Chapter 7
Capacity

7.1 Capacity

The notion of *capacity* appears in potential theory. The abstract theory was formulated by Choquet in 1954. In this section, we denote by X a metric space, by $\mathcal{K}$ the class of compact subsets of X, and by $\mathcal{O}$ the class of open subsets of X.

Definition 7.1.1. A capacity on X is a function

$$\text{cap} : \mathcal{K} \to [0, +\infty] : K \to \text{cap}(K)$$

such that:

(C_1) (monotonicity.) For every $A, B \in \mathcal{K}$ such that $A \subset B$, $\text{cap}(A) \leq \text{cap}(B)$.
(C_2) (regularity.) For every $K \in \mathcal{K}$ and for every $a > \text{cap}(K)$, there exists $U \in \mathcal{O}$ such that $K \subset U$, and for all $C \in \mathcal{K}$ satisfying $C \subset U$, $\text{cap}(C) < a$.
(C_3) (strong subadditivity.) For every $A, B \in \mathcal{K}$,

$$\text{cap}(A \cup B) + \text{cap}(A \cap B) \leq \text{cap}(A) + \text{cap}(B).$$

The Lebesgue measure of a compact subset of $\mathbb{R}^N$ is a capacity.

We denote by cap a capacity on X. We extend the capacity to the open subsets of X.

Definition 7.1.2. The capacity of $U \in \mathcal{O}$ is defined by

$$\text{cap}(U) = \sup\{\text{cap}(K) : K \in \mathcal{K} \text{ and } K \subset U\}.$$

Lemma 7.1.3. *Let $A, B \in \mathcal{O}$ and $K \in \mathcal{K}$ be such that $K \subset A \cup B$. Then there exist $L, M \in \mathcal{K}$ such that $L \subset A$, $M \subset B$, and $K = L \cup M$.*

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Proof. The compact sets $K \setminus A$ and $K \setminus B$ are disjoint. Hence there exist disjoint open sets U and V such that $K \setminus A \subset U$ and $K \setminus B \subset V$. It suffices to define $L = K \setminus U$ and $M = K \setminus V$. □

Proposition 7.1.4. *(a) (monotonicity.) For every $A, B \in \mathcal{O}$ such that $A \subset B$,* $\mathrm{cap}(A) \leq \mathrm{cap}(B)$.
(b) (regularity.) For every $K \in \mathcal{K}$, $\mathrm{cap}(K) = \inf\{\mathrm{cap}(U) : U \in \mathcal{O}$ *and* $U \supset K\}$.
(c) (strong subadditivity.) For every $A, B \in \mathcal{O}$,

$$\mathrm{cap}(A \cup B) + \mathrm{cap}\, A \cap B) \leq \mathrm{cap}(A) + \mathrm{cap}(B).$$

Proof. (a) Monotonicity is clear.
(b) Let us define $\mathrm{Cap}(K) = \inf\{\mathrm{cap}(U) : U \in \mathcal{O}$ and $U \supset K\}$. By definition, $\mathrm{cap}(K) \leq \mathrm{Cap}(K)$. Let $a > \mathrm{cap}(K)$. There exists $U \in \mathcal{O}$ such that $K \subset U$ and for every $C \in \mathcal{K}$ satisfying $C \subset U$, $\mathrm{cap}(C) < a$. Hence $\mathrm{Cap}(K) \leq \mathrm{cap}(U) < a$. Since $a > \mathrm{cap}(K)$ is arbitrary, we conclude that $\mathrm{Cap}(K) \leq \mathrm{cap}(K)$.
(c) Let $A, B \in \mathcal{O}$, $a < \mathrm{cap}(A \cup B)$, and $b < \mathrm{cap}(A \cap B)$. By definition, there exist $K, C \in \mathcal{K}$ such that $K \subset A \cup B$, $C \subset A \cap B$, $a < \mathrm{cap}(K)$, and $b \leq \mathrm{cap}(C)$. We can assume that $C \subset K$. The preceding lemma implies the existence of $L, M \in \mathcal{K}$ such that $L \subset A$, $M \subset B$, and $K = L \cup M$. We can assume that $C \subset L \cap M$. We obtain by monotonicity and strong subadditivity that

$$\begin{aligned} a + b \leq \mathrm{cap}(K) + \mathrm{cap}(C) &\leq \mathrm{cap}(L \cup M) + \mathrm{cap}(L \cap M) \\ &\leq \mathrm{cap}(L) + \mathrm{cap}(M) \leq \mathrm{cap}(A) + \mathrm{cap}(B). \end{aligned}$$

Since $a < \mathrm{cap}(A \cup B)$ and $b < \mathrm{cap}(A \cap B)$ are arbitrary, the proof is complete. □

We extend the capacity to all subsets of X.

Definition 7.1.5. The capacity of a subset A of X is defined by

$$\mathrm{cap}(A) = \inf\{\mathrm{cap}(U) : U \in \mathcal{O} \text{ and } U \supset A\}.$$

By regularity, the capacity of compact subsets is well defined.

Proposition 7.1.6. *(a) (monotonicity.) For every $A, B \subset X$,* $\mathrm{cap}(A) \leq \mathrm{cap}(B)$.
(b) (strong subadditivity.) For every $A, B \subset X$,

$$\mathrm{cap}(A \cup B) + \mathrm{cap}(A \cap B) \leq \mathrm{cap}(A) + \mathrm{cap}(B).$$

Proof. (a) Monotonicity is clear.
(b) Let $A, B \subset X$ and $U, V \in \mathcal{O}$ be such that $A \subset U$ and $B \subset V$. We have

$$\mathrm{cap}(A \cup B) + \mathrm{cap}(A \cap B) \leq \mathrm{cap}(U \cup V) + \mathrm{cap}(U \cap V) \leq \mathrm{cap}(U) + \mathrm{cap}(V).$$

It is easy to conclude the proof. □

Proposition 7.1.7. *Let* (K_n) *be a decreasing sequence in* $\mathcal{K}$. *Then*

$$\operatorname{cap}\left(\bigcap_{n=1}^{\infty} K_n\right) = \lim_{n\to\infty} \operatorname{cap}(K_n).$$

Proof. Let $K = \bigcap_{n=1}^{\infty} K_n$ and $U \in \mathcal{O}$, $U \supset K$. By compactness, there exists m such that $K_m \subset U$. We obtain, by monotonicity, $\operatorname{cap}(K) \leq \lim_{n\to\infty} \operatorname{cap}(K_n) \leq \operatorname{cap}(U)$. It suffices then to take the infimum with respect to U. □

Lemma 7.1.8. *Let* (U_n) *be an increasing sequence in* $\mathcal{O}$. *Then*

$$\operatorname{cap}\left(\bigcup_{n=1}^{\infty} U_n\right) = \lim_{n\to\infty} \operatorname{cap}(U_n).$$

Proof. Let $U = \bigcup_{n=1}^{\infty} U_n$ and $K \in \mathcal{K}, K \subset U$. By compactness, there exists m such that $K \subset U_m$. We obtain by monotonicity $\operatorname{cap}(K) \leq \lim_{n\to\infty} \operatorname{cap}(U_n) \leq \operatorname{cap} U$. It suffices then to take the supremum with respect to K. □

Theorem 7.1.9. *Let* (A_n) *be an increasing sequence of subsets of* X. *Then*

$$\operatorname{cap}\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \operatorname{cap}(A_n).$$

Proof. Let $A = \bigcup_{n=1}^{\infty} A_n$. By monotonicity, $\lim_{n\to\infty} \operatorname{cap}(A_n) \leq \operatorname{cap}(A)$. We can assume that $\lim_{n\to\infty} \operatorname{cap}(A_n) < +\infty$. Let $\varepsilon > 0$ and $a_n = 1 - 1/(n+1)$. We construct, by induction, an increasing sequence $(U_n) \subset \mathcal{O}$ such that $A_n \subset U_n$ and

$$\operatorname{cap}(U_n) \leq \operatorname{cap}(A_n) + \varepsilon\, a_n. \qquad (*)$$

When $n = 1$, $(*)$ holds by definition. Assume that $(*)$ holds for n. By definition, there exists $V \in \mathcal{O}$ such that $A_{n+1} \subset V$ and

$$\operatorname{cap}(V) \leq \operatorname{cap}(A_{n+1}) + \varepsilon(a_{n+1} - a_n).$$

We define $U_{n+1} = U_n \cup V$, so that $A_{n+1} \subset U_{n+1}$. We obtain, by strong subadditivity,

$$\begin{aligned}
\operatorname{cap}(U_{n+1}) &\leq \operatorname{cap}(U_n) + \operatorname{cap}(V) - \operatorname{cap}(U_n \cap V) \\
&\leq \operatorname{cap}(A_n) + \varepsilon\, a_n + \operatorname{cap}(A_{n+1}) + \varepsilon(a_{n+1} - a_n) - \operatorname{cap}(A_n) \\
&= \operatorname{cap}(A_{n+1}) + \varepsilon\, a_{n+1}.
\end{aligned}$$

It follows from (∗) and the preceding lemma that

$$\operatorname{cap}(A) \le \operatorname{cap}\left(\bigcup_{n=1}^{\infty} U_n\right) = \lim_{n\to\infty} \operatorname{cap}(U_n) \le \lim_{n\to\infty} \operatorname{cap}(A_n) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the proof is complete. □

Corollary 7.1.10 (Countable subadditivity). *Let (A_n) be a sequence of subsets of X. Then* $\operatorname{cap}\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} \operatorname{cap}(A_n)$.

Proof. Let $B_k = \bigcup_{n=1}^{k} A_k$. We have

$$\operatorname{cap}\left(\bigcup_{n=1}^{\infty} A_n\right) = \operatorname{cap}\left(\bigcup_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} \operatorname{cap}(B_k) \le \sum_{n=1}^{\infty} \operatorname{cap}(A_n).$$ □

Definition 7.1.11. The outer Lebesgue measure of a subset of $\mathbb{R}^N$ is defined by

$$m^*(A) = \inf\{m(U) : U \text{ is open and } U \supset A\}.$$

7.2 Variational Capacity

In order to define *variational capacity*, we introduce the space $\mathcal{D}^{1,p}(\mathbb{R}^N)$.

Definition 7.2.1. Let $1 \le p < N$. On the space

$$\mathcal{D}^{1,p}(\mathbb{R}^N) = \{u \in L^{p^*}(\mathbb{R}^N) : \nabla u \in L^p(\mathbb{R}^N; \mathbb{R}^N)\},$$

we define the norm

$$\|u\|_{\mathcal{D}^{1,p}(\mathbb{R}^N)} = \|\nabla u\|_p.$$

Proposition 7.2.2. *Let $1 \le p < N$.*

(a) The space $\mathcal{D}(\mathbb{R}^N)$ is dense in $\mathcal{D}^{1,p}(\mathbb{R}^N)$.
(b) (Sobolev's inequality.) There exists $c = c(p, N)$ such that for every $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$,

$$\|u\|_{p^*} \le c\|\nabla u\|_p.$$

(c) The space $\mathcal{D}^{1,p}(\mathbb{R}^N)$ is complete.

Proof. The space $\mathcal{D}(\mathbb{R}^N)$ is dense in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ with the norm $\|u\|_{p^*} + \|\nabla u\|_p$. The argument is similar to that of the proof of Theorem 6.1.10.

Sobolev's inequality follows by density from Lemma 6.4.2. Hence for every $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$,

$$\|\nabla u\|_p \le \|u\|_{p^*} + \|\nabla u\|_p \le (c+1)\|\nabla u\|_p.$$

Let (u_n) be a Cauchy sequence in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. Then $u_n \to u$ in $L^{p^*}(\mathbb{R}^N)$, and for $1 \le k \le N$, $\partial_k u_n \to v_k$ in $L^p(\mathbb{R}^N)$. By the closing lemma, for $1 \le k \le N$, $\partial_k u = v_k$. We conclude that $u_n \to u$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. □

Proposition 7.2.3. *Every bounded sequence in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ contains a subsequence converging in $L^1_{\text{loc}}(\mathbb{R}^N)$ and almost everywhere on $\mathbb{R}^N$.*

Proof. Cantor's diagonal argument will be used. Let (u_n) be bounded in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. By Sobolev's inequality, for every $k \ge 1$, (u_n) is bounded in $W^{1,1}(B(0,k))$. Rellich's theorem and Proposition 4.2.10 imply the existence of a subsequence $(u_{1,n})$ of (u_n) converging in $L^1(B(0,1))$ and almost everywhere on $B(0,1)$. By induction, for every k, there exists a subsequence $(u_{k,n})$ of $(u_{k-1,n})$ converging in $L^1(B(0,k))$ and almost everywhere on $B(0,k)$. The sequence $v_n = u_{n,n}$ converges in $L^1_{\text{loc}}(\mathbb{R}^N)$ and almost everywhere on $\mathbb{R}^N$. □

Definition 7.2.4. Let $1 \le p < N$ and let K be a compact subset of $\mathbb{R}^N$. The capacity of degree p of K is defined by

$$\text{cap}_p(K) = \inf\left\{\int_{\mathbb{R}^N} |\nabla u|^p dx : u \in \mathcal{D}^{1,p}_K(\mathbb{R}^N)\right\},$$

where

$$\mathcal{D}^{1,p}_K(\mathbb{R}^N) = \{u \in \mathcal{D}^{1,p}(\mathbb{R}^N) : \text{there exists } U \text{ open such that } K \subset U \text{ and } \chi_U \le u \text{ almost everywhere}\}.$$

Theorem 7.2.5. *The capacity of degree p is a capacity on $\mathbb{R}^N$.*

Proof. (a) Monotonicity is clear by definition.

(b) Let K be compact and $a > \text{cap}_p(\mathbb{R}^N)$. There exist $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ and U open such that $K \subset U, \chi_U \le u$ almost everywhere, and $\int_{\mathbb{R}^N} |\nabla u|^p dx < a$. For every compact set $C \subset U$, we have

$$\text{cap}_p(C) \le \int_{\mathbb{R}^N} |\nabla u|^p dx < a,$$

so that cap_p is regular.

(c) Let A and B be compact sets, $a > \text{cap}_p(A)$, and $b > \text{cap}_p(B)$. There exist $u, v \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ and U and V open sets such that $A \subset U$, $B \subset V, \chi_U \le u$, and $\chi_V \le v$ almost everywhere and

$$\int_{\mathbb{R}^N} |\nabla u|^p dx < a, \quad \int_{\mathbb{R}^N} |\nabla v|^p dx < b.$$

Since $\max(u,v) \in \mathcal{D}^{1,p}_{A\cup B}(\mathbb{R}^N)$ and $\min(u,v) \in \mathcal{D}^{1,p}_{A\cap B}(\mathbb{R}^N)$, Corollary 6.1.14 implies that

$$\int_{\mathbb{R}^N} |\nabla \max(u,v)|^p dx + \int_{\mathbb{R}^N} |\nabla \min(u,v)|^p = \int_{\mathbb{R}^N} |\nabla u|^p dx + \int_{\mathbb{R}^N} |\nabla v|^p dx \le a + b.$$

We conclude that

$$\operatorname{cap}_p(A \cup B) + \operatorname{cap}_p(A \cap B) \le a + b.$$

Since $a > \operatorname{cap}_p(A)$ and $b > \operatorname{cap}_p(B)$ are arbitrary, cap_p is strongly subadditive. □

The variational capacity is finer than the Lebesgue measure.

Proposition 7.2.6. *There exists a constant $c = c(p,N)$ such that for every $A \subset \mathbb{R}^N$,*

$$m^*(A) \le c \operatorname{cap}_p(A)^{N/(N-p)}.$$

Proof. Let K be a compact set and $u \in \mathcal{D}^{1,p}_K(\mathbb{R}^N)$. It follows from Sobolev's inequality that

$$m(K) \le \int_{\mathbb{R}^N} |u|^{p^*} dx \le c \left(\int_{\mathbb{R}^N} |\nabla u|^p dx \right)^{p^*/p}.$$

By definition,

$$m(K) \le c \operatorname{cap}_{p,\mathbb{R}^N}(K)^{N/(N-p)}.$$

To conclude, it suffices to extend this inequality to open subsets of $\mathbb{R}^N$ and to arbitrary subsets of $\mathbb{R}^N$. □

The variational capacity differs essentially from the Lebesgue measure.

Proposition 7.2.7. *Let K be a compact set. Then*

$$\operatorname{cap}_p(\partial K) = \operatorname{cap}_p(K).$$

Proof. Let $a > \operatorname{cap}_p(\partial K)$. There exist $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ and an open set U such that $\partial K \subset U, \chi_U \le u$ almost everywhere, and

$$\int_{\mathbb{R}^N} |\nabla u|^p dx < a.$$

Let us define $V = U \cup K$ and $v = \max(u, \chi_V)$. Then $v \in \mathcal{D}^{1,p}_K(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} |\nabla v|^p dx \le \int_{\mathbb{R}^N} |\nabla u|^p dx,$$

so that $\text{cap}_p(K) < a$. Since $a > \text{cap}_p(\partial K)$ is arbitrary, we obtain

$$\text{cap}_p(K) \le \text{cap}_p(\partial K) \le \text{cap}_p(K). \qquad \square$$

Example. Let $1 \le p < N$ and let B be a closed ball in $\mathbb{R}^N$. We deduce from the preceding propositions that

$$0 < \text{cap}_p(B) = \text{cap}_p(\partial B).$$

Theorem 7.2.8. *Let $1 < p < N$ and U an open set. Then*

$$\text{cap}_p(U) = \inf\left\{\int_{\mathbb{R}^N} |\nabla u|^p dx : u \in \mathcal{D}^{1,p}(\mathbb{R}^N), \chi_U \le u \textit{ almost everywhere}\right\}.$$

Proof. Let us denote by $\text{Cap}_p(U)$ the second member of the preceding equality. It is clear by definition that $\text{cap}_p(U) \le \text{Cap}_p(U)$.

Assume that $\text{cap}_p(U) < \infty$. Let (K_n) be an increasing sequence of compact subsets of U such that $U = \bigcup_{n=1}^{\infty} K_n$, and let $(u_n) \subset \mathcal{D}^{1,p}(\mathbb{R}^N)$ be such that for every n, $\chi_{K_n} \le u_n$ almost everywhere and

$$\int_{\mathbb{R}^N} |\nabla u_n|^p dx \le \text{cap}_p(K_n) + 1/n.$$

The sequence (u_n) is bounded in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. By Proposition 7.2.3, we can assume that $u_n \to u$ in $L^1_{\text{loc}}(\mathbb{R}^N)$ and almost everywhere. It follows from Sobolev's inequality that $u \in L^{p^*}(\mathbb{R}^N)$. Theorem 6.1.7 implies that

$$\int_{\mathbb{R}^N} |\nabla u|^p dx \le \varliminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^p dx \le \lim_{n\to\infty} \text{cap}_p(K_n) \le \text{cap}_p(U).$$

(By Theorem 7.1.9, $\lim_{n\to\infty} \text{cap}_p(K_n) = \text{cap}_p(U)$.) Since almost everywhere, $\chi_U \le u$, we conclude that $\text{Cap}_p(U) \le \text{cap}_p(U)$. $\square$

Corollary 7.2.9. *Let $1 < p < N$, and let U and V be open sets such that $U \subset V$ and $m(V \setminus U) = 0$. Then $\text{cap}_p(U) = \text{cap}_p(V)$.*

Proof. Let $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ be such that $\chi_U \le u$ almost everywhere. Then $\chi_V \le u$ almost everywhere. $\square$

Corollary 7.2.10 (Capacity inequality). *Let $1 < p < N$ and $u \in \mathcal{D}(\mathbb{R}^N)$. Then for every $t > 0$,*

$$\text{cap}_p(\{|u| > t\}) \le t^{-p} \int_{\mathbb{R}^N} |\nabla u|^p dx.$$

Proof. By Corollary 6.1.14, $|u|/t \in \mathcal{D}^{1,p}(\mathbb{R}^N)$. $\square$

Definition 7.2.11. Let $1 \le p < N$. A function $v : \mathbb{R}^N \to \mathbb{R}$ is quasicontinuous of degree p if for every $\varepsilon > 0$, there exists an ω-open set such that $\text{cap}_p(\omega) \le \varepsilon$ and $v\big|_{\mathbb{R}^N \setminus \omega}$ is continuous. Two quasicontinuous functions of degrees p, v, and w are equal quasi-everywhere if $\text{cap}_p(\{|v - w| > 0\}) = 0$.

Proposition 7.2.12. *Let $1 < p < N$ and let v and w be quasicontinuous functions of degree p and almost everywhere equal. Then v and w are quasi-everywhere equal.*

Proof. By assumption, $m(A) = 0$, where $A = \{|v - w| > 0\}$, and for every n, there exists an ω_n-open set such that $\text{cap}_p(\omega_n) \le 1/n$ and $|v - w|\big|_{\mathbb{R}^N \setminus \omega_n}$ is continuous. It follows that $A \cup \omega_n$ is open. We conclude, using Corollary 7.2.9, that

$$\text{cap}_p(A) \le \text{cap}_p(A \cup \omega_n) = \text{cap}_p(\omega_n) \to 0, \quad n \to \infty. \qquad \square$$

Proposition 7.2.13. *Let $1 < p < N$ and $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$. Then there exists a function v quasicontinuous of degree p and almost everywhere equal to u.*

Proof. By Proposition 7.2.2, there exists $(u_n) \subset \mathcal{D}(\mathbb{R}^N)$ such that $u_n \to u$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. Using Proposition 7.2.3, we can assume that $u_n \to u$ almost everywhere and

$$\sum_{k=1}^{\infty} 2^{kp} \int_{\mathbb{R}^N} |\nabla(u_{k+1} - u_k)|^p dx < \infty.$$

We define

$$U_k = \{|u_{k+1} - u_k| > 2^{-k}\}, \quad \omega_m = \bigcup_{k=m}^{\infty} U_k.$$

Corollary 7.2.10 implies that for every k,

$$\text{cap}_p(U_k) \le 2^{kp} \int_{\mathbb{R}^N} |\nabla(u_{k+1} - u_k)|^p dx.$$

It follows from Corollary 7.1.10 that for every m,

$$\text{cap}_p(\omega_m) \le \sum_{k=m}^{\infty} 2^{kp} \int_{\mathbb{R}^N} |\nabla(u_{k+1} - u_k)|^p dx \to 0, \quad m \to \infty.$$

We obtain, for every $x \in \mathbb{R}^N \setminus \omega_m$ and every $k \ge j \ge m$,

$$|u_j(x) - u_k(x)| \le 2^{1-j},$$

so that (u_n) converges simply to v on $\mathbb{R}^N \setminus \bigcap_{m=1}^{\infty} \omega_m$. Moreover, $v\big|_{\mathbb{R}^N \setminus \omega_m}$ is continuous, since the convergence of (u_n) on $\mathbb{R}^N \setminus \omega_m$ is uniform. For $x \in \bigcap_{m=1}^{\infty} \omega_m$, we define $v(x) = 0$. Since by Proposition 7.2.6, $m(\omega_m) \to 0$, we conclude that $u = v$ almost everywhere. $\square$

7.3 Functions of Bounded Variations

A function is of *bounded variation* if its first-order derivatives, in the sense of distributions, are bounded measures.

Definition 7.3.1. Let Ω be an open subset of $\mathbb{R}^N$. The divergence of $v \in C^1(\Omega;\mathbb{R}^N)$ is defined by

$$\operatorname{div} v = \sum_{k=1}^{N} \partial_k v_k.$$

The total variation of $u \in L^1_{\text{loc}}(\Omega)$ is defined by

$$\|Du\|_\Omega = \sup\left\{\int_\Omega u \operatorname{div} v\, dx : v \in \mathcal{D}(\Omega;\mathbb{R}^N), \|v\|_\infty \le 1\right\},$$

where

$$\|v\|_\infty = \sup_{x\in\Omega}\left(\sum_{k=1}^{N}(v_k(x))^2\right)^{1/2}.$$

Theorem 7.3.2. *Let (u_n) be such that $u_n \to u$ in $L^1_{\text{loc}}(\Omega)$. Then*

$$\|Du\|_\Omega \le \varliminf_{n\to\infty} \|Du_n\|_\Omega.$$

Proof. Let $v \in \mathcal{D}(\Omega;\mathbb{R}^N)$ be such that $\|v\|_\infty \le 1$. We have, by definition,

$$\int_\Omega u \operatorname{div} v\, dx = \lim_{n\to\infty}\int_\Omega u_n \operatorname{div} v\, dx \le \varliminf_{n\to\infty}\|Du_n\|_\Omega.$$

It suffices then to take the supremum with respect to v. □

Theorem 7.3.3. *Let $u \in W^{1,1}_{\text{loc}}(\Omega)$. Then the following properties are equivalent:*

(a) $\nabla u \in L^1(\Omega;\mathbb{R}^N)$;
(b) $\|Du\|_\Omega < \infty$.

In this case,

$$\|Du\|_\Omega = \|\nabla u\|_{L^1(\Omega)}.$$

Proof. (a) Assume that $\nabla u \in L^1(\Omega;\mathbb{R}^N)$. Let $v \in \mathcal{D}(\Omega;\mathbb{R}^N)$ be such that $\|v\|_\infty \le 1$. It follows from the Cauchy–Schwarz inequality that

$$\int_\Omega u \operatorname{div} v\, dx = -\int_\Omega \sum_{k=1}^{N} v_k \partial_k u\, dx \le \int_\Omega |\nabla u| dx.$$

Hence $\|Du\|_\Omega \le \|\nabla u\|_{L^1(\Omega)}$.

Theorem 4.3.11 implies the existence of $(w_n) \subset \mathcal{D}(\Omega; \mathbb{R}^N)$ converging to ∇u in $L^1(\Omega; \mathbb{R}^N)$. We can assume that $w_n \to \nabla u$ almost everywhere on Ω. Let us define

$$v_n = w_n / \sqrt{|w_n|^2 + 1/n}.$$

We infer from Lebesgue's dominated convergence theorem that

$$||\nabla u||_{L^1(\Omega)} = \int_\Omega |\nabla u| dx = \lim_{n\to\infty} \int_\Omega v_n \cdot \nabla u \, dx \leq ||Du||_\Omega.$$

(b) Assume that $||Du||_\Omega < \infty$ and define

$$\omega_n = \{x \in \Omega : d(x, \partial\Omega) > 1/n \text{ and } |x| < n\}.$$

Then by the preceding step, we obtain

$$||\nabla u||_{L^1(\omega_n)} = ||Du||_{\omega_n} \leq ||Du||_\Omega < \infty.$$

Levi's theorem ensures that $\nabla u \in L^1(\Omega; \mathbb{R}^N)$. □

Example. There exists a function everywhere differentiable on $[-1, 1]$ such that $||Du||_{]-1,1[} = +\infty$. We define

$$\begin{aligned} u(x) &= 0, & x &= 0, \\ &= x^2 \sin \tfrac{1}{x^2}, & 0 &< |x| \leq 1. \end{aligned}$$

We obtain

$$\begin{aligned} u'(x) &= 0, & x &= 0, \\ &= 2x \sin \tfrac{1}{x^2} - \tfrac{2}{x} \cos \tfrac{1}{x^2}, & 0 &< |x| \leq 1. \end{aligned}$$

The preceding theorem implies that

$$+\infty = \lim_{n\to\infty} ||u'||_{L^1(]1/n,1[)} \leq ||Du||_{]-1,1[}.$$

Indeed,

$$2 \int_0^1 |\cos \frac{1}{x^2}| \frac{dx}{x} = \int_1^\infty |\cos t| \frac{dt}{t} = +\infty.$$

The function u has no weak derivative!

Example (Cantor function). There exists a continuous nondecreasing function with almost everywhere zero derivative and positive total variation. We use the notation of the last example of Sect. 2.2. We consider the Cantor set C corresponding to $\ell_n = 1/3^{n+1}$. Observe that

$$m(C) = 1 - \sum_{j=0}^{\infty} 2^j / 3^{j+1} = 0.$$

We define on $\mathbb{R}$,

$$u_n(x) = \left(\frac{3}{2}\right)^n \int_0^x \chi_{C_n}(t)dt.$$

It is easy to verify by symmetry that

$$||u_{n+1} - u_n||_\infty \leq \frac{1}{3}\frac{1}{2^{n+1}}.$$

By the Weierstrass test, (u_n) converges uniformly to the *Cantor function* $u \in C(\mathbb{R})$. For $n \geq m$, $u_n' = 0$ on $\mathbb{R} \setminus C_m$. The closing lemma implies that $u' = 0$ on $\mathbb{R} \setminus C_m$. Since m is arbitrary, $u' = 0$ on $\mathbb{R} \setminus C$. Theorems 7.3.2 and 7.3.3 ensure that

$$||Du||_{\mathbb{R}} \leq \varliminf_{n\to\infty} ||u_n'||_{L^1(\mathbb{R})} = 1.$$

Let $v \in \mathcal{D}(\mathbb{R})$ be such that $||v||_\infty = 1$ and $v = -1$ on $[0, 1]$ and integrate by parts:

$$\int_{\mathbb{R}} v'u\, dx = \lim_{n\to\infty} \int_{\mathbb{R}} v'u_n\, dx = -\lim_{n\to\infty} \int_{\mathbb{R}} vu_n' dx = \lim_{n\to\infty} \left(\frac{3}{2}\right)^n m(C_n) = 1.$$

We conclude that $||Du||_{\mathbb{R}} = 1$. The function u has no weak derivative.

Definition 7.3.4. Let Ω be an open subset of $\mathbb{R}^N$. On the space

$$BV(\Omega) = \{u \in L^1(\Omega) : ||Du||_\Omega < \infty\},$$

we define the norm

$$||u||_{BV(\Omega)} = ||u||_{L^1(\Omega)} + ||Du||_\Omega$$

and the distance of strict convergence

$$d_S(u, v) = ||u - v||_{L^1(\Omega)} + \Big|||Du||_\Omega - ||Dv||_\Omega\Big|.$$

Remark. It is clear that convergence in norm implies strict convergence.

Example. The space $BV(]0, \pi[)$, with the distance of strict convergence, is not complete. We define on $]0, \pi[$,

$$u_n(x) = \frac{1}{n}\cos nx,$$

so that $u_n \to 0$ in $L^1(]0, \pi[)$. By Theorem 7.3.3, for every n,

$$||Du_n||_{]0,\pi[} = \int_0^\pi |\sin nx| dx = 2.$$

Hence $\lim_{j,k\to\infty} d_S(u_j, u_k) = \lim_{j,k\to\infty} ||u_j - u_k||_{L^1(]0,\pi[)} = 0$. If $\lim_{n\to\infty} d_S(u_n, v) = 0$, then $v = 0$. But $\lim_{n\to\infty} d_S(u_n, 0) = 2$. This is a contradiction.

Proposition 7.3.5. *The normed space $BV(\Omega)$ is complete.*

Proof. Let (u_n) be a Cauchy sequence on the normed space $BV(\Omega)$. Then (u_n) is a Cauchy sequence in $L^1(\Omega)$, so that $u_n \to u$ in $L^1(\Omega)$.

Let $\varepsilon > 0$. There exists m such that for $j, k \geq m$, $||D(u_j - u_k)||_\Omega \leq \varepsilon$. Theorem 7.3.2 implies that for $k \geq m$, $||D(u_k - u)|| \leq \underline{\lim}_{j\to\infty} ||D(u_j - u_k)||_\Omega \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $||D(u_k - u)||_\Omega \to 0$, $k \to \infty$. □

Lemma 7.3.6. *Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ be such that $||Du||_{\mathbb{R}^N} < \infty$. Then*

$$||\nabla(\rho_n * u)||_{L^1(\mathbb{R}^N)} \leq ||Du||_{\mathbb{R}^N} \text{ and } ||Du||_{\mathbb{R}^N} = \lim_{n\to\infty} ||\nabla(\rho_n * u)||_{L^1(\mathbb{R}^N)}.$$

Proof. Let $v \in \mathcal{D}(\mathbb{R}^N; \mathbb{R}^N)$ be such that $||v||_\infty \leq 1$. It follows from Proposition 4.3.15 that

$$\int_{\mathbb{R}^N} (\rho_n * u) \operatorname{div} v \, dx = \int_{\mathbb{R}^N} u \sum_{k=1}^N \rho_n * \partial_k v_k dx = \int_{\mathbb{R}^N} u \sum_{k=1}^N \partial_k(\rho_n * v_k) dx.$$

The Cauchy–Schwarz inequality implies that for every $x \in \mathbb{R}^N$,

$$\sum_{k=1}^N (\rho_n * v_k(x))^2 = \sum_{k=1}^N \left(\int_{\mathbb{R}^N} \rho_n(x-y) v_k(y) dy \right)^2 \leq \sum_{k=1}^N \int_{\mathbb{R}^N} \rho_n(x-y)(v_k(y))^2 dy \leq 1.$$

Hence we obtain

$$\int_{\mathbb{R}^N} (\rho_n * u) \operatorname{div} v \, dx \leq ||Du||_{\mathbb{R}^N},$$

and by Theorem 7.3.3, $||\nabla(\rho_n * u)||_{L^1(\mathbb{R}^N)} \leq ||Du||_{\mathbb{R}^N}$.

By the regularization theorem, $\rho_n * u \to u$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Theorems 7.3.2 and 7.3.3 ensure that

$$||Du||_{\mathbb{R}^N} \leq \underline{\lim}_{n\to\infty} ||\nabla(\rho_n * u)||_{L^1(\mathbb{R}^N)}.$$ □

Theorem 7.3.7. *(a) For every $u \in BV(\mathbb{R}^N)$, $(\rho_n * u)$ converges strictly to u.*

(b) (Gagliardo–Nirenberg inequality.) Let $N \geq 2$. There exists $c_N > 0$ such that for every $u \in BV(\mathbb{R}^N)$,

$$||u||_{L^{N/(N-1)}(\mathbb{R}^N)} \leq c_N ||Du||_{\mathbb{R}^N}.$$

Proof. (a) Proposition 4.3.14 and the preceding lemma imply the strict convergence of $(\rho_n * u)$ to u.

(b) Let $N \geq 2$. We can assume that $\rho_{n_k} * u \to u$ almost everywhere on $\mathbb{R}^N$. It follows from Fatou's lemma and Sobolev's inequality in $\mathcal{D}^{1,1}(\mathbb{R}^N)$ that

$$\|u\|_{N/(N-1)} \leq \underline{\lim}_{k\to\infty} \|\rho_{n_k} * u\|_{N/(N-1)} \leq c_N \lim_{n\to\infty} \|\nabla(\rho_{n_k} * u)\|_1 = c_N \|Du\|_{\mathbb{R}^N}. \qquad \square$$

7.4 Perimeter

The *perimeter* of a smooth domain is the total variation of its characteristic function.

Theorem 7.4.1. *Let Ω be an open subset of $\mathbb{R}^N$ of class C^1 with a bounded boundary Γ. Then*

$$\int_\Gamma d\gamma = \|D\chi_\Omega\|_{\mathbb{R}^N}.$$

Proof. Let $v \in \mathcal{D}(\mathbb{R}^N; \mathbb{R}^N)$ be such that $\|v\|_\infty \leq 1$. The divergence theorem and the Cauchy–Schwarz inequality imply that

$$\int_\Omega \operatorname{div} v \, dx = \int_\Gamma v \cdot n \, d\gamma \leq \int_\Gamma |v| \, |n| d\gamma \leq \int_\Gamma d\gamma.$$

Taking the supremum with respect to v, we obtain $\|D\chi_\Omega\|_{\mathbb{R}^N} \leq \int_\Gamma d\gamma$.

We use the notation of Definition 9.2.1 and define

$$U = \{x \in \mathbb{R}^N : \nabla\varphi(x) \neq 0\},$$

so that $\Gamma \subset U$. The theorem of partitions of unity ensures the existence of $\psi \in \mathcal{D}(U)$ such that $0 \leq \psi \leq 1$ and $\psi = 1$ on Γ. We define

$$\begin{aligned} v(x) &= \psi(x)\nabla\varphi(x)/|\nabla\varphi(x)|, & x \in U \\ &= 0, & x \in \mathbb{R}^N \setminus U. \end{aligned}$$

It is clear that $v \in \mathcal{K}(\mathbb{R}^N; \mathbb{R}^N)$ and for every $\gamma \in \Gamma$, $v(\gamma) = n(\gamma)$. For every $m \geq 1$, $w_m = \rho_m * v \in \mathcal{D}(\mathbb{R}^N; \mathbb{R}^N)$. We infer from the divergence and regularization theorems that

$$\lim_{m\to\infty} \int_\Omega \operatorname{div} w_m \, dx = \lim_{m\to\infty} \int_\Gamma w_m \cdot n \, d\gamma = \int_\Gamma n \cdot n \, d\gamma = \int_\Gamma d\gamma.$$

By definition, $\|v\|_\infty \leq 1$, and by the Cauchy–Schwarz inequality,

$$\sum_{k=1}^{N}(\rho_m * v_k(x))^2 = \sum_{k=1}^{N}\left(\int_{\mathbb{R}^N}\rho_m(x-y)v_k(y)dy\right)^2 \leq \sum_{k=1}^{N}\int_{\mathbb{R}^N}\rho_m(x-y)(v_k(y))^2dy \leq 1.$$

We conclude that $\int_\Gamma d\gamma \leq \|D\chi_\Omega\|_{\mathbb{R}^N}$. □

The preceding theorem suggests a functional definition of the perimeter due to De Giorgi.

Definition 7.4.2. Let A be a measurable subset of $\mathbb{R}^N$. The perimeter of A is defined by $p(A) = \|D\chi_A\|_{\mathbb{R}^N}$.

The proof of the *Morse–Sard theorem* is given in Sect. 9.3.

Theorem 7.4.3. *Let Ω be an open subset of $\mathbb{R}^N$ and $u \in C^\infty(\Omega)$. Then the Lebesgue measure of*

$$\{t \in \mathbb{R} : \textit{there exists } x \in \Omega \textit{ such that } u(x) = t \textit{ and } \nabla u(x) = 0\}$$

is equal to zero.

Theorem 7.4.4. *Let $1 < p < \infty$, $u \in L^p(\Omega)$, $u \geq 0$, and $g \in L^{p'}(\Omega)$. Then*

(a) $\int_\Omega g\, u\, dx = \int_0^\infty dt \int_{u>t} g\, dx;$

(b) $\|u\|_p \leq \int_0^\infty m(\{u > t\})^{1/p} dt;$

(c) $\|u\|_p^p = \int_0^\infty m(\{u > t\}) p t^{p-1} dt.$

Proof. (a) We deduce from Fubini's theorem that

$$\begin{aligned}\int_\Omega g\, u\, dx &= \int_\Omega dx \int_0^\infty g\, \chi_{u>t}\, dt \\ &= \int_0^\infty dt \int_\Omega g\, \chi_{u>t}\, dx \\ &= \int_0^\infty dt \int_{u>t} g\, dx.\end{aligned}$$

(b) If $\|g\|_{p'} = 1$, we obtain from Hölder's inequality that

$$\int_\Omega g\, u\, dx \leq \int_0^\infty m(\{u > t\})^{1/p} dt.$$

It suffices then to take the supremum.

(c) Define $f(t) = t^p$. It follows from Fubini's theorem that

$$\begin{aligned}
\|u\|_p^p &= \int_\Omega dx \int_0^u f'(t)dt \\
&= \int_\Omega dx \int_0^\infty \chi_{u>t} f'(t)dt \\
&= \int_0^\infty dt \int_\Omega \chi_{u>t} f'(t)dx \\
&= \int_0^\infty m(\{u > t\}) f'(t)dt.
\end{aligned}$$

□

Theorem 7.4.5 (Coarea formula). *Let $u \in \mathcal{D}(\mathbb{R}^N)$ and $f \in C^1(\mathbb{R}^N)$. Then*

$$\int_{\mathbb{R}^N} f\, |\nabla u|\, dx = \int_0^\infty dt \int_{|u|=t} f\, d\gamma.$$

Moreover, for every open subset Ω of $\mathbb{R}^N$,

$$\int_\Omega |\nabla u|\, dx = \int_0^\infty dt \int_{|u|=t} \chi_\Omega\, d\gamma.$$

Proof. By the Morse–Sard theorem, for almost every $t \in \mathbb{R}$,

$$u(x) = t \Longrightarrow \nabla u(x) \neq 0.$$

Hence for almost every $t > 0$, the open sets $\{u > t\}$ and $\{u < -t\}$ are smooth.

We infer from Lemma 6.1.1, Theorem 7.4.4, and the divergence theorem that for every $v \in C^1(\mathbb{R}^N; \mathbb{R}^N)$,

$$\begin{aligned}
\int_{\mathbb{R}^N} \nabla u \cdot v\, dx &= -\int_{\mathbb{R}^N} u \operatorname{div} v\, dx \\
&= -\int_0^\infty dt \int_{u>t} \operatorname{div} v\, dx + \int_0^\infty dt \int_{u<-t} \operatorname{div} v\, dx \\
&= \int_0^\infty dt \int_{|u|=t} v \cdot \frac{\nabla u}{|\nabla u|} d\gamma.
\end{aligned}$$

Define

$$v_n = f\, \nabla u / \sqrt{|\nabla u|^2 + 1/n}.$$

Lebesgue's dominated convergence theorem implies that

$$\int_{\mathbb{R}^N} f\,|\nabla u|\,dx = \lim_{n\to\infty}\int_{\mathbb{R}^N} \nabla u\cdot v_n\,dx = \lim_{n\to\infty}\int_0^\infty dt\int_{|u|=t} v_n\cdot\frac{\nabla u}{|\nabla u|}d\gamma = \int_0^\infty dt\int_{|u|=t} f\,d\gamma.$$

Define

$$\omega_n = \{x\in\Omega\ :\ d(x,\partial\Omega) > 1/n \text{ and } |x| < n\}.$$

For all n, there exists $\varphi_n \in \mathcal{D}(\omega_{n+1})$ such that $0 \le \varphi_n \le 1$ and $\varphi_n = 1$ on ω_n. Levi's monotone convergence theorem implies that

$$\int_\Omega |\nabla u| dx = \lim_{n\to\infty}\int_{\mathbb{R}^N}\varphi_n|\nabla u|dx = \lim_{n\to\infty}\int_0^\infty dt\int_{|u|=t}\varphi_n d\gamma = \int_0^\infty dt\int_{|u|=t}\chi_\Omega\,d\gamma. \quad \square$$

Lemma 7.4.6. *Let $1 \le p < N$, let K be a compact subset of $\mathbb{R}^N$, and $a > \text{cap}_p(K)$. Then there exist V open and $v \in \mathcal{D}(\mathbb{R}^N)$ such that $K \subset V$, $\chi_V \le v$, and $\int_\Omega |\nabla v|\,dx < a$.*

Proof. By assumption, there exist $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ and U open such that $K \subset U$, $\chi_U \le u$, and

$$\int_{\mathbb{R}^N}|\nabla u|^p dx < a.$$

There exists V open such that $K \subset V \subset\subset U$. For m large enough, $\chi_V \le w = \rho_m * u$ and

$$\int_{\mathbb{R}^N}|\nabla w|^p dx < a.$$

Let $\theta_n(x) = \theta(|x|/n)$ be a truncating sequence. For n large enough, $\chi_V \le v = \theta_n w$ and

$$\int_{\mathbb{R}^N}|\nabla v|^p dx < a. \quad \square$$

Theorem 7.4.7. *Let $N \ge 2$ and let K be a compact subset of $\mathbb{R}^N$. Then*

$$\text{cap}_1(K) = \inf\{p(U) : U \textit{ is open and bounded, and } U \supset K\}.$$

Proof. We denote by $\text{Cap}_1(K)$ the second member of the preceding equality. Let U be open, bounded, and such that $U \supset K$. Define $u_n = \rho_n * \chi_U$. For n large enough, $u \in \mathcal{D}_K^{1,1}(\mathbb{R}^N)$. Lemma 7.3.6 implies that for n large enough,

$$\text{cap}_1(K) \le \int_{\mathbb{R}^N}|\nabla u_n|dx \le \|D\chi_U\|_{\mathbb{R}^N} = p(U).$$

Taking the infimum with respect to U, we obtain $\text{cap}_1(K) \le \text{Cap}_1(K)$.

Let $a > \text{cap}_1(K)$. By the preceding lemma, there exist V open and $v \in \mathcal{D}(\mathbb{R}^N)$ such that $K \subset V$, $\chi_V \le v$ and $\int_{\mathbb{R}^N}|\nabla v|dx < a$. We deduce from the Morse–Sard theorem and from the coarea formula that

$$\mathrm{Cap}_1(K) \le \int_0^1 dt \int_{v=t} d\gamma \le \int_0^\infty dt \int_{v=t} d\gamma = \int_{\mathbb{R}^N} |\nabla v| dx < a.$$

Since $a > \mathrm{cap}_1(K)$ is arbitrary, we conclude that $\mathrm{Cap}_1(K) \le \mathrm{cap}_1(K)$. □

7.5 Comments

The book by Maz'ya ([51]) is the main reference on functions of bounded variations and on capacity theory. The beautiful proof of the coarea formula (Theorem 7.4.5) is due to Maz'ya. The derivative of the function of unbounded variation in Sect. 7.3 is Denjoy–Perron integrable (since it is a derivative); see *Analyse, fondements techniques, évolution* by J. Mawhin ([49]).

7.6 Exercises for Chap. 7

1. Let $1 \le p < N$. Then

$$\lambda p + N < 0 \Leftrightarrow (1 + |x|^2)^{\lambda/2} \in W^{1,p}(\mathbb{R}^N),$$
$$(\lambda - 1)p + N < 0 \Leftrightarrow (1 + |x|^2)^{\lambda/2} \in \mathcal{D}^{1,p}(\mathbb{R}^N).$$

2. What are the interior and the closure of $W^{1,1}(\Omega)$ in $BV(\Omega)$?
3. Let $u \in L^1_{\mathrm{loc}}(\Omega)$. The following properties are equivalent:
   (a) $\|Du\|_\Omega < \infty$;
   (b) there exists $c > 0$ such that for every $\omega \subset\subset \Omega$ and every $y \in \mathbb{R}^N$ such that $|y| < d(\omega, \partial\Omega)$

$$\|\tau_y u - u\|_{L^1(\omega)} \le c|y|.$$

4. (Relative variational capacity.) Let Ω be an open bounded subset of $\mathbb{R}^N$ (or more generally, an open subset bounded in one direction). Let $1 \le p < \infty$ and let K be a compact subset of Ω. The capacity of degree p of K relative to Ω is defined by

$$\mathrm{cap}_{p,\Omega}(K) = \inf\left\{ \int_\Omega |\nabla u|^p dx : u \in W_K^{1,p}(\Omega) \right\},$$

where

$$W_K^{1,p}(\Omega) = \{u \in W_0^{1,p}(\Omega) : \text{there exists } \omega \text{ such that } K \subset \omega \subset\subset \Omega \text{ and } \chi_\omega \le u \text{ a.e. on } \Omega\}.$$

Prove that the capacity of degree p relative to Ω is a capacity on Ω.

5. Verify that

$$\text{cap}_{p,\Omega}(K) = \inf\left\{\int_\Omega |\nabla u|^p dx : u \in \mathcal{D}_K(\Omega)\right\},$$

where

$$\mathcal{D}_K(\Omega) = \{u \in \mathcal{D}(\Omega) : \text{there exists } \omega \text{ such that } K \subset \omega \subset\subset \Omega \text{ and } \chi_\omega \le u\}.$$

6. (a) If $\text{cap}_{p,\Omega}(K) = 0$, then $m(K) = 0$. *Hint*: Use Poincaré's inequality.
   (b) If $p > N$ and if $\text{cap}_{p,\Omega}(K) = 0$, then $K = \phi$. *Hint*: Use the Morrey inequalities.
7. Assume that $\text{cap}_{p,\Omega}(K) = 0$. Then for every $u \in \mathcal{D}(\Omega)$, there exists $(u_n) \subset \mathcal{D}(\Omega \setminus K)$ such that $|u_n| \le |u|$ and $u_n \to u$ in $W^{1,p}(\Omega)$.
8. (Dupaigne–Ponce, 2004.) Assume that $\text{cap}_{1,\Omega}(K) = 0$. Then $W^{1,p}(\Omega \setminus K) = W^{1,p}(\Omega)$. *Hint*: Consider first the bounded functions in $W^{1,p}(\Omega \setminus K)$.
9. For every $u \in BV(\mathbb{R}^N)$,

$$\|D|u|\|_{\mathbb{R}^N} \le \|Du^+\|_{\mathbb{R}^N} + \|Du^-\|_{\mathbb{R}^N} = \|Du\|_{\mathbb{R}^N}.$$

   Hint: Consider a sequence $(u_n) \subset W^{1,1}(\mathbb{R}^N)$ such that $u_n \to u$ strictly in $BV(\mathbb{R}^N)$.
10. Let $u \in L^1(\Omega)$ and $f \in \mathcal{BC}^1(\Omega)$. Then

$$\|D(fu)\|_\Omega \le \|f\|_\infty \|Du\|_\Omega + \|\nabla f\|_\infty \|u\|_{L^1(\Omega)}.$$

11. (Cheeger constant.) Let Ω be an open bounded domain in $\mathbb{R}^N$ and define

$$h(\Omega) = \inf\{p(\omega)/m(\omega) : \omega \subset\subset \Omega \text{ and } \omega \text{ is of class } C^1\}.$$

   Then for $1 \le p < \infty$ and every $u \in W_0^{1,p}(\Omega)$,

$$\left(\frac{h(\Omega)}{p}\right)^p \int_\Omega |u|^p dx \le \int_\Omega |\nabla u|^p dx.$$

   Hint: Assume first that $p = 1$ and apply the coarea formula to $u \in \mathcal{D}(\Omega)$.
12. Let $u \in W^{1,1}(\Omega)$. Then

$$\int_\Omega \sqrt{1+|\nabla u|^2}dx = \sup\left\{\int_\Omega (v_{N+1} + u\sum_{k=1}^N \partial_k u_k)dx : u \in \mathcal{D}(\Omega;\mathbb{R}^{N+1}), \|u\|_\infty \le 1\right\}.$$