Domain Decomposition Scheme for First-Order Evolution Equations with Nonselfadjoint Operators

Petr Vabishchevich and Petr Zakharov

Abstract Domain decomposition iterative methods and implicit schemes are usually used for solving evolution equations. An alternative approach is based on constructing non-iterative method based on special schemes of splitting into subdomains. Such regional-additive schemes are based on the general theory of additive operator-difference schemes. Domain decomposition analogues of the classical schemes of alternating direction method, locally one-dimensional schemes, factored schemes, and regularized vector-additive schemes are used here. The main results in the literature are obtained for time-dependent problems with selfadjoint second-order elliptic operators. This paper discusses the Cauchy problem for first-order evolution equations with nonnegative nonselfadjoint operators in a finite-dimensional Hilbert space. Based on the partition of unity, we have constructed nonnegativity preserving decomposition operators for the respective operator term in the equation. We construct unconditionally stable additive domain decomposition schemes based on the principle of regularization of operator-difference schemes and vector-additive schemes.

Keywords First-order evolution equations • Parabolic partial differencial equation • Domain decomposition method • Difference scheme.

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P. Vabishchevich (✉)
Nuclear Safety Institute of RAS, 52, B. Tulskaya, Moscow, 115191, Russia
e-mail: vabishchevich@gmail.com

P. Zakharov
North-Eastern Federal University, 58, Belinskogo, Yakutsk, 677000, Russia
e-mail: zapetch@gmail.com

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1 Introduction

Domain decomposition methods are often used for the numerical solution of boundary value problems for partial differential equations on parallel computers. The theory of the domain decomposition (DD) methods is mostly developed for stationary problems [11, 12, 24, 25]. Numerous sequential and parallel algorithms for overlapping and nonoverlapping DD methods are developed and analysed in conjunction with such problems.

Domain decomposition methods for unsteady problems are based on two approaches [14]. In the first approach, standard implicit approximation in time is used. After that, domain decomposition methods developed for steady-state problems can be applied for solving the discrete problem on the new time level. In the case of optimal DD iterative methods, the number of iterations does not depend on space and time discretization steps [3, 4]. In the second approach, non-iterative domain decomposition algorithms are constructed for unsteady problems. In some cases, this can be interpreted as performing at each time step only one iteration of the Schwarz alternating method for the approximate solution of boundary value problems for second-order parabolic equation [6, 7]. We also construct a special scheme of splitting into subdomains (regional-additive schemes [26, 27]).

The construction of regional-additive schemes and the investigation of their convergence are based on the general theory of the splitting schemes [10, 13, 34]. Most interesting for the practice is the situation when the operator is split into a sum of three or more noncommutative nonselfadjoint operators. In the case of such a multicomponent splitting, stable additive splitting schemes are constructed based on the concept of additive approximation. Furthermore, additively averaged summarized approximation schemes are interesting, when we focus on parallel computers. In the class of splitting schemes with full approximation [19], we point to the vector-additive schemes, when the original equation is transformed into a system of similar equations [1, 2, 31]. The most suitable approach for constructing additive regularized operator-difference schemes for multicomponent splitting [18, 23] is the one in which the stability is achieved due to perturbations of the operators of the difference scheme.

A domain decomposition scheme is defined by a decomposition of the computational domain and by defining the splitting of the operator. To construct the decomposition operators when solving BVP for PDEs, it is convenient to use a partition of unity for the computational domain [5, 8, 16, 26, 28, 29, 33]. In the overlapping DD methods, a function is associated with each subdomain, and this function takes value between zero and one. Domain decomposition methods for unsteady convection-diffusion problems are studied in the works [17, 20, 30]. In the extreme case, the width of the overlap of the subdomains is equal to the space discretization step. In this case the regionally additive schemes can be interpreted as nonoverlapping domain decomposition schemes, where the exchange is achieved by setting proper boundary conditions for each of the subdomain. Research results on domain decomposition method for unsteady boundary value problems are summarized in the books [14, 19]. From the more recent studies, we mention [32], where DD schemes which are more suitable for computer implementation are presented.

In this paper, we construct a domain decomposition schemes for first-order evolution equations with general nonnegative operator in a finite-dimensional Hilbert space. Decomposition operators are constructed separately for the selfadjoint and for the skew-symmetric part of the operator. The splitting is based on partition of unity in the appropriate spaces. We propose two classes of unconditionally stable regionally additive regularized schemes, and we consider vector-additive operator-difference domain decomposition scheme.

2 The Cauchy Problem for First-Order Evolution Equations

Let H be finite-dimensional real Hilbert space of grid functions, in which the scalar product and the norm are $(\cdot,\cdot)$ $\|\cdot\|$, respectively. Consider a time independent and nonnegative in H grid operator A:

$$A \geq 0, \quad \frac{d}{dt}A = A\frac{d}{dt}. \tag{1}$$

Let us denote by E the identity operator in H. We seek a solution to the Cauchy problem

$$\frac{du}{dt} + Au = f(t), \quad 0 < t \leq T, \tag{2}$$

$$u(0) = u^0. \tag{3}$$

The problem (1)–(3) is obtained after a finite-difference approximation in space of initial boundary value problems (IBVP) for second-order partial differential equations (PDEs). Similar systems of ordinary differential equations arise when finite element method (FEM) or finite volume method (FVM) are used for space discretization.

Let us give a standard a priori estimate for the problem (1)–(3). We take a scalar product in H of the Eq. (2) and u. In view of (1) we arrive at

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 \leq (f,u). \tag{4}$$

Taking into account

$$(f,u) \leq \|f\|\|u\|,$$

from (4) we obtain

$$\frac{d}{dt}\|u\| \leq \|f\|.$$

Using the Gronwall lemma, we obtain the desired estimate

$$\|u\| \leq \|u^0\| + \int_0^t \|f(\theta)\| d\theta, \quad (5)$$

which expresses the stability of the solution to the initial data and right-hand side.

The scope of this work is to present discretizations in time for the Eq. (2). Our discretizations belong to the class of the two-layer schemes. Let τ be the time step and let $y^n = y(t^n)$, $t^n = n\tau$, $n = 0,1,\dots,N$, $N\tau = T$. Equation (2) is approximated by a two-level weighted scheme as follows:

$$\frac{y^{n+1} - y^n}{\tau} + A(\sigma y^{n+1} + (1-\sigma)y^n) = \varphi^n, \quad n = 0,1,\dots,N-1, \quad (6)$$

where, for example, $\varphi^n = f(\sigma t^{n+1} + (1-\sigma)t^n)$. It is supplemented by the initial condition

$$y^0 = u^0. \quad (7)$$

Difference scheme (6), (7) has approximation error $\mathcal{O}(\tau^2 + (\sigma - 0.5)\tau)$. An analogy of (5) for the discretized in time function reads as follows:

$$\|y^{n+1}\| \leq \|y^n\| + \tau \|\varphi^n\|, \quad n = 0,1,\dots,N-1. \quad (8)$$

We prove the following theorem.

Theorem 1. *The difference scheme (1), (6), (7) is unconditionally stable for* $\sigma \geq 0.5$*, and the estimate (8) holds for the solution of the above difference equation.*

Proof. Let us rewrite (6) in the form

$$y^{n+1} = Sy^n + \tau(E + \sigma\tau A)^{-1}\varphi^n, \quad (9)$$

where

$$S = (E + \sigma\tau A)^{-1}(E - (1-\sigma)\tau A) \quad (10)$$

is the operator of the transition to a new time level. From (9) we have

$$\|y^{n+1}\| = \|S\| \, \|y^n\| + \tau \|(E + \sigma\tau A)^{-1}\varphi^n\|. \quad (11)$$

For the last term on the right side of (11), in the class of operators (1), under natural conditions $\sigma \geq 0$, we have

$$\|(E + \sigma\tau A)^{-1}\varphi^n\| \leq \|\varphi^n\|.$$

Let us show that if $\sigma \geq 0.5$, for nonnegative operator A, it holds

$$\|S\| \leq 1. \quad (12)$$

In real Hilbert space H, the inequality (12) is equivalent to [9] the fulfilment of the operator inequality

$$SS^* \leq E.$$

In view of (10), this inequality takes the form

$$(E+\sigma\tau A)^{-1}(E-(1-\sigma)\tau A)(E-(1-\sigma)\tau A^*)(E+\sigma\tau A^*)^{-1} \leq E.$$

Multiplying this inequality on the left by $(E+\sigma\tau A)^{-1}$ and on the right by $(E+\sigma\tau A^*)^{-1}$, we obtain

$$(E-(1-\sigma)\tau A)(E-(1-\sigma)\tau A^*) \leq (E+\sigma\tau A)(E+\sigma\tau A^*).$$

It follows from here that

$$\tau(A+A^*)+(\sigma^2-(1-\sigma)^2)\tau^2 AA^* \geq 0.$$

This inequality holds for nonnegative operators A with $\sigma \geq 0.5$. In view of (12), from (11), we have obtained the required estimate (8). □

3 Decomposition Operators

To better understand the formal structure of the operators of the domain decomposition, we give a typical example. We consider a model nonstationary convection-diffusion problem with time-independent (but space-dependent) diffusion coefficient and velocity. The convective term below is written in the so-called (see, e.g., [21]) symmetric form. In a bounded domain Ω, the unknown function $u(\boldsymbol{x},t)$ satisfies the following equation:

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{1}{2}\sum_{\alpha=1}^{m}\left(v_\alpha(\boldsymbol{x})\frac{\partial u}{\partial x_\alpha} + \frac{\partial}{\partial x_\alpha}(v_\alpha(\boldsymbol{x})u)\right) \\ - \sum_{\alpha=1}^{m}\frac{\partial}{\partial x_\alpha}\left(k(\boldsymbol{x})\frac{\partial u}{\partial x_\alpha}\right) = f(\boldsymbol{x},t), \quad \boldsymbol{x}\in\Omega, \quad 0<t<T, \end{aligned} \tag{13}$$

in which $k(\boldsymbol{x}) \geq \kappa > 0$, $\boldsymbol{x} \in \Omega$. Equation (13) is supplemented with homogeneous Dirichlet boundary conditions

$$u(\boldsymbol{x},t) = 0, \quad \boldsymbol{x}\in\partial\Omega, \quad 0<t<T. \tag{14}$$

In addition, we define the initial condition

$$u(\boldsymbol{x},0) = u^0(\boldsymbol{x}), \quad \boldsymbol{x}\in\Omega. \tag{15}$$

We will consider the set of functions $u(\boldsymbol{x},t)$, satisfying the boundary conditions (14). Let us write the above unsteady convection-diffusion problem in the form of differential-operator equation

$$\frac{du}{dt} + \mathcal{A}u = f(t), \quad 0 < t < T. \tag{16}$$

We consider the Cauchy problem for the evolution equation (16):

$$u(0) = u^0. \tag{17}$$

Let us explicitly specify the diffusive and convective operators and rewrite (16) in the following form:

$$\mathcal{A} = \mathcal{C} + \mathcal{D}. \tag{18}$$

The diffusion operator stands for

$$\mathcal{D}u = -\sum_{\alpha=1}^{m} \frac{\partial}{\partial x_\alpha}\left(k(\boldsymbol{x})\frac{\partial u}{\partial x_\alpha}\right).$$

On the set of functions (14) in $\mathcal{H} = \mathcal{L}_2(\Omega)$, the diffusion operator $\mathcal{D}$ is selfadjoint and positive definite:

$$\mathcal{D} = \mathcal{D}^* \geq \kappa\delta\mathcal{E}, \quad \delta = \delta(\Omega) > 0, \tag{19}$$

where $\mathcal{E}$ is the identity operator in $\mathcal{H}$.

The convective transport operator $\mathcal{C}$ is defined by the expression

$$\mathcal{C}u = \frac{1}{2}\sum_{\alpha=1}^{m}\left(v_\alpha(\boldsymbol{x})\frac{\partial u}{\partial x_\alpha} + \frac{\partial}{\partial x_\alpha}(v_\alpha(\boldsymbol{x})u)\right).$$

For any $v_\alpha(\boldsymbol{x})$, the operator $\mathcal{C}$ is skew-symmetric in $\mathcal{H}$:

$$\mathcal{C} = -\mathcal{C}^*. \tag{20}$$

Taking into account the representation (18), from (19), (20), it follows that $\mathcal{A}>0\ \ \mathcal{H}$.

A domain decomposition scheme for this problem will be associated with the partition of unity of the computational domain Ω. Let the domain Ω consists of p (possibly overlapping) separate subdomains

$$\Omega = \Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_p.$$

With each separate subdomain Ω_α, $\alpha = 1,2,\ldots,p$, we associate function $\eta_\alpha(\boldsymbol{x})$, $\alpha = 1,2,\ldots,p$, such that

$$\eta_\alpha(\boldsymbol{x}) = \begin{cases} > 0, \ \boldsymbol{x} \in \Omega_\alpha, \\ \ 0, \quad \boldsymbol{x} \notin \Omega_\alpha, \end{cases} \quad \alpha = 1,2,\dots,p, \tag{21}$$

where

$$\sum_{\alpha=1}^{p} \eta_\alpha(\boldsymbol{x}) = 1, \quad \boldsymbol{x} \in \Omega. \tag{22}$$

In view of (21), (22) from (18), we obtain the representation

$$\mathcal{A} = \sum_{\alpha=1}^{p} \mathcal{A}_\alpha, \quad \mathcal{A}_\alpha = \mathcal{C}_\alpha + \mathcal{D}_\alpha, \quad \alpha = 1,2,\dots,p, \tag{23}$$

in which

$$\mathcal{D}_\alpha u = -\sum_{\alpha=1}^{m} \frac{\partial}{\partial x_\alpha}\left(k(\boldsymbol{x})\eta_\alpha(\boldsymbol{x})\frac{\partial u}{\partial x_\alpha}\right),$$

$$\mathcal{C}_\alpha u = \frac{1}{2}\sum_{\alpha=1}^{m}\left(v_\alpha(\boldsymbol{x})\eta_\alpha(\boldsymbol{x})\frac{\partial u}{\partial x_\alpha} + \frac{\partial}{\partial x_\alpha}(v_\alpha(\boldsymbol{x})\eta_\alpha(\boldsymbol{x})u)\right).$$

Similarly to (19), (20), it holds for the subdomain operators:

$$\mathcal{D}_\alpha = \mathcal{D}_\alpha^* \geq 0, \quad \mathcal{C}_\alpha = -\mathcal{C}_\alpha^*, \quad \alpha = 1,2,\dots,p. \tag{24}$$

Due to (24), the operators in the splitting (23) satisfy

$$\mathcal{A}_\alpha \geq 0, \quad \alpha = 1,2,\dots,p, \tag{25}$$

and the selfadjoint part of the operator $\mathcal{A}$ splits into sum of nonnegative selfadjoint operators, and the skew-symmetric operator splits into sum of skew-symmetric operators.

The diffusive transport operator $\mathcal{D}$ is conveniently represented as

$$\mathcal{D} = \mathcal{G}^*\mathcal{G}, \quad \mathcal{G} = k^{1/2}\,\mathrm{grad}, \quad \mathcal{G}^* = -\,\mathrm{div}\,k^{1/2}, \tag{26}$$

with $\mathcal{G} : \mathcal{H} \to \tilde{\mathcal{H}}$, where $\tilde{\mathcal{H}} = (\mathcal{L}_2(\Omega))^p$ is the corresponding Hilbert space of vector functions. Using these notations, operators $\mathcal{D}_\alpha, \ \alpha = 1,2,\dots,p$ can be written as

$$\mathcal{D}_\alpha = \mathcal{G}^*\eta_\alpha\mathcal{G}, \quad \alpha = 1,2,\dots,p. \tag{27}$$

Similarly, each of $\mathcal{C}_\alpha, \ \alpha = 1,2,\dots,p$ has the representation

$$\mathcal{C}_\alpha = \frac{1}{2}(\eta_\alpha\mathcal{C} + \mathcal{C}\eta_\alpha), \quad \alpha = 1,2,\dots,p. \tag{28}$$

The advantage of the notations (27), (28) is that diffusion and convection operators have clearly visible structure in the subdomains defined by the splitting (21), (22), and it is easy to verify if (24) is satisfied.

A similar consideration can be given for the operator of the general problem defined by (2), (3). Let us discuss it with some details. Let us select the selfadjoint and the skew-symmetric part of the operator A:

$$A = C + D, \quad C = \frac{1}{2}(A - A^*), \quad D = \frac{1}{2}(A + A^*). \tag{29}$$

The nonnegative operator D can be written as

$$D = G^* G, \tag{30}$$

in which $G : H \to \tilde{H}$. Let E and $\tilde{E}$ be identity operators in the spaces H and $\tilde{H}$, respectively, and let the following partitions of unity define the decomposition of the domain

$$\sum_{\alpha=1}^{p} \chi_\alpha = E, \quad \chi_\alpha \ge 0, \quad \alpha = 1, 2, \ldots, p, \tag{31}$$

$$\sum_{\alpha=1}^{p} \tilde{\chi}_\alpha = \tilde{E}, \quad \tilde{\chi}_\alpha \ge 0, \quad \alpha = 1, 2, \ldots, p. \tag{32}$$

In analogy with (23)–(25), we use the splitting

$$A = \sum_{\alpha=1}^{p} A_\alpha, \quad A_\alpha \ge 0, \quad \alpha = 1, 2, \ldots, p, \tag{33}$$

in which

$$A_\alpha = C_\alpha + D_\alpha, \quad D_\alpha = D_\alpha^* \ge 0, \quad C_\alpha = -C_\alpha^*, \quad \alpha = 1, 2, \ldots, p. \tag{34}$$

Based on (32), we set

$$D_\alpha = G^* \tilde{\chi}_\alpha G, \quad \alpha = 1, 2, \ldots, p. \tag{35}$$

The presentation of the terms in the antisymmetric part is based on (31):

$$C_\alpha = \frac{1}{2}(\chi_\alpha C + C \chi_\alpha), \quad \alpha = 1, 2, \ldots, p. \tag{36}$$

Such an additive representation is a discrete analogue of (27), (28), and it is interpreted as respective version of the domain decomposition.

4 Regularized Domain Decomposition Schemes

Various splitting schemes can be used solving the Cauchy problem for Eqs. (2), (3). The transition to a new time level is based on the solution p separate subtasks, each of which is based on solving a problem with individual operators $A_\alpha,\ \alpha = 1,2,\dots,p$. Taking into account the structure of the operators (see (34)–(36)), the presented splitting schemes belong to the class of regionally additive schemes and are based on consistent application of non-iterative domain decomposition schemes.

Currently, the principle of regularization of difference schemes is being considered as a basic methodological principle for improving the difference schemes [13]. The construction of unconditionally stable additive-difference schemes [19], based on the principle of regularization, will be implemented here in the following ways:

1. A simple difference scheme (called here *generating* difference scheme) is constructed for the original problem. This scheme does usually not possess the desired properties. For example, in the construction of additive schemes, the generating scheme can be only conditionally stable or even can be completely unstable.
2. The difference scheme is rewritten in a form for which the stability conditions are known.
3. Quality of the scheme (e.g., its stability) is improved due to perturbations of the operators of the difference scheme, at the same time preserving the possibility for its computational implementation as an additive scheme.

Let us now illustrate the above methodology by a particular case study. Applied to the problem (2), (3), we choose as a generating scheme the simple explicit scheme

$$\frac{y^{n+1} - y^n}{\tau} + Ay^n = \varphi^n, \quad n = 0,1,\dots,N-1, \tag{37}$$

which is complemented by the initial conditions (7). This scheme stable (see the proof of Theorem 1) if the inequality

$$A + A^* - \tau AA^* \geq 0 \tag{38}$$

is fulfilled. The inequality (38) with $D > 0$ imposes restrictions on the time step, i.e., the scheme (29), (37) is conditionally stable. Note also that if $D = 0$, the scheme (29), (37) is absolutely unstable. Taking into account the splitting (33), we refer to the scheme under consideration as to a scheme from the class of additive schemes.

In the construction of additive schemes, we can consider also an alternative variant, using as generating scheme the more general scheme (6), (7), which is not additive, but which is unconditionally stable for $\sigma \geq 0.5$. In this latter case, the perturbation is applied just in order to obtain an additive scheme while preserving the property of unconditional stability.

Regularization of difference schemes for improving the stability range (in the construction of splitting schemes) can be achieved via perturbation of the operator A. Another way is related to perturbation of the finite-difference approximation of the time derivative term. In the construction of additive schemes, it is convenient to work with the transition operator S, writing down the generating scheme (37) as

$$y^{n+1} = Sy^n + \tau\varphi^n, \quad n = 0,1,\dots,N-1. \tag{39}$$

In the case of (37), we have

$$S = E - \tau A. \tag{40}$$

A regularized scheme based on the perturbation of the operator S has the form

$$y^{n+1} = \tilde{S}y^n + \tau\varphi^n, \quad n = 0,1,\dots,N-1. \tag{41}$$

Let us formulate general conditions on $\tilde{S}$.

The generating scheme (39), (40) has first-order approximation in time, and to preserve this order of approximation, we impose on $\tilde{S}$ the following condition:

$$\tilde{S} = E - \tau A + \mathcal{O}(\tau^2). \tag{42}$$

The scheme (41) is stable in the sense of the estimate (8) provided that the following inequality holds:

$$\|\tilde{S}\| \le 1. \tag{43}$$

Additionally, it should be noted that we seek for additive regularization scheme, where the transition to a new time level is achieved via solving individual subproblems for the operators $A_\alpha,\ \alpha = 1,2,\dots,p$ in the decomposition (33).

The first class of regularized splitting schemes considered here is based on the following additive representation of the transition operator of the generating scheme:

$$S = \frac{1}{p}\sum_{\alpha=1}^{p} S_\alpha, \quad S_\alpha = E - p\tau A_\alpha, \quad \alpha = 1,2,\dots,p.$$

We use a similar additive representation for the transition operator in the regularized scheme

$$\tilde{S} = \frac{1}{p}\sum_{\alpha=1}^{p} \tilde{S}_\alpha, \quad \alpha = 1,2,\dots,p. \tag{44}$$

The individual terms $\tilde{S}_\alpha,\ \alpha = 1,2,\dots,p$ are based on perturbations of the operators $A_\alpha,\ \alpha = 1,2,\dots,p$. In analogy with (10), we set

$$\tilde{S}_\alpha = (E + \sigma p\tau A_\alpha)^{-1}(E - (1-\sigma)p\tau A_\alpha), \quad \alpha = 1,2,\dots,p. \tag{45}$$

If $\sigma \geq 0.5$ (see proof of Theorem 1) we have

$$\|\tilde{S}_\alpha\| \leq 1, \quad \alpha = 1,2,\dots,p.$$

In view of (44), this provides fulfilment of the stability conditions (43).

Accounting for

$$\tilde{S}_\alpha = E - p\tau(E + \sigma p\tau A_\alpha)^{-1} A_\alpha, \quad \alpha = 1,2,\dots,p$$

the regularized additive scheme (41), (44), (45) can be rewritten in the form

$$\frac{y^{n+1} - y^n}{\tau} + \sum_{\alpha=1}^{p} (E + \sigma p\tau A_\alpha)^{-1} A_\alpha y^n = \varphi^n, \quad n = 0,1,\dots,N-1. \tag{46}$$

Comparing to the generating scheme (33), (37), we see that the regularization in this case is achieved by perturbation of A. The outcome of our consideration is the following theorem.

Theorem 2. *The additive-difference scheme (7), (41), (44), (45) is unconditionally stable for* $\sigma \geq 0.5$, *and stability estimate (8) holds for its solution.*

The computational implementation of the scheme (7), (46) can be carried out as follows. We set

$$y^{n+1} = \frac{1}{p} \sum_{\alpha=1}^{p} y_\alpha^{n+1}, \quad \varphi^n = \sum_{\alpha=1}^{p} \varphi_\alpha^n.$$

In this case, we obtain

$$\frac{y_\alpha^{n+1} - y^n}{p\tau} + (E + \sigma p\tau A_\alpha)^{-1} A_\alpha y^n = \varphi_\alpha^n, \quad \alpha = 1,2,\dots,p \tag{47}$$

for the individual components of the approximate solution at the new time level y_α^{n+1}, $\alpha = 1,2,\dots,p$. The scheme (47) can be rewritten as

$$\frac{y_\alpha^{n+1} - y^n}{p\tau} + A_\alpha y^n(\sigma y_\alpha^{n+1} + (1-\sigma) y^n) = (E + \sigma p\tau A_\alpha)\varphi_\alpha^n.$$

In this form we can interpret the scheme (47) as a variant of the additive-averaged component splitting scheme [19].

Another class of regularized splitting schemes instead of additive (see (44)), exploits multiplicative representation of the transition operator:

$$\tilde{S} = \prod_{\alpha=1}^{p} \tilde{S}_\alpha, \quad \alpha = 1,2,\dots,p. \tag{48}$$

Taking into account (42), we have

$$S = \prod_{\alpha=1}^{p} S_\alpha + \mathcal{O}(\tau^2), \quad S_\alpha = E - \tau A_\alpha, \quad \alpha = 1,2,\dots,p.$$

Similarly to (45), we set

$$\tilde{S}_\alpha = (E + \sigma\tau A_\alpha)^{-1}(E - (1-\sigma)\tau A_\alpha), \quad \alpha = 1,2,\dots,p. \tag{49}$$

Under the standard restrictions $\sigma \geq 0.5$, the regularized scheme (41), (48), (49) is stable.

Theorem 3. *The additive-difference scheme (7), (41), (48), (49) is unconditionally stable for* $\sigma \geq 0.5$, *and the stability estimate (8) holds for its solution.*

Let us discuss a possible computer implementation of the constructed regularized scheme. We introduce auxiliary quantities $y^{n+\alpha/p}$, $\alpha = 1,2,\dots,p$. Taking into account (41), (48), these are defined from the equations

$$y^{n+\alpha/p} = \tilde{S}_\alpha y^{n+(\alpha-1)/p}, \quad \alpha = 1,2,\dots,p-1,$$

$$y^{n+1} = \tilde{S}_p y^{n+(p-1)/p} + \tau\varphi^n. \tag{50}$$

Similar to (47), we obtain from (50)

$$\frac{y^{n+\alpha/p} - y^{n+(\alpha-1)/p}}{\tau} + (E + \sigma\tau A_\alpha)^{-1} A_\alpha y^{n+(\alpha-1)/p} = \varphi_\alpha^n, \tag{51}$$

where

$$\varphi_\alpha^n = \begin{cases} 0, & \alpha = 1,2,\dots,p-1, \\ \varphi^n, & \alpha = p. \end{cases}$$

We write the scheme (51) as

$$\frac{y^{n+\alpha/p} - y^{n+(\alpha-1)/p}}{\tau} + A_\alpha(\sigma y^{n+\alpha/p} + (1-\sigma) y^{n+(\alpha-1)/p}) = \tilde{\varphi}_\alpha^n, \tag{52}$$

in which

$$\tilde{\varphi}_\alpha^n = (E + \sigma\tau A_\alpha)\varphi_\alpha^n, \quad \alpha = 1,2,\dots,p.$$

Scheme (52) can be considered as a special version of the standard component-wise splitting scheme [10, 13, 34]. However, those schemes are additive approximation schemes, while the constructed here scheme is a full approximation one. Regularized scheme (41), (44), (45), built on the additive representation (44) of the transition operator, is more suitable for parallel computations, compared to the regularized schemes (41), (48), (49) which is based on the multiplicative representation (48).

5 Vector Schemes for Domain Decomposition

Difference schemes for nonstationary problems can often be regarded as appropriate iterative methods for approximate solution of stationary problems. The introduced above regularized additive schemes are based on perturbation of the operator A in the producing scheme (37). Such schemes, as well as the standard additive component-wise splitting schemes, are not suitable for constructing iterative methods for solving stationary equations. Better opportunities in this direction are provided by the vector-additive schemes [1, 31].

Instead of a single unknown $u(t)$, we consider p unknowns $u_\alpha,\ \alpha = 1,2,\dots,p$, which are to be determined from the system

$$\frac{du_\alpha}{dt} + \sum_{\beta=1}^{p} A_\beta u_\beta = f(t), \quad \alpha = 1,2,\dots,p, \quad 0 < t \le T. \tag{53}$$

The following initial conditions are used for the system of equations (53)

$$u_\alpha(0) = u^0, \quad \alpha = 1,2,\dots,p, \tag{54}$$

which follow from (2). Obviously, each function is a solution of (2), (3), (33). Approximate solution of (2), (3), (33) will be constructed on the basis of difference schemes for the vector problem (53), (54).

To solve the problem (53), (54), we use the following two-level scheme:

$$\frac{y_\alpha^{n+1} - y_\alpha^n}{\tau} + \sum_{\beta=1}^{\alpha} A_\beta y_\beta^{n+1} + \sum_{\beta=\alpha+1}^{p} A_\beta y_\beta^n = \varphi^n,$$

$$\alpha = 1,2,\dots,p, \quad n = 0,1,\dots,N-1, \tag{55}$$

complemented with the initial conditions

$$y_\alpha(0) = u^0, \quad \alpha = 1,2,\dots,p. \tag{56}$$

The computational implementation of this scheme is connected with a consecutive inversion of operators $E + \tau A_\alpha,\ \alpha = 1,2,\dots,p$.

Theorem 4. *The vector-additive difference scheme (33), (55), (56) is unconditionally stable, and stability estimate holds for its components*

$$\|y_\alpha^{n+1}\| \le \|y_\alpha^n\| + \tau\|\varphi^0 - Au^0\| + \tau\sum_{k=1}^{n}\tau\left\|\frac{\varphi^k - \varphi^{k-1}}{\tau}\right\|,$$

$$\alpha = 1,2,\dots,p, \quad n = 0,1,\dots,N-1, \tag{57}$$

is valid.

Proof. The analysis of the vector scheme (55), (56) will be carried out following the work [22]. □

We emphasize that the above stability estimates (57) are obtained for each individual component y_α^{n+1}, $\alpha = 1,2,\dots,p$. Each of them or their linear combination

$$y^{n+1} = \sum_{\alpha=1}^{p} c_\alpha y_\alpha^{n+1}, \quad c_\alpha = \text{const} \geq 0, \quad \alpha = 1,2,\dots,p$$

can be regarded as an approximate solution to our problem (2), (3), (33) at time $t = t^{n+1}$.

6 Model Problem

The performance of the considered domain decomposition schemes is illustrated considering a simple example for numerical solution of the boundary value problem for parabolic equation. Consider a rectangular domain

$$\Omega = \{\, \boldsymbol{x} \mid \boldsymbol{x} = (x_1,x_2),\ 0 < x_\alpha < l_\alpha,\ \alpha = 1,2\}.$$

The following boundary value problem

$$\frac{\partial u}{\partial t} = \sum_{\alpha=1}^{2} \frac{\partial^2 u}{\partial x_\alpha^2}, \quad \boldsymbol{x} \in \Omega, \quad 0 < t < T, \tag{58}$$

$$u(\boldsymbol{x},t) = 0, \quad \boldsymbol{x} \in \partial\Omega, \quad 0 < t < T, \tag{59}$$

$$u(\boldsymbol{x},0) = u^0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega \tag{60}$$

is to be solved in Ω.

We introduce a uniform rectangular grid in Ω:

$$\bar{\omega} = \{\boldsymbol{x} \mid \boldsymbol{x} = (x_1,x_2), \quad x_\alpha = i_\alpha h_\alpha, \quad i_\alpha = 0,1,\dots,N_\alpha, \quad N_\alpha h_\alpha = l_\alpha\}$$

and let ω be the set of internal nodes ($\bar{\omega} = \omega \cup \partial\omega$). For grid functions $y(\boldsymbol{x}) = 0$, $\boldsymbol{x} \in \partial\omega$, we define Hilbert space $H = L_2(\omega)$ with the scalar product and norm

$$(y,w) \equiv \sum_{\boldsymbol{x}\in\omega} y(\boldsymbol{x})w(\boldsymbol{x})h_1 h_2, \quad \|y\| \equiv (y,y)^{1/2}.$$

After spatial approximations of the problem (58), (59), we arrive at the differential-difference equation:

$$\frac{dy}{dt} + Ay = 0, \quad \boldsymbol{x} \in \omega, \quad 0 < t < T, \tag{61}$$

in which

$$
\begin{aligned}
Ay = & -\frac{1}{h_1^2}(y(x_1+h_1,x_2) - 2y(x_1,x_2) + y(x_1-h_1,x_2)) \\
& -\frac{1}{h_2^2}(y(x_1,x_2+h_2) - 2y(x_1,x_2) + y(x_1,x_2-h_2)), \quad \boldsymbol{x} \in \omega.
\end{aligned} \tag{62}
$$

In the space H the operator A is selfadjoint and positive definite [13, 15]:

$$
A = A^* \geq (\delta_1 + \delta_2)E, \quad \delta_\alpha = \frac{4}{h_\alpha^2} \sin^2 \frac{\pi h_\alpha}{2 l_\alpha}, \quad \alpha = 1, 2. \tag{63}
$$

Taking into account (60), Eq. (62) is supplemented with the initial condition

$$
y(\boldsymbol{x}, 0) = u^0(\boldsymbol{x}), \quad \boldsymbol{x} \in \omega. \tag{64}
$$

For simplicity, the DD operator in the investigated problem (61)–(64) is constructed without the explicit separation of the operator G and G and the space $\tilde{H}$, focusing on the decomposition (21), (22). We set

$$
\begin{aligned}
A_\alpha y = & -\frac{1}{h_1^2}\eta_\alpha(x_1+0.5h_1,x_2)(y(x_1+h_1,x_2) - y(x_1,x_2)) \\
& + \frac{1}{h_1^2}\eta_\alpha(x_1-0.5h_1,x_2)(y(x_1,x_2) - y(x_1-h_1,x_2)) \\
& - \frac{1}{h_2^2}\eta_\alpha(x_1,x_2+0.5h_2)(y(x_1,x_2+h_2) - y(x_1,x_2)) \\
& + \frac{1}{h_2^2}\eta_\alpha(x_1,x_2-0.5h_2)(y(x_1,x_2) - y(x_1,x_2-h_2)), \\
& \alpha = 1, 2, \ldots, p.
\end{aligned} \tag{65}
$$

In view of (21), (22) we have

$$
A = \sum_{\alpha=1}^{p} A_\alpha, \quad A_\alpha = A_\alpha^*, \quad \alpha = 1, 2, \ldots, p. \tag{66}
$$

Thus, we are in a class of additive schemes (33), for which we construct different additive schemes.

Numerical calculations are carried out for the problem (58)–(60) in the unit square ($l_1 = l_2 = 1$) when the solution has the form

$$
u(\boldsymbol{x}, t) = \sin(n_1 \pi x_1) \sin(n_2 \pi x_2) \exp(-\pi^2 (n_1^2 + n_2^2) t) \tag{67}
$$

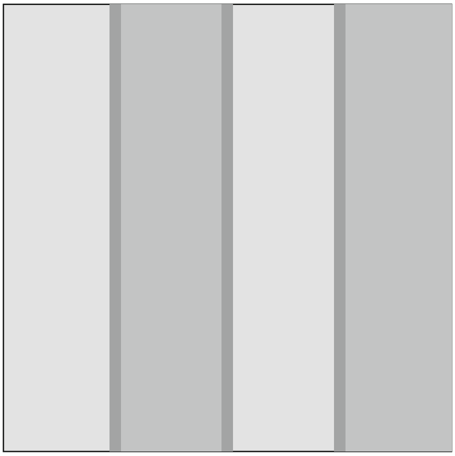

Fig. 1 Domain decomposition

for natural n_1 and n_2. We use this solution to set the initial conditions (60). The domain is decomposed into four overlapping subdomains (see Fig. 1). The disconnected subdomains can be considered as one subdomain, and the decomposition in Fig. 1 can be considered as a decomposition into two subdomains and described by two functions: $\eta_\alpha = \eta_\alpha(x_1),\ \alpha = 1,2$.

Overlapping and nonoverlapping domain decomposition methods can be constructed for problems of type (58)–(60). Methods without overlap require formulation of interface conditions at the common boundaries. Here we consider overlapping DD and therefore do not need to formulate such conditions. However, the proposed here schemes have straightforward extension for the case of nonoverlapping DD.

A fundamental question in DD methods, especially in their parallel implementation, is the exchange of calculated data between different subdomains. The usual explicit schemes can serve as reference in order to explain the exchange challenges. In this case, the domain decomposition can be associated with certain subsets of grid nodes: $\omega_\alpha,\ \alpha = 1,2$, where $\omega = \omega_1 \cup \omega_2$. In the case of (58)–(60) (seven point stencil in space), the transition to a new level in time for the explicit scheme is associated with the use of solution values at the boundary nodes (here we mean the boundary of each subdomain). We need to transfer the calculated data volume $\sim \partial\omega_\alpha,\ \alpha = 1,2$. In solving numerically the problem (61)–(64), we can consider two possibilities for minimal overlap of the subdomains. In our case, the first one corresponds to allocating the inter-subdomain boundary along the grid nodes with integer numbers; the second one is allocating interface lines along nodes with non-integer numbering.

The variant with division along integer-numbered nodes is displayed in Fig. 2. Let the decomposition be carried out in the variable x_1, i.e. $\theta = x_1$. Decomposition of the domain held by the node $\theta = \theta_i$. Given this decomposition, the operator (65)

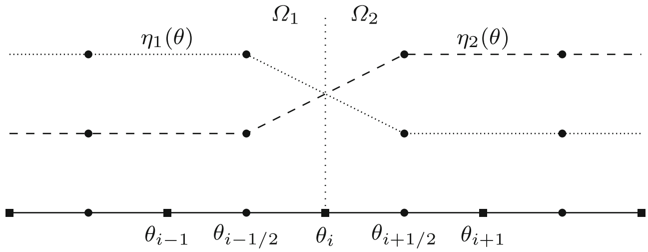

Fig. 2 Decomposition in integer nodes

is written in the form

$$A_1 y = \frac{1}{h_1^2}(y(x_1,x_2) - y(x_1 - h_1,x_2)) - \frac{1}{2h_2^2}(y(x_1,x_2+h_2) - 2y(x_1,x_2) + y(x_1,x_2-h_2)),$$

$$A_2 y = -\frac{1}{h_1^2}(y(x_1+h_1,x_2) - y(x_1,x_2)) - \frac{1}{2h_2^2}(y(x_1,x_2+h_2) - 2y(x_1,x_2) + y(x_1,x_2-h_2)), \quad x_1 = \theta_i.$$

This decomposition can be associated with Neumann boundary conditions as exchange boundary conditions. Relationship between the individual subdomains is minimal and they can exchange data with $\theta = \theta_i$. This case can be identified by the decomposition operators (32) as follows:

$$R(\tilde{\chi}_\alpha) = [0,1], \quad \alpha = 1,2,\ldots,p. \tag{68}$$

The values of $\eta_\alpha(x_1 \pm 0.5h_1,x_2)$, $\eta_\alpha(x_1,x_2 \pm 0.5h_1)$, $\alpha = 1,2$ for (65), (67) are equal to 0 or 1.

The second possibility, which is associated with decomposition along the non-integer nodes, is illustrated in Fig. 3. In this case, instead of (68), we have

$$R(\tilde{\chi}_\alpha) = [0,1/2,1], \quad \alpha = 1,2,\ldots,p. \tag{69}$$

In the node $\theta = \theta_i$, difference approximation is used with less twice the flux. With regard to the case in the decomposition of the variable x_1, operators decomposition (65) is

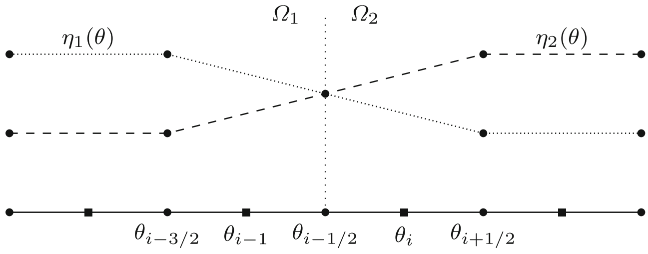

Fig. 3 Decomposition of a half-integer nodes

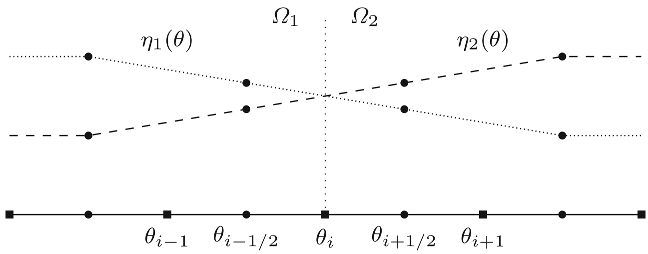

Fig. 4 Decomposition in integer nodes with a width of overlap $3h$

$$
\begin{aligned}
A_1 y = & \frac{1}{2h_1^2}(y(x_1,x_2) - y(x_1 - h_1, x_2)) \\
& - \frac{1}{4h_2^2}(y(x_1,x_2+h_2) - 2y(x_1,x_2) + y(x_1,x_2-h_2)),
\end{aligned}
$$

$$
\begin{aligned}
A_2 y = & -\frac{1}{h_1^2}(y(x_1+h_1,x_2) - y(x_1,x_2)) + \frac{1}{2h_1^2}(y(x_1,x_2) - y(x_1-h_1,x_2)) \\
& - \frac{3}{4h_2^2}(y(x_1,x_2+h_2) - 2y(x_1,x_2) + y(x_1,x_2-h_2)), \quad x_1 = \theta_i.
\end{aligned}
$$

For the calculations in Ω_1 (see Fig. 3), we use half of the flux at the node $\theta = \theta_i$. Thus, when using the domain decomposition method, the exchanges are minimal and coincide with the exchanges in the implementation of the explicit scheme.

The decomposition variants (68), (69) presented above correspond to the case of minimum overlapping of the subdomains. At the discrete level, the width of overlap is determined by the mesh size, h and $2h$, respectively. Similar variants are built for larger overlap of the subdomains. In particular, for the decomposition variant in Fig. 4, we have

$$
R(\tilde{\chi}_\alpha) = [0, 1/3, 2/3, 1], \quad \alpha = 1, 2, \ldots, p. \tag{70}
$$

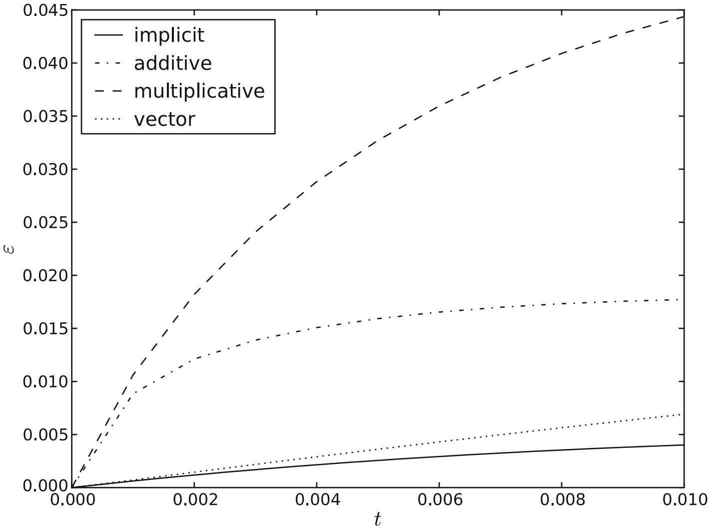

Fig. 5 Accuracy at $N_1 = N_2 = 32$, $N = 10$

In this case the volume of the data exchange is increased, but on the other hand, the transition from one subdomain to another is much smoother. The latter allows us to expect higher accuracy of the approximate solution. Let us present the numerical results obtained in solving (58)–(60). Recall that the exact solution is given by (67) for $n_1 = 2$, $n_2 = 1$ at $T = 0.01$. Square grid $N_1 = N_2$ is used. Regularized fully implicit ($\sigma = 1$) scheme based on additive perturbation (scheme (7), (41), (45), (45)) and based on multiplicative perturbation (scheme (7), (41), (48), (49)) is used, as well as vector-additive scheme (33), (55), (56). The results are compared with the finite-difference solution, which we obtain by using the implicit scheme (1), (6), (7) with $\sigma = 1$ (i.e., scheme without splitting). The errors of the approximate solutions are measured as $\varepsilon(t^n) = \|y^n(\boldsymbol{x}) - u(\boldsymbol{x}, t^n)\|$ on a single time step.

In the case of the decomposition (68) (the width of the overlay is h), the grid space of $N_1 = N_2 = 32$ and grid on time $N = 10$ ($\tau = 0.001$), the error norms of the difference solution using different decomposition schemes are shown in Fig. 5. Figures 6–8 show the local error at the final time. The error is localized in areas of overlap, and for vector decomposition scheme, it is much lower than for the additive and multiplicative versions of regularized additive schemes.

With an increase in the grid space, the error of approximate solution of domain decomposition schemes in comparison with the implicit scheme grows (Fig. 9). In this case, the width of the overlap is reduced by half.

The influence of the width of the overlap is shown in Fig. 10. When using the decomposition (70), there is a substantial increase in the accuracy of the approximate solution compared to the decomposition (68) (compare Figs. 5 and 10).

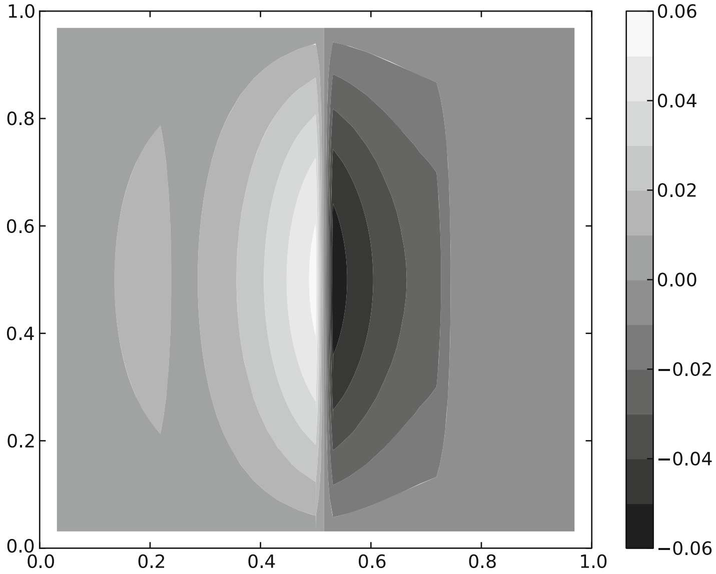

Fig. 6 Error of scheme (7), (41), (48), (49)

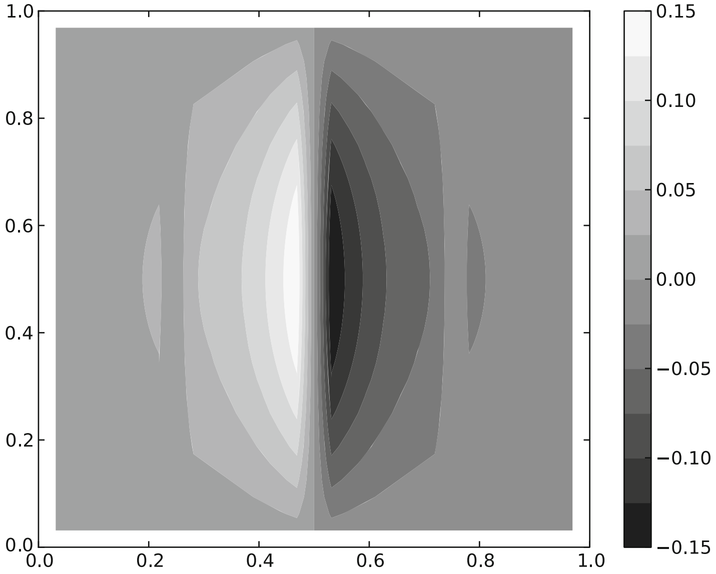

Fig. 7 Error of scheme (7), (41), (45), (45)

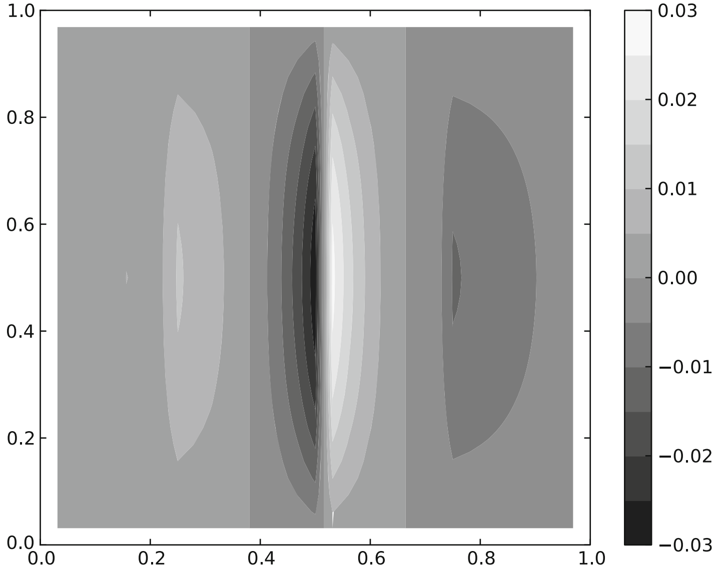

Fig. 8 Error of scheme (33), (55), (56)

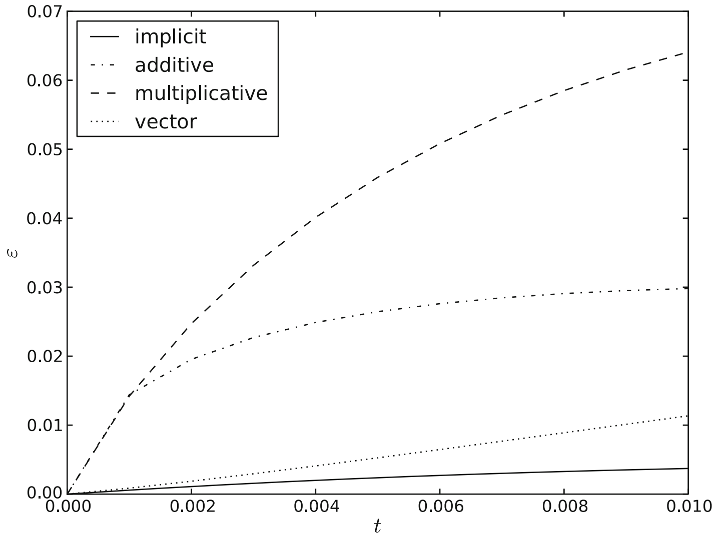

Fig. 9 The error at $N_1 = N_2 = 64$, $N = 10$

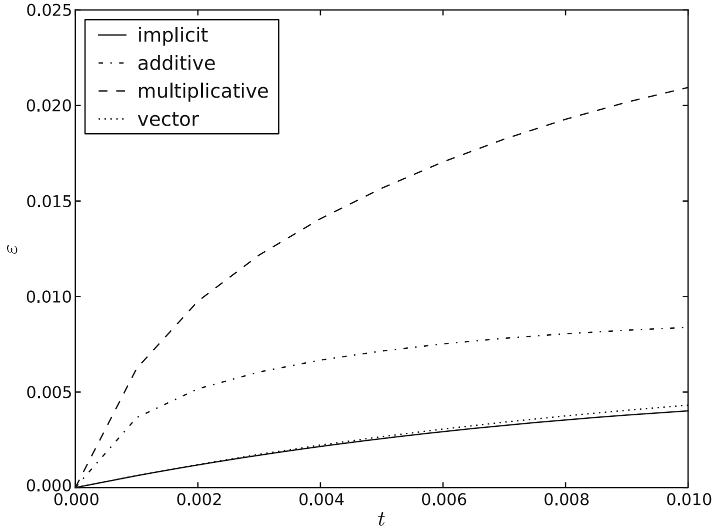

Fig. 10 The error at $N_1 = N_2 = 32$ and $N = 10$ and decomposition $R = [0, 1/3, 2/3, 1]$

7 Conclusions

1. In this paper we have constructed domain decomposition operators for solving evolution problems. The splitting of the common nonselfadjoint nonnegative finite-dimensional operator is carried out separately for its selfadjoint and skew-symmetric parts. This preserves the property of nonnegativity for the operator terms associated with each of the subdomains.
2. Unconditionally stable regularized additive schemes for the Cauchy problem for first-order evolution equations are constructed by splitting problem operators into sum of nonselfadjoint nonnegative operators. This regularization be based on the principles of regularization of operator-difference schemes with perturbation of the transition operator of the explicit scheme. Variants with regularization based on additive and multiplicative splitting are presented, the relationship between the new schemes and the classical additive schemes with summarized approximation (additively averaged schemes and standard component-wise splitting schemes) is discussed.
3. Among the splitting schemes for evolution equations, the vector additive schemes with full approximation are emphasized. They are based on the transition to a system of similar problems in each component with the special organization for computing the approximate solution at the new time level.
4. Numerical simulations for IBVP for a parabolic problem in a rectangular domain are performed. Calculations demonstrate the capabilities of the suggested domain decomposition schemes. The best results in terms of accuracy are demonstrated by the vector-additive scheme of domain decomposition.

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