extremal number is $O(n^{3/2})$. Erdős also proved (and it follows from [86] as well) that $\mathrm{ex}(n, F(3,1)) = O(n^{3/2})$.

Theorem 8.13 (Füredi [103]). $\mathrm{ex}(n, F(k,t)) = O(n^{3/2})$.

Remark [N] 18. The above theorem was extendend by Alon, Krivelevich and Sudakov. They proved a conjecture of Erdős and myself according to which

Theorem 8.14 ([217]). If L is a bipartite graph with the colour classes A and B and all the vertices of B have degree at most r then

$$\mathrm{ex}(n, L) = O(n^{2-(1/r)}).$$

Further sources to read: Füredi, Simonovits: [188], Simonovits [159].

9. Supersaturated Graphs, Rademacher Type Theorems

Almost immediately after Turán's result, Rademacher proved the following nice theorem (unpublished, see [50]) :

Theorem 9.1 (Rademacher Theorem). If $e(G_n) > \left[\frac{n^2}{4}\right]$ then G_n contains at least $[\frac{n}{2}]$ triangles.

This is sharp: adding an edge to (the smaller class of) $T_{n,2}$ we get $[\frac{n}{2}]$ K_3's. Erdős generalized this result by proving the following two basic theorems [50]:

Theorem 9.2. There exists a positive constant $c_1 > 0$ such that if $e(G_n) > \left[\frac{n^2}{4}\right]$, then G_n contains an edge e with at least $c_1 n$ triangles on it.

Theorem 9.3 (Generalized Rademacher Theorem). There exists a positive constant $c_2 > 0$ such that if $0 < k < c_2 n$ and $e(G_n) > \left[\frac{n^2}{4}\right] + k$, then G_n contains at least $k\left[\frac{n}{2}\right]$ copies of K_3.

Lovász and I proved the conjecture of Erdős that $c_2 = \frac{1}{2}$ [126]. For further results see Moser and Moon [138], Bollobás [13, 14], and [126, 127].

Remark [N] 19. In our paper [127] we have formulated a general theorem for the possible maximum value of triangles, or, more generally, of K_{p+1}'s in a supersaturated graph G_n. Our Stability approach went slightly further than what we published, however, was not enough to solve the general problem, not even for K_3. The first beakthrough came from Fisher [255], Fisher-Ryan [256]. In the last two decades several important results were achieved, by Razborov (who created his "flag algebras [312] in extremal graph theory" to solve such problems). Recently – after several steps, – (e.g. Nikiforov [305]) Reiher [315] finally proved our conjecture.

Erdős also proved the following theorem, going into the other direction.

Theorem 9.4 (Erdős [61]). If $e(G_n) = \left[\frac{n^2}{4}\right] - \ell$ and G_n contains at least one triangle, then it contains at least $[\frac{n}{2}] - \ell - 1$ triangles.

(Of course, we may assume that $0 \le \ell \le [\frac{n}{2}] - 3$.)

The General Supersaturated Case

Working with Erdős on multigraph and digraph extremal problems, Brown and I needed some generalizations of some theorems of Erdős [53, 59]. The results below are direct generalizations of some theorems of Erdős. To avoid proving the theorems in a setting narrower than what might be needed later, Brown and I formulated our results in the "most general, still reasonable"[21] form.

Definition 9.5 (Directed multi-hypergraphs [33]). A directed (r, q)–multi-hypergraph has a set V of vertices, a set $\mathcal{H}$ of directed hyperedges, i.e. ordered r–tuples, and a multiplicity function $\mu(\mathbf{H}) \le q$ (the multiplicity of the ordered hyperedge) $\mathbf{H} \in \mathcal{H}$.

We shall return to the multigraph and digraph problems later, here I formulate only some simpler facts. The extremal graph problems directly generalize to directed multi-hypergraphs with bounded hyper-edge-multiplicity:

> Given a family $\mathcal{L}$ of excluded directed (r, q)–multi-hypergraphs, we may ask the maximum number of directed hyperedges (counted with multiplic-

[21] Of course, this notion does not exist.

ity) a directed (r,q)–multi-hypergraph $\mathbf{G}_n$ can have without containing forbidden sub-multi-hypergraphs from $\mathcal{L}$. The maximum is again denoted by $\operatorname{ex}(n,\mathcal{L})$.

Let $\mathbf{L}$ be a directed (r,q)–multi-hypergraph and $\mathbf{L}[t]$ be obtained from $\mathbf{L}$ by replacing each vertex v_i of $\mathbf{L}$ by a set X_i of t independent vertices, and forming a directed multihyperedge $(y_1,\dots,y_r)$ of multiplicity μ if $y_1 \in X_{i_1},\dots,y_r \in X_{i_r}$ and the corresponding $(v_{i_1},\dots,v_{i_r})$ is a directed hyperedge of multiplicity μ in $\mathbf{L}$.

Theorem 9.6 (**Brown–Simonovits** [**33**]).

$$\operatorname{ex}(n,\mathbf{L}[t]) - \operatorname{ex}(n,\mathbf{L}) = o(n^r).$$

Again, the influence of Erdős is very direct: the above theorem is a direct generalization of his result in [59].

Theorem 9.7 (**Brown–Simonovits** [**33**]). Let $\mathcal{L}$ be an arbitrary family of (r,q)-hypergraphs, and $\gamma = \lim \frac{\operatorname{ex}(n,\mathcal{L})}{n^r}$, as $n \to \infty$.[22] There exists a constant $c_2 = c_2(\mathcal{L},\varepsilon)$ such that, if

$$e(\mathbf{G}_n) \ge (\gamma+\varepsilon)n^r$$

and n is sufficiently large, then there exists some $\mathbf{L} \in \mathcal{L}$ for which $\mathbf{G}_n$ contains at least $c_2 n^{v(\mathbf{L})}$ copies of this $\mathbf{L}$.

Remark [N] 20. This part is only a very short introduction into a very fast developing new area. Here we mention only a few results.

1. One part of this new area is when the problem is *non-degenerate* and we are noticeably above the Turán threshold. Such a general result is the above Theorem 9.7.
2. Lovász and Simonovits, "still in the ancient times", proved a conjecture of Erdős on the critical value of c in Theorem 9.3. On counting triangles, (or K_p's) in supersaturated graphs, see e.g., Fisher [255], Fisher-Ryan [256], Razborov [313], Nikiforov [304], and Reiher [315];
3. The third type of results in this field is related to the bipartite excluded graphs, Erdős-Simonovits conjecture [159], which has several forms and asserts in weaker or stronger forms that among the graphs with $e(G_n) > C \cdot \operatorname{ex}(n,L)$ the random graphs have the least copies of L. Sidorenko [328, 329] reformulated these counting questions to inequlities on integrals and formulated the "Sidorenko's conjecture" which is one of the central topics nowadays in the theory of graph limits. For more details, see Simonovits [159], Erdős–Simonovits [88], Füredi and Simonovits [188], Lovász [193], Hatami [270]. . .

[22] The limit exists, basically because of [117].

10. Typical K_p-Free Graphs: The Erdős-Kleitman-Rothschild Theory

Erdős, Kleitman and Rothschild [75] started investigating the following problem:

How many labelled graphs not containing L exist on n vertices?

Denote this number by $M(n,L)$. We have a trivial lower bound on $M(n,L)$: take any fixed extremal graph S_n and take all the $2^{\mathrm{ex}(n,L)}$ subgraphs of it:

$$M(n,L) \geq 2^{\mathrm{ex}(n,L)}.$$

In some sense it is irrelevant if we count *labelled* or *unlabelled* graphs. The number of labelled graphs is at most $n!$ times the number of unlabelled graphs and $\mathrm{ex}(n,L) \geq \left[\frac{n^2}{4}\right]$ for all non-degenerate cases, (and $\mathrm{ex}(n,L) \geq cn^{1+\alpha}$ for all the *non-tree-non-forest* cases). So, if we are satisfied with rough estimates, we may say: counting only labelled graphs is not a real restriction here.

Strictly speaking, this problem is not an extremal graph problem, neither a supersaturated graph problem. However, the answer to the question shows that this problem is in surprisingly strong connection with the corresponding extremal graph problem.

Theorem 10.1 (Erdős-Kleitman-Rothschild [75]). *The number of K_p–free graphs on n vertices and the number of $p-1$–chromatic graphs on n vertices are in logarithm asymptotically equal: For every $\varepsilon(n) \to 0$ there exists an $\eta(n) \to 0$ such that if $M(n,K_p,\varepsilon)$ denotes the number of graphs of n vertices and with at most εn^p subgraphs K_p, then*

$$\mathrm{ex}(n,K_p) \leq \log M(n,K_p,\varepsilon) \leq \mathrm{ex}(n,K_p) + \eta n^2.$$

In other word, we get "almost all of them" by simply taking all the $(p-1)$–chromatic graphs.

Remark [N] 21. Counting some structures may be interesting for many mathematicians, on its own, however there is an extra nice feature in Theorem 10.1, namely, that one gets some feeling for the typical L-free structures.

Graphs were not the first structures to be considered this way. Here I mention only one result of Kleitman and Rothschild [281] which asserts that–in some sense–the typical posets are described by a random bipartite graph.

Let us return to the typical structure of an L-free graph. Erdős conjectured that most of them are very similar to the subgraphs of an extremal graph (for L):

Conjecture 10.2 (Erdős). *If $\chi(L) > 2$, then*

$$M(n,L) = 2^{\mathrm{ex}(n,L)+o(\mathrm{ex}(n,L))}.$$

This was proved by Erdős, Frankl and Rödl [68].

The corresponding question for bipartite graphs is unsolved. Even for the simplest non trivial case, i.e. for C_4 the results are not satisfactory. This is not so surprising. All these problems are connected with random graphs, where for low edge-density the problems often become much more difficult. Kleitman and Winston [118] showed that

$$M(n, C_4) \le 2^{cn\sqrt{n}},$$

but the best value of the constant c is unknown. Erdős conjectured that

$$M(n, L) = 2^{(1+o(1))\mathrm{ex}(n,L)}.$$

Then the truth should be, of course

$$M(n, C_4) = 2^{((1/2)+o(1))n\sqrt{n}}.$$

Further sources to read: Kolaitis, Prömel and Rothschild extended the sharper estimates known for complete graphs to graphs L with critical edges [288]. Kleitman and David Wilson obtained results similar to the Kleitman–Winston Theorem [282]. Balogh and Samotij extended these results to general complete bipartite graphs [230], [231]. The original results of Erdős, Kleitman, and Rothschild, or, more generally, of Erdős, Frankl, and Rödl were sharpened (better error terms, typical structure, see e.g. Balogh, Bollobás, and Simonovits [223, 222, 224]) and extended to other structures, e.g., to hypergraphs, see e.g., Nagle and Rödl [302] and Person and Schacht [306].

I finish with a recent problem of Erdős.

Problem 10.3 (Erdős). *Determine or estimate the number of* **maximal** *triangle-free graphs on n vertices.*

Some explanation. In the Erdős–Kleitman–Rothschild case the number of bipartite graphs was large enough to give a logarithmically sharp estimate. Here $K(a, n-a)$ are the maximal *bipartite* graphs, their number is negligible. This is why the situation becomes less transparent.

Remark [N] 22. Since those early results many new results were proved in this field.

(a) Sharper estimates were proved for $M(n, L)$ [223]; actually, in some cases the typical structures were also described, [224]
(b) For the first difficult case, to estimate $M(n, L)$ Kleitman and Winston proved that

$$M(n, C_4) < 2^{cn\sqrt{n}} \tag{6}$$

11. Induced Subgraphs

One could ask, why do we always speak of *not necessarily induced* subgraphs. What if we try to exclude *induced* copies of L? If we are careless, we immediately run into a complete nonsense. If L is not a complete graph and we ask:

> What is the maximum number of edges a G_n can have without having an induced copy of L?

the answer is the trivial $\binom{n}{2}$ and the only extremal graph is K_n. So let us give up this question for a short while and try to attack the corresponding *counting problem* which turned out in the previous section to be in a strong connection with the extremal problem.

> How many labelled graphs not containing induced copies of L are on n vertices?

Denote this number by $M^*(n,L)$. Prömel and Steger succeeded in describing $M^*(n,L)$. They started with the case of C_4 and proved that almost all G_n not containing an induced C_4 have the following very specific structure. They are **split graphs** which means that they are obtained by taking a K_m and $(n-m)$ further independent points and joining them to K_m arbitrarily. (Trivially, these graphs contain no induced C_4's.)

Theorem 11.1 (Prömel–Steger [141]). If $\mathbf{S}_n^*$ is the family of split graphs, then

$$\frac{M^*(n,C_4)}{|\mathbf{S}_n^*|} \to 1 \quad \text{as} \quad n \to \infty.$$

This implies, by a result of Prömel [140], that

Corollary 11.2. There exist two constants, $c_{even} > 0$ and $c_{odd} > 0$, such that

$$\frac{M^*(n,C_4)}{2^{n^2/4+n-(1/2)n\,\log\,n}} \to \begin{cases} c_{even} & \text{for even } n, \\ c_{odd} & \text{for odd } n. \end{cases}$$

Can one generalize this theorem to arbitrary excluded induced subgraphs? To answer this question, first Prömel and Steger generalized the notion of chromatic number.

Definition 11.3. Let $\tau(L)$ be the largest integer k for which there exists an integer $j \in [0,k-1]$ such that no $k-1$–chromatic graph in which j colour–classes are replaced by cliques contains L as an induced subgraph.

Clearly, if $\sigma(L)$ denotes the clique covering number, (= the minimum number of complete subgraphs of L to cover all the vertices of L) then

Lemma 11.4 (Prömel–Steger). $\chi(L), \sigma(L) \le \tau(L) \le \chi(L) + \sigma(L)$.

Now, taking a $T_{n,p}$ for $p = \tau(L) - 1$ and replacing j appropriate classes of it (in the above definition) by complete graphs and then deleting arbitrary edges of the $T_{n,p}$ we get graphs not having induced L's:

$$M^*(n, H) \geq 2^{(1-\frac{1}{\tau-1})\binom{n}{2}+o(n^2)}.$$

This yields the lower bound in

Theorem 11.5 (Prömel–Steger). *Let H be a fixed nonempty subgraph with $\tau \geq 3$. Then*

$$M^*(n, H) = 2^{(1-\frac{1}{\tau-1})\binom{n}{2}+o(n^2)}.$$

Definition 11.6. Given a sample graph L, call G_n "good" if there exists a *fixed* subgraph $U_n \subseteq \overline{G_n}$ (= the complementary graph of G_n) such that whichever way we add some edges of G_n to U_n, the resulting U' contains no induced copies of L. $\mathrm{ex}^*(n, H)$ denotes the maximum number of edges such a G_n can have.

Example 11.7. In case of C_4, any bipartite graph $G(A, B)$ is "good", since taking all the edges in A, no edges in B and some edges from $G(A, B)$ we get a U_n not containing C_4 as an induced subgraph.

Theorem 11.8 (Prömel–Steger [143]).

$$\mathrm{ex}^*(n, L) = \left(1 - \frac{1}{\tau - 1}\right)\binom{n}{2} + o(n^2).$$

Thus Prömel and Steger convincingly showed that there is a possibility to generalize ordinary extremal problems and the corresponding counting problems to induced subgraph problems. For further information, see [142, 143, 233, 225].

Remark [N] 23. The whole theory described in the previous section which could be called Erdős–Kleitman–Rosthschild theory (or perhaps, Erdős–Frankl–Rödl theory) has analogs in case of the induced graphs, e.g. Theorem 11.5. Several interesting results were obtained lately, we mention just a few of them below.

This subarea started with Alekseev [210] and Bollobás, and Thomason [235][23]. (See also [236, 237].)

N. Alon, J. Balogh, B. Bollobás, and R. Morris [213] improve these results, by describing the typical structure when excluding an induced L.

Balogh and Butterfield [226] characterized those graphs L, satisfying that

> For almost all L-freee graphs G_n, $V(G_n)$ can be partitioned into a independent sets and b complete graphs, where $a+b = \tau(L)$ (=the coloring number of L).

[23] This volume!

So, in some sense, they define the "weakly-edge-color-critical" graphs and extend the "ordinary" results to this case as well.

Here I skip many further interesting results.

12. The Number of Disjoint Complete Graphs

There are many problems where instead of ensuring many K_{p+1}'s (see Sect. 9) we would like to ensure many *edge-disjoint* or *vertex-disjoint* copies of K_{p+1}. Let us start with the case of *vertex-disjoint* copies.

If G_n is a graph from which one can delete $s-1$ vertices so that the resulting graph is p-chromatic, then G_n cannot contain s vertex-disjoint copies of K_{p+1}. This is sharp: let $H_{n,p,s} := T_{n-s+1,p} \otimes K_{s-1}$. Then $H_{n,p,s}$ has the most edges among the graphs from which one can delete $s-1$ vertices to get a graph of chromatic number at most p. Further:

Theorem 12.1 (Moon [137]). Among all the graph not containing s vertex-independent K_{p+1}'s $H_{n,p,s}$ has the most edges, assumed that $n > n_0(p,s)$.

This theorem was first proved by Erdős and Gallai for $p=1$, then for K_3 by Erdős [51], and then it was generalized for arbitrary p by J. W. Moon, and finally, a more general theorem was proved by me [155]. This more general theorem contained the answer to Turán's two "Platonic" problem: it guaranteed that $H_{n,2,6}$ is the only extremal graph for the dodecahedron graph and $H_{n,3,3}$ for the icosahedron, if n is sufficiently large. For related more general results see Simonovits [198].

We get a slightly different result, if we look for edge-independent complete graphs. Clearly, if one puts k edges into the first class of $T_{n,p}$, then one gets k edge-independent K_p's as long as $k < cn$. One would conjecture that this is sharp. As long as k is fixed, the general theorems of [155] provide the correct answer. If we wish to find the maximum number of edge-independent copies of K_{p+1} for

$$e(G_n) = \left(1 - \frac{1}{p}\right)\binom{n}{2} + k,$$

for $k = k(n) \to \infty$, the problem changes in character, see e.g. recent papers of Győri [109, 111]. We mention just one theorem here:

Theorem 12.2 (Győri [110]). Let $e(G_n) = e(T_{n,p}) + k$, $(p \geq 3)$, where $k \leq 3\lfloor \frac{n+1}{p} \rfloor - 5$. Then G_n contains k edge-independent K_{p+1}'s, assumed that $n > n_0(p)$.

For $p = 2$ (for triangles) the result is different in style, see [109].

13. Extremal Graph Problems Connected to Pentagonlike Graphs

A lemma of Erdős asserts that each graph G_n can be turned into a bipartite graph by deleting at most half of its edges. (Above: Theorem 3.7.) The proof of this triviality is as follows. Take a bipartite $H_n \subseteq G_n$ of maximum number of edges. By the maximality, each $x \in V(G_n)$ having degree $d(x)$ in G_n must have degree $\geq \frac{1}{2}d(x)$ in H_n. Summing the degrees in both graphs we get $e(H_n) \geq \frac{1}{2}e(G_n)$. This estimate is sharp for random graphs of edge probability $p > 0$, in asymptotical sense. Now, our first question is if this estimate can be improved in cases when we know some extra information on the structure of the graph, say, excluding triangles in G_n. The next theorem asserts that this is not so. Let $D(G_n)$ denote the minimum number of edges one has to delete from G_n to turn it into a bipartite graph.

Theorem 13.1 (Erdős [56]). For every $\varepsilon > 0$ there exists a constant $c = c_\varepsilon > 0$ such that for infinitely many n, there exists a G_n for which $K_3 \not\subseteq G_n$, $e(G_n) > c_\varepsilon n^2$, and

$$D(G_n) > \left(\frac{1}{2} - \varepsilon\right) e(G_n).$$

Conjecture 13.2 (Erdős). If $K_3 \not\subseteq G_n$, then one can delete (at most) $n^2/25$ edges so that the remaining graph is bipartite.

Let us call a graph G_n *pentagonlike* if its vertex-set V can be partitioned into $V_1, \dots, V_5$ so that $x \in V_i$ and $y \in V_j$ are joined iff $i - j \equiv \pm 1 \mod 5$. The pentagonlike graph $Q_n := C_5[n/5]$ shows that, if true, this conjecture is sharp. The conjecture is still open, in spite of the fact that good approximations of its solutions were obtained by Erdős, Faudree, Pach and Spencer. This conjecture is proven for $e(G_n) \geq \frac{n^2}{5}$ (see below) and the following (other) weakening is also known, [67]:

Theorem 13.3. If $K_3 \not\subseteq G_n$ then

$$D(G_n) \leq \frac{n^2}{18 + \delta}.$$

for some (explicite) constant $\delta > 0$.

In fact, Erdős, Faudree, Pach, and Spencer [67] proved that

Theorem 13.4. For every triangle-free graph G with n vertices and m edges

$$D(G_n) \leq \max\left\{\frac{1}{2}m - \frac{2m(2m^2 - n^3)}{n^2(n^2 - 2m)}\ ,\ m - \frac{4m^2}{n^2}\right\} \tag{7}$$

Since the second term of (13.1) decreases in $[\frac{1}{8}n^2, \frac{1}{2}n^2]$, and its value is exactly $\frac{1}{25}n^2$ for $m = \frac{1}{5}n^2$, therefore (7) twice implies that if $e(G_n) > \frac{1}{5}n^2$, and $K_3 \not\subseteq G_n$, then $D(G_n) \leq \frac{1}{25}n^2$. By Theorem 3.7, trivially, if $e(G_n) \leq$

$\frac{2}{25}n^2$, then $D(G_n) \le \frac{1}{25}n^2$. However, the general conjecture is still open: it is unsettled in the middle interval $\frac{2n^2}{25} < e(G_n) < \frac{n^2}{5}$.

The next theorem of Erdős, Győri and myself [70] states that if $e(G_n) > \frac{1}{5}n^2$, then the pentagon-like graphs need the most edges to be deleted to become bipartite. (This is sharper than the earlier results, since it provides also information on the near-extremal structure.)

Theorem 13.5. If $K_3 \not\subseteq G_n$ and $e(G_n) \ge \frac{n^2}{5}$, then there is a pentagonlike graph H_n^* with at least the same number of edges: $e(G_n) \le e(H_n^*)$, for which $D(G_n) \le D(H_n^*)$.

Remark [N] 24. Not so long ago a similar question of Erdős was solved. If we consider $C_5[n/5]$, this does not contain K_3 and contains $\approx (\frac{n}{5})^5$ copies of C_5. Erdős conjectured that

Conjecture 13.6 (Erdős). If $K_3 \not\subseteq G_n$, then G_n has at most $\approx (\frac{n}{5})^5$ copies of C_5.

Andrzej Grzesik [267] and Hatami, Hladký, Král, and Razborov [271] proved this, independently.

14. Problems on the Booksize of a Graph

We have already seen a theorem of Erdős, stating that if a graph has many edges, then it has an edge e with cn triangles on it. Such configurations are usually called books. The existence of such edges is one of the crucial tools Erdős used in many of his graph theorems. Still, it was a longstanding open problem, what is the proper value of this constant c above. Without going into details we just mention three results:

Theorem 14.1 (Edwards [43, 44]). If $e(G_n) > \left[\frac{n^2}{4}\right]$, then G_n has an edge with $[n/6]+1$ triangles containing this edge.

This is sharp. The theorem would follow if we knew that there exists a $K_3 = (x,y,z)$ for which the sum of the degrees, $d(x)+d(y)+d(z) > \frac{3n}{2}$. Indeed, at least $\frac{n}{6}$ vertices would be joined to the same pair, say, to xy. Another paper of Edwards contains results of this type, but only for $e(G_n) > \frac{1}{3}n^2$. Let $\Delta_r = \Delta_r(G_n)$ denote the maximum of the sums of the degrees in a $K_r \subseteq G_n$. (For instance, in a random graph R_n $\Delta_r(R_n) \approx r \cdot \frac{2e(R_n)}{n}$.)

Theorem 14.2 (Edwards [43]). If $\frac{1}{r}\Delta_r > \left(1 - \frac{1}{r+1}\right)n$, $n \ge 1$, then

$$\frac{1}{r+1}\Delta_{r+1} \ge \frac{2e(G_n)}{n}.$$

This theorem says that if G_n has enough edges to ensure a K_{r+1}, then it also contains a K_{r+1} whose vertex-degree-sum is as large as it should be by averaging. Erdős, Faudree and Győri have improved Theorem 14.1 if we replace the edge-density condition by the corresponding degree-condition. Among others, they have shown that

Theorem 14.3 (Erdős–Faudree-Győri [66]). There exists a $c > 0$ such that if the minimum degree of G_n is at least $[n/2] + 1$, then G_n contains an edge with $[n/6] + cn$ triangles containing this edge.

15. Digraph/Multigraph Extremal Graph Problems

We have already seen supersaturated extremal graph theorems on multi-digraphs. Here we are interested in simple asymptotically extremal sequences for digraph extremal problems.

Multigraph or digraph extremal problems are closely related and in some sense the digraph problems are the slightly more general ones. So we shall restrict ourselves to digraph extremal problems. A digraph extremal problem means that some q is given and we consider the class of digraphs where loops are excluded and any two vertices may be joined by at most q arcs in one direction and by at most q arcs of the opposite direction. This applies to both the excluded graphs and to the graphs on n vertices the edges of which should be maximized. So our problem is:

> Fix the multiplicity bound q described above. A family $\mathcal{L}$ of digraphs is given and $\text{ex}(n, \mathcal{L})$ denotes the maximum number of arcs a digraph $\mathbf{D}_n$ can have under the condition that it contains no $\mathbf{L} \in \mathcal{L}$ and satisfies the multiplicity condition. Determine or estimate $\text{ex}(n, \mathcal{L})$.

The Digraph and Multigraph Extremal graph problems first occur in a paper of Brown and Harary [32]. They described fairly systematically all the cases of small forbidden multigraphs or digraphs. Next Erdős and Brown extended the investigation to the general case, finally I joined the "project". Our papers [28, 29, 30] and [31] describes fairly well the situation $q = 1$ for digraphs (which is roughly equivalent with $q = 2$ for multigraphs). We thought that our results can be extended to all q but Sidorenko [149] and then Rödl and Sidorenko [146] ruined all our hopes. One of our main results was in a somewhat simplified form:

Theorem 15.1. Let $q = 1$ and $\mathcal{L}$ be a given family of excluded digraphs. Then there exists a matrix $A = (a_{ij})$ of r rows and columns, depending only on $\mathcal{L}$, such that there exists a sequence $(\mathbf{S}_n)$ of asymptotically extremal graphs for $\mathcal{L}$ whose vertex-set V can be partitioned into $V_1, \dots, V_r$ so that for $1 \le i < j \le r$, a $v \in V_i$ is joined to a $v' \in V_j$ by an arc of this direction iff the corresponding matrix-element $a_{ij} = 2$; further, the subdigraphs spanned by the V_i's are either independent sets or tournaments, depending on whether $a_{ii} = 0$ or 1.

One crucial tool in our research was a *density* notion for matrices. We associated with every matrix A a quadratic form and maximized it over the standard simplex:

$$g(A) = \max \left\{ uAu^T \ : \ \sum u_i = 1, \ u_i \geq 0. \right\}$$

The matrices are used to characterize some generalizations of graph sequences like $(T_{n,p})_{n>n_0}$ of the general theory for ordinary graphs, and $g(A)$ measures the edge-density of these structures: replaces $(1 - \frac{1}{p})$ of the Erdős–Stone–Simonovits theorem.

Definition 15.2. A matrix A is called *dense* if for every submatrix B' of symmetric position $g(B') < g(B)$. In other words, B is minimal for $g(B) = \lambda$.

We conjectured that – as described below – the numbers $g(B)$ are of *finite multiplicity* and *well ordered* if the matrices are dense:

Conjecture 15.3. If q is fixed, then for each λ there are only finitely many dense matrices B with $g(B) = \lambda$. Further, if (B_n) is a sequence of matrices of bounded integer entries then $(g(B_n))$ cannot be strictly monotone decreasing.

One could wonder how one arrives at such conjectures, but we do not have the space to explain that here. Similar matrices (actually, multigraph extremal problems) occur when one attacks Turán–Ramsey problems, see [72, 73, 74].

Our conjecture was disproved by Sidorenko and Rödl [146]. As a consequence, while we feel that the case $q = 1$ (i.e. the case of digraphs where any two points can be joined only by one arc of each direction) is sufficiently well described, for $q > 1$ the problem today seems to be fairly hopeless. Multidigraphs have also been considered by Katona in [116], where he was primarily interested in continuous versions of Turán-type extremal problems.

Remark [N] 25. (a) Many further details of this chapter can be found in Brown–Simonovits [240].

(b) Since the original version of this paper the theory of Graph Limits (see e.g. [193]) emerged and many phenomena observed there have some (much simpler) analogues in the theory of multigraph extremal problems.

(c) Multigraph (or coloured multigraph) extremal problems – combined with stability methods – were used in solving the Hypergraph Extremal problem, where the Fano Plane was excluded, see de Caen-Füredi [241], Füredi and Simonovits [261] and Keevash and Sudakov [278].

An interesting feature of these solutions was that one did not have to assume the bounded multiplicity in the multigraph problem corresponding to the Fano hypergraph problem: it followed from the conditions.

Also, in some sense very special multigraph extremal problems were used in the proof of the main Ramsey–Turán theorems of Erdős, Hajnal, Sós, and Szemerédi [72].

16. Erdős and Nassredin

Let me finish this paper with an anecdote. Nassredin, the hero of many middle-east jokes, stories (at least this is how we know it in Budapest), once met his friends who were eager to listen to his speech. "Do you know what I wish to speak about" Nassredin asked them. "No, we don't" they answered. "Then why should I speak about it" said Nassredin and left.[24] Next time the friends really wanted to listen to the clever and entertaining Nassredin. So, when Nassredin asked the audience "Do you know what I want to speak about", they answered: "YES, we do" . "Then why should I speak about it" said Nassredin again and went home. The third time the audience decided to be more clever. When Nassredin asked them "Do you know what I will speak about", half of the people said "YES" the other half said "NO". Nassredin probably was lasy to speak: "Those who know what I wanted to tell you should tell it to the others" he said and left again.[25]

I am in some sense in Nassredin's shoes. How could I explain on 30 or 50 pages the influence of Erdős on Extremal Graph Theory to people who do not know it. And why should I explain to those who know it. Yet I think, Nassredin did not behave in the most appropriate way. So I tried – as I promised – to illustrate on some examples this enourmous influence of Paul. I do not think it covered half the topics and I have not tried to be too systematic.

Long Live Paul Erdős![26]

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[24] I am not saying I follow his logic, but this is how the story goes.

[25] One can easily find funny stories about Nassredin on the Internet, e.g., right now I have found the above story, in slightly different words at
http://sovyatnik.editboard.com/t855-hodscha-nassredin

[26] Paul died 3 years later.

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Applications of the Probabilistic Method to Partially Ordered Sets

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This paper is dedicated to Paul Erdős with appreciation for his impact on mathematics and the lives of mathematicians all over the world.

Summary There are two central themes to research involving applications of probabilistic methods to partially ordered sets. The first of these can be described as the study of random partially ordered sets. Among the specific models which have been studied are: random labelled posets; random t-dimensional posets; and the transitive closure of random graphs. A second theme concentrates on the adaptation of random methods so as to be applicable to general partially ordered sets. In this paper, we concentrate on the second theme. Among the topics we discuss are fibers and co-fibers; the dimension of subposets of the subset lattice; the dimension of posets of bounded degree; and fractional dimension. This last topic leads to a discussion of Ramsey theoretic questions for probability spaces.

1. Introduction

Probabilistic methods have been used extensively throughout combinatorial mathematics, so it no great surprise to see that researchers have applied these techniques with great success to finite partially ordered sets. One central theme to this research is to define appropriate definitions of a *random poset*, and G. Brightwell's excellent survey article [1] provides a summary of work in this direction.

A second theme involves the application of random methods to more general classes of posets. After this brief introductory section, we present four examples of this theme. The first example is quite elementary and involves

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fibers and co-fibers, concepts which generalize the notions of chains and antichains. The principal result here is an application of random methods to provide a non-trivial upper bound on the minimum size of fibers.

Our second example is more substantial. It involves the dimension of subposets of the subset lattice, an instance in which many of the classic techniques and results pioneered by Paul Erdős play major roles. The third example involves an application of the Lovász Local Lemma and leads naturally to the the investigation of the dimension of a random poset of height two.

Our last example involves fractional dimension for posets—an area where there are many attractive open problems. This topic leads to natural questions involving Ramsey theory for probability spaces.

The remainder of this section is a very brief condensation of key ideas and notation necessary for the remaining five sections. In this article, we consider a *partially ordered set* (or *poset*) $\mathbf{P} = (X, P)$ as a discrete structure consisting of a set X and a reflexive, antisymmetric and transitive binary relation P on X. We call X the *ground set* of the poset $\mathbf{P}$, and we refer to P as a *partial order* on X. The notations $x \le y$ in P, $y \ge x$ in P and $(x, y) \in P$ are used interchangeably, and the reference to the partial order P is often dropped when its definition is fixed throughout the discussion. We write $x < y$ in P and $y > x$ in P when $x \le y$ in P and $x \ne y$. When $x, y \in X$, $(x, y) \notin P$ and $(y, x) \notin P$, we say x and y are *incomparable* and write $x \parallel y$ in P.

Although we are concerned almost exclusively with *finite* posets, i.e., those posets with finite ground sets, we find it convenient to use the familiar notation $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ to denote respectively the reals, rationals, integers and positive integers equipped with the usual orders. Note that these four infinite posets are *total* orders; in each case, any two distinct points are comparable. Total orders are also called *linear* orders, or *chains.* We use $\mathbf{n}$ to denote an n-element chain with the points labelled as $0 < 1 < \cdots < n-1$.

A subset $A \subseteq X$ is called an *antichain* if no two distinct points in A are comparable. We also use $\mathbf{P} + \mathbf{Q}$ to denote the disjoint sum of $\mathbf{P}$ and $\mathbf{Q}$.

In the remainder of this article, we will assume that the reader is familiar with the basic concepts for partially ordered sets, including maximal and minimal elements, chains and antichains, sums and cartesian products, comparability graphs and Hasse diagrams. For additional background information on posets, the reader is referred to the author's monograph [23], the survey article [14] on dimension by Kelly and Trotter and the author's survey articles [21,22,25] and [26]. Another good source of background information on posets is Brightwell's general survey article [2].

2. Fibers and Co-fibers

The classic theorem of Dilworth [4] asserts that a poset $\mathbf{P} = (X, P)$ of width n can be partitioned into n chains. Also, a poset of height h can be partitioned into h antichains. For graph theorists, these results can be translated into the simple statement that comparability graphs are perfect. Against this backdrop, researchers have devoted considerable energy to generalizations of the concepts of chains and antichains. Here is one such example.

Let $\mathbf{P} = (X, P)$ be a poset. Lone and Rival [18] called a subset $A \subseteq X$ a *co-fiber* if it intersects every non-trivial maximal chain in $\mathbf{P}$. Let $\text{cof}(\mathbf{P})$ denote the least m so that $\mathbf{P}$ has a co-fiber of cardinality m. Then let $\text{cof}(n)$ denote the maximum value of $\text{cof}(\mathbf{P})$ taken over all n-element posets. In any poset, the set A_1 consisting of all maximal elements which are not minimal elements and the set A_2 of all minimal elements which are not maximal are both co-fibers. As $A_1 \cap A_2 = \emptyset$, it follows that $\text{cof}(n) \leq \lfloor n/2 \rfloor$. On the other hand, the fact that $\text{cof}(n) \geq \lfloor n/2 \rfloor$ is evidenced by a height 2 poset with $\lfloor n/2 \rfloor$ minimal elements each of which is less than all $\lceil n/2 \rceil$ maximal elements. So $\text{cof}(n) = \lfloor n/2 \rfloor$ (this argument appears in [18]).

Dually, a subset $B \subseteq X$ is called a *fiber* if it intersects every non-trivial maximal antichain. Let $\text{fib}(\mathbf{P})$ denote the least m so that $\mathbf{P}$ has a fiber of cardinality m. Then let $\text{fib}(n)$ denote the maximum value of $\text{fib}(\mathbf{P})$ taken over all n-element posets. Trivially, $\text{fib}(n) \geq \lfloor n/2 \rfloor$, and Lone and Rival asked whether equality holds.

In [6], Duffus, Sands, Sauer and Woodrow showed that if $\mathbf{P} = (X, P)$ is an n-element poset, then there exists a set $F \subseteq X$ which intersects every 2-element maximal antichain so that $|F| \leq \lfloor n/2 \rfloor$. However, B. Sands then constructed a 17-point poset in which the smallest fiber contains 9 points. This construction was generalized by R. Maltby [19] who proved that for every $\epsilon > 0$, there exist a n_0 so that for all $n > n_0$ there exists an n-element poset in which the smallest fiber has at least $(8/15 - \epsilon)n$ points.

From above, there is no elementary way to see that there exists a constant $\alpha > 0$ so that $\text{fib}(n) < (1 - \alpha)n$. However, this is an instance where random methods provided real insights into the truth. In the remainder of this paper, we use the notation $[n]$ to denote the n-element set $\{1, 2, \ldots, n\}$. (No order is implied on $[n]$, except for the natural order on positive integers.)

Theorem 1. *Let $\mathbf{P} = (X, P)$ be a poset with $|X| = n$. Then X contains a fiber of cardinality at most $4n/5$. Consequently,* $\text{fib}(n) \leq 4n/5$.

Proof. Let $C \subseteq X$ be a maximum chain. Then $X - C$ is a fiber. So we may assume that $|C| < n/5$. Label the points of C as $x_1 < x_2 < \cdots < x_t$, where $t = |C| < n/5$. Next we define two different partitions of $X - C$. First, for each $i \in [t]$, set $U_i = \{x \in X - C\colon i$ is the least integer for which $x \parallel x_i\}$. Then set $D_i = \{x \in X - C\colon i$ is the largest integer for which $x \parallel x_i\}$.

Then for each subset $S \subseteq [t-1]$, define

$$B(S) = C \cup (\cup\{D_i : i \in S\}) \cup (\cup\{U_{i+1} : i \notin S\})$$

Note that for each $i \in [t-1]$, the maximality of C implies that $D_i \cap U_{i+1} = \emptyset$.

Claim 1. *For every subset $S \subseteq [t-1]$, $B(S)$ is a fiber.*

Proof. Let $S \subseteq [t-1]$ and let A be a non-trivial maximal antichain. We show that $A \cap B(S) \neq \emptyset$. This intersection is nonempty if $A \cap C \neq \emptyset$, so we may assume that $A \cap C = \emptyset$. Now the fact that C is a maximal chain implies that every point of C is comparable with one or more points of A. However, no point of C can be greater than one point of A and less than another point of A. Also, x_1 can only be less than points in A, and x_t can only be greater than points in A. It follows that $t \geq 2$ and that there is an integer $i \in [t-1]$ and points a, $a' \in A$ for which $x_i < a$ in P and $x_{i+1} > a'$ in P. Clearly, $a' \in D_i$ and $a \in U_{i+1}$. If $i \in S$, then $D_i \subset B(S)$, and if $i \notin S$, then $U_{i+1} \subset B(S)$. In either case, we conclude that $A \cap B(S) \neq \emptyset$. □

Claim 2. *The expected cardinality of B(S) with all subsets $S \subseteq [t-1]$ equally likely is $t + 3(n-t)/4$.*

Proof. Note that $C \subseteq B(S)$, for all S. For each element $x \in X - C$, let i and j be the unique integers for which $x \in D_i$ and $x \in U_j$. Then $j \neq i+1$. It follows that the probability that x belongs to $B(S)$ is exactly $3/4$. □

To complete the proof of the theorem, we note that there is some $S \subseteq [t-1]$ for which the fiber $B(S)$ has at most $t + 3(n-t)/4$ points. However, $t < n/5$ implies that $t + 3(n-t)/4 < 4n/5$. □

The preceding theorem remains an interesting (although admittedly elementary) illustration of applying random methods to *general* partially ordered sets. Characteristically, it shows that an n-point poset has a fiber containing at most $4n/5$ points without actually producing the fiber. Furthermore, this is also an instance in which the constant provided by random methods can be improved by another approach.

The following result is due to Duffus, Kierstead and Trotter [5].

Theorem 2 (Duffus, Kierstead and Trotter). *Let $\mathbf{P} = (X, P)$ be a poset and let $\mathcal{H}$ be the hypergraph of non-trivial maximal antichains of $\mathbf{P}$. Then the chromatic number of $\mathcal{H}$ is at most* 3.

Theorem 2 shows that $\text{fib}(n) \leq 2n/3$, since whenever $X = B_1 \cup B_2 \cup B_3$ is a 3-coloring of the hypergraph $\mathcal{H}$ of non-trivial maximal antichains, then the union of any two of $\{B_1, B_2, B_3\}$ is a fiber. Quite recently, Lone [17] has obtained the following interesting result providing a better upper bound for posets with small width.

Theorem 3 (Lonc). *Let $\mathbf{P} = (X, P)$ be a poset of width 3 and let $|X| = n$. $\mathbf{P}$ has a fiber of cardinality at most $11n/18$.*

I am still tempted to assert that $\lim_{n\to\infty} \text{fib}(n)/n = 2/3$.

3. Dimension Theory

When $\mathbf{P} = (X, P)$ is a poset, a linear order L on X is called a *linear extension* of P when $x < y$ in L for all x, $y \in X$ with $x < y$ in P. A set $\mathcal{R}$ of linear extensions of P is called a *realizer* of $\mathbf{P}$ when $P = \cap\mathcal{R}$, i.e., for all x, y in X, $x < y$ in P if and only if $x < y$ in L, for every $L \in \mathcal{R}$. The minimum cardinality of a realizer of $\mathbf{P}$ is called the *dimension* of $\mathbf{P}$ and is denoted $\dim(\mathbf{P})$.

It is useful to have a simple test to determine whether a family of linear extensions of P is actually a realizer. The first such test is just a reformulation of the definition. Let $\mathrm{inc}(\mathbf{P}) = \mathrm{inc}(X, P)$ denote the set of all incomparable pairs in $\mathbf{P}$. Then a family $\mathcal{R}$ of linear extensions of P is a realizer of P if and only if for every $(x, y) \in \mathrm{inc}(X, P)$, there exist distinct linear extensions L, $L' \in \mathcal{R}$ so that $x > y$ in L and $y > x$ in L'.

Here is a more useful test. Call a pair $(x, y) \in X \times X$ a *critical pair* if:

1. $x \parallel y$ in P;
2. $z < x$ in P implies $z < y$ in P, for all $z \in X$; and
3. $w > y$ in P implies $w > x$ in P, for all $w \in X$.

The set of all critical pairs of $\mathbf{P}$ is denoted $\mathrm{crit}(\mathbf{P})$ or $\mathrm{crit}(X, P)$. Then it is easy to see that a family $\mathcal{R}$ of linear extensions of P is a realizer of P if and only if for every critical pair (x, y), there is some $L \in \mathcal{R}$ with $x > y$ in L. We say that a linear order L on X *reverses* (x, y) if $x > y$ in L. So the dimension of a poset is just the minimum number of linear extensions required to reverse all critical pairs.

For each $n \geq 3$, let $\mathbf{S}_n$ denote the height 2 poset with n minimal elements $a_1, a_2, \ldots, a_n$, n maximal elements $b_1, b_2, \ldots, b_n$ and $a_i < b_j$, for $i, j \in [n]$ and $j \neq i$. The poset $\mathbf{S}_n$ is called the standard example of an n-dimensional poset. Note that $\dim(\mathbf{S}_n)$ is at most n, since $\mathrm{crit}(\mathbf{S}_n) = \{(a_i, b_i) : i \in [n]\}$ and n linear extensions suffice to reverse the n critical pairs in $\mathrm{crit}(\mathbf{S}_n)$. On the other hand, $\dim(\mathbf{S}_n) \geq n$, since no linear extension can reverse more than one critical pair.

4. The Dimension of Subposets of the Subset Lattice

For integers k, r and n with $1 \leq k < r < n$, let $\mathbf{P}(k, r; n)$ denote the poset consisting all k-element and all r-element subsets of $\{1, 2, \ldots, n\}$ partially ordered by inclusion. For simplicity, we use $\dim(k, r; n)$ to denote the dimension of $\mathbf{P}(k, r; n)$.

Historically, most researchers have concentrated on the case $k = 1$. In a classic 1950 paper in dimension theory, Dushnik [7] gave an exact formula for $\dim(1, r; n)$, when $r \geq 2\sqrt{n}$.

Theorem 4 (Dushnik). *Let n, r and j be positive integers with $n \geq 4$ and $2\sqrt{n} - 2 \leq r < n - 1$. If j is the unique integer with $2 \leq j \leq \sqrt{n}$ for which*

$$\left\lfloor \frac{n - 2j + j^2}{j} \right\rfloor \leq k < \left\lfloor \frac{n - 2(j-1) + (j-1)^2}{j-1} \right\rfloor,$$

then $\dim(1, r; n) = n - i + 1$.

No general formula for $\dim(1, r; n)$ is known when r is relatively small in comparison to n, although some surprisingly tight estimates have been found. Here is a very brief overview of this work, beginning with an elementary reformulation of the problem. When L is a linear order on X, $S \subset X$ and $x \in X - S$, we say $x > S$ in L when $x > s$ in L, for every $s \in S$.

Proposition 1. $\dim(1, r; n)$ *is the least t so that there exist t linear orders $L_1, L_2, \ldots L_t$ of $[n]$ so that for every r-element subset $S \subset [n]$ and every $x \in [n] - S$, there is some $i \in [t]$ for which $x > S$ in L_i.*

Spencer [20] used this proposition to estimate $\dim(1, 2; n)$. First, he noted that by the Erdős-Szekeres theorem, if $n > 2^{2^t}$ and $\mathcal{R}$ is any set of t linear orders on $[n]$, then there exists a 3-element set $\{x, y, z\} \subset [n]$ so that for all $L \in \mathcal{R}$, either $x < y < z$ in L or $x > y > z$ in L. Thus $\dim(1, 2; n) > t$ when $n > 2^{2^t}$. On the other hand, if $n \leq 2^{2^t}$, then there exists a family $\mathcal{R}$ of t linear orders on $[n]$ so that for every 3-element subset $S \subset [n]$ and every $x \in S$, there exists some $L \in \mathcal{R}$ so that either $x < S - \{x\}$ in L or $x > S - \{x\}$ in L. Then let S be the family of $2t$ linear orders on X determined by adding to $\mathcal{R}$ the duals of the linear orders in $\mathcal{R}$. Clearly, the $2t$ linear orders in S satisfy the requirements of Proposition 1 when $r = 2$, and we conclude:

Theorem 5 (Spencer). *For all $n \geq 4$,*

$$\lg \lg n < \dim(1, 2; n) \leq 2 \lg \lg n.$$

Spencer [20] then proceeded to determine a more accurate upper bound for $\dim(1, 2; n)$ using a technique applicable to larger values of r. Let t be a positive integer, and let $\mathcal{F}$ be a family of subsets of $[t]$. Then let r be an integer with $1 \leq r \leq t$. We say $\mathcal{F}$ is *r-scrambling* if $|\mathcal{F}| \geq r$ and for every sequence $(A_1, A_2, \ldots, A_r)$ of r distinct sets from $\mathcal{F}$ and for every subset $B \subseteq [r]$, there is an element $\alpha \in [t]$ so that $\alpha \in A_\beta$ if and only if $\beta \in B$. We let $M(r, t)$ denote the maximum size of a r-scrambling family of subsets of $[t]$. Spencer then applied the Erdős/Ko/Rado theorem to provide a precise answer for the size of $M(2, t)$.

Theorem 6 (Spencer). $M(2, t) = \binom{t-1}{\lfloor \frac{t-2}{2} \rfloor}$*, for all $t \geq 4$.*

As a consequence, Spencer observed that

$$\lg \lg n < \dim(1, 2; n) \leq \lg \lg n + (\tfrac{1}{2} + o(1)) \lg \lg \lg n.$$

Almost 20 years later, Füredi, Hajnal, Rödl and Trotter [13] were able to show that the upper bound in this inequality is tight, i.e.,

$$\dim(1,2;n) = \lg\lg n + (\tfrac{1}{2} + o(1))\lg\lg\lg n.$$

For larger values of r, Spencer used random methods to produce the following bound.

Theorem 7 (Spencer). *For every $r \geq 2$, there exists a constant $c = c_r > 1$ so that $M(r,t) > c^t$.*

Proof. Let p be a positive integer and consider the set of all sequences of length p whose elements are subsets of $[t]$. There are 2^{pt} such sequences. The number of such sequences which fail to be r-scrambling is easily seen to be at most

$$\binom{p}{r} 2^r (2^r - 1)^t 2^{(p-r)t}.$$

So at least one of these sequences is a r-scrambling family of subsets of $[t]$ provided $\binom{p}{r} 2^r (2^r-1)^t 2^{(p-r)t} < 2^{pt}$. Clearly this inequality holds for $p > c^t$ where $c = c_r \sim e^{\frac{1}{r2^r}}$ is a constant larger than 1. □

Here's how the concept of scrambling families is used in provide upper bounds for $\dim(q,r;n)$.

Theorem 8 (Spencer). *If $p = M(r,t)$ and $n = 2^p$, then $\dim(1,r;n) \leq t$.*

Proof. Let $\mathcal{F}$ be an r-scrambling family of subsets of $[t]$, say $\mathcal{F} = \{A_1, A_2, \ldots, A_p\}$ where $p = M(r,t)$. Then set $n = 2^p$ and let $Q_1, Q_2, \ldots, Q_n$ be the subsets of $[p]$. For each $\alpha \in [t]$, define a linear order L_α on the set $[n]$ by the following rules. Let x and y be distinct integers from $[n]$ and let $u = \min((Q_x - Q_y) \cup (Q_y - Q_x))$. Set $x > y$ in L_α if either

1. $\alpha \in A_u$ and $u \in Q_x - Q_y$, or
2. $\alpha \notin A_u$ and $u \in Q_y - Q_x$.

It is not immediately clear why L_α is a linear order on $[n]$ for each $\alpha \in [t]$, but it is easy to check that this is so. Now let S be an r-element subset of $[n]$ and let $x \in [n] - S$. We must show that $x > S$ in L_α for some $\alpha \in [t]$. For each $y \in S$, let $u_y = \min((Q_x - Q_y) \cup (Q_y \cup Q_x))$ and then consider the family $\{A_{u_y} : y \in S\}$. Since $\mathcal{F}$ is a r-scrambling family of subsets of $[t]$, there exists some $\alpha \in [t]$ such that $\alpha \in A_{u_y}$ if and only if $u_y \in Q_x$. It follows from the definition of L_α that $x > S$ in L_α. □

By paying just a bit of attention to constants, the preceding results of Spencer actually yield the following upper bound on $\dim(1,r;n)$.

Theorem 9 (Spencer). *For all $r \geq 2$, $\dim(1,r;n) \leq (1+o(1))\frac{1}{\lg e} r 2^r \lg\lg n$.*

Of course, this bound is only meaningful if r is relatively small in comparison to n, but in this range, it is surprisingly tight. The following lower bound is a quite recent result due to Kierstead.

Theorem 10 (Kierstead). *If* $2 \leq r \leq \lg\lg n - \lg\lg\lg n$, *then*

$$\frac{(r+2-\lg\lg n+\lg\lg\lg n)^2 \lg n}{32\ \lg(r+2-\lg\lg n+\lg\lg\lg n)} \leq \dim(1,r;n).$$

We will return to the issue of estimating $\dim(1,r;n)$ in the next section.

5. The Dimension of Posets of Bounded Degree

Given a poset $\mathbf{P}=(X,P)$ and a point $x \in X$, define the *degree* of x in $\mathbf{P}$, denoted $\deg_{\mathbf{P}}(x)$, as the number of points in X which are comparable to x, This is just the degree of the vertex x in the associated comparability graph. Then define $\Delta(\mathbf{P})$ as the maximum degree of $\mathbf{P}$. Finally, define $\mathrm{Dim}(k)$ as the maximum dimension of a poset $\mathbf{P}$ with $\Delta(\mathbf{P}) \leq k$. Rödl and Trotter were the first to prove that $\mathrm{Dim}(k)$ is well defined. Their argument showed that $\mathrm{Dim}(k) \leq 2k^2+2$. It is now possible to present a very short argument for this result by first developing the following idea due to Füredi and Kahn [12].

For a poset $\mathbf{P}=(X,P)$ and a point $x \in X$, let $U(x)=\{y \in X : y > x \text{ in } P\}$ and let $U[x]=U(x)\cup\{x\}$. Dually, let $D(x)=\{y \in X : y < x \text{ in } P\}$ and $D[x]=D(x)\cup\{x\}$. The following proposition admits an elementary proof. In fact, something more can be said, and we will comment on this in the next section.

Proposition 2 (Füredi and Kahn). *Let* $\mathbf{P}=(X,P)$ *be a poset and let* L *be any linear order on* X*. Then there exist a linear extension* L' *of* P *so that if* (x,y) *is a critical pair and* $x > D[y]$ *in* L*, then* $x > y$ *in* L'*, so that* $x > D[y]$ *in* L'*.*

Theorem 11 (Rödl and Trotter). *If* $\mathbf{P}=(X,P)$ *is a poset with* $\Delta(\mathbf{P}) \leq k$, *then* $\dim(\mathbf{P}) \leq 2k^2+2$.

Proof. Define a graph $\mathbf{G}=(X,E)$ as follows. The vertex set X is the ground set of $\mathbf{P}$. The edge set E contains those two element subsets $\{x,y\}$ for which $U[x]\cap U[y] \neq \emptyset$. Clearly, the maximum degree of a vertex in $\mathbf{G}$ is at most k^2. Therefore, the chromatic number of $\mathbf{G}$ is at most k^2+1. Let $t=k^2+1$ and let $X=X_1 \cup X_2 \cup \ldots \cup X_t$ be a partition of X into subsets which are independent in $\mathbf{G}$. Then for each $i \in [t]$, let L_i be any linear order on X with $X_i > X - X_i$ in L_i. Finally, define L_{t+i} to be any linear order on X so that:

1. $X_i > X - X_i$ in L_{t+i}, and
2. The restriction of L_{t+i} to X_i is the dual of the restriction of L_i to X_i.

We claim that for every critical pair $(x, y) \in \text{crit}(\mathbf{P})$, if $x \in X_i$, then either $x > D[y]$ in L_i or $x > D[y]$ in L_{t+i}. This claim follows easily from the observation that any two points of $D[y]$ are adjacent in $\mathbf{G}$ so that $|D[y] \cap X_i| \leq 1$. □

Füredi and Kahn [12] made a dramatic improvement in the upper bound for in $\text{Dim}(k)$ by applying the Lovász Local Lemma [9]. We sketch their argument which begins with an application of random methods to provide an upper bound for $\dim(1, r; n)$. In this sketch, we make no attempt to provide the best possible constants.

Theorem 12 (Füredi and Kahn). *Let r and n be integers with $1 < r < n$. If t is an integer such that*

$$n\binom{n-1}{r}\left(\frac{r}{r+1}\right)^t < 1, \tag{1}$$

then $\dim(1, r; n) \leq t$. *In particular,* $\dim(1, r; n) \leq r(r+1)\log(n/r)$.

Proof. Let t be an integer satisfying the inequality given in the statement of the theorem. Then let $\{L_i : i \in [t]\}$ be a sequence of t random linear orders on X. The expected number of pairs (x, S) where S is an r-element subset of $[n]$, $x \in [n] - S$ and there is no $i \in [t]$ for which $x > S$ in L_i is exactly what the left hand side of this inequality is calculating. It follows that this quantity is less than one, so the probability that there are no such pairs is positive. This shows that $\dim(1, r; n) \leq t$. The estimate $\dim(1, r; n) \leq r(r+1)\log(n/r)$ follows easily. □

Theorem 13 (Füredi and Kahn). *If* $\mathbf{P} = (X, P)$ *is a poset for which* $\Delta(\mathbf{P}) \leq k$, *then* $\dim(\mathbf{P}) \leq 100k \log^2 k$, *i.e.,* $\text{Dim}(k) \leq 100k \log^2 k$.

Proof. The inequality $\dim(\mathbf{P}) \leq 100k \log^2 k$ follows from the preceding theorem if $k \leq 1{,}000$, so we may assume that $k > 1{,}000$. Set $m = \lceil k/\log k \rceil$ and $r = \lceil 9 \log k \rceil$. Using the Lovász Local Lemma, we see that there exists a partition $X = Y_1 \cup Y_2 \ldots \cup Y_m$, with $|D[x] \cap Y_i| \leq r$, for every $x \in X$. Now fix $i \in [m]$, let $q = rk + 1$ and let $s = \dim(1, r; q)$. We construct a family $\mathcal{R}_\rangle = \{\mathcal{L}_{\rangle,|} : | \in [\in \int]\}$ as follows.

Let $\mathbf{G}$ be the graph on X defined in the proof of Theorem 11. Then let $\mathbf{G}_i$ be the subgraph induced by Y_i. Now it is easy to see that any point of Y_i is adjacent to at most rk other points in Y_i in the graph $\mathbf{G}_i$. It follows that the chromatic number of $\mathbf{G}_i$ is at most $rk + 1$. Let $Y_i = Y_{i,1} \cup \ldots \cup Y_{i,q}$ be a partition into subsets each of which is independent in $\mathbf{G}_i$. Then let $\mathcal{R} = \{\mathcal{M}_| : | \in [\int]\}$ be a family of linear orders of $[q]$ so that for every r-element subset $S \subset [q]$ and every $x \in [q] - S$, there is some $j \in [s]$ for which $x > S$ in M_j.

Then for each $j \in [s]$, define $L_{i,j}$ as any linear order for which:

1. $Y_i > X - Y_i$ in $L_{i,j}$ and

2. If $a < b$ in M_j, then $Y_{i,a} < Y_{i,b}$ in $L_{i,j}$.

Finally, for each $j \in [s]$, define $L_{i,s+j}$ as any linear order for which:

1. $Y_i > X - Y_i$ in $L_{i,s+j}$,
2. If $a < b$ in M_j, then $Y_{i,a} < Y_{i,b}$ in $L_{i,j}$ and
3. If $a \in [q]$, then the restriction of $L_{i,s+j}$ to $Y_{i,a}$ is the dual of the restriction of $L_{i,j}$ to $Y_{i,a}$.

Next we claim that if (x, y) is a critical pair and $x \in Y_i$, then there is some $j \in [2s]$ so that $x > D[y]$ in $L(i, j)$. To see this observe that any two points in $D[y]$ are adjacent in $\mathbf{G}$ so at most r points in $D[y]$ belong to Y_i, and all points of $D[y] \cap Y_i$ belong to distinct subsets in the partition of Y_i into independent subsets. Let $x \in Y_{i,j0}$. Then there exists some $j \in [s]$ so that $j_0 > j$ in M_j whenever $j \neq j_0$ and $D[y] \cap Y_{i,j} \neq \emptyset$. It follows that either $x > D[y]$ in $L(i, j)$ or $x > D[y]$ in $L(i, s + j)$.

Finally, we note that $s = \dim(1, r; q) \leq r(r+1) \log(q/r)$, so that $\dim(\mathbf{P}) \leq 100k \log^2 k$ as claimed. □

There are two fundamentally important problems which leap out from the preceding inequality limiting the dimension of posets of bounded degree, beginning with the obvious question: Is the inequality $\mathrm{Dim}(k) = O(k \log^2 k)$ best possible? However, the details of the proof also suggest that the inequality could be improved if one could provide a better upper bound than $\dim(1, \log k; k) = O(\log^3 k)$. Unfortunately, the second approach will not yield much as Kierstead [15] has recently provided the following lower bound.

Theorem 14 (Kierstead). *If* $\lg \lg n - \lg \lg \lg n \leq r \leq 2^{\lg^{1/2} n}$, *then*

$$\frac{(r + 2 - \lg \lg n + \lg \lg \lg \lg n)^2 \lg n}{32 \lg(r + 2 - \lg \lg n + \lg \lg \lg n)} \leq \dim(1, r; n) \leq \frac{2k^2 \lg^2 n}{\lg^2 k}. \quad (2)$$

As a consequence, it follows that $\dim(1, \log k; k) = \Omega(\log^3 k / \log \log k)$. So the remaining challenge is to provide better lower bounds on $\mathrm{Dim}(k)$. Random methods seem to be our best hope. Here is a sketch of the technique used by Erdős, Kierstead and Trotter [8] to show that $\mathrm{Dim}(k) = \Omega(k \log k)$.

For a fixed positive integer n, consider a random poset $\mathbf{P}_n$ having n minimal elements $a_1, a_2, \ldots, a_n$ and n maximal elements $b_1, b_2, \ldots, b_n$. The order relation is defined by setting $a_i < b_j$ with probability $p = p(n)$; also, events corresponding to distinct min-max pairs are independent.

Erdős, Kierstead and Trotter then determine estimates for the expected value of the dimension of the resulting random poset. The arguments are far too complex to be conveniently summarized here, as they make non-trivial use of correlation inequalities. However, the following theorem summarizes the lower bounds obtained in [8].

Theorem 15 (Erdős, Kierstead and Trotter).

1. *For every* $\epsilon > 0$*, there exists* $\delta > 0$ *so that if*

$$\frac{\log^{1+\epsilon} n}{n} < p \leq \frac{1}{\log\, n},$$

then

$$\dim(\mathbf{P}) > \delta pn \log pn, \text{ for almost all } \mathbf{P}.$$

2. *For every* $\epsilon > 0$*, there exist* $\delta, c > 0$ *so that if*

$$\frac{1}{\log\, n} \leq p < 1 - n^{-1+\epsilon},$$

then

$$\dim(\mathbf{P}) > \max\{\delta n, n - \frac{cn}{p\,\log\, n}\}, \text{ for almost all } \mathbf{P}.$$

The following result is then an easy corollary.

Corollary 1 (Erdős, Kierstead and Trotter). *For every* $\epsilon > 0$*, there exists* $\delta > 0$ *so that if*

$$n^{-1+\epsilon} < p \leq \frac{1}{\log\, n},$$

then

$$\dim(\mathbf{P}) > \delta\Delta(\mathbf{P}) \log\, n, for\ almost\ all\ \mathbf{P}.$$

Summarizing, we now know that

$$\Omega(k\,\log\, k) = D(k) = O(k\,\log^2 k). \tag{3}$$

It is the author's opinion that the upper bound is more likely to be correct and that the proof of this assertion will come from investigating the dimension of a slightly different model of random height 2 posets. For integers n and k with k large but much smaller than n, we consider a poset with n minimal points and n maximal points. However, the comparabilities come from taking k random matchings.

The techniques used by Erdős, Kierstead and Trotter in [8] break down when $p = o(\log\, n/n)$. But this is just the point at which we can no longer guarantee that the maximum degree is $O(pn)$.

6. Fractional Dimension and Ramsey Theory for Probability Spaces

In many instances, it is useful to consider a fractional version of an integer valued combinatorial parameter, as in many cases, the resulting LP relaxation sheds light on the original problem. In [3], Brightwell and Scheinerman proposed to investigate fractional dimension for posets. This concept has already produced some interesting results, and many appealing questions have been raised. Here's a brief sketch of some questions with immediate connections to random methods.

Let $\mathbf{P} = (X, P)$ be a poset and let $\mathcal{F} = \{\mathcal{M}_\infty, \dots, \mathcal{M}_\sqcup\}$ be a multiset of linear extensions of P. Brightwell and Scheinerman [3] call $\mathcal{F}$ a *k-fold realizer* of P if for each incomparable pair (x, y), there are at least k linear extensions in $\mathcal{F}$ which reverse the pair (x, y), i.e., $|\{i : 1 \leq i \leq t, x > y \text{ in } M_i\}| \geq k$, The *fractional dimension* of $\mathbf{P}$, denoted by $\mathrm{fdim}(\mathbf{P})$, is then defined as the least real number $q \geq 1$ for which there exists a k-fold realizer $\mathcal{F} = \{M_1, \dots, M_t\}$ of P so that $k/t \geq 1/q$ (it is easily verified that the least upper bound of such real numbers q is indeed attained and is a rational number). Using this terminology, the *dimension* of $\mathbf{P}$ is just the least t for which there exists a 1-fold realizer of P. It follows immediately that $\mathrm{fdim}(\mathbf{P}) \leq \dim(\mathbf{P})$, for every poset $\mathbf{P}$.

Note that the standard example of an n-dimensional poset also has fractional dimension n. Brightwell and Scheinerman [3] proved that if $\mathbf{P}$ is a poset and $|D(x)| \leq k$, for all $x \in X$, then $\mathrm{fdim}(\mathbf{P}) \leq k + 2$. They conjectured that this inequality could be improved to $\mathrm{fdim}(\mathbf{P}) \leq k + 1$. This was proved by Felsner and Trotter [10], and the argument yielded a much stronger conclusion, a result with much the same flavor as Brooks' theorem for graphs.

Theorem 16 (Felsner and Trotter). *Let k be a positive integer, and let $\mathbf{P}$ be any poset with $|D(x)| \leq k$, for all $x \in X$. Then $\mathrm{fdim}(\mathbf{P}) \leq k + 1$. Furthermore, if $k \geq 2$, then $\mathrm{fdim}(\mathbf{P}) < k + 1$ unless one of the components of $\mathbf{P}$ is isomorphic to S_{k+1}, the standard example of a poset of dimension $k + 1$.*

We do not discuss the proof of this result here except to comment that it requires a strengthening of Proposition 2, and to note that it implies that the fractional dimension of the poset $\mathbf{P}(1, r; n)$ is $r + 1$. Thus a poset can have large dimension and small fractional dimension. However, there is one elementary bound which limits dimension in terms of fractional dimension.

Theorem 17. *If $\mathbf{P} = (X, P)$ is a poset with $|X| = n$ and $\mathrm{fdim}(\mathbf{P}) = q$, then $\dim(\mathbf{P}) \leq (2 + o(1))q \log n$.*

Proof. Let $\mathcal{F}$ be a multi-realizer of P so that $\mathrm{Prob}_{\mathcal{F}}[x > y] \geq 1/q$, for every critical pair $(x, y) \in \mathrm{crit}(\mathbf{P})$. Then take t to be any integer for which

$$n(n-1)(1 - 1/q)^t < 1.$$

Then let $\{L_1, \dots, L_t\}$ be a sequence of length t in which the linear extensions in $\mathcal{F}$ are equally likely to be chosen. Then the expected number of critical pairs which are not reversed is less than one, so the probability that we have a realizer of cardinality t is positive. □

Felsner and Trotter [10] derive several other inequalities for fractional dimension, and these lead to some challenging problems as to the relative tightness of inequalities similar to the one given in Theorem 16. However, the subject of fractional dimension has produced a number of challenging problems which are certain to require random methods in their solutions. Here is two such problems, one of which has recently been solved.

A poset $\mathbf{P} = (X, P)$ is called an *interval order* if there exists a family $\{[a_x, b_x] : x \in X\}$ of non-empty closed intervals of $\mathbb{R}$ so that $x < y$ in P if and only if $b_x < a_y$ in $\mathbb{R}$. Fishburn [11] showed that a poset is an interval order if and only if it does not contain $\mathbf{2}+\mathbf{2}$ as a subposet. The interval order $\mathbf{I}_n$ consisting of all intervals with integer endpoints from $\{1, 2, \ldots, n\}$ is called the *canonical interval order.*

Although posets of height 2 can have arbitrarily large dimension, this is not true for interval orders. For these posets, large height is a prerequisite for large dimension.

Theorem 18 (Füredi, Hajnal, Rödl and Trotter). *If $\mathbf{P} = (X, P)$ is an interval order of height n, then*

$$\dim(\mathbf{P}) \le \lg\lg n + (1/2 + o(1)) \lg\lg\lg n. \tag{4}$$

The inequality in the preceding theorem is best possible.

Theorem 19 (Füredi, Hajnal, Rödl and Trotter). *The dimension of the canonical interval order satisfies*

$$\dim(\mathbf{I}_n) = \lg\lg n + (1/2 + o(1)) \lg\lg\lg n. \tag{5}$$

Although interval orders may have large dimension, they have bounded fractional dimension. Brightwell and Scheinerman [3] proved that the dimension of any finite interval order is less than 4, and they conjectured that for every $\epsilon > 0$, there exists an interval order with dimension greater than $4 - \epsilon$. We believe that this conjecture is correct, but confess that our intuition is not really tested. For example, no interval order is known to have fractional dimension greater than 3.

Motivated by the preceding inequalities and the known bounds on the dimension and fractional dimension of interval orders and the posets $\mathbf{P}(1, r; n)$, Brightwell asked whether there exists a function $f : \mathbb{Q} \to \mathbb{R}$ so that if $\mathbf{P} = (X, P)$ is a poset with $|X| = n$ and $\operatorname{fdim}(\mathbf{P}) = q$, then $\dim(\mathbf{P}) \le f(q) \lg\lg n$. If such a function exists, then the family $P(1, r; n)$ shows that we would need to have $f(q) = \Omega(2^q)$.

But we will show that there is no such function. The argument requires some additional notation and terminology. Fix integers n and k with $1 \le k < n$. We call an ordered pair (A, B) of k-element sets a (k, n)-*shift pair* if there exists a $(k+1)$-element subset $C = \{i_1 < i_2 < \cdots < i_{k+1} \subseteq \{1, 2, \ldots, n\}$ so that $A = \{i_1, i_2, \ldots, i_k\}$ and $B = \{i_2, i_3, \ldots, i_{k+1}$. We then define the (k, n)-*shift graph* $S(k, n)$ as the graph whose vertex set consists of all k-element subsets of $\{1, 2, \ldots, n\}$ with a k-element set A adjacent to a k-element set B exactly when (A, B) is a (k, n)-shift pair. Note that the $(1, n)$ shift graph $\mathbf{S}(1, n)$ is just a complete graph. It is customary to call a $(2, n)$-shift graph just a shift graph; similarly, a $(3, n)$-shift graph is called a *double shift graph.* The formula for the chromatic number of the $(2, n)$-shift graph $\mathbf{S}(2, n)$ is a folklore result of graph theory: $\chi(\mathbf{S}(2, n)) = \lceil \lg n \rceil$. Several researchers in graph theory have told me that this result is due to Andras Hajnal, but Andras says that it is not. In any case, it is an easy exercise.

The following construction exploits the properties of the shift graph to provide a negative answer for Brightwell's question.

Theorem 20. *For every $m \geq 3$, there exists a poset $\mathbf{P} = (X, P)$ so that*

1. $|X| = m^2$;
2. $\dim(X, P) \geq \lg m$; *and*
3. $\mathrm{fdim}(X, P) \leq 4$.

Proof. The poset $\mathbf{P} = (X, P)$ is constructed as follows. Set $X = \{x(i, j) : 1 \leq i, j \leq m\}$, so that $|X| = m^2$. The partial order P is defined by first defining $x(i, j_1) < x(i, j_2)$ in P, for each $i \in [m]$ whenever $1 \leq j_1 < j_2 \leq m$. Furthermore, for each $i \in [m]$, $x(i_1, j_1) < x(i_2, j_2)$ in P if and only if $(i_2 - i_1) + (j_2 - j_1) > m$.

We now show that $\dim(X, P) \geq \lg m$. Note first that for each i, j with $1 \leq i < j \leq m$, $x(i, j - i) \parallel x(j, m)$. Let $\dim(X, P) = t$, and let $\mathcal{R} = \{\mathcal{L}_\infty, \mathcal{L}_\in, \ldots, \mathcal{L}_\sqcup\}$ be a realizer of P. For each i, j with $1 \leq i < j \leq m$, choose an integer $\phi(\{i, j\}) = \alpha \in \{1, 2, \ldots, t\}$ so that $x(i, j - i) > x(j, m)$ in L_α. We claim that ϕ is a proper coloring of the $(2, m)$ shift graph $S(1, m)$ using t colors, which requires that $\dim(X, P) = t \geq \chi(\mathbf{S}(2, m) = \lceil \lg m \rceil$. To see that ϕ is a proper coloring, let i, j and k be integers with $1 \leq i < j < k \leq m$, let $\phi(\{i, j\}) = \alpha$ and let $\phi(\{j, k\}) = \beta$. If $\alpha = \beta$, then $x(i, j - i) > x(j, m)$ in L_α and $x(j, k - j) > x(k, m)$ in L_α. Also, $x(j, m) > x(j, k - j)$ in P. However, since $(k - i) + (m - j + i) > m$, it follows that $x(k, m) > x(i, j - i)$ in P, so that $x(k, m) > x(i, j - i)$ in L_α. Thus,

$$x(i, j - i) > x(j, m) > x(j, k - j) > x(k, m) > x(i, j - i) \text{ in } P \qquad (6)$$

The inequalities in equation 6 cannot all be true. The contradiction shows that ϕ is a proper coloring of the shift graph $\mathbf{S}(2, m)$ as claimed. In turn, this shows that $\dim(X, P) \geq \lceil \lg m \rceil$.

Finally, we show that $\mathrm{fdim}(X, P) \leq 4$. For each element $x \in X$, let p_1 and p_2 be the natural projection maps defined by $p(x) = i$ and $p_2(x) = j$ when $x = x(i, j)$. Next, we claim that for each subset $A \subset [m]$, there exists a linear extension $L(A)$ of P so that $x > y$ in $L(A)$ if:

1. $x \parallel y$ in P;
2. $p_1(x) \in A$ and $p_1(y) \notin A$.

To show that such linear extensions exist, we use the alternating cycle test (see Chap. 2 of [23]). Let $A \subseteq [m]$, and let $S(A) = \{(x, y) \in X \times X : x \parallel y$ in $P, p_1(x) \in A$ and $p_1(y) \notin A\}$. Now suppose that $\{(u_k, v_k) : 1 \leq k \leq p\} \subseteq S(A)$ is an alternating cycle of length p, i.e., $u_k \parallel v_k$ and $u_k \leq v_{k+1}$ in

P, for all $k \in [p]$ (subscripts are interpreted cyclically). Let $k \in [m]$. Then $p_1(u_k) \in A$ and $p_1(v_{k+1}) \notin A$. It follows that $u_k < v_{k+1}$ in P, for each $k \in [p]$. It follows that $p_1(v_{k+1} - p_1(u_k) + p_2(v_{k+1}) - p_2(u_k) > m$. Also, we know that $m \geq p_1(v_k) - p_1(u_k) + p_2(v_k) - p_2(u_k)$. Thus $p_1(v_{k+1}) + p_2(v_{k+1}) > p_1(v_k) + p_2(v_k)$. Clearly, this last inequality cannot hold for all $k \in [p]$. The contradiction shows that $S(A)$ cannot contain any alternating cycles. Thus the desired linear extension $L(A)$ exists.

Finally, we note that if we take $\mathcal{F} = \{\mathcal{L}(A) : \mathcal{A} \subseteq [\Updownarrow]\}$ and set $s = |\mathcal{F}|$, then $x > y$ in at least $s/4$ of the linear extensions in $\mathcal{F}$, whenever $x \parallel y$ in P. To see this, observe that there are exactly $2^s/4$ subsets of $[m]$ which contain $p_1(x)$ but do not contain $p_1(y)$. This shows that $\text{fdim}(X, P) \leq 4$ as claimed. It also completes the proof of the theorem. □

Now we turn our attention to the double shift graph. If $\mathbf{P} = (X, P)$ is a poset, a subset $D \subseteq X$ is called a *down set*, or an *order ideal*, if $x \leq y$ in P and $y \in D$ always imply that $x \in P$. The following result appears in [13] but may have been known to other researchers in the area.

Theorem 21. *Let n be a positive integer. Then the chromatic number of the double shift graph* $\mathbf{S}(3, n)$ *is the least t so that there are at least n down sets in the subset lattice* $\mathbf{2}^t$.

The problem of counting the number of down sets in the subset lattice $\mathbf{2}^t$ is a classic problem and is traditionally called Dedekind's problem. Although no closed form expression is known, relatively tight asymptotic formulas have been given. For our purposes, the estimate provided by Kleitman and Markovsky [16] suffices. Theorem 21, coupled with the estimates from [16] permit the following surprisingly accurate estimate on the chromatic number $\chi(\mathbf{S}(3, n))$ of the double shift graph [13].

Theorem 22 (Füredi, Hajnal, Rödl and Trotter).

$$\chi(\mathbf{S}(3, n)) = \lg \lg n + (1/2 + o(1)) \lg \lg \lg n.$$

Now that we have introduced the double shift graph, the following elementary observation can be made [13].

Proposition 3. *For each* $n \geq 3$, $\dim(1, 2; n) \geq \chi(\mathbf{S}(3, n))$, *and* $\dim(\mathbf{I}_n) \geq \chi(\mathbf{S}(3, n))$.

Although the original intent was to investigate questions involving the fractional dimension of posets, Trotter and Winkler [27] began to attack a Ramsey theoretic problem for probability spaces which seems to have broader implications. Fix an integer $k \geq 1$, and let $n \geq k + 1$. Now suppose that Ω is a probability space containing an event E_s for every k-element subset $S \subset \{1, 2, \ldots, n\}$. We abuse terminology slightly and use the notation $\text{Prob}(S)$ rather than $\text{Prob}(E_S)$.

Now let $f(\Omega)$ denote the minimum value of $\text{Prob}(A\overline{B})$, taken over all (k,n)-shift pairs (A,B). Note that we are evaluating the probability that A is true and B is false. Then let $f(n,k)$ denote the maximum value of $f(\Omega)$ and let $f(k)$ denote the limit of $f(n,k)$ as n tends to infinity.

Even the case $k = 1$ is non-trivial, as it takes some work to show that $f(1) = 1/4$. However, there is a natural interpretation of this result. Given a sufficiently long sequence of events, it is inescapable that there are two events, A and B with A occurring before B in the sequence, so that

$$\text{Prob}(A\overline{B}) < \frac{1}{4} + \epsilon.$$

The $\frac{1}{4}$ term in this inequality represents coin flips. The ϵ is present because, for finite n, we can always do slightly better than tossing a fair coin.

For $k = 2$, Trotter and Winkler [27] show that $f(2) = 1/3$. Note that this is just the fractional chromatic number of the double shift graph. This result is also natural and comes from taking a random linear order L on $\{1, 2, \dots, n\}$ and then saying that a 2-element set $\{i, j\}$ is *true* if $i < j$ in L. Trotter and Winkler conjecture that $f(3) = 3/8$, $f(4) = 2/5$, and are able to prove that $\lim_{k\to\infty} f(k) = 1/2$. They originally conjectured that $f(k) = k/(2k+2)$, but they have since been able to show that $f(5) \geq \frac{27}{64}$ which is larger than $\frac{5}{12}$.

As an added bonus to this line of research, we are beginning to ask natural (and I suspect quite important) questions about patterns appearing in probability spaces.

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III. Infinity

Introduction

Paul Erdős was always interested in infinity. One of his earliest results is an infinite analogue of (the then very recent) Menger's theorem (which was included in a classical book of his teacher Denes Kőnig). Two out of his earliest three combinatorial papers are devoted to infinite graphs. According to his personal recollections, Erdős always had an interest in "large cardinals" and his earliest work on this subject are joint papers with A. Tarski from the end of 1930s. These interests evolved over the years into the Giant Triple Paper, with the Partition Calculus forming a field rightly called here Erdősian Set Theory.

We wish to thank András Hajnal for a beautiful paper which perhaps best captures the special style and spirit of Erdős' mathematics. We solicited another two survey papers as well. An extensive survey was written by Peter Cameron on the seemingly simple subject of the infinite random graph, which describes the surprising discovery of Rado and Erdős-Rényi finding many new fascinating connections and applications. The paper by Peter Komjáth deals with another (this time more geometrical) aspect of Erdősian set theory. In addition, the research articles by Shelah and Křížž complement the broad scope of today's set theory research, while the paper of Aharoni looks at another pre-war Erdős conjecture. It is only with pleasure that we can remark that the Erdős–Menger problem has ben solved:

R. Aharoni, E. Berger, "Menger's Theorem for infinite graphs". Inventiones Mathematicae 176 (2009), 1–62.

In 1995/1996, when the contents of these volumes was already crystallising, we asked Paul Erdős to isolate a few problems, both recent and old, for each of the eight main parts of this book. To this part on infinity he contributed the following collection of problems and comments.

Erdős in his own words

I have nearly 50 joint papers with Hajnal on set theory and many with Rado and Fodor and many triple papers and I only state a few samples of our results. The first was my result with Dushnik and Miller.

Let $m \geq \aleph_0$. Then

$$m \to (m, \aleph_0)^2_2.$$

I use the arrow notation invented by R. Rado—in human language: If one colors the pairs of a set of power $m \geq \aleph_0$ by two colors either in color 1 there is a complete graph of power m or in color 2 an infinite complete graph. Hajnal, Rado and I nearly completely settled $m \to (n, q)^2_2$ but the results are very technical and can be found in our joint triple paper and in our book. In one of our joint papers, Hajnal and I proved that a graph G of chromatic number $\aleph_1$ contains all finite bipartite graphs and with Shelah and Hajnal we proved that it contains all sufficiently large odd cycles (Hajnal and Komjáth have sharper results).

Hajnal and I have quite a few results on property B. A family of sets $\{A_\alpha\}$ has property B if there is a set C which meets each of the sets A_α and contains none of them. This definition is due to Miller. It is now more customary to call the family two chromatic. Here is a sample of our many results: Let $\{A_\alpha\}$ be a family of $\aleph_k$ countable sets with $|A_\alpha \cap A_\beta| \leq n$. Then there is a set S which meets each A_α in a set of size $(k+1)n+1$. The result is best possible. If $k \geq \omega$ then there is a set for which $|S \cap A_\alpha| < \aleph_0$. We have to assume the generalized hypothesis of the continuum.

Todorčevič proved with $c = \aleph_1$ that one can color the edges of the complete graph on $|S| = \aleph_1$ with $\aleph_1$ colors so that every $S_1 \subset S$, $|S_1| = \aleph_1$, contains all colors.

We wrote two papers on solved and unsolved problems in set theory. Most of them have been superseded in many cases because undecidability raised its ugly head (according to many: its pretty head). Here is a problem where, as far as we know, no progress has been made. One can divide the triples of a set of power $2^{2^{\aleph_0}}$ into t classes so that every set of power $\aleph_1$ contains a triple of both classes. On the other hand if we divide the triples of a set of power $(2^{2^{\aleph_0}})^+$ into two classes there is always a set of size $\aleph_1$ all whose triples are in the same class. If $S = 2^{2^{\aleph_0}}$, can we divide the triples into two classes so that every subset of size $\aleph_l$ should contain a $K(4)$ of both classes (or more generally a homogeneous subset of size $\aleph_0$)? I offer 500 dollars for clearing up this problem.

Erdős–Galvin–Hajnal problem. Let G have chromatic number $\aleph_1$. Can one color the edges by 2 (or $\aleph_0$, or $\aleph_1$) colors so that if we divide the vertices into $\aleph_0$ classes there always is a class which contains all the colors? Todorčevič proved this if G is the complete graph of $\aleph_1$ vertices—in the general case Hajnal and Komjáth have some results.

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So much P. Erdős. The paper with Hajnal and Shelah is:

P. Erdős, A. Hajnal, S. Shelah, Topics in topology (Proc. Colloq. Keszthely, 1972), Colloq. Math. J. Bolyai Soc. 8, North Holland, 1974, 243–245.

A. Hajnal is of course one of the most frequent collaborators of P. Erdős. His more recent survey is:

A. Hajnal, On the chromatic number of graphs and set systems. PIMS Distinguished Chair Lectures, University of Calgary, 2004.

The Rado graph (in today's terminology, the universal ultra homogeneous graph) is also the countable random graph considered by Erdős and Rényi. The recent development here was motivated by the connection of ultra homogeneous structures and Ramsey classes (see the preceding part on Ramsey theory) and on the other side by the connections to model theory and mathematical logic. This is described briefly in the update of Peter Cameron's paper.

A Few Remarks on a Conjecture of Erdős on the Infinite Version of Menger's Theorem

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Summary. We discuss a few issues concerning Erdős' conjecture on the extension of Menger's theorem to infinite graphs. A key role is given to a lemma to which the conjecture can probably be reduced. The paper is intended to be expository, so rather than claim completeness of proofs, we chose to prove the reduction only for graphs of size $\aleph_1$. We prove the lemma (and hence the $\aleph_1$ case of the conjecture) in two special cases: graphs with countable out-degrees, and graphs with no unending paths. We also present new versions of the proofs of the (already known) cases of countable graphs and graphs with no infinite paths. A main tool used is a transformation converting the graph into a bipartite graph.

1. Introduction

1.1. The Problem

Kőnig's classical book [7] contains a proof of an infinite version of Menger's theorem:

Theorem 1. *Given any two disjoint sets, A and B, of vertices in any graph, the minimal cardinality of an A-B separating set of vertices is equal to the maximal number of vertex-disjoint A-B paths.*

The proof was by a very young mathematician, Paul Erdős. In fact, it is not very hard: Let $\mathcal{F}$ be a maximal set (with respect to inclusion) of vertex disjoint paths, and let S be the set of all vertices participating in paths from $\mathcal{F}$. Clearly, then, S is A-B-separating. If $\mathcal{F}$ is infinite, then $|S| = |\mathcal{F}|$ and the theorem is proved. If $\mathcal{F}$ is finite then so is S, and then techniques from the finite case can be applied. But this proof shows that Theorem 1 is not really the right extension to the infinite case: the separating set is of the right cardinality, but still it is obviously too large. It is not clear when did Erdős form the "right" conjecture, which is:

Conjecture 1. *For any two vertex sets A and B in a graph, there exists a set $\mathcal{F}$ of disjoint A-B paths and an A-B separating set of vertices S, such that S consists of the choice of precisely one vertex from each path in $\mathcal{F}$.*

Definition 1. *A pair $(\mathcal{F}, S)$ as in the conjecture is called* orthogonal.

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As is well known, one can restrict the considerations to digraphs. The undirected case follows by the customary device of replacing each edge by a pair of oppositely directed edges.

The main developments so far have been a proof of the countable bipartite case [8] (where, of course, it is assumed that A and B are the two sides of the graph); the case of countable graphs containing no infinite paths [9]; the general bipartite case [2]; the case of general graphs containing no infinite paths [1], and the countable case [3].

In [1] it was realized that in the absence of infinite paths, the conjecture can be reduced to the bipartite case. The simple transformation leading to the reduction is one of the main themes of the present paper. We shall follow it a bit further, and see how it can be used in other cases.

In fact, already in [9] it was realized that the main difference between the bipartite case and the general one stems from the possible existence of infinite paths. The reason is that in the construction of $\mathcal{F}$, while trying to reach B from A you may end up with infinite paths instead of A-B paths. This obstacle was overcome in [3] in the countable case, but the proof there relies very heavily on countability.

1.2. Warps, Waves and Hindrances

A triple $\Gamma = (G, A, B)$, where $G = G(\Gamma)$ is a digraph, and $A = A(\Gamma)$, $B = B(\Gamma)$ are subsets of $V(G)$, is called a *web*. A *warp* is a set of disjoint paths (the term is taken from weaving). If the initial points of the warp are all in A then it is called A-*starting*.

The initial vertex of a path P (if such a vertex exists) is denoted by $in(P)$, and its terminal vertex by $ter(P)$. The vertex set of P is denoted by $V(P)$ and its edge set by $E(P)$. For a family $\mathcal{P}$ of paths, we write $V[\mathcal{P}]$, $E[\mathcal{P}]$, $in[\mathcal{P}]$, $ter[\mathcal{P}]$ for the vertex set, edge set, initial points set, and terminal points set of $\mathcal{P}$, respectively. A *linkage* is a warp $\mathcal{L}$ such that $in[\mathcal{L}] = A$ and $ter[\mathcal{L}] \subseteq B$. A web is called *linkable* if it contains a linkage.

Given a graph G and a subset R of its vertices, we write $G[R]$ for the subgraph of G induced by R. For any set T we write $G - T = G[V(G) \setminus T]$. By $\Gamma - T$ we denote the web $(G - T, A \setminus T, B \setminus T)$. Given a path P (possibly in a super-web of Γ) we write $\Gamma - P$ for $\Gamma - V(P)$. For an A-starting warp $\mathcal{W}$, we write $\Gamma/\mathcal{W}$ for the web $(G - (V[\mathcal{W}] \setminus ter[\mathcal{W}]), (A \setminus \ in[\mathcal{W}]) \cup ter[\mathcal{W}], B)$. (So, it is the web obtained by moving the points of $in[\mathcal{W}]$ along the paths of $\mathcal{W}$, to serve as new source points.) For a warp consisting of a single path P we shall write Γ/P for $\Gamma/\{P\}$.

If a warp $\mathcal{W}$ is A-starting and $ter[\mathcal{W}]$ is A-B separating then $\mathcal{W}$ is called a *wave*. The *trivial* wave is the set of all singleton paths of the form $(a), a \in A$. A *hindrance* is a wave $\mathcal{W}$ such that $in[\mathcal{W}] \neq A$. Obviously, a hindrance is a non-trivial wave. A web is called *hindered* if it contains a hindrance. It is called *loose* if it contains no non-trivial wave (and then, of course, it is also unhindered).

For a wave $\mathcal{W}$ we write $\Gamma\angle\mathcal{W}$ for the web $(G-(V[\mathcal{W}]\setminus ter[\mathcal{W}]), ter[\mathcal{W}], B)$ (the difference between this and $\Gamma/\mathcal{W}$ is that points in $A\setminus in[\mathcal{W}]$ are not taken here as source points).

Let $\mathcal{W}$ be a wave, and let T be the set of vertices in V which are separated by $ter[\mathcal{W}]$ from B. We write $\Gamma[\mathcal{W}]$ for the web $(G[T], A, ter[\mathcal{W}])$.

If $P_1, P_2, \ldots, P_k$ are paths and x_i, $1 \leq i < k$ are vertices such that $x_i \in V(P_i) \cap V(P_{i+1})$ then $P_1x_1P_2x_2 \ldots x_{k-1}P_k$ denotes the path (if indeed it is a path) obtained by going along P_1 until reaching x_1, then switching to P_2, then switching at x_2 to P_3, and so forth. If the sequence ends with a vertex, rather than a path, the path intended is the one ending at that vertex. So, for example, P_x means the part of P up to and including x. The same goes for sequences starting with a vertex. If $x = ter(P) = in(Q)$ then PQ denotes the path PxQ.

A useful operation between waves is the following: let $\mathcal{U}$, $\mathcal{W}$ be waves. We write $\mathcal{U} \uparrow \mathcal{W}$ for the warp $\{PxQ : P \in \mathcal{U}, Q \in \mathcal{W}, x = ter(P) \in V(Q)$ and $V(xQ) \cap V[\mathcal{U}] = \{x\}\}$. It was proved in [3] that $\mathcal{U} \uparrow \mathcal{W}$ is a wave.

There are two natural orders defined between waves: write $\mathcal{W} \leq \mathcal{U}$ if $\mathcal{U}$ is an extension of $\mathcal{W}$, i.e., every path $P \in \mathcal{U}$ is a continuation of some path Q from $\mathcal{W}$ such that $V(P) \setminus V(Q) \cup V[\mathcal{W}] = \emptyset$. As usual, "$<$" means "$\leq$ and not equal". Note that if $\mathcal{U}$ is obtained from $\mathcal{W}$ by the deletion of some paths then $\mathcal{W} < \mathcal{U}$.

Let $\mathcal{W} \preceq \mathcal{U}$ if $ter[\mathcal{W}]$ is separated from B by $ter[\mathcal{U}]$. It is straightforward to see that both relations are indeed partial orders, and that $\mathcal{W} \leq \mathcal{U}$ implies $\mathcal{W} \preceq \mathcal{U}$. In [3] the following was proved:

Lemma 1. *In any web Γ there exists a maximal wave in the order $\leq$. If $\mathcal{W}$ is such a maximal wave then $\Gamma\angle\mathcal{W}$ is loose.*

We shall often use this lemma without explicit mention.

Also the following is easy:

Lemma 2. *If $\mathcal{W} \not\preceq \mathcal{U}$ then $\mathcal{U} \uparrow \mathcal{W} \succ \mathcal{U}$.*

In [3] it was shown that Conjecture 1 is equivalent to the following conjecture:

Conjecture 2. *An unhindered web is linkable.*

The proof of the equivalence is not hard. One direction is even trivial: assuming Conjecture 1, if $\mathcal{F}$ is not a linkage then the warp $\{Ps : P \in \mathcal{F},\ s \in S$ and $s \in V(P)\}$ is a hindrance.

For the proof of the other direction, let $\mathcal{W}$ be a $<$-maximal wave in Γ. By Lemma 1 the web $\Gamma\angle\mathcal{W}$ is loose. Assuming Conjecture 2 there exists then a linkage $\mathcal{L}$ in $\Gamma\angle\mathcal{W}$. Taking the concatenation of $\mathcal{W}$ and $\mathcal{L}$ as $\mathcal{F}$ and $ter[\mathcal{W}]$ as S in Conjecture 1 yields then the conjecture.

It is Conjecture 2 which we shall try to solve. As already mentioned, the countable case of the conjecture is known [3]. The key lemma there is:

Lemma 3. *If Γ is an unhindered countable web then for every $a \in A$ there exists an $a - B$ path P such that $\Gamma - P$ is unhindered.*

(The countable case of Conjecture 2 follows immediately: if Γ is unhindered, then by the lemma one can link the vertices a_i of A into B by paths P_i one by one, while keeping the web $\Gamma - \bigcup\{V(P_j) : j < i\}$ unhindered.)

In [4] a slight generalization of the countable case was proved:

Lemma 4. *Conjecture 2 is true for webs Γ in which there exists an A-B warp $\mathcal{W}$ such that $in[A] \setminus in[\mathcal{W}]$ is countable.*

Note that the condition in Lemma 4 is not hereditary, that is, it is not passed on to sub-webs. Hence the proof that Conjecture 2 implies Conjecture 1 does not show immediately that Conjecture 1 is true for webs as in the above lemma. But this was proved in [5]:

Lemma 5. *Conjecture 1 is true for webs satisfying the condition of Lemma 4.*

We believe that Lemma 3 is true in all webs:

Conjecture 3. *Lemma 3 is true in any web.*

The main purpose of this paper is to prove a reduction of Conjecture 1 to Conjecture 3. But in order not go into too technical details we shall do it only for webs of size $\aleph_1$.

We shall prove Conjecture 3 (for graphs of size $\aleph_1$) in two special cases: webs with countable out-degrees, and webs with no unending paths. We shall therefore be able to prove Conjectures 1 and 2 for these two classes. In both proofs we shall use extensively the bipartite conversion, both technically and for inspiration. While not absolutely necessary, this will prove to be economical. The countable outdegrees case includes, of course, the case of countable webs, and the proof presented here is therefore also a new proof for this case (although the main ideas are similar to those of the old one, that in [3]). The new proof is more natural when considered in the language of the bipartite conversion.

1.3. The Bipartite Conversion

With any web Γ we associate a bipartite web $b(\Gamma) = (b(G), b(A), b(B))$ in the following way. Every vertex v in $V \setminus A$ is assigned a vertex v'' in $V(b(G))$, and every vertex $v \in V \setminus B$ is assigned a vertex v'. (So, vertices in $V \setminus (A \cup B)$ are assigned two vertices each.) The edge set $E(b(G))$ is defined as $\{(x', y'') : (x, y) \in E(\Gamma)\} \cup \{(x', x'') : x \in V \setminus (A \cup B)\}$. The "source set" of $b(\Gamma)$, which will be denoted by $b(A)$, is just $\{v' : v \in V \setminus B\}$. The "destination set", $b(B)$, is $\{v'' : v \in V \setminus A\}$.

A linkage $\mathcal{F}$ in Γ corresponds in a natural way to a linkage $b(\mathcal{F}) = \{(x', y'') : (x, y) \in E[\mathcal{F}]\} \cup \{(x', x'') : x \notin V[\mathcal{F}]\}$ in $b(\Gamma)$. Conversely, to a linkage J in $b(\Gamma)$ (linkages in bipartite webs are just matchings, hence they are denoted by capital letters, rather than script letters), there corresponds a warp $w(J)$ in Γ, defined by $E[w(J)] = \{(x, y) : (x', y'') \in J,\ x \neq y\}$. (In general, backwards transformations from $b(\Gamma)$ to Γ will be denoted by w.) We write $\tilde{w}(J)$ for the set of those paths in $w(J)$ which begin at A.

The following lemma is very easy to verify:

Lemma 6. *Let J be a linkage in $b(\Gamma)$. Then $in[w(J)] = in[\tilde{w}(J)] = A$. If a path in $w(J)$ has an endpoint then this point belongs to B. Hence, if $\tilde{w}(J)$ contains no undendinq paths, it is a linkage.*

Let now $\mathcal{W}$ be a wave in Γ. Define $b(\mathcal{W}) = \{(x', y'') : (x,\ y) \in E[\mathcal{W}]\} \cup \{(x') : x \in ter[\mathcal{W}]\} \cup \{(x', x'') : x \notin V[\mathcal{W}]\}$. It is not hard to see that $b(\mathcal{W})$ is a wave in $b(\Gamma)$. Conversely, a wave I in $b(\Gamma)$ corresponds to a warp $w(I)$ in Γ, defined by $E[w(I)] = \{(x, y) : (x', y'') \in E[I]\}$, and the singleton paths in $w(I)$ are the singletons (a), where $a \in A$ and $(a') \in I$.

Lemma 7. *$ter[w(I)]$ is A-B-separating. Every path in $w(I)$ either starts in A or is non-starting. Hence, if $w(I)$ contains no non-starting paths, it is a wave in Γ. In the latter case, if I is a hindrance, so is $w(I)$.*

Proof. Let $P = x_1 x_2 \ldots x_n$ be an A-B path in Γ. If P is not met by $ter[w(I)]$ then $(x_1') \notin I$. Since the edge (x_1', x_2'') is covered by $ter[I]$, it follows that $x_2'' \in ter[I]$. Let k be the last index for which $x_k'' \in ter[I]$. If $x_k \notin ter[w(I)]$ then there exists y such that $(x_k', y'') \in I$. Then the edge $(x_k',\ x_{k+1}'')$ necessarily meets $ter[I]$ at x_{k+1}'', contradicting the choice of k. This shows that $ter[w(I)]$ is A-B-separating.

Let now $Q \in w(I)$. If Q is a singleton x, then by the definition of $w(I)$ we have $x \in A$, so Q starts at A. If not, choose a vertex $v = v_0 \in V(Q) \setminus ter(Q)$. Then $v_0' \notin ter[I]$, and hence the edge (v_0', v_0'') is covered at v_0''. Hence there exists an edge $(v_0'',\ v_1') \in I$, and then $(v_1, v_0) \in E(Q)$. Repeating the same argument for v_1, we obtain an edge $(v_2,\ v_1) \in E(Q)$. This process is either infinite, in which case Q is non-starting, or terminates when some v_k is in A (and then the edge (v_k'', v_k') just does not exist.) This proves the second part of the lemma.

The fact that if I is a hindrance then so is $w(I)$ is obvious. □

From the above it follows that if Γ does not contain unending paths then it is linkable if and only if $b(\Gamma)$ is, and if it contains no non-starting path then it is hindered if and only if $b(\Gamma)$ is. Since Conjecture 2 is true for bipartite graphs [2], this proves:

Theorem 2. *Conjecture 2, and hence also Conjecture 1, are true for graphs with no infinite paths.*

This was proved in [1] in a different way.

2. Safely Linking One Point

For $x \in V \setminus A$ we write $\Gamma \vdash x$ for the web $w(b(\Gamma) - x'')$. In the language of webs, rather than that of bipartite graphs, it is the web obtained upon adding x to A, and deleting all edges going into x. For a subset X of $V \setminus A$ we denote by $\Gamma \vdash X$ the web $w(b(\Gamma) - \{x'' : x \in X\})$.

A key fact in the discussion to follow is:

Lemma 8. *If Γ is unhindered, $v \in V \setminus A$ and $\Gamma \vdash v$ is hindered, then there exists in Γ a wave $\mathcal{U}$ such that v is separated by $ter[\mathcal{U}]$ from B (possibly $v \in ter[\mathcal{U}]$).*

Proof. Let $\mathcal{H}$ be a hindrance in $\Gamma \vdash v$. If v is hindered in $\mathcal{H}$ (i.e., $v \notin in[\mathcal{H}]$), then $\mathcal{H}$ is a hindrance also in Γ. If not, then let u be the terminal point of the path R in $\mathcal{H}$ which starts at v, and let h be a point in A hindered by $\mathcal{H}$. Let $\mathcal{H}' = \mathcal{H} \setminus \{R\}$. If there is no $\mathcal{H}'$-alternating path in Γ starting at h and ending at u then the set of edges in $E[\mathcal{H}']$ participating in $\mathcal{H}'$-alternating paths starting at h forms a hindrance in Γ (see [3] for a detailed proof of this fact). Hence, we may assume that there exists an $\mathcal{H}'$-alternating path Q starting at h and ending at u. Applying Q to $\mathcal{H}$ (that is, taking the warp whose edge set is $E[\mathcal{H}]\Delta E(Q)$) yields then the desired wave $\mathcal{U}$. □

Corollary 1. *If Γ is loose and x is a vertex in $V \setminus A$ from which B is reachable then $\Gamma \vdash x$ is unhindered.*

Lemma 9. *If Γ is hindered then so is $\Gamma \vdash v$.*

Proof. Let $\mathcal{H}$ be a hindrance in Γ. If $v \notin V[\mathcal{H}]$ then $\mathcal{H} \cup \{(v)\}$ is a hindrance in $\Gamma \vdash v$. If v lies on some path $H \in \mathcal{H}$ then $\mathcal{H} \setminus \{H\} \cup \{vH\}$ is a hindrance in $\Gamma \vdash v$. □

Lemma 10. *If Γ is unhindered and $\mathcal{W}$ is a wave in Γ then for every path $P \in \mathcal{W}$ the web Γ/P is unhindered.*

Proof. Suppose that there exists a hindrance $\mathcal{H}$ in Γ/P. Let $\mathcal{J} = \mathcal{W} \setminus \{P\}$, $p = ter(P)$, let Q be the path in $\mathcal{H}$ starting at p, if such exists, and let $\mathcal{K} = \mathcal{H} \setminus \{Q\}$ ($\mathcal{K} = \mathcal{H}$ if $p \notin in[\mathcal{H}]$). Let $\mathcal{L} = \mathcal{K} \uparrow \mathcal{J}$. Assume first that $p \notin in[\mathcal{H}]$. We claim that then $\mathcal{L}$ is a hindrance in Γ, contrary to the assumption that Γ is unhindered. To see this, it is enough to show that $\mathcal{L}$ is a wave, since obviously $in[\mathcal{L}] \neq A$. But this, in turn, is obvious, since in this case $\mathcal{L}$ is just $\mathcal{H} \uparrow \mathcal{W}$, which is a wave.

Assume next that $p \in in[\mathcal{H}]$. Let $\mathcal{H}' = \mathcal{H} \setminus \{Q\} \cup PQ$ and $\mathcal{N} = \mathcal{H}' \uparrow \mathcal{W}$. We claim that $\mathcal{N}$ is a wave (and then, obviously, a hindrance.) To see this, take an A-B path R. Let t be the last point on R which lies on a path from $V[\mathcal{W}]$. Then, since $\mathcal{W}$ is a wave, $t \in ter[\mathcal{W}]$. Hence $t \notin V(P - ter(P))$. But $\mathcal{N}' = \mathcal{H} \uparrow \mathcal{J}$ is a wave in $\Gamma - (P - ter(P))$, and hence the path tR contains a vertex from $ter[\mathcal{N}']$ and hence also from $ter[\mathcal{N}]$. □

Lemma 11. *If* Γ *is unhindered and* $a \in A$ *then there exists a non-trivial finite path* P *starting at* a *such that* Γ/P *is unhindered.*

Proof. Let $\mathcal{W}$ be a maximal wave in Γ. If $\mathcal{W}$ contains a non-trivial path P starting at a then, by Lemma 10, P is the desired path. If not, then, since Γ is unhindered, there exists a vertex v which is connected to a in $\Gamma\angle\mathcal{W}$ and from which B is reachable. We claim that $P = av$ is the desired path. This will clearly follow if we show that $\Gamma \vdash v$ is unhindered. If this is not the case, then by Lemma 8 there exists a wave $\mathcal{U}$ in Γ which separates v from B. Since $\mathcal{W}$ does not separate v from B, $W \not\preceq \mathcal{U}$. But, by Lemma 2, this contradicts the maximality of $\mathcal{W}$. □

Let S be an A-B-separating set of vertices in Γ. We shall denote by $A - part(S)$ the set of all vertices which are separated by S from B, and by $B - part(S)$ the set of all vertices which are separated by S from A (Note that S is contained in both, and so are possibly other vertices.) We write Γ_S for the web $(G[A-part(S)], A, S)$ and Γ^S for the web $(G[B-part(S)], S, B)$. If $\mathcal{W}$ is a wave we write $\Gamma_{\mathcal{W}}$ for $\Gamma_{ter}[\mathcal{W}]$ and $\Gamma^{\mathcal{W}}$ for $\Gamma_{ter}[\mathcal{W}]$.

Lemma 12. *If* S *is* A-B*-separating and* Γ *is hindered then at least one of* Γ_S *or* Γ^S *is hindered.*

Proof. Let $\mathcal{H}$ be a hindrance in Γ. Let $\mathcal{X}$ be the set of all maximal initial parts of paths in $\mathcal{H}$ which are contained in Γ_S, and let $\mathcal{Y}$ be the set of all paths of the form sH, where $s \in S$, $H \in \mathcal{H}$ and s is the last vertex on H belonging to S.

Suppose that $\mathcal{X}$ is not a hindrance in Γ^S. Then there exists a vertex $s \in S$ which is not separated from A by $ter[\mathcal{X}]$. Then the warp $\mathcal{Y} \cup \{(t) : t \in S \setminus \{s\} \setminus in[\mathcal{Y}]\}$ is a hindrance in Γ_S (since a path from s to B in Γ^S which avoids $S \setminus \{s\} \cup ter[\mathcal{Y}]$ could be combined with a path in Γ_S from A to s which avoids $ter[\mathcal{X}]$ to yield a contradiction to the fact that $\mathcal{H}$ is a hindrance.) □

Lemma 13. *If* Γ *has no unending paths, and if it is unhindered, then for any* $a \in A$ *there exists an* a-B *path* P *such that* $\Gamma - P$ *is unhindered.*

Proof. By Lemma 11 there exists a non-trivial path $P_1 = Q_1$ starting at a such that Γ/Q_1 is unhindered. Apply the lemma again, to Γ/Q_1, to obtain a path $P_2 = Q_1Q_2$ in Γ such that Γ/P_2 is unhindered. Since Γ does not contain unending paths, this process must halt at a certain point, and this can happen only when P_k reaches B, for some k. Then P_k is the desired path. □

The following theorem was (essentially) proved in [3]. But we would like to present here a somewhat different proof. The advantage of the new proof is that it makes better use of the bipartite conversion. It differs from the proof in [3] in that it $\vdash$-s vertices, rather than deletes them. Since it is the $\vdash$-ing operation, and not deleting, which is used in the discussion of the uncountable case in the following sections, this may indicate that there is a better chance of extending the proof to the uncountable case.

Theorem 3. *Let Γ be an unhindered web, and let $a \in A$ be such that there are only countably many $a-B$ paths. Then there exists an $a-B$ path P such that $\Gamma - P$ is unhindered.*

Proof. Enumerate the $a - B$ paths as $(P_1,\ P_2,\ \ldots)$. We shall choose inductively webs Γ_i, waves $\mathcal{W}_i$ in Γ_i, vertices v_i warps $\mathcal{U}_i$ and sub-warps $\tilde{\mathcal{U}}_i$ of $\mathcal{U}_i (i < \omega)$, as follows. Let $\Gamma_0 = \Gamma$, $\mathcal{W}_0$ and $\mathcal{U}_0$ the trivial wave in Γ, and $v_0 = a$.

Let $\mathcal{W}_1$ be a maximal wave in Γ_0, and let $\Delta_1 = \Gamma_0 \angle \mathcal{W}_1$. By Lemma 1 Δ_1 is loose. Let $\mathcal{U}_1 = \mathcal{W}$ and let $\tilde{\mathcal{U}}_1$ be the set of those paths in $\mathcal{U}_1$ which do not start at a. Now let i_1 be first such that P_{i_1} does not meet $V[\tilde{\mathcal{U}}_1]$, and let v_1 be the first vertex on P_{i_1} which does not belong to $V[\mathcal{U}_1]$. Let (u_1, v_1) be the edge of P_{i_1} preceding v_1. Then $u_1 \in ter[\mathcal{W}_1]$, and since Δ_1 is loose, by Corollary 1 $\Gamma_1 \stackrel{def}{=} \Delta_1 \vdash v_1$ is unhindered.

Let $\mathcal{W}_2$ be a maximal wave in Γ_1, and define $\Delta_2 = \Gamma_1 \angle \mathcal{W}_2$. Let $\mathcal{U}_2$ be the concatenation of $\mathcal{W}_1$ and $\mathcal{W}_2$. Clearly, then, $\mathcal{U}_2$ is a wave in Γ_2. Let $\tilde{\mathcal{U}}_2$ be the set of those paths in $\mathcal{U}_2$ which do not start at v_0 or v_1. Let i_2 be first such that P_{i_2} does not meet $V[\tilde{\mathcal{U}}_2]$, and let v_2 be the first vertex on P_{i_2} which does not belong to $V[\mathcal{U}_2]$. Let $(u_2,\ v_2)$ be the edge on P_{i_2} which precedes v_2. Since $\mathcal{U}_2$ is a wave in Γ_2, $u_2 \in ter[\mathcal{U}_2]$. Hence, by Lemma 8 $\Gamma_2 \stackrel{def}{=} \Delta_2 \vdash v_2$ is unhindered.

Continuing in this way, we obtain after ω steps a wave $\mathcal{U}$ in the web $\Gamma_\omega \stackrel{def}{=} \Gamma \vdash \{v_0, v_1, \ldots\}$, which is the concatenation of all warps $\mathcal{U}_i$. It is divided into two parts: $\mathcal{Z}$, which consists of those paths in $\mathcal{U}$ which begin at some v_i, and $\tilde{\mathcal{U}} = \mathcal{U} \setminus \mathcal{Z}$, which is the "limit" of all $\tilde{\mathcal{U}}_i$ (i.e its edge set is the union of the edge sets of all $\tilde{\mathcal{U}}_i$,) and whose starting points are all in $A \setminus \{a\}$.

Assertion 1. *If some path in $\mathcal{Z}$ ends in B (in particular if some v_i is in B,) then the theorem holds.*

Proof. Suppose that some path $Q_0 \in \mathcal{Z}$ ends in B. Let k_0 be such that $Q_0 \in \mathcal{W}_{k_0}$ and let $v_{k_1} = in(Q_0)$ (possibly $Q_0 = (v_{k_1})$). Let u_{k_1} be as in the construction above. Then $u_{k_1} = ter(Q_1)$ for some path $Q_1 \in \mathcal{W}_{k_1}$ and $in(Q_1) = v_{k_2}$ for some $k_2 < k_1$. Let u_{k_2} be as in the construction above, and take a path $Q_2 \in \mathcal{W}_{k_2}$ which terminates at u_{k_2} and starts at v_{k_3} for some $k_3 < k_2$. Continuing this way we get a sequence of triples (v_{k_i}, u_{k_i}, Q_i), where the k_i are descending. Eventually, $k_j = 0$ for some j.

Let $P = Q_j Q_{j-1} \ldots Q_1 Q_0$. We claim that P is the desired path in the theorem. Assume, to the contrary, that $\Gamma - P$ is hindered.

Write $\Pi = \Gamma \vdash \{v_0, v_1, \ldots, v_{k_i}\}$. Let $\Xi_0 = \Pi_{\mathcal{U}_{k_0}}$, and for each $i \leq j$ let $\Xi_i = (\Xi_{i-1})_{\mathcal{U}_{k_i}}$ and $\Psi_i = (\Xi_{i-1})^{\mathcal{U}_{k_i}}$. By Lemma 9 the web $\Pi - P$ is hindered. Since Q_0 is a path in the wave $\mathcal{W}_{k_0}$ in Γ_{k_1}, by Lemma 10 Γ_{k_1}/Q_0

is unhindered. Hence also $\Psi_1 - P$, which is equal to $\Gamma_{k_1} - Q$, is unhindered. It thus follows that $\Xi_1 - P$ is hindered.

Again, by Lemma 12 this implies that either $\Xi_2 - P$ is hindered or $\Psi_2 - P$ is hindered. By the same argument as above, the second of these possibilities would imply that Γ_{k_2}/Q_1 id hindered, in contradiction to Lemma 10. Hence $\Xi_2 - P$ is hindered. Continuing this way we obtain that $\Xi_j - P$, which is just the trivial web Γ_A (i.e., the web having A as both its source set and its target set, and no edges,) is hindered, which is obviously false. This proves the assertion. □

We shall show that $\tilde{\mathcal{U}}$ is a wave. Since $a \notin in[\tilde{\mathcal{U}}]$, this will mean that it is actually a hindrance, contradicting the assumption that Γ is unhindered. In order to show that $\tilde{\mathcal{U}}$ is a wave, consider an A-B path T. Since $\mathcal{W}_1$ is a wave in Γ, T meets $ter[\mathcal{W}_1]$. Let x be the last vertex on T which belongs to $ter[\mathcal{W}_i]$ for some i. If $x \notin ter[\mathcal{U}]$ then there exists $j > i$ such that $x \in in[\mathcal{W}_j]$. But since $\mathcal{W}_j$ is a wave in Γ_i, the path xT contains a vertex in $ter[\mathcal{W}_j]$, contrary to the choice of i. Thus $x \in ter[\mathcal{U}]$. It remains to show the impossibility of $x \in ter[\mathcal{Z}]$. Suppose that $x = ter(Q)$ for some $Q \in \mathcal{Z}$. Let $in(Q) = v_m$, and suppose that v_m lies on the path P_j. Then the path $P_j v_m Q x T$ is one of the paths P_k. But, by the choice of x, the part xT of this path does not contain a vertex from any $V[\mathcal{W}_n]$, and hence when the turn of P_k arrived, we would have chosen from it all vertices of xT, one by one, as v_p-s. But then $ter(T) = v_p$ for some p, contradicting Assertion 1. □

One case in which the conditions of Theorem 3 hold is that in which the outdegrees of all vertices are countable, so for webs with this property the theorem is known for all vertices a. Also, clearly, Lemma 3 is an immediate corollary. As already mentioned, the countable case of Conjecture 1 follows from this lemma directly. So the theorem yields a new proof of the countable case of Conjecture 1.

3. $\aleph_1$-Hindrances

The key to the solution of the uncountable bipartite case of Erdős' conjecture was in the definition of higher order hindrances, i.e., κ-hindrances for regular uncountable cardinalities κ. The course of the proof there was proving first that a bipartite web is linkable if it does not contain a hindrance or a κ-hindrance for some regular uncountable cardinal κ [6], and then proving that the existence of a κ-hindrance implies the existence of a hindrance (in the old sense [2]).

The notion of higher order hindrances can be carried over to general webs. Since the aim of this paper is to present general techniques rather than complete proofs, we shall be satisfied with presenting the notion of an $\aleph_1$-hindrance, which means that the proofs will apply only to webs of size $\aleph_1$.

The reader may take on faith the fact that the notion and proofs can be extended to general cardinalities, and he is referred to [6, 2] for details in the bipartite case.

For an ordinal $\eta \leq \aleph_1$ an η-*ladder* is a sequence $L = (R_\alpha : \alpha < \eta)$, where each "rung" R_α of the ladder is of one of three possible types (the webs Γ^α mentioned in them are to be defined below)

(1) A vertex $v_\alpha \in V(\Gamma^\alpha) \setminus A(\Gamma^\alpha)$
(2) A hindrance $\mathcal{R}_\alpha$, in Γ^α.
(3) An infinite path R_α, which is the concatenation of paths in hindrances $\mathcal{R}_\beta, \beta < \alpha$ appearing in rungs of type (2), and has not appeared as yet as a rung of type (3) (i.e it is not R_γ for any $\gamma < \alpha, \gamma$ of type (3).)

The ordinal η is called the *height* of the ladder. (If special mention of the ladder L is necessary, we shall write $\eta(L)$, a remark applying to all notation referring to ladders.) The set of α-s of type (i) is denoted by Φ_i. The webs Γ^α are defined inductively:

If $\alpha \in \Phi_1$ then $\Gamma^{\alpha+1}$ is defined as $\Gamma^\alpha \vdash v_\alpha$.
If $\alpha \in \Phi_2$ then $\Gamma^{\alpha+1} = \Gamma^\alpha \angle R_\alpha$.
If $\alpha \in \Phi_3$ then $\Gamma^{\alpha+1}$ is obtained from Γ^α by deleting $in(R_\alpha)$ from its $'A$-set (for the motivation of this definition see below, after the definition of Γ_α.)

For limit α, Γ^α is defined as the "limit" of the webs Γ^β, $\beta < \alpha$, in an obvious way: its $'A'$ set is the set of all those vertices which belong to $A(\Gamma^\beta)$ for cofinally many β-s in α, and similarly for the definition of its vertex and edge sets. Its $'B'$ set is just B.

Alongside with the the webs Γ^α we define warps $\mathcal{Y}_\alpha = \mathcal{Y}_\alpha(L)$ for each $\alpha < \eta$. Loosely speaking, $\mathcal{Y}_\alpha$ is the concatenation of all hindrances $\mathcal{R}_\beta, \beta < \alpha$. More precisely, we let

$$E[\mathcal{Y}_\alpha] = \bigcup\{E[\mathcal{R}_\beta] : \beta < \alpha, \beta \in \Phi_2\} \setminus \bigcup\{E[\mathcal{R}_\beta] : \beta < \alpha, \beta \in \Phi_3\}.$$

The warp $\mathcal{Y}_\alpha$ is not defined solely by its edge set, because it may contain also singleton paths. It will be defined, however, once the set of its initial points is defined. We let $in[\mathcal{Y}_\alpha]$ be the set of all vertices in $A \cup \{v_\beta : \beta < \alpha\}$ which are not hindered in any hindrance $\mathcal{R}_\beta, \beta \in \Phi_2, \beta < \alpha$ and are not $in(R_\beta)$ for any $\beta \in \Phi_3, \beta < \alpha$. Also write $\mathcal{Y}_\eta = \mathcal{Y}(= \mathcal{Y}(L))$.

The following is easily proved by induction:

Lemma 14. *$ter[\mathcal{Y}_\alpha]$ separates A from B.*

From the inductive definition of the webs Γ^α it is easy to see that Γ^α is essentially $\Gamma^{ter[\mathcal{Y}_\alpha]}$, the difference being that the vertex set of the latter may be larger—there may be vertices separated by $ter[\mathcal{Y}_\alpha]$ from A, which are therefore in the vertex set of $\Gamma^{ter[\mathcal{Y}_\alpha]}$, but are deleted in the process of forming Γ^α. But for our purposes we may regard these two webs as being the same.

Let $\Gamma_\alpha = \Gamma_{ter[\mathcal{Y}_\alpha]}$.

For $v \in V \setminus A$ let $\rho(v) = \rho_L(v)$ be the first ordinal ρ for which $v \in V(\Gamma_\rho)$.

Rungs of type (3) appear in ladders since the initial point of an infinite path in $\mathcal{Y}$ has lost hope of being linked to B by $\mathcal{Y}$, while it is separated from B by $B(\Gamma_\alpha)$ (Lemma 14). So, it is a "problematic" vertex, which has to be specially taken care of. This complication does not arise, of course, in the case of webs with no unending paths, and as we shall see in Theorem 6 also not in the other main case discussed in this paper, namely that of webs with countable outdegrees.

Notation 1. *Let $F(L) = \cup\{R_\alpha : \alpha \in \Phi_1\}$. For each $\alpha \in \Phi_2$ let HD_α be the set of points in $A(\Gamma^\alpha)$ which are hindered by R_α. We also write HDA_α for the set of points in A which are the initial points of paths in $\mathcal{Y}_\alpha$ terminating at HD_α.*

For a limit ordinal α let $\mathcal{NF}_\alpha$ be the set of all infinite paths in $\mathcal{Y}_\alpha$ starting at A.

Also let $NF_\alpha = in[\mathcal{NF}_\alpha]$.

Definition 2. *If L is of height $\aleph_1$ and Φ_2 or Φ_3 are stationary in $\aleph_1$ then L is called an $\aleph_1$-hindrance.*

One of the two main ingredients of the proof of Conjecture 1 goes over to general webs:

Theorem 4. *If Γ contains an $\aleph_1$-hindrance then it is hindered.*

Proof. In [2] the theorem was proved for bipartite webs. We shall use this case of the theorem, but not directly. One has to go a bit into the proof in order to make it applicable to general webs.

Let $L = (R_\alpha : \alpha < \aleph_1)$ be an $\aleph_1$-hindrance. Assume, first, that $\Phi_2(L)$ is stationary. Then L corresponds to an $\aleph_1$-hindrance $N = (T_\alpha : \alpha < \aleph_1)$ in $b(\Gamma)$. By the main theorem of [2], N gives rise to a hindrance H in $b(\Gamma)$. We would like to show that H can be chosen so that $w(H)$ is a hindrance in Γ, i.e., so that it does not contain non-starting paths.

Let $M = b(E[\mathcal{Y}(L)])$, i.e., the set of edges in $b(\Gamma)$ corresponding to the edges in paths in the warp $\mathcal{Y}$. Then $M = E[\mathcal{Y}(N)]$, and it is a matching in $b(\Gamma)$. We write $F = F(L)$ and $BF = F(N)$. Obviously $BF = b(F)$

The basic concept in the proof in [2] is that of *popularity.*

Given a set of vertices U in any graph, a set $\mathcal{P}$ of paths in the graph is called *U-joined* if every two members of $\mathcal{P}$ meet only within U. A subset U of $B(b(\Gamma))$ is called *popular* if there exists a U-joined set $\{Z_\beta : \beta \in \Psi\}$ of paths, where each Z_β is an M-alternating path from some vertex u'' of U to some vertex $v' \in HD_\beta(N)$ and $\Psi \subseteq \Phi_2$ is stationary.

By the usual abuse of notation we shall say that a vertex $u'' \in B(b(\Gamma))$ is popular if $\{u''\}$ is popular.

We shall now follow a construction given in [2].

Let U_0 be the set of all unpopular vertices in BF (that is, the set of all unpopular vertices $u''_\alpha, \alpha \in \Phi_1$.) Let K_1 be the set of all vertices x' connected to them, U_1 the set of all unpopular vertices in $M[K_1]$, K_2 the set of all unpopular vertices connected to vertices in U_1, U_2 the set of unpopular vertices in $M[K_2]$, and so forth. Let, eventually, $U = \bigcup\{U_i : i < \omega\}, K = \bigcup\{K_i : i < \omega\}$, and $\Delta = b(\Gamma) - U - K$.

Assertion 2. *All points in $A(\Delta)$ are connected in $b(\Gamma)$ only to points from $B(\Delta)$.*

Proof. This is clear, since by definition K is the set of points connected in $b(\Gamma)$ to U. □

Let $M' = M_{|B(\Delta)}$ and $FR = B(\Delta) \setminus \bigcup M$ ("FR" stands for "free"). Clearly, $FR = BF \setminus U$.

The points in FR were popular in Γ. The main idea of the proof in [2] is that they still remain so in Δ, although Δ is formed from Γ by the removal of vertices. In a somewhat informal manner, this can be put in:

Assertion 3. *Every point p in FR is still popular in Δ, namely it has a p-joined set of M'-alternating paths to "stationarily many" hindered points belonging to $A(\Delta)$ (i.e., points in $A(\Delta) \setminus \bigcup M'$.)*

A little more formal, and stronger, is the following:

Assertion 4. *Let $p \in FR$ and let $\{X_\alpha : \alpha \in \Psi\}$ be a p-joined set of M-alternating paths, where Ψ is stationary and each path X_α goes from p to HD_α. Then $\{\beta : K \text{ meets } X_\beta\}$ is non-stationary.*

In [2] Assertion 4 was derived from an even stronger one, which we shall also use here:

Assertion 5. *The entire set U is unpopular.*

To see how Assertion 4 follows from Assertion 5, assume that the set Θ of β-s for which K meets X_β (say, at a point z_β) is stationary. For each $\beta \in \Theta$ choose a point $u_\beta \in U$ connected to z_β. Then the set of paths $\{u_\beta z_\beta X_\beta : \beta \in \Theta\}$ would show that U is popular.

Another immediate corollary of Assertion 5 is:

Assertion 6. *The set of α-s for which $HD_\alpha \cap A(\Delta) \neq \emptyset$ is stationary.*

This would clearly follow also from Assertion 3 if we knew that $FR \neq \emptyset$. But this is not guaranteed.

In the bipartite case, Assertion 3 is sufficient for the proof of Theorem 4. In fact, all that is used is that every point $p \in FR$ has a p-joined set of size $\aleph_1$ of M'-alternating paths to points in $A(\Delta) \setminus \bigcup M'$. Since $|FR| \leq \aleph_1$, these alternating paths can be used to match all points of $B(\Delta)$ one by one into $A(\Delta)$. One then uses Fodor's Lemma and Assertion 6 to prove that this matching is strictly into. By Assertion 2 this matching provides the desired hindrance in $b(\Gamma)$.

An observation which will be used below is that $\mathcal{Y}$-alternating paths in Γ are in one to one correspondence with M-alternating paths in $b(\Gamma)$, with the natural correspondence: an M-alternating path Q corresponds to a $\mathcal{Y}$-alternating path $w(Q)$ whose edge set is $w(E(Q))$. In fact, this can be used as a definition of a $\mathcal{Y}$-alternating path. (A detail worth noting is that when a $\mathcal{Y}$-alternating path Q lingers for more than one edge on a path from $\mathcal{Y}$, its corresponding path alternates between edges of M and edges of the form (x', x''), which are not in M.)

Write $S = w(FR)$ and $\mathcal{Y}' = w(M')$. Then $\mathcal{Y}'$ consists of fragments of paths from $\mathcal{Y}$. Let $\mathcal{Z}$ be the set of paths in $\mathcal{Y}'$ which start at A. Each $s \in S$ has $\aleph_1$ many $\mathcal{Y}'$-alternating paths meeting only at s. Listing the points of S in a sequence of length at most ω_1, we can link them one by one using their $\mathcal{Y}'$-alternating paths, and sticking to the rule that a path in $\mathcal{Y}$ is never used in alternating paths pertaining to more than one point from S.

The trouble is that if we are not careful in the application of the alternating paths, the warp which results may contain non-starting paths. To overcome this difficulty we prove:

Assertion 7. *Let $s \in S$ and let $\{X_\beta : \beta \in \Psi\}$ be an s-joined set of $\mathcal{Y}'$-alternating paths, each X_β leading to HD_β, and Ψ being stationary. Then there exists a stationary subset Ψ' of Ψ such that for each $\beta \in \Psi'$ the paths from $\mathcal{Y}'$ which X_β meets are all in $\mathcal{Z}$.*

Proof. We shall prove something even stronger, namely that the set of those β-s for which X_β meets a path not in $\mathcal{Z}$ is non-stationary. This set consists of two (possibly overlapping) parts. The first, Ξ_1, consists of β-s for which X_β meets a path $Q_\beta \in \mathcal{Y}$ which does not start at A. The second, Ξ_2, consists of β-s for which X_β meets a path, say Y_β, at a point y_β such that there exists a point $k_\beta \in K$ preceding y_β on Y_β.

For each $\beta \in \Xi_1$ the initial point of Q_β is a vertex $v = R_\alpha$ for some $\alpha < \beta$. Hence, if Ξ_1 were stationary, by Fodor's Lemma $\aleph_1$ many X_β-s would share the same Q_β. This is impossible, since the paths X_β are disjoint and each Q_β is finite.

Assume next that Ξ_2 is stationary. For each $\beta \in \Xi_2$ choose a vertex $u''_\beta \in U$ connected to k'_β and witnessing the fact that $k' \in K$. Consider then the paths $J_\beta = u_\beta k_\beta Y_{\beta y_\beta} X_\beta$. The paths $b(J_\beta)$ in $b(\Gamma)$ then show that U is popular, contradicting Assertion 5. □

Let $(s_\delta : \delta < \nu \prec \aleph_1)$ be a listing of all element of S. Choose inductively for each s_δ a $\mathcal{Z}$-alternating path P_δ from s_δ to some $HD_{\alpha(\delta)}(L)$ which meets only paths in $\mathcal{Z}$ which were not met by any $P_\zeta, \zeta < \delta$, does not contain edges from $E[\mathcal{Y}] \setminus E[\mathcal{Z}]$, and such that $\alpha(\delta) \neq \alpha(\zeta)$ for $\zeta < \delta$. Such choice is possible, by Assertion 7. Let $\mathcal{H}$ be the warp resulting from the application of all P_δ-s to $\mathcal{Y}$. Since the paths P_δ correspond (by the correspondence b) to M'-alternating paths in Δ, $ter[\mathcal{H}] = w(B(\Delta))$ and $in[\mathcal{H}] \subseteq w(A(\Delta))$. By Assertion 2 this means that $H = b(E[\mathcal{H}])$ is a wave in $b(\Gamma)$. By Fodor's Lemma there exists at least one $\alpha \in \Phi_2$ such that no path P_δ reaches $HD_\alpha(N)$. (In fact, there is no need to invoke Fodor's Lemma for this, just choose the paths P_δ so that this condition is fulfilled.) Hence H is, in fact, a hindrance. Since the paths P_δ only use paths from $\mathcal{Z}$ to alternate on, and distinct ones at that, the paths in $\mathcal{H}$ all start at A. Hence, by Lemma 7, $\mathcal{H}$. is the hindrance desired in the theorem.

The case that Φ_3 is stationary is similar, but for simplicity of presentation it is better to work directly in Γ. For each $\alpha \in \Phi_3$ let P_α be the infinite path which constitutes the rung R_α. A vertex $w \in F$ is called *popular* if for a stationary subset Ψ of Φ_3 there exist $\mathcal{Y}$-alternating paths $Q_\psi, \psi \in \Psi$ from w to P_ψ. (We demand that such a path ends when it reaches some vertex in P_ψ). The proof then goes along the same lines as in the case of Φ_2 stationary. □

We conjecture that also the other main ingredient of the proof in the bipartite case goes over to general webs, namely:

Conjecture 4. *If $|V(\Gamma)| \leq \aleph_1$ and Γ contains no hindrance and no $\aleph_1$-hindrance then Γ is linkable.*

We are able to prove the conjecture only in the cases in which Conjecture 3 is known. So, we have:

Theorem 5. *Conjecture 2 (and hence also Conjecture 1) is true for webs of size $\aleph_1$ with no unending paths or with countable outdeqrees.*

Proof. By Lemma 5 the theorem holds for webs in which A or B are countable. Hence we may assume that $|A| = |B| = \aleph_1$. Enumerate $V \setminus A$ as $(v_0 : \theta < \aleph_1)$. Define inductively the rungs R_α of a ladder L as follows. Let $\Gamma^0 = \Gamma$. Suppose that R_δ have been defined for all $\delta < \alpha$. Let Γ^α be defined as in the definition of ladders made above. First priority in the choice of rungs is given to type (3). That is, if there exists an infinite path in $\mathcal{NF}_\alpha$ which is not equal to R_β for any $\beta < \alpha$, then choose such a path to be R_α.

Second priority is given to type (2). So, if there does not exist an infinite path as above, but there exists a hindrance in Γ^α, choose R_α to be such a hindrance.

If no path or hindrance as above exist, choose R_α to be the first v_0 in $V \setminus V(\Gamma_\alpha)$.

We proceed in this definition until all vertices in V have been covered. This, obviously, happens for some $\eta \leq \aleph_1$ For each $v \in V \setminus A$ write $\alpha(v)$ for the first α for which $v \in V(\Gamma_\alpha)$.

Assume first that $\eta(L) < \aleph_1$. Then the warp $\mathcal{W}$ consisting of all paths in $\mathcal{Y}(L)$ starting at A satisfies the condition that $A \setminus in[\mathcal{W}]$ is countable. Moreover, since all points of $V(\Gamma) \setminus A$ appear in the ladder L, $ter[\mathcal{Y}(L)]$ must cover all but countably many vertices in B. It follows that Γ satisfies the condition of Lemma 4, and hence also Conjecture 2.

Thus we may assume that $\eta(L) < \aleph_1$. By the assumption that Γ is $\aleph_1$-unhindered, there exists a club set Π such that $\Pi \cap (\Phi_2 \cup \Phi_3) = \emptyset$.

The idea is to use Π to link Γ. Informally, this is done as follows: we link one point from A to B by a path P, using Lemma 13 and Theorem 3. Let π_1 be the first element of Π for which all vertices in P are contained in Γ_{π_1}. We wish to take care of the points in A which are not linked by $\mathcal{Y}_{\pi_1}$ into $B(\Gamma_{\pi_1})$, There are only countably many of those, and we link them one by one to B. At each step we take a possibly larger $\pi_i \in \Pi$ so that Γ_{π_i} contains the new paths. We add to the set of problematic points (those which have to be taken care of) all points in A which are hindered in some R_α, $\pi_i > \alpha \in \Pi_2$, and also NF_α, if α is a limit ordinal. After ω steps we have taken care of all problematic points in Γ_π, where π is the limit of all π_i. Since $\pi \in \Pi$, the web Γ^π is unhindered. Also, $NF_\pi = \emptyset$. Hence we can turn over a new leaf—try to link points in Γ^π. By the club-ness of Π we can join all the partial linkages obtained in this procedure to obtain a linkage of A.

Formally all this is a bit more cumbersome:

Clearly, NF_α is countable for every $\alpha < \aleph_1$. For otherwise the rungs of L are of type (3) for all $\alpha < \beta < \aleph_1$, making L an $\aleph_1$-hindrance. Similarly, HD_α is countable, or else we could choose the rungs R_β to be of type (2) from α onwards.

List the points of A as $a_\theta, \theta < \aleph_1$.

For a subset Z of $V \setminus A$ let $\pi(Z)$ be the minimal ordinal $\pi \in \Pi$ such that $\pi > \rho(z)$ for every $z \in Z$.

By Lemma 13 and Theorem 3 there exists an $a_0 - B$ path P_0 such that $\Delta_1 = \Gamma - P_0$ is unhindered. Let $\pi_1^1 = \pi(V(P_0))$, Let K_1 be the set of all vertices in $A \setminus \{a_0\}$ which are initial vertices of paths from $V[\mathcal{Y}(L)]$ which are met by P_0, and let

$$D_1 = K_1 \cup HDA_{\pi_1^1}(L) \cup NF_{\pi_1^1} \cup \{a_1\}$$

(For the definition of HDA see Notation 2.) Clearly, D_1 is countable, so we can order it in an ω-sequence d_1. Let v_1 be the first vertex in this list. There exists then a $v_1 - B$ path P_1 such that $\Delta_2 = \Delta_1 - P_1$ is unhindered. Let $\pi_1^2 = \pi(V[\{P_0, P_1\}])$. Let K_2 be the set of vertices from $A \setminus \{v_0, v_1\}$ (where $v_0 = a_0$) which are initial vertices of paths from $\mathcal{Y}$ which are met by P_0, P_1, and let

$$D_2 = K_2 \cup HDA_{\pi_1^2} \cup (NF_{\pi_1^2} \setminus NF_{\pi_1^2}) \cup \{a_2\}$$

Intersperse D_2 in the even numbered places of d_1, to obtain a list d_2. Let v_2 be the first element in d_2 which does not belong to $\{v_0, v_1\}$. Choose a $v_2 - B$ path P_2 such that $\Gamma_3 = \Gamma_2 - P_1$ is unhindered. We continue in this way, and obtain sequences (v_i), (π_1^i), (P_i) of length ω. Let $\pi_1 = \sup\{\pi_1^i : i < \omega\}$. The interspersing of the sets D_i, in the sequence of points to be linked to B is done so as to guarantee that each point is reached at some stage (so, for example, D_3 is interspersed in every third place, say, in d_2). Then, after ω steps, when the dust has settled, the warp N_0 consisting of $\{P_i : i < \omega\}$ together with all paths in $\mathcal{Y}_{\pi_1}$ which connect A to $B(\Gamma_{\pi_1})$ and which do not meet any P_i, links A into $B(\Gamma_{\pi_1})$.

Now repeat the procedure with Γ^{π_1} replacing Γ. This is possible, since $\pi_1 \in \Pi$, and hence Γ^{π_1} is unhindered. This produces a linkage N_2 of $B(\Gamma^{\pi_1})$ into $B(\Gamma^{\pi_2})$ for some $\pi_1 < \pi_2 \in \Pi$.

Continuing this way, we obtain an $\aleph_1$-sequence of ordinals $\pi_\zeta \in \Pi$, and linkages $\mathcal{N}_\zeta$ of $A(\Gamma_{\pi_\zeta})$ into $B(\Gamma_{\pi_{\zeta+1}})$. For limit ζ we just define $\pi_\zeta = \sup\{\pi_\beta : \beta < \zeta\}$. Since Π is closed, $\pi_\zeta \in \Pi$.

At each step ζ, the first a_θ in the list of elements of A which have not been linked as yet to B is not hindered in Γ_{π_ζ}, since this web is not hindered. So, it is the starting point of a path $Q \in \mathcal{Y}_{\pi_\zeta}$. This path is not infinite, since if it is then, since $\Pi \cap \Phi_3 = \emptyset$, $Q \in \mathcal{NF}_\alpha$ for some $\alpha < \pi_\zeta$. But then, by our construction, a_θ has been linked already to B at a previous stage. Let $a'_\theta = ter(Q)$. Then $a'_\theta \in A(\Gamma_{\pi_\zeta})$. We put it first in the first list of vertices to be linked, so that after the ζ-th stage a_θ is linked to B. This guarantees that after $\aleph_1$ steps all vertices of A are linked to B. □

Remark 1. *The true content of the theorem, and what is actually proved, is that Conjecture 4 can be reduced to Conjecture 3 for webs of size $\aleph_1$. We chose the above presentation since it states absolute results, rather than a result about implication. In fact, both cases mentioned in the theorem have simpler proofs. For webs with no unending path one can construct a ladder as in the proof of the theorem, and then link its 'slices' (defined by the club set Π), and these linkages join up to form a linkage of Γ, since no unending path has been generated. In the case of countable outdegrees one can make do with a simpler kind of $\aleph_1$-hindrances, as follows:*

Call a hindrance degenerate *if all of its paths are singletons. A point in A hindered by a degenerate hindrance is just one from which B is unreachable. An $\aleph_1$-hindrance is called degenerate if all of it rungs R_α, $\alpha \in \Phi_2$, are degenerate. Note that in such an $\aleph_1$-hindrance $\mathcal{Y} = \emptyset$, and hence there are no rungs of type (3). Similar arguments to those above (only simpler) show the following.*

Theorem 6. *If $|V(\Gamma)| \leq \aleph_1$, the outdegrees of all vertices in Γ are countable, and Γ does not contain a hindrance or a degenerate $\aleph_1$-hindrance, then it is linkable.*

Of course, this follows from the previous theorems, since not being hindered suffices. What it really means is that the above proof can be simplified in this case. A version which does add new information, however, is this:

Theorem 7. *A web of size $\aleph_1$ with countable outdegrees is linkable if and only if every countable subset of A is linkable, and it does not contain a degenerate $\aleph_1$-hindrance.*

4. A Remark on Necessary Conditions for Linkability

Conjecture 2 will yield (if true) a sufficient condition for linkability in webs. The proof of the bipartite case of the conjecture gave, in fact, more than just a sufficient condition—a necessary and sufficient condition is known in this case.

Podewski and Steffens [8] defined (in somewhat different terms) a special type of waves. A wave $\mathcal{W}$ is called *tight* if there is no warp $\mathcal{U}$ such that $in[\mathcal{U}] = in[\mathcal{W}]$, $ter[\mathcal{U}] \subset ter[\mathcal{W}]$ and $ter[\mathcal{U}] \neq ter[\mathcal{W}]$. An *obstruction* is a tight hindrance. In [8] it was proved that a countable bipartite graph is linkable if and only if it is unobstructed. $\aleph_1$-obstructions are defined by taking the rungs R_α, $\alpha \in \Phi_2$ to be obstructions, rather than hindrances. Again, it was proved [6] that a bipartite graph of size $\aleph_1$ is linkable if and only if it does not contain an obstruction or an $\aleph_1$-obstruction.

It is not even completely clear what should be the formulation of the necessary and sufficient condition for linkability in webs. Transferring the notions of tight waves and obstructions verbatim does not work, as the following example shows:

Let $A^1 = \{a_i^1 : i < \omega\} A^2 = \{a_i^2 : i < \omega\}$, $B = \{b_i : i < \omega\}$ and $C = \{c_i : i < \omega\}$. Also let $A = A^1 \cup A^2$ and $V(G) = A \cup B \cup C$. Let $E(G) = A^2 \times C \cup C \times B \cup \{(a_i^1,\ c_i) : i < \omega\} \cup \{(a_i^2,\ b_i) : i < \omega\}$. Then the web $\Delta = (G, A, B)$ does not contain an obstruction in the above sense, and yet it is not linkable. What does exist in Δ is a tight wave $\mathcal{T}$ with a proper subwave $\mathcal{T}'$, which means to say that a proper subset of $ter[\mathcal{T}]$ is A-B-separating (take $\mathcal{T} = \{(a_i^2,\ b_i) : i < \omega\} \cup \{(a_i^1,\ c_i) : i < \omega\}$). It is possible that this is the right notion for an "obstruction" in the case of webs. It will be interesting (and possibly of value for the proof of the general conjecture) to show that the non-existence of obstructions in this sense is sufficient for linkability in countable webs.

Podewski and Steffens [8] noted that in a bipartite web Γ there exists a maximal linkable subset of $A(\Gamma)$ (the set of initial points of a maximal tight wave is such a set). The web Δ above provides a counterexample to this statement in the case of countable webs: it is not hard to show that a subset N of A is linkable if and only if $A \setminus N$ is infinite, and clearly there is no maximal subset N of A satisfying this condition.

Author's Note, 2013 Edition

Since the publication of this paper, the Erdős-Menger conjecture was solved:

Aharoni, Ron and Berger, Eli (2009). "Menger's Theorem for infinite graphs", Inventiones Mathematicae 176: 162. doi:10.1007/s00222-008-0157-3.

As foreseen in the present paper, the key was the proof of Lemma 3 for general (namely, uncountable) graphs. But the way from the lemma to the theorem proved to be more intricate for cardinalities of size larger than $\aleph_1$ than for the $\aleph_1$ case (which is done in this paper).

As in the present paper, another key role was played by the bipartite conversion, that provides good insight into Menger's theorem in the infinite case, as it does in the finite case.

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The Random Graph

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Summary. Erdős and Rényi showed the paradoxical result that there is a unique (and highly symmetric) countably infinite random graph. This graph, and its automorphism group, form the subject of the present survey.

1. Introduction

In 1963, Erdős and Rényi [27] showed:

Theorem 1. *There exists a graph R with the following property. If a countable graph is chosen at random, by selecting edges independently with probability $\frac{1}{2}$ from the set of 2-element subsets of the vertex set, then almost surely (i.e., with probability 1), the resulting graph is isomorphic to R.*

This theorem, on first acquaintance, seems to defy common sense—a random process whose outcome is predictable. Nevertheless, the argument which establishes it is quite short. (It is given below.) Indeed, it formed a tailpiece to the paper of Erdős and Rényi, which mainly concerned the much less predictable world of finite random graphs. (In their book *Probabilistic Methods in Combinatorics*, Erdős and Spencer [28] remark that this result "demolishes the theory of infinite random graphs.")

I will give the proof in detail, since it underlies much that follows. The key is to consider the following property, which a graph may or may not have:

($*$) *Given finitely many distinct vertices $u_1, \ldots, u_m, v_1, \ldots, v_n$, there exists a vertex z which is adjacent to $u_1, \ldots, u_m$ and nonadjacent to $v_1, \ldots, v_n$.*

Often I will say, for brevity, "z is correctly joined". Obviously, a graph satisfying ($*$) is infinite, since z is distinct from all of $u_1, \ldots, u_m, v_1, \ldots, v_n$. It is not obvious that any graph has this property. The theorem follows from two facts:

Fact 1. *With probability 1, a countable random graph satisfies ($*$).*

Fact 2. *Any two countable graphs satisfying ($*$) are isomorphic.*

Proof (of Fact 1). We have to show that the event that ($*$) fails has probability 0, i.e., the set of graphs not satisfying ($*$) is a null set. For this, it is enough to show that the set of graphs for which ($*$) fails for

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some given vertices $u_1, \ldots, u_m, v_1, \ldots, v_n$ is null. (For this deduction, we use an elementary lemma from measure theory: the union of countably many null sets is null. There are only countably many values of m and n, and for each pair of values, only countably many choices of the vertices $u_1, \ldots, u_m, v_1, \ldots, v_n$.) Now we can calculate the probability of this set. Let $z_1, \ldots, z_N$ be vertices distinct from $u_1, \ldots, u_m, v_1, \ldots, v_n$. The probability that any z_i is not correctly joined is $1 - \frac{1}{2^{m+n}}$; since these events are independent (for different z_i), the probability that none of $z_1, \ldots, z_N$ is correctly joined is $(1 - \frac{1}{2^{m+n}})^N$. This tends to 0 as $N \to \infty$; so the event that no vertex is correctly joined does have probability 0. □

Note that, at this stage, we know that graphs satisfying (∗) exist, though we have not constructed one—a typical "probabilistic existence proof". Note also that "probability $\frac{1}{2}$" is not essential to the proof; the same result holds if edges are chosen with fixed probability p, where $0 < p < 1$. Some variation in the edge probability can also be permitted.

Proof (of Fact 2). Let Γ_1 and Γ_2 be two countable graphs satisfying (∗). Suppose that f is a map from a finite set $\{x_1, \ldots, x_n\}$ of vertices of Γ_1 to Γ_2, which is an isomorphism of induced subgraphs, and x_{n+1} is another vertex of Γ_1. We show that f can be extended to x_{n+1}. Let U be the set of neighbours of x_{n+1} within $\{x_1, \ldots, x_n\}$, and $V = \{x_1, \ldots, x_n\} \setminus U$. A potential image of x_{n+1} must be a vertex of Γ_2 adjacent to every vertex in $f(U)$ and nonadjacent to every vertex in $f(V)$. Now property (∗) (for the graph Γ_2) guarantees that such a vertex exists.

Now we use a model-theoretic device called "back-and-forth". (This is often attributed to Cantor [20], in his characterization of the rationals as countable dense ordered set without endpoints. However, as Plotkin [58] has shown, it was not used by Cantor; it was discovered by Huntington [44] and popularized by Hausdorff [35].)

Enumerate the vertices of Γ_1 and Γ_2, as $\{x_1, x_2, \ldots\}$ and $\{y_1, y_2, \ldots\}$ respectively. We build finite isomorphisms f_n as follows. Start with $f_0 = \emptyset$. Suppose that f_n has been constructed. If n is even, let m be the smallest index of a vertex of Γ_1 not in the domain of f_n; then extend f_n (as above) to a map f_{n+1} with x_m in its domain. (To avoid the use of the Axiom of Choice, select the correctly-joined vertex of Γ_2 with smallest index to be the image of x_m.) If n is odd, we work backwards. Let m be the smallest index of a vertex of Γ_2 which is not in the range of f_n; extend f_n to a map f_{n+1} with y_m in its range (using property (∗) for Γ_1).

Take f to be the union of all these partial maps. By going alternately back and forth, we guaranteed that every vertex of Γ_1 is in the domain, and every vertex of Γ_2 is in the range, of f. So f is the required isomorphism. □

The graph R holds as central a position in graph theory as $\mathbb{Q}$ does in the theory of ordered sets. It is surprising that it was not discovered long before the 1960s! Since then, its importance has grown rapidly, both in its own right, and as a prototype for other theories.

Remark 1. Results of Shelah and Spencer [65] and Hrushovski [42] suggest that there are interesting countable graphs which "control" the first-order theory of finite random graphs whose edge-probabilities tend to zero in specified ways. See Wagner [76], Winkler [77] for surveys of this.

2. Some Constructions

Erdős and Rényi did not feel it necessary to give an explicit construction of R; the fact that almost all countable graphs are isomorphic to R guarantees its existence. Nevertheless, such constructions may tell us more about R. Of course, to show that we have constructed R, it is necessary and sufficient to verify condition $(*)$.

I begin with an example from set theory. The downward Löwenheim-Skolem theorem says that a consistent first-order theory over a countable language has a countable model. In particular, there is a countable model of set theory (the *Skolem paradox*).

Theorem 2. *Let M be a countable model of set theory. Define a graph M^* by the rule that $x \sim y$ if and only if either $x \in y$ or $y \in x$. Then M^* is isomorphic to R.*

Proof. Let $u_1, \ldots, u_m, v_1, \ldots, v_n$ be distinct elements of M. Let $x = \{v_1, \ldots, v_n\}$ and $z = \{u_1, \ldots, u_m, x\}$. We claim that z is a witness to condition $(*)$. Clearly $u_i \sim z$ for all i. Suppose that $v_j \sim z$. If $v_j \in z$, then either $v_j = u_i$ (contrary to assumption), or $v_j = x$ (whence $x \in x$, contradicting the Axiom of Foundation). If $z \in v_j$, then $x \in z \in v_j \in x$, again contradicting Foundation. □

Note how little set theory was actually used: only our ability to gather finitely many elements into a set (a consequence of the Empty Set, Pairing and Union Axioms) and the Axiom of Foundation. In particular, the Axiom of Infinity is not required. Now there is a familiar way to encode finite subsets of $\mathbb{N}$ as natural numbers: the set $\{a_1, \ldots, a_n\}$ of distinct elements is encoded as $2^{a_1} + \cdots + 2^{a_n}$. This leads to an explicit description of R: the vertex set is $\mathbb{N}$; x and y are adjacent if the x^{th} digit in the base 2 expansion of y is a 1 or *vice versa*. This description was given by Rado [59, 60].

The next construction is more number-theoretic. Take as vertices the set $\mathbb{P}$ of primes congruent to 1 (mod 4). By quadratic reciprocity, if $p, q \in \mathbb{P}$, then $\left(\frac{p}{q}\right) = 1$ if and only if $\left(\frac{q}{p}\right) = 1$. (Here "$\left(\frac{p}{q}\right) = 1$" means that p is a quadratic residue $\pmod q$.) We declare p and q adjacent if $\left(\frac{p}{q}\right) = 1$.

Let $u_1, \ldots, u_m, v_1, \ldots, v_n \in \mathbb{P}$. Choose a fixed quadratic residue a_i (mod u_i) (for example, $a_i = 1$), and a fixed non-residue b_j (mod v_j). By the Chinese Remainder Theorem, the congruences

$$x \equiv 1 \pmod 4, \quad x \equiv a_i \pmod{u_i}, \quad x \equiv b_j \pmod{v_j},$$

have a unique solution $x \equiv x_0 \pmod{4u_1 \dots u_m v_1 \dots v_n}$. By Dirichlet's Theorem, there is a prime z satisfying this congruence. So property $(*)$ holds.

A set S of positive integers is called *universal* if, given $k \in \mathbb{N}$ and $T \subseteq \{1, \dots, k\}$, there is an integer N such that, for $i = 1, \dots, k$,

$$N + i \in S \quad \text{if and only if} \quad i \in T.$$

(It is often convenient to consider binary sequences instead of sets. There is an obvious bijection, under which the sequence σ and the set S correspond when $(\sigma_i = 1) \Leftrightarrow (i \in S)$—thus σ is the characteristic function of S. Now a binary sequence σ is universal if and only if it contains every finite binary sequence as a consecutive subsequence.)

Let S be a universal set. Define a graph with vertex set $\mathbb{Z}$, in which x and y are adjacent if and only if $|x - y| \in S$. This graph is isomorphic to R. For let $u_1, \dots, u_m, v_1, \dots, v_n$ be distinct integers; let l and L be the least and greatest of these integers. Let $k = L - l + 1$ and $T = \{u_i - l + 1 : i = 1, \dots, m\}$. Choose N as in the definition of universality. Then $z = l - 1 - N$ has the required adjacencies.

The simplest construction of a universal sequence is to enumerate all finite binary sequences and concatenate them. But there are many others. It is straightforward to show that a random subset of $\mathbb{N}$ (obtained by choosing positive integers independently with probability $\frac{1}{2}$) is almost surely universal. (Said otherwise, the base 2 expansion of almost every real number in $[0, 1]$ is a universal sequence.)

Of course, it is possible to construct a graph satisfying $(*)$ directly. For example, let Γ_0 be the empty graph; if Γ_k has been constructed, let Γ_{k+1} be obtained by adding, for each subset U of the vertex set of Γ_k, a vertex $z(U)$ whose neighbour set is precisely U. Clearly, the union of this sequence of graphs satisfies $(*)$.

3. Indestructibility

The graph R is remarkably stable: if small changes are made to it, the resulting graph is still isomorphic to R. Some of these results depend on the following analogue of property $(*)$, which appears stronger but is an immediate consequence of $(*)$ itself.

Proposition 1. *Let $u_1, \dots, u_m, v_1, \dots, v_n$ be distinct vertices of R. Then the set*

$$Z = \{z : z \sim u_i \textit{ for } i = 1, \dots, m; z \nsim v_j \textit{ for } j = 1, \dots n\}$$

is infinite; and the induced subgraph on this set is isomorphic to R.

Proof. It is enough to verify property $(*)$ for Z. So let $u'_1, \ldots, u'_k, v'_1, \ldots, v'_l$ be distinct vertices of Z. Now the vertex z adjacent to $u_1, \ldots, u_n, u'_1, \ldots, u'_k$ and not to $v_1, \ldots, v_n, v'_1, \ldots, v'_l$, belongs to Z and witnesses the truth of this instance of $(*)$ there. □

The operation of *switching* a graph with respect to a set X of vertices is defined as follows. Replace each edge between a vertex of X and a vertex of its complement by a non-edge, and each such non-edge by an edge; leave the adjacencies within X or outside X unaltered. See Seidel [64] for more properties of this operation.

Proposition 2. *The result of any of the following operations on R is isomorphic to R:*

(a) Deleting a finite number of vertices;
(b) Changing a finite number of edges into non-edges or vice versa;
(c) Switching with respect to a finite set of vertices.

Proof. In cases (a) and (b), to verify an instance of property $(*)$, we use Proposition 1 to avoid the vertices which have been tampered with. For (c), if $U = \{u_1, \ldots, u_m\}$ and $V = \{v_1, \ldots, v_n\}$, we choose a vertex outside X which is adjacent (in R) to the vertices of $U \setminus X$ and $V \cap X$, and non-adjacent to those of $U \cap X$ and $V \setminus X$. □

Not every graph obtained from R by switching is isomorphic to R. For example, if we switch with respect to the neighbours of a vertex x, then x is an isolated vertex in the resulting graph. However, if x is deleted, we obtain R once again! Moreover, if we switch with respect to a random set of vertices, the result is almost certainly isomorphic to R.

R satisfies the *pigeonhole principle*:

Proposition 3. *If the vertex set of R is partitioned into a finite number of parts, then the induced subgraph on one of these parts is isomorphic to R.*

Proof. Suppose that the conclusion is false for the partition $X_1 \cup \ldots \cup X_k$ of the vertex set. Then, for each i, property $(*)$ fails in X_i, so there are finite disjoint subsets U_i, V_i of X_i such that no vertex of X_i is "correctly joined" to all vertices of U_i, and to none of V_i. Setting $U = U_1 \cup \ldots \cup U_k$ and $V = V_1 \cup \ldots \cup V_k$, we find that condition $(*)$ fails in R for the sets U and V, a contradiction. □

Indeed, this property is characteristic:

Proposition 4. *The only countable graphs Γ which have the property that, if the vertex set is partitioned into two parts, then one of those parts induces a subgraph isomorphic to Γ, are the complete and null graphs and R.*

Proof. Suppose that Γ has this property but is not complete or null. Since any graph can be partitioned into a null graph and a graph with no isolated vertices, we see that Γ has no isolated vertices. Similarly, it has no vertices joined to all others.

Now suppose that Γ is not isomorphic to R. Then we can find $u_1, \ldots, u_m$ and $v_1, \ldots, v_n$ such that $(*)$ fails, with $m+n$ minimal subject to this. By the preceding paragraph, $m+n > 1$. So the set $\{u_1, \ldots, v_n\}$ can be partitioned into two non-empty subsets A and B. Now let X consist of A together with all vertices (not in B) which are not "correctly joined" to the vertices in A; and let Y consist of B together with all vertices (not in X) which are not "correctly joined" to the vertices in B. By assumption, X and Y form a partition of the vertex set. Moreover, the induced subgraphs on X and Y fail instances of condition $(*)$ with fewer than $m+n$ vertices; by minimality, neither is isomorphic to Γ, a contradiction. □

Finally:

Proposition 5. *R is isomorphic to its complement.*

For property $(*)$ is clearly self-complementary.

4. Graph-Theoretic Properties

The most important property of R (and the reason for Rado's interest) is that it is *universal*:

Proposition 6. *Every finite or countable graph can be embedded as an induced subgraph of R.*

Proof. We apply the proof technique of Fact 2; but, instead of back-and-forth, we just "go forth". Let Γ have vertex set $\{x_1, x_2, \ldots\}$, and suppose that we have a map $f_n : \{x_1, \ldots, x_n\} \to R$ which is an isomorphism of induced subgraphs. Let U and V be the sets of neighbours and non-neighbours respectively of x_{n+1} in $\{x_1, \ldots, x_n\}$. Choose $z \in R$ adjacent to the vertices of $f(U)$ and nonadjacent to those of $f(V)$, and extend f_n to map x_{n+1} to z. The resulting map f_{n+1} is still an isomorphism of induced subgraphs. Then $f = \bigcup f_n$ is the required embedding. (The point is that, going forth, we only require that property $(*)$ holds in the target graph.) □

In particular, R contains infinite cliques and cocliques. Clearly no finite clique or coclique can be maximal. There do exist infinite maximal cliques and cocliques. For example, if we enumerate the vertices of R as $\{x_1, x_2, \ldots\}$, and build a set S by $S_0 = \emptyset, S_{n+1} = S_n \cup \{x_m\}$ where m is the least index of a vertex joined to every vertex in S_n, and $S = \bigcup S_n$, then S is a maximal clique.

Dual to the concept of induced subgraph is that of *spanning subgraph*, using all the vertices and some of the edges. Not every countable graph is a spanning subgraph of R (for example, the complete graph is not). We have the following characterization:

Proposition 7. *A countable graph Γ is isomorphic to a spanning subgraph of R if and only if, given any finite set $\{v_1, \ldots, v_n\}$ of vertices of Γ, there is a vertex z joined to none of $v_1, \ldots, v_n$.*

Proof. We use back-and-forth to construct a bijection between the vertex sets of Γ and R, but when going back from R to Γ, we only require that *nonadjacencies* should be preserved. □

This shows, in particular, that every infinite locally finite graph is a spanning subgraph (so R contains 1-factors, one- and two-way infinite Hamiltonian paths, etc.). But more can be said.

The argument can be modified to show that, given any non-null locally finite graph Γ, any edge of R lies in a spanning subgraph isomorphic to Γ. Moreover, as in the last section, if the edges of a locally finite graph are deleted from R, the result is still isomorphic to R. Now let $\Gamma_1, \Gamma_2, \ldots$ be given non-null locally finite countable graphs. Enumerate the edges of R, as $\{e_1, e_2, \ldots\}$. Suppose that we have found edge-disjoint spanning subgraphs of R isomorphic to $\Gamma_1, \ldots, \Gamma_n$. Let m be the smallest index of an edge of R lying in none of these subgraphs. Then we can find a spanning subgraph of $R - (\Gamma_1 \cup \cdots \cup \Gamma_n)$ containing e_m and isomorphic to Γ_{n+1}. We conclude:

Proposition 8. *The edge set of R can be partitioned into spanning subgraphs isomorphic to any given countable sequence of non-null countable locally finite graphs.*

In particular, R has a 1-factorization, and a partition into Hamiltonian paths.

5. Homogeneity and Categoricity

We come now to two model-theoretic properties of R. These illustrate two important general theorems, the Engeler–Ryll-Nardzewski–Svenonius theorem and Fraïssé's theorem. The context is first-order logic; so a *structure* is a set equipped with a collection of relations, functions and constants whose names are specified in the language. If there are no functions or constants, we have a *relational structure*. The significance is that any subset of a relational structure carries an induced substructure. (In general, a substructure must contain the constants and be closed with respect to the functions.)

Let M be a relational structure. We say that M is *homogeneous* if every isomorphism between finite induced substructures of M can be extended to an automorphism of M.

Proposition 9. *R is homogeneous.*

Proof. In the proof of Fact 2, the back-and-forth machine can be started with any given isomorphism between finite substructures of the graphs Γ_1 and Γ_2, and extends it to an isomorphism between the two structures. Now, taking Γ_1 and Γ_2 to be R gives the conclusion. □

Fraïssé [30] observed that $\mathbb{Q}$ (as an ordered set) is homogeneous, and used this as a prototype: he gave a necessary and sufficient condition for the existence of a homogeneous structure with prescribed finite substructures. Following his terminology, the *age* of a structure M is the class of all finite structures embeddable in M. A class $\mathcal{C}$ of finite structures has the *amalgamation property* if, given A, B_1, $B_2 \in \mathcal{C}$ and embeddings $f_1 : A \to B_1$ and $f_2 : A \to B_2$, there exists $C \in \mathcal{C}$ and embeddings $g_1 : B_1 \to C$ and $g_2 : B_2 \to C$ such that $f_1 g_1 = f_2 g_2$. (Less formally, if the two structures B_1, B_2 have isomorphic substructures A, they can be "glued together" so that the copies of A coincide, the resulting structure C also belonging to the class $\mathcal{C}$.) We allow $A = \emptyset$ here.

Theorem 3. *(a) A class C of finite structures (over a fixed relational language)is the age of a countable homogeneous structure M if and only if $\mathcal{C}$ is closed under isomorphism, closed under taking induced substructures, contains only countably many non-isomorphic structures, and has the amalgamation property.*

(b) If the conditions of (a) are satisfied, then the structure M is unique up to isomorphism.

A class $\mathcal{C}$ having the properties of this theorem is called a *Fraïssé class*, and the countable homogeneous structure M whose age is $\mathcal{C}$ is its *Fraïssé limit*. The class of all finite graphs is Fraïssé class; its Fraïssé limit is R. The Fraïssé limit of a class $\mathcal{C}$ is characterized by a condition generalizing property $(*)$: *If A and B are members of the age of M with $A \subseteq B$ and $|B| = |A| + 1$, then every embedding of A into M can be extended to an embedding of B into M.*

In the statement of the amalgamation property, when the two structures B_1, B_2 are "glued together", the overlap may be larger than A. We say that the class $\mathcal{C}$ has the *strong amalgamation property* if this doesn't occur; formally, if the embeddings g_1, g_2 can be chosen so that, if $b_1 g_1 = b_2 g_2$, then there exists $a \in A$ such that $b_1 = a f_1$ and $b_2 = a f_2$. This property is equivalent to others we have met.

Proposition 10. *Let M be the Fraïssé limit of the class $\mathcal{C}$, and $G = \mathrm{Aut}(M)$. Then the following are equivalent:*

(a) $\mathcal{C}$ has the strong amalgamation property;
(b) $M \setminus A \cong M$ for any finite subset A of M;
(c) The orbits of G_A on $M \setminus A$ are infinite for any finite subset A of M, where G_A is the setwise stabiliser of A.

See Cameron [10], El-Zahar and Sauer [24].

A structure M is called $\aleph_0$-*categorical* if any countable structure satisfying the same first-order sentences as M is isomorphic to M. (We must specify countability here: the upward Löwenheim–Skolem theorem shows that, if M is infinite, then there are structures of arbitrarily large cardinality which satisfy the same first-order sentences as M.)

Proposition 11. *R is $\aleph_0$-categorical.*

Proof. Property ($*$) is not first-order as it stands, but it can be translated into a countable set of first-order sentences $\sigma_{m,n}$ (for $m, n \in \mathbb{N}$), where $\sigma_{m,n}$ is the sentence

$$(\forall u_1..u_m v_1..v_n)\left(\begin{pmatrix}(u_1 \neq v_1)\& \dots \& \\ (u_m \neq v_n)\end{pmatrix} \to (\exists z)\begin{pmatrix}(z \sim u_1)\& \dots \&(z \sim u_m)\& \\ \neg(z \sim v_1)\& \dots \&\neg(z \sim v_n)\end{pmatrix}\right). \square$$

Once again this is an instance of a more general result. An *n-type* in a structure M is an equivalence class of n-tuples, where two tuples are equivalent if they satisfy the same (n-variable) first-order formulae. Now the following theorem was proved by Engeler [25], Ryll-Nardzewski [61] and Svenonius [68]:

Theorem 4. *For a countable first-order structure M, the following conditions are equivalent:*

(a) M is $\aleph_0$-categorical;
(b) M has only finitely many n-types, for every n;
(c) The automorphism group of M has only finitely many orbits on M^n, for every n.

Note that the equivalence of conditions (a) (axiomatizability) and (c) (symmetry) is in the spirit of Klein's *Erlanger Programm*. The fact that R satisfies (c) is a consequence of its homogeneity, since $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ lie in the same orbit of $\mathrm{Aut}(R)$ if and only if the map $(x_i \to y_i)$ $(i = 1, \dots, n)$ is an isomorphism of induced subgraphs, and there are only finitely many n-vertex graphs.

Remark 2. The general definition of an n-type in first-order logic is more complicated than the one given here: roughly, it is a maximal set of n-variable formulae consistent with a given theory. I have used the fact that, in an $\aleph_0$-categorical structure, any n-type is *realized* (i.e., satisfied by some tuple)—this is a consequence of the Gödel–Henkin completeness theorem and the downward Löwenheim–Skolem theorem. See Hodges [39] for more details.

Some properties of R can be deduced from either its homogeneity or its $\aleph_0$-categoricity. For example, Proposition 6 generalizes. We say that a countable relational structure M is *universal* (or *rich for its age*, in Fraïssé's terminology [31]) if every countable structure N whose age is contained in that of M (i.e., which is *younger* than M) is embeddable in M.

Theorem 5. *If M is either $\aleph_0$-categorical or homogeneous, then it is universal.*

The proof for homogeneous structures follows that of Proposition 6, using the analogue of property (∗) described above. The argument for $\aleph_0$-categorical structures is a bit more subtle, using Theorem 5.4 and König's Infinity Lemma: see Cameron [11].

6. First-Order Theory of Random Graphs

The graph R controls the first-order theory of finite random graphs, in a manner I now describe. This theory is due to Glebskii et al. [34], Fagin [29], and Blass and Harary [4]. A property P holds in *almost all finite random graphs* if the proportion of N-vertex graphs which satisfy P tends to 1 as $N \to \infty$. Recall the sentences $\sigma_{m,n}$ which axiomatize R.

Theorem 6. *Let θ be a first-order sentence in the language of graph theory. Then the following are equivalent:*

(a) θ holds in almost all finite random graphs;
(b) θ holds in the graph R;
(c) θ is a logical consequence of $\{\sigma_{m,n} : m, n \in \mathbb{N}\}$.

Proof. The equivalence of (b) and (c) is immediate from the Gödel–Henkin completeness theorem for first-order logic and the fact that the sentences $\sigma_{m,n}$ axiomatize R.

We show that (c) implies (a). First we show that $\sigma_{m,n}$ holds in almost all finite random graphs. The probability that it fails in an N-vertex graph is not greater than $N^{m+n}(1 - \frac{1}{2^{m+n}})^{N-m-n}$, since there are at most N^{m+n} ways of choosing $m+n$ distinct points, and $(1 - \frac{1}{2^{m+n}})^{N-m-n}$ is the probability that no further point is correctly joined. This probability tends to 0 as $N \to \infty$.

Now let θ be an arbitrary sentence satisfying (c). Since proofs in first-order logic are finite, the deduction of θ involves only a finite set Σ of sentences $\sigma_{m,n}$. It follows from the last paragraph that almost all finite graphs satisfy the sentences in Σ; so almost all satisfy θ too.

Finally, we show that not (c) implies not (a). If (c) fails, then θ doesn't hold in R, so $(\neg\theta)$ holds in R, so $(\neg\theta)$ is a logical consequence of the sentences $\sigma_{m,n}$. By the preceding paragraph, $(\neg\theta)$ holds in almost all random graphs. □

The last part of the argument shows that there is a zero-one law:

Corollary 1. *Let θ be a sentence in the language of graph theory. Then either θ holds in almost all finite random graphs, or it holds in almost none.*

It should be stressed that, striking though this result is, most interesting graph properties (connectedness, hamiltonicity, etc.) are not first-order, and most interesting results on finite random graphs are obtained by letting the probability of an edge tend to zero in a specified manner as $N \to \infty$, rather than keeping it constant (see Bollobás [5]). Nevertheless, we will see a recent application of Theorem 6 later.

7. Measure and Category

When the existence of an infinite object can be proved by a probabilistic argument (as we did with R in Sect. 1), it is often the case that an alternative argument using the concept of Baire category can be found. In this section, I will sketch the tools briefly. See Oxtoby [55] for a discussion of measure and Baire category.

In a topological space, a set is *dense* if it meets every nonempty open set; a set is *residual* if it contains a countable intersection of open dense sets. The *Baire category theorem* states:

Theorem 7. *In a complete metric space, any residual set is non-empty.*

(The analogous statement for probability is that a set which contains a countable intersection of sets of measure 1 is non-empty. We used this to prove Fact 1.)

The simplest situation concerns the space $2^{\mathbb{N}}$ of all infinite sequences of zeros and ones. This is a probability space, with the "coin-tossing measure"—this was the basis of our earlier discussion—and also a complete metric space, where we define $d(x, y) = \frac{1}{2^n}$ if the sequences x and y agree in positions $0, 1, \ldots, n-1$ and disagree in position n. Now the topological concepts translate into combinatorial ones as follows. A set S of sequences is open if and only if it is *finitely determined*, i.e., any $x \in S$ has a finite initial segment such that all sequences with this initial segment are in S. A set S is dense if and only if it is *always reachable*, i.e., any finite sequence has a continuation lying in S. Now it is a simple exercise to prove the Baire category theorem for this space, and indeed to show that a residual set is dense and has cardinality $2^{\aleph_0}$. We will say that "almost all sequences have property P (in the sense of Baire category)" if the set of sequences which have property P is residual.

We can describe countable graphs by binary sequences: take a fixed enumeration of the 2-element sets of vertices, and regard the sequence as the characteristic function of the edge set of the graph. This gives meaning to the phrase "almost all graphs (in the sense of Baire category)". Now, by analogy with Fact 1, we have:

Fact 3. *Almost all countable graphs (in the sense of either measure or Baire category) have property (∗).*

The proof is an easy exercise. In fact, it is simpler for Baire category than for measure—no limit is required!

In the same way, almost all binary sequences (in either sense) are universal (as defined in Sect. 2).

A binary sequence defines a path in the binary tree of countable height, if we start at the root and interpret 0 and 1 as instructions to take the left or right branch at any node. More generally, given any countable tree, the set of paths is a complete metric space, where we define the distance between two paths to be $\frac{1}{2^n}$ if they first split apart at level n in the tree. So the concept of Baire category is applicable. The combinatorial interpretation of open and dense sets is similar to that given for the binary case.

For example, the age of a countable relational structure M can be described by a tree: nodes at level n are structures in the age which have point set $\{0, 1, \ldots, n-1\}$, and nodes X_n, X_{n+1} at levels n and $n+1$ are declared to be adjacent if the induced structure of X_{n+1} on the set $\{0, 1, \ldots, n-1\}$ is X_n. A path in this tree uniquely describes a structure N on the natural numbers which is younger than M, and conversely. Now Fact 3 generalizes as follows:

Proposition 12. *If M is a countable homogeneous relational structure, then almost all countable structures younger than M are isomorphic to M.*

It is possible to formulate analogous concepts in the measure-theoretic framework, though with more difficulty. But the results are not so straightforward. For example, almost all finite triangle-free graphs are bipartite (a result of Erdős, Kleitman and Rothschild [26]); so the "random countable triangle-free graph" is almost surely bipartite. (In fact, it is almost surely isomorphic to the "random countable bipartite graph", obtained by taking two disjoint countable sets and selecting edges between them at random.)

A structure which satisfies the conclusion of Proposition 12 is called *ubiquitous* (or sometimes *ubiquitous in category*, if we want to distinguish measure-theoretic or other forms of ubiquity). Thus the random graph is ubiquitous in both measure and category. See Bankston and Ruitenberg [2] for further discussion.

8. The Automorphism Group

8.1. General Properties

From the homogeneity of R (Proposition 9), we see that it has a large and rich group of automorphisms: the automorphism group $G = \mathrm{Aut}(R)$ acts transitively on the vertices, edges, non-edges, etc.—indeed, on finite configurations of any given isomorphism type. In the language of permutation groups, it is a rank 3 permutation group on the vertex set, since it has three orbits on ordered pairs of vertices, viz., equal, adjacent and non-adjacent pairs. Much more is known about G; this section will be the longest so far.

First, the cardinality:

Proposition 13. $|\,\mathrm{Aut}(R)| = 2^{\aleph_0}$.

This is a special case of a more general fact. The automorphism group of any countable first-order structure is either at most countable or of cardinality

$2^{\aleph_0}$, the first alternative holding if and only if the stabilizer of some finite tuple of points is the identity.

The normal subgroup structure was settled by Truss [71]:

Theorem 8. Aut(R) *is simple.*

Truss proved a stronger result: if g and h are two non-identity elements of Aut(R), then h can be expressed as a product of five conjugates of g or g^{-1}. (This clearly implies simplicity.) Recently Macpherson and Tent [49] gave a different proof of simplicity which applies in more general situations.

Truss also described the cycle structures of all elements of Aut(R).

A countable structure M is said to have the *small index property* if any subgroup of Aut(M) with index less than $2^{\aleph_0}$ contains the pointwise stabilizer of a finite set of points of M; it has the *strong small index property* if any such subgroup lies between the pointwise and setwise stabilizer of a finite set (see Truss [72]). Hodges et al. [40] and Cameron [13] showed:

Theorem 9. *R has the strong small index property.*

The significance of this appears in the next subsection. It is also related to the question of the reconstruction of a structure from its automorphism group. For example, Theorem 9 has the following consequence:

Corollary 2. *Let Γ be a graph with fewer than $2^{\aleph_0}$ vertices, on which* Aut(R) *acts transitively on vertices, edges and non-edges. Then Γ is isomorphic to R (and the isomorphism respects the action of* Aut(R)*).*

8.2. Topology

The symmetric group Sym(X) on an infinite set X has a natural topology, in which a neighbourhood basis of the identity is given by the pointwise stabilizers of finite tuples. In the case where X is countable, this topology is derived from a complete metric, as follows. Take $X = \mathbb{N}$.

Let $m(g)$ be the smallest point moved by the permutation g. Take the distance between the identity and g to be $\max\{2^{-m(g)}, 2^{-m(g^{-1})}\}$. Finally, the metric is translation-invariant, so that $d(f, g) = d(fg^{-1}, 1)$.

Proposition 14. *Let G be a subgroup of the symmetric group on a countable set X. Then the following are equivalent:*

(a) G is closed in Sym(X)*;*
(b) G is the automorphism group of a first-order structure on X;
(c) G is the automorphism group of a homogeneous relational structure on X.

So automorphism groups of homogeneous relational structures such as R are themselves topological groups whose topology is derived from a complete metric.

In particular, the Baire category theorem applies to groups like $\mathrm{Aut}(R)$. So we can ask: is there a “typical” automorphism? Truss [73] showed the following result.

Theorem 10. *There is a conjugacy class which is residual in* $\mathrm{Aut}(R)$. *Its members have infinitely many cycles of each finite length, and no infinite cycles.*

Members of the residual conjugacy class (which is, of course, unique) are called *generic automorphisms* of R. I outline the argument. Each of the following sets of automorphisms is residual:

(a) Those with no infinite cycles;
(b) Those automorphisms g with the property that, if Γ is any finite graph and f any isomorphism between subgraphs of Γ, then there is an embedding of Γ into R in such a way that g extends f.

(Here (a) holds because the set of automorphisms for which the first n points lie in finite cycles is open and dense.) In fact, (b) can be strengthened; we can require that, if the pair (Γ, f) extends the pair (Γ_0, f_0) (in the obvious sense), then any embedding of Γ_0 into R such that g extends f_0 can be extended to an embedding of Γ such that g extends f. Then a residual set of automorphisms satisfy both (a) and the strengthened (b); this is the required conjugacy class.

Another way of expressing this result is to consider the class $\mathcal{C}$ of finite structures each of which is a graph Γ with an isomorphism f between two induced subgraphs (regarded as a binary relation). This class satisfies Fraïssé's hypotheses, and so has a Fraïssé limit M. It is not hard to show that, as a graph, M is the random graph R; arguing as above, the map f can be shown to be a (generic) automorphism of R.

More generally, Hodges et al. [40] showed that there exist “generic n-tuples” of automorphisms of R, and used this to prove the small index property for R; see also Hrushovski [41]. The group generated by a generic n-tuple of automorphisms is, not surprisingly, a free group; all its orbits are finite. In the next subsection, we turn to some very different subgroups.

To conclude this section, we revisit the strong small index property. Recall that a neighbourhood basis for the identity consists of the pointwise stabilisers of finite sets. If the strong small index property holds, then every subgroup of small index (less than $2^{\aleph_0}$) contains one of these, and so is open. So we can take the subgroups of small index as a neighbourhood basis of the identity. So we have the following reconstruction result:

Proposition 15. *If M is a countable structure with the strong small index property (for example, R), then the structure of* $\mathrm{Aut}(M)$ *as topological group is determined by its abstract group structure.*

8.3. Subgroups

Another field of study concerns small subgroups. To introduce this, we reinterpret the last construction of R in Sect. 2. Recall that we took a universal set $S \subseteq \mathbb{N}$, and showed that the graph $\Gamma(S)$ with vertex set $\mathbb{Z}$, in which x and y are adjacent whenever $|x-y| \in S$, is isomorphic to R. Now this graph admits the "shift" automorphism $x \mapsto x+1$, which permutes the vertices in a single cycle. Conversely, let g be a cyclic automorphism of R. We can index the vertices of R by integers so that g is the map $x \mapsto x+1$. Then, if $S = \{n \in \mathbb{N} : n \sim 0\}$, we see that $x \sim y$ if and only if $|x-y| \in S$, and that S is universal. A short calculation shows that two cyclic automorphisms are conjugate in $\mathrm{Aut}(R)$ if and only if they give rise to the same set S. Since there are $2^{\aleph_0}$ universal sets, we conclude:

Proposition 16. *R has $2^{\aleph_0}$ non-conjugate cyclic automorphisms.*

(Note that this gives another proof of Proposition 13.)

Almost all subsets of $\mathbb{N}$ are universal—this is true in either sense discussed in Sect. 7. The construction preceding Proposition 16 shows that graphs admitting a given cyclic automorphism correspond to subsets of $\mathbb{N}$; so almost all "cyclic graphs" are isomorphic to R. What if the cyclic permutation is replaced by an arbitrary permutation or permutation group? The general answer is unknown:

Conjecture 1. *Given a permutation group G on a countable set, the following are equivalent:*

(a) Some G-invariant graph is isomorphic to R;
(b) A random G-invariant graph is isomorphic to R with positive probability.

A random G-invariant graph is obtained by listing the orbits of G on the 2-subsets of the vertex set, and deciding randomly whether the pairs in each orbit are edges or not. We cannot replace "positive probability" by "probability 1" here. For example, consider a permutation with one fixed point x and two infinite cycles. With probability $\frac{1}{2}$, x is joined to all or none of the other vertices; if this occurs, the graph is not isomorphic to R. However, almost all graphs for which this event does not occur are isomorphic to R. It can be shown that the conjecture is true for the group generated by a single permutation; and Truss' list of cycle structures of automorphisms can be re-derived in this way.

Another interesting class consists of the *regular* permutation groups. A group is *regular* if it is transitive and the stabilizer of a point is the identity. Such a group G can be considered to act on itself by right multiplication. Then any G-invariant graph is a *Cayley graph* for G; in other words, there is a subset S of G, closed under inverses and not containing the identity, so that x and y are adjacent if and only if $xy^{-1} \in S$. Now we can choose a *random Cayley graph* for G by putting inverse pairs into S with probability $\frac{1}{2}$. It is not true that, for every countable group G, a random Cayley graph for G is almost surely isomorphic to R. Necessary and sufficient conditions can be given; they are somewhat untidy. I will state here a fairly general sufficient condition.

A *square-root set* in G is a set

$$\sqrt{a} = \{x \in G : x^2 = a\};$$

it is *principal* if $a = 1$, and *non-principal* otherwise.

Proposition 17. *Suppose that the countable group G cannot be expressed as the union of finitely many translates of non-principal square-root sets and a finite set. Then almost all Cayley graphs for G are isomorphic to R.*

This proposition is true in the sense of Baire category as well. In the infinite cyclic group, a square-root set has cardinality at most 1; so the earlier result about cyclic automorphisms follows. See Cameron and Johnson [15] for further details.

8.4. Overgroups

There are a number of interesting overgroups of $\mathrm{Aut}(R)$ in the symmetric group on the vertex set X of R.

Pride of place goes to the *reducts*, the overgroups which are closed in the topology on $\mathrm{Sym}(X)$ (that is, which are automorphism groups of relational structures which can be defined from R without parameters). These were classified by Simon Thomas [69].

An *anti-automorphism* of R is an isomorphism from R to its complement; a *switching automorphism* maps R to a graph equivalent to R by switching. The concept of a *switching anti-automorphism* should be clear.

Theorem 11. *There are exactly five reducts of R, viz.: $A = \mathrm{Aut}(R)$; the group D of automorphisms and anti-automorphisms of R; the group S of switching automorphisms of R; the group B of switching automorphisms and anti-automorphisms of R; and the symmetric group.*

Remark 3. The set of all graphs on a given vertex set is a $\mathbb{Z}_2$-vector space, where the sum of two graphs is obtained by taking the symmetric difference of their edge sets. Now complementation corresponds to adding the complete graph, and switching to adding a complete bipartite graph. Thus, it follows from Theorem 11 that, if G is a closed supergroup of $\mathrm{Aut}(R)$, then the set of all images of R under G is contained in a coset of a subspace $W(G)$ of this vector space. (For example, $W(B)$ consists of all complete bipartite graphs and all unions of at most two complete graphs.) Moreover, these subspaces are invariant under the symmetric group. It is remarkable that the combinatorial proof leads to this algebraic conclusion.

Here is an application due to Cameron and Martins [18], which draws together several threads from earlier sections. Though it is a result about finite random graphs, the graph R is inextricably involved in the proof.

Let $\mathcal{F}$ be a finite collection of finite graphs. For any graph Γ, let $\mathcal{F}(\Gamma)$ be the hypergraph whose vertices are those of Γ, and whose edges are the subsets which induce graphs in $\mathcal{F}$. To what extent does $\mathcal{F}(\Gamma)$ determine Γ?

Theorem 12. *Given $\mathcal{F}$, one of the following possibilities holds for almost all finite random graphs Γ:*

(a) $\mathcal{F}(\Gamma)$ determines Γ uniquely;
(b) $\mathcal{F}(\Gamma)$ determines Γ up to complementation;
(c) $\mathcal{F}(\Gamma)$ determines Γ up to switching;
(d) $\mathcal{F}(\Gamma)$ determines Γ up to switching and/or complementation;
(e) $\mathcal{F}(\Gamma)$ determines only the number of vertices of Γ.

I sketch the proof in the first case, that in which $\mathcal{F}$ is not closed under either complementation or switching. We distinguish two first-order languages, that of graphs and that of hypergraphs (with relations of the arities appropriate for the graphs in $\mathcal{F}$). Any sentence in the hypergraph language can be "translated" into the graph language, by replacing "E is an edge" by "the induced subgraph on E is one of the graphs in $\mathcal{F}$".

By the case assumption and Theorem 11, we have $\mathrm{Aut}(\mathcal{F}(R)) = \mathrm{Aut}(R)$. Now by Theorem 4, the edges and non-edges in R are 2-types in $\mathcal{F}(R)$, so there is a formula $\phi(x, y)$ (in the hypergraph language) such that $x \sim y$ in R if and only if $\phi(x, y)$ holds in $\mathcal{F}(R)$. If ϕ^* is the "translation" of ϕ, then R satisfies the sentence

$$(\forall x, y)((x \sim y) \leftrightarrow \phi^*(x, y)).$$

By Theorem 6, this sentence holds in almost all finite graphs. Thus, in almost all finite graphs, Γ, vertices x and y are joined if and only if $\phi(x, y)$ holds in $\mathcal{F}(\Gamma)$. So $\mathcal{F}(\Gamma)$ determines Γ uniquely.

By Theorem 11, $\mathrm{Aut}(\mathcal{F}(R))$ must be one of the five possibilities listed; in each case, an argument like the one just given shows that the appropriate conclusion holds.

There are many interesting overgroups of $\mathrm{Aut}(R)$ which are not closed, some of which are surveyed (and their inclusions determined) in a forthcoming paper of Cameron et al. [16]. These arise in one of two ways.

First, we can take automorphism groups of non-relational structures, such as hypergraphs with infinite hyperedges (for example, take the hyperedges to be the subsets of the vertex set which induce subgraphs isomorphic to R), or topologies or filters (discussed in the next section). Second, we may weaken the notion of automorphism. For example, we have a chain of subgroups

$$\mathrm{Aut}(R) < \mathrm{Aut}_1(R) < \mathrm{Aut}_2(R) < \mathrm{Aut}_3(R) < \mathrm{Sym}(V(R))$$

with all inclusions proper, where

- $\mathrm{Aut}_1(R)$ is the set of permutations which change only finitely many adjacencies (such permutations are called *almost automorphisms* of R);
- $\mathrm{Aut}_2(R)$ is the set of permutations which change only finitely many adjacencies at any vertex of R;
- $\mathrm{Aut}_3(R)$ is the set of permutations which change only finitely many adjacencies at all but finitely many vertices of R.

All these groups are *highly transitive*, that is, given any two n-tuples $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_n)$ of distinct vertices, there is an element of the relevant group carrying the first tuple to the second. This follows from $\mathrm{Aut}_1(R)$ by the indestructibility of R. If R_1 and R_2 are the graphs obtained by deleting all edges within $\{v_1, \ldots, v_n\}$ and within $\{w_1, \ldots, w_n\}$ respectively, then R_1 and R_2 are both isomorphic to R. By homogeneity of R, there is an isomorphism from R_1 to R_2 mapping $(v_1, \ldots, v_n)$ to $(w_1, \ldots, w_n)$; clearly this map is an almost-automorphism of R.

Indeed, any overgroup of R which is not a reduct preserves no non-trivial relational structure, and so must be highly transitive.

9. Topological Aspects

There is a natural way to define a topology on the vertex set of R: we take as a basis for the open sets the set of all finite intersections of vertex neighbourhoods. It can be shown that this topology is homeomorphic to $\mathbb{Q}$ (using the characterization of $\mathbb{Q}$ as the unique countable, totally disconnected, topological space without isolated points, due to Sierpińiski [66], see also Neumann [54]). Thus:

Proposition 18. $\mathrm{Aut}(R)$ *is a subgroup of the homeomorphism group of* $\mathbb{Q}$.

This is related to a theorem of Mekler [50]:

Theorem 13. *A countable permutation group G is embeddable in the homeomorphism group of $\mathbb{Q}$ if and only if the intersection of the supports of any finite number of elements of G is empty or infinite.*

Here, the support of a permutation is the set of points it doesn't fix. Now of course $\mathrm{Aut}(R)$ is not countable; yet it does satisfy Mekler's condition. (If x is moved by each of the automorphisms $g_1, \ldots, g_n$, then the infinitely many vertices joined to x but to none of $xg_1, \ldots, xg_n$ are also moved by these permutations.)

The embedding in Proposition 18 can be realised constructively: the topology can be defined directly from the graph. Take a basis for the open sets to be the sets of witnesses for our defining property $(*)$; that is, sets of the form

$$Z(U, V) = \{z \in V(R) : (\forall u \in U)(z \sim u) \wedge (\forall v \in V)(z \not\sim v)\}$$

for finite disjoint sets U and V. Now given $u \neq v$, there is a point $z \in Z(\{u\},\{v\})$; so the open neighbourhood of z is open and closed in the topology and contains u but not v. So the topology is totally disconnected. It has no isolated points, so it is homeomorphic to $\mathbb{Q}$, by Sierpiński's Theoreom.

There is another interesting topology on the vertex set of R, which can be defined in three different ways. Let B be the "random bipartite graph", the graph with vertex set $X \cup Y$ where X and Y are countable and disjoint, where edges between X and Y are chosen randomly. (A simple modification of the Erdős–Rényi argument shows that there is a unique graph which occurs with probability 1.) Now consider the following topologies on a countable set X:

$\mathcal{T}$: point set $V(R)$, sub-basic open sets are open vertex neighbourhoods.
$\mathcal{T}^*$: points set $V(R)$, sub-basic open sets are closed vertex neighbourhoods.
$\mathcal{T}^\dagger$: points are one bipartite block in B, sub-basic open sets are neighbourhoods of vertices in the other bipartite block.

Proposition 19. *(a) The three topologies defined above are all homeomorphic.*
(b) The homeomorphism groups of these topologies are highly transitive.

Note that the topologies are homeomorphic but not identical. For example, the identity map is a continuous bijection from $\mathcal{T}^*$ to $\mathcal{T}$, but is not a homeomorphism.

10. Some Other Structures

10.1. General Results

As we have seen, R has several properties of a general kind: for example, homogeneity, $\aleph_0$-categoricity, universality, ubiquity. Much effort has gone into studying, and if possible characterizing, structures of other kinds with these properties. (For example, they are all shared by the ordered set $\mathbb{Q}$.)

Note that, of the four properties listed, the first two each imply the third, and the first implies the fourth. Moreover, a homogeneous structure over a finite relational language is $\aleph_0$-categorical, since there are only finitely many isomorphism types of n-element structure for each n. Thus, homogeneity is in practice the strongest condition, most likely to lead to characterizations.

A major result of Lachlan and Woodrow [48] determines the countable homogeneous graphs. The graphs H_n in this theorem are so-called because they were first constructed by Henson [36].

Theorem 14. *A countable homogeneous graph is isomorphic to one of the following:*

(a) The disjoining union of m complete graphs of size n, where $m, n \leq \aleph_0$ and at least one of m and n is $\aleph_0$;
(b) Complements of (a);

(c) The Fraïssé limit H_n of the class of K_n-free graphs, for fixed $n \geq 3$;
(d) Complements of (c);
(e) The random graph R.

The result of Macpherson and Tent [49] shows that the automorphism groups of the Henson graphs are simple. It follows from Proposition 15 that $\mathrm{Aut}(R)$ is not isomorphic to $\mathrm{Aut}(H_n)$. Herwig [38] showed that these automorphism groups are all pairwise non-isomorphic; see also Barbina and Macpherson [3].

Other classes in which the homogeneous structures have been determined include finite graphs (Gardiner [32]), tournaments (Lachlan [47]—surprisingly, there are just three), digraphs (Cherlin [21] (there are uncountably many, see Henson [37]), posets (Schmerl [62]) and permutations (Cameron[12]). In the case of posets, Droste [23] has characterizations under weaker assumptions.

For a number of structures, properties of the automorphism group, such as normal subgroups, small index property, or existence of generic automorphisms, have been established.

A theorem of Cameron [8] determines the reducts of $\mathrm{Aut}(\mathbb{Q})$:

Theorem 15. *There are just five closed permutation groups containing the group* $\mathrm{Aut}(\mathbb{Q})$ *of order-preserving permutations of* $\mathbb{Q}$*, viz.:* $\mathrm{Aut}(\mathbb{Q})$*; the group of order preserving or reversing permutations; the group of permutations preserving a cyclic order; the group of permutations preserving or reversing a cyclic order; and* $\mathrm{Sym}(\mathbb{Q})$*.*

However, there is no analogue of Theorem 12 in this case, since there is no Glebskii–Blass–Fagin–Harary theory for ordered sets. ($\mathbb{Q}$ is dense; this is a first-order property, but no finite ordered set is dense.)

Simon Thomas [70] has determined the reducts of the random k-uniform hypergraph for all k.

Since my paper with Paul Erdős concerns sum-free sets (Cameron and Erdős [14]), it is appropriate to discuss their relevance here. Let H_n be the Fraïssé limit of the class of K_n-free graphs, for $n \geq 3$ (see Theorem 14). These graphs were first constructed by Henson [36], who also showed that H_3 admits cyclic automorphisms but H_n does not for $n > 3$. We have seen how a subset S of $\mathbb{N}$ gives rise to a graph $\Gamma(S)$ admitting a cyclic automorphism: the vertex set is $\mathbb{Z}$, and $x \sim y$ if and only if $|x - y| \in S$. Now $\Gamma(S)$ is triangle-free if and only if S is *sum-free* (i.e., $x, y \in S \Rightarrow x + y \notin S$). It can be shown that, for almost all sum-free sets S (in the sense of Baire category), the graph $\Gamma(S)$ is isomorphic to H_3; so H_3 has $2^{\aleph_0}$ non-conjugate cyclic automorphisms. However, the analogue of this statement for measure is false; and, indeed, random sum-free sets have a rich and surprising structure which is not well understood (Cameron [9]). For example, the probability that $\Gamma(S)$ is bipartite is approximately 0.218. It is conjectured that a random sum-free set S almost never satisfies $\Gamma(S) \cong H_3$. In this direction, Schoen [63] has shown that, if $\Gamma(S) \cong H_3$, then S has density zero.

The Henson K_n-free graphs H_n, being homogeneous, are ubiquitous in the sense of Baire category: for example, the set of graphs isomorphic to H_3 is residual in the set of triangle-free graphs on a given countable vertex set (so H_3 is ubiquitous, in the sense defined earlier). However, until recently, no measure-theoretic analogue was known. We saw after Proposition 12 that a random triangle-free graph is almost surely bipartite! However, Petrov and Vershik [57] recently managed to construct an exchangeable measure on graphs on a given countable vertex set which is concentrated on Henson's graph. More recently, Ackerman, Freer and Patel [1] showed that the construction works much more generally: the necessary and sufficient condition turns out to be the strong amalgamation property, which we discussed in Sect. 5.

Universality of a structure M was defined in a somewhat introverted way in Sect. 5: M is universal if every structure younger than M is embeddable in M. A more general definition would start with a class $\mathcal{C}$ of structures, and say that $M \in \mathcal{C}$ is *universal* for $\mathcal{C}$ if every member of $\mathcal{C}$ embeds into M. For a survey on this sort of universality, for various classes of graphs, see Komjath and Pach [46]. Two early negative results, for the classes of locally finite graphs and of planar graphs, are due to De Bruijn (see Rado [59]) and Pach [56] respectively.

10.2. The Urysohn Space

A remarkable example of a homogeneous structure is the celebrated Urysohn space, whose construction predates Fraïssé's work by more than two decades. Urysohn's paper [74] was published posthumously, following his drowning in the Bay of Biscay at the age of 26 on his first visit to western Europe (one of the most romantic stories in mathematics). An exposition of the Urysohn space is given by Vershik [75].

The Urysohn space is a complete separable metric space $\mathbb{U}$ which is universal (every finite metric space is isometrically embeddable in $\mathbb{U}$) and homogeneous (any isometry between finite subsets can be extended to an isometry of the whole space). Since $\mathbb{U}$ is uncountable, it is not strictly covered by the Fraïssé theory, but one can proceed as follows. The set of finite *rational* metric spaces (those with all distances rational) is a Fraïssé class; the restriction to countable distances ensures that there are only countably many non-isomorphic members. Its Fraïssé limit is the so-called *rational Urysohn space* $\mathbb{U}_Q$. Now the Urysohn space is the completion of $\mathbb{U}_Q$.

Other interesting homogeneous metric spaces can be constructed similarly, by restricting the values of the metric in the finite spaces. For example, we can take integral distances, and obtain the *integral Urysohn space* $\mathbb{U}_Z$. We can also take distances from the set $\{0, 1, 2, \ldots, k\}$ and obtain a countable homogeneous metric space with these distances. For $k = 2$, we obtain precisely the path metric of the random graph R. (Property $(*)$ guarantees that, given two points at distance 2, there is a point at distance 1 from both; so, defining two points to be adjacent if they are at distance 1, we obtain a graph whose path metric is the given metric. It is easily seen that this graph is isomorphic to R.)

Note that R occurs in many different ways as a reduct of $\mathbb{U}_Q$. Split the positive rationals into two dense subsets A and B, and let two points v, w be adjacent if $d(v, w) \in A$; the graph we obtain is R.

A study of the isometry group of the Urysohn space, similar to that done for R, was given by Cameron and Vershik [19]. The automorphism group is not simple, since the isometries which move every point by a bounded distance form a non-trivial normal subgroup.

10.3. KPT Theory

I conclude with a brief discussion of a dramatic development at the interface of homogeneous structures, Ramsey theory, and topological dynamics.

The first intimation of such a connection was pointed out by Nešetřil [51]; in [52] he suggested using this connection to characterize Ramsey classes, see also [43]. We use the notation $\binom{A}{B}$ for the set of all substructures of A isomorphic to B. A class $\mathcal{C}$ of finite structures is a *Ramsey class* if, given a natural number r and a pair A, B of structures in $\mathcal{C}$, there exists a structure $C \in \mathcal{C}$ such that, if $\binom{C}{A}$ is partitioned into r classes, then there is an element $B' \in \binom{C}{B}$ for which $\binom{B'}{A}$ is contained in a single class. In other words, if we colour the A-substructures of C with r colours, then there is a B-substructure of C, all of whose A-substructures belong to the same class. Ramsey's classical theorem asserts that the class of finite sets is a Ramsey class.

Theorem 16. *A hereditary isomorphism-closed Ramsey class is a Fraïssé class.*

There are simple examples which show that a good theory of Ramsey classes can only be obtained by making the objects rigid. The simplest way to do this is to require that a total order is part of the structure. Note that, if a Fraïssé class has the strong amalgamation property, than we may adjoin to it a total order (independent of the rest of the structure) to obtain a new Fraïssé class. We refer to *ordered structures* in this situation. Now the theorem above suggests a procedure for finding Ramsey classes: take a Fraïssé class of ordered structures and test the Ramsey property. A number of Ramsey classes, old and new, arise in this way: ordered graphs, K_n-free graphs, metric spaces, etc. Indeed, if we take an ordered set and "order" it as above to obtain a set with two orderings, we obtain the class of *permutation patterns*, which is also a Ramsey class: see Cameron [7], Böttcher and Foniok [13], Sokić [67] and, for a more general result, Bodirsky [6].

The third vertex of the triangle was quite unexpected.

A *flow* is a continuous action of a topological group G on a topological space X, usually assumed to be a compact Hausdorff space. A topological group G admits a unique *minimal flow*, or universal minimal continuous action on a compact space X. (Here *minimal* means that X has no non-empty proper closed G-invariant subspace, and *universal* means that it can be mapped onto any minimal G-flow.)

The group G is said to be *extremely amenable* if its minimal flow consists of a single point.

The theorem of Kechris, Pestov and Todorcevic [45] asserts:

Theorem 17. *Let X be a countable set, and G a closed subgroup of* $\mathrm{Sym}(X)$*. Then G is extremely amenable if and only if it is the automorphism group of a homogeneous structure whose age is a Ramsey class of ordered structures.*

As a simple example, the theorem shows that $\mathrm{Aut}(\mathbb{Q})$ (the group of order-preserving permutations of $\mathbb{Q}$ is extremely amenable (a result of Pestov).

The fact that the two conditions are equivalent allows information to be transferred in both directions between combinatorics and topological dynamics. In particular, known Ramsey classes such as ordered graphs, ordered K_n-free graphs, ordered metric spaces, and permutation patterns give examples of extremely amenable groups.

The theorem can also be used in determining the minimal flows for various closed subgroups of $\mathrm{Sym}(X)$. For example, the minimal flow for $\mathrm{Sym}(X)$ is the set of all total orderings of X (a result of Glasner and Weiss [33]).

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Paul Erdős' Set Theory

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1. An Apology

Paul Erdős has published more than one hundred research papers in set theory. It is my rough estimate that these contain more than one thousand theorems, many having an interest in their own right. Although most of his problems and results have a combinatorial flavour, and the subject now known as "combinatorial set theory" is one he helped to create, it is also true to say that his work has had a very important impact upon the direction of research in many parts of present day set theory. Whole theories have developed out of basic questions which he formulated.

This (relatively) short note does not, and is not intended to, give a methodical survey of set theory or combinatorial set theory or even of Erdős' work in set theory. I shall simply write about some of the ideas as I learned them during our cooperation over many years, some of the highlights, and some of the outstanding results. In many cases it will not be possible to give a detailed discussion of the present day status of some of the problems I shall mention. If the reader considers that my own name occurs too frequently in this note, I can only offer the excuse that we have published more than 50 joint papers, mostly in set theory, and I probably know these papers better than the rest of his work.

2. Early Days and Some Philosophy

Paul was a child mathematical prodigy, and he started to discover outstanding original results in number theory as a first year undergraduate. We are familiar with the names of mathematicians who influenced his early work in number theory and analysis, Pál Veress, Fejér, Davenport, Mordell to mention just a few. But this is not true for set theory. Paul told me that he learned the basics of set theory from his father, a well educated high-school teacher, and he soon became fascinated with "Cantor's paradise". However, he discovered set theory as a subject for research by himself.

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Paul, who has always refrained from seriously formulating any kind of philosophy, was (and still is) the ultimate Platonist. $\aleph_{\omega_{\omega+1}+1}$ existed for him just as surely as 3, the smallest odd prime. He was driven by the same compulsive search for "truth" whether he was thinking about inaccessible cardinals or twin primes. Moreover, he could switch from one subject to the other in an instant. All questions which admit a relevant answer in finite combinatorics should also be asked and answered in set theory, and vice-versa. A large part of his greatness lies in the fact that he really did find the relevant questions. It was this attitude which led him to his first encounter with (actual) infinity.

In 1931, as a first year undergraduate student attending the graph theory course of Dénes König, he proved a generalization of Menger's theorem for infinite graphs. This only appeared in 1936 at the end of König's book on graph theory. In 1936 he wrote a paper jointly with Tibor Gallai and Endre Vázsonyi having a similar character; they gave necessary and sufficient conditions for an infinite graph to have an Euler line [1, 2].

The next paper I have to mention [ESz], about finite combinatorics, was written with George Szekeres in 1935. They rediscovered the finite version of Ramsey's theorem and proved a fundamental Ramsey-type result of finite character: *If G is a graph having $\binom{k+\ell-2}{k-1}$ vertices* ($k, \ell \geq 3$)*, then either G contains a complete graph on k vertices or there is an independent set of ℓ vertices.* From then on he always had in mind possible generalizations of Ramsey's theorem, and so became the creator of both finite and infinite Ramsey theory.

3. Infinite Ramsey Theory: Early Papers

The Ramsey theorem is about partitions of (finite) k-element subsets of ω (the set of non-negative integers), and in the mid-1930s Erdős began to speculate about partitioning the countable sets. He corresponded with Richard Rado in Cambridge, England about this problem and Rado proved the first theorem saying that "nothing can be said in this case". The result appeared only much later, in the early 1950s, in a sequence of joint papers by them [9, 12, 13, 18]. I will return to this later.

Erdős' first real set-theoretic result appeared in a paper of Dushnik and Miller [DM]. The theorem, now known as the Erdős; Dushnik, Miller theorem, says that *for an infinite cardinal κ, if a graph on κ vertices does not contain an infinite complete subgraph, then there is an independent set of vertices of size κ.* This was the first "unbalanced" generalization of Ramsey's theorem. Once the result is formulated, the verification for regular κ is a fairly easy exercise. Erdős proved it for singular κ, and his proof, which required a good technical knowledge of the set theory of those days, is included in the Dushnik-Miller paper.

Soon after, in 1942, he proved in [3] the basic theorems of infinite Ramsey theory. Let $[X]^r$ denote the set of r-element subsets of X, and let $f : [X]^r \to \gamma$

be an r-partition of γ colours on X. A set $H \subseteq X$ is *homogeneous* for f in the colour $\nu < \gamma$ if $f(Y) = \nu$ for all $Y \in [H]^r$. More than 10 years later in joint work with Paul, [12], Richard Rado introduced the partition symbol

$$\kappa \to (\lambda_\nu)^r_{\nu<\gamma}$$

to denote the following assertion: *for any r-partition $f : [\kappa]^r \to \gamma$ there are $\nu < \gamma$ and $H \subseteq \kappa$ such that $|H| = \lambda_\nu$ and H is homogeneous for colour ν.* The negation of this is denoted by replacing the arrow by a crossed arrow, $\nrightarrow$. If $\lambda_\nu = \lambda$ for all $\nu < \gamma$, the notation $\kappa \to (\lambda)^r_\gamma$ is used; this is called the "balanced" partition symbol.

Using this notation, Ramsey's theorem states

$$\omega \to (\omega)^r_k \text{ for } 1 \leq r, k < w.$$

The Erdős; Dushnik, Miller theorem says, for any infinite cardinal κ,

$$\kappa \to (\kappa, \aleph_0)^2.$$

The result of Rado I did not state says, for every κ,

$$\kappa \nrightarrow (\aleph_0)^{\aleph_0}_2.$$

Using these notations, the results proved by Erdős in the 1942 paper are the following:

(i) $(2^\lambda)^+ \to (\lambda^+)^2_\lambda$.
(ii) $2^\lambda \nrightarrow (3)^2_\lambda$.
(iii) Assuming the generalized continuum hypothesis (GCH in what follows) $\aleph_{\alpha+2} \to (\aleph_{\alpha+2},\ \aleph_{\alpha+1})^2$.
(iv) $2^\lambda \nrightarrow (\lambda^+)^2_2$.

He attributes (iv) to Sierpiński, who proved it for $\lambda = \aleph_0$. He also mentions that the obvious $\nrightarrow$ relation (ii) was pointed out to him by Gödel in conversation.

Of course, the partition symbol was not used in that early paper. In fact Erdős was always slightly resistant to its use. Later, when forced, he did sometimes write the symbol, but I have never seen him read it. When we were discussing such relations he frequently asked me in a complaining voice to "state it in human language".

The observant reader would already have noted that (iii) was an attempt to find the right generalization of the Erdős; Dushnik, Miller theorem. We will return to this topic later.

4. His "Remarks"

As yet we are still in 1943, and in that year two more significant papers appeared. First I want to say a few words about [5], "Some remarks on Set Theory". I quote the first sentence: "This paper contains a few disconnected results on the theory of sets." The "Remarks" became a series. Eleven of them appeared in all. The fifth and sixth were written jointly with Géza Fodor, the seventh to ninth and eleventh with me, and the tenth with Michael Makkai. Several of these papers contain set-theoretical results about Euclidean spaces, Hamel bases and other objects that are familiar to analysts and combinatorialists. The editors and I have decided that this typical Erdős genre deserves a separate treatment, and this will be given by Péter Komjáth in this volume. However, I cannot resist mentioning the first theorem in the first of these papers, the Erdős-Sierpiński duality principle. This generalization of an earlier result of Sierpiński states: *Assuming the continuum hypothesis (CH) there is a surjective map* $f : \mathbb{R} \to \mathbb{R}$ *which interchanges sets of Lebesgue measure zero and sets of first category.*

Stating this theorem allows me an opportunity to say something about his attitude towards the generalized continuum hypothesis (GCH) and mathematical logic in general. It should be remembered that in 1943 Gödal's proof of the consistency of GCH was quite new. Erdős always knew and appreciated and applied these results. He was happy to have them as a more-or-less justified tool to prove new theorems, and if he could not solve a set theory problem he always tried to solve it assuming GCH. On the other hand, later on he was always uneasy and disappointed if one of his favourite problems turned out to be independent, and he would remark "independence has raised its ugly head".

5. Large Cardinals: The Erdős-Tarski Paper

The cardinal κ, has the property $P(\kappa)$ if there is a field of sets which contains a family of λ pairwise disjoint sets for every $\lambda < \kappa$, but which does not contain such a family of size κ. Much to their surprise, Erdős and Tarski [4] proved that *for limit cardinals* κ, $P(\kappa)$ holds if and only if κ, is an uncountable inaccessible. First of all it was surprising that such a seemingly harmless problem should involve inaccessible cardinals which, in those days, had "hardly been born". The second surprise was, and this is explicitly mentioned in the paper, that the negation of $P(2^{\aleph_0})$ could not be proved in ZFC since it was generally believed that it would be proved consistent that $2^{\aleph_0}$ is inaccessible. (Indeed, this was one of the first corollaries of Cohen's method.)

In their paper, they formulated several properties of inaccessible cardinals and they mentioned quite a few connections between these properties in footnotes (without proofs). For example, they knew that if κ is measurable then it has the tree property and that this implies that $\kappa \to (\kappa)^r_\lambda$ holds for all $\lambda < \kappa$ and $r < \omega$. Of course, they also knew that the simplest Ramsey-type theorem $\kappa \to (\kappa)^2_2$ is false if κ is not strongly inaccessible. Later, in 1960 when Tarski, using a result of Hanf, proved that "small" inaccessibles are not measurable, and the theory of large cardinals was created, it became necessary to publish these classical proofs. A new Erdős-Tarski paper (written by Donald Monk) appeared in 1961.

It is of interest to note that Erdős and Tarski made a historical mistake in [4]. It was vaguely speculated that it may turn out to be at least consistent that all strongly inaccessible cardinals are measurable. This probably postponed the discovery of the true situation for almost 20 years. I cannot say how seriously Tarski believed this (I did try to ask him), but Erdős quite happily accepted this hypothesis in the spirit I described in §2. In our joint work from 1956 to 1960 we investigated every combinatorial property of strongly inaccessible cardinals under the assumption that they are measurable, although we did always mention that this was an assumption. However, as was so often the case for Paul, in the end this turned out to be quite fortunate. I will come back to this in §9 and §10.

6. Set Mappings and Compactness

Erdős visited Hungary in 1948 for the first time after the Second World War. Very likely it was during this visit that he recalled an old problem of Paul Turán. Let f be a set mapping on a set X, i.e. $f : X \to \mathbb{P}(X)$ (the power set of X) such that $x \notin f(x)$. We say that f is of order λ if $|f(x)| < \lambda$ for $x \in X$. A subset $S \subset X$ is independent for f if for all $x, y \in S$, $y \notin f(x)$. For combinatorialists, a set mapping of type λ is just a loop-free digraph having out degrees $< \lambda$. Turán was interested in the case when $X = \mathbb{R}$ and $f(x)$ is finite, i.e., f is of order ω, and he asked if there is a free set of power $2^{\aleph_0}$. A young Hungarian, Dezső Lázár who was killed during the war, proved this and he also proved that *if f is of order $\lambda < \kappa \geq \omega$ then there is a free set of size κ provided κ is a regular cardinal.* Ruziewicz conjectured that this is true for any $\kappa \geq \omega$.

Erdős proved this conjecture assuming GCH in the second of his "Remarks" in 1950 [10]. It remained an open question for ten more years if GCH is really needed. In 1960, I proved the result in ZFC in [H 1] where more history of the problem can be found.

Typically, even before proving the result he conjectured that if λ is an infinite cardinal and f is a set mapping of order λ on any set X, then X is the union of λ independent sets, i.e. the digraph has chromatic number at most λ. This was proved by Géza Fodor [F]. Erdős investigated the problem for finite λ in a paper with N.G. de Bruijn [11]. A little reflection will convince the reader that if the underlying set is finite then it is the union $2\lambda - 1$ independent sets (and this is best possible). To show that this is true for arbitrary X they proved that *for any $k < \omega$ if every finite subgraph of a graph G has chromatic number of at most k then G also has chromatic number at most k.*

The reader may say that this is a consequence of either Tychonov's theorem on the product of compact spaces or Gödel' s compactness theorem, but this is how compactness was introduced to infinite combinatorics. Let me point out that it would be quite difficult to find a proof of the set mapping theorem not using compactness. I do not know of any.

Erdős continued the investigation of set mappings with Géza Fodor in [19] and [21]. I want to mention one of their theorems, which later proved to be useful in applications.

Assume f is a set mapping of order $\lambda < \kappa \geq \omega$ on κ. Let $\tau < \kappa$ and let $X_\alpha (\alpha < \tau)$ be a sequence of subsets of κ each of size κ. There is a set S free for f which meets each X_α in a set of size κ. For singular κ their proof used GCH, but my method yields this in ZFC as well.

7. A Partition Calculus in Set Theory

The partition calculus was developed by Erdős and Rado in the early 1950s. The long paper [18] contains all the results they had proved up until then. Their first discovery was that the partition relation $\kappa \to (\lambda_\nu)^r_{\nu<\gamma}$ made sense for order types as well as cardinals, or even a mixture of these. This led them to a great variety of new problems, some simple and some difficult, but requiring different methods. Let me mention just a few of these. For what countable ordinals α does $\alpha \to (\alpha, 3)^2$ hold? For what $\alpha < \omega_1$ does $\lambda \to (\alpha)^2_k$ hold, where k is finite and λ is the order type of the reals? They proved the pleasing result that $\eta \to (\aleph_0, \eta)^2$, where η is the order type of the rationals, but for what other countable types is this true? They also noticed that the proof of the Erdős; Dushnik, Miller theorem in the special case $\kappa = \omega_1$ actually gives the slightly stronger fact that $\omega_1 \to (\omega_1, \omega+1)^2$, and then a natural question is, what about $\omega_1 \to (\omega_1, \omega+2)^2$?

They proved a great many partial results and isolated the most important problems. We cannot collect all their results and problems here, instead I shall discuss some of the important new discoveries. One of these is the *positive stepping up lemma* which can be stated as follows: *If κ is a cardinal and $\kappa \to (\lambda_\nu)^r_{\nu<\gamma}$ holds, then* $(2^{<\kappa})^+ \to (\lambda_\nu + 1)^{r+1}_{\nu<\gamma}$. Here μ^+ denotes the smallest cardinal greater than μ, and $2^{<\kappa} = \sum\{2^\mu : \mu < \kappa \text{ a cardinal}\}$. Let $\exp_n(\kappa)$ denote the n-times iterated exponentiation (i.e. $\exp_0(\kappa) = \kappa$ and $\exp_{n+1}(\kappa) = 2^{\exp_n(\kappa)}$.) Since $2^{<\kappa^+} = 2^\kappa$, and $\kappa^+ \to (\kappa^+)^1_\kappa$ just expresses the fact that κ^+ is a regular cardinal, we obtain by induction the

Erdős-Rado Theorem:

$$(\exp_n(\kappa))^+ \to (\kappa^+)^{n+1}_\kappa.$$

In particular, we have $(2^\kappa)^+ \to (\kappa^+)^2_\kappa, (2^{2^\kappa})^+ \to (\kappa^+)^3_\kappa$ etc. Quite often in the literature only the first of these (case $n = 2$) is referred to as the Erdős-Rado theorem, but we have seen that this was already proved in 1943. There can be very few theorems in set theory which have received so many "simplified" proofs as the Erdős-Rado theorem. But in the early 1950s there was no pressing-down lemma, and elementary substructures and chains had not been introduced into set theory. Erdős and Rado used the so-called *ramification method*. Let me outline this in a simple case. Let $f : [X]^2 \to \gamma$ be a 2-partition of X with γ colours. Pick an $x_0 \in X$. The rest of the points can be split into γ parts according to the colour of $f(\{x_0, y\})$. Repeat this in each part and continue transfinitely. We get a tree or ramification system as they called it. At the α stage we will have $|\gamma|^{|\alpha|}$ parts. If X has large enough cardinality, the tree we get will have large height. The points picked along a branch of this tree will form a *prehomogeneous set*, i.e., the colour of a pair $\{x_\alpha, x_\beta\}$ will depend only upon the point, say x_α, which was chosen first. This method is fairly hard to write down formally, but it is really quite intuitive. It was elaborated in great detail in the Erdős-Rado papers and was used for several years to obtain positive partition relations for the case when the underlying set has regular cardinality. Although most of the important proofs have been streamlined to "linear" ones, there is really no algorithm for this translation , and the intuition behind ramification serves as a good tool to obtain new results.

They also discovered polarized partition relations. The symbol

$$\binom{\kappa}{\lambda} \to \binom{\kappa_\nu}{\lambda_\nu}^{1,1}_{\nu<\gamma}$$

means the following: whenever $f : \kappa \times \lambda \to \nu$ is a colouring with γ colours, then there are $\nu < \gamma$, $K \in [\kappa]^{\kappa_\nu}$ and $L \in [\lambda]^{\lambda_\nu}$ such that $K \times L$ is homogeneous for f in the colour ν. For combinatorialists, this is just Ramsey for complete bipartite graphs, and the reader can easily formulate the generalization to s-partite graphs.

However, this is not just formalism. It turned out that quite a few problems about polarized partitions are basic questions in set theory. As an illustration, they proved

$$\binom{\aleph_1}{\aleph_0} \to \binom{\aleph_1}{\aleph_0}, \binom{\aleph_0}{\aleph_0}^{1,1}.$$

In "human language" this says: *if A_α $(\alpha < \omega_1)$ are arbitrary subsets of ω, then either the intersection of $\aleph_1$ of them is infinite, or the union of $\aleph_0$ of them has an infinite complement.* They attributed the negative relation

$$CH \Rightarrow \binom{\aleph_1}{\aleph_0} \not\to \binom{\aleph_1}{\aleph_1}, \binom{\aleph_0}{\aleph_0}^{1,1}.$$

to Sierpiński who, of course, proved this in a different context.

There is one more important partition relation I should mention, and this realized Paul's old wish to have a Ramsey theorem for something more than just k-element sets. They introduced the symbol $\kappa \to (\lambda)^{<\omega}_\gamma$ to denote the following statement: *for every sequence $f_n : [\kappa]^n \to \gamma$ of n-partitions of κ with γ colours, there is a subset $H \subset \kappa$ of cardinality λ which is simultaneously homogeneous for each f_n.* They only proved that $2^{\aleph_0} \not\to (\aleph_0)^{<\omega}_2$ with a clever ad-hoc construction, and they asked if $\kappa \to (\aleph_0)_2^{<\omega}$ can be true for any cardinal κ? We turn back to the discussion of this important symbol in §9 and §10.

There is one more type of partition theorem which I should have mentioned earlier. In [9] they proved the first *canonical Ramsey theorem*, and this also led to a long sequence of investigations, improvements and generalizations. Just to give the flavour of what this is all about, I will state just one special case. Let $f : [\omega]^2 \to \gamma$ be a 2-*partition of* ω *with any number of colours. Then there is an infinite subset* $H \subseteq \omega$ *such that either* H *is homogeneous for some colour, or all pairs in* H *have different colours, or* H *is prehomogeneous (the colour of a pair depends only on the least element), or* H *is endhomogeneous (the colour of a pair depends only on the largest element).*

8. My First Encounter with Paul

Erdős visited Hungary in 1955 for the first time since the country had become a member of the Eastern block. He was an Hungarian citizen traveling on an Hungarian passport, and he could not have returned earlier if he wanted to leave again. But in the "liberalized" atmosphere of 1956 the Academy was allowed to elect him as a member and the government granted him a special diplomatic type of passport which allowed him to come and go whenever he wished. This was a great opportunity for young Hungarian mathematicians who had heard of him only by name (of course, it was impossible for us to travel to the West before 1956).

At that time I was a graduate student of László Kalmár in Szeged (a small town on the south-eastern border of Hungary). Paul travelled around the country in 1956 and came to visit the mathematics department at the University of Szeged. He had already corresponded with Géza Fodor who was then a young assistant professor in the department. I was introduced to him as "a promising young man" studying set theory, and soon we were left alone in Professor Kalmár's office sitting in two enormous armchairs facing each other over a coffee table. I thought he was very old—he was 43 years old and I was 25. I felt very honoured, and a little embarrassed, to be left alone with this famous man. I did not know then that he had met most of his young collaborators in a similar way. He first asked me what were my interests in set theory. I was then writing my thesis on a subject which later was called relative constructibility, and I was quite proud of it. So I started to explain my results with some enthusiasm. He listened to me very politely, and when I had finished he asked "and are you interested in normal set theory as well?" Of course, we were not on first-name terms then and so the question was phrased in a very polite form of Hungarian that is used for addressing a stranger. But it was clear that it was a genuine inquiry and he meant no harm.

Earlier I had thought of a problem when I heard about all the set-mapping results from Géza, What if we investigate set-mappings of more variables, and I asked if there would be large free subsets in this case too? To tell the truth, as a student of Kalmár, I was trained to think like a logician. So what I had in mind was this: if X is a large set and to every finite subset $V \subseteq X$ we associate a countable $F(V) \subset X$ such that $F(V) \cap V = \emptyset$, is there a large independent set? I even had the vague idea that if $F(V)$ is the Skolem hull of V in some structure, then an "independent set" would really deserve its name.

Luckily, Paul liked that one! It started a furious activity and the conversation became more fluent and colloquial. The first thing I learned from him (and this took quite a while) was that he would not start to think about the general case. He first wanted to know what happened for set mappings defined on pairs. He proved several lemmas and some partial results and stated a few conjectures. He then suddenly remembered that there was something else he had to do and he called Géza. Quite close to the mathematics building in Szeged stands a rather ugly cathedral built in the 1930s with two high towers. It turned out that he "must" climb the 300 odd stairs to the top! Géza had earlier agreed to accompany him, and he then gently began to persuade me to come along too. I had by then lived for 2 years in Szeged, and I had never had the slightest difficulty in resisting any pressure to visit the tower, especially since the surrounding countryside is absolutely flat and so there was not very much to see. However, much to my own surprise, I could not resist this invitation. Climbing those stairs more results and conjectures were formulated by him while, at the same time, he was complaining that he felt a little dizzy.

That day ended with dinner at Kalmár's house where the conversation continued mainly about set mappings, but sometimes interrupted with some of his comments on "Sam and Joe". When we parted, it was almost as from an old friend—there was a joint-paper half ready, which could be completed by correspondence.

9. Our First Joint Paper

The notation used for discussing set-mapping problems is not standardized as in the case of partition relations. A set-mapping of *order* λ and *type* μ on κ is a function $f : [\kappa]^\mu \to [\kappa]^{<\lambda}$ such that $f(x) \cap x = \emptyset$ for all $x \in \text{dom}(f)$, and a set $S \subseteq \kappa$ is free if $f(x) \cap S = \emptyset$ for all $x \in [S]^\mu$. I shall denote by $\text{Free}(\kappa, \lambda, \mu, \nu)$ the following assertion: *For every set-mapping of order* λ *and type* μ *on* κ *there is a free set of cardinality* ν. Likewise, $\text{Free}(\kappa, \lambda, < \mu, \nu)$ denotes the corresponding assertion when $f : [\kappa]^{<\mu} \to [\kappa]^{<\lambda}$.

Our very first result was to prove that $\text{Free}(\exp_{n-l}(\lambda)^+, \lambda, n, \lambda^+)$ holds for every infinite λ. The proof of this uses the Erdős-Rado theorem which I had learned during my first conversation with Paul. I also learned that it was not known if the Erdős-Rado theorem is best possible, for example it was not known then if

$$2^{2^{\aleph_0}} \nrightarrow (\aleph_0)^3_2$$

holds. Assuming GCH we could prove $\neg\,\mathrm{Free}(\aleph_1, 2, 2, \aleph_1)$ so that our theorem is best possible for $n = 2$, but for $n > 2$ any progress seemed to lie far in the future. I will return to this in the section on the negative stepping-up method we developed with Rado, but let me say now that there are only consistency results to show that the theorem is best possible.

I was therefore surprised when shortly afterwards I received a letter from Paul (who was visiting Israel) claiming that $\mathrm{Free}(\kappa, \lambda, n, \kappa)$ holds for $n < \omega$ and any uncountable limit cardinal $\kappa > \lambda$. The proof used GCH and I realized that it only worked for singular κ (the real theorem is for κ a singular strong limit cardinal, i.e. $2^\tau < \kappa$ for $\tau < \kappa$.) We now know that this is a special case of a general "canonization" theorem proved later with Rado, but which Paul discovered in at least two other interesting contexts before the general theorem was formulated. I wrote to tell him that I could not see how the proof works for regular limit κ. He replied by return of post that I was right, but the theorem was true since we may use the "measure hypothesis" from Erdős-Tarski, and he wrote down a proof that $\mathrm{Free}(\kappa, \lambda, n, \kappa)$ holds for finite n and $\lambda < \kappa$, an uncountable measurable cardinal. I have to say that during my studies I had read the Erdős-Tarski paper, but either I skipped the footnotes or I did not recognize the significance of the remarks. However, after reading the letter, I did understand the strength of the hypothesis, and the same day proved that it implies $\mathrm{Free}(\kappa, \lambda, < \omega, \kappa)$. Later, when the paper was actually written, we realized that the proof actually gave the stronger result that

$$\kappa \rightarrow (\kappa)^{<\omega}_\lambda$$

holds for $\lambda < \kappa$ if $\kappa > \omega$ is a measurable. This is perhaps one of our best-known joint theorems, and I will say more about this in the next section.

This brings me to our first joint oversight. Although our joint paper with Rado did not appear until 1965, I already had a weak form of the negative stepping-up lemma in 1957 (in fact it was because of this that we decided to write the triple paper on partition relations even though Erdős and Rado had already obtained a great many new unpublished results.)

I told Paul that the negative stepping-up gives us that

$$\kappa \nrightarrow (\omega)^{<\omega}_2 \Rightarrow 2^\kappa \nrightarrow (\omega)^{<\omega}_2.$$

He immediately pointed out that it is easy "to go through" singular cardinals, and we put into the paper the remark that $\kappa \nrightarrow (\omega)^{<\omega}_2$ holds for all κ, less than the first strongly inaccessible cardinal $\kappa_0 > \omega$. We only realized later [35], after we learned the Hanf-Tarski results, that this almost trivially implies that

$$\kappa_0 \nrightarrow (\omega + 1)^{<\omega}_2,$$

and therefore, by our theorem, it follows that κ_0 is not measurable. Unfortunately, this argument is not very strong, we could never make it work beyond the first fixed point in the sequence of inaccessibles.

10. Erdős Cardinals and the Strength of $\kappa \to (\lambda)_2^{<\omega}$

The real strength of the statement $\kappa \to (\lambda)_2^{<\omega}$ was discovered by the Berkeley school in the early 1960s. Dana Scott first proved that the existence of a measurable cardinal contradicts Gödel's axiom of constructibility ($V = L$). Soon afterwards Gaiffman and Rowbottom proved that it also implies that ω has only countably many constructible sets, and more generally Rowbottom proved that $\kappa \to (\omega_1)_2^{<\omega}$ implies that ω_1 is inaccessible in L.

Rowbottom also generalized our theorem with Paul. According to the Kiesler-Tarski paper [KT] it was Dana Scott who introduced the notion of a normal measure (a κ-complete 0-1 measure on κ satisfying the pressing down lemma, i.e. if f is regressive on a subset of measure one, then it is constant on a set of measure one.) It was known that if κ is measurable then it carries a normal measure, and Rowbottom proved that, if there is a normal measure on κ and $f : [\kappa]^n \to \lambda(n < w, \lambda < \kappa)$, then there is a homogeneous set of measure one. Of course, this immediately yields our theorem, and the proof actually becomes easier.

I happened to spend 1964 at Berkeley with Tarski's group and gave a course of lectures on Erdős-Rado set theory. I could not have been too successful as a lecturer, more than 20 people attended the first lecture, and in the end I was left with an audience of three—two students, Reinhardt and Silver, and a young assistant professor Donald Monk. I told them everything I knew about ordinary partition theorems and the little I knew about $\kappa \to (\lambda)_2^{<\omega}$. Silver apparently got interested and his thesis [Si], which appeared in 1966, contained some fantastic discoveries.

First he realized that the real strength of $\kappa \to (\lambda)_2^{<\omega}$ is that it yields, for any given structure on κ, a set of indiscernibles having order type λ (i.e. a set of ordinals such that any two similarly ordered n-tuples satisfy the same formulas.) Using this he proved that the smallest κ satisfying $\kappa \to (\omega)_2^{<\omega}$ must be very large, for example there must be many weakly compact cardinals less than κ. He showed that, for $\alpha < \omega_1$, if $\kappa \to (\alpha)_2^{<\omega}$ holds, then it is true in L, and finally he proved that if $\kappa \to (\omega_1)_2^{<\omega}$ holds, then $O^{\#}$ exists, which means that L is very small, and this expresses the real strength of $\kappa \to (\omega_1)_2^{<\omega}$. I am not willing to write down the technicalities of this here.

Let me remind the reader that one of the few consequences of the axiom of constructibility which Gödel himself had noticed was that there is an uncountable analytic complement (the complement of a continuous image of a Borel set) which has no perfect subset. It was Solovay who proved that if $\kappa \to (\omega_1)_2^{<\omega}$ holds for some κ then such a set cannot exist. This was the first application in descriptive set theory. It is stated in Descriptive Set

Theory, the 1978 book of Moschovakis, that all applications of the existence of measurable cardinals in descriptive set theory come from a κ satisfying $\kappa \to (\omega_1)_2^{<\omega}$. All this shows that such cardinals deserve a special name, and the story I have written down shows that they are quite rightly called Erdős cardinals.

11. The Early Sixties. A Long Chapter

After 1956, Paul came home to visit his mother every year. He usually spent some months in Budapest where I also lived at that time. When he was at home I went to work with him at their apartment two or three times a week. Mrs. Erdős was not only a devoted mother to Paul, but she was also an efficient secretary and would keep a record of his publications and look after his papers. When I visited them she would make us coffee and then leave us alone to "work". Our meetings had no prepared agenda, sometimes we went through earlier proofs, sometimes we had to read a manuscript or proof sheets, but the main point of our conversations was always the discovery of new problems and to start thinking about them. Paul was fantastically fast in both making and understanding proofs and finding the new questions. Though I usually made some notes, they were never quite satisfactory. We both needed to rely on our memories. This was quite a good fit, he always remembered the theorems and then I could scrape together the old proofs. I think now that these visits were real highlights in my life.

Now I have to change strategy. I cannot continue telling the results paper by paper, and in any case they were not proved in the order of publication. Starting around 1957 or 1958, we agreed to write a triple paper with Rado on the partition calculus and the three of us set aside everything which we thought belonged there. Already in 1960 I visited Rado in Reading to work on the triple paper, carrying with me an almost completed manuscript. So, in this long section I will open subsections about the results of these years with an indication of where they appeared.

11.1. Canonization

Let $f : [X]^r \to \gamma$ be an r partition of length γ of X. Let $\langle Y_\alpha : \alpha < \varphi \rangle$ be a sequence of disjoint subsets of X , $Y = \bigcup_{\alpha<\varphi} Y_\alpha$ For a subset $v \in [Y]^r$ there is a number $s(v) \le r$ an increasing sequence $\alpha(v) = \langle \alpha_i(v) : i < s(v) \rangle$ of ordinals and a sequence $r(v) = \langle r_i(v) : i < s(v) \rangle$ of integers, defining the position of v in the partition $Y = \bigcup_{\alpha<\varphi} Y_\alpha$, so that $\sum_{i<s(v)} r_i(v) = r$ and $|v \cap Y_{\alpha_i(v)}| = r_i(v)$ for $i < s(v)$. Two r-element sets $v, v' \in [Y]^r$ have the same position if $\alpha(v) = \alpha(v')$ and $r(v) = r(v')$. f is *canonical with respect to the sequence* $(Y_\alpha : \alpha < \varphi)$ if for any two v, v' having the same position

$$f(v) = f(v')$$

The "canonization" theorem of [43] tells us that: *there is an integer k_r such that whenever $\langle X_\alpha : \alpha < \varphi\rangle$ is a sequence of subsets of X with fast enough increasing cardinalities,*

$$|X_\alpha| > \exp_{k_r}(|\bigcup_{\beta<\alpha} X_\beta|),$$

then there is a disjointed sequence $Y_\alpha \subset X_\alpha (\alpha < \varphi)$ also with fast increasing cardinalities, $|Y_\alpha| > |\bigcup_{\beta<\alpha} X_\beta|$, such that $f : [x]^r \to \gamma$ is canonical with respect to the sequence $\langle Y_\alpha : \alpha < \varphi\rangle$, provided $\gamma, \varphi < |X_0|$.

One corollary of this is the following. Assume κ is a singular strong limit cardinal then $\kappa \to (\kappa, \lambda_\nu)^2_{\nu<\gamma}$ if and only if $\mathrm{cf}(\kappa) \to (\mathrm{cf}(\kappa), \lambda_\nu)^2_{\nu<\gamma}$. The 'only if' part comes easily using a canonical partition, and the 'if' part uses the canonization theorem. The reader should remember the Erdős; Dushnik, Miller theorem $\kappa \to (\kappa, \aleph_0)^2$. Now the above result tells us, at least with GCH, for which singular cardinals κ the relation $\kappa, \to (\kappa, \aleph_1)^2$ holds. For example, $\aleph_{\omega_1} \not\to (\aleph_{\omega_1}, \aleph_1)^2$ but $\aleph_{\omega_2} \to (\aleph_{\omega_2}, \aleph_1)^2$. By now I do not really have to tell the reader that this is the form discovered by Paul.

It would be nice to have a necessary and sufficient condition for the case of arbitrary singular κ. We knew that for a singular κ, say with $\mathrm{cf}(\kappa) = (2^{\aleph_0})^+$, to have $\kappa, \to (\kappa, \aleph_1)^2$ it is necessary to have $\lambda^{\aleph_0} < \kappa$ for $\lambda < \kappa$ We repeatedly asked if this is sufficient. It was proved by Shelah and Stanley in the 1980s that this is consistently false [SS1].

When preparing the material of our book with Attila Máté and Richard Rado [100] where we tried to discuss the ordinary partition relation is ZFC, we isolated the following problem. Assume there is an increasing sequence of integers $n_k : k < \omega$ such that

$$\aleph_\omega < 2^{\aleph_{n_0}} < \ldots < 2^{\aleph_{n_k}} < \ldots$$

Does it follow that $2^{<\aleph_\omega} \to (\aleph_\omega)^2_2$ holds? Clearly our canonization does not work in this case. Shelah proved this with a new type of canonization theorem [S1], and parts of his results are given in the book. Further uses of "canonization" will be mentioned later.

One last remark. It is interesting to see how combinatorial ideas do pop up in different topics. When Shelah obtained with such miraculous speed his celebrated result on the bound in van der Waerden's theorem, he was already the best expert on canonization, and one of the main lemmas in his proof is indeed a (finite) canonization theorem.

11.2. Square Brackets

Sierpiński proved $2^{\aleph_0} \not\to (\aleph_1)^2_2$ by well ordering the continuum and defining a partition of the pairs into two classes, so that a pair ordered in the same way in both the natural ordering and the well-ordering belongs to the first class.

Paul told me that he formulated a generalization of this in 1956 with the following question. Can one split the pairs of reals into three classes so that every subset of size $\aleph_1$ (or $2^{\aleph_0}$) contains a pair from each class?

He soon proved this assuming CH. We discovered that whenever a partition relation fails, one can ask for a corresponding weaker property, and in [43] we introduced the following square bracket relation

$$\kappa \to [\lambda_\nu]^r_{\nu<\gamma}.$$

This means that for every $f : [\kappa]^r \to \gamma$ there is a $\nu < \gamma$ and a subset $H \subset \kappa$, $|H| = \lambda_\nu$ so that f does not take the value ν on the r-tuples of H.

It is worthwhile to formulate separately the negation of the "balanced" form of this (when all the λ_ν are equal). Thus $\kappa \nrightarrow [\lambda]^r_\gamma$ means that there is an $f : [\kappa]^r \to \gamma$ such that all subsets of size λ are completely inhomogeneous i.e. f takes all possible values on the r-tuples of any set of size λ.

Clearly we needed some test cases. We proved that $2^\kappa = \kappa^+$ mplies $\kappa^+ \nrightarrow [\kappa^+]^2_{\kappa^+}$, and only later did we realize that this was also known to Sierpińiski in a different context. But probably the nicest result was the following:

If κ is a strong limit cardinal of cofinality ω, then $\kappa \to [\kappa]^2_3$.

(Note that $\kappa \nrightarrow (\kappa, (cf(\kappa))^+)2$ and $\kappa \nrightarrow [\kappa]^2_2$ is a trivial corollary.) This follows from the "canonization" theorem of the previous section. Indeed it gives a stronger result. Under the above conditions on κ, for every $f : [\kappa]^2 \to \gamma$, $\gamma < \kappa$, there is a set $H \in [\kappa]^\kappa$ such that f takes at most two different values on the pairs of H.

So we introduced a third symbol, the strong square bracket. Let γ, δ be cardinals. $\kappa \to [\lambda]^r_{\gamma,\delta}(\kappa \to [\lambda]^r_{\gamma,<\delta})$ means that for every r-partition $f : [\kappa]^r \to \gamma$ with γ-colors, there is a subset $H \subset \kappa$ of size λ such that f takes at most δ (fewer than δ) values on the r-tuples of H.

So the above theorem says that $\kappa \to [\kappa]^2_{\gamma,2}$ for singular strong limit κ of cofinality ω and $\gamma < \kappa$. We used this symbol to ask if $\aleph_2 \to [\aleph_1]^2_{\aleph_1,\aleph_0}$? Paul thought this was an old question of Ulam, but later we discovered that it is equivalent to a well-known model theoretical conjecture of C.C. Chang.

In §12, I will discuss the effect of our 1967 problem paper [67], but this is a good place to write down the present status of some of the square bracket problems stated in that paper. Let me begin with an innocent but very nice result of Fred Galvin

$$\eta \to [\eta]^2_3.$$

Many generalizations of this were published later. Galvin and Shelah proved $2^{\aleph_0} \nrightarrow [2^{\aleph_0}]^2_{\aleph_0}$ and $\mathrm{cf}(2^{\aleph_0}) \nrightarrow [\mathrm{cf}(2^{\aleph_0})]^2_{\aleph_0}$ in 1968 [GS], they also proved some weak results like $\aleph_1 \nrightarrow [\aleph_1]^2_4$ and $\aleph_1 \nrightarrow [\aleph_1]^3_{\aleph_1}$ for the case when the underlying set has cardinality $\aleph_1$.

Again we had a false feeling. Although we did not state it explicitly, we clearly believed that $\aleph_1 \nrightarrow [\aleph_1]^2_{\aleph_1}$ could not be proved in ZFC. But in 1987 Stevo Todorčević proved us wrong [T1]. He proved in ZFC that

$$\kappa^+ \nrightarrow [\kappa^+]^2_{\kappa^+}$$

holds for every regular κ. This was extended by Todorčević and Shelah for more successors, and inaccessibles. This was certainly one of the most significant discoveries in set theory in the 1980s requiring entirely new methods. I will come back to this for a moment in the next section. However, this still leaves open the question whether

$$2^{\aleph_0} \nrightarrow [\aleph]^2_3$$

holds? Shelah [S2] proved that $2^{\aleph_0} \to [\aleph_1]^2_3$ is really consistent with ZFC. In his model $2^{\aleph_0}$ is quite large so it is still possible, but quite unlikely, that

$$2^{\aleph_0} = \aleph_2 \Rightarrow \aleph_2 \nrightarrow [\aleph_1]^2_3.$$

11.3. Jónsson Algebras-Negative Relations with Infinite Exponents

A Jónsson algebra is an infinite algebra A with countably many finitary operations such that all proper subalgebras have cardinality strictly less than IAI. The question is, for what infinite cardinals κ is there a Jónsson algebra of cardinality κ? I mention this here because of a connection with the square brackets. As pointed out by Shelah much later in the game, there is a Jónsson algebra on κ if and only if $\kappa \nrightarrow [\kappa]^{<\omega}_\kappa$ holds.

I heard the problem from Tarski in 1964 and when I returned to Hungary and met Paul, we immediately had some remarks about this which we published in [45]. First we proved that if there is a Jónsson algebra on κ, then there is also one on κ^+, and hence there is one on $\aleph_n$ for $n < \omega$. We also proved that $2^\kappa = \kappa^+$ implies that there is a Jónsson algebra on κ^+ since we knew that $2^\kappa = \kappa^+ \Rightarrow \kappa^+ \nrightarrow [\kappa^+]^2_{\kappa^+}$.

I must also mention that it was already proved by Kiesler and Rowbottom that there is a Jónsson algebra on every κ if $V = L$ [KR].

It was a metatheorem for the two of us because of Rado's theorem that "nothing is true for infinite exponents". So we proved already in [24] that Free$(\kappa, 2, \aleph_0, \aleph_0)$ fails for every κ and in [43] we strengthened Rado's result to $\kappa \nrightarrow [\aleph_0]^{\aleph_0}_{2^{\aleph_0}}$. In this spirit we also proved in the Jónsson algebra paper that there is an infinitary Jónsson algebra on every κ in other words

$$\kappa \nrightarrow [\kappa]^{\aleph_0}_\kappa$$

This became one of our best used theorems. Kunen used it for a simple proof of his famous theorem that there is no nontrivial elementary embedding of the universe into itself (disproving the hoped for existence of Reinhardt cardinals) and Solovay used it in his proof that GCH holds at every singular strong limit cardinal above a strongly compact cardinal.

Our "metatheorem" is not quite true since Foreman and Magidor [FM] recently proved that it is consistent that $\aleph_3 \to [\aleph_2]^{\aleph_0}_{\aleph_2}$. It goes without saying that Erdős always assumed the axiom of choice, and I would not even mention this except that it happens that partition relations with infinite exponents may be true if we do not assume the axiom of choice, and indeed they became an important tool of set theory e.g. in investigations concern ing the Axiom of Determinacy and its consequences to descriptive set theory.

Back to the previous section, Shelah [S5] recently published quite a few theorems extending the class of cardinals κ for which there is a Jónsson algebra and then later proving the stronger result $\kappa \nrightarrow [\kappa]^2_\kappa$. I presently do not know of any instance of the result where $\kappa^+ \nrightarrow [\kappa^+]^{<\omega}_{\kappa^+}$ is true but $\kappa^+ \nrightarrow [\kappa^+]^2_{\kappa^+}$ is not known.

11.4. Negative Stepping-Up

This result published in [43] says that, if $r \geq 2$, $\kappa \geq \omega$ and $\kappa \nrightarrow (\lambda_\nu)^r_{\nu<\gamma}$, then

$$2^\kappa \nrightarrow (\lambda_\nu + 1)^{r+1}_{\nu<\gamma}$$

provided the sequence λ_ν satisfies certain simple conditions. The simplest of these is that two of them are infinite and one of them is regular. There are about six more conditions to cover relevant cases. These conditions become less restrictive as r grows, and there is no condition at all for $r \geq 5$.

But even the one just stated tells us that the Erdős-Rado theorem of §7 is best possible, i.e.

$$\exp_{n-1}(\kappa) \nrightarrow (\kappa^+)^n_2$$

for $n \geq 2$, since the result $2^\kappa \nrightarrow (\kappa^+)^2_2$ can be lifted by induction on n.

Let me state another example. We know that if $\kappa \nrightarrow (\kappa^+)^2_2$ then $\kappa \nrightarrow (\kappa, 4)^3$. This should imply $2^\kappa \nrightarrow (\kappa, 5)^4$, but to get this, a special argument is needed say if κ is singular.

Maybe the negative stepping up is true without any conditions at all on the λ_ν, but to the best of my knowledge this is still wide open. There are cases where we do not know what happens without GCH for $n = 2$. Let me explain this with an example. It is very easy to see that $\aleph_\omega^{\aleph_0} \nrightarrow (\aleph_{\omega+1}, (\aleph_0)_{\aleph_0})^2$, but this should still be true if the $\aleph_0$ entries are replaced by 3's, and indeed we did prove this with GCH

$$\aleph_{\omega+1} \nrightarrow (\aleph_{\omega+1}, (3)_{\aleph_0})^2.$$

To stick my neck out again, it seems inconceivable to prove this in ZFC, but no consistency proofs are known in the other direction.

Now a trivial canonization lifts this say to the first singular cardinal with cofinality $\aleph_{\omega+1}$ i.e. to $\aleph_{\omega_{\omega+1}} \nrightarrow (\aleph_{\omega_{\omega+1}}, (3)^2_{\aleph_0})$ and this should be stepped up to

$$\aleph_{\omega_{\omega+1}+1} \nrightarrow (\aleph_{\omega_{\omega+1}}, (4)_{\aleph_0})^3.$$

Unfortunately in this case, for $r = 2$, only one of the entries is infinite and even that is singular. So we had nothing to cover this case and it was stated as one of our open problems for a long time. I thought Shelah and Stanley had a proof of this $\not\rightarrow$ from GCH, but I understand that it is still open.

A more significant problem is that our result suggests a negative stepping-up, for square brackets and set mappings as well.

It was recognized early in the game that for square brackets this is consistently false even assuming GCH. For example, $2^{\aleph_0} = \aleph_1 \Rightarrow \aleph_1 \not\rightarrow [\aleph_1]^2_{\aleph_1}$, but it is very easy to see that $\aleph_2 \not\rightarrow [\aleph_1]^3_{\aleph_1}$ implies $\aleph_2 \not\rightarrow [\aleph_1]^2_{\aleph_1,\aleph_0}$, the negation of Chang's conjecture, which was proved to be consistent in an early paper of Silver.

Stevo Todorčević worked out stepping-up methods from combinatorial principles known to hold in L, which do give the stepping-up for square brackets and for set mappings in most cases. See [T2] and also [HK1] for more history.

Let me conclude this section with two more interesting recent results of Todorčević which show the present direction of research in this area. He proved that $\aleph_2 \rightarrow [\aleph_1]^3_{\aleph_1}$ is equivalent to Chang's conjecture in ZFC (without assuming CH), and $\aleph_2 \not\rightarrow [\aleph_1]^3_{\aleph_0}$ is true in ZFC [T3]. See also [57]. This is a very deep result, but Erdős had a hand in initiating of this type of theorem as well, in [81] we remarked that the stepping-up method yields $2^{\aleph_1} \not\rightarrow [\aleph_1]^3_4$ in ZFC.

11.5. Polarized Partition Relations

While working on the triple paper [43], we had to draw the line somewhere, and we decided that we will only include results for polarized partitions of the form

$$\binom{\kappa}{\lambda} \rightarrow \binom{\kappa_0}{\lambda_0}, \binom{\kappa_1}{\lambda_1}^{1,1}$$

and we gave a number of results assuming GCH. Some of the results inherent in the methods were only stated in the second problem paper [81]. But the simplest problem we isolated was if

$$2^{\aleph_0} = \aleph_1 \Rightarrow \binom{\aleph_2}{\aleph_1} \rightarrow \binom{\mu}{\aleph_1}, \binom{\nu}{\aleph_1}^{1,1}$$

holds for $\aleph_0 \leq \mu, \nu \leq \aleph_1$.

One of the first results proved after our problem list became public was due to Karel Prikry [P]. I state a special case:

$$\binom{\aleph_2}{\aleph_0} \not\rightarrow \binom{\aleph_0}{\aleph_1}^{1,1} \text{ or even } \binom{\aleph_2}{\aleph_0} \not\rightarrow \begin{bmatrix}\aleph_0 \\ \aleph_1\end{bmatrix}^{1,1}_{\aleph_1}$$

is consistent with ZFC and GCH.

Later, Richard Laver [L] proved that relative to a very large cardinal it is consistent with GCH that there is an ω_1 complete ideal I on ω_1 having the following strong saturation property: Given $F \subset I^+$, the complement of I, $|F| = \aleph_2$ (i.e. $\aleph_2$ large subsets of $\aleph_1$ there is an $F^1 \subset F$, $|F^1| = \aleph_2$, such that the intersection of any countably many sets in F^1 is in I^+. This easily yields

$$\binom{\aleph_2}{\aleph_1} \to \binom{\aleph_1}{\aleph_1}_2^{1,1} \text{ even } \binom{\aleph_2}{\aleph_1} \to \binom{\aleph_1}{\aleph_1}_{\aleph_0}^{1,1},$$

and it was one of the first corollaries of Jensen's morasses, that Prikry's result holds in L.

For lack of space and time, we did not include polarized partitions in the book [100] so there is no comprehensive account in the literature about the recent results. Let me state one problem of the form $\binom{\aleph_{\alpha+1}}{\aleph_\alpha} \to (\ \cdot\ )^{1,1}$ which is unsolved and for which there are no consistency results either. Does GCH imply

$$\binom{\aleph_{\omega_1+1}}{\aleph_{\omega_1}} \to \binom{\aleph_{\omega_1}}{\aleph_{\omega_1}}_2^{1,1}?$$

A small hope here is an unpublished remark of Shelah from 1989. Assume $\langle \kappa_\alpha : \alpha < \omega_1 \rangle$ *is an increasing sequence of measurable cardinals*, $\kappa = \sup_\alpha \kappa_\alpha$ *and* $2^\kappa = \kappa^+$ *then*

$$\binom{\kappa^+}{\kappa} \to \binom{\kappa}{\kappa}_\tau^{1,1} \quad \text{holds.}$$

Added in proof (March 1995). In September 1994, Shelah proved the following striking result. Assume $\kappa > \mathrm{cf}(\kappa)$, κ is strong limit and $2^\kappa > \kappa^+$. Then

$$\binom{\kappa^+}{\kappa} \to \binom{\kappa}{\kappa}_\tau^{1,1} \quad \text{holds for } \tau < \kappa.$$

Writing up the second problem paper [81], I realized that our theorem in [43] yielding $\binom{\aleph_2}{\aleph_2} \to \binom{\aleph_1}{\aleph_1}_2^{1,1}$ from CH can be generalized to give

$$2^{\aleph_0} = \aleph_1 \Rightarrow \binom{\aleph_2}{\aleph_2} \to \binom{\aleph_1}{\aleph_1}_3^{1,1}$$

but it is consistent with CH that

$$\binom{\aleph_2}{\aleph_2} \not\to \binom{\aleph_1}{\aleph_1}_4^{1,1}$$

holds. A recent still unpublished result of J. Baumgartner using a new kind of argument, says that assuming CH (but no more of GCH)

$$\binom{\aleph_3}{\aleph_2} \to \binom{\aleph_1}{\aleph_1}_{\aleph_0}^{1,1}$$

holds.

11.6. Property B and Incompactness

Our second major joint paper [33] is about the following property of families of sets, F : *There is a set B which meets every element of F but does not contain any member of F as a subset.* This means, in a terminology introduced later, that the chromatic number of F is two. Before stating some results I want to tell how we came across property **B**. Property **B** was actually discovered by Felix Bernstein in 1908. He proved that for every $\kappa \geq \omega$, if F is a family of size κ of sets of size κ then F has this property. (He used it to get a subset $B \subset \mathbb{R}$, $|B| = |\mathbb{R} \setminus B| = 2^{\aleph_0}$ and such that neither B nor $\mathbb{R} \setminus B$ contains a perfect subset of $\mathbb{R}$.

In those years I often visited Erdős at the summer house of the Academy in Mátraháza (a summer resort in the mountains), where he used to spend part of the summer with his mother. The place was reserved for members of the Academy and I was still young, so I had to find a place in the village for a couple of days. But I did get decently fed in the summer house during the day time. Usually there were other visitors or regular inhabitants to also work with Paul, and he would do this simultaneously. He led his usual life there, alternately proving, conjecturing, playing chess, ping pong, bridge, or walking to mountain tops. It was his habit to stop playing abruptly, when the rest of us were warming up to the game, and to return to work. In those days he went to bed around ten o-clock, but he woke up early, between four or five in the morning, so it was actually safer for me not to be living too close.

There was a vague plan to write a book on set theory and I arrived with a number of old journals. One of them was the 1937 volume of the Comptes Rendus Varsowie containing a long paper of Tarski, "Ideale in Vollständigen Mengenkörper", in which we wanted to find something for the planned book. Erdős volunteered to look it up. I had something else to do and I left him alone for a while. When I returned, he was excitedly reading. But not Tarski's paper, it was a forgotten paper [Mil] of an American set theorist E.W. Miller, which was next to Tarski's paper in the same volume. (Yes, the same as in Dushnik-Miller.) Miller proved that, for n finite, if F is a family of infinite sets and any two members of F intersect in at most n elements, then F has property **B**. "Reading" meant reading the statements and trying to figure out the proofs. After a while, I gave up and started reading the paper in detail.

The proof was by a cardinal induction on $\kappa = |F|$, the size of the family, and for a given κ, the underlying set was split into the increasing continuous union of κ smaller sets $\{A_\alpha : \alpha < \kappa\}$, each A_α, being closed with respect to certain operations. In this case, for each $n+1$ element set, there is at most one set containing it, and the elements of this (possibly non-existent) set were the values of these operations. Then the induction hypothesis was applied to the families $F|A_\alpha$. This is called nowadays, the method of elementary chains. Miller actually proved that F has the stronger property $\mathbf{B}(< \aleph_0)$, i.e. there is a set B which meets each element A of F in a finite set.

Paul's only comment was: "You see there are still things we do not know," and before we actually read all the details, he started to ask questions. What if the sets are only almost disjoint (have finite intersections)? There is a counter-example on the second page of Miller's paper, and I tried to return to the details. "Yes" he said, "but we should then assume that the sets are bigger." So, instead of collecting data for the book, we wrote a long paper.

Let me state a special case of one of the main results: *Assume GCH. If F is a family of very strongly almost disjoint sets of size $\aleph_2$, i.e. $|A \cap B| < \aleph_0$ for $A \neq B \in F$, then F has property $\boldsymbol{B}(< \aleph_2)$. More importantly, if F consists of sets of size $\aleph_1$ just strongly almost disjoint, i.e. $|A \cap B| < \aleph_0$ for $A \neq B$ in F, then F still has property $\boldsymbol{B}(< \aleph_2)$ provided $|F| \leq \aleph_\omega$.*

The reason why the proof broke down for $\aleph_{\omega+1}$ was quite clear. In the generalization of Miller's proof we had to use infinitary operations, and alas $\aleph_\omega^{\aleph_0}$ is greater than $\aleph_\omega$ no matter what we assume.

We both felt that this is a real hard-core problem and we tried to find other methods. In doing so we formulated the following statement $(*)$: *For $\alpha < \aleph_{\omega+1}$ there is a partition of $\alpha = \cup_{n<\omega} S_{\alpha,n}$ into countably many pieces such that $|S_{\alpha,n}| \leq \aleph_n$ and for any $\alpha < \aleph_{\omega+1}$ with $\mathrm{cf}(\alpha) = \omega_1$ there is an increasing sequence $(\alpha_\nu : \nu < \omega_1)$ of ordinals $\alpha_\nu \to \alpha$ such that for each $n < \omega$ the sequences $\{S_{\alpha_\nu,n} : \nu < \omega_1\}$ are increasing as well.*

Of course, we could not prove this, but we could deduce from it the theorem for $|F| = \aleph_{\omega+1}$.

Later a young German set theorist W. Donder pointed out that our statement is an easy corollary of Jensen's $\square_{\aleph_\omega}$ and as a corollary of this statement and some obvious generalizations, the theorem for families of sets of size $\aleph_1$ is true in L.

In 1986, in a paper with Juhász and Shelah [HJS], we proved that *it is consistent, relative to a super compact cardinal, that there is a family of size $\aleph_{\omega+1}$ of strongly almost disjoint sets of size $\aleph_1$ not having property $\boldsymbol{B}$ and also GCH holds in the model.*

Paul was of course immediately asking if in Miller's theorem $\mathbf{B}(< \aleph_0)$ can be replaced by $\mathbf{B}(k)$ with some $k < \omega$. Let us consider families F of countably infinite sets, such that for any two $A \neq B \in F$, $|A \cap B| \leq n < \omega$. First we proved that *for countable families F, F has property* $\mathbf{B}(n+1)$ *but not necessarily* $\mathbf{B}(n)$. Then much to our surprise, we proved using GCH that, if $|F| = \aleph_k$, $k < \omega$ *then F must have property* $\mathbf{B}((k+1)n+1)$ *but not necessarily* $\mathbf{B}((k+1)n)$. The reason for the surprise was, that these were strong incompactness results saying that there is a family of size $\aleph_{k+1}$ of countable sets not having property $\mathbf{B}((k+1)n+1)$ but every subfamily of size $\aleph_k$ has this property and such incompactness results were not then in the literature (but we already knew of the Hanf-Tarski result by the time we finished the paper). At the end of the paper we gave a long list of incompactness problems for $\aleph_2$ which were later solved by different authors. Eventually, Paul's persistent interest in these problems led to Shelah's celebrated compactness theorem for singular cardinals [S3].

I just state here one of the problems, the fate of which I will describe in §10.9. *Does there exist a graph on $\aleph_2$ vertices having uncountable chromatic number, such that all subgraphs of size $\aleph_1$ are at most $\aleph_0$ chromatic?*

Finally, let me mention that due to the interest of Paul, property **B** had an even bigger career in finite combinatorics. But fortunately, this is not the subject of this note.

11.7. Chromatic Number

In our paper [42] we discovered r-shift graphs. Reference [42] is "Some remarks on set theory IX". Its subject is a general problem involving reals, so I hope it fits into Komjáth's paper. But I must mention that a few years ago, Fremlin and Talagrand obtained some very interesting results solving most of the problems stated there [FT].

The vertices of the κ, r-shift graph $G(\kappa, r)$, $2 \le r < \omega$ are the r-tuples of κ or rather the increasing sequences $\{\alpha_0, \dots, \alpha_{r-1}\}, \alpha_0 < \dots < \alpha_{r-1} < \kappa$ and we join $\{\alpha_0, \alpha_1, \dots, \alpha_{r-1}\}$ and $\{\alpha_1, \alpha_2, \dots, \alpha_r\}$. We proved, as a corollary of Ramsey's theorem or the Erdős-Rado theorem, that these graphs have large chromatic number and that they do not contain odd circuits of length less than $r + 2$.

It was an early result of finite graph theory that there exist graphs having large chromatic number and not containing a K_3 (see [46] for historical references). I think $G(n, 2)$ is the simplest example of this and we were both surprised that this was not known earlier. Paul was always interested in this problem. He proved in 1959 using his probability method that for all $k < \omega$ and $r < \omega$ there are graphs of chromatic number $\ge k$ and of girth $\ge r$ (not containing circuits of length $< r$). He was always interested in infinitary generalizations and in [22] he proved with Rado that, for $\kappa \ge \omega$ there is a K_3-free graph on κ of chromatic number κ.

These results again suggested the wrong generalization, but this time we were not defeated. In [46] we proved that a graph not containing C_4 (a circuit of length 4) has chromatic number at most $\aleph_0$. In fact, we proved a much stronger result. We defined col(G) the coloring number of G as the smallest cardinal κ such that the vertex set of G has a well ordering such that for each vertex x the number of edges having x as the larger element is smaller that κ. This concept was later introduced in finite combinatorics under a different name as well (G is k-degenerate if col$(G) \le k + 1$) [Bo]. Obviously, chr$(G) \le$ col(G) and we proved that *if G does not contain a complete bipartite graph $K_{k,\aleph_1}$, for every $k < \omega$ then* col$(G) \le \aleph_0$. We used the cardinal induction method described in the previous section. Again, the problem arose, what can be said if only larger complete bipartite graphs are excluded? Let me again state a special case of our result. *Assume GCH. If G*

does not contain a $K_{\aleph_0,\aleph_3}$ *then* $\mathrm{chr}(G) \leq \aleph_2$ *and if* G *does not contain a* $K_{\aleph_0,\aleph_2}$ *then* $\mathrm{chr}(G) \leq \aleph_1$ *provided* $|G| \leq \aleph_\omega$.

The situation is analogous to the one described in the previous section. It is consistent that the second clause of the theorem is true for every G e.g. if $V = L$, and it is consistent (relative to a super compact cardinal) that the result strongly fails for $\aleph_{\omega+1}$ i.e. there exists a graph G on $\aleph_{\omega+1}$ of chromatic number $\aleph_2$ not containing a $K_{\aleph_0,\aleph_0}$. This was shown in our paper with Juhász and Shelah mentioned in 10.6. The construction of this example from the one described there is a combinatorial argument, which uses that in the model we have many instances of $\diamondsuit$ (the diamond principle).

I have to mention that we also introduced generalized Specker graphs to show that for $\kappa \geq \omega$ there are graphs of *size* κ, with chromatic number κ having large odd girth.

There are quite a few generalizations of our theorem for $\mathrm{col}(G) > \aleph_0$ but I do not state these here, instead I offer the references [K 1, HK3] and [101]. Let me mention one typical Erdős question: Does $\mathrm{chr}(G) > \aleph_0$ imply that G contains all large odd circuits, say of length $2k+1$ for $k \geq k_0$ for some k_0. Note that this is a typical problem where it is the chromatic number that has to be large as $\mathrm{col}(K_{\aleph_0,\aleph_1}) = \aleph_1$. Later we proved this with Shelah in [79].

Rado asked if the de Bruijn-Erdős compactness theorem for finite chromatic number extends to finite coloring numbers. As the definition of the coloring number involves a well-ordering this can not be expected. Indeed we disproved it, but a surprising result of [46] is that still there is a uniform bound. We proved: *If* $\mathrm{col}(G') \leq k (2 \leq k < \omega)$ *for every finite subgraph* G' *of* G *then* $\mathrm{col}(G) \leq 2k-2$, and there is a countable graph to show that this is best possible for each k.

There is an important finite theorem hidden at the end of [46]. We proved there, using the probabilistic method, that *for every* r*,* s*,* k*, there are* r*-uniform hypergraphs of chromatic number greater than* k *and girth greater than* s. In fact, defined on some n-element set, they do not contain an independent set of size n^{1-d} for some $d > 0$. This fit logically into the line of thought of [46] and it did not occur to us that no finite combinatorialist will look at, much less read, a 40 page paper full of alephs, to find an interesting probabilistic argument on the 35th page.

11.8. Another Miss

We first met Eric Milner in 1958 at the IMC meeting in Edinburgh. He was a former student of Rado and was working in Singapore. Rado interested him in partition problems and he settled one of their problems about countable ordinals [M1]. That was enough to induce Paul to visit and work with him in Singapore in 1960. He returned from there to Budapest with a new interesting problem which I solved and this began a long collaboration between the three of us. The Milners' returned to England in 1961 and Eric joined Rado at Reading. On my way back from Berkeley to Budapest in 1965, I stayed in Reading for a month with them discussing a long half-finished manuscript. Although our long joint papers only appeared a few years later, in 1965 we were already deeply involved in our joint work and we thought it would probably help if all three of us could be together at the same place and at the same time. So it was arranged that Eric should visit us during the summer of 1965 to spend a week at the summer house of the Writer's Union in Szigliget on Lake Balaton. Eric arrived with an interesting question about transversals, and as a consequence, instead of regularly working on manuscripts, we wrote another shorter paper [56] which became our first joint work to appear. As a side issue in that paper we proved the following theorem: *Let* $\lambda > \mathrm{cf}(\lambda) = \kappa > \omega$, *and let* $\lambda_\alpha(\alpha < \kappa)$ *be an increasing continuous sequence of cardinals cofinal in* λ, *and assume that* $\tau^\kappa < \lambda$ *for* $\tau < \lambda$. *If* S *is a stationary subset of* κ *and* $\mathcal{F} \subset \Pi_{\alpha \in S}\lambda_\alpha$ *is an almost disjoint set of transversals (i.e.* $|\{\alpha \in S : f(\alpha) = g(\alpha)\}| < \kappa for f \neq g \in \mathcal{F}$*), then* $|\mathcal{F}| \leq \lambda$.

Eric made notes of our results and wrote it up and the pap er appeared in 1968. We forgot the whole thing, and the paper seems to have gone unnoticed. Even in 1967 when we wrote the first problems paper with Paul, where our intention was to write down all our interesting problems, we omitted any mention of this. However, it seems we were not the only blind ones. During the summer of 1971 Adrian Matthias organized a large conference on set theory in Cambridge, England. Karel Prikry was one of the invited speakers and he gave a talk on a generalization of Jensen's work on Kurepa families. He discovered the following result: *Under the assumptions of our theorem, if* $\mathcal{H} \subseteq \mathbb{P}(\lambda)$ *is a Kurepa family in the sense that* $|\mathcal{H}|\lambda_\alpha| \leq \lambda_\alpha$ *for* $\alpha \in S$ *(*S *a stationary subset of* κ*), then* $|\mathcal{H} \leq \lambda$. $(\mathcal{H}|\lambda_\alpha = \{H \cap \lambda_\alpha : H \in \mathcal{H}\}.)$

He told me this result the day before his lecture and it sounded vaguely familiar. But it took me the whole day to realize that this was just our earlier theorem applied to the sets $\mathcal{H}|\lambda_\alpha$ in place of λ_α. I managed to get a copy of our paper to give to Prikry before the lecture. Now there were about one hundred set-theorists in attendance, including all the leading ones, when Karel stated our result in a totally digestible form. But nobody asked, what happens if we replaced λ_α by λ_α^+? I suppose the psychological barrier was too strong. In 1974, just before the ICM in Vancouver, I was visiting Eric again in Calgary (he moved there in 1967), when I received a preprint of Silver 's ingenious discovery that: *if* $\lambda > \mathrm{cf}(\lambda) > \omega$ *and if* $2^\tau = \tau^+$ *on a stationary set of cardinals* $\tau < \lambda$, *then* $2^\lambda = \lambda^+$. At the same time I received a preprint from Prikry giving a combinatorial proof of Silver 's result. Prikry told me that the instant he saw Silver 's manuscript it dawned on him that the only thing needed was to lift our old result with λ_α^+ in place of λ_α. Of course this requires a non-trivial argument. Baumgartner and Jensen also found elementary proofs of Silver 's result without remembering our theorem. But the real miss, and so uncharacteristic of Paul, was not to have asked the question!

11.9. Incompactness for the Chromatic Number

In 1966, assuming CH, we solved the problem on chromatic numbers stated in §10.6. We proved in [54] that there is a graph of chromatic number at least $\aleph_1$ on $(2^{\aleph_0})^+$ vertices all of whose subgraphs of cardinality at most $2^{\aleph_0}$ have chromatic number at most $\aleph_0$. This also comes with a story and some advice. During a working session at Paul's apartment, we were talking about something totally unrelated to chromatic number and compactness. In the middle of an attempted proof, we found that the pairs of $\mathbb{R}$ are colored with countably many colors and our proof would be finished if there was a monochromatic increasing path of length 2, IP_2, i.e. a triple $x_1 < x_2 < x_3$ with $\{x_1, x_2\}$ and $\{x_2,\ x_3\}$ having the same color. Unfortunately there was not, and I tried to get another proof. But Paul started to insist that we should know for what order types θ,

$$\theta \to (IP_2)^2_\omega$$

holds. We parted unsuccessful in both attempts. But on the way home, I could not help thinking about his question. I remembered an old idea of Sierpiński which easily implied that there is a $\not\to$ for every θ of cardinality $|\theta| \le 2^{\aleph_0}$. Then I saw in a flash that this just says that all subgraphs of size $2^{\aleph_0}$ of our shift graph $G((2^{\aleph_0})^+, 2)$ have chromatic number $\le \aleph_0$. As this was a 100 dollar problem, I immediately called Paul when I arrived home. (I think eventually I got only \$50 for it, but with some reason). The advice is this: just answer his questions, you have time later to ponder if it is important or not.

This was the status of the problem when we published it in [67]. Let me tell some later developments. First Jim Baumgartner proved with a forcing argument that it is consistent that *GCH holds and there is an* $\aleph_2$*-chromatic graph on* $\aleph_2$ *vertices, such that the chromatic number of every subgraph of size* $\aleph_1$ *is at most* $\aleph_0$ [B1].

M. Foreman and R. Laver proved that: *it is consistent with GCH relative to a large cardinal that every graph on* $\aleph_2$*, all of whose* $\aleph_1$ *subgraphs are* $\aleph_0$*-chromatic is at most* $\aleph_1$*-chromatic* [FL].

Finally, Shelah proved, after improving his own results several times, that it is consistent (true in L) that: *for every regular non-weakly compact* κ*, there is a* κ*-chromatic graph on* κ *all of whose subgraphs of size less that* κ *are* $\aleph_0$*-chromatic* [S4].

We also invented an interesting graph in [54]. Let $C(\omega_2, \omega)$ be a graph whose vertices are the elements of ${}^{\omega_2}\omega$, i.e., ω_2-sequence of integers and we join two sequences if they are eventually different. We proved, and these are obvious facts, *that every subgraph of size* $\aleph_1$ *of* $C(\omega_2, \omega)$ *has chromatic number* $\le \aleph_0$*, and moreover, that every graph of size* $\aleph_2$ *having this property embeds into* $C(\omega_2, \omega)$. However, the chromatic number of $C(\omega_2, \omega)$ in ZFC is a mystery. Our result implies that CH$\Rightarrow$ $\mathrm{chr}(C(\omega_2, \omega)) \le \aleph_1$ and Péter Komjáth proved this from the weaker assumption $2^{\aleph_0} \le \aleph_2$, and he proved it consistent with GCH that $\text{chr}(C(\omega_2, \omega)) = \aleph_3$ [K 2]. On the other hand Foreman proved it is consistent relative to a large cardinal that $\text{chr}(C(\omega_2, \omega)) \leq \aleph_1$ [Fo]. It seems to be out of the question with the present methods to prove that our graph is consistently $\aleph_0$-chromatic.

11.10. Decomposition of Graphs

I just want to mention our paper [53] which appeared in 1967 but was written about 2 years earlier. In this paper we raised problems of the following type. Does there exist graphs G not containing a K_λ, for some cardinal λ, such that for every vertex partition or edge partition with few colors say a monochromatic K_τ appears. In present notation: For what λ, τ, γ is there a K_λ-free graph G such that

$$G \to (K_\tau)^1_\gamma \text{ or } G \to (K_\tau)^2_\gamma$$

holds? We had some results but I just want to restate two edge partition problems from the paper, one of them finite.

Does there exist a finite K_4-free G with $G \to (K_3)^2_2$? This was solved by Folkmann [Fol] affirmatively, but the question became one of the starting points of structural Ramsey theory (see §15).

The infinitary problem is the following. Does there exist a K_4-free graph G of cardinality $(2^{\aleph_0})^+$ such that $G \to (K_3)^2_\omega$ holds? As far as I remember, this was the last set theory problem Paul offered a prize for (it was worth \$250.) Shelah later proved this to be consistent, but I will speak about the status of this kind of problem in a more general context in §15.

12. Δ-Systems and More Set Mappings

Δ-Systems were introduced in a paper of Erdős and Rado which appeared in 1960 [26]. A family F of sets is a Δ-system if there is a set D, the kernel of F, such that $A \cap B = D$ for all $A \neq B \in F$. The paper set the task to determine $\Delta(\kappa, \lambda) = \delta$ the smallest cardinal for which every family F, of sets of size κ and cardinality δ contains a Δ-system of size $\lambda \geq 3$. As it is well known, for finite κ, the problem is still unsolved. A \$1,000 reward is offered by Paul for the proof or disproof of the conjecture that for some $c > 0$

$$\Delta(\kappa, 3) < c^\kappa \text{ for } \kappa < \omega$$

However, for $\kappa \geq \omega$, Erdős and Rado settled the problem completely. Although some of the details were only cleared up in their second paper [60] on the subject, the main upper bounds were already obtained in [26]. One of the main results says that *if $\kappa < \delta = \text{cf}(\delta)$, $|F| = \delta \geq \omega$ and δ is inaccessible from κ, i.e. $\sigma^\kappa < \delta$ for $\sigma < \delta$ then F contains a Δ-system of size δ*. This is probably the most frequently used theorem of set theory, since it is the simplest tool to prove that certain partially ordered sets satisfy certain chain conditions. If $\langle P, \preceq\rangle$ is a partially ordered set $p, q \in P$ are *incompatible* if there is no $r \in P$ with $r \preceq p, q$ and P satisfies the κ-*chain condition* if every subset of pairwise incompatible elements has cardinality $\leq \kappa$. It was already an important element of Cohen's proof of the independence of the continuum hypothesis, that finite 0, 1 sequences from any index set, ordered by reverse inclusion satisfy the $\aleph_0$-chain condition. Cohen and his early followers did not know the Erdős-Rado theorem and they proved it for the special cases they needed. But soon it was discovered by logicians, and it is invoked almost any time forcing is used.

There is another theorem of Erdős and Specker [30], I should have mentioned in §5, which is used almost as often in forcing arguments to establish chain conditions as the Δ-system theorem. Assume $f : \kappa \to \mathbb{P}(\kappa)$ is an ordinary set mapping. In §5 we saw that, if If $|f(x)| < \tau < \kappa$ for some cardinal $\tau < \kappa$ then there is a free set of size κ and κ is the union of τ free sets. Now if κ is a successor cardinal λ^+ then the assumption that the type $\mathrm{tp}(f(x)) < \xi < \kappa^+$ for some fixed $\xi < \lambda^+$ is weaker than the assumption that $|f(x)| < \tau$ for some cardinal $\tau < \kappa$ but by the Erdős-Specker theorem, this still implies the existence of a free set of size κ; however, Fodor's theorem does not apply since the graph induced by f can be λ^+-chromatic. Most of the time that we want to construct or force an object on λ^+ such that each subset of size λ^+ contains a subset of size λ^+ of certain kind, but the whole set is not the union of λ sets of this kind, then the Erdős-Specker result is the first thing to remember.

13. The Unsolved Problems in Set Theory [67]

In 1967 the first major post Cohen conference was held at UCLA. By that time, Cohen's method was generally known and developed, and the aim of the conference was to bring together all experts of set theory and to collect and make public all the fantastic new results available. We were both invited, Paul was there, but I could not make it. (It was the only time I did not get a passport from the Hungarian authorities.) The organizers convinced Paul that instead of mentioning a few interesting problems as the spirit moves him, he should write up all the difficult problems he came across during his work in combinatorial style set theory. He immediately promised that *we* will do it in a joint paper. This time we worked hard and fast. A mimeographed version of the manuscript containing 82 problems (or groups of problems really) was ready in the same year and we sent a copy to everyone we knew and who we thought would be interested. It included all the problems I mentioned in the previous sections and quite a few more. A large number of (*then*) young mathematicians started to work on these, and produced solutions either by applying the newly developed methods of independence proofs or simply divising new combinatorial methods. The paper only appeared 4 years later in 1971, and by that time the status of most of the problems had changed. We tried to keep the manuscript up to date by adding remarks, but in 1971 we decided to write a second problem paper [81] which contained the status of the problems up until that time.

It would clearly be impossible to write a similar survey today. In the previous sections, I tried to show on selected topics how the Erdős problems generated new questions and results and how they became integral parts of modern set theory, and how many of them are still alive. In this section I can only mention the status of a few more which I omitted earlier.

I did not finish the story of set mappings of type $< \omega$. Shortly after our problem paper was distributed Jim Baumgartner proved in his thesis [B2] that if $V = L$ then $\mathrm{Free}(\kappa, 2, <\omega, \aleph_0)$ is equivalent to $\kappa \to (\aleph_0)_2^{<\omega}$ but on the other hand, it is still open if it is consistent relative to a large cardinal that $\mathrm{Free}(\aleph_\omega, 2, <\omega, \aleph_0)$ holds, or more strongly, there is no Jónsson algebra on $\aleph_\omega$. As I already mentioned, $\mathrm{Free}(\aleph_\omega, 2, <\omega, \aleph_0)$ was our first joint problem. We already suspected at Kalmár's supper that it will be hard, but probably not quite as hard as it turned out to be.

I am afraid I have mentioned too many problems which led to independence results, so here is a difficult theorem of Shelah and Stanley solving one of our problems in ZFC:

$$(2^{\aleph_0})^+.\omega \to ((2^{\aleph_0})^+\omega, n)^2$$

for $n < \omega$ [SS2]. It is another matter that they also proved $\omega_3.\omega_1 \to (\omega_3.\omega_1, 3)^2$ to be independent of ZFC and GCH.

Erdős proved with Alaoglu in 1950 in [5] that if κ is smaller than the first weakly inaccessible cardinal greater than $\aleph_0$, then one can not have $\aleph_0$ σ-additive 0, 1 measures so that every subset of S is measurable with respect to one of them. Erdős attributes the question to Stanislaw Ulam but he got the first result. We asked if $\aleph_0$ can be replaced by $\aleph_1$ here? Prikry proved it to be consistent, but this question became the forerunner of so many questions in the theory of large cardinals that I do not dare to write about later developments in detail.

Instead, here are some evergreen problems from the theory of ordinary partition relations for ordinals.

1. $\omega^\omega \to (\omega^\omega, 3)^2$ was proved by C.C. Chang [C] and $\omega^\omega \to (\omega^\omega, n)^2 n < \omega$ was proved by E.C. Milner [M2] and independently by Jean Larson [Lar]. But $\omega^{\omega^2} \to (\omega^{\omega^2}, 3)^2$ or $\omega^{\omega^\alpha} \to (\omega^{\omega^\alpha}, 3)^2$ seem to be as hard as ever. Here of course α^β means ordinal exponentiation.
2. Does there exist an α with $\alpha \to (\alpha, 3)^2$ such that $\alpha \nrightarrow (\alpha, 4)^2$?
3. I proved with Jim Baumgartner in 1970 [BH] that $\Phi \to (\omega)^1_\omega \Rightarrow \forall_\alpha < \omega_1 \forall k < \omega \Phi \to (\alpha)^2_k$. But for exponents >2 very little is known. For example, $\omega_1 \to (\alpha, 4)^3$ is still open for $\alpha < \omega_1$. The world record is presently held by Milner and Prikry; they proved this for $\alpha \leq \omega.2+1$. See [MP].

4. Is $\omega_2 \to (\alpha)^2_2$ for $\alpha < \omega_2$ consistent with GCH? I proved the consistency of $\omega_2 \not\to (\omega_1+\omega)^2_2$ and it follows from the existence of Laver's ideal mentioned in §10.5 that $\omega_2 \to (\omega_1.2)^2_2$.
5. It follows from a recent result of Baumgartner, myself and Todorčević [BHT] that GCH⇒ $\omega_3 \to (\omega_2+\xi)^2_k$ for $\xi < \omega_1$ and $k < \omega$ but $\omega_3 \to (\omega_2+2)^2_\omega$ is still open. See [BHT] for many new problems arising from our results.

14. Paradoxical Decompositions

Erdős has 12 major joint papers with Eric Milner, nine of those were written by the three of us. These are from a later period so the results and problems are more technical than the ones I described earlier, it is out of question to give a list of them. I want to speak about one idea which features in quite a few of them.

It was always clear that Ramsey's theorem is a generalization of the pigeonhole principle of Dedekind. When partition relations $\kappa \to (\lambda_\nu)^r_{\nu<\gamma}$ were formally introduced, it became apparent that the pigeonhole principle is just a partition relation for cardinals with exponent $r = 1$. For example, $nk+1 \to (n+1)^1_k$ for the finite case with k boxes, and $\aleph_0 \to (\aleph_0)^1_k$ for $k < \omega$, and more generally, $\kappa \to (\kappa)^1_\lambda$ for $\lambda < \mathrm{cf}(\kappa)$, $\kappa \geq \omega$. It was discovered by Milner and Rado in [MR] which appeared in 1965 that the pigeonhole principle does not work the same way for ordinals. They proved that for any $\kappa \geq \omega$

(i) $\xi \not\to (\kappa^n)^1_{n<\omega}$ if $\xi < \kappa^+$ and as a corollary of this $\xi \not\to (\kappa^\omega)^1_\omega$ for $\xi < \kappa^+$.

Here again α^β denotes ordinal exponentiation. This phenomena, often called the *Milner-Rado paradox*, has to be kept in mind, just because it is so contrary to one's first intuition. When partition relations proliferated it was discovered that this (as almost anything) can be written as a polarized partition relation:

(ii) $\begin{pmatrix}\omega\\ \xi\end{pmatrix} \not\to \begin{pmatrix}1 & \omega\\ \kappa^\omega, & 1\end{pmatrix}^{1,1}$ for $\xi < \kappa^+$

and also as a square bracket relation:

(iii) $\xi \not\to [\kappa^\omega]^1_{\aleph_0,<\aleph_0}$ for $\xi < \kappa^+$. In [63] we have investigated the polarized partition relation

$$\begin{pmatrix}\kappa\\ \xi\end{pmatrix} \not\to \begin{pmatrix}1 & \delta\\ \sigma, & \tau\end{pmatrix}^{1,1} \text{ for } \kappa = \omega \text{ and } \kappa = \omega_1,\ \xi < \omega_2.$$

We gave a complete discussion, relying heavily on the form (iii) of the paradox in the case $\kappa = \omega_1$, i.e.

(iv) $\xi \not\to [\omega_1^\omega]^1_{\aleph_0,<\aleph_0}$ for $\xi < \omega_2$.

When we tried to lift our results to higher cardinals we realized that we would need to generalize (iv) to

(v) $\xi \not\to [\kappa_2^{\omega_1}]^1_{\aleph_1,\aleph_0}$ for $\xi < \omega_3$.

We already discovered in 1967, that this will not be possible in ZFC, but we only wrote down the results which we called the $\aleph_2$-phenomenon in our 1978 paper [93] relying heavily on other people's results. See [93] for references.

Since this is not so well known, I will write down the $\aleph_2$-phenomenon as it relates to (v).

A.1 $\xi \not\to [\kappa_2^{\omega_1}]^1_{\aleph_0,\aleph_0}$ *holds for* $\xi < \omega_2^{\omega_2}$

A.2 *If* $2^{\aleph_1} = \aleph_2$ *then for some* $\xi_0 < \omega_3$, $\xi_0 \to [\omega_2^{\omega_1}]^1_{\aleph_1,\aleph_0}$

A.3 *It is consistent with* $2^{\aleph_1} = \aleph_3$ *that* $\xi \to [\omega_2^{\omega_1}]^1_{\aleph_1,\aleph_0}$ *holds for* $\xi < \omega_3$

A.4 $\omega_2^{\omega_2} \to [\omega_2^{\omega}]^1_{\aleph_1,\aleph_0}$ *and* $\omega_2^{\omega_2} \not\to [\omega_2^{\omega}]^1_{\aleph_1,\aleph_0}$ *are both consistent with ZFC and GCH.* (The $\not\to$ holds e.g. in L while the $\to$ follows from Chang's conjecture.)

All this happens because a counterexample establishing the $\not\to$ is really a sequence $\{A^\xi_\alpha : \alpha < \omega_1\} \subset \xi$ such that the order type $\text{tp}(\bigcup_{\beta<\alpha}, A_\beta) < \omega_2^{f_\xi(\alpha)+1}$ for a function $f_\xi : \omega_1 \to \omega_1$ and, for $\zeta < \xi$, f_ζ must be smaller than f_ξ in some well known ordering of these functions. In fact, this was the reason why we asked all the problems 19A–19E in the unsolved problems paper, about the relation of the transversal hypothesis and the Kurepa hypothesis.

Problem 19D was slightly out of the line there. Typically, Paul asked something that was quite new: are there $2^{\aleph_1}$ almost disjoint, stationary subsets of ω_1? It is easy to see the consistency of a 'yes' answer, it is true e.g. in L, however the consistency of a 'no' answer with CH is not completely proved. Foreman, Magidor and Shelah proved in [FMS] that 'no' follows from a consistent set-theoretical principle called Martin's Maximum (MM), but MM implies that $2^{\aleph_0} = \aleph_2$. They also proved it consistent with CH that there is a stationary subset of ω_1 on which the nonstationary ideal is $\aleph_2$-saturated. All these very difficult consistency proofs of course are relative to the existence of some large cardinals.

15. A Mistake and Its Consequences

In §12 of our paper [46] about chromatic numbers, we claimed a false theorem. I just state a special case. Let $\mathcal{H} = (h, H)$ be a 3-uniform hypergraph (i.e. $H \subset [h]^3$) such that every pair $e \in [h]^2$ is in at most countably many elements of H. Then we claimed that the chromatic number of $\mathcal{H}$ is at most $\aleph_0$.

As we know now, this is true if $|h| \leq \aleph_1$ and false for a triple system of cardinality $(2^{\aleph_0})^+$. Now I have to disclose a not so surprising secret. Paul actually wrote up some of our joint papers, but these were the short ones. For the long ones it was my job to prepare the manuscript, but we always read the manuscript and even the proof sheets together. The trouble was that he often got bored with mechanical work like this, and he made up new conjectures and theorems and insisted that we should include them by adding remarks—even to the galley proofs. Taking the responsibility, I think I was the one who overlooked that the cardinal induction method breaks down from $\aleph_1$ to $\aleph_2$ in this case. Anyway, if the theorem was really true, the whole structure of the paper should have been changed but fortunately we did not have time for that.

As usual, I forgot the theorem, but Paul did not. I got a phone call from him from abroad about 4 years after the paper had appeared. He was trying to tell the proof of it to Bruce Rothschild, and got stuck. They soon discovered a counter-example. Let $(2^{\aleph_0})^+ = \kappa$, $h = [\kappa]^2$, $H = \{\{\{\alpha,\beta\},\{\beta,\gamma\},\{\alpha,\gamma\}\} : \alpha < \beta < \gamma < \kappa\}$, Clearly, any two elements of H have at most one element in common, and the chromatic number is at least $\aleph_1$ by the Erdős-Rado theorem. We wrote a triple paper [76] about it. However, Paul got interested in this question: *what kind of finite triple systems must appear in an* $\aleph_1$-*chromatic triple system*? I think the first question was the 6/3, i.e., are there three triples with empty intersection such that each pair has exactly one point in common? (This question for triple systems made quite a splash in finite combinatorics as well.) Fred Galvin came up with a negative answer. Later Fred spent the academic year 1972–1973 in Budapest, and the three of us started to work on this problem and we asked the same question for triple systems not containing large independent sets. Unfortunately, we did not find a general answer, maybe there isn't one, every time that we constructed a large chromatic system avoiding concrete finite systems, Paul ingeniously invented new ones for which the construction did not work. The motivation behind this was the following. Clearly there is a cardinal κ with the following property. *If for a finite triple system* $\mathcal{H}$ *there is a* $(> \aleph_0)$-*chromatic triple system* $\mathcal{K}$ *not containing* $\mathcal{H}$, *then there is one of cardinality at most* κ. Cardinals which satisfy a condition like this are quite often impossible to determine, and such was the case with this problem. For two triples with a common edge the number we found is $(2^{\aleph_0})^+$. With GCH all our examples had cardinality $\aleph_2$ associated with them. We ended up with a concisely written paper almost 90 pages long [85] containing some really good theorems, but which remained relatively unknown. Again, it is not possible to give a list of the results, but I do want to mention one concept and problem from the paper that I really like.

We constructed large chromatic r-uniform hypergraphs by induction on r, and to support the induction from r to $r+1$, we needed the r-tuple system $\mathcal{H}$, to have a stronger property than $\mathrm{chr}(\mathcal{H}) > \aleph_0$.

Let $\mathcal{H}_\nu = \langle h, H_\nu \rangle (\nu < \varphi)$, be a system of r-uniform hypergraphs on the same vertex set h. The system has *simultaneous chromatic number* $> \aleph_0$ if, for every partition of the vertex set $h = \bigcup_{n<\omega} h_n$ into $\aleph_0$ parts, there is an $n < \omega$ such that h_n contains "edges" from each $\mathcal{H}_\nu$ for $\nu < \varphi$.

We say that a $(>\aleph_0)$-chromatic $\mathcal{H} = \langle h, H\rangle$ *splits to* δ *parts*, if there is a disjoint partition $H = \bigcup_{\nu<\delta} H_\nu$ so that the system $\mathcal{H}_\nu = \langle h, H_\nu\rangle (\nu < \delta)$ has simultaneous chromatic number $> \aleph_0$. We proved that quite a few known $(>\aleph_0)$-chromatic graphs split to $\aleph_1$ parts and these served as a basis of our induction process.

In those days, before Todorčević's result, we only knew with CH that $K_{\aleph_1}$ splits to $\aleph_1$-parts. Still, as we did not find anything that does not split, we asked the question: *Is it true that every* $(>\aleph_0)$*-chromatic graph splits to two (or* $\aleph_1$*) parts?*

This problem as it stands is still unsolved. With Péter Komjáth I have some unpublished partial results. Here are two of them.

(1) *It is consistent that every* $\aleph_1$*-chromatic graph splits into* $\aleph_1$ *parts.*
(2) *It is consistent relative to a measurable cardinal, that there is a* $(>\aleph_0)$*-chromatic graph which does not split into* $\aleph_1$ *parts.* (We do not know this for two parts.)

16. Structural Ramsey Theory

As I already mentioned in §10.10, we asked the first questions of the following type: Does there exist a K_4-free graph G such that $G \to (K_3)^2_2$.

The following type of generalization appeared first in Deuber's paper [D]. Let G, H be graphs; H embeds into G if G has an induced subgraph isomorphic to H. With present day partition calculus notation, we say $G \rightarrowtail (H)^2_{\kappa,\lambda}$ if for arbitrary colorings $k : G \to \kappa, \ell : [g]^2 \setminus G \to \lambda$ of the edges of G with κ colors and non-edges with λ-colors, there is an induced subgraph $H' \subset G$ isomorphic to H such that k and ℓ are constant on the edges and on the non-edges of H' respectively.

Deuber proved that *for all finite* H *and* $k < \omega$*, there is a finite* G *with* $G >\!\!\rightarrowtail (H)^2_{k,k}$ and the combination of the two types of questions Paul raised became the starting points of the Nešetřil-Rödl type structural Ramsey theory.

With Erdős and Pósa we proved the first infinitary result of this kind [88]. The paper appeared in the volume of the Keszthely conference held for Paul's 60th birthday in 1973, and this volume contains the first Něsetřil Rödl paper on the subject. The finitary theory developed very fast. The problem was generalized for coloring of substructures of a fixed kind instead of coloring pairs, but fortunately I do not have to give an account of this. I just want to say that this was not done in the infinitary case because here some basic problems are still open.

In the paper with Erdős and Pósa we proved that *for every countable* H *and* $k < \omega$ *there is a* G $(|G| = 2^{\aleph_0})$ *such that* $G \rightarrowtail (H)^2_{k,k}$ and asked if this holds true for countably many colors, or for larger H.

We discovered a decade later with Péter Komjáth that *it is consistent to have* $|H| = \omega_1$ *and* $G \not\rightarrowtail (H)^2_{2,1}$ *for every* G [HK2] and Shelah proved that *it is consistent that for all* H *and* γ *there is a* G *with* $G \rightarrowtail (H)^2_{\gamma,\gamma}$ [S6].

In 1989, [H 2] I proved *in ZFC that for all finite* H *and arbitrary* γ *there is a* G *with* $G \rightarrowtail (H)^2_{\gamma,\gamma}$ but the problem of countable H and countable γ is open (though the $\rightarrowtail$ is consistent by Shelah's result). The answer may turn out to be to Paul's liking (a theorem in ZFC) but I am sure it will be very difficult.

Shelah generalized his consistency results for K_r-free H as well, but at this point I feel I have to stop and refer the reader to a recent survey paper of mine on this subject [H 3].

17. Applications of Partition Relations in Set Theoretical Topology

In the last 30 years, set theoretical topology became a major area of research as shown e.g. in the Handbook of Set Theoretical Topology. The reason for this is that the new methods of set theory (forcing, large cardinals) made it possible to study topological spaces for what they are, namely set theoretical objects. The point I want to make is that, although Erdős did not take an active part in most of this, combinatorial set theory which he created is one of the major tools in this development.

This happens not just through the applications of positive theorems. There are of course some famous ones. Being closest to the fire, with István Juhász we showed, for example, as a consequence of $(2^{2^{\aleph_0}})^+ \to (\aleph_1)^2_4$ that, *every Hausdorff space of cardinality* $(2^{2^{\aleph_0}})^+$ *has discrete subspaces of size* $\aleph_1$. Also, as a consequence of the canonization theorem of §10 that, *the spread (the supremum of the sizes of discrete subspaces) is attained in a Hausdorff space if this supremum is a singular strong limit cardinal.*

More importantly, there are literally dozens and dozens of examples obtained as strengthenings of negative partition relations which would never have turned up in their present form without a detailed analysis of these relations. Let me try to make this clear with an example. I already mentioned Prikry 's consistency proof of

$$\binom{\aleph_2}{\aleph_1} \not\to \binom{\aleph_0}{\aleph_1}^{1,1}_2 .$$

To state it in "human language" (but already a little twisted for my purposes), it means that there is a sequence $\{f_\alpha : \alpha < \omega_2\} \subset^{\omega_1} 2$ such that for all countable $I \in [\omega_2]\aleph_0$ there is a $\nu(I) < \omega_1$ such that for $\nu(I) < \nu < \omega_1$ there are $\alpha_0, \alpha_1 \in I$ with $f_{\alpha_0}(\nu) = 0$ and $f_{\alpha_1}(\nu) = 1$.

When in [HJ] with Juhász we discovered HFD's (hereditarily finally dense sets) and proved the consistency of the existence of a hereditarily separable space of power $\aleph_2$ (assuming $2^{\aleph_0} = \aleph_1$) we only had to change the last clause of the above statement: there is a $\nu(I) < \omega_1$ such that for every finite set F with $\nu(I) < F < \omega_1$ and every 0, I-function ϵ defined on F there is an $\alpha \in I$ such that for all $\nu \in F$, $f_\alpha(\nu) = \epsilon(\nu)$. And now $\{f_\alpha : \alpha < \omega_2\}$ is a hereditarily separable subspace of cardinality $\aleph_2$ of $D(2)^{\omega_1}$.

18. A Final Apology

I feel that I should stop at this point. One reason is that this is the 100th page of my handwritten manuscript, but there are other reasons. Paul has continued to work on set theory, stating new and old problems in the numerous problem papers he published. Our last major set theory paper with Jean Larson [109] appeared in 1993. It would not really be appropriate for me to speculate on the reactions that these latest problems may provoke, for we lack the perspective. It is also true, that his interest in set theory is slightly diminished, he does not like the technical problems which already in the assumptions involve consistency results. But he triumphantly continues to carry the flag of Georg Cantor.

I also have some doubts about my manuscript. It is as if I have been trying to sketch a rain forest, but with only enough time and ability to draw the trunks of what I thought to be the largest trees. Paul's real strength is in the great variety of those hundreds of small questions which he has asked that have given some real insights into so many different topics. I can only admire his inventiveness and thank him for everything he has given us.

Finally, I also wish to thank our old friend Eric Milner for helping me to prepare this paper.

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Set Theory: Geometric and Real

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1. Introduction

In this Chapter we consider P. Erdős' research on what can be called as the borderlines of set theory with some of the more classical branches of mathematics as geometry and real analysis. His continuing interest in these topics arose from the world view in which the prime examples of sets are those which are subsets of some Euclidean spaces. 'Abstract' sets of arbitrary cardinality are of course equally existing. Paul only uses his favorite game for inventing new problems; having solved some problems find new ones by adding and/or deleting some structure on the sets currently under research. A good example is the one about set mappings. This topic was initiated by P. Turán who asked if a finite set $f(x)$ is associated to every point x of the real line does there necessarily exist an infinite free set, i.e., when $x \notin f(y)$ holds for any two distinct elements. Clearly the underlying structure has nothing to do with the question and eventually a nice theory emerged which culminated in the results of Erdős, G. Fodor, and A. Hajnal. But Paul and his collaborators kept returning to the original setup when the condition is e.g. changed to: let $f(x)$ be nowhere dense, etc. Several nice and hard results have recently been proved. (See Sect. 8 in this Chapter.)

But this is not Paul's secret yet. He has the gift to ask the right questions and-always-ask the right person.

2. Sierpińiski's Decomposition

One direction of theorems and problems originated from W. Sierpińiski's famous decomposition theorem [34]. This paradoxical statement says that CH (the continuum hypothesis) holds if and only if $\mathbf{R}^2$ can be decomposed as $A \cup B$ where A, B have countable intersection with every horizontal, resp. vertical line. (See [14] for a lucid description of Paul Erdős' astonishment when first hearing on this result.) This shows that the generalization of Fubini's theorem $\int\int f(x,y)\,dx\,dy = \int\int f(x,y)\,dy\,dx$ may be false if we only

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assume the existence of the two double integrals and not the measurability of f (take the characteristic function of A). H. Friedman showed that it can also be true (if one adds $\aleph_2$ random reals) (see [21]). Later, C. Freiling [20] proved that if there is a counter-example to this statement then necessarily there is one of the Sierpińiski type, i.e., then $\mathbf{R}^2 = A \cup B$ such that the intersection of A (or B) with every horizontal (vertical) line is a measure zero set. See [36] which is a very thorough survey paper on this result and several variants. Erdős [11] generalizes this as follows. Suppose the lines of $\mathbf{R}^2$ are decomposed into two classes (and CH holds, of course). Then a decomposition as above exists with $A \cap \ell$ countable if ℓ is in the first class, and $B \cap \ell$ is countable if ℓ is in the second class.

3. Linear Equations

An old result of R. Rado states that every $\mathbf{R}^n$ in fact every vector space over $\mathbf{Q}$, the rationals, is the union of countably many pieces none containing a 3-element arithmetic progression. This observation initiated several different research areas. In [18] Erdős and Kakutani showed that every vector space of cardinal at most $\aleph_1$ is the union of countably many bases and this is sharp. Various other results hold if one want to cover a linear space with countably many sets omitting the solutions of some linear equations (and not all as in the above case) [27].

One way of proving the above mentioned Erdős-Kakutani result is via a combinatorial lemma of Erdős and Hajnal which also gives that if the continuum hypothesis fails and $\mathbf{R}$ is decomposed into countably many pieces then one of them contains $a, a+x, b, b+x$ for some a, b, x, $x \neq 0$. Erdős then asked if the following complementary result holds. If CH holds then $\mathbf{R}^n$ can be written as the union of countably many sets none containing the same distance twice. For $n = 1$ this is a very special case of the quoted Erdős-Kakutani result. For $n = 2$ this was established by R. O. Davies [7] and finally for the general case by K. Kunen [29].

4. Euclidean Ramsey Theory

Perhaps the central topic in our discussion is the one that may be called Euclidean Ramsey theory. This deals with decompositions of $\mathbf{R}^n$ where a configuration similar to a given one is excluded. Though in the finite version, i.e., when the number of classes is finite, there are several positive results (given by P. Erdős, R.L. Graham, P. Frankl, V. Rödl, and others), the infinite case abounds in counterexamples.

To begin with, if the excluded configuration is infinite then there is always a decomposition of $\mathbf{R}^n$ into two classes with no copy of the configuration in one class. This nice observation which uses an earlier result of P. Erdős and A. Hajnal is well hidden in [15]. As for the case of of finite configurations

J. Ceder already in 1969 proved that $\mathbf{R}^2$ can be colored with countably many colors without a monocolored equilateral triangle [4]. The tricky proof uses that $\mathbf{C}$, the complex plane is a vector space over the countable field $\mathbf{Q}(\sqrt{-3})$. In this vector space any node of an equilateral triangle is a convex linear combination of the other two. The proof now goes like R. Rado's quoted proof. It "defines" the coloring of a vector from its coordinates over a Hamel-basis.

P. Erdős asked if Ceder's result holds for $\mathbf{R}^n$, as well. Obviously, the above proof won't generalize. The more classic inductive argument, in the spirit of Davies' results, makes possible to show that a similar statement holds for regular tetrahedra in $\mathbf{R}^3$, see [23]. For regular simplices in $\mathbf{R}^n$ this was shown by J. Schmerl [31]. Then he extended his result to equilateral triangles [32]. Finally, adding some really nice and deep arguments from higher dimensional calculus and logic J. Schmerl was able to show that P. Erdős's conjecture is true, there is a countable decomposition of every $\mathbf{R}^n$ with no isosceles triangles in one part [33].

One might think that excluding right triangles is similar to the case of the isosceles triangles. In fact, it was shown in [19] that CH is actually equivalent to the existence of a countable decomposition of $\mathbf{R}^2$ not containing the three nodes of of a right triangle in one piece. Interestingly enough the 'only if' part is an easy corollary of an above quoted Erdős-Hajnal result, the 'if' part is rather complicated.

Another variant, also raised by P. Erdős, if one can exclude those triangles with rational, nonzero areas. Clearly, we cannot multicolor triangles with zero area, and the statement is obvious for one given value of area. In unpublished work K. Kunen showed that if CH holds then there is a decomposition of $\mathbf{R}^2$ omitting rational areas as asked. In [28] we extended this to a broad class of configurations namely we showed that there is, under CH, a count able decomposition of $\mathbf{R}^n$ with no different points $\overline{a_1}, \ldots \overline{a_t}$, satisfying any $p(\overline{a_1}, \ldots \overline{a_t}) = 0$ where p is a polynomial with rational coefficients such that $p(\overline{a_1}, \ldots \overline{a_t}) \neq 0$ holds for every $\overline{a} \in \mathbf{R}^n$. (Again, this is trivial for one such polynomial.) This has recently been extended to ZFC proofs in [33] by J. Schmerl.

Paul Erdős asked if the following asymmetric variant of the above quoted result of [15] is true. $\mathbb{R}$ can be decomposed into two pieces, A and B such that A omits 3-element while B omits infinite arithmetic progressions. This was proved (for arbitrary vector spaces) by J. E. Baumgartner [3] using ideas somewhat similar to the proofs of the results of Rado, Ceder, and Schmerl.

5. The Hilbert Space

The situation changes radically if we replace $\mathbf{R}^n$ with the Hilbert space ℓ^∞ of infinite real vectors $(x_0, x+1, \ldots)$ with $\sum x_i^2$ finite. An observation due to Erdős, Kakutani, Oxtoby, L.M. Kelly, Nordhaus, and possibly many others is that in this case there are continuum many points with pairwise rational distance. As it is easy not to find a proof I sketch one. Work in the Hilbert space where an orthonormal basis is $\{b_s\}$ where s can be any finite 0-1 sequence. To every infinite 0-1 sequence z associate $a(z) = \sum \lambda_b b_{z|n}$ where $z|n$ denotes the string of the first n terms of z and $\lambda_n = \sqrt{3} \cdot 2^{-(n+l)}$. If $z \neq z'$ first differ at the $(n+1)$-st position then $(a(z) - a(z'))2 = \sum\{\lambda_i^2 : i > n = 4^{-n}\}$ so the distance between $a(z)$ and $a(z')$ is 2^{-n}. It is easy to see that every triangle in this construction is isosceles. I don't know if there is a similar (or any) construction of continuum many points such that all three-element subsets form a triangle with nonzero rational area. (See some problems and several remarks on this topic in [12].)

6. Games

Yet another variant of these problems is to decompose the spaces by games. Let V be the κ-dimensional space over the rationals and let two players alternatively choose previously unchosen vectors in a transfinite series of steps. At limit stages the first player chooses and he has to cover an arithmetic progress ion as long as possible. Erdős and Hajnal proved that the second player can always prevent her opponent from selecting an infinite arithmetic progression. If the second player is allowed to select $\aleph_0$ vectors she can prevent the first player from selecting a 3-element (4-element) AP if $\kappa \leq \aleph_1 (\kappa \leq \aleph_2)$ but the first player can always cover a 3-element arithmetic progression if $\kappa \geq \aleph_2$. This was generalized by Fred Galvin and Zsiga Nagy who showed that the first player can cover an $(n+2)$-element AP for $\kappa = \aleph_n$ but his opponent can prevent him from selecting longer.

7. Large Subsets

Another type of problems asked repeatedly by P. Erdős is the following. Let X be a subset of the n-dimensional Euclidean space with some infinite (usually uncountable) cardinal κ Given a property P can one always find a subset $Y \subseteq X$ of cardinal κ with property P. Already in [9] (see also [13, 5]) Erdős proves this if P is the property that no distance occurs twice. Notice that this is a property depending on finite sets. In [9] it is also proved that if κ is regular then there is a Y such that all $r \leq n$-dimensional simplices formed by $r+1$ elements of Y are of different area (but this fails for e.g., $\aleph_\omega$).

8. Set Mappings

As I already sketched in the introduction the theory of set mappings originated from a question of P. Turán concerning a problem in approximation theory. As usual, a set mapping is any function f on a set X such that $f(x)$ is a subset of X excluding x. A subset $Y \subseteq X$ is free if $x \notin f(y)$ for $x, y \in Y$.

Turán inquired about free sets if $X = \mathbf{R}$ and $f(x)$ is always finite. If one allows $f(x)$ to be countable then there may be no 2-element free subset if $|X| \leq \aleph_1$, a moment's reflection shows that this is just a reformulation of Sierpińiski's statement about the paradoxical decomposition of $\mathbf{R}^2$. P. Erdős proved [10] that if $f(x)$ is nowhere dense for $x \in \mathbf{R}$ then there is always an infinite free set. Bagemihl extended this to finding a dense free set [2]. Hechler proved [22] that under CH there is a set mapping f for which an uncountable free set does not exist and $f(x)$ is an ω-sequence converging to x! Uri Abraham observed that this may not occur if MA_{ω_1} holds and the existence of an uncountable free set if $f(x)$ is assumed to be nowhere dense is consistent with and independent of MA_{ω_1} (see [1]).

C. Freiling investigated what happens if a set mapping is defined on a set of size $\aleph_2$. One of his results can be formulated as follows. If $A \subseteq \mathbf{R}$ has cardinal $\leq \aleph_2$ and f is a set mapping on A with no free subsets of size 2 such that $f(x)$ is always of first category then either A is the union of $\aleph_1$ first category sets or every subset of A of cardinal $< |A|$ is of first category. It is easy to see that in either case there is a well order $\prec$ of A in which every proper initial segment is first category. In recent work the author (of this paper) proved that it is consistent that there is a set $A \subseteq \mathbf{R}$ with $|A| = \aleph_3$ with a set mapping as above but with no well order with first category segments [28].

In [16] Erdős and Hajnal proved that if f is a set mapping on $\mathbf{R}$ such that $f(x)$ is always of measure zero and not everywhere dense then there is a free pair but not necessarily a free triplet. They also proved that if $f(x)$ is bounded and of outer measure at most 1 then for every $k < \omega$ there is a free set of size k and asked if an infinite free set exists. This was finally proved in [30].

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On Order-Perfect Lattices

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Summary. We investigate the property of certain well-founded orderings to have a chain of maximal ordinal length. We show that Heyting algebras and countable modular lattices have this property, but we also present an example of a "well-behaved" lattice which does not have it. We prove a general necessary and sufficient condition for modular lattices to have the property in a hereditary form. This hereditary form is called *order-perfectness*, being analogous to perfectness of finite graphs. Certain well-known theorems of D.H.J. de Jongh, R. Parikh, D. Schmidt, E.C. Milner, N. Sauer, and N. Zaguia turn out to be order-perfectness results.

1. Introduction

In every ordering of a set, there are maximal chains with respect to inclusion. In well-founded orderings, chains have ordinal type. However, different maximal chains may have different ordinal types and a question arises as to whether or not there is a chain of maximal order type. This need not be true in general. It turns out to be interesting to investigate well-founded orderings A such that A itself and also all the sets of the form $\{x \in A \mid x < a\}$ for an $a \in A$ have a chain of maximal order type. We call such orderings *order-perfect*. A motivation for this terminology arises in Sect. 2. Of course, each finite ordering is order-perfect. In fact, each *AWPO* is order-perfect. This is a special case of Theorem 1.2 from Milner and Sauer [3]. Also, the special case of it for countable orderings is equivalent to a Theorem by D. Schmidt [5].

The aim of this paper is to prove order-perfectness for different kinds of orderings, namely for certain classes of lattices. To indicate the motivation, let us take an example. For a *WPO* A, the set $\downarrow A$ of all downward closed subsets is well-founded (the converse is also true). Maximal chains in $\downarrow A$ are in 1-1-correspondence with linear extensions of A. As is well known (see [2]), for each *WPO* A there exists a linear extension of maximal ordinal type, and so $\downarrow A$ is order-perfect. This is a special case of our result saying that each well-founded completely distributive lattice is order-perfect.

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More generally, we present a necessary and sufficient condition for a well-founded modular lattice to be order-perfect. This condition is true for Heyting algebras (cf. [1]) as well as for countable modular lattices. We do not know if there is a natural common generalization of these two facts. In Sect. 4 we show that one fairly natural candidate fails (the counter-example is a complete sublattice of $\mathrm{Vect}(\mathbb{R}^\omega)$).

Problem 1. *Is there a class $\mathcal{C}$ of lattices which can be characterized by equations, includes both Heyting algebras and countable modular lattices and has the property that each well-founded lattice from $\mathcal{C}$ is order-perfect?*

To conclude this section, let us remark that we call our main concept order-perfectness to avoid confusion with the work of Milner and Pouzet [4] who consider perfectness of infinite comparability graphs in the usual graph-theoretical sense.

2. Preliminaries: Orderings

Definition 1. *In this paper, $\oplus$ resp. $+$ denotes the natural (resp. ordinal) sum of ordinal numbers. For a well-ordered set K (isomorphic to an ordinal) a subset $S \subseteq K$ is called closed if*

$$(T \subseteq S \text{ and } T \text{ has a supremum in } K) \rightarrow \sup T \in S.$$

Now let A be an ordering. The symbol $\overline{A}$ designates the ordering obtained from A by adding a new greatest element. A chain (resp. strict, decreasing, strictly decreasing chain) in A is a sequence $(a_\alpha)_{\alpha\in\lambda}$ ($\lambda \in \mathrm{Ord}$) such that $\alpha < \beta$ implies $a_\alpha \leq a_\beta$ (resp. $a_\alpha < a_\beta$, $a_\alpha \geq a_\beta$, $a_\alpha > a_\beta$). The length $|c|$ of a chain $c = (a_\alpha)_{\alpha\in\lambda}$ is the ordinal type of the set $\{a \in \lambda \mid \beta < \alpha \rightarrow a_\beta < a_\alpha\}$. A chain $(a_\alpha)_{\alpha\in\lambda}$ is continuous, if $a_{\sup I} = \sup\{a_i \mid i \in I\}$ for any I such that $I \cup \{\sup I\} \subseteq \lambda$; in particular, the right hand supremum (the unique least upper bound) is required to exist. An antichain in A is a set of pairwise incomparable elements.

The ordering A is *well-founded* (abb. WF) if it has no infinite strictly decreasing chain. A mapping $\varphi : A \rightarrow B$, where A, B are orderings, is called *monotone* resp. *strictly monotone* if it satisfies $a \leq b \rightarrow \varphi(a) \leq \varphi(b)$ (resp. $a < b \rightarrow \varphi(a) < \varphi(b)$). For well-founded A we define

$$\mu(A) = \min\{\gamma \mid \exists\varphi : A \rightarrow \gamma \text{ strictly monotone}\}$$

$$\overline{\lambda}(A) = \sup\{\gamma \mid \exists\varphi : \gamma \rightarrow A \text{ strictly monotone}\}.$$

Observe that $\overline{\lambda}(A) \leq \mu(A)$. We write $\lambda(A)$ instead of $\overline{\lambda}(A)$ if this supremum as attained (i.e., if there is a maximum).

Put $\downarrow A = \{B \subseteq A \mid a \in B \& b \leq a \rightarrow b \in B\}$. Then $\downarrow A$ is ordered by inclusion. For $a \in A$ we denote by $\Downarrow a$ (resp.$\downarrow a$) the induced ordering on

$\{b \in A \mid b < a\}$ (resp.$\{b \in A \mid b \leq a\}$) . An ordering A is called *order-perfect* if we have

$$\lambda(\Downarrow a) = \mu(\Downarrow a) \text{ for any } a \in \overline{A}.$$

(In particular, the left hand sides are required to exist). We call A *stratified*, if for each $a \in A$ all maximal chains in $\Downarrow A$ have the same ordinal length. For a poset A we say that $a \in A$ covers $b \in A$ if we have $a > b$ and if there is no $c \in A$ such that $b < c < a$.

Let A be a partial ordering. A (finite or infinite) sequence (a_i) in A is called *bad* if we have

$$i < j \to a_i \nleq a_j.$$

The ordering A is called a *well-partial-ordering* (abb. WPO) if it contains no infinite bad sequence. We denote by $A^{<\omega}$ the set of all finite sequences in A and put

$$\mathrm{Bad}\, A = \{(a_0, \ldots, a_n) \in A^{<\omega} \mid i < j \to a_i \nleq a_j\}.$$

Thus, the empty sequence belongs to $\mathrm{Bad}\, A$. A mapping $\varphi : \mathrm{Bad}\, A \to \gamma + 1 (\gamma \in \mathrm{Ord})$ is called a *character* of A if we have

$$\varphi(\lambda x) < \varphi(\lambda) \text{ whenever } \lambda x \in \mathrm{Bad}\, A.$$

Put

$$c(A) = \min\{\gamma \mid \exists \text{ a character } \varphi : \mathrm{Bad}\, A \to \gamma + 1\}$$

$$d(A) = \max\{\gamma \mid \gamma \text{ is isomorphic to a linear extension of } A\}.$$

(Let us recall that an *extension* means a stronger ordering on the same set.) An extension of a WPO is always WPO (such a statement is not true for WF orderings). Thus, a linear extension of the WPO A is a well-ordering. The existence of a number $d(A)$ was proved in [2].

Now let A be WF, $a \in A$. Define the *height* $h(a)$ of an element $a \in A$ by $h(a) = \overline{\lambda}(\Downarrow a)$. (The reader should be warned that the terminology in the literature differs and some authors use the word "height" in a different meaning.) We call A *almost well-partial ordering* (AWPO) if for any sequence $(a_i)_{i \in \omega}$ in A there exist $i < j$ with

$$a_i \leq a_j \text{ or } h(a_i) \geq h(a_j).$$

For an ordering A and a set $M \subseteq A$ put

$$A_M = \{x \in A \mid (\forall_y \in M) y \nleq x\}.$$

For a subset $B \subset \mathrm{Ord}$ put

$$M(B) = \sup\{x + 1 \mid x \in B\}.$$

Thus, $M(B)$ is the least ordinal greater than all members of B.

Proposition 1. *Let A be WF. Then A is order-perfect if and only if all the numbers $\lambda(A)$, $\lambda(\Downarrow a)$ $(a \in A)$ exist.*

Proof. If the numbers $\lambda(A), \lambda(\Downarrow a)$ exist, then we have

$$\lambda(\Downarrow a) = M(\{\lambda(\Downarrow b) \mid b < a\})$$
$$\lambda(A) = M(\{\lambda(\Downarrow a) \mid a \in A\}).$$

We can easily check, however, that analogous recurrent relations hold for u. □

Remark 1. *Of course, not all WF orderings have to be order-perfect. Put, e.g.,* $A_\alpha = \mathrm{Bad}\,\alpha$ *with*

$$a \leq b \equiv (\exists c) a = bc.$$

Then we have

$$\overline{\lambda}(A_\alpha) = \omega, \mu(A_\alpha) = \alpha.$$

Proposition 2. *For each WPO A we have*

$$c(A) + 1 = \mu(\downarrow A) = \lambda(\downarrow A) = d(A) + 1.$$

In particular, $c(A) = d(A)$.

Proof. Once we know that $d(A)$ always exists (see [2] or Sect. 3 below), the last two equalities are trivial. We present a proof of the first one: $\leq$: If $\chi :\downarrow A \to \mu(\downarrow A)$ is strictly monotone then the mapping $\psi : \mathrm{Bad}\,A \to \mu(\downarrow A)$ given by $\psi(a_1 \dots a_n) = \varphi(A_{\{a_1 \dots a_n\}})$ is a character. $\geq$: If

$$\psi : \mathrm{Bad}\,A \to \gamma + 1$$

is a character, then define $\varphi :\downarrow A \to \gamma + 1$ by

$$\varphi(B) = \min\{\psi(T) \mid T \text{ is a bad sequence in } A \setminus B\}.$$

To see that φ is strictly monotone, let B, $C \in\downarrow A$, $B \subset C$. There is an element

$$c \in C \setminus B.$$

let T be a bad sequence in $A \setminus C$ with

$$\varphi(C) = \psi(T).$$

The concatenation Tc is a bad sequence, since $C \in\downarrow A$. Thus,

$$\varphi(B) \leq \psi(Tc) < \psi(T) = \varphi(C).$$

□

Theorem 1 (**Milner and Sauer [3]**). *Each AWPO A is order-perfect.*

Lemma 1. *Let I, J, K be well-ordered sets, $K = I \cup J$. Then*

(a) $c(K) \leq c(I) \oplus c(J)$
(b) If $\min K \in J$, $|I| \geq \omega$ *and I, J are closed in K then* $c(K) < c(I) \oplus c(J)$.

Proof. (a) Is a special case of a theorem proved in [2].

(b) Suppose the contrary. Without loss of generality, $K = \kappa$ is an ordinal. Let λ be the largest limit ordinal with

$$\lambda \leq \kappa$$

By our assumptions,

$$I \cap \lambda \neq \emptyset \quad \& \quad J \cap \lambda \neq \emptyset. \tag{$*$}$$

There exist $\alpha < \lambda$, $\beta \leq \lambda$ such that β is sum-closed and

$$\alpha + \beta = \lambda.$$

Put

$$\beta' = \{\alpha + i \mid i \in \beta\}.$$

We will show that

$$\text{One of the sets } I \cap \lambda, J \cap \lambda \text{ has a maximum.} \tag{$**$}$$

Indeed, suppose ($**$) false. Since both I and J are closed in K, the sets

$$I \cap \beta', J \cap \beta'$$

are cofinal in β'. We conclude that

$$c(I) \oplus c(J) \geq \alpha + 2\beta' > K.$$

Thus, ($**$) is proved.

Now let, say, $\epsilon = \max(I \cap \lambda)$. Define a mapping $\varphi : \kappa \to \kappa + 1$ by

$$\varphi(\beta) = \beta \text{ for } \beta < \epsilon$$
$$\varphi(\epsilon) = \lambda$$
$$\varphi(\beta) = \beta - 1 \text{ for } \epsilon < \beta < \epsilon + \omega$$
$$\varphi(\beta) = \beta \text{ for } \epsilon + \omega \leq \beta < \lambda$$
$$\varphi(\beta) = \beta + 1 \text{ for } \lambda \leq \beta.$$

Observe that we have

$$\kappa + 1 = \varphi(I) \cap \varphi(J)$$
$$c(\varphi(I)) = c(I) \text{ and } c(\varphi(J)) = c(J).$$

We conclude that

$$c(I) \oplus c(J) \geq c(\varphi(I)) \oplus c(\varphi(J)) \geq c(\varphi(I) \cup \varphi(J)) \geq \kappa + 1.$$

□

3. Lattices: Positive Results

Observation 1. *Let A be a WF lattice. Then $\overline{A}$ is a complete lattice.*

Proof. Since A has finite meets and is well-founded, it has no empty meets.□

Conditions 1. *We say that a WF ordering A* satisfies CC *(continuity condition), if there exists a chain $(a_\kappa)_{i\in cf\overline{\lambda}(A)}$ such that*

$$\sup_i h(a_i) = \overline{\lambda}(A).$$

A WF-ordering A is said to satisfy HCC *if for each interval $\langle b, a)$, $b, a \in \overline{A})$, such that $\overline{\lambda}(\langle b, a))$ is sum-closed and of cofinality $> \omega$, satisfies CC.*

We say that a lattice A satisfies the condition WHA *(weak Heyting algebra) if for any set I, $|I| < |A|$, and elements $(b_i)_{i\in I}$, a of A we have*

$$(\bigvee_{i\in I} b_i) \wedge a = \bigvee_{i\in I} (b_i \wedge a). \qquad (+)$$

In particular, the joins in question are required to exist.

Theorem 2. *A modular WF lattice is order-perfect if and only if it satisfies HCC.*

Proof will be given below. Until the end of the proof, A is always a well founded modular lattice.

Lemma 2. *Let $(c_i)_{i<\lambda}$ be a chain in A, $a \in A$, $M = \{i < \lambda / c_i \not\leq a\}$. Then we have*

$$|(c_i)_{i<\lambda}| \leq |(c_i \wedge a)_{i<\lambda}| \oplus |(c_i \vee a)_{i\in M}|. \qquad (+)$$

Moreover, we have

$$|(c_i)_{i<\lambda}| < |(c_i \wedge a)_{i<\lambda}| \oplus |(c_i \vee a)_{i\in M}| \qquad (++)$$

whenever

$$|M| \geq \omega. \qquad (+++)$$

Remark 2. *The condition (+++) for (++) cannot be weakened. Indeed, consider $A = \omega + 2$, $\lambda = \omega + 2$, $c_i = i$ $(i \in A)$, $a = \omega$, $I = \{w + I\}$, $|(c_i)_{i<\lambda}| = \omega + 2$, $|(c_i \wedge a)_{i\in\lambda}| = \omega + 1$, $|(c_i \vee a)_{i<\lambda} = 1$ and equality occurs in (+).*

Proof. Let $b = (b_i)_{i<\kappa}$ be a chain in A. Put

$$I(b) = \{i < \kappa | (\forall j < i) b_j < b_i\}.$$

Obviously,

$$|b| = c(I(b)).$$

By modularity of A,

$$I((c_i)_{i<\lambda}) = I((c_i \wedge a)_{i<\lambda}) \cup I((c_i \vee a)_{i\in M}).$$

Moreover, obviously, the right-hand sets are closed in the left-hand one and

$$0 \in I((c_i \wedge a)_{i<\lambda}).$$

The statement of our lemma now directly follows from Lemma 1. □

Lemma 3. *If we have $b < a$ and there is a chain (c_i) of length α in $\Downarrow a$ then there are chains of lengths β, γ in $\Downarrow b$, $\langle b, a)$, respectively, such that*

$$\beta \oplus \gamma \geq \alpha.$$

Proof. Assume the chain (c_i) in $\Downarrow a$ is maximal hence continuous. We distinguish two possibilities:

(a) $|(c_i \vee b \mid c_i \not\leq b)| \geq \omega$. Then take the chains $(c_i \wedge b \mid c_i \not\geq b)$, $(c_i \wedge b | c_i \not\leq b)$ and use Lemma 2(++).
(b) $|(c_i \vee b \mid c_i \not\leq b)| < \omega$. Then take the chain $(c_i \wedge b \mid c_i \not\geq b)$ and the concatenation of b and $(c_i \wedge b \mid c_i \not\leq b)$ and use Lemma 2(+) together with the commutativity of addition of finite ordinals. □

Remark 3. *Trying out a few examples, one can see that the cases (a), (b) in the proof of Lemma 3 are really governed by different principles. A less careful analysis, however, would cause only a discrepancy by 1, which is still sufficient for the proof of Theorem 2.*

Lemma 4. *For $b < a \in A$ we have*

$$\overline{\lambda}(\Downarrow a) \leq \overline{\lambda}(\Downarrow b) \oplus \overline{\lambda}(\langle b, a)). \tag{+}$$

Moreover, if $\lambda(\Downarrow b)$ exists we have

$$\overline{\lambda}(\Downarrow b) + \overline{\lambda}(\langle b, a)) \leq \overline{\lambda}(\Downarrow a). \tag{++}$$

Proof. (++) is trivial, (+) follows from Lemma 3. □

Theorem 3. *If A (a modular WF lattice) is order-perfect then any interval $\langle b, a)$, $a, b \in \overline{A}$, is order-perfect.*

Proof. First let $\overline{\lambda}(\langle b, a))$ be sum-closed. We have a chain (a_i) in $\Downarrow a$ of length $\lambda(\Downarrow a)$. Thus, there are chains in $\Downarrow b, \langle b, a)$ of lengths β, γ, respectively, $\beta \oplus \gamma \geq \lambda(\Downarrow a)$. By Lemma 4(++),

$$\overline{\lambda}(\Downarrow b) + \overline{\lambda}(\langle b, a)) \leq \lambda(\Downarrow a).$$

Since $\beta \leq \lambda(\Downarrow b)$, we have

$$\lambda(\Downarrow b) \oplus \gamma \leq \lambda(\Downarrow a).$$

Since $\overline{\lambda}(\langle b, a))$ is sum-closed, we have $\overline{\lambda}(\langle b, a)) \leq \gamma$ and hence

$$\overline{\lambda}(\langle b, a)) = \gamma.$$

Now let $\alpha = \overline{\lambda}(\langle b, a))$ not be sum-closed. We proceed by induction on a. First realize that $b < c < a \to \overline{\lambda}(\langle b, c)) < \overline{\lambda}(\langle b, a))$. In fact $\lambda(\langle b,\ c))$ exists by the inductional hypothesis; use Lemma 4(++). Now let β be the last summand of α (in the same sense as in the proof of Lemma 1). Let (a_i) be a chain in $\langle b, a)$ of length $> \alpha - \beta$ (the least number κ s.t. $\kappa + \beta \geq \alpha$). Put $c = a_{\alpha-\beta}$. We have $\overline{\lambda}(\langle c, a)) = \beta$: $\leq$ is obvious and $\geq$ follows from the fact that $\overline{\lambda}(\langle b, c)) < \overline{\lambda}(\langle b, a))$ and

$$\overline{\lambda}(\langle b, c)) \oplus \overline{\lambda}(\langle c, a)) \geq \overline{\lambda}(\langle b, a)) \tag{+}$$

(by Lemma 4(+)). Now $\lambda(\langle b, c))$ exists by the inductional hypothesis and $\overline{\lambda}(\langle c, a))$ exists by the first part of the proof. By (+), we have

$$\lambda(\langle b, c)) + \lambda(\langle c, a)) \geq \overline{\lambda}(\langle b, a))$$

(since $\lambda(\langle c, a)) = \beta$). □

Lemma 5. *Let a, $b \in A$. Then we have*

$$h(a) \oplus h(b) \geq h(a \vee b).$$

Proof. Let (c_i) be a chain in $\Downarrow a \vee b$ of length α. By Lemma 3, there are chains in $\Downarrow a\langle a, a \vee b)$ of lengths β, γ, respectively, such that

$$\beta \oplus \gamma \geq \alpha.$$

As is well known, $\langle a, a \vee b)$ is isomorphic to $\langle a, a \wedge b)$ via "$\wedge b$". Thus, $\gamma \leq h(b)$.□

Remark 4. *The discrepancy between natural and ordinal sum prevents us to obtain an exact result as in the finite case.*

Lemma 6. *Let $\overline{\lambda}(\langle a, b))$ be sum-closed and of cofinality $\omega, a, b \in \overline{A}$. Then $\langle a, b)$ satisfies CC.*

Proof. Choose $a_i \in \langle a, b)$ with $\overline{\lambda}(\langle a, b_i)) \nearrow \overline{\lambda}(\langle a, b))$ and put $a_i = b_0 \vee \ldots \vee b_i$. We have $a_i < b$ by Lemma 5 and the sum-closedness. □

Proof of Theorem 2. Necessity follows from Theorem 3. In order to prove sufficiency, we introduce the following statement

S: If A is modular and satisfies HCC and $a \in A$ then $\lambda(\Downarrow a)$ exists.

This implies our result by Proposition 1, since neither modularity nor HCC is violated by adding a new greatest element.

We shall prove S by induction on $h(a)$.

Case 1. *$h(a)$ is not sum-closed. Then let $\alpha + \beta = h(a)$, α, $\beta < h(a)$, β the last summand of $h(a)$. Then there exists a $b < a$ with*

$$h(b) \geq \alpha.$$

We may assume $h(b) < h(a)$ (otherwise we replace a by b), hence $\lambda(\Downarrow b)$ exists. By Lemma 4,

$$\overline{\lambda}(\langle b, a)) = \beta. \tag{+}$$

The inductional hypothesis applied to $a \in \langle b, a)$ (a modular lattice satisfying HCC) yields

$$\lambda(\langle b, a)) = \beta. \tag{++}$$

Now by (+),(++), there exist strict chains

$$(c_i)_{i\in\alpha} \text{ in } \Downarrow b$$

$$(d_i)_{i\in\beta} \text{ in } \langle b, a).$$

The desired chain is obtained by concatenation.

Case 2. $\lambda = \overline{\lambda}(\Downarrow a)$ *is sum-closed. Then either by HCC or by Lemma 6 we have a strict chain $(c_i)_{i\in cf\lambda}$ in $\Downarrow a$ with*

$$\lambda(\Downarrow c_i) \nearrow \lambda.$$

By Lemma 4(+) and the sum-closedness of λ we have

$$\lim_{j:j\geq i j\in cf\lambda} \overline{\lambda}(\langle c_i, c_j)) = \lambda$$

and thus

$$\lim_{j:j\geq i j\in cf\lambda} \lambda(\langle c_i, c_j)) = \lambda$$

by the inductional hypothesis. Now take an increasing sequence $i_\alpha(\alpha \in \mathrm{cf}\lambda)$ such that

$$\lim_{\alpha\in cf\lambda} \lambda(\langle c_{i_\alpha}, c_{i_{\alpha+1}})) = \lambda.$$

Now if $(c_i^\alpha)_{i\in\lambda(\langle c_{i_\alpha},\ e_{i_{\alpha+1}}))}$ is a strict chain, we obtain a chain of length λ in $\Downarrow a$ by concatenation. □

Theorem 4. *If $\overline{A}$ satisfies WHA and is WF then A satisfies CC (and hence, by hereditarity, also HCC).*

Proof. Let $\overline{A} = \langle 0, 1)$, $\lambda = \overline{\lambda}(A)$. If there is a sequence $(c_i)_{i\in cf\lambda}$ such that for $\beta < cf\lambda$

$$\bigvee_{i\in\beta} c_i < 1,$$

the statement is proved. So assume the contrary. We will prove that following fact:

$$T: \text{For any } c < 1 \text{ we have } \sup\{h(b) | c \not\leq b\} < \lambda.$$

Really, since A is WF, we can write

$$c = \bigvee_{i \in \beta} c_i$$

where $\beta < cf\lambda$ and c_i are irreducible to sums of size $< cf\lambda$. Now for any $i \in \beta$ the set

$$M_i = \{h(b)c_i \not\leq b\}$$

has supremum $< \lambda$: Otherwise there are an $a_\alpha \not\geq c_i$, $\alpha \in cf\lambda$, such that $h(a_\alpha) \nearrow \lambda$. Thus, there is a $\gamma < cf\lambda$ such that

$$\bigvee_{\alpha \in \lambda} a_\alpha = 1.$$

Thus, by WHA,

$$\bigvee_{\alpha \in \gamma} (c_i \wedge a_\alpha) = c_i.$$

contradicting the assumed irreducibility of c_i. Now

$$\sup_{\imath \in \beta} \sup M_i < \lambda.$$

However, T clearly enables us to choose a sequence (c_i) with (+), contradicting the original assumption. □

Remark 5. *We have shown that each WF WHA is order-perfect. On the other hand, each countable modular WF lattice is order-perfect, since it satisfies HCC by default.*

On might look for weakenings of these conditions. It is fairly easy to find an example of a distributive complete well-founded lattice which is not order-perfect: Take, for instance, the set

$$A = \{\varphi : \omega_1 \setminus \{0\} \to \omega_1 || \omega_1 \setminus \varphi^{-1}(0)| < \omega \& (\forall \alpha \in \omega_1 \setminus \{0\})\varphi(\alpha) < \alpha\}.$$

and put

$$\varphi \leq \psi \text{ iff } (\forall \alpha \in \omega_1 \setminus \{0\})\varphi(\alpha) \leq \psi(\alpha).$$

This makes A a well-founded distributive lattice. $\overline{A}$ is a well-founded complete distributive lattice. Obviously,

$$\overline{\lambda}(A) = \omega_1$$

while there is no uncountable chain. An observation of this kind has been first made in Zaguia [6].

This example teaches us that the infinite joins (at least of cardinality $< |A|$) indeed have to be "tightened up" in some way. On the other hand, the (finite) distributivity is obviously not needed.

Note that WHA is equivalent to distributivity plus Lemma 2(+) for *chains* (b_i). One might suggest *modularity* and Lemma 2(+) for chains (b_i). In the next section we shall show, however, that this condition is not sufficient for order-perfectness. In fact, we shall find a complete well-founded sublattice of $\mathrm{Vect}(\mathbb{R}^{(\omega)})$ which is not perfect. This example, because of the strong condition on infinite joins is much more interesting than the one mentioned above.

4. Lattices: A Negative Result

Theorem 5. *There exists a complete sublattice P of* $\mathrm{Vect}(\mathbb{R}^{(\omega)})$ *which is WF but not order-perfect.*

(In this paper, $(\mathbb{R}^{(A)})$ means the free $\mathbb{R}$-modul over A. $\mathrm{Vect}(X)$ stands for the lattice of all submodules of X.)

Lemma 7. *For any $\alpha < w_1$ there exists a WPO B_α such that $\lambda(B_\alpha) = \omega$, $c(B_\alpha) \geq \alpha$, B_α is stratified and*

$$|\{x \in B_\alpha | h(x) = n\}| = n + 1. \qquad (+)$$

Proof. Let $\varphi : \alpha \to \omega$ be a bijection. Put $B_\alpha = \{(\alpha, k) | \varphi(\alpha) \leq k\}$. Now let α, $k < (\beta, m)$ if and only if $\alpha \leq \beta$ and $k < m$. Clearly $\lambda(B_\alpha) = \omega$ and B_α is stratified (since $h(\alpha, k) = k$). For the same reason B_α, satisfies (+) (since φ is a bijection). Now B_α is WPO, since it is the intersection of three (linear) well-orderings: The lexicographical one prefering first coordinate, the lexicographical one prefering second coordinate and the ordering $\prec$ given by

$$(\alpha, k) \prec (\beta, m) \text{ iff } k < m \text{ or } (k = m \text{ and } \beta < \alpha).$$

□

Construction 1. *In the sequel, we will assume that we are given disjoint sets $B_\alpha | \alpha \in \omega_1$ satisfying the conditions of Lemma 7. Let*

$$\mathbb{R}^{(\omega)} = \bigoplus_{i \in \omega} V_i, \quad \dim V_i = 2^{i+1}(i+1)!$$

Let $\{v(\alpha,) | \alpha \in \omega_1, c \in B_\alpha, h(c) = i\}$ be vectors of V_i in general position. (Each subset of cardinality $\leq \dim V_i$ is linearly independent.) For a vector $v \in V_i$ define a subspace $\partial v \subseteq V_{i-1}$ as follows:

If there exists a (in the sequel minimal with res pect to inclusion) set

$$\{v(\alpha_1, c_1), \ldots, v(\alpha_n, c_n)\}$$

such that $v \in \langle v(\alpha_1, c_1), \ldots, v(\alpha_n, c_n)\rangle$ (the linear hull) and

$$n \leq 2^i(i+1)!$$

then put

$$\partial v = \langle v(\alpha_i, d) | i \leq n \text{ and } c_i \text{ covers } d\rangle.$$

Otherwise put

$$\partial v = V_{i-1}.$$

Because of the general position of $v(\alpha, c), \partial$ *is defined correctly. Put for a subspace* $V \subseteq V_i$

$$\partial V = \langle \partial v | v \in V \rangle.$$

Now let

$$P \subseteq \bigoplus_{i \in \omega} \mathrm{Vect} V_i$$

be the set of all subspaces W *satisfying*

$$\partial(W \cap V_i) \subseteq W.$$

We will show that P *is the desired lattice.*

Lemma 8. *P is a meet-closed subset of* $\mathrm{Vect}(\mathbb{R}^{(\omega)})$.

Proof. It suffices to realize that a space W belongs to P if and only if it satisfies a set of conditions of the form

$$a \in W \to b \in W.$$

Such conditions are preserved by intersection. □

Lemma 9. *Let* $v \in V_i$, $\langle v(\beta_1, d_1), \ldots, v(\beta_m, d_m) \rangle \ni v$. *Then we have*

$$\partial v \subseteq \langle v(\beta_i, d) | i \leq m, d_i \text{ covers } d \rangle.$$

Proof. First realize that if $m \geq 2^i(i+1)!$ then at least $2^i i!$ of the numbers β_i are different (by Lemma 7(+) and the Dirichlet principle). Thus, the vectors $v(\beta_j, d)$ such that d_j covers d are distinct and hence they generate V_{i-1}. If, on the other hand, $n < 2^i(1+i)!$ then (by the general position) $\{v(\beta_1, d_1), \ldots, v(\beta_m, d_m)\}$ contains the set $\{v(\alpha_1, c_1), \ldots, v(\alpha_n, c_n)\}$ from the definition of ∂v. □

Lemma 10. *P is a complete sublattice of* $\mathrm{Vect}(\mathbb{R}^{(\omega)})$.

Proof. Now it suffices to show that P is closed under joins. This, however follows from the relation

$$\partial(\bigvee V_i) = \bigvee \partial V_i$$

which is an easy consequence of Lemma 7. □

Observation 2. *Let* $B \in\downarrow B_\alpha$. *Then*

$$V_B = (\{v(\alpha, c) | c \in B\}) \in P.$$

Thus, P contains a chain of arbitrary countable length. On the other hand, it does not contain an uncountable one, since $\mathrm{Vect}(\mathbb{R}^{(\omega)})$ *does not.*

Lemma 11. *Let $V \in P$ and let an $i \leq k$ exist such that $V \not\supseteq V_i$. Then there exist only finitely many different a with*

$$v(\alpha, c) \in V$$

for some $c \in B_\alpha, h(c) \geq k$.

Proof. Because of the stratifiedness of B_α, V contains for each such α a vector $v(\alpha, d)$ with $h(d) = i$, If there were infinitely many such ones then $V_i \subseteq V$ by the general position. □

Lemma 12. *P is WF.*

Proof. Let $(W_j)_{j\in\omega}$ be a strictly decreasing chain in P. Let, without loss of generality, $W_0 \not\supseteq V_k$. Now let

$$\{\alpha_1, \ldots, \alpha_n\} = \{\alpha | (\exists v(\alpha, c))h(c) \geq k \& v(\alpha, c) \in W_0\}.$$

Put $B_j^i = \{c \mid v(\alpha_i, c) \in W_j\}$, $i \leq n$, $j \in w$. By the definition of P clearly $B_j^i \in\downarrow B_{\alpha_i}$. Moreover, we have

$$B_0^i \supseteq B_1^i \supseteq \ldots.$$

Since B_{α_i}, is WPO, it holds without loss of generality that

$$B_0^i = B_1^i = \ldots.$$

Since we have $j \neq j'$ with

$$W_j \cap \bigoplus_{i\in k} V_i = W_j' \cap \oplus_{i\in k} V_i,$$

we have $W_j = W_j'$. A contradiction. □

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The PCF Theorem Revisited

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Dedicated to Paul Erdős

Summary *The pcf theorem (of the possible cofinability theory) was proved for reduced products* $\prod_{i<\kappa} \lambda_i/I$, *where* $\kappa < \min_{i<\kappa} \lambda_i$. *Here we prove this theorem under weaker assumptions such as* $\mathrm{wsat}(I) < \min_{i<\kappa} \lambda_i$, *where* $\mathrm{wsat}(I)$ *is the minimal* θ *such that* κ *cannot be divided to* θ *sets* $\notin I$ *(or even slightly weaker condition). We also look at the existence of exact upper bounds relative to* $<_I$ *(*$<_I$*-eub) as well as cardinalities of reduced products and the cardinals* $T_D(\lambda)$. *Finally we apply this to the problem of the depth of ultraproducts (and reduced products) of Boolean algebras.*

1. Introduction

An aim of the pcf theory is to answer the question, what are the possible cofinalities (pcf) of the partial orders $\prod_{i<\kappa} \lambda_i/I$, where $\mathrm{cf}(\lambda_i) = \lambda_i$, for different ideals I on κ. For a quick introduction to the pcf theory see [11], and for a detailed exposition, see [8] and more history. In §1 and §2 we generalize the basic theorem of this theory by weakening the assumption $\kappa < \min_{i<\kappa} \lambda_i$ to the assumption that I extends a fixed ideal I^* with $\mathrm{wsat}(I^*) < \min_{i<\kappa} \lambda_i$, where $\mathrm{wsat}(I^*)$ is the minimal θ such that κ cannot be divided to θ sets $\notin I^*$ (not just that the Boolean algebra $\mathcal{P}(\kappa)/I^*$ has no θ pairwise disjoint non zero elements). So §1, §2 follow closely [8, Ch. I = [10]], [8, II 3.1], [8, VIII §1]. It is interesting to note that some of (as presented in courses and see a forthcoming survey of Kojman) those proofs which look to be superseded when by [13, §1] we know that for regular $\theta < \lambda$, $\theta^+ < \lambda \Rightarrow \exists$ stationary $S \in I[\lambda]$, $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$, give rise to proofs here which seem necessary. Note $\mathrm{wsat}(I^*) \leq |\operatorname{Dom}(I^*)|^+$ (and $\mathrm{reg}_*(I^*) \leq |\operatorname{Dom}(I^*)|^+$) so [8, I §1, §2, II §1, VII 2.1, 2.2, 2.6] are really a special case of the proofs here.

During the 1960s the cardinalities of ultraproducts and reduced products were much investigated (see Chang and Keisler [1]). For this the notion "regular filter" (and (λ, μ)-regular filter) were introduced, as: if $\lambda_i \geq \aleph_0$,

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D a regular ultrafilter (or filter) on κ then $\prod_{i<\kappa} \lambda_i/D = (\limsup_D \lambda_i)^\kappa$. We reconsider these problems in §3 (again continuing [8]). We also draw a conclusion on the depth of the reduced product of Boolean algebras partially answering a problem of Monk; and make it clear that the truth of the full expected result is translated to a problem on pcf. On those problems on Boolean algebras see Monk [6]. In this section we include known proofs for completeness.

Let us review the paper in more details. In 1.1, 1.2 we give basic definition of cofinality, true cofinality, $\mathrm{pcf}(\bar{\lambda})$ and $J_{<\lambda}[\bar{\lambda}]$ where usually $\bar{\lambda} = \langle \lambda_i : i < \kappa \rangle$ a sequence of regular cardinals, I^* a fixed ideal on κ such that we consider only ideals extending it (and filter disjoint to it). Let $\mathrm{wsat}(I^*)$ be the first θ such that we cannot partition κ to θ I^*-positive set (so they are pairwise disjoint, not just disjoint modulo I^*). In 1.3, 1.4 we give the basic properties. In Lemma 1 we phrase the basic property enabling us to do anything: $(1.5(*))$: $\liminf_{I^*}(\bar{\lambda}) \geq \theta \geq \mathrm{wsat}(I^*)$, $\prod \bar{\lambda}/I^*$ is θ^+-directed; we prove that $\prod \bar{\lambda}/J_{<\lambda}[\bar{\lambda}]$ is λ-directed. In 1.6, 1.8 we deduce more properties of $\langle J_{<\lambda}[\bar{\lambda}] : \lambda \in \mathrm{pcf}(\bar{\lambda}) \rangle$ and in 1.7 deal with $<_{J_{<\lambda}[\bar{\lambda}]}$-increasing sequence $\langle f_\alpha : \alpha < \lambda \rangle$ with no $<_{J_{<\lambda}[\bar{\lambda}]}$-bound in $\prod \bar{\lambda}$. In 1.9 we prove $\mathrm{pcf}(\bar{\lambda})$ has a last element and in 1.10, 1.11 deal with the connection between the true cofinality of $\prod_{i<\kappa} \lambda_i/D^*$ and $\prod_{i<\sigma} \mu_i/E$ when $\mu_i =: \mathrm{tcf}(\prod_{i<\kappa} \lambda_i/D_i)$ and D^* is the E-limit of the D_i's.

In 2.1 we define normality of λ for $\bar{\lambda}$: $J_{\leq\lambda}[\bar{\lambda}] = J_{<\lambda}[\bar{\lambda}] + B_\lambda$ and we define semi-normality: $J_{\leq}[\bar{\lambda}] = J_{<\lambda}[\bar{\lambda}] + \{B_\alpha : \alpha < \lambda\}$ where $B_\alpha/J_{<\lambda}[\bar{\lambda}]$ is increasing. We then (in 2.2) characterize semi normality (there is a $<_{J_{<\lambda}[\bar{\lambda}]}$-increasing $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$ cofinal in $\prod \bar{\lambda}/D$ for every ultrafilter D (disjoint to I^* of course) such that $\mathrm{tcf}(\prod \bar{\lambda}/D) = \lambda$) and when semi normality implies normality (if some such $\bar{f}$ has a $<_{J_{<\lambda}[\bar{\lambda}]}$-eub).

We then deal with continuity system $\bar{a}$ and $<_{J_{<\lambda}[\bar{\lambda}]}$-increasing sequence obeying $\bar{a}$, in a way adapted to the basic assumption $(*)$ of 1.5.

Here as elsewhere if $\min(\bar{\lambda}) \geq \theta^+$ our life is easier than when we just assume $\mathrm{limsup}_{I^*}(\bar{\lambda}) \geq \theta$, $\prod \bar{\lambda}/I^*$ is θ^+-directed (where $\theta \geq \mathrm{wsat}(I^*)$ of course). In 2.3 we give the definitions, in 2.4 we quote existence theorem, show existence of obedient sequences (in 2.5), essential uniqueness (in 2.7) and better consequence to 1.7 (in the direction to normality). We define (2.9) generating sequence and draw a conclusion (2.10(1)). Now we get some desirable properties: in 2.8 we prove semi normality, in 2.10(2) we compute $\mathrm{cf}(\prod \bar{\lambda}/I^*)$ as $\max \mathrm{pcf}(\bar{\lambda})$. Next we relook at the whole thing: define several variants of the pcf-th (Definition 6). Then (in 2.12) we show that e.g. if $\min(\bar{\lambda}) > \theta^+$, we get the strongest version (including normality using 2.6, i.e. obedience). Lastly we try to map the implications between the various properties when we do not use the basic assumption 1.5 $(*)$ (in fact there are considerable dependence, see 2.13, 2.14).

In 3.1, 3.3 we present measures of regularity of filters, in 3.2 we present measures of hereditary cofinality of $\prod \bar{\lambda}/D$: allowing to decrease $\bar{\lambda}$ and/or increase the filter. In 3.4–3.8 we try to estimate reduced products of cardinalities $\prod_{i<\kappa} \lambda_i/D$ and in 3.9 we give a reasonable upper bound by hereditary cofinality ($\leq (\theta^\kappa/D + \mathrm{hcf}_{D,\theta}(\prod_{i<\kappa} \lambda_I))^{<\theta}$ when $\theta \geq \mathrm{reg}_\otimes(D)$).

In 3.10–3.11 we return to existence of eub's and obedience and in 3.12 draw conclusion on "downward closure". On $T_D(f)$, starting with Galvin and Hajnal [2] see [8].

In 3.13–3.14 we estimate $T_D(\bar\lambda)$ and in 3.15 try to translate it more fully to pcf problem (countable cofinality is somewhat problematic (so we restrict ourselves to $T_D(\bar\lambda) > \mu = \mu^{\aleph_0}$). We also mention $\aleph_1$-complete filters; (3.16, 3.17) and see what can be done without relaying on pcf (3.20)).

Now we deal with depth: define it (3.18, see 3.19), give lower bound (3.22), compute it for ultraproducts of interval Boolean algebras of ordinals (3.24). Lastly we connect the problem "does $\lambda_i < \mathrm{Depth}^+(B_i)$ for $i < \kappa$ implies $\mu < \mathrm{Depth}^+(\prod_{i<\kappa} B_i/D)$" at least when $\mu > 2^\kappa$ and $(\forall \alpha < \mu)[|\alpha|^{\aleph_0} < \mu]$, to a pcf problem (in 3.26). This is continued in [16].

In the last section we phrase a reason why 1.5(∗) works (see 4.1), analyze the case we weaken 1.5(∗) to $\liminf_{I^*}(\bar\lambda) \geq \theta \geq \mathrm{wsat}(I^*)$ proving the pseudo pcf-th (4.3).

2. Basic pcf

2.1. Notation

I, J denote ideals on a set $\mathrm{Dom}(I)$, $\mathrm{Dom}(J)$ resp., called its domain (possibly $\bigcup_{A \in I} A \subset \mathrm{Dom}\, I$). If not said otherwise the domain is an infinite cardinal denoted by κ and also the ideal is proper i.e. $\mathrm{Dom}(I) \notin I$. Similarly D denotes a filter on a set $\mathrm{Dom}\, D$; we do not always distinguish strictly between an ideal on κ and the dual filter on κ. Let $\bar\lambda$ denote a sequence of the form $\langle \lambda_i : i < \kappa \rangle$. We say $\bar\lambda$ is regular if every λ_i is regular, $\min \bar\lambda = \min\{\lambda_i : i < \kappa\}$ (of course also in $\bar\lambda$ we can replace κ by another set), and let $\prod \bar\lambda = \prod_{i<\kappa} \lambda_i$; usually we are assuming $\bar\lambda$ is regular. Let I^* denote a fixed ideal on κ. Let $I^+ = \mathcal{P}(\kappa) \setminus I$ (similarly $D^+ = \{A \subseteq \kappa : \kappa \setminus A \notin D\}$), let

$$\liminf_I \bar\lambda = \min\{\mu : \{i < \kappa : \lambda_i \leq \mu\} \in I^+\} \quad \text{and}$$

$$\limsup_I \bar\lambda = \min\{\mu : \{i < \kappa : \lambda_i > \mu\} \in I\} \quad \text{and}$$

$$\mathrm{atom}_I \bar\lambda = \{\mu : \{i : \lambda_i = \mu\} \in I^+\}.$$

For a set A of ordinals with no last element, $J_A^{\mathrm{bd}} = \{B \subseteq A : \sup(B) < \sup(A)\}$, i.e. the ideal of bounded subsets. Generally, if $\mathrm{inv}(X) = \sup\{|y| :\models \varphi[X, y]\}$ then $\mathrm{inv}^+(X) = \sup\{|y|^+ :\models \varphi[X, y]\}$, and any y such that $\models \varphi[X, y]$ is a witness for $|y| \leq \mathrm{inv}(X)$ (and $|y| < \mathrm{inv}^+(X)$), and it exemplifies this. Let $\bar A^*_\theta[\bar\lambda] = \langle A^*_\alpha : \alpha < \theta \rangle = \langle A^*_{\theta,\alpha}[\bar\lambda] : \alpha < \theta \rangle$ be defined by: $A^*_\alpha = \{i < \kappa : \lambda_i > \alpha\}$. Let Ord be the class of ordinals.

Definition 1. *(1) For a partial order* P:*

(a) P is λ-directed if: for every $A \subseteq P, |A| < \lambda$ there is $q \in P$ such that $\bigwedge_{p \in A} p \leq q$, and we say: q is an upper bound of A;
(b) P has true cofinality λ if there is $\langle p_\alpha : \alpha < \lambda \rangle$ cofinal in P, i.e.: $\bigwedge_{\alpha<\beta} p_\alpha < p_\beta$ and $\forall_q \in P[\bigvee_{\alpha<\lambda} q \leq p_\alpha]$ [and one writes $\mathrm{tcf}(P) = \lambda$ for the minimal such λ] (note: if P is linearly ordered it always has a true cofinality but e.g. $(\omega, <) \times (\omega_1, <)$ does not).
(c) P is called endless if $\forall p \in P \exists q \in P[q > p]$ (so if P is endless, in clauses (a), (b), (d) above we can replace $\leq$ by $<$).
(d) $A \subseteq P$ is a cover if: $\forall p \in P \exists q \in A[p \leq q]$; we also say "$A$ is cofinal in P".
(e) $\mathrm{cf}(P) = \min\{|A| : A \subseteq P$ is a cover$\}$.
(f) We say that, in P, p is a lub (least upper bound) of $A \subseteq P$ if:

(α) p is an upper bound of A (see (a))
(β) If p' is an upper bound of A then $p \leq p'$.

(2) If D is a filter on S, α_s (for $s \in S$) are ordinals, $f, g \in \prod_{s\in S} \alpha_s$, then: $f/D < g/D$, $f <_D g$ and $f < g \mod D$ all mean $\{s \in S : f(s) < g(s)\} \in D$. Also if f, g are partial functions from S to ordinals, D a filter on S __then__ $f < g \mod D$ means $\{i \in \mathrm{Dom}(D) : i \notin \mathrm{Dom}(f)$ __or__ $f(i) < g(i)$ (so both are defined)$\}$ belongs to D. We write $X = A \mod D$ if $\mathrm{Dom}(D) \setminus [(X \setminus A) \cup (A \setminus X)]$ belongs to D. Similarly for $\leq$, and we do not distinguish between a filter and the dual ideal in such notions. So if J is an ideal on κ and $f, g \in \prod \bar{\lambda}$, then $f < g \mod J$ iff $\{i < \kappa : \neg f(i) < g(i)\} \in J$. Similarly if we replace the α_s's by partial orders.
(3) For $f, g : S \to$ Ordinals, $f < g$ means $\bigwedge_{s\in S} f(s) < g(s)$; similarly $f \leq g$. So $(\prod \bar{\lambda}, \leq)$ is a partial order, we denote it usually by $\prod \bar{\lambda}$; similarly $\prod f$ or $\prod_{i<\kappa} f(i)$.
(4) If I is an ideal on $\kappa, F \subseteq {}^\kappa\mathrm{Ord}$, we call $g \in {}^\kappa\mathrm{Ord}$ an $\leq_I$-eub (exact upper bound) of F if:

(α) g is an $\leq_I$-upper bound of F (in ${}^\kappa\mathrm{Ord}$)
(β) If $h \in {}^\kappa\mathrm{Ord}, h <_I \mathrm{Max}\{g, 1\}$ __then__ for some $f \in F, h < \max\{f, 1\} \mod I$.
(γ) If $A \subseteq \kappa$, $A \neq \emptyset \mod I$ and $[f \in F \Rightarrow f \restriction A =_I 0_A$, i.e. $\{i \in A : f(i) \neq 0\} \in I]$ then $g \restriction A =_J 0_A$.

(5)(a) We say the ideal I (on κ) is θ-weakly saturated if κ cannot be divided to θ pairwise disjoint sets from I^+ (which is $\mathcal{P}(\kappa) \setminus I$)
(b) $\mathrm{wsat}(I) = \min\{\theta : I$ is $\theta -$ weakly saturated$\}$

*actually we do not require $p \leq q \leq p \Rightarrow p = q$ so we should say quasi partial order

Remark 1.1A.

(1) Concerning 1.1(4), note: $g' = \text{Max}\{g, 1\}$ means $g'(i) = \text{Max}\{g(i), 1\}$ for each $i < \kappa$; if there is $f \in F$, $\{i < \kappa : f(i) = 0\} \in I$ we can replace $\text{Max}\{g, 1\}$, $\text{Max}\{f, 1\}$ by g, f respectively in clause (β) and omit clause (γ).
(2) Considering $\prod_{i<\kappa} f(i)$, $<_I$ formally if $(\exists i) f(i) = 0$ then $\prod_{i<\kappa} f(i) = \emptyset$; but we usually ignore this, particularly when $\{i : f(i) = 0\} \in I$.

Definition 2. *Below if* Γ *is "a filter disjoint to* I*", we write* I *instead* Γ.

(1) For a property Γ *of ultrafilters:*

$$\text{pcf}_\Gamma(\bar\lambda) = \text{pcf}(\bar\lambda, \Gamma) = \{\text{tcf}(\prod \bar\lambda/D) : D \textit{ is an ultrafilter on } \kappa \text{ satisfying } \Gamma\}$$

(so $\bar\lambda$ *is a sequence of ordinals, usually of regular cardinals, note: as* D *is an ultrafilter,* $\prod \bar\lambda/D$ *is linearly ordered hence has true cofinality).*

(1A) More generally, for a property Γ *of ideals on* κ *we let* $\text{pcf}_\Gamma(\bar\lambda) = \{\text{tcf}(\prod \bar\lambda/J) : J$ *is an ideal on* κ satisfying Γ *such that* $\prod \bar\lambda/J$*has true cofinality*$\}$*. Similarly below.*

(2) $J_{<\lambda}[\bar\lambda, \Gamma] = \{B \subseteq \kappa$*: for no ultrafilter* D *on* κ *satisfying* Γ *to which* B *belongs, is* $\text{tcf}(\prod \bar\lambda/D) \geq \lambda\}$.

(3) $J_{\leq\lambda}[\bar\lambda, \Gamma] = J_{<\lambda^+}[\bar\lambda, \Gamma]$.

(4) $\text{pcf}_\Gamma(\bar\lambda, I) = \{\text{tcf}(\prod \bar\lambda/D)$ *:*D *a filter on* κ *disjoint to* I *satisfying* $\Gamma\}$.

(5) If $B \in I^+, \text{pcf}_I(\bar\lambda \restriction B) = \text{pcf}_{I+(\kappa\setminus B)}(\bar\lambda)$ *(so if* $B \in I$ *it is* $\emptyset$*), also* $J_{<\lambda}(\bar\lambda \restriction B, I) \subseteq \mathcal{P}(B)$ *is defined similarly.*

(6) If $I = I^*$ *we may omit it, similarly in (2), (4).*

If $\Gamma = \Gamma_{I^*} = \{D : D$ *a filter on* κ *disjoint to* $I^*\}$ *we may omit it.*

Remark. We mostly use $\text{pcf}(\bar\lambda), J_{<\lambda}[\bar\lambda]$.

Claim 1.3.

(0) $(\prod \bar\lambda, <_J)$ and $(\prod \bar\lambda, \leq_J)$ are endless (even when each λ_i is just a limit ordinal);
(1) $\min(\text{pcf}_I(\bar\lambda)) \geq \liminf_I(\bar\lambda)$ for $\bar\lambda$ regular;
(2) (i) If $B_1 \subseteq B_2$ are from I^+ <u>then</u> $\text{pcf}_I(\bar\lambda \restriction B_1) \subseteq \text{pcf}_I(\bar\lambda \restriction B_2)$;
(ii) If $I \subseteq J$ <u>then</u> $\text{pcf}_J(\bar\lambda) \subseteq \text{pcf}_I(\bar\lambda)$; and
(iii) For $B_1, B_2 \subseteq \kappa$ we have $\text{pcf}_I(\bar\lambda \restriction (B_1 \cup B_2)) = \text{pcf}_I(\bar\lambda \restriction B_1) \bigcup \text{pcf}_I(\bar\lambda \restriction B_2)$. Also
(iv) $A \in J_{<\lambda}[\bar\lambda \restriction (B_1 \cup B_2)] \Leftrightarrow A \cap B_1 \in J_{<\lambda}[\bar\lambda \restriction B_1]$ & $A \cap B_2 \in J_{<\lambda}[\bar\lambda \restriction B_2]$
(v) If $A_1, A_2 \in I^+, A_1 \cap A_2 = \emptyset, A_1 \cup A_2 = \kappa$, and $\text{tcf}(\prod \bar\lambda \restriction A_\ell, <_I) = \lambda$ for $\ell = 1, 2$ <u>then</u> $\text{tcf}(\prod \bar\lambda, <_I) = \lambda$; and if the sequence $\bar f = \langle f_\alpha : \alpha < \lambda\rangle$ witness both assumptions then it witness the conclusion.
(3) (i) If $B_1 \subseteq B_2 \subseteq \kappa$, B_1 finite and $\bar\lambda$ regular <u>then</u>

$$\text{pcf}_I(\bar\lambda \restriction B_2) \setminus \text{Rang}(\bar\lambda \restriction B_1) \subseteq \text{pcf}_I(\bar\lambda \restriction (B_2 \setminus B_1)) \subseteq \text{pcf}_I(\bar\lambda \restriction B_2)$$

(ii) If in addition $i \in B_1 \Rightarrow \lambda_i < \min(\mathrm{Rang}[\bar\lambda \restriction (B_2 \setminus B_1)])$, <u>then</u> $\mathrm{pcf}_I(\bar\lambda \restriction B_2) \setminus \mathrm{Rang}(\bar\lambda \restriction B_1) = \mathrm{pcf}_I(\bar\lambda \restriction (B_2 \setminus B_1))$.

(4) Let $\bar\lambda$ be regular (i.e. each λ_i is regular);

(i) If $\theta = \liminf_I \bar\lambda$ <u>then</u> $\prod \bar\lambda / I$ is θ-directed
(ii) If $\theta = \liminf_I \bar\lambda$ is singular <u>then</u> $\prod \bar\lambda / I$ is θ^+-directed
(iii) If $\theta = \liminf_I \bar\lambda$ is inaccessible (i.e. a limit regular cardinal), the set $\{i < \kappa : \lambda_i = \theta\}$ is in the ideal I and for some club E of θ, $\{i < \kappa : \lambda_i \in E\} \in I$ <u>then</u> $\prod \bar\lambda / I$ is θ^+-directed. We can weaken the assumption to "I is not weakly normal for $\bar\lambda$" (defined in the next sentence). Let "I is not medium normal for $(\theta, \bar\lambda)$" mean: for some $h \in \prod \bar\lambda$, for no $j < \theta$ is $\{i < \kappa : \lambda_i \le \theta \Rightarrow h(i) < j\} = \kappa \mod I$; and let "$I$ is not weakly normal for (θ, λ)" mean: for some $h \in \prod \bar\lambda$, for no $\zeta < \liminf_I(\bar\lambda) = \theta$, is $\{i < \kappa : \lambda_i \le \theta \Rightarrow h(i) < \zeta\} \in I^+$.
(iv) If $\{i : \lambda_i = \theta\} = \kappa \mod I$ and I is medium normal for $\bar\lambda$ <u>then</u> $(\prod \bar\lambda, <_I)$ has true cofinality θ.
(v) If $\prod \bar\lambda / I$ is θ-directed <u>then</u> $\mathrm{cf}(\prod \bar\lambda / I) \ge \theta$ and $\min \mathrm{pcf}_I(\prod \bar\lambda) \ge \theta$.
(vi) $\mathrm{pcf}_I(\bar\lambda)$ is non empty set of regular cardinals. [see part (7)]

(5) Assume $\bar\lambda$ is regular and: if $\theta' =: \limsup_I(\bar\lambda)$ is regular <u>then</u> I is not medium normal for $(\theta', \bar\lambda)$. <u>Then</u> $\mathrm{pcf}_I(\bar\lambda) \not\subseteq (\limsup_I(\bar\lambda))^+$; in fact for some ideal J extending $I, \prod \bar\lambda / J$ is $(\limsup_I(\bar\lambda))^+$-directed.

(6) If D is a filter on a set S and for $s \in S$, α_s is a limit ordinal <u>then</u>:

(i) $\mathrm{cf}(\prod_{s \in S} \alpha_s, <_D) = \mathrm{cf}(\prod_{s \in S} \mathrm{cf}(\alpha_s), <_D) = \mathrm{cf}(\prod_{s \in S} (\alpha_s, <)/D)$, and
(ii) $\mathrm{tcf}(\prod_{s \in S} \alpha_s, <_D) = \mathrm{tcf}(\prod_{s \in S} (\mathrm{cf}(\alpha_s), <_D)) = \mathrm{tcf}(\prod_{s \in S} (\alpha_s, <)/D)$.

In particular, if one of them is well defined, <u>then</u> so are the others. This is true even if we replace α_s by linear orders or even partial orders with true cofinality.

(7) If D is an ultrafilter on a set S, λ_s a regular cardinal, <u>then</u> $\theta =: \mathrm{tcf}(\prod_{s \in S} \lambda_s, <_D)$ is well defined and $\theta \in \mathrm{pcf}(\{\lambda_s : s \in S\})$.

(8) If D is a filter on a set S, for $s \in S$, λ_s is a regular cardinal, $S^* = \{\lambda_s : s \in S\}$ and

$$E =: \{B : \quad B \subseteq S^* \text{ and } \{s : \lambda_s \in B\} \in D\}$$

and $\lambda_s > |S|$ or at least $\lambda_s > |\{t : \lambda_t = \lambda_s\}|$ for any $s \in S$ <u>then</u>:

(i) E is a filter on S^*, and if D is an ultrafilter on S <u>then</u> E is an ultrafilter on S^*.
(ii) S^* is a set of regular cardinals and
if $s \in S \Rightarrow \lambda_s > |S|$ <u>then</u> $(\forall \lambda \in S^*) \lambda > |S^*|$,
(iii) $F = \{f \in \prod_{s \in S} \lambda_s : s = t \Rightarrow f(s) = f(t)\}$ is a cover of $\prod_{s \in S} \lambda_s$,
(iv) $\mathrm{cf}(\prod_{s \in S} \lambda_s / D) = \mathrm{cf}(\prod S^* / E)$ and $\mathrm{tcf}(\prod_{s \in S} \lambda_s / D) = \mathrm{tcf}(\prod S^* / E)$.

(9) Assume I is an ideal on $\kappa, F \subseteq {}^\kappa\mathrm{Ord}$ and $g \in {}^\kappa\mathrm{Ord}$. If g is a $\le_I$-eub of F <u>then</u> g is a $\le_I$-lub of F.

(10) $\sup \mathrm{pcf}_I(\bar\lambda) \le |\prod \bar\lambda/I|$

(11) If I is an ideal on S and $(\prod_{s\in S}\alpha_s, <_I)$ has true cofinality λ as exemplified by $\bar f = \langle f_\alpha : \alpha < \lambda\rangle$ then the function $\langle \alpha_s : s \in S\rangle$ is a $<_I$ -eub (hence $<_I$-lub) of $\bar f$.

(12) The inverse of (11) holds: if I is an ideal on S and $f_\alpha \in {}^S\mathrm{Ord}$ for $\alpha < \lambda = \mathrm{cf}(\lambda)$, $\langle f_\alpha : \alpha < \lambda\rangle$ is $<_I$-increasing with $<_I -$ eub f then $\mathrm{tcf}(\prod_i f(i), <_I) = \mathrm{tcf}(\prod \mathrm{cf}[f(i)], <_I) = \lambda$.

(13) If $I \subseteq J$ are ideals on κ then

  (a) $\mathrm{wsat}(I) \ge \mathrm{wsat}(J)$
  (b) $\liminf_I(\bar\lambda) \le \liminf_J(\bar\lambda)$
  (c) If $\lambda = \mathrm{tcf}(\prod_{i<\kappa}\lambda_i, <_I)$ then $\lambda = \mathrm{tcf}(\prod_{i<\kappa}\lambda_i, <_J)$

(14) If f_1, f_2 are $<_I$-lub of F then $f_1 =_I f_2$.

Proof. They are all very easy, e.g.

(0) We shall show $(\prod \bar\lambda, <_J)$ is endless (assuming, of course, that J is a proper ideal on κ). Let $f \in \prod \bar\lambda$, then $g =: f + 1$ (defined $(f+1)(\gamma) = f(\gamma) + 1$) is in $\prod \bar\lambda$ too as each λ_α being an infinite cardinal is a limit ordinal and $f < g \mod J$.

(5) Let $\theta' =: \limsup_I(\bar\lambda)$ and define

$$J =: \{A \subseteq \kappa : \text{ for some } \theta < \theta', \{i < \kappa : \lambda_i > \theta \text{ and } i \in A\} \text{ belongs to } I\}.$$

Clearly J is an ideal on κ extending I (and $\kappa \notin J$) and $\limsup_J(\bar\lambda) = \liminf_J(\bar\lambda) = \theta'$.

Case 1: θ' is singular
By part (4) clause (ii), $\prod \bar\lambda/J$ is $(\theta')^+$-directed and we get the desired conclusion.
Case 2: θ' is regular.
Let $h \in \prod \lambda$ witness that "I is not medium normal for $(\theta', \bar\lambda)$" and let

$$J^* = \{A \subseteq \kappa : \text{ for some } j < \theta' \text{ we have } \{i \in A : h(i) < j\} = A \mod I\}.$$

Note that if $A \in J$ then for some $\theta < \theta'$, $A' =: \{i \in A : \theta_i > \theta\} \in I$ hence the choice $j =: \theta$ witness $A \in J^*$. So $J \subseteq J^*$. Also $J^* \subseteq \mathcal{P}(\kappa)$ by its definition. J^* is closed under subsets (trivial) and under union [why? assume A_0, $A_1 \in J^*$, $A = A_0 \cup A_1$; choose $j_0, j_1 < \theta'$ such that $A'_\ell =: \{i \in A_\ell : h(i) < j_\ell\} = A_\ell \mod I$, so $j =: \max\{j_0, j_1\} < \theta$ and $A' = \{i \in A : h(i) < j\} = A \mod I$; so $A \in J^*$]. Also $\kappa \notin J^*$ [why? as h witness that I is not medium normal for $(\theta', \bar\lambda)$]. So together J^* is an ideal on κ extending I. Now J^* is not weakly normal for $(\theta, \bar\lambda)$, as witnessed by h. Lastly $\prod \bar\lambda/J^*$ is $(\theta')^+$-directed (by part (4) clause (iii)), and so $\mathrm{pcf}_J(\bar\lambda)$ is disjoint to $(\theta')^+$.

(9) Let us prove g is a $\le_I$-lub of F in $({}^\kappa\mathrm{Ord}, \le_I)$. As we can deal separately with $I + A$, $I + (\kappa\backslash A)$ where $A =: \{i : g(i) = 0\}$, and the later case is trivial we can assume $A = \emptyset$. So assume g is not a $\le_I$-lub, so there is an upper bound g' of F, but not $g \le_I g'$. Define $g'' \in {}^\kappa\mathrm{Ord}$:

$$g''(i) = \begin{cases} 0 & \text{if } g(i) \le g'(i) \\ g'(i) & \text{if } g'(i) < g(i) \end{cases}.$$

Clearly $g'' <_I g$. So, as g in an $\le_I$ - eub of F for I, there is $f \in F$ such that $g'' <_I \max\{f, 1\}$, but $B =: \{i : g'(i) < g(i)\} \ne \emptyset \mod I$ (as not $g \le_I g'$) so $g' \restriction B = g'' \restriction B <_I \max\{f, 1\} \restriction B$. But we know that $f \le_I g'$ (as g' is an upper bound of F) hence $f \restriction B \le_I g' \restriction B$, so by the previous sentence necessarily $f \restriction B =_I 0_B$ hence $g' \restriction B =_I 0_B$; as g' is a $\le_I$-upper bound of F we know $[f' \in F \Rightarrow f' \restriction B =_I 0_B]$, hence by (γ) of Definition 1(4) we have $g \restriction B =_I 0_B$, a contradiction to $B \notin I$ (see above). $\square_{1.3}$

Remark 1.3A. In 1.3 we can also have the straight monotonicity properties of

$$\mathrm{pcf}_I(\prod \bar\lambda, \Gamma) \text{ in } \Gamma \text{ and in } I.$$

Claim 1.4:

(1) $J_{<\lambda}[\bar\lambda]$ is an ideal (of $\mathcal{P}(\kappa)$ i.e. on κ, but the ideal may not be proper).
(2) If $\lambda \le \mu$, then $J_{<\lambda}[\bar\lambda] \subseteq J_{<\mu}[\bar\lambda]$
(3) If λ is singular, $J_{<\lambda}[\bar\lambda] = J_{<\lambda^+}[\bar\lambda] = J_{\le\lambda}[\bar\lambda]$
(4) If $\lambda \notin \mathrm{pcf}(\bar\lambda)$, <u>then</u> we have $J_{<\lambda}[\bar\lambda] = J_{\le\lambda}[\bar\lambda]$.
(5) If $A \subseteq \kappa$, $A \notin J_{<\lambda}[\bar\lambda]$, and $f_\alpha \in \prod \bar\lambda \restriction A$, $\langle f_\alpha : \alpha < \lambda\rangle$ is $<_{J_{<\lambda}[\bar\lambda]}$-increasing cofinal in $(\prod \bar\lambda \restriction A)/J_{<\lambda}[\bar\lambda]$ <u>then</u> $A \in J_{\le\lambda}[\bar\lambda]$. Also this holds when we replace $J_{<\lambda}[\bar\lambda]$ by any ideal J on κ, $I^* \subseteq J \subseteq J_{\le\lambda}[\bar\lambda]$.
(6) The earlier parts hold for $J_{<\lambda}[\bar\lambda, \Gamma]$ too.

Proof. Straight.

Lemma 1. *Assume*

$(*)$ $\bar\lambda$ is regular and

(α) $\min \bar\lambda > \theta \ge \mathrm{wsat}(I^*)$ (see 1.1(5)(b)) or at least
(β) $\liminf_{I^*}(\bar\lambda) \ge \theta \ge \mathrm{wsat}(I^*)$, and $\prod \bar\lambda/I^*$ is θ^+-directed.[**]

If λ is a cardinal $\ge \theta$, and $\kappa \notin J_{<\lambda}[\bar\lambda]$ <u>then</u> $(\prod \bar\lambda, <_{J_{<\lambda}[\bar\lambda]})$ is λ-directed (remember: $J_{<\lambda}[\bar\lambda] = J_{<\lambda}[\bar\lambda, I^*]$).

Proof. Note: if $f \in \prod \bar\lambda$ then $f < f + 1 \in \prod \bar\lambda$, (i.e. $(\prod \bar\lambda, <_{J_\lambda[\bar\lambda]})$ is endless) where $f+1$ is defined by $(f+1)(i) = f(i)+1$. Let $F \subseteq \prod \bar\lambda$, $|F| < \lambda$, and we shall prove that for some $g \in \prod \bar\lambda$ we have $(\forall f \in F)(f \le g \mod J_{<\lambda}[\bar\lambda])$, this suffices. The proof is by induction on $|F|$. If $|F|$ is finite, this is trivial. Also if $|F| \le \theta$, when (α) of $(*)$ holds it is easy: let $g \in \prod \bar\lambda$ be $g(i) = \sup\{f(i) : f \in F\} < \lambda_i$; when (β) of $(*)$ holds use second clause of (β). So assume

[**] note if $\mathrm{cf}(\theta) < \theta$ then "θ^+-directed" follows from "θ-directed" which follows from "$\liminf_{I^*}(\bar\lambda) \ge \theta$, i.e. first part of clause (β) implies clause (β). Note also that if clause (α) holds then $\prod \bar\lambda/I^*$ is θ^+-directed (even $(\prod \bar\lambda, <)$ is θ^+-directed), so clause (α) implies clause (β).

$|F| = \mu, \theta < \mu < \lambda$ so let $F = \{f^0_\alpha : \alpha < \mu\}$. By the induction hypothesis we can choose by induction on $\alpha < \mu$, $f^1_\alpha \in \prod \bar{\lambda}$ such that:

(a) $f^0_\alpha \leq f^1_\alpha \mod J_{<\lambda}[\bar{\lambda}]$
(b) For $\beta < \alpha$ we have $f^1_\beta < f^1_\alpha \mod J_{<\lambda}[\bar{\lambda}]$.

If μ is singular, there is $C \subseteq \mu$ unbounded, $|C| = \mathrm{cf}(\mu) < \mu$, and by the induction hypothesis there is $g \in \prod \bar{\lambda}$ such that for $\alpha \in C$, $f^1_\alpha \leq g \mod J_{<\lambda}[\bar{\lambda}]$. Now g is as required: $f^0_\alpha \leq f^1_\alpha \leq f^1_{\min(C\setminus\alpha)} \leq g \mod J_{<\lambda}[\bar{\lambda}]$. So without loss of generality μ is regular. Let us define $A^*_\varepsilon =: \{i < \kappa : \lambda_i > |\varepsilon|\}$ for $\varepsilon < \theta$, so $\varepsilon < \zeta < \theta \Rightarrow A^*_\zeta \subseteq A^*_\varepsilon$ and $\varepsilon < \theta \Rightarrow A^*_\varepsilon = \kappa \mod I^*$. Now we try to define by induction on $\varepsilon < \theta, g_\varepsilon, \alpha_\varepsilon = \alpha(\varepsilon) < \mu$ and $\langle B^\varepsilon_\alpha : \alpha < \mu\rangle$ such that:

(i) $g_\varepsilon \in \prod \bar{\lambda}$
(ii) For $\varepsilon < \zeta$ we have $g_\varepsilon \restriction A^*_\zeta \leq g_\zeta \restriction A^*_\zeta$
(iii) For $\alpha < \mu$ let $B^\varepsilon_\alpha =: \{i < \kappa : f^1_\alpha(i) > g_\varepsilon(i)\}$
(iv) For each $\varepsilon < \theta$, for every $\alpha \in [\alpha_{\varepsilon+1}, \mu), B^\varepsilon_\alpha \neq B^{\varepsilon+1}_\alpha \mod J_{<\lambda}[\bar{\lambda}]$.

We cannot carry this definition: as letting $\alpha(*) = \sup\{\alpha_\varepsilon : \varepsilon < \theta\}$, then $\alpha(*) < \mu$, since $\mu = \mathrm{cf}(\mu) > \theta$. We know that $B^\varepsilon_{\alpha(*)} \cap A^*_{\varepsilon+1} \neq B^{\varepsilon+1}_{\alpha(*)} \cap A^*_{\varepsilon+1}$ for $\alpha < \theta$ (by (iv) and as $A^*_{\varepsilon+1} = \kappa \mod I^*$ and $I^* \subseteq J_{<\lambda}[\bar{\lambda}]$) and $B^\varepsilon_{\alpha(*)} \subseteq \kappa$ (by (iii)) and $[\varepsilon < \zeta \Rightarrow B^\zeta_{\alpha(*)} \cap A^*_\zeta \subseteq B^\varepsilon_{\alpha(*)}]$ (by (ii)), together $\langle A^*_{\varepsilon+1} \cap (B^\varepsilon_{\alpha(*)} \setminus B^{\varepsilon+1}_{\alpha(*)}) : \varepsilon < \theta\rangle$ is a sequence of θ pairwise disjoint members of $(I^*)^+$, a contradiction*** to the definition of $\theta = \mathrm{wsat}(I^*)$.

Now for $\varepsilon = 0$ let g_i be f^1_0 and $\alpha_\varepsilon = 0$.

For ε limit let $g_\varepsilon(i) = \bigcup_{\zeta<\varepsilon} g_\zeta(i)$ for $i \in A^*_\varepsilon$ and zero otherwise (note: $g_\varepsilon \in \prod \bar{\lambda}$ as $\varepsilon < \theta, \lambda_i > \varepsilon$ for $i \in A^*_\varepsilon$ and $\bar{\lambda}$ is a sequence of regular cardinals) and let $\alpha_\varepsilon = 0$. For $\varepsilon = \zeta + 1$, suppose that g_ζ hence $\langle B^\zeta_\alpha : \alpha < \mu\rangle$ are defined. If $B^\zeta_\alpha \in J_{<\lambda}[\bar{\lambda}]$ for unboundedly many $\alpha < \mu$ (hence for every $\alpha < \mu$) then g_ζ is an upper bound for $F \mod J_{<\lambda}[\bar{\lambda}]$ and the proof is complete. So assume this fails, then there is a minimal $\alpha(\varepsilon) < \mu$ such that $B^\zeta_{\alpha(\varepsilon)} \notin J_{<\lambda}[\bar{\lambda}]$. As $B^\zeta_{\alpha(\varepsilon)} \notin J_{<\lambda}[\bar{\lambda}]$, by Definition 2(2) for some ultrafilter D on κ disjoint to $J_{<\lambda}[\bar{\lambda}]$ we have $B^\zeta_{\alpha(\varepsilon)} \in D$ and $\mathrm{cf}(\prod \bar{\lambda}/D) \geq \lambda$. Hence $\{f^1_\alpha/D : \alpha < \mu\}$ has an upper bound h_ε/D where $h_\varepsilon \in \prod \bar{\lambda}$. Let us define $g_\varepsilon \in \prod \bar{\lambda}$:

$$g_\varepsilon(i) = \mathrm{Max}\{g_\zeta(i), h_\varepsilon(i)\}.$$

Now (i), (ii) hold trivially and B^ε_α is defined by (iii). Why does (iv) hold (for ζ) with $\alpha_{\zeta+1} = \alpha_\varepsilon =: \alpha(\varepsilon)$? Suppose $\alpha(\varepsilon) \leq \alpha < \mu$. As $f^1_{\alpha(\varepsilon)} \leq f^1_\alpha$

***in fact note that for no $B_\varepsilon \subseteq \kappa(\varepsilon < \theta)$ do we have: $B_\varepsilon \neq B_{\varepsilon+1} \mod I^*$ and $\varepsilon < \zeta < \theta \Rightarrow B_\varepsilon \cap A_\zeta \subseteq B_\zeta$ where $A_\zeta = \kappa \mod I^*$ (e.g. $A_\zeta = A^*_\zeta$)

mod $J_{<\lambda}[\bar\lambda]$ clearly $B^\zeta_{\alpha(\varepsilon)} \subseteq B^\zeta_\alpha \mod J_{<\lambda}[\bar\lambda]$. Moreover $J_{<\lambda}[\bar\lambda]$ is disjoint to D (by its choice) so $B^\zeta_{\alpha(\varepsilon)} \in D$ implies $B^\zeta_\alpha \in D$.

On the other hand B^ε_α is $\{i < \kappa : f^1_\alpha(i) > g_\varepsilon(i)\}$ which is equal to $\{i \in \bar\lambda : f^1_\alpha(i) > g_\zeta(i), h_\varepsilon(i)\}$ which does not belong to D (h_ε was chosen such that $f^1_\alpha \leq h_\varepsilon \mod D$). We can conclude $B^\varepsilon_\alpha \notin D$, whereas $B^\zeta_\alpha \in D$; so they are distinct $\mod J_{<\lambda}[\bar\lambda]$ as required in clause (iv).

Now we have said that we cannot carry the definition for all $\varepsilon < \theta$, so we are stuck at some ε; by the above ε is successor, say $\varepsilon = \zeta + 1$, and g_ζ is as required: an upper bound for F modulo $J_{<\lambda}[\bar\lambda]$. $\square_{1.5}$

Lemma 2. *: If* $(*)$ *of 1.5,* D *is an ultrafilter on* κ *disjoint to* I^* *and* $\lambda = \mathrm{tcf}(\prod\bar\lambda, <_D)$*, <u>then</u> for some* $B \in D$*,* $(\prod\bar\lambda \restriction B, <_{J_{<\lambda}[\bar\lambda]})$ *has true cofinality* λ*. (So* $B \in J_{\leq\lambda}[\bar\lambda] \backslash J_{<\lambda}[\bar\lambda]$ *by 1.4(5).)*

Proof. By the definition of $J_{<\lambda}[\bar\lambda]$ clearly we have $D \cap J_{<\lambda}[\bar\lambda] = \emptyset$.

Let $\langle f_\alpha/D : \alpha < \lambda\rangle$ be increasing unbounded in $\prod\bar\lambda/D$ (so $f_\alpha \in \prod\bar\lambda$). By 1.5 without loss of generality $(\forall\beta < \alpha)(f_\beta < f_\alpha \mod J_{<\lambda}[\bar\lambda])$.

Now 1.6 follows from 1.7 below: its hypothesis clearly holds. If $\bigwedge_{\alpha<\lambda} B_\alpha = \emptyset \mod D$, (see (A) of 1.7) then (see (D) of 1.7) $J \cap D = \emptyset$ hence (see (D) of 1.7) g/D contradicts the choice of $\langle f_\alpha/D : \alpha < \lambda\rangle$. So for some $\alpha < \lambda, B_\alpha \in D$; by (C) of 1.7 and 1.4(5) we get the desired conclusion. $\square_{1.6}$

Lemma 3. *: Suppose* $(^*)$ *of 1.5,* $\mathrm{cf}(\lambda) > \theta$*,* $f_\alpha \in \prod\bar\lambda$*,* $f_\alpha < f_\beta \mod J_{<\lambda}[\bar\lambda]$ *for* $\alpha < \beta < \lambda$*, and there is no* $g \in \prod\bar\lambda$ *such that for every* $\alpha < \lambda, f_\alpha < g \mod J_{<\lambda}[\bar\lambda]$*. <u>Then</u> there are* B_α *(for* $\alpha < \lambda$*) such that:*

- *(A)* $B_\alpha \subseteq \kappa$ *and for some* $\alpha(*) < \lambda : \alpha(*) \leq \alpha < \lambda \Rightarrow B_\alpha \notin J_{<\lambda}[\bar\lambda]$
- *(B)* $\alpha < \beta \Rightarrow B_\alpha \subseteq B_\beta \mod J_{<\lambda}[\bar\lambda]$ *(i.e.* $B_\alpha \backslash B_\beta \in J_{<\lambda}[\bar\lambda]$*)*
- *(C) For each* α*,* $\langle f_\beta \restriction B_\alpha : \beta < \lambda\rangle$ *is cofinal in* $(\prod\bar\lambda \restriction B_\alpha, <_{J_{<\lambda}[\bar\lambda]})$ *(better restrict yourselves to* $\alpha \geq \alpha(*)$ *(see (A)) so that necessarily* $B_\alpha \notin J_{<\lambda}[\bar\lambda]$*);.*
- *(D) For some* $g \in \prod\bar\lambda, \bigwedge_{\alpha<\lambda} f_\alpha \leq g \mod J$ *where* $J = J_{<\lambda}[\bar\lambda] + \{B_\alpha : \alpha < \lambda\}$*; in fact*
- *(D)*$^+$ *For some* $g \in \prod\bar\lambda$ *for every* $\alpha < \lambda$*, we have* † $f_\alpha \leq g \mod (J_{<\lambda}[\bar\lambda] + B_\alpha)$*, in fact* $B_\alpha = \{i < \kappa : f_\alpha(i) > g(i)\}$
- *(E) If* $g \leq g' \in \prod\bar\lambda$*, then for arbitrarily large* $\alpha < \lambda$*:*

$$\{i < \kappa : [g(i) \geq f_\alpha(i) \Leftrightarrow g'(i) \geq f_\alpha(i)]\} = \kappa \mod J_{<\lambda}[\bar\lambda]$$

  (hence for every large enough $\alpha < \lambda$ *this holds)*
- *(F) If* δ *is a limit ordinal* $< \lambda$*,* f_δ *is a* $\leq_{J_{<\lambda}[\bar\lambda]}$*-lub of* $\{f_\alpha : \alpha < \delta\}$ *<u>then</u>* B_δ *is a lub of* $\{B_\alpha : \alpha < \delta\}$ *in* $\mathcal{P}(\kappa)/J_{<\lambda}[\bar\lambda]$*.*

†Of course, $B_\alpha = \kappa \mod J_{<\lambda}(\bar\lambda)$, this becomes trivial.

Proof of 1.7. Remember that for $\varepsilon < \theta$, $A^*_\varepsilon = \{i < \kappa : \lambda_i > |\varepsilon|\}$ so $A^*_\varepsilon = \kappa \mod I^*$ and $\varepsilon < \zeta \Rightarrow A^*_\zeta \subseteq A^*_\varepsilon$. We now define by induction on $\varepsilon < \theta$, $g_\varepsilon, \alpha(\varepsilon) < \lambda$, $\langle B^\varepsilon_\alpha : \alpha < \lambda\rangle$ such that:

(i) $g_\varepsilon \in \prod \bar\lambda$
(ii) For $\zeta < \varepsilon$, $g_\zeta \restriction A^*_\varepsilon \leq g_\varepsilon \restriction A^*_\varepsilon$
(iii) $B^\varepsilon_\alpha =: \{i \in \kappa : f_\alpha(i) > g_\varepsilon(i)\}$
(iv) If $\alpha(\varepsilon) \leq \alpha < \lambda$ then $B^\varepsilon_\alpha \neq B^{\varepsilon+1}_\alpha \mod J_{<\lambda}[\bar\lambda]$

For $\varepsilon = 0$ let $g_\varepsilon = f_0$, and $\alpha(\varepsilon) = 0$.

For ε limit let $g_\varepsilon(i) = \bigcup_{\zeta<\varepsilon} g_\zeta(i)$ if $i \in A^*_\varepsilon$ and zero otherwise; now

$$[\zeta < \varepsilon \Rightarrow g_\zeta \restriction A^*_\varepsilon \leq g_\varepsilon \restriction A^*_\varepsilon]$$

holds trivially and $g_\varepsilon \in \prod \bar\lambda$ as each λ_i is regular and $[i \in A^*_\varepsilon \Leftrightarrow \lambda_i > \varepsilon]$), and let $\alpha(\varepsilon) = 0$.

For $\varepsilon = \zeta + 1$, if $\{\alpha < \lambda : B^\zeta_\alpha \in J_{<\lambda}[\bar\lambda]\}$ is unbounded in λ, then g_ζ is a bound for $\langle f_\alpha : \alpha < \lambda\rangle \mod J_{<\lambda}[\bar\lambda]$, contradicting an assumption. Clearly

$$\alpha < \beta < \lambda \Rightarrow B^\zeta_\alpha \subseteq B^\zeta_\beta \mod J_{<\lambda}[\bar\lambda]$$

hence $\{\alpha < \lambda : B^\zeta_\alpha \in J_{<\lambda}[\bar\lambda]\}$ is an initial segment of λ. So by the previous sentence there is $\alpha(\varepsilon) < \lambda$ such that for every $\alpha \in [\alpha(\varepsilon), \lambda)$, we have $B^\zeta_\alpha \notin J_{<\lambda}[\bar\lambda]$ (of course, we may increase $\alpha(\varepsilon)$ later). If $\langle B^\zeta_\alpha : \alpha < \lambda\rangle$ satisfies the desired conclusion, with $\alpha(\varepsilon)$ for $\alpha(*)$ in (A) and g_ζ for g in (D), (D)$^+$ and (E), we are done. Now among the conditions in the conclusion of 1.7, (A) holds by the definition of B^ζ_α and of $\alpha(\varepsilon)$, (B) holds by B^ζ_α's definition as $\alpha < \beta \Rightarrow f_\alpha < f_\beta \mod J_{<\lambda}[\bar\lambda]$, (D)$^+$ holds with $g = g_\zeta$ by the choice of B^ζ_α hence also clause (D) follows. Lastly if (E) fails, say for g', then it can serve as g_ε. Now condition (F) follows immediately from (iii) (if (F) fails for δ, then there is $B \subseteq B^\zeta_\delta$ such that $\bigwedge_{\alpha<\delta} B^\zeta_\alpha \subseteq B \mod J_{<\lambda}[\bar\lambda]$ and $B^\zeta_\delta \setminus B \notin J_{<\lambda}[\bar\lambda]$); now the function $g^* =: (g_\zeta \restriction (\kappa\setminus B)) \cup (f_\delta \restriction B)$ contradicts "f_δ is a $\leq_{J_{<\lambda}[\bar\lambda]}$-lub of $\{f_\alpha : \alpha < \delta\}$", because: $g^* \in \prod \bar\lambda$ (obvious), $\neg(f_\delta \leq g^* \mod J_{<\lambda}[\bar\lambda])$ [why? as $B^\zeta_\delta \setminus B \notin J_{<\lambda}[\bar\lambda]$ and $g^* \restriction (B^\zeta_\delta \setminus B) = g_\zeta \restriction (B^\zeta_\delta \setminus B) < f_\delta \restriction (B^\zeta_\delta \setminus B)$ by the choice of B^ζ_δ], and for $\alpha < \delta$ we have:

$$f_\alpha \restriction B \leq_{J_{<\lambda}[\bar\lambda]} f_\delta \restriction B = g^* \restriction B \quad \text{and}$$

$$f_\alpha \restriction (\kappa\setminus B) \leq_{J_{<\lambda}[\bar\lambda]} g_\zeta \restriction (\kappa\setminus B) = g^* \restriction (\kappa\setminus B)$$

(the $\leq_{J_{<\lambda}[\bar\lambda]}$ holds as $(\kappa\setminus B) \cap B^\zeta_\alpha \in J_{<\lambda}[\bar\lambda]$ and the definition of B^ζ_α). So only clause (C) (of 1.7) may fail, without loss of generality for $\alpha = \alpha(\varepsilon)$. I.e. $\langle f_\beta \restriction B^\zeta_{\alpha(\varepsilon)} : \beta < \lambda\rangle$ is not cofinal in $(\prod \bar\lambda \restriction B^\zeta_{\alpha(\varepsilon)}, <_{J_{<\lambda}[\bar\lambda]})$. As this sequence of functions is increasing w.r.t. $<_{J_{<\lambda}[\bar\lambda]}$, there is $h_\alpha \in \prod(\bar\lambda \restriction B^\zeta_{\alpha(\varepsilon)})$ such that for no $\beta < \lambda$ do we have $h_\alpha \leq f_\beta \restriction B^j_{\alpha(\varepsilon)} \mod J_{<\lambda}[\bar\lambda]$. Let $h'_\varepsilon = h_\varepsilon \cup 0_{(\kappa\setminus B^\zeta_{\alpha(\varepsilon)})}$

and $g_\varepsilon \in \prod \bar\lambda$ be defined by $g_\varepsilon(i) = \mathrm{Max}\{g_\zeta(i), h'_\varepsilon(i)\}$. Now define B^ε_α by (iii) so (i), (ii), (iii) hold trivially, and we can check (iv).

So we can define $g_\varepsilon, \alpha(\varepsilon)$ for $\varepsilon < \theta$, satisfying (i)–(iv). As in the proof of 1.5, this is impossible: because (remembering $\mathrm{cf}(\lambda) = \lambda > \theta$) letting $\alpha(*) =: \bigcup_{\varepsilon<\theta} \alpha(\varepsilon) < \lambda$ we have: $\langle B^\varepsilon_{\alpha(*)} \cap A^*_\zeta : \varepsilon < \zeta\rangle$ is $\subseteq$-decreasing, for each $\zeta < \theta$, and $A^*_\varepsilon = \kappa \mod I^*$ and $B^{\varepsilon+1}_{\alpha(*)} \neq B^\varepsilon_{\alpha(*)} \mod J_{<\lambda}[\bar\lambda]$ so $\langle B^\varepsilon_{\alpha(*)} \cap A^*_{\varepsilon+1} \backslash B^{\varepsilon+1}_{\alpha(*)} : \varepsilon < \theta\rangle$ is a sequence of θ pairwise disjoint members of $(J_{<\lambda}[\bar\lambda])^+$ hence of $(I^*)^+$ which give the contradiction to $(*)$ of 1.5; so the lemma cannot fail. $\square_{1.7}$

Lemma 4. *: Suppose $(*)$ of 1.5.*

(1) For every $B \in J_{\le\lambda}[\bar\lambda] \backslash J_{<\lambda}[\bar\lambda]$, we have:

$$(\prod \bar\lambda \restriction B, <_{J_{<\lambda}[\bar\lambda]}) \text{ has true cofinality } \lambda (\text{hence } \lambda \text{ is regular}).$$

(2) If D is an ultrafilter on κ, disjoint to I^*, <u>then</u> $\mathrm{cf}(\prod \bar\lambda/D)$ is $\min\{\lambda : D \cap J_{\le\lambda}[\bar\lambda] \neq \emptyset\}$.

(3) (i) For λ limit $J_{<\lambda}[\bar\lambda] = \bigcup_{\mu<\lambda} J_{<\mu}[\bar\lambda]$ hence

(ii) For every λ, $J_{<\lambda}[\bar\lambda] = \bigcup_{\mu<\lambda} J_{\le\mu}[\bar\lambda]$.

(4) $J_{\le\lambda}[\bar\lambda] \neq J_{<\lambda}[\bar\lambda]$ <u>iff</u> $J_{\le\lambda}[\bar\lambda] \backslash J_{<\lambda}[\bar\lambda] \neq \emptyset$ <u>iff</u> $\lambda \in \mathrm{pcf}(\bar\lambda)$.

(5) $J_{\le\lambda}[\bar\lambda]/J_{<\lambda}[\bar\lambda]$ is λ-directed (i.e. if $B_\gamma \in J_{\le\lambda}[\bar\lambda]$ for $\gamma < \gamma^*, \gamma^* < \lambda$ <u>then</u> for some $B \in J_{\le\lambda}[\bar\lambda]$ we have $B_\gamma \subseteq B \mod J_{<\lambda}[\bar\lambda]$ for every $\gamma < \gamma^*$.)

Proof. (1) Let

$$J = \{B \subseteq \kappa : B \in J_{<\lambda}[\bar\lambda] \text{ or } B \in J_{\le\lambda}[\bar\lambda] \backslash J_{<\lambda}[\bar\lambda] \text{ and } (\prod \bar\lambda \restriction B, <_{J_{<\lambda}[\bar\lambda]}) \text{ has true cofinality } \lambda\}.$$

By its definition clearly $J \subseteq J_{\le\lambda}[\bar\lambda]$; it is quite easy to check it is an ideal (use 1.3(2)(v)). Assume $J \neq J_{\le\lambda}[\bar\lambda]$ and we shall get a contradiction. Choose $B \in J_{\le\lambda}[\bar\lambda] \backslash J$; as J is an ideal, there is an ultrafilter D on κ such that: $D \cap J = \emptyset$ and $B \in D$. Now if $\mathrm{tcf}(\prod \bar\lambda/D) \ge \lambda^+$, then $B \notin J_{\le\lambda}[\bar\lambda]$ (by the definition of $J_{\le\lambda}[\bar\lambda]$); contradiction. On the other hand if $F \subseteq \prod \bar\lambda, |F| < \lambda$ then there is $g \in \prod \bar\lambda$ such that $(\forall f \in F)(f < g \mod J_{<\lambda}[\bar\lambda])$ (by 1.5), so $(\forall f \in F)[f < g \mod D]$ (as $J_{<\lambda}[\bar\lambda] \subseteq J$, $D \cap J = \emptyset$), and this implies $\mathrm{cf}(\prod \bar\lambda/D) \ge \lambda$. By the last two sentences we know that $\mathrm{tcf}(\prod \bar\lambda/D)$ is λ. Now by 1.6 for some $C \in D$, $(\prod(\bar\lambda \restriction C), <_{J_{<\lambda}[\bar\lambda]})$ has true cofinality λ, of course $C \cap B \subseteq C$ and $C \cap B \in D$ hence $C \cap B \notin J_{<\lambda}[\bar\lambda]$. Clearly if $C' \subseteq C, C' \notin J_{<\lambda}[\bar\lambda]$ then also $(\prod \bar\lambda \restriction C', <_{J_{<\lambda}[\bar\lambda]})$ has true cofinality λ, hence by the last sentence without loss of generality $C \subseteq B$; hence by 1.4(5) we know that $C \in J_{\le\lambda}[\bar\lambda]$ hence by the definition of J we have $C \in J$. But this contradicts the choice of D as disjoint from J.

We have to conclude that $J = J_{\le\lambda}[\bar\lambda]$ so we have proved 1.8(1).

(2) Let λ be minimal such that $D \cap J_{\leq\lambda}[\bar{\lambda}] \neq \emptyset$ (it exists as by 1.3(10) $J_{<(\prod \bar{\lambda})^+}[\bar{\lambda}] = \mathcal{P}(\kappa)$) and choose $B \in D \cap J_{\leq\lambda}[\bar{\lambda}]$. So $[\mu < \lambda \Rightarrow B \notin J_{\leq\mu}[\bar{\lambda}]]$ (by the choice of λ) hence by 1.8(3)(ii) below, we have $B \notin J_{<\lambda}[\bar{\lambda}]$. It similarly follows that $D \cap J_{<\lambda}[\bar{\lambda}] = \emptyset$. Now $(\prod \bar{\lambda} \restriction B, <_{J_{<\lambda}[\bar{\lambda}]}$) has true cofinality λ by 1.8(1). As we know that $B \in D \cap J_{\leq\lambda}[\bar{\lambda}]$, and $J_{<\lambda}[\bar{\lambda}] \cap D = \emptyset$; clearly we have finished the proof.

(3)(i) Let $J =: \bigcup_{\mu<\lambda} J_{<\mu}[\bar{\lambda}]$. Now J is an ideal by 1.4(1)+(2) and $(\prod \bar{\lambda}, <_J)$ is λ-directed; i.e. if $\alpha^* < \lambda$ and $\{f_\alpha : \alpha < \alpha^*\} \subseteq \prod \bar{\lambda}$, then there exists $f \in \prod \bar{\lambda}$ such that

$$(\forall \alpha < \alpha^*)(f_\alpha < f \mod J).$$

[Why? if $\alpha^* < \theta^+$ as $(*)$ of 1.5 holds, this is obvious, suppose not; λ is a limit cardinal, hence there is μ^* such that $\alpha^* < \mu^* < \lambda$. Without loss of generality $|\alpha^*|^+ < \mu^*$. By 1.5, there is $f \in \prod \bar{\lambda}$ such that $(\forall \alpha < \alpha^*)(f_\alpha < f \mod J_{<\mu^*}[\bar{\lambda}])$. Since $J_{<\mu^*}[\bar{\lambda}] \subseteq J$, it is immediate that

$$(\forall \alpha < \alpha^*)(f_\alpha < f \mod J).]$$

Clearly $\bigcup_{\mu<\lambda} J_{<\mu}[\bar{\lambda}] \subseteq J_{<\lambda}[\bar{\lambda}]$ by 1.4(2). On the other hand, let us suppose that there is $B \in (J_{<\lambda}[\bar{\lambda}] \setminus \bigcup_{\mu<\lambda} J_{<\mu}[\bar{\lambda}])$. Choose an ultrafilter D on κ such that $B \in D$ and $D \cap J = \emptyset$. Since $(\prod \bar{\lambda}, <_J)$ is λ-directed and $D \cap J = \emptyset$, one has $\mathrm{tcf}(\prod \bar{\lambda}/D) \geq \lambda$, but $B \in D \cap J_{<\lambda}[\bar{\lambda}]$, in contradiction to Definition 2(2).

(3)(ii) If λ limit—by part (i) and 1.4(2); if λ successor—by 1.4(2) and Definition 2(3).

(4) Easy.

(5) Let $\langle f^\gamma_\alpha : \alpha < \lambda \rangle$ be $<_{J_{<\lambda}[\bar{\lambda}]+(\kappa \setminus B_\gamma)}$-increasing and cofinal in $\prod \bar{\lambda}$ (for $\gamma < \gamma^*$). Let us choose by induction on $\alpha < \lambda$ a function $f_\alpha \in \prod \bar{\lambda}$, as a $<_{J_{<\lambda}[\bar{\lambda}]}$-bound to $\{f_\beta : \beta < \alpha\} \cup \{f^\gamma_\alpha : \gamma < \gamma^*\}$, such f_α exists by 1.5 and apply 1.7 to $\langle f_\alpha : \alpha < \lambda \rangle$, getting $\langle B'_\alpha : \alpha < \lambda \rangle$, now B'_α for α large enough is as required. $\square_{1.8}$

2.2. Conclusion

If $(*)$ of 1.5, then $\mathrm{pcf}(\bar{\lambda})$ has a last element.

Proof. This is the minimal λ such that $\kappa \in J_{\leq\lambda}[\bar{\lambda}]$. λ exists, since $\lambda^* =: |\prod \bar{\lambda}| \in \{\lambda : \kappa \in J_{\leq\lambda}[\bar{\lambda}]\} \neq \emptyset$ and by 1.4(2); and $\lambda \in \mathrm{pcf}(\bar{\lambda})$ by 1.8(4) and $\lambda = \max pcf(\bar{\lambda})$ by 1.4(7)+ 1.8(4). $\square_{1.9}$

Claim 1.10: Suppose $(^*)$ of 1.5 holds. Assume for $j < \sigma$, D_j is a filter on κ extending $\{\kappa \setminus A : A \in I^*\}$, E a filter on σ and $D^* = \{B \subseteq \kappa : \{j < \sigma : B \in D_j\} \in E\}$ (a filter on κ). Let $\mu_j =: \mathrm{tcf}(\prod \bar{\lambda}, <_{D_j})$ be well defined for $j < \sigma$, and assume further $\mu_j > \sigma + \theta$ (where θ is from $(*)$ of 1.5).

Let

$$\lambda = \mathrm{tcf}(\prod \bar{\lambda}, <_{D^*}), \mu = \mathrm{tcf}(\prod_{j<\sigma} \mu_j, <_E).$$

Then $\lambda = \mu$ (in particular, if one is well defined, than so is the other).

Proof. Wlog $\sigma \geq \theta$ (otherwise we can add $\mu_j =: \mu_0, D_j =: D_o$ for $j \in \theta \setminus \sigma$, and replace σ by θ and E by $E' = \{A \subseteq \theta : A \cap \sigma \in E\}$). Let $\langle f^j_\alpha : \alpha < \mu_j \rangle$ be an $<_{D_j}$-increasing cofinal sequence in $(\prod \overline{\lambda}, <_{D_j})$.

Now $\ell = 0$, 1, for each $f \in \prod \overline{\lambda}$, define $G_\ell(f) \in \prod_{j<\sigma} \mu_j$ by $G_\ell(f)(j) = \min\{\alpha < \mu_j$: if $\ell = 1$ then $f \leq f^j_\alpha \mod D_j$ and if $\ell = 0$ then: not $f^j_\alpha < f \mod D_j\}$ (it is well defined for $f \in \prod \overline{\lambda}$ by the choice of $\langle f^j_\alpha : \alpha < \mu_j \rangle$).

Note that for f^1, $f^2 \in \prod \overline{\lambda}$ and $\ell < 2$ we have:

$$\begin{aligned} f^1 \leq f^2 \mod D^* &\Leftrightarrow B(f^1, f^2) =: \{i < \kappa : f^1(i) \leq f^2(i)\} \in D^* \\ &\Leftrightarrow A(f^1, f^2) =: \{j < \sigma : B(f^1, f^2) \in D_j\} \in E \\ &\Leftrightarrow \text{ for some } A \in E, \text{ for every } i \in A \text{ we have } f^1 \leq_{D_i} f^2 \\ &\Rightarrow \text{ for some } A \in E \text{ for every } i \in A \text{ we have} \\ &\qquad G_\ell(f^1)(i) \leq G_\ell(f^2)(i) \\ &\Leftrightarrow G_\ell(f^1) \leq G_\ell(f^2) \mod E. \end{aligned}$$

So

$\otimes_1$ G_ℓ is a mapping from $(\prod \overline{\lambda}, \leq_{D^*})$ into $(\prod_{j<\sigma} \mu_j, \leq_E)$ preserving order.
Next we prove that

$\otimes_2$ For every $g \in \prod_{j<\sigma} \mu_j$ for some $f \in \prod \overline{\lambda}$, we have $g \leq G_0(f) \mod E$.

[Why? Note that $\min\{\mu_j : j < \sigma\} \geq \sigma^+ \geq \theta^+$ and $J_{\leq\theta}[\overline{\lambda}] \subseteq J_{\leq\sigma}[\overline{\lambda}]$. By 1.5 we know $(\prod \overline{\lambda}, <_{J_{\leq\sigma}[\overline{\lambda}]})$ is σ^+-directed, hence for some function $f \in \prod \overline{\lambda}$:

$(*)_1$ For $j < \sigma$ we have $f^j_{g(j)} < f \mod J_{\leq\sigma}[\overline{\lambda}]$.

We here assumed $\sigma < \mu_j$, hence $J_{\leq\sigma}[\overline{\lambda}] \subseteq J_{<\mu_j}[\overline{\lambda}]$ (by 1.4(2)) but $J_{<\mu_j}[\overline{\lambda}]$ is disjoint to D_j by the definition of $J_{<\mu_j}[\overline{\lambda}]$ (by 1.8(2)+1.3(13)(c)) so together with $(*)_1$:

$(*)_2$ For $j < \sigma$, $f^j_{g(j)} < f \mod D_j$.

So by the definition of G_0 for every $j < \sigma$ we have $g(j) < G_0(f)(j)$ hence clearly $g < G_0(f)$.]

$\otimes_3$ For $f \in \prod \overline{\lambda}$ we have $G_0(f) \leq G_1(f)$ [Why? read the definitions]

$\otimes_4$ If $f_1, f_2 \in \prod \overline{\lambda}$ and $G_1(f_1) <_E G_0(f_2)$ then $f_1 <_{D^*} f_2$

[Why? as $G_1(f_1) <_E G_0(f_2)$ there is $B \in E$ such that: $j \in B \Rightarrow G_1(f_1)(j) < G_0(f_2)(j)$ so for each $j \in B$ we have $f_1, \leq_{D_j} f^j_{G_1(f_1)(j)}$ (by the definition of $G_1(f_1)$) and $f^j_{G_1(f_1)(j)} <_{D_j} f_2$ (as $G_1(f_1)(j) < G_0(f_2)(j)$ and the definition of $G_0(f_2)(j)$) so together $f_1 <_{D_j} f_2$. So $A(f_1, f_2) = \{i < \kappa : f_1(i) < f_2(i)\}$ satisfies: $A(f_1, f_2) \in D_j$ for every $j \in B$ but B was chosen in E, hence $A(f_1, f_2) \in D^*$ (by the definition of D^*) hence $f_1 <_{D^*} f_2$ as required]

Now first assume $\lambda = \mathrm{tcf}(\prod \overline{\lambda}, <_{D^*})$ is well defined, so there is a sequence $\overline{f} = \langle f_\alpha : \alpha < \lambda \rangle$ of members of $\prod \overline{\lambda}$, $<_{D^*}$-increasing and cofinal. So $\langle G_0(f_\alpha) : \alpha < \lambda \rangle$ is $\leq_E$-increasing in $\prod_{j<\sigma} \mu_j$ (by $\otimes_1$), for every $g \in \prod_{j<\sigma} \mu_j$ for some $f \in \prod \overline{\lambda}$ we have $g \leq_E G_0(f)$ (why? by $\otimes_2$), but by the choice of $\bar{f}$ for some $\beta < \lambda$ we have $f <_{D^*} f_\beta$ hence by $\otimes_1$ we have $g \leq_E G_0(f) \leq_E G_0(f_\beta)$, so $\langle G_0(f_\alpha) : \alpha < \lambda \rangle$ is cofinal in $(\prod_{j<\sigma} \mu_j, <_E)$. Also for every $\alpha < \lambda$, applying the previous sentence to $G(f_\alpha) + 1 (\in \prod_{j<\sigma} \mu_j)$ we can find $\beta < \lambda$ such that $G(f_\alpha) + 1 \leq_E G(f_\beta)$, so $G(f_\alpha) \leq_E G(f_\alpha)$ so for some club C of λ, $\langle G_0(f_\alpha) : \alpha \in C \rangle$ is $<_E$-increasing cofinal in $(\prod_{j<\sigma} \mu_j, <_E)$. So if λ is well defined then $\mu = \mathrm{tcf}(\prod_{j<\sigma} \mu_j, <_E)$ is well defined and equal to λ.

Lastly assume that μ is well defined i.e. $\prod_{j<\sigma} \mu_j / E$ has true cofinality μ, let $\overline{g} = (g_\alpha : \alpha < \mu)$ exemplifies it. Choose by induction on $\alpha < \mu$, a function f_α and ordinals $\beta_\alpha, \gamma_\alpha$ such that

(i) $f_\alpha \in \prod \overline{\lambda}$ and $\beta_\alpha < \mu$ and $\gamma_\alpha < \mu$
(ii) $g_{\beta_\alpha} <_E G_0(f_\alpha) \leq {}_E G_1(f_\alpha) <_E g_{\gamma_\alpha}$ (so $\beta_\alpha < \gamma_\alpha$)
(iii) $\alpha_1 < \alpha_2 < \mu \Rightarrow \gamma_{\alpha_1} < \beta_{\alpha_2}$ (so $\beta_\alpha \geq \alpha$)

In stage α, first choose $\beta_\alpha = \bigcup\{\gamma_{\alpha_1} + 1 : \alpha_1 < \alpha\}$, then choose $f_\alpha \in \prod \overline{\lambda}$ such that $g_{\beta_\alpha} + 1 <_E G_0(f_\alpha)$ (possible by $\otimes_2$) then choose γ_α such that $G_1(f_\alpha) <_E g_{\gamma_\alpha}$. Now $G_0(f_\alpha) \leq_E G_1(f_\alpha)$ by $\otimes_3$. By $\otimes_4$ we have $\alpha_1 < \alpha_2 \Rightarrow f_{\alpha_1} <_{D^*} f_{\alpha_2}$. Also if $f \in \prod \overline{\lambda}$ then $G_1(f) \in \prod_{j<\sigma} \mu_j$ hence by the choice of $\bar{g}$, for some $\alpha < \mu$ we have $G_1(f) <_E g_\alpha$ but $\alpha \leq \beta_\alpha$ so $G_1(f) <_E g_\alpha \leq_E G_0(f_\alpha)$ hence by $\otimes_4$, $f <_{D^*} f_\alpha$. Altogether, $\langle f_\alpha : \alpha < \mu \rangle$ exemplifies that $(\prod \overline{\lambda}, <_{D^*})$ has true cofinality μ, so λ is well defined and equal to μ. $\square_{1.11}$

2.3. Conclusion

If $(*)$ of 1.5 holds, and $\sigma, \bar{\mu} = \langle \mu_j : j < \sigma \rangle, \langle D_j : j < \sigma \rangle$ are as in 1.10 and $\sigma + \theta < \min(\bar{\mu})$, and J is an ideal on σ and I an ideal on κ such that $I^* \subseteq I \subseteq \{A \subseteq \kappa :$ for some $B \in J$ for every $j \in \sigma \setminus A$ we have $B \notin D_j\}$ (e.g. $I = I^*$) <u>then</u> $\mathrm{pcf}_J(\{\mu_j : j < \sigma\}) \subseteq \mathrm{pcf}_I(\overline{\lambda})$.

Proof. Let E be an ultrafilter on σ disjoint to J then we can define an ultrafilter D^* on κ as in 1.10, so clearly D^* is disjoint to I and we apply 1.10. $\square_{1.11}$

3. Normality of $\boldsymbol{\lambda} \in \mathbf{pcf}(\bar{\boldsymbol{\lambda}})$ for $\bar{\boldsymbol{\lambda}}$

Having found those ideals $J_{<\lambda}[\overline{\lambda}]$, we would like to know more. As $J_{<\lambda}[\overline{\lambda}]$ is increasing continuous in λ, the question is how $J_{<}[\overline{\lambda}], J < [\overline{\lambda}]$ are related.

The simplest relation is $J_{<\lambda} + [\overline{\lambda}] = J_{<\lambda}[\overline{\lambda}] + B$ for some $B \subseteq \kappa$, and then we call λ normal (for $\overline{\lambda}$) and denote $B = B_\lambda[\overline{\lambda}]$ though it is unique only modulo $J_{<\lambda}[\overline{\lambda}]$. We give a sufficient condition for existence of such B, using this in 2.8; giving the necessary definition in 2.3 and needed information in 2.4, 2.5, 2.6; lastly 2.7 is the essential uniqueness of cofinal sequences in appropriate $\prod \overline{\lambda}/I$.

Definition 3. *(1) We say* $\lambda \in \mathrm{pcf}(\overline{\lambda})$ *is normal (for* $\overline{\lambda}$*) if for some* $B \subseteq \kappa$, $J_{\leq\lambda}[\overline{\lambda}] = J_{<\lambda}[\overline{\lambda}] + B$.

(2) We say $\lambda \in \mathrm{pcf}(\bar{\lambda})$ *is semi-normal (for* $\bar{\lambda}$*) if there are* B_α *for* $\alpha < \lambda$ *such that:*

(i) $\alpha < \beta \Rightarrow B_\alpha \subseteq B_\beta \mod J_{<\lambda}[\overline{\lambda}]$ *and*

(ii) $J_{\leq\lambda}[\overline{\lambda}] = J_{<\lambda}[\overline{\lambda}] + \{B_\alpha : \alpha < \lambda\}$.

(3) We say λ *is normal if every* $\lambda \in \mathrm{pcf}(\bar{\lambda})$ *is normal for* $\overline{\lambda}$*. Similarly for semi normal.*

(4) In (1), (2), (3) instead $\overline{\lambda}$ *we can say* $(\overline{\lambda}, I)$ *or* $\prod \overline{\lambda}/I$ *or* $(\prod \overline{\lambda}, <_I)$ *if we replace* I^* *by* I *(an ideal on Dom(*$\overline{\lambda}$*)).*

3.1. Fact

Suppose (*) of 1.5 and $\lambda \in \mathrm{pcf}(\overline{\lambda})$. Now:

(1) λ is semi-normal for $\overline{\lambda}$ iff for some $F = \{f_\alpha : \alpha < \lambda\} \subseteq \prod \overline{\lambda}$ we have: $[\alpha < \beta \Rightarrow f_\alpha < f_\beta \mod J_{<\lambda}[\overline{\lambda}]]$ and for every ultrafilter D over κ disjoint to $J_{<\lambda}[\overline{\lambda}]$, F is unbounded in $(\prod \lambda, <_D)$ whenever $\mathrm{tcf}(\prod \overline{\lambda}, <_D) = \lambda$.

(2) In 2.1(2), without loss of generality, we may assume that
either: $B_\alpha = B_0 \mod J_{<\lambda}[\overline{\lambda}]$ (so λ is normal)
or: $B_\alpha \neq B_\beta \mod J_{\leq\lambda}[\overline{\lambda}]$ for $\alpha < \beta < \lambda$ so λ is not normal.

(3) Assume λ is semi normal for $\overline{\lambda}$. Then λ is normal for $\overline{\lambda}$ <u>iff</u> for some F as in part (1) (of 2.2), F has a $<_{J_{<\lambda}[\overline{\lambda}]}$-exact upper bound $g \in \prod_{i<\kappa}(\lambda_i + 1)$ and then $B =: \{i < \kappa : g(i) = \lambda_i\}$ generates $J_{\leq\lambda}[\overline{\lambda}]$ over $J_{<\lambda}[\overline{\lambda}]$.

(4) If λ is semi normal for $\overline{\lambda}$ then for some $\bar{f} = \langle f_\alpha : \alpha < \lambda\rangle$, $\bar{B} = \langle B_\alpha : \alpha < \lambda\rangle$ we have: $\bar{B}$ is increasing modulo $J_{<\lambda}[\overline{\lambda}]$, $J_{\leq_\lambda}[\overline{\lambda}] = J_{<\lambda}[\overline{\lambda}] + \{B_\alpha : \alpha < \lambda\}$, and the sequences $\langle f_\alpha : \alpha < \lambda\rangle$ is $<_{J_{<\lambda}[\overline{\lambda}]}$-increasing and $\bar{f}$, $\bar{B}$ are as in 1.7.

Proof. (1) For the direction $\Rightarrow$, given $\langle B_\alpha : \alpha < \lambda\rangle$ as in Definition 3(2), for each $\alpha < \lambda$, by 1.8(1) we have $(\prod \overline{\lambda} \restriction B_\alpha, <_{J_{<\lambda}[\overline{\lambda}]})$ has true cofinality λ, and let it be exemplified by $\langle f^\alpha_\beta : \beta < \lambda\rangle$. By 1.5 we can choose by induction on $\gamma < \lambda$ a function $f_\gamma \in \prod \overline{\lambda}$ such that: $\beta, \gamma \le \alpha \Rightarrow f^\alpha_\beta \le_{J_{<\lambda}[\overline{\lambda}]} f_\gamma$ and $\beta < \gamma \Rightarrow f_\beta <_{J_{<\lambda}[\overline{\lambda}]} f_\gamma$.

Now $F =: \{f_\alpha : \alpha < \lambda\}$ is as required. [Why? First, obviously $\alpha < \beta \Rightarrow f_\alpha < f_\beta \mod J_{<\lambda}[\overline{\lambda}]$. Second, if D is an ultrafilter on κ disjoint to I^* and $(\prod \overline{\lambda}, <_D)$ has true cofinality λ, then by 1.6 for some $B \in J_{\le_\lambda}[\overline{\lambda}] \setminus J_{<\lambda}[\overline{\lambda}]$ we have $B \in D$, so for some $\alpha < \lambda, B \subseteq B_\alpha \mod J_{<\lambda}[\overline{\lambda}]$ hence $B_\alpha \in D$. As $f^\alpha_\beta \le_{J_{<\lambda}[\overline{\lambda}]} f_\beta$ for $\beta \in [\alpha, \lambda)$ clearly F is cofinal in $(\prod \overline{\lambda}, <_D)$.]

The other direction, $\Leftarrow$ follows from 1.7 applied to $F = \{f_\alpha : \alpha < \lambda\}$. [Why? we get there $\langle B_\alpha : \alpha < \lambda \rangle, B_\alpha \in J_{\le_\lambda}[\overline{\lambda}]$ increasing modulo $J_{<\lambda}[\overline{\lambda}]$ so $J =: J_{<\lambda}[\overline{\lambda}] + \{B_\alpha : \alpha < \lambda\} \subseteq J_{<\lambda}[\overline{\lambda}]$.

If equality does not hold then for some ultrafilter D over κ, $D \cap J = \emptyset$ but $D \cap J_{<\lambda}[\overline{\lambda}] \ne \emptyset$ so by clause (D) of 1.7, F is bounded in $\prod \lambda / D$ whereas by 1.8(1),(2), $\mathrm{tcf}(\prod \overline{\lambda}, <_D) = \lambda$ contradicting the assumption on F.]

(2) Because we can replace $\langle B_\alpha : \alpha < \lambda \rangle$ by $\langle B_{\alpha_i} : i < \lambda \rangle$ whenever $\langle \alpha_i : i < \lambda \rangle$ is non decreasing, non eventually constant.

(3) If λ is normal for $\overline{\lambda}$, let $B \subseteq \kappa$ be such that $J_{\le_\lambda}[\overline{\lambda}] = J_{<\lambda}[\overline{\lambda}] + B$. By 1.8(1) we know that $(\prod(\overline{\lambda} \restriction B), <_{J_{<\lambda}[\overline{\lambda}]})$ has true cofinality λ, so let it be exemplified by $\langle f^0_\alpha : \alpha < \lambda \rangle$. Let $f_\alpha = f^0_\alpha \cup 0_{(\kappa \setminus B)}$ for $\alpha < \lambda$ and let $g \in {}^\kappa\mathrm{Ord}$ be defined by $g(i) = \lambda_i$ if $i \in B$ and $g(i) = 0$ if $i \in \kappa \setminus B$. Now $\langle f_\alpha : \alpha < \lambda \rangle$, g are as required by 1.3(11).

Now suppose $\langle f_\alpha : \alpha < \lambda \rangle$ is as in part (1) of 2.2 and g is a $<_{J_{<\lambda}[\overline{\lambda}]}$-eub of F, $g \in \prod_{j<\kappa}(\lambda_i + 1)$ and $B = \{i : g(i) = \lambda_i\}$. Let D be an ultrafilter on κ disjoint to $J_{<\lambda}[\overline{\lambda}]$. If $B \in D$ then for every $f \in \prod \overline{\lambda}$, let $f' = (f \restriction B) \cup 0(\kappa \setminus B)$, now necessarily $f' < \max\{g, 1\}$ (as $[i \in B \Rightarrow f'(i) < \lambda_i = g(i)]$ and $[i \in \kappa \setminus B \Rightarrow f'(i) = 0 \le g < 1]$), hence (see Definition 2(4)) for some $\alpha < \lambda$ we have $f' < \max\{f_\alpha, 1\} \mod J_{<\lambda}[\overline{\lambda}]$ hence for some $\alpha < \lambda$, $f' \le f_\alpha \mod J_{<\lambda}[\overline{\lambda}]$ hence $f \le f' \le f_\alpha \mod D$; also $\alpha < \beta \Rightarrow f_\alpha < f_\beta \mod D$, hence together $\langle f_\alpha : \alpha < \lambda \rangle$ exemplifies $\mathrm{tcf}(\prod \overline{\lambda}, <_D) = \lambda$. If $B \notin D$ then $\kappa \setminus B \in D$ so $g' = g \restriction (\kappa \setminus B) \cup 0_B = g \mod D$ and $\alpha < \lambda \Rightarrow f_\alpha <_D f_{\alpha+1} \le_D g =_D g'$, so $g' \in \prod \overline{\lambda}$ exemplifies F is bounded in $(\prod \overline{\lambda}, <_D)$ so as F is as in 2.2(1), $\mathrm{tcf}(\prod \overline{\lambda}, <_D) = \lambda$ is impossible. As D is disjoint to $J_{<\lambda}[\overline{\lambda}]$, necessarily $\mathrm{tcf}(\prod \overline{\lambda}, <_D) > \lambda$. The last two arguments together give, by 1.8(2) that $J_{\le_\lambda}[\overline{\lambda}] = J_{<\lambda}[\overline{\lambda}] + B$ as required in the definition of normality.

(4) Should be clear. $\square_{2.2}$

We shall give some sufficient conditions for normality.

Remark. In the following definitions we slightly deviate from [8, Ch. I = [10]]. The ones here are perhaps somewhat artificial but enable us to deal also with case (β) of 1.5$(*)$. I.e. in Definition 4 below we concentrate on the first θ elements of an a_α and for "obey" we also have $\bar{A}^* = \langle A_\alpha : \alpha < \theta \rangle$ and we want to cover also the case θ is singular.

Definition 4. *Let there be given regular λ, $\theta < \mu < \lambda$, μ possibly an ordinal, $S \subseteq \lambda$, $\sup(S) = \lambda$ and for simplicity S is a set of limit ordinals or at least have no two successive members.*

(1) We call $\bar{a} = \langle a_\alpha : \alpha < \lambda \rangle$ a continuity condition for (S, μ, θ) (or is an (S, μ, θ)-continuity condition) if: S is an unbounded subset of $\lambda, a_\alpha \subseteq \alpha, \mathrm{otp}(a_\alpha) < \mu$, and $[\beta \in a_\alpha \Rightarrow a_\beta = a_\alpha \cap \beta]$ and, for every club E of λ, for some[‡] *$\delta \in S$ we have $\theta = \mathrm{otp}\{\alpha \in a_\delta : \mathrm{otp}(a_\alpha) < \theta$ and for no $\beta \in a_\delta \cap \alpha$ is $(\beta, \alpha) \cap E = \emptyset\}$. We say $\bar{a}$ is continuous in S^* if $\alpha \in S^* \Rightarrow \alpha = \sup(a_\alpha)$.*

(2) Assume $f_\alpha \in {}^\kappa\mathrm{Ord}$ for $\alpha < \lambda$ and $\bar{A}^* = \langle A^*_\alpha : \alpha < \theta \rangle$ be a decreasing sequence of subsets of κ such that $\kappa \setminus A^*_\alpha \in I^*$. We say $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$ obeys $\bar{a} = \langle a_\alpha : \alpha < \lambda \rangle$ for $\bar{A}^*$ if:

(i) For $\beta \in a_\alpha$, if $\varepsilon =: \mathrm{otp}(a_\alpha) < \theta$ then we have $f_\beta \restriction A^*_\varepsilon \leq f_\alpha \restriction A^*_\varepsilon$ (note: $\bar{A}^*$ determine θ).

(2A) Let κ, λ, I^* be as usual. We say f obeys $\bar{a}$ for $\bar{A}^*$ continuously on S^* if: $\bar{a}$ is continuous in S^* and $\bar{f}$ obeys $\bar{a}$ for $\bar{A}^*$ and in addition $S^* \subseteq S$ and for $\alpha \in S^*$ (a limit ordinal) we have $f_\alpha = f_{a_\alpha}$ from (2B), i.e. for every $i < \kappa$ we have $f_\alpha(i) = \sup\{f_\beta(i) : \beta \in a_\alpha\}$ when $|a_\alpha| < \lambda_i$.

(2B) For given $\bar{\lambda} = \langle \lambda_i : i < \kappa \rangle, \bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$ where $f_\alpha \in \prod \bar{\lambda}$ and $a \subseteq \lambda$, and θ let $f_a \in \prod \bar{\lambda}$ be defined by: $f_a(i)$ is 0 if $|a| \geq \lambda_i$ and $\cup\{f_\alpha(i) : \alpha \in a\}$ if $|a| < \lambda_i$.

(3) Let (S, θ) stands for $(S, \theta + 1, \theta)$; (λ, μ, θ) stands for "(S, μ, θ) for some unbounded subset S of λ" and (λ, θ) stands for $(\lambda, \theta + 1, \theta)$.

If each A^*_α is κ, then we omit "for $\bar{A}^*$" (but θ should be fixed or said).

(4) We add to "continuity condition" (in part (1)) the adjective "weak" ["θ-weak"] if "$\beta \in a_\alpha \Rightarrow a_\beta = a_\alpha \cap \beta$" is replaced by "$\alpha \in S \& \beta \in a_\alpha \Rightarrow (\exists \gamma < \alpha)[a_\alpha \cap \beta \subseteq a_\gamma \& \gamma < \min(a_\alpha \setminus (\beta + 1)) \& [|a_\alpha \cap \beta| < \theta \Rightarrow |a_\gamma| < \theta]]$" [but we demand that γ exists only if $\mathrm{otp}(a_\alpha \cap \beta) < \theta$]. (Of course a continuity condition is a weak continuity condition which is a θ-weak continuity condition.)

Remark 2.3A. There are some obvious monotonicity implications, we state below only 2.4(3).

3.2. Fact

(1) Let $\theta_r = \begin{cases} \theta & \mathrm{cf}(\theta) = \theta \\ \theta^+ & \mathrm{cf}(\theta) < \theta \end{cases}$ and assume $\lambda = \mathrm{cf}(\lambda) > \theta_r^+$. Then for some stationary $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta_r\}$, there is a continuity condition $\bar{a}$ for

[‡]Note: if $\mathrm{otp}(a_\delta) = \theta$ and $\delta = \sup(a_\delta)$ (holds if $\delta \in S$, $\mu = \theta + 1$ and $\bar{a}$ continuous in S (see below)) and $\delta \in acc(E)$ then δ is as required.

(S, θ_r); moreover, it is continuous in S and $\delta \in S \Rightarrow \text{otp}(a_\delta) = \theta_r$; so for every club E of λ for some $\delta \in S, \forall \alpha, \beta[\alpha < \beta \ \& \ \alpha \in \alpha_\delta \ \& \ \beta \in a_\delta \rightarrow (\alpha, \beta) \cap E \neq \emptyset]$.

(2) Assume $\lambda = \theta^{++}$, then for some stationary $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \text{cf}(\theta)\}$ there is a continuity condition for $(S, \theta + 1, \theta)$. (In fact continuous in S and $[\delta \in S \Rightarrow a_\delta$ closed in $\delta]$ and $[\alpha \in a_\delta$ and $\delta \in S \Rightarrow a_\alpha = a_\delta \cap \alpha]$.)

(3) If $\bar{a}$ is a (λ, μ, θ_1)-continuity condition and $\theta_1 \geq \theta$ then there is a $(\lambda, \theta + 1, \theta)$-continuity condition.

Proof. (1) By [13, §1].
(2) By [Sh351, 4.4(2)] and[§] [8, III 2.14(2), clause (c), pp. 135–7].
(3) Check. $\square_{2.4}$

Remark 2.4A. Of course also if $\lambda = \theta^+$ the conclusion of 2.4(2) may well hold. We suspect but do not know that the negation is consistent with ZFC.

3.3. Fact

Suppose (*) of 1.5, $f_\alpha <\in \prod \bar{\lambda}$ for $\alpha < \lambda, \lambda = \text{cf}(\lambda) > \theta$ (of course $\kappa = \text{Dom}(\bar{\lambda})$) and $\bar{A}^* = \bar{A}^*[\bar{\lambda}]$ is as in the proof of 1.5, i.e., $A^*_\alpha = \{i < \kappa : \lambda_i > \alpha\}$. Then

(1) Assume $\bar{a}$ is a θ-weak continuity condition for $(S, \theta), \lambda = \sup(S)$, then we can find $\bar{f}' = \langle f'_\alpha : \alpha < \lambda \rangle$ such that:

  (i) $f'_\alpha \in \prod \bar{\lambda}$,
  (ii) For $\alpha < \lambda$ we have $f_\alpha \leq f'_\alpha$
  (iii) For $\alpha < \beta < \lambda$ we have $f'_\alpha <_{J_{<\lambda}[\bar{\lambda}]} f'_\beta$
  (iv) $\bar{f}'$ obeys $\bar{a}$ for $\bar{A}^*$

(2) If in addition $\min(\bar{\lambda}) \geq \mu, S^* \subseteq S$ are stationary subsets of λ but $\bar{a}$ is a continuity condition for (S, μ, θ) and $\bar{a}$ is continuous on S^* then we can find $\bar{f}' = \langle f'_\alpha : \alpha < \lambda \rangle$ such that

  (i) $f'_\alpha \in \prod \bar{\lambda}$
  (ii) For $\alpha \in \lambda \setminus S^*$ we have $f_\alpha \leq f'_\alpha$ and $\alpha = \beta + 1 \in \lambda \setminus S^* \ \& \ \beta \in S^* \Rightarrow f_\beta \leq f'_\alpha$
  (iii) For $\alpha < \beta < \lambda$ we have $f'_\alpha <_{J_{<\lambda}[\bar{\lambda}]} f'_\beta$
  (iv) $\bar{f}'$ obeys $\bar{a}$ for $\bar{A}^*$ continuously on S^*; moreover 2.3(2)(i) can be strengthened to $\beta \in a_\alpha \Rightarrow f_\beta < f_\alpha$.

[§]the definition of B^α_i in the proof of [8, III 2.14(2)] should be changed as in [Sh351, 4.4(2)]

(3) Suppose $\langle f'_\alpha : \alpha < \lambda\rangle$ obeys $\bar a$ continuously on S^* and satisfies 2.5(2)(ii) (and 2.5(2)'s assumption holds). If $g_\alpha \in \prod \bar\lambda$ and $\langle g_\alpha : \alpha < \lambda\rangle$ obeys $\bar a$ continuously on S^* and $[\alpha \in \lambda \setminus S^* \Rightarrow g_\alpha \le f_\alpha]$ then $\bigwedge_\alpha g_\alpha \le f'_\alpha$.

(4) If $\zeta < \theta$, for $\varepsilon < \zeta$ we have $\bar f^\varepsilon = \langle f^\varepsilon_\alpha : \alpha < \lambda\rangle$, where $f^\varepsilon_\alpha \in \prod \bar\lambda$, then in 2.5(1) (and 2.5(2)) we can find $\bar f'$ as there for all $\bar f^\varepsilon$ simultaneously. Only in clause (ii) we replace $f_\alpha \le f'_\alpha$ by $f_\alpha \restriction A^*_\zeta \le f'_\alpha \restriction A^*_\zeta$ (and $f_\beta \le f'_\alpha$ by $f_\beta \restriction A^*_\zeta \le f'_\alpha \restriction A^*_\zeta$).

Proof. Easy (using 1.5 of course).

CLAIM 2.5A: In 2.5 we can replace "(*) from 1.5" by "$\prod \bar\lambda / J_{<\lambda}[\bar\lambda]$ is λ-directed and $\liminf_{I^*}(\bar\lambda) \ge \theta$".

CLAIM 2.6: Assume (*) of 1.5 and let $\bar A^*$ be as there,

(1) In 1.7, if $\langle f_\alpha : \alpha < \lambda\rangle$ obeys some (S,θ)-continuity condition or just a θ-weak one for $\bar A^*$ (where $S \subseteq \lambda$ is unbounded) then we can deduce also: (G) the sequence $\langle B_\alpha / J_{<\lambda}[\bar\lambda] : \alpha < \lambda\rangle$ is eventually constant.

(2) If $\theta^+ < \lambda$ then $J_{\le\lambda}[\bar\lambda]/J_{<\lambda}[\bar\lambda]$ is λ^+-directed (hence if λ is semi normal for $\bar\lambda$ then it is normal to λ).

Proof. (1) Assume not, so for some club E of λ we have

$(*)$ $\alpha < \delta < \lambda \& \delta \in E \Rightarrow B_\alpha \ne B_\delta \mod J_{<\lambda}[\bar\lambda]$.

As $\bar a$ is a θ-weak (S,θ)-continuity condition, there is $\delta \in S$ such that $b =: \{\alpha \in a_\delta : \mathrm{otp}(a_\delta \cap \alpha) < \theta$ and for no $\beta \in a_\delta \cap \alpha$ is $(\beta,\alpha) \cap E = \emptyset$ and for some $\gamma < \alpha, a_\alpha \cap \beta \subseteq a_\gamma$ and $\gamma < \min(a_\alpha \setminus (\beta+1))$ and $|a_\gamma| < \theta\}$ has order type θ. Let $\{\alpha_\varepsilon : \varepsilon < \theta\}$ list b (increasing with ε). So for every $\varepsilon < \theta$ there is $\gamma_\varepsilon \in (\alpha_\varepsilon, \alpha_{\varepsilon+1}) \cap E$, and let $\beta_\varepsilon < \alpha_{\varepsilon+1}$ be such that $a_\delta \cap \alpha_\varepsilon \subseteq a_{\beta_\varepsilon}$ and $\mathrm{otp}(a_{\beta_\varepsilon} \cap \alpha_\varepsilon) < \theta$; by shrinking b and renaming wlog $\beta_\varepsilon < \gamma_\epsilon$ and $\alpha_\varepsilon \in a_{\beta_\varepsilon}$. Let $\xi(\varepsilon) =: \mathrm{otp}(a_{\beta_\varepsilon})$. Lastly let $B^0_\varepsilon =: \{i < \kappa : f_{\alpha_\varepsilon}(i) < f_{\beta_\varepsilon}(i) < f_{\gamma_\varepsilon}(i) < f_{\alpha_{\varepsilon+1}}(i)\}$, clearly it is $= \kappa \mod I^*$ and let (remember $(*)$ above) $B^*_\varepsilon =: A^*_{\xi(\varepsilon)+1} \cap (B_{\gamma_\varepsilon} \setminus B_{\beta_\varepsilon}) \cap B^0_\varepsilon$, now $B_{\alpha_\varepsilon} \subseteq B_{\beta_\varepsilon} \subseteq B_{\gamma_\varepsilon} \mod J_{<\lambda}[\bar\lambda]$ by clause (B) of 1.7, and $B_{\gamma_\varepsilon} \ne B_{\beta_\varepsilon} \mod J_{<\lambda}[\bar\lambda]$ by $(*)$ above hence $B_{\gamma_\varepsilon} \setminus B_{\beta_\varepsilon} \ne \emptyset \mod J_{<\lambda}[\bar\lambda]$. Now $B^0_\varepsilon, A^*_{\xi(\varepsilon)+1} = \kappa \mod I^*$ by the previous sentence and by 1.5$(*)$ which we are assuming respectively and $I^* \subseteq J_{<\lambda}[\bar\lambda]$ by the later's definition; so we have gotten $B^*_\varepsilon \ne \emptyset \mod J_{<\lambda}[\bar\lambda]$. But for $\varepsilon < \zeta < \theta$ we have $B^*_\varepsilon \cap B^*_\zeta = \emptyset$, for suppose $i \in B^*_\varepsilon \cap B^*_\zeta$, so $i \in A^*_{\xi(\varepsilon)+1}$ and also $f_{\gamma_\varepsilon}(i) < f_{\alpha_{\varepsilon+1}}(i) \le f_{\beta_\zeta}(i)$ (as $i \in B^0_\varepsilon$ and as $\alpha_{\varepsilon+1} \in a_{\beta_\zeta}$ & $i \in A^*_{\xi(\zeta)+1}$ respectively); now $i \in B^*_\varepsilon$ hence $i \in B_{\gamma_\varepsilon}$ i.e., (where g is from 1.7 clause $(\mathrm{D})^+) f_{\gamma_\varepsilon}(i) > g(i)$ hence (by the above) $f_{\beta_\zeta}(i) > g(i)$ hence $i \in B_{\beta_\zeta}$ hence $i \notin B^*_\zeta$, contradiction. So $\langle B^*_\varepsilon : \varepsilon < \theta\rangle$ is a sequence of θ pairwise disjoint members of $(J_{<\lambda}[\bar\lambda])^+$, contradiction.

(2) The proof is similar to the proof of 1.8(5), using 2.6(1) instead 1.7 (and $\bar a$ from 2.4(1) if $\lambda > \theta^+_r$ or 2.4(2) if $\lambda = \theta^{++}$). $\square_{2.6}$

We note also (but shall not use):

Claim 2.7: Suppose $(*)$ of 1.5 and

(a) $f_\alpha \in \prod \bar{\lambda}$ for $\alpha < \lambda, \lambda \in \mathrm{pcf}(\bar{\lambda})$ and $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$ is $<_{J_{<\lambda}[\bar{\lambda}]}$-increasing
(b) $\bar{f}$ obeys $\bar{a}$ continuously on S^*, where $\bar{a}$ is a continuity condition for (S, θ) and $\lambda = \sup(S)$ (hence $\lambda > \theta$ by the last phrase of 2.3(1))
(c) J is an ideal on κ extending $J_{<\lambda}[\bar{\lambda}]$, and $\langle f_\alpha / J : \alpha < \lambda \rangle$ is cofinal in $(\prod \bar{\lambda}, <_J)$ (e.g. $J = J_{<\lambda}[\bar{\lambda}] + (\kappa \setminus B), B \in J_{\leq\lambda}[\bar{\lambda}] \setminus J_{<\lambda}[\bar{\lambda}]$).
(d) $\langle f'_\alpha : \alpha < \lambda \rangle$ satisfies (a), (b) above.
(e) $f_\alpha \leq f'_\alpha$ for $\alpha \in \lambda \setminus S^*$ (alternatively: $\langle f'_\alpha : \alpha < \lambda \rangle$ satisfies (c)).
(f) If $\delta \in S'$ then J is $\mathrm{cf}(\delta)$-indecomposable (i.e. if $\langle A_\varepsilon : \varepsilon < \mathrm{cf}(\delta) \rangle$ is a $\subseteq$-increasing sequence of members, of J then $\bigcup_{\varepsilon < \mathrm{cf}(\delta)} A_\varepsilon \in J$).

Then:

(A) The set

$$\{\delta < \lambda : \text{if } \delta \in S^* \text{ and } \mathrm{otp}(a_\delta) = \theta \text{ then } f'_\delta = f_\delta \mod J\}$$

contains a club of λ.

(B) The set

$$\{\delta < \lambda : \text{if } \alpha \in S \text{ and } \delta = \sup(\delta \cap a_\alpha) \text{ and } \mathrm{otp}(\alpha \cap a_\delta) = \theta$$
$$\text{then } f'_{\alpha \cap a_\delta} = f_{\alpha \cap a_\delta} \mod J\}$$

contains a club of λ.

Proof. We concentrate on proving (A). Suppose $\delta \in S^*$, and $f_\delta \neq f'_\delta \mod J$. Let

$$A_{1,\delta} = \{i < \kappa : f_\delta(i) < f'_\delta(i)\}$$
$$A_{2,\delta} = \{i < \kappa : f_\delta(i) > f'_\delta(i)\},$$

So $A_{1,\delta} \cup A_{2,\delta} \in J^+$, suppose first $A_{1,\delta} \in J^+$. By Definition 4(2A), for every $i \in A_{1,\delta}$ for every large enough $\alpha \in a_\delta$, $f_\delta(i) < f'_\alpha(i)$, say for $\alpha \in a_\delta \setminus \beta_i$. As J is $\mathrm{cf}(\delta)$-indecomposable for some $\beta < \alpha$ we have $\{i < \kappa : \beta_i < \beta\} \in J^+$ so $f_\delta \restriction A_{1,\delta} < f'_\beta \restriction A_{1,\delta}$ (and $\beta < \delta$). Now by clause (c), $E =: \{\delta < \lambda :$ for every $\beta < \delta$ we have $f'_\beta < f_\delta \mod J\}$ is a club of λ, and so we have proved

$$\delta \in E \Rightarrow A_{1,\delta} \in J.$$

If $\bigwedge_{\alpha < \lambda} f_\alpha \leq f'_\alpha$ (first possibility in clause (e) implies it) also $A_{2,\delta} \in J$ hence for no $\delta \in S^* \cap E$ do we have $f_\delta \neq f'_\delta \mod J$. If the second possibility of clause (e) holds, we can interchange $\bar{f}$, $\bar{f}'$ hence $[\delta \in E \Rightarrow A_{2,\delta} \in J]$ and we are done. $\square_{2.7}$

We now return to investigating the $J_{<\lambda}[\bar{\lambda}]$, first without using continuity conditions.

Lemma 5. *Suppose* $(*)$ *of 1.5 and* $\lambda = \mathrm{cf}(\lambda) \in \mathrm{pcf}(\overline{\lambda})$. *Then* λ *is semi normal for* $\overline{\lambda}$.

Proof. We assume λ is not semi normal for $\overline{\lambda}$ and eventually get a contradiction. Note that by our assumption $(\prod \overline{\lambda}, <_I)$ is θ^+-directed hence $\min \mathrm{pcf}_I(\overline{\lambda}) \geq \theta^+$ (by 1.3(4)(v)) hence let us define by induction on $\xi \leq \theta, \bar{f}^\xi = \langle f^\xi_\alpha : \alpha < \lambda \rangle$, B_ξ and D_ξ such that:

(I) (i) $f^\xi_\alpha \in \prod \overline{\lambda}$
(ii) $\alpha < \beta < \lambda \Rightarrow f^\xi_\alpha \leq f^\xi_\beta \mod J_{<\lambda}[\overline{\lambda}]$
(iii) $\alpha < \lambda \& \xi < \theta \Rightarrow f^\xi_\alpha \leq f^\theta_\alpha \mod J_{<\lambda}[\overline{\lambda}]$
(iv) For $\zeta < \xi < \theta$ and $\alpha < \lambda : f^\zeta_\alpha \restriction A^*_\xi \leq f^\xi_\alpha \restriction A^*_\xi$

(II) (i) D_ξ is an ultrafilter on κ such that: $\mathrm{cf}(\prod \overline{\lambda}/D_\xi) = \lambda$
(ii) $\langle f^\xi_\alpha/D_\xi : \alpha < \lambda \rangle$ is not cofinal in $\prod \overline{\lambda}/D_\xi$
(iii) $\langle f^{\xi+1}_\alpha/D_\xi : \alpha < \lambda \rangle$ is increasing and cofinal in $\prod \overline{\lambda}/D_\xi$; moreover
(iii)$^+$ $B_\xi \in D_\xi$ and $\langle f^{\xi+1}_\alpha/D_\xi : \alpha < \lambda \rangle$ is increasing and cofinal in $\prod \overline{\lambda}/(J_{<\lambda}[\overline{\lambda}] + (\kappa \setminus B_\xi))$
(iv) $f^{\xi+1}_0/D_\xi$ is above $\{f^\xi_\alpha/D_\xi : \alpha < \lambda\}$.

For $\xi = 0$. No problem. [Use 1.8(1) + (4)].

For ξ *limit* $< \theta$. Let $g^\xi_\alpha \in \prod \overline{\lambda}$ be defined by $g^\xi_\alpha(i) = \sup\{f^\zeta_\alpha(i) : \zeta < \xi\}$ for $i \in A^*_\xi$ and $f^\xi_\alpha(i) = 0$ else, (remember that $\kappa \setminus A^*_\xi \in I^*$). Then choose by induction on $\alpha < \lambda, f^\xi_\alpha \in \prod \overline{\lambda}$ such that $g^\xi_\alpha \leq f^\xi_\alpha$ and $\beta < \alpha \Rightarrow f_\beta < f_\alpha \mod J_{<_\lambda}[\overline{\lambda}]$. This is possible by 1.5 and clearly the requirements (I)(i),(ii),(iv) are satisfied. Use 2.2(1) to find an appropriate D_ξ (i.e. satisfying II (i)+(ii)). Now $\langle f^\xi_\alpha : \alpha < \lambda \rangle$ and D_ξ are as required. (The other clauses are irrelevant.)

For $\xi = \theta$. Choose f^θ_α by induction of α satisfying I(i), (ii),(iii) (possible by 1.5).

For $\xi = \zeta + 1$. Use 1.6 to choose $B_\zeta \in D_\zeta \cap J_{\leq\lambda}[\overline{\lambda}] \setminus J_{<\lambda}[\overline{\lambda}]$. Let $\langle g^\xi_\alpha : \alpha < \lambda \rangle$ be cofinal in $(\prod \overline{\lambda}, <_{D_\xi})$ and even in $(\prod \overline{\lambda}, <_{J_{<\lambda}[\overline{\lambda}]+(\kappa \setminus B_\xi)})$ and without loss of generality $\bigwedge_{\alpha<\lambda} f^\zeta_\alpha/D_\zeta < g^\xi_0/D_\zeta$ and $\bigwedge_{\alpha<\lambda} f^\zeta_\alpha \restriction A^*_\xi \leq g^\xi_\alpha \restriction A^*_\xi$. We get $\langle f^\xi_\alpha : \alpha < \lambda \rangle$ increasing and cofinal $\mod (J_{<\lambda}[\overline{\lambda}] + (\kappa \setminus B_\xi))$ such that $g^\xi_\alpha \leq f^\xi_\alpha$ by 1.5 from $\langle g^\xi_\alpha : \alpha < \lambda \rangle$. Then get D_ξ as in the case "ξ limit".

So we have defined the f^ξ_α's and D_ξ's. Now for each $\xi < \theta$ we apply (II) (iii)$^+$ for $\langle f^{\xi+1}_\alpha : \alpha < \lambda \rangle$, $\langle f^\theta_\alpha : \alpha < \lambda \rangle$. We get a club C_ξ of λ such that:

$$\alpha < \beta \in C_\xi \Rightarrow f^\theta_\alpha \restriction B_\xi < f^{\xi+1}_\beta \restriction B_\xi \mod J_{<\lambda}[\overline{\lambda}] \qquad (*)$$

So $C =: \bigcap_{\xi<\theta} C_\xi$ is a club of λ. By 2.2(1) applied to $\langle f^\theta_\alpha : \alpha < \lambda \rangle$ (and the assumption "λ is not semi-normal for $\overline{\lambda}$") there is $g \in \prod \overline{\lambda}$ such that

$$\neg g \leq f^\theta_\alpha \mod J_{<\lambda}[\overline{\lambda}] \text{ for } \alpha < \lambda \qquad (*)_1$$

(not used) and by 1.5 wlog

$$f^\xi_0 < g \mod J_{<\lambda}[\overline{\lambda}] \text{ for } \xi < \theta \qquad (*)_2$$

For each $\xi < \theta$, by II (iii), (iii)$^+$ for some $\alpha_\xi < \lambda$ we have

$$g \restriction B_\xi < f^{\xi+1}_{\alpha_\xi} \restriction B_\xi \mod J_{<\lambda}[\overline{\lambda}] \qquad (*)_3$$

Let $\alpha(*) = \sup_{\xi<\theta} \alpha_\xi$, so $\alpha(*) < \lambda$ and so

$$g \restriction B_\xi < f^{\xi+1}_{\alpha(*)} \restriction B_\xi \mod J_{<\lambda}[\overline{\lambda}] \qquad (*)_4$$

For $\zeta < \theta$ let $B^*_\zeta = \{i \in A^*_\zeta : g(i) < f^\zeta_{\alpha(*)}(i)\}$. By $(*)_4$, $B^*_{\xi+1} \in D_\xi$; by (II)(iv)+$(*)_2$ we know $B^*_\xi \notin D_\xi$, hence $B^*_\xi \neq B^*_{\xi+1} \mod D_\xi$ hence $B^*_\xi \neq B^*_{\xi+1} \mod J_{<\lambda}[\overline{\lambda}]$.

On the other hand by (I)(iv) for each $\zeta < \theta$ we have $\langle B^*_\xi \cap A^*_\zeta : \xi \leq \zeta\rangle$ is $\subseteq$-increasing and (as $A^*_\zeta = \kappa \mod J_{<\lambda}[\overline{\lambda}]$ for each $\zeta < \theta$) hence by I(iv) we have $\langle B^*_\xi / I^* : \xi < \theta\rangle$ is $\subseteq$-increasing, and by the previous sentence $B^*_\xi \neq B^*_{\xi+1} \mod J_{<\lambda}[\overline{\lambda}]$ hence $\langle B^*_\xi / I^* : \xi < \theta\rangle$ is strictly $\subseteq$-increasing. Together clearly $\langle B^*_{\xi+1} \cap A^*_{\xi+1} \setminus B^*_\xi : \xi < \theta\rangle$ is a sequence of θ pairwise disjoint members of $(J_{<\lambda}[\overline{\lambda}])^+$, hence of $(I^*)^+$; contradiction to $\theta \geq \mathrm{wsat}(I^*)$. $\square_{2.8}$

Definition 5. *(1) We say $\langle B_\lambda : \lambda \in \mathfrak{c}\rangle$ is a generating sequence for $\bar{\lambda}$ if:*

- *(i) $B_\lambda \subseteq \kappa$ and $\mathfrak{c} \subseteq \mathrm{pcf}(\bar{\lambda})$*
- *(ii) $J_{\leq\lambda}[\overline{\lambda}] = J_{<\lambda}[\overline{\lambda}] + B_\lambda$ for each $\lambda \in \mathfrak{c}$*

(2) We call $\bar{B} = \langle B_\lambda : \lambda \in \mathfrak{c}\rangle$ smooth if:

$$i \in B_\lambda \& \lambda_i \in \mathfrak{c} \Rightarrow B_{\lambda_i} \subseteq B_\lambda.$$

(3) We call $\bar{B} = \langle B_\lambda : \lambda \in \mathrm{Rang}(\overline{\lambda})\rangle$ closed if for each λ

$$B_\lambda \supseteq \{i < \kappa : \lambda_i \in \mathrm{pcf}(\overline{\lambda} \restriction B_\lambda)\}$$

3.4. Fact

Assume $(*)$ of 1.5.

(1) Suppose $\mathfrak{c} \subseteq \mathrm{pcf}(\overline{\lambda})$, $\bar{B} = (B_\lambda : \lambda \in \mathfrak{c}\rangle$ is a generating sequence for $\overline{\lambda}$, and $B \subseteq \kappa$, $\mathrm{pcf}(\overline{\lambda} \restriction B) \subseteq \mathfrak{c}$ then for some finite $\mathfrak{d} \subseteq \mathfrak{c}, B \subseteq \bigcup_{\mu\in\mathfrak{d}} B_\mu \mod I^*$.

(2) $\mathrm{cf}(\prod \overline{\lambda}/I^*) = \max \mathrm{pcf}(\overline{\lambda})$

Remark 2.10A. For another proof of 2.10(2) see 2.12(2) + 2.12(4) and for another use of the proof of 2.10(2) see 2.14(1).

Proof. (1) If not, then $I = I^* + \{B \cap \bigcup_{\mu\in\mathfrak{d}} B_\mu : \mathfrak{d} \subseteq \mathfrak{c}, \mathfrak{d} \text{ finite}\}$ is a family of subsets of κ, closed under union, $B \notin I$, hence there is an ultrafilter D on κ. disjoint from I to which B belongs. Let $\mu =: \mathrm{cf}(\prod_{i<\kappa} \lambda_i / D)$; necessarily $\mu \in \mathrm{pcf}(\overline{\lambda} \restriction B)$, hence by the last assumption of 2.10(1) we have $\mu \in \mathfrak{c}$. By 1.8(2) we know $B_\mu \in D$ hence $B \cap B_\mu \in D$, contradicting the choice of D.

(2) The case $\theta = \aleph_0$ is trivial (as $\mathrm{wsat}(I^*) \leq \aleph_0$ implies $\mathcal{P}(\kappa)/I^*$ is a Boolean algebra satisfying the $\aleph_0$-c.c. (as here we can subtract) hence this Boolean algebra is finite hence also $\mathrm{pcf}(\overline{\lambda})$ is finite) so we assume $\theta > \aleph_0$. For $B \in (I^*)^+$ let $\lambda(B) = \max \mathrm{pcf}_{I^* \restriction B}(\overline{\lambda} \restriction B)$.

We prove by induction on λ that for every $B \in (I^*)^+, \mathrm{cf}(\prod \overline{\lambda}, <_{I^* + (\kappa \setminus B)}) = \lambda(B)$ when $\lambda(B) \leq \lambda$; this will suffice (use $B = \kappa$ and $\lambda = |\prod_{i<\kappa} \lambda_i|^+$). Given B let $\lambda = \lambda(B)$, by notational change wlog $B = \kappa$. By 1.9, $\mathrm{pcf}(\prod \overline{\lambda})$ has a last element, necessarily it is $\lambda =: \lambda(B)$. Let $\langle f_\alpha : \alpha < \lambda \rangle$ be $<_{J_{<\lambda}[\overline{\lambda}]}$ increasing cofinal in $\prod \overline{\lambda} / J_{<\lambda}[\overline{\lambda}]$, it clearly exemplifies $\max \mathrm{pcf}(\overline{\lambda}) \leq \mathrm{cf}(\prod \overline{\lambda} / I^*)$. Let us prove the other inequality. For $A \in J_{<\lambda}[\overline{\lambda}] \setminus I^*$ choose $F_A \subseteq \prod \overline{\lambda}$ which is cofinal in $\prod \overline{\lambda} / (I^* + (\kappa \setminus A))$, $|F_A| = \lambda(A) < \lambda$ (exists by the induction hypothesis). Let χ be a large enough regular, and we now choose by induction on $\varepsilon < \theta, N_\varepsilon, g_\varepsilon$ such that:

(A) (i) $N_\varepsilon \prec (H(\chi), \in, <^*_\chi)$
  (ii) $||N_\varepsilon|| = \lambda$
  (iii) $\langle N_\varepsilon : \xi \leq \varepsilon \rangle \in N_{\varepsilon+1}$
  (iv) $\langle N_\varepsilon : \varepsilon < \theta \rangle$ is increasing continuous
  (v) $\{\varepsilon : \varepsilon \leq \lambda + 1\} \subseteq N_0, \{\overline{\lambda}, I^*\} \in N_0, \langle f_\alpha : \alpha < \lambda \rangle \in N_0$ and the function $A \mapsto F_A$ belongs to N_0.

(B) (i) $g_\varepsilon \in \prod \overline{\lambda}$ and $g_\varepsilon \in N_{\varepsilon+1}$
  (ii) For no $f \in N_\varepsilon \cap \prod \overline{\lambda}$ does $g_\varepsilon <_{I^*} f$
  (iii) $\zeta < \varepsilon \& \lambda i > |\varepsilon| \Rightarrow g_\zeta(i) < g_\varepsilon(i)$.

There is no problem to define N_ε, and if we cannot choose g_ε this means that $N_\varepsilon \cap \prod \overline{\lambda}$ exemplifies $\mathrm{cf}(\prod \overline{\lambda}, <) \leq \lambda$ as required. So assume $\langle N_\varepsilon, g_\varepsilon : \varepsilon < \theta \rangle$ is defined. For each $\varepsilon < \theta$ for some $\alpha(\varepsilon) < \lambda, g_\varepsilon < f_{\alpha(\varepsilon)} \mod J_{<\lambda}[\overline{\lambda}]$ hence $\alpha(\varepsilon) \leq \alpha < \lambda \Rightarrow g_\varepsilon <_{J_{<\lambda}[\overline{\lambda}]} f_\alpha$. As $\lambda = \mathrm{cf}(\lambda) > \theta$, we can choose $\alpha < \lambda$ such that $\alpha > \bigcup_{\varepsilon<\theta} \alpha(\varepsilon)$. Let $B_\varepsilon = \{i < \kappa : g_\varepsilon(i) \geq f_\alpha(i)\}$; so for each $\xi < \theta$ we have $\langle B_\varepsilon \cap A^*_\xi : \varepsilon \leq \xi \rangle$ is increasing with ε, (by clause (B)(iii)), hence as usual as $\theta \geq \mathrm{wsat}(I^*)$ (and $\theta > \aleph_0$) we can find $\varepsilon(*) < \theta$ such that $\bigwedge_n B_{\varepsilon(*)+n} = B_{\varepsilon(*)} \mod I^*$ [why do we not demand $\varepsilon \in (\varepsilon(*), \theta) \Rightarrow B_\varepsilon = B_{\varepsilon(*)} \mod I^*$? as θ may be singular]. Now as $g_{\varepsilon(*)} \in N_{\varepsilon(*)+1}$ and $f_\alpha \in N_0 \prec N_{\varepsilon(*)+1}$ clearly, by its definition, $B_{\varepsilon(*)} \in N_{\varepsilon(*)+1}$ hence $F_{B_{\varepsilon(*)}} \in N_{\varepsilon(*)+1}$ Now:

$$g_{\varepsilon(*)+1} \restriction (\kappa \setminus B_{\varepsilon(*)}) =_{I^*} g_{\varepsilon(*)+1} \restriction (\kappa \setminus B_{\varepsilon(*)+1}) < f_\alpha \restriction (\kappa \setminus B_{\varepsilon(*)+1})$$
$$=_{I^*} f_\alpha \restriction (\kappa \setminus B_{\varepsilon(*)})$$

[why first equality and last equality? as $B_{\varepsilon(*)+1} = B_{\varepsilon(*)} \mod I^*$, why the $<$ in the middle? by the definition of $B_{\varepsilon(*)+1}$].

But $g_{\varepsilon(*)+1} \restriction B_{\varepsilon(*)} \in \prod_{i \in B_{\varepsilon(*)}} \lambda_i$, and $B_{\varepsilon(*)} \in J_{<\lambda}[\overline{\lambda}]$ as $g_\varepsilon < f_{\alpha(\varepsilon)} \leq f_\alpha \mod J_{<\lambda}[\overline{\lambda}]$ so for some $f \in F_{B_{\varepsilon(*)}} \subseteq \prod \overline{\lambda}$ we have $g_{\varepsilon(*)+1} \restriction B_{\varepsilon(*)} < f \restriction B_{\varepsilon(*)} \mod I^*$. By the last two sentences

$$g_{\varepsilon(*)+1} < \max\{f, f_\alpha\} \mod I^* \qquad (*)$$

Now $f_\alpha \in N_{\varepsilon(*)+1}$ and $f \in N_{\varepsilon(*)+1}$ (as $f \in F_{B_{\varepsilon(*)}}, |F_{B_{\varepsilon(*)}}| \leq \lambda, \lambda + 1 \subseteq N_{\varepsilon(*)+1}$ the function $B \mapsto F_B$ belongs to $N_0 \prec N_{\varepsilon(*)+1}$ and $B_{\varepsilon(*)} \in N_{\varepsilon(*)+1}$ as $\{g_{\varepsilon(*)}, f_\alpha\} \in N_{\varepsilon(*)+1}$) so together

$$\max\{f, f_\alpha\} \in N_{\varepsilon(*)+1}; \qquad (**)$$

But $(*)$, $(**)$ together contradict the choice of $g_{\varepsilon(*)+1}$ (i.e. clause (B)(ii)). $\square_{2.10}$

Definition 6. *(1) We say that I^* satisfies the pcf-th for (the regular) $(\overline{\lambda}, \theta)$ if $\prod \overline{\lambda}/I^*$ is θ-directed and $(\prod \overline{\lambda}, <_{J_{<\lambda}[\overline{\lambda}]})$ is λ-directed for each λ and we can find $\langle B_\lambda : \lambda \in \mathrm{pcf}_{I^*}(\overline{\lambda})\rangle$, such that:*

$B\lambda \subseteq \kappa, J_{<\lambda}[\overline{\lambda}, I^*] = I^* + \{B_\mu : \mu \in \lambda \cap \mathrm{pcf}_{I^*}(\overline{\lambda})\}, B_\lambda \notin J_{<\lambda}[\overline{\lambda}, I^*]$ and $\prod(\overline{\lambda} \restriction B_\lambda)/J_{<\lambda}[\overline{\lambda}, I^*]$ has true cofinality λ (so $B_\lambda \in J_{\leq\lambda}[\overline{\lambda}] \setminus J_{<\lambda}[\overline{\lambda}]$ and $J_{\leq\lambda}[\overline{\lambda}] = J_{<\lambda}[\overline{\lambda}] + B_\lambda$).

(1A) We say that I^* satisfies the weak pcf-th for $(\overline{\lambda}, \theta)$ if $(\prod \overline{\lambda}, <_{I^*})$ is θ-directed

each $(\prod \overline{\lambda}, <_{J_{<\lambda}[\overline{\lambda}]})$ is λ-directed and

there are $B_{\lambda,\alpha} \subseteq \kappa$ for $\alpha < \lambda \in \mathrm{pcf}_{I^*}(\overline{\lambda})$ such that

$$\alpha < \beta < \mu \in \mathrm{pcf}_{I^*}(\bar{\lambda}) \Rightarrow B_{\mu,\alpha} \subseteq B_{\mu,\beta} \mod J_{<\mu}[\bar{\lambda}, I^*]$$

$$J_{<\lambda}[\overline{\lambda}] = I^* + \{B_{\mu,\alpha} : a < \mu < \lambda, \mu \in \mathrm{pcf}_{I^*}(\bar{\lambda})\}$$

and

$$(\prod(\overline{\lambda} \restriction B_{\mu,\alpha}), <_{J_{<\lambda}[\overline{\lambda}]}) \text{ has true cofinality } \lambda$$

(1B) We say that I^* satisfies the weaker pcf-th for $(\bar{\lambda}, \theta)$ if $(\prod \bar{\lambda}, <_{I^*})$ is θ-directed and each $(\prod \overline{\lambda}, <_{J_{<\lambda}[\overline{\lambda}]})$, is λ-directed and for any ultrafilter D on κ disjoint to $J_{<\theta}[\overline{\lambda}]$ letting $\lambda = \mathrm{tcf}(\prod \overline{\lambda}, <_D)$ we have: $\lambda \geq \theta$ and for some $B \in D \cap J_{\leq\lambda}[\overline{\lambda}] \setminus J_{<\lambda}[\overline{\lambda}]$, the partial order $(\prod(\overline{\lambda} \restriction B), <_{J_{<\lambda}[\overline{\lambda}]})$ has true cofinality λ.

(1C) We say that I^* satisfies the weakest pcf-th for $(\bar{\lambda}, \theta)$ if $(\prod \overline{\lambda}, <_{I^*})$ is θ-directed and $(\prod \overline{\lambda}, <_{J_{<\lambda}[\overline{\lambda}]})$ is λ-directed for any $\lambda \geq \theta$

(1D) Above we write $\bar{\lambda}$ instead $(\bar{\lambda}, \theta)$ when we mean

$$\theta = \sup\{\theta : (\prod \bar{\lambda}, <_{I^*}) \text{ is } \theta^+ - \text{directed}\}.$$

(2) We say that I^* satisfies the pcf-th for θ if for any regular $\bar{\lambda}$ such that $\liminf_{I^*}(\bar{\lambda}) \geq \theta$, we have: I^* satisfies the pcf-th for $\bar{\lambda}$. We say that I^* satisfies the pcf-th above μ (above μ^-) if it satisfies the pcf-th for $\bar{\lambda}$ with $\liminf_{I^*}(\bar{\lambda}) > \mu$ (with $\{i : \lambda_i \geq \mu\} = \kappa \mod I^*$). Similarly (in both cases) for the weak pcf-th and the weaker pcf-th.

(3) Given I^*, θ let $J_\theta^{\rm pcf} = \{A \subseteq \kappa : A \in I^*$ or $A \notin I^*$ and $I^* + (\kappa \setminus A)$ satisfies the pcf-theorem for $\theta\}$.

$$J_\theta^{\rm wsat} =: \{A \subseteq \kappa : {\rm wsat}(I^* \restriction A) \leq \theta \text{ or } A \in I^*\};$$

similarly $J_\theta^{\rm wpcf}$; we may write, $J_\theta^x[I^*]$.

(4) We say that I^* satisfies the pseudo pcf-th for $\bar\lambda$ <u>if</u> for every ideal I on κ extending I^*, for some $A \in I^+$ we have $(\prod(\bar\lambda \restriction A), <_I)$ has a true cofinality.

CLAIM 2.12:

(1) If $(*)$ of 1.5 then I^* satisfies the weak pcf-th for $(\bar\lambda, \theta^+)$.
(2) If $(*)$ of 1.5 holds, and $\prod \bar\lambda / I^*$ is θ^{++}-directed (i.e. $\theta^+ < \min \bar\lambda$) or just there is a continuity condition for (θ^+, θ) <u>then</u> I^* satisfies the pcf-th for $(\bar\lambda, \theta^+)$.
(3) If I^* satisfy the pcf-th for $(\bar\lambda, \theta)$ <u>then</u> I^* satisfy the weak pcf-th for $(\bar\lambda, \theta)$ which implies that I^* satisfies the weaker pcf-th for $(\bar\lambda, \theta)$, which implies that I^* satisfies the weakest pcf-th for $(\bar\lambda, \theta)$.

Proof. (1) Let appropriate $\bar\lambda$ be given. By 1.5, 1.8 most demands holds, but we are left with seminormality. By 2.8, if $\lambda \in {\rm pcf}(\bar\lambda)$, then $\bar\lambda$ is semi normal for λ. This finishing the proof of (1).

(2) Let $\lambda \in {\rm pcf}(\bar\lambda)$ and let $\bar f, \bar B$ be as in 2.2(4). By 2.4(1) + (2) there is $\bar a$, a (λ, θ)-continuity condition; by 2.5(1) wlog $\bar f$ obeys $\bar a$, by 2.6(1) the relevant B_α / I^* are eventually constant which suffices by 2.2(2).

(3) Should be clear. $\square_{2.12}$

CLAIM 2.13: Assume $(\prod \bar\lambda, <_{I^*})$ is given (but possibly $(*)$ of 1.5 fails).

(1) If $I^*, \bar\lambda$ satisfies (the conclusion of) 1.6, <u>then</u> $I^*, \bar\lambda$ satisfy (the conclusions of) 1.8(1), 1.8(2), 1.8(3), 1.8(4), 1.9.
(1A) If I^* satisfies the weaker pcf-th for $\bar\lambda$ <u>then</u> they satisfy the conclusions of 1.6 and 1.5.
(2) If $I^*, \bar\lambda$ satisfies (the conclusion of) 1.5 <u>then</u> $I^*, \bar\lambda$ satisfies (the conclusion of) 1.10.
(2A) If I^* satisfies the weakest pcf-th for $\bar\lambda$ <u>then</u> $I^*, \bar\lambda$ satisfy the conclusion of 1.5.
(3) If $I^*, \bar\lambda$ satisfies 1.5, 1.6 <u>then</u> $I^*, \bar\lambda$ satisfies 2.2(1) (for 2.2(2) – no assumptions).
(4) If $I^*, \bar\lambda$ satisfies 1.8(1), 1.8(2) <u>then</u> $I^*, \bar\lambda$ satisfies 2.2(3) when we interpret "seminormal" by the second phrase of 2.2(1)
(5) If $I^*, \bar\lambda$ satisfies 1.8(2) <u>then</u> $I^*, \bar\lambda$ satisfies 2.10(1).
(6) If $I^* \bar\lambda$ satisfy 1.8(1) + 1.8(3)(i) <u>then</u> $I^*, \bar\lambda$ satisfies 1.8(2)
(7) If $I^*, \bar\lambda$ satisfies 1.8(1) + 1.8(2) and is semi normal <u>then</u> 2.10(2) holds i.e.

$$\mathrm{cf}(\prod \bar\lambda, <_{I^*}) \leq \sup {\rm pcf}_{I*}(\lambda).$$

(8) If $I^*, \bar\lambda$ satisfies $1.5 + 1.6$ <u>then</u> they satisfy 2.10(2).

Proof. (1) We prove by parts.

Proof of 1.8(2). Let $\lambda = \mathrm{tcf}(\prod \bar\lambda/D)$; by the definition of $J_{<\lambda}[\bar\lambda]$, clearly $D \cap J_{<\lambda}[\bar\lambda] = \emptyset$. Also by 1.6 for some $B \in D$ we have $\lambda = \mathrm{tcf}(\prod(\bar\lambda \restriction B), <_{J_{<\lambda[\bar\lambda]}})$, so by the previous sentence $B \notin J_{<\lambda}[\bar\lambda]$, and by 1.4(5) we have $B \in J_{\leq\lambda}[\bar\lambda]$, together we finish.

Proof of 1.8(1). Rep eat the proof of 1.8(1) replacing the use of 1.5 by 1.8(2).

Proof of 1.8(3)(i). Let $J =: \bigcup_{\mu<\lambda} J_{<\mu}[\bar\lambda]$, so $J \subseteq J_{<\lambda}[\bar\lambda]$ is an ideal because $\langle J_{<\mu}[\bar\lambda] : \mu < \lambda\rangle$ is $\subseteq$-increasing (by 1.4(2)), if equality fail choose $B \in J_{<\lambda}[\bar\lambda] \setminus J$ and choose D an ultrafilter on κ, disjoint to J to which B belongs. Now if $\mu = \mathrm{cf}(\mu) < \lambda$ then $\mu^+ < \lambda$ (as λ is a limit cardinal) and $\mu = \mathrm{cf}(\mu)$ & $\mu^+ < \lambda \Rightarrow D \cap J_{\leq\mu}[\bar\lambda] = D \cap J_{<\mu^+}[\bar\lambda] = \emptyset$ hence by 1.8(2) we have $\mu \neq \mathrm{cf}(\prod \bar\lambda/D)$. Also if $\mu = \mathrm{cf}(\mu) \geq \lambda$ then $D \cap J_{<\mu}[\bar\lambda] \subseteq D \cap J_{<\lambda}[\bar\lambda] \neq \emptyset$ hence by 1.8(2) we have $\mu \neq \mathrm{cf}(\prod \bar\lambda/D)$. Together contradict ion by 1.3(7).

Proof of 1.8(3)(ii). Follows.

Proof of 1.8(4). Follows.

Proof of 1.9. As in 1.9.

(1A) Check.
(2) Read the proof of 1.10.
(2A) Check.
(3) The direction $\Rightarrow$ is proved directly as in the proof of 2.2(1) (where the use of 1.8(1) is justified by 2.13(1)).

So let us deal with the direction $\Leftarrow$. So assume $\bar f = \langle f_\alpha : \alpha < \lambda\rangle$ is a sequence of members of $\prod \bar\lambda$ which is $<_{J_{<\lambda}[\bar\lambda]}$-increasing such that for every ultrafilter D on κ disjoint to $J_{<\lambda}[\bar\lambda]$ we have: $\lambda = \mathrm{tcf}(\prod \bar\lambda, < D)$ iff $\bar f$ is unbounded (equivalently cofinal) in $(\prod \bar\lambda, <_D)$. By (the conclusion of) 1.5 wlog $\bar f$ is $<_{J_{<\lambda}[\bar\lambda]}$-increasing.

By 1.5 there is $g \in \prod \bar\lambda$ such that $f_\alpha < g \mod J_{\leq\lambda}[\bar\lambda]$ for each $\alpha < \lambda$, and let $B_\alpha =: \{i < \kappa : g(i) \leq f_\alpha(i)\}$. Hence $B_\alpha \in J_{\leq\lambda}[\bar\lambda]$ (by the previous sentence) and $\langle B_\alpha/J_{<\lambda}[\bar\lambda] : \alpha < \lambda\rangle$ is $\subseteq$-increasing (as $\langle f_\alpha : \alpha < \lambda\rangle$ is $<_{J_{<\lambda}[\bar\lambda]}$-increasing). Lastly if $B \in J_{\leq\lambda}[\bar\lambda]$, but $B \setminus B_\alpha \notin J_{<\lambda}[\bar\lambda]$ for each $\alpha < \lambda$, let D be an ultrafilter on κ, disjoint to $J_{<\lambda}[\bar\lambda] + \{B_\alpha : \alpha < \lambda\}$ but to which B belongs, so $\mathrm{tcf}(\prod \bar\lambda, <_D) = \lambda$ (by 1.8(2) which holds by 2.13(1)) but $\{f_\alpha/D : \alpha < \lambda\}$ is bounded by g/D (as $f_\alpha/D \leq g/D$ by the definition of B_α), contradiction. So the sequence $\langle B_\alpha : \alpha < \lambda\rangle$ is as required.

(4)–(6) Left to the reader.

(7) Let for $\lambda \in \mathrm{pcf}(\bar{\lambda})$, $\langle B_i^\lambda : i < \lambda \rangle$ be such that $J_{\leq\lambda}[\bar{\lambda}] = J_{<\lambda}[\bar{\lambda}] + \{B_i^\lambda : i < \lambda\}$ (exists by seminormality; we use only this equality). Let $\langle f_\alpha^{\lambda,i} : \alpha < \lambda \rangle$ be cofinal in $(\prod(\bar{\lambda} \restriction B_i^\lambda), <_{J,\lambda[\bar{\lambda}]})$, it exists by 1.8(1). Let F be the closure of $\{f_\alpha^{\lambda,i} : \alpha < \lambda, i < \lambda, \lambda \in \mathrm{pcf}(\bar{\lambda})\}$, under the operation $\max\{g, h\}$. Clearly $|F| \leq \sup \mathrm{pcf}(\bar{\lambda})$, so it suffice to prove that F is a cover of $(\prod \bar{\lambda}, <_{I^*})$. Let $g \in \prod \bar{\lambda}$, if $(\exists f \in F)(g \leq f)$ we are done, if not

$$I = \{A \cup \{i < \kappa : f(i) > g(i)\} : f \in F, A \in I^*\}$$

is $\aleph_0$-directed, $\kappa \notin I$, so there is an ultrafilter D on κ disjoint to I, (so $f \in F \Rightarrow f <_D g$) and let $\lambda = \mathrm{tcf}(\prod \bar{\lambda}/D)$, so by 1.8(2) we have $D \cap J_{\leq\lambda}[\bar{\lambda}] \setminus J_{<\lambda}[\bar{\lambda}] \neq \emptyset$, hence for some $i < \lambda, B_i^\lambda \in D$, and we get contradiction to the choice of the $\{f_\alpha^{\lambda,\alpha} : \alpha < \lambda\} (\subseteq F)$.

(8) Repeat the proof of 2.10(2) (only using $J = \{A \subseteq \kappa :$ if $A \notin J_{<\lambda}[\bar{\lambda}]$ then $\mathrm{cf}(\prod \bar{\lambda}/I^*) \leq \lambda\}$; if $\kappa \notin J$ let D be an ultrafilter on κ disjoint to J, and use 1.6). $\square_{2.14}$

Claim 2.14: If I^* satisfies pseudo pcf-th <u>then</u>

(1) We can find $\langle (J_\zeta, \theta_\zeta) : \zeta < \zeta^* \rangle$, ζ^* a successor ordinal such that $J_0 = I^*, J_{\zeta+1} = \{A \subseteq \kappa :$ if $A \not\subseteq J_\zeta$ then $\mathrm{tcf}(\prod(\bar{\lambda} \restriction A), <_{J_\zeta}) = \theta_\zeta\}$ and for no $A \in (J_\zeta)^+$ does $(\prod(\bar{\lambda} \restriction A), <_{J_\zeta})$ has true cofinality which is $< \theta_\zeta$.

(2) If I^* satisfies the weaker pcf-th for $\bar{\lambda}$ <u>then</u> I^* satisfies the pseudo pcf-th for $\bar{\lambda}$.

Proof. (1) Check (we can also present those ideals in other ways).
(2) Check. $\square_{2.14}$

4. Reduced Products of Cardinals

We characterize here the cardinalities $\prod_{i<\kappa} \lambda_i/D$ and $T_D(\langle \lambda_i : i < \kappa \rangle)$ using pcf's and the amount of regularity of D (in 3.1–3.4). Later we give sufficient conditions for the existence of $<_D$-eub or $<_D$-eub. Remember the old result of Kanamori [3] and Ketonen [5]: for D an ultrafilter the sequence $\langle \alpha/D : \alpha < \kappa \rangle$ (i.e. the constant functions) has a $<_D$-eub if $\mathrm{reg}(D) < \kappa$; and see [8, III 3.3] (for filters). Then we turn to depth of ultraproducts of Boolean algebras.

The questions we would like to answer are (restricting ourselves to "$\lambda_i \geq 2^\kappa$" or "$\lambda_i \geq 2^{2^\kappa}$" and D an ultrafilter on κ will be good enough).

Question A: What can be $\mathrm{Car}_D =: \{\prod_{i<\kappa} \lambda_i/D : \lambda_i$ a cardinal for $i < \kappa\}$ i.e. characterize it by properties of D; (or at least $\mathrm{Car}_D \setminus 2^\kappa$) (for D a filter also $T_D(\prod \lambda_i)$ is natural).

Question B: What can be $\mathrm{DEPTH}_D^+ = \{\mathrm{Depth}^+(\prod_{i<\kappa} \lambda_i/D) : \lambda_i$a regular cardinal$\}$ (at least $\mathrm{DEPTH}_D^+ \setminus 2^\kappa$, see Definition 10).

If D is an $\aleph_1$-complete ultrafilter, the answer is clear. For D a regular ultrafilter on $\kappa, \lambda_i \geq \aleph_0$ the answer to question A is known [1] in fact it was the reason for defining "regularity of filters" (for $\lambda_i < \aleph_0$ see [9, Sh-a, VI §3 Th. 3.12 and pp. 357–370] better [Sh-c, VI §3] and Koppleberg [4].) For D a regular ultrafilter on κ, the answer to the question is essentially completed in 3.22(1), the remaining problem can be answered by *pp* (see [8]) except the restriction $(\forall_\alpha < \lambda)(|\alpha|^{\aleph_0} < \lambda)$, which can be removed if the cov = *pp* problem is completed (see [8], [AG]). So the problem is for the other ultrafilters D, on which we give a reasonable amount on information translating to a pcf problem, some times depending on the pcf theorem.

Definition 7. *(1) For a filter D let reg$(D) = \min\{\theta : D$ is not $\theta-$regular$\}$ (see below).*

(2) A filter D is θ-regular if there are $A_\varepsilon \in D$ for $\varepsilon < \theta$ such that the intersection of any infinitely many A_ε-s' is empty.

(3) For a filter D let

$$\text{reg}_*(D) = \min\{\theta : \textit{ there are no } A_\varepsilon \in D^+ \textit{ for } \varepsilon < \theta \textit{ such that } \textit{no } i < \kappa \textit{ belongs to infinitely many } A_\varepsilon{}'\text{s}\}$$

and

$$\text{reg}_\otimes(D) = : \{\theta : \textit{ there are no } A_\varepsilon \in D^+ \textit{ for } \varepsilon < \theta \textit{ such that } : \varepsilon < \zeta \Rightarrow A_\zeta \subseteq A_\varepsilon \mod D \textit{ and no } i < \kappa \textit{ belongs to infinitely many } A_\varepsilon{}'\text{s}\}.$$

(4) $\text{reg}^\sigma(D) = \min\{\theta : D$ is not $(\theta, \sigma)-$regular$\}$ where "D is (θ, σ)-regular" means that there are $A_\varepsilon \in D$ for $\alpha < \theta$ such that the intersection of any σ of them is empty. Lastly $\text{reg}^\sigma_*(D)$, $\text{reg}^\sigma_\otimes(D)$ are defined similarly using $A_\varepsilon \in D^+$. Of course reg(I) etc. means reg(D) where D is the dual filter.

Definition 8. *(1) Let*

$$\text{htcf}_{D,\mu}(\prod \gamma_i) = \sup\{\text{tcf}(\prod_{i<\kappa} \lambda_i/D) : \mu \leq \lambda_i = \text{cf}\lambda_i \leq \gamma_i \textit{ for } i < \kappa \textit{ and } \text{tcf}(\prod \lambda_i/D) \textit{ is well defined}\} \textit{ and}$$

$$\text{hcf}_{D,\mu}(\prod_{i<\kappa} \gamma_i) = \sup\{\text{cf}(\prod_{i<\kappa} \lambda_i/D) : \mu \leq \lambda_i = \text{cf}\lambda_i \leq \gamma_i\};$$

if $\mu = \aleph_0$ we may omit it.

(2) For E a family of filters on κ let $\text{htcf}_{E,\mu}(\prod_{i<\kappa} < \alpha_i)$ be

$$\sup\{\text{tcf}(\prod_{i<\kappa} \lambda_i/D) : D \in E \textit{ and } \mu \leq \lambda_i = \text{cf}\lambda_i \leq \alpha_i \textit{ for } i < \kappa \textit{ and } \text{tcf}(\prod_{i<\kappa} \lambda_i/D) \textit{ is well defined}\}.$$

Similarly for $\mathrm{hcf}_{E,\mu}$ *(using cf instead tcf).*

(3) $\mathrm{hcf}^*_{D,\mu}(\prod_{i<\kappa}\alpha_i)$ *is* $\mathrm{hcf}_{E,\mu}(\prod_{i<\kappa}\alpha_i)$ *for* $E = \{D' : D'$ *a filter on* κ *extend ing* $D\}$. *Similarly for* $\mathrm{htcf}^*_{D,\mu}$.

(4) *When we write* I *e.g. in* $hcf_{I,\mu}$ *we mean* $\mathrm{hcf}_{D,\mu}$ *where* D *is the dual filter.*

CLAIM 3.3:

(1) $\mathrm{reg}(D)$ is always regular
(2) If $\theta < \mathrm{reg}_*(D)$ <u>then</u> some filter extending D is θ-regular.
(3) $\mathrm{wsat}(D) \leq \mathrm{reg}_*(D)$
(4) $\mathrm{reg}(D) \leq \mathrm{reg}_{\otimes}(D) \leq \mathrm{reg}_*(D)$
(5) $\mathrm{reg}_*(D) = \min\{\theta :$ no ultrafilter D_1 on κ extending D is $\theta-$ regular$\}$
(6) If $D \subseteq E$ are filters on κ then:
   (a) $\mathrm{reg}(D) \leq \mathrm{reg}(E)$
   (b) $\mathrm{reg}_*(D) \geq \mathrm{reg}_*(E)$

Proof. Should be clear. E.g. (2) let $\langle u_\varepsilon : \varepsilon < \theta\rangle$ list the finite subsets of θ, and let $\{A_\varepsilon : \varepsilon < \theta\} \subseteq D^+$ exemplify "$\theta < \mathrm{reg}_*(D)$". Now let $D^* =: \{A \subseteq \kappa :$ for some finite $u \subseteq \theta,$ for every $\varepsilon < \theta$ we have:$u \subseteq u_\varepsilon \Rightarrow A_\varepsilon \subseteq A \mod D\}$, and let $A^*_\varepsilon = \bigcup\{A_\zeta : \varepsilon \in u_\zeta\}$. Now D^* is a filter on κ extending D and for $\varepsilon < \theta$ we have $A^*_\varepsilon \in D$. Finally the intersection of $A^*_{\varepsilon_0} \cap A^*_{\varepsilon_1} \cap \ldots$ for distinct $\varepsilon_n < \theta$ is empty, because for any member j of it we can find $\zeta_n < \theta$ such that $j \in A_{\zeta_n}$ and $\varepsilon_n \in u_{\zeta_n}$. Now <u>if</u> $\{\zeta_n : n < \omega\}$ is infinite then there is no such j by the choice of $\langle A_\varepsilon : \varepsilon < \theta\rangle$, and <u>if</u> $\{\zeta_n : n < \omega\}$ is finite then wlog $\bigwedge_{n,\omega}\zeta_n = \zeta_0$ contradicting "u_{ζ_0} is finite" as $\bigwedge_{n<\omega}\varepsilon_n \in u_{\zeta_n}$. Lastly $\emptyset \notin D^*$ because $A^*_\varepsilon \neq \emptyset \mod D$. $\square_{3.3}$

4.1. Observation

$|\prod_{i<\kappa}\lambda_i/I| \geq |\aleph_0^\kappa/I|$ holds when $\bigwedge_{1<\kappa}\lambda_i \geq \aleph_0$.

4.2. Observation

(1) $|\prod_{i<\kappa}\lambda_i/I| \geq \mathrm{htcf}^*_I(\prod_{i<\kappa}\lambda_i)$.
(2) If I^* satisfies the pcf-th for $\overline{\lambda}$ or even the weaker pcf-th for $\overline{\lambda}$ (see Definition 6) <u>then</u>: $\mathrm{cf}(\prod\overline{\lambda}/I^*) = \max\mathrm{pcf}_{I^*}(\overline{\lambda})$.
(3) If I^* satisfies the pcf-th for μ for and $\min(\overline{\lambda}) \geq \mu$ <u>then</u>

$$\mathrm{hcf}_{D,\mu}(\prod\overline{\lambda}) = \mathrm{hcf}^*_{D,\mu}(\prod\overline{\lambda}) = \mathrm{htcf}^*_{D,\mu}(\prod\overline{\lambda})$$

whenever D is disjoint to I^*.
(4) $\mathrm{hcf}_{E,\mu}(\prod_{i<\kappa}\lambda_i) = \mathrm{hcf}^*_{E,\mu}(\prod_{i<\kappa}\lambda_i)$.
(5) $\prod_{i<\kappa}\lambda_i/I \geq \mathrm{hcf}_{I,\mu}(\prod_{i<\kappa}\lambda_i) = \mathrm{hcf}^*_{I,\mu}(\prod_{i<\kappa}\lambda_i) \geq \mathrm{htcf}^*_{I,\mu}(\prod_{i<\kappa}\lambda_i)$ and $\mathrm{hcf}_{I,\mu}(\prod_{i<\kappa}\lambda_i) \geq \mathrm{htcf}_{I,\mu}(\prod_{i<\kappa}\lambda_i)$.

Remark 3.5A. In 3.5(3) concerning $\mathrm{htcf}_{D,\mu}$ see 3.10.

Proof. (1) By the definition of htcf^*_I it suffices to show $|\prod_{i<\kappa}\lambda_i/I| \geq \mathrm{tcf}(\prod \lambda'_i/I')$, when I' is an ideal on κ extending $I, \lambda'_i = \mathrm{cf}\lambda'_i \leq \lambda_i$ for $i<\kappa$ and $\mathrm{tcf}(\prod_{i<\kappa}\lambda'_i/I')$ is well defined. Now $|\prod_{i<\kappa}\lambda_i/I| \geq |\prod_{i<\kappa}\lambda'_i/I| \geq |\prod_{i<\kappa}\lambda'_i/I'| \geq \mathrm{cf}(\prod_{i<\kappa}\lambda'_i/I')$, so we have finished.

(2) By 2.13(IA)clearly I^*, $\bar\lambda$ satisfies 1.5, 1.6 hence by 2.13(1), (2) also 1.8(1), (2), (3), (4) and 1.9 and 1.10. Now by 2.13(8) also (the conclusion of) 2.10(2) holds which is what we need.

(3) Left to the reader (see Definition 6(2) and part (2)).

(4), (5) Check. $\square_{3.5}$

CLAIM 3.6: If $\lambda = |\prod_{i<\kappa}\lambda_i/I|$ (and $\lambda_i \geq \aleph_0$ and, of course, I an ideal on κ) and $\theta < \mathrm{reg}(I)$ <u>then</u> $\lambda = \lambda^\theta$.

Proof. For each $i<\kappa$, let $\langle \eta^i_\alpha : \alpha < \lambda_i\rangle$ list the finite sequences from λ_i. Let $M_i = (\lambda_i, F_i, G_i)$ where $F_i(\alpha) = \ell g(\eta^i_\alpha), G_i(\alpha,\beta)$ is $\eta^i_\alpha(\beta)$ if $\beta < \ell g(\eta^i_\alpha)(= F_i(\alpha))$, and $F(\alpha,\beta) = 0$ otherwise; let $M = \prod_{i<\kappa} M_i/I$ so $\|M\| = |\prod \lambda_i/I|$ and let $M = (\prod \lambda_i/I, F, G)$. Let $\langle A_i : i < \theta\rangle$ exemplifies I is θ-regular. Now

$(*)_1$ We can find $f \in {}^\kappa\omega$ and $f_\varepsilon \in \prod_{i<\kappa} f(i)$ for $\varepsilon < \theta$ such that: $\varepsilon < \zeta < \theta \Rightarrow f_\varepsilon <_I f_\zeta$ [just for $i<\kappa$ let $w_i = \{\varepsilon < \theta : i \in A_\varepsilon\}$, it is finite and let $f(i) = |w_i|+1$ and $f_\varepsilon(i) = |\varepsilon \cap w_i| \leq f(i)$, and note $\varepsilon < \zeta \,\&\, i \in A_\varepsilon \cap A_\zeta \Rightarrow f_\varepsilon(i) < f_\zeta(i)$].

$(*)_2$ For every sequence $\bar g = \langle g_\varepsilon : \varepsilon < \theta\rangle$ of members of $\prod_{i<\kappa}\lambda_i$, there is $h \in \prod_{i<\kappa}\lambda_i$ such that $\varepsilon < \theta \Rightarrow M \models F(h/I, f_\varepsilon/I) = g_\varepsilon/I$ [why? let, in the notation of $(*)_1, h(i)$ be such that $\eta^i_{h(i)} = \langle g_\varepsilon(i) : \varepsilon \in w_i\rangle$ (in the natural order)].

So in M, every θ-sequence of members is coded using f/I, f_ε/I (for $\varepsilon < \theta$) by at least one member so $\|M\|^\theta = \|M\|$, but $\|M\| = |\prod_{i<\kappa}\lambda_i/I|$ hence we have proved 3.6. $\square_{3.6}$

4.3. Fact

(1) For D a filter on κ, $\langle A_i, A_2\rangle$ a partition of κ and (non zero) cardinals λ_i for $i<\kappa$ we have

$$|\prod_{i<\kappa}\lambda_i/D| = |\prod_{i<\kappa}\lambda_i/(D+A_1)| \times |\prod_{i<\kappa}\lambda_i/(D+A_2)|$$

(note: $|\prod_{i<\kappa}\lambda_i/\mathcal{P}(\kappa)| = 1$).

(2) $D^{[\mu]} =: \{A \subseteq \kappa : |\prod_{i<\kappa}\lambda_i/(D+(\kappa\setminus A))| < \mu\}$ is a filter on κ (μ an infinite cardinal of course) and if $\aleph_0 \leq \mu \leq \prod_{i<\kappa}\lambda_i/D$ <u>then</u> $D^{[\mu]}$ is a proper filter.

(3) If $\lambda \leq |\prod_{i<\kappa}\lambda_i/I|$, ($\lambda_i$ infinite, of course, I an ideal on κ) and $A \in I^+ \Rightarrow |\prod_{i\in A}\lambda_i/I| \geq \lambda$ and $\sigma < \mathrm{reg}_*(I)$ <u>then</u> $|\prod \lambda_i/I| \geq \lambda^\sigma$

Proof. Check (part (3): by the proof of 3.3(2) we can find $A_\varepsilon \in I^+$ for $\varepsilon < \sigma$ such that for finite $u \subseteq \sigma$, $\cap_{\varepsilon \in u} A_\varepsilon \in I^+$ and continue as in the proof of 3.6).

CLAIM 3.8: If $D \subseteq E$ are filters on κ then

$$|\prod_{i<\kappa} \lambda_i/D| \leq |\prod_{i<\kappa} \lambda_i/E| + \sup_{A \in E \setminus D} |\prod_{i<\kappa} \lambda_i/(D+(\kappa\setminus A))| + (2^\kappa/D) + \aleph_0.$$

We can replace $2^\kappa/D$ by $|\mathcal{P}|$ if $\mathcal{P}$ is a maximal subset of E such that $A \neq B \in \mathcal{P} \Rightarrow (A \setminus B) \cup (B \setminus A) \neq \emptyset \mod D$.

Proof. Think.

Lemma 6. $|\prod_{i<\kappa} \lambda_i/D| \leq (\theta^\kappa/D + \mathrm{hcf}_{D,\theta}(\prod_{i<\kappa} \lambda_i))^{<\theta}$ *(see Definition 8(1)) provided that:*

$$\theta \geq \mathrm{reg}_{\otimes}(D) \qquad (*)$$

Remark 3.9A. (1) If $\theta = \theta_1^+$, we can replace θ^κ/D by θ_1^κ/D. In general we can replace θ^κ/D by $\sup\{\prod_{i<\kappa} f(i)/D : f \in \theta^\kappa\}$.

(2) If D satisfies the pcf-th above θ (see 2.11(1A), 2.12(2) then by 3.5(3) we can use htcf* (sometime even htcf, see 3.10). But by 3.7(1) we can ignore the $\lambda_i \leq \theta$, and when $i < \kappa \Rightarrow \lambda_i > \theta$ we know that 1.5$(^*)(\alpha)$ holds by 3.3(3).

Proof. Let $\lambda = \theta^\kappa/D + \mathrm{hcf}_{D,\theta}(\prod_{i<\kappa} \lambda_i)$. Let for $\zeta < \theta, \mu_\zeta =: \lambda^{|\zeta|}$ i.e. $\mu_\zeta =: (\theta^\kappa/D + \mathrm{hcf}_{D,\theta} \prod_{i<\kappa} \lambda_i)^{|\zeta|}$, clearly $\mu_\zeta = \mu_\zeta^{|\zeta|}$. Let $\chi =\beth_8 (\sup_{i<\kappa} \lambda_i)^+$ and $N_\zeta \prec (H(\chi), \in, <^*_\chi)$ be such that $\|N_\zeta\| = \mu_\zeta, N^{\leq|\zeta|} \subseteq N_\zeta, \lambda + 1 \subseteq N_\zeta$ and $\{D, \langle \lambda_i : i < \kappa\rangle\} \in N_\zeta$ and $[\varepsilon < \zeta \Rightarrow N_\varepsilon \prec N_\zeta]$. Let $N = \cup\{N_\zeta : \zeta < \theta\}$. Let $g^* \in \prod_{i<\kappa} \lambda_i$ and we shall find $f \in N$ such that $g^* = f \mod D$, this will suffice. We shall choose by induction on $\zeta < \theta, f^e_\zeta (e < 3)$ and $\bar{A}^\zeta$ such that:

(a) $f^e_\zeta \in \prod_{i<\kappa}(\lambda_i + 1)$
(b) $f^1_\zeta \in N_\zeta$ and $f^2_\zeta \in N_\zeta$.
(c) $\bar{A}^\zeta = \langle A^\zeta_i : i < \kappa\rangle \in N_\zeta$.
(d) $\lambda_i \in A^\zeta_i \subseteq \lambda_i + 1$, $|A^\zeta_i| \leq |\zeta| + 1$, and $\langle A^\zeta_i : \zeta < \theta\rangle$ is increasing continuous (in ζ).
(e) $f^0_\zeta(i) = \min(A^\zeta_i \setminus g^*(i))$; note: it is well defined as $g^*(i) < \lambda_i \in A^\zeta_i$
(f) $f^1_\zeta = f^0_\zeta \mod D$
(g) $g^* < f^2_\zeta < f^1_\zeta \mod (D + \{i < \kappa : g^*(i) \neq f^1_\zeta(i)\})$.
(h) $f^2_\zeta(i) \in A^{\zeta+1}_i$

So assume everything is defined for every $\varepsilon < \zeta$. If $\zeta = 0$, let $A^\zeta_i = \{\lambda_i\}$, if ζ limit $A^\zeta_i = \bigcup_{\varepsilon<\zeta} A^\varepsilon_i$, for $\zeta = \varepsilon + 1, A^\zeta_i$ will be defined in stage ε. So arriving to ζ, $\bar{A}^\zeta$ is well defined and it belongs to N_ζ: for $\zeta = 0$ check, for $\zeta = \varepsilon + 1$, done in stage ε, for ζ limit it belongs to N_ζ as we have $N^{\leq|\zeta|}_\zeta \subseteq N_\zeta$ and:

$\xi < \zeta \Rightarrow N_\xi \prec N_\zeta$ and $\bar{A}^\xi \in N_\xi$. Now use clause (e) to define f^0_ζ/D. As $\langle A^\zeta_i : i < \kappa\rangle \in N_\zeta, |A^\zeta_i| \leq |\zeta| + 1 < \theta$ and $\theta^\kappa/D \leq \lambda < \lambda + 1 \subseteq N_\eta$, clearly $|\prod_{i<\kappa} |A^\zeta_i|/D| \leq \lambda$ hence $\{f/D : f \in \prod_{i<\kappa} A^\zeta_i\} \subseteq N_\zeta$ hence $f^0_\zeta/D \in N_\zeta$ hence there is $f^1_\zeta \in N_\zeta$ such that $f^1_\zeta \in f^0_\zeta/D$ i.e. clause (f) holds. As $g^* \leq f^0_\zeta$ clearly $g^* \leq f^1_\zeta \mod D$, let $y^\zeta_0 =: \{i < \kappa : g^*(i) \geq f^1_\zeta(i)\}$, $y^\zeta_1 =: \{i < \kappa : i \notin y^\zeta_0\ and \mathrm{cf}(f^1_\zeta(i)) < \theta\}$ $y^\zeta_2 =: \kappa \setminus y^\zeta_0 \setminus y^\zeta_1$. So $\langle y^\zeta_e : e < 3\rangle$ is a partition of κ and $g^* < f^1_\zeta \mod (D + y^\zeta_e)$ for $e = 1, 2$.

Let $y^\zeta_4 = \{i < \kappa : \mathrm{cf}(f^1_\zeta(i)) \geq \theta\}$ so $f^1_\zeta \in N_\zeta$, and $\theta = N_\zeta$ hence $y^\zeta_4 \in N_\zeta$, so $(\prod_{i<\kappa} f^1_\zeta(i), <_{D+y^\zeta_4}) \in N_\zeta$. Clearly $y^\zeta_2 \subseteq y^\zeta_4 \subseteq y^\zeta_0 \cup y^\zeta_2$. Now

$$\begin{aligned}\mathrm{cf}(\prod\nolimits_{i<\kappa} f^1_\zeta(i), <_{D+y^\zeta_4}) &\leq \mathrm{hcf}_{D+y^\zeta_4,\theta}(\prod\nolimits_{i<\kappa} \lambda_i) \\ &\leq \mathrm{hcf}_{D,\theta}(\prod\nolimits_{i<\kappa} \lambda_i) \subseteq \lambda < \lambda + 1 \subseteq N_\zeta\end{aligned}$$

hence there is $F \in N_\zeta$, $|F| \leq \lambda$, $F \subseteq \prod_{i\in y^\zeta_4} f^1_\zeta(i)$ such that:

$$(\forall g)[g \in \prod_{i\in y^\zeta_4} f^1_\zeta(i) \Rightarrow (\exists f \in F)(g < f \mod (D + y^\zeta_4))]$$

As $\lambda + 1 \subseteq N$ necessarily $F \subseteq N_\zeta$. Apply the property of F to $(g^* \restriction y^\zeta_2) \cup 0_{(\kappa\setminus y^\zeta_2)}$ and get $f^\zeta_4 \in F \subseteq N_\zeta$ such that $g^* < f^\zeta_4 \mod (D + y^\zeta_2)$. Now use similarly $\prod_{i<\kappa} \mathrm{cf}(f^1_\zeta(i))/(D + (\kappa \setminus y^\zeta_4)) \leq |\theta^\kappa/D| \leq \lambda$; by the proof of 3.7(1) there is a function $f^2_\zeta \in N_\zeta \cap \prod_{i<\kappa} f^1_\zeta(i)$ such that $g^* \restriction (y^\zeta_1 + y^\zeta_2) < f^2_\zeta \mod D$. Let $A^{\zeta+1}_i$ be: $A^{\zeta+1}_i \cup \{f^2_\zeta(i)\}$.

It is easy to check clauses (g), (h). So we have carried the definition. Let

$$X_\zeta =: \{i < \kappa : f^0_{\zeta+1}(i) < f^0_\zeta(i)\}.$$

Note that by the choice of f^1_ζ, $f^1_{\zeta+1}$ we know $X_\zeta = y^\zeta_1 \cup y^\zeta_2 \mod D$, if this last set is not D-positive then $g^* \geq f^1_\zeta \mod D$, hence $g^*/D = f^1_\zeta/D \in N_\zeta$, contradiction, so $y^\zeta_1 \cup y^\zeta_2 \neq \emptyset \mod D$ hence $X_\zeta \in D^+$. Also $\langle (y^\zeta_1 \cup y^\zeta_2)/D : \zeta < \theta\rangle$ is $\subseteq$-decreasing hence $\langle X_\zeta/D : \zeta < \theta\rangle$ is $\subseteq$-decreasing.

Also if $i \in X_{\zeta_1} \cap X_{\zeta_2}$ and $\zeta_1 < \zeta_2$ then $f^0_{\zeta_2}(i) \leq f^0_{\zeta_1+1}(i) < f^0_{\zeta_1}(i)$ (first inequality: as $A^{\zeta_1+1}_i \subseteq A^{\zeta_2}_i$ and clause (e) above, second inequality by the definition of X_{ζ_1}, hence for each ordinal i the set $\{\zeta < \theta : i \in X_\zeta\}$ is finite). So $\theta < \mathrm{reg}_\otimes(D)$, contradiction to the assumption $(*)$. $\square_{3.9}$

Note we can conclude

Claim 3.9B:

$\prod_{i<\kappa} \lambda_i/D = \sup\{(\prod_{i<\kappa} f(i))^{<\mathrm{reg}_\otimes(D_1)} + \mathrm{hcf}_{D_1}(\prod_{i<\kappa} \lambda_i)^{<\mathrm{reg}_\otimes(D_1)} : D_1$ is a filter on κ extending D such that

$$A \in D_1^+ \Rightarrow \prod_{i<\kappa} \lambda_i/(D_1 + A) = \prod_{i<\kappa} \lambda_i/D_1$$

$$\text{and } f \in \theta^\kappa, f(i) \leq \lambda_i\}$$

Proof. The inequality $\geq$ should be clear by 3.7(3). For the other direction let μ be the right side cardinality and let $D_0 = \{\kappa \setminus A : A \subseteq \kappa$ and if $A \in D^+$ then $\prod_{i<\kappa} \lambda_i/(D + A) \leq \mu\}$, so we know by 3.7(2) that D_0 is a filter on κ extending D. If $\emptyset \in D_0$ we are done so assume not. Now $\mu \geq 2^\kappa/D$ (by the term $(\prod_i f(i)/D_0)^{<\text{reg}_\otimes(D_1)}$) so by 3.8 we have $\prod_{i<\kappa} \lambda_i/D_0 > \mu$ (use 3.8 with D, D_0 here corresponding to D, E there). Now the same holds for $D_0 + A$ for every $A \in D_0^+$. Also $A \subseteq B \subseteq \kappa$ and $A \in D_0^+ \Rightarrow \prod_{i<\kappa} \lambda_i/(D_0 + A) \leq \prod_{i<\kappa} \lambda_i/(D_1 + B)$ so for some $B \in D_0^+$, $D_1 =: D_0 + B$ satisfies the requirement inside the definition of μ, so $\mu \geq \text{hcf}_{D_1}(\prod_{i<\kappa} \lambda_1)^{<\text{reg}_\otimes}(D_1)$.

By 3.9 (see 3.9A(1)) we get a contradiction. $\square_{3.9B}$

Next we deal with existence of $<_D$-eub.

Claim 3.10:

(1) Assume D a filter on $\kappa, g^*_\alpha \in {}^\kappa\text{Ord}$ for $\alpha < \delta, \bar{g}^* = \langle g^*_\alpha : \alpha < \delta \rangle$ is $\leq_D$-increasing, and

$$\text{cf}(\delta) \geq \theta \geq \text{reg}_*(D). \qquad (*)$$

Then at least one of the following holds:

(A) $\langle g^*_\alpha : \alpha < \delta \rangle$ has a $<_D$-eub $g \in {}^\kappa\text{Ord}$; moreover $\theta \leq \liminf_D \langle \text{cf}[g(i)] : i < \kappa \rangle$

(B) $\text{cf}(\delta) = \text{reg}_*(D)$

(C) For some club C of δ and some $\theta_1 < \theta$ and $\gamma_i < \theta_1^+$ and $w_i \subseteq \text{Ord}$ of order type γ_i for $i < \kappa$, there are $f_\alpha \in \prod_{i<\kappa} w_i$ (for $\alpha \in C$) such that $f_\alpha(i) = \min(w_i \setminus g^*_\alpha(i))$ and $\alpha \in C \& \beta \in C \& \alpha < \beta \Rightarrow f_\alpha \leq_D f_\beta \& \neg f_\alpha =_D f_\beta \& \neg f_\alpha \leq_D g^*_\beta \& g^*_\alpha \leq f_\alpha$.

(2) In (C) above if for simplicity D is an ultrafilter we can find $w_i \subseteq \text{Ord}$, $\text{otp}(w_i) = \gamma_i$, $\langle \alpha_\xi : \xi < \text{cf}(\delta) \rangle$ increasing continuous with limit δ, and $h_\varepsilon \in \prod_{i<\kappa} w_i$ such that $f_{\alpha_\varepsilon} <_D h_\varepsilon <_D f_{\alpha_{\varepsilon+1}}$ moreover, $\bigwedge_{i<\kappa} \gamma_i < \omega$.

Proof. (1) Let $\sigma = \text{reg}_*(D)$. We try to choose by induction on $\zeta < \sigma, g_\zeta, f_{\alpha,\zeta}$ (for $\alpha < \delta$), $\bar{A}^\zeta$, α_ζ such that

(a) $\bar{A}^\zeta = \langle A^\zeta_i : i < \kappa \rangle$.

(b) $A^\zeta_i = \{f_{\alpha_\varepsilon,\varepsilon}(i), g_\varepsilon(i) : \varepsilon < \zeta\} \cup \{[\sup_{\alpha<\delta} g^*_\alpha(i)] + 1\}$.

(c) $f_{\alpha,\zeta}(i) = \min(A^\zeta_i \setminus g^*_\alpha(i))$ (and $f_{\alpha,\zeta} \in {}^\kappa\text{Ord}$, of course).

(d) α_ζ is the first α, $\bigcup_{\varepsilon<\zeta}, \alpha_\varepsilon < \alpha < \delta$ such that $[\beta \in [\alpha, \delta) \Rightarrow f_{\beta,\zeta} = f_{\alpha,\zeta}$ mod $D]$ if there is one.

(e) $g_\zeta \leq f_{\alpha_\zeta,\zeta}$ moreover $g_\zeta < \max\{f_{\alpha_\zeta,\zeta}, 1_\kappa\}$ but for no $\alpha < \delta$ do we have $g_\zeta < \max\{g^*_\alpha, 1\}$ mod D.

Let ζ^* be the first for which they are not defined (so $\zeta^* \le \sigma$). Note

$$\varepsilon < \xi < \zeta^* \& \alpha_\xi \le \alpha < \delta \Rightarrow f_{\alpha_\varepsilon,\varepsilon} =_D f_{\alpha,\varepsilon} \& f_{\alpha,\xi} \le f_{\alpha,\varepsilon} \& f_{\alpha,\xi} \neq_D f_{\alpha,\varepsilon}. \quad (*)$$

[Why last phrase? applying clause (e) above, second phrase with α, ε here standing for α, ζ there we get $A_0 =: \{i < \kappa : \max\{g^*_\alpha(i), 1\} \le g_\varepsilon(i)\} \in D^+$ and applying clause (e) above first phrase with ε here standing for ζ there we get $A_1 = \{i < \kappa : g_\varepsilon(i) < f_{\alpha,\varepsilon}(i)$ or $g_\varepsilon(i) = 0 = f_{\alpha,\varepsilon}(i)\} \in D$, hence $A_0 \cap A_1 \in D^+$, and $g_\varepsilon(i) > 0$ for $i \in A_0 \cap A_1$ (even for $i \in A_0$). Also by clause (c) above $g^*_\alpha(i) \le g_\varepsilon(i) \Rightarrow f_{\alpha,\xi}(i) \le g_\varepsilon(i)$. Now by the last two sentences $i \in A_0 \cap A_1 \Rightarrow g^*_\alpha(i) \le g_\varepsilon(i) < f_{\alpha,\varepsilon}(i) \Rightarrow f_{\alpha,\xi}(i) < g_\varepsilon(i) < f_{\alpha,\varepsilon}(i)$, together $f_{\alpha,\xi} \neq_D f_{\alpha,\varepsilon}$ as required]

Case A. $\zeta^* = \sigma$ and $\bigcup_{\zeta<\sigma} \alpha_\zeta < \delta$. Let $\alpha(*) = \bigcup_{\zeta<\sigma} \alpha_\zeta$, for $\zeta < \sigma$ let $y_\zeta = \{i < \kappa : f_{\alpha(*),\zeta}(i) \neq f_{\alpha(*),\zeta+1}(i)\} \neq \emptyset \mod D$. Now for $i < \kappa, \langle f_{\alpha(*),\zeta}(i) : \zeta < \sigma\rangle$ is non increasing so i belongs to finitely many y_ζ's only, so $\langle y_\zeta : \zeta < \sigma\rangle$ contradict $\sigma \ge \mathrm{reg}_*(D)$.

Case B. $\zeta^* = \sigma$ and $\bigcup_{\zeta<\sigma} \alpha_\zeta = \delta$. So possibility (B) of Claim 3.10 holds.

Case C. $\zeta^* < \sigma$.

Still $A^{\zeta^*}_i (i < \kappa), f_{\alpha,\zeta^*}(\alpha < \delta)$ are well defined.

Subcase C1. α_{ζ^*} cannot be defined.

Then possibility C of 3.10 holds (use $w_i =: A^{\zeta^*}_i, f_\beta = f_{\alpha_{\zeta^*+\beta},\zeta^*}$).

Subcase C2. α_{ζ^*} can be defined.

Then $f_{\alpha_{\zeta^*},\zeta^*}$ is a $<_D$-eub of $\langle g^*_\alpha : \alpha < \delta\rangle$ as otherwise there is g_{ζ^*} as required in clause (e). Now $f_{\alpha^*_\zeta,\zeta^*}$ is almost as required in possibility (A) of Claim 3.10 only the second phrase is missing. If for no $\theta_1 < \theta, \{i < \kappa : \mathrm{cf}[f_{\alpha_{\zeta^*},\zeta^*}(i)] \le \theta_1\} \in D^+$, then possibility (A) holds.

So assume $\theta_1 < \theta$ and $B =: \{i < \kappa : \aleph_0 \le \mathrm{cf}[f_{\alpha_{\zeta^*},\zeta^*}(i)] \le \theta_1\}$ belongs to D^+, we shall try to prove that possibility (C) holds, thus finishing. Now we choose w_i for $i < \kappa$: for $i \in \kappa$ we let $w^0_i =: \{f_{\alpha_{\zeta^*},\zeta^*}(i), [\sup_{\alpha<\delta} g^*_\alpha(i)] + 1\}$, for $i \in B$ let w^1_i be an unbounded subset of $f_{\alpha_{\zeta^*},\zeta^*}(i)$ of order type $\mathrm{cf}[f_{\alpha_{\zeta^*},\zeta^*}(i)]$ and for $i \in \kappa \setminus B$ let $w^1_i = \emptyset$, lastly let $w_i = w^0_i \cup w^1_i$, so $|w_i| \le \theta_1$ as required in possibility (C). Define $f_\alpha \in {}^\kappa\mathrm{Ord}$ by $f_\alpha(i) = \min(w_i \setminus g^*_\alpha(i))$ (by the choice of w^0_i it is well defined). So $\langle f_\alpha : \alpha < \delta\rangle$ is $\le_D$-increasing; if for some $\alpha^* < \delta$, for every $\alpha \in [\alpha^*, \delta)$ we have $f_\alpha/D = f_{\alpha^*}/D$, we could define $g_{\zeta^*} \in {}^\kappa\mathrm{Ord}$ by:

$g_{\zeta^*} \restriction B = f_{\alpha^*}$ (which is $< f_{\alpha_{\zeta^*},\zeta^*}$),

$g_{\zeta^*} \restriction (\kappa \setminus B) = 0_{\kappa\setminus B}$

Now g_{ζ^*} is as required in clause (e) so we get contradiction to the choice of ζ^*. So there is no $\alpha^* < \delta$ as above so for some club C of δ we have $\alpha < \beta \in C \Rightarrow f_\alpha \neq_D f_\beta$, so we have actually proved possibility (C).

(2) Easy (for $\bigwedge_i \gamma_i < \omega$, wlog $\theta = \mathrm{reg}_*(D)$ but $\mathrm{reg}_*(D) = \mathrm{reg}(D)$ so $\theta_1 = \mathrm{reg}(D)$). $\square_{3.10}$

Claim 3.11:

(1) In 3.10(1), if $\lambda = \delta = \mathrm{cf}(\lambda)$, $\bar{g}^*$ obeys $\bar{a}$ ($\bar{a}$ as in 2.1), $\bar{a}$ a θ-weak (S,θ)-continuity condition, $S \subseteq \lambda$ unbounded, then clause (C) of 3.10 implies: (C)$'$ there are $\theta_1 < \mathrm{reg}_*(D)$ and $A_\varepsilon \in D^+$ for $\varepsilon < \theta$ such that the intersection of any θ_1^+ of the sets A_ε is empty (equivalently $i < \kappa \Rightarrow (\exists^{\leq \theta_1} \varepsilon)[i \in A_\varepsilon]$ (reminds (σ, θ_1^+)-regularity of ultrafilters).

(2) We can in 3.10(1) weaken the assumption $(*)$ to $(*)'$ below if in the conclusion we weaken clause (A) to (A)$'$ where

$(*)'$ $\mathrm{cf}(\delta) \geq \theta \geq \mathrm{reg}(D)$

(A)$'$ There is a $\leq_D$-upper bound f of $\{g^*_\alpha : \alpha < \delta\}$ such that no $f' <_D f$ (of course $f' \in {}^\kappa\mathrm{Ord}$) is a $\leq_D$-upper bound of $\{g^*_\alpha : \alpha < \delta\}$ and $\theta \leq \liminf_D \langle \mathrm{cf}[f(i)] : i < \kappa \rangle$

(3) If $g^*_\alpha \in {}^\kappa\mathrm{Ord}$, $\langle g^*_\alpha : \alpha < \delta \rangle$ is $<_D$-increasing and $f \in {}^\kappa\mathrm{Ord}$ satisfies (A)$'$ above and

$(*)''$ $\mathrm{cf}(\delta) \geq \mathrm{wsat}(D)$ and for some $A \in D$ for every $i < \kappa$, $\mathrm{cf}(f(i)) \geq \mathrm{wsat}(D)$ <u>then</u> for some $B \in D^+$ we have $\prod_{i<\kappa} \mathrm{cf}[f(i)]/(D+B)$ has true cofinality $\mathrm{cf}(\delta)$.

Remark. Compare with 2.6.

Proof. (1) By the choice of $\bar{a} = \langle a_\alpha : \alpha < \lambda \rangle$ as C (in clause (c) of 3.11(1)) is a club of λ, we can find $\beta < \lambda$ such that letting $\langle \alpha_\varepsilon : \varepsilon < \theta \rangle$ list $\{\alpha \in a_\beta : \mathrm{otp}(\alpha \cap a_\beta) < \theta\}$ (or just a subset of it) we have $(\alpha_\varepsilon, \alpha_{\varepsilon+1}) \cap C \neq \emptyset$.

Let $\gamma_\varepsilon \in (\alpha_\varepsilon, \alpha_{\varepsilon+1}) \cap C$, and $\xi_\varepsilon \in (\alpha_\varepsilon, \alpha_{\varepsilon+1})$ be such that $\{\alpha_\zeta : \zeta \leq \varepsilon\} \subseteq a_{\xi_\varepsilon}$, and as we can use $(\alpha_{2_\varepsilon} : \varepsilon < \theta)$, wlog $\xi_\varepsilon < \gamma_\varepsilon$. For $\zeta < \theta$ let $B_\zeta = \{i < \kappa : f_{\alpha_\zeta}(i) < f_{\beta_\zeta}(i) < f_{\gamma_\zeta}(i) < f_{\alpha_\zeta+1}(i)$ and $\sup\{f_{\alpha_\xi}(i)+1 : \xi < \zeta\} < \sup\{f_{\alpha_\xi}(i)+1 : \xi < \zeta+1\}$.

(2) In the proof of 3.10 we replace clause (e) by

(e$'$) $g_\zeta \leq f_{\alpha_\zeta,\zeta}$ and for $\alpha < \delta$ we have $f_\alpha \leq g_\zeta \mod D$

(3) By 1.8(1) $\square_{3.11}$

Claim 3.12:

(1) Assume $\lambda = \mathrm{tcf}(\prod \bar{\lambda}/D)$ and $\mu = \mathrm{cf}(\mu) < \lambda$ <u>then</u> there is $\bar{\lambda}' <_D \bar{\lambda}, \bar{\lambda}'$ a sequence of regular cardinals and $\mu = \mathrm{tcf}(\prod \bar{\lambda}'/D)$ provided that

$$\mu > \mathrm{reg}_*(D), \min(\bar{\lambda}) > \mathrm{reg}_*^{\sigma^+}(D) \text{ whenever } \sigma < \mathrm{reg}_*(D) \qquad (*)$$

(2) Let I^* be the ideal dual to D, and assume $(*)$ above. If $(*)(\alpha)$ of 1.5 holds and μ is semi-normal (for $(\bar{\lambda}, I^*)$) <u>*then*</u> it is normal.

Proof. Part (2) follows from part (1) by 2.2(3). Let us prove (1).

Case 1. $\mu < \liminf_D(\bar{\lambda})$

We let

$$\lambda' = \begin{cases} \mu & \text{if } \mu < \lambda_i \\ 1 & \text{if } \mu \geq \lambda_i \end{cases}$$

and we are done.

Case 2. $\liminf + D(\bar{\lambda}) \geq \theta \geq \operatorname{reg}_*(D)$, $\mu > \theta$, and $(\forall \sigma < \operatorname{reg}_*(D))[\operatorname{reg}^\sigma_*(D) < \theta]$.

Let $\theta =: \operatorname{reg}_*(D)$. There is an unbounded $S \subseteq \mu$ and an (S,θ)-continuity system $\bar{a}$ (see 2.4). As $\prod \bar{\lambda}/D$ has true cofinality $\lambda, \lambda > \mu$ clearly there are $g^*_\alpha \in \prod \bar{\lambda}$ for $\alpha < \mu$ such that $\bar{g}^*_\alpha = \langle g^*_\alpha : \alpha < \mu\rangle$ obeys $\bar{a}$ for $\bar{A}^*[\bar{\lambda}]$ (exists as $\theta \leq \liminf_D(\bar{\lambda})$).

Now if in claim 3.10(1) for $\bar{g}^*$ possibility (A) holds, we are done. By 3.11(1) we get that for some $\sigma < \operatorname{reg}_*(D)$ we have $\operatorname{reg}^\sigma_*(I) \geq \mu$, contradiction.

Case 3. $\liminf_D(\bar{\lambda}) \geq \theta \geq \operatorname{reg}_*(D)$, $\mu \geq \theta$, and $(\forall \sigma < \operatorname{reg}_*(D))[\operatorname{reg}^\sigma_*(D) < \theta]$.

Like the proof of [8, Ch. II 1.5B] using the silly square. $\square_{3.12}$

* * *

We turn to other measures of $\prod \bar{\lambda}/D$.

Definition 9. *(a)* $T^0_D(\bar{\lambda}) = \sup\{|F| : F \subseteq \prod \bar{\lambda}$ *and* $f_1 \neq f_2 \in F \Rightarrow f_1 \neq_D f_2\}$.

(b) $T^1_D(\bar{\lambda}) = \min\{|F| :$

- *(i)* $F \subseteq \prod \bar{\lambda}$
- *(ii)* $f_1 \neq f_2 \in F \Rightarrow f_1 \neq_D f_2$
- *(iii)* F *maximal under (i)+(ii)*$\}$

(c) $T^2_D(\bar{\lambda}) = \min\{|F| : F \subseteq \prod \bar{\lambda}$ *and for every* $f_1 \in \prod \bar{\lambda}$, *for some* $f_2 \in F$ *we have* $\neg f_1 \neq_D f_2\}$.

(d) If $T^0_D(\bar{\lambda}) = T^1_D(\bar{\lambda}) = T^2_D(\bar{\lambda})$ *then let* $T_D(\bar{\lambda}) = T^l_D(\bar{\lambda})$ *for* $l < 3$.

(e) For $f \in {}^\kappa\mathrm{Ord}$ *and* $\ell < 3$ *let* $T^l_D(f)$ *means* $T^l_D(\langle f(\alpha) : \alpha < \kappa\rangle)$.

Theorem 1. *(0) If* $D_0 \subseteq D_1$ *are filters on* κ *then* $T^\ell_{D_0}(\bar{\lambda}) \leq T^\ell_{D_1}(\bar{\lambda})$ *for* $\ell = 0,2$. *Also if* $\kappa = A_0 \cup A_1, A_0 \in D^+$, *and* $A_1 \in D^+$ *then* $T^\ell_D(\bar{\lambda}) = \min\{T^\ell_{D+A_0}(\bar{\lambda}), T^\ell_{D+A_1}(\bar{\lambda})\}$ *for* $\ell = 0,2$.

(1) $\operatorname{htcf}_D(\prod \bar{\lambda}) \leq T^2_D(\bar{\lambda}) \leq T^1_D(\bar{\lambda}) \leq T^0_D(\bar{\lambda})$

(2) If $T^0_D(\bar{\lambda}) > |\mathcal{P}(\kappa)/D|$ *or just* $T^0_D(\bar{\lambda}) > \mu$, *and* $\mathcal{P}(\kappa)/D$ *satisfies the* μ^+*-c.c.* *<u>then</u>* $T^0_D(\bar{\lambda}) = T^1_D(\bar{\lambda}) = T^2_D(\bar{\lambda})$ *so the supremum in 3.13(a) is obtained (so e.g.* $T^0_D(\bar{\lambda}) > 2^\kappa$ *suffice)*

(3) $T^0_D(\bar{\lambda})^{<\operatorname{reg} D} = T^0_D(\bar{\lambda})$ *(each* λ_i *infinite of course).*

(4) $[\operatorname{htcf}_D \prod_{i<\kappa} f(i)] \leq T^2_D(f) \leq [\operatorname{htcf}_D \prod_{i<\kappa} f(i)]^{<\theta} + \operatorname{reg}(D)^\kappa/D$ *where* $\theta = \operatorname{reg}_*(D)$ *in fact* $\theta = \operatorname{reg}(D) + \operatorname{wsat}(D)$ *suffice*

(5) If D *is an ultrafilter* $|\prod \bar{\lambda}/D| = T^e_D(\bar{\lambda})$ *for* $e \leq 2$.

(6) In (4), if $\bigwedge_{i<\kappa} f(i) \geq 2^\kappa$ *(or just* $(\operatorname{reg}(D)+2)^\kappa/D \leq \min_{i<\kappa} f(i)$*), <u>then</u>* $[\operatorname{htcf}_D \prod_{i<\kappa} f(i)]^{<\operatorname{reg} D} \leq T^0_D(f)$

(7) If the sup in the definition of $T^0_D(\bar{\lambda})$ *is not obtained then it has cofinality* $\geq \operatorname{reg}(D)$ *and even is regular.*

Proof. (0) Check.

(1) First assume $\mu =: T^2_D(\bar{\lambda}) < \text{htcf}_D(\prod \bar{\lambda})$; then we can find $\mu^* = \text{cf}(\mu^*) \in (\mu, \text{htcf}_D(\prod \bar{\lambda})]$ and $\bar{\mu} = \langle \mu_i : i < \kappa \rangle$, a sequence of regular cardinals, $\bigwedge_{i<\kappa} \mu_i \leq \lambda_i$ such that $\mu^* = \text{tcf}(\prod \bar{\mu}/D)$ and let $\langle f_\alpha : \alpha < \mu^* \rangle$ exemplify this. Now let F exemplify $\mu = T^2_D(\bar{\lambda})$, for each $g \in F$ let

$$g' \in \prod_{i<\kappa} \mu_i \text{ be } : g'(i) = \begin{cases} g(i) & \text{if } g(i) < \mu_i \\ 0 & \text{otherwise.} \end{cases}$$

So there is $\alpha(g) < \mu^*$ such that $g' <_D f_{\alpha(g)}$. Let $\alpha^* = \sup\{\alpha(g) : g \in F\}$, now $\alpha^* < \mu^*$ (as $\mu^* = \text{cf } \mu^* > \mu = |F|$). So $g \in F \Rightarrow g \neq_D f_{\alpha^*}$, contradiction. So really $T^2_D(\bar{\lambda}) \leq \text{htcf}_D(\prod \bar{\lambda})$ as required.

If F exemplifies the value of $T^1_D(\bar{\lambda})$, it also exemplifies $T^2_D(\bar{\lambda}) \leq |F|$ hence $T^2_D(\bar{\lambda}) \leq T^1_D(\bar{\lambda})$.

Lastly if F exemplifies the value of $T^1_D(f)$ it also exemplifies $T^0_D(\bar{\lambda}) \geq |F|$, so $T^1_D(\bar{\lambda}) \leq T^0_D(\bar{\lambda})$.

(2) Let μ be $|\mathcal{P}(\kappa)/D|$ or at least μ is such that the Boolean algebra $\mathcal{P}(\kappa)/D$ satisfies the μ^+-c.c. Assume that the desired conclusion fails so $T^2_D(\bar{\lambda}) \leq T^0_D(\bar{\lambda})$, so there is $F_0 \subseteq \prod \bar{\lambda}$, such that $[f_1 \neq f_2 \in F_0 \Rightarrow f_1 \neq_D f_2]$, and $|F_0| > T^2_D(\bar{\lambda}) + \mu$ (by the definition of $T^0_D(\bar{\lambda})$). Also there is $F_2 \subseteq \prod \bar{\lambda}$ exemplifying the value of $T^2_D(\bar{\lambda})$. For every $f \in F_0$ there is $g_f \in F_2$ such that $\neg f \neq_D g_f$ (by the choice of F_2). As $|F_0| > T^2_D(\bar{\lambda}) + \mu$ for some $g \in F_2, F^* =: \{f \in F_0 : g_f = g\}$ has cardinality $> T^2_D(f) + \mu$. Now for each $f \in F^*$ let $A_f = \{i < \kappa : f(i) = g(i)\}$, clearly $A_f \in D^+$. Now $f \mapsto A_f/D$ is a function from F^* into $\mathcal{P}(\kappa)/D$, hence, if $\mu = |\mathcal{P}(\kappa)/D|$, it is not one to one (by cardinality consideration) so for some $f' \neq f''$ from F^* (hence form F_0) we have $A_{f'}/D = A_{f''}/D$; but so

$$\begin{aligned}\{i < \kappa : f'(i) = f''(i)\} &\supseteq \{i < \kappa : f'(i) = g(i)\} \cap \{i < \kappa : f''(i) = g(i)\} \\ &= A_{f'}/D\end{aligned}$$

hence is $\neq \emptyset \mod D$, so $\neg f' \neq_D f''$, contradiction the choice of F_0. If $\mu \neq |\mathcal{P}(\kappa)/D|$ (as $F^* \subseteq F_0$ by the choice of F_0) we have :

$$f_1 \neq f_2 \in F^* \Rightarrow A_{f_1} \cap A_{f^2} = \emptyset \mod D$$

so $\{A_f : f \in F^*\}$ contradicts "the Boolean algebra $\mathcal{P}(\kappa)/D$ satisfies the μ^+-c.c.".

(3) Assume that $\theta < \text{reg}(D)$ and[¶] $\mu \leq^+ T^0_D(\bar{\lambda})$. As $\mu \leq^+ T^0_D(\bar{\lambda})$ we can find $f_\alpha \in \prod \bar{\lambda}$ for $\alpha < \mu$ such that $[\alpha < \beta \Rightarrow f_\alpha \neq_D f_\beta]$. Also (as $\theta < \text{reg}(D)$) we can find $\{A_\varepsilon : \varepsilon < \theta\} \subseteq D$ such that for every $i < \kappa$ the set $w_i =: \{\varepsilon < \theta : i \in A_\varepsilon\}$ is finite. Now for every function $h : \theta \to \mu$ we define g_h, a function with domain κ:

[¶] $\leq^+$ means here that the right side is a supremum, right bigger than the left or equal but the supremum is obtained

$$g_h(i) = \{(\varepsilon, f_{h(\varepsilon)}(i)) : \varepsilon \in w_i\}$$

So $|\{g_h(i) : h \in {}^\theta\mu\}| \leq (\lambda_i)^{|w_i|} = \lambda_i$, and if $h_1 \neq h_2$ are from ${}^\theta\mu$ then for some $\varepsilon < \theta$, $h_1(\varepsilon) \neq h_2(\varepsilon)$ so $B_{h_1,h_2} = \{i : f_{h_1(\varepsilon)}(i) \neq f_{h_2(\varepsilon)}(in)\} \in D$ that is $B_{h_1,h_2} \cap A_\varepsilon \in D$ so

$\otimes_1$ If $i \in B_{h_1,h_2} \cap A_\varepsilon$ then $\varepsilon \in w_i$, so $g_{h_1}(i) \neq g_{h_2}(i)$.
$\otimes_2$ $B_{h_1,h_2} \cap A_\varepsilon \in D$

So $\langle g_h : h \in {}^\theta\mu\rangle$ exemplifies $T^0_D(\bar\lambda) \geq \mu^\theta$. If the supremum in the definition of $T^0_D(\bar\lambda)$ is obtained we are done. If not then $T^0_D(\bar\lambda)$ is a limit cardinal, and by the proof above:

$$[\mu < T^0_D(\bar\lambda) \quad \& \quad \theta < \text{reg}(D) \Rightarrow \mu^\theta < T^0_D(\bar\lambda)].$$

So if $T^0_D(\bar\lambda)$ has cofinality $\geq \text{reg}(D)$ we are done; otherwise let it be $\sum_{\varepsilon<\theta} \mu_\varepsilon$ with $\mu_\varepsilon < T^0_D(\bar\lambda)$ and $\theta < \text{reg}D$. Note that by the previous sentence $T^0_D(\bar\lambda)^\theta = T^0_D(\bar\lambda)^{<\text{reg}(D)} = \prod_{\varepsilon<\theta} \mu_\varepsilon$, and let $\{f^\varepsilon_\theta : \alpha < \mu_\varepsilon\} \subseteq \prod \bar\lambda$ be such that $[\alpha < \beta \Rightarrow f^\varepsilon_\alpha \neq_D f^\varepsilon_D]$ and repeat the previous proof with $f^\varepsilon_{h(\varepsilon)}$ replacing $f_{h(\varepsilon)}$.

(4) *For the first inequality.* Assume it fails so $\mu =: T^2_D(f) < \text{htcf}_D(\prod_{i<\kappa} f(i))$ hence for some $g \in \prod_{i<f(i)}(f(i)+1), \text{tcf}(\prod_{i<\kappa} g(i), <_D)$ is λ with $\lambda = \text{cf}(\lambda) > \mu$. Let $\langle f_\alpha : \alpha < \lambda\rangle$ exemplifies this. Let F be as in the definition of $T^2_D(f)$, now for each $h \in F$, there is $\alpha(h) < \lambda$ such that

$$\{i < \kappa : \text{ if } h(i) < g(i) \text{ then } h(i) < f_{\alpha(g)}(i)\} \in D.$$

Let $\alpha^* = \sup\{\alpha(h) + 1 : h \in F\}$, now $f_{\alpha^*} \in \prod_{i<\kappa} f(i)$ and $h \in F \Rightarrow h \neq_D f_{\alpha^*}$ contradicting the choice of F.

For the second inequality. Repeat the proof of 3.9 except that here we prove $F =: \bigcup_{\zeta<\theta}(N_\zeta \cap \prod_{i<\kappa} f(i))$ exemplifies $T^2_D(f) \leq \lambda$. So let $g^* \in \prod_{i<\kappa} \lambda_i$, and we should find $f \in N$ such that $(g^* \neq_D f)$; we replace clause (g) in the proof by $(g)' g^* < f^2_{\zeta+1} < f^1_\zeta \mod D$

the construction is for $\zeta < \text{reg}(D)$ and if we are stuck in ζ then $\neg f^1_\zeta \neq_D g^*$ and so we are done.

(5) Straightforward.

(6) Note that all those cardinals are $\geq 2^\kappa$ and $2^\kappa \geq \text{reg}(D)^\kappa/D$. Now write successively inequalities from (2), (4), (1) and (3):

$$T^0_D(f) = T^2_D(f) \leq [\text{htcf}_D \prod_{i<\kappa} f(i)]^{<\text{reg}(D)} \leq [T^0_D(f)]^{<\text{reg}(D)} = T^0_D(f).$$

(7) See proof of part (3). Moreover, let $\mu = \sum_{\varepsilon<\tau} \mu_\varepsilon, \tau < T^0_D(\bar\lambda), \mu_\varepsilon < T^0_D(\bar\lambda)$ as exemplified by $\{f_\varepsilon : \varepsilon < \tau\}, \{f^\varepsilon_\alpha : \alpha < \mu_\varepsilon\}$ respectively. Let g_α be: if $\sum_{\varepsilon<\zeta} \mu_\varepsilon < \alpha < \sum_{\varepsilon\leq\zeta} \mu_\varepsilon$ then $g_\alpha(i) = (f_\varepsilon(i), f^\varepsilon_\alpha(i))$. So $\{g_\alpha : \alpha < \mu\}$ show: if $T^0_D(\bar\lambda)$ is singular then the supremum is obtained. $\square_{3.14}$

Claim 1. *Assume D is a filter on $\kappa, f \in {}^{\kappa}Ord$, $\mu^{\aleph_0} = \mu$ and $2^\kappa < \mu, T_D(f)$, (see Definition 9(d) and Theorem 1(2)) and $\mathrm{reg}_*(D) = \mathrm{reg}(D)$. If $\mu < T_D(f)$ then for some sequence $\bar{\lambda} \leq f$ of regulars, $\mu^+ = \mathrm{tcf}(\prod \bar{\lambda}/D)$, or at least*

(∗) *There are $\langle\langle \lambda_{i,n} : n < n_i\rangle : i < \kappa\rangle$, $\lambda_{i,n} = \mathrm{cf}(\lambda_{i,n}) < f(i)$ and a filter D^* on $\bigcup_{i<\kappa}\{i\} \times n_i$ such that: $\mu^+ = \mathrm{tcf}(\prod_{(i,n)} \lambda_{i,n}/D^*)$ and $D = \{A \subseteq \kappa : \bigcup_{i\in A}\{i\} \times n_i \in D^*\}$.*
Also the inverse is true.

Remark 3.15A.

(1) It is not clear whether the first possibility may fail. We have explained earlier the doubtful role of $\mu^{\aleph_0} = \mu$.
(2) We can replace μ^+ by any regular μ such that $\bigwedge_{\alpha<\mu} |\alpha|^{\aleph_0} < \mu$ and then we use 3.14(4) to get $\mu \leq^+ T_D(f)$.
(3) The assumption $2^\kappa < \mu$ can be omitted.

Proof. The inverse should be clear (as in the proof of 3.6, by 3.14(3)).
Wlog $f(i) > 2^\kappa$ for $i < \kappa$, and trivially $(\mathrm{reg}(D))^\kappa/D \leq 2^\kappa$, so by 3.14(4)

$$T_D(f) \leq [\mathrm{htcf}_D(\prod_{i<\kappa} f(i)]^{<\mathrm{reg}_*(D)}.$$

If $\mu < \mathrm{htcf}_D(\prod_{i<\kappa} f(i))$ we are done (by 3.12(1)), so assume $\mathrm{htcf}_D(\prod_{i<\kappa} f(i)) \leq \mu$, but we have assumed $\mu < T_D(f)$ so by 3.14(4) as $\mathrm{reg}_*(D) = \mathrm{reg}(D)$ we have $\mu^{<\mathrm{reg}(D)} \geq \mu^+$. Let $\chi \leq \mu$ be minimal such that $\bigvee_{\theta<\mathrm{reg}(D)} \chi^\theta \geq \mu$, and let $\theta =: \mathrm{cf}(\chi)$ so, as $\mu > 2^\kappa$ we know $\chi^{\mathrm{cf}\chi} = \chi^{<\mathrm{reg}(D)} = \mu^{<\mathrm{reg}(D)} \geq \mu^+, \chi > 2^\kappa, \bigwedge_{\alpha<\chi} |\alpha|^{<\mathrm{reg}(D)} < \chi$. By the assumption $\mu = \mu^{\aleph_0}$ we know $\theta > \aleph_0$ (of course θ is regular). By [8, VIII 1.6(2), IX 3.5] and [Sh513, 6.12] there is a strictly increasing sequence $\langle \mu_\varepsilon : \varepsilon < \theta\rangle$ of regular cardinals with limit χ such that $\mu^+ = \mathrm{tcf}(\prod_{\varepsilon<\theta} \mu_\varepsilon/J^{\mathrm{bd}}_\theta)$.

As clearly $\chi \leq \mathrm{htcf}_D(\prod_{i<\kappa} f(i))$, by 2.12(1) there is for each $\varepsilon < \theta$, a sequence $\bar{\lambda}^\varepsilon = \langle \lambda^\varepsilon_i : i < \kappa\rangle$ such that $\lambda^\varepsilon_i = \mathrm{cf}(\lambda^\varepsilon_i) \leq f(i)$, and $\mathrm{tcf}(\prod_{i<\kappa} \lambda^\varepsilon_i/D) = \mu_\varepsilon$, also wlog $\lambda^\varepsilon_i > 2^\kappa$. Let $\langle A_\varepsilon : \varepsilon < \theta\rangle$ exemplify $\theta < \mathrm{reg}(D)$ and $n_i = |\{\varepsilon < \theta : i \in A_\varepsilon\}|$ and $\{\lambda_{i,n} : n < \omega\}$ enumerate $\{\lambda^\varepsilon_i : \varepsilon \text{ satisfies } i \in A_\varepsilon\}$, so we have gotten (∗). $\square_{3.15}$

4.4. Conclusion

Suppose D is an $\aleph_1$-complete filter on κ and $\mathrm{reg}_*(D) = \mathrm{reg}(D)$. If $\lambda_i \geq 2^\kappa$ for $i < \kappa$ and $\sup_{A\in D+} T_{D+A}(\bar{\lambda}) > \mu^{\aleph_0}$ then for some $\lambda'_i = \mathrm{cf}(\lambda'_i) \leq \lambda_i$ we have

$$\sup_{A\in D+} \mathrm{htcf}_{D+A}(\prod_{i<\kappa} \lambda'_i) > \mu.$$

4.5. Conclusion

Let D be an $\aleph_1$-complete filter on κ and $\text{reg}_*(D) = \text{reg}(D)$. If for $i < \kappa, B_i$ is a Boolean algebra and $\lambda_i < \text{Depth}^+(B_i)$ (see below) and

$$2^\kappa < \mu^{\aleph_0} < \sup_{A \in D+} T_{D+A}(\bar{\lambda})$$

then $\mu^+ < \text{Depth}^+(\prod_{i<\kappa} B_i/D)$

Proof. Use 3.25 below and 3.16 above.

Definition 10. *For a partial order P (e.g. a Boolean algebra) let* $\text{Depth}^+(P) = \min\{\lambda :$ *we cannot find $a_\alpha \in P$ for $\alpha < \lambda$ such that $\alpha < \beta \Rightarrow a_\alpha <_P a_\beta$*$\}$.

4.6. Discussion

(1) We conjecture that in 3.16 (and 3.17) the assumption "D is $\aleph_0$-complete" can be omitted. See [16].
(2) Note that our results are for $\mu = \mu^{\aleph_0}$ only; to remove this we need first to improve the theorem on $pp = cov$ (i.e. to prove $\text{cf}(\lambda) = \aleph_0 < \lambda \Rightarrow pp(\lambda) = cov(\lambda, \lambda, \aleph_1, 2)$ (or $\sup\{pp(\mu) : \text{cf}\mu = \aleph_0 < \mu < \lambda\} = \text{cf}(S_{\leq \aleph_0}(\lambda), \subseteq)$ (see [8], [14], §1]), which seems to me a very serious open problem (see [8, Analitic guide, 14]).
(3) In 3.17, if we can find $f_\alpha \in \prod_{i<\kappa} \lambda_i$ for $\alpha < \lambda : [\alpha < \beta < \lambda \Rightarrow f_\alpha \leq f_\beta \mod D]$ and $\neg f_\alpha =_D f_{\alpha+1}$ then $\lambda < \text{Depth}^+(\prod_{i<\kappa} B_i/D)$. But this do es not help for λ regular $> 2^\kappa$.
(4) We can approach 3.15 differently, by 3.20–3.23 below.

Claim 3.20: If $2^{2^\kappa} \leq \mu < T_D(\overline{\lambda})$, (or at least $2^{|D|+\kappa} \leq \mu < T_D(\overline{\lambda})$) and $\mu^{<\theta} = \mu$, then for some θ-complete filter $E \subseteq D$ we have $T_E(\overline{\lambda}) > \mu$.

Proof. Wlog θ is regular (as $\mu^{<\theta} = \mu$ &, $\text{cf}(\theta) < \theta \Rightarrow \mu^{<\theta^+} = \mu$). Let $\{f_\alpha : \alpha < \mu^+\} \subseteq \prod \bar{\lambda}$, be such that $[\alpha < \beta \Rightarrow f_\alpha \neq_D f_\alpha]$. We choose by induction on $\zeta, \alpha_\zeta < \mu^+$ as follows: α_ζ is the minimal ordinal $\alpha < \mu^+$ such that $E_{\zeta,\alpha} \subseteq D$ where $E_{\zeta,\alpha}$ = the θ-complete filter generated by

$$\{\{i < \kappa : f_{\alpha_\varepsilon}(i) \neq f_\alpha(i)\} : \varepsilon < \zeta\}$$

(note: each generator of $E_{\zeta,\alpha}$ is in D but not necessarily $E_{\zeta,\alpha} \subseteq D$!).

Let α_ζ be well defined if $\zeta < \zeta^*$, clearly $\varepsilon < \zeta \Rightarrow \alpha_\varepsilon < \alpha_\zeta$. Now if $\zeta^* < \mu^+$, then clearly $\alpha^* = \bigcup_{\zeta<\zeta^*} \alpha_\zeta < \mu^+$ and for every $\alpha \in (a^*, \mu^+), E_{\zeta^*,\alpha} \nsubseteq D$, so for every such a there are $A_\alpha \in D^+$ and $a_\alpha \in [\zeta^*]^{<\theta}$ such that $A_\alpha = \bigcup_{\varepsilon \in a_\alpha} \{i < \kappa : f_{\alpha_\varepsilon}(i) = f_\alpha(i)\}$. But for every $A \in D^+$, $a \in [\zeta^*]^{<\theta}$ we have

$$\{\alpha : \alpha \in (\alpha^*, \mu^+), A_\alpha = A, a_\alpha = a\} \subseteq \{a : f_\alpha \restriction A \in \prod_{i<\kappa} \{f_{\alpha_\varepsilon}(i) : \varepsilon \in a_\alpha\}\},$$

hence has cardinality $\leq \theta^\kappa \leq 2^\kappa < \mu$. Also $|[\zeta^*]^{<0}| \leq \mu^{<0} < \mu^+, |D^+| \leq 2^\kappa < \mu^\kappa$ so we get easy contradiction.

So $\zeta^* = \mu^+$, but the number of possible E's is $\leq 2^{2^\kappa}$, hence for some E we have $|\{\varepsilon < \mu^+ : E_{\varepsilon,\alpha_\varepsilon} = E\}| = \mu^+$. Necessarily $E \subseteq D$ and E is θ-complete, and $\{f_{\alpha_\varepsilon} : \varepsilon < \mu^+$, and $E_{\alpha_\varepsilon} = E\}$ exemplifies $T_E(\bar\lambda) > \mu$, so E is as required. $\square_{3.20}$

4.7. Fact

1. In 3.20 we can replace μ^+ by μ^* if $2^{2^\kappa} < \mathrm{cf}(\mu^*) \leq \mu^* \leq T^0_D(\bar\lambda)$ and $\bigwedge_{\alpha<\mu^*} |\alpha|^{<\theta} < \mu^*$.

Proof. The same proof as 3.20.

Claim 3.22:

(1) If $2^\kappa < |\prod \bar\lambda/D|$, D an ultrafilter on $\kappa, \mu = \mathrm{cf}(\mu) \leq |\prod \bar\lambda/D|, \bigwedge_{i<\kappa} |i|^{\aleph_0} < \mu$, and D is regular <u>then</u> $\mu < \mathrm{Depth}^+(\prod_{i<\kappa} \lambda_i/D)$

(2) Similarly for D just a filter but $A \in D^+ \Rightarrow \prod \bar\lambda/(D+A) = \prod \bar\lambda/D$.

Proof.

(1) Wlog $\lambda =: \lim_D \bar\lambda = \sup(\bar\lambda)$, so $|\prod \bar\lambda/D| = \lambda^\kappa$ (see 3.6, by [1]). If $\mu \leq \lambda$ we are done; otherwise let $\chi = \min\{\chi : \chi^\kappa = \lambda^\kappa\}$, so $\chi^{\mathrm{cf}(\chi)} = \lambda^\kappa, \mathrm{cf}(\chi) \leq \kappa$ but $\lambda < \mu \leq \lambda^\kappa$ hence $\lambda^{\aleph_0} < \mu$ hence $\mathrm{cf}(\chi) > \aleph_0$, also by χ's minimality $\bigwedge_{i<\chi} |i|^{\mathrm{cf}\chi} \leq |i|^\kappa < \chi$, and remember $\chi < \mu = \mathrm{cf}\ \mu \leq \chi^{\mathrm{cf}\chi}$ so by [8,. VIII 1.6(2)] there is $\langle \mu_\varepsilon : \varepsilon < \mathrm{cf}(\chi)\rangle$ strictly increasing sequence of regular cardinals with limit χ, $\prod_{\varepsilon<\mathrm{cf}(\chi)} \mu_\varepsilon/J^{bd}_{\mathrm{cf}\chi}$ has true cofinality μ. Let $\chi_\varepsilon = \sup\{\mu_\zeta : \zeta < \varepsilon\} + 2^\kappa$, let $i : \kappa \Rightarrow \mathrm{cf}(\chi)$ be $i(i) = \sup\{\varepsilon + 1 : \lambda_i \geq \chi_\varepsilon\}$. If there is a function $h \in \prod_{i<\kappa} i(i)$ such that $\bigwedge_{j<\mathrm{cf}(\chi)} \{i < \kappa : h(i) < j\} = \emptyset \mod D$ then $\prod_{i<\kappa} \mu_{h(i)}/D$ has true cofinality μ as required; if not (D, i) is weakly normal (i.e. there is no such h- see [13]). But for D regular, D is $\mathrm{cf}(\chi)$-regular, some $\langle A_\varepsilon : \varepsilon < \mathrm{cf}(\kappa)\rangle$ exemplifies it and $h(i) = \max\{\varepsilon : \varepsilon < i(i)$ and $i \in A_\varepsilon\}$ (maximum over a finite set) is as required.

(2) Similarly using $\lambda =: \liminf_D(\bar\lambda)$. $\square_{3.22}$

4.8. Discussion

1. In 3.20 (or 3.21) we can apply [12, §6] so $\mu = \mathrm{tcf}(\prod \bigcup_{i<\mu} a_i/D^*)$, where $D = \{A \subseteq \kappa : \bigcup_{i\in A} a_i \in D^*\}$ and each a_i, is finite.

See also in 3.15.

Claim 3.24: If D is a filter on κ, B_i is the interval Boolean algebra on the ordinal α_i, and $|\prod_{i<\kappa} \alpha_i/D| > 2^\kappa$ <u>then</u> for regular μ we have: $\mu < \mathrm{Depth}^+(\prod_{i<\kappa} B_i/D)$ iff for some $\mu_i \leq \alpha_i$ (for $i < \kappa$) and $A \in D^+$, the true cofinality of $\prod_{i<\kappa} \mu_i/(D+A)$ is well defined and equal to μ.

Proof. The $\Leftarrow$ (i.e. if direction) is clear. For the $\Rightarrow$ direction assume μ is regular $<$ Depth$^+(\prod_{i<\kappa} B_i/D)$ so there are $f_\alpha \in \prod_{i<\kappa} B_i$, such that $\prod_{i<\kappa} B_i/D \models f_\alpha/D < f_\beta/D$ for $\alpha < \beta$.

Wlog $\mu > 2^\kappa$. Let $f_\alpha(i) = \bigcup [j_{\alpha,i,2\ell}, j_{\alpha,i.2\ell+1})$ where $j_{\alpha,i,\ell} < j_{\alpha,i,\ell+1} < \alpha_i$ for $\ell < 2n(\alpha,i)$. As $\mu = \mathrm{cf}(\mu) > 2^\kappa$ wlog $n_{\alpha,i} = n_i$. By [14], 6.60] (see more [Sh513, 6.1]) we can find $A \subseteq A^* =: \{(i,\ell) : i < \kappa, \ell < 2n_\alpha\}$ and $\langle \gamma^*_{i,\ell} : i < \kappa, \ell < 2n_i \rangle$ such that $(i,\ell) \in A \Rightarrow \gamma^*_{i,\ell}$ is a limit ordinal and

(∗) For every $f \in \prod_{(i,\ell)\in A} \gamma^*_{i,\ell}$ and $\alpha < \mu$ there is $\beta \in (\alpha,\mu)$ such that

$$(i,\ell) \in A^*\ A \Rightarrow j_{\alpha,i,\ell} = \gamma^*_{i,\ell}$$

$$(i,\ell) \in A \Rightarrow f(i,\ell) < j_{\beta,i,\ell} < \gamma^*_{i,\ell}$$

$$(i,\ell) \in A \Rightarrow \mathrm{cf}(\gamma^*_{i,\ell}) > 2^\kappa$$

Let $\ell(i) = \max\{\ell < 2n(i) : (i,\ell) \in A\}$ and let $B = \{i : \ell(i) \text{ well defined}\}$. Clearly $B \in D^+$ (otherwise we can find $\alpha < \beta < \mu$ such that $f_\alpha/D = f_\beta/D$, contradiction). For $(i,\ell) \in A$ define $\beta^*_{i,\ell}$ by $\beta^*_{i,\ell} = \sup\{\gamma^*_{j,m} + 1 : (j,m) \in A^*$ and $\gamma^*_{j,m} < \gamma^*_{i,\ell}\}$. Now $\beta^*_{i,\ell} < \gamma^*_{i,\ell}$ as $\mathrm{cf}(\gamma^*_{i,\ell}) > 2^\kappa$. Let

$$Y = \{\alpha < \mu : \text{ if } (i,\ell) \in A^* \setminus A\ then j_{\alpha,i,\ell} = \gamma^*_{i,\ell}$$
$$\text{and if } (i,\ell) \in A \text{ then } \beta^*_{i,\ell} < j_{\alpha,\ell,i} < \gamma^*_{i,\ell}\}$$

Let $B_1 = \{i \in B : \ell(i) \text{ is odd}\}$. Clearly $B_1 \subseteq B$ and $B \setminus B_1 = \emptyset \mod D$ (otherwise as in $(*)_1$, $(*)_2$ below get contradiction) hence $B_1 \in D^+$. Now $(*)_1$ for $\alpha < \beta$ from Y we have

$$\langle j_{\alpha,i,\ell(i)} : i \in B_1 \rangle \le \langle j_{\beta,i,\ell(i)} : i \in B_1 \rangle \mod (D \restriction B_1)$$

[Why? as f_α/D was non decreasing in $\prod_{i<\kappa} B_i/D$]

$(*)_2$ for every $\alpha \in Y$ for some $\beta, \alpha < \beta \in Y$ we have

$$\langle j_{\alpha,i,\ell(i)} : i \in B_1 \rangle < \langle j_{\beta,i,\ell(i)} : i \in B_1 \rangle \mod (D \restriction B_1)$$

[Why? by (∗) above]

Together for some unbounded $Z \subseteq Y$, $\langle \langle j_{\alpha,i,\ell,\ell(i)} : i \in B_1 \rangle/(D \restriction B_1) : \alpha \in Z \rangle$ is $<_{D\restriction B_1}$-increasing, so it has $a <_{(D\restriction B_1)}$-eub (as $\mu > 2^\kappa$ see 3.10, and more in [8, II §1]), say $\langle j^*_i : i \in B_1 \rangle$ hence $\prod_{i\in B_1} j^*_i/(D \restriction B_1)$ has true cofinality μ by 1.3(12) and clearly $j^*_i \le \gamma^*_{i,\ell(i)} \le \alpha_i$, so we have finished. $\square_{3.24}$

Claim 3.25 If D is a filter on κ, B_i a Boolean algebra, $\lambda_i <$ Depth$^+(B_i)$ <u>then</u>

(a) Depth$(\prod_{i<\kappa} B/D) \ge: \sup_{A\in D^+} \mathrm{tcf}(\prod_{i<\kappa} \lambda_i(D + A))$ (i.e. on the cases tcf is well defined).
(b) Depth$^+(\prod_{i<\kappa} B/D)$ is $\ge$: Depth$^+(\mathcal{P}(\kappa)/D)$ and is at least

$$\sup\{[\mathrm{tcf}(\prod_{i<\kappa} \lambda'_i/(D + A))]^+ : \lambda'_i < \mathrm{Depth}^+(B_i), A \in D^+\}.$$

Proof. Check.

CLAIM 3.26: Let D be a filter on $\kappa, \langle\lambda_i : i < \kappa\rangle$ a sequence of cardinals and $2^\kappa < \mu = \mathrm{cf}(\mu)$. Then $(\alpha) \Leftrightarrow (\beta) \Rightarrow (\gamma) \Rightarrow (\delta)$, and if $(\forall_\sigma < \mu)(\sigma^{\aleph_0} < \mu)$ and $\mathrm{reg}_*(D) = \mathrm{reg}(D)$ we also have $(\gamma) \Leftrightarrow (\delta)$ where

- (α) If B_i is a Boolean algebra, $\lambda_i < \mathrm{Depth}^+(B_i)$ <u>then</u> $\mu < \mathrm{Depth}^+(\prod_{i<\kappa} B_i/D)$
- (β) There are $\mu_i = \mathrm{cf}(\mu_i) \leq \lambda_i$ for $i < \kappa$ and $A \in D^+$ such that $\mu = \mathrm{tcf}(\prod \mu_i/(D + A))$
- (γ) There are $\langle\langle\lambda_{i,n} : n < n_i\rangle : i < \kappa\rangle, \lambda_{i,n} = \mathrm{cf}(\lambda_{i,n}) < \lambda_i, A^* \in D^+$ and a filter D^* on $\bigcup_{i<\kappa}\{i\} \times n_i$ such that:
$$\mu = \mathrm{tcf}(\prod_{(i,n)} \lambda_{i,n}/D^*) \text{ and } D + A^* = \{A \subseteq \kappa : \text{ the set } \bigcup_{i\in A}\{i\} \times n_i$$
belongs to $D^*\}$.
- (δ) For some $A \in D^+$, $\mu \leq T_{D+A}(\langle\lambda_i : i < \kappa\rangle)$

Remark. So the question whether $(\alpha) \Leftrightarrow (\delta)$ assuming $(\forall_\sigma < \mu)(\sigma^{\aleph_0} < \mu)$ is equivalent to $(\beta) \leftrightarrow (\delta)$ which is a "pure" pcf problem.

Proof. Note $(\gamma) \Rightarrow (\delta)$ is easy (as in 3.15, i.e. as in the proof of 3.6, only easier). Now $(\beta) \Rightarrow (\gamma)$ is trivial and $(\beta) \Rightarrow (\alpha)$ by 3.25. Next $(\alpha) \Rightarrow (\beta)$ holds as we can use (α) for $B_i =$: the interval Boolean algebra of the order λ_i and use 3.24. Lastly assume $(\forall_\sigma < \mu)(\sigma^{\aleph_0} < \mu)$ and $\mathrm{reg}_*(D) = \mathrm{reg}(D)$, now $(\gamma) \Leftrightarrow (\delta)$ by 3.15. $\square_{3.26}$

Discussion. We would like to have (letting B_i denote Boolean algebra)
$$\mathrm{Depth}^{(+)}(\prod_{i<\kappa} B_i/D) \geq \prod_{i<\kappa} \mathrm{Depth}^{(+)}(B_i)/D$$
if D is just filter we should use T_D and so by the problem of attainment (serious by Magidor and Shelah [15]), we ask

$\otimes$ For D an ultrafilter on κ, does $\lambda_i < \mathrm{Depth}^+(B_i)$ for $i < \kappa$ implies
$$\prod_{i<\kappa} \lambda_i/D < \mathrm{Depth}^+(\prod_{i<\kappa} B_i/D)$$
at least when $\lambda_i > 2^\kappa$;

$\otimes'$ For D a filter on κ, does $\lambda_i < \mathrm{Depth}^+(B_i)$ for $i < \kappa$ implies, assuming $\lambda_i > 2^\kappa$ for simplicity,
$$T_D(\langle\lambda_i : i < \kappa\rangle) < \mathrm{Depth}^+(\prod_{i<\kappa} B_i/D)$$
As explained in 3.26 this is a pcf problem.

In [16] we deal with this under reasonable assumption (e.g. $\mu = \chi^+$ and $\chi = \chi^{\aleph_0}$). We also deal with a variant, changing the invariant (closing under homomorphisms, see [6]).

5. Remarks on the Conditions for the pcf Analysis

We consider a generalization whose interest is not so clear.

Claim 4.1: Suppose $\overline{\lambda} = \langle \lambda_i : i < \kappa \rangle$ is a sequence of regular cardinals, and θ is a cardinal and I^* is an ideal on κ; and H is a function with domain κ. We consider the following statements:

$(**)_H$ $\liminf_{I^*}(\overline{\lambda}) \geq \theta \geq \mathrm{wsat}(I^*)$ and H is a function from κ to $\mathcal{P}(\theta)$ such that:

(a) For every $\varepsilon < \theta$ we have $\{i < \kappa : \varepsilon \in H(i)\} = \kappa \mod I^*$
(b) For $i < \kappa$ we have $\mathrm{otp}(H(i)) \leq \lambda_i$ or at least $\{i < \kappa : |H(i)| \geq \lambda_i\} \in I^*$

$(**)^+$ Similarly but

$(b)^+$ For $i < \kappa$ we have $\mathrm{otp}(H(i)) < \lambda_i$

(1) In 1.5 we can replace the assumption $(*)$ by $(**)_H$ above.
(2) Also in 1.6, 1.7, 1.8, 1.9, 1.10, 1.11 we can replace 1.5$(*)$ by $(**)_H$.
(3) Suppose in Definition 4(2) we say $\bar{f}$ obeys $\bar{a}$ for H (instead of for $\bar{A}*$) if

(i) For $\beta \in a_\alpha$, such that $\varepsilon =: \mathrm{otp}(a_\alpha) < \theta$ we have

$$\mathrm{otp}(a_\beta), \mathrm{otp}(a_\beta) \in H(i) \Rightarrow f_\beta(i) \leq f_\alpha(i)$$

and in 2.3(2A), $f_\alpha(i) = \sup\{f_\beta(i) : \beta \in a_\alpha \text{ and } \mathrm{otp}(a_\beta), \mathrm{otp}(a_\alpha) \in H(i)\}$.

Then we can replace 1.5$(*)$ by $(**)_H$ in 2.5, 2.5A , 2.6; and replace 1.5$(*)$ by

$(**)^+_H$ In 2.7 (with the natural changes).

Proof. (1) Like the proof of 1.5, but defining the g_ε's by induction on ε we change requirement (ii) to

(ii)′ If ζ_ε, and $\{\zeta, \varepsilon\} \subseteq H(i)$ then $g_\zeta(i) < g_\varepsilon(i)$.

We can not succeed as

$$\langle (B^\varepsilon_{\alpha(*)} \setminus B^{\varepsilon+1}_{\alpha(*)}) \cap \{i < \kappa : \varepsilon, \varepsilon + 1 \in H(i)\} : \varepsilon < \theta \rangle$$

is a sequence of θ pairwise disjoint member of $(I^*)^+$.

In the induction, for ε limit let $g_\varepsilon(i) < \cup\{g_\zeta(i) : \zeta \in H(i)$ and $\varepsilon \in H(i)\}$ (so this is a union at most $\mathrm{otp}(H(i) \cap \varepsilon)$ but only when $\varepsilon \in H(i)$ hence is $< \mathrm{otp}(H(i)) \leq \lambda_i$).

(2) The proof of 1.6 is the same, in the proof of 1.7 we again replace (ii) by (ii)$'$. Also the proof of the rest is the same.

(3) Left to the reader. $\square_{4.1}$

We want to see how much weakening $(*)$ of 1.5 10 "$\liminf_{I^*}(\bar\lambda) \geq \theta \geq \mathrm{wsat}(I^*)$ suffices. If θ singular or $\liminf_{I^*}(\bar\lambda) > \theta$ or just $(\prod(\bar\lambda), <_{I^*})$ is θ^+-directed then case (β) of 1.5 applies. This explains $(*)$ of 4.2 below.

Claim 4.2: Suppose $\overline{\lambda} = \langle \lambda_i : i < \kappa\rangle, \lambda_i = \mathrm{cf}(\lambda_i), I^*$ an ideal on κ and

$$\liminf_I(\overline{\lambda}) = \theta \geq \mathrm{wsat}(I^*), \qquad \theta \text{ regular} \qquad (*)$$

Then we can define a sequence $\bar J = \langle J_\zeta : \zeta < \zeta(*)\rangle$ and an ordinal $\zeta(*) \leq \theta^+$ such that

(a) $\bar J$ is an increasing continuous sequence of ideals on κ.

(b) $J_0 = I^*, J_{\zeta+1} =: \{A : A \subseteq \kappa, \mathit{and} : A \in J_\zeta$ or we can find $h : A \to \theta$ such that $\lambda_i > h(i)$ and $\varepsilon < \theta \Rightarrow \{i : h(i) < \varepsilon\} \in J_\zeta\}$

(c) For $\zeta < \zeta(*)$ and $A \in J_{\zeta+1} \setminus J_\zeta$, the pair $(\prod \bar\lambda, J_\zeta + (\kappa \setminus A))$ (equivalently $(\prod \overline{\lambda} \restriction A, J_\zeta \restriction A))$ satisfies condition 1.5$(*)$ (case (β) hence its consequences, (in particular it satisfies the weak pcf-th for θ).

(d) If $\kappa \notin \cup_{\zeta<\zeta^*})J_\zeta$ then $(\prod \overline{\lambda}, \cup_{\zeta<\zeta^*})J_\zeta)$ has true cofinality θ.

Proof. Straight. (We define J_ζ for $\zeta \leq \theta^+$ by clause (b) for $\zeta = 0$, ζ successor and as $\bigcup_{\varepsilon<\zeta} J_\varepsilon$ for ζ limit. Clause (c) holds by claim 4.4 below. It should be clear that $J_{\theta^++1} = J_{\theta^+}$, and let $\zeta(*) = \min\{\zeta : J_{\zeta+1} = \bigcup_{\varepsilon<\zeta} J_\varepsilon\}$ so we are left with checking clause (d). If $A \in J^+_{\zeta(*)}, h \in \prod_{i\in A} \lambda_i$, choose by induction on $\zeta < \theta, \varepsilon(\zeta) < \theta$ increasing with ζ such that $\{i < \kappa : h(i) \in (\varepsilon(\zeta), \varepsilon(\zeta+1)) \in J^+_{\zeta(*)}$). If we succeed we contradict $\theta \geq \mathrm{wsat}(I^*)$ as θ is regular. So for some $\zeta < \theta, \varepsilon(\zeta)$ is well defined but not $\varepsilon(\zeta + 1)$. As $J_{\zeta(*)} = J_{\zeta(*)+1}$, clearly $\{i < \kappa : h(i) \leq \varepsilon(\zeta)\} = \kappa \mod J_{\zeta(*)}$. So $g_\varepsilon(i) = \begin{cases} \varepsilon & \text{if } \varepsilon < \lambda_i \\ 0 & \text{if } \varepsilon \geq \lambda_i \end{cases}$ exemplifies $\mathrm{tcf}(\prod \bar\lambda / J_{\zeta(*)}) = \theta$. $\square_{4.2}$

Now:

5.1. Conclusion

Under the assumptions of 4.2, I^* satisfies the pseudo pcf-th (see Definition 6(4)).

Claim 4.4: Under the assumption of 4.2, if J is an ideal on κ extending I^* the following conditions are equivalent

(a) For some $h \in \prod \bar\lambda$, for every $\varepsilon < \theta$ we have $\{i \in A : h(i) < \varepsilon\} \in J$

(b) $(\prod \bar\lambda, <_{J+(\kappa\setminus A)})$ is θ^+-directed.

Proof. (a) $\Rightarrow$ (b)

Let $f_\zeta \in \prod \bar\lambda$ for $\zeta < \theta$, we define $f^* \in \prod \bar\lambda$ by

$$f^*(i) = \sup\{f_\zeta(i) + 1 : \zeta < h(i)\}.$$

Now $f^*(i) < \lambda_i$ as $h(i) < \lambda_i = \mathrm{cf}(\lambda_i)$ and $f_\zeta \restriction A <_J f^* \restriction A$ as $\{i \in A : h(i) < \zeta\} \in J$.

(b) $\Rightarrow$ (a)

Let f_ζ be the following function with domain κ:

$$f_\zeta(i) = \begin{cases} \zeta & \text{if } \zeta < \lambda_i \\ 0 & \text{if } \zeta \geq \lambda_i \end{cases}$$

As $\liminf_{I^*} \geq \theta$, clearly $\varepsilon < \zeta \Rightarrow f_\varepsilon <_{I^*} f_\zeta$ and of course $f_\zeta \in \prod \bar\lambda$. By our assumption (b) there is $h \in \prod \bar\lambda$ such that $\zeta < \theta \Rightarrow f_\zeta \restriction A < h \restriction A$ mod J. Clearly h is as required. $\square_{4.4}$

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Paul Erdős: The Master of Collaboration

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Over a span of more than 60 years, Paul Erdős took the art of collaborative research in mathematics to heights never before achieved. In this brief look at his collaborative efforts, we will explore the breadth of Paul's interests, the company he kept, and the influence of his collaboration in the mathematical community. Rather than focusing on the mathematical content of his work or the man himself, we will see what conclusions can be drawn by looking mainly at publication lists. Thus our approach will be mostly bibliographical, rather than either mathematical or biographical. The data come mainly from the bibliography in this present volume and records kept by *Mathematical Reviews (MR)* [15]. Additional useful sources of information include *The DBLP Computer Science Bibliography* (a database of articles in computer science) [3], *Zentralblatt* [19], the *Jahrbuch* [14], various necrological articles too numerous to list, and personal communications. Other articles on these topics can be found in [4, 5, 7, 8, 9, 10, 11, 16].

Paul became a legend during his long and productive life, and his fame (as well as his genius and eccentricity) spread beyond the circles of research mathematicians. We find a popular documentary film about him [2], articles in general circulation magazines [13, 18] (as well as in mathematical publications—see [1] for a wonderful example), two popular biographies [12, 17], and graffiti (e.g., his quotation that a mathematician is a device for turning coffee into theorems). But even within the academic (and corporate research) community, his style and output have created a lot of folklore.

The reader is probably familiar with the concept of *Erdős number*, defined inductively as follows. Paul has Erdős number 0. For each $n \geq 0$, a person not yet assigned an Erdős number who has written a joint mathematical paper with a person having Erdős number n has Erdős number $n + 1$. Anyone who is not assigned an Erdős number by this process is said to have Erdős number ∞. Thus a person's Erdős number is just the distance from that person to Paul Erdős in the *collaboration graph* C, in which two authors are joined by an edge if they have published joint research. For example, Albert Einstein has Erdős number 2, because he did not collaborate with Paul Erdős, but he did publish joint research with Ernst Straus, who was one of Paul's major collaborators. Purists can argue over how to count papers with more than two authors, but here we will adopt the liberal attitude that each of the $\binom{k}{2}$ pairs of authors in a k-author paper are adjacent in C. Of course one

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need not restrict the collaboration graph to the field to mathematics (more about this below).

A common variant is to give a person who has written $p > 0$ papers with Paul the Erdős number $1/p$. András Sárközy, with $\frac{1}{62}$ and András Hajnal, with $\frac{1}{56}$ have the smallest positive Erdős numbers under this definition, followed in order by Faudree, Schelp, Sós, Rousseau, Rényi, Turán, Szemerédi, Graham, Burr, Spencer, Pomerance, Simonovits, Pach, Straus, Nicolas, Nathanson, Rado, Bollobás, Milner, Gyárfás, Selfridge, Piranian, Hall, Chung, Joó, Rödl, and Reddy, who all have a value under 0.1. (This list is based on data from 2010, but most of the other statistics in this paper are based on data from 1995.)

The author maintains lists of coauthors of Paul Erdős and coauthors of these coauthors (i.e., all people with Erdős number not exceeding 2) and updates these lists periodically. They are available as part of the *Erdős Number Project* website [6]. (Difficulties in author identification, among other problems, surely make these lists and other data discussed in this article less than 100 % accurate, but we believe that the number of errors is not large.)

As was pointed out in [10], the average number of authors per research article in the mathematical sciences increased dramatically over Paul Erdős's lifetime. (One can speculate whether his existence is part of the reason for this.) Specifically, the fraction of all authored items reviewed in *MR* having two or more authors has increased, as a function of time. While over 90 % of all papers in 1940 (when *MR* began) were the work of just one mathematician, today only about half of them are solo works. In the same period, the fraction of two-author papers has risen from under 10 % to about one third. Also, in 1940 there were virtually no papers with three authors, let alone four or more; now about 10 % of all papers in the mathematical sciences have three or more authors, including about 2 % with four or more.

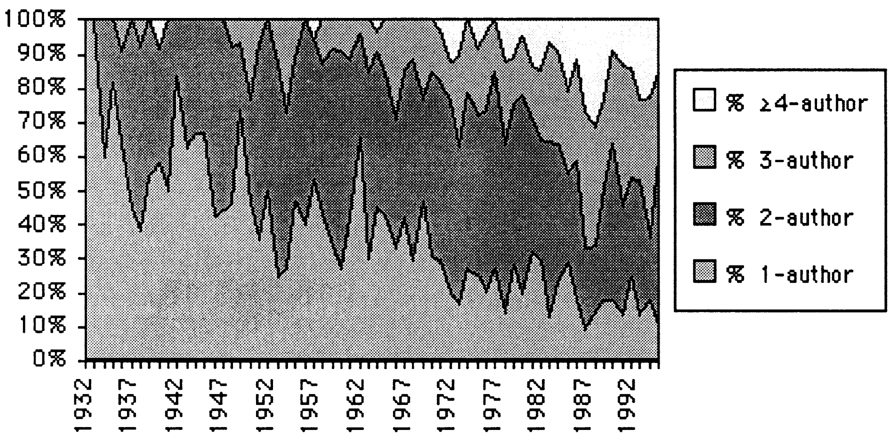

Fig. 1 The number of authors on Paul Erdős's papers over the years.

The same trend can be seen in Paul's work, but with an even greater amount of collaboration. The graph in Fig. 1 shows the fraction of 1-, 2-, 3-, and $\geq$ 4-author items in Paul's publication list, year by year through 1995. (Almost all of these are research papers. The rest are books, articles about people, or other writings.) For reference, Fig. 2 shows the absolute sizes we are talking about—the number of publications year by year. (The last two counts—for 1994 and 1995—are probably too low, because of incomplete publication data at the time the figures were constructed.) Cumulatively (Fig. 3), fewer than one third of Paul's 1,400 works completed by 1995 were solo ventures. In fact, the mean number of authors (including Erdős) is almost exactly two.

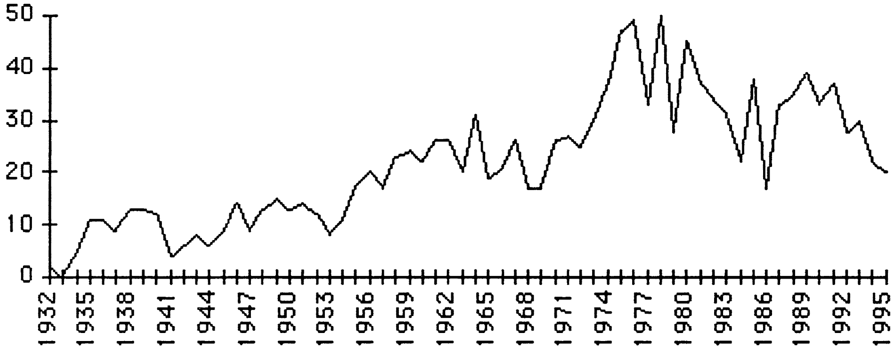

Fig. 2 The number of papers by Paul Erdős over the years.

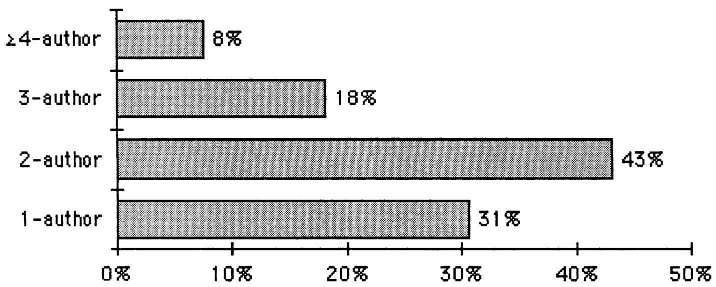

Fig. 3 The fraction of Paul Erdős's papers with different numbers of authors.

Paul's mode of operation (dating from his departure from a permanent position at Notre Dame around 1954) was unique among mathematicians. Rather than staying at a home institution (research institute or academic department), he was constantly on the move, visiting mathematicians at conferences and research centers around the world. He often spent some summer months in Budapest, where he was a member of the Hungarian

Academy of Sciences and where he could work with several of his most prolific collaborators. Some of his favorite haunts included Memphis, Tennessee, the New York City area, and other places too numerous to list. He was a permanent fixture at the annual Southeastern Combinatorics Conference (in Boca Raton, Florida, or Baton Rouge, Louisiana) and other regular meetings in his various fields. For example, in the few months around the time this article was originally being written, Paul reported having been (or planning to go) at least to Atlanta, Memphis, three cities in Texas, New Jersey, New Haven, Baton Rouge, Colorado, France, Germany, Kalamazoo, and Pennsylvania, in that order.

As he met and worked with ever-increasing numbers of people on his travels, it is not surprising that Paul added new coauthors every year since 1936. (Only two—George Szekeres and Pál Turán—published with him before that, in 1934.) Figure 4 shows how the cumulative number of coauthors increased, while Fig. 5 shows the discrete time derivative of this function. Paul usually left the actual writing up of the papers to his collaborators—partly, he said, because he did not type.

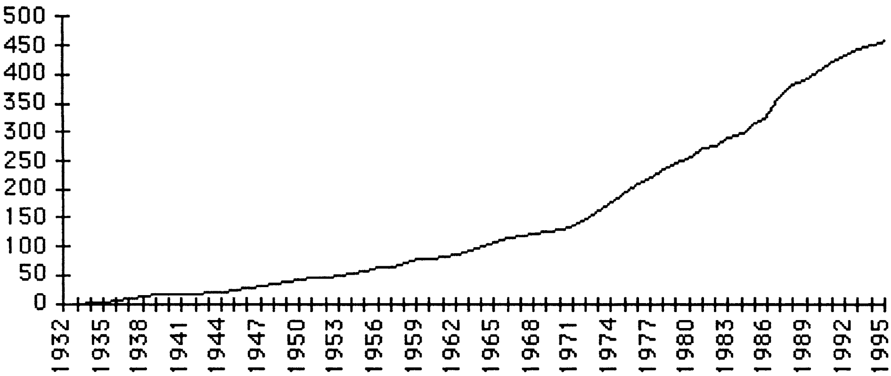

Fig. 4 The cumulative number of Paul Erdős's coauthors as a function of time.

As the present collection shows, Paul Erdős's papers span many branches of mathematics and exploit relationships among them. (One fine example of the latter is his application of probability to combinatorics.) *Mathematical Reviews* currently has about 60 broad subject classifications, ranging from "Mathematical Logic and Foundations" through "Information and Communication, Circuits" (plus a section on history and biography), one of which it assigns to every item it records as its primary subject area. This list has varied slightly over time, with some categories being added or discarded as *MR* tries to keep up with current trends. Paul's works, although often spanning two or more subject areas and therefore difficult to pinpoint into one category, have been given primary classification in about 40 % of these subject areas or their equivalent predecessors. They include not only the two main areas of

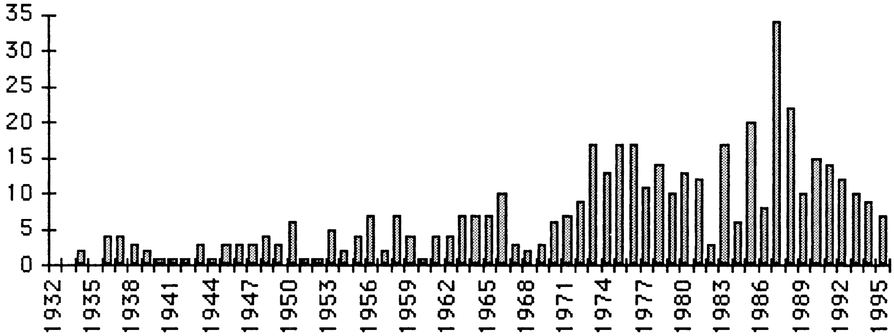

Fig. 5 The number of new coauthors added each year.

number theory and combinatorics, and substantial work in approximation theory, geometry, set theory, and probability theory, but also papers in mathematical logic and foundations, lattices and ordered algebraic structures, linear algebra, group theory, topological groups, polynomials, measure and integration, functions of a complex variable, finite differences and functional equations, sequences and series, Fourier analysis, functional analysis, general and algebraic topology, statistics, numerical analysis, computer science, and information theory.

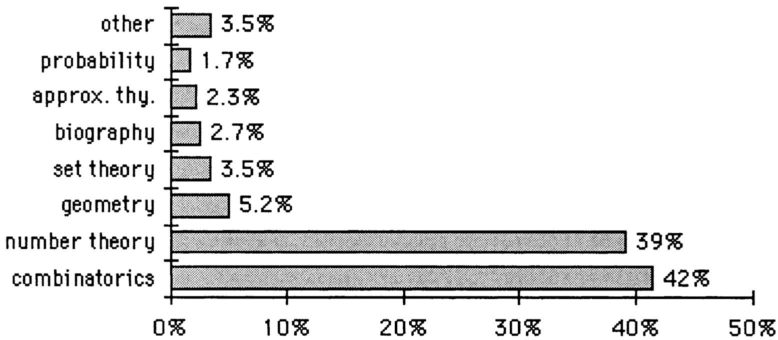

Fig. 6 Paul Erdős's papers since 1979 by broad category.

The chart in Fig. 6 shows the fractions of Paul's publications reviewed in *MR* from 1980 to 1995 in the various categories. Such a tabulation was easy to do, because the old-style *MR* review number included the category code as the first two digits following the colon. Figure 7 is a less accurate pie chart covering all years. Taken together, these graphics suggest a slight trend toward increased work in combinatorics (including graph theory) in the later years, with a comparable decline in the output in number theory. Indeed, nearly all Paul's early papers were in number theory (61 of the first 64, by one count, covering the period 1932–1939).

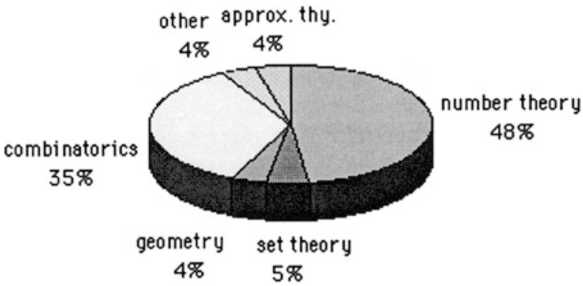

Fig. 7 Paul Erdős's papers by broad category (approximate).

Since Erdős's coauthors work in such varied fields, one would expect the set of people with Erdős number 2, 3, or a little higher to range over essentially all of mathematics. Indeed, this is the case. All Fields Medalists (through 2010) are in the Erdős component of the collaboration graph, with Erdős number at most 5. (The Erdős component of C is just the connected component that contains Erdős.) This group includes people working in theoretical physics; for instance, there is the path Edward Witten–Sidney Coleman–Daniel J. Kleitman–Paul Erdős. Thus one can conjecture that many (if not most) physicists are also in the Erdős component, as are, therefore, many (or most) scientists in general. The large number of applications of graph theory and statistics to the social sciences might also lead one to suspect that many researchers in other academic areas are included as well.

It is interesting to explore the publication lists (or at least the coauthor lists) of Erdős's coauthors, to see how much collaboration went on after Paul left town. Let E_1 be the subgraph of C induced by people with Erdős number 1. According to data collected through 1995, E_1 contains 458 vertices and 1,218 edges; thus an average Erdős coauthor collaborated with over 5 other Erdős coauthors. (The median, as opposed to the mean, of this statistic is only 3, however. Its standard deviation is about 6, and it takes values over 30 in four cases—Ron Graham, Frank Harary, Vojtěch Rödl, and Joel Spencer.) There are only 40 isolated vertices in E_1 (less than 9 %), and three components with two vertices each. The remaining 412 vertices in E_1 induce a connected subgraph. Paul's style seemed to rub off.

Looking at it more broadly, we find that people with Erdős number 1 have a mean of 20 other collaborators (median 15, standard deviation 22), and only six of them collaborated with no one except Erdős. As of 1995, five of them had over 100 coauthors (Frank Harary, Saharon Shelah, Ron Graham, Noga Alon, and Dan Kleitman).

Another 4,546 people as of 1995 felt Paul's influence second-hand—by doing joint research with one of the honored 458. (These numbers increased to 9,267 and 511, respectively, as of 2010.) Three quarters of the people with Erdős number 2 have only one coauthor with Erdős number 1 (i.e., each such person has a unique path of length 2 to Erdős in C). However, their mean number of Erdős number 1 coauthors is 1.5 (standard deviation 1.2), and the count ranges as high as 13 (for Dwight Duffus and Linda Lesniak).

The bibliography in the original edition of this volume listed about 1,400 papers, but it was incomplete, especially with regard to works that appeared near the end of Paul's life (and posthumously—a paper with Florian Luca and Carl Pomerance was published in 2008). An addendum is available on the *Erdős Number Project* website; the total as of 2010 stood at 1,525. Most of the papers published since 1939 appear in the *Mathematical Reviews* database. Reviews of Paul Erdős's papers appeared in every volume that *MR* published during his lifetime, including a review of a joint paper with Tibor Gallai on page 1 of volume 1, written by George Pólya. It is interesting to note that the second most prolific writer in the *MR* database (as of 1995) is Leonard Carlitz, with about 735 items. Carlitz had Erdős number 2 (via seven different coauthors) and wrote the *MR* review of several of Paul's papers. In all, nearly 500 different people reviewed Erdős's papers for *MR*. Paul wrote for *MR* as well and had over 700 reviews to his credit.

Readers with additions or corrections to any of the information in this article, the accompanying bibliography, or the *Erdős Number Project* website are urged to communicate with the author (`grossman@oakland.edu`).

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This bibliography was prepared by Jerrold Grossman (Oakland University, Rochester, Michigan), updating previous bibliographies, most recently the list prepared by Dezső Miklós in *Combinatorics, Paul Erdős is Eighty, Volume 1, Bolyai Society Mathematical Studies*, pp. 471–527, János Bolyai Mathematical Society, Budapest, 1993. Thanks are due not only to Dezső Miklós (Mathematical Institute, Hungarian Academy of Sciences) and the János Bolyai Mathematical Society, but also to László Babai (University of Chicago), Ronald Graham (AT&T Research), Patrick Ion (*Mathematical Reviews*), and Jaroslav Nešetřil (Charles University, Prague) for additional advice and data, *Zentralblatt für Mathematik und Ihre Grenzgebiete* for providing the Zbl numbers and making other additions and corrections, Springer-Verlag for the preparation of this list in its final form, and Paul Erdős for several helpful conversations.

Generally, the bibliographic style of *Mathematical Reviews* has been followed, at least approximately. Coauthor names, if any, are given in parentheses near the end of each entry. The *Mathematical Reviews* review number (**MR**) is included with each item for which it exists; otherwise, the *Current Mathematical Publications* control number (**CMP**) is included when it exists. Similarly, the **Zbl** numbers are provided where possible. This list incorporates corrections of some errors and omissions that appeared in the 1993 list (for example, the omission of Turán as a coauthor of the last paper listed for 1935). In some cases items have been shifted to a different year to reflect more accurately the actual publication date. Items have necessarily been renumbered in order to maintain a year-by-year list, alphabetical by title. *Please send further additions or corrections to* grossman@oakland.edu.

This list as it existed in early 1996 was included in *The Mathematics of Paul Erdős*, R. L. Graham and J. Nešetřil, eds., Volume II, pp. 477–573 (Springer, 1997). Updates beyond that are posted on the World Wide Web site of the Erdős Number Project: http://www.oakland.edu/enp. That site also contains more complete versions of the names of the coauthors.

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Note: The total number of items in this list is 1,525.

Postscript

It is fitting that Paul Erdős himself should have the last word. As is amply illustrated in this collection, Paul's profound influence on so many mathematicians and fields of mathematics through his prolific research and incessant traveling is destined to leave a legacy that may never be equalled. Here then are some very special lines written by Paul for the Postscript of these volumes:

"Determine or estimate as well as you can the number of solutions in positive integers of

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_k} = 1, \quad x_1 < x_2 < \cdots x_k.$$

Can the squares be decomposed as the finite union of sum-free sets (or Sidon) sequences? Apologies if this is trivial or trivially false. Let $x_1, \ldots, x_n$, be n points in the plane with at most t on a line (t fixed, n large). $f(n;t)$ is defined to be size of the largest subset with no three on a line. We can only prove

$$c_2\sqrt{\frac{n}{t}} < f(n;t) < \frac{c_1 n}{t}.$$

The lower bound is easy by the greedy algorithm. Füredi showed

$$f(n;t) > g(n)\sqrt{\frac{n}{t}}$$

where $g(n) \to \infty$ slowly. What is the truth here?

(Erdős–Turán.) Let $p_{n+1} - p_n = d_n$. Prove that for infinitely many n,

$$d_n > d_{n+1} > d_{n+2} \text{ or } d_n < d_{n+1} < d_{n+2}.$$

This surely holds since if not, then there is an n for which

$$d_n \geq d_{n+1}, d_{n+1} \leq d_{n+2}, d_{n+2} \geq d_{n+3}, \ldots, \text{ etc.}$$

R.L. Graham et al. (eds.), *The Mathematics of Paul Erdős II*,
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I offer 100 dollar for a proof and 25000 dollar for a counterexample. This is of course a joke since the conjecture surely holds.

If I live I hope to have some more conjectures and even proofs. Will there be a celebration for my 90th birthday or only a meeting for my memory. May my theorems and problems live forever

My mother was very glad to read the many eulogies written for my 50-th birthday. I am only sorry that my mother and father are not reading this volume (if you believe in survival after death then you can believe that perhaps they are reading it). Let me add another problem: In 1934 Turan and I proved (Amer Math Monthly 1934): Let $a_1 < a_2 < \dots < a_n$ be any set of n integers. Then the number of distinct prime factors of $\prod_{1 \le i < j \le n} (a_i + a_j)$ is greater than $c \log n$. It does not have to be greater than $\frac{cn}{\log n}$ (trivially). Try to improve both the upper and lower bounds. I offer 500 dollars for this.

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The preceding lines are some of the last lines written by Paul Erdős. They not only represent a fitting conclusion to these volumes but they capture Paul's style and his vision of life as a scholar, a style which influenced us all from around the world, a world which will not be the same without him.