Efficient Solvers for Some Classes of Time-Periodic Eddy Current Optimal Control Problems

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Abstract In this paper, we present and discuss the results of our numerical studies of preconditioned MinRes methods for solving the optimality systems arising from the multiharmonic finite element approximations to time-periodic eddy current optimal control problems in different settings including different observation and control regions, different tracking terms, as well as box constraints for the Fourier coefficients of the state and the control. These numerical studies confirm the theoretical results published by the first author in a recent paper.

Keywords Time-periodic eddy current optimal control problems • Multiharmonic finite element discretization • MinRes solver • Preconditioners

Mathematics Subject Classification (2010): 49J20, 65T40, 65M60, 65F08

1 Introduction

This work is devoted to the study of efficient solution procedures for the following time-periodic eddy current optimal control problem: Minimize the functional

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$$J(\mathrm{y},\mathrm{u}) = \frac{\alpha}{2}\int_{\Omega_1\times(0,T)} |\mathrm{y}-\mathrm{y_d}|^2 \mathrm{dx}\,\mathrm{dt} + \frac{\beta}{2}\int_{\Omega_1\times(0,T)} |\mathrm{curl}\,\mathrm{y}-\mathrm{y_c}|^2 \mathrm{dx}\,\mathrm{dt} + \frac{\lambda}{2}\int_{\Omega_2\times(0,T)} |\mathrm{u}|^2 \mathrm{dx}\,\mathrm{dt}, \tag{1}$$

subject to the state equations

$$\begin{cases} \sigma\dfrac{\partial \mathrm{y}}{\partial t} + \mathrm{curl}(\nu\,\mathrm{curl}\,\mathrm{y}) = \mathrm{u}, & \text{in } \Omega\times(0,T),\\ \mathrm{div}(\sigma\mathrm{y}) = 0, & \text{in } \Omega\times(0,T),\\ \mathrm{y}\times\mathrm{n} = 0, & \text{on } \partial\Omega\times(0,T),\\ \mathrm{y}(0) = \mathrm{y}(T), & \text{in } \Omega, \end{cases} \tag{2}$$

where Ω is a bounded, simply connected Lipschitz domain with the boundary $\partial\Omega$. The domains Ω_1 and Ω_2 are nonempty Lipschitz subdomains of Ω, i.e., $\Omega_1,\Omega_2 \subset \Omega \subset \mathbb{R}^3$. The reluctivity $\nu \in L^\infty(\Omega)$ and the conductivity $\sigma \in L^\infty(\Omega)$ are supposed to be uniformly positive, i.e.,

$$0 < \nu_{\min} \le \nu(\mathrm{x}) \le \nu_{\max}, \quad \text{and} \quad 0 < \sigma_{\min} \le \sigma(\mathrm{x}) \le \sigma_{\max}, \qquad \mathrm{x}\in\Omega.$$

We mention that the electric conductivity σ vanishes in regions consisting of nonconducting materials. In order to fulfill the assumption made above on the uniform positivity of σ, one can replace $\sigma(x)$ by $\max\{\varepsilon,\sigma(x)\}$ with some suitably chosen positive ε; see, e.g., [10, 12] for more details. We here assume that the reluctivity ν is independent of $|\mathrm{curl}\,\mathrm{y}|$, i.e., we only consider linear eddy current problems. The regularization parameter λ also representing a weight for the cost of the control is assumed to be a suitably chosen positive real number. The weight parameters α and β are nonnegative. In fact, we only study the cases $(\alpha = 1, \beta = 0)$ and $(\alpha = 0, \beta = 1)$. The functions $\mathrm{y_d}$ and $\mathrm{y_c}$ from $L_2((0,T),\mathrm{L}_2(\Omega))$ are the given desired state and the desired curl of the state, respectively.

The problem setting (1)–(2) has been analyzed in [11, 12], wherein, due to the time-periodic structure, a time discretization in terms of a truncated Fourier series, also called multiharmonic approach, is used. In [12], we consider the special case of a fully distributed optimal control problem for tracking some $\mathrm{y_d}$ in the complete computational domain, i.e., $\Omega_1 = \Omega_2 = \Omega$ and $\beta = 0$ in (1), whereas [11] is devoted to the various other settings including different observation and control regions, different tracking terms, as well as box constraints for the Fourier coefficients of the state and the control. Similar optimal control problems for time-periodic parabolic equations and their numerical treatment by means of the multiharmonic finite element method (FEM) have recently been considered in [9] and [8]. Other approaches to time-periodic parabolic optimal control problems have been discussed in [1]. There are many publications on optimal control problems with PDE constraints given by initial-boundary value problems for parabolic equations; see, e.g., [14] for a comprehensive presentation. There are less publications on optimal control problems where initial-boundary value problems for eddy current equations are considered as PDE constraints; see, e.g., [15,16], where one can also find interesting applications. The multiharmonic approach allows us to switch from the time domain to the frequency domain and, therefore, to replace a time-dependent problem by a system of time-independent problems for the Fourier coefficients. Since we are here interested in studying robust solvers, this special time discretization technique justifies the following assumption: Let us assume that the desired states $\mathrm{y_d}$ and $\mathrm{y_c}$ are multiharmonic, i.e., $\mathrm{y_d}$ and $\mathrm{y_c}$ have the form of a truncated Fourier series:

$$
\begin{aligned}
\mathrm{y_d} &= \sum_{k=0}^{N} \mathrm{y}^{\mathrm{c}}_{\mathrm{d,k}} \cos(k\omega t) + \mathrm{y}^{\mathrm{s}}_{\mathrm{d,k}} \sin(k\omega t), \\
\mathrm{y_c} &= \sum_{k=0}^{N} \mathrm{y}^{\mathrm{c}}_{\mathrm{c,k}} \cos(k\omega t) + \mathrm{y}^{\mathrm{s}}_{\mathrm{c,k}} \sin(k\omega t).
\end{aligned}
\tag{3}
$$

Consequently, the state y and the control u are multiharmonic as well and, therefore, have a representation in terms of a truncated Fourier series with the same number of modes N, i.e.,

$$
\begin{aligned}
\mathrm{y} &= \sum_{k=0}^{N} \mathrm{y}^{\mathrm{c}}_{\mathrm{k}} \cos(k\omega t) + \mathrm{y}^{\mathrm{s}}_{\mathrm{k}} \sin(k\omega t), \\
\mathrm{u} &= \sum_{k=0}^{N} \mathrm{u}^{\mathrm{c}}_{\mathrm{k}} \cos(k\omega t) + \mathrm{u}^{\mathrm{s}}_{\mathrm{k}} \sin(k\omega t).
\end{aligned}
\tag{4}
$$

Using the multiharmonic representation of $\mathrm{y_d}$, $\mathrm{y_c}$, y, and u, the minimization problem (1)–(2) can be stated in the frequency domain: Minimize the functional

$$
\begin{aligned}
J_N = \frac{1}{2} \sum_{k=0}^{N} \Big[\sum_{j \in \{c,s\}} \Big[\alpha \int_{\Omega_1} |\mathrm{y}^{\mathrm{j}}_{\mathrm{k}} - \mathrm{y}^{\mathrm{j}}_{\mathrm{d,k}}|^2 \mathrm{dx} + \beta \int_{\Omega_1} |\operatorname{curl} \mathrm{y}^{\mathrm{j}}_{\mathrm{k}} - \mathrm{y}^{\mathrm{j}}_{\mathrm{c,k}}|^2 \mathrm{dx} \\
+ \lambda \sum_{j \in \{c,s\}} \int_{\Omega_2} |\mathrm{u}^{\mathrm{j}}_{\mathrm{k}}|^2 \mathrm{dx} \Big] \Big],
\end{aligned}
\tag{5a}
$$

subject to the state equation

$$
\begin{cases}
k\omega\sigma \mathrm{y}^{\mathrm{s}}_{\mathrm{k}} + \operatorname{curl}(\nu \operatorname{curl} \mathrm{y}^{\mathrm{c}}_{\mathrm{k}}) = \mathrm{u}^{\mathrm{c}}_{\mathrm{k}}, & \text{in } \Omega, k = 1, \ldots, N, \\
-k\omega\sigma \mathrm{y}^{\mathrm{c}}_{\mathrm{k}} + \operatorname{curl}(\nu \operatorname{curl} \mathrm{y}^{\mathrm{s}}_{\mathrm{k}}) = \mathrm{u}^{\mathrm{s}}_{\mathrm{k}}, & \text{in } \Omega, k = 1, \ldots, N, \\
\operatorname{curl}(\nu \operatorname{curl} \mathrm{y}^{\mathrm{c}}_{0}) = \mathrm{u}^{\mathrm{c}}_{0}, & \text{in } \Omega, \\
\mathrm{y}^{\mathrm{c}}_{\mathrm{k}} \times \mathrm{n} = \mathrm{y}^{\mathrm{s}}_{\mathrm{k}} \times \mathrm{n} = 0, & \text{on } \partial\Omega, k = 1, \ldots, N, \\
\mathrm{y}^{0}_{\mathrm{k}} \times \mathrm{n} = 0, & \text{on } \partial\Omega,
\end{cases}
\tag{5b}
$$

completed by the divergence constraints

$$\begin{cases} k\omega\,\mathrm{div}(\sigma \mathrm{y}_\mathrm{k}^\mathrm{c}) = 0, & \text{in } \Omega, k = 1,\dots,N, \\ k\omega\,\mathrm{div}(\sigma \mathrm{y}_\mathrm{k}^\mathrm{s}) = 0, & \text{in } \Omega, k = 1,\dots,N, \\ \quad \mathrm{div}(\sigma \mathrm{y}_0^\mathrm{c}) = 0, & \text{in } \Omega. \end{cases} \tag{5c}$$

Additionally, we add control constraints associated to the Fourier coefficients of the control u, i.e.,

$$\begin{aligned} \underline{\mathrm{u}}_\mathrm{k}^\mathrm{c} \le \mathrm{u}_\mathrm{k}^\mathrm{c} \le \overline{\mathrm{u}}_\mathrm{k}^\mathrm{c}, &\quad \text{a.e. in } \Omega, k = 0,1,\dots,N, \\ \underline{\mathrm{u}}_\mathrm{k}^\mathrm{s} \le \mathrm{u}_\mathrm{k}^\mathrm{s} \le \overline{\mathrm{u}}_\mathrm{k}^\mathrm{s}, &\quad \text{a.e. in } \Omega, k = 1,\dots,N, \end{aligned} \tag{5d}$$

and state constraints associated to the Fourier coefficients of the state y, i.e.,

$$\begin{aligned} \underline{\mathrm{y}}_\mathrm{k}^\mathrm{c} \le \mathrm{y}_\mathrm{k}^\mathrm{c} \le \overline{\mathrm{y}}_\mathrm{k}^\mathrm{c}, &\quad \text{a.e. in } \Omega, k = 0,1,\dots,N, \\ \underline{\mathrm{y}}_\mathrm{k}^\mathrm{s} \le \mathrm{y}_\mathrm{k}^\mathrm{s} \le \overline{\mathrm{y}}_\mathrm{k}^\mathrm{s}, &\quad \text{a.e. in } \Omega, k = 1,\dots,N. \end{aligned} \tag{5e}$$

This minimization problem is typically solved by deriving the corresponding optimality system, which fortunately decouples in terms of the mode k. The decoupled systems are then discretized in space by means of the FEM. Since even the simple box constraints (5d)–(5e) give rise to nonlinear optimality systems, we apply a primal–dual active set strategy (semi-smooth Newton) approach for their solution [5]. The resulting procedure is summarized in Algorithm 1.

Algorithm 1: Primal–dual active set strategy

```
Input: number of modes N, initial guesses x^(k,0) ∈ R^n (k = 0,...,N) .
Output: approximate solution x^(k,l) ∈ R^n (k = 0,...,N).
for k ← 0 to N do
  | Determine the active sets E^c_{k,0} and E^s_{k,0};
end
Set l := 0;
while not converged do
  | for k ← 0 to N do
  |   | Compute b_E^(k,l+1), A_E^(k,l+1);
  |   | Solve A_E^(k,l+1) x^(k,l+1) = b_E^(k,l+1);
  |   | Determine the active sets E^c_{k,l+1} and E^s_{k,l+1};
  | end
  | Set l := l+1;
end
```

The specific structure of the Jacobi matrix $\mathcal{A}_\mathcal{E}^{(k,l+1)}$ depends on the actual computational setting. In our applications, the matrix $\mathcal{A}_\mathcal{E}^{(k,l+1)}$ has either the form $\mathcal{A}_1$ (cf. (6a)) or the form $\mathcal{A}_2$, cf. (6b). It is clear that the efficient and parameter-robust solution of the $(N+1)$ linear systems of equations at each semi-smooth Newton step is essential for the efficiency of the proposed method. For further details we refer to [11].

2 Parameter-Robust and Efficient Solution Procedures

In order to discretize the problems in space, we use the edge (Nédélec) finite element space $\mathcal{ND}_0^0(\mathcal{T}_h)$, that is a conforming finite element subspace of $\mathrm{H}_0(\mathrm{curl},\Omega)$, and the nodal (Lagrange) finite element space $\mathcal{S}_0^1(\mathcal{T}_h)$, that is a conforming finite element subspace of $H_0^1(\Omega)$. Let $\{\varphi_\mathrm{i}\}_{i=1,N_h}$ and $\{\psi_i\}_{i=1,M_h}$ denote the usual edge basis of $\mathcal{ND}_0^0(\mathcal{T}_h)$ and the usual nodal basis of $\mathcal{S}_0^1(\mathcal{T}_h)$, respectively. We are now in the position to define the following FEM matrices:

$$
\begin{aligned}
(\mathrm{K}_\nu)_{ij} &= (\nu\,\mathrm{curl}\,\varphi_\mathrm{i}, \mathrm{curl}\,\varphi_\mathrm{j})_{0,\Omega},\\
(\mathrm{M}_{\sigma,\mathrm{k}\omega})_{ij} &= k\omega(\sigma\varphi_\mathrm{i}, \varphi_\mathrm{j})_{0,\Omega},\\
(\mathrm{M})_{ij} &= (\varphi_\mathrm{i}, \varphi_\mathrm{j})_{0,\Omega},\\
(\mathrm{D}_{\sigma,\mathrm{k}\omega})_{ij} &= k\omega(\sigma\varphi_\mathrm{i}, \nabla\psi_j)_{0,\Omega},
\end{aligned}
$$

where $(\cdot,\cdot)_{0,\Omega}$ denotes the inner product in $\mathrm{L}_2(\Omega)$. Throughout this paper we are repeatedly faced with the following two types of system matrices:

$$
\mathcal{A}_1 = \begin{pmatrix} * & 0 & \mathrm{K}_\nu & -\mathrm{M}_{\sigma,\mathrm{k}\omega} \\ 0 & * & \mathrm{M}_{\sigma,\mathrm{k}\omega} & \mathrm{K}_\nu \\ \mathrm{K}_\nu & \mathrm{M}_{\sigma,\mathrm{k}\omega} & -\lambda^{-1}* & 0 \\ -\mathrm{M}_{\sigma,\mathrm{k}\omega} & \mathrm{K}_\nu & 0 & -\lambda^{-1}* \end{pmatrix} \tag{6a}
$$

$$
\mathcal{A}_2 = \begin{pmatrix} * & 0 & \mathrm{K}_\nu & -\mathrm{M}_{\sigma,\mathrm{k}\omega} & 0 & 0 & {\mathrm{D}_{\sigma,\mathrm{k}\omega}}^T & 0 \\ 0 & * & \mathrm{M}_{\sigma,\mathrm{k}\omega} & \mathrm{K}_\nu & 0 & 0 & 0 & {\mathrm{D}_{\sigma,\mathrm{k}\omega}}^T \\ \mathrm{K}_\nu & \mathrm{M}_{\sigma,\mathrm{k}\omega} & -\lambda^{-1}* & 0 & {\mathrm{D}_{\sigma,\mathrm{k}\omega}}^T & 0 & 0 & 0 \\ -\mathrm{M}_{\sigma,\mathrm{k}\omega} & \mathrm{K}_\nu & 0 & -\lambda^{-1}* & 0 & {\mathrm{D}_{\sigma,\mathrm{k}\omega}}^T & 0 & 0 \\ 0 & 0 & \mathrm{D}_{\sigma,\mathrm{k}\omega} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{D}_{\sigma,\mathrm{k}\omega} & 0 & 0 & 0 & 0 \\ \mathrm{D}_{\sigma,\mathrm{k}\omega} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathrm{D}_{\sigma,\mathrm{k}\omega} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6b}
$$

Therein, the placeholder $*$ stands for a symmetric and positive semi-definite matrix, that actually depends on the considered setting (cf. Table 1). We refer to problems described by matrices of the types $\mathcal{A}_1$ and $\mathcal{A}_2$ as *Formulation OC-FEM 1* and *Formulation OC-FEM 2*, respectively. In fact, the system matrices $\mathcal{A}_1$

and $\mathcal{A}_2$ are symmetric and indefinite and have a two- or threefold saddle point structure, respectively. Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are symmetric, the corresponding systems can be solved by a preconditioned minimal residual (MinRes) method (cf. [13]). Typically, the convergence rate of any iterative Krylov subspace method applied to the unpreconditioned system deteriorates, with respect to the mesh size h, the parameters $k = 0,1,\dots,N$ and ω involved in the spectral time discretization and the problem parameters ν, σ, and λ (cf. also Tables 2 and 3). Therefore, preconditioning is an important issue.

The proper choice of parameter-robust and efficient preconditioners has been addressed by the authors in [11, 12]. While for equations with system matrices of type (6a), we propose to use the preconditioner

$$\mathcal{C} := \text{diag}\left(\sqrt{\lambda}\mathrm{F}, \sqrt{\lambda}\mathrm{F}, \frac{1}{\sqrt{\lambda}}\mathrm{F}, \frac{1}{\sqrt{\lambda}}\mathrm{F}\right), \tag{7}$$

with the block $\mathrm{F} = \mathrm{K}_\nu + \mathrm{M}_{\sigma,\mathrm{k}\omega} + 1/\sqrt{\lambda}\mathrm{M}$; for equations with system matrices of type (6b), we advise to use the preconditioner

$$\mathcal{C}_M = \text{diag}\left(\sqrt{\lambda}\mathrm{F}, \sqrt{\lambda}\mathrm{F}, \frac{1}{\sqrt{\lambda}}\mathrm{F}, \frac{1}{\sqrt{\lambda}}\mathrm{F}, \frac{1}{\sqrt{\lambda}}\mathrm{S_J}, \frac{1}{\sqrt{\lambda}}\mathrm{S_J}, \sqrt{\lambda}\mathrm{S_J}, \sqrt{\lambda}\mathrm{S_J}\right), \tag{8}$$

where $\mathrm{S_J} = \mathrm{D}_{\sigma,\mathrm{k}\omega}{}^T\mathrm{F}^{-1}\mathrm{D}_{\sigma,\mathrm{k}\omega}$. In a MinRes setting, the quality of the preconditioners $\mathcal{C}$ and $\mathcal{C}_M$, used for the system matrices $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, is in general determined by the condition number κ_1 or κ_2 of the preconditioned system, defined as follows:

$$\kappa_1 := \|\mathcal{C}^{-1}\mathcal{A}_1\|_{\mathcal{C}}\|\mathcal{A}_1^{-1}\mathcal{C}\|_{\mathcal{C}} \quad \text{and} \quad \kappa_2 := \|\mathcal{C}_M^{-1}\mathcal{A}_2\|_{\mathcal{C}_M}\|\mathcal{A}_2^{-1}\mathcal{C}_M\|_{\mathcal{C}_M}. \tag{9}$$

In Table 1, we list the theoretical results that have been derived for different settings of (5) in [11, 12]. We especially want to point out that the bounds for the condition numbers are at least uniform in the space discretization parameter h as well as the time discretization parameters ω and N. This has the important consequence that the proposed preconditioned MinRes method converges within a few iterations, independent of the discretization parameters that are directly related to the size of the system matrices.

3 Numerical Validation

The main aim of this paper is to verify the theoretical proven convergence rates by numerical experiments. We consider an academic test problem of the form (1)–(2) or rather (5) in the unit cube $\Omega = (0,1)^3$ and report on various numerical test for various computational settings and varying parameters. Since we are here only

Table 1 Condition number estimates for different settings. Here (σ) denotes robustness with respect to $\sigma \in \mathbb{R}^+$

Test case	α	β	Domains	Equations	Condition number estimate
I	1	0	$\Omega_1 = \Omega_2$	(5a)–(5b)	$\kappa_1 \leq \sqrt{3} \neq c(h, \omega, N, \sigma, \nu, \lambda)$
II	1	0	$\Omega_1 = \Omega_2$	(5a)–(5c)	$\kappa_2 \leq \sqrt{3}(1+\sqrt{5}) \neq c(h, \omega, N, \sigma, \nu, \lambda)$
III	0	1	$\Omega_1 = \Omega_2$	(5a)–(5c)	$\kappa_2 \leq c \neq c(h, \omega, N, (\sigma))$
IV	1	0	$\Omega_1 \neq \Omega_2$	(5a)–(5c)	$\kappa_2 \leq c \neq c(h, \omega, N, (\sigma), \Omega_1, \Omega_2)$
V	1	0	$\Omega_1 = \Omega_2$	(5a)–(5d)	$\kappa_2 \leq c \neq c(h, \omega, N, (\sigma), \text{index sets})$
VI	1	0	$\Omega_1 = \Omega_2$	(5a)–(5b) + (5e)	$\kappa_1 \leq c \neq c(h, \omega, N, \sigma, \nu, \lambda, \text{index sets})$

interested in the study of the robustness of the solver, it is obviously sufficient to consider the solution of the system corresponding to the block of the mode $k = 1$. The numerical results presented in this section were attained using ParMax.[1] We demonstrate the robustness of the block-diagonal preconditioners with respect to the involved parameters. Therefore, for the solution of the preconditioning equations arising from the diagonal blocks F, we use the sparse direct solver UMFPACK,[2] that is very efficient for several thousand unknowns in the case of three-dimensional problems [2–4]. For numerical tests, where the diagonal blocks are replaced by an auxiliary space preconditioner [6, 7], we refer the reader to [10] and [12].

3.1 Test Case I

Tables 2–5 provide the number of MinRes iterations needed for reducing the initial residual by a factor of 10^{-8}. These experiments demonstrate the independence of the MinRes convergence rate of the parameters ω, σ, λ and the mesh size h for all computed constellations. Indeed, the number of iterations is bounded by 28, that is very close to the theoretical bound 30 given by the condition number estimate $\sqrt{3}$. We mention that varying ω also covers the variation of $k\omega$ in terms of k. Furthermore, in Tables 2 and 3, we also report the number of unpreconditioned MinRes iterations, that are necessary for reducing the initial residual by a factor of 10^{-8}. The large number of iterations in the unpreconditioned case underlines the importance of appropriate preconditioning.

3.2 Test Case II

Table 6 provides the number of MinRes iterations needed for reducing the initial residual by a factor 10^{-8}. These experiments demonstrate the independence of the

[1] http://www.numa.uni-linz.ac.at/P19255/software.shtml.

[2] http://www.cise.ufl.edu/research/sparse/umfpack/.

Table 2 Formulation OC-FEM 1 for test case I. Number of MinRes iterations for $DOF = 2,416$, $\nu = \sigma = 1$, and different values of λ and ω. $[\cdot]$ denotes the number of MinRes iterations without preconditioner

$\lambda \setminus \omega$	10^{-10}	10^{-8}	10^{-6}	10^{-4}	10^{-2}	1	10^{2}	10^{4}	10^{6}	10^{8}	10^{10}
10^{-10}	7	7	7	7	7	7	7	7	7	6	4
	[587]	[587]	[586]	[587]	[587]	[587]	[587]	[591]	[485]	[263]	[116]
10^{-6}	21	21	21	21	21	21	20	12	6	4	4
	[373]	[373]	[373]	[373]	[373]	[373]	[373]	[263]	[116]	[114]	[114]
10^{-2}	20	20	20	20	20	20	20	12	6	4	4
	[1,134]	[1,134]	[1,134]	[1,136]	[1,135]	[1,134]	[227]	[114]	[114]	[114]	[114]
1	10	10	10	10	10	14	20	12	6	4	4
	[2,349]	[2,351]	[2,349]	[2,350]	[2,350]	[2,274]	[222]	[114]	[114]	[114]	[114]
10^{2}	6	6	6	6	8	10	20	12	6	4	4
	[2,688]	[2,681]	[2,696]	[2,667]	[3,291]	[2,494]	[224]	[114]	[114]	[114]	[114]
10^{6}	4	4	4	6	6	10	20	12	6	4	4
	[1,152]	[1,159]	[3,434]	[4,697]	[4,867]	[2,493]	[222]	[114]	[114]	[114]	[114]
10^{10}	2	4	4	4	4	10	20	12	6	4	4
	[1,157]	[1,163]	[4,937]	[5,881]	[4,791]	[2,501]	[224]	[114]	[114]	[114]	[114]

Table 3 Formulation OC-FEM 1 for test case I. Number of MinRes iterations for $DOF = 16,736$, $\nu = \sigma = 1$, and different values of λ and ω. $[\cdot]$ denotes the number of MinRes iterations without preconditioner. [-] indicates that MinRes did not converge within 10,000 iterations

$\lambda \setminus \omega$	10^{-10}	10^{-8}	10^{-6}	10^{-4}	10^{-2}	1	10^{2}	10^{4}	10^{6}	10^{8}	10^{10}
10^{-10}	9	9	9	9	9	9	9	10	6	4	4
	[708]	[708]	[708]	[708]	[708]	[708]	[708]	[711]	[578]	[308]	[134]
10^{-6}	21	21	21	21	21	21	20	18	6	4	4
	[825]	[824]	[825]	[825]	[825]	[825]	[824]	[307]	[134]	[132]	[132]
10^{-2}	18	18	18	18	18	20	22	20	6	4	4
	[6,698]	[6,669]	[6,696]	[6,698]	[6,690]	[6,676]	[1,095]	[132]	[132]	[132]	[132]
1	10	10	10	10	10	14	22	20	6	4	4
	[-]	[-]	[-]	[-]	[-]	[-]	[1,094]	[132]	[132]	[132]	[132]
10^{2}	6	6	6	6	8	10	22	20	6	4	4
	[-]	[-]	[-]	[-]	[-]	[-]	[1,094]	[132]	[132]	[132]	[132]
10^{6}	4	4	4	6	6	10	22	20	6	4	4
	[7,365]	[7,547]	[-]	[-]	[-]	[-]	[1,094]	[132]	[132]	[132]	[132]
10^{10}	2	4	4	4	4	10	22	20	6	4	4
	[7,381]	[1,545]	[-]	[-]	[-]	[-]	[1,094]	[132]	[132]	[132]	[132]

MinRes convergence rate of the parameters ω, σ, λ and the mesh size h since the number of iterations is bounded by 88 for all computed constellations. The condition number estimate from Table 1 yields 106 as a bound for the maximal number of iterations.

Table 4 Formulation OC-FEM 1 for test case I. Number of MinRes iterations for $DOF = 124,096$, $\nu = \sigma = 1$, and different values of λ and ω

$\lambda \setminus \omega$	10^{-10}	10^{-8}	10^{-6}	10^{-4}	10^{-2}	1	10^{2}	10^{4}	10^{6}	10^{8}	10^{10}
10^{-10}	13	13	13	13	13	13	13	13	8	4	4
10^{-8}	21	21	21	21	21	21	21	17	8	4	4
10^{-6}	21	21	21	21	21	21	21	20	8	4	4
10^{-4}	20	20	20	20	20	20	28	22	8	4	4
10^{-2}	16	16	16	16	16	18	22	22	8	4	4
1	10	10	10	10	10	12	20	22	8	4	4
10^{2}	6	6	6	6	8	10	20	22	8	4	4
10^{4}	4	4	4	6	6	10	20	22	8	4	4
10^{6}	4	4	4	4	6	10	20	22	8	4	4
10^{8}	2	4	4	4	6	10	20	22	8	4	4
10^{10}	3	4	4	4	4	10	20	22	8	4	4

Table 5 Formulation OC-FEM 1 for test case I. Number of MinRes iterations for $DOF = 124,096$, $\omega = \sigma = 1$, and different values of λ and ν

$\lambda \setminus \nu$	10^{-10}	10^{-8}	10^{-6}	10^{-4}	10^{-2}	1	10^{2}	10^{4}	10^{6}	10^{8}	10^{10}
10^{-10}	2	2	3	3	5	13	21	16	6	4	3
10^{-8}	2	2	3	4	7	21	20	10	4	4	3
10^{-6}	2	3	3	5	13	21	16	6	4	4	4
10^{-4}	2	3	4	7	21	20	10	6	4	4	4
10^{-2}	3	4	6	13	21	18	8	4	4	6	6
1	4	4	8	17	28	12	6	4	6	6	9
10^{2}	4	4	8	20	22	10	6	4	6	6	8
10^{4}	4	4	8	22	20	10	6	4	4	4	8
10^{6}	4	4	8	22	20	10	4	4	4	4	8
10^{8}	4	4	8	22	20	10	4	4	4	4	8
10^{10}	4	4	8	22	20	10	4	2	4	4	8

Table 6 Formulation OC-FEM 2 for test case II. Number of MinRes iterations for $\nu = \sigma = 1$, different values of λ and ω, and $DOF = 19,652$ / $\mathbf{143,748}$

$\lambda \setminus \omega$	10^{-10}	10^{-6}	10^{-2}	1	10^{2}	10^{6}	10^{10}
10^{-10}	21 / **27**	19 / **25**	17 / **25**	17 / **25**	17 / **25**	12 / **16**	10 / **10**
10^{-6}	33 / **32**	33 / **32**	33 / **32**	33 / **32**	29 / **33**	10 / **14**	8 / **8**
10^{-2}	22 / **20**	22 / **20**	26 / **23**	31 / **29**	34 / **35**	14 / **16**	10 / **10**
1	12 / **12**	14 / **14**	14 / **14**	14 / **14**	24 / **24**	10 / **12**	8 / **8**
10^{2}	11 / **11**	13 / **13**	13 / **13**	18 / **18**	34 / **34**	14 / **16**	10 / **10**
10^{6}	13 / **13**	13 / **15**	21 / **21**	28 / **30**	56 / **58**	22 / **24**	14 / **14**
10^{10}	31 / **46**	34 / **65**	33 / **33**	42 / **42**	80 / **88**	30 / **38**	16 / **16**

3.3 Test Case III

Numerical results for the observation of the magnetic flux density are reported in Tables 7–9. The robustness with respect to the space and time discretization

Table 7 Observation of the magnetic flux density B in Formulation OC-FEM 2 for test case *III*. Number of MinRes iterations for $\nu = \sigma = \lambda = 1$ and for different values of ω and various *DOF*

	ω										
DOF	10^{-10}	10^{-8}	10^{-6}	10^{-4}	10^{-2}	1	10^{2}	10^{4}	10^{6}	10^{8}	10^{10}
500	13	13	14	14	14	16	23	12	9	8	7
2,916	11	12	13	13	13	15	29	16	10	8	8
19,652	11	11	12	12	12	14	30	21	11	8	8
143,748	11	11	12	12	12	14	28	27	13	8	8

Table 8 Observation of the magnetic flux density B in Formulation OC-FEM 2 for test case *III*. Number of MinRes iterations for $\sigma = \omega = 1$, different values of λ and ν, and $DOF = 19{,}652/\mathbf{143{,}748}$. [-] indicates that MinRes did not converge within 10,000 iterations

$\lambda \setminus \nu$	10^{-10}	10^{-6}	10^{-2}	1	10^{2}	10^{6}	10^{10}
10^{-10}	174 / **325**	175 / **326**	175 / **327**	213 / **411**	290 / **505**	14 / **14**	8 / **8**
10^{-6}	146 / **289**	146 / **289**	177 / **359**	215 / **392**	58 / **53**	8 / **10**	8 / **8**
10^{-2}	272 / **543**	272 / **543**	306 / **523**	55 / **52**	13 / **13**	9 / **8**	13 / **15**
1	290 / **543**	290 / **541**	240 / **325**	14 / **14**	8 / **8**	8 / **8**	12 / **14**
10^{2}	475 / **948**	479 / **941**	83 / **79**	18 / **18**	12 / **12**	14 / **14**	26 / **36**
10^{6}	193 / **688**	195 / **680**	55 / **55**	28 / **30**	18 / **18**	24 / **26**	360 / **[-]**
10^{10}	36 / **56**	39 / **55**	84 / **88**	42 / **42**	26 / **26**	50 / **54**	[-] / **[-]**

Table 9 Observation of the magnetic flux density B in Formulation OC-FEM 2 for test case *III*. Number of MinRes iterations for $\nu = \sigma = \omega = 1$ and for different values of λ and various *DOF*

	λ										
DOF	10^{-10}	10^{-8}	10^{-6}	10^{-4}	10^{-2}	1	10^{2}	10^{4}	10^{6}	10^{8}	10^{10}
500	36	36	37	39	40	16	19	26	30	36	44
2,916	115	113	121	121	55	15	18	24	28	38	44
19,652	213	214	215	195	55	14	18	24	28	36	42
143,748	411	402	392	265	52	14	18	24	30	36	42

parameters h and ω is demonstrated in Table 7. Table 8 describes the non-robust behavior with respect to the parameters λ and ν. In Table 9 we observe that for large mesh sizes, good iteration numbers are observed even for small λ. Nevertheless, for fixed λ, the iteration numbers are growing with respect to the involved degrees of freedom.

The next experiment demonstrates that robustness with respect to the time discretization parameter ω cannot be achieved by using the preconditioner $\mathcal{C}$ in *Formulation OC-FEM 1*. In Table 10 the number of MinRes iteration needed for reducing the initial residual by a factor of 10^{-8} is displayed. In Table 11, the same experiment as in Table 8 is performed, but using *Formulation OC-FEM 1* instead of *Formulation OC-FEM 2*. Indeed, comparing Table 7 with Table 10 and Table 8 with Table 11 clearly shows that it is essential to work with *Formulation OC-FEM 2*. Besides the robustness with respect to the frequency ω, that is related to the time discretization parameters, we additionally observe better iteration numbers with respect to the regularization parameter λ in the interesting region $0 < \lambda < 1$.

Table 10 Observation of the magnetic flux density B in Formulation OC-FEM 1 for test case *III*. Number of MinRes iterations for $\nu = \sigma = \lambda = 1$ and for different values of ω and various *DOF*. [-] indicates that MinRes did not converge within 10,000 iterations

	ω										
DOF	10^{-10}	10^{-8}	10^{-6}	10^{-4}	10^{-2}	1	10^{2}	10^{4}	10^{6}	10^{8}	10^{10}
392	4,133	[-]	46	20	16	15	21	9	5	4	3
2,416	[-]	[-]	64	29	15	13	27	12	6	4	4
16,736	[-]	[-]	102	28	15	13	26	18	7	4	4
124,096	[-]	[-]	28	13	12	26	24	9	5	4	4

Table 11 Observation of the magnetic flux density B in Formulation OC-FEM 1 for test case *III*. Number of MinRes iterations for $DOF = 16,736$, $\sigma = \omega = 1$, and different values of λ and ν. [-] indicates that MinRes did not converge within 10,000 iterations

$\lambda \setminus \nu$	10^{-10}	10^{-8}	10^{-6}	10^{-4}	10^{-2}	1	10^{2}	10^{4}	10^{6}	10^{8}	10^{10}
10^{-10}	739	901	1,073	1,140	1,462	1,153	1,548	182	32	19	[-]
10^{-6}	357	361	357	385	478	607	96	17	10	9	18
10^{-2}	234	234	234	253	279	50	9	6	7	6	9
1	260	260	260	259	214	13	7	5	6	6	8
10^{2}	462	462	469	440	76	11	6	4	6	6	7
10^{6}	79	79	79	73	21	10	4	4	4	4	6
10^{10}	10	10	9	19	22	10	4	3	4	4	6

Table 12 Different control and observation domains in Formulation OC-FEM 2 / **OC-FEM 1** for test case IV. Number of MinRes iterations for $\nu = \sigma = \lambda = 1$ and for different values of ω and various *DOF*

	ω						
DOF	10^{-10}	10^{-6}	10^{-2}	1	10^{2}	10^{6}	10^{10}
2,916	19 / **34**	20 / **67**	23 / **52**	30 / **30**	30 / **22**	12 / **6**	8 / **4**
19,652	19 / **32**	20 / **82**	24 / **51**	30 / **30**	32 / **22**	12 / **6**	8 / **4**
143,748	19 / **29**	19 / **83**	23 / **48**	29 / **30**	32 / **20**	14 / **8**	8 / **4**

3.4 Test Case VI

In this subsection we consider a numerical example with different observation and control domains Ω_1 and Ω_2, i.e., $\Omega_1 = \Omega = (0,1)^3$ and $\Omega_2 = (0.25, 0.75)^3$. Let us mention that we have to ensure that Ω_1 and Ω_2 are resolved by the mesh. The corresponding numerical results are documented in Tables 12–14. Robustness with respect to the space and time discretization parameters h and ω is demonstrated in Table 12. Table 13 describes the non-robust behavior with respect to the parameters λ and ν. Table 12 in combination with Table 14 indicates that, for the *Formulation OC-FEM 1* in combination with the preconditioner $\mathcal{C}$, robustness with respect to the frequency ω, that is related to the time discretization parameters, cannot be obtained. Here, we want to mention that the good iteration numbers observed in Table 12 are caused by the special choice of $\lambda = 1$.

Table 13 Different control and observation domains in Formulation OC-FEM 2 / **OC-FEM 1** for test case IV. Number of MinRes iterations for $DOF = 19{,}652$ / $\mathbf{16{,}736}$, $\sigma = \omega = 1$, and different values of λ and ν. [-] indicates that MinRes did not converge within 10,000 iterations

$\lambda \setminus \nu$	10^{-10}	10^{-6}	10^{-2}	1	10^{2}	10^{6}	10^{10}
10^{-10}	1,038 / **34**	661 / **36**	[-] / **2,701**	[-] / **[-]**	[-] / **983**	49 / **60**	9 / **[-]**
10^{-6}	342 / **31**	363 / **32**	6,843 / **2,630**	7,142 / **828**	619 / **81**	26 / **41**	8 / **73**
10^{-2}	188 / **29**	209 / **37**	607 / **169**	204 / **61**	114 / **43**	79 / **37**	106 / **47**
1	40 / **19**	41 / **22**	52 / **39**	30 / **30**	26 / **25**	26 / **22**	26 / **24**
10^{2}	41 / **10**	42 / **11**	70 / **22**	40 / **13**	26 / **12**	22 / **11**	28 / **10**
10^{6}	24 / **6**	30 / **6**	76 / **22**	38 / **10**	24 / **6**	26 / **6**	414 / **6**
10^{10}	22 / **4**	34 / **6**	148 / **22**	46 / **10**	44 / **4**	68 / **4**	[-] / **6**

Table 14 Different control and observation domains in Formulation OC-FEM 1 for test case IV. Number of MinRes iterations for $DOF = 16{,}736$, $\sigma = \nu = 1$, and different values of λ and ω. [-] indicates that MinRes did not converge within 10,000 iterations

$\lambda \setminus \omega$	10^{-10}	10^{-8}	10^{-6}	10^{-4}	10^{-2}	1	10^{2}	10^{4}	10^{6}	10^{8}	10^{10}
10^{-10}	9,338	9,347	9,346	9,340	[-]	[-]	2,630	66	11	6	4
10^{-6}	571	571	571	1,075	983	828	169	20	6	4	4
10^{-2}	49	49	122	103	81	61	22	20	6	4	4
1	32	33	82	67	51	30	22	20	6	4	4
10^{2}	23	112	60	46	43	13	22	20	6	4	4
10^{6}	[-]	46	41	39	12	10	22	20	6	4	4
10^{10}	[-]	58	37	12	6	10	22	20	6	4	4

3.5 Test Case V

Numerical results for the case of state constraints imposed on the Fourier coefficients are presented in Tables 15, 16. Here we choose $15{,}512$ random points as the active sets $\mathcal{E}^c$ and $\mathcal{E}^s$ and solve the resulting Jacobi system. The dependence of the MinRes convergence rate on the Moreau–Yosida regularization parameter ε is demonstrated in Table 15. Table 16 clearly demonstrates the robustness with respect to the parameters λ and ω. We refer the reader to [11] for a detailed description of the treatment of state constraints via the Moreau–Yosida regularization. Furthermore, we mention that the presence of constrains imposed on the control Fourier coefficients finally results in (linearized) systems with system matrices having the same structure as the system matrix arising from the case of different observation and control domains.

4 Summary and Conclusion

We demonstrated in many numerical experiments that the preconditioners derived and analyzed in [12] and [11] lead to parameter-robust and efficient solvers in many

Table 15 State constraints in Formulation OC-FEM 1 for test case VI. Number of MinRes iterations for $\nu = \sigma = \omega = 1$, different values of λ and ε, and $DOF = 16,736$ / $\mathbf{124,096}$. [-] indicates that MinRes did not converge within 10,000 iterations

$\lambda \setminus \varepsilon$	10^{-10}	10^{-6}	10^{-2}	1	10^{2}	10^{6}	10^{10}
10^{-10}	88 / **142**	59 / **94**	31 / **46**	17 / **22**	9 / **13**	9 / **13**	9 / **13**
10^{-6}	992 / **3,275**	612 / **1,930**	220 / **372**	36 / **35**	21 / **21**	21 / **21**	21 / **21**
10^{-2}	[-] / **[-]**	[-] / **[-]**	351 / **383**	29 / **29**	20 / **18**	20 / **18**	20 / **18**
1	[-] / **[-]**	[-] / **[-]**	191 / **206**	24 / **24**	16 / **16**	14 / **13**	14 / **12**
10^{2}	[-] / **[-]**	[-] / **[-]**	120 / **124**	13 / **13**	12 / **12**	10 / **10**	10 / **10**
10^{6}	[-] / **[-]**	5,882 / **6,619**	12 / **11**	10 / **10**	10 / **10**	10 / **10**	10 / **10**
10^{10}	[-] / **[-]**	162 / **167**	10 / **10**	10 / **10**	10 / **10**	10 / **10**	10 / **10**

Table 16 State constraints in Formulation OC-FEM 1 for test case VI. Number of MinRes iterations for $DOF = 124,096$, $\nu = \sigma = \varepsilon = 1$, and different values of λ and ω

$\lambda \setminus \omega$	10^{-10}	10^{-8}	10^{-6}	10^{-4}	10^{-2}	1	10^{2}	10^{4}	10^{6}	10^{8}	10^{10}
10^{-10}	22	22	22	22	22	22	22	22	12	6	4
10^{-6}	35	35	35	35	35	35	35	22	8	4	4
10^{-2}	30	30	30	30	30	29	22	22	8	4	4
1	20	20	20	20	20	24	20	22	8	4	4
10^{2}	16	16	16	16	18	13	20	22	8	4	4
10^{6}	13	13	14	18	12	10	20	22	8	4	4
10^{10}	13	13	16	12	6	10	20	22	8	4	4

practically important cases. Therefore, we reported on a broad range of numerical experiments, that confirm the theoretical convergence rates. Consequently, the multiharmonic finite element discretization technique in combination with efficient and parameter-robust solvers leads to a very competitive method. Furthermore, we want to mention that due to the decoupling nature of the frequency domain equations with respect to the individual modes, a parallelization of the proposed method is straightforward (cf. Algorithm 1).

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