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Complex Systems and Society

Modeling and Simulation

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Preface

Socio-economic sciences are undergoing a great conceptual change. Little by little economists and sociologists have started to understand the need to introduce new and more sophisticated mathematical models in their fields of study. New quantitative approaches are required: models that merge mathematics and physics on the one hand, and economics and sociology on the other hand, have proven to be useful in explaining phenomena of the complex world we live in.

This monograph goes in such a direction: it aims at developing a mathematical approach toward the modeling of socio-economic systems, composed of a large number of interacting agents, both in the case of spatial homogeneity and on networks. The contents focus on living complex systems, for which the derivation of mathematical tools requires tackling several difficulties arising along the following conceptual path:

- Identification of the main complexity features that characterize the systems under consideration. The first consequent step is developing a strategy for representing the state of the system, with the aim of reducing the global complexity while keeping, however, the distinctive features.
- Derivation of mathematical structures suitable for describing, via properly specified mathematical models, the evolution in time of the variables selected for representing the state of the system.
- Modeling of specific socio-economic systems on the basis of a phenomenological interpretation of the microscopic interactions among the composing entities. This step may involve multiscale issues.
- Validation of models by investigating their ability to depict emerging behaviors observed in real systems. In some cases, models may even describe trends not yet revealed by empirical data, thereby suggesting new perspectives for interpreting the genesis of emerging behaviors.

This project calls for the development of new mathematical tools. In this monograph we specifically refer to the approach by the kinetic theory for active particles, KTAP for short, which has already been applied to various fields of life sciences and social sciences. It has been proven that the mathematical structures that formalize this theory include, as particular cases, some well known models of the kinetic theory for classical particles. The main difference here is that interactions among particles are described as stochastic games rather than by deterministic causality principles (analogous to the laws of classical mechanics).

This monograph has three parts. The first part, encompassing Chaps. 1 and 2, is devoted to methodological insights into the complexity features of socio-economic systems and to the derivation of mathematical modeling tools. The second part, encompassing Chaps. 3 and 4, focuses on applications. The third part, consisting of Chap. 5, offers a critical analysis and looks forward at to research perspectives, including the application of the proposed methods to a large variety of social systems. In more detail, chapter contents are as follows:

Chapter 1 presents the aims of the monograph and provides an assessment of relevant complexity features of living systems in general, and social systems in particular. One of the main features is the ability of interacting entities to express behavioral strategies, which are modified according to the state and strategy of other entities. Next, the chapter offers a concise literature survey of modeling approaches, particularly those that are close to the cultural context of this monograph.

Chapter 2 deals with the derivation of mathematical structures, which can act as a background paradigm for the subsequent construction of models. It shows how complex systems can be properly represented by suitable variables, some of them directly related to the aforementioned expression of a behavioral strategy, and thereby described in a probabilistic/statistical way by means of distribution functions over such variables.

Chapter 3 applies the mathematical tools derived in Chap. 2 to the dynamics of social competition in nations. Social interactions can modify the distribution of wealth among the individuals according to both cooperative and competitive strategies. Such interactions can be partly controlled by welfare policies, so a goal of mathematical modeling is predicting the large-scale consequences of the latter. A hallmark of the proposed model is that microscopic dynamics are modeled by nonlinearly additive interactions.

Chapter 4 develops various simulations, which explore prototypical scenarios of welfare policy. Simulations aim at both a parameter sensitivity analysis and assessing the effect of different actions of a hypothetical government on the dynamics of wealth redistribution. This detailed analysis contributes to the necessary background for the developments proposed in the last chapter.

Chapter 5 is devoted to research perspectives concerning both modeling and analytical issues. More precisely, it discusses possible generalizations of the modeling approach presented in the preceding chapters to a variety of different social contexts; for instance, opinion formation related to political competition for leadership, which can be fostered by both communication among individuals and external actions, including some aspects of interactions on networks. Finally, it critically analyzes the contribution of mathematics to social sciences, having in mind the ambitious goal of constructing a mathematical theory of social systems. We hope this forms a useful prelude to the future development of the monograph into an exhaustive book.

All chapters are concluded by a critical analysis, proposed with a twofold goal: focusing on developments needed for improving the efficacy of the proposed methods, as well as envisaging further applications, possibly in fields different from those treated in this monograph.

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N. Bellomo dedicates this monograph to Luigi Salvadori, a master and a friend.

Contents

Chapter 1
The Role of Individual Behaviors in Socio-Economic Sciences

Abstract This chapter provides an assessment of the relevant complexity features of social systems, focusing on the ability of individuals to express strategic behaviors that determine their interactions with other individuals. It also offers a concise survey of various modeling methods, which pertain to the cultural context of this monograph. Finally, this chapter critically assesses the effectiveness of the different mathematical approaches in capturing the complexity features of social systems.

1.1 Introduction

The dynamics of social and economic systems are primarily influenced by individual behaviors, by which living entities express, rationally or irrationally, a certain strategy for achieving their own well-being. These systems are often complex: the dynamics of a few entities do not directly lead to those of the entire system, because the latter manifest themselves only upon scaling up the effects of individual interactions at a collective level. At large scales, collective behaviors appear, which are apparently coordinated but are actually *self-organized*. That is, collective behaviors emerge spontaneously without the action of any external organizing principle. Individuals are typically not even aware of the group behavior to which they are contributing with their autonomous strategies.

Self-organized trends can be extremely difficult to control. Individual behaviors can be partly modified by external actions, but the resulting collective effects can be hard to predict heuristically. Sudden deviations from the usual standards may occur, leading to even highly unpredictable events with dramatic collective consequences, as is well-demonstrated nowadays by recent events in our societies. An occurrence of this type, which is of paramount interest in social sciences in general, and in economics in particular, is a so-called *black swan*, which is an extreme and largely unpredictable event at a collective level, originating from apparently rational and controlled individual behaviors [149, 150].

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It is very difficult to describe social dynamics using mathematics. Nevertheless, the scientific community agrees that this is an important goal, of modern applied mathematics, albeit not yet achieved. In this respect, the interplay between mathematics and social sciences is essential to the understanding of these kinds of phenomena.

Mathematics has developed rather sophisticated qualitative and quantitative tools for studying inert matter, where causality principles can be generally applied. On the other hand, the modeling of living matter cannot rely on direct cause–effect links, because the *active* ability of individuals to develop behavioral strategies, and to adapt them to various contexts, leads to observable effects resulting from causes that are often not evident. The lack of invariance principles has been well highlighted in biology [93, 119] (for example, in the theory of evolution [120]), and in many other fields. Similar arguments can be proposed as far as the interplay between mathematics and socio-economic sciences is concerned.

The importance of developing quantitative methods for studying social systems, and possibly controlling their complexity, is well documented in various essays [54, 102], and also in consideration of the impact that mathematical and physical sciences can have on society.

This monograph pursues this goal by proposing advanced mathematical tools, along with related models, which aim at describing real behaviors in groups or societies of interacting individuals. The lack of invariance principles, the heterogeneous behavior of individuals, and the ability of the latter to actively develop specific strategies are issues that the proposed modeling approach is able to cope with.

Numerous recent events have contributed to raise awareness of the need for better mathematical models able to take into account the hallmarks of socio-economic phenomena. In particular, events like the global financial crisis (beginning in 2007) and the Arab Spring revolts in the Middle East and Northern Africa (beginning in 2010) have demonstrated how collective behaviors can result in unexpected outcomes. Econometric and statistical methods traditionally used in economics and social sciences are simply not sufficient to capture the complex interactions underlying these kinds of phenomena. Therefore, better understanding and modeling of such phenomena is definitely a crucial challenge for applied mathematical sciences [9, 147].

After this introduction, Sect. 1.2 presents an overview of the complexity features of living systems in general, and of social systems in particular, which should somehow be retained in a modeling approach that aims to be realistic. Section 1.3 provides a brief survey of some mathematical tools that have been developed for dealing with the class of systems under consideration. It also introduces, though not yet at a formal level, the mathematical ideas of the approach proposed in this monograph. Finally, Sect. 1.4 critically analyzes the potential of mathematical approaches in light of the complexity features discussed in the preceding sections.

In order to unify the terminology, throughout this chapter we will refer to individuals, or, more generally, to socio-economic actors, as *active particles*, or simply particles. Technical aspects and definitions of these terms will be given in more detail in Chap. 2.

1.2 Complexity Aspects of Social and Economic Systems

As already mentioned, the leading idea of the modeling approach to social systems proposed in this monograph is that they have to be regarded as complex systems. For this reason, it is important as a starting point to extract the main phenomenological features that can be ascribed to the complexity of the kind of systems under consideration.

As reported in [37] and in the bibliography cited therein, the scientific community agrees that the ability of active particles to express a strategy through nonlinearly additive interactions is one of the main complexity issues. Although formal definitions will be given in following chapters, it is worth anticipating that the expression *nonlinearly additive* refers to the fact that, while such interactions finally add with one another to produce their global effect (hence the additivity), pairwise interaction rules may nevertheless be correlated to the local collective state of the system, thereby also involving particles other than the interacting ones (hence the nonlinearity).

The behavioral strategy inspiring the interactions can be rational or irrational and focused on a well-defined goal. Furthermore, when the strategy is rational, it may not be the best possible one; in particular it can be influenced by contingencies possibly leading to a behavior in contrast with the primary goal. In economic theory, it is indeed well known that practical conditions may constrain the best optimal solution, leading to a second-best option different from the theoretically optimal one, which is often not realizable. For further details, we refer interested readers to the literature about “the second best theory” [115]. In addition, human beings are not perfectly rational. They exhibit “bounded rational” behaviors, which can give rise to emergent collective irrational behaviors [15, 104, 105]. In some extreme cases, mostly ruled by irrational behaviors, interactions may generate outcomes rather distant from any best solution, as in the case of panic. Another example is the so-called “information asymmetry” extensively studied in economics and contract theory. In socio-economic transactions, different parties often do not possess the same information, as supposed in traditional economic models. This may imply imperfect solutions and unexpected outcomes even in simple economic models.

Bearing in mind that the concepts of individual strategy and interactions are at the core of the complexity of the systems we are concerned with, we proceed now to the identification of a few phenomenological and methodological aspects which, according to our perspective, should act as guidelines for the derivation of mathematical models. The proposed selection does not claim to be exhaustive. On the contrary, it is limited to ten specific items so as to avoid an excess of concepts, considering also that the reductionism of the mathematical approach imposes the retention of only elements that are absolutely essential to understanding the system.

1. *Strategy*. Active particles develop and update their behavioral strategies on the basis of the transient state of the system in which they operate. This includes, in particular, the assessment of the strategy expressed by other active particles.

Generally, such a strategy is inspired by rational principles. However, irrational behaviors cannot be excluded, also because of the possible unpredictable emergence of interaction-driven behaviors.

2. *Heterogeneity*. Strategic ability is heterogeneously distributed among the active particles. This distribution, however, may be modified by the reciprocal interactions among the particles, as well as by those between active particles and the outer environment. For instance, the *learning ability* of living agents plays an important role in such modifications.
3. *Equilibria*. Living systems typically operate not in equilibrium. Social and economic systems, in particular, are driven to states of "ever-changing-equilibria" by the unending search for personal benefits.
4. *Nonlinear and nonlocal interactions*. Interactions involving active particles are generally nonlinear because, as already mentioned, one-to-one interaction rules can be modified by the milieu in which particles operate. Furthermore, interactions do not necessarily require the interacting particles to be physically in contact, because living systems can develop inner communication procedures. In some cases, interactions are *metric*; that is, they involve all particles within a properly defined *interaction neighborhood*. It is worth stressing that the notion of distance implied by such a neighborhood need not be the physical spatial one. Depending on the characterization of the state of the particles with respect to the main dynamics of the system, it can also be a social-cognitive distance [1, 5, 91, 159]. In other cases interactions are defined *topologically*; that is, they are based on the number of neighbors that each particle chooses to interact with simultaneously, no matter how far away they are. Such a selection can be determined, for instance, by the ability of active particles to process information and communicate with other particles [64, 107].
5. *Stochastic games*. Interacting particles *play a game* at each interaction according to the behavioral strategy they express. As a payoff of such a game, they update their strategy, which can influence future interactions. Interactions are regarded as *stochastic* games, in the sense that only the probabilities of payoffs are known. This is convenient in view of the intrinsic uncertainty of their output, especially in the presence of irrational behaviors.
6. *Learning and evolution*. Active particles are able to learn from past experience. Consequently, their behavioral strategies evolve in time, which in turn produces qualitative changes in the dynamics of interactions. Out of such an evolution, a Darwinian-like selection of the particles most suited to the socio-economic context can occur. Learning, adaptation, and selection are often related to interaction with the outer environment, which evolves in time due to both its natural trend and the interplay with the hosted systems.
7. *Space and networks*. Active particles may either move freely in space or occupy fixed positions, thereby communicating over networks. In the latter case, the frequency of the interactions depends on the network topology and on a distance in state among the particles rather than on the spatial metric distance. We will come back to this concept in Chap. 2, in connection with the derivation of mathematical structures.

8. *Multiscale issues*. The concept of active particle is related to the observation and representation scale. For instance, in socio-economic systems an active particle may be identified with either a single individual, or a group of interest, or a social class depending on the kind of interactions being studied. In all cases, active particles are the minimal (viz., atomic) entities of the system; hence they define the *microscopic* scale. Often the latter is not observable individually, whereas collective behaviors are observable at a larger *macroscopic* scale.
9. *Emerging behaviors*. Interactions involving active particles produce collective behaviors, which are usually so different from those of the single active particles that they may appear as autonomous expressions of the behavior of the system as a whole. As a matter of fact, they are the visible collective complex effect of much simpler causes taking place at smaller scales. In order to really contribute to the understanding of these phenomena, mathematical models must address these causes in a *multiscale* perspective.
10. *Need for complexity reduction*. Complex systems often contain a large number of components, so that modeling and computational methods have to face an excessive number of equations. Therefore, a reduction of complexity by means of proper mathematical approaches is needed to deal with them at a practical level.

The considerations reported above are, of course, still quite general. As such, they have to be made more precise when referred to specific socio-economic systems along with their own interaction dynamics. A deeper investigation will be initiated in the next chapter, in connection with the development of mathematical tools.

1.3 The Contribution of Mathematics to Social Sciences

In recent years, a radical philosophical change has been taking place in economic disciplines leading to a reconciliation among economics, sociology, and psychology, thanks to new cognitive approaches toward economics in general [20, 21, 82, 110]. New branches of economics are emerging, and they are much more linked to sociology and psychology than economics was in the past. Starting from the concept of bounded rationality [145], critiques of the traditional assumption of rational collective behavior [87, 103, 151] led to the idea of economics as a discipline highly affected by either rational or irrational individual trends, reactions, and interactions. This innovative point of view promoted the image of an economy as an evolving complex system [20], where heterogeneous individuals [109] interact to produce emerging unpredictable outcomes. In this context, the development of new mathematical descriptions, able to capture the complex evolving features of socio-economic systems, is a challenging, though difficult, perspective, which calls for an evolved interplay between mathematics and social sciences. Such a goal is increasingly acknowledged by the community of social scientists. As stated at the beginning of this chapter, recent events such as the financial crisis and political transitions in the developing world have contributed to establishing a general consensus about the need for new mathematical models; see [108], among others.

An important milestone for this program is the assessment of theoretical paradigms, and related models, which can act as conceptual background for a unified mathematical approach to a variety of social systems characterized by common complex features. More generally, it is worth remarking that a unified list of common features of complex systems would enable mathematical tools currently used in different fields of life sciences to be synergistically evolved and transferred, with proper adaptations, from one field to the other.

Before deriving mathematical tools of the Kinetic Theory for Active Particles (KTAP methods), which constitute the central subject of this monograph, let us briefly survey some of the already existing attempts to combine social sciences and mathematical modeling, namely: agent-based models, game theory, population dynamics, and social networks. At the end of this concise overview, we will also anticipate some aspects of the approach in order to outline the mathematical context of this monograph with respect to other parallel approaches.

Agent-based models are computational models in which interactions among individuals (*agents*) are simulated with the final aim of finding equilibria or emerging phenomena. This approach combines elements of game theory, multi-agent systems, and Monte Carlo methods for introducing randomness. Models are structured into sets of behavioral rules (which appeal to concepts such as bounded rationality, personal benefit, and social status, among other specific sociological assumptions) that describe the simultaneous behavior of several agents in order to recreate and predict the global trends of the system. These models have been recently used to describe a great variety of network-structured phenomena, such as the Internet, terrorism, traffic jams, financial crises, consumer behaviors, the spread of epidemics, and social segregation. Some key references can be found in [21, 82]. Computational methods are treated in [75].

Game theory is a branch of applied mathematics that is largely used nowadays in social sciences and economics. It describes the behavior of individuals (*players*), who design specific strategies for determining the best response to other players' choices. The final aim is usually to find equilibria in the set of all possible strategies. Different kinds of equilibria have been defined, among which the most popular one is perhaps the *Nash equilibrium*. It is defined as a set of individual strategies such that it is *not* advantageous for any player to make a unilateral change in strategy. It is worth stressing that game theory is heavily grounded on the assumption of rational players. One of the first fields of application of this theory was the analysis of economic competition [122]. Afterwards, it was applied to the modeling of biological, social, and even telecommunication interactions. The most notable references can be found in [88, 132]. More recent contributions have involved the development of evolutionary games [123, 126]; see also [138, 139] and the bibliography cited therein. These new theories study how player strategies evolve in time due to selective processes, which can lead to clustering of the players into different groups depending on their fitness for the outer environment. The time evolution of game dynamics can be modeled in terms of differential games [57–59], where players apply a control action over basic dynamics modeled by ordinary differential equations in order to increase their payoffs.

Population dynamics studies how the number of individuals in one or several interacting populations changes under the action of biological and environmental processes. The point of view is super-macroscopic: the elementary entities are the populations themselves as a whole, whose rise and decline over time are investigated. Models are classically stated in terms of systems of ordinary differential equations, whose unknowns are the sizes of the various populations. The main aspects considered in the modeling of population evolution are: birth rate, growth rate, and mortality rate, possibly triggered by cooperative or competitive interactions among the populations. Population dynamics has been a dominant branch of mathematical applications to biology for more than 200 years. In particular, in the early nineteenth century it was widely applied to demographic investigations. For a complete survey, we refer interested readers to [156].

Population dynamics with internal structure introduces an additional variable (besides the number of individuals) describing an inner characteristic of the population, which is supposed to play a role in the emergence of collective behaviors (for instance, the ages of the individuals, their fitness for the outer environment, their social status); see e.g., [80, 133, 161]. These models are formalized using systems of partial differential equations; hence classical models without internal structure can be generally recovered via a formal integration over the internal variable. The mathematical formalization of this approach is arguably due to Webb [161].

Social networks are an extremely helpful tool for studying the role of connections in determining and constraining social behaviors and their evolution. Starting from Milgram's experiment [121] and from the celebrated papers developing the concept of *small-world phenomena* [160] and *scale-free networks* [29], it is possible to find, in the recent literature, many socio-economic phenomena addressed by this approach [67, 81, 89, 90]. Indeed, networks seem to be an ideal tool for modeling contagion, percolation, and diffusion; see [27, 32, 142], among others.

Methods of statistical mechanics, together with kinetic and game theory, have been placed in the framework of a unified approach by Helbing [94], who has the great merit of having understood that individual microscopic interactions need to be modeled by a game-theoretical approach. Further recent developments have been proposed in [97, 101, 164], and also in [102], which looks ahead at the fascinating perspective of modeling the global dynamics of modern societies. These ideas constitute the conceptual basis of the approach promoted in this monograph and briefly introduced in the following paragraph.

Generalized kinetic theory and game theoretical tools are at the basis of a recently developed modeling approach to socio-economic complex systems [7, 8], which rests on KTAP methods [34]. The cited papers offer various hints for revisiting the theory by introducing the concepts of stochastic games and network structure from a multiscale viewpoint. According to these methods, social systems are described in terms of interacting particles (such as consumers, firms, and institutions) described by a specific socio-economic state (for instance, their individual wealth).The latter is mathematically accounted for by a dedicated variable, generically named *activity*, which characterizes the microscopic state of the system. The global state is instead described by a *distribution function* over the microscopic state, whose statistical moments recover macroscopic observable quantities (such as, for example, the total number of individuals, the average wealth, and similar quantities). The evolution of the system is mainly determined by interactions among the particles, which modify the distribution of the activity. Interactions are regarded as stochastic games, in the sense that their outputs, which depend on the states of the interacting pairs, are specified by probabilities. These methods borrow and extend mathematical tools from the various approaches described above. As in the agent-based framework, models derived from this approach are characterized by multi-agent interactions defined by appealing to phenomenological sociological features. On the other hand, KTAP mathematical tools are more refined, in that they are generally formalized in terms of systems of integro-differential equations amenable to mathematical and computational analysis. Furthermore, as in game-theoretical models, the evolution of the system depends upon a game strategy, which determines the particle response to interactive encounters. However, it is worth pointing out that, unlike the game-theoretical approach, the KTAP approach does not primarily aim at assessing any best strategy among a certain number of predefined ones. Instead, it studies the evolution in time of the strategies expressed by active particles, taking into account their heterogeneous distribution and the possible influence of behaviors that are not strictly rational, with the final aim of depicting the large-scale distribution of behavioral trends. Compared to population dynamics, it can be noticed that in the KTAP approach the distribution of the internal variable, i.e., the activity, evolves according to mainly stochastic principles implied in the aforesaid game-theoretically-inspired interaction dynamics. This seems indeed to be an essential ingredient for modeling social systems in the framework of behavioral economics. Finally, interactions can give rise to a network structure among the particles. In this respect, it is worth mentioning that KTAP methods can be extended to account for a decomposition of the system into parts, called *functional subsystems*, which can be regarded as micro-systems structuring the overall system as a network of subsystems [7, 8]. As we will discuss in the next chapter, the approach can include specific features of social networks via a proper modeling of the interactions among different subsystems.

The mathematical approaches summarized above provide useful complementary ways of modeling living systems in different fields of life sciences and social sciences. The borders among these disciplines display, in practical applications, various flexibilities: methods and tools of each of them can be applied together for a better development of mathematical models and more general mathematical approaches.

1.4 Critical Analysis

Looking ahead at the contents of the next chapter, we outline some guidelines that can be followed to design the mathematical tools suitable for the challenging goal of describing socio-economic systems by mathematical equations from a complex-system perspective.

- Identification of the main complexity features that characterize the system under consideration.
- Development of a procedure for representing the state of the system by reducing the overall complexity, while keeping its essential features.
- Derivation of mathematical structures suitable for modeling the evolution in time of the variables charged to describe the state of the system.
- Modeling of specific socio-economic phenomena through the characterization of some objects of the aforesaid mathematical structures (such as, for example, parameters and functions) on the basis of a phenomenological (and in some cases multiscale) interpretation of the microscopic interaction dynamics.
- Validation of models; in particular, assessment of their ability to depict emerging behaviors observed in the real system. In some cases, models may even describe behaviors not yet revealed by empirical data, thereby suggesting phenomenological conjectures to be tested and possibly confirmed or rejected.

The first issue has already been treated in this chapter, fostering the idea that new mathematical tools stemming from the approach under consideration can be developed consistently with the discussed complexity features of socio-economic systems. Additional observations are inspired by the very recent monograph [24]:

- The title and the introduction explain *why society is a complex matter*. This statement is based on some considerations about what complexity is and how (or if) our society is predictable. Subsequently the author explains his point of view concerning the substantial differences between modeling either classical or complex systems and anticipates the important concept that different types of societies, from cells to ants and humans, exhibit analogous complexity features.
- The various contributions that follow focus on specific applications to an extremely broad range of social systems. Among others: vehicular traffic, human crowds, economic and financial systems, mobility and spread of epidemics. Next, it is shown how each specific system has a relevant impact on society as a whole. For example, to stay with the main topic of our present monograph, the reaction of societies to financial crises, and to what extent cooperation can contribute to minimize consequent damage.
- Particularly important is the final contribution by Helbing [96], who suggests that a prospective overview of future societies can be obtained from the joint effort of a large organization of scientists devoted to tackling the various interconnected systems under consideration by formalized approaches based on "hard" sciences: mathematics, physics, and computer science in a virtuous interplay.

The main idea conveyed by Ball [24] is that our societies need to be regarded as complex systems, and thus traditional tools used in the study of classical systems of inert matter cannot be directly applied. Actually, it is not simply a matter of technically improving, or updating, such tools. A new science needs to be developed in order to capture the complexity features of living matter [162]. In this respect, the present monograph is perfectly aligned with this cultural movement. In particular, the methodological approach promoted here is potentially able to account for the complexity features presented in the preceding sections, and so it can be viewed as a new way of approaching social sciences by means of mathematical modeling. Clearly such an ability cannot simply be claimed but needs to be properly supported by qualitative and quantitative analytical investigations. We stress that they are not primarily pursued in this monograph, because here the main goal is to introduce the new ideas discussed above at a prospective (though already formally sound) level, consistent with the spirit of a Springer-Briefs monograph. Still, it is worth mentioning that connections of KTAP methods with the classical kinetic theory have been analyzed in [17], where it has been shown that structures of the approach under consideration include, as particular cases, known models of the kinetic theory for classical particles. The main difference is that interactions are modeled by stochastic games rather than by laws of classical mechanics. This aspect has stimulated a deep investigation of the links with Markov processes [112]. Further speculations are offered in [106].

Chapter 2
Mathematical Tools for Modeling Social Complex Systems

Abstract This chapter deals with the derivation of mathematical structures suitable for constructing models of phenomena of interest in social sciences. The reference framework is the approach of the Kinetic Theory for Active Particles (KTAP), which uses distribution functions over the microscopic states of the individuals composing the system under consideration. Modeling includes: the strategic behavior of active particles from a stochastic game perspective; a Darwinian-like evolution of the particles, which learn from past experience and evolve their strategy in time; and hints about small-network dynamics, in particular particle interactions within and among the nodes of the network. A critical analysis is finally proposed in order to assess the consistency of the mathematical tools with the main features of complexity.

2.1 Introduction

This chapter presents the concepts underlying the KTAP approach, which has been selected as the mathematical tool for deriving specific models of socio-economic phenomena. The contents of the chapter focus, in particular, on the assumptions needed for interpreting complexity and deriving consistent mathematical structures, with the aim of reducing both the actual complexity of the real world systems and the technical ones of its mathematical description.

The KTAP approach was originally proposed to model biological systems, in particular competition between immune systems and pathogens [44]. Next it was revisited in the framework of social cooperation and competition dynamics [48, 49], and afterward extended to include more advanced modeling structures [6–8]. However, the approach requires further improvements in order to match the complexity features discussed in Chap. 1. In particular, we recall nonlinear interactions and network topology, the latter also implying communication and possibly transitions from node to node of the network. The literature on this topic is constantly growing, as documented in [10, 13, 28, 74, 148, 158], among other sources.

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Additionally, it is worth mentioning that the assumption of a constant total number of individuals, which usually is tacitly made in the modeling of socio-economic systems, may be valid only for short time periods. In the long run, birth and death processes, as well as inlets from the outer environment, have to be taken into account, which requires a further evolution of modeling structures.

Before tackling technical issues it is worth discussing in some detail the proper role played by *mathematical structures* in the derivation of particular models. It is known [119] that the modeling of living systems cannot take advantage of field theories, as in the case of inert matter. This concept was well presented in the celebrated book by Schrödinger [141]. Therefore heuristic approaches are generally adopted, relying mainly on personal intuitions of the modelers. A more rigorous approach can be developed by grounding models on the preliminary derivation of abstract mathematical structures consistent with the complexity features presented in Chap. 1, which, as a matter of fact, can serve as mathematical/theoretical guidelines.

The advantage of such an approach over the various heuristic strategies discussed in the literature is that if, at some point, a sound field theory becomes available then it can be implemented, at the proper level, in the mathematical structures, thereby guiding the deduction of more targeted models. In other words, the mathematical structures provide modelers with a rigorous conceptual framework virtually independent of (overly) specific heuristic intuitions. As such, they act as a mathematical theory for the construction of models, which can receive (with due control) external phenomenological insights while not being forced to pursue them.

In general, a modeling approach aiming at identifying proper background mathematical structures for complex systems (such as those treated in this monograph) should cope with the following issues:

- Understanding the links between system dynamics and their complexity features.
- Deriving a general mathematical structure that offers a conceptual framework for the derivation of specific models.
- Designing specific models, corresponding to specific classes of systems, by complementing the mathematical structure with suitable models of individual-based interactions according to a detailed interpretation of the dynamics at the microscale.
- Validating models by comparison of their predictions with empirical data.

Apparently, these are standard issues in all fields of applied mathematics. Nevertheless, it is worth pointing out that in the case of inert matter a background field theory is available. For instance, Newtonian mechanics establishes balance equations for mass, momentum, and energy. When physical conditions reveal an inadequacy of classical mechanical laws, more refined theories, such as relativistic and quantum mechanics, can improve the aforesaid field theories thereby contributing to more targeted mathematical models.

More substantial differences arise when dealing with living matter, for no background theory that generally supports the derivation of models exists yet. Moreover, the great heterogeneity that characterizes living systems induces stochastic features that cannot be neglected, nor simply replaced by noise. Rather, they should be inserted in the mathematical structures as hallmarks of the modeling approach.

In this context, the so-called Kinetic Theory for Active Particles (KTAP) has been developed in the last decade to model large systems of interacting entities. Generally they are *living* entities; hence they are termed *active* particles. The conceptual guidelines that inspire the KTAP approach can be listed in detail as follows:

- The system is separated into *functional subsystems* constituted by active particles featuring a microstate called *activity*, which is collectively expressed in each subsystem. It may represent a behavioral characteristic of the particles to be selected according to the specific microscopic dynamics considered in the system, and particularly within that subsystem.
- The state of each functional subsystem is expressed by a time-dependent distribution function over the microscopic states of the active particles.
- The time evolution of the distribution function is triggered by *microscopic interactions* of active particles within and among subsystems. Interactions are modeled by *stochastic games* whose payoffs are probabilistic.
- An equation for the time evolution of the distribution function is obtained by a balance of particles within elementary volumes of the space of microstates, inflow and outflow dynamics of particles being related to interactions at the microscopic scale.

Readers who are more interested in applications should be patient with the cold language of equations characterizing this chapter. Applications will appear soon, starting with the next chapter. On the other hand, as the discussion above should have highlighted, this preliminary path through mathematical tools is a necessary step toward a more rigorous approach to the interplay between mathematical sciences and living systems [40].

The ideas just outlined are organized in the chapter's following sections. In detail, Sect. 2.2 deals with introductory issues concerning the representation of socio-economic systems, which possibly also apply to the inner structure of each node of a small social network. Section 2.3 then yields the mathematical structures at the core of the KTAP approach, focusing on *closed* systems; i.e., systems that do not interact with the outer environment. Subsequently, the structures are extended in Sect. 2.4 to *open* systems, confining attention to the modeling of known external actions. Section 2.5 concerns a technical modification of the mathematical structures, however relevant for modeling purposes, oriented to systems with *discrete* microscopic states (viz., activity). Section 2.6 proposes a general analysis and various hints toward the modeling of interactions at the microscale. The contents also include detailed insight into different sources of nonlinearity that might characterize the interaction dynamics. Section 2.7 outlines some general ideas about the search for solutions to the mathematical models previously derived, also from an approximate numerical point of view, and shows how numerical solutions can provide useful feedback for a deeper understanding of the phenomenology of complex systems. Finally, Sect. 2.8 presents a critical analysis centered around both the ability of the mathematical tools to capture the complexity features presented in Chap. 1 and the general problem of reducing and handling the real world complexity via mathematical approaches.

2.2 Complexity Reduction and Mathematical Representation

As already mentioned, socio-economic systems are understood as ensembles of interacting active particles, which can be either individuals or aggregates clustered by common organization and aims and are, in any case, the atomic (viz., minimal) entities of the system. Their microscopic state (the activity) which also identifies the behavioral strategy to be used in game-type interactions, will be denoted by a variable u belonging to a domain D_u.

Let us consider, at first, particles at a specific node of a social network. That is, we focus on the inner representation of a single node, without explicitly considering links with other nodes. Therefore, in the following discussion, expressions such as "the (global) system" will refer to the node as the universe. In general, it is reasonable to assume that the system is composed of different types of active particles, which express different activities. Aiming at a complexity reduction, it is then convenient to decompose the system into *functional subsystems* composed of active particles that collectively express the same activity.

In principle, the activity therefore identifies a different microscopic feature of active particles in each subsystem, although it is always denoted by the same letter u. Nevertheless, the above decomposition can also be applied if the activity is the same but each subsystem has a different way to express it.

As we will see in the following chapters, u can usually be considered to be a scalar variable. As an example, it may be (an indicator of) the wealth of the active particles. For this reason, it is convenient to think of its domain as a subset of the real line; $D_u \subseteq \mathbb{R}$. More precisely, if we agree to use the letter p to label the various functional subsystems, then in view of the above discussion, we need to distinguish among different domains $D_u^p \subseteq \mathbb{R}$, where p runs from 1 to the total number of functional subsystems, say $m \geq 1$.

The expression of the activity is, in general, heterogeneously distributed within the functional subsystems. Accordingly, the large scale (viz., collective) representation of each of the latter is provided by a *distribution function*:

$$f^p = f^p(t, u) : [0, T_{\max}] \times D_u^p \to [0, +\infty), \quad p = 1, \ldots, m,$$

$T_{\max} > 0$ being a certain final time, possibly $+\infty$. The meaning of the distribution function is that $f^p(t, u)\,du$ is the (infinitesimal) number of active particles of the p-th subsystem which, at time t, are expressing an activity located in the infinitesimal volume du centered at u. Under suitable integrability assumptions, the number of active particles of the p-th subsystem at time t is therefore given by:

$$N^p[f^p](t) := \int_{D_u^p} f^p(t, u)\,du,$$

where square brackets are used, henceforth throughout the monograph, when it is necessary to stress that there is a functional dependence of some quantities on the distribution function f^p (here, for instance, N^p is a linear operator over f^p).

If the N^p's are constant in time then each f^p can be normalized with respect to the corresponding N^p at $t = 0$ and understood as a probability density:

$$\int_{D_u^p} f^p(t,u)\,du = 1, \quad \forall p = 1,\ldots,m, \quad \forall t \in [0, T_{\max}].$$

In this case, ℓ-th order moments of the probability distribution f^p can be defined as follows:

$$\mathbb{E}_\ell^p[f^p](t) := \int_{D_u^p} u^\ell f^p(t,u)\,du, \quad \ell = 0,1,2,\ldots;$$

notice in particular that $N^p(t) = \mathbb{E}_0^p(t)$.

The variance of the distribution is:

$$Var^p[f^p](t) := \int_{D_u^p} |u - \mathbb{E}_1^p[f^p](t)|^2 f^p(t,u)\,du,$$

which provides a measure of the local microscopic oscillations of the system with respect to an average macroscopic description.

Alternatively, if particle transitions among functional subsystems occur, then each f^p is no longer a probability distribution. Nonetheless, due to the hypothesis that the system is closed, which entails that birth/death processes are disregarded, we have:

$$\sum_{p=1}^{m} \int_{D_u^p} f^p(t,u)\,du = \text{constant in } t,$$

hence the normalization can be performed with respect to the total number of active particles of the system. The expression for the moments of each distribution function is technically modified as follows:

$$\mathbb{E}_\ell^p[f^p](t) := \frac{1}{N^p[f^p](t)} \int_{D_u^p} u^\ell f^p(t,u)\,du,$$

which also applies when N^p varies in time because of birth/death events within the p-th subsystem.

It is worth mentioning that this representation does not include any variable related to space among those charged to describe the microscopic state of the active particles. This is because we consider either spatially homogeneous systems or systems in which active particles communicate independently of their localization; e.g., via media. However, when a node is embedded in a social network structure of, say, $M \geq 2$ nodes, additional notation is needed for representing the global system, which now coincides with the network. For instance, the distribution functions of each node can be labeled with two indices:

$$f^{pq} = f^{pq}(t,u) : [0, T_{\max}] \times D_u^{pq} \to [0, +\infty),$$

where $q = 1, \ldots, M$ identifies the node. In principle, the number of functional subsystems may vary from node to node: $m = m(q)$. Also notice that the domain of the activity has been doubly labeled, in order to account for the fact that the variable u may identify different microscopic characteristics of the active particles in each node and each subsystem of a node.

At this point, it is useful to consider a few examples which show how the above representation works in practice. The same examples will be used later on to show the application of the approach at a more practical level.

Example 2.1 (Secessionist trends rising up on a regional basis in a given country [7, 8]). Proceeding from the outermost to the innermost level of description:

- the country can be viewed as a network of M regions;
- each region can be understood as a node q of the network;
- within each region, two functional subsystems $p = 1, 2$ can be identified, corresponding to individuals (viz., active particles) in favor of secession, or against secession, respectively;
- the activity u of the former individuals can represent their inclination for secession, and can be conventionally assumed to be positive. Hence $D_u^{1q} = [0, +\infty)$ for all $q = 1, \ldots, M$;
- the activity, still denoted by u, of the latter individuals can represent instead their aversion to secession, and can be conventionally assumed to be negative. Hence $D_u^{2q} = (-\infty, 0]$ for all $q = 1, \ldots, M$.

It can be questioned whether partitioning each node into two functional subsystems is really useful. In fact, a single distribution function, defined over $D_u = \mathbb{R}$, may be sufficient to capture the population opinion in each region, by agreeing that the more positive (respectively, negative) the value of u, the more in favor of (respectively, against) secession the population is. However, this simplification cannot be adopted as a general rule, because the detail of representation needed also depends on the specific interaction rules that are established within and among the nodes.

Example 2.2 (Process of democratization of a dictatorship in a given country [3, 11, 146]). In this case, a possible representation of the system is as follows:

- the country itself can be viewed as a network of $M = 4$ components of the society;
- the components of the society, namely the dictator ($q = 1$), the ministers ($q = 2$), the parliament ($q = 3$), and the citizens ($q = 4$), are the nodes of the network;
- each node is not further subdivided into functional subsystems; thus, for each of them, a single distribution function defined over $D_u = \mathbb{R}$ is sufficient to describe the trends of the corresponding component of the society. In particular, it can be decided that $u > 0$ corresponds to support for the dictatorship, whereas $u < 0$ corresponds to dissent.

Different representations are possible, some of which are actually equivalent to the proposed one. For instance, the country can also be viewed as an isolated node and the various social components as its functional subsystems, thereby not emphasizing the network structure.

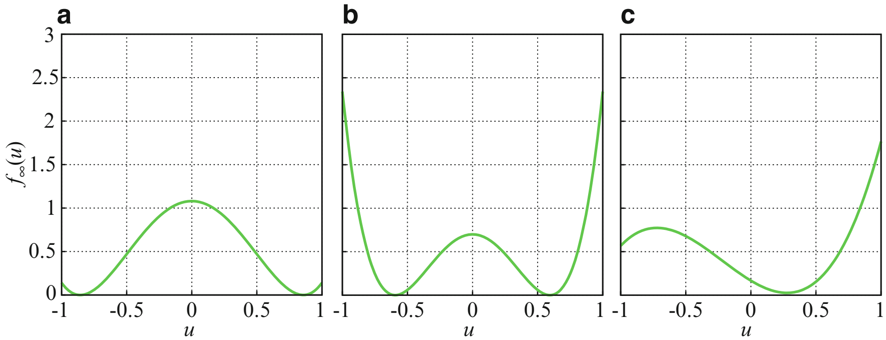

Fig. 2.1 Wealth distribution functions corresponding to various possible social profiles. (**a**) Society with a balanced welfare where middle classes (about $u = 0$) dominate. (**b**) Society strongly radicalized into poor (about $u = -1$) and wealthy (about $u = 1$) classes. (**c**) Asymmetric social profile indicating that most of the population is distributed in the poor classes ($-1 \leq u < 0$) with however a non-negligible presence of very wealthy ones (about $u = 1$)

It is worth stressing that the use of a distribution function over the activity variable improves the representation of the global state of the system with respect to a purely deterministic average representation. In fact, the use of random dependent variables not only allows one to compute moments, but also to detail more precisely how active particles are distributed over the possible microscopic states. For instance, in problems in which the activity represents the wealth status of the particles, such a representation can provide both mean and variance, as deterministic representations might also do, and also additional descriptions such as clustering on certain social states.

Let us be more precise about this issue. Looking ahead at the contents of the next chapter, let us consider dynamics of wealth redistribution assuming that $u \in D_u = [-1, 1]$ is the wealth variable. Negative values close to -1 correspond to poor social classes, and positive values close to 1 correspond to wealthy social classes. Middle classes are located about $u = 0$. Moreover, let us suppose that a certain mathematical model can provide the asymptotic distribution function $f_\infty = f_\infty(u)$ representing the stationary wealth distribution reached for large times (ideally, $t \to +\infty$). Several configurations are possible, such as those shown in Fig. 2.1a, b. Both distributions are symmetric with respect to $u = 0$, and thus they have the same zero mean. However, the former depicts a society with a significant presence of the middle class whereas the latter depicts a society radicalized into poor and wealthy classes, with a very limited presence of middle classes. Such a detail would not be caught by purely average deterministic representations.

The symmetry of these asymptotic distributions can be broken, for instance, by either political choices (to be regarded as external actions on the system) or by asymmetric initial conditions; see Fig. 2.1c.

2.3 Mathematical Structures Toward Modeling

In this section we consider the derivation of the evolution equations for the distribution functions introduced in Sect. 2.2. From this section on, to fix the ideas, we disregard a possible network structure of the system; i.e., we concentrate on the equations valid for each isolated node. The extension to networks is worth being developed with reference to specific applications, which can allow the cumbersome notation required in the general case to be conveniently reduced. We feel confident that readers will be able to draw inspiration from the contents of the following sections to also model interactions among active particles belonging to different interconnected nodes.

As already stressed, the approach is based on the assumption that the evolution of the activity distribution depends on interactions among the active particles taking place at the microscopic scale, the effects of which are specified probabilistically (*stochastic games*). The evolution equation for the distribution function of the p-th functional subsystem is obtained by a balance of particles playing games in the space of microscopic states. It can be expressed in the following general form:

$$\frac{\partial f^p}{\partial t} = J^p[\mathbf{f}], \tag{2.1}$$

where $\mathbf{f} = (f^1, \ldots, f^m)$ and J^p is the p-th *interaction operator*, which depends in principle on all distribution functions because interactions can involve particles of both the same and of different functional subsystems.

The operator J^p will include, in general, both *conservative* and *non-conservative* interactions. The former do not change the total number of particles of the system, whereas the latter account for possible birth/death of particles in the various subsystems. Accordingly:

$$J^p[\mathbf{f}](t,u) = C^p[\mathbf{f}](t,u) + P^p[\mathbf{f}](t,u), \tag{2.2}$$

C standing for *conservative* and P for *proliferative* (agreeing that one may also have *negative* proliferation). Let us now detail the two types of interactions.

2.3.1 Conservative Interactions

Roughly speaking, Eq. (2.1) describes the evolution in time of the distribution of the microscopic state u. Reasoning microscopically, imagine freezing u and asking for the net number of active particles per unit time that reach state u. Such a number is determined by two concomitant facts:

- It is *increased* whenever an active particle, approaching a game-type interaction with a strategy different from u, changes the latter to u in consequence of the interaction.

- It is *decreased* whenever an active particle, approaching a game-type interaction with strategy u, obtains as a payoff a new strategy different from u.

Summarizing, the conservative operator C^p includes both a *gain* and a *loss* contribution of active particles with state u. For this reason, it is technically written as:

$$C^p[\mathbf{f}](t,u) = \sum_{p_*,p^*=1}^{m} \int_{D_u^{p_*}\times D_u^{p^*}} \eta^{p_*p^*}(u_*,u^*)\mathscr{C}^{p_*p^*}(u_*\to u;p|u_*,u^*) \times f^{p_*}(t,u_*)f^{p^*}(t,u^*)\,du_*\,du^* - f^p(t,u)\sum_{p^*=1}^{m}\int_{D_u^{p^*}} \eta^{pp^*}(u,u^*)f^{p^*}(t,u^*)\,du^*, \tag{2.3}$$

where:

- $\eta^{p_*p^*}(u_*,u^*)$ is the *interaction rate*, namely the frequency of the interactions (or encounters) among particles of the p_*-th subsystem with state $u_* \in D_u^{p_*}$ and particles of the p^*-th subsystem with state $u^* \in D_u^{p^*}$.
- $\mathscr{C}^{p_*p^*}(u_*\to u;p|u_*,u^*)$ is the probability density that a particle of the p_*-th subsystem, playing a game with strategy $u_* \in D_u^{p_*}$ against a particle of the p^*-th subsystem with strategy u^*, shifts to the p-th subsystem getting simultaneously the new strategy $u \in D_u^p$. The collection $\{\mathscr{C}^{p_*p^*}\}$ of all such transition probabilities is called the *table of games*.
- The following probability density property holds true:

$$\sum_{p=1}^{m}\int_{D_u^p}\mathscr{C}^{p_*p^*}(u_*\to u;p|u_*,u^*)\,du = 1, \qquad \begin{array}{l}\forall u_*\in D_u^{p_*},\, u^*\in D_u^{p^*}\\ \forall p_*,p^*=1,\ldots,m,\end{array} \tag{2.4}$$

which basically means that every game event produces a payoff within the set D_u^p of admissible strategies for the subsystem p.

The structure (2.3) of the conservative interaction operator takes into account possible transitions of particles across the subsystems. In the particular case that they are not allowed, the same expression of the operator is still formally valid with:

$$\mathscr{C}^{p_*p^*}(u_*\to u;p|u_*,u^*) = \mathscr{C}^{pp^*}(u_*\to u|u_*,u^*)\delta_{p_*p},$$

where: $\mathscr{C}^{pp^*}(u_*\to u|u_*,u^*)$ is now the probability that a particle already belonging to the target subsystem p, with pre-interaction strategy $u_* \in D_u^p$, changes its state to $u \in D_u^p$ because of an interaction with a particle of the p^*-th subsystem playing with strategy $u^* \in D_u^{p^*}$; and $\delta_{p_*p} = 1$ if $p_* = p$, $\delta_{p_*p} = 0$ otherwise is the Kronecker's delta.

A straightforward calculation shows that property (2.4) is the one ensuring the conservation of the total number of particles of the system under the interactions described by the operators C^p. Indeed, it follows from Eq. (2.3) that:

$$\sum_{p=1}^{m} \int_{D_u^p} C^p[\mathbf{f}](t,u)\,du = 0, \quad \forall t \in [0, T_{\max}],$$

whence Eq. (2.1), in the absence of the operator P^p in J^p, yields:

$$\frac{d}{dt} \sum_{p=1}^{m} \int_{D_u^p} f^p(t,u)\,du = \frac{d}{dt} \sum_{p=1}^{m} N^p(t) = 0.$$

If, moreover, particle transitions among subsystems are not possible, then a similar calculation shows that the operators C^p keep the number of particles within each subsystem constant.

Before concluding the discussion about conservative interactions, it is worth recalling the following terminology, which is very customary in the KTAP approach:

- A particle possessing the target state u is called a *test* particle, and is taken to be representative of a generic entity of the system.
- A particle with pre-interaction strategy u_*, which may obtain the target state after a game event, is called a *candidate* particle.
- A particle with strategy u^*, which triggers the interaction, is called a *field* particle.

Hence the structure of conservative interactions can be rephrased by saying that a candidate particle can obtain, with a certain probability, the state of a test particle, whereas the latter can lose it with a certain probability, because of stochastic games with field particles.

2.3.2 Non-conservative Interactions

The non-conservative operator P^p of the p-th subsystem is written following reasoning formally similar to that which led to the gain term of the conservative operator:

$$P^p[\mathbf{f}](t,u) = \sum_{p_*,p^*=1}^{m} \int_{D_u^{p_*} \times D_u^{p^*}} \eta^{p_* p^*}(u_*, u^*) \mathscr{P}^{p_* p^*}(u_*, u^*; u, p) \times f^{p_*}(t, u_*) f^{p^*}(t, u^*)\, du_*\, du^*, \tag{2.5}$$

where:

- $\mathscr{P}^{p_* p^*}(u_*, u^*; u, p)$ is the net *birth/death rate* of a test particle in the p-th subsystem due to an interaction between a candidate particle in the p_*-th subsystem and a field particle in the p^*-th subsystem. In particular, a birth event occurs when $\mathscr{P}^{p_* p^*}(u_*, u^*; u, p) > 0$ while a death event occurs when $\mathscr{P}^{p_* p^*}(u_*, u^*; u, p) < 0$

The birth/death rate is required to satisfy

$$\mathscr{P}^{p_* p^*}(u_*, u^*; u, p) \geq 0 \quad \forall p_* \neq p, \forall u_* \neq u,$$

so that death events can at most take place in the same functional subsystem and state of the candidate particle.

The framework depicted by Eqs. (2.1)–(2.3), and (2.5) offers the mathematical environment for the derivation of models of socio-economic systems. Specific ones are obtained by particularization of the terms η, $\mathscr{C}$, and $\mathscr{P}$, which describe particle dynamics at the microscopic scale. This is indeed the purpose of the following chapters, which will further clarify the concepts just introduced by referring specifically to the modeling of welfare distribution dynamics.

Before concluding this section, we further discuss the table of games $\mathscr{C}$. It is worth stressing that this term of the equations models the game that active particles play at each interaction. As observed in [66], the table of games can be inspired by classical game theory applied to socio-economic systems [122] but it can also include more recent developments addressing evolutionary features of games [79, 88, 100, 123, 126, 138]. In addition, it can account for qualitative and quantitative differences of game rules in different functional subsystems.

A final definition, heuristically anticipated in Chap. 1, which can now be made more precise, is that of linearly vs. nonlinearly additive interactions concerning the terms η, $\mathscr{C}$, and $\mathscr{P}$. As Eqs. (2.3), (2.5) clearly show, pairwise interaction events among candidate and field particles add up to give rise to a variation of the distribution of active particles over the values of the activity. This is indeed the meaning of the integrations over $D_u^{p_*} \times D_u^{p^*}$ in the previous equations. However, if the interaction rate, the table of games, and the birth/death rates depend only on the pre-interaction states of the particles, so that each interaction event is not affected by the presence of field particles other than the one the candidate particle is interacting with, then the final output is the linear superposition of the actions individually applied by the field particles. In this case, hence, interactions are said to be *linearly additive*.

If, on the contrary, each interaction event also depends on field particles other than the one the candidate is interacting with, possibly in an aggregate manner, then the final output is not the linear superposition of the individual actions applied by the field particles. Interactions are therefore said to be *nonlinearly additive*. In practice, in the nonlinearly additive case the terms η, $\mathscr{C}$, and $\mathscr{P}$ do not only depend on the state of the interacting pairs but also on the distribution functions. Recent papers [39] suggest, for instance, using low order moments. This amounts to assuming that each individual does not only feel the state of the interacting companion but also an aggregate effect induced by suitable moments of the particle distribution. This topic will be further discussed in Sect. 2.6.

2.4 Open Systems

We now generalize the mathematical structures presented in Sect. 2.3 to the case of open systems; that is, systems whose particles interact also with the outer environment. More specifically, the latter are understood as members of an additional category of active particles, called *field agents*, that express an activity $w \in D_w \subseteq \mathbb{R}$ by which they can affect the expression of the activity u in the inner system. For the sake of generality, it is convenient to assume that field agents are also organized in a certain number $A \geq 1$ of functional subsystems labeled by $\alpha = 1, \dots, A$. The activity distribution of the field agents in the α-th functional subsystem is given by the distribution function:

$$g^\alpha = g^\alpha(t, w) : [0, T_{\max}] \times D_w^\alpha \to [0, +\infty), \tag{2.6}$$

which can be thought of as normalized to a probability density if the number of field agents does not change in time. In the following we assume that the g^α's are known functions, which means that the state of the outer system is prescribed *a priori*.

When applying their action on the inner system, field agents are assumed to trigger conservative interactions with active particles. Consequently, the evolution equation (2.1) for the f^p's can be rewritten as follows:

$$\frac{\partial f^p}{\partial t} = J^p[\mathbf{f}] + Q^p[\mathbf{f}, \mathbf{g}], \tag{2.7}$$

where $\mathbf{g} = (g^1, \dots, g^A)$ and Q^p is the p-th *inner–outer conservative operator*. The formal expression of the latter recalls that of the conservative component C^p of J^p:

$$\begin{aligned} Q^p[\mathbf{f}, \mathbf{g}](t, u) = \sum_{\alpha=1}^{A} \sum_{p_*=1}^{m} \int_{D_u^{p_*} \times D_w^\alpha} & \hat{\eta}^{p_*\alpha}(u_*, w) \mathscr{Q}^{p_*\alpha}(u_* \to u; p|u_*, w) \\ & \times f^{p_*}(t, u_*) g^\alpha(t, w) \, du_* \, dw \\ - f^p(t, u) \sum_{\alpha=1}^{A} \int_{D_w^\alpha} & \hat{\eta}^{p\alpha}(u, w) g^\alpha(t, w) \, dw, \end{aligned} \tag{2.8}$$

where:

- $\hat{\eta}^{p_*\alpha}(u_*, w)$ is the *inner–outer interaction rate*; that is, the frequency at which candidate particles of the inner system interact with field agents of the outer environment.
- $\mathscr{Q}^{p_*\alpha}(u_* \to u; p|u_*, w)$ is the probability that candidate particles get the test state, shifting simultaneously to the p-th test functional subsystem, upon playing games with field agents. The collection $\{\mathscr{Q}^{p_*\alpha}\}$ of all such transition probabilities is the

inner–outer table of games, which is required to satisfy the probability density property:

$$\sum_{p=1}^{m} \int_{D_u^p} \mathscr{Q}^{p_*\alpha}(u_* \to u; p|u_*, w)\, du = 1, \quad \begin{array}{l} \forall u_* \in D_u^{p_*}, w \in D_w^{\alpha} \\ \forall p_* = 1, \dots, m, \\ \forall \alpha = 1, \dots A, \end{array}$$

whence:

$$\sum_{p=1}^{m} \int_{D_u^p} Q^p[\mathbf{f}, \mathbf{g}](t, u)\, du = 0, \quad \forall t \in [0, T_{\max}],$$

which expresses conservation of inner–outer interactions. Generalizing to the non-conservative case is straightforward, based on properties of the operator P^p discussed earlier; hence this is left as an exercise for the reader.

The mathematical framework (2.7) and (2.8) is derived under the assumption that the action of the outer environment on the inner system is applied through stochastic games. However, deterministic actions applied directly to the test particle can also be considered, which produce a stream effect in the space of the microscopic states. The mathematical translation of this idea is a conservative advection term in the evolution equation for f^p:

$$\frac{\partial f^p}{\partial t} + \frac{\partial}{\partial u}(K^p[\mathbf{g}] f^p) = J^p[\mathbf{f}], \tag{2.9}$$

where $K^p[\mathbf{g}]$ is the global microscopic action applied by field agents on the test particle. Usually, it takes the form of a *mean field action*:

$$K^p[\mathbf{g}](t, u) = \sum_{\alpha=1}^{A} \int_{D_w^{\alpha}} \mathscr{K}^{p\alpha}(u, w) g^{\alpha}(t, w)\, dw, \tag{2.10}$$

$\mathscr{K}^{p\alpha} : D_u^p \times D_w^{\alpha} \to \mathbb{R}$ being the *interaction kernel* for pairs of test particles and field agents belonging to the p-th and α-th inner and outer functional subsystem, respectively. Notice that, in this case, individuals do not adopt any stochastic game strategy for deciding the output of their interaction. The latter is indeed given deterministically by the interaction kernel, once the states of the interacting subjects are known.

In principle, both stochastic and deterministic inner–outer interactions can be taken into account. The corresponding mathematical structure is then:

$$\frac{\partial f^p}{\partial t} + \frac{\partial}{\partial u}(K^p[\mathbf{g}] f^p) = J^p[\mathbf{f}] + Q^p[\mathbf{f}, \mathbf{g}]. \tag{2.11}$$

2.5 Systems with Discrete States

In applications to living complex systems, it is not always practical, or even possible, to identify a continuous distribution of microscopic states of the active particles. Indeed, the activity variable often refers to originally non-numerical and qualitative characteristics, such as e.g., individual opinions, political preferences, wealth status, which need to be transformed to quantitative information in the mathematical approach. In these cases, it might be convenient to reason in terms of *activity classes* roughly representative of the social structure of the system. Technically, this amounts to assuming that u is, in each functional subsystem, a discrete variable with only a finite number of possible values in a lattice:

$$I_u^p = \{u_1^p, u_2^p, \dots, u_n^p\} \subset D_u^p, \quad p = 1, \dots, m.$$

Each u_i^p is called an activity class of the p-th functional subsystem. As in the continuous case set forth in the preceding sections, it identifies a possible strategy by which active particles can approach game-type interaction events. The difference is that now the number of admissible strategies is finite, as is usually the case in classical game theory.

The corresponding formal expression of the distribution function is

$$f^p(t, u) = \sum_{i=1}^{n} f_i^p(t)\delta_{u_i^p}(u), \tag{2.12}$$

where $\delta_{u_i^p}$ is the Dirac distribution centered at u_i^p and $f_i^p(t) = f^p(t, u_i^p)$ in the sense of distributions. The function $f_i^p : [0, T_{\max}] \to [0, +\infty)$, for $i = 1, \dots, n$ and $p = 1, \dots, m$, is the distribution function of the i-th activity class in the p-th functional subsystem.

Using the representation (2.12), the formulas given in Sect. 2.2 for the moments of the distribution can be straightforwardly restated in the discrete activity case. For instance, the number of particles of the p-th subsystem (zeroth-order moment) is given by:

$$N^p[\mathbf{f}^p](t) = \sum_{i=1}^{n} f_i^p(t),$$

whereas ℓ-th order moments, $\ell \geq 1$, are computed as:

$$\mathbb{E}_\ell^p[\mathbf{f}^p](t) = \sum_{i=1}^{n} \left(u_i^p\right)^\ell f_i^p(t).$$

Analogously, the evolution equations for the f_i^p's are technically deduced by substituting the representation (2.12) into Eqs. (2.2), (2.3), (2.5), (2.7), (2.8), after reinterpreting the latter in the sense of distributions. This yields:

$$\frac{df_i^p}{dt} = J_i^p[\mathbf{F}] + Q_i^p[\mathbf{F}, \mathbf{G}] = C_i^p[\mathbf{F}] + P_i^p[\mathbf{F}] + Q_i^p[\mathbf{F}, \mathbf{G}], \tag{2.13}$$

where now

$$\mathbf{F} = \{f_i^p\}_{\substack{i=1,\dots,n \\ p=1,\dots,m}}, \qquad \text{and} \qquad \mathbf{G} = \{g_i^\alpha\}_{\substack{i=1,\dots,\nu \\ \alpha=1,\dots,A}},$$

and the operators on the right side of Eq. 2.13 are:

$$C_i^p[\mathbf{F}] = \sum_{p_*,\, p^*=1}^{m} \sum_{h,k=1}^{n} \eta_{hk}^{p_* p^*} \mathscr{C}_{hk}^{p_* p^*}(i, p) f_h^{p_*} f_k^{p^*} - f_i^p \sum_{p^*=1}^{m} \sum_{k=1}^{n} \eta_{ik}^{pp^*} f_k^{p^*}$$

$$P_i^p[\mathbf{F}] = \sum_{p_*,\, p^*=1}^{m} \sum_{h,k=1}^{n} \eta_{hk}^{p_* p^*} \mathscr{P}_{hk}^{p_* p^*}(i, p) f_h^{p_*} f_k^{p^*}$$

$$Q_i^p[\mathbf{F}, \mathbf{G}] = \sum_{\alpha=1}^{A} \sum_{p_*=1}^{m} \sum_{h=1}^{n} \sum_{j=1}^{\nu} \hat{\eta}_{hj}^{p_*\alpha} \mathscr{Q}_{hj}^{p_*\alpha}(i, p) f_h^{p_*} g_j^\alpha - f_i^p \sum_{\alpha=1}^{A} \sum_{j=1}^{\nu} \hat{\eta}_{ij}^{p\alpha} g_j^\alpha,$$

The symbols above have an intuitive meaning, that we explicitly detail, however, for the sake of clarity:

- $\eta_{hk}^{p_* p^*}$ is the *interaction rate* between a candidate particle in the h-th activity class of the p_*-th subsystem and a field particle in the k-th activity class of the p^*-th subsystem. An analogous interpretation holds for $\hat{\eta}_{ij}^{p\alpha}$, which refers to inner-outer interactions among candidate particles and field agents.
- $\mathscr{C}_{hk}^{p_* p^*}(i, p)$ is the probability that a candidate particle in the h-th activity class of the p_*-th subsystem shifts to the test activity class i of the test subsystem p after playing a game with a field particle in the k-th activity class of the p^*-th subsystem. The set of all such transition probabilities forms the *table of games* $\{\mathscr{C}_{hk}^{p_* p^*}(i, p)\}$, which models the game played by active particles. It satisfies the probability density property:

  $$\sum_{p=1}^{m} \sum_{i=1}^{n} \mathscr{C}_{hk}^{p_* p^*}(i, p) = 1, \quad \forall h, k = 1, \dots, n, \ \forall p_*, p^* = 1, \dots, m,$$

  which guarantees the conservation of inner interactions described by the operator C_i^p.
- $\mathscr{P}_{hk}^{p_* p^*}(i, p)$ is the *net birth/death rate* of active particles for proliferative inner interactions. In order for deaths to occur only in the functional subsystem and activity class of candidate particles, the following assumption is made:

  $$\mathscr{P}_{hk}^{p_* p^*}(i, p) \geq 0 \quad \forall h \neq i, \ \forall p_* \neq p.$$

- $\mathscr{Q}_{hj}^{p_*\alpha}(i, p)$ is the probability that a candidate particle in the h-th activity class of the p_*-th functional subsystem shifts to the test activity class i of the test subsystem p when playing a game with a field agent in the j-th activity class of the α-th outer functional subsystem. It is also assumed that field agents are grouped into a finite number $\nu \geq 1$ of activity classes over the lattice

$$I_w^\alpha = \{w_1^\alpha, w_2^\alpha, \dots, w_\nu^\alpha\} \subset D_w^\alpha$$

and represented by a set of $A \times \nu$ known distribution functions $g_j^\alpha = g_j^\alpha(t) : [0, T_{\max}] \to [0, +\infty)$ such that the distribution function (2.6) is recovered, for each $\alpha = 1, \dots, A$, as:

$$g^\alpha(t, w) = \sum_{j=1}^{\nu} g_j^\alpha(t)\delta_{w_j^\alpha}(w),$$

with $g_j^\alpha(t) = g^\alpha(t, w_j^\alpha)$ in the sense of distributions. The set of all of the above inner-outer transition probabilities constitutes the *inner-outer table of games* $\{\mathscr{Q}_{hj}^{p_*\alpha}(i, p)\}$, which is required to satisfy the probability density property:

$$\sum_{p=1}^{m}\sum_{i=1}^{n} \mathscr{Q}_{hj}^{p_*\alpha}(i, p) = 1, \qquad \begin{array}{l} \forall h = 1, \dots, n, \ \forall j = 1, \dots, \nu \\ \forall p_* = 1, \dots, m, \ \forall \alpha = 1, \dots, A \end{array} \tag{2.14}$$

in order for the interactions described by the operator Q_i^p to be conservative.

2.6 Microscopic Interactions and Sources of Nonlinearity

Mathematical models can be derived from the mathematical structures presented in Sects. 2.3–2.5 by devising descriptions of the interactions at the microscale, for instance at the levels of the interaction rate and of the table of games.

As already mentioned, theoretical tools from game theory can be used. However, the literature does not yet offer a unified systematic approach. Hence, heuristic methods are generally applied for each specific case. Actually, recent contributions [38, 43] give some hints that will be revisited in this section.

An important issue is nonlinear interactions, when particles are not simply subject to the superposition of binary actions but are also affected by the aggregate state of neighboring individuals. In this case, the interaction terms have to be viewed as operators over the distribution function, whereby further nonlinearities are introduced in the right-hand sides of the relevant equations.

For instance, the interaction rate may be assumed to decay with the distance between the states of the interacting particles as well as with the distance between the distribution functions of the subsystems they belong to. Such an assumption would imply a higher interaction frequency for similar particles belonging to similarly distributed subsystems. In other cases, dissimilar particles carrying out different functions may be more likely to interact with high frequency, as happens, for example, in hiding-learning processes [72] such as chasing and/or escaping dynamics involving criminals and detectives [143].

Another source of nonlinearity in the interaction dynamics can be the actual activity subdomain where interactions are effective, which may not coincide with the whole D_u^p. In other words, candidate particles may interact only with some field particles selected on a distribution-dependent basis (one then speaks of *topological* interactions). This idea originates from conjectures made by physicists about the dynamics of swarms [25], which can be interestingly transferred to social sciences for addressing swarming behaviors such as those treated in [142, 143, 165].

Finally, a great source of nonlinearities in the evolution equations is definitely the table of games. Indeed, as already pointed out, candidate particles can also modify their interaction strategy depending on some (local) aggregate state of the field particles they interact with, which is duly described by suitable moments of the distribution function.

Some technical arguments about these topics are presented in the following paragraphs, without claiming to be exhaustive. For expository purposes the focus will be on closed systems. Technical generalizations to open systems are left to interested readers.

2.6.1 Interaction Rate

The frequency of the interactions among candidate and field particles belonging to the p_*-th and p^*-th functional subsystems, respectively, is modeled by the term $\eta^{p_* p^*}$. Following [38], the latter can be assumed to decay with the distance between the interacting particles. In the linear case, such a distance depends only on the microstates of the interacting pairs, hence it is given by $|u_* - u^*|$. Conversely, in the nonlinear case it can also depend on the distribution functions f^{p_*} and f^{p^*}, particularly on $\|f^{p_*} - f^{p^*}\|$. Here, $\|\cdot\|$ is a suitable norm such as, for example, the uniform L^∞-norm or the mean L^1-norm, also depending on the physics of the system under consideration (which might suggest appropriate ways of measuring the distance between different configurations of the system). This concept is based on the idea that two systems with close distributions are *affine*, hence they tend to interact with higher frequency.

The interaction rate can also include a dependence on the actual interaction domain. If particles interact only within a certain distance or with a predefined number of other particles then $\eta^{p_* p^*}$ is zero outside the respective bounds. In turn, the latter can be linked to the distribution functions f^{p_*}, f^{p^*} themselves.

2.6.2 Table of Games

The table of games $\mathscr{C}^{p_* p^*}$ can be modeled by relating the payoffs of the interactions to:

- *Cooperative/competitive* games, in which candidate particles try to profit from the state of field particles in order to consolidate their well-being or to fairly redistribute wealth.
- *Hiding-learning* dynamics, in which attempts by candidate particles to improve their state are balanced by a tendency to reduce social distances thus produced (learning process).

In general, it may be argued that the occurrence of either type of game depends, once again, on a microstate-based distance between the interacting particles. Moreover, if the distance between the configurations of the subsystems that particles belong to is involved, further nonlinearities are brought into the problem.

2.6.3 Inner Reorganization of Functional Subsystems

Some social systems, for instance political parties, are characterized by a (small) *critical size for survival*. That is, if the size $N^p[f^p]$ of a certain functional subsystem falls below a critical threshold then particles may prefer to migrate to other functional subsystem or to aggregate in brand new ones. Similarly, a (large) critical size can exist such that, when it is overcome, particles are again induced to migrate to other subsystems for avoiding a kind of depersonalization due to "overcrowding" of their original subsystem.

In both cases, the two sizes are generally not constant but depend on the global state of the system. Therefore disappearance, splitting, or creation of functional subsystems can be additional sources of nonlinearity in the equations.

The previous arguments drop a hint that interactions at the microscale can be strongly characterized by various types of nonlinearities. Consequently, in most cases the evolution equations feature further nonlinearities besides the standard quadratic one due to the product of the distribution functions in the terms C^p and P^p.

The mathematical structures proposed in the preceding sections are still valid but the following notations are worth being introduced for stressing such *constitutive* nonlinearities:

$$\eta^{p_* p^*}[\mathbf{f}](u_*, u^*), \quad \mathscr{C}^{p_* p^*}[\mathbf{f}](u_* \to u; p|u_*, u^*), \quad \text{and} \quad \mathscr{P}^{p_* p^*}[\mathbf{f}](u_*, u^*; u, p).$$

This notation should also be extended to the domain of interaction, when it depends on the distribution functions: $D_u = D_u[\mathbf{f}]$. See [43] for a detailed analysis of the related convolution problems.

According to our perspective as advanced in this monograph, the modeling approach cannot be oversimplified by neglecting nonlinearly additive interactions with the only aim of pursuing analytical results. On the contrary, no matter how difficult the analytical treatment of the resulting equations may appear, most of the existing literature should probably be revisited under this new perspective.

2.7 On the Solution of Mathematical Problems

The application of the mathematical structures presented in the previous sections to real social phenomena generates *mathematical problems*. The latter can be of essentially two types:

- *Initial-value problems*, namely those generated by Eqs. (2.1), (2.7), and (2.13) linked to *initial conditions*:

$$f^p(0,u) = f_0^p(u), \quad u \in D_u^p,\ p = 1, \dots, m \quad \text{for Eqs. (2.1), (2.7)}$$

or

$$f_i^p(0) = f_{0i}^p, \quad i = 1, \dots, n,\ p = 1, \dots, m \quad \text{for Eq. (2.13),}$$

where $f_0^p : D_u^p \to [0, +\infty)$, $f_{0i}^p \in [0, +\infty)$ are, respectively, known functions and numerical values prescribed for describing the distribution of the active particles over the activity u or the activity classes u_i^p at the initial time $t = 0$.

- *Initial/boundary-value problems*, namely those generated by Eqs. (2.9) and (2.11) linked to initial conditions analogous to those discussed above and, in addition, to conditions at the boundary of D_u^p due to the flux term $\partial_u(K^p[\mathbf{g}]f^p)$. These are of the form:

$$f^p(t, u \in \partial D_u^p) = \varphi^p(t), \quad p = 1, \dots, m,$$

$\varphi^p : [0, T_{\max}] \to [0, +\infty)$ being known functions. These conditions provide the values of the distribution functions f^p on the boundary of D_u^p at all times. They do not necessarily have to be prescribed on the whole boundary of D_u^p but possibly only on subsets of ∂D_u^p, depending on the structure of the advection speed $K^p[\mathbf{g}]$. If D_u^p is unbounded then boundary conditions are replaced by suitable properties of decay to zero of the corresponding p-th distribution function at infinity for ensuring its integrability with respect to u.

In all cases, the mathematical problem consists of computing the solution, being either

$$\{f^p(t,u)\}_{p=1}^m, \quad \forall\, u \in D_u^p, \quad \forall\, t \in [0, T_{\max}],$$

or

$$\{f_i^p(t)\}_{\substack{i=1,\ldots,n\\ p=1,\ldots,m}}, \quad \forall\, t \in [0, T_{\max}],$$

starting from the input data required by the model.

The numerical solution of such problems is generally not a difficult task, at least in the absence of mean-field fluxes K^p. In fact, collocation methods can be used to transform activity-continuous integro-differential equations into ordinary differential equations in time, which can then be discretized by the most appropriate computational schemes, with consideration of stability and accuracy. The technique is described in [41], where it is also shown how boundary conditions can be implemented directly into the system of ODEs. If the activity variable is discrete, models immediately take the structure of a system of ordinary differential equations.

Simulations should be supported by a qualitative analysis of the mathematical problems. Existence and uniqueness of solutions to initial-value problems in the absence of non-conservative interactions have been studied in [19] by extending a previous study about linearly additive interactions [16] to nonlinearly additive interactions. Proliferative terms require additional studies, as they may even induce bifurcation phenomena. This issue has been carefully addressed in [44] referring specifically to models of competition between the immune system and pathogens. Analogous investigations in the case of social systems are not yet available; see Chap. 5 for a critical analysis of this issue.

The use of models for simulating real systems also motivates further challenging analytical investigations not necessarily limited to a qualitative analysis of the solutions to mathematical problems. Other challenging issues are, for instance, understanding the links between the KTAP approach and Markov processes [112] or classical kinetic theories [26]. These aspects are quite well mastered in the case of models with linearly additive interactions [17], whereas they are still an open problem in the case of nonlinearly additive interactions.

2.8 Critical Analysis

In this section we analyze how far the mathematical structures proposed in this chapter are able to capture the complexity features of social systems discussed in Chap. 1. Readers should be aware that, for the moment, only preliminary considerations can be made. A more exhaustive analysis has to be deferred to the next chapters, after seeing specific models in action on well defined applications. In the following, we proceed by keywords reminiscent of crucial complexity aspects dealt with so far.

- *Emerging behaviors and validation*. Mathematical models provide the time evolution of the distribution function, which in some cases is a probability density (when the number of active particles is constant in time), over the activity variable. This output potentially depicts both microscopic details and average macroscopic quantities. Validation of models should be based on their ability to describe actually observed emerging behaviors. A successful model may also be expected to predict, under special circumstances, events which have never been observed before.

- *Strategy, organization ability, and heterogeneity*. All of these features are variously linked to the activity variable u, which expresses the (non-mechanical) state of the active particles, namely the behavioral strategy they apply when interacting in a game-type fashion with other particles or with external agents. The heterogeneous distribution of such a variable among the active particles is expressed, within each functional subsystem, by the distribution function f^p, which evolves in time. The activity can be though of, at the microscopic scale, as a *random variable* attached to each active particle. The KTAP approach then studies the evolution in time of its distribution at the mesoscopic scale. Reference to random variables is necessary in order to model the partly irrational (viz., stochastic) as opposed to rational (viz., deterministic) behavior of the active particles, which may not react in the same way even if placed in similar conditions.
- *Interactions by stochastic games*. Microscopic interactions among active particles are modeled by the terms η, $\mathscr{C}$, and $\mathscr{P}$, namely the interaction rate, the table of games, and the net birth/death rate, respectively. Particularly important is the table of games, which describes interactions as stochastic games. In more detail, it encodes, in probabilistic terms, the (conservative) interaction rules, hence the game played by active particles. The distribution function then changes as a result of a change in the activity of the particles after the game they play when interacting.
- *Reducing the complexity generated by a large variety of components*. Splitting the whole system into functional subsystems is a way to reduce the technical complexity induced by a large number of variables. Such a decomposition can possibly also be viewed as associated with a secondary activity variable, say v, which takes only m discrete values $v_1, v_2, \dots, v_m$. According to this interpretation, the p-th functional subsystem would thus group all active particles which collectively express the value v_p of the activity v. This point of view is conceptually convenient when the primary activity variable u has the same meaning in all subsystems but the rules at the basis of the microscopic interactions depend on the specific function v carried out by different groups of active particles. Alternatively, the decomposition in functional subsystems can be thought of as a way to have a scalar activity variable u in each of them rather than a vector-valued one $\mathbf{u} = (u_1, u_2, \dots, u_m)$ when the various components u_p, $p = 1, \dots, m$, have different meanings in each subsystem.
- *Mutations and selections*. These events are respectively related to transitions across functional subsystems, accounted for by the table of games, and to proliferative interactions, modeled by the operators P^p. Their modeling requires,

in general, that the influence of the outer environment on the conservative interactions be carefully analyzed.

- *Multiscale essence*. The modeling of aforementioned terms η, $\mathscr{C}$, and $\mathscr{P}$ is generally obtained by a mainly phenomenological approach at the *microscopic* scale. Due to the behavioral individuality of the active particles composing the system, this approach is probably better than trying to directly model the collective behavior of groups of particles at the *macroscopic* scale by means of *constitutive relationships*. Next, ensemble dynamics are predicted by the model and can be studied *a posteriori*, once the evolution of the distribution function is available at the *mesoscopic* scale. Phenomenological guidelines for modeling the terms η, $\mathscr{C}$, and $\mathscr{P}$ cannot however be given in full generality. Specific applications can instead indicate, each time, some reasonable ones, thereby also suggesting possible technical developments of the mathematical structures.

Chapter 3
Modeling Cooperation and Competition in Socio-Economic Systems

Abstract This chapter shows how the mathematical tools derived in Chap. 2 can be profitably exploited for modeling social interaction dynamics. The focus is on cooperative and competitive games among the members of a social population, which result in a modification of the well-being of the individuals due to a redistribution of their global wealth. External actions related to welfare policies are also considered in the modeling approach.

3.1 Introduction

Starting from this chapter, we consider the practical implementation of the mathematical structures set forth in Chap. 2 toward the modeling of specific social scenarios. In particular, we select as a case study the cooperation and competition dynamics at the basis of the redistribution of wealth among the members of a social community, which can affect the well-being and social cohesion of populations. This is a widely studied topic in socio-economic sciences [31, 128]; see also the bibliographical references reported in Chap. 1, the mathematical treatment of which by the KTAP approach was initiated in [48]. Next, in [8] the guidelines for more refined models, taking advantage of a decomposition in functional subsystems, have been introduced, in this way promoting a systemic approach to social sciences, that one may call *system sociology*. A few applications are developed in [7, 49]. In addition, it is worth recalling that, as already mentioned in Chap. 2, other conceivable topics that might be addressed by analogous approaches are criminal behaviors [143], migration phenomena [23], and opinion formation [78, 152].

As a matter of fact, the application of the KTAP methods to social systems has been so far limited to linearly additive interactions. On the other hand, several papers have pointed out the necessity of taking the next step toward nonlinearly additive interactions when dealing with living complex systems; for example, see [37]. Two recent mathematical letters [35, 72] have suggested prospective ideas in this direction, and a recent paper [39] has put them into practice in the context

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of rare social events, the so-called *black swans* [149]. By resting on the contents of [39], the present chapter derives exploratory models aimed at understanding how different socio-economic policies can affect the wealth redistribution among the members of a social community. The focus is on modeling issues, specifically on the specialization of the general mathematical structures to well-defined contexts. The assessment, through numerical simulations, of the predictive ability of models is the subject of the next chapter.

Motivations for studyng this type of social dynamics also come from the recent literature [135], where the authors argue that there exists an interplay between increases in wealth and unethical behaviors. In particular, they present various empirical data referred to significant case studies. Actually, we do not aim at thoroughly addressing the issues proposed in [135], which would require knowledge of psychology and social sciences far beyond ours. Rather, we aim at showing that mathematical sciences can provide useful exploratory, and in some cases also predictive, tools for a deeper interpretation of social phenomena. The reasonings developed here and in the next chapter also take advantage of recent studies focused on the role of selfishness and cooperation in managing the social dynamics of society [144].

A detailed account of the contents of this chapter is as follows. Section 3.2 offers a qualitative discussion about cooperative and competitive stochastic games related to altruistic and selfish social attitudes, respectively, with the idea that they can serve as a paradigm for the modeling of social interaction dynamics. Section 3.3 brings such a discussion to a quantitative level by designing the main model elements accordingly, such as the interaction rate and especially the table of games. In doing so, specific sources of nonlinearly additive interactions are considered in view of their significance for modeling. Section 3.4 then suggests a few technical developments of the model, with a forward look at additional improvements that will be discussed later in Chap. 5. Finally, Sect. 3.5 proposes a critical analysis of the contents of the chapter, partly preparatory to the numerical simulations developed in Chap. 4.

3.2 Cooperative and Competitive Stochastic Games

As we have seen in Chap. 2, the derivation of mathematical models according to the KTAP approach requires a detailed characterization of the microscopic interaction dynamics, which can be assimilated to nonlinearly additive stochastic games. The game rules depend, in general, on the specific system under consideration; however, some broad guidelines can be extracted from, for example, references [88, 95, 123].

The approach we propose in this chapter is based on game dynamics that can be classified as either *cooperative* or *competitive*. In particular, active particles are assumed to play a game at each interaction by choosing one of these two game regimes according to the strategy (viz., activity) they individually express.

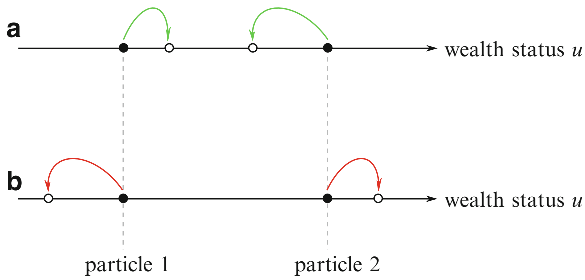

Fig. 3.1 Pictorial illustration of (**a**) cooperative and (**b**) competitive game dynamics between two active particles. *Black* and *white bullets* denote the pre-interaction and post-interaction states, respectively, of the particles

It is worth anticipating that, in the socio-economic application of wealth redistribution we are concerned with, the activity of the particles coincides with their wealth status. As usual, the payoff of the game is a new wealth status (the test one), that a candidate particle gains, with a certain probability, after an interaction with a field particle depending on the kind of game they played. Bearing this in mind, let us describe in more detail what cooperation and competition rules consist of.

Cooperation: The candidate particle either increases its wealth status, by benefiting from a field particle with higher status, or decreases it, by supporting a field particle with lower status. After the interaction, the states of the particles become closer together than before the interaction (see Fig. 3.1a).

Competition: The candidate particle either further decreases its wealth status, by yielding to a field particle with higher status, or further increases it, by getting the better of a field particle with lower status. After the interaction, the states of the particles become farther apart than before the interaction (see Fig. 3.1b).

These interaction models are somehow classical in game theory; see, for example, [33, 101, 138]. They have been transposed to the KTAP approach in the previously cited paper [48], where the authors have also proposed a criterion for active particles to switch between cooperative and competitive dynamics based on their relative activity state. Specifically, a distance between the activities of the interacting pairs is defined, whose low and high values, with respect to a certain *critical threshold*, trigger cooperation and competition, respectively, in such a way that individuals with similar wealth status tend to compete, whereas those with significantly different wealth status tend to cooperate. The critical threshold is a key element of the model. It can either be constant in time, as in [48], or depend on the time evolution of the wealth distribution itself, as in [39]. The second option seems to be more realistic, for individuals can adapt their game dynamics to the evolving circumstances in which they operate, possibly also learning from past experiences.

It is worth pointing out that a non-constant critical threshold typically induces nonlinearly additive interactions, especially when the threshold variability is linked to inner characteristics of the systems, such as, for example, the gap between wealthy and poor individuals or social classes; cf. [39].

In general, game dynamics have to ensure the conservation of some global quantities of the system. A basic requirement is that they preserve the total number of active particles. Moreover, we observe that the cooperative and competitive rules

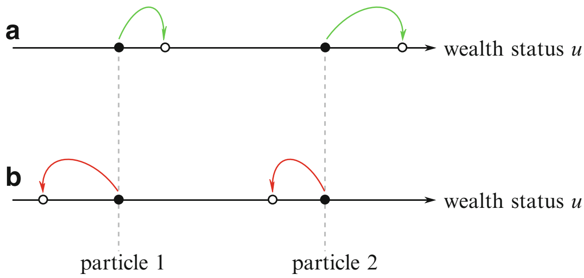

Fig. 3.2 Game dynamics which do not preserve, in general, the average wealth of the society. (**a**) Wealth-productive dynamics. (**b**) Wealth-dissipative dynamics

described above are consistent with the preservation of the average wealth of the society, if the impoverishment of a particle and the simultaneous enrichment of its interacting companion are of the same extent. Other dynamics can instead be markedly wealth-productive (see Fig. 3.2a) or wealth-dissipative (see Fig. 3.2b). For instance, the average wealth is often increased by social agreement and reduced by social conflicts because such situations can enhance or worsen, respectively, the productive ability of a society. In this case, game dynamics follow mixed rules. Often they are linked to interactions with the outer environment, possibly in a network context, as in the case of competition among countries at an international level.

3.3 Modeling Socio-Economic Interactions

From the interaction paradigm discussed in Sect. 3.2, we now derive a specific model of socio-economic dynamics grounded on the mathematical frameworks introduced in Chap. 2. In particular, we focus on *discrete activity models*, considering that in the real world it is more customary to deal with *social classes* when the discriminant is individual wealth. Additionally, we assume that the whole society coincides with a single functional subsystem grouping all individuals of a certain country or regional area, and that the system is closed. Therefore, no action is applied on the active particles by field agents. Consequently, the reference mathematical structure for this specific scenario is:

$$\frac{df_i}{dt} = \sum_{h,k=1}^{n} \eta_{hk}[\mathbf{f}]\mathscr{C}_{hk}[\mathbf{f}](i) f_h f_k - f_i \sum_{k=1}^{n} \eta_{ik}[\mathbf{f}] f_k, \tag{3.1}$$

where we have omitted the index of the functional subsystem in order to simplify the notation. We have instead indicated explicitly the dependence of the interaction rate and the table of games on the collection of distribution functions $\mathbf{f} := (f_1, \dots, f_n)$ according to the notation introduced in Sect. 2.6 of Chap. 2. Indeed, the model presented in the following includes certain nonlinearly additive interactions.

In practice, Eq. (3.1) is what Eq. (2.13) reduces to in the presence of conservative interactions only within a unique functional subsystem. The conservation of the total number of active particles, obtained from the probability density property (2.14):

$$\sum_{i=1}^{n} \mathscr{C}_{hk}[\mathbf{f}](i) = 1, \quad \forall\, h, k = 1, \ldots, n, \ \forall\, \mathbf{f}, \tag{3.2}$$

implies:

$$\sum_{i=1}^{n} f_i(t) = \text{constant}, \quad \forall\, t \in [0, T_{\max}],$$

hence, up to normalization with respect to the total number of active particles of the system, the distribution function $f_i = f_i(t) : [0, T_{\max}] \to [0, 1]$ can be understood as the probability that the test particle be in wealth class i at time t.

In the following, we consider the detailed modeling of the terms of Eq. (3.1) devoted to the mathematical description of the interactions among active particles.

3.3.1 Activity Lattice

As already stated, the activity u of the particles is, in this context, an indicator of their wealth status; more precisely, from a discrete-state point of view, of their social wealth class. Assuming conventionally that poor social classes are identified by a negative activity and wealthy ones by a positive activity, the following uniformly spaced activity lattice can be introduced:

$$\begin{aligned} I_u &= \{u_1 = -1, \ldots, u_{\frac{n+1}{2}} = 0, \ldots, u_n = 1\}, \\ u_i &= \frac{2}{n-1} i - \frac{n+1}{n-1}, \quad i = 1, \ldots, n \end{aligned} \tag{3.3}$$

with odd n, so that a middle "neutral" class $u_{\frac{n+1}{2}} = 0$ exists.

3.3.2 Interaction Rate

Following [39], we consider two different rates of interaction corresponding to cooperative and competitive dynamics:

$$\eta_{hk}[\mathbf{f}] = \begin{cases} \eta_0 & \text{if } |k-h| \le \gamma[\mathbf{f}] \quad \text{(competition)} \\ \mu\eta_0 & \text{if } |k-h| > \gamma[\mathbf{f}] \quad \text{(cooperation),} \end{cases} \tag{3.4}$$

where $\mu \in (0, 1)$ is a parameter and $\eta_0 > 0$ allows one to scale the time variable in Eq. (3.1). Model (3.4) basically assumes that in a cooperative regime interactions are less frequent (i.e., individuals are less reactive) than in a competitive one. The value $\gamma \geq 0$ is the previously mentioned *critical threshold*, which triggers either type of dynamics according to the distance between the activities of the interacting particles. We will henceforth denote it by $\gamma[\mathbf{f}]$ in order to stress the possible functional dependence on the probability distribution $\mathbf{f}$ in the case of nonlinearly additive interactions. We will discuss it in detail at the end of this section.

Remark 3.1. The distance between the activities of the interacting particles is evaluated, in Eq. (3.4) as well as in forthcoming ones, in terms of the distance between the indices h and k of the respective activity classes. This is possible, and indeed customary, because the lattice (3.3) is uniformly spaced, so that, denoting the lattice's constant step by $\Delta u = \frac{2}{n-1}$, one obtains:

$$|h-k| = \frac{1}{\Delta u}|u_k - u_h|.$$

In other words, the index distance is directly proportional to the actual activity distance, and the constant factor can be duly rescaled. In general, if the activity lattice has a variable step then the correct way of evaluating the activity distance is via $|u_k - u_h|$ and the threshold $\gamma[\mathbf{f}]$ has to be thought of as being rescaled accordingly.

A more general model for the interaction rate is obtained by assuming that, within the same game regime, η_{hk} decreases with the distance between the activity classes of the interacting particles. For instance:

$$\eta_{hk}[\mathbf{f}] = \begin{cases} \eta_0\, e^{-a|k-h|} & \text{if } |k-h| \leq \gamma[\mathbf{f}] \quad \text{(competition)} \\ \mu\, \eta_0\, e^{-a|k-h|} & \text{if } |k-h| > \gamma[\mathbf{f}] \quad \text{(cooperation),} \end{cases}$$

where $a \geq 0$ is a parameter (for $a = 0$ one recovers Eq. (3.4) as a special case).

3.3.3 Table of Games

As mentioned in Sect. 3.2, in the absence of production and dissipation of wealth, cooperative and competitive stochastic games preserve the average wealth of the society; that is, the first order moment of the probability distribution is constant in time:

$$\mathbb{E}_1[\mathbf{f}](t) = \sum_{i=1}^{n} u_i f_i(t) = \text{constant in } t,$$

so that $\mathbb{E}_1[\mathbf{f}](t) = \mathbb{E}_1[\mathbf{f}](0)$ for all $t \in (0, T_{\max}]$. In order for the solutions of Eq. (3.1) to satisfy this property, further conditions, besides (3.2), must be imposed on the table of games. Computations with Eq. (3.1) verify that the following conditions are sufficient:

- Symmetric interaction rate: $\eta_{hk} = \eta_{kh}$ for all $h, k = 1, \ldots, n$ (notice that the interaction rates previously examined do indeed satisfy this condition).
- *Quasi-fair* stochastic games:

$$\sum_{i=1}^{n} u_i \mathscr{C}_{hk}[\mathbf{f}](i) = u_h + \sigma_{hk}, \quad \forall h, k = 1, \ldots, n, \tag{3.5}$$

where $\{\sigma_{hk}\}_{h,k=1,\ldots,n}$ is an antisymmetric tensor; i.e., $\sigma_{hk} = -\sigma_{kh}$ for all h, k.

Condition (3.5) has an inspiring interpretation in the context of game theory. The expected payoff of the candidate particle (that is, the wealth class it shifts to in average after the game (cf. the left-hand side)) is required not to differ substantially from the pre-interaction wealth class u_h (cf. the right-hand side). In detail, if Eq. (3.5) holds with $\sigma_{hk} = 0$ for all h and k, then the game described by the table $\{\mathscr{C}_{hk}(i)\}$ is perfectly fair, since active particles are not expected either to gain or lose, on average, from the interactions. If instead $\sigma_{hk} \neq 0$ then single games between specific candidate and field particles can be biased but the overall bias vanishes because

$$\sum_{h,k=1}^{n} \sigma_{hk} = 0.$$

Condition (3.5) can either be relaxed, in case of wealth-dissipative or wealth-productive interactions, or reinforced by also requiring the conservation of higher-order moments. An empirical rule may be that conservation has to be guaranteed for average quantities playing a role in the table of games.

After the above preliminary considerations, we now enter into the details of a possible modeling of the transition probabilities $\mathscr{C}_{hk}[\mathbf{f}](i)$ for the specific application at hand. A general guideline is that the main features of the interactions should be captured by limiting as much as possible the number of model parameters. One parameter is welcome for simulating different interaction regimes. More parameters need to be carefully related to well-defined and (one hopes) empirically quantifiable aspects of the interactions. Accordingly, we sketch a minimal table of games (taken from [39] with minor modifications) in which the number n of wealth classes is the only parameter. Next we discuss how a second parameter can be conveniently introduced.

Competitive Interactions

We begin by examining the case in which a candidate particle, with strategy u_h, and a field particle, with strategy u_k, play a competitive game. The condition for this to happen is:

$$|k-h| \leq \gamma[\mathbf{f}],$$

which means that the wealth classes they belong to are sufficiently close for their individual interests to conflict. We need to distinguish a few sub-cases.

1. Particles with the same state: $\boxed{h = k}$

$$\mathscr{C}_{hh}[\mathbf{f}](i) = \begin{cases} 1 & \text{if } i = h \\ 0 & \text{otherwise.} \end{cases}$$

This type of interaction does not modify the activity of the candidate particle, which, with unit probability, maintains the state u_h. The corresponding game is the trivial one in which players simply ignore each other.

2. Particles with different states: $\boxed{h \neq k}$, in particular:

2.1 Candidate particle at the boundary of the activity lattice: $\boxed{h = 1, n}$

$$\mathscr{C}_{1k}[\mathbf{f}](i) = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{otherwise,} \end{cases} \qquad \mathscr{C}_{nk}[\mathbf{f}](i) = \begin{cases} 1 & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases}$$

In this case the candidate particle also maintains, with unit probability, its pre-interaction state due to the lack of further lower or higher wealth classes that it can possibly access. Consider, for example, the case $h = 1$. The competitive dynamics would imply that the candidate particle yields to a necessarily wealthier ($k > 1$) field particle, which is impossible because no lower class than u_1 exists in the lattice I_u. An analogous argument holds in the case $h = n$, when the candidate particle should in principle get the better of a definitely poorer ($k < n$) field particle.

2.2 Candidate particle inside the activity lattice: $\boxed{1 < h < n}$, in particular:

2.2.1 Candidate particle poorer than the field particle: $\boxed{h < k}$

$$\boxed{k < n}, \quad \mathscr{C}_{hk}[\mathbf{f}](i) = \begin{cases} \alpha_{hk} & \text{if } i = h - 1 \\ 1 - \alpha_{hk} & \text{if } i = h \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{k = n}, \quad \mathscr{C}_{hn}[\mathbf{f}](i) = \begin{cases} 1 & \text{if } i = h \\ 0 & \text{otherwise.} \end{cases}$$

In this case, competitive dynamics are such that the candidate particle can lose one wealth class with probability $\alpha_{hk} \in (0, 1)$ or, at best, stay in the same pre-interaction class with the complementary probability $1 - \alpha_{hk}$. Nevertheless, if the field particle is in the highest class u_n then no competition takes place and the candidate particle simply remains, with unit probability, in its pre-interaction class. This is necessary in order for the average wealth to be conserved. Indeed, exchanging h with k, we see that the field particle would not have the possibility of a corresponding increase in its wealth class when playing the game from the candidate side.

The probability α_{hk} can be related in turn to the distance between the wealth classes of the interacting particles, in such a way that the larger the distance the stronger the competition. For instance:

$$\alpha_{hk} = \frac{|k-h|}{n-1}, \tag{3.6}$$

which depends only on the parameter n.

2.2.2 Candidate particle wealthier than the field particle: $\boxed{h > k}$

$$\boxed{k=1}, \quad \mathscr{C}_{h1}[\mathbf{f}](i) = \begin{cases} 1 & \text{if } i = h \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{k>1}, \quad \mathscr{C}_{hk}[\mathbf{f}](i) = \begin{cases} 1-\alpha_{hk} & \text{if } i = h \\ \alpha_{hk} & \text{if } i = h+1 \\ 0 & \text{otherwise.} \end{cases}$$

Game dynamics in this case are conceptually analogous to those of the previous case except for the fact that now competition can cause the candidate particle to gain a wealth class, still with probability α_{hk}. If, however, the field particle is in the lowest class u_1 then the candidate particle remains in its pre-interaction wealth class to ensure average wealth conservation (to show this, exchange h with k and repeat the same reasoning as before).

Cooperative Interactions

Candidate and field particles play a cooperative game if:

$$|k-h| > \gamma[\mathbf{f}];$$

that is, if their respective wealth classes are so far apart that their individual interests do not conflict. In this case, h and k are necessarily distinct; in addition, the extreme classes u_1 and u_n need not be treated separately, since game dynamics are such that the poorest and wealthiest particles can only enhance and worsen, respectively, their states. Therefore, only two sub-cases need to be distinguished.

3. Candidate particle poorer than the field particle: $\boxed{h < k}$

$$\mathscr{C}_{hk}[\mathbf{f}](i) = \begin{cases} 1-\alpha_{hk} & \text{if } i = h \\ \alpha_{hk} & \text{if } i = h+1 \\ 0 & \text{otherwise.} \end{cases}$$

   In this case, the candidate particle can gain a wealth class with probability α_{hk} increasing with the distance $|k-h|$ (cf. Eq. (3.6)), thus implying stronger cooperation between distant particles. The other possible output of the game is that the candidate particle stays in its pre-interaction class with complementary probability $1-\alpha_{hk}$.

4. Candidate particle wealthier than the field particle: $\boxed{h > k}$

$$\mathscr{C}_{hk}[\mathbf{f}](i) = \begin{cases} \alpha_{hk} & \text{if } i = h-1 \\ 1-\alpha_{hk} & \text{if } i = h \\ 0 & \text{otherwise.} \end{cases}$$

   In this case, the candidate particle can lose a wealth class with the same probability α_{hk} as that by which it can gain one in the case above. This balance is consistent with the conservation of average wealth.

We conclude this part by recording a possible generalization of the probability α_{hk} (cf. Eq. (3.6)):

$$\alpha_{hk} = \varepsilon \frac{|k-h|}{n-1}, \tag{3.7}$$

where $\varepsilon \in (0, 1)$ is a constant playing the role of a genuine parameter featured by the table of games. Different game scenarios can then be simulated by duly tuning the parameter ε.

3.3.4 Critical Threshold

The expression of both the interaction rate and the transition probabilities calls for the definition of the critical threshold $\gamma[\mathbf{f}]$, which discriminates between the cooperative and competitive game regimes. Probably the simplest option is to take $\gamma[\mathbf{f}]$ constant with respect to $\mathbf{f}$; cf. [48]. In this case, the resulting model features linearly additive interactions. A more refined alternative is to allow for a non-constant $\gamma[\mathbf{f}]$ linked to the evolution of the socio-economic conditions of the society under consideration; cf. [39]. As we will see in a moment, this implies dealing with nonlinearly additive interactions.

Let us introduce the (normalized) numbers N^-, N^+ of poor and wealthy particles at a given time t:

$$N^-[\mathbf{f}](t) := \sum_{i=1}^{\frac{n-1}{2}} f_i(t), \qquad N^+[\mathbf{f}](t) := \sum_{i=\frac{n+3}{2}}^{n} f_i(t),$$

which we use to define the following measure of the *social gap*:

$$S[\mathbf{f}] := N^-[\mathbf{f}] - N^+[\mathbf{f}]. \tag{3.8}$$

This quantity can be understood as a macroscopic thermometer of the social tension due to economic issues: the higher $S[\mathbf{f}]$ the larger the number of poor active particles, which can result in a stronger social conflict because poor individuals tend to be involved in a "battle of the have-nots". Conversely, the lower $S[\mathbf{f}]$ the larger the number of wealthy active particles, which typically induces low levels of social conflict because wealthy individuals tend preferentially to preserve their common benefits. The critical threshold $\gamma[\mathbf{f}]$ can therefore depend on $S[\mathbf{f}]$ according to the idea that the larger the social gap, namely the social tension, the stronger the competitive behavior of the individuals (large $\gamma[\mathbf{f}]$) and vice versa.

Technically, since $S[\mathbf{f}]$ is bounded between -1 and 1 due to $0 \le N^{\pm}[\mathbf{f}] \le 1$ and $N^-[\mathbf{f}] + N^+[\mathbf{f}] \le 1$ for all $\mathbf{f}$, the following model for $\gamma[\mathbf{f}]$ is proposed; cf. [39]:

$$\gamma[\mathbf{f}] = \left\lfloor \frac{2\gamma_0(S[\mathbf{f}]^2 - 1) - n(S_0 + 1)(S[\mathbf{f}]^2 - S_0)}{2(S_0^2 - 1)} + \frac{n}{2} S[\mathbf{f}] \right\rfloor, \tag{3.9}$$

where $\lfloor \cdot \rfloor$ denotes the integer part (floor). It stems from the following analytical assumptions inspired by the previous qualitative discussion.

- The dependence of $\gamma[\mathbf{f}]$ on $S[\mathbf{f}]$ is polynomial of second order except for the integer part, which is taken *a posteriori* considering that only integer values of $\gamma[\mathbf{f}]$ are meaningful since the wealth class distance $|k-h|$ is an integer. A second order polynomial is chosen as its three degrees of freedom allow for the three conditions below to be satisfied.
- $\gamma[\mathbf{f}] = n$ for $S[\mathbf{f}] = 1$, in such a way that when the society consists only of non-wealthy individuals ($N^-[\mathbf{f}] = 1$, $N^+[\mathbf{f}] = 0$) the interaction dynamics are of full competition (in fact $|k-h| \le n$ all h, k).
- $\gamma[\mathbf{f}] = 0$ for $S[\mathbf{f}] = -1$, so that, conversely, when the society consists only of wealthy individuals ($N^-[\mathbf{f}] = 0$, $N^+[\mathbf{f}] = 1$) the interaction dynamics are of full cooperation.
- $\gamma[\mathbf{f}] = \gamma_0$ for $S[\mathbf{f}] = S_0 \in (-1, 1)$, the latter being the initial value of S fixed by the initial distribution $\mathbf{f}(t=0)$ of active particles over the various wealth classes and $\gamma_0 \in \{1, 2, \ldots, n-1\}$ the corresponding critical threshold which fixes a reference value for $\gamma[\mathbf{f}]$.

Model (3.9) is, of course, not the only conceivable one for the critical threshold. The evolution of $\gamma[\mathbf{f}]$ can be linked to average quantities different from $S[\mathbf{f}]$, which provide additional or complementary information about the society's economic bias. For instance:

$$\mathscr{S}_\ell[\mathbf{f}] := \sum_{i=1}^{\frac{n-1}{2}} (-u_i)^\ell f_i(t) - \sum_{i=\frac{n+3}{2}}^{n} u_i^\ell f_i(t),$$

which for $\ell = 0$ coincides with $S[\mathbf{f}]$ whereas for $\ell > 0$ is a measure of the distortion of the actual wealth distribution with respect to a balanced symmetric one.

We finally observe that a constant critical threshold, say coinciding with the reference value γ_0, can be interpreted as the action of a government aimed at imposing a certain welfare policy with controlled levels of cooperation and competition. Within such a perspective, a variable critical threshold means instead that there is no control by the government on the socio-economic dynamics, which can allow the wealthiest social classes to impose selfish market rules. We will come back to these two scenarios in Chap. 4 by investigating, through numerical simulations, their respective effects on the socio-economic profile of the simulated society.

Remark 3.2. The critical threshold can also be understood as a measure of how fair (or unfair) the behavior of wealthy classes is. This delicate issue is the subject of recent speculations [135], which aim at understanding to what extent unethical behaviors can be sources of wealth.

3.4 Technical Improvements

The model presented in the previous sections can be further developed and improved in view of its applications to different social dynamics, such as, for example, competition for a secession [7], learning processes [72], and opinion formation [152], possibly over networks. Some of them were already mentioned in Chap. 2.

In this section we discuss, in particular, the following issues:

- Additional sources of nonlinearity, besides the non-constant critical threshold, in the interaction dynamics.
- Models with several interacting functional subsystems.
- Models with continuous activity of the particles.

The reference framework is still a spatially homogeneous closed system, meaning that the microscopic state of the individuals is fully characterized by the variable u alone and cannot be perturbed by any action of external field agents.

3.4.1 Additional Sources of Nonlinearity in the Interactions

The table of games illustrated in Sect. 3.3 features nonlinearly additive interactions due to the dependence of γ on the probability distribution $\mathbf{f}$ through $S[\mathbf{f}]$. An additional nonlinearity is produced by assuming that active particles do not only play binary games with one another but are also involved in a game with the *stream*, namely a certain mean trend of the system. Such dynamics imply a bias of the transition probabilities by some average quantities, for instance the mean value of the wealth, which can give rise to different game outputs, or even to different game regimes, depending on the states of the interacting pairs with respect to the collective mean trend. As an example, still remaining in the context of socio-economic systems, the effect of competition between two active particles within threshold (i.e., such that $|k-h| \leq \gamma[\mathbf{f}]$) may be attenuated or enhanced if the particles are both wealthier or both poorer, respectively, than the average population.

Instead, in problems of opinion formation the stream effect may consist of a trend of the individuals toward the average collective opinion, which modifies the output of their one-to-one interactions.

It is worth mentioning that the stream effect can also be introduced into the equations as a mean field action applied directly on the test particle by a certain number of (possibly all) other field particles. The corresponding mathematical structure is similar to Eqs. (2.9)–(2.10) except for the fact that the term K is computed using the distribution function of the active particles themselves rather than that of field agents. Nevertheless, this approach is analytically feasible only in the case of a continuous activity variable, because the derivative with respect to u produced by the mean field flux is not directly compatible with nontrivial (weak) solutions containing Dirac distributions in u.

Other nonlinearities can be induced by the assumption that active particles do not interact with all other particles of the system but only with a subset of them, corresponding to the idea that individuals can retain and process only a finite maximum amount of information at a time. Each test particle thus has an *interaction neighborhood*, say a ball $B_R(u_i) = \{u_k \in I_u : |u_k - u_i| \leq R\}$ centered in its activity class u_i and with variable radius $R = R[\mathbf{f}](t, u_i) > 0$, which encompasses the (possibly normalized) maximum number, say $N_{\max}$, of other particles it can interact with simultaneously:

$$R[\mathbf{f}](t, u_i) = \min \left\{ r > 0 : \sum_{k : |u_k - u_i| \leq r} f_k(t) \geq N_{\max} \right\}.$$

In this case one speaks of *topological interactions*, i.e., interactions with a fixed number of group mates independent of the group density and of the distance between the interacting states. As the formula above demonstrates, this implies a dependence of the interaction neighborhood $B_R(u_i)$ on the distribution functions of the active particles. The prototype of such peculiar dynamics, which has already received some attention from the mathematical modeling side [43], is the flock behavior observed in swarms of starlings [25].

3.4.2 Models with Several Interacting Functional Subsystems

Various social groups can interact in the same population under the same social rules imposed by a common government. However, they can develop different ways to react to such rules. This suggests that the population can be conveniently partitioned into functional subsystems in which the activity variable is possibly the same but interactions produce different effects according to specific factors of each subsystem. The partition can be inspired by various elements, such as, among others, ethnic or religious differences or the various work expertise of the active particles (e.g., farmers, workers, employees, intellectuals).

The corresponding mathematical structure, still in the framework of conservative interactions considered in this chapter, is a special case of Eq. (2.13), that we report here for completeness:

$$\frac{df_i^p}{dt} = \sum_{p_*,p^*=1}^{m} \sum_{h,k=1}^{n} \eta_{hk}^{p_*p^*}[\mathbf{f}]\mathscr{C}_{hk}^{p_*p^*}[\mathbf{f}](i,p) f_h^{p_*} f_k^{p^*} - f_i^p \sum_{p^*=1}^{m} \sum_{k=1}^{n} \eta_{ik}^{pp^*}[\mathbf{f}] f_k^{p^*}$$

for $p = 1, \ldots, m$.

Now both the interaction rate and the transition probabilities also depend in principle on the pair of interacting subsystems. For instance, the former can depend on the distance (in norm) of the respective distribution functions, which says how similar or dissimilar the social profiles are in the two subsystems. Conversely, the latter can account for the fact that cooperation and competition may prevail in certain functional subsystems simply because of a more or less aggressive way of their members to express their behavioral strategy. As an example, one may consider that some categories of workers are defended in a more or less aggressive way by their representative labor unions.

The modeling approach can provide a description of the trends shown by each subsystem and, consequently, indications for correcting governing policies when necessary. Collective dynamics can also identify interesting emerging trends, such as the predominance of some functional subsystems over others or the emergence/disappearance of groups of interest.

3.4.3 Models with Continuous Activity Variable

The generalization of the model presented in this chapter to the case of continuous activity variable can take advantage of Eq. (2.1), with the interaction operator on the right side containing conservative contributions only; cf. Eq. (2.2). With one functional subsystem, the model equation can be written as:

$$\frac{\partial f}{\partial t}(t,u) = \int_{-1}^{1}\int_{-1}^{1} \eta[\mathbf{f}](u_*,u^*)\mathscr{C}[\mathbf{f}](u_* \to u|u_*,u^*)f(t,u_*)f(t,u^*)\,du_*\,du^*$$
$$- f(t,u)\int_{-1}^{1} \eta[\mathbf{f}](u,u^*)f(t,u^*)\,du^*.$$

The social gap S is computed as:

$$S[\mathbf{f}](t) = \int_{-1}^{0} f(t,u)\,du - \int_{0}^{1} f(t,u)\,du$$

and, similarly, higher order social distortion measures can be written as:

$$\mathscr{S}_\ell[\mathbf{f}](t) = \int_{-1}^{0} (-u)^\ell f(t,u)\,du - \int_{0}^{1} u^\ell f(t,u)\,du.$$

The derivation of models consists in looking for appropriate expressions of the terms η and $\mathscr{C}$, for which conceptual lines analogous to those of Sect. 3.3 can be followed. In particular, the continuous-activity counterparts of conditions (3.2) and (3.5) are, respectively:

$$\int_{-1}^{1} \mathscr{C}[\mathbf{f}](u_* \to u|u_*,u^*)\,du = 1$$

and

$$\int_{-1}^{1} u\mathscr{C}[\mathbf{f}](u_* \to u|u_*,u^*)\,du = u_* + \sigma(u_*,u^*)$$

with antisymmetric σ; i.e., $\sigma(u_*,u^*) = -\sigma(u^*,u_*)$, for all $u_*, u^* \in [-1, 1]$.

3.5 Critical Analysis

The mathematical model presented in this chapter has been derived, according to the guidelines of Chap. 2, by inserting models of the microscopic interaction dynamics into the general mathematical structure designed as a conceptual basis.

It can be rapidly shown, with tutorial aims, that the model can describe different asymptotic trends of the wealth distribution as a consequence of different critical thresholds γ. For instance, Fig. 3.3 shows the asymptotic configurations reached with $n = 9$ social classes interacting with a constant critical threshold that is either small ($\gamma = 2$) or relatively large ($\gamma = 6$). The starting configuration is the uniform one over the wealth classes; i.e., $f_{0i} = f_i(t = 0) = \frac{1}{9}$ for all $i = 1, \ldots, 9$. In the first

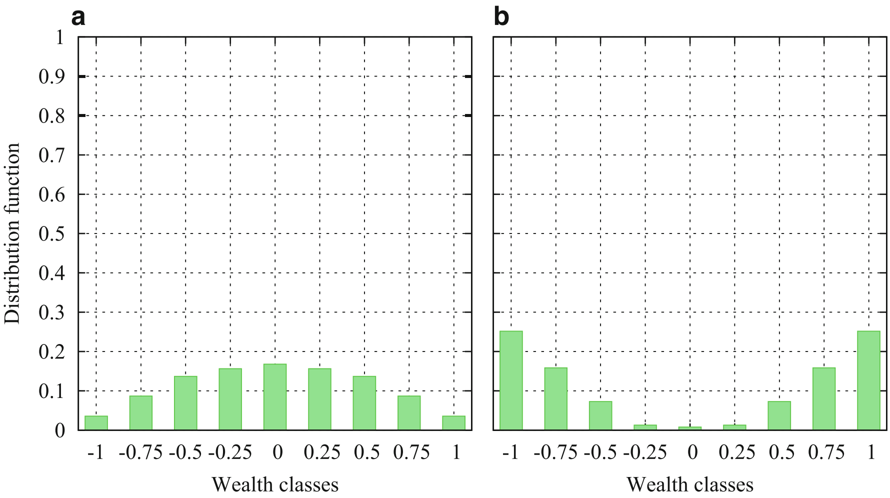

Fig. 3.3 Asymptotic configurations of the wealth distribution predicted by the model in the cases of: (**a**) strong social cooperation, and (**b**) strong social competition

case the model predicts that the society evolves toward a middle-class-dominated configuration, due to cooperation. In the second case the model predicts instead that it evolves toward a radicalization of the competition, a hallmark of which is the almost complete absence of any middle class. Figure 3.4 additionally shows the influence of initial conditions on asymptotic behavior (for a fixed constant critical threshold $\gamma = 3$). Of course, these simulations have to be considered only as a preliminary illustrative step, which will be followed by a deeper investigation in the next chapter.

In particular, looking ahead at the numerical simulations that will be developed in Chap. 4, we feel confident in claiming that a careful interpretation of the results of targeted simulations can contribute to additional refinements of models, especially as far as the description of the interactions at the microscopic scale is concerned.

The basic idea is that the predictive ability of models can be improved by further detailing and/or enriching the mathematical description of the game dynamics of the active particles. On the contrary, it cannot be naively enhanced by increasing the number of phenomenological parameters of the equations, for this procedure simply leads to what could be called a "fitting ability" of the models. On the contrary, in the specific field considered in this monograph, the validity of a model should be assessed in terms of its ability to predict emerging collective trends, qualitatively observed in real conditions, from the characterization of individual behaviors. This is, in our opinion, the conceptually correct way to capture the complexity of a system by a mathematical model. Such a consideration does not only apply to "stable" trends, namely those that recur under similar circumstances, but also to trends susceptible of large deviations in consequence of even small changes in the causes that generate them, up to extremely rare events hardly predictable until they arise for the first time.

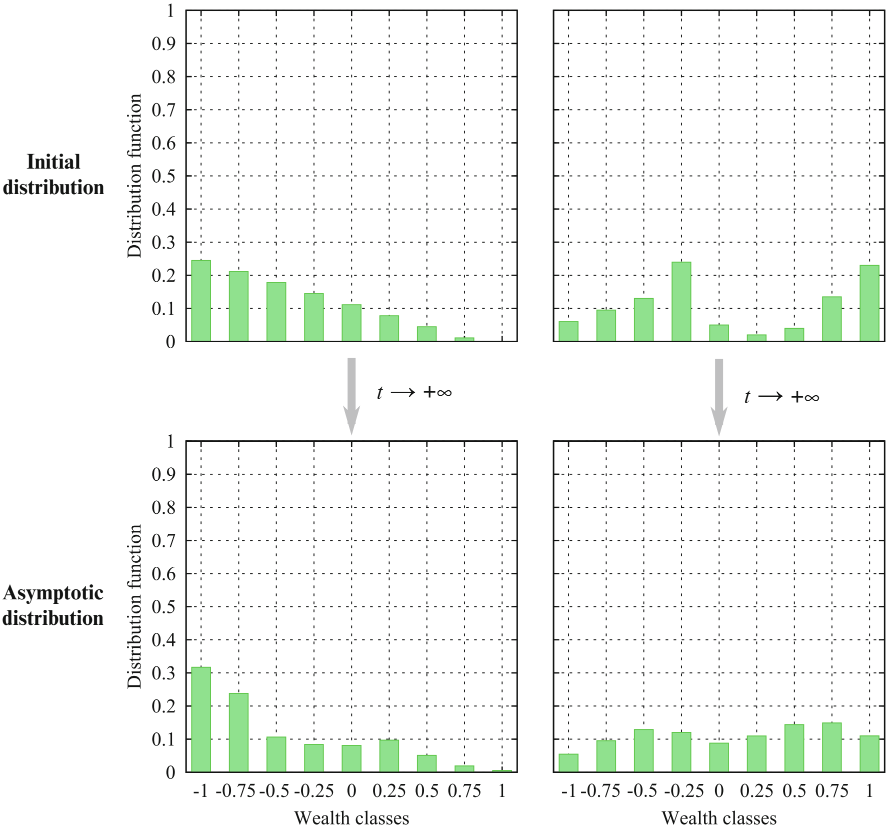

Fig. 3.4 Effect of initial conditions on the asymptotic distributions predicted by the model

Probably the most complex collective trends are those emerging in *panic conditions*. The reason is that panic appears suddenly and spontaneously, often as a consequence of barely conceivable events. In addition, once widespread panic occurs it usually entails dramatic changes in the microscopic interaction rules followed by active particles, which typically are no longer inspired by fully rational principles. In order to construct models of these phenomena, stochastic game theory can be a valuable alternative (however still in need of systematic development) to classical game theory, which is essentially grounded on the assumption of rational players.

Panic conditions can also be the basis for the onset of rare events known by the evocative name of *black swans* [149]. A challenging goal of mathematical models is to predict, at least qualitatively, the conditions for their emergence, as well as to identify suitable early-identification signals (also called *tips* in the specialized literature) preceding their full development. Nevertheless, as shown in [39], such kinds of predictions are mostly possible if the modeled microscopic games are sufficiently rich to allow for self-enhanced effects on an individual behavioral basis. This implies considering multi-strategy games with related interplays among the various strategies expressed by active particles (such as, for example, their wealth status and their support/opposition to government welfare policy). We defer to Chap. 5 a more in-depth discussion of this topic.

Chapter 4
Welfare Policy: Applications and Simulations

Abstract This chapter is devoted to the investigation, through targeted numerical experiments, of various social scenarios predicted by the model presented in Chap. 3 in consequence of different simulated welfare policies. Qualitative simulations are developed with a mainly exploratory purpose, especially in order to test the ability of the model to account for the emergence of nontrivial collective average trends out of the probabilistic description of microscopic individual interactions. To this aim, a parameter sensitivity analysis is performed, which guides the organization of the simulations and the critical assessment of their results.

4.1 Introduction

Numerical simulations can serve either quantitative or qualitative purposes. An example of their use in the first case is as tools to support engineering design in industrial practice, often with reference to consolidated models of well-understood physical phenomena (such as, for example, fluid dynamics or solid mechanics). In the second case they directly support the theoretical development of new mathematical models, by enabling one to test the predictive ability of the latter with respect to their reference applications. Comparisons with experimental data, when available, can also be made; however, this is typically done mainly for contributing to the validation of models rather than for pursuing immediate practical implications. In particular, when dealing with complex systems the qualitative exploratory role is often the major one in numerical simulations, especially as far as their ability to simulate emerging collective behaviors is concerned. Indeed, this is an important indication of the validity of the models.

Emerging behaviors in living complex systems are the macroscopic outcome of often quite simple behavioral rules applied individually by group members when interacting with one another and with the outer environment. The effects of such rules are then amplified by the superposition of numerous interaction events taking place simultaneously all over the group, which lead to characteristic

G. Ajmone Marsan et al., *Complex Systems and Society: Modeling and Simulation*, SpringerBriefs in Mathematics, DOI 10.1007/978-1-4614-7242-1_4,

patterns (in the space of the microscopic states of the individuals) clearly visible by adopting an aggregate (viz., collective) point of view. Heuristic interpretations of such outcomes do not generally provide a satisfactory explanation for the way the studied system behaves. Indeed, as should be clear by now, the real causes pertain to the microscopic scale of single individuals: as such, they are neither directly visible nor often immediately recognizable at large scales, since collective behaviors can be qualitatively much different from the causes that generated them.

In addition, the aforesaid amplification effect produced by interactions results, in some cases, in large deviations in observable effects even starting from nearly the same causes. Mathematical models, duly supported by exploratory numerical simulations, can therefore be valuable tools for unraveling the tangled network of interactions and testing the real cause-effect links, thereby shedding light on the essential dynamics of the system.

As a matter of fact, emerging behaviors can rarely be predicted by purely analytical methods. One such case concerns collective *clusterings*; i.e., when the states of all individuals shrink in number to a unique one or to a small set of limit states. Examples are, among others, the *chemotactic aggregation* of cell populations, the *rendez-vous* of agents such as robots in coordinated movement, the *consensus* of individuals in opinion formation problems, and the *flocking* of swarms.

When emerging behaviors give rise to more complicated self-organized patterns, precise analytic characterizations of the latter may no longer be feasible. In this case, targeted qualitative simulations play a major role in assessing the validity and the potential of mathematical models. Actually, most emerging behaviors in living complex systems are of this second type. We recall, for instance, the evolving shapes of swarms of birds during collective migrations or during the attack of predators [25, 62]; lane formation and the oscillatory patterns at a bottleneck in pedestrian counter-flows [98]; and the alternate passage of clusters of vehicles at unregulated crossroads [99]. For such applications, mathematical approaches grounded on the concept of complex systems, and particularly focused on the relationship between small and large scale effects [42, 69, 70, 134], have already proved to be successful in explaining the spontaneous emergence of typical self-organized patterns.

Numerical simulations can also serve speculative purposes. For instance, they can indicate new pathways of experimental investigation when they result in evidence of emerging behaviors not yet empirically observed. New experiments can then be designed to confirm or reject the speculative assumptions of mathematical models. Such a bidirectional feedback between experimenting and simulating can profitably contribute to compose the big picture, particularly in the case of incomplete, or even missing, empirical knowledge. A useful reference for the empirical interpretation of the results of numerical simulations is [128].

In the spirit of the considerations above, this chapter is devoted to numerical simulations of the model of welfare policy introduced in Chap. 3, with special emphasis on a parameter sensitivity analysis. The core of the chapter comprises the next two sections. In more detail, Sect. 4.2 discusses the specific type of sensitivity analysis to be developed for each of the main ingredients of the model, and Sect. 4.3

accordingly offers a variety of targeted simulations followed by their interpretation. Finally, Sect. 4.4 briefly sketches possible ideas for improving the model in light of the findings of the preceding sections.

4.2 Brainstorming Toward Parameter Sensitivity Analysis

The model of welfare dynamics presented in Chap. 3 is characterized by a few phenomenological parameters concerning the initial wealth status of the population, the critical threshold discriminating between cooperative and competitive behaviors, and the interaction rate and transition probabilities contained in the table of games. All of them have, in principle, a relevant impact on the qualitative behavior of the solutions of the model. It is worth stressing again that, in designing the model, constant attention has been devoted to keeping the total number of parameters as small as possible (one parameter only for each effect accounted for by the equations), which now turns out to be advantageous for performing a parameter sensitivity analysis.

In the following, we examine in detail the parameters featured by the aforementioned terms of the equations as a preliminary step toward the organization of targeted numerical simulations. In particular, we focus on the role played by the initial wealth status and the critical threshold, which, as we will see, are amenable to a more direct interpretation in terms of social status of the population and welfare policy of the government.

4.2.1 Initial Wealth Status

The initial wealth status of the population is described by the set of distribution functions $\mathbf{f} = \{f_i\}_{i=1}^n$ at time $t = 0$, which has to be prescribed as an initial condition for Eq. (3.1). An important parameter characterizing such a distribution is the *average wealth*, hereafter denoted by U_0:

$$U_0 := \mathbb{E}_1[\mathbf{f}](0) = \sum_{i=1}^{n} u_i f_i(0).$$

Notice that $-1 \leq U_0 \leq 1$. Owing to condition (3.5), this quantity is conserved in time, i.e.,

$$\sum_{i=1}^{n} u_i f_i(t) = U_0, \qquad \forall t \in (0, T_{\max}].$$

The asymptotic scenarios predicted by the model are expected to be sensitive to the value of U_0, because societies that are poor or wealthy (on average) have different interpretations of cooperation and competition among individuals, especially in the absence of an externally imposed strong welfare policy. Nevertheless, it can be observed that U_0 does not completely characterize the initial distribution. Indeed,

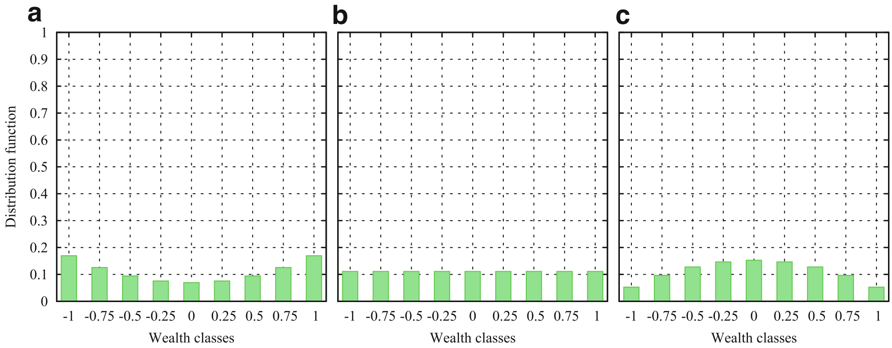

Fig. 4.1 Three possible profiles of wealth distribution featuring the same value of average wealth $U_0 = 0$ but decreasing variance. (**a**) Predominance of extreme wealth classes, $\mathrm{Var}[\mathbf{f}](t=0) \sim 0.54$. (**b**) Uniform distribution, $\mathrm{Var}[\mathbf{f}](t=0) \sim 0.42$. (**c**) Predominance of middle wealth classes, $\mathrm{Var}[\mathbf{f}](t=0) \sim 0.3$

the same initial average wealth can be obtained with different distribution profiles as Fig. 4.1 demonstrates. In order to also take into account the bias of the initial distribution, higher order statistical quantities might be used, such as the variance:

$$\mathrm{Var}[\mathbf{f}](t=0) := \mathbb{E}_2[\mathbf{f}](0) - (\mathbb{E}_1[\mathbf{f}](0))^2 = \sum_{i=1}^{n} u_i^2 f_i(0) - U_0^2$$

or even the third order moment, which gives information about the distortion of the distribution with respect to a symmetric balanced one. However, higher order moments different from U_0 are, in general, *not* conserved in time, which makes them questionable as synthetic indicators for classifying different case studies. Their conservation could be enforced by further constraints on the table of games, which however would make the modeling of the latter perhaps too artificial and difficult to link to clear empirical facts. Therefore, we will essentially rely on U_0 for synthetically characterizing the initial wealth status and the evolution at future times it gives rise to.

4.2.2 Controlled vs. Free Social Competition

The evolution of the system for prescribed initial conditions depends on the alternation of cooperative and competitive games played by active particles.

Fig. 4.2 (continued) ($U_0 = -0.1$) about 40 % of active particles end up in the two lowest wealth classes ($u_1 = -1$, $u_2 = -0.75$) and the social gap tends to the asymptotic value $S_\infty = 0.098 > 0$ (predominance of poor individuals). Conversely, in a society that is wealthy on average ($U_0 = 0.1$) the symmetric scenario arises, with about 40 % of active particles ending up in the two highest wealth classes ($u_8 = 0.75$, $u_9 = 1$) and the social gap tending invariably to the asymptotic value $S_\infty = -0.098 < 0$ (predominance of wealthy individuals)

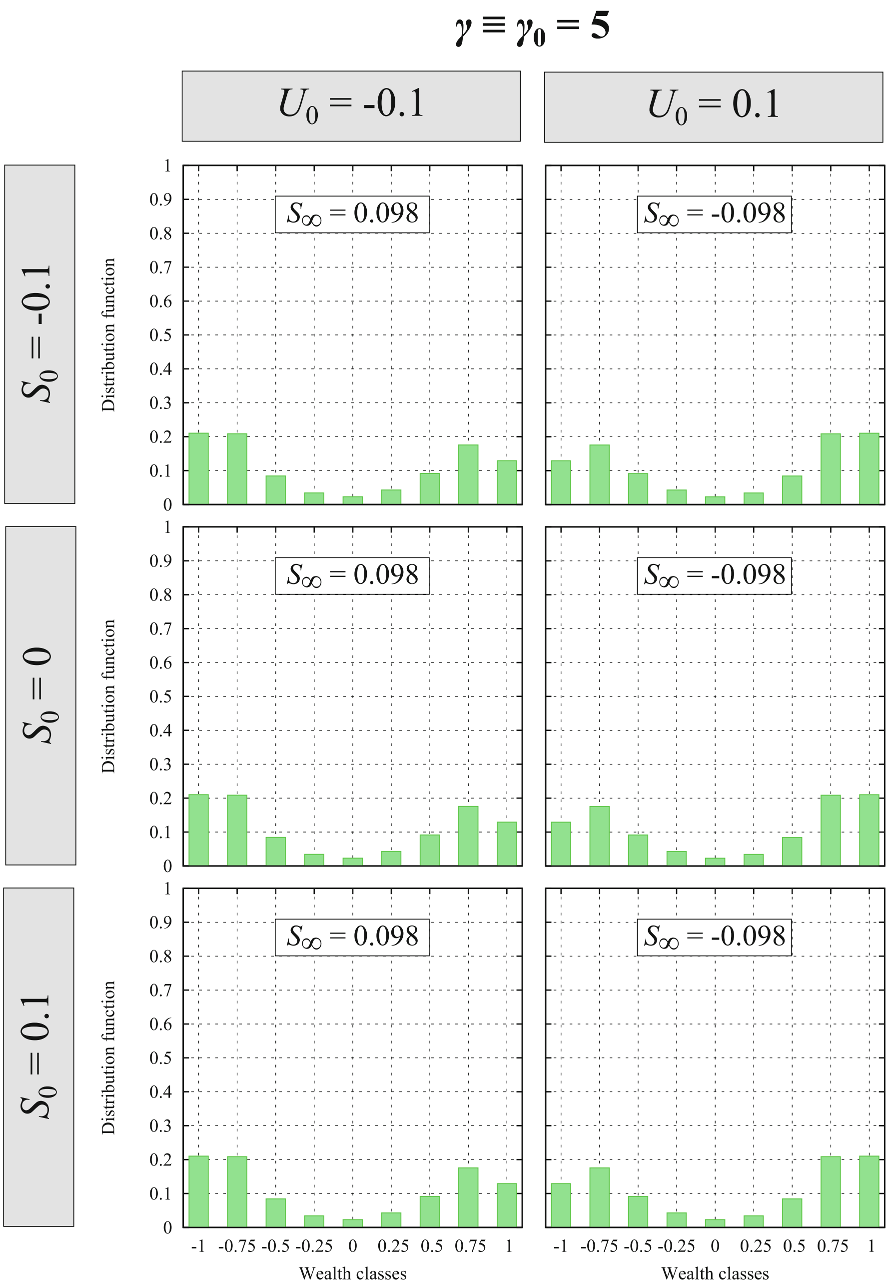

Fig. 4.2 Asymptotic profiles of the wealth distribution with a fixed critical threshold $\gamma \equiv \gamma_0 = 5$. In all of the considered cases, the shape of the final distribution is independent of the initial bias (as determined by the sign of the initial social gap S_0), being only affected by the average wealth U_0. In general, due to the moderately competitive welfare policy imposed by $\gamma_0 = 5$, active particles tend to concentrate in the extreme classes ($u_1 = -1$, $u_2 = -0.75$, $u_3 = -0.5$ on the one hand, and $u_7 = 0.5$, $u_8 = 0.75$, $u_9 = 1$ on the other hand). In particular, in a society that is poor on average

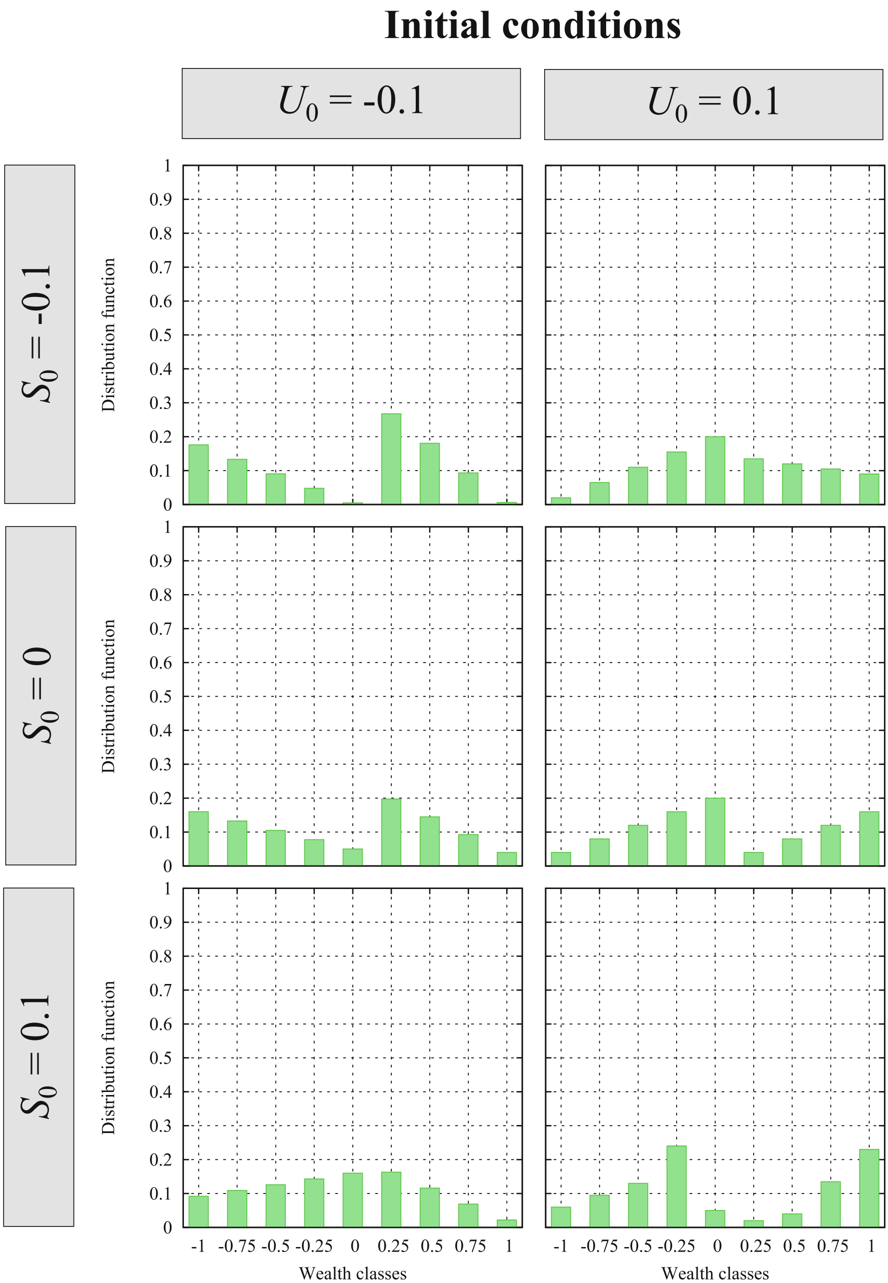

Fig. 4.3 Wealth distributions at the initial time $t = 0$ used for the various simulations presented in this chapter. Two cases are considered: that of a society that is poor on average ($U_0 = -0.1$) and that of a society that is wealthy on average ($U_0 = 0.1$), respectively. For each of them, three representative scenarios are studied, corresponding to different initial biases of the wealth distribution: predominance of wealthy individuals (negative initial social gap $S_0 = -0.1$), balance

The discriminating parameter for such game regimes is the critical threshold γ, which can either be constant or evolve with the system according to a properly designed model such as the one given by Eq. (3.9).

A constant threshold γ can be interpreted as a control (operated, for example, by the government) on the level of admissible social competition according to a precise welfare policy. For instance, it can be argued that a poor society needs a mainly cooperative welfare policy in order to avoid an increase of poor individuals, whereas a wealthier society can sustain higher levels of competition, thereby allowing individuals to take more care of their individual interests. Conversely, a non-constant threshold $\gamma = \gamma[\mathbf{f}]$, which varies on the basis of the instantaneous distribution of the system, can correspond to a regime of mainly free (viz. self-determined) social competition, due, for example, to weak action by the central government.

Numerical simulations are expected to provide perspectives on the different possible evolutions in the two cases. In particular, with reference to Eq. (3.9), interesting observations can be made by fixing an initial condition, which features a certain average wealth U_0 and a social gap S_0, and then comparing the asymptotic configurations (if any) reached for a fixed threshold $\gamma \equiv \gamma_0$ (namely, the reference value corresponding to the given social gap S_0), and for a non-constant $\gamma[\mathbf{f}]$ starting from γ_0.

It is worth noticing that weak government control is not the only interpretation for a non-constant critical threshold. In fact, different concomitant causes, such as for instance ethical implications, can break or promote social cooperation even in a basically controlled welfare regime; see, for example, [101, 135]. Of course, model (3.9) for γ does not pretend to be as accurate as realistic modeling of such phenomena would require. Nonetheless, in the present simplified context it can give useful preliminary hints to be properly developed within targeted research programs or specific applied studies.

4.3 Numerical Simulations of Selected Case Studies

Simulations are obtained by numerically solving the system of equations (3.1) duly supplemented by initial conditions:

$$f_i(0) = f_{0i}, \qquad i = 1, \dots, n,$$

Fig. 4.3 (continued) between wealthy and poor individuals (zero initial social gap $S_0 = 0$), and predominance of poor individuals (positive initial social gap $S_0 = 0.1$). Notice that the initial distributions corresponding to the pairs $(U_0, S_0) = (-0.1, 0.1)$ and $(U_0, S_0) = (0.1, -0.1)$ have quite a smooth profile, whereas the others feature sharp transitions across the middle class $u_5 = 0$. This qualitative difference can be understood by considering that the opposite signs of U_0 and S_0 imply either a poor society with predominance of poor individuals $(U_0 < 0, S_0 > 0)$ or a wealthy society with predominance of wealthy individuals $(U_0 > 0, S_0 < 0)$. Hence, in these cases the social gap is not a contrast to the average wealth status, whereas it is in all other cases where the f_{0i}'s have to be chosen subject to the conditions:

$$0 \leq f_{0i} \leq 1, \quad \forall i = 1, \ldots, n$$
$$\sum_{i=1}^{n} f_{0i} = 1.$$

Each f_{0i} represents the initial percentage of active particles in the i-th wealth class. Alternatively, one can think of f_{0i} as the probability that the test particle is initially in the i-th class. The resulting initial-value problem is well-posed, and is also well-posed in the case of nonlinearly additive interactions; that is, the solution $\{f_i(t)\}_{i=1}^n$ exists and is unique for all $t \in (0, +\infty)$, and moreover it depends continuously on the initial data. The proof of these claims relies on fixed point arguments in suitable Banach spaces. In particular, the proof makes use of the conservation of the zeroth and first-order moments implied by properties (3.2), (3.5). The interested reader is referred to [19] for technical details.

Equation (3.1) can be easily discretized by any of the standard computational schemes for ordinary differential equations, such as Runge-Kutta schemes or even the simpler explicit Euler scheme:

$$f_i^{\mathrm{j}+1} = f_i^{\mathrm{j}} + \Delta t \left(\sum_{h,k=1}^{n} \eta_{hk}[\mathbf{f}^{\mathrm{j}}] \mathscr{C}_{hk}[\mathbf{f}^{\mathrm{j}}](i) f_h^{\mathrm{j}} f_k^{\mathrm{j}} - f_i^{\mathrm{j}} \sum_{k=1}^{n} \eta_{ik}[\mathbf{f}^{\mathrm{j}}] f_k^{\mathrm{j}} \right), \quad \mathrm{j} = 0, 1, 2, \ldots,$$

which imposes tighter constraints on the time step Δt for reaching comparable accuracy but whose practical implementation is definitely more immediate.

In the following simulations, a few parameters, which will not be the subject of the sensitivity analysis, are fixed once and for all. They are:

- The number of wealth classes (cf. Eq. (3.3)), which is set to $n = 9$ (thus the middle neutral class is $u_5 = 0$).
- The coefficients of the interaction rate as in Eq. (3.4). In particular, $\mu = 0.3$ is chosen whereas η_0 is hidden in the time scale by a proper rescaling of the variable t (technically, η_0 can be tuned so as to speed up the convergence of the solution to possible steady states).
- The fundamental transition probability α_{hk} appearing in the table of games, which is taken as in Eq. (3.6), thereby being fully defined by the aforementioned parameters.

4.3.1 Influence of the Initial Condition

This set of simulations aims at testing the influence of different initial profiles of the distribution of active particles on the collective trend exhibited by the system. For such a purpose, the critical threshold γ is fixed to the following value:

$$\gamma \equiv \gamma_0 = 5$$

and will not be varied, considering that the interplay between initial conditions and γ, either constant or non-constant, will be specifically analyzed later.

The simulated scenarios, which will also serve as a reference for the following case studies, are classified in terms of the average wealth U_0 and the initial social gap S_0. Specifically, the following cases are considered:

(I) Poor society with $U_0 < 0$.
(II) Wealthy society with $U_0 > 0$.

and for each of them the following sub-cases are investigated:

(i) Initial prevalence of wealthy individuals, $S_0 < 0$ (recall the definition of S given by Eq. (3.8)).
(ii) Initial balance between wealthy and poor individuals, $S_0 = 0$.
(iii) Initial prevalence of poor individuals, $S_0 > 0$,

The sub-cases allow one to test the effect of different initial profiles for the same initial average wealth.

Figure 4.2 shows the configurations of the distribution of active particles over the wealth classes reached asymptotically in time, starting from the initial conditions depicted in Fig. 4.3. Essential remarks about the observed trends are reported in the caption, whereas general comparative comments are deferred to the last part of this section.

4.3.2 Influence of a Constant Critical Threshold

As already stated, a constant critical threshold γ can be regarded as an expression of government policy, which regulates social welfare dynamics by means of targeted actions such as passing specific laws. Although the parameter γ provides only a very limited representation of such issues, the analysis of its influence on the asymptotic trend of the wealth distribution is useful for assessing the descriptive/predictive ability of the model.

Simulations are therefore now developed by choosing the following two constant values of the critical threshold:

$$\gamma \equiv \gamma_0 = 2, \qquad \gamma \equiv \gamma_0 = 7,$$

which are smaller and greater than the value used in the previous case study to allow for further comparisons. Notice that $\gamma_0 = 2$ corresponds to a largely cooperative welfare policy, whereas $\gamma_0 = 7$ corresponds to a strongly competitive one, considering that the minimum and maximum class distances with $n = 9$ wealth classes are 1 and 8, respectively. For both values of γ_0, simulations are organized as before; cf. Figs. 4.4 and 4.5. In particular, the same set of initial conditions (cf. Fig. 4.3), with corresponding average wealth U_0 and initial social gap S_0, is used.

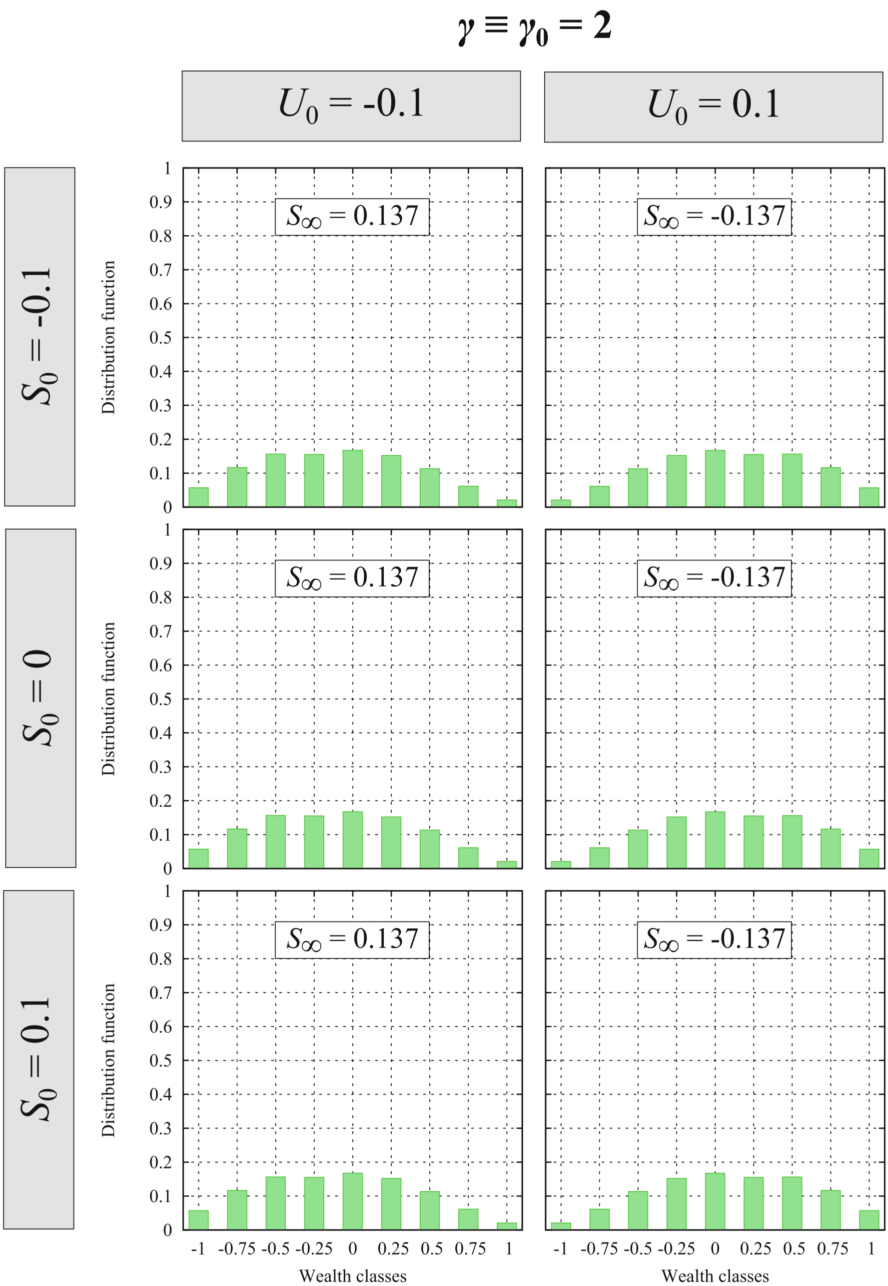

Fig. 4.4 Asymptotic profiles of the wealth distribution for a fixed critical threshold $\gamma \equiv \gamma_0 = 2$. As in the cases illustrated in Fig. 4.2, the final shape of the distribution is independent of its initial bias estimated using S_0. In addition, the markedly cooperative welfare policy imposed by $\gamma_0 = 2$ gives rise to a high concentration of active particles in the middle wealth classes: about 75 % of particles are in the classes from $u_3 = -0.5$ to $u_7 = 0.5$ in both cases $U_0 = \pm 0.1$. Consequently,

4.3.3 Influence of a Variable Critical Threshold

We stress again that a variable critical threshold γ can be understood as due to weak action by the central government, which mainly leaves rules of cooperation and competition among various social classes to the market. In some respects, particles are then free to set, either consciously or unconsciously, the regime of game rules for chasing their own well-being. In the present context, such a choice is modeled by the function $\gamma[\mathbf{f}] = \gamma[\mathbf{f}](S)$ (cf. Eq. (3.9)) according to the inspiring principles discussed in the last part of Sect. 3.3. The latter can be roughly summarized by saying that wealthy social classes tend to profit from their wealth against poor ones by increasing $\gamma[\mathbf{f}]$ (i.e., strengthening social competition) as the social gap S increases. Of course this is not the only possible model for $\gamma[\mathbf{f}]$; different, possibly more refined, dynamics of spontaneous social competition can certainly be envisaged. The present one is however sufficient to serve exploratory purposes, particularly to assess the descriptive ability of the mathematical structures presented in this monograph.

Parallel to the previous case studies, simulations (cf. Figs. 4.6 and 4.7) are performed for the same initial conditions and values of U_0, S_0, γ_0, simply letting $\gamma[\mathbf{f}]$ evolve in time according to the already cited Eq. (3.9).

4.3.4 Overview of the Whole Set of Simulations

Simulations have shown that the wealth distribution has a trend to an asymptotic profile in both cases of constant and non-constant critical threshold γ. In particular, this implies that the critical threshold ultimately settles on an equilibrium value; this is also true for free social competition rules. In addition, the asymptotic social gap S_∞ invariably has the opposite sign with respect to the average wealth U_0. This can be understood as an *emergent self-organized collective trend* of the system, which spontaneously tends to a coherent "social polarization" (predominance of poor individuals in a society that is poor on average and of wealthy ones in a society that is wealthy on average) even for incoherent initial conditions.

Tables 4.1 and 4.2 summarize the results of the proposed case studies, focusing especially on the attenuation or sharp increase of possible initial bias in the wealth distribution. These events are indeed of some interest, as they can be related to successful or disastrous, respectively, welfare policies. In the worst case, they can even foreshadow extreme unpredictable consequences well-studied in

Fig. 4.4 (continued) the asymptotic distributions are always concave. In addition, in a society that is poor on average ($U_0 = -0.1$) about 17 % of active particles end up in the two lowest wealth classes ($u_1 = -1$, $u_2 = -0.75$) and the remaining 8 % in the two highest ones, with a social gap tending asymptotically to $S_\infty = 0.137 > 0$. On the other hand, in a society that is wealthy on average ($U_0 = 0.1$) the long-time scenario is symmetric with a social gap tending to $S_\infty = -0.137 < 0$

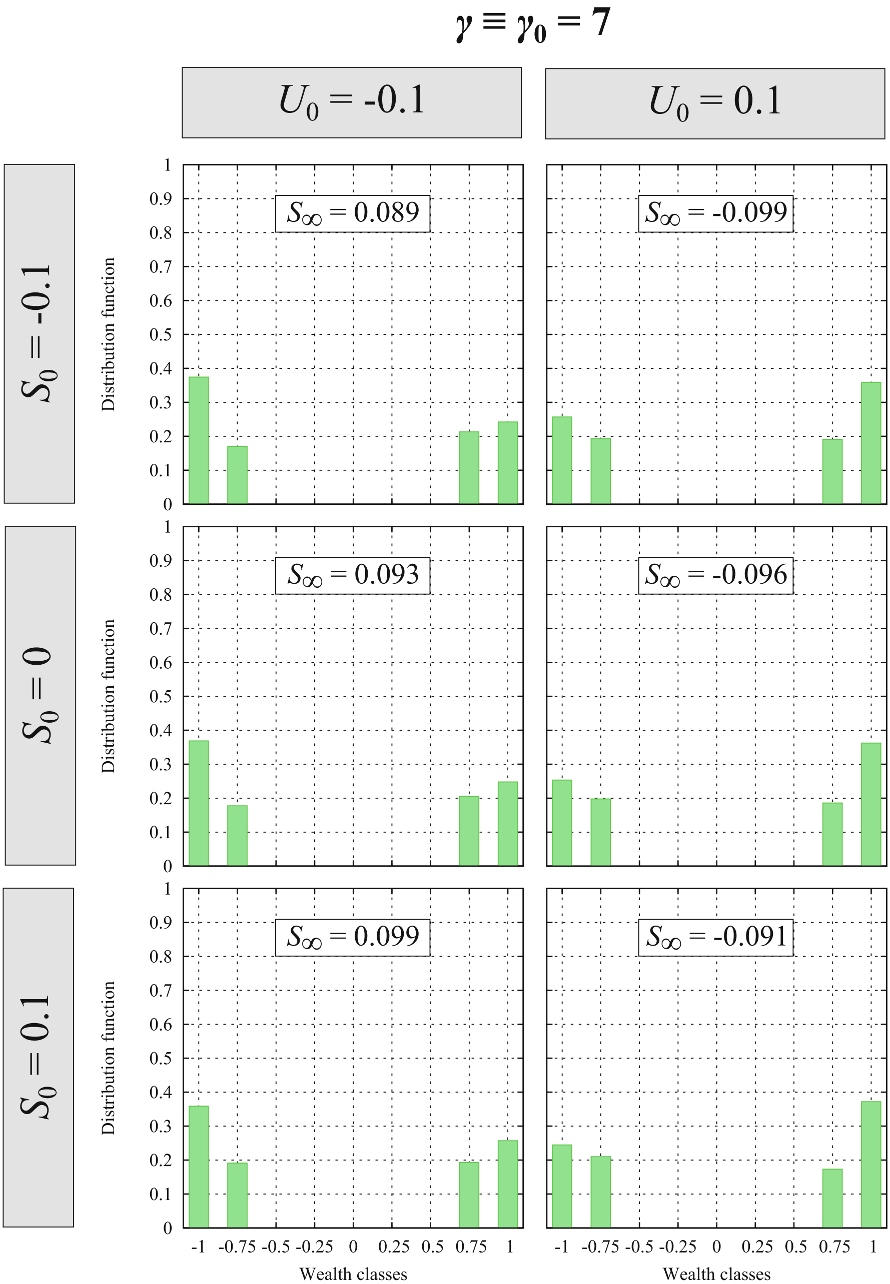

Fig. 4.5 Asymptotic profiles of the wealth distribution for a fixed critical threshold $\gamma \equiv \gamma_0 = 7$. As in the cases illustrated in Figs. 4.2 and 4.4, the qualitative asymptotic profile of the distribution is again independent of the initial social gap S_0. In this case, the strongly competitive welfare policy imposed by $\gamma_0 = 7$ makes middle wealth classes (from $u_3 = -0.5$ to $u_7 = 0.5$) disappear for both $U_0 = \pm 0.1$, which produces a sharp clustering of active particles in the lowest classes ($u_1 = -1$, $u_2 = -0.75$) and highest classes ($u_8 = 0.75$, $u_9 = 1$). In particular, in a society that is poor on

the specialized literature [2, 3, 149], which recently have been tackled from the mathematical side in a preliminary study [39] by means of suitable elaborations of the ideas and models presented in this monograph.

4.4 Critical Analysis

This chapter has demonstrated that numerical simulations are an integral part of the process of mathematical modeling. On the one hand, they can confirm the solidity of models, and hence of the hypotheses and of the mathematical structures that the latter rely upon, if they are successful in reproducing the hallmarks of real world phenomena. On the other hand, they can also provide evidence for limitations of models, thereby contributing to assessing the ranges of validity of the underlying mathematical approaches and motivating further research for improvements.

The simulations presented in this chapter have shown that stochastic game theories, in the framework of the KTAP approach, can be a valuable tool for representing some aspects of human behavior by mathematical equations. They provide a mathematically sound and empirically coherent framework for dealing with situations in which deterministic causality principles do not strictly apply, as in the case of classical physical laws of inert matter. More generally, we observe that game-theory inspired approaches [14, 77, 101, 118, 130, 155] are permeating various fields distinct from the one treated in this monograph. A remarkable example is *evolutionary dynamics*, where several contributions of this type are already available [83, 123, 125, 138, 153, 154].

At the same time, the simulations of this chapter have not been able to depict a large class of interesting phenomena having to do with the *bounded rationality* and general unpredictability of human behaviors: the emergence of extreme events not easily conceivable from the very beginning, the so-called *black swans* [149], which we already mentioned in previous chapters. As a matter of fact, all of the qualitative asymptotic scenarios obtained from the simulations were somehow roughly predictable beforehand from the given initial conditions and parameters. Therefore, simulations were useful to confirm, or explain in more detail, the expected influence of qualitative differences in such conditions on the final outcomes. In order to

Fig. 4.5 (continued) average ($U_0 = -0.1$) about 55 % of active particles end up in the lowest classes, with a social gap tending asymptotically to $S_\infty \sim 0.09 > 0$. On the other hand, in a society that is wealthy on average the asymptotic scenario is symmetric, with a social gap tending, in the long run, to $S_\infty \sim -0.09 < 0$. It is worth noticing, also in view of the following simulations with non-constant critical threshold γ, that, here as in the previous simulations (cf. the aforementioned Figs. 4.2 and 4.4), the asymptotic social gap S_∞ and the average wealth U_0 invariably have opposite signs. This can be understood as an *emergent self-organized collective trend* of the system, which tends to spontaneously restore a kind of coherent "social polarization" (predominance of poor individuals, $S_\infty > 0$, in a society that is poor on average, $U_0 < 0$, and vice versa), possibly inverting it with respect to the initial condition

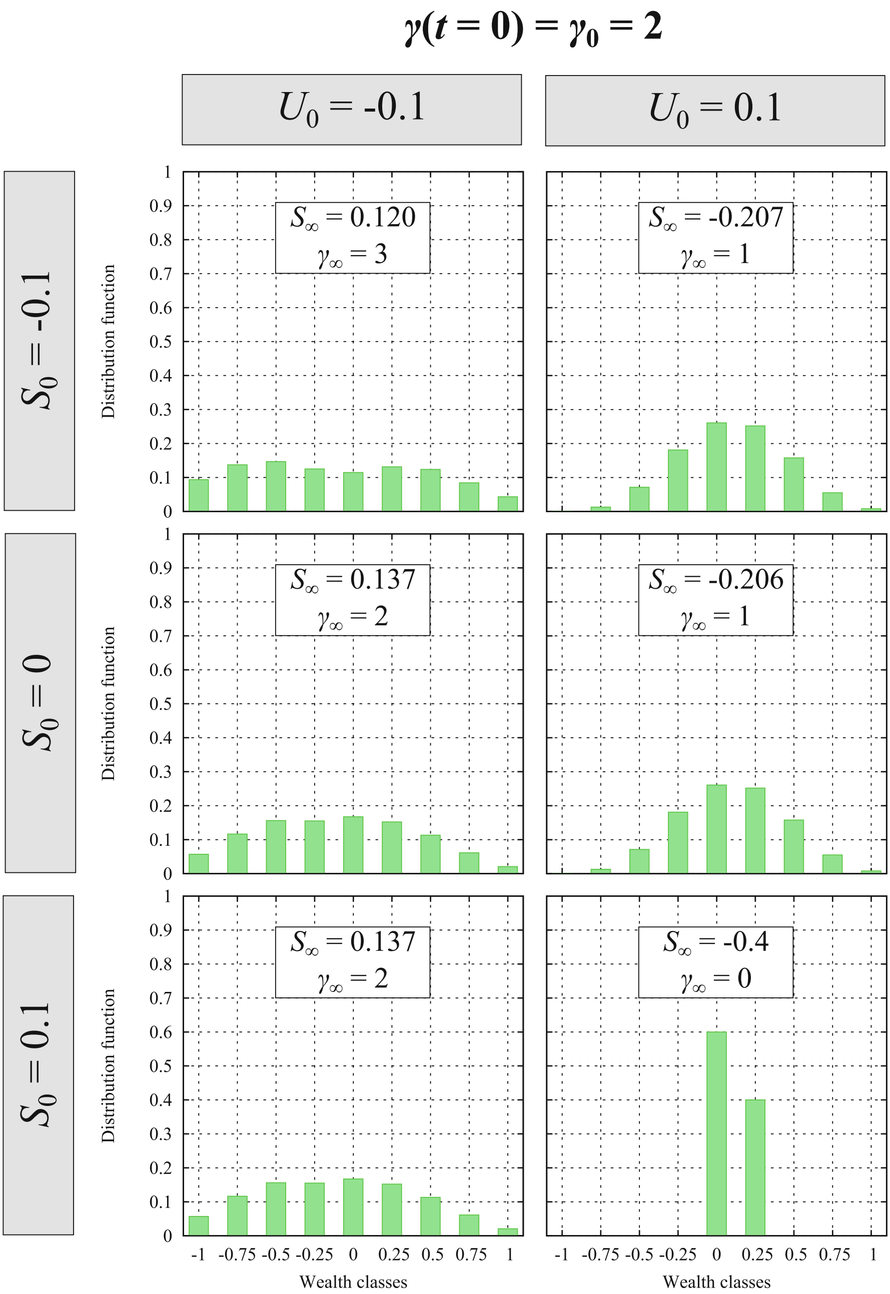

Fig. 4.6 Asymptotic profiles of the wealth distribution under a non-constant critical threshold starting from $\gamma_0 = 2$ and then evolving according to Eq. (3.9). The basically cooperative social behavior induced by such a small γ_0 produces, in the long run, mostly concave distribution profiles, which, in a society that is poor on average ($U_0 = -0.1$), are quite similar to those obtained in the corresponding case of fixed critical threshold (cf. the left column in Fig. 4.4). In particular,

chase the black swan, a more elaborate modeling of stochastic games is probably needed. For instance, active particles can be assumed to play games with multiple complementary strategies, whose complex interplay can generate, under specific circumstances, unpredictable events. Interested readers are referred to [39] for a preliminary attempt to place these ideas into a mathematical form.

An additional issue to be properly analyzed is the role of external actions (cf. Sect. 2.4). The present simulations only account for the influence of the threshold $\gamma[\mathbf{f}]$, which can be understood as the effect of the presence of external actors (such as the government) not explicitly modeled. A further development should consider specific actions from the outer environment, possibly including active interactions between the inner and the outer systems. Further prospective hints will be given in the next chapter.

Fig. 4.6 (continued) for an initially incoherent social gap ($S_0 = -0.1 < 0$) a mild increase of competition (from $\gamma_0 = 2$ to the asymptotic value $\gamma_\infty = 3$) is sufficient for self-organization to restore a coherent gap ($S_\infty = 0.120 > 0$) with minor effects on the asymptotic distribution profile. On the other hand, major differences are observed in a society that is wealthy on average ($U_0 = 0.1$, cf. also the right column in Fig. 4.4). Under game rules freely left to social dynamics, if the initial social gap is unbiased ($S_0 = 0$) then self-organization restores a coherent social gap ($S_\infty = -0.206 < 0$) via a mild increase of cooperation (from $\gamma_0 = 2$ to $\gamma_\infty = 1$). Instead, for a social gap initially already negative ($S_0 = -0.1$), the mild increase of cooperation here is due to the tendency of wealthy individuals to cooperate to preserve their common benefits (as modeled by Eq. (3.9)). Finally, if the initial gap is incoherent (i.e., positive, $S_0 = 0.1$) these two effects overlap, giving rise to an asymptotic distribution strongly clustered in the middle and to a vanishing asymptotic critical threshold, which implies no competition at all among active particles

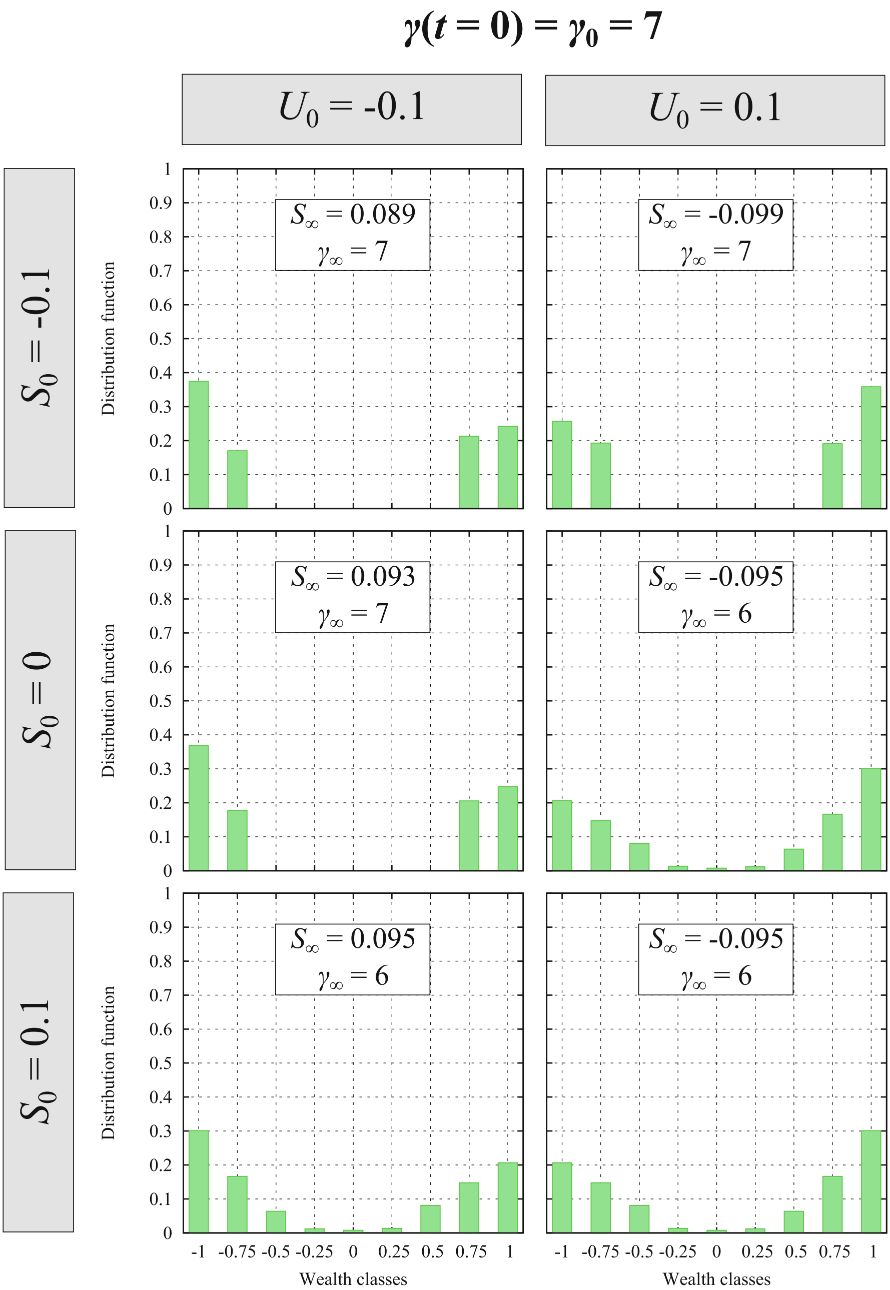

Fig. 4.7 Asymptotic profiles of the wealth distribution under a non-constant critical threshold starting from $\gamma_0 = 7$ and then evolving according to Eq. (3.9). The basically competitive social behavior induced by such a large γ_0 produces, in the long run, convex distribution profiles, which, in a society that is poor on average ($U_0 = -0.1$), are quite similar to those obtained in the corresponding case of fixed critical threshold (cf. the left column in Fig. 4.5). In particular, for

Table 4.1 Summary of the results of simulations in the case of constant critical threshold; cf. Figs. 4.2–4.5

		$U_0 = -0.1$	$U_0 = 0.1$
		Concave	Concave
	$\gamma_0 = 2$	All classes present	All classes present
		Initial bias reversed	Initial bias unchanged
		Convex	Convex
$S_0 = -0.1$	$\gamma_0 = 5$	All classes present	All classes present
		Initial bias reversed	Initial bias unchanged
		Strongly convex	Strongly convex
	$\gamma_0 = 7$	No middle classes	No middle classes
		Initial bias reversed	Initial bias unchanged
		Concave	Concave
	$\gamma_0 = 2$	All classes present	All classes present
		Emerging bias > 0	Emerging bias < 0
		Convex	Convex
$S_0 = 0$	$\gamma_0 = 5$	All classes present	All classes present
		Emerging bias > 0	Emerging bias < 0
		Strongly convex	Strongly convex
	$\gamma_0 = 7$	No middle classes	No middle classes
		Emerging bias > 0	Emerging bias < 0
		Concave	Concave
	$\gamma_0 = 2$	All classes present	All classes present
		Initial bias unchanged	Initial bias reversed
		Convex	Convex
$S_0 = 0.1$	$\gamma_0 = 5$	All classes present	All classes present
		Initial bias unchanged	Initial bias reversed
		Strongly convex	Strongly convex
	$\gamma_0 = 7$	No middle classes	No middle classes
		Initial bias unchanged	Initial bias reversed

For each of the triples (U_0, S_0, γ_0) examined, the table reports a concise qualitative overview of the most striking features of the asymptotic distribution, such as: convexity/concavity (denoting tendency to cluster in the extreme and middle classes, respectively), possible absence of some wealth classes in the long-run distribution, and trend in the asymptotic social gap S_∞ with respect to the initial one S_0.

Fig. 4.7 (continued) both an initially incoherent and an initially unbiased social gap ($S_0 = -0.1 < 0$, $S_0 = 0$, respectively) the basic competition level fixed by γ_0 is sufficient by itself for self-organization to restore a coherent social polarization ($S_\infty > 0$ with unchanged asymptotic critical threshold $\gamma_\infty = 7$). Conversely, in a society that is wealthy on average ($U_0 = 0.1$; cf. also the right column in Fig. 4.5) self-organization restores a coherent social gap (when needed, i.e., for $S_0 = 0$ and $S_0 = 0.1$) by slightly lowering the level of social competition (from $\gamma_0 = 7$ to $\gamma_\infty = 6$), which induces a less clustered asymptotic distribution in which all wealth classes are present, although most active particles still concentrate in the extreme ones

Table 4.2 Summary of the results of simulations in the case of variable critical threshold; cf. Figs. 4.6 and 4.7

		$U_0 = -0.1$	$U_0 = 0.1$
		Almost concave	Concave
	$\gamma_0 = 2$	All classes present	Extreme classes almost absent
		Initial bias reversed	Initial bias stressed
$S_0 = -0.1$		More competition	More cooperation
		Strongly convex	Strongly convex
	$\gamma_0 = 7$	No middle classes	No middle classes
		Initial bias reversed	Initial bias unchanged
		Initial threshold unchanged	Initial threshold unchanged
		Concave	Concave
	$\gamma_0 = 2$	All classes present	Extreme classes almost absent
		Emerging bias > 0	Emerging bias < 0
$S_0 = 0$		Initial threshold unchanged	More cooperation
		Strongly convex	Convex
	$\gamma_0 = 7$	No middle classes	All classes present
		Emerging bias > 0	Emerging bias < 0
		Initial threshold unchanged	More cooperation
		Concave	Strongly concave
	$\gamma_0 = 2$	All classes present	Only non-negative middle classes present
		Initial bias unchanged	Initial bias strongly reversed
$S_0 = 0.1$		Initial threshold unchanged	Much more cooperation
		Convex	Convex
	$\gamma_0 = 7$	All classes present	All classes present
		Initial bias unchanged	Initial bias reversed
		More cooperation	More cooperation

For each of the triples (U_0, S_0, γ_0) examined, the table reports a concise qualitative overview of the most striking features of the asymptotic distribution, in the same spirit as Table 4.1. In this case, an indication about the asymptotic critical threshold is also given in terms of more or less cooperative/competitive behaviors of active particles with respect to the initial level of social competition fixed by γ_0.

Chapter 5
Forward Look at Research Perspectives

Abstract This chapter presents some on research perspectives. Various topics are treated focusing on the following issues: further analysis of the modeling of welfare policy in the case of interactions in a network and in open systems; generalization of the modeling approach to various systems of social sciences, for instance opinion formation; modeling the interplay of different types of dynamics also viewed as a tool for predicting rare events; and analytic problems posed by the application of models to the study of social phenomena.

5.1 Introduction

The mathematical tools presented in Chap. 2 have been applied to modeling the dynamics related to social and economic policies that a government can develop to affect the trend of wealth distribution toward a planned direction. The specific case study, presented in Chaps. 3 and 4, has shown that models can predict various aspects of the dynamics and analyze the influence of the parameters of the model on the aforesaid trend. This case study should be regarded as a preliminary attempt, to be further generalized to enlarge the variety of phenomena described. This exercise can possibly improve the predictive ability of models. In addition, applications generate interesting analytic problems that offer applied mathematicians challenging goals to be properly achieved within a program of research.

This chapter refers to the identification of research perspectives related to the aforementioned topics. The style will be somewhat different from that of preceding chapters. In fact, rather than dealing exhaustively with the issues presented, it focuses on new models and mathematical problems generated by the study of real socio-economic problems. Some hints toward research directions follow. Moreover, additional bibliography is reported to offer interested readers a sufficiently broad list of references. In particular, we wish to mention the following web pages, which

G. Ajmone Marsan et al., *Complex Systems and Society: Modeling and Simulation*,
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offer profitable suggestions for developing mathematical approaches to studying socio-economic systems:

http://www.oecd.org/about/secretarygeneral

http://ineteconomics.org/research_note

In this chapter we also revisit the fact that one of the goals, perhaps the most important one, is the design of models that have the ability to predict rare, seemingly unpredictable, events such as so-called *black swans* [149, 150]. The idea of pursuing this challenging goal is given in a recent paper [39], which has shown that extreme events can be generated by an interplay of different types of dynamics; in the specific case studied, the interplay is between welfare policies and support or opposition to a certain political regime.

Reader may be somewhat disappointed by the very introductory stage at which the contents of this chapter are presented. Nevertheless, it is useful to recall that this monograph is presented according to the spirit of "Springer Briefs", hence as an introduction, by no means exhaustive, to possible research lines currently not yet thoroughly developed, which might stimulate specific research programs.

Bearing all of the above in mind, the following topics have been selected, according to the authors' preferences, for looking ahead at research perspectives:

- Further analysis on the modeling of welfare dynamics involving interactions over networks and within open systems.
- Generalizations of the modeling approach to a variety of new studies, including opinion formation, democratic transitions, and political instability.
- Modeling the interplay of different types of dynamics, also viewed as a source that can generate rare events.
- Analytic problems related to the applications of models.

These topics are treated in the following sections, and the last section concludes the monograph.

5.2 Welfare and Well-being Policies

As already mentioned, the contents of Chaps. 3 and 4 cannot be considered exhaustive; indeed various specific problems and generalizations are omitted. A few of these are presented in this section, followed by some hints for pursuing them.

5.2.1 Interactions Over Networks

Active particles interact, in most cases, over networks [27, 29, 32, 160]. In general, the identification of the functional subsystems playing the game depends on the localization of the nodes. The simplest case is when each functional subsystem corresponds to a node.

Interactions over networks can induce substantial modifications to the dynamics studied in Chaps. 3 and 4. For instance, the prevalence of selfishness or altruism, as well as the overall wealth distribution, may be affected by the network topology of interaction. In addition, the modeling of the interaction rate depends entirely on the structure of the network. In the case of welfare dynamics, the analysis of Chap. 4 provides evidence that the trend of the system depends on the overall wealth of a society. Therefore it may happen that the same rules cannot be applied with the same advantage to every society—for example, every country of a given continent. As an example, imposing the rules of the wealthiest country to other countries can have a negative effect. Rules and laws must account for individual characteristics of each place, and therefore must be adapted accordingly [68, 129].

An interesting research perspective consists of the development of models in which functional subsystems localized in a node interact over networks of living systems having a self-organizing ability; see e.g., [99]. Moreover, the study of networks can involve the modeling of other interesting types of dynamics such as migration phenomena from less developed countries to wealthier ones [111].

The mathematical tools proposed in Chap. 2 can be technically generalized, while still maintaining a low computational complexity at least for small networks. The advantage is that nonlinearly additive interactions can be taken into account, thereby improving mean field descriptions when necessary. On the other hand, the need for reducing computational complexity arises in the case of large networks. Perhaps the clustering of nodes that exhibit the same features can contribute to tackling this delicate problem.

5.2.2 Modeling Open Systems

All models mentioned so far refer to closed systems, such as countries and networks of countries closed to any external action. However, it can be argued that economic and political phenomena can be subject to important modifications induced by external actions, which modify the interaction rules at the microscopic scale. Particularly interesting is the case of external actions causing quantitative modifications of time-asymptotic configurations and qualitative dynamical behaviors.

The mathematical structures presented in Sect. 2.4 provide the tools to deal with this problem. Specific applications are known [51] in the case of binary interactions. However, the modeling approach should be complemented with a deep analysis of the generation of internal forces. The book by Helbing [95] offers a valuable contribution to this issue. The modeling approach should take into account the action of the outer environment on the whole system, both at the macroscopic and at the particle scales. Some relevant studies, for instance in the field of opinion formation [52], can be found in the literature. However, a systematic study able to model the influence of real external actions on the dynamics of the system is still missing. Transferring the mathematical tools of Chap. 2 to these new concepts appears to be a challenging research problem.

5.2.3 Understanding Ethical and Unethical Behaviors

A recent paper [135] has analyzed to what extent unethical behaviors can be sources of richness. This interesting topic can be approached using the model presented in Chap. 3, with the expectation of further and deeper analysis of the interplay between the critical threshold γ and the average wealth of the society. In fact, the various simulations produced in Chaps. 3 and 4 have clearly shown that the resulting asymptotic configurations are sensitive to both of them. Again, this suggests that the rules valid for a rich country cannot be applied to countries with less wealth.

5.3 Toward Additional Applications

The mathematical approach under consideration has been developed, since the pioneer paper [48], in various fields of social sciences. Applications have been made not only to the dynamics of wealth redistribution but also in other contexts, such as opinion formation [50], taxation systems [46,53], competition for secession [7], and behavioral economics [6]. Additional studies make use of methods of mathematical kinetic theory, specifically mean field equations. We note, among other applications, decision making [65], market dynamics [55,116], opinion formation [12], and more recently migration phenomena [23,111].

Therefore, we can state that there exists a quite extensive literature in the field applying similar, though technically different, approaches. In general, understanding a certain system contributes to generalizing the approach to other systems. This line of thought suggests also examining systems in other fields of life sciences, such as mathematical biology [44] or epidemics with virus mutations [73].

As we have seen, the modeling approach first requires assessment of the functional subsystems and the types of active particles that play the game, along with the specific activities they express. Subsequently, the developed strategy has to be identified, in such a way that interactions at the microscopic scale can be properly modeled. In most cases, it is also useful to speculate about the lower and higher scales, when they can be identified, as well as on conceivable networks related to the specific system under consideration. This process can be analyzed in the case of some specific applications briefly reported in the following paragraphs.

However, before dealing with technical issues it is worth stressing that the various fields of application reported in the following paragraphs have been selected among several conceivable ones according to the authors' preferences. The aforementioned screening can be viewed as a very preliminary step toward modeling. Obviously, the derivation of models should constantly face the complexity features of the system under consideration. The remarks concluding each specific presentation are possibly valid for all of them.

5.3.1 Voting Dynamics

Voting dynamics have been widely studied in the literature by political scientists and economists [30,76,117], with a wide range of techniques borrowed from qualitative analysis, statistics, game theory, and mathematical modeling. Understanding the way in which different groups of individuals interact in making decisions about whom to vote for in national or local elections is crucial in democratic regimes. A wide range of voting mechanisms exist, strongly dependent on various countries and political systems. In addition, different voting systems (such as, for example, electronic voting and mail-in voting) [45] and news and information media [86] may affect election outcomes, making voting dynamics a complex phenomenon, whose aggregate outcomes, namely the results of elections, depends on the sub-dynamics of several concurrent factors.

Consistent with the approach of Chap. 2, we now identify some basic modeling features for this kind of system.

- *Microscopic entities*: individuals in a nation.
- *Microscopic state*: inclination to vote for a certain party in an election.
- *Lower scale*: individual political opinions.
- *Higher scale*: aggregates of opinions in a nation.
- *Networks of interaction*: spatial interactions among voters in cities, districts, and regional areas.
- *External actions for open systems*: actions of parties by means of various types of media.

Remark 5.1. Voting dynamics are generally studied in the framework of closed systems. However, international networks can play a role in the game by driving public opinion toward the trend of economically stronger countries. In some cases these interactions can generate a domino effect.

5.3.2 Diffusion of Technological Innovations

Knowledge transfer has largely been studied in the economics literature related to innovation adoption and diffusion [107]. Knowledge spill-overs and the diffusion of new ideas are clearly linked to interactions of individuals in firms and other institutions, over time and space [4,22,91]. The diffusion of innovations may also be induced by networks of firms located in different geographical areas, as well as by market structures and dynamics, and external actions related to the kinds of policies implemented by governments either domestically or internationally.

- *Microscopic entities*: firms.
- *Microscopic state*: technological stage in firms.
- *Lower scale*: staff desires to improve the quality of their activity.
- *Higher scale*: population of firms in a given area.
- *Networks of interaction*: networks of firms at a regional or national level.
- *External actions for open systems*: innovation policy.

Remark 5.2. The diffusion of technology has a relevant interplay with other social dynamics, starting with economic growth or decay. Therefore, it is important that the modeling approach takes into account the hints of the following section.

5.3.3 Migration Phenomena

Migration flows have great socio-economic impacts on countries and regions from which and to which they occur. Migrations have been extensively studied in the literature by sociologists and economists [56, 63, 71]. They occur as a consequence of several phenomena: level of wealth and safety of the country of origin of migrants, wars and discrimination, and natural and environmental disasters, to mention but a few examples. Migrants may modify the socio-economic and cultural context of cities, regions, and countries, and therefore a better understanding of the dynamics generating and generated by migrations would help us understand how this phenomenon contributes to shape socio-economic development.

- *Microscopic entities*: individuals and/or families.
- *Microscopic state*: tendency to migrate.
- *Lower scale*: level of wealth of individuals and families.
- *Higher scale*: level of development of a country.
- *Networks of interaction*: networks of displacements.
- *External actions for open systems*: social policies fostering and/or preventing migration.

Remark 5.3. The modeling approach of [111] is based on the mathematical approach presented in this monograph, whereas a different technique is used in [23], where suitable developments of Hamiltonian mechanics are applied.

5.3.4 Democratic Transitions

The dynamics of the transition from a dictatorship to a democracy are still not clear as they can be influenced by a very complex interplay of factors. Political transitions have often occurred in the past century especially in political regimes outside the Western World, which have been studied from several points of view by scholars in political science and economics. The latter studied, in particular, which economic variables and factors could either foster or prevent the occurrence of political transitions.

These complex dynamics can lead to unpredictable events that cause sudden shocks in political regimes (as in the recent case of the "Arab Spring"), and whose origins still need to be understood clearly. For a comprehensive overview of these topics, one can refer to [2, 3, 131].

- *Microscopic entities*: supporters or opponents of a dictatorship.
- *Microscopic state*: political attitude.
- *Lower scale*: level of support or dissent toward a dictatorship.
- *Higher scale*: average collective behaviors.
- *Networks of interaction*: small areas of a town or a region.
- *External actions for open systems*: international support for or dissent toward a regime, media impact.

Remark 5.4. The modeling approach can be developed by either using one functional system only or partitioning the whole system into different subsystems characterized by different styles in pursuing their strategies. When the strategy is different, possibly even antagonistic, the identification of several functional subsystems is often mandatory, as happens for instance for the system briefly described in the following paragraph.

5.3.5 Spread and Evolution of Criminality

Criminality may originate because of multiple factors related to social, economic, and political conditions and it clearly has a great impact on people's well-being. Social segregation can affect criminality rates and their patterns. The problem of criminality in urban agglomerations has been studied by sociologists interested in understanding the dynamics of crime and insecurity [60, 137, 157]. More recently, some mathematical models on this topic have emerged [85]. Criminality and social segregation are interconnected complex phenomena, which could be better understood by means of sophisticated mathematical modeling intended to capture their patterns of evolution.

- *Microscopic entities*: individuals subdivided into different functional subsystems; for example, criminals and police officers.
- *Microscopic state*: criminal ability in the first subsystem and ability to apprehend criminals in the second one.
- *Lower scale*: psychological attitude to criminality.
- *Higher scale*: average collective behaviors.
- *Networks of interaction*: small areas of a town or of a region.
- *External actions for open systems*: international crime legislation.

Remark 5.5. The modeling approach might divide the whole system into several functional subsystems characterized by different levels of criminal and detective ability. Possible transitions across the levels should be included in the model.

5.4 On the Interplay Among Different Dynamics

The modeling approach proposed in previous chapters is based on the idea that the activity variable is a scalar or, when it is a vector, that the whole system can be decomposed into subsystems such that each of them is characterized by a scalar activity variable only. However, this type of decomposition is not always technically feasible. Moreover, systems where the interplay involves different activity variables appears to be an interesting topic worthy of investigation.

It has been shown [39] how welfare dynamics based on a selfish attitude of specific social classes can lead, for certain parameter values, to the clustering of the population in the extreme wings of support/opposition to a regime. Moreover, in these extreme situations, even subgroups of the wealthy population become doubtful about the regime, though they were initially in favor of it.

The interplay between different factors appears to be crucial in several types of dynamics. For instance, in [3] the interplay between dictatorial and democratic trends linked to economic issues is analyzed; in [33] the evolution of biodiversity related to mutualistic networks is modeled (also see [136]). Similarly, it is possible to look at the influence of social policies, including welfare policies, on the growth of criminal behaviors [143].

More generally, an overview of social sciences shows that the interplay of different dynamics can have an important influence on the collective behaviors of social systems and, more specifically, on their asymptotic trends. Indeed, this is what has been shown in [39]. Therefore, a natural question arises: how can this interesting topic be further studied and understood? A straightforward application of the methods presented in Chap. 2 suggests that the use of vector activity variables gives, at a technical level, the desired result. On the other hand, this approach significantly increases the difficulty of modeling individual interactions. A useful alternative is offered in [39]. It consists of assuming the sequentiality of the dynamics; for instance first welfare policy and then political dynamics. In this way the output of the first-level dynamics becomes an input for the second-level dynamics. Future activity in the field may clarify which is the most appropriate strategy.

Developing these perspectives requires deeper insight into game theory and evolution. A fundamental reference in this context is [124], which proposes qualitative analogies between biological and social systems. The role of Darwinian selection is deeply analyzed in [113]; also see [125, 130].

The case studies addressed in the previous section indicate, specifically, some conceivable interplays. For instance, the study of opinion formation has an important interplay with the expression of political preferences in voting dynamics, whereas diffusion of technology may have an immediate influence on the dynamics of wealth distribution and life conditions. Heterogeneity needs to be carefully taken into account, also considering that small groups of individuals (representing a minority of the whole population) can have an important impact on the global dynamics of social systems, as documented in [84].

In general, it can be stated that the selection of interplays that effectively have a role in the game depends on the specific goal of the modeling approach. Different choices correspond to different goals. Interplays should provide *early-warning signals*, which indicate changes in the trend of the collective behaviors exhibited by the systems under consideration [127, 140]. A quantitative example is offered in [39] for the interplay between welfare policy and support/opposition to a certain political regime. Specifically, it is shown that when welfare policies tend to be oriented against the well-being of citizens, and when this trend is not controlled by the government but is simply left to spontaneous competition within the population, early signals can be detected which anticipate a radicalization of opposition to the regime.

5.5 Analytical Problems

The application of models to real social phenomena generates interesting analytical problems, which can stimulate further challenging investigations for applied mathematicians.

As we have seen, mathematical problems are stated, in the case of discrete activity variables, as initial-value problems for a nonlinear system of ordinary differential equations. If the activity variable is continuous, initial-value problems refer to systems of integro-differential equations. The qualitative analysis can focus on the following issues:

- Well-posedness of initial-value problems.
- Existence and uniqueness of equilibrium configurations and their stability properties.
- Dependence of the qualitative behaviors of solutions on the parameters of the model, in particular on the initial conditions.
- Analytical problems for open systems.
- Multiscale issues.
- Models with spatial structure.
- Further developments of game theory.

Some results are already known in the literature, generally for models featuring linearly additive interactions. Their extension to the case of nonlinearly additive interactions may not be immediate but is necessary due to the much greater interest of this class of models.

Bearing this in mind, let us briefly sketch some aspects of the issues mentioned above.

5.5.1 Existence of Solutions

This problem was first addressed in [16] for systems of integro-differential equations with linear interactions and in the absence of external actions. It is not a difficult problem, considering that the interaction operator is locally Lipschitz continuous and that, due to the conservation of mass, the L^1 norm of the distribution function is preserved in time. More recent works involve generalization to open systems [18] and qualitative analysis in the case of nonlinear interactions [19].

The generalization of these results to the case of systems of ordinary differential equations generated by discrete activity variables is immediate as documented in [48]. Additional difficulties have to be tackled if the model includes proliferative events, which however have not been treated in this monograph.

5.5.2 Equilibrium Configurations and Their Dependence on the Model Parameters

Existence, but not uniqueness, of equilibrium solutions has already been studied in [16] and further generalized in [19]. On the other hand proof of uniqueness seems to be a difficult problem, although simulations presented in Chap. 3 suggest that the equations, at least in the case of closed systems, show a trend toward a unique asymptotic configuration, which appears to be numerically stable. If the activity variable is discrete then the proof of uniqueness and stability has been obtained for models with a relatively small number of activity classes [49]; however, the proof has not yet been extended to the general case.

Moreover, as shown by the specific examples treated in Chaps. 3 and 4, although the equations always show a trend to an asymptotic equilibrium configuration, the shape of such a configuration depends on initial conditions. More precisely, it depends, in the specific model dealt with in this monograph, on the initial mean value of the wealth but apparently not on the shape of the initial distribution. This amazing result is not well understood and analytical proofs are not available to support such a numerical insight. Partial results are known for models of opinion formation with discrete states [47]; on the other hand, further analysis is welcome for understanding the role played by various parameters on the aforementioned equilibrium configurations.

5.5.3 Open Systems

Most of the literature concerned with analytical problems, such as those briefly sketched above, is limited to closed systems. On the other hand, the role of external actions can be of paramount importance if it refers either to actions at the macroscopic scale or to the influence of agents acting at the microscopic scale. The formal structure to be used in this type of modeling approach was given in Chap. 2.

Since very limited activity has been developed in this field, we simply bring this topic to the attention of readers and stress its importance for applications. The main difficulty consists in modeling external actions in terms of agents and of the games they play with active particles. In some cases interactions can modify the outer environment. This issue is well documented in earth sciences [163] and should probably also be accounted for in the case of social dynamics.

5.5.4 Multiscale Problems

The modeling approach has shown that the analysis of dynamics at the microscopic scale can be transferred to a statistical description of collective behaviors. The conceivable applications summarized in the previous section have shown that for each system it is possible to look at a lower submicroscopic scale and at a higher macroscopic scale. The interplay between different scales generates modeling and analytical problems of great interest for applications. The link between submicroscopic and microscopic scales implies the necessity to model the interplay between the games at the level of individuals and the inner dynamics of the latter. On the other hand, looking for collective dynamics at the macroscopic scale means obtaining an aggregate characterization of the system behavior, for instance via suitable asymptotic approaches or averaging techniques, stemming from, but not necessarily focused on, individualities.

5.5.5 Models with Spatial Structure

The mathematical models studied in this monograph have been derived by assuming that the dynamics in space were limited to interactions involving different nodes of a network, whereas spatial dynamics within each node were neglected. This assumption is not always valid. In fact, social interactions produce, in some cases, aggregation and fragmentation phenomena that are localized in space.

An example is offered by the study of criminal behaviors: the knowledge of the localizations of aggregation spots can contribute to organizing the fight against criminals [143]. Furthermore, as observed in [111], in the case of migrations the spatial distribution of communities of migrants can also be a useful detail for the study of the system.

The modeling of spatial dynamics can take advantage of kinetic-type descriptions, where the localization of active particles is included in the distribution function as a further microstate. The derivation of models at higher scales then needs to be obtained from the underlying description delivered by such kinetic models. It is possible that the approach reviewed in [36] can be properly developed in this direction for addressing the study of social systems. The conceptual difficulty consists in modeling spatial dynamics related to nonlocal games [113]. Some hints toward this specific goal might be extracted from the study of swarms [25].

5.5.6 Further Developments of Game Theory

The modeling approach proposed in this monograph has been constantly referred to game-theoretical ideas, which have been used to model nonlinear interactions within a general framework of generalized kinetic equations. Recent literature reflects different approaches according to different ways of treating individual-based interactions and of inserting them in different classes of evolution equations. Among others, we mention here evolutionary games in the framework of statistical mechanics [95], differential games [57, 58] in the framework of controlled differential equations, and mean field games [92, 114]; see also the recent special issue [61].

5.6 Conclusions

This monograph has shown how suitable generalizations of the kinetic theory for active particles can be applied to model a variety of social and economic systems. The first part of the monograph has focused on the derivation of mathematical tools, and the second part on applications and research perspectives. We feel confident in stating that the indications given as possible research perspectives will generate interesting results from the point of view of both modeling and analytical problems.

The application of the mathematical tools discussed in this monograph to a broader set of socio-economic systems appears to be quite a natural perspective, whereas analysis of the interplay involving different activity variables is more challenging. The indication that the latter can lead to extreme radicalization suggests continuing along the research line outlined in [39] in other fields of life sciences as well. Concerning this, it is worth stressing again that interest in new analytical problems generated by such a modeling approach, some of which are definitely challenging to tackle, is not only due to their intrinsic technical difficulty but, first and foremost, to their immediate applicability.

Some concluding arguments can address the big problem of looking for a mathematical theory of social systems. The first step toward this challenging goal should be the development of mathematical tools suitable for capturing the most relevant general complexity features of such systems. Addressing such an issue in a satisfactory way is, by itself, an extremely challenging task. We certainly do not claim that the search for mathematical tools is completed with the contents of Chap. 2. We simply claim that a preliminary approach has been proposed, which is waiting for further refinements and improvements.

The main positive aspect of the proposed approach is that it has introduced, within a unified framework, a class of equations that includes the following specific features:

- Heterogeneous distribution of the ability of individuals to pursue specific goals. Heterogeneity can have an important influence in determining the output of interactions and hence the overall dynamics.
- Nonlinearly additive interactions, along with related learning processes, treated in terms of stochastic games. This opens up the possibility of going beyond the limitations of rational players and classical game theory.
- The ability to describe social behaviors within a multiscale perspective, which includes, in particular, interaction rules at the microscale and collective trends at a more aggregate statistical level.

The various arguments presented in Chap. 1 motivate the search for a unified mathematical structure, to be regarded as a first step toward the derivation of a mathematical theory of social systems. Such a structure is required to include all paradigms of the complexity of the class of systems under consideration, so that it can compensate, at least partially, for the lack of fundamental background theories that is currently typical of living systems.

These structures can be technically improved by including the ability to describe additional phenomena. Nevertheless, the validity of a model is related to its success in modeling interactions at the microscopic scale by an appropriate phenomenological interpretation of social reality. It is possible that mathematical methods such as those reviewed in [36] can derive macroscopic averaged behaviors from the underlying description delivered by the kinetic theory for active particles.

However, the most significant step toward a mathematical theory of social systems might be a deeper understanding of the dynamics at the submicroscopic scale, which are responsible for the games played by individuals at the microscopic scale. This implies in turn a deeper understanding of the psychological mechanisms that generate individual strategies.

Even if this is an extremely difficult goal to be achieved, intermediate results are interesting and the analytical problems generated by this attempt are definitely challenging, and hence worth tackling.

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