Chapter 6
Degree Theory

6.1 Introduction

Many applications, including some bifurcation problems of functional differential equations, lead to the problem of finding all zeros of a mapping $f\colon U \subseteq X \to X$, where X is some (real) Banach space. In this type of nonlinear problem, we are interested in the solutions of

$$f(x) = 0, \quad x \in U. \tag{6.1}$$

In most cases, it turns out that it is too much to ask to determine the zeros analytically and explicitly. Hence one looks for a more qualitative study of the zeros, such as the number, location, and multiplicity.

To illustrate this and to motivate the topological degree, we consider the case $f \in \mathscr{H}(\mathbb{C})$, where $\mathscr{H}(\mathbb{C})$ denotes the set of holomorphic functions on a domain $U \subset \mathbb{C}$. Recall that the winding number of a path $\gamma\colon [0,1] \to \mathbb{C}$ around a point $z_0 \in \mathbb{C}$ is defined by

$$n(\gamma, z_0) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{dz}{z - z_0} \in \mathbb{Z}. \tag{6.2}$$

It gives the number of times that z_0 is encircled, taking orientation into account (that is, encirclings in opposite directions are counted with opposite signs).

In particular, if we pick $f \in \mathscr{H}(\mathbb{C})$, we compute (assuming $0 \notin f(\gamma)$)

$$n(f(\gamma), 0) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_k n(\gamma, z_k)\alpha_k, \tag{6.3}$$

where z_k denotes zeros of f, and α_k their respective multiplicity. Moreover, if γ is a Jordan curve encircling a simply connected domain $U \subset \mathbb{C}$, then $n(\gamma, z_k) = 0$ if $z_k \notin U$ and $n(\gamma, z_k) = 1$ if $z_k \in U$. Hence $n(f(\gamma), 0)$ counts the number of zeros inside U.

Let us also recall how we compute complex integrals along complicated paths using homotopy invariance (see [23, 240, 241]). In this approach, we look for a

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simpler path along which the integral can be computed that is homotopic to the original one. In particular, if $f\colon \gamma \to \mathbb{C}\setminus\{0\}$ and $g\colon \gamma \to \mathbb{C}\setminus\{0\}$ are homotopic, we have $n(f(\gamma),0) = n(g(\gamma),0)$ (which is known as Rouché's theorem). More explicitly, we need to find a mapping g for which $n(g(\gamma),0)$ can be computed and a homotopy $H\colon [0,1]\times\gamma \to C\setminus\{0\}$ such that $H(0,z) = f(z)$ and $H(1,z) = g(z)$ for $z \in \gamma$. For example, to see how many zeros of $f(z) = \frac{1}{2}z^6 + z - \frac{1}{3}$ lie inside the unit circle, we consider $g(z) = z$. Then $H(t,z) = (1-t)f(z) + tg(z)$ is the required homotopy, since $|f(z) - g(z)| < |g(z)|$, $|z| = 1$, implying $H(t,z) \neq 0$ on $[0,1]\times\gamma$. Hence $f(z)$ has one zero inside the unit circle.

To summarize, given a (sufficiently smooth) domain U with enclosing Jordan curve ∂U, we have defined a degree $\deg(f,U,z_0) = n(f(\partial U),z_0) = n(f(\partial U) - z_0, 0) \in \mathbb{Z}$ that counts the number of solutions of $f(z) = z_0$ inside U. The invariance of this degree with respect to certain deformations of f allow us to explicitly compute $\deg(f,U,z_0)$ even in nontrivial cases. Degree theory has been developed for various classes of mappings, not all of which are mentioned in the chapter. For relevant results on topological degree, see, for example, [24, 25, 177–182, 191–195]. Moreover, similar ideas also appears in the definitions of Fuller index. See, for example, Chow and Mallet-Paret [69].

6.2 The Brouwer Degree

In 1912, Brouwer [47] introduced the so-called Brouwer degree in $\mathbb{R}^n$. See Brouwder [46], Alexander et al. [8–10], Chow et al. [71], Krasnosel'skii [191], Sieberg [265] for historical developments. In this section, we introduce Brouwer degree theory. Throughout this section, U will be a bounded open subset of $\mathbb{R}^n$. For $f \in C^1(U,\mathbb{R}^n)$, the Jacobi matrix of f at $x \in U$ is $f'(x) = (\frac{\partial f_j(x)}{\partial x_i})_{1\le i,j\le n}$, and the Jacobi determinant of f at $x \in U$ is

$$J_f(x) = \det f'(x).$$

The set of regular values is

$$\mathrm{RV}(f) = \{y \in \mathbb{R}^n : J_f(x) \neq 0 \text{ for all } x \in f^{-1}(y)\}.$$

Its complement $\mathrm{CV}(f) = \mathbb{R}^n \setminus \mathrm{RV}(f)$ is called the set of critical values. Set $C^r(\bar{U},\mathbb{R}^n) = \{f \in C^r(U,\mathbb{R}^n) : d^j f \in C(\bar{U},\mathbb{R}^n) \text{ for all } 0 \le j \le r\}$ and

$$D_y^r(\bar{U},\mathbb{R}^n) = \{f \in C^r(\bar{U},\mathbb{R}^n) : y \notin f(\partial U)\},$$
$$D_y^0(\bar{U},\mathbb{R}^n) = \{f \in C(\bar{U},\mathbb{R}^n) : y \notin f(\partial U)\}$$

for $y \in \mathbb{R}^n$.

Lemma 6.1 (Sard's lemma). *Let $U \subset \mathbb{R}^n$ be open and $f \in C^1(U,\mathbb{R}^n)$. Then $\mu_n(f(\mathrm{CV}(f))) = 0$, where μ_n is the n-dimensional Lebesgue measure.*

A function deg that assigns each $f \in D^0_y(U,\mathbb{R}^n)$, $y \in \mathbb{R}^n$, a real number $\deg(f,U,y)$ will be called a degree if it satisfies the following conditions:

(BD1) *(translation invariance)* $\deg(f,U,y) = \deg(f-y,U,0)$.
(BD2) *(normalization)* $\deg(\mathbb{I},U,y) = 1$ if $y \in U$, where $\mathbb{I}$ denotes the identity operator when the space involved is clear.
(BD3) *(additivity)* If U_1 and U_2 are open, disjoint subsets of U such that $y \notin f(U \setminus (U_1 \cup U_2))$, then $\deg(f,U,y) = \deg(f,U_1,y) + \deg(f,U_2,y)$.
(BD4) *(homotopy invariance)* If $H : [0,1] \times \bar{U} \to \mathbb{R}^n$ is continuous, so that $y \notin H(t,\partial U)$ for evert $t \in [0,1]$, and $f = H(0,\cdot), g = H(1,\cdot)$, then $\deg(f,U,y) = \deg(g,U,y)$.

To compute the degree of a nonsingular matrix, we need the following lemma.

Lemma 6.2. *Two nonsingular matrices $M_1, M_2 \in \mathrm{GL}(n)$ are homotopic in $GL(n)$ if and only if* $\operatorname{sgn}\det M_1 = \operatorname{sgn}\det M_2$.

Using this lemma, we can prove the following theorem.

Theorem 6.1. *Suppose $f \in D^1_y(\bar{U},\mathbb{R}^n)$ and $y \notin \mathrm{CV}(f)$. Then a degree satisfying* (BD1)–(BD4) *satisfies*

$$\deg(f,U,y) = \sum_{x \in f^{-1}(y)} \operatorname{sgn} J_f(x), \tag{6.4}$$

where the sum is finite.

In fact, the determinant formula (6.4) can be extended to all $f \in D^0_y(\bar{U},\mathbb{R}^n)$, that is, $\deg(f,U,y)$ as defined in (6.4) is locally constant with respect to both y and f. In particular, we have the following result.

Theorem 6.2. *There is a unique degree* deg *satisfying* (BD1)–(BD4). *Moreover, for each given $f \in D^0_y(\bar{U},\mathbb{R}^n)$, we have*

$$\deg(f,U,y) = \sum_{x \in \tilde{f}^{-1}(y)} \operatorname{sgn} J_{\tilde{f}}(x), \tag{6.5}$$

where $\tilde{f} \in D^2_y(\bar{U},\mathbb{R}^n)$ is sufficiently close to f (with respect to the sup-norm topology in $C^r(\bar{U},\mathbb{R}^n)$), and $y \in \mathrm{RV}(\tilde{f})$, and the above calculation is independent of the choice of $\tilde{f}$.

To extend the formula (6.4) to all $f \in D^0_y(\bar{U},\mathbb{R}^n)$, we first note that $\varepsilon := \min\{|f(x)-y| : x \in \partial U\} > 0$, and then apply the Weierstrass theorem to obtain $\tilde{g} \in C^2(\bar{U},R^n)$, so that $\max\{|f(x)-\tilde{g}(x)| : x \in \bar{U}\} < \varepsilon/2$. We then use Sard's theorem to find a regular value $y_0 \in \mathbb{R}^n$ of $\tilde{g}$ such that $|y-y_0| < \varepsilon/2$. We then define $g : \bar{U} \to \mathbb{R}^n$ as $g(x) = \tilde{g}(x) - y_0$. Then $g \in C^2(\bar{U};\mathbb{R}^n), \max\{|g(x)-f(x)|\} < \varepsilon$, and 0 is a regular value of g. We can define

$$\deg(f,U,y) = \sum_{x \in g^{-1}(0)} \operatorname{sgn} J_g(x),$$

and we need to check that this definition is independent of the choice of such g.

6.3 The Leray–Schauder Degree

In 1934, Leray and Schauder [207] generalized Brouwer degree theory to an infinite Banach space and established the so-called Leray–Schauder degree. It turns out that the Leray–Schauder degree is a powerful tool in proving various existence results for nonlinear differential equations (see, for example, [89, 90]). The objective of this section is to extend the Brouwer degree to general Banach spaces. We first extend the Brouwer degree to general finite-dimensional spaces.

Let X be a (real) Banach space of dimension n, and let n be an isomorphism between X and $\mathbb{R}^n$. Then for $f \in D_y(\bar{U},X)$, $U \subset X$ open, $y \in X$, we can define

$$\deg(f,U,y) = \deg(\phi \circ f \circ \phi^{-1}, \phi(U), \phi(y)), \tag{6.6}$$

provided this definition is independent of the basis chosen. To see this, let ψ be a second isomorphism. Then $A = \psi \circ \phi^{-1} \in \mathrm{GL}(n)$. Abbreviate $f^* = \phi \circ f \circ \phi^{-1}$, $y^* = \phi(y)$, and pick $\tilde{f}^* \in D^1_y(\phi(\bar{U}),\mathbb{R}^n)$ in the same component of $D_y(\phi(\bar{U}),\mathbb{R}^n)$ as f^* such that $y^* \in \mathrm{RV}(f^*)$. Then $A \circ \tilde{f}^* \circ A^{-1} \in D^1_y(\psi(U),R^n)$ is the same component of $D_y(\psi(\bar{U}),\mathbb{R}^n)$ as $A \circ f^* \circ A^{-1} = \psi \circ f \circ \psi$ (since A is also a homeomorphism) and

$$J_{A\circ\tilde{f}^*\circ A^{-1}}(Ay^*) = \det(A) J_{\tilde{f}^*}(y^*) \det(A^{-1}) = J_{\tilde{f}^*}(y^*) \tag{6.7}$$

by the chain rule. Thus we have $\deg(\psi \circ f \circ \psi^{-1}, \psi(U), \psi(y)) = \deg(\phi \circ f \circ \phi^{-1}, \phi(U), \phi(y))$, and our definition is independent of the basis chosen. In addition, it inherits all properties from the mapping degree in $\mathbb{R}^n$. Note also that the reduction property holds if $\mathbb{R}^m$ is replaced by an arbitrary subspace X_1, since we can always choose $\phi\colon X \to \mathbb{R}^n$ such that $\phi(X_1) = \mathbb{R}^m$.

Our next aim is to tackle the infinite-dimensional case. The general idea is to approximate F by finite-dimensional operators (in the same spirit as we approximated continuous f by smooth functions). To do this, we need to know which operators can be approximated by finite-dimensional operators. Hence we have to recall some basic facts first.

Let X and Y be Banach spaces and $U \subset X$. An operator $F\colon U \subset X \to Y$ is called finite-dimensional if its range is finite-dimensional. In addition, it is called *compact* (completely continuous) if it is continuous and maps bounded sets into relatively compact ones. The set of all compact operators is denoted by $\mathscr{C}(U,Y)$, and the set of all compact finite-dimensional operators is denoted by $\mathscr{F}(U,Y)$. Both sets are normed linear spaces, and we have $\mathscr{F}(U,Y) \subseteq \mathscr{C}(U,Y) \subseteq C(U,Y)$. If U is compact, then $\mathscr{C}(U,Y) = C(U,Y)$ (since the continuous image of a compact set is compact), and if $\dim(Y) < \infty$, then $\mathscr{F}(U,Y) = \mathscr{C}(U,Y)$. In particular, if $U \subset \mathbb{R}^n$ is bounded, then $\mathscr{F}(U,Y) = \mathscr{C}(U,\mathbb{R}^n) = C(U,\mathbb{R}^n)$.

For $U \subset X$, we set $\mathscr{D}_y(\bar{U},X) = \{F \in \mathscr{C}(\bar{U},X)\colon y \notin (\mathbb{I}+F)(\partial U)\}$ and $\mathscr{F}_y(\bar{U},X) = \{F \in \mathscr{F}(\bar{U},X)\colon y \notin (\mathbb{I}+F)(\partial U)\}$. Note that for $F \in \mathscr{D}_y(\bar{U},X)$, we have $\rho = \mathrm{dist}(y, (\mathbb{I}+F)(\partial U)) > 0$, since $\mathbb{I}+F$ maps closed sets to closed sets.

Pick $F_1 \in \mathscr{F}(\bar{U}, X)$ such that $|F - F_1| < \rho$. Hence, $F_1 \in \mathscr{F}_y(\bar{U}, X)$. Next, let X_1 be a finite-dimensional subspace of X such that $F_1(U) \subset X_1$, $y \in X_1$, and set $U_1 = U \cap X_1$. Then we have $F_1 \in \mathscr{F}_y(\bar{U}_1, X_1)$, and we may define

$$\deg(\mathbb{I} + F, U, y) = \deg(\mathbb{I} + F_1, U_1, y). \tag{6.8}$$

It is easy to verify that this definition is independent of F_1 and X_1.

Theorem 6.3. *Let U be a bounded open subset of a (real) Banach space X and let $F \in \mathscr{F}_y(\bar{U}, X)$, $y \in X$. Then the following hold.*

(i) $\deg(\mathbb{I} + F, U, y) = \deg(\mathbb{I} + F - y, U, 0)$.
(ii) $\deg(\mathbb{I}, U, y) = 1$ *if* $y \in U$.
(iii) If $U_{1,2}$ are open, disjoint subsets of U such that $y \notin f(\bar{U} \setminus (U_1 \cup U_2))$, then $\deg(\mathbb{I} + F, U, y) = \deg(\mathbb{I} + F, U_1, y) + \deg(\mathbb{I} + F, U_2, y)$.
(iv) If H: $[0,1] \times \bar{U} \to X$ and y: $[0,1] \to X$ are both continuous such that $H(t) \in D_{y(t)}(U, \mathbb{R}^n)$, $t \in [0,1]$, then $\deg(\mathbb{I} + H(0), U, y(0)) = \deg(\mathbb{I} + H(1), U, y(1))$.

6.4 Global Bifurcation Theorem

As in Sect. 5.1, we study the nonlinear parameter-dependent problem

$$F(u, \alpha) = 0, \tag{6.9}$$

where F: $E \times \mathbb{R} \to X$ is a C^1-map such that $F(0, \alpha) = 0$ for all $\alpha \in \mathbb{R}$, and $E \subseteq X$ is an open neighborhood of 0 (possibly $E = X$). Note that (6.9) has the trivial solution for all values of α. We shall now consider the question of bifurcation from this trivial branch of solutions and demonstrate the existence of global branches of nontrivial solutions bifurcating from the trivial branch. If $X = \mathbb{R}^n$, then we use the Brouwer degree; if X is an infinite-dimensional (real) Banach space, then we assume that $F(x, \alpha) = x + f(x, \alpha)$ and that f: $E \times \mathbb{R} \to X$ is completely continuous. Thus for $F(\cdot, \alpha)$, the Leray–Schauder degree is applicable. The application of degree theory to bifurcation theory goes back to Krasnosel'skii [191]. Global bifurcation theorem of the following type were first proved by Rabinowitz [251]. Several generalizations have been given by Ize et al. [177–182], Krawcewicz et al. [192–195], and Nussbaum et al. [232–236].

Theorem 6.4. *Let there exist a, $b \in \mathbb{R}$ with $a < b$ such that $u = 0$ is an isolated solution of (6.9) for $\alpha = a$ and $\alpha = b$, where a and b are not bifurcation points. Furthermore, assume that*

$$\deg(F(\cdot, a), B_r(0), 0) \neq \deg(F(\cdot, b), B_r(0), 0), \tag{6.10}$$

where $B_r(0) = \{u \in E\colon \|u\| < r\}$ is an isolating neighborhood of the trivial solution. Let

$$\mathscr{S} = \overline{\{(u, \alpha)\colon (u, \alpha) \text{ solves (6.9) with } u \neq 0\}} \cup \{0\} \times [a, b],$$

and let $\mathscr{C}$ be the maximal connected subset of $\mathscr{S}$ that contains $\{0\} \times [a, b]$. Then either *(i)* $\mathscr{C}$ *is unbounded in* $E \times \mathbb{R}$*,*

or else

(ii) $\mathscr{C} \cap \{0\} \times (\mathbb{R} \setminus [a,b]) \neq \emptyset$*.*

Proof. Define a class $\mathscr{U}$ of subsets of $E \times \mathbb{R}$ as follows:

$$\mathscr{U} = \{\Omega \subset E \times \mathbb{R} \colon \Omega = \Omega_0 \cup \Omega_\infty\},$$

where $\Omega_0 = B_r(0) \times [a,b]$, and Ω_∞ is a bounded open subset of $(E \setminus \{0\}) \times \mathbb{R}$. We shall first show that (6.9) has a nontrivial solution $(u,\alpha) \in \partial\Omega$ for every such $\Omega \in \mathscr{U}$. To accomplish this, let us consider the following sets:

$$\begin{cases} K = F^{-1}(0) \cap \bar{\Omega}, \\ A = \{0\} \times [a,b], \\ B = F^{-1}(0) \cap \{\partial\Omega \setminus [B_r(0) \times \{a\} \cup B_r(0) \times \{b\}]\}. \end{cases} \tag{6.11}$$

We observe that K may be regarded as a compact metric space, and A and B are compact subsets of K. We hence may apply Whyburn's lemma to deduce that either there exists a continuum in K connecting A to B, or else there is a separation K_A, K_B of K, with $A \subset K_A$, $B \subset K_B$. If the latter holds, we may find open sets U, V in $E \times \mathbb{R}$ such that $K_A \subset U$, $K_B \subset V$, with $U \cap V = \emptyset$. We let $\Omega^* = \Omega \cap (U \cup V)$ and observe that $\Omega^* \in \mathscr{U}$. It follows by construction that there are no nontrivial solutions of (6.9) that belong to $\partial\Omega^*$; this, however, is impossible, since it would imply, by the generalized homotopy and the excision principle of the Leray–Schauder degree, that $\deg(F(\cdot,a),B_r(0),0) = \deg(F(\cdot,b),B_r(0),0)$, contradicting (6.10). We hence have that, for each $\Omega \in \mathscr{U}$, there is a continuum C of solutions of (6.9) that intersects $\partial\Omega$ in a nontrivial solution.

We assume now that neither of the alternatives of the theorem holds, that is, we assume that $\mathscr{C}$ is bounded and $\mathscr{C} \cap \{0\} \times (\mathbb{R} \setminus [a,b]) = \emptyset$. In this case, we may, using the boundedness of $\mathscr{C}$, construct a set $\Omega \in \mathscr{U}$ containing no nontrivial solutions in its boundary, thus arriving once more at a contradiction. □

6.5 $\mathbb{S}^1$-Equivariant Degree

Let $\mathbb{E}$ be a real isometric Banach representation of the group $G = \mathbb{S}^1$. The isotypical direct sum decomposition is denoted by

$$\mathbb{E} = \mathbb{E}_0 \oplus \mathbb{E}_1 \oplus \cdots \oplus \mathbb{E}_k \oplus \cdots, \tag{6.12}$$

where $\mathbb{E}_0 = \mathbb{E}^G \stackrel{\text{def}}{=} \{x \in \mathbb{E}; gx = x \text{ for all } g \in G\}$ is the subspace of G-fixed points, and for $k \geq 1$, $x \in \mathbb{E}_k \setminus \{0\}$ implies that G_x, the isotropy group of x, is $\mathbb{Z}_k \stackrel{\text{def}}{=} \{g \in G; g^k = 1\}$. Throughout this section, we assume the following:

(SD1) For each integer $k = 0, 1, \ldots$, the subspace $\mathbb{E}_k$ is of finite dimension.

All subspaces $\mathbb{E}_k$, $k \geq 1$, admit a natural structure of complex vector spaces such that an $\mathbb{R}$-linear operator $A \colon \mathbb{E}_k \to \mathbb{E}_k$ is G-equivariant if and only if it is $\mathbb{C}$-linear with respect to this complex structure. Therefore, by choosing a $\mathbb{C}$ basis in $\mathbb{E}_k$, $k \geq 1$, we can define an isomorphism between the group of all G-equivariant automorphisms of $\mathbb{E}_k$, denoted by $GL_G(\mathbb{E}_k)$, and the general linear group $GL(m_k, \mathbb{C})$, where $m_k = \dim_{\mathbb{C}} \mathbb{E}_k$.

Let $\mathbb{F}$ be another Banach isometric representation of G, and $L\colon \mathbb{E} \to \mathbb{F}$ a given equivariant linear bounded Fredholm operator of index zero. We say that an equivariant compact operator $K\colon \mathbb{E} \to \mathbb{F}$ is an *equivariant compact resolvent* of L if $L+K\colon \mathbb{E} \to \mathbb{F}$ is an isomorphism. We shall denote by $CR^G(L)$ the set of all equivariant compact resolvents of L, and assume that

(SD2) $\quad CR^G(L) \neq \emptyset$.

In what follows, a point of the Banach space $\mathbb{E} \times \mathbb{R}^2$ is denoted by (x, λ) with $x \in \mathbb{E}$ and $\lambda \in \mathbb{R}^2$, and the action of G on $\mathbb{E} \times \mathbb{R}^2$ is defined by $g(x, \lambda) = (gx, \lambda)$ for every $g \in G$.

We consider a G-equivariant continuous map $f\colon \mathbb{E} \times \mathbb{R}^2 \to \mathbb{F}$ such that

$$f(u, \lambda) = Lu - Q(u, \lambda), \quad (u, \lambda) \in \mathbb{E} \times \mathbb{R}^2, \tag{6.13}$$

where $Q\colon \mathbb{E} \times \mathbb{R}^2 \to \mathbb{F}$ is a completely continuous map and the following assumption is satisfied:

(SD3) There exists a two-dimensional submanifold $M \subset \mathbb{E}_0 \times \mathbb{R}^2$ such that (i) $M \subset f^{-1}(0)$; (ii) if $(u_0, \lambda_0) \in M$, then there exist an open neighborhood U_{λ_0} of λ_0 in $\mathbb{R}^2$, an open neighborhood U_{u_0} of u_0 in $\mathbb{E}_0$, and a C^1-map $\eta\colon U_{\lambda_0} \to \mathbb{E}_0$ such that $M \cap (U_{u_0} \times U_{\lambda_0}) = \{(\eta(\lambda), \lambda); \lambda \in U_{\lambda_0}\}$.

In relation to the bifurcation problem of (6.13), we consider the structure of the set of solutions to the following equation:

$$f(u, \lambda) = 0, \ (u, \lambda) \in \mathbb{E} \times \mathbb{R}^2. \tag{6.14}$$

All points $(u, \lambda) \in M$ are called *trivial solutions* of (6.13) or (6.14), and all other solutions in $f^{-1}(0) \setminus M$ are called *nontrivial solutions*. A point $(u_0, \lambda_0) \in M$ is called a *bifurcation point* if in every neighborhood of $(u_0, \lambda_0) \in M$ there is a nontrivial solution for (6.14).

Equation (6.14) can be transformed into the equivariant fixed-point problem

$$u = (L+K)^{-1} \circ [K + Q(\cdot, \lambda)](u), \quad (u, \lambda) \in \mathbb{E} \times \mathbb{R}^2. \tag{6.15}$$

Let $\mathscr{F}(u, \lambda) = u - (L+K)^{-1} \circ [Q(\cdot, \lambda) + K](u)$, $(u, \lambda) \in \mathbb{E} \times \mathbb{R}^2$. Then (6.14) is equivalent to the equation

$$\mathscr{F}(u, \lambda) = 0, (u, \lambda) \in \mathbb{E} \times \mathbb{R}^2. \tag{6.16}$$

The idea of finding nontrivial solutions to (6.16) in an open G-invariant neighborhood $\mathscr{U} \subseteq \mathbb{E} \times \mathbb{R}^2$ of $(u_0, \lambda_0) \in M$ is based on an *auxiliary function* ψ to (6.16), which is introduced to distinguish nontrivial solutions from trivial solutions. Here $\mathscr{U}$ is said to be G-invariant if $(gx, \lambda) \in \mathscr{U}$ for all $g \in G$, $(x, \lambda) \in \mathscr{U}$. An auxiliary function to (6.16) on the set $\mathscr{U}$ is an equivariant function (i.e.,

$\psi(gx) = g\psi(x)$ for all $g \in G$ and $x \in \overline{\mathscr{U}}$, where $\overline{\mathscr{U}}$ denotes the closure of $\mathscr{U}$; here and in what follows G acts on $\mathbb{R}^2$ trivially) $\psi : \overline{\mathscr{U}} \subset \mathbb{E} \times \mathbb{R}^2 \to \mathbb{R}$ satisfying $\psi(u,\lambda) < 0$ for all $(u,\lambda) \in \overline{\mathscr{U}} \cap M$. Then every solution to the system

$$\begin{cases} \mathscr{F}(u,\lambda) = 0, \\ \psi(u,\lambda) = 0, \end{cases} \quad (u,\lambda) \in \overline{\mathscr{U}}, \tag{6.17}$$

is a nontrivial solution to (6.13). This leads to the equivariant map $\mathscr{F}_\psi : \overline{\mathscr{U}} \to \mathbb{E} \times \mathbb{R}$ defined by

$$\mathscr{F}_\psi(u,\lambda) = (\mathscr{F}(u,\lambda), \psi(u,\lambda)),\ (u,\lambda) \in \overline{\mathscr{U}}, \tag{6.18}$$

and the problem of finding a nontrivial solution to (6.13) in $\mathscr{U}$ can be reduced to the problem of finding a solution to the equation $\mathscr{F}_\psi(u,\lambda) = 0$ in $\mathscr{U}$, which may be solved by the so-called $\mathbb{S}^1$-*equivariant degree* as a topological invariant associated with the problem (6.17).

To describe the definition and basic properties of $\mathbb{S}^1$-degree, we assume that V is an isometric Hilbert representation of $G = \mathbb{S}^1$. If U is an open bounded invariant subset of $V \oplus \mathbb{R}$ (where $\mathbb{S}^1$ acts trivially on $\mathbb{R}$) and $F\colon (\overline{U}, \partial U) \to (V, V \setminus \{0\})$ is an equivariant compact vector field on $\overline{U}$, then there is defined the $\mathbb{S}^1$-equivariant degree of F with respect to U, which is a sequence of integers

$$\mathbb{S}^1\text{-}\deg(F,U) := \{\deg_k(F,U)\}_{k=1}^{\infty} \in \bigoplus_{k=1}^{\infty} \mathbb{Z}$$

such that $\deg_k(F,U) \neq 0$ for only a finite number of indices k. The basic properties of $\mathbb{S}^1$-deg are as follows (see [24, 25, 112, 180, 194] for details):

(i) *Existence:* If $\mathbb{S}^1\text{-}\deg(F,U) := \{\deg_k(F,U)\}_{k=1}^{\infty} \neq 0$, i.e., there exists $k \in \mathbb{N}$ such that $\deg_k(F,U) \neq 0$, then $F^{-1}(0) \cap U^H \neq \emptyset$, where $H = \mathbb{Z}_k$ and

$$U^H := \{v \in U : gv = v \text{ for any } g \in H\}.$$

(ii) *Additivity:* If U_1 and U_2 are two open invariant subsets of U such that $U_1 \cap U_2 = \emptyset$ and $F^{-1}(0) \cap U \subseteq U_1 \cup U_2$, then $\mathbb{S}^1\text{-}\deg(F,U) = \mathbb{S}^1\text{-}\deg(F,U_1) + \mathbb{S}^1\text{-}\deg(F,U_2)$.

(iii) *Homotopy invariance:* If $\mathscr{H}\colon (\overline{U}, \partial U) \times [0,1] \to (V, V \setminus \{0\})$ is an $\mathbb{S}^1$-equivariant homotopy of compact vector fields, then $\mathbb{S}^1\text{-}\deg(\mathscr{H}_0, U) = \mathbb{S}^1\text{-}\deg(\mathscr{H}_1, U)$, where $\mathscr{H}_t(\theta) = \mathscr{H}(t,\theta)$ for $t \in [0,1]$ and $\theta \in \overline{U}$.

(iv) *Contraction:* Suppose that W is another isometric Hilbert representation of $\mathbb{S}^1$ and let Ω be an open, bounded, invariant subset of W such that $0 \in \Omega$. Define $\Phi\colon \overline{\Omega} \times \overline{U} \to W \oplus V$ by $\Phi(y,x,t) = (y, F(x,t))$. Then $\mathbb{S}^1\text{-}\deg(\Phi, U) = \mathbb{S}^1\text{-}\deg(F,U)$.

Now we return to the problem (6.17). If the mapping $\mathscr{F}_\psi : \overline{\mathscr{U}} \to \mathbb{E} \times \mathbb{R}$ has no solution on $\partial\mathscr{U}$ and $\mathscr{F} : \mathscr{U} \to \mathbb{E}$ is a condensing field (i.e., $\pi - \mathscr{F}$ is a condensing map, where $\pi : \mathscr{U} \to \mathbb{E}$ is the natural projection on $\mathbb{E}$), then the $\mathbb{S}^1$-equivariant degree $\mathbb{S}^1$-deg$(\mathscr{F}_\psi, \mathscr{U})$ is well defined, and its nontriviality implies the existence of solutions of $\mathscr{F}_\psi(u,\lambda)=0$ in $\mathscr{U}$. Global continuation of the branch of nontrivial solutions (solutions in $f^{-1}(0)\setminus M$) bifurcating from (u_0,λ_0) can be characterized by the above $\mathbb{S}^1$-degree at all bifurcation points along the closure of the branch if such a branch is bounded in $\mathbb{E}\times\mathbb{R}^2$ (the so-called Fuller space).

To describe precisely this $\mathbb{S}^1$-degree-based bifurcation theory, we need some additional information about: (i) the construction of the open neighborhood $\mathscr{U}$, (ii) the auxiliary function ψ, (iii) the computation of $\mathbb{S}^1\text{-deg}(\mathscr{F}_\psi,\mathscr{U})$.

If $\mathscr{F}(u,\lambda)$ is differentiable with respect to u, we are able to define *singular points* of system (6.16) through its linearization at the trivial solutions of (6.14). This is unfortunately not so for state-dependent DDEs. Therefore, we shall distinguish two cases.

6.5.1 Differentiability Case

Throughout this subsection, we further assume that at all points $(u_0,\lambda_0)\in M$, the derivative $D_uf(u_0,\lambda_0)\colon\mathbb{E}\to\mathbb{F}$ of f with respect to u exists and is continuous on M. We say that $(u_0,\lambda_0)\in M$ is $\mathbb{E}$-*singular* if $D_uf(u_0,\lambda_0)\colon\mathbb{E}\to\mathbb{F}$ is not an isomorphism. An $\mathbb{E}$-singular point (u_0,λ_0) is *isolated* if there are no other $\mathbb{E}$-singular points in some neighborhood of (u_0,λ_0). It follows from the implicit function theorem that if (u_0,λ_0) is a bifurcation point, then (u_0,λ_0) is an $\mathbb{E}$-singular point.

We start with the construction of the open neighborhood $\mathscr{U}$. We consider the open neighborhood of $(u_0,\lambda_0)\in M$ defined by

$$B_M(u_0,\lambda_0;r,\rho)\stackrel{\text{def}}{=}\{(u,\lambda)\in\mathbb{E}\times\mathbb{R}^2:|\lambda-\lambda_0|<\rho,\|u-\eta(\lambda)\|<r\},\tag{6.19}$$

where $r>0$ is chosen such that

(i) $\mathscr{F}(u,\lambda)\neq 0$ for all $(u,\lambda)\in\overline{B_M(u_0,\lambda_0;r,\rho)}$ such that $|\lambda-\lambda_0|=\rho,\|u-\eta(\lambda)\|\neq 0$;
(ii) (u_0,λ_0) is the only $\mathbb{E}$-singular point of $\mathscr{F}$ in $\overline{B_M(u_0,\lambda_0;r,\rho)}$.

We call $B_M(u_0,\lambda_0;r,\rho)$ a *special neighborhood* of $\mathscr{F}$ determined by r and ρ. The existence of a special neighborhood $\mathscr{U}\stackrel{\text{def}}{=}B_M(u_0,\lambda_0;r,\rho)$ follows from the assumption that the $\mathbb{E}$-singular point (u_0,λ_0) of $\mathscr{F}$ is isolated. Note that the equivariant version of Dugundji's extension theorem (see [193, p. 197]) implies that there exists a continuous $\mathbb{S}^1$-equivariant function $\theta\colon\overline{\mathscr{U}}\to\mathbb{R}$ such that

(i) $\theta(\eta(\lambda),\lambda)=-|\lambda-\lambda_0|$ for all $(\eta(\lambda),\lambda)\in\overline{\mathscr{U}}\cap M$;
(ii) $\theta(u,\lambda)=r$ if $\|u-\eta(\lambda)\|=r$.

Such a function θ is called a *completing function* (or Ize's function). Clearly, if θ is a completing function, then $\psi_\delta(u,\lambda)\stackrel{\text{def}}{=}\theta(u,\lambda)-\delta$ is negative on the subset of trivial solutions $\mathscr{U}\cap M$, provided that $\delta>0$. So ψ_δ is an auxiliary function to (6.16). For $\delta>0$ small enough, we can define $\mathscr{F}_{\psi_\delta}:\overline{\mathscr{U}}\to\mathbb{E}\times\mathbb{R}$ by

$$\mathscr{F}_{\psi_\delta}(u,\lambda)\stackrel{\text{def}}{=}(\mathscr{F}(u,\lambda),\psi_\delta(u,\lambda)),$$

and define the $\mathbb{S}^1$-equivariant degree $\mathbb{S}^1\text{-}\deg(\mathscr{F}_{\psi_\delta}, \mathscr{U})$. By the homotopy invariance of the $\mathbb{S}^1$-degree, $\mathbb{S}^1\text{-}\deg(\mathscr{F}_{\psi_\delta}, \mathscr{U}) = \mathbb{S}^1\text{-}\deg(\mathscr{F}_\theta, \mathscr{U})$. Therefore, the nontriviality of $\mathbb{S}^1\text{-}\deg(\mathscr{F}_\theta, \mathscr{U})$ implies the existence of a nontrivial solution of (6.14) in $\mathscr{U}$.

We now turn to the computation of $\mathbb{S}^1\text{-}\deg(\mathscr{F}_\theta, \mathscr{U})$. We identify $\mathbb{R}^2$ with $\mathbb{C}$, and for sufficiently small $\rho > 0$, we define $\alpha : D \to M$, $D \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z| \le 1\}$, by

$$\alpha(z) = (\eta(\lambda_0 + \rho z), \lambda_0 + \rho z) \in \mathbb{E}_0 \times \mathbb{R}^2.$$

Since we have assumed that $(x_0, \lambda_0) = (\eta(\lambda_0), \lambda_0) \in M$ is an isolated $\mathbb{E}$-singular point, it is clear that we can choose sufficiently small $\rho > 0$ such that $\alpha(D)$ contains only one $\mathbb{E}$-singular point, namely (x_0, λ_0). Consequently, the formula $\Psi(z) \stackrel{\text{def}}{=} D_u\mathscr{F}(\alpha(z)), z \in \mathbb{S}^1 \subseteq D$, defines a continuous map $\Psi : \mathbb{S}^1 \to GL_G(\mathbb{E})$, which has the decomposition (see [88] for details) $\Psi = \Psi_0 \oplus \Psi_1 \oplus \cdots \oplus \Psi_k \oplus \cdots$, where $\Psi_k = \Psi|_{\mathbb{E}_k} : \mathbb{S}^1 \to GL_G(\mathbb{E}_k)$ for $k = 1, 2, \cdots$ and $\Psi_0 : \mathbb{S}^1 \to GL(\mathbb{E}_0)$, with $GL(\mathbb{E}_0)$ the set of linear automorphisms of $\mathbb{E}_0$. We now define

$$\begin{cases} \varepsilon_0(u_0, \lambda_0) = \operatorname{sgn}\det \Psi_0(z), \\ \mu_k(u_0, \lambda_0) = \deg_B(\det_{\mathbb{C}}[\Psi_k]), k = 1, 2, \cdots, \\ \mu(u_0, \lambda_0) = \{\mu_k(u_0, \lambda_0)\} \in \oplus_{k=1}^{\infty}\mathbb{Z}. \end{cases} \tag{6.20}$$

It is clear that ε_0 does not depend on the choice of $z \in \mathbb{S}^1$.

We need one more notion, the crossing number, to calculate $\deg_B(\det_{\mathbb{C}}[\Psi_k])$:

Lemma 6.3 ([88]). *Suppose $\alpha_0, \beta_0, \delta, \varepsilon$ are given numbers with $\alpha_0, \delta, \varepsilon > 0$. Let $\Omega \stackrel{\text{def}}{=} (0, \alpha_0) \times (\beta_0 - \varepsilon, \beta_0 + \varepsilon) \subseteq \mathbb{R}^2$. Assume that $H : [\sigma_0 - \delta, \sigma_0 + \delta] \times \bar{\Omega} \to \mathbb{R}^2$ is a continuous function satisfying:*

(i) $H(\sigma, \alpha, \beta) \neq 0$ for all $\sigma \in [\sigma_0 - \delta, \sigma_0 + \delta]$ and $(\alpha, \beta) \in \partial\Omega \setminus \{(0, \beta); \beta \in (\beta_0 - \varepsilon, \beta_0 + \varepsilon)\}$;
(ii) if $(\alpha, \beta) \in \Omega$ and $H_{\sigma_0 \pm \delta}(\alpha, \beta) = 0$, then $\alpha \neq 0$.

Let $\Omega_1 \stackrel{\text{def}}{=} (\sigma_0 - \delta, \sigma_0 + \delta) \times (\beta_0 - \varepsilon, \beta_0 + \varepsilon)$ and define the function $\Psi_H : \bar{\Omega}_1 \to \mathbb{R}^2$ by $\Psi_H(\sigma, \beta) = H(\sigma, 0, \beta)$, for $\sigma \in [\sigma_0 - \delta, \sigma_0 + \delta]$, and $\beta \in [\beta_0 - \varepsilon, \beta_0 + \varepsilon]$. Then $\Psi_H(\sigma, \beta) \neq 0$ for $(\sigma, \beta) \in \partial\Omega_1$ and $\deg_B(\Psi_H, \Omega_1) = \gamma$, where γ is the crossing number given by

$$\gamma \stackrel{\text{def}}{=} \deg_B(H_{\sigma_0 - \delta}, \Omega) - \deg_B(H_{\sigma_0 + \delta}, \Omega).$$

Lemma 6.4. *Let $\mathscr{U} = B_M(u_0, \lambda_0; r', \rho) \subseteq \mathbb{E} \times \mathbb{R}^2$ be a special neighborhood of $\mathscr{F}$, and θ a completing function. Then the $\mathbb{S}^1$ degree $\mathbb{S}^1\text{-}\deg(\mathscr{F}_\theta, \mathscr{U})$ is well defined, and*

$$\mathbb{S}^1\text{-}\deg(\mathscr{F}_\theta, \mathscr{U}) = \varepsilon_0 \cdot \mu(u_0, \lambda_0).$$

That is,

$$\mathbb{S}^1\text{-}\deg_k(\mathscr{F}_\theta, \mathscr{U}) = \varepsilon_0 \cdot \mu_k(u_0, \lambda_0), k = 1, 2, \cdots,$$

where $\mu(u_0, \lambda_0)$ is defined by (6.20).

By Lemma 6.4, we have the following local bifurcation theorem of Krasnosel'skii type [191].

Theorem 6.5. *Suppose that $f\colon \mathbb{E}\oplus\mathbb{R}^2 \to \mathbb{F}$ is a G-equivariant continuous map that is continuously differentiable with respect to x at points $(x,\lambda)\in M$ and satisfies* (SD1)–(SD3). *If $(u_0,\lambda_0)\in M$ is an isolated $\mathbb{E}$-singular point such that $\varepsilon_0\mu_k(u_0,\lambda_0)\neq 0$ for some $k\geq 1$, then (u_0,λ_0) is a bifurcation point of* (6.13). *More precisely, there exists a sequence (u_n,λ_n) of nontrivial solutions to* (6.13) *such that the isotropy group of u_n contains $\mathbb{Z}_k$ and $(u_n,\lambda_n)\to(u_0,\lambda_0)$ as $n\to\infty$.*

We remark that the above results hold when $\mathbb{R}^2$ is replaced by an open subset of $\mathbb{R}^2$. Geba and Marzantowicz [111] established the following global bifurcation theorem of Rabinowitz type [251] by applying the $\mathbb{S}^1$-degree theory due to Dylawerski, Geba, Jodel, and Marzantowicz [85].

Theorem 6.6. *Suppose that $f\colon \mathbb{E}\oplus\mathbb{R}^2\to\mathbb{F}$ is as in Theorem 6.5 and suppose further that M is complete and every $\mathbb{E}$-singular point in M is isolated. Let $\mathscr{S}(f)$ denote the closure of the set of all nontrivial solutions of* (6.13). *Then for each bounded component C of $\mathscr{S}(f)$, the set $C\cap M$ is a finite set, i.e.,*

$$C\cap M=\{(u_1,\lambda_1),(u_2,\lambda_2),\cdots,(u_q,\lambda_q)\},$$

and

$$\sum_{i=1}^{q}\mathbb{S}^1\text{-}\deg(\mathscr{F}_{\theta_i},\mathscr{U}_i)=\sum_{i=1}^{q}\varepsilon_i\cdot\mu(u_i,\lambda_i)=0,$$

where $\mathscr{U}_i$ is a special neighborhood of (u_i,λ_i), θ_i is a completing function defined on $\overline{\mathscr{U}}_i$, and ε_i and $\mu(u_i,\lambda_i)$ are defined by (6.20).

Proof. If C is a bounded component of $\mathscr{S}(f)$, then every point of $C\cap M$ is a bifurcation point that is also a $\mathbb{E}$-singular point of f. Since every $\mathbb{E}$-singular point of f is isolated and M is complete, $C\cap M$ is a bounded and closed subset of $\mathbb{E}_0\times\mathbb{R}^2\supset M$. By (SD1), $\mathbb{E}_0\times\mathbb{R}^2$ is finite-dimensional, and hence $C\cap M$ is compact. Therefore, $C\cap M$ is a finite set.

Choose $r,\rho>0$ sufficiently small that for each $i=1,2,\cdots,q$, $U_i=B_M(u_i,\lambda_i;r,\rho)$ is a special neighborhood of (u_i,λ_i) for f and $U_i\cap U_j=\emptyset$ if $i\neq j$. Let $U=U_1\cup U_2\cup\cdots\cup U_q$ and find a bounded open set $\Omega_1\subset\mathbb{E}\times\mathbb{R}^2$ such that $C\setminus U\subseteq\Omega_1$ and $\Omega_1\bigcap M=\emptyset$. Put $\Omega_2=U\cup\Omega_1$. Then $C\subseteq\Omega_2$. We can (e.g., see [193, p. 174]) find an open invariant subset $\Omega\subseteq\mathbb{E}\times\mathbb{R}^2$ such that $C\subseteq\Omega\subseteq\Omega_2$ and $\partial\Omega\cap\mathscr{S}(f)=\emptyset$.

Note that Ω is an open, bounded, invariant subset. We now choose $r_0\in(0,r)$ and $\rho_0\in(0,\rho)$ such that for every $i=1,2,\cdots,q$, we have

(i) $B_M(u_i,\lambda_i;r_0,\rho_0)\subseteq\Omega$;
(ii) $\mathscr{U}_i\overset{\text{def}}{=}B_M(u_i,\lambda_i;r_0,\rho_0)$ is a special neighborhood of (u_i,λ_i) for f.

Set $\mathscr{U} = \mathscr{U}_1 \cup \mathscr{U}_2 \cup \cdots \cup \mathscr{U}_q$ and

$$\partial \mathscr{U}_{r_0} \stackrel{\text{def}}{=} \{(u, \lambda) \in \bar{\Omega} : \|u - \eta(\lambda)\| = r_0, (\eta(\lambda), \lambda) \in \overline{\mathscr{U}} \cap M\}.$$

We note that $r_0 > 0$, and define an invariant function by

$$\theta(u, \lambda) = \begin{cases} |\lambda - \lambda_i|^{\frac{\|u-\eta(\lambda)\|-r_0}{r_0}} + \|u - \eta(\lambda)\|, & \text{if } (u, \lambda) \in \bar{\mathscr{U}_i}, \\ r_0, & \text{if } (u, \lambda) \in C \setminus \mathscr{U}. \end{cases} \tag{6.21}$$

Now, $\mathscr{U}_i$ is a special neighborhood, and hence we have $(C \setminus \mathscr{U}) \cap \bar{\mathscr{U}_i} = C \cap \partial \mathscr{U}_i \subseteq \partial \mathscr{U}_{r_0}$, where we have $\theta(u, \lambda) = r_0$. Then by (6.21), $\theta(u, \lambda)$ is continuous on $(C \setminus \mathscr{U}) \cap \bar{\mathscr{U}_i}$. Also, by the construction of $\mathscr{U}_i$, we have $\bar{\mathscr{U}_i} \cap \bar{\mathscr{U}_j} = \emptyset$ if $i \neq j$. Therefore, $\theta : C \cup \bar{\mathscr{U}} \to \mathbb{R}$ is continuous.

By the equivariant version of Dugundji's extension theorem (see [193, p. 197]), we can extend $\theta : C \cup \bar{\mathscr{U}} \to \mathbb{R}$ to a continuous invariant function $\theta : \overline{\Omega} \to \mathbb{R}$ such that

(iii) $\theta(u, \lambda) = -|\lambda - \lambda_i|$ if $(u, \lambda) \in \overline{\mathscr{U}}_i \bigcap M$;
(iv) $\theta(u, \lambda) = r_0$ if $(u, \lambda) \in (C \setminus \mathscr{U}) \cup \partial \mathscr{U}_{r_0}$.

Let $\mathscr{F}_\theta(u, \lambda) = (\mathscr{F}(u, \lambda), \theta(u, \lambda))$. Then $\mathscr{F}_\theta^{-1}(0) = \mathscr{F}^{-1}(0) \bigcap \theta^{-1}(0)$. By (iii), we know that $\mathscr{F}_\theta^{-1}(0) \subseteq C$. Since $C \cap \partial \Omega = \emptyset$, $\mathscr{F}_\theta^{-1}(0) \cap \partial \Omega = \emptyset$. Therefore, $\mathbb{S}^1$-$\deg(\mathscr{F}_\theta, \Omega)$ is well defined.

We now construct a homotopy $H : \overline{\Omega} \times [0, 1] \to \mathbb{E} \times \mathbb{R}$ as follows:

$$H(u, \lambda, \alpha) = (\mathscr{F}(u, \lambda), (1 - \alpha)\theta(u, \lambda) - \alpha\rho_0), (u, \lambda, \alpha) \in \overline{\Omega} \times [0, 1].$$

Note that trivial solutions $(u, \lambda) \in \bar{\Omega}$ outside $\mathscr{S}(f)$ are contained in $\bar{\mathscr{U}_i} \cap M$ for some $i = 1, 2, \cdots, q$, and by (iii), we have

$$(1 - \alpha)\theta(u, \lambda) - \alpha\rho_0 = -(1 - \alpha)|\lambda - \lambda_i| - \alpha\rho_0 < 0.$$

Then by the fact that $\partial \Omega \cap \mathscr{S}(f) = \emptyset$, we have $H(u, \lambda, \alpha) \neq 0$ for all $(u, \lambda, \alpha) \in \partial \Omega \times [0, 1]$. Note that θ is invariant and $\mathscr{F}$ is equivariant. So H is an $\mathbb{S}^1$-homotopy. Since $H(u, \lambda, 0) = \mathscr{F}_\theta(u, \lambda)$ and $H(u, \lambda, 1) = (\mathscr{F}(u, \lambda), -\rho_0) \neq 0$ for all $(u, \lambda) \in \overline{\Omega} \times [0, 1]$, by the existence and homotopy invariance of the $\mathbb{S}^1$-degree, we have $\mathbb{S}^1$-$\deg(\mathscr{F}_\theta, \Omega) = 0$. But (i)–(iv) imply that $\mathscr{F}_\theta^{-1}(0) \subseteq C \cap \mathscr{U}$. Then it follows from the excision property of the $\mathbb{S}^1$-degree that

$$\mathbb{S}^1\text{-}\deg(\mathscr{F}_\theta, \mathscr{U}) = \mathbb{S}^1\text{-}\deg(\mathscr{F}_\theta, \Omega) = 0.$$

On the other hand, by the additivity property of the $\mathbb{S}^1$-degree, we have

$$\sum_{i=1}^{q} \mathbb{S}^1\text{-}\deg(\mathscr{F}_\theta, \mathscr{U}_i) = \mathbb{S}^1\text{-}\deg(\mathscr{F}_\theta, \mathscr{U}) = 0.$$

Let $\theta_i(u,\lambda) = \theta(u,\lambda)|_{\overline{\mathscr{U}}_i}$. Note that $\mathscr{U} \subseteq \overline{\Omega}$ implies that $((C\setminus\mathscr{U})\cup\partial\mathscr{U}_{r_0})\bigcap\overline{\mathscr{U}}_i = \partial\mathscr{U}_i\cap\partial\mathscr{U}_{r_0}$. Then $\theta_i(u,\lambda)$ is a completing function on $\overline{\mathscr{U}}_i$, and we have

$$\sum_{i=1}^{q}\mathbb{S}^1\text{-}\deg(\mathscr{F}_{\theta_i},\mathscr{U}_i) = \mathbb{S}^1\text{-}\deg(\mathscr{F}_{\theta},\mathscr{U}) = 0.$$

Therefore, it follows from Lemma 6.4 that

$$\sum_{i=1}^{q}\varepsilon_i\cdot\mu(u_i,\lambda_i) = \sum_{i=1}^{q}\mathbb{S}^1\text{-}\deg(\mathscr{F}_{\theta_i},\mathscr{U}_i) = 0.$$

The proof is complete. □

6.5.2 Nondifferentiability Case

If $f(u,\lambda)$ is not differentiable with respect to u, then we need to justify that the formal linearization can be utilized to detect the local Hopf bifurcation and to describe the global continuation of periodic solutions for such a system with state-dependent delay. Our approach to this justification of formal linearization is through a simple homotopy argument. Namely, we will consider the equation

$$\tilde{\mathscr{F}}(u,\lambda) = 0, \quad (u,\lambda)\in\bar{\mathscr{U}}, \tag{6.22}$$

for an $\mathbb{S}^1$-equivariant C^1-map $\tilde{\mathscr{F}}:\overline{\mathscr{U}}\to\mathbb{E}$ that is $\mathbb{S}^1$-homotopic to $\mathscr{F}$ in a sense to be detailed below. For the functional-analytic setting of the Hopf bifurcation of state-dependent DDEs, such a C^1-map is attained by extending a linear operator obtained through the formal linearization from a C^1-space to a C-space, an idea previously used by Eichmann [86] and Mallet-Paret–Nussbaum–Paraskevopoulos [215]. To be more precise, we assume that such a C^1-map is given by

$$\tilde{\mathscr{F}}(u,\lambda) = u-(L+K)^{-1}\circ[\tilde{Q}(\cdot,\lambda)+K](u), \quad (u,\lambda)\in\bar{\mathscr{U}}, \tag{6.23}$$

where $\tilde{Q}:\overline{\mathscr{U}}\to\mathbb{E}$ is an $\mathbb{S}^1$-equivariant C^1-map and

(SD4) $M\subseteq\tilde{\mathscr{F}}^{-1}(0)$, and for every $\lambda\in\mathbb{R}^2$, $(L+K)^{-1}\circ(\tilde{Q}(\cdot,\lambda)+K):\mathbb{E}\to\mathbb{E}$ is a condensing map.

By the implicit function theorem, if $(u_0,\lambda_0)\in M$ is a bifurcation point of system (6.23), then the derivative $D_u\tilde{\mathscr{F}}(u_0,\lambda_0)$, which is G-equivariant, is not an automorphism of $\mathbb{E}$. Therefore, all bifurcation points of (6.23) are contained in the set

$$\Lambda \overset{\text{def}}{=} \{(u,\lambda)\in M : D_u\tilde{\mathscr{F}}(u,\lambda)\notin GL_G(\mathbb{E})\}.$$

Let (u_0,λ_0) be an isolated $\mathbb{E}$-singular point of $\tilde{\mathscr{F}}$. To tie the $\mathbb{S}^1$-equivariant degree of $\mathscr{F}$ to that of $\tilde{\mathscr{F}}$, we assume that:

(SD5) We can choose the constants $r > 0$ and $\rho > 0$ such that $B_M(u_0, \lambda_0; r, \rho)$ is a special neighborhood of $\tilde{\mathscr{F}}$ and there exists $0 < r' \leq r$ such that $\mathscr{F}(u, \lambda) \neq 0$ for all $(u, \lambda) \in B_M(u_0, \lambda_0; r', \rho)$ with $|\lambda - \lambda_0| = \rho$ and $\|u - \eta(\lambda)\| \neq 0$.

If ψ is an auxiliary function to (6.16), then by the construction of the $\mathbb{S}^1$-degree and the assumptions (SD2), (SD4), and (SD5), there exists a special neighborhood $\mathscr{U} \stackrel{\text{def}}{=} B_M(u_0, \lambda_0; r', \rho)$ of $\tilde{\mathscr{F}}$ such that the continuous G-equivariant maps $\mathscr{F}_\psi$ and $\tilde{\mathscr{F}}_\psi$ are nonzero on the boundary of $\mathscr{U}$, and therefore both $\mathbb{S}^1$-deg$(\tilde{\mathscr{F}}_\psi, \mathscr{U})$ and $\mathbb{S}^1$-deg$(\mathscr{F}_\psi, \mathscr{U})$ are well defined.

For a completing function θ defined on $\mathscr{U}$, if $\tilde{\mathscr{F}}_\theta = (\tilde{\mathscr{F}}, \theta)$ is homotopic to $\mathscr{F}_\theta$ on $\mathscr{U}$, then the homotopy invariance of the $\mathbb{S}^1$-degree ensures that $\mathbb{S}^1$-deg$(\mathscr{F}_\theta, \mathscr{U}) = \mathbb{S}^1$-deg$(\tilde{\mathscr{F}}_\theta, \mathscr{U})$. On the other hand, we can follow the approach presented in the previous subsection to calculate $\mathbb{S}^1$-deg$(\tilde{\mathscr{F}}_\theta, \mathscr{U})$.

Finally, in order to exclude bifurcation of solutions of (6.22) in $\mathbb{E}_0 \times \mathbb{R}^2$, we assume that

(SD6) $D_u\tilde{\mathscr{F}}(u_0, \lambda_0)|_{\mathbb{E}_0} : \mathbb{E}_0 \to \mathbb{E}_0$ is an isomorphism.

Theorem 6.7 ([170]). *Assume that* (SD1)–(SD6) *hold and let* $\mathscr{U} = B_M(u_0, \lambda_0; r', \rho) \subseteq \mathbb{E} \times \mathbb{R}^2$ *be a special neighborhood for* $\tilde{\mathscr{F}}$ *and* θ *a completing function. If* $\tilde{\mathscr{F}}_\theta$ *is homotopic to* $\mathscr{F}_\theta$ *on* $\overline{\mathscr{U}}$ *and there exists* $k \geq 1$ *such that* $\mathbb{S}^1$-$\deg_k(\tilde{\mathscr{F}}_\theta, \mathscr{U}) \neq 0$, *then* (u_0, λ_0) *is a bifurcation point for (6.13). That is, there exists a sequence of nontrivial solutions* (u_n, λ_n) *of (6.13) such that the isotropy group of* u_n *contains* $\mathbb{Z}_k$ *and* $(u_n, \lambda_n) \to (u_0, \lambda_0)$ *as* $n \to \infty$.

For global bifurcation, we assume further that both $\mathscr{F}$ and $\tilde{\mathscr{F}}$ are defined on $\mathbb{E} \times \mathbb{R}^2$, and that:

(SD7) Every bifurcation point of (6.13) is a $\mathbb{E}$-singular point of $\tilde{\mathscr{F}}$.
(SD8) $\tilde{\mathscr{F}}_\theta$ is homotopic to $\mathscr{F}_\theta$ on some special neighborhood $\mathscr{U}$ of each isolated $\mathbb{E}$-singular point of $\tilde{\mathscr{F}}$, where θ is a completing function defined on $\mathscr{U}$.

Now we can state the following global bifurcation theorem of Rabinowitz type.

Theorem 6.8 ([170]). *Assume that* (SD1)–(SD8) *hold and* (SD5)–(SD6) *hold for every* $\mathbb{E}$*-singular point* (u_0, λ_0) *of* $\tilde{\mathscr{F}}$. *Assume further that every* $\mathbb{E}$*-singular point of* $\tilde{\mathscr{F}}$ *in* M *is isolated and* M *is complete. Let* $\mathscr{S}$ *denote the closure of the set of all nontrivial solutions of (6.13). Then for each bounded component* C *of* $\mathscr{S}$, *the set* $C \cap M$ *is a finite set, i.e.,* $C \cap M = \{(u_1, \lambda_1), (u_2, \lambda_2), \cdots, (u_q, \lambda_q)\}$, *and*

$$\sum_{i=1}^{q} \mathbb{S}^1\text{-}\deg(\tilde{\mathscr{F}}_{\theta_i}, \mathscr{U}_i) = 0,$$

where $\mathscr{U}_i$ *is a special neighborhood of* (u_i, λ_i), *and* θ_i *is a completing function defined on* $\overline{\mathscr{U}}_i$.

6.6 Global Hopf Bifurcation Theory of DDEs

In this section, we employ the $\mathbb{S}^1$-equivariant degree to establish global Hopf bifurcations for general functional differential equations of mixed type with two parameters. We state our theory in a very general setting to allow for mixed type to ensure that the general theory can be used to address the issue of global bifurcations of bifurcated periodic solutions with additional features, such as spatial–temporal symmetry, for systems of DDEs with special symmetries.

Let X denote the Banach space of bounded continuous mappings $x\colon \mathbb{R} \to \mathbb{R}^n$ equipped with the supremum norm. For reasons mentioned above, we will consider functional differential equations with both delayed and advanced arguments. Therefore, for $x \in X$ and $t \in \mathbb{R}$, we will use x^t to denote an element in X defined by $x^t(s) = x(t+s)$ for $s \in \mathbb{R}$.

Consider the functional differential equation

$$\dot{x}(t) = F(x^t, \alpha, p) \tag{6.24}$$

parameterized by two real numbers $(\alpha, p) \in \mathbb{R} \times \mathbb{R}_+$, where $\mathbb{R}_+ = (0, \infty)$, and $F\colon X \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^n$ is completely continuous. Identifying the subspace of X consisting of all constant mappings with $\mathbb{R}^n$, we obtain a mapping $\hat{F} = F|_{\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_+} : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^n$. We require the following assumption:

(GHB1) $\hat{F}$ is twice continuously differentiable.

Denote by $\hat{x}_0 \in X$ the constant mapping with the value $x_0 \in \mathbb{R}^n$. We call $(\hat{x}_0, \alpha_0, p_0)$ a *stationary solution* of (6.24) if $\hat{F}(x_0, \alpha_0, p_0) = 0$. We assume that:

(GHB2) At each stationary solution $(\hat{x}_0, \alpha_0, p_0)$, the derivative of $\hat{F}(x, \alpha, p)$ with respect to the first variable x, evaluated at $(\hat{x}_0, \alpha_0, p_0)$, is an isomorphism of $\mathbb{R}^n$.

Under (GHB1)–(GHB2), for each stationary solution $(\hat{x}_0, \alpha_0, p_0)$, there exist $\varepsilon_0 > 0$ and a continuously differentiable mapping $y\colon B_{\varepsilon_0}(\alpha_0, p_0) \to \mathbb{R}^n$ such that $\hat{F}(y(\alpha, p), \alpha, p) = 0$ for $(\alpha, p) \in B_{\varepsilon_0}(\alpha_0, p_0) = (\alpha_0 - \varepsilon_0, \alpha_0 + \varepsilon_0) \times (p_0 - \varepsilon_0, p_0 + \varepsilon_0)$.

We need the following smoothness condition:

(GHB3) $F(\varphi, \alpha, p)$ is differentiable with respect to φ, and the $n \times n$ complex matrix function $\Delta_{(\hat{y}(\alpha,p),\alpha,p)}(\lambda)$ is continuous in $(\alpha, p, \lambda) \in B_{\varepsilon_0}(\alpha_0, p_0) \times \mathbb{C}$. Here, for each stationary solution $(\hat{x}_0, \alpha_0, p_0)$, we have $\Delta_{(\hat{x}_0,\alpha_0,p_0)}(\lambda) = \lambda \mathrm{Id} - DF(\hat{x}_0, \alpha_0, p_0)(\mathrm{e}^{\lambda \cdot}\mathrm{Id})$, where $DF(\hat{x}_0, \alpha_0, p_0)$ is the complexification of the derivative of $F(\varphi, \alpha, p)$ with respect to φ, evaluated at $(\hat{x}_0, \alpha_0, p_0)$.

For easy reference, we will again call $\Delta_{(\hat{x}_0,\alpha_0,p_0)}(\lambda)$ the *characteristic matrix* and the zeros of $\det \Delta_{(\hat{x}_0,\alpha_0,p_0)}(\lambda) = 0$ the *characteristic values* of the stationary solution $(\hat{x}_0, \alpha_0, p_0)$. So (GHB2) is equivalent to assuming that 0 is not a characteristic value of any stationary solution of (6.24).

Definition 6.1. A stationary solution $(\hat{x}_0, \alpha_0, p_0)$ is called a *center* if it has purely imaginary characteristic values of the form $\mathrm{i}m\frac{2\pi}{p_0}$ for some positive integer m. A center $(\hat{x}_0, \alpha_0, p_0)$ is said to be *isolated* if (i) it is the only center in some neighborhood of $(\hat{x}_0, \alpha_0, p_0)$; (ii) it has only finitely many purely imaginary characteristic values of the form $\mathrm{i}m\frac{2\pi}{p_0}$, m an integer.

Assume now that $(\hat{x}_0, \alpha_0, p_0)$ is an isolated center. Let $J(\hat{x}_0, \alpha_0, p_0)$ denote the set of all positive integers m such that $\mathrm{i}m\frac{2\pi}{p_0}$ is a characteristic value of $(\hat{x}_0, \alpha_0, p_0)$. We assume that there exists $m \in J(\hat{x}_0, \alpha_0, p_0)$ such that:

(GHB4) There exist $\varepsilon \in (0, \varepsilon_0)$ and $\delta \in (0, \varepsilon_0)$ such that on $[\alpha_0 - \delta, \alpha_0 + \delta] \times \partial\Omega_{\varepsilon, p_0}$, $\det \Delta_{(\hat{y}(\alpha,p),\alpha,p)}(u + \mathrm{i}m\frac{2\pi}{p}) = 0$ if and only if $\alpha = \alpha_0$, $u = 0$, $p = p_0$, where $\Omega_{\varepsilon, p_0} = \{(u, p) : 0 < u < \varepsilon, p_0 - \varepsilon < p < p_0 + \varepsilon\}$.

Let

$$H^{\pm}(\hat{x}_0, \alpha_0, p_0)(u, p) = \det \Delta_{(\hat{y}(\alpha_0 \pm \delta, p), \alpha_0 \pm \delta, p)} \left(u + \mathrm{i}m\frac{2\pi}{p} \right).$$

Then (GHB4) implies that $H_m^{\pm}(\hat{x}_0, \alpha_0, p_0) \neq 0$ on $\partial\Omega_{\varepsilon, p_0}$. Consequently, the integer

$$\gamma_m(\hat{x}_0, \alpha_0, p_0) = \deg_B(H_m^{-}(\hat{x}_0, \alpha_0, p_0), \Omega_{\varepsilon, p_0}) - \deg_B(H_m^{+}(\hat{x}_0, \alpha_0, p_0), \Omega_{\varepsilon, p_0})$$

is well defined.

Definition 6.2. $\gamma_m(\hat{x}_0, \alpha_0, p_0)$ is called the mth *crossing number* of $(\hat{x}_0, \alpha_0, p_0)$.

We will show that $\gamma_m(\hat{x}_0, \alpha_0, p_0) \neq 0$ implies the existence of a local bifurcation of periodic solutions with periods near p_0/m. More precisely, we have the following:

Theorem 6.9. *Assume that* (GHB1)–(GHB3) *are satisfied, and that there exist an isolated center* $(\hat{x}_0, \alpha_0, p_0)$ *and an integer* $m \in J(\hat{x}_0, \alpha_0, p_0)$ *such that* (GHB4) *holds and* $\gamma_m(\hat{x}_0, \alpha_0, p_0) \neq 0$. *Then there exists a sequence* $(\alpha_k, p_k) \in \mathbb{R} \times \mathbb{R}_+$ *such that*

(i) $\lim_{k\to\infty}(\alpha_k, p_k) = (\alpha_0, p_0)$;
(ii) at each $(\alpha, p) = (\alpha_k, p_k)$, (6.24) *has a nonconstant periodic solution* $x_k(t)$ *with period* p_k/m;
(iii) $\lim_{k\to\infty} x_k(t) = \hat{x}_0$, *uniformly for* $t \in \mathbb{R}$.

To describe the global continuation of the local bifurcation obtained in Theorem 6.9, we need to assume that:

(GHB5) All centers of (6.24) are isolated and (GHB4) holds for each center $(\hat{x}_0, \alpha_0, p_0)$ and each $m \in J(\hat{x}_0, \alpha_0, p_0)$.
(GHB6) For each bounded set $W \subseteq X \times \mathbb{R} \times \mathbb{R}_+$, there exists a constant $l > 0$ such that $|F(\varphi, \alpha, p) - F(\psi, \alpha, p)| \leq l \sup_{s\in R} |\varphi(s) - \psi(s)|$ for $(\varphi, \alpha, p), (\psi, \alpha, p) \in W$.

Theorem 6.10. *Let*

$$\Sigma(F) = Cl\{(x, \alpha, p);\ x \text{ is a } p\text{-periodic solution of } (6.24)\} \subset X \times \mathbb{R} \times \mathbb{R},$$
$$N(F) = \{(\hat{x}, \alpha, p); F(\hat{x}, \alpha, p) = 0\}.$$

Assume that $(\hat{x}_0, \alpha_0, p_0)$ *is an isolated center satisfying the conditions in Theorem 6.9. Denote by* $C(\hat{x}_0, \alpha_0, p_0)$ *the connected component of* $(\hat{x}_0, \alpha_0, p_0)$ *in* $\Sigma(F)$*. Then either*

(i) $C(\hat{x}_0, \alpha_0, p_0)$ *is unbounded, or*
(ii) $C(\hat{x}_0, \alpha_0, p_0)$ *is bounded,* $C(\hat{x}_0, \alpha_0, p_0) \cap N(F)$ *is finite, and*

$$\sum_{(\hat{x},\alpha,p)\in C(\hat{x}_0,\alpha_0,p_0)\cap N(F)} \gamma_m(\hat{x}, \alpha, p) = 0 \tag{6.25}$$

for all $m = 1, 2, \ldots$*, where* $\gamma_m(\hat{x}, \alpha, p)$ *is the mth crossing number of* $(\hat{x}, \alpha, p)$ *if* $m \in J(\hat{x}, \alpha, p)$*, and zero otherwise.*

Proof of Theorems 6.9 and 6.10: Put $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$, $\mathbb{E} = L^1(\mathbb{S}^1; \mathbb{R}^n)$, $\mathbb{F} = L^2(\mathbb{S}^1; \mathbb{R}^n)$. Define $L\colon \mathbb{E} \to \mathbb{F}$ and $Q\colon \mathbb{E} \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{F}$ by

$$Lz = \dot{z}(t), \quad Q(z, \alpha, p)(t) = \frac{p}{2\pi} F(z_{t,p}, \alpha, p),$$

where

$$z_{t,p}(\theta) = z\left(t + \frac{2\pi}{p}\theta\right), \ \theta \in R.$$

Clearly, $x(t)$ is a p-periodic solution of (6.24) if and only if $z(t) = x(\frac{p}{2\pi}t)$ is a solution in $\mathbb{E}$ of the operator equation $Lz = Q(z, \alpha, p)$.

The representations $\mathbb{E}$ and $\mathbb{F}$ are isometric Hilbert representations of the group $\mathbb{S}^1$, where $\mathbb{S}^1$ acts by shifting the argument. With respect to these $\mathbb{S}^1$-actions, L is an equivariant bounded linear Fredholm operator of index zero with an equivariant compact resolvent K, and Q is an $\mathbb{S}^1$-equivariant compact mapping. Moreover, at $(\hat{y}(\alpha, p), \alpha, p)$ with $(\alpha, p) \in \mathscr{D} \stackrel{\text{def}}{=} (\alpha_0 - \delta, \alpha_0 + \delta) \times (p_0 - \varepsilon, p_0 + \varepsilon)$, the derivative of Q with respect to the first variable is given by

$$D_z Q(\hat{y}(\alpha, p), \alpha, p) z(t) = \frac{p}{2\pi} DF(\hat{y}(\alpha, p), \alpha, p) z_{t,p}.$$

Identifying $\partial\mathscr{D}$ with $\mathbb{S}^1$, since $(\hat{x}_0, \alpha_0, p_0)$ is an isolated center, we can easily show that the mapping $\mathrm{Id} - (L+K)^{-1}[K + D_z F(\hat{y}(\alpha, p), \alpha, p)]$ is an isomorphism of $\mathbb{E}$ and that the mapping $\Psi\colon \mathbb{S}^1 \to GL(\mathbb{E})$ defined by

$$(\alpha, p) \in \partial\mathscr{D} \cong \mathbb{S}^1 \to \mathrm{Id} - (L+K)^{-1}[K + D_z F(\hat{y}(\alpha, p), \alpha, p)] \in GL(\mathbb{E})$$

is continuous.

The representation $\mathbb{E}$ has the well-known isotypical decomposition $\mathbb{E} = \bigoplus_{k=0}^{\infty} \mathbb{E}_k$, where $\mathbb{E}_0 \cong \mathbb{R}^n$ and for each $k \geq 1$, $\mathbb{E}_k$ is spanned by $\cos(kt)\varepsilon_j$ and $\sin(kt)\varepsilon_j$, $1 \leq j \leq n$, where $\{\varepsilon_1, \ldots, \varepsilon_n\}$ is the standard basis of $\mathbb{R}^n$. So we have $\Psi(\alpha, p)\mathbb{E}_k \subseteq \mathbb{E}_k$. Let $\Psi_k(\alpha, p) = \Psi(\alpha, p)|_{\mathbb{E}_k}$. It is not difficult to show that

$$\Psi_k(\alpha, p) = \frac{p}{\mathrm{i}2k\pi} \Delta_{(\hat{y}(\alpha,p),\alpha,p)}\left(\mathrm{i}k\frac{2\pi}{p}\right).$$

Let

$$\varepsilon = \operatorname{sign}\det \Psi_0(\alpha, p), \qquad (\alpha, p) \in \partial\mathscr{D},$$
$$n_k(\hat{x}_0, \alpha_0, p_0) = \varepsilon \deg_B(\det \Psi_k(\cdot), \mathscr{D}), \quad k = 1, 2, \ldots.$$

Then one can show, as in Erbe, Geba, Krawcewicz, and Wu [112], that $\gamma_k(\hat{x}_0, \alpha_0, p_0) = n_k(\hat{x}_0, \alpha_0, p_0)$, and therefore Theorems 6.9 and 6.10 are simply immediate consequences of Theorems 6.5 and 6.6 with $M = \{(\hat{x}_0, \alpha_0, p_0) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_+; F(\hat{x}_0, \alpha_0, p_0) = 0\}$. This completes the proof. □

For ease of applications, we describe below the local and global Hopf bifurcation theory for parameterized DDEs. Let $X = C_{n,\tau}$ and consider the following functional differential equation:

$$\dot{x}(t) = F(x_t, \alpha) \tag{6.26}$$

with parameter $\alpha \in \mathbb{R}$, $F: X \times \mathbb{R} \to \mathbb{R}^n$ is completely continuous.

Identifying the subspace of X consisting of all constant mappings with $\mathbb{R}^n$, we obtain a mapping $\hat{F} = F|_{\mathbb{R}^n \times \mathbb{R}}: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$. We now describe conditions (GHB1)–(GHB6) in relatively simple form:

(SGHB1) $\hat{F}$ is twice continuously differentiable.

Denote by $\hat{x}_0 \in X$ the constant mapping with the value $x_0 \in \mathbb{R}^n$. We call $(\hat{x}_0, \alpha_0)$ a *stationary solution* of (6.26) if $\hat{F}(x_0, \alpha_0) = 0$. We assume that:

(SGHB2) At each stationary solution $(\hat{x}_0, \alpha_0)$, the derivative of $\hat{F}(x, \alpha)$ with respect to the first variable x, evaluated at $(\hat{x}_0, \alpha_0)$, is an isomorphism of $\mathbb{R}^n$.

Under (SGHB1)–(SGHB2), for each stationary solution $(\hat{x}_0, \alpha_0)$, there exist $\varepsilon_0 > 0$ and a continuously differentiable mapping y: $B_{\varepsilon_0}(\alpha_0) \to \mathbb{R}^n$ such that $\hat{F}(y(\alpha), \alpha) = 0$ for $\alpha \in B_{\varepsilon_0}(\alpha_0) = (\alpha_0 - \varepsilon_0, \alpha_0 + \varepsilon_0)$.

We need the following smoothness condition:

(SGHB3) $F(\varphi, \alpha)$ is differentiable with respect to φ, and the $n \times n$ complex matrix function $\Delta_{(\hat{y}(\alpha),\alpha)}(\lambda)$ is continuous in $(\alpha, \lambda) \in B_{\varepsilon_0}(\alpha_0) \times \mathbb{C}$. Here, for each stationary solution $(\hat{x}_0, \alpha_0)$, we have $\Delta_{(\hat{x}_0,\alpha_0)}(\lambda) = \lambda \mathrm{Id} - DF(\hat{x}_0, \alpha_0)(e^{\lambda \cdot}\mathrm{Id})$, where $DF(\hat{x}_0, \alpha_0)$ is the complexification of the derivative of $F(\varphi, \alpha)$ with respect to φ, evaluated at $(\hat{x}_0, \alpha_0)$.

For easy reference, we will again call $\Delta_{(\hat{x}_0,\alpha_0)}(\lambda)$ the *characteristic matrix* and the zeros of $\det \Delta_{(\hat{x}_0,\alpha_0)}(\lambda) = 0$ the *characteristic values* of the stationary solution $(\hat{x}_0, \alpha_0)$. So (SGHB2) is equivalent to assuming that 0 is not a characteristic value of any stationary solution of (6.26).

The concepts of isolated centers and crossing numbers are now simplified as follows:

Definition 6.3. A stationary solution $(\hat{x}_0, \alpha_0)$ is called a *center* if it has purely imaginary characteristic values $\pm i\beta_0$ for some positive $\beta_0 > 0$. A center $(\hat{x}_0, \alpha_0)$ is said to be *isolated* if it is the only center in some neighborhood of $(\hat{x}_0, \alpha_0)$.

Assume that $(\hat{x}_0, \alpha_0)$ is an isolated center. We assume that:

(SGHB4) There exist $\varepsilon \in (0, \varepsilon_0)$ and $\delta \in (0, \varepsilon_0)$ such that on $[\alpha_0 - \delta, \alpha_0 + \delta] \times \partial\Omega_{\varepsilon, p_0}$, $\det \Delta_{(\hat{y}(\alpha), \alpha)}(u + \mathrm{i}\beta) = 0$ if and only if $\alpha = \alpha_0$, $u = 0$, $\beta = \beta_0$, where $\Omega_{\varepsilon, \beta_0} = \{(u, p) : 0 < u < \varepsilon, \beta_0 - \varepsilon < \beta < \beta_0 + \varepsilon\}$.

Let

$$H^{\pm}(\hat{x}_0, \alpha_0)(u, \beta) = \det \Delta_{(\hat{y}(\alpha_0 \pm \delta), \alpha_0 \pm \delta)}(u + \mathrm{i}\beta).$$

Then (SGHB4) implies that $H^{\pm}(\hat{x}_0, \alpha_0, \beta_0) \neq 0$ on $\partial\Omega_{\varepsilon, \beta_0}$. Consequently, the integer

$$\gamma(\hat{x}_0, \alpha_0, \beta_0) = \deg_B(H^{-}(\hat{x}_0, \alpha_0, \beta_0), \Omega_{\varepsilon, \beta_0}) - \deg_B(H^{+}(\hat{x}_0, \alpha_0, \beta_0), \Omega_{\varepsilon, \beta_0})$$

is well defined; it is called the first *crossing number* of $(\hat{x}_0, \alpha_0, \beta_0)$.

The local Hopf bifurcation theory below shows that $\gamma(\hat{x}_0, \alpha_0, \beta_0) \neq 0$ implies the existence of a local bifurcation of periodic solutions with periods near $2\pi/\beta_0$. More precisely, we have the following theorem:

Theorem 6.11. *Assume that* (SGHB1)–(SGHB3) *are satisfied, and that there exists an isolated center* $(\hat{x}_0, \alpha_0)$ *such that* (SGHB4) *holds and* $\gamma(\hat{x}_0, \alpha_0, \beta_0) \neq 0$. *Then there exists a sequence* $(\alpha_k, \beta_k) \in \mathbb{R} \times \mathbb{R}_+$ *such that*

(i) $\lim_{k\to\infty}(\alpha_k, \beta_k) = (\alpha_0, \beta_0)$*;*
(ii) at each $\alpha = \alpha_k$*,* (6.26) *has a nonconstant periodic solution* $x_k(t)$ *with a period* $\frac{2\pi}{\beta_k}$*;*
(iii) $\lim_{k\to\infty} x_k(t) = \hat{x}_0$*, uniformly for* $t \in \mathbb{R}$*.*

The global Hopf bifurcation theorem can now be stated as follows:

(SGHB5) All centers of (6.26) are isolated and (SGHB4) holds for each center $(\hat{x}_0, \alpha_0)$ with the corresponding β_0.
(SGHB6) For each bounded set $W \subseteq X \times \mathbb{R}$, there exists a constant $l > 0$ such that $|F(\varphi, \alpha) - F(\psi, \alpha)| \leq l \sup_{s\in R} |\varphi(s) - \psi(s)|$ for $(\varphi, \alpha), (\psi, \alpha) \in W$.

Theorem 6.12. *Set*

$$\Sigma(F) = Cl\{(x, \alpha, \beta);\ x \text{ is a } 2\pi/\beta\text{-periodic solution of } (6.26)\} \subset X \times \mathbb{R} \times \mathbb{R},$$
$$N(F) = \{(\hat{x}, \alpha, \beta); F(\hat{x}, \alpha) = 0, \det \Delta_{(\hat{y}(\alpha), \alpha)}(\mathrm{i}\beta) = 0\}.$$

Assume that $(\hat{x}_0, \alpha_0, \beta_0$ *is an isolated center satisfying the conditions in Theorem 6.11. Denote by* $C(\hat{x}_0, \alpha_0, \beta_0)$ *the connected component of* $(\hat{x}_0, \alpha_0, \beta_0)$ *in* $\Sigma(F)$*. Then either*

(i) $C(\hat{x}_0, \alpha_0, \beta_0)$ *is unbounded, or*
(ii) $C(\hat{x}_0, \alpha_0, \beta_0)$ *is bounded,* $C(\hat{x}_0, \alpha_0, \beta_0) \cap N(F)$ *is finite, and*

$$\sum_{(\hat{x}, \alpha, \beta) \in C(\hat{x}_0, \alpha_0, \beta_0) \cap N(F)} \gamma(\hat{x}, \alpha, \beta) = 0, \tag{6.27}$$

where $\gamma(\hat{x}, \alpha, \beta)$ *is the crossing number of* $(\hat{x}, \alpha, \beta)$*.*

6.7 Application to a Delayed Nicholson Blowflies Equation

6.7.1 The Nicholson Blowflies Equation

Gurney et al. [142] proposed the following simple-looking delay differential equation to explain the oscillatory behavior of the observed sheep blowfly *Lucilia cuprina* population in the experimental data collected by the Australian entomologist Nicholson [231]:

$$N'(t) = f(N(t-\tau)) - \gamma N(t)$$

with $f(N) = pNe^{-\alpha N}$, where $N(t)$ denotes the population of sexually mature adults at time t, p is the maximum possible per capita egg production rate, $1/\alpha$ is the population size at which the whole population reproduces at its maximum rate. In the model, τ is the generation time, or the time from egg to sexually mature adult, and γ is the per capita mortality rate of adults. This model is now commonly called Nicholson's blowflies equation. It was used by Oster and Ipatkchi [239] for the development of an insect population, and its modifications have been intensively studied in the literature of theoretical biology and delay differential equations. Notably, it has been shown that a unique positive equilibrium of the model is globally asymptotically stable (with respect to nonnegative and nontrivial initial conditions) for every $\tau \geq 0$, provided that $1 < p/\gamma < e^2$ (see, for example, [198]). In the case $p/\gamma > e^2$, the positive equilibrium loses its local stability, and Hopf bifurcations occur at an unbounded sequence of critical values. In the next subsection, we introduce the work of Wei and Li [294] that uses the global Hopf bifurcation theorem coupled with Bendixson's criterion for higher dimensional ordinary differential equations to establish the existence of periodic solutions when the delay τ is not necessarily near the local Hopf bifurcation values.

6.7.2 The Global Hopf Bifurcation Theorem of Wei–Li

In this subsection, we consider the equation

$$N'(t) = -\gamma N(t) + pN(t-\tau)\mathrm{e}^{-aN(t-\tau)}, t \geq 0. \tag{6.28}$$

We introduce the theorem of Wei–Li [294] that shows that under the assumption $p > \gamma \mathrm{e}^2$, as the delay τ increases, the positive equilibrium N^* loses its stability, a sequence of Hopf bifurcations occurs at N^*, and these periodic solutions persist for τ far away from these Hopf bifurcation values. Wei and Li established this theorem using a global Hopf bifurcation result (Theorem 6.12). A key step in establishing the global extension of the local Hopf branch at $\tau = \tau_0$ is to show that (6.28) has no nonconstant periodic solutions of period 4. This is accomplished by applying a higher-dimensional Bendixson criterion for ordinary differential equations due to Li and Muldowney [210].

The positive equilibrium $N^* = \frac{1}{a}\log\frac{p}{\gamma}$ of (6.28) exists if and only if $a > 0$ and $p > \gamma$. These relations are assumed throughout this section. Set $N(t) = N^* + \frac{1}{a}y(t)$. Then $x(t)$ satisfies

$$y'(t) = -\gamma y(t) - a\gamma N^*\left[1 - \mathrm{e}^{-y(t-\tau)}\right] + \gamma y(t-\tau)\mathrm{e}^{-y(t-\tau)}. \tag{6.29}$$

The linearization of (6.29) at $y = 0$ is

$$Y'(t) = -\gamma Y(t) - \gamma[aN^* - 1]\,Y(t-\tau),$$

whose characteristic equation is

$$\lambda = \gamma - \gamma[aN^* - 1]\,\mathrm{e}^{-\lambda\tau}. \tag{6.30}$$

For $\tau = 0$, the only root of (6.30) is $\lambda = -aN^* < 0$, since $p > \gamma$. For $\omega \neq 0$, $\mathrm{i}\omega$ is a root of (6.30) if and only if

$$\mathrm{i}\omega = -\gamma - \gamma[aN^* - 1]\,(\cos\omega\tau - \mathrm{i}\sin\omega\tau).$$

Separating the real and imaginary parts, we obtain

$$\begin{aligned}\gamma(aN^* - 1)\cos\omega\tau &= -\gamma,\\ \gamma(aN^* - 1)\sin\omega\tau &= \omega,\end{aligned}$$

which leads to

$$\gamma^2(aN^* - 1)^2 = \gamma^2 + \omega^2,$$

namely,

$$\omega = \pm\gamma\sqrt{aN^*(aN^* - 2)}.$$

This is possible if and only if $aN^* > 2$, or equivalently, if $p > \gamma\mathrm{e}^2$.

For $p > \gamma\mathrm{e}^2$, let

$$\tau_k = \frac{1}{\gamma\sqrt{aN^*(aN^* - 2)}}\left[\sin^{-1}\left(\frac{\sqrt{aN^*(aN^* - 2)}}{aN^* - 1}\right) + 2k\pi\right],$$

$k = 0, 1, 2, \cdots$. Set

$$\omega_0 = \gamma\sqrt{aN^*(aN^* - 2)}. \tag{6.31}$$

Let $\lambda_k = \alpha_k(\tau) + \mathrm{i}\,\omega_k(\tau)$ denote a root of (6.30) near $\tau = \tau_k$ such that $\alpha_k(\tau_k) = 0$, $\omega_k(\tau_k) = \omega_0$. Obviously, ${\alpha_k}'(\tau_k) > 0$. Therefore, we have obtained that when $\gamma < p \leq \gamma\mathrm{e}^2$, all roots of the characteristic equation (6.30) have negative real parts; when $p > \gamma\mathrm{e}^2$, (6.30) has a pair of simple imaginary roots $\pm i\,\omega_0$ at $\tau = \tau_k,\ k = 0, 1, 2, \cdots$. Furthermore, if $\tau \in [0, \tau_0)$, then all roots of (6.30) have negative real part; if $\tau = \tau_0$, then all roots of (6.30) except $\pm i\,\omega_0$ have negative real part; and if $\tau \in (\tau_k,\ \tau_{k+1})$ for $k = 0, 1, 2, \cdots$, then (6.30) has $2(k+1)$ roots with positive real part. In particular, we have shown that under the condition $p > \gamma e^2$, $N = N^*$ is asymptotically stable for $\tau \in [0, \tau_0)$ and unstable for $\tau > \tau_0$. Furthermore, (6.28) undergoes a Hopf bifurcation at N^* when $\tau = \tau_k$, for $k = 0, 1, 2, \cdots$.

Let $x(t) = y(\tau t)$. Then (6.29) becomes

$$x'(t) = -\gamma\tau\left[x(t) + aN^*(1 - e^{-x(t-1)}) - x(t-1)e^{-x(t-1)}\right]. \tag{6.32}$$

Lemma 6.5. *All periodic solutions to (6.32) are uniformly bounded.*

Proof. Let $x(t)$ be a nonconstant periodic solution to (6.32), and let $x(t_1) = M, x(t_2) = m$ be its maximum and minimum, respectively. Then, $x'(t_1) = x'(t_2) = 0$, and by (6.32),

$$M = x(t_1 - 1)e^{-x(t_1-1)} - aN^*[1 - e^{-x(t_1-1)}], \tag{6.33}$$

$$m = x(t_2 - 1)e^{-x(t_2-1)} - aN^*[1 - e^{-x(t_2-1)}]. \tag{6.34}$$

We claim that $x(t_1 - 1) < 0$ and $x(t_2 - 1) > 0$. In fact, if $x(t_1 - 1) = 0$, then (6.33) implies $M = 0$, and thus $m < 0$ and $x(t_2 - 1) \le 0$. Using (6.34), we know that $x(t_2 - 1) < 0$, and thus

$$m > x(t_2 - 1)e^{-x(t_2-1)},$$

which contradicts the fact that m is the minimum. If $x(t_1 - 1) > 0$, then by (6.33), we arrive at

$$M \le M - aN^*(1 - e^{-x(t_1-1)}) < M,$$

a contradiction. Therefore, $x(t_1 - 1) < 0$. A similar argument shows that $x(t_2 - 1) > 0$. Therefore, we have $m < 0$ and $M > 0$. Again by (6.33) and (6.34), we have

$$m > aN^*[e^{-M} - 1] > -aN^*. \tag{6.35}$$

Also by (6.33), we have

$$\begin{aligned} M &= -aN^* + (x(t_1 - 1) + aN^*)e^{-x(t_1-1)} \\ &= -aN^* + e^{aN^*}(x(t_1 - 1) + aN^*)e^{-(x(t_1-1)+aN^*)} \\ &\le -aN^* + e^{aN^*}e^{-1} = -aN^* + e^{aN^*-1}. \end{aligned} \tag{6.36}$$

Here we have used the fact that $x(t_1 - 1) + aN^* > m + aN^* > 0$ and that $xe^{-x} < e^{-1}$ for $x \ge 0$. Relations (6.35) and (6.36) imply uniform boundedness of the periodic solutions. □

Lemma 6.6. *Assume that $\gamma e^2 < p < \sqrt{2}\gamma e^2$. Then (6.32) has no periodic solutions of period 4.*

Proof. Let $x(t)$ be a periodic solution to (6.32) of period 4. Set $u_j(t) = x(t - j + 1)$, $j = 1, 2, 3, 4$. Then $u(t) = (u_1(t), u_2(t), u_3(t), u_4(t))$ is a periodic solution of the following system of ordinary differential equations:

$$\begin{aligned} u_1'(t) &= -\gamma\tau\,[u_1(t) + aN^*(1 - \mathrm{e}^{-u_2(t)}) - u_2(t)\mathrm{e}^{-u_2(t)}],\\ u_2'(t) &= -\gamma\tau\,[u_2(t) + aN^*(1 - \mathrm{e}^{-u_3(t)}) - u_3(t)\mathrm{e}^{-u_3(t)}],\\ u_3'(t) &= -\gamma\tau\,[u_3(t) + aN^*(1 - \mathrm{e}^{-u_4(t)}) - u_4(t)\mathrm{e}^{-u_4(t)}],\\ u_4'(t) &= -\gamma\tau\,[u_4(t) + aN^*(1 - \mathrm{e}^{-u_1(t)}) - u_1(t)\mathrm{e}^{-u_1(t)}], \end{aligned} \tag{6.37}$$

whose orbit belongs to the region

$$G = \{\, u \in \mathbb{R}^4 : \quad \bar{m} < |u_k| < \bar{M}, \quad k = 1,2,3,4 \,\}, \tag{6.38}$$

where $\bar{m}$ and $\bar{M}$ are a pair of uniform bounds for periodic solutions of (6.32) obtained in Lemma 6.5. To rule out 4-periodic solutions of (6.32), it suffices to prove the nonexistence of nonconstant periodic solutions of (6.37) in the region G. To do the latter, we use a general Bendixson's criterion in higher dimensions developed in Li and Muldowney [210]. More specifically, we will apply Corollary 3.5 in [210]. The Jacobian matrix $J = J(u)$ of (6.37), for $u \in \mathbb{R}^4$, is

$$J(u) = -\gamma\tau \begin{pmatrix} 1 & f(u_2) & 0 & 0 \\ 0 & 1 & f(u_3) & 0 \\ 0 & 0 & 1 & f(u_4) \\ f(u_1) & 0 & 0 & 1 \end{pmatrix},$$

where

$$f(v) = (aN^* + v - 1)\,\mathrm{e}^{-v}. \tag{6.39}$$

The second additive compound matrix $J^{[2]}(u)$ of $J(u)$ is (see [103] and [226])

$$J^{[2]}(u) = -\gamma\tau \begin{pmatrix} 2 & f(u_3) & 0 & 0 & 0 & 0 \\ 0 & 2 & f(u_4) & f(u_2) & 0 & 0 \\ 0 & 0 & 2 & 0 & f(u_2) & 0 \\ 0 & 0 & 0 & 2 & f(u_4) & 0 \\ -f(u_1) & 0 & 0 & 0 & 2 & f(u_3) \\ 0 & -f(u_1) & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Choose a vector norm in $\mathbb{R}^6$ as

$$|(x_1,x_2,x_3,x_4,x_5,x_6)| = \max\{\,\sqrt{2}|x_1|,\ |x_2|,\ \sqrt{2}|x_3|,\ \sqrt{2}|x_4|,\ |x_5|,\ \sqrt{2}|x_6|\,\}.$$

Then with respect to this norm, the Lozinskiĭ measure $\mu(J^{[2]}(u)$ of the matrix $J^{[2]}(u)$ is, see [73],

$$\begin{aligned} &\mu(J^{[2]}(u)) = \\ &\max\{\,\sqrt{2}\gamma\tau(-\sqrt{2} + |f(u_3)|),\ \sqrt{2}\gamma\tau(-\sqrt{2} + |f(u_4)|/2 + |f(u_2)|/2),\\ &\quad \sqrt{2}\gamma\tau(-\sqrt{2} + |f(u_2)|),\ \sqrt{2}\gamma\tau(-\sqrt{2} + |f(u_4)|),\\ &\quad \sqrt{2}\gamma\tau(-\sqrt{2} + |f(u_1)|/2 + |f(u_3)|/2),\ \sqrt{2}\gamma\tau(-\sqrt{2} + |f(u_1)|)\,\}. \end{aligned} \tag{6.40}$$

By Corollary 3.5 in [210], system (6.37) has no periodic orbits in G if $\mu(J^{[2]}(u)) < 0$ for all $u \in G$. From (6.40), we see that $\mu(J^{[2]}(u)) < 0$ if and only if

$$|f(u_j)| < \sqrt{2}, \quad j = 1,2,3,4, \tag{6.41}$$

for $u \in G$.

To establish (6.41), we first use the assumption $e^{aN^*} = p/\gamma < \sqrt{2}e^2$ to improve the lower bound m given in Lemma 6.5. In (6.34), we now have

$$x(t_2-1)+aN^* < M+aN^* < e^{aN^*-1} < \sqrt{2}e.$$

Using the fact that the function xe^{-x} is monotonically decreasing for $x > 1$ and that $x(t_2-1)+aN^* > 1$, we have

$$\begin{aligned} m &= -aN^* + e^{aN^*}(x(t_2-1)+aN^*)e^{-(x(t_2-1)+aN^*)} \\ &> -aN^* + e^{aN^*}\sqrt{2}e\, e^{-\sqrt{2}e} > -aN^* + 2e^2\sqrt{2}e\, e^{-\sqrt{2}e} \\ &= -aN^* + 2\sqrt{2}e^{3-\sqrt{2}e}. \end{aligned}$$

Therefore, $u \in G$ satisfies

$$|u_i| > -aN^* + 2\sqrt{2}e^{3-\sqrt{2}e}.$$

For $\delta = 2\sqrt{2}e^{3-\sqrt{2}e} > 1$, we can verify

$$|f(-aN^*+\delta)| = e^{aN^*-\delta}|\delta-1| = e^{aN^*-2}e^{2-\delta}(\delta-1) < e^{aN^*-2}.$$

From the graph of $f(v)$, we know that $f(v)$ has a global maximum $e^{aN^*-2} = e^{-2}p/\gamma$. Therefore, for $u \in G$,

$$|f(u_k)| \le \max\{e^{aN^*-2}, |f(-aN^*+\delta)|\} \le e^{aN^*-2} = \frac{p}{\gamma}e^{-2} < \sqrt{2},$$

and (6.41) is satisfied, completing the proof. □

Lemma 6.7. *Assume that $\gamma e^2 < p$. Then (6.32) has no periodic solutions of period 1 or 2.*

Proof. First note that every nonconstant 1-periodic solution $u(t)$ of (6.32) is also a nonconstant periodic solution of the ordinary differential equation

$$u'(t) = -\gamma\tau(1-e^{-u(t)})(u(t)+aN^*). \tag{6.42}$$

A simple phase-line analysis shows that (6.42) has no nonconstant periodic solutions.

As in the proof of Lemma 6.6, if $u(t)$ is a periodic solution of (6.32) of period 2, then $u_1(t) = u(t)$ and $u_2(t) = u(t-1)$ are periodic solutions of the system of ordinary differential equations

$$\begin{aligned} u_1'(t) &= -\gamma\tau\,[u_1(t) + aN^*(1 - \mathrm{e}^{-u_2(t)}) - u_2(t)\mathrm{e}^{-u_2(t)}] \\ u_2'(t) &= -\gamma\tau\,[u_2(t) + aN^*(1 - \mathrm{e}^{-u_1(t)}) - u_1(t)\mathrm{e}^{-u_1(t)}]. \end{aligned} \tag{6.43}$$

Let $(P(u_1,u_2),\, Q(u_1,u_2))$ denote the vector field of (6.43). Then

$$\frac{\partial P}{\partial u_1} + \frac{\partial Q}{\partial u_2} = -2\gamma\tau < 0$$

for all (u_1,u_2). Thus the classical Bendixson's negative criterion implies that (6.43) has no nonconstant periodic solutions. □

Theorem 6.13. *Suppose that* $\gamma\mathrm{e}^2 < p < \sqrt{2}\gamma\mathrm{e}^2$ *holds. Then for each* $\tau > \tau_k,\ k = 0,1,2,\cdots$, *(6.32) has at least* $k+1$ *periodic solutions.*

Proof. First note that

$$F(x_t,\tau) \stackrel{\text{def}}{=} -\gamma\tau[x(t) + aN^*(1 - \mathrm{e}^{x(t-1)}) - y(t-1)\mathrm{e}^{-x(t-1)}]$$

satisfies hypotheses (SGHB1), (SGHB2), and (SGHB3) of Sect. 6.6, with

$$\begin{aligned} (\hat{x}_0, \alpha_0) &= (0, \tau_k), \\ \Delta_{(0,\tau_k)}(z) &= z + \tau\gamma + \tau\gamma[aN^* - 1]\mathrm{e}^{-z}. \end{aligned}$$

It can also be verified that $(0, \tau_k)$ are isolated centers with the corresponding imaginary characteristic values $\pm\mathrm{i}\tau_k\omega_0$. We have shown that there exist $\varepsilon > 0$, $\delta > 0$, and a smooth curve $z : (\tau_k - \delta,\, \tau_k + \delta) \to \mathbb{C}$ such that $\Delta(z(\tau)) = 0$, $|z(\tau) - \mathrm{i}\tau_k\omega_0| < \varepsilon$ for all $\tau \in [\tau_k - \delta,\, \tau_k + \delta]$, and

$$z(\tau_k) = \mathrm{i}\tau_k\omega_0, \qquad \frac{d\mathrm{Re}z(\tau)}{d\tau}\bigg|_{\tau=\tau_k} > 0.$$

Set $\beta_k = \tau_k\omega_0$ and let

$$\Omega_\varepsilon = \{(0,\beta)\ :\ 0 < u < \varepsilon,\ |\beta - \beta_k| < \varepsilon\}.$$

Clearly, if $|\tau - \tau_k| \le \delta$ and $(u,p) \in \partial\Omega_\varepsilon$ such that $\Delta_{0,\tau)}(u + \mathrm{i}\beta) = 0$, then $\tau = \tau_k$, $u = 0$, and $\beta = \beta_k$. This satisfies assumption (SGHB4) in Sect. 6.6. Moreover, if we put

$$H^{\pm}(0,\tau_k)(u,\beta) = \Delta_{(0,\tau_k\pm\delta)}(u + \mathrm{i}\beta),$$

then we have the cross number

$$\begin{aligned} \gamma(0,\tau_k) = \deg_B&(H^-(0,\tau_k,\tau_k\omega_0),\, \Omega_\varepsilon) \\ &- \deg_B(H^+(0,\tau_k,\tau_k\omega_0),\, \Omega_\varepsilon) = -1. \end{aligned}$$

By Theorem 6.12, we conclude that the connected component $C(0,\tau_k,\tau_k\omega_0)$ through $(0,\tau_k,\tau_k\omega_0)$ in $\Sigma(F)$ is nonempty. Meanwhile, we have

$$\sum_{(\hat{x},\tau,\beta)\in C(0,\tau_k,\tau_k\omega_0)\cap N(F)} \gamma(\hat{y},\tau,T) < 0,$$

and hence $C(0,\tau_k,\tau_k\omega_0)$ is unbounded.

Lemma 6.5 implies that the projection of $C(0,\tau_k,\tau_k\omega_0)$ onto the x-space is bounded. It can be verified using a phase-line analysis that when $\tau = 0$, (6.32) has no nonconstant periodic solutions. Therefore, the projection of $C(0,\tau_k,\tau_k\omega_0)$ onto the τ-space is bounded below. From the definitions of τ_k and ω_0, we obtain

$$\tau_k\omega_0 = \sin^{-1}\left(\frac{\sqrt{aN^*(aN^*-2)}}{aN^*-1}\right) + 2k\pi \tag{6.44}$$

for $k \geq 0$. Also, we know that $\sin\omega_0\tau_k > 0$ and $\cos\omega_0\tau_k < 0$, for $k \geq 0$. Hence

$$\frac{\pi}{2} < \omega_0\tau_0 < \pi, \quad \text{and} \quad 2\pi < \omega_0\tau_k < (2k+1)\pi,\ k \geq 1.$$

Therefore

$$2 < \frac{2\pi}{\tau_0\omega_0} < 4, \quad \text{and} \quad \frac{1}{k+1} < \frac{2\pi}{\omega_0\tau_k} < 1,\ k \geq 1. \tag{6.45}$$

Applying Lemmas 6.6 and 6.7, we know that $2 < 2\pi/\beta < 4$ if $(x,\tau,\beta) \in C(0,\tau_0,\tau_0\omega_0)$, and that $1/(k+1) < 2\pi/\beta < 1$ if $(x,\tau,\beta) \in C(0,\tau_k,\tau_k\omega_0)$ for $k \geq 1$. This shows that in order for $C(0,\tau_k,\tau_k\omega_0)$ to be unbounded, its projection onto the τ-space must be unbounded. Consequently, the projection of $C(0,\tau_k,\tau_k\omega_0)$ onto the τ-space includes $[\tau_k,\infty)$. This shows that for each $\tau > \tau_k$, (6.32) has $k+1$ nonconstant periodic solutions, completing the proof of the theorem. □

Remark 6.1. (i) From the proof of Theorem 6.13, we know that the first global Hopf branch contains periodic solutions of period between 2 and 4. These are the slowly oscillating periodic solutions. See [13, 60, 197, 291] for more details about the existence of slowly oscillating periodic solutions in delay differential equations. The τ_k branches, for $k \geq 1$, since the periods are less than 1, contain fast-oscillating periodic solutions.

(ii) For $k \geq 1$,

$$\frac{1}{k+1} < \frac{2\pi}{\tau_k\omega_0} < 1$$

automatically holds. The bounds on the period $2\pi/\beta$ for $(x,\tau,\beta)\in C(0,\tau_k,\tau_k\omega_0)$ hold without resort to Lemma 6.6. Thus, the global extension of the τ_k-branch for $k \geq 1$ can be proved without the restriction $p < \sqrt{2}\gamma e^2$.

6.7.3 Nicholson's Blowflies Equation Revisited: Onset and Termination of Nonlinear Oscillations

In [264], the authors reexamined the Nicholson's blowflies model with natural death rate explicitly incorporated into the delay feedback, obtaining the following delay differential equation with a delay-dependent coefficient

$$N'(t) = e^{-\delta\tau} f(N(t-\tau)) - \gamma N(t), \tag{6.46}$$

where $\delta > 0$ is the death rate of the immature population, and $f(N) = pNe^{-\alpha N}$. One can derive this, as was done in [82, 222], from a structured population model for $u(t,a)$ (the population density at age a and time t) as follows:

$$\frac{\partial}{\partial t}u(t,a) + \frac{\partial}{\partial a}u(t,a) = -\mu(a)u(t,a),$$

with the stage-specific mortality rate

$$\mu(a) = \begin{cases} \gamma, & t > \tau, \\ \delta, & t < \tau. \end{cases}$$

A simple application of the integration along characteristic lines leads to the model equation for the mature population $N(t) = \int_\tau^\infty u(t,a)da$ with the Ricker's-type birth function f.

The additional term $e^{-\delta\tau}$ is the probability of the immature population surviving τ time units before becoming mature. This addition, as shown in [264], leads to rather different dynamics for model (6.46): as the delay τ increases, the positive equilibrium loses its stability and undergoes local Hopf bifurcations at a *finite even number* of critical values, and as τ passes a critical threshold, the positive equilibrium regains its stability. In other words, as τ keeps increasing and passes another threshold value, the positive equilibrium disappears, and the species becomes extinct (the zero solution is globally asymptotically stable). Shu, Wang, and Wu [264] also observed the coexistence of multiple stable periodic solutions.

As we did in the last subsection, Shu, Wang, and Wu [264] considered the delay a bifurcation parameter and examined the onset and termination of Hopf bifurcations of periodic solutions from a positive equilibrium. They proved that the model has only a finite number of Hopf bifurcation values and that these branches of Hopf bifurcations are paired, so that the existence of periodic solutions with specific oscillation frequencies occurs only in bounded delay intervals. The bifurcation analysis then guided some numerical simulations to identify ranges of parameters for coexisting multiple attractive periodic solutions.

6.8 Rotating Waves and Circulant Matrices

We have noticed that a key step in applying the global Hopf bifurcation theory is to exclude the existence of nonconstant periodic solutions with a certain prescribed period, normally the integer multiplier of the delay if the delay is constant. A general approach outlined in [237] is as follows: If one assumes that $y(t)$ is a periodic solution of a prototype equation $x'(t) = f(x(t), x(t-\tau))$ for some scalar function f, of period $m\tau$ for a certain integer m, and defines $u_j(t) = y(t-(j-1)\tau))$ for $1 \le j \le m$, one then discovers that $u(t) = (u_1(t), \ldots, u_m(t))$ satisfies a cyclic system of ordinary differential equations $u'(t) = g(u(t))$, and we shall show that solutions of such a cyclic system satisfy $\lim_{t\to\infty}|u(t)| = 0$ or ∞, and the key step in proving the latter statement will be the construction of appropriate Lyapunov functions for the cyclic system. This will normally require the estimation of the spectral radius of a so-called circulant matrix. If we linearize this cyclic system at the trivial solution, we are led to a linear system with a real circulant matrix. Here and in what follows, an $n\times n$ matrix is called circulant if its (i,j)-element is given by a_{j-i+1} for n real numbers $a_1,\cdots a_n$. This matrix will be written as $A = \mathrm{circ}(a_1,a_2,\cdots a_n)$. For such a matrix, it was shown in [237] that

$$\inf\{\langle Ay,y\rangle : y\in\mathbb{R}^n, \sum_{j=1}^{n} y_j^2 = 1\} = \min\{\mathrm{Re}\Big(\sum_{j=1}^{n} a_j z^{j-1}\Big) : z\in\mathbb{C}; z^n = 1\}.$$

In this section, we will demonstrate the use of the approach outlined by Nussbaum based on the above-mentioned spectral property of circulant matrices.

We consider the following partial NFDE:

$$\begin{aligned}\frac{\partial}{\partial t}[u(t,x)-qu(t-\tau,x)] = d\frac{\partial^2}{\partial x^2}[u(t,x)-qu(t,x)]\\ -au(t,x)-aqu(t-\tau,x)-g[u(t,x)-qu(t-\tau,x)],\end{aligned} \tag{6.47}$$

where $x\in\mathbb{S}^1$, a,d,τ are positive constants, $g:\mathbb{R}\to\mathbb{R}$ is continuously differentiable with $g(0)=0$, $q\in(0,1)$ is the bifurcation parameter. This partial NFDE can be obtained from the coupled lossless transmission line NFDE introduced in Sect. 5.9 by letting the number of coupled oscillators go to infinity.

We are interested in the Hopf bifurcation of rotating waves from the trivial solution. Rotating wave solutions are solutions that satisfy

$$u(t,x) = u(t+\frac{p}{2\pi}x,0), \quad u(t+p,x) = u(t,x), \quad (t,x)\in\mathbb{R}\times\mathbb{S}^1, \tag{6.48}$$

where $p>0$ is a constant.

Let $y(t) = u(t\frac{p}{2\pi},0)$. Then using the spatiotemporal relation (6.48) of the rotating waves, we can show that u is a rotating wave if and only if y is a 2π-periodic solution of an NFDE with two parameters (q,p). This two-parameter NFDE is very much similar to (6.24), and a global Hopf bifurcation has been established (see [193] for details). Here we describe how Wu and Xia [306] applied this theory to establish the existence of rotating waves, and how this is related to circulant matrices.

Let $g'(0) = -\gamma$ and assume that $0<\gamma<a$ in what follows. The characteristic equation of (6.47) at the trivial solution takes the form

$$(\lambda+dk^2+a-\gamma)\mathrm{e}^{\lambda\tau} - q(\lambda+dk^2-a-\gamma) = 0, \quad k\geq 1. \tag{6.49}$$

Letting $\lambda = \mathrm{i}\beta$ in (6.49), we get

$$\begin{cases} -(dk^2+a-\gamma)\cos\beta\tau+\beta\sin\beta\tau = q(a+\gamma-dk^2),\\ \beta\cos\beta\tau+(dk^2+a-\gamma)\sin\beta\tau = q\beta,\end{cases}$$

or equivalently,

$$\begin{cases} \tan(\beta\tau) = \dfrac{2\alpha\beta}{\beta^2-(a+dk^2-\gamma)(a-dk^2+\gamma)}, \\ q^2 = \dfrac{\beta^2+(a-\gamma+dk^2)^2}{\beta^2+(a+\gamma-dk^2)^2}. \end{cases} \tag{6.50}$$

It is easy to show that for a real number $\beta > 0$, the second equation of (6.50) has a solution $q \in (0,1)$ only if

$$dk^2 < \gamma. \tag{6.51}$$

Therefore, there are only finitely many $k \geq 1$ such that (6.50) has a pair of purely imaginary solutions.

For each fixed $k \geq 1$ such that $dk^2 < \gamma$, we can easily show graphically that there exists a sequence of positive numbers $\beta_{k,1} < \beta_{k,2} < \cdots$ such that the first equation of (6.50) is satisfied by $\beta_{k,j}, j = 1,2,\ldots$. Substituting this $\beta_{k,j}$ into the second equation of (6.50) gives

$$q_{k,j} = \sqrt{\frac{\beta_{k,j}^2+(a-\gamma+dk^2)^2}{\beta_{k,j}^2+(a+\gamma-dk^2)^2}}. \tag{6.52}$$

Therefore, we can conclude that the set $\{(q,p) \in (0,1)\times(0,\infty)$; (6.49) has a solution $\mathrm{i}(2\pi/p)m$ for some $m \geq 1\}$ is discrete.

Let $\lambda = \lambda(q)$ be a smooth curve of zeros of (6.49) such that $\lambda(q_{k,j}) = \mathrm{i}\beta_{k,j}$. Differentiating (6.49) with respect to q, we get

$$\lambda'(q)\mathrm{e}^{\lambda} + \tau(\lambda+dk^2+a-\gamma)\mathrm{e}^{\lambda\tau}\lambda'(q) = \lambda+dk^2-\gamma-a+q\lambda'(q).$$

That is,

$$\lambda'(q) = \frac{\lambda+dk^2-\gamma-a}{\tau(\lambda+dk^2+a-\gamma)\mathrm{e}^{\lambda\tau}+\mathrm{e}^{\lambda}-q}.$$

This leads to

$$\begin{aligned} &\mathrm{sgnRe}\lambda'(q)|_{q=q_{k,j}} \\ &= \mathrm{sgnRe}\frac{1}{\lambda'(q)}|_{q=q_{k,j}} \\ &= \mathrm{sgn}\left\{\tau + \frac{2a\beta_{k,j}^2}{[(dk^2+a-\gamma)^2+\beta_{k,j}^2][(dk^2-\gamma-a)^2+\beta_{k,j}^2]}\right\} = 1 > 0. \end{aligned}$$

From the definition of the crossing number in Sect. 6.6, we can see that this will be crucial in ruling out bounded connected components of rotating waves of (6.47).

For the sake of later application, let us look at the location of $\beta_0 = \beta_{1,1}$. We assume that

$$0 < d < \gamma. \tag{6.53}$$

Then β_0 is the first positive solution of

$$\tan(\beta\tau) = \frac{2a\beta}{\beta^2 - (a-\gamma+d)(a+\gamma-d)}, \tag{6.54}$$

and hence $i\beta_0$ is a solution of (6.49) with $k=1$ and

$$q_0 = q_{1,1} = \sqrt{\frac{\beta_0^2 + (a-\gamma+d)^2}{\beta_0^2 + (a+\gamma-d)^2}}. \tag{6.55}$$

Lemma 6.8. *If*

$$\frac{\pi}{2\tau} < \sqrt{(a+\gamma-d)(a-\gamma+d)}, \tag{6.56}$$

then $\pi/2\tau < \beta_0 < \sqrt{(a+\gamma-d)(a-\gamma+d)}$, *and hence*

$$\frac{2\pi}{\sqrt{(a+\gamma-d)(a-\gamma+d)}} < \frac{2\pi}{\beta_0} < 4\tau. \tag{6.57}$$

In particular, if

$$\frac{\pi}{2\tau} < \sqrt{(a+\gamma-d)(a-\gamma+d)} < \frac{\pi}{\tau}, \tag{6.58}$$

then

$$2\tau < \frac{2\pi}{\beta_0} < 4\tau. \tag{6.59}$$

In order to apply the global bifurcation theorem to establish the global existence of rotating waves, we need to obtain a priori bounds for rotating waves. Assume that $u(t,x)$ is a rotating wave of (6.47) satisfying (6.48). Let $[u(t_0,x_0) - qu(t_0-\tau,x_0)]^2$ be the maximum value of $[u(t,x) - qu(t-\tau,x)]^2$ over $\mathbb{R}\times\mathbb{S}^1$. Then

$$\begin{aligned}
0 &= \frac{\partial}{\partial t}[u(t_0,x_0) - qu(t_0-\tau,x_0)]^2 \\
&= 2[u(t_0,x_0) - qu(t_0-\tau,x_0)]\frac{\partial}{\partial t}[u(t_0,x_0) - qu(t_0-\tau,x_0)], \\
0 &= \frac{\partial}{\partial x}[u(t_0,x_0) - qu(t_0-\tau,x_0)]^2 \\
&= 2[u(t_0,x_0) - qu(t_0-\tau,x_0)]\frac{\partial}{\partial x}[u(t_0,x_0) - qu(t_0-\tau,x_0)], \\
0 &\le \frac{\partial^2}{\partial x^2}[u(t_0,x_0) - qu(t_0-\tau,x_0)]^2 \\
&= 2\{\frac{\partial}{\partial x}[u(t_0,x_0) - qu(t_0-\tau,x_0)]\}^2 \\
&\quad + 2[u(t_0,x_0) - qu(t_0-\tau,x_0)]\frac{\partial^2}{\partial x^2}[u(t_0,x_0) - qu(t_0-\tau,x_0)].
\end{aligned}$$

Without loss of generality, we may assume that $u(t_0,x_0)-qu(t_0-\tau,x_0)\neq 0$. Therefore, from (6.47) it follows that

$$\begin{aligned}&[u(t_0,x_0)-qu(t_0-\tau,x_0)]\{-au(t_0,x_0)-aqu(t_0-\tau,x_0)\\&-g[u(t_0,x_0)-qu(t_0-\tau,x_0)]\}\geq 0.\end{aligned}$$

That is,

$$\begin{aligned}&-2aqu(t_0-\tau,x_0)[u(t_0,x_0)-qu(t_0-\tau,x_0)]\\&\quad\geq\{a[u(t_0,x_0)-qu(t_0-\tau,x_0)]\\&\qquad+g[u(t_0,x_0)-qu(t_0-\tau,x_0)]\}[u(t_0,x_0)-qu(t_0-\tau,x_0)].\end{aligned}\tag{6.60}$$

Note that

$$|u(t,x)-qu(t-\tau,x)|\leq|u(t_0,x_0)-qu(t_0-\tau,x_0)|,\quad t\in\mathbb{R},\quad x\in\mathbb{S}^1$$

implies

$$|u(t,x)|\leq\frac{1}{1-q}|u(t_0,x_0)-qu(t_0-\tau,x_0)|,\quad t\in\mathbb{R},\quad x\in\mathbb{S}^1.\tag{6.61}$$

Therefore, by (6.60), we obtain

$$a+\frac{g[u(t_0,x_0)-qu(t_0-\tau,x_0)]}{u(t_0,x_0)-qu(t_0-\tau,x_0)}\leq\frac{2aq}{1-q}.\tag{6.62}$$

If we assume that

$$\lim_{z\to\infty}\frac{g(z)}{z}=\infty,\tag{6.63}$$

then (6.62) implies the existence of $Q=Q(2aq/(1-q))$, so that

$$|u(t_0,x_0)-qu(t_0-\tau,x_0)|\leq Q,$$

and hence from (6.61), it follows that

$$|u(t,x)|\leq\frac{1}{1-q}Q(\frac{2aq}{1-q}),\quad t\in\mathbb{R},\quad x\in\mathbb{S}^1.\tag{6.64}$$

Summarizing the above discussion, we get the following.

Lemma 6.9. *If (6.63) is satisfied, then there exists a nondecreasing function $Q:(0,\infty)\to(0,\infty)$ such that every rotating wave $u(t,x)$ of (6.47) satisfies $|u(t,x)|\leq(1/(1-q)Q(2aq/(1-q))$ for $t\in\mathbb{R}$ and $x\in\mathbb{S}^1$. In particular, for fixed $q^*\in(0,1)$, the set of rotating waves of (6.47) corresponding to $q\in[0,q^*)$ is uniformly bounded in the sup-norm.*

Now we try to exclude nontrivial 4τ-periodic rotating waves. Assume that $u(t,x)$ is a nontrivial rotating wave of (6.47) satisfying (6.48) with $p=4\tau$. Then

$$u(t,0)=u(t+4\tau,0),$$
$$u(t,x)=u(t-\tfrac{4\tau}{2\pi}x,0)=u(t-\tfrac{2}{\pi}x,0),\quad t\in\mathbb{R},\quad x\in\mathbb{S}^1.$$

So, $v(t)=u(t,0)$ satisfies

$$\begin{aligned}&\frac{d}{dt}[v(t)-qv(t-\tau)]\\ &=\left(\frac{2\tau}{\pi}\right)^2 d\frac{\mathrm{d}^2}{\mathrm{d}t^2}[v(t)-qv(t-\tau)]\\ &\quad -a[v(t)-qv(t-\tau)]-2aqv(t-\tau)-g[v(t)-qv(t-\tau)],\end{aligned} \tag{6.65}$$

$t\in\mathbb{R}$. Let

$$\begin{cases} x_1(t)=v(t)-qv(t-\tau),\\ x_2(t)=v(t-\tau)-qv(t-2\tau),\\ x_3(t)=v(t-2\tau)-qv(t-3\tau),\\ x_4(t)=v(t-3\tau)-qv(t).\end{cases} \tag{6.66}$$

Then

$$\begin{pmatrix} v(t-\tau)\\ v(t-2\tau)\\ v(t-3\tau)\\ v(t)\end{pmatrix}=\frac{1}{1-q^4}B\begin{pmatrix} x_1(t)\\ x_2(t)\\ x_3(t)\\ x_4(t)\end{pmatrix}, \tag{6.67}$$

where we have the following circulant matrix:

$$B=\begin{pmatrix} q^3 & 1 & q & q^2\\ q^2 & q^3 & 1 & q\\ q & q^2 & q^3 & 1\\ 1 & q & q^2 & q^3\end{pmatrix}.$$

Substituting (6.66) and (6.67) into (6.65), we get

$$\dot{x}_i=\left(\frac{2\tau}{\pi}\right)^2 d\ddot{x}_i-ax_i-\frac{2aq}{1-q^4}(Bx)_i-g(x_i),\quad 1\le i\le 4.$$

Its similarity to the Liénard equation suggests a transformation that leads to an equivalent system,

$$\begin{cases}\dot{x}_i=y_i+\left(\frac{\pi}{2\tau}\right)^2\frac{1}{d}x_i,\\ \dot{y}_i=\left(\frac{\pi}{2\tau}\right)^2\frac{1}{d}\left[ax_i+\frac{2aq}{1-q^4}(Bx)_i+g(x_i)\right],\quad 1\le i\le 4,\end{cases} \tag{6.68}$$

and a related Lyapunov function,

$$V=\sum_{i=1}^{4}\left[\frac{1}{2}y_i^2-\left(\frac{\pi}{2\tau}\right)^2\frac{1}{d}\int_0^{x_i}g(s)ds-ax_iy_i-\frac{2aq}{1-q^4}y_i(Bx)_i\right].$$

The derivative of V along solutions of (6.68) is given by

$$\begin{aligned}
\dot{V} =& -a\sum_{i=1}^{4} y_i^2 - \frac{2aq}{1-q^4}\sum_{i=1}^{4} y_i(By)_i \\
&- \left[(\frac{\pi}{2\tau})^2\frac{1}{d}\right]^2 \sum_{i=1}^{4} x_i g(x_i) - \frac{a}{d}\left(\frac{\pi}{2\tau}\right)^2 \sum_{i=1}^{4} x_i g(x_i) \\
&- \frac{a^2}{d}\sum_{i=1}^{4}\left(\frac{\pi}{2\tau}\right)^2 x_i^2 - \frac{4a^2q}{1-q^4}\left(\frac{\pi}{2\tau}\right)^2\frac{1}{d}\sum_{i=1}^{4} x_i(Bx)_i \\
&- \left(\frac{2aq}{1-q^4}\right)^2\left(\frac{\pi}{2\tau}\right)^2\frac{1}{d}\sum_{i=1}^{4}(Bx)_i(Bx)_i - \frac{2aq}{1-q^4}\left(\frac{\pi}{2\tau}\right)^2\frac{1}{d}\sum_{i=1}^{4} g(x_i)(Bx)_i.
\end{aligned}$$

We need the following lemma.

Lemma 6.10. $\sum_{i=1}^4 z_i(Bz)_i \geq -(1-q)(1+q^2)\sum_{i=1}^4 z_i^2, \quad z_i \in \mathbb{R}, \quad 1 \leq i \leq 4.$

Proof. Using the aforementioned spectral property of circulant matrices, we have $\sum_{i=1}^4 z_i(Bz)_i \geq \Gamma\sum_{i=1}^4 z_i^2, z_i \in \mathbb{R}, \leq i \leq 4$, where

$$\begin{aligned}
\Gamma &= \min\{\mathrm{Re}\sum_{j=1}^{4} a_j z^{j-1} : z^4 = 1, a_1 = q^3, a_2 = 1, a_3 = q, a_4 = q^2\} \\
&= \min\{\mathrm{Re}\big(q^3 + e^{\mathrm{i}(2\pi/4)j} + qe^{\mathrm{i}(4\pi/4)j} + q^2e^{\mathrm{i}(6\pi/4)j} : j = 0,1,2,3\} \\
&= \min\{(1+q)(1+q^2), -q(1-q^2), -(1-q)(1+q^2)\} \\
&= -(1-q)(1+q^2).
\end{aligned}$$

□

We also need to compute the eigenvalues of B^TB. While this can be done directly, Wu and Xia [306] have presented an approach that can be extended to general circular matrices.

Lemma 6.11. *The minimal eigenvalue of B^TB is $\lambda_{\min}(B^TB) = (1-q^4)^2/(1+q)^2$, and the maximal eigenvalue of B^TB is $\lambda_{\max}(B^TB) = (1-q^4)^2/(1-q)^2$.*

Proof. Let

$$v_j = (1, e^{\mathrm{i}(\pi/2)j}, e^{\mathrm{i}(2\pi/2)j}, e^{\mathrm{i}(3\pi/2)j}), j = 0,1,2,3.$$

It can be shown that v_j is an eigenvector of B corresponding to the eigenvalue

$$\alpha_j = e^{\mathrm{i}(\pi/2)j}(1 + qe^{\mathrm{i}(\pi/2)j} + q^2e^{\mathrm{i}(2\pi/2)j} + q^3e^{\mathrm{i}(3\pi/2)j}) = e^{\mathrm{i}(\pi/2)j}\frac{1-q^4}{1-qe^{\mathrm{i}(\pi/2)j}}$$

and an eigenvector of B^T corresponding to the eigenvector

$$\beta_j = e^{-\mathrm{i}(\pi/2)j}(1 + qe^{-\mathrm{i}(\pi/2)j} + q^2e^{-\mathrm{i}(2\pi/2)j} + q^3e^{-\mathrm{i}(3\pi/2)j}) = e^{-\mathrm{i}(\pi/2)j}\frac{1-q^4}{1-qe^{-\mathrm{i}(\pi/2)j}}.$$

Assume that $x \in \mathbb{C}^4$ is an eigenvector of $B^T B$ corresponding to an eigenvalue $\lambda \in \mathbb{C}$. Then $x = a_0 v_0 + a_1 v_1 + a_2 v_2 + a_3 v_3$, and $B^T Bx = \lambda x$ is equivalent to

$$\sum_{j=0}^{3} \alpha_j \beta_j a_j v_j = \lambda \sum_{j=0}^{3} a_j v_j,$$

from which it follows that $\lambda = \alpha_j \beta_j$ for some $j = 0, 1, 2, 3$. Therefore, all eigenvalues of $B^T B$ are given by

$$\frac{(1-q^4)^2}{(1-qe^{\mathrm{i}(\pi/2)j})(1-qe^{-\mathrm{i}(\pi/2)j})}, \; j = 0, 1, 2, 3,$$

from which the conclusion follows. □

We also note the following result; see [306]

Lemma 6.12. *Assume that*

$$-K \le \frac{g(x)}{x}, \quad g(-x) = -g(x) \quad for \quad x \neq 0, \tag{6.69}$$

$$\frac{g(x)}{x} \quad \text{is nondecreasing in} \quad x \in (0, \infty). \tag{6.70}$$

Let $x_i(t)$, $i = 1, \ldots, 4$, be given by (6.66). Then

$$\left| \frac{g(x_i(t))}{x_i(t)} \right| \le \max \left\{ K, \frac{a(3q-1)}{1-q} \right\}. \tag{6.71}$$

We now return to the estimation of $\dot{V}$. Using Lemma 6.11, we get

$$\sum_{i=1}^{4} (Bx)_i (Bx)_i \ge \lambda_{\min}(B^T B) \sum_{i=1}^{4} x_i^2 = \frac{(1-q^4)^2}{(1+q)^2} \sum_{i=1}^{4} x_i^2.$$

By Lemma 6.12, we have

$$\begin{aligned} \left| \sum_{i=1}^{4} g(x_i)(Bx)_i \right| &\le \sqrt{\sum_{i=1}^{4} g^2(x_i)} \sqrt{\sum_{i=1}^{4} x_i^2 \lambda_{\max}(B^T B)} \\ &\le \frac{(1-q^4)^2}{(1+q)^2} \max \left\{ K, \frac{a(3q-1)}{1-q} \right\} \sum_{i=1}^{4} x_i^2. \end{aligned}$$

Therefore, using Lemma 6.10, we get

$$\begin{aligned} \dot{V} \le &-a \sum_{i=1}^{4} \left[1 - \frac{2q(1-q)(1+q^2)}{1-q^4} \right] y_i \\ &- \left(\frac{\pi}{2\tau} \right)^2 \frac{1}{d} \left\{ (\frac{1}{d} \left(\frac{\pi}{2\tau} \right)^2 + a \right) \sum_{i=1}^{4} x_i g(x_i) \\ &- \sum_{i=1}^{4} \frac{1}{d} \left(\frac{\pi}{2\tau} \right)^2 \left\{ a^2 \left[1 - \frac{4q}{1-q^4} (1-q)(1+q^2) \right. \right. \end{aligned}$$

$$
\begin{aligned}
&+ \frac{4q^2}{(1-q^4)^2}\frac{(1-q^4)^2}{(1+q)^2}\bigg] x_i^2 \\
&- \frac{2aq}{1-q^4}\frac{(1-q^4)^2}{(1-q)^2}\max\left\{K, \frac{a(3q-1)}{1-q}\right\}\sum_{i=1}^{4} x_i^2\bigg\} \\
\le & -a\sum_{i=1}^{4}\left(\frac{1-q}{1+q}\right) y_i^2 \\
&-\left(\frac{\pi}{2\tau}\right)^2\frac{1}{d}\sum_{i=1}^{4}\left[a^2\left(\frac{1-q}{1+q}\right)^2-\left(\frac{1}{d}\left(\frac{\pi}{2\tau}\right)^2+a\right)K\right. \\
&\left.-\frac{2aq(1+q)(1+q^2)}{1-q}\max\left\{K, \frac{a(3q-1)}{1-q}\right\}\right] x_i^2.
\end{aligned}
$$

Consequently, if we assume that

$$
0 \le q < 1-\delta \quad \text{for some} \quad \delta \in (0,1), \tag{6.72}
$$

$$
\frac{1}{4}a^2\delta^2 > \left[\frac{1}{d}(\frac{\pi}{2\tau})^2 + a\right]K + \frac{8a(1-\delta)\max\left\{K, \frac{4a}{\delta}\right\}}{\delta}, \tag{6.73}
$$

then $\dot{V}$ is a strictly negative function of $(x_1,\dots,x_4,y_1,\dots,y_4)$ unless $x_i = y_i = 0$ for $1 \le i \le 4$. Therefore, under assumptions (6.69), (6.70), (6.72), and (6.73), system (6.68) has no nontrivial periodic solution. This implies that system (6.47) has no nontrivial rotating wave of period 4τ. That is, we have proved the following.

Lemma 6.13. *Under assumptions (6.69), (6.70), (6.72), and (6.73), the partial neutral functional differential equation (6.47) has no nontrivial 4π-periodic rotating wave for $q \in [0, 1-\delta)$.*

We can then use global bifurcation theory to obtain the following result, for which we refer to [306] for more details of the proof.

Theorem 6.14. *Assume that*

(i) $g'(0) = -\gamma, d < \gamma < a, \pi/2 < \sqrt{(a+\gamma-d)(a-\gamma+d)}$;
(ii) $\inf_{y\neq 0} g(y)/y > -a, \lim_{y\to\infty} g(y)/y = \infty$;
(iii) $g(-y) = -g(y)$ *for* $y \in \mathbb{R}$ *and* $g(y)/y$ *is nondecreasing in* $y \in (0,\infty)$;
(iv) *there exist constants* $\delta \in (0,1)$ *and* $K \ge 0$ *such that*

$$
\begin{gathered}
-K \le g(x)/x \text{ for } x \neq 0, \\
\tfrac{1}{4}a^2\delta^2 > \left[\tfrac{1}{d}\left(\tfrac{\pi}{2}\right)^2 + a\right]K + 8a\left(\tfrac{1-\delta}{\delta}\right)\max\left\{K, \tfrac{4a}{\delta}\right\},
\end{gathered}
$$

and

$$
q_0 \overset{\text{def}}{=} \sqrt{\frac{\beta_0^2 + (a+\gamma-d)^2}{\beta_0^2 + (a-\gamma+d)^2}} < 1-\delta,
$$

where β_0 *is the first solution in* $((\pi/2\tau), \sqrt{(a+\gamma-d)(a-\gamma+d)})$ *of the equation*

$$\tan(\beta\tau) = \frac{2a\beta}{\beta^2-(a+\gamma-d)(a-\gamma+d)}.$$

Then for each $q \in (q_0, 1-\delta)$*, system (6.47) has a rotating wave with a period less than* 4*. If, in addition, we assume*

(iv)
$$\frac{\pi}{2} < \sqrt{(a+\gamma-d)(a-\gamma+d)} < \frac{\pi}{\tau},$$

then for each $q \in (q_0, 1-\delta)$*, system (6.47) has a slowly oscillating rotating wave, that is, a rotating wave with a period in* $(2\tau, 4\tau)$*.*

6.9 State-Dependent DDEs

State-dependent DDEs arises from a number of applications such as electrodynamics, automatic and remote control, machine cutting, neutral networks, population biology, mathematical epidemiology, and economics. They describe the evolution of systems in which the rate of change depends on the history of the rate, and the delay depends on the system's status in a complicated manner, such as by an explicit or implicit algebraic equation or a differential or integral equation.

Early results on the existence of periodic solutions for state-dependent DDEs include work by Smith [269] that considered bifurcations of periodic solutions from a stationary state for a system of integral equations with state-dependent delay, and work on the existence of periodic solutions by Nussbaum, Mallet-Paret, and Paraskevopoulos [215]. These studies address the aspect of global continuation of Hopf bifurcations of periodic solutions, especially the existence of periodic solutions in which the bifurcation parameter is away from the critical value where a local Hopf bifurcation is born. The work of Nussbaum et al. [215] focuses on important prototype classes of state-dependent delay differential equations with negative feedback and provides some detailed information on slowly oscillating periodic solutions. Here we introduce the work [170, 171] to provide a general tool and framework for studying the Hopf bifurcation problem, and in particular, the global continuation of local bifurcation of periodic solutions of the following parameterized state-dependent DDEs from an equivariant-degree point of view:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{\tau}(t) \end{pmatrix} = \begin{pmatrix} f(x(t), x(t-\tau(t)), \sigma) \\ g(x(t), \tau(t), \sigma) \end{pmatrix}, \tag{6.74}$$

where $x \in \mathbb{R}^N$, $\tau \in \mathbb{R}$, $t \in \mathbb{R}$ and $\sigma \in \mathbb{R}$, $f : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$, and $g : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given maps. A stationary state of (6.74) with parameter σ is a vector $(x, \tau) \in \mathbb{R}^N \times \mathbb{R}$ such that $f(x, x, \sigma) = 0$ and $g(x, \tau, \sigma) = 0$.

The major problem in developing such a global Hopf bifurcation theory for the system of state-dependent DDEs (6.74) is that in the spaces of continuous periodic functions $C_T(\mathbb{R};\mathbb{R}^N) = \{x \in C(\mathbb{R};\mathbb{R}^N) : x(t+T) = x(t) \text{ for all } t \in \mathbb{R}\}$ and $C_T(\mathbb{R};\mathbb{R}) = \{\tau \in C(\mathbb{R};\mathbb{R}) : \tau(t+T) = \tau(t) \text{ for all } t \in \mathbb{R}\}$ with a fixed period $T > 0$, the composition operator

$$\begin{aligned} \chi : C_T(\mathbb{R};\mathbb{R}^N) \times C_T(\mathbb{R};\mathbb{R}) \to C_T(\mathbb{R};\mathbb{R}^N), \\ \chi(x, \tau)(t) = x(t-\tau(t)), t \in \mathbb{R}, \end{aligned} \tag{6.75}$$

is generally not a C^1 (continuously differentiable) map with respect to τ in the supremum norm. This causes difficulty in formulating linearization at a stationary state, and such a linearization is usually necessary in the functional-analytic setting for Hopf bifurcation problems in which a topological index such as an $\mathbb{S}^1$-equivariant degree can be calculated and applied to investigate the birth and continuation of periodic solutions bifurcating from a stationary state.

In [72], a system of auxiliary equations obtained through a formal linearization technique was used in the study of local stability of state-dependent DDEs in the space of continuously differentiable functions. This formal linearization technique is only heuristic and can be described in the following way: the state-dependent delay $\tau(t)$ in $x(t-\tau(t))$ is first fixed at a given stationary state, and then the resulting nonlinear system with frozen constant delay is linearized. Other applications of systems of auxiliary equations obtained through a formal linearization process can be found in [26, 45, 156] and [157]. None of these results is sufficient for us to develop a global Hopf bifurcation theory based on the $\mathbb{S}^1$-equivariant degree for state-dependent DDEs (6.74). However, the above-mentioned results strongly indicate that the system of auxiliary equations obtained through the heuristic technique of formal linearization can be used to detect the local Hopf bifurcation and to describe its global continuation for state-dependent DDEs.

In this section, we use the homotopy invariance property of the $\mathbb{S}^1$-equivariant degree to relate the Hopf bifurcation problem of (6.74) to the change of stability of stationary states of the corresponding system of auxiliary equations obtained through formal linearization. As such, much of the effort has been dedicated to justifying that the detection of Hopf bifurcation can be achieved through the formal linearization technique: the state-dependent delay $\tau(t)$ in $x(t-\tau(t))$ is first fixed at a given stationary state; then the resulting nonlinear system with frozen constant delay is linearized. This linearization technique is used in the functional-analytic setting that converts the Hopf bifurcation problem of system (6.74) to solving an operator equation (6.13) involving $\mathbb{S}^1$-equivariant maps with two parameters, in the space of periodic functions with a fixed period. Implicitly used is the C^1-smoothness of the operator defined in Lemma 6.17 in the space $\mathbb{E}$ (the space of periodic functions with fixed period 2π). The formal linearization leads to this operator naturally in the space of continuously differentiable periodic functions with period 2π, and the fact that this operator can be extended to a bounded operator in the space $\mathbb{E}$ is essential in our homotopy argument. This technique of extending the linearized operator of a state-dependent delay differential equation from the space C^1 to the space C has previously been used in other contexts; see, for example, Mallet-Paret–Nussbaum–Paraskevopoulos [215], Krisztin [196], Walther [290], and the survey paper by Hartung–Krisztin–Walther–Wu [158].

6.9.1 Local Hopf Bifurcation

We turn to the Hopf bifurcation of (6.74), with its solution denoted by $u(t) = (x(t), \tau(t))$. Denote by $C(\mathbb{R};\mathbb{R}^N)$ the normed space of continuous functions from $\mathbb{R}$ to $\mathbb{R}^N$ equipped with the usual supremum norm $\|x\| = \sup_{t\in\mathbb{R}} |x(t)|$ for $x \in C(\mathbb{R};\mathbb{R}^N)$, where $|\cdot|$ denotes the Euclidean norm. We also denote by $C^1(\mathbb{R};\mathbb{R}^N)$ the normed space of continuously differentiable bounded functions from $\mathbb{R}$ to $\mathbb{R}^N$ equipped with the usual C^1 norm

$$\|x\|_{C^1} = \max\{\sup_{t\in\mathbb{R}} |x(t)|, \sup_{t\in\mathbb{R}} |\dot{x}(t)|\}$$

for $x \in C(\mathbb{R};\mathbb{R}^N)$. For a stationary state (u_0, τ_0) of (6.74) with the parameter σ_0, we say that (u_0, σ_0) is a *Hopf bifurcation point* of system (6.74) if there exist a sequence $\{(u_k, \sigma_k, T_k)\}_{k=1}^{+\infty} \subseteq C(\mathbb{R};\mathbb{R}^{N+1}) \times \mathbb{R}^2$ and $T_0 > 0$ such that

$$\lim_{k\to+\infty} \|(u_k, \sigma_k, T_k) - (u_0, \sigma_0, T_0)\|_{C(\mathbb{R};\mathbb{R}^{N+1})\times\mathbb{R}^2} = 0,$$

and (u_k, σ_k) is a nonconstant T_k-periodic solution of system (6.74).

We assume that:

(SHB1) The map $f\colon \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \ni (\theta_1, \theta_2, \sigma) \to f(\theta_1, \theta_2, \sigma) \in \mathbb{R}^N$ and the map $g\colon \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \ni (\gamma_1, \gamma_2, \sigma) \to g(\gamma_1, \gamma_2, \sigma) \in \mathbb{R}$ are C^2 (twice continuously differentiable).

(SHB2) There exists $L_0 > 0$ such that $g(\gamma_1, \gamma_2, \sigma) < \frac{L_0}{L_0+1}$ for $\gamma_1 \in \mathbb{R}^N$, $\gamma_2 \in \mathbb{R}$, $\sigma \in \mathbb{R}$.

In what follows, we write $\partial_i f = \frac{\partial}{\partial \theta_i} f$ for $i = 1, 2$, and similarly we define $\partial_i g$ for $i = 1, 2$.

We shall study the Hopf bifurcation of (6.74) through its formal linearization. We assume that for a fixed $\sigma_0 \in \mathbb{R}$, $(x_{\sigma_0}, \tau_{\sigma_0})$ (or abusing notation, $(x_{\sigma_0}, \tau_{\sigma_0}, \sigma_0)$) is a stationary state of (6.74). That is,

$$f(x_{\sigma_0}, x_{\sigma_0}, \sigma_0) = 0, \quad g(x_{\sigma_0}, \tau_{\sigma_0}, \sigma_0) = 0.$$

We also assume that

(SHB3) $(\frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2}) f(\theta_1, \theta_2, \sigma)|_{\sigma=\sigma_0, \theta_1=\theta_2=x_{\sigma_0}}$ is nonsingular and

$$\frac{\partial}{\partial\gamma_2} g(\gamma_1, \gamma_2, \sigma)|_{\sigma=\sigma_0, \gamma_1=x_{\sigma_0}, \gamma_2=\tau_{\sigma_0}} \neq 0.$$

This assumption implies that there exist $\varepsilon_0 > 0$ and a C^1-smooth curve $(\sigma_0 - \varepsilon_0,\, \sigma_0 + \varepsilon_0) \ni \sigma \mapsto (x_\sigma,\, \tau_\sigma) \in \mathbb{R}^{N+1}$ such that $(x_\sigma,\, \tau_\sigma)$ is the unique stationary state of (6.74) in a small neighborhood of $(x_{\sigma_0},\, \tau_{\sigma_0})$ for σ close to σ_0.

We now consider, for $\sigma \in (\sigma_0 - \varepsilon_0,\, \sigma_0 + \varepsilon_0)$, the following formal linearization of system (6.74) at the stationary point $\eta(\sigma) = (x_\sigma, z_\sigma)$:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{\tau}(t) \end{pmatrix} = \begin{bmatrix} \partial_1 f(\sigma) & 0 \\ \partial_1 g(\sigma) & \partial_2 g(\sigma) \end{bmatrix} \begin{pmatrix} x(t) - x_\sigma \\ \tau(t) - \tau_\sigma \end{pmatrix} + \begin{bmatrix} \partial_2 f(\sigma) & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x(t-\tau_\sigma) - x_\sigma \\ \tau(t-\tau_\sigma) - \tau_\sigma \end{pmatrix}, \tag{6.76}$$

where

$$\partial_1 f(\sigma) \stackrel{\text{def}}{=} \partial_1 f(x_\sigma,\, \tau_\sigma,\, \sigma),\, \partial_2 f(\sigma) \stackrel{\text{def}}{=} \partial_2 f(x_\sigma,\, \tau_\sigma,\, \sigma),$$
$$\partial_1 g(\sigma) \stackrel{\text{def}}{=} \partial_1 g(x_\sigma,\, \tau_\sigma,\, \sigma),\ \partial_2 g(\sigma) \stackrel{\text{def}}{=} \partial_2 g(x_\sigma,\, \tau_\sigma,\, \sigma).$$

Then we obtain the following characteristic equation of the linear system corresponding to the inhomogeneous linear system (6.76):

$$\det \Delta_{(x_\sigma,\, \tau_\sigma,\, \sigma)}(\omega) = 0, \tag{6.77}$$

where $\Delta_{(x_\sigma,\, \tau_\sigma,\, \sigma)}(\omega)$ is an $(N+1) \times (N+1)$ complex matrix defined by

$$\Delta_{(x_\sigma,\, \tau_\sigma,\, \sigma)}(\omega) = \omega I - \begin{bmatrix} \partial_1 f(\sigma) & 0 \\ \partial_1 g(\sigma) & \partial_2 g(\sigma) \end{bmatrix} - \begin{bmatrix} \partial_2 f(\sigma) & 0 \\ 0 & 0 \end{bmatrix} \mathrm{e}^{-\omega \tau_\sigma}. \tag{6.78}$$

A solution ω_0 to the characteristic equation (6.77) is called a *characteristic value* of the stationary state $(x_{\sigma_0},\, \tau_{\sigma_0},\, \sigma_0)$. We call $(x_{\sigma_0},\, \tau_{\sigma_0},\, \sigma_0)$ a nonsingular stationary state if and only if zero is not a characteristic value of $(x_{\sigma_0},\, \tau_{\sigma_0},\, \sigma_0)$. Here $(x_{\sigma_0},\, \tau_{\sigma_0},\, \sigma_0)$ is a *center* if the set of nonzero purely imaginary characteristic values of $(x_{\sigma_0},\, \tau_{\sigma_0},\, \sigma_0)$ is nonempty and discrete. We call $(x_{\sigma_0},\, \tau_{\sigma_0},\, \sigma_0)$ an *isolated center* if it is the only center in some neighborhood of $(x_{\sigma_0},\, \tau_{\sigma_0},\, \sigma_0)$ in $\mathbb{R}^{N+1} \times \mathbb{R}$.

If $(x_{\sigma_0},\, \tau_{\sigma_0},\, \sigma_0)$ is an isolated center of (6.76), then there exist $\beta_0 > 0$ and $\delta \in (0,\, \varepsilon_0)$ such that

$$\det \Delta_{(x_{\sigma_0},\, \tau_{\sigma_0},\, \sigma_0)}(\mathrm{i}\beta_0) = 0, \quad \det \Delta_{(x_\sigma,\, \tau_\sigma,\, \sigma)}(\mathrm{i}\beta) \neq 0, \tag{6.79}$$

for every $\sigma \in (\sigma_0 - \delta,\, \sigma_0) \cup (\sigma_0,\, \sigma_0 + \delta)$ and $\beta \in (0, +\infty)$. Hence, we can choose constants $\alpha_0 = \alpha_0(\sigma_0, \beta_0) > 0$ and $\varepsilon = \varepsilon(\sigma_0, \beta_0) > 0$ such that the closure of the set $\Omega \stackrel{\text{def}}{=} (0, \alpha_0) \times (\beta_0 - \varepsilon, \beta_0 + \varepsilon) \subset \mathbb{R}^2 \cong \mathbb{C}$ contains no other zero of $\det \Delta_{(x_{\sigma_0},\, \tau_{\sigma_0},\, \sigma_0)}(\cdot)$. The quantity $p_0 = 2\pi/\beta_0$ is called the *virtual period* associated with the center $(x_{\sigma_0},\, \tau_{\sigma_0},\, \sigma_0)$. We note that $\det \Delta_{(x_\sigma,\, \tau_\sigma,\, \sigma)}(\omega)$ is analytic in ω and is continuous in σ. If $\delta > 0$ is small enough, then there is no zero of $\det \Delta_{(x_{\sigma_0 \pm \delta},\, \tau_{\sigma_0 \pm \delta},\, \sigma_0 \pm \delta)}(\omega)$ in $\partial\Omega$. So we can define the number

$$\gamma_\pm(x_{\sigma_0},\, \tau_{\sigma_0},\, \sigma_0,\, \beta_0) = \deg_B(\det \Delta_{(x_{\sigma_0 \pm \delta},\, \tau_{\sigma_0 \pm \delta},\, \sigma_0 \pm \delta)}(\cdot),\, \Omega),$$

and the crossing number of $(x_{\sigma_0}, \tau_{\sigma_0}, \sigma_0, \beta_0)$ as

$$\gamma(x_{\sigma_0}, \tau_{\sigma_0}, \sigma_0, \beta_0) = \gamma_-(x_{\sigma_0}, \tau_{\sigma_0}, \sigma_0, \beta_0) - \gamma_+(x_{\sigma_0}, \tau_{\sigma_0}, \sigma_0, \beta_0). \tag{6.80}$$

This crossing number counts the number of characteristic values (with multiplicities) escaping from the region Ω as α increases and crosses α_0. Define the function $H : [\sigma_0 - \delta, \sigma_0 + \delta] \times \overline{\Omega} \to \mathbb{R}^2 \simeq \mathbb{C}$ by

$$H(\alpha, u, \beta) \stackrel{\text{def}}{=} \det \Delta_{(x_\sigma, \tau_\sigma, \sigma)}(u + \mathrm{i}\beta),$$

and

$$\deg(\Psi_H, \mathscr{D}(\sigma_0, \beta_0)) = \gamma(x_{\sigma_0}, \tau_{\sigma_0}, \sigma_0, \beta_0), \tag{6.81}$$

where Ψ_H: $\mathscr{D}(\sigma_0, \beta_0) \to \mathbb{R}^2 \simeq \mathbb{C}$ is defined by $\Psi_H(\sigma, \beta) = \det \Delta_{(x_\sigma, \tau_\sigma, \sigma)}(\mathrm{i}\beta)$ and $\mathscr{D}(\sigma_0, \beta_0) = (\sigma_0 - \delta, \sigma_0 + \delta) \times (\beta_0 - c, \beta_0 + c)$.

Let $\mathbb{E} \stackrel{\text{def}}{=} C_{2\pi}(\mathbb{R}; \mathbb{R}^n)$ be the normed space of continuous 2π-periodic functions from $\mathbb{R}$ to $\mathbb{R}^n$ equipped with the usual supremum norm. Then $\mathbb{S}^1$ acts on $\mathbb{E}$ by argument shift. Namely, for $\xi = e^{\mathrm{i}\nu} \in \mathbb{S}^1$, $u \in \mathbb{E}$, $(\xi u)(t) \stackrel{\text{def}}{=} u(t + \nu)$. For the isotypical direct sum decomposition (6.12) of $\mathbb{E}$, we see that $\mathbb{E}_0 \cong \mathbb{R}^n$ and for each $k \geq 1$, $\mathbb{E}_k$ is spanned by $\cos(kt)\varepsilon_j$ and $\sin(kt)\varepsilon_j$, $1 \leq j \leq n$, where $\{\varepsilon_1, \ldots, \varepsilon_n\}$ is the standard basis of $\mathbb{R}^n$. Therefore, $\mathbb{E}_k$, $k \geq 0$, are real $2n$-dimensional and so satisfy (SD1) of Sect. 6.5. To formulate the Hopf bifurcation problem as a fixed-point problem in the space of continuous functions of period 2π, we normalize the period of the $2\pi/\beta$-periodic solution (x, τ) in (6.74) by $(x(t), \tau(t)) = (y(\beta t), z(\beta t))$ and obtain

$$\dot{u}(t) = Q(u, \sigma, \beta)(t), \tag{6.82}$$

where $u = (y, z)^T$ and $(\sigma, \beta) \in \mathscr{D}(\sigma_0, \beta_0)$, and

$$Q(u, \sigma, \beta)(t) = \frac{1}{\beta} \begin{bmatrix} f(y(t), y(t - \beta z(t)), \sigma) \\ g(y(t), z(t), \sigma) \end{bmatrix}.$$

Correspondingly, (6.76) is transformed into

$$\dot{u}(t) = \tilde{Q}(u, \sigma, \beta)(t), \tag{6.83}$$

where $\tilde{Q}$: $\mathbb{E} \times \mathscr{D}(\sigma_0, \beta_0) \to \mathbb{E}$ is defined by

$$\tilde{Q}(u, \sigma, \beta)(t) = \frac{1}{\beta} \begin{bmatrix} \partial_1 f(\sigma)(y(t) - y_\sigma) + \partial_2 f(\sigma)(y(t - \beta z_\sigma) - y_\sigma) \\ \partial_1 g(\sigma)(y(t) - y_\sigma) + \partial_2 g(\sigma)(z(t) - z_\sigma) \end{bmatrix}$$

and $(y_\sigma, z_\sigma) = \eta(\sigma) = (x_\sigma, \tau_\sigma)$.

Before we state and prove the local Hopf bifurcation theorem, we need some technical preparations. We denote by $C^1_{2\pi}(\mathbb{R}; \mathbb{R}^{N+1})$ the Banach space of 2π-periodic and continuously differentiable functions equipped with the C^1 norm

$$\|x\|_{C^1} = \max\{\sup_{t \in [0, 2\pi]} |x(t)|, \sup_{t \in [0, 2\pi]} |\dot{x}(t)|\}.$$

Lemma 6.14. *Let* $L\colon C^1_{2\pi}(\mathbb{R};\mathbb{R}^{N+1}) \to \mathbb{E}$ *and* $K\colon \mathbb{E} \to \mathbb{R}^{N+1}$ *be defined by*

$$Lu(t) = \dot{u}(t), \quad Ku(t) = \frac{1}{2\pi}\int_0^{2\pi} u(t)dt$$

for $t \in \mathbb{R}$*. Then* $L+K$ *has a compact inverse* $(L+K)^{-1} : \mathbb{E} \to \mathbb{E}$*, which is given by*

$$(L+K)^{-1}(v)(t) = \int_0^t v(s)ds + \frac{1}{2\pi}\int_0^{2\pi}(1-\pi-t+s)\,v(s)ds.$$

This lemma can be found in [170] and is omitted here. It follows from Lemma 6.14 that $(L+K)^{-1}\circ[Q(\cdot,\alpha,\beta)+K] : \mathbb{E}\to\mathbb{E}$ and $(L+K)^{-1}\circ[\tilde{Q}(\cdot,\alpha,\beta)+K] : \mathbb{E}\to\mathbb{E}$ are completely continuous. That is, (SD2) and (SD4) are satisfied. Thus, finding a $2\pi/\beta$-periodic solution for the system (6.74) is equivalent to finding a solution of the following fixed-point problem:

$$u = (L+K)^{-1}\left[Q(u,\sigma,\beta)+K(u)\right], \tag{6.84}$$

where $(u,\sigma,\beta)\in\mathbb{E}\times\mathbb{R}\times\mathbb{R}_+$. Define the maps $\mathscr{F} : \mathbb{E}\times\mathbb{R}\times\mathbb{R}_+ \to \mathbb{E}$ and $\tilde{\mathscr{F}} : \mathbb{E}\times\mathbb{R}\times\mathbb{R}_+ \to \mathbb{E}$ by

$$\mathscr{F}(u,\sigma,\beta) \stackrel{\text{def}}{=} u-(L+K)^{-1}\left[Q(u,\sigma,\beta)+K(u)\right],$$
$$\tilde{\mathscr{F}}(u,\sigma,\beta) \stackrel{\text{def}}{=} u-(L+K)^{-1}\left[\tilde{Q}(u,\sigma,\beta)+K(u)\right],$$

which are equivariant compact fields. Finding a $2\pi/\beta$-periodic solution of system (6.74) is equivalent to finding the solution of the problem

$$\mathscr{F}(u,\sigma,\beta)=0, \quad (u,\sigma,\beta)\in\mathbb{E}\times\mathbb{R}\times\mathbb{R}_+.$$

It is an easy exercise to verify the following results.

Lemma 6.15. *For* $\sigma\in\mathbb{R}$ *and* $\beta>0$*, the map* $Q(\cdot,\sigma,\beta):\mathbb{E}\to\mathbb{E}$ *defined by (6.82) is continuous.*

Lemma 6.16. *If system (6.76) has a nonconstant periodic solution with period* $T>0$*, then there exists an integer* $m\geq 1$*,* $m\in\mathbb{N}$ *such that* $\pm \mathrm{i}m2\pi/T$ *are characteristic values of the stationary state* $(x_\sigma,\tau_\sigma,\sigma)$*.*

Lemma 6.17. *Assume* (SHB1)–(SHB3) *hold. If* $B_M(u_0,\sigma_0,\beta_0;r,\rho)\subseteq\mathbb{E}\times\mathbb{R}^2$ *is a special neighborhood of* $\tilde{\mathscr{F}}$*, where* $0<\rho<\beta_0$*, then there exists* $r'\in(0,r]$ *such that the neighborhood*

$$B_M(u_0,\sigma_0,\beta_0;r',\rho)=\{(u,\sigma,\beta):\|u-\eta(\sigma)\|<r',|(\sigma,\beta)-(\sigma_0,\beta_0)|<\rho\}$$

satisfies $\dot{u}(t)\not\equiv Q(u,\sigma,\beta)$ *for* $(u,\sigma,\beta)\in\overline{B_M(u_0,\sigma_0,\beta_0;r',\rho)}$ *with* $u=(y,z)^T\neq\eta(\sigma)$ *and* $|(\sigma,\beta)-(\sigma_0,\beta_0)|=\rho$*.*

Proof. Suppose not. Then for all $0 < r' \leq r$, there exists (u, σ, β) such that $0 < \|u - \eta(\sigma)\| < r'$, $|(\sigma, \beta) - (\sigma_0, \beta_0)| = \rho$ and $\dot{u}(t) = Q(u, \sigma, \beta)$ for $t \in \mathbb{R}$. Then there exists a sequence of nonconstant periodic solutions $\{(u_k, \sigma_k, \beta_k) = (y_k, z_k, \sigma_k, \beta_k)\}_{k=1}^{\infty}$ such that

$$\lim_{k\to+\infty} \|u_k - \eta(\sigma_k)\| = 0,\ |(\sigma_k, \beta_k) - (\sigma_0, \beta_0)| = \rho, \tag{6.85}$$

and

$$\dot{u}_k(t) = \frac{1}{\beta_k}\begin{pmatrix} f(y_k(t), y_k(t-\beta_k z_k(t)), \sigma_k) \\ g(y_k(t), z_k(t), \sigma_k) \end{pmatrix} \text{ for } t \in \mathbb{R}. \tag{6.86}$$

Note that $0 < \rho < \beta_0$ implies that $\beta_k \geq \beta_0 - \rho > 0$ for every $k \in \mathbb{N}$. Also, since the sequence $\{(\sigma_k, \beta_k)\}_{k=1}^{\infty}$ belongs to a bounded neighborhood of (σ_0, β_0) in $\mathbb{R}^2$, there exists a subsequence, denoted by $\{(\sigma_k, \beta_k)\}_{k=1}^{\infty}$, that converges to (σ^*, β^*), so that $|(\sigma^*, \beta^*) - (\sigma_0, \beta_0)| = \rho$ and $\beta^* > 0$. Without loss of generality, we denote this sequence by $\{(\sigma_k, \beta_k)\}_{k=1}^{\infty}$. Our strategy here is to show that the system

$$\dot{v}(t) = \frac{1}{\beta^*}\begin{bmatrix} \partial_1 f(\sigma^*) & 0 \\ \partial_1 g(\sigma^*) & \partial_2 g(\sigma^*) \end{bmatrix} v(t) + \frac{1}{\beta^*}\begin{bmatrix} \partial_2 f(\sigma^*) & 0 \\ 0 & 0 \end{bmatrix} v(t - \beta^* z_{\sigma^*}) \tag{6.87}$$

has a nonconstant periodic solution, which contradicts the assumption that $u_0 = (y_{\sigma_0}, z_{\sigma_0})^T$ is the only center of (6.83) in $\overline{B_M(u_0, \sigma_0, \beta_0; r, \rho)}$.

Put

$$v_k(t) = \frac{u_k(t) - \eta(\sigma_k)}{\|u_k - \eta(\sigma_k)\|}.$$

Then we have

$$\begin{aligned} \dot{v}_k(t) =& \frac{1}{\beta_k}\int_0^1 \begin{bmatrix} \partial_1 f_k(\sigma_k, s)(t) & 0 \\ \partial_1 g_k(\sigma_k, s)(t) & \partial_2 g_k(\sigma_k, s)(t) \end{bmatrix} ds\, v_k(t) \\ &+ \frac{1}{\beta_k}\int_0^1 \begin{bmatrix} \partial_2 f_k(\sigma_k, s)(t) & 0 \\ 0 & 0 \end{bmatrix} ds\, v_k(t - \beta_k z_k(t)), \end{aligned} \tag{6.88}$$

where

$$\partial_j f_k(\sigma_k, s)(t) \overset{\text{def}}{=} \partial_j f(y_{\sigma_k} + s(y_k(t) - y_{\sigma_k}), y_{\sigma_k} + s(y_k(t - z_k(t)) - y_{\sigma_k}), \sigma_k)),$$
$$\partial_j g_k(\sigma_k, s)(t) \overset{\text{def}}{=} \partial_j g(y_{\sigma_k} + s(y_k(t) - y_{\sigma_k}), z_{\sigma_k} + s(z_k(t) - z_{\sigma_k}), \sigma_k)$$

for all $j = 1, 2$. We claim that there exists a convergent subsequence of $\{v_k\}_{k=1}^{+\infty}$. Indeed, by (6.85), we know that $\{(z_k, \beta_k)\}_{k=1}^{+\infty}$ is uniformly bounded in $C(\mathbb{R}; \mathbb{R}) \times \mathbb{R}$, and hence $\lim_{t\to+\infty}[t - \beta_k z_k(t)] = +\infty$. Then, we have

$$\|v_k\| = 1, \quad \|v_k(\cdot - \beta_k z_k(\cdot))\| = 1.$$

Recall that $\partial_i f(\sigma^*)$ and $\partial_i g(\sigma^*)$, $i=1,2$, are defined in (6.76). By (6.85), we know that $(y_{\sigma_k}+s(y_k(t)-y_{\sigma_k}), y_{\sigma_k}+s(y_k(t-z_k(t))-y_{\sigma_k}), \sigma_k)$ converges to the stationary state $(x_{\sigma^*}, \tau_{\sigma^*}, \sigma^*)$ in $C(\mathbb{R};\mathbb{R})\times\mathbb{R}$ uniformly for all $s\in[0,1]$. By (SHB1), we know that $f(\theta_1, \theta_2, \sigma)$ is C^2 in (θ_1, θ_2) and $\partial_1 f(\theta_1, \theta_2, \sigma)$ is C^1 in σ. Also, by (6.85), the sequence $\{(u_k, \beta_k, \sigma_k)\}_{k=1}^{+\infty}$ is uniformly bounded in $C(\mathbb{R};\mathbb{R}^{N+1})\times\mathbb{R}^2$. Then we obtain

$$\begin{cases} \lim_{k\to+\infty}\|\partial_j f_k(\sigma_k, s)-\partial_j f(\sigma^*)\|=0, \\ \lim_{k\to+\infty}\|\partial_j g_k(\sigma_k, s)-\partial_j g(\sigma^*)\|=0 \end{cases} \tag{6.89}$$

uniformly for $s\in[0,1]$, $j=1,2$. It is clear from (6.89) that $\|\partial_j f_k(\sigma_k, s)\|$ and $\|\partial_j g_k(\sigma_k, s)\|$ $(j=1,2)$ are all uniformly bounded for all $k\in\mathbb{N}$ and $s\in[0,1]$. Then it follows from (6.88) that there exists a constant $\tilde{L}_2>0$ such that $\|\dot{v}_k\|<\tilde{L}_2$ for every $k\in\mathbb{N}$. By the Arzelà–Ascoli theorem, there exists a convergent subsequence $\{v_{k_j}\}_{j=1}^{+\infty}$ of $\{v_k\}_{k=1}^{+\infty}$. That is, there exists $v^*\in\{v\in V:\|v\|=1\}$ such that

$$\lim_{j\to+\infty}\|v_{k_j}-v^*\|=0. \tag{6.90}$$

By the integral mean value theorem, we have

$$\begin{aligned} &|v_{k_j}(t-\beta_{k_j}z_{k_j}(t))-v_{k_j}(t-\beta^* z_{\sigma^*})| \\ =&\left|\int_0^1 \dot{v}_{k_j}(t-\theta(\beta_{k_j}z_{k_j}(t)-\beta^* z_{\sigma^*}))d\theta(\beta_{k_j}z_{k_j}(t)-\beta^* z_{\sigma^*})\right| \\ \le&\|\dot{v}_{k_j}\|\cdot|\beta_{k_j}z_{k_j}(t)-\beta^* z_{\sigma^*}| \\ \le&\tilde{L}_2(\beta_{k_j}|z_{k_j}(t)-z_{\sigma^*}|+|\beta_{k_j}-\beta^*|z_{\sigma^*}). \end{aligned} \tag{6.91}$$

By (6.85) and (6.91), we have

$$\lim_{j\to+\infty}\|v_{k_j}(\cdot-\beta_{k_j}z_{k_j}(\cdot))-v_{k_j}(\cdot-\beta^* z_{\sigma^*})\|=0. \tag{6.92}$$

Therefore, it follows from (6.90) and (6.92) that

$$\lim_{j\to+\infty}\|v_{k_j}(\cdot-\beta_{k_j}z_{k_j}(\cdot))-v^*(\cdot-\beta^* z_{\sigma^*})\|=0. \tag{6.93}$$

It follows from (6.85), (6.89), (6.90), and (6.93) that the right-hand side of (6.88) converges uniformly to the right-hand side of (6.87). Therefore, v^* is differentiable and satisfies (6.87). Moreover, we have

$$\lim_{k\to+\infty}|\dot{v}_k(t)-\dot{v}^*(t)|=0.$$

Since by (SHB3), the matrix

$$\begin{bmatrix} \partial_1 f(\sigma^*)+\partial_2 f(\sigma^*) & 0 \\ \partial_1 g(\sigma^*) & \partial_2 g(\sigma^*) \end{bmatrix}$$

is nonsingular, $v = 0$ is the only constant solution of (6.87). Also, we have $v^* \in \{v \in V : \|v\| = 1\}$, $\|v^*\| \neq 0$. Therefore, $(v^*(t), \sigma^*, \beta^*)$ is a nonconstant periodic solution of the linear equation (6.87). Then by Lemma 6.16, $(\eta(\sigma^*), \sigma^*, \beta^*)$ is also a center of (6.83) in $\overline{B_M(u_0, \sigma_0, \beta_0; r, \rho)}$. This contradicts the assumption that $B_M(u_0, \sigma_0, \beta_0; r, \rho)$ is a special neighborhood of (6.82). This completes the proof. □

As preparation for the proof of the local Hopf bifurcation theorem, we need the following lemma.

Lemma 6.18. *Assume that* (SHB1)–(SHB3) *hold. If* $\mathscr{U} = B_M(u_0, \sigma_0, \beta_0; r, \rho) \subseteq \mathbb{E} \times \mathbb{R}^2$ *is a special neighborhood of* $\tilde{\mathscr{F}}$ *with* $0 < \rho < \beta_0$*, then there exists* $r' \in (0, r]$ *such that* $\mathscr{F}_\theta = (\mathscr{F}, \theta)$ *and* $\tilde{\mathscr{F}}_\theta = (\tilde{\mathscr{F}}, \theta)$ *are homotopic on* $\overline{B_M(u_0, \sigma_0, \beta_0; r', \rho)}$*, where* θ *is a completing function defined on* $\overline{B_M(u_0, \sigma_0, \beta_0; r', \rho)}$*.*

Proof. Since $\mathscr{U} = B_M(u_0, \sigma_0, \beta_0; r, \rho) \subseteq \mathbb{E} \times \mathbb{R}^2$ is a special neighborhood of $\tilde{\mathscr{F}}$ with $0 < \rho < \beta_0$, then by Lemma 6.17, both $\mathscr{F}_\theta = (\mathscr{F}, \theta)$ and $\tilde{\mathscr{F}}_\theta = (\mathscr{F}, \theta)$ are $\mathscr{U}$-admissible.

Suppose that the conclusion is not true. Then for every $r' \in (0, r]$, $\mathscr{F}_\theta = (\mathscr{F}, \theta)$ and $\tilde{\mathscr{F}}_\theta = (\mathscr{F}, \theta)$ are not homotopic on $\overline{B_M(u_0, \sigma_0, \beta_0; r', \rho)}$. That is, every homotopy map between $\mathscr{F}_\theta$ and $\tilde{\mathscr{F}}_\theta$ has a zero on the boundary of $\overline{B_M(u_0, \sigma_0, \beta_0; r', \rho)}$. In particular, the linear homotopy $h(\cdot, \alpha) \overset{\text{def}}{=} \alpha \mathscr{F}_\theta + (1-\alpha)\tilde{\mathscr{F}}_\theta = (\alpha \mathscr{F} + (1-\alpha)\tilde{\mathscr{F}}, \theta)$ has a zero on the boundary of $B_M(u_0, \sigma_0, \beta_0; r', \rho)$, where $\alpha \in [0, 1]$.

Note that $\theta(u, \sigma, \beta) < 0$ if $\|u - \eta(\sigma)\| = r'$. Then there exist (u, σ, β) and $\alpha \in [0, 1]$ such that $\|u - \eta(\sigma)\| < r'$, $|(\sigma, \beta) - (\sigma_0, \beta_0)| = \rho$ and

$$H(u, \sigma, \beta, \alpha) \overset{\text{def}}{=} \alpha \mathscr{F} + (1-\alpha)\tilde{\mathscr{F}} = 0. \tag{6.94}$$

Since $r' > 0$ is arbitrary in the interval $(0, r]$, there exists a nonconstant sequence $\{(y_k, z_k, \sigma_k, \beta_k, \alpha_k)\}_{k=1}^{\infty}$ of solutions of (6.94) such that

$$\lim_{k \to +\infty} \|u_k - \eta(\sigma_k)\| = 0, \ |(\sigma_k, \beta_k) - (\sigma_0, \beta_0)| = \rho, 0 \leq \alpha_k \leq 1, \tag{6.95}$$

and

$$\dot{u}_k = \alpha_k Q(u_k, \sigma_k, \beta_k) + (1-\alpha_k)\tilde{Q}(u_k, \sigma_k, \beta_k, \text{ for all } k \in \mathbb{N}. \tag{6.96}$$

Note that $0 < \rho < \beta_0$ implies that $\beta_k \geq \beta_0 - \rho > 0$ for every $k \in \mathbb{N}$. From (6.95), we know that $\{(\sigma_k, \beta_k, \alpha_k)\}_{k=1}^{\infty}$ belongs to a compact subset of $\mathbb{R}^3$. Therefore, there exist a convergent subsequence, denoted still by $\{(\sigma_k, \beta_k, \alpha_k)\}_{k=1}^{\infty}$ without loss of generality, and $(\sigma^*, \beta^*, \alpha^*) \in \mathbb{R}^3$ such that $\beta^* \geq \beta_0 - \rho > 0$, $\alpha^* \in [0, 1]$ and

$$\lim_{k \to +\infty} |(\sigma_k, \beta_k, \alpha_k) - (\sigma^*, \beta^*, \alpha^*)| = 0.$$

Similarly to the proof of Lemma 6.17, we can show that the system

$$\dot{v}(t) = \frac{1}{\beta^*}\begin{bmatrix} \partial_1 f(\sigma^*) & 0 \\ \partial_1 g(\sigma^*) & \partial_2 g(\sigma^*) \end{bmatrix} v(t) + \frac{1}{\beta^*}\begin{bmatrix} \partial_2 f(\sigma^*) & 0 \\ 0 & 0 \end{bmatrix} v(t-\beta^* z_{\sigma^*})$$

with $\partial_i f(\sigma^*)$, $\partial_i g(\sigma^*)$, $i = 1,2$, defined after (6.76), has a nonconstant periodic solution, which contradicts the assumption that $B_M(u_0,\, \sigma_0,\, \beta_0; r,\, \rho)$ is a special neighborhood that contains an isolated center of (6.83). This completes the proof. □

Now we are able to state and prove the local Hopf bifurcation theorem.

Theorem 6.15. *Assume that* (SHB1)–(SHB3) *hold. Let* $(x_{\sigma_0},\, \tau_{\sigma_0},\, \sigma_0)$ *be an isolated center of system (6.76). If the crossing number defined by (6.80) satisfies*

$$\gamma(x_{\sigma_0},\, \tau_{\sigma_0},\, \sigma_0,\, \beta_0) \neq 0,$$

then there exists a bifurcation of nonconstant periodic solutions of (6.74) *near* $(x_{\sigma_0}, \tau_{\sigma_0},\, \sigma_0)$*. More precisely, there exists a sequence* $\{(x_n,\, \tau_n,\, \sigma_n, \beta_n)\}$ *such that* $\sigma_n \to \sigma_0$*,* $\beta_n \to \beta_0$ *as* $n \to \infty$*, and* $\lim_{n\to\infty} \|x_n - x_{\sigma_0}\| = 0$*,* $\lim_{n\to\infty} \|\tau_n - \tau_{\sigma_0}\| = 0$*, where*

$$(x_n,\, \tau_n,\, \sigma_n) \in C(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}$$

is a nonconstant $2\pi/\beta_n$*-periodic solution of system* (6.74)*.*

Proof. By (SHB1), we know that the linear operator $\tilde{Q}$ is continuous. By Lemma 6.15, we know that $Q(\cdot,\, \sigma,\, \beta) \colon \mathbb{E} \to \mathbb{E}$ is continuous. Moreover, as stated before, by Lemma 6.14, (SD2) and (SD4) are satisfied. Since $(u_0,\, \sigma_0) \stackrel{\text{def}}{=} (x_{\sigma_0},\, \tau_{\sigma_0},\, \sigma_0)$ is an isolated center of system (6.76) with a purely imaginary characteristic value $i\beta_0$, $\beta_0 > 0$, $(u_0,\, \sigma_0,\, \beta_0) \in \mathbb{E} \times \mathbb{R} \times (0, +\infty)$ is an isolated $\mathbb{E}$-singular point of $\tilde{\mathscr{F}}$. One can define the following two-dimensional submanifold $M \subset \mathbb{E}^G \times \mathbb{R} \times (0, +\infty)$ by

$$M \stackrel{\text{def}}{=} \{(\eta(\sigma),\, \sigma,\, \beta) : \sigma \in (\sigma_0 - \delta,\, \sigma_0 + \delta),\, \beta \in (\beta_0 - \varepsilon,\, \beta_0 + \varepsilon)\}$$

such that the point $(\eta(\sigma_0),\, \sigma_0,\, \beta_0)) = (u_0,\, \sigma_0,\, \beta_0)$ is the only $\mathbb{E}$-singular point of $\tilde{\mathscr{F}}$ in M, which is the set of trivial solutions to the system (6.76); it satisfies assumption (SD3).

Moreover, $(u_0,\, \sigma_0,\, \beta_0) \in \mathbb{E} \times \mathbb{R} \times (0, +\infty)$ is an isolated $\mathbb{E}$-singular point of $\tilde{\mathscr{F}}$. That is, for $\rho > 0$ sufficiently small, the linear operator $D_u\tilde{\mathscr{F}}(\eta(\sigma),\, \sigma,\, \beta) : \mathbb{E} \to \mathbb{E}$ with $|(\sigma,\, \beta) - (\sigma_0,\, \beta_0)| < \rho$ is not an isomorphism only if $(\sigma,\, \beta) = (\sigma_0,\, \beta_0)$. Then by the implicit function theorem, there exists $r > 0$ such that for all $(u,\, \sigma,\, \beta) \in \mathbb{E} \times \mathbb{R} \times (0, +\infty)$ with $|(\sigma,\, \beta) - (\sigma_0,\, \beta_0)| = \rho$ and $0 < \|u - \eta(\sigma)\| \le r$, we have $\tilde{\mathscr{F}}(u,\, \sigma,\, \beta) \neq 0$. Then the set $B_M(u_0,\, \sigma_0,\, \beta_0; r, \rho)$ defined by

$$\{(u,\, \sigma,\, \beta) \in \mathbb{E} \times \mathbb{R} \times (0, +\infty) : |(\sigma,\, \beta) - (\sigma_0,\, \beta_0)| < \rho, \|u - \eta(\sigma)\| < r\}$$

is a special neighborhood for $\tilde{\mathscr{F}}$. By Lemma 6.17, there exists a special neighborhood $\mathscr{U} = B_M(u_0, \sigma_0, \beta_0; r', \rho)$ such that $\mathscr{F}$ and $\tilde{\mathscr{F}}$ are nonzero for $(u, \sigma, \beta) \in \overline{B_M(u_0, \sigma_0, \beta_0; r', \rho)}$ with $u \neq \eta(\sigma)$ and $|(\sigma, \beta) - (\sigma_0, \beta_0)| = \rho$. That is, (SD5) is satisfied.

Let θ be a completing function on $\mathscr{U}$. It follows from Lemma 6.18 that $(\mathscr{F}, \theta)$ is homotopic to $(\tilde{\mathscr{F}}, \theta)$ on $\mathscr{U}$.

For $(\sigma, \beta) \in \mathscr{D}(\sigma_0, \beta_0)$, we denote by $\Psi(\sigma, \beta)$ the map $D_u\tilde{\mathscr{F}}(u(\sigma), \sigma, \beta) : \mathbb{E} \to \mathbb{E}$. It is easy to see that $\Psi(\sigma, \beta)(\mathbb{E}_k) \subset \mathbb{E}_k$ for all $k = 0, 1, 2, \cdots$. Define $\Psi_k : \mathscr{D}(\sigma_0, \beta_0) \to L(\mathbb{E}_k, \mathbb{E}_k)$ by

$$\Psi_k(\sigma, \beta) \overset{\text{def}}{=} \Psi(\sigma, \beta)|_{\mathbb{E}_k}.$$

Thus, the matrix representation $[\Psi_k]$ of $\Psi_k(\sigma, \beta)$ $\{e^{ik\cdot}\varepsilon_j\}_{j=1}^{N+1}$ is given by

$$\frac{1}{ik\beta}\Delta_{(u(\sigma),\sigma)}(ik\beta).$$

For the application of Theorem 6.7, we now show that there exists some $k \in \mathbb{Z}$, $k \geq 1$, such that $\varepsilon_0\mu_k(u(\sigma_0), \sigma_0, \beta_0) = \varepsilon_0 \deg_B(\det_{\mathbb{C}}[\Psi_k]) \neq 0$, where $\varepsilon_0 = \operatorname{sgn}\det\Psi_0(\sigma, \beta)$ for $(\sigma,\beta) \in \mathscr{D}(\sigma_0, \beta_0)$. For a constant map $v_0 \in \mathbb{E}_0$,

$$\Psi_0(\sigma, \beta)v_0 = -\frac{1}{\beta}\begin{bmatrix} \partial_1 f(\sigma) + \partial_2 f(\sigma) & 0 \\ \partial_1 g(\sigma) & \partial_2 g(\sigma) \end{bmatrix} v_0.$$

Then by (SHB3), we have $\varepsilon_0 \neq 0$, and therefore (SD6) is satisfied. In view of (6.81), we have

$$\mu_1(u(\sigma_0), \sigma_0, \beta_0) = \gamma(x_{\sigma_0}, \tau_{\sigma_0}, \sigma_0, \beta_0) \neq 0,$$

which by Theorem 6.7, implies that $(u(\sigma_0), \sigma_0, \beta_0)$ is a bifurcation point of the system (6.82). Consequently, there exists a sequence of nonconstant periodic solutions $(u_n, \sigma_n, \beta_n) = (x_n, \tau_n, \sigma_n, \beta_n)$ such that $\sigma_n \to \sigma_0$, $\beta_n \to \beta_0$ as $n \to \infty$, and $(x_n(t), \tau_n(t))$ is a $2\pi/\beta_n$-periodic solution of (6.74) such that $\lim_{n\to+\infty}\|(x_n, \tau_n) - (x_{\sigma_0}, \tau_{\sigma_0})\| = 0$. □

Remark 6.2. A local Hopf bifurcation theory for FDEs with state-dependent delays was developed by Eichmann [86], where the existence of a local Hopf bifurcation is guaranteed by a transversality condition. This transversality implies that the crossing number defined by (6.80) is not zero, and hence the existence of a local Hopf bifurcation is also established in Theorem 6.15. Note that even in the case of a constant delay, one can have nontrivial crossing number while the transversality condition is not satisfied. Note also that the work of Eichmann gives more information about the local Hopf bifurcation such as smoothness of the bifurcation curve with respect to the parameter.

6.9.2 Global Bifurcation

To use Theorem 6.8 to describe the maximal continuation of bifurcated periodic solutions with large amplitudes when the bifurcation parameter σ is far away from the bifurcation value, we need to prove that there is a lower bound for the periods of periodic solutions of system (6.74).

Lemma 6.19 (Vidossich [287]). *Let X be a Banach space and $v : \mathbb{R} \to X$ a p-periodic function with the following properties:*

(i) $v \in L^1_{loc}(\mathbb{R}, X)$*;*
(ii) there exists $U \in L^1([0, \frac{\mathsf{p}}{2}]; \mathbb{R}_+)$ *such that* $|v(t) - v(s)| \le U(t-s)$ *for almost every (in the sense of the Lebesgue measure)* $s, t \in \mathbb{R}$ *such that* $s \le t$, $t - s \le \frac{\mathsf{p}}{2}$*;*
(iii) $\int_0^{\mathsf{p}} v(t)\, dt = 0$.

Then

$$\mathsf{p}\, \|v\|_{L^\infty} \le 2 \int_0^{\frac{\mathsf{p}}{2}} U(t)\, dt.$$

We make the following assumption on system (6.74):

(SHB4) There exist constants $L_f > 0$, $L_g > 0$ such that

$$|f(\theta_1, \theta_2, \sigma) - f(\overline{\theta}_1, \overline{\theta}_2, \sigma)| \le L_f(|\theta_1 - \overline{\theta}_1| + |\theta_2 - \overline{\theta}_2|)$$
$$|g(\gamma_1, \gamma_2, \sigma) - g(\overline{\gamma}_1, \overline{\gamma}_2, \sigma)| \le L_g(|\gamma_1 - \overline{\gamma}_1| + |\gamma_2 - \overline{\gamma}_2|)$$

for every $\theta_1, \theta_2, \overline{\theta}_1, \overline{\theta}_2, \gamma_1, \overline{\gamma}_1 \in \mathbb{R}^N$, $\gamma_2, \overline{\gamma}_2 \in \mathbb{R}$, $\sigma \in \mathbb{R}$.

Lemma 6.20. *Assume that system (6.74) satisfies the assumption* (SHB4)*. If $u = (x, \tau)$ is a nonconstant periodic solution of (1.1), then the minimal period of u satisfies*

$$\mathsf{p} \ge \frac{4(|\dot{x}|_{L^\infty} + |\dot{\tau}|_{L^\infty})}{(2L_f + L_g)|\dot{x}|_{L^\infty} + L_g|\dot{\tau}|_{L^\infty} + L_f|\dot{x}|_{L^\infty}|\dot{\tau}|_{L^\infty}}.$$

Moreover, suppose $g(x, \tau, \sigma)$ satisfies

(SHB5) *for every $\sigma \in \mathbb{R}$, there exists $L_0 > 0$ such that $-L_0 \le g(x, \tau, \sigma) < 1$ for all $(x, \tau) \in \mathbb{R}^{N+1}$.*

Then the minimal period p of u satisfies

$$\mathsf{p} \ge \frac{4}{\max\{L_0, 1\} + 2(L_f + L_g)}.$$

Proof. Let $v(t) = \dot{u}(t)$. Then $\int_0^{\mathsf{p}} v(t)dt = 0$, since $u(t)$ is a p-periodic solution. For $s \le t$, by (SHB4) and the integral mean value theorem, we have

$$\begin{aligned}|v(t) - v(s)| &\le |\dot{x}(t) - \dot{x}(s)| + |\dot{\tau}(t) - \dot{\tau}(s)| \\ &\le L_f(|x(t) - x(s)| + |x(t - \tau(t)) - x(s - \tau(s))|)\end{aligned}$$

$$
\begin{aligned}
&+L_g(|x(t)-x(s)|+|\tau(t)-\tau(s)|)\\
&\le L_f|\dot{x}|_{L^\infty}(t-s)+L_f|\dot{x}|_{L^\infty}(t-s+|\tau(t)-\tau(s)|)\\
&\quad +L_g|\dot{x}|_{L^\infty}(t-s)+L_g|\dot{\tau}|_{L^\infty}(t-s)\\
&\le \left[(2L_f+L_g)|\dot{x}|_{L^\infty}+L_g|\dot{\tau}|_{L^\infty}+L_f|\dot{x}|_{L^\infty}\cdot|\dot{\tau}|_{L^\infty}\right](t-s).
\end{aligned}
$$

Let

$$
U(t)=\left[(2L_f+L_g)|\dot{x}|_{L^\infty}+L_g|\dot{\tau}|_{L^\infty}+|\dot{x}|_{L^\infty}\cdot|\dot{\tau}|_{L^\infty}\right]t.
$$

Then by Lemma 6.19, we obtain

$$
\mathsf{p}|(\dot{x},\,\dot{\tau})|_{L^\infty}\le 2\int_0^{\frac{\mathsf{p}}{2}}U(t)dt=\frac{\mathsf{p}^2}{4}\left[(2L_f+L_g)|\dot{x}|_{L^\infty}+L_g|\dot{\tau}|_{L^\infty}+|\dot{x}|_{L^\infty}\cdot|\dot{\tau}|_{L^\infty}\right].
$$

Therefore,

$$
\mathsf{p}\ge\frac{4|(\dot{x},\,\dot{\tau})|_{L^\infty}}{(2L_f+L_g)|\dot{x}|_{L^\infty}+L_g|\dot{\tau}|_{L^\infty}+L_f|\dot{x}|_{L^\infty}|\dot{\tau}|_{L^\infty}}.
$$

Moreover, if $-L_0\le g(x(t),\,\tau(t),\,\sigma)<1$, then

$$
|\dot{x}|_{L^\infty}\cdot|\dot{\tau}|_{L^\infty}\le\max\{L_0,\,1\}|\dot{x}|_{L^\infty},
$$

and hence

$$
\begin{aligned}
\mathsf{p}&\ge\frac{4|(\dot{x},\,\dot{\tau})|_{L^\infty}}{(2L_f+L_g)|\dot{x}|_{L^\infty}+L_g|\dot{\tau}|_{L^\infty}+\max\{L_0,\,1\}|\dot{x}|_{L^\infty}}\\
&\ge\frac{4|(\dot{x},\,\dot{\tau})|_{L^\infty}}{(2L_f+L_g)|(\dot{x},\,\dot{\tau})|_{L^\infty}+L_g|(\dot{x},\,\dot{\tau})|_{L^\infty}+\max\{L_0,\,1\}|(\dot{x},\,\dot{\tau})|_{L^\infty}}\\
&=\frac{4}{\max\{L_0,\,1\}+2(L_f+L_g)}.
\end{aligned}
$$

□

The following result was first established by Mallet-Paret and Yorke [216] for ordinary differential equations and was extended to neutral equations by Wu [301].

Lemma 6.21. *Suppose that system (6.74) satisfies* (SHB1)–(SHB2) *and* (SHB4)–(SHB5). *Assume further that there exists a sequence of real numbers* $\{\sigma_k\}_{k=1}^{\infty}$ *such that:*

(i) For each k, system (6.74) with $\sigma=\sigma_k$ *has a nonconstant periodic solution* $u_k=(x_k,\,\tau_k)\in C(\mathbb{R};\mathbb{R}^{N+1})$ *with minimal period* $T_k>0$*;*
(ii) $\lim_{k\to\infty}\sigma_k=\sigma_0\in\mathbb{R}$, $\lim_{k\to\infty}T_k=T_0<\infty$, *and* $\lim_{k\to\infty}\|u_k-u_0\|=0$, *where* $u_0:\mathbb{R}\to\mathbb{R}^{N+1}$ *is a constant map with the value* $(x_0,\,\tau_0)$.

Then $(u_0,\,\sigma_0)$ *is a stationary state of (6.74), and there exists* $m\ge 1$, $m\in\mathbb{N}$ *such that* $\pm\mathrm{i}m2\pi/T_0$ *are the roots of the characteristic equation (6.77) with* $\sigma=\sigma_0$.

Proof. By Lemma 6.20, we conclude that $T_k \geq \frac{4}{\max\{L_0,1\}+2(L_f+L_g)}$ and therefore $T_0 \geq \frac{4}{\max\{L_0,1\}+2(L_f+L_g)} > 0$.

Now we show that (u_0, σ_0) is a stationary state of (6.74). Since by (ii), $\lim_{k\to\infty} \sigma_k = \sigma_0$ and $\lim_{k\to\infty} \|u_k - u_0\| = 0$, we have only to show that the derivatives $\{\dot{u}_k\}_{k=1}^{+\infty}$ converge uniformly to the right-hand side of system (6.74). That is,

$$\|f(x_k, x_k(\cdot - \tau_k), \sigma_k) - f(x_0, x_0, \sigma_0)\| + \|g(x_k, \tau_k, \sigma_k) - g(x_0, \tau_0, \sigma_0)\| \to 0 \quad (6.97)$$

as $k \to +\infty$. Note that we have used $f(x_k, x_k(\cdot - \tau_k), \sigma_k)$ to denote the function $f(x_k(\cdot), x_k(\cdot - \tau_k), \sigma_k)$. In fact, it follows from (SHB1) and assumption (ii) that $\lim_{k\to\infty} \|g(x_k, \tau_k, \sigma_k) - g(x_0, \tau_0, \sigma_0)\| = 0$. Moreover, by the integral mean value theorem, we have $\|f(x_k, x_k(\cdot - \tau_k), \sigma_k) - f(x_0, x_0, \sigma_0)\| \to 0$ as $k \to +\infty$. This completes the proof of (6.97). Therefore, $(u_0, \sigma_0) = (x_{\sigma_0}, \tau_{\sigma_0}, \sigma_0)$ is the stationary state of (6.74) with $\sigma = \sigma_0$.

Next, we show that the linear system

$$\dot{v}(t) = \begin{bmatrix} \partial_1 f(\sigma_0) & 0 \\ \partial_1 g(\sigma_0) & \partial_2 g(\sigma_0) \end{bmatrix} v(t) + \begin{bmatrix} \partial_2 f(\sigma_0) & 0 \\ 0 & 0 \end{bmatrix} v(t - \tau_0) \quad (6.98)$$

has a nonconstant periodic solution.

For $\rho \in (0, 1)$, define

$$\varepsilon_{k,\rho} = \max_{t\in\mathbb{R}} |u_k(t + \rho T_k) - u_k(t)|,$$
$$v_k(t) = \varepsilon_{k,\rho}^{-1} [u_k(t + \rho T_k) - u_k(t)].$$

Then $\|v_k\| = 1$, and $v_k(t) \overset{\text{def}}{=} (y_k(t), z_k(t))$ satisfies

$$\dot{v}_k(t) = \begin{bmatrix} \partial_1 f(\sigma_0) & 0 \\ \partial_1 g(\sigma_0) & \partial_2 g(\sigma_0) \end{bmatrix} v_k(t) + \begin{bmatrix} \partial_2 f(\sigma_0) & 0 \\ 0 & 0 \end{bmatrix} v_k(t - \tau_0) + \begin{pmatrix} \delta_{1k}(t) \\ \delta_{2k}(t) \end{pmatrix},$$

where

$$\begin{cases} \delta_{1k}(t) = \varepsilon_{k,\rho}^{-1} [f(x_k(t+\rho T_k), x_k(t+\rho T_k - \tau_k(t+\rho T_k)), \sigma_k) \\ \qquad - f(x_k(t), x_k(t - \tau_k(t)), \sigma_k) - \partial_1 f(\sigma_0)(x_k(t+\rho T_k) - x_k(t)) \\ \qquad - \partial_2 f(\sigma_0)(x_k(t+\rho T_k - \tau_0) - x_k(t - \tau_0))], \\ \delta_{2k}(t) = \varepsilon_{k,\rho}^{-1} [g(x_k(t+\rho T_k), \tau_k(t+\rho T_k), \sigma_k) - g(x_k(t), \tau_k(t), \sigma_k) \\ \qquad - \partial_1 g(\sigma_0)(x_k(t+\rho T_k) - x_k(t)) - \partial_2 g(\sigma_0)(\tau_k(t+\rho T_k) - \tau_k(t))]. \end{cases}$$

Using the integral mean value theorem, we can show that $|\delta_{1k}(t)| \to 0$, $|\delta_{2k}(t)| \to 0$ as $k \to +\infty$ uniformly for $t \in \mathbb{R}$. This, together with the fact that $\|v_k\| = 1$, implies that there exists $\tilde{L}_6 > 0$ such that $\|\dot{v}_k\| \leq \tilde{L}_6$ for all $k \in \mathbb{N}$. Also, by assumption (ii), the set of periods $\{T_k\}_{k=1}^{+\infty}$ is bounded. Then by the Arzelà–Ascoli theorem, $\{v_k\}_{k=1}^{+\infty}$ has a convergent subsequence, denoted by $\{v_{k_j}\}_{j=1}^{+\infty}$. Let

$$v_\rho(t) = \lim_{j\to+\infty} v_{k_j}(t).$$

Then v_ρ is a periodic solution of (6.98) with period T_0. Since $\|v_k\| = 1$ and the average value of each v_k is zero, the same is true for v_ρ. So v_ρ is a nonconstant T_0-periodic solution of (6.98). Then by Lemma 6.16, there exists $m \geq 1, m \in \mathbb{N}$, such that $\pm \mathrm{i} m 2\pi/T_0$ are characteristic values of (6.77). This completes the proof. □

Now we can describe the relation between $2\pi/\beta_k$ and the minimal period of u_k in Theorem 6.15.

Theorem 6.16. *Assume that* (SHB1)–(SHB5) *hold. In Theorem 6.15, every limit point of the minimal period of $u_k = (x_k, \tau_k)$ as $k \to +\infty$ is contained in the set*

$$\left\{ \frac{2\pi}{(n\beta_0)} : \pm \mathrm{i} m n \beta_0 \text{ are characteristic values of } (u_0, \sigma_0), m, n \geq 1, m, n \in \mathbb{N} \right\}.$$

Moreover, if $\pm \mathrm{i} m n \beta_0$ are not characteristic values of (u_0, σ_0) for any integers $m, n \in \mathbb{N}$ such that $mn > 1$, then $2\pi/\beta_k$ is the minimal period of $u_k(t)$ and $2\pi/\beta_k \to 2\pi/\beta_0$ as $k \to \infty$.

Proof. Let T_k denote the minimal period of $u_k(t)$. Then there exists a positive integer n_k such that $2\pi/\beta_k = n_k T_k$. Since $T_k \leq 2\pi/\beta_k \to 2\pi/\beta_0$ as $k \to \infty$, there exist a subsequence $\{T_{k_j}\}_{j=1}^{\infty}$ and T_0 such that $T_0 = \lim_{j\to\infty} T_{k_j}$. Since $2\pi/\beta_{k_j} \to 2\pi/\beta_0$, $T_{k_j} \to T_0$ as $j \to \infty$, n_{k_j} is identical to a constant n for k large enough. Therefore, $2\pi/\beta_0 = nT_0$. Thus $T_{k_j} \to 2\pi/(n\beta_0)$ as $j \to \infty$. By Lemma 6.21, $\pm \mathrm{i} m 2\pi/T_0 = \pm \mathrm{i} m n \beta_0$ are characteristic values of (u_0, σ_0) for some $m \geq 1, m \in \mathbb{N}$.

Moreover, if $\pm \mathrm{i} m n \beta_0$ are not characteristic values of (u_0, σ_0) for any integers $m \in \mathbb{N}$ and $n \in \mathbb{N}$ with $mn > 1$, then $m = n = 1$. Therefore, for k large enough, $n_{k_j} = 1$ and $2\pi/\beta_k = T_k$ is the minimal period of $u_k(t)$ and $2\pi/\beta_k \to 2\pi/\beta_0$ as $k \to \infty$. This completes the proof. □

The following lemma shows that we can locate all possible Hopf bifurcation points of system (6.74) with state-dependent delay at the centers of its corresponding formal linearization.

Lemma 6.22. *Assume that* (SHB1)–(SHB3) *hold. If (u_0, σ_0) is a Hopf bifurcation point of system (6.74), then it is a center of (6.76).*

Proof. If (u_0, σ_0) is a Hopf bifurcation point of system (6.74), then there exist a sequence $\{(u_k, \sigma_k, T_k)\}_{k=1}^{+\infty} \subseteq C(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2$ and $T_0 \geq 0$ such that $\lim_{k\to+\infty} \|(u_k, \sigma_k, T_k) - (u_0, \sigma_0, T_0)\| = 0$, where (u_k, σ_k) is a nonconstant T_k-periodic solution of system (6.74). Using a similar argument to that in the proof of Lemma 6.17, we see that the system

$$\dot{v}(t) = \begin{bmatrix} \partial_1 f(\sigma_0) & 0 \\ \partial_1 g(\sigma_0) & \partial_2 g(\sigma_0) \end{bmatrix} v(t) + \begin{bmatrix} \partial_2 f(\sigma_0) & 0 \\ 0 & 0 \end{bmatrix} v(t - \tau_{\sigma_0}) \qquad (6.99)$$

has a nonconstant periodic solution v^*. Therefore, $(v^* + u_0, \sigma_0)$ is a nonconstant periodic solution of (6.76). Then, by Lemma 6.16, (u_0, σ_0) is a center of (6.76). □

Now we are able to consider the global Hopf bifurcation problem of system (6.74). Letting $(x(t), \tau(t)) = (y(\frac{2\pi}{\mathsf{p}}t), z(\frac{2\pi}{\mathsf{p}}t))$, we can reformulate the problem as a problem of finding 2π-periodic solutions to the following equation:

$$\dot{u}(t) = Q(u(t), \sigma, 2\pi/\mathsf{p}), \tag{6.100}$$

where $u(t) = (y(t), z(t))$. Accordingly, the formal linearization (6.76) becomes

$$\dot{u}(t) = \tilde{Q}(u(t), \sigma, 2\pi/\mathsf{p}). \tag{6.101}$$

Using the same notation as in the proof of Theorem 6.15, we can define $\mathscr{N}_0(u, \sigma, \mathsf{p}) = Q(u, \sigma, 2\pi/\mathsf{p}), \tilde{\mathscr{N}_0}(u, \sigma, \mathsf{p}) = \tilde{Q}(u, \sigma, 2\pi/\mathsf{p})$. Then the system

$$Lu = \mathscr{N}_0(u, \sigma, \mathsf{p}),\ \mathsf{p} > 0, \tag{6.102}$$

is equivalent to (6.100), and

$$Lu = \tilde{\mathscr{N}_0}(u, \sigma, \mathsf{p}),\ \mathsf{p} > 0, \tag{6.103}$$

is equivalent to (6.101). Let $\mathscr{S}$ denote the closure of the set of all nontrivial periodic solutions of system (6.102) in the space $\mathbb{E} \times \mathbb{R} \times \mathbb{R}_+$, where $\mathbb{R}_+$ is the set of all nonnegative real numbers. It follows from Lemma 6.20 that the constant solution $(u_0, \sigma_0, 0)$ does not belong to this set. Consequently, we can assume that problem (6.102) is well posed on the whole space $\mathbb{E} \times \mathbb{R}^2$, in the sense that if $\mathscr{S}$ exists in $\mathbb{E} \times \mathbb{R}^2$, then it must be contained in $\mathbb{E} \times \mathbb{R} \times \mathbb{R}_+$.

On the other hand, assume that (SHB3) holds at every center of (6.103). Then from the proof of Theorem 6.15, we know that the assumptions (SHB1–SHB3) are sufficient for the systems (6.102) and (6.103) to satisfy the conditions (SD1)–(SD6). Also, under the same assumptions, Lemma 6.22 implies (SD7), and Lemma 6.18 implies (SD8). Then by Theorem 6.8, we obtain the following global Hopf bifurcation theorem for system (6.102) with state-dependent delay.

Theorem 6.17. *Suppose that system (6.74) satisfies* (SHB1)–(SHB5) *and* (SHB3) *holds at every center of (6.103). Assume that all the centers of (6.103) are isolated. Let $\mathscr{M}$ be the set of trivial periodic solutions of (6.102) and suppose that $\mathscr{M}$ is complete. If $(u_0, \sigma_0, \mathsf{p}_0) \in \mathscr{M}$ is a bifurcation point, then either the connected component $C(u_0, \sigma_0, \mathsf{p}_0)$ of $(u_0, \sigma_0, \mathsf{p}_0)$ in $\mathscr{S}$ is unbounded, or*

$$C(u_0, \sigma_0, \mathsf{p}_0) \cap \mathscr{M} = \{(u_0, \sigma_0, \mathsf{p}_0), (u_1, \sigma_1, \mathsf{p}_1), \cdots, (u_q, \sigma_q, \mathsf{p}_q)\},$$

where $\mathsf{p}_i \in \mathbb{R}_+$, $(u_i, \sigma_i, \mathsf{p}_i) \in \mathscr{M}$, $i = 0, 1, 2, \cdots, q$. Moreover, in the latter case, we have

$$\sum_{i=0}^{q} \varepsilon_i \gamma(u_i, \sigma_i, 2\pi/\mathsf{p}_i) = 0,$$

where $\gamma(u_i, \sigma_i, 2\pi/\mathsf{p}_i)$ is the crossing number of $(u_i, \sigma_i, \mathsf{p}_i)$ defined by (6.80) and

$$\varepsilon_i = \operatorname{sgn}\det\begin{bmatrix} \partial_1 f(\sigma_i) + \partial_2 f(\sigma_i) & 0 \\ \partial_1 g(\sigma_i) & \partial_2 g(\sigma_i) \end{bmatrix}.$$

Definition 6.4. Let $\mathscr{C}$ be a connected component of the closure of all nonconstant periodic solutions of (6.74) in the Fuller space $C(\mathbb{R};\mathbb{R}^{N+1})\times\mathbb{R}^2$. We call $\mathscr{C}$ a continuum of *slowly oscillating periodic solutions* if for every $(x,\tau,\sigma,p)\in\mathscr{C}$, there exists $t_0\in\mathbb{R}$ such that $p>\tau(t_0)>0$. Similarly, we call $\mathscr{C}$ a continuum of *rapidly oscillating periodic solutions* if for every $(x,\tau,\sigma,p)\in\mathscr{C}$, there exists $t_0\in\mathbb{R}$ such that $0<p<\tau(t_0)$.

Theorem 6.17 shows that for a given trivial solution (x^*,τ^*,σ^*) with virtual period p^*, either the connected component $C(x^*,\tau^*,\sigma^*,p^*)$ has finitely many bifurcation points with the sum of $\mathbb{S}^1$-equivariant degrees being zero or $C(x^*,\tau^*,\sigma^*,p^*)$ is unbounded in the Fuller space $C(\mathbb{R};\mathbb{R}^{N+1})\times\mathbb{R}^2$. Therefore, if global persistence of periodic solutions when the parameter is far away from the local Hopf bifurcation value σ^* is desired, we should find conditions to ensure that the connected component $C(x^*,\tau^*,\sigma^*,p^*)$ of Hopf bifurcation is unbounded in the Fuller space $C(\mathbb{R};\mathbb{R}^{N+1})\times\mathbb{R}^2$ and $C(x^*,\tau^*,\sigma^*,p^*)$ will not blow up to infinity at any given σ in the norm of the Fuller space $C(\mathbb{R};\mathbb{R}^{N+1})\times\mathbb{R}^2$. That is, there exists a continuous function $M:\mathbb{R}\ni\sigma\to M(\sigma)>0$ such that for every $(x,\tau,\sigma,p)\in C(x^*,\tau^*,\sigma^*,p^*)$, we have

$$\|(x,\tau,p)\|_{C(\mathbb{R};\mathbb{R}^{N+1})\times\mathbb{R}}\le M(\sigma). \tag{6.104}$$

To achieve this goal, we shall give some sufficient geometric conditions ensuring the uniform boundedness of all possible periodic solutions (x,τ,σ) of (6.74), that is, we show that there exists a continuous function $M_1:\mathbb{R}\ni\sigma\to M_1(\sigma)>0$ such that for every $(x,\tau,\sigma,p)\in C(x^*,\tau^*,\sigma^*,p^*)$, we have

$$\|(x,\tau)\|_{C(\mathbb{R};\mathbb{R}^{N+1})}\le M_1(\sigma). \tag{6.105}$$

Then we seek a continuous function $M_2:\mathbb{R}\ni\sigma\to M_2(\sigma)>0$ such that for every $(x,\tau,\sigma,p)\in C(x^*,\tau^*,\sigma^*,p^*)$, we have

$$|p|\le M_2(\sigma). \tag{6.106}$$

6.9.3 Uniform Bounds for Periods of Periodic Solutions in a Connected Component

In order, by way of contradiction, to exclude certain values of the period of the periodic solutions in a given connected component, we need some analytic properties of an interval map under the following assumptions:

(SHB6) For every $(\sigma,\tau)\in\mathbb{R}^2$, $\frac{\partial g}{\partial\tau}(x_\sigma,\tau,\sigma)\neq 0$.

(SHB7) $\frac{\partial g}{\partial x}(x,\tau,\sigma)f(x,x,\sigma)\neq 0$ for $(x,\tau,\sigma)\in\mathbb{R}^{N+1}\times\mathbb{R}$ such that $x\neq x_\sigma$ and $g(x,\tau,\sigma)=0$.

In this section, to avoid notational complications, we use superscripts to denote function compositions, e.g., $l^j(t)$ denotes the jth composition of l evaluated at time t.

The following result can be found in [171].

Lemma 6.23. *Suppose that* (6.74) *satisfies* (SHB1)–(SHB2) *and* (SHB6)–(SHB7) *and* (x, τ, σ_0) *is a nonconstant periodic solution of* (6.74). *If* (x, τ) *is* $\tau(t_0)$*-periodic and if* $\tau(t_0) \neq \tau_{\sigma_0}$, *then the function* $l(t) = t - \tau(t) + \tau(t_0)$ *defined on* $[t_0, t_0 + \tau(t_0)]$ *satisfies the following properties:*

(a) $l(t)$ *is a self-mapping on* $[t_0, t_0 + \tau(t_0)]$.
(b) $l(t)$ *has only finitely many fixed points* $\{t_i\}_{i=1}^n$ *in* $[t_0, t_0 + \tau(t_0)]$ *with* $t_i < t_{i+1}$ *for every* $i \in \{1, 2, \cdots, n-1\}$.
(c) *For every* $t \in (t_i, t_{i+1}) \subseteq [t_0, t_0 + \tau(t_0)]$,

$$\lim_{j \to +\infty} l^j(t) = \begin{cases} t_i, & \text{if there exists } \bar{t} \in [t_i, t_{i+1}] \text{ such that } \bar{t} > l(\bar{t}), \\ t_{i+1}, & \text{if there exists } \bar{t} \in [t_i, t_{i+1}] \text{ such that } \bar{t} < l(\bar{t}). \end{cases}$$

(d) *Let* $\{t_{i_k}\}_{k=1}^{k_0} \subseteq \{t_i\}_{i=1}^n$ *be all the fixed points such that* $\lim_{j \to +\infty} l^j(t) = t_{i_k}$ *for every* $t \in [t_{i_k}, t_{i_k+1})$. *Then for* $\delta > 0$ *small enough,*

$$\lim_{j \to +\infty} \sup_{t \in [t_{i_k}, t_{i_k+1} - \delta]} |l^j(t) - t_{i_k}| = 0,$$

$$\lim_{j \to +\infty} \sup_{t \in [t_i + \delta, t_{i+1}],\, t_i \in \{t_1, t_2, \cdots, t_n\} \setminus \{t_{i_k}\}_{k=1}^{k_0}} |l^j(t) - t_{i+1}| = 0.$$

(e) *Let* $h(t) = t - \tau(t)$. *Then* $l^j(t) = h^j(t) + j\tau(t_0)$ *for every* $t \in [t_0, t_0 + \tau(t_0)]$ *and* $j \in \mathbb{N}$;
(f) $h^j(t + \tau(t_0)) = h^j(t) + \tau(t_0)$ *for all* $t \in \mathbb{R}$ *and* $j \in \mathbb{N}$.

Recall that $C(x^*, \tau^*, \sigma^*, p^*)$ denotes the connected component of the closure of all the nonconstant periodic solutions of system (6.74) bifurcated at $(x^*, \tau^*, \sigma^*, p^*)$ in the Fuller space $C(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2$. We hope to exclude, for each periodic solution $(x_0, \tau_0, \sigma_0, p_0)$, certain values of the period. To be specific, we find an open interval I and a small open neighborhood $U \ni (x_0, \tau_0, \sigma_0, p_0)$ such that every $(x, \tau, \sigma, p) \in U \cap C(x^*, \tau^*, \sigma^*, p^*)$ satisfies $\tau(t) \neq mp$ for all $t \in I$ and $m \in \mathbb{N}$. Then we will glue up these local exclusions to a global upper bound for the period along the rescaled (by period normalization) connected component $C(y^*, z^*, \sigma^*, p^*)$.

We first consider the periods of the solutions in a neighborhood of a periodic solution that does not assume a certain period.

Lemma 6.24. *If a solution* $(x_0, \tau_0, \sigma_0, p_0) \in C(x^*, \tau^*, \sigma^*, p^*)$ *satisfies* $\tau_0(t_0) \neq mp_0$ *for some* $t_0 \in \mathbb{R}$ *and for all* $m \in \mathbb{N}$, *then there exist an open neighborhood* $I \ni t_0$ *and an open neighborhood* $U \ni (x_0, \tau_0, \sigma_0, p_0)$ *in* $C(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2$ *such that every solution* $(x, \tau, \sigma, p) \in U \cap C(x^*, \tau^*, \sigma^*, p^*)$ *satisfies* $\tau(t) \neq mp$ *for all* $m \in \mathbb{N}$ *and* $t \in I$.

Proof. By way of contradiction, we suppose that for every open interval $I \ni t_0$ and every open neighborhood $U \ni (x_0, \tau_0, \sigma_0, p_0)$ in $C(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2$, there exist $t \in I$, $m \in \mathbb{N}$, and a periodic solution $(x, \tau, \sigma, p) \in U \cap C(x^*, \tau^*, \sigma^*, p^*)$ such that $\tau(t) = mp$. Then there exist sequences $\{(x_k, \tau_k, \sigma_k, p_k, t_k)\}_{k=1}^{+\infty} \subseteq U \cap C(x^*, \tau^*, \sigma^*, p^*)$ and $\{m_k : m_k \in \mathbb{N}\}_{k=1}^{+\infty}$ such that

$$\begin{cases} \tau_k(t_k) = m_k p_k, \\ \lim_{k \to +\infty} (x_k, \tau_k, \sigma_k, p_k, t_k) = (x_0, \tau_0, \sigma_0, p_0, t_0). \end{cases} \tag{6.107}$$

Without loss of generality, we assume $m_k \to m_0 \in \mathbb{N}$ as $k \to +\infty$ (otherwise, we take a subsequence). Then it follows from (6.107), (SHB2), and (SHB5) that

$$m_0 = \lim_{k \to +\infty} m_k = \lim_{k \to +\infty} \frac{\tau_k(t_k)}{p_k} = \frac{\tau_0(t_0)}{p_0}. \tag{6.108}$$

Therefore, we have $\tau_0(t_0) = m_0 p_0$, which is a contradiction to the assumption. □

We note that for a nonconstant periodic solution (x, τ, σ) of system (6.74), it is allowed that $\tau(t)$ assume its stationary value τ_σ, or even $\tau(t) = \tau_\sigma$ for all $t \in \mathbb{R}$. Ruling out these cases turns out to be crucial for us to exclude certain values of periods of the periodic solutions.

Now we consider the periods of the periodic solutions in a neighborhood of a given nonconstant periodic solution in the Fuller space for which the delay τ-component is not equal to the corresponding stationary value at some time t. We need the following condition:

(SHB8) (i) $f(0, 0, \sigma) = 0$ for all $\sigma \in \mathbb{R}$;
(ii) $xf(x, x, \sigma)$ is positive (or negative) if $f(x, x, \sigma) \neq 0$.

Theorem 6.18. *Suppose that system (6.74) satisfies* (SHB6)–(SHB8). *Let $(x_0, \tau_0, \sigma_0, p_0)$ be a nonconstant periodic solution in $C(x^*, \tau^*, \sigma^*, p^*)$. If $\tau_0(t_0) \neq \tau_{\sigma_0}$ for some t_0, then there exist an open interval I and an open neighborhood U of $(x_0, \tau_0, \sigma_0, p_0)$ in $C(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2$ such that every solution (x, τ, σ, p) in $U \cap C(x^*, \tau^*, \sigma^*, p^*)$ satisfies $\tau(t) \neq mp$ for all $m \in \mathbb{N}$ and $t \in I$.*

Proof. We first show that there exist an open neighborhood U of $(x_0, \tau_0, \sigma_0, p_0)$ and an open neighborhood I_0 of t_0 such that $\tau(t) \neq \tau_{\sigma_0}$ for every $(x, \tau, \sigma, p) \in U \cap C(x^*, \tau^*, \sigma^*, p^*)$ and $t \in I_0$.

By way of contradiction, suppose that for every neighborhood $\tilde{I}$ of t_0 and neighborhood U of $(x_0, \tau_0, \sigma_0, p_0)$, there exist $t \in \tilde{I}$ and a nonconstant solution $(x, \tau, \sigma, p) \in U \cap C(x^*, \tau^*, \sigma^*, p^*)$ such that $\tau(t) = \tau_{\sigma_0}$. Then there exist a sequence of periodic solutions $\{(x_k, \tau_k, \sigma_k, p_k)\}_{k=1}^{+\infty}$ and $\{t_k\}_{k=1}^{+\infty}$ such that

$$\begin{cases} \tau_k(t_k) = \tau_{\sigma_k}, \\ \lim_{k \to +\infty} (x_k, \tau_k, \sigma_k, p_k, t_k) = (x_0, \tau_0, \sigma_0, p_0, t_0). \end{cases}$$

This, together with assumption (SHB2), implies that

$$
\begin{aligned}
|\tau_k(t_k) - \tau_0(t_0)| &\le |\tau_k(t_k) - \tau_k(t_0)| + |\tau_k(t_0) - \tau_0(t_0)| \\
&\le |t_k - t_0| + \sup_{t\in\mathbb{R}} \|\tau_k - \tau_0\| \\
&\to 0 \ \text{ as } k \to +\infty.
\end{aligned}
$$

Therefore, we have

$$\tau_0(t_0) = \lim_{k\to+\infty} \tau_k(t_k) = \lim_{k\to+\infty} \tau_{\sigma_k} = \tau_{\sigma_0}.$$

This is a contradiction to the assumption that $\tau_0(t_0) \neq \tau_{\sigma_0}$, and hence the claim is proved.

If $(x_0, \tau_0, \sigma_0, p_0)$ satisfies $\tau_0(t_0) \neq mp_0$ for all $m \in \mathbb{N}$, then the existence of I and U is followed from Lemma 6.24. Otherwise, $(x_0, \tau_0, \sigma_0, p_0)$ is $\tau_0(t_0)$-periodic. Let Γ_{σ_0} be the nonempty solution set of the equation $f(x, x, \sigma_0) = 0$ for $x \in \mathbb{R}^N$. Then by (SHB6), for every $x \in \Gamma_{\sigma_0}$, τ_{σ_0} is the unique solution of $g(x, \tau, \sigma_0) = 0$ for $\tau \in \mathbb{R}$. Now we distinguish two cases:

Case 1. $x_0(t_0) = x_{\sigma_0}$ for some $x_{\sigma_0} \in \Gamma_{\sigma_0}$. Since $\tau_0(t_0) \neq \tau_{\sigma_0}$, by system (6.74) and by (SHB6), we have

$$
\begin{cases}
\dot{x}_0(t_0) = f(x_{\sigma_0}, x_{\sigma_0}, \sigma_0) = 0, \\
\dot{\tau}_0(t_0) = g(x_{\sigma_0}, \tau_0(t_0), \sigma_0) \neq 0.
\end{cases}
\tag{6.109}
$$

Without loss of generality, we suppose $\dot{\tau}_0(t) > 0$ for t in some open neighborhood of t_0. Then, by the continuity and local monotonicity of $\tau_0(t)$, there exists $\delta > 0$ small enough that

$$0 < \tau_0(t) - \tau_0(t_0) < p_{\min}, t \in (t_0, t_0 + \delta),$$

where $p_{\min} > 0$ is the minimal period of (x_0, τ_0). Then $\tau_0(t) \neq m p_{\min}$ for every $m \in \mathbb{N}$. Therefore, (x_0, τ_0) is not $\tau_0(t)$-periodic for all $t \in (t_0, t_0 + \delta)$. So we have $\tau_0(t) \neq mp_0$ for all $t \in (t_0, t_0 + \delta)$ and $m \in \mathbb{N}$.

By Lemma 6.24, for every $t^* \in (t_0, t_0 + \delta)$, there exist an open interval I of t^* and an open neighborhood U of $(x_0, \tau_0, \sigma_0, p_0)$ in $C(\mathbb{R}; \mathbb{R}^{N+1}) \times \mathbb{R}^2$ such that every solution (x, τ, σ, p) in $U \cap C(x^*, \tau^*, \sigma^*, p^*)$ satisfies $\tau(t) \neq mp$ for all $m \in \mathbb{N}$ and $t \in I$.

Case 2. $x_0(t_0) \neq x_\sigma$ for every $x_\sigma \in \Gamma_{\sigma_0}$. By Lemma 6.23 (c), there are finitely many fixed points $\{t_i\}_{i=1}^n$ of $l(t) = t - \tau_0(t) + \tau_0(t_0)$ in $[t_0, t_0 + \tau_0(t_0)]$ that are in ascending order (we assume in the proof that all the sequences of the fixed points of l are in ascending order). And we can let the subsequence $\{t_{i_k}\}_{k=1}^{k_0} \subseteq \{t_i\}_{i=1}^n$ be all the fixed points such that $\lim_{j\to+\infty} l^j(t) = t_{i_k}$ for every $t \in [t_{i_k}, t_{i_k+1})$. Note that $\tau_0(t_i) = \tau_0(t_0)$ and $\tau_0(t_0) \neq \tau_{\sigma_0}$ implies that $\tau_0(t_i) \neq \tau_{\sigma_0}$ for all $i \in \{1, 2, \cdots, n\}$. If $x_0(t_{i_0}) = x_{\sigma_0}$ for some $i_0 \in \{1, 2, \cdots, n\}$ and for some $x_{\sigma_0} \in \Gamma_{\sigma_0}$. Then the conclusion follows by Case 1 with t_0 replaced by t_{i_0}.

Now we exclude that $x_0(t_i) \neq x_\sigma$ for every $i \in \{0, 1, 2, \cdots, n\}$ and for every $x_\sigma \in \Gamma_{\sigma_0}$. Assume that the contrary is true. We want to obtain a contradiction under the assumption that (x_0, τ_0) is $\tau_0(t_0)$-periodic.

For $\delta > 0$ small enough, we consider the following compact subset I_δ of $[t_0, t_0 + \tau_0(t_0)]$:

$$I_\delta = \bigcup_{t_{i_k} \in \{t_{i_1}, t_{i_2}, \cdots, t_{i_{k_0}}\}} [t_{i_k}, t_{i_k+1} - \delta] \bigcup \bigcup_{t_i \in \{t_1, t_2, \cdots, t_n\} \setminus \{t_{i_k}\}_{k=1}^{k_0}} [t_i + \delta, t_i].$$

Note that for each interval $[t_i, t_{i+1}]$, only one of the endpoints is the limit of $\lim_{j\to+\infty} l^j(t)$ for every $t \in (t_i, t_{i+1})$. Note also that when δ goes to zero, I_δ goes to $[t_0, t_0 + \tau_0(t_0)]$ in the sense of Lebesgue measure.

Now for $\delta > 0$ small enough, we introduce the following piecewise constant function $\chi(t)$ on the compact subset I_δ of $[t_0, t_0 + \tau_0(t_0)]$:

$$\chi(t) = \begin{cases} t_{i_k}, & \text{if } t \in [t_{i_k}, t_{i_k+1} - \delta],\, t_{i_k} \in \{t_{i_k}\}_{k=1}^{k_0}, \\ t_{i+1}, & \text{if } t \in [t_i + \delta, t_{i+1}],\, t_i \in \{t_1, t_2, \cdots, t_n\} \setminus \{t_{i_k}\}_{k=1}^{k_0}. \end{cases}$$

Since the number of intervals with endpoints the fixed points of $l(t)$ is finite, it is clear from Lemma 6.23 (d) that

$$\lim_{j\to+\infty} \sup_{t\in I_\delta} |l^j(t) - \chi(t)| = 0. \tag{6.110}$$

Note that $(x(t), \tau(t))$ is a periodic solution of system (6.74). There exists $\tilde{M} > 0$ such that $|\dot{x}(t)| \le \tilde{M}$ for every $t \in [t_0, t_0 + \tau(t_0)]$. Let I_i with $i \in \{1, 2, \cdots, n\}$ be the subinterval of I_δ that is either $[t_{i-1}, t_i - \delta]$ or $[t_{i-1} + \delta, t_i]$. Then we have $\chi(t) = t_{i-1}$ or $\chi(t) = t_i$ for $t \in I_i$, and hence we have

$$x_0(\chi(t)) = x_0(t_{i-1}) \text{ or } x_0(\chi(t)) = x_0(t_i) \text{ for every } t \in I_i. \tag{6.111}$$

Since $x_0(t_i) \ne x_\sigma$ for every $i \in \{0, 1, 2, \cdots, n\}$ and for every $x_\sigma \in \Gamma_{\sigma_0}$, by (6.111), we have

$$x_0(\chi(t)) \notin \Gamma_{\sigma_0} \text{ for every } t \in I_\delta. \tag{6.112}$$

By (6.110), for every $\varepsilon > 0$, there exists $N_0 > 0$ large enough that

$$\sup_{t\in I_\delta} |l^j(t) - \chi(t)| \le \varepsilon, \text{ for every } j > N_0. \tag{6.113}$$

Let $(x_j(t), \tau_j(t)) = (x_0(h^j(t)), \tau_0(h^j(t))$ for $j = 0, 1, 2, \cdots$, where we define $h^0(t) = t$. Then by Lemma 6.23 (e), we have $(x_j(t), \tau_j(t)) = (x_0(l^j(t)), \tau_0(l^j(t))$. Note that I_δ is composed of finitely many subintervals. By applying the integral mean value theorem to each subinterval of I_δ and by (6.113), we have for every $j > N_0$ that

$$\sup_{t\in I_\delta} |x_0(l^j(t)) - x_0(\chi(t))| \le \sup_{t\in I_\delta} |\dot{x}_0(t)| \sup_{t\in I_\delta} |l^j(t) - \chi(t)| \le \tilde{M}\varepsilon. \tag{6.114}$$

Differentiating $x_j(t)$ for $j = 1, 2, \cdots$, we can obtain from system (6.74) that

$$\dot{x}_j(t) = \prod_{m=0}^{j-1}(1 - g(x_m(t),\, \tau_m(t)),\, \sigma_0) f(x_j(t), x_{j+1}(t),\, \sigma_0). \tag{6.115}$$

Since $g(x,\, \tau,\, \sigma) < 1$, we have

$$\prod_{m=0}^{j-1}(1 - g(x_m(t),\, \tau_m(t)),\, \sigma_0) > 0, t \in \mathbb{R}. \tag{6.116}$$

Also by (ii) of (SHB8), $xf(x,\, x,\, \sigma_0) > 0$ as long as $x \notin \Gamma_{\sigma_0}$. Then by (6.112) we have

$$x_0(\chi(t)) f(x_0(\chi(t)), x_0(\chi(t)),\, \sigma_0) > 0 \tag{6.117}$$

for every $t \in I_\delta$. By (6.114), (6.117), and by the continuity of f, it follows that there exists $N_1 > N_0$ such that

$$x_j(t) f(x_j(t), x_{j+1}(t),\, \sigma_0) > 0 \text{ for } j > N_1 \text{ and } t \in I_\delta. \tag{6.118}$$

Therefore, for every $t \in I_\delta$ and $j > N_1$, by (6.115), (6.116), and (6.118), we have

$$x_j(t) \cdot \dot{x}_j(t) = \prod_{m=0}^{j-1}(1 - g(x_m(t),\, \tau_m(t)),\, \sigma_0) x_j(t) f(x_j(t), x_{j+1}(t),\, \sigma_0) > 0. \tag{6.119}$$

Since $\delta > 0$ is arbitrary and I_δ goes to I in measure as $\delta \to 0$, by the continuity of $x_j \cdot \dot{x}_j$, we have $x_j(t) \cdot \dot{x}_j(t) \geq 0$ for every $t \in I$ and $j > N_1$. By (6.119), we know that $x_j \cdot \dot{x}_j \not\equiv 0$ on I with $j > N_1$. Therefore, $x_j \cdot x_j$ is a nonconstant increasing continuous function. But this is impossible, since $x_j \cdot x_j$ is continuous and periodic. This completes the proof. □

We now consider the periods of nonconstant periodic solutions, where the delay coincides with the corresponding stationary value for every $t \in \mathbb{R}$.

Lemma 6.25. *Suppose system (6.74) satisfies* (SHB7)*. Let $(x,\, \tau,\, \sigma,\, p)$ be a nonconstant p-periodic solution of system (6.74). If $\tau(t) = \tau_\sigma$ for every $t \in \mathbb{R}$, then $(x,\, \tau,\, \sigma,\, p)$ is not τ_σ-periodic.*

Proof. Suppose, by way of contradiction, that $(x,\, \tau,\, \sigma,\, p)$ is τ_σ-periodic. If $\tau(t) = \tau_\sigma$ for every $t \in \mathbb{R}$, then we have

$$\begin{cases} \dot{x}(t) = f(x(t), x(t),\, \sigma), \\ \quad 0 = \dot{\tau}(t) = g(x(t),\, \tau_\sigma,\, \sigma). \end{cases} \tag{6.120}$$

It follows from (6.120) that

$$\ddot{\tau}(t) = \frac{\partial g}{\partial x}(x(t),\, \tau_\sigma,\, \sigma) \cdot f(x(t), x(t),\, \sigma) = 0. \tag{6.121}$$

Then by (SHB7) and (6.121), $x(t) = x_\sigma$ for every $t \in \mathbb{R}$. Thus, (x, τ, σ, p) is a constant periodic solution of (6.74). This is a contradiction. □

We now formulate our next assumption:

(SHB9) For every Hopf bifurcation point $(x, \tau, \sigma, p) \in C(x^*, \tau^*, \sigma^*, p^*)$, $mp \neq \tau$ for every $m \in \mathbb{N}$.

Theorem 6.19. *Assume that system (6.74) satisfies* (SHB6)–(SHB9). *Then for every solution* $(x_0, \tau_0, \sigma_0, p_0) \in C(x^*, \tau^*, \sigma^*, p^*)$*, there exist an open interval* I *and an open neighborhood* $U \ni (x_0, \tau_0, \sigma_0, p_0)$ *such that every solution*

$$(x, \tau, \sigma, p) \in U \cap C(x^*, \tau^*, \sigma^*, p^*)$$

satisfies $\tau(t) \neq mp$ *for all* $m \in \mathbb{N}$ *and* $t \in I$.

Proof. For a given $\sigma_0 \in \mathbb{R}$, if $(x_0, \tau_0, \sigma_0, p_0) \in C(x^*, \tau^*, \sigma^*, p^*)$ is a constant periodic solution, then it is a Hopf bifurcation point of system (6.74) (See Lemma 6.21). Thus the existence of an open interval I and an open neighborhood $U \ni (x_0, \tau_0, \sigma_0, p_0)$ follows immediately from (SHB9) and Lemma 6.24.

If $(x_0, \tau_0, \sigma_0, p_0) \in C(x^*, \tau^*, \sigma^*, p^*)$ is a nonconstant periodic solution and $\tau_0(t) = \tau_{\sigma_0}$ for all $t \in \mathbb{R}$, then by Lemma 6.25, $(x_0, \tau_0, \sigma_0, p_0)$ is not τ_{σ_0}-periodic. The conclusion is implied by Lemma 6.24.

If $(x_0, \tau_0, \sigma_0, p_0)$ is a nonconstant periodic solution and $\tau_0(t) \neq \tau_{\sigma_0}$ for some $t \in \mathbb{R}$, then the conclusion follows from Theorem 6.18. □

We now start the process that uses the local exclusion of periods developed above to construct a uniform upper bound for periods of solutions in the Fuller space. To achieve this goal, we need to "glue" the local exclusion of periods along the connected component. Now we shall show that (6.106) is valid, provided that (6.105) holds.

Theorem 6.20. *Let* $C(y^*, z^*, \sigma^*, p^*)$ *be a connected component of the closure of all the nonconstant periodic solutions of system (6.100), bifurcated from* $(y^*, z^*, \sigma^*, p^*)$ *in the Fuller space* $C(\mathbb{R}/2\pi; \mathbb{R}^{N+1}) \times \mathbb{R}^2$*. Suppose that system (6.74) satisfies* (SHB6)–(SHB9). *Then for every* $(y_0, z_0, \sigma_0, p_0) \in C(y^*, z^*, \sigma^*, p^*)$*, there exist an open interval* I *and an open neighborhood* $U \ni (y_0, z_0, \sigma_0, p_0)$ *such that* $mp \neq z(t)$ *for every solution* $(y, z, \sigma, p) \in U \cap C(y^*, z^*, \sigma^*, p^*)$*,* $m \in \mathbb{N}$ *and* $t \in I$.

Proof. Note that $p > 0$ for every solution (y, z, σ, p) in $C(y^*, z^*, \sigma^*, p^*)$. We show that the mapping

$$\begin{aligned} \iota : C(y^*, z^*, \sigma^*, p^*) &\to C(x^*, \tau^*, \sigma^*, p^*) \\ (y(\cdot), z(\cdot), \sigma, p) &\to \left(y\left(\frac{2\pi}{p}\cdot\right), z\left(\frac{2\pi}{p}\cdot\right), \sigma, p \right) \end{aligned} \tag{6.122}$$

is continuous, where $C(x^*, \tau^*, \sigma^*, p^*) \subseteq C(\mathbb{R};\mathbb{R}^{N+1}) \times \mathbb{R}^2$. Indeed, if

$$\lim_{n\to+\infty} \|(y_n(\cdot), z_n(\cdot), \sigma_n, p_n) - (y_0(\cdot), z_0(\cdot), \sigma_0, p_0)\|_{C(\mathbb{R}/2\pi;\mathbb{R}^{N+1})\times\mathbb{R}^2} = 0,$$

then we have

$$\begin{aligned}
&\|\iota(y_n(\cdot), z_n(\cdot), \sigma_n, p_n) - \iota(y_0(\cdot), z_0(\cdot), \sigma_0, p_0)\|_{C(\mathbb{R};\mathbb{R}^{N+1})\times\mathbb{R}} \\
&= |y_n\left(\frac{2\pi}{p_n}\cdot\right) - y_0\left(\frac{2\pi}{p_0}\cdot\right)|_C + |z_n\left(\frac{2\pi}{p_n}\cdot\right) - z_0\left(\frac{2\pi}{p_0}\cdot\right)|_C \\
&\quad + |\sigma_n - \sigma_0| + |p_n - p_0| \\
&\le |y_n - y_0|_C + 2\pi|\dot{y}_0|\left|\frac{1}{p_n} - \frac{1}{p_0}\right| + |z_n - z_0|_C + 2\pi|\dot{z}_0|\left|\frac{1}{p_n} - \frac{1}{p_0}\right| \\
&\quad + |\sigma_n - \sigma_0| + |p_n - p_0| \\
&\to 0 \text{ as } n \to +\infty,
\end{aligned}$$

where $|\cdot|_C$ denotes the supremum norm in either $C(\mathbb{R}/2\pi;\mathbb{R}^N)$ or $C(\mathbb{R}/2\pi;\mathbb{R})$. Therefore, $C(x^*, \tau^*, \sigma^*, p^*)$ is a connected component of periodic solutions of (6.74).

Let $(x_0, \tau_0, \sigma_0, p_0) = \iota(y_0, z_0, \sigma_0, p_0) \in C(x^*, \tau^*, \sigma^*, p^*)$. Then by Theorem 6.19, there exist an open interval I' and an open neighborhood $U' \ni (x_0, \tau_0, \sigma_0, p_0)$ such that every solution $(x, \tau, \sigma, p) \in U' \cap C(x^*, \tau^*, \sigma^*, p^*)$ satisfies $\tau(t) \neq mp$ for all $m \in \mathbb{N}$ and $t \in I'$.

Since ι is continuous, we can choose an open set $U \subseteq C(\mathbb{R}/2\pi;\mathbb{R}^{N+1}) \times \mathbb{R}^2$ small enough that $(y_0, z_0, \sigma_0, p_0) \in U \subseteq \iota^{-1}(U')$ and the open set

$$I \stackrel{\text{def}}{=} \bigcap_{\{p:(y,z,\sigma,p)\in U\}} \frac{p}{2\pi} \cdot I'$$

is nonempty. Then by the definition of ι, $mp \neq z(t)$ for every $(y, z, \sigma, p) \in U \cap C(y^*, z^*, \sigma^*, p^*)$, $m \in \mathbb{N}$, and $t \in I$. □

Lemma 6.26 (The generalized intermediate value theorem [227]). *Let $f : X \to Y$ be a continuous map from a connected space X to a linearly ordered set Y with order topology. If $a, b \in X$ and $y \in Y$ lies between $f(a)$ and $f(b)$, then there exists $x \in X$ such that $f(x) = y$.*

Definition 6.5. Let $C(y^*, z^*, \sigma^*, p^*)$ be a connected component of the closure of all the nonconstant periodic solutions of system (6.100), bifurcated from $(y^*, z^*, \sigma^*, p^*)$ in the Fuller space $C(\mathbb{R}/2\pi;\mathbb{R}^{N+1}) \times \mathbb{R}^2$. Let $I \subset \mathbb{R}$ be an interval and U a subset in $C(y^*, z^*, \sigma^*, p^*)$. We call $I \times (U \cap C(y^*, z^*, \sigma^*, p^*))$ a delay-period disparity set if every solution

$$(y, z, \sigma, p) \in U \cap C(y^*, z^*, \sigma^*, p^*)$$

satisfies $mp \neq z(t)$ for every $t \in I$ and $m \in \mathbb{N}$. We call $I \times (U \cap C(y^*, z^*, \sigma^*, p^*))$ a delay-period disparity set at $(t_0, y_0, z_0, \sigma_0, p_0)$ if $(t_0, y_0, z_0, \sigma_0, p_0) \in I \times (U \cap C(y^*, z^*, \sigma^*, p^*))$.

In the remainder of this subsection, the following assumption is sometimes needed:

(SHB10) Every periodic solution (x, τ, σ) of (6.74) satisfies $\tau(t) > 0$ for every $t \in \mathbb{R}$.

Lemma 6.27. *Suppose that system (6.74) satisfies* (SHB6)–(SHB7) *and* (x, τ, σ) *is a nonconstant periodic solution. If*

(i) $\tau \not\equiv \tau_\sigma$ *and there exists* $t_0 \in \mathbb{R}$ *such that* $\tau(t_0) = \tau_\sigma$*, and*
(ii) (x, τ) *is* τ_σ*-periodic,*

then there exists $t_1 \in \mathbb{R}$ *such that* $\tau(t_1) > \tau_\sigma$*.*

Proof. We prove the result by contradiction. Suppose that

$$\tau(t) \leq \tau_\sigma \text{ for every } t \in \mathbb{R}. \tag{6.123}$$

Then since $\tau \not\equiv \tau_\sigma$, there exists $t^* \in \mathbb{R}$ such that $\tau(t^*) < \tau_\sigma$. We can choose a maximal interval $[a, b] \subset \mathbb{R}$ that contains t^* in the sense that

$$\tau(t) < \tau_\sigma \text{ for any } t \in (a, b), \tag{6.124}$$

$$\tau(t) = \tau_\sigma \text{ for any } t = a \text{ and } t = b. \tag{6.125}$$

If $\dot{\tau}(a) \neq 0$ or $\dot{\tau}(b) \neq 0$, then it follows from the local monotonicity of $\tau(t)$ (at a or b) that there exists $t_1 \in \mathbb{R}$ in some neighborhood of a or b such that $\tau(t_1) > \tau_\sigma$. This is a contradiction to (6.123).

If $\dot{\tau}(a) = \dot{\tau}(b) = 0$, then we have

$$g(x(a), \tau_\sigma, \sigma) = g(x(b), \tau_\sigma, \sigma) = 0. \tag{6.126}$$

We distinguish the following two cases:

Case 1. $x(a) \neq x_\sigma$ or $x(b) \neq x_\sigma$. Without loss of generality, we suppose $x(a) \neq x_\sigma$. Then by (ii), we have

$$\ddot{\tau}(a) = \frac{\partial g}{\partial x}(x(a), \tau_\sigma, \sigma) f(x(a), x(a), \sigma). \tag{6.127}$$

It follows from (SHB7), (6.126), and (6.127) that $\ddot{\tau}(a) \neq 0$. Therefore, we have that $\dot{\tau}(t)$ is strictly monotonic in some neighborhood of a. Hence there exists $t_1 \in \mathbb{R}$ such that $\tau(t_1) > \tau_\sigma$. This is also a contradiction to (6.123).

Case 2. $x(a) = x(b) = x_\sigma$. By (S5), we have $\frac{\partial g}{\partial \tau}(x_\sigma, \tau_\sigma, \sigma) \neq 0$. Without loss of generality, we assume that

$$\frac{\partial g}{\partial \tau}(x_\sigma, \tau_\sigma, \sigma) < 0. \tag{6.128}$$

Then by (6.124), (6.126), (6.128), and the continuity of $x(t)$ and $\tau(t)$, we can choose $\varepsilon > 0$ small enough that

$$\dot{\tau}(t) = g(x(t), \tau(t), \sigma) > 0 \text{ for every } t \in (a, a+\varepsilon) \cup (b-\varepsilon, b). \tag{6.129}$$

Therefore, we have $\tau(a) < \tau(a+\varepsilon)$. That is, there exists $t_1 = a+\varepsilon$ such that $\tau(a) = \tau_\sigma < \tau(t_1)$. This is a contradiction to (6.123). The proof is complete. □

Lemma 6.28. *Suppose that (6.74) satisfies* (SHB6)–(SHB10). *Let $C(y^*, z^*, \sigma^*, p^*)$ be a connected component of the closure of all the nonconstant periodic solutions of system (6.100), bifurcated from $(y^*, z^*, \sigma^*, p^*)$ in the Fuller space $C(\mathbb{R}/2\pi; \mathbb{R}^{N+1}) \times \mathbb{R}^2$. Let $I \subset \mathbb{R}$ be an open interval and $\bar{v} \stackrel{\text{def}}{=} (\bar{y}, \bar{z}, \bar{\sigma}, \bar{p}) \in C(y^*, z^*, \sigma^*, p^*)$. If there is no delay-period disparity set at $(t, \bar{u})$ for any $t \in I$, then*

(i) there exists $m \in \mathbb{N}$ such that $m\bar{p} = \bar{z}(t) = z_{\bar{\sigma}}$ for every $t \in I$;
(ii) $\bar{v}$ is a nonconstant solution with $\bar{z}(t) = z_{\bar{\sigma}}$ for every $t \in I$;
(iii) there exist an open interval $I' \subseteq \mathbb{R}$ and an open neighborhood U' of $\bar{v}$ such that $I' \times (U' \cap C(y^*, z^*, \sigma^*, p^*))$ is a delay-period disparity set with $\bar{v} \in U' \cap C(y^*, z^*, \sigma^*, p^*)$, and the inequality $z_{\bar{\sigma}} < \bar{z}(t)$ holds for every $t \in I'$.

Proof. (i) By Definition 6.5, for every $t \in I$, there exists $m \in \mathbb{N}$ such that $\bar{z}(t) = m\bar{p}$. Note that $\bar{z}(t)$ is continuous, $\bar{z}(t) = m\bar{p}$ for every $t \in I$. Then for every $t \in I$, we have

$$\dot{\bar{y}}(t) = \frac{\bar{p}}{2\pi} f(\bar{y}(t), \bar{y}(t), \bar{\sigma}), \tag{6.130}$$

$$\dot{\bar{z}}(t) = \frac{\bar{p}}{2\pi} g(\bar{y}(t), m\bar{p}, \bar{\sigma}) = 0. \tag{6.131}$$

By (6.131), we have

$$\ddot{\bar{z}}(t) = \frac{\bar{p}^2}{4\pi^2} \frac{\partial g}{\partial x}(\bar{y}(t), m\bar{p}), \bar{\sigma}) \cdot f(\bar{y}(t), \bar{y}(t), \bar{\sigma}) = 0. \tag{6.132}$$

By (SHB7), (6.131), and (6.132), we have $\bar{y}(t) = y_{\bar{\sigma}}$ on I. Hence by (SHB6) and by (6.131), we have $\bar{z}(t) = z_{\bar{\sigma}} = m\bar{p}$ on I. This finishes the proof of (i).

(ii) Note that the stationary solutions of (6.74) and (6.100) are equal. That is, $(x_\sigma, \tau_\sigma) = (y_\sigma, z_\sigma)$ for every $\sigma \in \mathbb{R}$.
If $\bar{v}$ is a constant solution, then by (i) we have $\bar{z}(t) = z_{\bar{\sigma}} = m\bar{p}$ and $\bar{y}(t) = y_{\bar{\sigma}}$ for all $t \in \mathbb{R}$. Then $(y_{\bar{\sigma}}, z_{\bar{\sigma}}, \bar{\sigma}, \bar{p})$ is a bifurcation point in $C(y^*, z^*, \sigma^*, p^*)$ that satisfies $z_{\bar{\sigma}} = m\bar{p}$ for some $m \in \mathbb{N}$. This contradicts assumption (SHB9). So $\bar{v}$ is a nonconstant solution with $\bar{z}(t) = z_{\bar{\sigma}}$ for all $t \in I$.

(iii) Now we show that there exists $t_0 \in \mathbb{R}$ such that $\bar{z}(t_0) \neq z_{\bar{\sigma}}$. If not, that is, if $\bar{z}(t) = z_{\bar{\sigma}}$ for all $t \in \mathbb{R}$, then

$$(\bar{x}(\cdot), \bar{\tau}(\cdot), \bar{\sigma}) \stackrel{\text{def}}{=} (\bar{y}(\frac{2\pi}{\bar{p}}\cdot), \bar{z}(\frac{2\pi}{\bar{p}}\cdot), \bar{\sigma}) = (\bar{y}(\frac{2\pi}{\bar{p}}\cdot), z_{\bar{\sigma}}, \bar{\sigma}) = (\bar{y}(\frac{2\pi}{\bar{p}}\cdot), \tau_{\bar{\sigma}}, \bar{\sigma})$$

is a solution of (6.74). Then by Lemma 6.25, $(\bar{x}, \bar{\tau})$ is not $\tau_{\bar{\sigma}}$-periodic. Then we have $m\bar{p} \neq z_{\bar{\sigma}}$ for every $m \in \mathbb{N}$. This is a contradiction to (i).

Therefore, there exists $t_0 \in \mathbb{R}$ such that $\bar{z}(t_0) \neq z_{\bar{\sigma}}$. That is, $\bar{\tau}(\frac{\bar{p}}{2\pi}t_0) \neq \tau_{\bar{\sigma}}$. Note that by (i), $(\bar{x}, \bar{\tau})$ is $\tau_{\bar{\sigma}}$-periodic and $\bar{\tau}(t) = \tau_{\bar{\sigma}}$ on $\frac{\bar{p}}{2\pi}I$. Then by Lemma 6.27, there exists $t_1 \in \mathbb{R}$ such that

$$\bar{\tau}(t_1) > \tau_{\bar{\sigma}}. \tag{6.133}$$

By the continuity of $\bar{\tau}$ and by (6.133), there exists a finite interval $(a,b) \ni t_1$ such that for every $t \in (a,b)$,

$$\bar{\tau}(t) > \tau_{\bar{\sigma}}. \tag{6.134}$$

We claim that there exists $t_0 \in (a,b)$ such that $\bar{v}$ is not $\bar{\tau}(t_0)$-periodic. Indeed, if not, then $\bar{v}$ would be $\bar{\tau}(t)$-periodic for every $t \in (a,b)$. Then by the continuity of $\bar{\tau}$ and by (6.134), there would exist $t_1, t_2 \in (a,b)$ and an interval $(\bar{\tau}(t_1), \bar{\tau}(t_2))$ with $\bar{\tau}(t_2) > \bar{\tau}(t_1)$, so that $\bar{\tau}$ would be p-periodic for all $p \in (\bar{\tau}(t_1), \bar{\tau}(t_2))$. Hence $\bar{v}$ would be a constant solution. This is a contradiction to (ii), and the claim is proved.

Then we have $\bar{\tau}(t_0) \neq m\bar{p}$ for all $m \in \mathbb{N}$. By Lemma 6.24, there exist an open interval $I_1 \ni t_0$ and an open neighborhood $U_1 \ni (\bar{x}, \bar{\tau}, \bar{\sigma}, \bar{p})$ such that every solution (x, τ, σ, p) of (6.74) in $U_1 \cap C(x^*, \tau^*, \sigma^*, p^*)$ satisfies $\tau(t) \neq mp$ for all $m \in \mathbb{N}$ and $t \in I_1$. Note that $\bar{\tau}$ is continuous at $t = t_0$. We can therefore choose I_1 small enough that (6.134) holds for all $t \in I_1$.

Let ι be the continuous mapping defined by (6.122). Then we can choose an open set $U' \subseteq C(\mathbb{R}/2\pi; \mathbb{R}^{N+1}) \times \mathbb{R}^2$ small enough that $\bar{v} \in U' \subseteq \iota^{-1}(U_1)$ and

$$I' \stackrel{\text{def}}{=} \bigcap_{\{p:(y,z,\sigma,p)\in U'\}} \frac{p}{2\pi} \cdot I_1$$

is nonempty. It follows from the definition of ι that $mp \neq z(t)$ for every solution $(y, z, \sigma, p) \in U' \cap C(y^*, z^*, \sigma^*, p^*)$, $m \in \mathbb{N}$, and $t \in I'$. In particular, noting that (6.134) holds for all $t \in I_1$ and $I' \subseteq \frac{\bar{p}}{2\pi}I_1$, we have

$$\bar{z}(t) > z_{\bar{\sigma}} \tag{6.135}$$

for every $t \in I'$. This completes the proof. □

Now we are able to state our main result.

Theorem 6.21. *Let $C(y^*, z^*, \sigma^*, p^*)$ be a connected component of the closure of all the nonconstant periodic solutions of system (6.100), bifurcated from $(y^*, z^*, \sigma^*, p^*)$ in the Fuller space $C(\mathbb{R}/2\pi; \mathbb{R}^{N+1}) \times \mathbb{R}^2$. Suppose that (6.74) satisfies* (SHB6)–(SHB10). *If $p^* < z^*$, then for every $(y, z, \sigma, p) \in C(y^*, z^*, \sigma^*, p^*)$, $p < z(t)$ for some $t \in \mathbb{R}$.*

Proof. By Theorem 6.20 and (SHB9), there exist an open interval $I^* \subseteq \mathbb{R}$ and an open set U^* in $C(\mathbb{R}/2\pi; \mathbb{R}^{N+1}) \times \mathbb{R}^2$ such that $I^* \times (U^* \cap C(y^*, z^*, \sigma^*, p^*))$ is a delay-period disparity set with $(y^*, z^*, \sigma^*, p^*) \in U^*$.

Let $A^* \ni (y^*, z^*, \sigma^*, p^*)$ be a connected component of $(U^* \cap C(y^*, z^*, \sigma^*, p^*))$. Then $I^* \times A^*$ is connected in $\mathbb{R} \times C(\mathbb{R}/2\pi; \mathbb{R}^{N+1}) \times \mathbb{R}^2$. Define $S : \mathbb{R} \times C(\mathbb{R}/2\pi; \mathbb{R}^{N+1}) \times \mathbb{R}^2 \to \mathbb{R}$ by

$$S(t, y, z, \sigma, p) = p - z(t).$$

Note that we have $p^* < z^*$. Then it follows that $S(t, y^*, z^*, \sigma^*, p^*) = p^* - z^* < 0$. Note that S is continuous. By Lemma 6.26, we have

$$S(t, y, z, \sigma, p) = p - z(t) < 0 \tag{6.136}$$

for every $(t, y, z, \sigma, p) \in I^* \times A^*$, for otherwise, there would exist $(t_0, y_0, z_0, \sigma_0, p_0) \in I^* \times A^*$ such that $p_0 = z_0(t_0)$, which contradicts the fact that $I^* \times A^*$ is a subset of the forbidden range of delay $I^* \times (U^* \cap C(y^*, z^*, \sigma^*, p^*))$.

Now we show that there exists a sequence of connected subsets of $C(y^*, z^*, \sigma^*, p^*)$, denoted by $\{A_n\}_{n=1}^{n_0}$, $n_0 \in \mathbb{N}$ or $n_0 = +\infty$, that satisfies

(i) $A^* \subseteq A_1 \subset A_2 \subset \cdots \subset A_{n_0}$ and $\cup_{n=1}^{n_0} A_n = C(y^*, z^*, \sigma^*, p^*)$;
(ii) for every $(y, z, \sigma, p) \in A_n$ with $n \in \{1, 2, \cdots, n_0\}$, $p < z(t)$ at some $t \in \mathbb{R}$.

Let $A_1 \stackrel{\text{def}}{=} A^*$. If $A_1 = C(y^*, z^*, \sigma^*, p^*))$, then we are done by (6.136). If not, since the only sets that are both closed and open in the connected topological space $C(y^*, z^*, \sigma^*, p^*)$ are the empty set and the connected component $C(y^*, z^*, \sigma^*, p^*)$ itself, $A_1 \ni (y^*, z^*, \sigma^*, p^*)$ is not both closed and open. Then the boundary of A_1 in the sense of the relative topology induced by $C(y^*, z^*, \sigma^*, p^*)$ is nonempty. That is,

$$\partial A_1 \neq \emptyset. \tag{6.137}$$

Let $\bar{v} = (\bar{y}, \bar{z}, \bar{\sigma}, \bar{p}) \in \partial A_1$. If there exist $t_1 \in I_1 \stackrel{\text{def}}{=} I^*$ and a delay-period disparity set $I' \times (U' \cap C(y^*, z^*, \sigma^*, p^*))$ such that $(t_1, \bar{v}) \in \bar{I}' \times (U' \cap C(y^*, z^*, \sigma^*, p^*)$, and if $A_{\bar{v}} \ni \bar{v}$ is the connected component of $U' \cap C(y^*, z^*, \sigma^*, p^*)$, then it is clear that $A_1 \cup A_{\bar{v}}$ is connected. Since A_1 is closed, we have $\bar{p} < \bar{z}(t_1)$. Then by Lemma 6.26, we have

$$S(t, y, z, \sigma, p) = p - z(t) < 0 \text{ for every } (t, y, z, \sigma, p) \in I' \times A_{\bar{v}}. \tag{6.138}$$

If for every $t \in I_1$, there is no delay-period disparity set at $(t, \bar{u})$, then by Lemma 6.28, there exists a delay-period disparity set $I'' \times (U'' \cap C(y^*, z^*, \sigma^*, p^*)$ with $\bar{v} \in U' \cap C(y^*, z^*, \sigma^*, p^*)$ and

$$m\bar{p} = z_{\bar{\sigma}} < \bar{z}(t) \text{ for every } t \in I'' \text{ and } m \in \mathbb{N}. \tag{6.139}$$

Let $A_{\bar{v}} \ni \bar{v}$ be the connected component of $U'' \cap C(y^*, z^*, \sigma^*, p^*)$. It is clear that $A_1 \cup A_{\bar{v}}$ is connected. Then by (6.139) and Lemma 6.26,

$$S(t, y, z, \sigma, p) = p - z(t) < 0 \text{ for any } (t, y, z, \sigma, p) \in I'' \times A_{\bar{v}}. \tag{6.140}$$

By (6.138) and (6.140), we know that if $\bar{v} \in \partial A_1$, then there exists a delay-period disparity set $\tilde{I} \times (\tilde{U} \cap C(y^*, z^*, \sigma^*, p^*))$ with $A_{\bar{v}} \ni \bar{v}$ the connected component of $\tilde{U} \cap C(y^*, z^*, \sigma^*, p^*)$ such that

$$S(t, y, z, \sigma, p) = p - z(t) < 0 \text{ for any } (t, y, z, \sigma, p) \in \tilde{I} \times A_{\bar{v}}. \tag{6.141}$$

For every $\bar{v} \in \partial A_1$, we find a $A_{\bar{v}}$ satisfying (6.141). Then we define

$$A_2 = A_1 \cup \bigcup_{\bar{v} \in \partial A_1} A_{\bar{v}}.$$

It follows from (6.136), (6.138), and (6.140) that for every $(y, z, \sigma, p) \in A_2$, $p < z(t)$ for some $t \in \mathbb{R}$. Note that for every $\bar{v} \in \partial A_1$, $A_1 \cup A_{\bar{v}}$ is connected. Therefore, A_2 is connected.

Note that the existence of A_2 depends only on the fact that $\partial A_1 \neq \emptyset$, in the sense of the relative topology induced by $C(y^*, z^*, \sigma^*, p^*)$. Beginning with $n = 1$, we can always recursively construct a connected subset for each $n \geq 1$, $n \in \mathbb{N}$, with $\partial A_n \neq \emptyset$,

$$A_{n+1} = A_n \cup \bigcup_{\bar{v} \in \partial A_n} A_{\bar{v}}, \tag{6.142}$$

satisfying that for every $(y, z, \sigma, p) \in A_{n+1}$,

$$p < z(t) \text{ for some } t \in \mathbb{R}, \tag{6.143}$$

where $I_n \times (U_n \cap C(y^*, z^*, \sigma^*, p^*))$ is a delay-period disparity set at $(t, \bar{v}) \in I_n \times \partial A_n$ and $A_{\bar{v}}$ is the connected component of U_n.

If the construction in (6.142) stops at some $n_0 \in \mathbb{N}$ with $\partial A_{n_0} = \emptyset$, then $A_{n_0} = C(y^*, z^*, \sigma^*, p^*)$, and we are done. If not, then $n_0 = +\infty$, and we obtain a sequence of sets $\{A_n\}_{n=1}^{+\infty}$ that is a totally ordered family of sets with respect to the set inclusion relation $\subseteq$. Note that $\cup_{n=1}^{+\infty} A_n$ is the upper bound of $\{A_n\}_{n=1}^{+\infty}$. Then by Zorn's lemma, there exists a maximal element A_∞ for the sequence $\{A_n\}_{n=1}^{+\infty}$.

Now we show that $\partial A_\infty = \emptyset$, in the sense of the relative topology induced by $C(y^*, z^*, \sigma^*, p^*)$. Suppose not. Then there exist $\bar{v} \in \partial A_\infty$ and $A_{\bar{v}}$, which is the connected component of U_∞, where $I_\infty \times (U_\infty \times C(y^*, z^*, \sigma^*, p^*))$ is a delay-period disparity set at $(t, \bar{v}) \in I_\infty \times \partial A_\infty$. We distinguish two cases:

Case 1. $A_{\bar{v}} \setminus A_\infty = \emptyset$ for all $\bar{v} \in \partial A_\infty$. Then A_∞ is a connected component of $C(y^*, z^*, \sigma^*, p^*)$. Recall that $C(y^*, z^*, \sigma^*, p^*)$ itself is a connected component of the closure of all the nonconstant periodic solutions of system (6.100). So we have $A_\infty = C(y^*, z^*, \sigma^*, p^*)$. That is, $\partial A_\infty = \emptyset$. This is a contradiction.

Case 2. $A_{\bar{v}} \setminus A_\infty \neq \emptyset$. But this means that $A_\infty \subset A_\infty \cup A_{\bar{v}}$, which contradicts the maximality of A_∞.

These contradictions show that $\partial A_\infty = \emptyset$, and hence $A_\infty = C(y^*, z^*, \sigma^*, p^*)$. Therefore, (6.143) holds for all $(y, z, \sigma, p) \in C(y^*, z^*, \sigma^*, p^*)$. This completes the proof. □

Theorem 6.22. *Let $C(y^*, z^*, \sigma^*, p^*)$ be a connected component of the closure of all the nonconstant periodic solutions of system (6.100), bifurcated at $(y^*, z^*, \sigma^*, p^*)$ in the Fuller space $C(\mathbb{R}/2\pi; \mathbb{R}^{N+1}) \times \mathbb{R}^2$. Suppose that (6.74) satisfies (S5)–(S9). If there exists a continuous function $M_1 : \mathbb{R} \ni \sigma \to M_1(\sigma) > 0$ such that for every $(y, z, \sigma, p) \in C(y^*, z^*, \sigma^*, p^*)$, we have*

$$\|(y, z)\|_{C(\mathbb{R};\mathbb{R}^{N+1})} \le M_1(\sigma), \tag{6.144}$$

then $p^* < z^*$ implies that $p < M_1(\sigma)$ for every $(y, z, \sigma, p) \in C(y^*, z^*, \sigma^*, p^*)$.

Proof. By Theorem 6.21, we have, for every $(y, z, \sigma, p) \in C(y^*, z^*, \sigma^*, p^*)$, that $p < z(t)$ for some $t \in \mathbb{R}$. Then by (6.144), we have $p < M_1(\sigma)$. □

6.9.4 Uniform Boundedness of Periodic Solutions

We refer to [254] for the concepts of balanced, convex, and absorbing subsets and the Minkowski functional.

Lemma 6.29. *Let G be a convex absorbing subset of a locally convex linear topological space X that defines a Minkowski functional $p_G : X \to \mathbb{R}$ with $p_G(x) = \inf\{\alpha > 0 : \alpha^{-1}x \overset{\text{def}}{=} x/\alpha \in G\}$. For each $\gamma > 0$, define*

$$G^\gamma = \{x : p_G(x) < \gamma\}. \tag{6.145}$$

Then $x \in \partial G^\gamma$ if and only if $p_G(x) = \gamma$.

Proof. It is clear that $G^\gamma = \gamma G$. By linearity, the Minkowski functional $p_{G^\gamma} : X \to \mathbb{R}$ determined by G^γ is well defined. By (6.145) and by the definition of Minkowski functional, we have

$$\begin{aligned} x \in \partial G^\gamma &\Longleftrightarrow p_{G^\gamma}(x) = 1 \\ &\Longleftrightarrow \inf\{\alpha > 0 : x/\alpha \in G^\gamma\} = 1 \\ &\Longleftrightarrow \inf\{\alpha > 0 : p_G(x/\alpha) < \gamma\} = 1 \\ &\Longleftrightarrow \inf\{\alpha > 0 : p_G(x)/\gamma < \alpha\} = 1 \\ &\Longleftrightarrow p_G(x) = \gamma. \end{aligned}$$

□

Lemma 6.30. *Let G_1 and G_2 be convex absorbing subsets of locally convex linear topological spaces X_1 and X_2, respectively. Let the Minkowski functionals associated with G_1 and G_2 be $p_{G_1}(x)$ and $p_{G_2}(\tau)$, respectively. Then the Minkowski functional defined by $G = G_1 \times G_2$ exists and satisfies*

$$p_G(x, \tau) = \max\{p_{G_1}(x), p_{G_2}(\tau)\}.$$

Proof. The existence of $p_G(x,\tau)$ is clear from the definition of a Minkowski functional. Let $A=\{\alpha : x/\alpha \in G_1\}$, $B=\{\alpha : \tau/\alpha \in G_2\}$. Then it is clear that $\inf A\cap B \geq \inf A$ and $\inf A\cap B\geq \inf B$. It follows that $\inf A\cap B\geq \max\{\inf A, \inf B\}$, that is,

$$p_G(x,\tau)\geq \max\{p_{G_1}(x), p_{G_2}(\tau)\}. \tag{6.146}$$

On the other hand, if $\alpha_A=\inf A\geq \alpha_B=\inf B$, since G_1 and G_2 are absorbing, we have for every $\varepsilon>0$ that $\alpha_A+\varepsilon\in A$, $\alpha_A+\varepsilon\in B$. Therefore, $\inf A\cap B\leq \alpha_A+\varepsilon$. Similarly, if $\alpha_A=\inf A\leq \alpha_B=\inf B$, we have $\inf A\cap B\leq \alpha_B+\varepsilon$. Hence we obtain $\inf A\cap B\leq \max\{\alpha_A, \alpha_B\}+\varepsilon$. By the arbitrariness of $\varepsilon>0$, we get $\inf A\cap B\leq \max\{\alpha_A, \alpha_B\}$, that is,

$$p_G(x,\tau)\leq \max\{p_{G_1}(x), p_{G_2}(\tau)\}. \tag{6.147}$$

By (6.146) and (6.147), we have

$$p_G(x,\tau)= \max\{p_{G_1}(x), p_{G_2}(\tau)\}.$$

This completes the proof. □

An immediate corollary of Lemmas 6.29 and 6.30 is the following.

Corollary 6.1. *Let G_1 and G_2 be convex absorbing subsets of locally convex linear topological spaces X_1 and X_2, respectively. Let $p_{G_1}(x)$ and $p_{G_2}(\tau)$ be the Minkowski functionals associated with G_1 and G_2, respectively. Let $G=G_1\times G_2$, and for every $\gamma>0$, define*

$$G^\gamma=\{(x,\tau) : p_G(x,\tau)<\gamma\},$$
$$G_1^\gamma=\{x : p_{G_1}(x)<\gamma\},$$
$$G_2^\gamma=\{\tau : p_{G_2}(\tau)<\gamma\}.$$

Then $G^\gamma=G_1^\gamma\times G_2^\gamma$ and $\bar{G}^\gamma=\bar{G}_1^\gamma\times\bar{G}_2^\gamma$.

In this section, we use "·" to denote the usual inner product of a Euclidean space, and we use G^c and D^c to denote the complementary sets of G and D, respectively.

We can now state and prove the geometric conditions for uniform boundedness of the periodic solutions of (6.74) with $\sigma\in\Sigma$, where $\Sigma\subseteq\mathbb{R}$ is a given subset.

Theorem 6.23. *Suppose that $G_1\subset\mathbb{R}^N$ and $G_2\subset\mathbb{R}$ are bounded, balanced, convex, and absorbing open subsets with associated Minkowski functionals $p_{G_1}(x)$ and $p_{G_2}(\tau)$. Let $G=G_1\times G_2$ and $(\bar{x},\bar{\tau})=\frac{1}{p_G(x,\tau)}(x,\tau)\in\partial G$ for $(x,\tau)\neq 0$. Assume that there exists a vector-valued function $N:\partial G\setminus(\partial G_1\times\partial G_2)\to\mathbb{R}^{N+1}\setminus\{0\}$ satisfying*

(i) : $\overline{G}\subseteq U_1\cup U_2$, where

$$U_1=\bigcap_{(x,\tau)\in\partial G\setminus(\partial G_1\times\partial G_2)}\{(u,v) : N(x,\tau)\cdot(u-x,v-\tau)\leq 0\};$$

$$U_2 = \bigcap_{(x,\tau)\in\partial G_1\times\partial G_2} \{(u,v) : x\cdot(u-x)\le 0,\ \tau\cdot(v-\tau)\le 0\};$$

(ii) : *$N(\bar{x},\bar{\tau})\cdot(f(x,\tilde{x},\sigma),g(x,\tau,\sigma))$ is positive (or negative) for all $(x,\tau)\in G^c$ with $(\bar{x},\bar{\tau})\notin\partial G_1\times\partial G_2$, and all $(\tilde{x},\tau)\in\mathbb{R}^N\times\mathbb{R}$ with $p_G(\tilde{x},\tau)\le p_G(x,\tau)$ and $\sigma\in\Sigma$;*

(iii) : *$x\cdot f(x,\tilde{x},\sigma)$ and $\tau\cdot g(x,\tau,\sigma)$ are both positive (or negative) for all $(x,\tau)\in G^c$ with $(\bar{x},\bar{\tau})\in\partial G_1\times\partial G_2$, and all $(\tilde{x},\tau)\in\mathbb{R}^N\times\mathbb{R}$ with $p_G(\tilde{x},\tau)\le p_G(x,\tau)$ and $\sigma\in\Sigma$.*

Then the range of all the periodic solutions of (6.74) with $\sigma\in\Sigma$ is contained in G.

Remark 6.3. The prototype of the vector-valued function $N(x,\tau)$ is the (outer or inner) normal of G, which is not defined on $\partial G_1\times\partial G_2$. If G is a rectangle in a planar space, $\partial G_1\times\partial G_2$ are the four corner points of G. Conditions (ii)–(iii) of Theorem 6.23 require that the vector field determined by the right-hand side of system (6.74) have positive (or negative) inner product with respect to the normal of a given rectangle G, where the vector field is evaluated at $(x,\tau)\in\mathbb{R}^{N+1}$, which satisfies $(x,\tau)\in G^c$ and $p_G(\tilde{x},\tau)\le p_G(x,\tau)$.

Proof. Letting $(x,\tau)(t)=(y,z)(\beta t)$ with a normalization parameter $\beta>0$, we only need to consider the 2π-periodic solutions of the following system:

$$\begin{cases} \dot{y}(t)=\frac{1}{\beta}f(y(t),y(t-\beta z(t)),\sigma), \\ \dot{z}(t)=\frac{1}{\beta}g(y(t),z(t),\sigma), \end{cases} \tag{6.148}$$

where $x\in\mathbb{R}^N$ and $\tau\in\mathbb{R}$. It is clear that if $(x(t),\tau(t))$ and $(y(t),z(t))$ are solutions of (6.74) and (6.148), respectively, then $(x(t),\tau(t))\in G$ for all $t\in\mathbb{R}$ if and only if $(y(t),z(t))\in G$ for all $t\in\mathbb{R}$.

For simplicity, we denote $y(t-\beta z(t))$ by $\tilde{y}(t)$ for each solution $(y(t),z(t))$ of (6.148). Let $(\bar{y},\bar{z})$ be the positive constant multiple of (y,z) such that $(\bar{y},\bar{z})\in\partial G$. That is, for every $(y,z)\in\mathbb{R}^{N+1}\setminus\{0\}$, there exists $(\bar{y},\bar{z})\in\partial G$ such that $(y,z)=p_G(y,z)(\bar{y},\bar{z})$.

Suppose there exists a 2π-periodic solution of (6.148) such that $(y(t_0),z(t_0))\notin G$ for some $t_0\in[0,2\pi]$ and define the map $\gamma:\mathbb{R}\ni t\to p_G(y(t),z(t))\in\mathbb{R}$. Since $\mathbb{R}^{N+1}\ni(y,z)\mapsto p_G(y,z)\in\mathbb{R}$ and $\mathbb{R}\ni t\mapsto(y(t),z(t))\in\mathbb{R}^{N+1}$ are continuous, the map $\gamma: t\to p_G(y(t),z(t))$ is continuous and there exist $\gamma^*\ge 1$ and $t^*\in[0,2\pi]$ such that

$$\gamma^* = p_G(y(t^*),z(t^*)) = \max_{t\in[0,2\pi]} p_G(y(t),z(t)). \tag{6.149}$$

Then by Lemma 6.29 and (6.149), we have $(y(t^*),z(t^*))\in\partial G^{\gamma^*}$ and $G^{\gamma(t)}\subseteq G^{\gamma^*}$ for all $t\in\mathbb{R}$. Therefore, by Corollary 6.1, $(y(t),z(t))\in\bar{G}^{\gamma^*}=\bar{G}_1^{\gamma^*}\times\bar{G}_2^{\gamma^*}$ for all $t \in [0, 2\pi]$. In particular, by the periodicity of $(y(t), z(t))$, we obtain $(y(t - \beta z(t)), z(t)) \in \bar{G}^{\gamma^*}$ for all $t \in [0, 2\pi]$ and $\beta > 0$. Therefore, we have

$$p_G(y(t^* - \beta z(t^*), z(t^*)) \leq p_G(y(t^*), z(t^*)). \tag{6.150}$$

We first suppose that $(\bar{y}(t^*), \bar{z}(t^*)) = \frac{1}{p_G(y(t^*), z(t^*))}(y(t^*), z(t^*)) \in U_1$. Then by (6.149), (6.150), and assumption (ii), we have (we use the positivity assumption in the proof; the proof is similar if we use the negativity assumption; see Remark 6.4 for details)

$$N(\bar{y}(t^*), \bar{z}(t^*)) \cdot \left[\frac{1}{\beta} f(y(t^*), y(t^* - \beta z(t^*)), \sigma), \frac{1}{\beta} g(y(t^*), z(t^*), \sigma)\right] > 0. \tag{6.151}$$

Let us write

$$\begin{bmatrix} y(t^* + h) \\ z(t^* + h) \end{bmatrix} = \begin{bmatrix} y(t^*) \\ z(t^*) \end{bmatrix} + \begin{bmatrix} \int_0^1 \dot{y}(t^* + sh) ds\, h \\ \int_0^1 \dot{z}(t^* + sh) ds\, h \end{bmatrix}, \tag{6.152}$$

and choose $h > 0$ small enough that

$$N(\bar{y}(t^*), \bar{z}(t^*)) \cdot \left[\frac{1}{\beta} f(y(t), y(t - \beta z(t)), \sigma), \frac{1}{\beta} g(y(t), z(t), \sigma)\right] > 0 \tag{6.153}$$

for $t^* \leq t < t^* + h$. Then by (6.148), (6.152), and (6.153), we have

$$N(\bar{y}(t^*), \bar{z}(t^*)) \cdot (y(t^* + h) - y(t^*), z(t^* + h) - z(t^*)) > 0. \tag{6.154}$$

Now we distinguish the following two cases in order to deduce contradictions:
Case 1. If $(y(t^* + h), z(t^* + h)) \in \bar{G}$, then $\gamma^{*-1}(y(t^* + h), z(t^* + h)) \in \bar{G}$, since $\gamma^* \geq 1$. Also, we have $(y(t^*), z(t^*)) = (\gamma^* \bar{y}(t^*), \gamma^* \bar{z}(t^*))$ with $(\bar{y}(t^*), \bar{z}(t^*)) \in \partial G$. Then by assumption (i), we have

$$N(\bar{y}(t^*), \bar{z}(t^*)) \cdot \left(\gamma^{*-1} y(t^* + h) - \bar{y}(t^*), \gamma^{*-1} z(t^* + h) - \bar{z}(t^*)\right) \leq 0. \tag{6.155}$$

On the other hand, we have by (6.154),

$$\begin{aligned} 0 <& N(\bar{y}(t^*), \bar{z}(t^*)) \cdot (y(t^* + h) - y(t^*), z(t^* + h) - z(t^*)) \\ =& \gamma^* N(\bar{y}(t^*), \bar{z}(t^*)) \cdot \left(\gamma^{*-1} y(t^* + h) - \bar{y}(t^*), \gamma^{*-1} z(t^* + h) - \bar{z}(t^*)\right), \end{aligned} \tag{6.156}$$

which contradicts (6.155).
Case 2. If $(y(t^* + h), z(t^* + h)) \notin \bar{G}$, then by (6.149), we have

$$1 \leq \gamma_h = p_G(y(t^* + h), z(t^* + h)) \leq p_G(y(t^*), z(t^*)) = \gamma^*. \tag{6.157}$$

Also, we have $(y(t^*+h), z(t^*+h)) = \gamma_h(\bar{y}(t^*+h), \bar{z}(t^*+h))$ with $(\bar{y}(t^*+h), \bar{z}(t^*+h)) \in \partial G$. By the convexity of $\bar{G}$ and by the inequality $\gamma_h/\gamma^* \leq 1$, we have

$$\left(\frac{\gamma_h}{\gamma^*}\bar{y}(t^*+h),\, \frac{\gamma_h}{\gamma^*}\bar{z}(t^*+h)\right) \in \bar{G}.$$

Then by assumption (i), we have

$$N(\bar{y}(t^*), \bar{z}(t^*)) \cdot \left(\frac{\gamma_h}{\gamma^*}\bar{y}(t^*+h) - \bar{y}(t^*),\, \frac{\gamma_h}{\gamma^*}\bar{z}(t^*+h) - \bar{z}(t^*)\right) \leq 0. \tag{6.158}$$

On the other hand, we have by (6.154),

$$\begin{aligned} 0 <& N(\bar{y}(t^*), \bar{z}(t^*)) \cdot (y(t^*+h) - y(t^*), z(t^*+h) - z(t^*)) \\ =& \gamma^* N(\bar{y}(t^*), \bar{z}(t^*)) \cdot \left(\frac{\gamma_h}{\gamma^*}\bar{y}(t^*+h) - \bar{y}(t^*),\, \frac{\gamma_h}{\gamma^*}\bar{z}(t^*+h) - \bar{z}(t^*)\right), \end{aligned} \tag{6.159}$$

which contradicts (6.158).

Second, we suppose that $(\bar{y}(t^*), \bar{z}(t^*)) = \frac{1}{p_G(y(t^*), z(t^*))}(y(t^*), z(t^*)) \in U_2$. By assumption (iii), we have

$$\begin{cases} \bar{y}(t^*) \cdot \dfrac{1}{\beta} f(y(t^*), y(t^* - \beta z(t^*)), \sigma) > 0, \\ \bar{z}(t^*) \cdot \dfrac{1}{\beta} g(y(t^*), z(t^*), \sigma) > 0. \end{cases} \tag{6.160}$$

Therefore, we can choose $h > 0$ small enough that for $t^* \leq t < t^* + h$,

$$\begin{cases} \bar{y}(t^*) \cdot \dfrac{1}{\beta} f(y(t), y(t - \beta z(t)), \sigma) > 0, \\ \bar{z}(t^*) \cdot \dfrac{1}{\beta} g(y(t), z(t), \sigma) > 0. \end{cases} \tag{6.161}$$

Then by (6.148), (6.152), and (6.161), we have

$$\begin{cases} \bar{y}(t^*) \cdot (y(t^*+h) - y(t^*)) > 0, \\ \bar{z}(t^*) \cdot (z(t^*+h) - z(t^*)) > 0. \end{cases} \tag{6.162}$$

We distinguish the following two cases in order to deduce contradictions:

Case 1′. If $(y(t^*+h), z(t^*+h)) \in \bar{G}$, then $\gamma^{*-1}(y(t^*+h), z(t^*+h)) \in \bar{G}$, since $\gamma^* \geq 1$. Also, we have $(y(t^*), z(t^*)) = (\gamma^*\bar{y}(t^*), \gamma^*\bar{z}(t^*))$ with $(\bar{y}(t^*), \bar{z}(t^*)) \in \partial G$. Then by assumption (i), we have

$$\begin{cases} \bar{y}(t^*) \cdot (\gamma^{*-1} y(t^*+h) - \bar{y}(t^*)) \leq 0, \\ \bar{z}(t^*) \cdot (\gamma^{*-1} z(t^*+h) - \bar{z}(t^*)) \leq 0. \end{cases} \tag{6.163}$$

On the other hand, we have by (6.162),

$$\begin{cases} \bar{y}(t^*)\cdot(y(t^*+h)-y(t^*)) = \gamma^*\bar{y}(t^*)\cdot(\gamma^{*-1}y(t^*+h)-\bar{y}(t^*)) > 0, \\ \bar{z}(t^*)\cdot(z(t^*+h)-z(t^*)) = \gamma^*\bar{z}(t^*)\cdot(\gamma^{*-1}z(t^*+h)-\bar{z}(t^*)) > 0, \end{cases} \tag{6.164}$$

which contradicts (6.163).

Case $2'$**.** If $(y(t^*+h), z(t^*+h)) \notin \bar{G}$, then by (6.157) and the convexity of $\bar{G}$, we have

$$\left(\frac{\gamma_h}{\gamma^*}\bar{y}(t^*+h), \frac{\gamma_h}{\gamma^*}\bar{z}(t^*+h)\right) \in \bar{G},$$

where $\gamma_h = p_G(y(t^*+h), z(t^*+h))$. Then by assumption (i), we have

$$\begin{cases} \bar{y}(t^*)\cdot(\frac{\gamma_h}{\gamma^*}\bar{y}(t^*+h)-\bar{y}(t^*)) \le 0, \\ \bar{z}(t^*)\cdot(\frac{\gamma_h}{\gamma^*}\bar{z}(t^*+h)-\bar{z}(t^*)) \le 0. \end{cases} \tag{6.165}$$

On the other hand, we have by (6.162),

$$\begin{cases} \bar{y}(t^*)\cdot(y(t^*+h)-y(t^*)) = \gamma^*\bar{y}(t^*)\cdot(\frac{\gamma_h}{\gamma^*}\bar{y}(t^*+h)-\bar{y}(t^*)) > 0, \\ \bar{z}(t^*)\cdot(z(t^*+h)-z(t^*)) = \gamma^*\bar{z}(t^*)\cdot(\frac{\gamma_h}{\gamma^*}\bar{z}(t^*+h)-\bar{z}(t^*)) > 0, \end{cases} \tag{6.166}$$

which contradicts (6.165). Therefore, contradictions are obtained in all cases, and the proof is complete. □

Remark 6.4. If we use <0 instead of >0 in the inequality (6.151), we need to change (6.152) to be the difference between $(y(t^*), z(t^*))$ and $(y(t^*-h), z(t^*-h))$. That is,

$$\begin{bmatrix} y(t^*) \\ z(t^*) \end{bmatrix} = \begin{bmatrix} y(t^*-h) \\ z(t^*-h) \end{bmatrix} + \begin{bmatrix} \int_0^1 \dot{y}(t^*-sh)dsh \\ \int_0^1 \dot{z}(t^*-sh)dsh \end{bmatrix}.$$

Then the rest of the proof is similar.

Corollary 6.2. *Suppose that $G_1 \subset \mathbb{R}^N$ and $G_2 \subset \mathbb{R}$ are bounded, balanced, convex, and absorbing open subsets that define the Minkowski functionals $p_{G_1}(x)$ and $p_{G_2}(\tau)$. Suppose $N : \partial G \setminus (\partial G_1 \times \partial G_2) \to \mathbb{R}^{N+1} \setminus \{0\}$ is the outer normal of G. Fix $\sigma \in \Sigma$ and let $G = G_1 \times G_2$ and*

$$F_{\max}(x,\sigma) = \max_{\{\tilde{x}:\, p_{G_1}(\tilde{x}) \le p_{G_1}(x)\}} x \cdot f(x,\tilde{x},\sigma),$$

$$F_{\min}(x,\sigma) = \min_{\{\tilde{x}:\, p_{G_1}(\tilde{x}) \le p_{G_1}(x)\}} x \cdot f(x,\tilde{x},\sigma).$$

Then the range of all the periodic solutions of (6.74) are contained in G if either of the following conditions holds:

(*H1*) $F_{\max}(x,\sigma) < 0$ *for every* $x \in G_1^c$ *and* $\tau \cdot g(x,\tau) < 0$ *for every* $\tau \in G_2^c, x \in \mathbb{R}^N$.
(*H2*) $F_{\min}(x,\sigma) > 0$ *for every* $x \in G_1^c$ *and* $\tau \cdot g(x,\tau) > 0$ *for every* $\tau \in G_2^c, x \in \mathbb{R}^N$.

Proof. We prove the conclusions by applying Theorem 6.23. By Corollary 6.1, there exist Minkowski functionals $p_G(x,\tau)$, $p_{G_1}(x)$, and $p_{G_2}(\tau)$ defined on $\mathbb{R}^N \times \mathbb{R}$, $\mathbb{R}^N$, and $\mathbb{R}$, respectively. For every $(x,\tau) \in G^c$, let $(\bar{x},\bar{\tau}) = (x,\tau)/p_G(x,\tau) \in \partial G$. Recall that $N : \partial G \setminus (\partial G_1 \times \partial G_2) \to \mathbb{R}^{N+1} \setminus \{0\}$ is the outer normal of the convex set G. Then condition (i) of Theorem 6.23 is satisfied.

Suppose (H1) holds. Then we have

$$x \cdot f(x,\tilde{x},\sigma) < 0, \text{for all } (x,\tilde{x}) \in G_1^c \times \mathbb{R}^N \text{ with } p_{G_1}(\tilde{x}) \le p_{G_1}(x), \tag{6.167}$$

$$\tau \cdot g(x,\tau,\sigma) < 0, \text{ for all } \tau \in G_2^c, x \in \mathbb{R}^N. \tag{6.168}$$

For every $(x,\tau) \in G^c$ with $p_G(\tilde{x},\tau) \le p_G(x,\tau)$, let $(\bar{x},\bar{\tau}) = (x,\tau)/p_G(x,\tau) \in \partial G$. Note that $\partial G = (G_1 \times \partial G_2) \cup (\partial G_1 \times G_2) \cup (\partial G_1 \times \partial G_2)$. We distinguish the following three cases:

Case 1: If $(\bar{x},\bar{\tau}) \in G_1 \times \partial G_2$, then $N(\bar{x},\bar{\tau}) = (0,\tau)/p_G(x,\tau) \neq 0$ is an outer normal of G. We claim that $\tau \in G_2^c$ holds.

Indeed, since $\bar{x} \in G_1$, we have $p_{G_1}(\bar{x}) = p_{G_1}(x/p_G(x,\tau)) < 1$. Therefore, $p_{G_1}(x) < p_G(x,\tau)$. By Lemma 6.30, we know that $p_G(x,\tau) = \max\{p_{G_1}(x), p_{G_2}(\tau)\}$. Then we have $p_{G_1}(x) < p_{G_2}(\tau)$ and $p_G(x,\tau) = p_{G_2}(\tau) > 1$. Then by Lemma 6.29, we have $\tau \in G_2^c$.

Then by (6.168), we have

$$N(\bar{x},\bar{\tau}) \cdot (f(x,\tilde{x},\sigma), g(x,\tau,\sigma)) = \tau \cdot g(x,\tau,\sigma)/p_G(x,\tau) < 0.$$

Case 2: If $(\bar{x},\bar{\tau}) \in \partial G_1 \times G_2$, then $N(\bar{x},\bar{\tau}) = (x,0)/p_G(x,\tau) \neq 0$ is an outer normal of G. We claim that $x \in G_1^c$ and $p_{G_1}(\tilde{x}) \le p_{G_1}(x)$.

Indeed, since $\bar{\tau} \in G_2$, we have $p_{G_2}(\bar{\tau}) = p_{G_2}(\tau/p_G(x,\tau)) < 1$. Therefore, $p_{G_2}(\tau) < p_G(x,\tau)$. By Lemma 6.30, we know that $p_G(x,\tau) = \max\{p_{G_1}(x), p_{G_2}(\tau)\}$. Then we have $p_{G_2}(\tau) < p_{G_1}(x)$ and $p_G(x,\tau) = p_{G_1}(x) > 1$. Then by Lemma 6.29, we have $x \in G_1^c$. Moreover, it follows again by Lemma 6.30 that $p_G(\tilde{x},\tau) \le p_G(x,\tau)$ implies $p_{G_1}(\tilde{x}) \le p_{G_1}(x)$. This proves the claim.

By (6.167), we have

$$N(\bar{x},\bar{\tau}) \cdot (f(x,\tilde{x},\sigma), g(x,\tau,\sigma)) = x \cdot f(x,\tilde{x},\sigma)/p_G(x,\tau) < 0.$$

From Case 1 and Case 2, we know that $N(\bar{x},\bar{\tau}) \cdot (f(x,\tilde{x},\sigma), g(x,\tau,\sigma))$ is negative definite for all $(x,\tau) \in G^c$ and $\sigma \in \Sigma$ with $(\bar{x},\bar{\tau}) \notin \partial G_1 \times \partial G_2$, and all $(\tilde{x},\tau) \in \mathbb{R}^N \times \mathbb{R}$ with $p_G(\tilde{x},\tau) \le p_G(x,\tau)$. That is, condition (ii) of Theorem 6.23 is satisfied.
Case 3: If $(\bar{x},\bar{\tau}) \in \partial G_1 \times \partial G_2$, we claim that $(x,\tau) \in G_1^c \times G_2^c$ and $p_{G_1}(\tilde{x}) = p_{G_1}(x)$ hold.

Indeed, since $(\bar{x}, \bar{\tau}) \in \partial G_1 \times \partial G_2$, we have $p_{G_1}(\bar{x}) = p_{G_1}(x/p_G(x, \tau)) = 1$ and $p_{G_2}(\bar{\tau}) = p_{G_2}(\tau/p_G(x, \tau)) = 1$. Therefore, $p_G(x, \tau) = p_{G_1}(x) = p_{G_2}(\tau)$. Since $(x, \tau) \in G^c$, we have $p_{G_1}(x) = p_{G_2}(\tau) = p_G(x, \tau) > 1$. Then by Lemma 6.29, we have $(x, \tau) \in G_1^c \times G_2^c$. Moreover, it follows again by Lemma 6.30 that $p_G(\tilde{x}, \tau) \leq p_G(x, \tau)$ implies $p_{G_1}(\tilde{x}) \leq p_{G_1}(x)$. This proves the claim.

Then by (6.167) and (6.168), we have

$$x \cdot f(x, \tilde{x}, \sigma) < 0 \text{ and } \tau \cdot g(x, \tau, \sigma) < 0.$$

From Case 3, we know that $x \cdot f(x, \widetilde{x}, \sigma)$ and $\tau \cdot g(x, \tau, \sigma)$ are both negative definite for all $(x, \tau) \in G^c$ and $\sigma \in \Sigma$ with $(\overline{x}, \overline{\tau}) \in \partial G_1 \times \partial G_2$, and all $(\tilde{x}, \tau) \in \mathbb{R}^N \times \mathbb{R}$ with $p_G(\tilde{x}, \tau) \leq p_G(x, \tau)$. That is, condition (iii) of Theorem 6.23 is satisfied.

It follows from Theorem 6.23 that the range of all the periodic solutions of (6.74) with $\sigma \in \Sigma$ is contained in G. Similarly, if (H2) holds, we can obtain from Theorem 6.23 the same conclusion. This completes the proof. □

6.9.5 Global Continuation of Rapidly Oscillating Periodic Solutions: An Example

In this section, we illustrate the general results in the previous subsections by applying them to the study of the global continua of rapidly oscillating periodic solutions for the following differential equations with state-dependent delay:

$$\begin{cases} \dot{x}(t) = -\mu x(t) + \sigma^2 b(x(t-\tau(t))), \\ \dot{\tau}(t) = 1 - h(x(t)) \cdot (1 + \tanh \tau(t)), \end{cases} \tag{6.169}$$

where $\tanh(\tau) = (\mathrm{e}^{2\tau} - 1)/(\mathrm{e}^{2\tau} + 1)$ and $\mu > 0$ is a constant. We make the following assumptions:

(α_1) $b, h : \mathbb{R} \to \mathbb{R}$ are C^2 functions with $b'(0) = -1$;
(α_2) There exist $h_0 < h_1$ in $(1/2, 1)$ such that $h_1 > h(x) > h_0$ for all $x \in \mathbb{R}$;
(α_3) b is decreasing on $\mathbb{R}$;
(α_4) $xb(x) < 0$ for $x \neq 0$, and there exists a continuous function $M : \mathbb{R} \ni \sigma \to M(\sigma) \in (0, +\infty)$ such that

$$\frac{b(x)}{x} > -\frac{\mu}{\sigma^2}$$

for every $x \in \mathbb{R}$ with $|x| \geq M(\sigma)$;
(α_5) There exists $M_0 > 0$ such that $|b'(x)| < M_0$ for every $x \in \mathbb{R}$;
(α_6) $h'(x) = 0$ only if x satisfies $-\mu x + \sigma^2 b(x) = 0$.

Remark 6.5. We use $\tanh(\tau)$ just for the sake of simplicity. Other types of functions can be used with minor changes in our arguments below.

We start with the uniform boundedness of periodic solutions $(x(t), \tau(t))$ of (6.169).

Lemma 6.31. *Assume that* (α_1)–(α_4) *hold. Then the range of every periodic solution* (x,τ) *of* (6.169) *with* $\sigma\in\mathbb{R}$ *is contained in*

$$\Omega_1=(-M(\sigma),M(\sigma))\times\left(0,-\frac{\ln(2h_0-1)}{2}\right).$$

Proof. If $\sigma=0$, the only periodic solution is $\left(0,-\frac{\ln(2h(0)-1)}{2}\right)$, which is contained in Ω_1. Now we assume that $\sigma\neq 0$. If $x>0$, then by assumptions (α_3) and (α_4), we have

$$\max_{y\in\{y:|y|\le|x|\}} x\cdot(-\mu x+\sigma^2 b(y))=-\sigma^2x^2\left(\frac{\mu}{\sigma^2}-\frac{b(-x)}{x}\right)<0$$

for every $x\in\mathbb{R}$ with $x\ge M(\sigma)$. It follows that

$$\max_{y\in\{y:|y|\le|x|\}} x\cdot(-\mu x+\sigma^2 b(y))<0 \text{ for } x\ge M(\sigma).$$

Similarly, we have

$$\max_{y\in\{y:|y|\le|x|\}} x\cdot(-\mu x+\sigma^2 b(y))<0 \text{ for } x\le -M(\sigma).$$

Thus,

$$\max_{y\in\{y:|y|\le|x|\}} x\cdot(-\mu x+\sigma^2 b(y))<0 \text{ if } x\notin(-M(\sigma),M(\sigma)). \tag{6.170}$$

It is clear from (α_2) that for all $x\in\mathbb{R}$,

$$\lim_{\tau\to\pm\infty}\tau\cdot(1-h(x)(1+\tanh\tau))<0.$$

To obtain an upper bound for τ, where (x,τ) is a periodic solution of (6.169), we introduce the following change of variable:

$$z(t)=\tau(t)+\frac{\ln(2h_0-1)}{4}. \tag{6.171}$$

Then system (6.169) is transformed to

$$\begin{cases}\dot{x}(t)=-\mu x(t)+\sigma^2 b\left(x\left(t-z(t)+\frac{\ln(2h_0-1)}{4}\right)\right),\\ \dot{z}(t)=1-h(x(t))\left(1+\tanh\left(z(t)-\frac{1}{4}\ln(2h_0-1)\right)\right).\end{cases} \tag{6.172}$$

By (α_2) and the monotonicity of $\tanh\tau$, we have, for every $z\notin\left(\frac{\ln(2h_0-1)}{4},-\frac{\ln(2h_0-1)}{4}\right)$ and for all $x\in\mathbb{R}$,

$$z\cdot\left(1-h(x)\left(1+\tanh\left(z-\frac{1}{4}\ln(2h_0-1)\right)\right)\right)<0. \tag{6.173}$$

Thus it follows from Corollary 6.2, (6.170), and (6.173) that the range of all the periodic solutions (x,z) of (6.172) is contained in $(-M(\sigma),M(\sigma))\times\left(\frac{\ln(2h_0-1)}{4},-\frac{\ln(2h_0-1)}{4}\right)$. Then by (6.171), all periodic solutions (x,τ) of (6.169) with $\sigma\neq 0$ are contained in Ω_1. The proof is complete. □

Now we consider the global Hopf bifurcation problem of system (6.169) under the assumptions (α_1)–(α_6). By (α_4), $(x,\tau)=(0,\tau^*)$ is the only stationary solution of (6.169), where $\tau^*=-\frac{1}{2}\ln(2h(0)-1)>0$. Freezing the state-dependent delay $\tau(t)$ at τ^* for the term $x(t-\tau(t))$ of (6.169) and linearizing the resulting system with constant delay at the stationary solution $(0,\tau^*)$, we obtain the following formal linearization of system (6.169):

$$\begin{cases}\dot{X}(t)=-\mu X(t)-\sigma^2X(t-\tau^*),\\ \dot{T}(t)=-\rho X(t)-qT(t),\end{cases}\tag{6.174}$$

where

$$\rho=\frac{h'(0)}{h(0)},\ q=2-\frac{1}{h(0)}>0.\tag{6.175}$$

In the following, we regard σ as the bifurcation parameter. We obtain the characteristic equation of the linear system corresponding to (6.174):

$$(\lambda+\mu+\sigma^2\mathrm{e}^{-\tau^*\lambda})(\lambda+q)=0.\tag{6.176}$$

Since the zero of $\lambda+q=0$ is $-q$, which is real, Hopf bifurcation points are related to zeros of only the first factor $(\lambda+\mu+\sigma^2\mathrm{e}^{-\tau^*\lambda})$. To locate local Hopf bifurcation points, we let $\lambda=\mathrm{i}\beta$, $\beta>0$, in $\lambda+\mu+\sigma^2\mathrm{e}^{-\tau^*\lambda}=0$ and express the resulting equation in terms of its real and imaginary parts as

$$\begin{cases}\beta=\sigma^2\sin(\tau^*\beta),\\ \mu=-\sigma^2\cos(\tau^*\beta).\end{cases}\tag{6.177}$$

It is easy to verify the following lemma.

Lemma 6.32. *(i) All the positive solutions of (6.177) can be represented by an infinite sequence $\{\beta_n\}_{n=1}^{+\infty}$ that satisfies $0<\beta_1<\beta_2<\cdots<\beta_n<\cdots$, $\lim_{n\to+\infty}\beta_n=+\infty$, and*

$$\beta_n\in\left(\frac{(4n-3)\pi}{2\tau^*},\frac{(4n-2)\pi}{2\tau^*}\right)\ \textit{for}\ n\geq 1.$$

(ii) $\pm\mathrm{i}\beta_n$ are characteristic values of the stationary solution $(0,\tau^*,\sigma_n)$, where

$$\sigma_n=\pm(\beta_n^2+\mu^2)^{1/4}.$$

If $\sigma\neq\sigma_n$, then the stationary solution $(0,\tau^*,\sigma)$ has no purely imaginary characteristic value.

(iii) Let $\lambda_n(\sigma) = u_n(\sigma) + \mathrm{i}v_n(\sigma)$ *be the root of (6.176) for* σ *close to* σ_n *such that* $u_n(\sigma_n) + \mathrm{i}v_n(\sigma_n) = \mathrm{i}\beta_n$. *Then*

$$u_n'(\sigma)\Big|_{\sigma=\sigma_n} = \frac{2}{\sigma_n}\frac{(\mu^2+\beta_n^2)\tau^*+\mu}{(1+\mu\tau^*)^2+(\beta_n\tau^*)^2}.$$

Now we are able to state our main results.

Theorem 6.24. *Assume that* (α_1)–(α_6) *hold. Let* $\beta_n \in \left(\frac{(4n-3)\pi}{2\tau^*}, \frac{(4n-2)\pi}{2\tau^*}\right)$, $n \geq 1$, *be as given in (i) of Lemma 6.32. Let* $\sigma_n = \pm(\mu^2+\beta_n)^{1/4}$ *for* $n \geq 1$. *Then:*

(a) There exists an unbounded connected component $C\left(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n}\right)$ *of the closure of all the nonconstant periodic solutions of system (6.169), bifurcated from* $(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n})$ *in the Fuller space where* σ *satisfies* $\mathrm{sgn}(\sigma_n)\sigma > 0$.

(b) $(0, \tau^*, \sigma_1, \frac{2\pi}{\beta_1}) \notin C\left(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n}\right)$ *for every* $n \geq 2$.

(c) For every $n \geq 2$, *the projection of* $C\left(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n}\right)$ *onto the parameter space* $\mathbb{R}$ *is unbounded in* $(0, +\infty)$ *if* $\sigma_n > 0$ *and is unbounded in* $(-\infty, 0)$ *if* $\sigma_n < 0$.

Proof. (a) We apply Theorem 6.15. We first verify assumptions (SHB1)–(SHB3) and (SHB5). It is clear that (α_2) and (α_1) imply (SHB1), (SHB2), and (SHB5). Let us check (SHB3). Indeed, noticing that $\sigma_n = \pm(\mu^2+\beta_n^2)^{1/4}$, $b'(0) = -1$, and $\beta_n > 0$, we have

$$\left(\frac{\partial}{\partial\theta_1}+\frac{\partial}{\partial\theta_2}\right)\left[-\mu\theta_1+\sigma^2 b(\theta_2)\right]_{\sigma=\sigma_n,\,\theta_1=\theta_2=0} = -\mu-\sigma_n^2 < 0. \tag{6.178}$$

Also, it follows from $\tau^* = -\frac{\ln(2h(0)-1)}{2}$ that

$$\frac{\partial}{\partial\gamma_2}(1-h(\gamma_1)(1+\tanh(\gamma_2))\Big|_{\sigma=\sigma_n,\,\gamma_1=0,\,\gamma_2=\tau^*} = -h(0)\cdot\frac{4\mathrm{e}^{2\tau^*}}{(\mathrm{e}^{2\tau^*}+1)^2} < 0. \tag{6.179}$$

Therefore, condition (SHB3) is satisfied by system (6.169).

We note from Lemma 6.32 (i), (ii), and (iii) that every center (including those with $\sigma < 0$) of system (6.174) is isolated. We now calculate the crossing number of $(0, \tau^*, \sigma_n, \beta_n)$. Let $u_n(\sigma)+\mathrm{i}v_n(\sigma)$ be the characteristic value of (6.174) such that $u_n(\sigma_n)+\mathrm{i}v_n(\sigma_n) = \mathrm{i}\beta_n$. By (iv) of Lemma 6.32, we have

$$\begin{aligned}\frac{d}{d\sigma}u_n(\sigma)\Big|_{\sigma=\sigma_n} &= u_n'(\sigma_n)|_{\sigma=\sigma_n}\\ &= \frac{2}{\sigma_n}\frac{(\mu^2+\beta_n^2)\tau^*+\mu}{(1+\mu\tau^*)^2+(\beta_n\tau^*)^2}.\end{aligned} \tag{6.180}$$

That is, $\frac{d}{d\sigma}u_n(\sigma)\big|_{\sigma=\sigma_n}$ has the same sign as σ_n, since $\tau^* > 0$ and $\mu > 0$. We note from (6.80) that the crossing number $\gamma(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n})$ counts the difference, as σ varies from σ_n^- to σ_n^+, of the number of imaginary characteristic values with positive real parts in a small neighborhood of $\mathrm{i}\beta_n$ in the complex plane, where $\sigma_n^- < \sigma_n < \sigma_n^+$ are numbers in a small neighborhood of σ_n. Then by (6.180), the crossing number of the isolated center $(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n})$ in the Fuller space $C(\mathbb{R}; \mathbb{R}^2) \times \mathbb{R}^2$ satisfies

$$\gamma(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n}) = -\mathrm{sgn}(\sigma_n) \text{ for every } n \in \mathbb{N}. \tag{6.181}$$

Then by Theorem 6.15, there exists a connected component $C\left(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n}\right)$ of the closure of all the nonconstant periodic solutions of system (6.169), bifurcated from the stationary solution $(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n})$ in the Fuller space. Note that there is no nonconstant periodic solution for the system (6.169) if $\sigma = 0$, since in this case, x satisfies a scalar ordinary differential equation. Moreover, there is no bifurcation point at $\sigma = 0$. Therefore, $C\left(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n}\right)$ is located in the Fuller space where σ satisfies $\mathrm{sgn}(\sigma_n)\sigma > 0$.

To prove the unboundedness of $C\left(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n}\right)$ in the Fuller space, we apply the global Hopf bifurcation Theorem 6.17 to exclude the case that there are finitely many bifurcation points in $C\left(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n}\right)$.

Now we suppose there are finitely many bifurcation points $\{(0, \tau^*, \sigma_{n_j}, \frac{2\pi}{\beta_{n_j}})\}_{j=1}^q$, $q \in \mathbb{N}$, in $C\left(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n}\right)$. We know that $C\left(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n}\right)$ is located in the Fuller space where σ satisfies $\mathrm{sgn}(\sigma_n)\sigma > 0$. Then the bifurcation points $\{(0, \tau^*, \sigma_{n_j}, \frac{2\pi}{\beta_{n_j}})\}_{j=1}^q$ satisfy $\mathrm{sgn}(\sigma_n)\sigma_{n_j} > 0$ for all $j \in \{1, 2, \cdots, q\}$.

Let ε_{n_j} be the value of

$$\mathrm{sgn}\det\begin{bmatrix} \left(\frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2}\right)\tilde{f}(\theta_1, \theta_2, \sigma) & 0 \\ \frac{\partial}{\partial\gamma_1}\tilde{g}(\gamma_1, \gamma_2, \sigma) & \frac{\partial}{\partial\gamma_2}\tilde{g}(\gamma_1, \gamma_2, \sigma) \end{bmatrix}$$

evaluated at $(\theta_1, \theta_2, \sigma) = (0, 0, \sigma_{n_j})$ and $(\gamma_1, \gamma_2, \sigma) = (0, \tau^*, \sigma_{n_j})$, where

$$\tilde{f}(\theta_1, \theta_2, \sigma) = \left[-\mu\theta_1 + \sigma^2 b(\theta_2)\right], \quad \tilde{g}(\gamma_1, \gamma_2, \sigma) = (1 - h(\gamma_1)(1 + \tanh(\gamma_2)).$$

Then by (6.178) and (6.179), we have

$$\varepsilon_{n_j} = 1 \text{ for all } j = 1, 2, \cdots, q. \tag{6.182}$$

By (6.181) and (6.182), we have

$$\sum_{j=1}^q \varepsilon_{n_j}\gamma((0, \tau^*, \sigma_{n_j}, \frac{2\pi}{\beta_{n_j}}) = -q\,\mathrm{sgn}(\sigma_n) \neq 0. \tag{6.183}$$

Note that (α_5) and (α_6) implies (SHB4). Then by Theorem 6.17, (6.183) is a contradiction. The unboundedness of $C\left(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n}\right)$ follows.

(b) In order to verify assumption (SHB7), we claim that the virtual period p_n of every bifurcation point $(0,\, \tau^*,\, \sigma_n,\, 2\pi/\beta_n)$ satisfies

$$mp_n \neq \tau^* \text{ for every } m \in \mathbb{N}. \tag{6.184}$$

Suppose that there exist $m_0, n_0 \in \mathbb{N}$ such that $m_0 p_{n_0} = m_0 \cdot 2\pi/\beta_{n_0} = \tau^*$. We note that

$$\beta_n \in \left(\frac{(4n-3)\pi}{2\tau^*}, \frac{(4n-2)\pi}{2\tau^*} \right) \text{ for all } n \geq 1. \tag{6.185}$$

Then we have

$$4n_0 - 3 < 4m_0 < 4n_0 - 2.$$

This is a contradiction, and the claim is proved.

We note that by (6.185), a sufficient condition for $p_n = \frac{2\pi}{\beta_n} < \tau^*$, is that $\frac{2\pi}{\beta_n} < 4\tau^*/(4n-3) < \tau^*$, that is, $n \geq 7/4$. Therefore, every $(0,\, \tau^*,\, \sigma_n,\, p_n)$ with $n \geq 2$ is a bifurcation point of system (6.169) satisfying

$$p_n < \tau^* \text{ for all } n \geq 2. \tag{6.186}$$

For the bifurcation point $(0,\, \tau^*,\, \sigma_1,\, p_1)$, we can conclude from (6.185) that

$$2\tau^* < p_1 < 4\tau^*. \tag{6.187}$$

We want to obtain the uniform boundedness of the period in $C(0,\, \tau^*,\, \sigma_n,\, \frac{2\pi}{\beta_n})$ with $n \geq 2$. We only need to check the conditions (SHB6)–(SHB10) for applying Theorems 6.21 and 6.22.

It is clear that (α_4), (6.184), and (6.179) imply (SHB8), (SHB9), and (SHB6), respectively. Also we conclude from (SHB2), (SHB4), and Lemma 6.20 that

$$p > 0 \tag{6.188}$$

for every $(x,\, \tau,\, \sigma,\, p) \in C(0,\, \tau^*,\, \sigma_n,\, \frac{2\pi}{\beta_n})$. Also, by Lemma 6.31, we have

$$0 < \tau(t) < -\frac{1}{2}\ln(2h_0 - 1) \tag{6.189}$$

for every $t \in \mathbb{R}$, and hence (SHB10) is satisfied. To check (SHB7), we let

$$\begin{cases} 1 - h(x)(1 + \tanh\tau) = 0, \\ (1 + \tanh\tau)h'(x)\left(-\mu x + \sigma^2 b(x)\right) = 0. \end{cases} \tag{6.190}$$

Then by (α_1), (α_4), and (α_6), the solutions of (6.190) are stationary solutions of (6.169). This verifies (SHB7).

Therefore, we can use Theorems 6.21, 6.22, (6.186), (6.188), and (6.189) to conclude that there exists some $t \in \mathbb{R}$ such that

$$0 < p < \tau(t) < -\frac{1}{2}\ln(2h_0 - 1) \tag{6.191}$$

for every $(x, \tau, \sigma, p) \in C(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n})$ with $n \geq 2$. Then by (6.187) and (6.191), we know that $(0, \tau^*, \sigma_1, \frac{2\pi}{\beta_1}) \notin C\left(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n}\right)$ for every $n \geq 2$. This proves (b).

(c) Let Σ be the projection of $C\left(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n}\right)$ on the σ-parameter space $\mathbb{R}$. By (a), we know that $\Sigma \subseteq (0, +\infty)$ if $\sigma_n > 0$ and $\Sigma \subseteq (-\infty, 0)$ if $\sigma_n < 0$. By Lemma 6.31, we know that for every $\sigma \in \Sigma$, there exists a constant $M_n(\sigma) > 0$ such that

$$\|(x, \tau)\|_{C(\mathbb{R};\mathbb{R}^{N+1})} \leq M_n(\sigma), \tag{6.192}$$

where (x, τ, σ, p) is the solution associated with σ in $C\left(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n}\right)$ and $M_n : \mathbb{R} \ni \sigma \to M_n(\sigma) \in (0, +\infty)$ is a continuous function on $\mathbb{R}$.

We know from (6.191) that the projection of $C\left(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n}\right)$ on the p-parameter space $\mathbb{R}$ is bounded. If Σ is bounded, then it follows from (a) that the projection of $C\left(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n}\right)$ on the (x, τ)-space $C(\mathbb{R}; \mathbb{R}^{N+1})$ must be unbounded in the supremum norm. But by the continuity of M_n on $\mathbb{R}$ and by (6.192), the projection of $C\left(0, \tau^*, \sigma_n, \frac{2\pi}{\beta_n}\right)$ on the (x, τ)-space $C(\mathbb{R}; \mathbb{R}^{N+1})$ is uniformly bounded with respect to $\sigma \in \Sigma$. This is a contradiction, and the proof is complete. □

We conclude by noting that the global continuation of slowly oscillating periodic solutions is addressed in [172].