Spectral Coarse Spaces in Robust Two-Level Schwarz Methods

J. Willems

Abstract A survey of recently proposed approaches for the construction of spectral coarse spaces is provided. These coarse spaces are in particular used in two-level preconditioners. At the core of their construction are local generalized eigenvalue problems. It is shown that by means of employing these spectral coarse spaces in two-level additive Schwarz preconditioners one obtains preconditioned systems whose condition numbers are independent of the problem sizes and problem parameters such as (highly) varying coefficients. A unifying analysis of the recently presented approaches is given, pointing out similarities and differences. Some numerical experiments confirm the analytically obtained robustness results.

Keywords Spectral coarse space • Robust preconditioner • Two-level domain decomposition • Additive Schwarz • Multiscale problems

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1 Introduction

The robust preconditioning of linear systems of equations resulting from the discretization of partial differential equations is an important objective in the numerical analysis community. The importance arises due to an abundance of applications in the natural and engineering sciences, including, e.g., porous media flows in natural

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reservoirs or man-made materials and computational solid mechanics. In many practical situations the obtained discrete systems are too large to be solved by direct solvers in acceptable computational time. This leaves the class of iterative solvers as viable alternative. Nevertheless, since the convergence rates of iterative solvers generally depend on the condition numbers of the systems to be solved, suitable preconditioners are necessary to speed up convergence.

More precisely, one is typically faced with a situation where the condition number of the discrete system increases with the size of the problem (or equivalently with decreasing the mesh parameter) and may additionally deteriorate with specific problem parameters. Instances of such problem parameters are, e.g., (highly) varying coefficients or otherwise degenerate parameters. The latter may for instance be observed in linear elasticity in the almost incompressible case, i.e., when the Poisson ratio is close to $1/2$. In view of these two aspects one is therefore interested in the design of preconditioners that yield condition numbers of the preconditioned systems that are independent of mesh and problem parameters. In the following we refer to these preconditioners as *robust* with respect to mesh and problem parameters.

In the absence of degenerate problem parameters obtaining robust preconditioners with respect to the problem size has been successfully addressed for a variety of settings. Here we in particular mention various multilevel and multigrid methods (see e.g. [3, 18, 24, 26] and references therein) and domain decomposition methods (see e.g. [21, 23] and references therein). For problems with varying coefficients these methods remain to work robustly provided the coefficient variations are resolved by the coarsest grid.

However, even for two-level methods the situation is more complicated if the coarse mesh does not resolve the coefficient discontinuities. For certain classes of coefficients robustness of two-level preconditioners could be established by using a coarse space spanned by specially designed multiscale finite element functions (see e.g. [10, 17, 19]) or energy minimizing functions (see e.g. [25, 29]). The dimensions of these "exotic" coarse spaces are essentially given by the dimensions of corresponding standard coarse spaces. While this is desirable from the point of view of computational complexity, it can be shown that for general coefficient configurations the obtained coarse spaces cannot be rich enough to maintain robustness in all situations.

A two-level preconditioner for the scalar elliptic equation with highly varying coefficients that is robust for general coefficient configurations was presented in [15]. Here the authors use local generalized eigenvalue problems in the coarse space construction. More precisely, they consider a family of overlapping subdomains. On each of the subdomains a generalized eigenvalue problem is posed. The eigenfunctions corresponding to eigenvalues below a predefined threshold are then used for constructing the coarse space in the two-level preconditioner. The analysis of this preconditioner then shows that the condition number of the preconditioned system only depends on this predefined threshold, and is thus in particular independent of problem and mesh parameters. The approach of [15] is furthermore refined in [16]

where multiscale partition of unity functions are used to reduce the dimension of the coarse space while preserving the robustness of the preconditioner.

Here it should be noted that the idea of using local eigenvalue problems for the coarse space construction has previously been used in [6–8] leading to spectral element-based algebraic multigrid (ρAMGe) methods. More recently, in the framework of ρAMGe and focussing on the robustness with respect to coefficient variations, local generalized eigenvalue problems have been used to construct a tentative coarse space (see [5]). The actual coarse space used in [5] is then obtained from this tentative coarse space after a smoothed aggregation construction (see also [6]). A two-grid method similar to that of [5] is discussed in [20], where additionally advanced polynomial smoothers based on the best uniform polynomial approximation to x^{-1} are considered.

The concept of using local generalized eigenvalue problems in the coarse space construction of robust two-level preconditioners for the scalar elliptic equation with highly varying coefficients is put into a more general framework in [22]. Here the local generalized eigenfunctions corresponding to eigenvalues below a predefined threshold are employed to define functionals. These functionals are in turn used to specify constraints for minimization problems whose solutions are taken as coarse space basis functions. The framework of [22] is a generalization, since it allows for functional constraints not only originating from local generalized eigenproblems. In fact, it is shown that an alternative way for choosing the functional constraints is by specifying averages over suitably chosen, i.e., coefficient dependent, subdomains.

Another generalization of [15, 16] is the use of local generalized eigenvalue problems for the construction of robust preconditioners for abstract symmetric positive definite bilinear forms, which was considered in [12] and later on in [9]. The idea is to formulate the generalized eigenvalue problems only in terms of the abstract bilinear form. This generality makes the theory applicable to a variety of problems such as the scalar elliptic equation with isotropic or anisotropic coefficients, the stream function formulations of Stokes' and Brinkman's problem, the equations of linear elasticity, as well as equations arising in the solution of Maxwell's equations.

The main objective of the chapter at hand is to put the derivations of [12] and [9] in a common perspective, to emphasize their similarities and differences, and to relate them to the original works in [15, 16]. For this we restrict to analyzing the scalar elliptic equation with highly varying isotropic coefficients to keep the argument as simple as possible.

We remark that rather recently the approaches of [15, 16] and [12] have been generalized to multiple levels in [13] and [27], respectively. We note that due to the high computational cost involved in solving generalized eigenvalue problems the generalization to multiple levels provides an important step for keeping the sizes of these eigenvalue problems manageable for overall problem sizes that could hardly be coped with in a two-level framework. Nevertheless, for the sake of simplicity we refrain from including the analysis of these multilevel methods in our present exposition. Finally, for the sake of completeness, we note that these concepts of robust preconditioners have been applied to multiscale anisotropic problems (see [11] for a two-level and [28] for a multilevel method).

The remainder of this chapter is organized as follows. In Sect. 2 we outline the problem setting and formulate the abstract overlapping Schwarz preconditioner. Section 3 is devoted to different related approaches for constructing suitable coarse spaces resulting in robust preconditioners. In Sect. 4 we analyze the coarse space dimension and clarify differences and similarities between the various methods. In Sect. 5 we present some numerical results exemplifying the robustness of the obtained preconditioners before ending with some conclusions.

2 Problem Setting

In order to make our presentation as accessible as possible, we restrict to the following model problem posed in a bounded polyhedral domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$:

$$-\nabla \cdot (\kappa(\boldsymbol{x})\nabla u) = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1}$$

where $0 < \kappa_{\min} \leq \kappa \leq \kappa_{\max} < \infty$ and $f \in L^2(\Omega)$, with $L^2(\Omega)$ denoting the space of square integrable functions on Ω. It is well-known that the variational formulation of (1) is given by

$$\text{Find } u \in H_0^1(\Omega) \text{ such that } a_\Omega(u, v) = (f, v), \quad \forall v \in H_0^1(\Omega), \tag{2}$$

where $a_\omega(u, v) := \int_\omega \kappa(\boldsymbol{x})\nabla u \cdot \nabla v \, d\boldsymbol{x}$ for any $\omega \subset \Omega$, $(f, v) := \int_\Omega f u \, d\boldsymbol{x}$, and $H_0^1(\Omega)$ denotes the subspace of $L^2(\Omega)$ of functions with square integrable derivatives and zero trace on $\partial\Omega$.

Let $\mathcal{T}_h$ be a quasi-uniform triangulation of Ω with mesh parameter h. Corresponding to $\mathcal{T}_h$ let $\mathcal{V} \subset H_0^1(\Omega)$ be a (possibly higher order) Lagrange finite element space. The finite dimensional problem corresponding to (2) is then given by

$$\text{Find } u \in \mathcal{V} \text{ such that } a_\Omega(u, v) = (f, v), \quad \forall v \in \mathcal{V}. \tag{3a}$$

An equivalent operator notation reads

$$\text{Find } u \in \mathcal{V} \text{ satisfying } Au = F, \tag{3b}$$

where, with $\mathcal{V}'$ denoting the dual space of $\mathcal{V}$, $A : \mathcal{V} \to \mathcal{V}'$ is given by $< Au, v >:= a_\Omega(u, v)$, and $F \in \mathcal{V}'$ is defined by $< F, v >:= (f, v)$. Here $< \cdot, \cdot >$ denotes the duality pairing of $\mathcal{V}'$ and $\mathcal{V}$.

Our main objective in this chapter is to discuss robust two-level additive Schwarz preconditioners for solving (3). The term "robust" refers to the condition number of the preconditioned system being independent of the mesh parameter h and variations in κ.

Algorithm 1: Additive Schwarz preconditioner $M : \mathcal{V}' \to \mathcal{V}$ corresponding to $\{\mathcal{V}_0(\Omega_j^{(1)})\}_{j=1}^{n_\Omega^{(1)}}$ and $\mathcal{V}_H$.

Let $F \in \mathcal{V}'$.
Set $v \equiv 0 \in \mathcal{V}$.
for $j = 1, \ldots, n_\Omega^{(1)}$ **do**
  Compute $\psi \in \mathcal{V}_0(\Omega_j^{(1)})$ such that

$$a_{\Omega_j}(\psi, w) = F(w), \qquad \forall w \in \mathcal{V}_0(\Omega_j^{(1)}).$$

  $v \leftarrow v + \psi$
end for
Compute $\psi \in \mathcal{V}_H$ such that

$$a(\psi, w) = F(w), \qquad \forall w \in \mathcal{V}_H.$$

$v \leftarrow v + \psi$
return $MF := v$

To make this more precise let $\{\Omega_j^{(1)}\}_{j=1}^{n_\Omega^{(1)}}$ be a family of overlapping subdomains of Ω. For any $\omega \subset \Omega$ we define

$$\mathcal{V}(\omega) := \{v|_\omega \,|\, v \in \mathcal{V}\} \quad \text{and} \quad \mathcal{V}_0(\omega) := \{v \in \mathcal{V} \,|\, \mathrm{supp}(v) \subset \overline{\omega}\}.$$

Also, we identify functions in $\mathcal{V}_0(\omega)$ with their restrictions to ω, and we thus in particular have that $\mathcal{V}_0(\omega) \subset \mathcal{V}(\omega)$. Let $\mathcal{V}_H \subset \mathcal{V}$ be a coarse space whose construction is discussed in Sect. 3. The action of the two-level additive Schwarz preconditioner corresponding to $\mathcal{V}_0(\Omega_j^{(1)})$, $j = 1, \ldots, n_\Omega^{(1)}$ and $\mathcal{V}_H$ is given by Algorithm 1.

Applying M to (3b) yields the following preconditioned system

$$MAu = MF. \tag{4}$$

For $j = 1, \ldots, n_\Omega^{(1)}$ let $\mathcal{I}_j^{(1)} := \{i = 1, \ldots, n_\Omega^{(1)} \,|\, \Omega_i^{(1)} \cap \Omega_j^{(1)} \neq \emptyset\}$. Also, we set $n_{\mathcal{I}}^{(1)} := \max_{j=1,\ldots,n_\Omega^{(1)}} \#\mathcal{I}_j^{(1)}$. Using this notation it follows from [21, Lemma 2.51] that

$$\lambda_{\max}(MA) \leq n_{\mathcal{I}}^{(1)} + 1, \tag{5}$$

where $\lambda_{\max}(\cdot)$ denotes the largest eigenvalue. For establishing the robustness of our preconditioner it, therefore, suffices to derive a lower bound for $\lambda_{\min}(MA)$ independent of h and variations in κ. Here $\lambda_{\min}(\cdot)$ denotes the smallest eigenvalue.

Provided that there exists a constant $K > 0$ such that for any $v \in \mathscr{V}$ there exist $v_H \in \mathscr{V}_H$ and $v_j \in \mathscr{V}_0(\Omega_j^{(1)})$, $j = 1, \ldots, n_\Omega^{(1)}$ satisfying

$$v = v_H + \sum_{j=1}^{n_\Omega^{(1)}} v_j \qquad \text{and} \qquad a_\Omega(v_H, v_H) + \sum_{j=1}^{n_\Omega^{(1)}} a_\Omega(v_j, v_j) \leq K\, a_\Omega(v, v) \tag{6}$$

a standard result for abstract alternating Schwarz methods yields that

$$\lambda_{\min}(MA) \geq K^{-1}, \tag{7}$$

which together with (5) in particular implies the following result (see e.g. [21, Theorem 2.52]).

Theorem 2.1. *The condition number of the additive Schwarz preconditioned system* (4) *is bounded by* $K(n_{\mathscr{I}}^{(1)} + 1)$.

In view of Theorem 2.1 it is, therefore, sufficient to establish a stable decomposition (6) with a constant K independent of h and variations in κ. The crucial ingredient for obtaining such a robust bound is the careful design of the coarse space $\mathscr{V}_H$, which is described in the next section.

3 Spectral Coarse Space Construction

First, we need to introduce some further notation. Let $\{\Omega_j^{(2)}\}_{j=1}^{n_\Omega^{(2)}}$ be another overlapping decomposition of Ω, which may coincide with $\{\Omega_j^{(1)}\}_{j=1}^{n_\Omega^{(1)}}$. Let $\{\xi_j^{(1)}\}_{j=1}^{n_\Omega^{(1)}}$ and $\{\xi_j^{(2)}\}_{j=1}^{n_\Omega^{(2)}}$ be partition of unities subordinate to $\{\Omega_j^{(1)}\}_{j=1}^{n_\Omega^{(1)}}$ and $\{\Omega_j^{(2)}\}_{j=1}^{n_\Omega^{(2)}}$, respectively, such that $\operatorname{supp}(\xi_j^{(i)}) = \overline{\Omega}_j^{(i)}$ for $j = 1, \ldots, n_\Omega^{(i)}$. As a starting point of our derivations, we observe that for any $v_{H,j}^{(i)} \in \mathscr{V}(\Omega_j^{(i)})$, $j = 1, \ldots, n_\Omega^{(i)}$ we have the following two variants of a decomposition of v:

$$v = \underbrace{\sum_{j=1}^{n_\Omega^{(1)}} \xi_j^{(1)} v_{H,j}^{(1)}}_{=:v_H^{(1)}} + \sum_{j=1}^{n_\Omega^{(1)}} \underbrace{\xi_j^{(1)}(v - v_{H,j}^{(1)})}_{=:v_j^{(1)}}, \tag{8a}$$

$$v = \underbrace{\sum_{j=1}^{n_\Omega^{(2)}} \xi_j^{(2)} v_{H,j}^{(2)}}_{=:v_H^{(2)}} + \sum_{j=1}^{n_\Omega^{(1)}} \underbrace{\xi_j^{(1)}(v - v_H^{(2)})}_{=:v_j^{(2)}}. \tag{8b}$$

(8b) is the choice considered in [12], whereas (8a) is essentially the variant considered in [9]. Note that in the first variant there appears only one partition of unity, whereas in the second variant one has the freedom to choose two distinct partition of unities (see Remark 3.4). We now aim at choosing $v_{H,j}^{(i)}$ in such a way that the decompositions (8) are also stable, i.e., have a robust constant K in estimate (6). This approach eventually leads to the definition of a suitable coarse space $\mathscr{V}_H$.

Before proceeding with the actual derivations we note that $v_H^{(i)} = v - \sum_{j=1}^{n_\Omega^{(1)}} v_j^{(i)}$. Thus, by the definition of $n_{\mathscr{J}}^{(1)}$ and a strengthened Cauchy–Schwarz inequality we observe that

$$\begin{aligned} a_\Omega\Big(v_H^{(i)}, v_H^{(i)}\Big) &\leq 2\,a_\Omega(v,v) + 2\,a_\Omega\left(\sum_{j=1}^{n_\Omega^{(1)}} v_j^{(i)}, \sum_{j=1}^{n_\Omega^{(1)}} v_j^{(i)}\right) \\ &\leq 2\,a_\Omega(v,v) + 2\,n_{\mathscr{J}}^{(1)} \sum_{j=1}^{n_\Omega^{(1)}} a_\Omega\Big(v_j^{(i)}, v_j^{(i)}\Big)\,. \end{aligned} \tag{9}$$

Thus, for establishing the estimate in (6) with a robust constant K, it suffices to derive the following estimate

$$\sum_{j=1}^{n_\Omega^{(1)}} a_\Omega\Big(v_j^{(i)}, v_j^{(i)}\Big) \leq C\,a_\Omega(v,v)\,, \tag{10}$$

where C is a generic constant independent of h and variations in κ, i.e., we may disregard the term $a_\Omega(v_H, v_H)$ in the estimate of (6).

Considering the definition of $v_j^{(i)}$ in (8) we aim at choosing $v_{H,j}^{(i)}$ in such a way that (10) holds.

Remark 3.1. We would like to point out here that generally, due to the multiplication by partition of unity functions, we have that $v_H^{(i)}, v_j^{(i)} \notin \mathscr{V}$. This problem can be overcome by considering $I_h v_H^{(i)}$ and $I_h v_j^{(i)}$ instead, where I_h denotes the usual nodal interpolation associated with $\mathscr{V}$.

Another possibility which is proposed in [9] is the use of partition of identity operators in (8) instead of partition of unity functions. At the current place this modification indeed makes the argument more elegant. Nevertheless, this modification shifts the difficulty to the analysis relating the dimension of $\mathscr{V}_H$ to the geometry underlying the variations of κ. This issue will be further addressed in Sect. 4.2.

First we consider the case $i = 1$, i.e., (8a). We observe that

$$\sum_{j=1}^{n_\Omega^{(1)}} a_\Omega\Big(v_j^{(1)}, v_j^{(1)}\Big) = \sum_{j=1}^{n_\Omega^{(1)}} \underbrace{a_\Omega\Big(\xi_j^{(1)}(v - v_{H,j}^{(1)}), \xi_j^{(1)}(v - v_{H,j}^{(1)})\Big)}_{=:m_{\Omega_j^{(1)}}^{(1)}\left(v - v_{H,j}^{(1)}, v - v_{H,j}^{(1)}\right)}\,. \tag{11}$$

Similarly, but slightly more complicated, we obtain for the case $i = 2$, i.e., (8b),

$$\begin{aligned}
\sum_{k=1}^{n_\Omega^{(1)}} a_\Omega\Big(v_k^{(2)}, v_k^{(2)}\Big) &= \sum_{k=1}^{n_\Omega^{(1)}} a_\Omega\Big(\xi_k^{(1)}(v - v_H^{(2)}), \xi_k^{(1)}(v - v_H^{(2)})\Big) \\
&= \sum_{k=1}^{n_\Omega^{(1)}} a_\Omega\left(\xi_k^{(1)}(v - \sum_{j=1}^{n_\Omega^{(2)}} \xi_j^{(2)} v_{H,j}^{(2)}), \xi_k^{(1)}(v - \sum_{j=1}^{n_\Omega^{(2)}} \xi_j^{(2)} v_{H,j}^{(2)})\right) \\
&\le n_{\mathscr{I}}^{(2)} \sum_{k=1}^{n_\Omega^{(1)}} \sum_{j=1}^{n_\Omega^{(2)}} a_\Omega\Big(\xi_k^{(1)}\xi_j^{(2)}(v - v_{H,j}^{(2)}), \xi_k^{(1)}\xi_j^{(2)}(v - v_{H,j}^{(2)})\Big) \\
&= n_{\mathscr{I}}^{(2)} \sum_{j=1}^{n_\Omega^{(2)}} \underbrace{\sum_{k:\Omega_k^{(1)} \cap \Omega_j^{(2)} \neq \emptyset} a_{\Omega_j^{(2)}}\Big(\xi_j^{(2)}\xi_k^{(1)}(v - v_{H,j}^{(2)}), \xi_j^{(2)}\xi_k^{(1)}(v - v_{H,j}^{(2)})\Big)}_{=: m_{\Omega_j^{(2)}}^{(2)}\left(v - v_{H,j}^{(2)}, v - v_{H,j}^{(2)}\right)},
\end{aligned} \tag{12}$$

where $n_{\mathscr{I}}^{(2)}$ is defined analogously to $n_{\mathscr{I}}^{(1)}$ corresponding to the decomposition $\{\Omega_j^{(2)}\}_{j=1}^{n_\Omega^{(2)}}$.

In view of (11) and (12) it is therefore sufficient for satisfying (10) to choose $v_{H,j}^{(i)}$ in such a way that

$$m_{\Omega_j^{(i)}}^{(i)}\Big(v - v_{H,j}^{(i)}, v - v_{H,j}^{(i)}\Big) \le C\, a_{\Omega_j^{(i)}}(v, v)\,. \tag{13}$$

The following proposition (see e.g. [15, Sect. 3.3.1] or [12, Sect. 2]) is crucial for establishing (13) with a robust constant C.

Proposition 3.2. *Consider the following local generalized eigenvalue problem:*

$$\text{Find } (\varphi_{j,\lambda}^{(i)}, \lambda) \in \mathscr{V}(\Omega_j^{(i)}) \times \mathbb{R}_0^+ \text{ s.t. } a_{\Omega_j^{(i)}}\Big(w, \varphi_{j,\lambda}^{(i)}\Big) = \lambda\, m_{\Omega_j^{(i)}}^{(i)}\Big(w, \varphi_{j,\lambda}^{(i)}\Big)\ \forall w \in \mathscr{V}(\Omega_j^{(i)}). \tag{14}$$

For $v \in \mathscr{V}$ let $v_{H,j}^{(i)} := \Pi_j^{(i)} v \in \mathscr{V}(\Omega_j)$ be the $a_{\Omega_j^{(i)}}(\cdot,\cdot)$-orthogonal projection of $v|_{\Omega_j^{(i)}}$ onto those eigenfunctions corresponding to eigenvalues below a predefined "threshold" $\tau_\lambda^{-1} > 0$, i.e., $\Pi_j^{(i)} v \in \operatorname{span}\{\varphi_{j,\lambda}^{(i)} \,|\, \lambda < \tau_\lambda^{-1}\}$ satisfies

$$a_{\Omega_j^{(i)}}\Big(v - \Pi_j^{(i)} v, \varphi_{j,\lambda}^{(i)}\Big) = 0 \text{ for all } \lambda < \tau_\lambda^{-1}.$$

Then we have that

$$m_{\Omega_j^{(i)}}^{(i)}\Big(v - v_{H,j}^{(i)}, v - v_{H,j}^{(i)}\Big) \le \tau_\lambda\, a_{\Omega_j^{(i)}}\Big(v - v_{H,j}^{(i)}, v - v_{H,j}^{(i)}\Big) \le \tau_\lambda\, a_{\Omega_j^{(i)}}(v, v)\,. \tag{15}$$

Proof. The second inequality in (15) is obvious, since $\Pi_j^{(i)} v$ is the $a_{\Omega_j^{(i)}}(\cdot,\cdot)$-orthogonal projection of $v|_{\Omega_j^{(i)}}$.

Next, we note that $v|_{\Omega_j^{(i)}} - \Pi_j^{(i)} v = \sum_{\lambda \geq \tau_\lambda^{-1}} a_{\Omega_j^{(i)}}\left(v, \varphi_{j,\lambda}^{(i)}\right) \varphi_{j,\lambda}^{(i)}$. Thus,

$$\begin{aligned} m_{\Omega_j^{(i)}}^{(i)}\left(v - \Pi_j^{(i)} v, v - \Pi_j^{(i)} v\right) &= \sum_{\lambda \geq \tau_\lambda^{-1}} a_{\Omega_j^{(i)}}\left(v, \varphi_{j,\lambda}^{(i)}\right) m_{\Omega_j^{(i)}}^{(i)}\left(v - \Pi_j^{(i)} v, \varphi_{j,\lambda}^{(i)}\right) \\ &= \sum_{\lambda \geq \tau_\lambda^{-1}} \lambda^{-1} a_{\Omega_j^{(i)}}\left(v, \varphi_{j,\lambda}^{(i)}\right) a_{\Omega_j^{(i)}}\left(v - \Pi_j^{(i)} v, \varphi_{j,\lambda}^{(i)}\right) \\ &\leq \tau_\lambda \, a_{\Omega_j^{(i)}}\left(v - \Pi_j^{(i)} v, \sum_{\lambda \geq \tau_\lambda^{-1}} a_{\Omega_j^{(i)}}\left(v, \varphi_{j,\lambda}^{(i)}\right) \varphi_{j,\lambda}^{(i)}\right) \\ &= \tau_\lambda \, a_{\Omega_j^{(i)}}\left(v - \Pi_j^{(i)} v, v - \Pi_j^{(i)} v\right). \end{aligned}$$

□

Note that by choosing the threshold τ_λ we can essentially fix the constant C in estimate (13). Thus, C and therefore also K in (6) only depend on τ_λ and $n_{\mathscr{J}}^{(i)}$, $i = 1, 2$, but are in particular independent of h and variations in κ.

For the solvability of (14) it is also important to note that $m_{\Omega_j^{(i)}}^{(i)}(\cdot,\cdot)$ is positive definite on $\mathscr{V}(\Omega_j^{(i)})$, since $\operatorname{supp}(\xi_j^{(i)}) = \overline{\Omega}_j^{(i)}$ by assumption.

The considerations above suggest choosing the coarse space $\mathscr{V}_H^{(i)}$ as $\operatorname{span}\{\xi_j^{(i)} \varphi_{j,\lambda}^{(i)} \,|\, \lambda < \tau_\lambda^{-1},\ j = 1, \ldots, n_\Omega^{(i)}\}$. However, as indicated in Remark 3.1, this choice in general does not yield a subspace of $\mathscr{V}$. The following proposition resolves this issue by means of applying a nodal interpolation.

Proposition 3.3. *For $i = 1, 2$ let*

$$\mathscr{V}_H^{(i)} := \operatorname{span}\{I_h(\xi_j^{(i)} \varphi_{j,\lambda}^{(i)}) \,|\, \lambda < \tau_\lambda^{-1},\ j = 1, \ldots, n_\Omega^{(i)}\}, \tag{16}$$

where as above I_h denotes the nodal interpolation corresponding to $\mathscr{V}$. With $v_{H,j}^{(i)}$ as defined in Proposition 3.2 we have that

$$v = \underbrace{\sum_{j=1}^{n_\Omega^{(1)}} I_h(\xi_j^{(1)} v_{H,j}^{(1)})}_{=: v_{H,I}^{(1)}} + \sum_{j=1}^{n_\Omega^{(1)}} \underbrace{I_h(\xi_j^{(1)}(v - v_{H,j}^{(1)}))}_{=: v_{j,I}^{(1)}} \tag{17a}$$

and

$$v = \underbrace{\sum_{j=1}^{n_\Omega^{(2)}} I_h(\xi_j^{(2)} v_{H,j}^{(2)})}_{=:v_{H,I}^{(2)}} + \sum_{j=1}^{n_\Omega^{(1)}} \underbrace{I_h(\xi_j^{(1)}(v - v_H^{(2)}))}_{=:v_{j,I}^{(2)}}. \tag{17b}$$

Moreover, $v_{H,I}^{(i)} \in \mathscr{V}_H^{(i)}$ *and the decompositions* (17) *satisfy a stable decomposition property* (6) *with a constant* K *only depending on* $n_{\mathscr{I}}^{(i)}$, τ_λ, *and the shape regularity of* $\mathcal{T}_h$.

Proof. The identities (17) follow by the linearity of I_h and the fact that $I_h v = v$ for all $v \in \mathscr{V}$.

$v_{H,I}^{(i)} \in \mathscr{V}_H^{(i)}$ follows directly from the definitions of $v_{H,j}^{(i)}$ and $\mathscr{V}_H^{(i)}$.

For showing stability we need to reduce decompositions (17) to the case (8). Thus, it suffices to show that for any $v \in \mathscr{V}$ we have that

$$a_\Omega\Big(I_h(\xi_j^{(i)} v), I_h(\xi_j^{(i)} v)\Big) \leq C\, a_\Omega\Big(\xi_j^{(i)} v, \xi_j^{(i)} v\Big), \tag{18}$$

with a constant C only depending on the mesh regularity of $\mathcal{T}_h$. By [15, Proposition 15] (see also [4, Lemma 4.5.3]) we know that (18) is satisfied. □

Remark 3.4. Note that for the additive Schwarz preconditioner corresponding to the second variant of a stable decomposition, i.e., (17b), the local solves are carried out with respect to $\mathscr{V}_0(\Omega_j^{(1)})$, $j = 1, \ldots, n_\Omega^{(1)}$, whereas by the definition of $\mathscr{V}_H^{(2)}$ we see that the supports of the coarse basis functions are given by $\Omega_j^{(2)}$, $j = 1, \ldots, n_\Omega^{(2)}$. That is, the subdomains of the local solves do not need to coincide with the supports of the coarse basis functions.

This observation is in contrast to the first variant of a stable decomposition, i.e., (17a), where the support of the coarse basis functions is given by $\Omega_j^{(1)}$, $j = 1, \ldots, n_\Omega^{(1)}$, corresponding to the spaces of the local solves, i.e., $\mathscr{V}_0(\Omega_j^{(1)})$, $j, \ldots, n_\Omega^{(1)}$.

Remark 3.5. For actual numerical computations it is important to have a basis of $\mathscr{V}_H^{(i)}$ available. Definition (16) obviously provides a generating set of our spectral coarse space. Note, however, that even though the generalized eigenfunctions $\varphi_{j,\lambda}^{(i)}$ are mutually $a_{\Omega_j}(\cdot,\cdot)$ orthogonal, it is not clear that the generating set in (16) also constitutes a basis. In fact, in particular for anisotropic problems (see [28]) it is discussed that this generating set may not be minimal. Nevertheless, for simplicity we assume in the following that the set given in (16) constitutes a basis and refer to [28] for the more general situation.

4 Analysis of Spectral Coarse Space Dimensions

After establishing the robustness of the additive Schwarz preconditioner given by Algorithm 1 and utilizing the coarse spaces $\mathcal{V}_H^{(i)}$ it is important to analyze the dimension of these coarse spaces. This is in particular crucial for the overall computational complexity of the method, and it is generally desirable to keep the dimension of $\mathcal{V}_H^{(i)}$ as small as possible.

By construction the dimension of $\mathcal{V}_H^{(i)}$ is determined by the number of generalized eigenvalues below the threshold τ_λ^{-1} (see Proposition 3.2). We now investigate the number of these "small" eigenvalues for binary geometries and for different choices of subdomains and partition of unities. For this we first recall the well-known min–max/Courant–Fischer principle (see e.g. [14, Theorem 7.36]), which states that

$$\lambda_{j,k}^{(i)} = \min_{\mathcal{V}_k(\Omega_j^{(i)}) \subset \mathcal{V}(\Omega_j^{(i)})} \max_{v \in \mathcal{V}_k(\Omega_j^{(i)})} \frac{a_{\Omega_j^{(i)}}(v,v)}{m_{\Omega_j^{(i)}}^{(i)}(v,v)}, \tag{19}$$

where $\mathcal{V}_k(\Omega_j^{(i)})$ for $k \geq 1$ is a k-dimensional subspace of $\mathcal{V}(\Omega_j^{(i)})$ and $\lambda_{j,k}^{(i)}$ denotes the k-th eigenvalue of (14) sorted in increasing order accounting for multiplicity.

By our assumption of having a binary medium we know that $\overline{\Omega} = \overline{\Omega^p} \cup \overline{\Omega^s}$ such that

$$\kappa(\boldsymbol{x}) = \begin{cases} \kappa_{\max}, \ \boldsymbol{x} \in \Omega^s \\ \kappa_{\min}, \ \boldsymbol{x} \in \Omega^p, \end{cases}$$

which in particular means that the contrast $\kappa_{\max}/\kappa_{\min}$ is the problem parameter of interest.

For simplicity of the exposition we restrict to the case when $\{\Omega_j^{(2)}\}_{j=1}^{n_\Omega^{(2)}} = \{\Omega_j^{(1)}\}_{j=1}^{n_\Omega^{(1)}}$ and $\{\xi_j^{(2)}\}_{j=1}^{n_\Omega^{(2)}} = \{\xi_j^{(1)}\}_{j=1}^{n_\Omega^{(1)}}$. Thus, without any danger of confusion we may drop the superindices $^{(1)}$ and $^{(2)}$ distinguishing different families of subdomains and partition of unity functions. Note, however, that even with this simplification the bilinear forms $m_{\Omega_j}^{(1)}(\cdot,\cdot)$ and $m_{\Omega_j}^{(2)}(\cdot,\cdot)$ are not identical.

Furthermore, let $\Omega_j^p := \Omega^p \cap \Omega_j$ and similarly $\Omega_j^s := \Omega^s \cap \Omega_j$. Besides, we set

$$\Omega_j^{\text{int}} := \Omega_j \backslash \Big(\bigcup_{k \neq j} \overline{\Omega}_k\Big).$$

Note that we do not exclude the possibility that $\Omega_j^{\text{int}} = \emptyset$. Additionally, let $\Omega_{j,k}^s$, $k = 1, \ldots, L_j$ be the path-connected components of Ω_j^s, where we assume an ordering such that $\Omega_{j,k}^s \backslash \Omega_j^{\text{int}} \neq \emptyset$ for $k = 1, \ldots, \tilde{L}_j$, where $\tilde{L}_j \leq L_j$ is suitably chosen. If $\Omega_{j,k}^s \backslash \Omega_j^{\text{int}} = \emptyset$ for $k = 1, \ldots, L_j$ we set $\tilde{L}_j = 1$ and $\Omega_{j,1}^s = \Omega_j \backslash \Omega_j^{\text{int}}$. The diameter of the subdomains $\{\Omega_j\}_{j=1}^{n_\Omega}$ is assumed to be $\mathcal{O}(H)$, and the width of the overlaps of intersecting subdomains is assumed to be $\mathcal{O}(\delta)$. For a better understanding of these definitions we refer to Fig. 1.

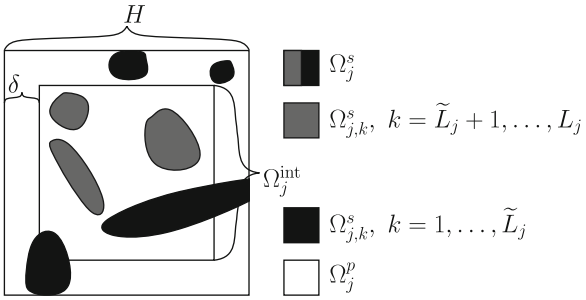

Fig. 1 Subdomain Ω_j with connected components of Ω_j^s and non-overlapping part Ω_j^{int}. In the present configuration $L_j = 7$ and $\tilde{L}_j = 4$

Let

$$\mathcal{V}_{\tilde{L}_j}^c(\Omega_j) := \{v \in \mathcal{V}(\Omega_j) \,|\, \int_{\Omega_{j,k}^s} v\, d\boldsymbol{x} = 0 \text{ for } k = 1,\ldots,\tilde{L}_j\}.$$

Obviously, any $\tilde{L}_j + 1$-dimensional subspace of $\mathcal{V}(\Omega_j)$ has a nontrivial intersection with $\mathcal{V}_{\tilde{L}_j}^c(\Omega_j)$. Thus, by (19) we see that there exists a $w \in \mathcal{V}_{\tilde{L}_j}^c(\Omega_j)$ such that

$$\lambda_{j,\tilde{L}_j+1}^{(i)} \geq \frac{a_{\Omega_j}(w,w)}{m_{\Omega_j}^{(i)}(w,w)}. \tag{20}$$

We first consider the case $(i) = (1)$ and note that by Schwarz' inequality

$$m_{\Omega_j}^{(1)}(w,w) = \int_{\Omega_j} \kappa(\nabla(\xi_j w))^2\, d\boldsymbol{x} \leq 2\int_{\Omega_j} \kappa w^2 (\nabla\xi_j)^2\, d\boldsymbol{x} + 2\underbrace{\int_{\Omega_j} \kappa\xi_j^2 (\nabla w)^2\, d\boldsymbol{x}}_{\leq a_{\Omega_j}(w,w)}. \tag{21}$$

Since $\xi_j \equiv 1$ in Ω_j^{int} we have that

$$\begin{aligned}
\int_{\Omega_j} \kappa w^2 (\nabla\xi_j)^2\, d\boldsymbol{x} &= \int_{\Omega_j\setminus\Omega_j^{\text{int}}} \kappa w^2 (\nabla\xi_j)^2\, d\boldsymbol{x} \\
&\leq C\delta^{-2} \int_{\Omega_j\setminus\Omega_j^{\text{int}}} \kappa w^2\, d\boldsymbol{x} \\
&\leq C\delta^{-2} \left(\sum_{k=1}^{\tilde{L}_j} \int_{\Omega_{j,k}^s} \kappa_{\max} w^2\, d\boldsymbol{x} + \int_{\Omega_j\setminus\Omega_j^{\text{int}}} \kappa_{\min} w^2\, d\boldsymbol{x} \right) \\
&\leq C \left(\frac{H}{\delta}\right)^2 \left(\sum_{k=1}^{\tilde{L}_j} \int_{\Omega_{j,k}^s} \kappa_{\max} (\nabla w)^2\, d\boldsymbol{x} + \int_{\Omega_j\setminus\Omega_j^{\text{int}}} \kappa_{\min} (\nabla w)^2\, d\boldsymbol{x} \right) \\
&\leq C \left(\frac{H}{\delta}\right)^2 a_{\Omega_j}(w,w)\,,
\end{aligned} \tag{22}$$

where we have used Poincaré's inequality, which is possible since $w \in \mathcal{V}^c_{\tilde{L}_j}(\Omega_j)$, and where C is independent of H, h, δ, and $\kappa_{\max}/\kappa_{\min}$.

Similarly, but again slightly more complicated, we obtain for $(i) = (2)$

$$\begin{aligned} m^{(2)}_{\Omega_j}(w,w) &= \sum_{k:\Omega_k\cap\Omega_j\neq\emptyset}\int_{\Omega_j}\kappa(\nabla(\xi_j\xi_k w))^2\,d\boldsymbol{x} \\ &\le 2\sum_{k:\Omega_k\cap\Omega_j\neq\emptyset}\int_{\Omega_j}\kappa w^2(\nabla(\xi_j\xi_k))^2+\kappa(\nabla w)^2(\xi_j\xi_k)^2\,d\boldsymbol{x} \\ &\le 4\int_{\Omega_j}\kappa w^2(\nabla\xi_j)^2\,d\boldsymbol{x}+4\sum_{k:\Omega_k\cap\Omega_j\neq\emptyset}\int_{\Omega_j\cap\Omega_k}\kappa w^2(\nabla\xi_k)^2\,d\boldsymbol{x}+2a_{\Omega_j}(w,w)\,. \end{aligned}$$

Noting that

$$\begin{aligned} \int_{\Omega_j\cap\Omega_k}\kappa w^2(\nabla\xi_k)^2\,d\boldsymbol{x} &\le C\delta^{-2}\begin{cases}\displaystyle\int_{\Omega_j\setminus\Omega_j^{\mathrm{int}}}\kappa w^2\,d\boldsymbol{x}, \text{ if } j=k\\ \displaystyle\int_{\Omega_j\cap\Omega_k}\kappa w^2\,d\boldsymbol{x}, \text{ if } j\neq k\end{cases} \\ &\le C\delta^{-2}\int_{\Omega_j\setminus\Omega_j^{\mathrm{int}}}\kappa w^2\,d\boldsymbol{x} \end{aligned}$$

we thus obtain by (22) that

$$m^{(2)}_{\Omega_j}(w,w)\le C\left(\frac{H}{\delta}\right)^2 a_{\Omega_j}(w,w)\,. \tag{23}$$

where C is again independent of H, h, δ, and $\kappa_{\max}/\kappa_{\min}$.

Combining (20), (21), and (22) on the one hand and (20) and (23) on the other hand we thus obtain

$$\lambda^{(i)}_{j,\tilde{L}_j+1}\ge C\left(\frac{\delta}{H}\right)^2. \tag{24}$$

Hence, choosing $\delta=\mathcal{O}(H)$ yields a lower bound of $\lambda^{(i)}_{j,\tilde{L}_j+1}$, which is independent of mesh parameters H and h as well as of the contrast $\kappa_{\max}/\kappa_{\min}$, which is our problem parameter of interest.

4.1 Choice of the Bilinear Form $m_{\Omega_j}(\cdot,\cdot)$

So far, we have carried out our analysis for the bilinear forms $m^{(i)}_{\Omega_j}(\cdot,\cdot)$, $i=1,2$, defined by (11) and (12), respectively. Now, we generalize this choice to any bilinear form $\overline{m}_{\Omega_j}(\cdot,\cdot)$ satisfying

$$m^{(i)}_{\Omega_j}(v,v)\le Ca_{\Omega_j}(v,v)+\overline{m}_{\Omega_j}(v,v)\quad \text{for any } v\in\mathcal{V}(\Omega_j) \tag{25}$$

for $i = 1$ or $i = 2$. Now, analogously to (14) consider the corresponding generalized eigenvalue problem

$$\text{Find } (\overline{\varphi}_{j,\lambda}, \lambda) \text{ such that } a_{\Omega_j^{(i)}}\Big(w, \overline{\varphi}_{j,\lambda}\Big) = \lambda\, \overline{m}_{\Omega_j^{(i)}}\Big(w, \overline{\varphi}_{j,\lambda}\Big) \text{ for all } w \in \mathscr{V}(\Omega_j^{(i)}).$$

In exactly the same way as (15) in Proposition 3.2 we then obtain

$$\overline{m}_{\Omega_j^{(i)}}(v - \overline{v}_{H,j}, v - \overline{v}_{H,j}) \leq \tau_\lambda\, a_{\Omega_j^{(i)}}(v - \overline{v}_{H,j}, v - \overline{v}_{H,j}) \leq \tau_\lambda\, a_{\Omega_j^{(i)}}(v, v)\,,$$

where $\overline{v}_{H,j} \in \mathscr{V}(\Omega_j)$ denotes the $a_{\Omega_j}(\cdot,\cdot)$-orthogonal projection of $v|_{\Omega_j}$ onto the span of those eigenfunctions $\overline{\varphi}_{j,\lambda}$ for which $\lambda \leq \tau_\lambda^{-1}$. Using (25) together with this estimate we therefore obtain

$$\begin{aligned} m_{\Omega_j}^{(i)}(v - \overline{v}_{H,j}, v - \overline{v}_{H,j}) &\leq C a_{\Omega_j}(v - \overline{v}_{H,j}, v - \overline{v}_{H,j}) + \overline{m}_{\Omega_j}(v - \overline{v}_{H,j}, v - \overline{v}_{H,j}) \\ &\leq (C + \tau_\lambda)\, a_{\Omega_j}(v - \overline{v}_{H,j}, v - \overline{v}_{H,j}) \\ &\leq (C + \tau_\lambda)\, a_{\Omega_j}(v, v)\,. \end{aligned}$$

That is, up to a change in the constant we obtain the same estimate as (15), which implies that in the coarse space construction of our robust preconditioner we may use $\overline{\varphi}_{j,\lambda}$ instead of $\varphi_{j,\lambda}^{(i)}$ in the definition of $\mathscr{V}_H^{(i)}$ (see (16)).

Looking at (21) we see that (25) is satisfied for $(i) = (1)$ and

$$\overline{m}_{\Omega_j}(v, w) := 2 \int_{\Omega_j} \kappa (\nabla \xi_j)^2 v w \, d\boldsymbol{x},$$

which is essentially the choice made in [15, 16].

4.2 *Partition of Unity vs. Partition of Identity*

As indicated in Remark 3.1 the authors of [9] advocate the use of partition of identity operators $\{\Xi_j\}_{j=1}^{n_\Omega}$ instead of partition of unity functions $\{\xi_j\}_{j=1}^{n_\Omega}$. We now elaborate on the changes that this modification necessitates in the analysis of the coarse space dimension.

In the following we consider the case when

$$\Xi_j v := I_h(\xi_j v),$$

where v is either an element of $\mathscr{V}$ or $\mathscr{V}(\Omega_j)$. Instead of $m_{\Omega_j}^{(1)}(\cdot,\cdot)$ given by (12) we then consider $a_{\Omega_j}(\Xi_j w, \Xi_j w)$ for which we obtain

$$a_{\Omega_j}(\Xi_j w, \Xi_j w) = a_{\Omega_j}(I_h(\xi_j w), I_h(\xi_j w)) \leq C a_{\Omega_j}(\xi_j w, \xi_j w) = C m_{\Omega_j}^{(1)}(w, w)\,,$$

where we have used estimate (18). The remainder of the analysis proceeds along the lines of (21) and (22).

We note that when using partition of unity functions estimate (18) is needed for establishing the stable decomposition property when employing a coarse space $\mathcal{V}_H^{(i)}$ defined in (16) (see Proposition 3.3). When using partition of identity operators the same estimate is necessary for analyzing the number of asymptotically small (w.r.t. the contrast $\kappa_{\max}/\kappa_{\min}$) generalized eigenvalues and thus the dimension of $\mathcal{V}_H^{(i)}$.

4.3 Choice of the Subdomains

Concerning the choice of the subdomains estimate (24) admits several observations. First of all we see that regardless of the choice of δ the lower bound for $\lambda_{j,\tilde{L}_j+1}^{(i)}$ is independent of the contrast $\kappa_{\max}/\kappa_{\min}$. Our computational experience confirms that this bound on the eigenvalue index is sharp in the sense that $\lambda_{j,k}^{(i)} \to 0$ as $\kappa_{\max}/\kappa_{\min} \to \infty$ for $k = 1, \ldots, \tilde{L}_j$. For (very) high-contrast problems one may therefore expect a "gap" in the spectrum and to recover $\tilde{L}_j$ "small" eigenvalues below τ_λ^{-1} provided this threshold is chosen to lie within this spectral gap.

These considerations imply the following tradeoff regarding the choice of δ. On the one hand one would like to choose δ small, e.g., $\delta = \mathcal{O}(h)$, in order to have Ω_j^{int} as large and thus $\tilde{L}_j$ as small as possible. The latter is desirable, since one is generally interested in a small dimensional coarse space $\mathcal{V}_H$.

On the other hand choosing δ (very) small leads to a (very) small lower bound in (24). In particular choosing a minimal overlap of one layer of fine cells $T \in \mathcal{T}_h$—or more generally $\delta = \mathcal{O}(h)$—results in a lower bound for $\lambda_{j,\tilde{L}_j+1}^{(i)}$ that depends on the mesh parameters and degenerates as $H/h \to \infty$. The occurrence of a spectral gap therefore depends on the relation of H/h and $\kappa_{\max}/\kappa_{\min}$. Hence for a given threshold τ_λ^{-1} and H/h sufficiently large one may in fact recover more "small" eigenvalues than $\tilde{L}_j$, which may ultimately result in a larger dimensional coarse space.

4.4 Eigenvalue Problems in Overlaps of Subdomains

For the case of $\Omega_j^{\text{int}} \neq \emptyset$ a further modification is suggested in [9], which results in a reduction of the number of degrees of freedom involved in the solution of the generalized eigenvalue problem (14). To achieve this one may proceed as follows:

Let

$$\tilde{\mathcal{V}}(\Omega_j) := \{v \in \mathcal{V}(\Omega_j) \,|\, a_{\Omega_j}(v, w) = 0 \; \forall w \in \mathcal{V}_0(\Omega_j^{\text{int}})\}.$$

Note that by construction we have that

$$\mathcal{V}(\Omega_j) = \mathcal{V}_0(\Omega_j^{\text{int}}) \oplus \tilde{\mathcal{V}}(\Omega_j) \quad \text{and} \quad \mathcal{V}_0(\Omega_j^{\text{int}}) \perp_a \tilde{\mathcal{V}}(\Omega_j) \tag{26}$$

and that for small overlaps δ the dimension of $\tilde{\mathscr{V}}(\Omega_j)$ may be much smaller than the dimension of $\mathscr{V}(\Omega_j)$.

Now, consider the following modification of the generalized eigenvalue problem (14) posed with respect to $\tilde{\mathscr{V}}(\Omega_j)$ instead of $\mathscr{V}(\Omega_j)$, i.e.,

$$\text{Find } (\tilde{\varphi}_{j,\lambda}^{(i)}, \lambda) \in \tilde{\mathscr{V}}(\Omega_j) \times \mathbb{R}_0^+ \text{ s.t. } a_{\Omega_j}\Big(w, \tilde{\varphi}_{j,\lambda}^{(i)}\Big) = \lambda\, m^{(i)}_{\Omega_j \setminus \Omega_j^{\text{int}}}\Big(w, \tilde{\varphi}_{j,\lambda}^{(i)}\Big) \;\; \forall w \in \tilde{\mathscr{V}}(\Omega_j), \tag{27}$$

where

$$m^{(1)}_{\Omega_j \setminus \Omega_j^{\text{int}}}(v, w) := a_{\Omega \setminus \Omega_j^{\text{int}}}(\xi_j v, \xi_j w)$$

and

$$m^{(2)}_{\Omega_j \setminus \Omega_j^{\text{int}}}(v, w) := \sum_{k:\Omega_k \cap \Omega_j \neq \emptyset} a_{\Omega_j \setminus \Omega_j^{\text{int}}}(\xi_j \xi_k v, \xi_j \xi_k w)\,,$$

respectively. Note that with these definitions we have

$$m^{(i)}_{\Omega_j}(v, w) = m^{(i)}_{\Omega_j \setminus \Omega_j^{\text{int}}}(v, w) + a_{\Omega_j^{\text{int}}}(v, w)\,. \tag{28}$$

Furthermore, for the solvability of (27) it is again important to note that $m^{(i)}_{\Omega_j \setminus \Omega_j^{\text{int}}}(\cdot, \cdot)$ is positive definite on $\tilde{\mathscr{V}}(\Omega_j)$. This follows from the fact that $\text{supp}(\xi_j) = \overline{\Omega}_j$ by assumption and $v|_{\Omega_j \setminus \Omega_j^{\text{int}}} \not\equiv 0$ for all $v \in \tilde{\mathscr{V}}(\Omega_j) \setminus \{0\}$ by construction.

According to the analysis in Sect. 3 we need to prove a statement analogous to that of Proposition 3.2.

Proposition 4.1. *For $v \in \mathscr{V}$ let $\tilde{v}_{H,j}^{(i)} := \tilde{\Pi}_j^{(i)} v \in \tilde{\mathscr{V}}(\Omega_j)$ be the $a_{\Omega_j}(\cdot,\cdot)$-orthogonal projection of $v|_{\Omega_j}$ onto those eigenfunctions of (27) corresponding to eigenvalues below $\tau_\lambda^{-1} > 0$, i.e., $\tilde{\Pi}_j^{(i)} v \in \text{span}\{\tilde{\varphi}_{j,\lambda}^{(i)} \,|\, \lambda < \tau_\lambda^{-1}\}$ satisfies*

$$a_{\Omega_j}\Big(v - \tilde{\Pi}_j^{(i)} v, \tilde{\varphi}_{j,\lambda}^{(i)}\Big) = 0 \text{ for all } \lambda < \tau_\lambda^{-1}.$$

Then we have that

$$m^{(i)}_{\Omega_j}\Big(v - \tilde{v}_{H,j}^{(i)}, v - \tilde{v}_{H,j}\Big) \leq (1 + \tau_\lambda)\, a_{\Omega_j}\Big(v - \tilde{v}_{H,j}^{(i)}, v - \tilde{v}_{H,j}^{(i)}\Big) \leq (1 + \tau_\lambda)\, a_{\Omega_j}(v, v)\,. \tag{29}$$

Proof. The second inequality in (29) is obvious for the same reason as the second inequality in (15).

By (28) we have that

$$\begin{aligned} &m^{(i)}_{\Omega_j}\Big(v - \tilde{\Pi}_j^{(i)} v, v - \tilde{\Pi}_j^{(i)}\Big) \\ &= m^{(i)}_{\Omega_j \setminus \Omega_j^{\text{int}}}\Big(v - \tilde{\Pi}_j^{(i)} v, v - \tilde{\Pi}_j^{(i)}\Big) + \underbrace{a_{\Omega_j^{\text{int}}}\Big(v - \tilde{\Pi}_j^{(i)} v, v - \tilde{\Pi}_j^{(i)} v\Big)}_{\leq a_{\Omega_j}\left(v - \tilde{\Pi}_j^{(i)} v, v - \tilde{\Pi}_j^{(i)} v\right)}. \end{aligned}$$

Thus, it remains to show that

$$m^{(i)}_{\Omega_j\setminus\Omega_j^{\mathrm{int}}}\Big(v-\tilde{\Pi}_j^{(i)}v,\, v-\tilde{\Pi}_j^{(i)}v\Big) \le \tau_\lambda\, a_{\Omega_j}\Big(v-\tilde{\Pi}_j^{(i)}v,\, v-\tilde{\Pi}_j^{(i)}v\Big)\,. \tag{30}$$

By a reasoning identical to that of Proposition 3.2 it follows that (30) holds for all $v \in \tilde{\mathscr{V}}(\Omega_j)$.

For general $v \in \mathscr{V}(\Omega_j)$ consider the unique $a_{\Omega_j}(\cdot,\cdot)$-orthogonal decomposition $v = v^{\mathrm{int}} + \tilde{v}$ with $v^{\mathrm{int}} \in \mathscr{V}_0(\Omega_j^{\mathrm{int}})$ and $\tilde{v} \in \tilde{\mathscr{V}}(\Omega_j)$. By the $a_{\Omega_j}(\cdot,\cdot)$-orthogonality of $\mathscr{V}_0(\Omega_j^{\mathrm{int}})$ and $\tilde{\mathscr{V}}(\Omega_j)$ and since $\tilde{\Pi}_j^{(i)}$ is an $a_{\Omega_j}(\cdot,\cdot)$-orthogonal projection onto a subspace of $\tilde{\mathscr{V}}(\Omega_j)$ it easily follows that $\tilde{\Pi}_j^{(i)}v = \tilde{\Pi}_j^{(i)}\tilde{v}$. Thus, and since $\mathrm{supp}(v^{\mathrm{int}}) \subset \overline{\Omega}_j^{\mathrm{int}}$ we have that

$$\begin{aligned}
m^{(i)}_{\Omega_j\setminus\Omega_j^{\mathrm{int}}}\Big(v-\tilde{\Pi}_j^{(i)}v,\, v-\tilde{\Pi}_j^{(i)}v\Big) &= m^{(i)}_{\Omega_j\setminus\Omega_j^{\mathrm{int}}}\Big(v^{\mathrm{int}}+\tilde{v}-\tilde{\Pi}_j^{(i)}\tilde{v},\, v^{\mathrm{int}}+\tilde{v}-\tilde{\Pi}_j^{(i)}\tilde{v}\Big)\\
&= m^{(i)}_{\Omega_j\setminus\Omega_j^{\mathrm{int}}}\Big(\tilde{v}-\tilde{\Pi}_j^{(i)}\tilde{v},\, \tilde{v}-\tilde{\Pi}_j^{(i)}\tilde{v}\Big)\\
&\le \tau_\lambda\, a_{\Omega_j}\Big(\tilde{v}-\tilde{\Pi}_j^{(i)}\tilde{v},\, \tilde{v}-\tilde{\Pi}_j^{(i)}\tilde{v}\Big)\\
&\le \tau_\lambda\, a_{\Omega_j}\Big(v^{\mathrm{int}}+\tilde{v}-\tilde{\Pi}_j^{(i)}v,\, v^{\mathrm{int}}+\tilde{v}-\tilde{\Pi}_j^{(i)}v\Big)\\
&= \tau_\lambda\, a_{\Omega_j}\Big(v-\tilde{\Pi}_j^{(i)}v,\, v-\tilde{\Pi}_j^{(i)}v\Big)\,,
\end{aligned}$$

where the first inequality holds, since (30) is satisfied for $v \in \tilde{\mathscr{V}}(\Omega_j)$, and the second inequality follows by $a_{\Omega_j}(\cdot,\cdot)$-orthogonality. □

In view of Proposition 4.1 we may perform the same reasoning as in Sect. 3 with $v^{(i)}_{H,j}$ replaced by $\tilde{v}^{(i)}_{H,j}$, and we thus obtain an additive Schwarz preconditioner with a coarse space given by $\tilde{\mathscr{V}}_H^{(i)} := \mathrm{span}\{I_h(\xi_j^{(i)}\tilde{\varphi}^{(i)}_{j,\lambda})\,|\,\lambda < \tau_\lambda^{-1},\ j = 1,\dots,n_\Omega^{(i)}\}$ yielding a condition number independent of problem and mesh parameters.

4.5 *Choice of the Partition of Unity*

So far, in the derivations of this section we have tacitly assumed that the choice of our partition of unity functions only depends on the subdomains $\{\Omega_j\}_{j=1}^{n_\Omega}$. According to estimate (24) this choice is certainly viable. As a matter of fact, it is necessary to have $\xi_j \equiv 1$ in Ω_j^{int}.

Nevertheless, in particular for large overlaps δ one may consider to choose $\{\xi_j\}_{j=1}^{n_\Omega}$ in a problem, i.e., κ, dependent way. The objective of such an approach, which was first considered in [16], is to reduce the number of asymptotically small (w.r.t. $\kappa_{\max}/\kappa_{\min}$) eigenvalues without introducing a degeneracy due to an increasingly smaller overlap δ.

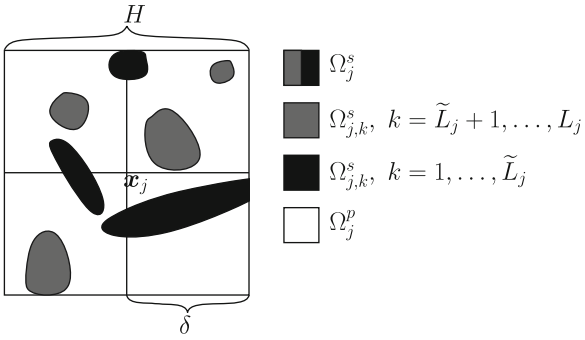

Fig. 2 Subdomain Ω_j with connected components of Ω_j^s. Due to the large overlap $\Omega_j^{\text{int}} = \emptyset$. In the present configuration $L_j = 7$ and $\tilde{L}_j = 3$

Let us consider a coarse grid $\mathcal{T}_H$ of cells obtained by agglomerating fine cells in $\mathcal{T}_h$ (cf. [26, Sect. 1.9] for a description of an agglomeration procedure). The agglomerate coarse cells are assumed to have diameters $\mathcal{O}(H)$. We consider an overlapping decomposition $\{\Omega_j\}_{j=1}^{n_\Omega}$ of Ω, where each subdomain Ω_j is associated with a coarse node $\boldsymbol{x}_j$ and is given by $\Omega_j := \mathit{interior}(\cup\{T \in \mathcal{T}_H \,|\, \boldsymbol{x}_j \in T\})$, i.e., the union of all cells $T \in \mathcal{T}_H$ containing this coarse node. Thus, we obviously have that $\delta = \mathcal{O}(H)$ and $\Omega_j^{\text{int}} = \emptyset$.

In the following we outline the construction of a multiscale partition of unity—henceforth denoted by $\{\xi_j^{\text{ms}}\}_{j=1}^{n_\Omega}$. Let ξ_j^{ms} satisfy $\nabla \cdot (\kappa \nabla \xi_j^{\text{ms}}) = 0$ in those $T \in \mathcal{T}_H$ for which $T \subset \overline{\Omega}_j$. Here we assume that $\xi_j^{\text{ms}}|_T$ satisfies suitable boundary conditions on ∂T, which are chosen in such a way that $\sum_{j=1}^{n_\Omega} \xi_j^{\text{ms}} \equiv 1$. One may for instance think of the boundary conditions as being given by the solutions of lower dimensional problems along the agglomerate edges constituting the boundary of T. Here we suppose that ξ_j^{ms} constructed in this way satisfies $0 \le \xi_j^{\text{ms}} \le 1$, which is guaranteed if the validity of a discrete maximum principle is assumed. For a more general situation we refer to [27, Sect. 5].

As above we denote by $\Omega_{j,k}^s$, $k = 1, \ldots, L_j$, the path-connected components of Ω_j^s. This time we assume an ordering such that those $\Omega_{j,k}^s$ are ordered first for which it holds that $\Omega_{j,k}^s \cap (\partial T \backslash \partial \Omega_j) \neq \emptyset$ for some $T \in \mathcal{T}_H$ with $T \subset \overline{\Omega}_j$. The number of these path-connected components of Ω_j^s is denoted by $\tilde{L}_j \le L_j$. We refer to Fig. 2 for a better understanding of the current setting. The idea of this construction is that $(\kappa \nabla \xi_j^{\text{ms}})|_{\Omega_{j,k}^s}$, $k = \tilde{L}_j + 1, \ldots, L_j$ is small, which seems desirable when looking at the definition of $\overline{m}_{\Omega_j}(\cdot,\cdot)$. More precisely, it is shown in [12, Sect. 5] that provided

$$\left\|\nabla \xi_j^{\text{ms}}\right\|_{L^\infty(\Omega_j)} \le CH^{-1} \quad \text{and} \quad \left\|\kappa_{\max} \nabla \xi_j\right\|_{L^\infty(\Omega_{j,k}^s)} \le CH^{-1},\ \forall k = \tilde{L}_j + 1, \ldots, L_j \tag{31}$$

we have that

$$\lambda_{j,\tilde{L}_j+1}^{(i)} \geq C > 0,$$

where C is independent of $\kappa_{\max}/\kappa_{\min}$, δ, H, and h. Although a rigorous analysis clarifying the question when (31) can be expected to hold is still a largely unsolved problem for general coefficients κ, the computational practice shows that using multiscale partition of unity functions as opposed to standard ones may significantly reduce the coarse space dimension, while maintaining the robustness of the overall preconditioner.

Remark 4.2. It should be noted here that the analysis above generalizes to different symmetric positive definite bilinear forms corresponding, e.g., to the equations of linear elasticity or the *curl–curl* equation with a positive L^2-term arising in the solution of Maxwell's equations (see [27]). The major difficulty in a rigorous, fully discrete analysis is the establishment of an estimate analogous to (18). Also, the construction of a suitable (multiscale) partition of unity/identity resulting in small dimensional coarse spaces has not been addressed in the literature, so far.

5 Numerical Experiments

We now turn to some numerical experiments to exemplify the robustness of two-level additive Schwarz preconditioners using spectral coarse spaces. To demonstrate the necessity of employing this spectral coarse space we also report numerical results for two-level additive Schwarz preconditioners using standard coarse spaces and coarse spaces spanned by multiscale finite element functions. More precisely, we consider the following four different cases

- $\mathcal{V}_H^{\mathrm{st}} := \mathrm{span}\{\xi_j \,|\, j = 1, \ldots, n_\Omega\}$ (cf. [21, Sect. 2.5.3]).
- $\mathcal{V}_H^{\mathrm{ms,st}} := \mathrm{span}\{\xi_j^{\mathrm{ms}} \,|\, j = 1, \ldots, n_\Omega\}$ (cf. [17]).
- $\mathcal{V}_H := \mathrm{span}\{I_h(\xi_j \varphi_{j,\lambda}) \,|\, \lambda < \tau_\lambda^{-1},\ j = 1, \ldots, n_\Omega\}$ (see (16)).
- $\mathcal{V}_H^{\mathrm{ms}} := \mathrm{span}\{I_h(\xi_j^{\mathrm{ms}} \varphi_{j,\lambda}) \,|\, \lambda < \tau_\lambda^{-1},\ j = 1, \ldots, n_\Omega\}$ (see Sect. 4.5),

with $\{\xi_j\}_{j=1}^{n_\Omega}$ a standard partition of unity and $\{\xi_j^{\mathrm{ms}}\}_{j=1}^{n_\Omega}$ as in Sect. 4.5. The subdomains are chosen as described in Sect. 4.5, and the bilinear form $m_{\Omega_j}(\cdot,\cdot)$ is chosen as $1/2\,\overline{m}_{\Omega_j}(\cdot,\cdot)$ in Sect. 4.1 with the standard partition of unity ξ_j and the multiscale partition of unity ξ_j^{ms}, respectively. The eigenvalue threshold is fixed by setting $\tau_\lambda = 2$.

On our computational domain $\Omega := (0,1)^2$ we use a 256×256 fine and a 16×16 coarse tensor grid. The problems under consideration are discretized using bilinear Lagrange finite elements. We emphasize that there is no essential difficulty in treating more realistic settings. In particular one can consider the case of an unstructured two- or three-dimensional fine grid and a corresponding coarse grid resulting from an agglomeration procedure as, e.g., outlined in [26, Sect. 1.9].

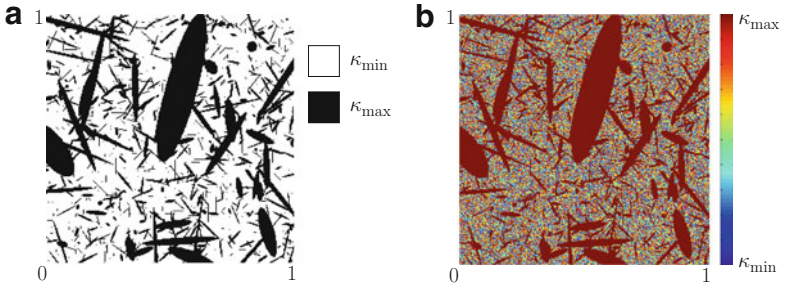

Fig. 3 κ for two random geometries. (**a**) κ for a binary random multiscale geometry. (**b**) Logarithmic plot of κ for a non-binary random multiscale geometry

We choose two different configurations for κ. The first geometry depicted in Fig. 3a is a binary one, i.e., κ only takes two values. As opposed to this, Fig. 3b shows a coefficient κ which assumes a multitude of values between $\kappa_{\min}$ and $\kappa_{\max}$. Although both geometries are artificial in the sense that they do not represent any concrete real life application, we consider them to be "hard" test problems. In particular the (highly) varying coefficients represent multiscale features, which is common in, e.g., reservoir simulations.

In our numerical experiments below we consider the cases $\kappa_{\min} = 1$ and $\kappa_{\max} = 1e1, \ldots, 1e6$ to test our preconditioner for robustness. We would also like to point out that the coefficient variations are not aligned with the coarse 16×16 grid.

For completeness we remark that our implementations are carried out in C++ using the deal.II finite element library (cf. [2]), which in turn uses the LAPACK software package (cf. [1]) for solving all appearing direct and eigenvalue problems.

In Table 1(1) we report the results obtained for the binary geometry shown in Fig. 3a. The table shows the condition numbers of the additive Schwarz preconditioned systems, where we employ the different choices of coarse spaces listed at the beginning of this section. The numbers reported in parentheses are the respective coarse space dimensions. As we can see, the condition numbers corresponding to the spaces $\mathcal{V}_H^{\mathrm{st}}$ and $\mathcal{V}_H^{\mathrm{ms,st}}$ increase quite substantially with increasing the contrast $\kappa_{\max}/\kappa_{\min}$. As opposed to this the preconditioners with the spectral coarse spaces $\mathcal{V}_H$ and $\mathcal{V}_H^{\mathrm{ms}}$ yield condition numbers which are robust with respect to the contrast. This robustness comes at the expense of having to solve local generalized eigenvalue problems, which of course can be done completely in parallel, and of having a larger dimensional coarse space, which is in particular pronounced for higher contrasts. We emphasize, however, that this increase in complexity can be significantly reduced by multiscale partition of unity functions, i.e., by using $\mathcal{V}_H^{\mathrm{ms}}$ instead of $\mathcal{V}_H$. For the highest considered contrast the dimension of the former is less than 3 times as large as the dimension of $\mathcal{V}_H^{\mathrm{st}}$ and $\mathcal{V}_H^{\mathrm{ms,st}}$, whereas for the latter the factor is close to 10. As indicated above the dimension of the spectral coarse spaces changes with increasing the contrast. Nevertheless, in coherence with our theory in Sect. 4 this increase appears to reach some saturation for very high contrasts.

Table 1 Condition numbers of the additive Schwarz preconditioned systems for the geometries shown in Fig. 3 with different contrasts $\kappa_{\max}/\kappa_{\min}$. In parentheses we report the coarse space dimension.

(1) Results for Fig. 3a

$\frac{\kappa_{\max}}{\kappa_{\min}}$	$\mathcal{V}_H^{\mathrm{st}}$	$\mathcal{V}_H^{\mathrm{ms,st}}$	$\mathcal{V}_H$	$\mathcal{V}_H^{\mathrm{ms}}$
1e1	4.7e0(225)	4.7e0(225)	4.7e0(279)	4.7e0(276)
1e2	1.2e1(225)	8.2e0(225)	4.9e0(570)	5.3e0(340)
1e3	7.6e1(225)	3.6e1(225)	4.6e0(1477)	5.2e0(547)
1e4	7.2e2(225)	3.4e2(225)	4.7e0(1995)	5.2e0(669)
1e5	6.1e3(225)	3.1e3(225)	4.8e0(2081)	5.2e0(668)
1e6	4.4e4(225)	2.8e4(225)	4.8e0(2093)	5.2e0(665)

(2) Results for Fig. 3b

$\frac{\kappa_{\max}}{\kappa_{\min}}$	$\mathcal{V}_H^{\mathrm{ms}}$
1e1	4.6e0(276)
1e2	4.7e0(273)
1e3	4.9e0(275)
1e4	4.9e0(306)
1e5	5.3e0(380)
1e6	5.4e0(461)

In order to not only test our theory for binary geometries we also consider the coefficient depicted in Fig. 3b. For this geometry we only report the results corresponding to the coarse space $\mathcal{V}_H^{\mathrm{ms}}$ in Table 1(2). As we can see, the condition numbers also behave robustly in this situation. Also, similarly to the binary geometry, we can observe the trend that increasing the contrast tends to increase the dimension of the coarse space. Nevertheless, even for the highest contrast $1e6$ the size of the coarse space is still rather manageable and in particular smaller than the corresponding one for the binary geometry.

We close this section by some comments regarding the computational complexity of the discussed domain decomposition methods using spectral coarse spaces. These remarks apply not only to the considered two-dimensional examples but also to the three-dimensional case.

The bottleneck of the discussed methods is the coarse space construction and in particular the solution of the local generalized eigenvalue problems. As indicated above these eigenvalue problems are solved using LAPACK. The algorithm implemented in the subroutine DSYGVX first reduces the generalized eigenvalue problems to standard ones by performing Cholesky decompositions. The resulting matrices are then reduced to Hessenberg tridiagonal form, which can be done by Householder transformations. A QR-algorithm employing Givens rotations can then be used to compute the actual eigenpairs. The overall complexity of this algorithm is cubic in the number of unknowns.

Even though the generalized eigenvalue problems can be solved in parallel, it may be unreasonably costly to construct a spectral coarse space, if one is only

interested in solving a single problem on a given geometry. However, if one needs to solve many problems on a single geometry, which, e.g., is the case when computing an approximate solution of a time-dependent problem by an implicit time-stepping scheme, constructing a spectral coarse space may be rather reasonable.

If one wants to solve many problems on a single geometry, it makes sense to distinguish between an offline phase, which in particular includes the construction of the spectral coarse space, and an online phase, which is the actual application of the preconditioner. As the computations in the offline phase are only carried out once, the computational cost of the online phase becomes the major concern. Considering the discussed methods we see that one iteration of a two-level algorithm with a coarse space given by $\mathscr{V}_H$ or $\mathscr{V}_H^{\mathrm{ms}}$ is about as expensive as one iteration of a two-level algorithm with a coarse space given by $\mathscr{V}_H^{\mathrm{st}}$ or $\mathscr{V}_H^{\mathrm{ms,st}}$. The only difference making the former somewhat more expensive than the latter is due to the increased coarse space dimension. Although this space dimension is inherently problem dependent, we note that $\dim(\mathscr{V}_H^{\mathrm{ms}})$ remains rather manageable for the considered examples. In view of drastically reduced condition numbers of the preconditioned systems, this slight increase in computational complexity for one iteration in the online phase seems justified. After all, the number of preconditioned conjugate gradient iterations needed to achieve a prescribed accuracy depend on the condition number of the preconditioned system, and one may therefore expect significant overall computational savings by employing two-level preconditioners using spectral coarse spaces.

6 Conclusions

We have given an overview of several recently proposed approaches for constructing spectral coarse spaces for robust preconditioners. For this we have developed a monolithic framework enabling us to detail the similarities and distinctions of the different methods and to discuss their advantages and shortcomings. In this context we have in particular related the more recent abstract works for general symmetric positive definite bilinear forms to the originally introduced concepts and ideas for the scalar elliptic equation. To show the applicability of the discussed analysis, we have presented some numerical examples to validate the theoretical results.

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