Keith R. Leatham *Editor*

Vital Directions for Mathematics Education Research

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Preface

This book grew out of a 2008–2009 lecture series (Scholars in Mathematics Education) at Brigham Young University in Provo, Utah. Seven prominent mathematics educators from the USA and Canada were invited to discuss what they viewed as vital issues facing mathematics education and what they saw as viable directions research in mathematics education could take to address these issues. Each presenter then wrote a chapter based on this premise and their presentation; these chapters make up the middle seven chapters of the book. The first and last chapters are from other prominent mathematics educators and were written in reaction to the middle seven chapters.

All of the issues raised in this book are related to the complexities of learning and teaching mathematics. The recommendations take the form of broad, overarching principles and ideas that cut across the field, garnished with specific and poignant examples. (Although the lectures were originally delivered to a U.S. audience, and thus the chapters often pull their examples from the state of education in the USA, the ideas speak to the international mathematics education community.) In this sense, this book differs from classical "research agenda projects," which seek to outline specific research questions that the field should address around a central topic. Rather, in this case, each chapter takes on vital issues in mathematics education that cut across many research agendas. The desired message is as follows: Here are vital issues facing mathematics education and here are some frameworks to direct and support research that will move us forward in addressing these issues.

Provo, UT, USA Keith R. Leatham

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Chapter 1
Reflections on a Portrait of Our Field

Steven R. Williams

Abstract After providing summaries of the seven core chapters that follow, I discuss some common themes that run through them. As a framework I use the four-component model of teacher, student, content, and environment, which continues to provide a useful way to talk about meta-issues in mathematics education. Chapters in this volume emphasize both the centrality of the acts of teaching that lie where the teacher and student components meet, and the importance of better understanding the ways in which students come to understand the content. These emphases and the educational problems that gave rise to them help paint a portrait of our field, and strongly suggest some necessary next steps. Although the seven core chapters place less emphasis on the need for careful attention to content and environment, they too need to play an important role as we move forward in our scholarly efforts.

My intention in this first chapter is to provide a brief overview of the seven core chapters that follow, and to offer some response to them. Providing individual summaries is relatively straightforward. However, the nature and origin of the chapters in this volume make the task of responding to them a daunting one. The chapters are each rich and substantial and represent a broad range of viewpoints and approaches, and so finding common themes to discuss is challenging. Moreover, asking a small number of leaders in the field of mathematics education to talk about what they see as a central problem hardly makes it fair to comment on what they chose *not* to discuss. Having pointed out the difficulty of these two tasks, I will nevertheless attempt both anyway.

Thus, following brief summaries of each core chapter, I discuss several themes common across them, and then explore a few related ideas. Those themes I have

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chosen to discuss are a product of my personal reactions to both the original talks and the associated chapters, and in particular the issues they prompted me to think about in the time since the colloquium series occurred.

I also admit that my comments are colored by the political and intellectual world in which I work, one that is likely similar to that of most U.S. mathematics educators. I think many of us would agree, for example, that public education is clearly in the national view right now, or perhaps “squarely in the political crosshairs,” is an even better description. In my own state, as well as my surrounding community, the Math Wars have quieted some in the past few years, but spirited skirmishes still break out. At the university level, there remains the usual tension between the views of mathematics educators and the views of research mathematicians on how best to solve the problems of mathematics education, as well as disagreements on exactly what those problems are. In many arenas, we find ourselves personally and professionally involved in a whirlwind of claims, counterclaims, and recommendations that I often find both frustrating and emotionally exhausting. I hope I can be pardoned, then, for an occasional remark addressing the political windmills at which I have been tilting for the past few years.

Finally, and most importantly, I express thanks to the chapter authors, both for their willingness to participate in the colloquium series and their willingness to contribute to this volume. I am confident that those who read it will find it as challenging and engaging as I did.

Summaries of the Seven Core Chapters

In Chap. 2, Mark Thames and Deborah Ball seek to reframe the problem of mathematics education in the United States, and to offer a solution. They point out that the decades-old problem of failing to adequately teach mathematics to most of our children takes on new urgency with the gaps in achievement between groups, increasing diversity, and higher expectations for all students. They present a vision of mathematical literacy consistent with the “strands” of Adding it Up (National Research Council, 2001), including the ability of students to understand both ideas and procedures, solve problems, reason about a wide variety of numerical and spatial information, model and communicate about mathematical situations, and think with and use data.

Thames and Ball next take on the components of the “high-quality instruction” that would be necessary to fulfill their vision of mathematical literacy for all students. Their components include coherent curriculum; a supportive learning environment; an educational infrastructure that aligns with curricula, assessments, teacher development, and policy; and skilled teaching. They illustrate the complexity of this last component by discussing a teaching vignette. Finally, they end their chapter with specific suggestions on how to move forward to implement their vision.

In Chap. 3, Jim Hiebert argues forcefully that changing mathematics instruction is both vitally important and frustratingly difficult. It is vitally important, he argues, because recent studies make it clear that students are not learning as much as they could. Moreover, teaching is the conduit through which every other change we might wish to make to an educational system (e.g., new curricula, standards, or assessments) will eventually reach the student. Despite its importance, however, Hiebert notes that we have been largely unsuccessful in changing teaching. He discusses three reasons for this failure: a lack of common goals, a propensity to confuse teaching with teachers (and teacher characteristics), and an under-appreciation of the cultural nature of teaching.

Hiebert closes his chapter with a discussion of what it would take to change teaching in the United States. Taking seriously its cultural nature, and assuming the presence of some common goals, Hiebert notes that teaching will likely change only when (1) there is consensus on goals for student learning; (2) we come to believe that good teaching can be learned, rather than it being an innate gift; and (3) teachers turn teaching into an object of study.

In Chap. 4, Pat Thompson explores inattention to *meaning* as a contributing factor to what he calls the "bad state of U.S. mathematics education" (Thompson, 2013, p. 57). After reviewing various approaches to defining *meaning*, he couples a Piagetian outlook with Pask's (1975, 1976) *conversation theory* to explain how students construct meanings through interaction with teachers, both when such meanings are central to the teachers' educational intentions and when they are not. In the absence of careful attention to meaning, Thompson claims that too few constraints exist on the student/teacher interaction to guard against students developing inappropriate understandings. Thompson gives several classroom examples of teaching with or teaching without attention to meaning, and ends with suggestions for making attention to mathematical meanings central in our schools.

Thompson's long-term strategy to accomplish this objective draws from Tucker (2011), and includes studying the specific mathematical meanings that high-achieving school systems want their students to attain; clarifying and garnering public support for those education goals; striving for equity in supporting systems and students who need assistance in attaining the desired mathematical meanings; monitoring quality; and making sure that these efforts cohere and mutually support each other.

Marty Simon begins Chap. 5 with an overview of how mathematics education has been limited in the United States by an absence of pedagogical theory. He argues that a good pedagogical theory, grounded in careful study of how students learn conceptually, could provide the kind of specific instruction to teachers, curriculum developers, and teacher educators that would promote such learning in students. After pointing out that recent comparisons of U.S. performance with that of other countries have neither focused on nor produced a useful pedagogical theory, Simon briefly discusses two "developing" pedagogical theories that show promise: *Realistic Mathematics Education* in the Netherlands, and the *Theory of Didactical Situations* in France. Still, he argues, "none of these efforts entail detailed understanding of students' mathematics conceptual learning processes and pedagogical theory derived from it" (Simon, 2013, p. 108). Simon ends his chapter with a discussion of how a theory of *conceptual learning and teaching of mathematics* could be developed, and with a review of his own recent work in this area, outlining the methodological challenges in such a development.

Guershon Harel's contribution in Chap. 6 focuses on the notion of *intellectual need*, one of the three pillars of his duality, need, and repeated reasoning (DNR) framework of instruction. His work in this area is motivated by his observations, over years of working in schools, that students feel "intellectually aimless" (Harel, 2013, p. 119) in mathematics classes. Thus, he attempts in his chapter to define intellectual (as opposed to psychological) need, illustrate how it looks in mathematical practice, discuss how its absence is detrimental to student learning, and suggest how its presence in teaching could lead to better learning.

Harel illustrates five intellectual needs specific to learning mathematics: the need for certainty, which leads to proof; the need for causality, which leads to explanation; the need for computation, which leads to quantification and algebraic manipulation; the need for communication, which leads to formulating and formalizing; and the need for structure, which leads to logical organization of mathematical knowledge. Harel also provides a clarifying discussion of the characteristics of intellectual need, and ends with pedagogical implications that flow directly from his theory.

In Chap. 7, Carolyn Kieran takes on the pervasive practice of separating conceptual and procedural aspects of mathematical knowledge. After reviewing the history of this "false dichotomy" (Kieran, 2013, p. 153), she draws on work from French scholars who distinguish among *task*, *technique*, and *theory*. From this viewpoint, technique (which includes knowledge of the procedural) has an epistemic role in building concepts, but also includes conceptual activity in its own right as it accomplishes tasks. One consequence of this perspective is that even automatic skills are enriched and updated by new conceptual knowledge. Indeed, she argues, "the interaction between the conceptual and the procedural is an ongoing recursive process" (p. 160).

As Kieran points out, algebra has suffered more from the separation of concept and procedure than any other school subject, with many mathematics educators turning to applications and real-world problems to give algebra a conceptual aspect. By way of contrast, Kieran includes examples taken from one of her algebra projects that illustrate both how algebraic techniques are conceptually understood and how existing technical facility is updated by new conceptual knowledge. She ends her chapter with a discussion of the broad implications the marriage of the conceptual and the procedural has for the teaching and learning of algebra.

In Chap. 8, Jeremy Kilpatrick writes not about a research agenda, but about the need to build a stronger culture of scholarly criticism. The title suggests the need for criticism from within our field, but not narrow criticism from those with a single, all-encompassing worldview—those he calls hedgehogs. Rather, Kilpatrick suggests that the need is for careful scrutiny of ideas from those who are willing to bring different critical lenses to the task.

Kilpatrick notes that there is no lack of "hedgehog" ideas in education, and provides some examples from both inside and outside the mathematics education community. He uses a historical analysis of the *Forum for Researchers* and more recent *Research Commentary* sections of the *Journal for Research in Mathematics Education* to argue that, as a field, we produce very little in the way of scholarly criticism.

He ends his contribution by discussing how we might grow more critical foxes within our field: by taking their work seriously enough to comment and offer critique; by encouraging them to become self-critical; and by building in them "both the courage to guess, and the courage to doubt our guess" (Kilpatrick, 1987, p. 330).

What Stands Out

In their chapter, Thames and Ball use a four-component model of instruction that highlights interactions among student, teacher, content, and environment. Using this model as a fairly dull tool, I can place the chapters by Hiebert and Thames and Ball closest to the *teacher* component; the chapters by Harel, Thompson, Simon, and Kieran somewhere near the *student* component; and the chapter by Kilpatrick floating above the model at a meta-level, which deals not with instruction but with how our field can evolve to better understand instruction. Again, this is a very rough categorization, since Harel, for example, certainly takes mathematical content into account as he discusses intellectual need. Furthermore, the implications for teaching are clear in all four chapters I have identified as being in the *student* category.

The authors were not chosen to be representative of our field, so it is difficult to say whether my categorization says anything about our field as a whole, but based on my past editorial experience, it seems about right, and also consistent with our history: more attention is typically paid to teaching and learning than to curricular, mathematical, environmental, or contextual factors. As a whole, the chapters may constitute an argument for where our efforts could be focused to shed more light on instruction.

The Centrality of Teaching

I am struck, for example, by the arguments by Hiebert and Thames and Ball regarding the pivotal nature of the act of teaching. Turning for a moment to the instructional model, it is easy to see that when components are considered more carefully, the great complexity of the educational process and its many fundamental aspects clearly emerge. There is curriculum (intended, enacted, hidden, etc.) as well as content (Jackson, 1992; Romberg & Kaput, 1999). A teacher is affected by certification programs, past education, and in-service development (Sowder, 2007), and brings to their instruction both knowledge [content knowledge, pedagogical knowledge, mathematical knowledge for teaching (Ball, Lubienski, & Mewborn, 2001; Hill, Sleep, Lewis, & Ball, 2007)] and beliefs (Leder, Pehkonen, & Törner, 2002; Philipp, 2007). There are building, district, and state curriculum guides and policies, as well as many other aspects of the environment. And of course, we could perform a similar breakdown for students. The vital point is, as both Hiebert and Thames and Ball point out, all the variables involving content, teachers, and policies ultimately affect students *only* through what the teacher does in the classroom, i.e., through the act of teaching. Thus, attempts to improve instruction by curriculum, characteristics of the teacher, and so on will not have much effect unless teaching changes also. Similarly, attempts to measure good teaching or good instructional environments by measuring these sorts of variables will likely fail. Indeed, this filtering of variables through the act of teaching probably explains the lack of correlation, discussed in these chapters, between student learning and seemingly commonsense measures of being a good teacher such as math classes taken and even completion of certification programs.[1]

This important theoretical point has implications for future research programs in our field: it is time to make sure that as we study such things as curricula, teacher education, and teacher development, we also study whether and how *teaching* is affected. Taking the centrality of teaching seriously also has practical value as we make time and resource decisions about research initiatives, projects, and political activities aimed at improving instruction.

Politics

This may be a good time to discuss a particular political windmill, and I will personalize it to a degree. As a department chair, various initiatives and invitations often come my way (from colleagues in neighboring departments or universities) to implement changes in programs, curricula, or other aspects of instruction. Three recent examples include the adoption of the Singapore curriculum (Ministry of Education, Singapore, 2006), the use of abaci as a primary learning tool in elementary school, and the credentialing of elementary school teachers based on a mathematics content exam. All three of these ideas have some political appeal in my state. Indeed, the adoption of a new curriculum is a popular "experiment" that often makes good newspaper copy (see, e.g., Hu, 2010). But such approaches are typically isolated from the other aspects of instruction we know to be equally vital. Beyond a few comments about what students are doing on the day the reporter visits, or the cost of "training" teachers to use the new curriculum, most of the focus is on the curriculum itself. In particular, rarely if ever is there a careful focus on teaching. All three initiatives also share some features common to many reforms being proposed by business, media, and government: they focus largely on changes intended to help teachers do their job better, and on blunt measures of teacher characteristics. Although not always explicit, there is usually an assumption that this aid will be a silver bullet—a quick and relatively simple solution—for our schools, reversing the "dismal" mathematics performance of our students.

Reading this volume reminds me of the scholarly tools we have to address these sorts of initiatives. The centrality of *teaching* as opposed to *teachers* discussed in

[1] And may I say, on behalf of all of us who spend their careers helping myriads of students through certification programs, "Ouch!"

the last section is one such tool. Indeed, I have considered printing on cards for easy distribution the following quote from Thames and Ball: "Tinkering with the curriculum only improves learning if the tinkering increases the chances of lessons getting taught well in classrooms by teachers" (2013, p. 36). In a more serious vein, their argument does give me more than a way of cynically dismissing overly simplistic approaches to instructional reform: it allows me to suggest principles whose application might actually help initiatives make some progress, such as studying *teaching* whenever you are trying out a new curriculum, certification program, or teacher development experience.

Another tool these chapters give me is the power of patience. Unfortunately, in our current society you neither get elected nor sell many newspapers by advocating a lot of hard work over an extended period of time in order to fix a problem. It is understandably more popular to suggest that immediate action—in particular, the immediate action you are proposing—is needed, and moreover that such action will bring swift resolution to the current crisis, whatever it might be. Thus there is a tendency to hope for and trust in silver bullets. As Hiebert notes, our culture is "addicted to quick fixes" (2013, p. 54). But these chapters remind us that silver bullets are very scarce. Because teaching is a cultural activity (Hiebert, 2013; Stigler & Hiebert, 1999), it takes significant time to effect changes in the basic practices and beliefs that surround it. Hiebert advocates for a slower, more evolutionary change consistent with teaching's fundamentally cultural nature. Thompson also notes the difficulty of changing cultural institutions and suggests that "30–100 years of concerted, purposeful effort" (2013, p. 89) will be needed to effect real change.

Finally, these chapters have the positive effect of getting my attention focused back on scholarship, and away from the windmills for a while. It occurs to me that, even if we found that adopting a certain practice (perhaps memorizing arithmetic facts to Gregorian chants) improved student performance, the fundamental question (after asking "What *kind* of performance?" and "W*hich students*?") would be, "Why does it work?" Our field, while motivated largely by trying to effect real changes in students' learning, is not a wholly practical one. We both rely on, and revel in, understanding of the underlying phenomena.

The Centrality of Learning

I place the four chapters by Harel, Thompson, Simon, and Kieran in the *student* category because although each has some definite implications for teaching, each also focused on some aspect of what would traditionally be called *learning*. In emphasizing the importance of *meaning*, Thompson sets a standard for the kind of mathematics learning that is needed, but is mostly missing, in our schools. In exposing the false dichotomy between conceptual and procedural knowledge, Kieran also exposes the richness of the meanings that can be developed for procedures, and their interrelation with concepts. Harel argues for a particular view of what makes learning meaningful, and subjects one part of that process to close analysis. Simon argues for the necessity of an adequate theory of learning (as opposed to a theory of states of understanding) as a basis for decisions about instruction, curriculum, teacher preparation and development, and so forth.

I address two points that strike me as of particular note from these chapters. The first is that, just as Hiebert and Thames and Ball taken together argue for the centrality of the teaching act as a focus for study, these four chapters can be taken as an argument that we need to move toward including a focus on the act of learning as we study other aspects of the fourfold model of instruction. This focus is clearly one of Simon's messages, but it is, I think, implied in the chapters by Harel, Thompson, and Kieran as well. Taken all together, perhaps this volume argues for more careful attention in our scholarly work to that point where the act of teaching and the act of learning meet.

The second point is that three of the four chapters on learning invoke Piaget in fundamental ways. Again, the authors here are not necessarily representative, but I think it is significant that so much of our best work on learning is grounded in Piaget's essential insights. My assumption is that Piaget's work being foundational to constructivism has something to do with this phenomenon, but I also assume that Piaget still provides us with the best example we have of a theory of *learning* as opposed to a theory of organized knowledge. Few other theories have provided the rich vocabulary and compelling constructs (assimilation, accommodation, reflective abstraction) that at least let us start down the path toward what Simon argues is critical to understanding of learning. This continued reliance on Piaget seems to suggest that, at least in our field, the "cognitive revolution" did not live up to the promises made to me as a graduate student (Davis, 1984; Schoenfeld, 1987). The problem of cognitive science being a "'transparent snapshot' psychology, in which mental processes are depicted at a given point in time" (Resnick & Ford, 1981, p. 244), still seems alive and well today, and is nicely explicated in Simon's chapter.

The Power of Shared Vision and Critical Thinking

A few other aspects of the chapters stand out to me. One is the suggestion in both Hiebert and Thames and Ball that a common curriculum, or at least a shared set of learning goals, is critical if teaching is to be improved. While recognizing both the political reality of the United States' long history of local decision making in schooling and the difficulty of reaching consensus, I heartily applaud the efforts to establish some common learning goals. I agree that without them, no real systemic progress will be made in improving instruction. Moreover, this suggestion agrees with the optimistic appraisal of our profession that I have had most of my career—that, given a set of agreed-upon goals, we are probably collectively smart enough to figure out how to meet them. As I will discuss later, I now feel there may be some goals that are beyond our power as a discipline to address. But I still believe there is a great deal of power in common goals, single-mindedly pursued.

Another aspect of these chapters that I particularly enjoy I will characterize as *mythbusting*—critically analyzing common wisdom, from both inside and outside of our field. Kieran's chapter is a wonderful example of such careful analysis, applied to the false (but somehow popular) dichotomy between procedural and conceptual knowledge (this issue is also raised by Hiebert and by Thames and Ball). Kieran provides strong arguments for the interrelatedness of conceptual and procedural knowledge in a way that should further our discipline's thinking and make us more wary of easy dichotomizing. Several examples of mythbusting occur in the context of Kilpatrick's *hedgehog ideas* as well as in numerous isolated statements throughout the other chapters. Among my favorites I mention (and invite the readers to look for as they read the volume) the silver bullet idea already discussed, the idea that today's students are learning less mathematics than their parents and grandparents, and the notion that teachers taking more mathematics courses improves mathematics teaching.

My Own (Hopefully Vulpine) Reflections

I hope it is clear that I find the chapters in this volume thoughtful, well crafted, and convincing. Yet for any such collection and for any reader, there will likely be a feeling that some things are missing or require more thought. To end my comments, I turn to three areas these chapters compel me to think about more carefully.

Riding the Dismal Bandwagon

As I mentioned before, public education is in the crosshairs of media and government, as well as reformers of various kinds from the public sector. And, as I implied, the reforms typically suggested are not focused on incremental changes and addressing fundamental questions, but are increasingly focused on one or two narrow ideas—usually involving testing and accountability—to achieve desired ends. Lying beneath it all is a seemingly universal belief that our schools are in desperate need of reform. For most of my career I have seen myself as a reformer, too. And of course, there is no reason to be a reformer if nothing needs reforming. Thankfully, it has proved quite easy to find support for more funding, more research, and more of whatever I am currently doing, by invoking international comparisons, declining or stagnant NAEP scores, or the latest government report. But recently, as I have noticed the company I am keeping as a reformer, I have tried to examine the evidence of educational decay a bit more carefully.

I was born the year Sputnik was launched, so it is safe to say that never in my lifetime has there been acknowledged satisfaction in the United States with the public schools. That launch was taken as a sign that the Russians had "beat us" in the race to space, and *Life* magazine ran a five-part series on the crisis in education that supposedly gave rise to it. The basic story line has been the same since then. *A Nation at Risk* (National Commission on Excellence in Education, 1983) fueled the rhetorical fire, again invoking the specter of our international competitiveness being directly affected by a mediocre school system. Business and government leaders use the same reasoning today. And it is very tempting for us as mathematics educators to get what mileage we can by jumping on the bandwagon, for at least a short ride.

Of course, others believe that the reports of the American education system's death have been greatly exaggerated. Berliner and Biddle (1995) and more recently Gerald Bracey (2009) have called into question many of the beliefs about public schools that are used by current reformers. Careful examination of the most recent PISA data suggests that the United States' mediocre ranking may have more to do with poverty than with schooling (National Education Access Network, Teachers College, Columbia University, 2011). My point is not that one or the other viewpoint is correct, but that as scholars, we need to be more critical of the claims made about the state of public schooling, especially in regard to international comparisons and the supposed threats to America's economic well-being. Policy issues have always been underrepresented as a research subject in our field, but they are becoming of increasing importance. Individually and collectively, we need to turn more scholarly attention to this area. Until that happens, we need to avoid awfulizing[2] the state of public education as an easy way to justify our research.

Learning and Teaching

It is clear that I have sympathy for viewing both the act of teaching and the act of learning as central to our instruction, because many other variables affect instruction only as they are filtered through teaching and learning acts. These chapters again remind me of the complexity of these two common activities. Simon is very forthcoming about the difficulty of studying learning as he envisions it. Similarly, the culturally embedded nature of teaching together with the dynamic nature of teachers' own cognitive acts make the study of teaching acts at least as difficult. As we continue to investigate these two fundamental acts, it is surely necessary to make simplifying assumptions, isolate interrelated variables, and perhaps build brick by brick (Begle & Gibb, 1980) our knowledge of how teaching and learning each happens. We have a lot of hard work ahead of us in understanding these two phenomena separately.

Nevertheless, teaching and learning must eventually be studied as an interacting pair if we are really to understand instruction. Our ability to do so will likely grow as we clarify each separately, but complete understanding will come only when they

[2] A term often used by psychologist Albert Ellis to describe a tendency to see things as much worse than they really are.

are studied together. To me focusing on these related activities, and eventually on the relatedness itself, seems to be a very difficult but a very important goal for the profession. For many reasons developing a robust theory of the complex system that is teaching and learning is a daunting task, and to some readers such theory may seem far removed from the realities of the school and classroom. Nevertheless, I view such theory as fundamental and would argue that, when fleshed out, it will almost certainly bear fruit. "There is nothing so practical as a good theory," Lewin (1951, p. 169) observed, and we have in this research program the possibility of a very important, very practical theory indeed.

Mathematics

As much as I applaud the call for a common set of learning goals (or even a common curriculum) for mathematics, I am not sure that such a call addresses the issue of mathematics content at a fundamental level. Elsewhere (Williams, 2008) I have suggested that we have, as a discipline, no single compelling version of what mathematics is. Certainly a common curriculum could be taken as a de facto common view of mathematics, but that begs the question of how such a curriculum would be decided upon and just whose mathematics would be the official version underlying it.

It is clear that we have a history in our field of recognizing different mathematics, beginning at least with Skemp's (1987) statement that "we are not talking about better or worse teaching of the same kind of mathematics…. There are two effectively different subjects being taught under the same name, 'mathematics'" (p. 156). Richards (1991) distinguished among four "domains of discourse" in which mathematics is the subject matter: (a) Research Math, or the "spoken mathematics of the professional mathematician or the scientist;" (b) Inquiry Math, or "mathematics as it is used by mathematically literate adults;" (c) Journal Math, or the "language of mathematical publications and papers;" and (d) School Math, or "the discourse of the standard classroom in which mathematics is taught" (pp. 15–16). Ball et al. (2001) suggested that the mathematical knowledge needed for teaching is "not something a mathematician would have by virtue of having studied advanced mathematics" (2001, p. 448). Thus, although it is obvious that some kind of mathematical knowledge is needed to become a mathematics teacher, it is not clear whether that knowledge looks much like what is provided by most college-level mathematics courses. Finally, Sfard (1998) discussed the gulf between the views of mathematics commonly held by what she calls a "Typical Mathematician" and those held by a "mathematics education researcher" (p. 505). She concluded that

> the difference is too fundamental to be just dismissed or glossed over. Moreover, it seems that trying to fill in the gap in an attempt to make the two mathematics into one would be pointless. Indeed, we are faced here with a system of beliefs as distinct as those which separate incommensurable scientific paradigms, rivaling socio-economic doctrines or different religions. (p. 505)

These examples suggest that the different kinds of mathematical knowledge and practice distinguished above are different enough to be treated, educationally, as distinct sorts of activities, and therefore as distinctly different mathematics. My point here is that there is no one *mathematics* that can underlie a common curriculum, for the concepts, procedures, discourses, and habits of mind differ depending on the community of mathematical practice. It may be possible to choose an *optimal* mathematics, but the task will not be an easy one.

For example, it is tempting to choose a mathematics that optimizes students' abilities to function in our increasingly mathematized world. This seems to be the path taken by NCTM's (2000) *Principles and Standards for School Mathematics*:

> The underpinnings of everyday life are increasingly mathematical and technological. For instance, making purchasing decisions, choosing insurance or health plans, and voting knowledgeably all call for quantitative sophistication…. Just as the level of mathematics needed for intelligent citizenship has increased dramatically, so too has the level of mathematical thinking and problem solving needed in the workplace, in professional areas ranging from health care to graphic design. (p. 4)

However, the situation is muddied by what Labaree (1997) sees as the struggle between three separate goals for public education: democratic equality, social efficiency, and social mobility. For the first two goals, education is valuable because it is useful in helping make decisions or solve problems such as in the performance of a job. Labaree calls this *use value*, and it is consistent with the view of NCTM (2000) above. For the third goal, education is valuable because it can be exchanged for better social standing, for example, a higher paying job—what Labaree calls *exchange value*. Labaree argues that many problems in public education have roots in the struggle between these three goals. It is easy to see that, although parents have a sort of general notion of the "use value" of mathematics, the "exchange value" of mathematics, in terms of success on college placement tests and grade point average, is immediate and clear. In the absence of clear and compelling information about the kind of mathematical knowledge that will have use value for students, the kind that can be quickly exchanged for social goods is likely to loom much larger for parents, and will exert pressure to maintain traditional curricula that have worked to provide exchange value for parents.

Neither Labaree nor I have solutions to this problem, but it is likely that systemic changes in the mathematics education of students will not occur without focused scholarly attention to the nature of the mathematics that students *need* to learn. Even with a compelling model of how teaching and learning occur in the moment and over time, the questions of *what* mathematics is being taught and learned will still loom large and have a profound effect on what students and teachers can accomplish. My hope (and maybe I'm letting my own little hedgehog in through the back door, here) is that scholars in our field will emerge to take this challenge seriously, and provide our field with fundamental knowledge about the mathematics that is most important to teach and learn.

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Chapter 2
Making Progress in U.S. Mathematics Education: Lessons Learned—Past, Present, and Future

Mark Hoover Thames and Deborah Loewenberg Ball

Abstract Critics have deplored the quality of U.S. mathematics education for over 50 years. Schemes to improve it disappoint in their outcomes. At the same time, much more is now known about the challenges of effective mathematics education and about what it takes to tackle them. The U.S. mathematics education community stands at a threshold where it could help the country take substantial steps forward if it deliberately learned from the past, clarified its best ideas, and developed strategies for moving those ideas into the public debate. This chapter characterizes the challenge and argues for action informed by current practice and past reforms.

Americans have long complained about the quality of mathematics education. This discontent was evident in the wake-up calls that spawned the "New Math" of the Sputnik era and in the warnings of *A Nation at Risk* (National Commission on Excellence in Education, 1983). It has grown as American students appear to fall further and further behind the students of other countries. Although complaints about schools differ, concerns about mathematics education consistently trouble the public. Common schemes for improving mathematics education (e.g., new curricula, high stakes assessment, and teacher incentives) have been overused with little lasting impact.

This chapter offers a redefinition of what might be referred to as the "mathematics education problem" and articulates a solution. If the discourse 10 years from now is to be something other than a refrain about why U.S. mathematics education does not work, a different strategy is needed. This chapter begins by clarifying the problem before drawing on lessons from past failures to propose a plan for improvement.

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Framing the Problem of Mathematics Education in the United States

In a recent National Science Foundation special report, *Math*: *What's the Problem*? (Zacharias, 2009), William Schmidt traces the poor mathematics achievement of U.S. students to the simple fact that the United States has not adequately taught its children mathematics for generations. This travesty has led to a situation in which it is acceptable for adults in U.S. society to say, "I'm not good at math," as if it were a joke or a badge of honor. Schmidt suggests that Americans have routinely communicated to their children that a few people have a "math gene," but most do not—a notion he claims is completely wrong and profoundly damaging. He says that while everyone may not excel in math, everyone can develop a strong mathematical foundation. In addition, the problem in the United States has a new urgency with features notably different from those of the past.

The new urgency stems from four pressing realities. First are the persistent gaps in achievement gains among different groups. African-American and Hispanic students in this country consistently score lower and exhibit lower achievement gains than their white and Asian-American counterparts, even when taking social class into account (Fryer & Levitt, 2006; Kao & Thompson, 2003; KewelRamani, Gilbertson, Fox, & Provasnik, 2007; Reardon & Galindo, 2009; Riegle-Crumb & Grodsky, 2010; Strutchens, Lubienski, McGraw, & Westbrook, 2004). Similar gaps are evident for students when comparisons are based on family income, again even when taking social class into account (Lubienski & Crane, 2010). And these gaps, associated with social class and race, are not shrinking. Likewise, there is a gap in achievement between U.S. students and their counterparts in similar countries. See, for instance, results of the Trends in International Mathematics and Science Study 2007 (Gonzales et al., 2008) and of the Programme for International Student Assessment 2009 (Organisation for Economic Co-operation and Development, 2010). It is appalling that a country with the resources and strengths of the United States, and built on principles of freedom, equality, and justice, has a system that educates its young people so poorly and so unevenly.

The second point contributing to the urgency complements the first. In this same system, in which the education of a diverse population of students is already a challenge and in which mathematics achievement can be predicted based on students' race and family income, the school population is changing dramatically. Drawing from the U.S. census, the Federal Interagency Forum on Child and Family Statistics (2010) reported the following: In 1972 about 80 % of the students in the United States were white and about 20 % were underrepresented minorities; currently the country is about 55 % white; and by 2023, according to its projections, white students in U.S. schools will be a minority. Although there is little change in the proportion of African-American students, there are large changes in the Hispanic population and in non-Asian-American Asian populations, a group whose achievement patterns are similar to Hispanics and African-Americans (Federal Interagency Forum on Child and Family Statistics, 2010; Zhao & Qiu, 2009).

A third source of urgency is language diversity. The Federal Interagency Forum on Child and Family Statistics (2010) reported that in 1979 about 9 % of U.S. students spoke a language other than English in the home and that this is now about 21 % of U.S. students. These language differences lead to a variety of challenges for teachers and students, but require careful consideration.[1] It is worth noting that many of these children learn to speak English well in school—only 5 % both speak a language other than English at home and have difficulty speaking English (Federal Interagency Forum on Child and Family Statistics, 2010). Having so many students come from homes where English is not the language used in the home creates a challenge for teachers in communicating with parents, made all the more difficult because, while the children often speak English, the parents often do not. The relationship of school to home is crucial to children's success, so the challenge of communicating with parents is rapidly becoming culturally and linguistically more complex. This, too, adds to the immediate needs for reconsidering the problem of mathematics education and for designing a system for improving it.

In addition to imperatives resulting from who is in school and how well they are being served, the country is also expecting more complex academic outcomes of all students than ever before. These increased expectations increase the demand on the education system and increase the need to find solutions to the mathematics education problem. For instance, state curriculum frameworks now specify goals that are considerably more challenging than in the past. As an example, the State of Michigan recently decided that in order to graduate from high school all students must pass a state-certified Algebra 2 course (this in a context in which 25 % of the entering ninth graders in Detroit, Lansing, Pontiac, Flint, and several other Michigan cities drop out before completing high school). What will happen over the next few years as the system expects students who are inadequately prepared and who typically have not taken this course to begin suddenly to not only take it but also to pass it? It may be a good idea to expect students to take Algebra 2, but a number of issues deserve both thoughtful consideration and public debate. States across the country are setting higher expectations though they have been unable to meet current expectations. In short, schools that are not doing well with their students are being asked to teach more mathematics to more students—this dramatic double rise in demands (of what is taught and who is taught) greatly adds to the urgency of the problem.

To point out the nature of the problem, below is a short "pretest" that highlights a number of prevailing myths about the condition of schools in the United States, myths that color common views of the problem (see Fig. 2.1). This simple pretest is

[1] We do not mean to imply that language diversity should be viewed as an impediment to teaching. Indeed, different languages provide additional resources for learning mathematics that often are not used well. For example, in Spanish some mathematical terms are much more comfortably related to the targeted mathematical meaning than are the English terms, yet programs often require that students go through awkward English terminology as they move from Spanish to English to mathematical language. Smarter instruction would make better use of the resources that Spanish-speaking children bring.

1. The U.S. mathematics education system used to educate our nation's young people much better than it does now.
2. The number of mathematics courses that a teacher has taken is a good predictor of how effective he or she will be.
3. Societal problems (e.g., inequality, poverty, the eroding family unit) are so overwhelming that schools cannot do their job.
4. Teacher education and mathematics curricula are similar to those taught 50 years ago.
5. College and university programs prepare teachers better than alternative routes into teaching.

Fig. 2.1 Pretest for identifying prevailing myths about the conditions of U.S. schools

meant to provide a sense of how these myths operate. It consists of five common statements, some supportable with evidence, some not. Which are myths and which are true?

The four pressing problems discussed above also provide an important metric for judging progress. Looking forward 10 years, a better education system would not produce achievement gaps between, on the one hand, underrepresented minority students and students living in poverty and, on the other hand, their white and middle-class counterparts. That students differ in their achievement is to be expected because people differ, but those differences should not be predictable based on ethnicity or family wealth. Ours is not an argument that everyone be treated the same or come out looking the same. It is that in an acceptable, equitable education system social identifiers would not be predictors of achievement gains. That is what we mean by eliminating the gap.

In addition, all students (every student) would have reliable access to high-quality mathematics instruction, no matter who they are or where they live (every year). The United States is far from this goal right now. Currently, in the United States, the likelihood that a child's teacher understands mathematics and can teach it skillfully to every student is low—and it is even lower in schools that serve under-represented, poor communities. No other occupation in the country is handled in this way. When people go to the dentist for a root canal, they expect the dentist to know what he or she is doing. If a hairdresser does not cut hair well, clients do not go back. The situation for teaching is different. The target for teaching needs to be high levels of achievement gains by all students, and high levels need to be maintained across transitions—from preschool to elementary, elementary to middle, middle to high school, and high school to college. In other words, the slippages and gaps so evident now need to be replaced with significant learning across social groups and across transitional points.

Commitment by the country to the importance of the mathematics education of all students needs to be demonstrated by the allocation of adequate human, fiscal, social, and political resources. It's easy for people to say that the situation needs to improve, but the current will and allocation of resources are insufficient for the goals and challenges described above. This is not simply an issue of money. It is about understanding the goals and challenges, and their implications, well enough to act effectively. Without changes in attitudes and understanding, U.S. citizens are unlikely to choose improvement strategies wisely or to rally the necessary will if a promising strategy were launched. To help mobilize citizens, the mathematics education community needs to develop strategic formulations of the key issues and effective ways to engage the broader society in their solution. Those hoping to contribute to improvement need to invest in developing a well-tuned campaign that helps educate people about the problem. Without such a campaign, little progress is likely to be made.

The metrics discussed above omit other reasonable criteria, such as adequate mathematical literacy for good citizenship or satisfaction among leaders in business and industry with the mathematical skills of people entering the job market. However, the metrics above are critical indicators of dynamics associated with the overall health of an education system. For this reason they deserve immediate attention. Our argument about the importance of these indicators is central to this chapter. However, to understand the problem fully, it is important first to have a clear sense of the goal. The next section lays out a vision of mathematical literacy and of the nature of quality mathematics instruction—because these factors are key to building effective improvement strategies.

A Vision of Mathematical Literacy

At the center of the problem is a vision of mathematical literacy, or of mathematical proficiency. Namely, what should a mathematically educated person know and be able to do? Experimenting with a vision of mathematical literacy that could be used with the general public, we propose that it would involve being able to:

- Understand and be able to use mathematical ideas and procedures
- Frame and solve problems involving quantity, space, and probability
- Interpret and reason about quantitative, probabilistic, and spatial information
- Use representations to model situations and communicate about mathematical ideas
- Think with and use data

Somewhat buried in this list is the idea of being able to use and manipulate the symbolic languages crucial to mathematics. Not being taught to use these languages fluently is a significant disservice to people. Perhaps this aspect of mathematical literacy is best embedded inside the five elements listed above, but perhaps it deserves its own place on the list.

Drawing from established mathematics education literature, another way to think about mathematical literacy is to use the concept of proficiency developed in *Adding It Up* (National Research Council, 2001). This National Research Council report suggests five strands for thinking about mathematical proficiency (see Fig. 2.2). It argues that being good, or skillful, at mathematics does not rely on any one of these strands alone, but relies on all five of them and on the interrelationship among them.

Procedural fluency can be thought of as computational skill in the lower grades, but it also involves being able to manipulate equations, use algorithms, and work

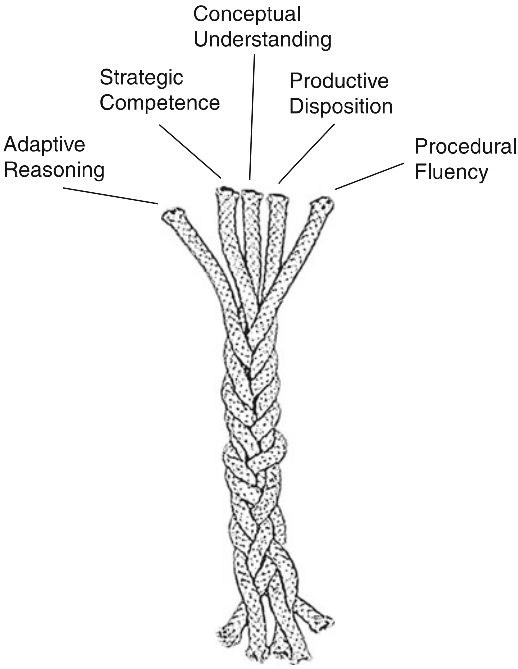


Fig. 2.2 Intertwined strands of mathematical proficiency (from Kilpatrick et al., 2001, p. 117)

quickly with things expressed in condensed form, with fluency and understanding. *Conceptual understanding* can be thought of in several related ways, but focuses on central ideas instead of procedures. *Adaptive reasoning* is the reasoning, proving, and explaining that are central to building mathematical knowledge. *Strategic competence* is skill in thinking about the way one decides to formulate problems, choose and use representations, or set up and manipulate equations or other symbolic forms. *Productive disposition* is the only strand that deals with the person, for instance with whether one sees oneself as someone who is capable of doing mathematics. Productive disposition also includes, perhaps even more importantly, seeing mathematics itself as a rational domain in which effort, learning, and work make it possible to be successful. The National Research Council report argued for the combination of these five strands based on summaries of a wide range of studies in the field. Taken together, these five strands offer a relatively succinct, yet accurate, way to think about mathematical literacy.

The point here is not simply about a collection of strands taken individually, but about the intertwined nature of the strands that make the rope, about the idea that if one only works on a single strand, one is unlikely to become fully mathematically literate. This feature of a vision of mathematical literacy is important because the history of mathematics education in this country has been dominated by pendulum swings back and forth between procedural fluency and conceptual understanding,

Two people are discussing the weather forecast. *Saturday: 50% chance of rain* *Sunday: 50% chance of rain* *One says, "Darn, what a bummer! I was planning to play golf on the weekend, but now there's a 100% chance of rain on the weekend."* *Is this right?*

Fig. 2.3 A "simple" probability problem

with little attention to the intertwined nature of these strands and to the importance of adaptive reasoning.

Schools in the United States do a poor job of preparing young people to reason about mathematics. This is true in the teaching of proof at the university level, but earlier versions exist in the lower grades, where few students are taught what constitutes a mathematical explanation. Instead, many students think that reporting the steps they used to solve a problem is an explanation, and they readily propose taking a vote if a debate ensues about a mathematical claim. Thus, a curriculum that would grow students' capacity to reason about and justify mathematical claims would look very different from current practice in schools. In other words, while members of the mathematics education community are busy arguing about the relative weight of procedural fluency and conceptual understanding, other key ingredients of mathematical literacy are being ignored. And, because these students become people who participate in and shape the public debate, their miseducation carries over into policies and perceptions of the broader society.

To illustrate these ideas about mathematical literacy, consider the problem in Fig. 2.3. People often laugh upon reading it, but consider both the answer and what people are likely to answer. Assuming that rain on the 2 days are independent events, what is the chance of rain? Why? And, how would you represent the problem in order to reason about it and communicate your answer clearly?

There are two reasons we give this problem. One is that it captures some of the sense of mathematical literacy described above, and a second is to propose it as an example one might give to people who need help appreciating the nature of the broader problem in mathematics education. Most people who read this chapter will be concerned about mathematics education, but many in the larger society do not think that mathematics education is a concern. We argue that this example provides a good start for raising key issues with people of different backgrounds and convictions.

Many people answer that they think 100 % is wrong and that an answer of 100 % is "funny," but then they often say, "Oh, obviously it's 50 %." If you then ask them to explain, they then provide one of a variety of explanations. Below are two different ways of explaining a correct answer for this problem. For each, there are two questions worth considering, one about the mathematical reasoning and the other about the representations used.

a

	Sunday: No Rain	Sunday: Rain
Saturday: No Rain		25%
Saturday: Rain		

b

	Sunday: No Rain	Sunday: Rain
Saturday: No Rain	25%	75%
Saturday: Rain		

Fig. 2.4 Two-by-two table representations of possible events indicating the chance of (**a**) no rain on Saturday and rain on Sunday; (**b**) rain on the weekend

The situation in the problem gives a weather forecast for each day and a person who wants to play golf on the weekend. The question is: What is the chance it will rain on the weekend? Different things could happen. It could not rain on Saturday, or it could rain on Saturday. And we know that the chances are 50/50. Likewise, it could not rain on Sunday, or it could rain on Sunday. This information can be represented in a two-by-two matrix of possible events for what could happen when you combine the 2 days. One possibility for the weekend would be that it could not rain on Saturday and then, after that, it could rain on Sunday. And, there are one in four chances of that happening, or 25 % (see Fig. 2.4a). Likewise, it could rain on Saturday and not rain on Sunday, with one in four chances of that happening. It could not rain on either day, or it could rain on both days, also with chances one in four. The question of what are the chances of rain on the weekend is really a question of whether it will rain at any time on the weekend. The only way it does not rain on the weekend is if it does not rain on either day. Hence, there is a 75 % chance of rain on the weekend (see Fig. 2.4b).

Next, consider a second representation and explanation, paying attention to its features and ways in which its features are similar or different from the previous representation. Remember that the purpose is to illustrate how to help a general audience to understand that the aims of mathematics education ought to be the ability to think well in everyday life using the tools of mathematics and to communicate and debate effectively with others. The second representation breaks down the problem in a different way. One could say that it could rain or not rain on Saturday, and that it could rain or not rain on Sunday, with a 50/50 chance for each. Thinking chronologically, the possibilities can be arrayed out as in Fig. 2.5a, where the first possibility is that it rains on Saturday and also on Sunday, and the second is that it rains on Saturday but not on Sunday, and so on. Then, if one reorganizes one's thinking to ask "What are the chances of rain on the weekend?" these different trees, or pathways, can lead to understanding that there is only one of four possible ways that it would not rain at all (see Fig. 2.5b).

These are two different ways, among many, of representing this problem. The underlying mathematical structure for both is the same, and in that sense they are not so different, but the very competency of recognizing them as the same is indeed a crucial part of the mathematical literacy we need in the United States. For many, these two explanations are quite different. Different ways of thinking lead to them, and the representations support different kinds of thinking.

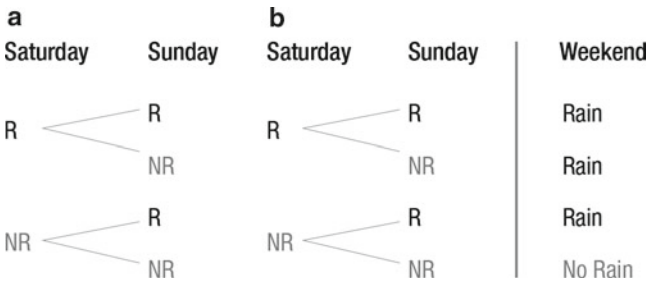


Fig. 2.5 Tree representations of (**a**) possible 2-day events; (**b**) possible 2-day events with associated outcomes

Returning to what it means to be mathematically literate, people need to be able to solve problems like this one, but they also need to be able to explain their answers, choose and use representations for addressing such problems, follow approaches different from their own, and communicate about and across approaches. They need to be able to recognize what it means to reason about something that is ordinary, such as questions that have to do with probabilistic reasoning about the weather. This probability problem illustrates that one has to understand some basic probabilistic concepts about the space of different possible outcomes for a particular problem, but that one must also understand how one represents the problem, how one frames the question, how one uses representations, and how one thinks with data.

The field of mathematics education needs problems such as this, ones that a very math-phobic or math-uninterested adult would appreciate and that might help to make more clear, in serious and nontrivial ways, what the consequences are that so many people think that the probability of weather, or of coin tosses, or of other problems with the same structure, is 100 or 50 %. Having compelling ways to represent the problem of mathematics education to the general public is crucial for moving from a concern held by a small minority to a widely shared concern necessary for real change. The mathematics education community needs to become better at communicating with the majority of people in this country, people who are not mathematically inclined or well educated, people beyond the immediate community. This is challenging for the very reason that the current system does not work well. Efforts to improve mathematics education suffer from the fact that most educated adults are not very well oriented to the problem of creating a mathematically literate society.

Thus, the problem of mathematics education in the United States is characterized by (a) severe underperformance and inequality in educating our nation's youth in mathematics; (b) no shared sense in the country—maybe even within the mathematics and mathematics education communities—about what mathematical literacy is and its importance; and (c) an enormous challenge of building a strategy for improvement in a country where so many people, including leaders and policy makers, are themselves not mathematically well educated. This is an unusual and extraordinary problem. No such problem exists when it comes to language literacy. In contrast to mathematics, leading policy makers can read and do appreciate that it is important for young people to read. Many people in positions of authority do not understand why mathematics educators think it is important for students to be better at mathematics—both our most able children and all of our children.

Before proposing ways to address the problem of mathematics education, it is essential that we are clear about what most influences student learning and how that influence is achieved. We move now to consider mathematics teaching in the United States.

Constituents of High-Quality Mathematics Instruction

Evident in the comments above is the idea that instruction in the classroom is key. With the word instruction, we don't mean quite the same thing as teaching. Instead we mean to suggest a systemic point—where the alignment of components of the system affords opportunities for students to learn. As described by Cohen, Raudenbush, and Ball (2003), it is in the interactions in the classroom that alignment occurs and it is these interactions that result in student learning (see Fig. 2.6). An instructional system is one in which teachers are interacting with the content and are representing it to students and where students are listening to one another, even in lecture classes, hearing answers from other students, and engaging with others in learning the content. Students hear and interact with their teachers and peers, and they interact with the content.

All of these dynamics shape what any particular presentation of content produces. One can produce a well-designed sequence of lessons on fractions, for

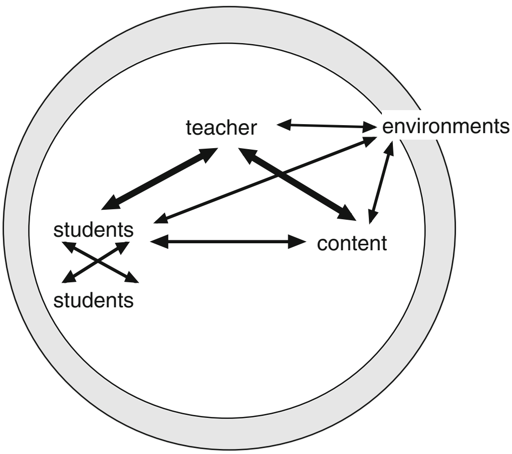


Fig. 2.6 Instruction as interaction of teacher, students, and content, in environments

instance, but one must also understand that there is nothing about that sequence of well-designed lessons that will predict instruction because it will depend on what particular students bring to the lessons, how the students interpret the teacher's use of those lessons, and how the teacher understands the lessons. Even a slightly different statement of a problem given on a worksheet or of a definition written on the board can change what students are thinking and learning. A student could ask a question in one classroom that is not asked in another, either creating a favorable opportunity or throwing everyone off. All of the interactions down the line can vary just slightly—like an accumulation of measurement errors—and result in very different lessons. Thus, it is impossible to determine from any specification of a lesson what instruction will actually be.

However, teachers are the ones who are charged with increasing the probability that the lessons one hopes will get taught do get taught and that students learn what they are supposed to learn. Primary responsibility cannot be assigned to the curriculum, and it cannot be assigned to students. Professionals should be held, and should expect to be held, accountable because they are the people with the skill to raise the probability that the lesson produces the outcomes it was designed to produce.

Of course, these interactions do not happen in a vacuum. All of them are influenced by the surrounding environment—by the values of the community, by the policy in the context, by testing, and by the principal in the school—but the double-edged arrows in Fig. 2.6 are intended to suggest that the environment does not simply bear down on schools. The environment is interpreted, and not uniformly. Two teachers working in the same school often interpret the pressures from the school board differently. A principal who says, "You must be on page 358 on April 11th," will not have every teacher in the school on page 358 on April 11th, because some teachers will say, "I know how to handle that principal. I can do it this way—I can teach and make sure I'm covering the content," and another will say, "I feel completely intimidated about what I'm being told, and will be sure to be on page 358 on April 11th." Regarding instruction, all of the interactions that constitute it are bidirectional and none fully determines any of the others.

From our experiences studying teaching and from teaching, we have developed a working hypothesis that four elements of high-quality mathematics instruction lie at the heart of the premise that teachers can raise the probability that the dynamics of instruction will produce the desired outcomes. The first has to do with having a *coherent mathematics curriculum*—the curriculum must be focused in a balanced way across the features of mathematics literacy, or strands of proficiency (see Fig. 2.2). The second element is a *supportive learning environment*. This includes characteristics both of the classroom itself, such as a careful use of language and the availability of public space for recording mathematical resources, and of the situation beyond the classroom, such as high-quality homework that connects the home to school and meaningful connections to students' out-of-school lives. The third element is *educational infrastructure*. This includes alignment among the curriculum, the assessment, the training of teachers, and the policy environment, as well as structural features that support successful instruction.

Unfortunately, right now the education system lacks infrastructure specifically for the support of instruction. Beginning teachers enter schools that provide little or no support for them to improve their skills. This makes teaching an anomaly among professional occupations. For instance, nurses are put into hospitals where it is assumed that the charge nurse and the other nurses on the shift will assist the beginning nurse in knowing which cases he or she is ready to handle alone and for which help is warranted. Inexperienced nurses are not assigned all of the difficult cases at the outset, as often happens in schools. Even in an occupation such as nursing, with similar scale and preparation to that of teaching, there is a system for taking novices and building their skill. In education, beginners are asked, haphazardly, to do work at all different levels of complexity. Infrastructure is lacking, but desperately needed.

The last element of high-quality mathematics instruction is *skilled teaching*, which is characterized by five features. Skilled teaching focuses on core concepts and skills. It is also culturally and linguistically sensitive. If we take as a given that a teacher's success is a matter of whether or not children learn, then teaching has to deal with the students who are in class. Because teaching is about relating content to students, teachers have to be sensitive linguistically and culturally to who their students are. They have to know which example will work best, where language matters, and so on. Third, students need to be active and engaged. This does not mean that students need to be talking or be in small groups. A student can be engaged and be in a lecture. Engagement is about whether students' minds are actively interacting with the content, and there are different ways to accomplish this (Dewey, 1965/1904). Perhaps more to the point, if a teacher is giving an elegant lecture and no one is following it, then that does not count as active engagement, or skilled teaching, but small group work where students are fooling around and not working on math problems also does not qualify as active engagement. Engagement does not rely on the physical organization; it depends on an intellectual connection.

The phrase equitable engagement is meant to point out that if you call on students, whether in a university lecture hall or in an elementary school classroom, and you only call on the people with the answers, this would not count as skilled teaching. Teaching a few people who already know what you are teaching does not constitute skilled practice. Skilled teaching requires the complexity of having people who do not understand the content actively thinking about and learning that content. That work is harder than simply coming up with a good explanation. (For an extended discussion of this issue, see Cohen, 2011.)

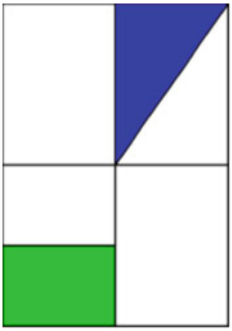
In addition, skilled teaching involves attention to mathematical language and reasoning. It is our impression that the role of mathematical language has been woefully underestimated in practical guidance given to teachers by the mathematics education community. Language is a key medium for teaching and learning. A number of theorists (e.g., Cazden, 1988; Vygotsky, 1986; Wittgenstein, 1958; and others) have focused on the importance of language in teaching and learning. Given that teaching and learning are language intensive, disciplinary language practices can offer resources for teaching mathematics. That mathematics educators express such affection for students' invented language while giving little attention to disciplinary language means that they are poorly positioning teachers to make good judgments about when it is okay for students to speak in a mathematically sloppy way and when it is not. This is not to say that first graders should always speak precisely. They cannot and should not be expected to do so. However, a teacher needs to know when it matters that students are implicitly defining something in their speech in a way that is going to lead to distorted understanding of mathematics within a year or two, and when what they are saying is an intermediate step in becoming competent and is not going to cause problems. Being able to recognize this difference is critically important.

As mentioned earlier, reasoning is important for student learning and is central to skillful teaching. As Ball and Bass (2003, p. 29) argue, mathematical reasoning is as fundamental to knowing and using mathematics as comprehension of text is to reading. As with the importance of disciplinary language practices, given that teaching and learning center on students' knowledge building, disciplinary knowledge-building practices offer resources as well. In addition to attending to students' mathematical language and mathematical reasoning as part of students' growing mathematical proficiency, attending to mathematical language and reasoning can support teachers in the language-intensive and reasoning-intensive work of instruction.

Last, it is important to understand that teaching is diagnostic work, whether teaching 500 students or 12. Given that teachers are responsible for the content and students, the skill of understanding whether students are "getting it" is at the heart of being able to teach skillfully. If a teacher flies blind, hoping that students are understanding, he or she is likely to stray far from where students actually are and is likely to greatly reduce the chances that students learn. For instance, in a mathematics departmental seminar at the University of Michigan, faculty did interviews of students who had received A's and B's in honors calculus courses, students whom faculty believed had done well in the calculus sequence, and they found that even with basic questions about the meaning of the derivative, students routinely gave wrong, curious, or even remarkable answers. One implication is that skillful teaching requires exam questions and methods of assessment that allow for finding out whether even very good students are missing major concepts and skills essential to mathematics literacy. Teaching that does not do that is effectively abdicating its defining responsibility.

The Work of Teaching

To get a clearer sense of skilled teaching, we next provide a short vignette of a lesson on fractions from the 2007 Elementary Mathematics Laboratory at the University of Michigan (adapted from Thames, 2009). The teaching described here is not meant to be good or not good, but is meant to illustrate in more detail the dynamics of instruction and to do so in a way that conveys the fact that there are



- What fraction of the big rectangle is shaded blue?
- What fraction of the big rectangle is shaded green?
- What fraction of the big rectangle is shaded altogether?

Fig. 2.7 *Blue–green rectangle* fraction problem (The *small shaded rectangle* is *green* and the *shaded triangle* is *blue*)

many teachers in the United States who know a great deal about how to teach. What happens in this vignette is not exceptional; events like this occur routinely across the country in many classrooms, where teachers manage emotions, coordinate activities, and focus student attention on mathematics. Our purpose is to point out the specialized work teachers do, and the extensive knowledge and skill they demonstrate. If mathematics education is to be improved, the image this country holds of skilled teaching cannot make teaching precious. The United States cannot put itself in a position where there are just a few people who can teach well. It needs to have four million people who can teach well, a fact that has major policy and practical implications.

The mathematical task given to students in this episode (see Fig. 2.7) was developed by researchers at the University of California, Berkeley, as part of their work at the Elementary Mathematics Laboratory and as part of a larger investigation of upper elementary students' learning of fractions. In using this problem, there are several questions a teacher would need to consider: What fundamental mathematical issue is the task designed to address? For the first question, what answer is a fourth grader who is just beginning to understand key ideas about fractions likely to produce? For instance, they might say one-half. Why? They might say one-sixth. Why?

An interesting point here is that U.S. school curricula do not always do a good job of helping students make the transition from a counting model to an area model. When considering the fraction of people in a room who are male, the size of the individual males is irrelevant, with attention given to just the *number* of males. In this problem about shaded regions of the rectangle, an answer of one-sixth is consistent with such a counting model. Saying that the blue part is one-sixth is a common, and sensible, initial answer at this point in children's learning about fractions. Unfortunately, as instruction begins to move into area models, it often fails to signal to students that the crucial issue now has to do with equal areas and not with equal numbers of parts. Furthermore, the phrase "equal parts" is often used in teaching fractions in the United States, pervasively, even in work on area models, which surfaces an important language issue, reinforcing the point we made earlier about the crucial role of language in teaching. Given the common use of "equal parts," it should not be surprising that many students would answer one-sixth, because one-sixth of the parts is shaded. In addition to the language demands of teaching that are implicated by this problem, we can also glimpse the kind of work involved in choosing well-designed problems that target key ideas, in this case the meaning of equal parts in an area model and the importance of paying careful attention to the whole.

This problem was used in a summer laboratory class with students who had finished fourth grade the previous spring and had been identified as struggling in mathematics (by their teachers and schools). The problem was used as a warm-up, written on the board for students to work on as they entered class. The episode represented here is from a whole-class discussion that occurred after discussing the previous day's work and before beginning the next major work. The 4-min interaction is between the teacher and a student who has an answer to the problem after having worked on it for a few minutes at the start of class. Consider what is involved in the work of teaching—in being responsible for designing and enacting instruction to support student learning. The point in describing the episode is to make clear why the strategy for improving mathematics education in the United States has to pay much more attention to this level of the work. Our argument is that the United States will not make improvements for students without worrying more about what instruction requires and about what the dynamics of instruction mean for improving the system.

Early in the class, the teacher asks for a volunteer to explain his or her thinking about the first question. A number of students raise their hands, but she lingers, encouraging more students to consider volunteering, in particular someone who has not spoken in whole group yet on this day. Mahluli volunteers for the first time. She calls on him, and he says the answer is one-half. When asked to explain, he says, "Because they both equal—they both equal—and one, one, half of it is shaded in and the other half is not." The teacher then asks him to go to the board to explain his answer. As he makes his way to the front of the class she engages another student, Doran, in explaining what Mahluli has said:

> Okay. Can you come up to the board and point and show us what you're looking at? Just—there's a diagram right there. Can you come up and show? Did everyone hear what Mahluli said? You should be thinking already about his reason. Who can repeat what Mahluli said? Okay. Well if you're listening carefully, you should always be able to tell what someone just said. Doran, what did he say?

Doran says that Mahluli is just looking at the rectangle and saying it is one-half—that Mahluli is not looking at the whole. As Doran starts to go on to explain what Mahluli has *not* done (to explain a "correct" solution to the problem) the teacher interrupts him, asking that he not go on to explain it yet, and she turns back to Mahluli to have him explain his thinking using a large poster of the figure that is stuck to the blackboard. Mahluli repeats his explanation, quickly pointing to the two triangles and saying that they are equal, so the shaded one is one-half. The teacher then suggests looking back at the working ideas the class generated the day before. On the board is a list labeled "working ideas" about fractions. It contains three points:

- Identify the whole
- Equal parts
- How many parts of the whole

She asks Mahluli what he is calling the whole and has him run his finger around the part he is calling the whole. Mahluli indicates that he is using the upper right rectangle as the whole. In keeping with the working ideas, the teacher has him indicate the equal parts and say how many of the parts are shaded. She then works with another student to reiterate Mahluli's explanation, checking with Mahluli that they are understanding him correctly, and then asks the class, "If Mahluli calls this the whole, is he right that that's one half?" Getting affirmatives, she then says, "Now the question asks you something a little bit different. So who can tell everybody what question we're trying to answer? What Mahluli did is right, but he used something different to be the whole." Referring back to the first question, the teacher asks Avery what she thinks is meant by "the big rectangle." Avery ventures, "The whole rectangle." The teacher responds, "What whole rectangle? You want to come up and show us? Mahluli, are you watching?" Avery uses her finger to trace around the outside of the full figure. The teacher then reiterates the first question, emphasizing the intended whole: "If you use the whole big rectangle to be the whole, how much is shaded blue?" Before inviting students to answer the intended question, she checks in with Mahluli, "Do you see the difference between the question you answered and this question?"

Mahluli: You gotta try to figure out of the whole square.
Teacher: Out of the whole rectangle. And you used what?
Mahluli: And I did half of the rectangle.
Teacher: You did a smaller part of the rectangle. Okay?

The class goes on to discuss why the answer is one-eighth.

This episode suggests some of what is involved in teaching—beforehand, during the class, and possibly later. It shows the importance of teachers' listening, investigating, and drawing-out skills. One thing is that teachers must try to figure out what students are saying and what they do and do not understand at the current point in instruction. A student remark can seem completely incorrect, but a teacher needs to figure out what exactly the student is thinking, and a teacher's precision in reading a student's remark informs the quality of the response. The episode above reveals the fluency a teacher needs in being able to hear the correct thinking in Mahluli's seemingly errant answer and then deciding whether this is the right time to make a point of it and what sort of correction can be made so that at least the public version of things is not incorrect. Many people would just hear Mahluli's answer as wrong and would not even understand what might have led him—and very likely other students—to misinterpret or fail to attend adequately to what whole is intended. The ability to hear students' broken speech about their emerging ideas in real-time instruction demands a degree of speed and fluency that are routinely underestimated in deliberation on teaching. In contrast to professional mathematics, where people may spend years on a single problem, mathematical problems of teaching often require quick judgment and prompt action. Teachers cannot spend years figuring out what a student is thinking, not even minutes. And mathematics educators responsible for teacher education have not thought very much about how to train teachers for mathematical fluency. If a teacher cannot think mathematically on his or her feet, the teacher is quite impaired in his or her ability to teach. Some things are predictable, others are not, and if a teacher cannot quickly say, "I think this is what's going on," then teaching becomes an awkward enterprise. Having to interview students every time something unexpected comes up is cost expensive in a way a teacher often cannot afford.

In sizing up a student's answer, a teacher also has to coordinate hearing a student's thinking with a sense of whether or not the mathematical point is crucial at the moment. In the class just described, students are just developing clarity about the notion of the whole and the meaning of equal areas (as area and not necessarily congruent). In this situation, it is probably important to invest in Mahluli's thinking, but in another situation it might not be wise to invest in Mahluli's use of the word "square" in place of "rectangle" or his statement about considering "half" of the big rectangle when it was one-fourth. In another situation, in response to his incorrect answer of one-half, it might be better to quickly say, "no, not one half," and move on. It depends. In the episode above, it is important to take up his thinking because it is on point for the lesson, represents a misunderstanding likely shared by other students, and provides an opportunity to develop explicit language about the key idea of the whole. This does not mean that taking up a wrong answer like this is always the right thing to do in teaching. Instead, the point here is to draw attention to the sort of diagnostic nature of the work—that Mahluli's answer represents a crucial mathematic hinge moment for opening up the topic and achieving the goals of the lesson.

Another thing to notice in this episode is that the teacher invited other students to think actively about Mahluli's reasons for saying that the blue triangle is one-half. Teachers need to maintain interest and engagement. They need to productively fill the few moments taken while Mahluli walks to the front of the class. Additionally, teachers are responsible for teaching all of the students in the class, even when probing the thinking of a single student. By setting an expectation that, after a student has spoken in a whole-class discussion, all of the other students should be able to repeat what was said, a teacher directs attention, sidesteps misbehavior, maintains mental engagement, and teaches students skills for productive participation in public discussions about mathematical ideas and issues.

The selection and design of tasks themselves, whether from a textbook or self-made, is another important domain of the work of teaching, as is setting up or framing the task. The task in the example above has students think not only about equal areas but also specifically about distinct shapes whose areas are not readily comparable. The shapes in the figure are deliberately different, which then means that there is some work to do to establish that each shaded region is one-eighth, or that the two of them together are one-fourth of the overall shape. The need to rely on a deduced equality of their areas maintains an intellectual honesty about an area model for fractions. This is a key mathematical feature of the task. Figuring out how to put the task in play so that the students get to the mathematical point is an important piece of mathematics teaching. A teacher can have a problem, can get all kinds of interesting things to come up, and can raise an interesting side trip during the lesson, but more importantly, the teacher must keep an eye on the goals for the lesson and must coordinate decisions with whatever students are doing to get to that goal. If a teacher does not accomplish this, and do so routinely, then the teacher is well off the mark of skillful teaching, whether in a college course or a first-grade classroom.

The pedagogical issues that can be mined from almost any 4-min episode of skillful teaching are nearly boundless. Teachers manage interpersonal dynamics, even in lectures, where a skillful choice of jokes, seriousness, and emphasis can greatly enhance student learning. They need to recognize and construct what is going on mathematically, where their use of silence can matter as much as their talk. There is much that could be discussed about the 4-min episode described above. Here is one possible list, only some of which have been mentioned:

- Selecting/designing tasks
- Teaching students what counts as "mathematics" and mathematical practice
- Making error a fruitful site for mathematical work
- Deciding what to clarify, what to make more precise, and what to leave in students' own language
- Attending to the ambiguity of "big rectangle"
- Listening to and interpreting students' responses
- Identifying and working toward the mathematical goal of the lesson

The point in describing skillful teaching in some detail is that, if mathematics teaching and learning are to be improved significantly, the mathematics education community must address the fact that the current system does not prepare people to teach at a reasonable level of skill. Our argument is that for any change to matter, be it curriculum, school finance, or high standards, it must necessarily change interactions in classrooms among teachers and students around content. Short of that, nothing changes. And teachers have primary responsibility for managing instructional interactions. Thus, skillful teaching and a system that adequately prepares a large number of people to teach skillfully are requirements of a system that adequately educates children.

Learning from the Past

Consider now the question of what can be done so that the story will be different in the near future. It is important to note that in the United States a great deal is expected of schools and calls for improvement are business as usual. The United States, as well as the U.S. mathematics education community, keeps gravitating to the same reform strategies, perhaps because those strategies represent a commonsense point of view (e.g., fix the curriculum; hold students, teachers, and schools accountable; overhaul the administration). Improving education is a big order and many past decrees have languished. Thus, it is important to ask what can be learned from past attempts so that the pattern may be changed.

Here are some of the most widely touted strategies for improving mathematics education in the United States:

- Teacher-proof instruction
- Install a more challenging curriculum
- Increase accountability
- Reorganize schools
- Pay teachers more
- Recruit talented teachers by lowering the barriers for entry

Notice that the word "mathematics" does not appear in any of these strategies, yet they dominate both conversations and policies aimed at improvement. They have been used for other subjects as well, but mathematics is arguably the subject that has received the most attention. That makes it the saddest story because more work has been done on mathematics, done on the part of the U.S. society and the professional community, than has been done with science, social studies, or other school subjects, yet the payoff for that investment has been small. The point in listing these six strategies is not to say that none of them is worthwhile; each has merit, but taken alone, none has accomplished much, or is likely to.

We argue that one reason for this is simple, and goes back to what we said about instruction. None of these strategies guarantees that instruction will be different because none gets directly at instruction, changing what happens between teachers and students in schools. For example, a teacher-proof instructional program could be implemented, but only those naïve about teaching could think there is a way to control completely what a teacher says to students. It is probably a good idea to give teachers more guidance, but the notion of teacher-proof instruction underestimates the complexity of the work. Likewise, the introduction of a challenging curriculum or increased accountability or reward, without attention to teacher capacity, is unlikely to change instructional dynamics. Schools are regularly reorganized without changing what happens in classrooms, and talented recruits would only be more effective if their presence systematically altered patterns of interaction—the nature and process of such a change are unclear. Taken individually and apart from a plan for impacting teaching and learning, it is not clear why any of the most common approaches to sweeping reform would change basic classroom interactions.

We can also look at past attempts to improve education, see what has impeded progress, and consider which factors are the ones we can effect. One impediment to progress is endless arguments about which matters more, skills or concepts, when they both matter. This is an unproductive argument. Nobody who knows much about mathematics or mathematics teaching believes that only one matters. Such debates in the United States need to stop and the focus needs to be turned to learning to teach both skills and concepts better and to subtler issues about their order within a pedagogical approach.

Another, perhaps more controversial, lesson from the past is that the lack of a central or a common curriculum is a major impediment. It is popular to talk about what goes on in Japan, China, Hong Kong, Taiwan, or Singapore, but one of the main differences is that each of these countries has a coherent, uniform curriculum. When children move within the United States, they go from one curriculum to another, often with quite different goals. Think what it is like to be a third grader whose parents move frequently. You are already having trouble with math. You show up at a new school district—it is a different book, with a different vocabulary, and a different set of representations. The situation is absurd. Fractions are no different in Idaho than they are in Utah, yet they are treated as though they were and students often experience them as though they were. Every state and every one of the 15,000 school districts in the United States does not need a different curriculum. Currently, each school district, to a large extent, makes its own decisions about what gets taught. Efforts such as the Common Core State Standards Initiative are meant to address this problem, but the United States, with its history of political commitment to local control of education, is still a long way from establishing a common curriculum coherently used throughout the country. It is precisely for a public debate like the one shaping up around the Common Core State Standards that the mathematics education community needs to develop tools to frame and clarify the nature of the problem.

In addition to the focus on local control of education in the United States, there is also a tendency toward frontier individualism with regard to teacher autonomy. Even within a single school, teachers hold different convictions about what is important to teach, what formulation of a concept is best to use, and how best to sequence instruction for a given topic. Even if an individual teacher makes wise choices, as students pass from grade to grade, the curricular confusion can be profound. This is not a formula for a coherent system for teaching mathematics.

Another impediment to progress is the inclination to persist with outdated and refuted ideas about "teacher quality," especially with respect to content knowledge. (See the National Math Panel Report (2008) for an appraisal of the issue of teacher content knowledge.) The focus tends to be on *teacher* quality, particularly when it comes to teachers' inadequate content knowledge. However, the issue is not teacher quality, but *teaching* quality (Gitomer, 2009). If teachers could be selected in ways that were predictive of how well those teachers would teach, then teacher quality could be taken as indicative of the quality of teaching, but because there are currently no effective ways of identifying the characteristics of teachers that will predict whether their teaching will be good, the focus should be on assessing the quality of teaching. Maybe someday, when more is known about which teacher characteristics account for shaping interactional dynamics and for variance in student learning, then measures of teacher characteristics could be useful, but in the end it is not the characteristics of a group of teachers that matter. Instead, it is what children do with mathematics that is of high quality, and that is more immediately related to teaching than it is to characteristics of teachers. The field needs more studies that focus on instructional dynamics and needs to develop tools for evaluating it.

Similarly, the United States continues to engage in pendulum swings from teacher-proofing schooling to presuming that, if obstacles are removed and professional community provided, teachers will grow improvement on their own. The focus should be on building an understanding of, and a capacity for, skillful teaching. What needs to be changed in the United States is the way teaching is done.

A final lesson to draw from that past has to do with the education of teachers in the United States. Teacher education, both preservice and in-service, persistently emphasizes things other than practice (e.g., reflection, beliefs, propositional knowledge, experience) (Hiebert, Morris, Berk, & Jansen, 2007). Poor teaching is inevitable in a system that teaches knowledge remote from the actual doing of teaching, strives to change what teachers believe, and spends time on having people reflect. Again, these things do matter. Teachers need to learn to be analytic about their teaching, and they need to learn to talk clearly about teaching. However, teaching is primarily a practice, something done. Thus, a professional training system that does not hold itself accountable for whether people can do the work is a bankrupt system. That is the current situation in the United States.

What to Do About the Problem of Mathematics Education in the United States

So, what do these lessons from the past imply for making real progress—for making it possible to design a strong instructional system? With an aging workforce, the next 3–4 years will see the largest incoming group of new teachers that this country has seen in a long time, even with the downturn in the economy. The notion of greater accountability for beginning practice is more acute than it has ever been. In a sense, this is an opportunity—if the country can figure out how to capitalize on it. From our understanding of what it would take to support the improvement of teaching, we argue that there are five strategic elements for designing a strong instructional system:

- Build a common mathematics curriculum
- Develop valid and reliable assessments coordinated to the curriculum
- Build a system of supplying skilled teachers to every school to teach that curriculum
- Center teacher licensure and training on practice
- Organize schools to support beginning teachers

In recognizing these five critical elements for designing a strong instructional system, it is important to note that instruction is foundational to the endeavor and that the country needs strategies that address this foundation. The dynamics of instruction are the educational core. They are what affect the children engaged in learning. Consequently, an effective strategy has to build a system that changes that set of transactions. If the strategy does not affect that core, then it is playing at the edges of the problem. Tinkering with the curriculum only improves learning if the tinkering increases the chances of lessons getting taught well in classrooms by teachers. It is good for programs to foster student confidence, as confidence may increase their motivation. However, by themselves such programs will not necessarily change the transactions in schools. What is needed are strategies that ensure that programs actually change the transactions of teaching and learning.

The five elements above offer promising tools for building a system capable of creating change. Establishing a common mathematics curriculum is at the heart because it provides coherence that enables everything else. There is, for example, evidence that, when professional education for teachers is situated in the curriculum that they have to teach, teachers are demonstrably more skillful than when professional development is based on guesses or generic versions of what teachers will teach (Cohen & Hill, 2001). Adopting a common curriculum would not only provide children with a more coherent experience as they move geographically between schools and vertically through the grades, but it would also provide critical infrastructure for high-quality teacher training. Currently, when prospective teachers are taught either about content or ways of teaching that content, instruction must be designed without knowledge of the books they will be using when they get a job, which severely handicaps what can be done. A system built with knowledge about what teachers will be teaching would be able to get much closer to assuring that teachers know mathematics and pedagogical practice well enough to deliver mathematics to their students. The fact that this information is missing leads to programs for teacher education that must constantly stretch, guess, and talk in generalities. As long as agreement on a common curriculum is presumed to be unobtainable, teacher education is likely to continue to be a story of eclectic improvisation. It may be that the United States is not ready for a common curriculum.[2]

In addition to a common curriculum, in a coherent education system common assessments are needed to track students' progress and these assessments need to be coordinated to the common curriculum. It is a direct result of not having a common curriculum that assessments in the United States are not coordinated to a curriculum but are, of necessity, curriculum-less (Cohen, 2011). Historically, the testing systems used in this country do not test the curriculum taught (because no shared curriculum exists across the contexts that assessment must serve). The prime example of this phenomenon is college admission testing established by the College Board, historically called the "Scholastic Aptitude Test" and specifically designed to create

[2]Other countries that have a common curriculum, and build around it, do not decide on the curriculum by political means, such as is the case in the United States with its politically mandated state curriculum standards. In other countries, governing bodies have professional authority and oversight for determining the common curriculum. Of course, achieving this necessary first step in the United States would require a great deal of discussion.

an evaluation tool that would work across the uninterpretable information about high school course-taking and grades. The choice to assess aptitude, however ambiguous the notion, was a result of not being able to assess actual achievement in a fair way—in a way that adequately accounted for variation in local curricula. However, this is only one extreme example. The Iowa Test of Basic Skills, the Stanford Achievement Test, TerraNova, and many others, because they are designed to be administered across schools, districts, and states in the U.S. context, make choices about what content to assess, in what amounts, and at what level. These decisions require some smoothing of the curricular terrain. As these tests are used for high stakes accountability, it is no wonder that, because they do not match the curriculum being taught, teachers feel pressed to teach to the test—in other words, to take the test as the curriculum. However, these tests are not curricula and are a poor replacement. Every major assessment of student achievement in the United States reveals this dilemma in one way or another, and this feature contributes to the overall weakness of the current system.

Even if there were some degree of agreement in some group about the content that students should know and even if that group developed assessments aligned to that content, those assessments would still not be coordinated with what is actually being taught in the current nonsystem of U.S. education. This is yet another example of the effects of a very decoupled, nonsystemic approach to teaching in the United States. Current assessment technologies, however, permit some new designs for how assessments can be delivered and for the reliability and validity of those assessments. For example, a progress-variable approach that views learning as progress toward higher levels of competence, instead of acquisition of more knowledge and skill, can be conceptualized, assessed, and informatively displayed (Wilson & Scalise, 2006). With recent psychometric advances, the possibility exists for building better assessments than ever before. If the country is to make progress on improving mathematics education, then the all-too-common aversion to assessment among professional educators and mathematics educators is untenable. Testing (in some form) is critical to education. The fact that current assessments are not ideal does not mean that good ones cannot be designed, ones that attend to cultural and linguistic equity, that assess what is taught and what matters, and that readily inform teaching and learning.

The third strategic area to target has to do with the teacher supply problem. Many people are fond of thinking of ways to get rid of teachers who do not teach well, but the United States does not have a system for supplying people to replace the teachers who would be fired. Much of the accountability rhetoric and the push for value-added measures are motivated by an agenda of finding the teachers whose students are not making gains and getting rid of them. Instead, the country needs to figure out how to supply schools with skilled teachers.

This goal will require several things. Ideally, the training system would need to be coordinated to the curriculum that teachers would teach. Physicians are not certified, for example, without being prepared to use the tools central to their practice. No medical school says, "You are going to be a surgeon and we cannot tell you what tools your hospital will have, but if they have this tool you do this, and maybe you will make your own, because we want you to be resourceful and creative." A common assignment in teacher education is to have prospective teachers develop their own lesson or unit. This activity is analogous to having surgeons make scalpels. Curriculum is the central tool of teaching. Teaching itself, in large part a matter of using a curriculum, is a skillful work. Attention needs to be given to training people in the demanding work of using a curriculum, not spent on a wilderness camp experience of making all of one's own tools from scratch.

Mathematics teacher educators need to agree that effective teacher preparation is about preparing teachers to use a relatively well-specified curriculum. Then, as professionals with standing in the larger society, they need to insist that a common curriculum is not optional and explain to others that it is essential to professional preparation for skillful teaching. Learning to teach with such a curriculum would include knowing when it is working and knowing when it needs to be supplemented. Blind use of a curriculum is not good, but teachers need to be prepared for professional and serious use of what is in fact the central tool of teaching and learning. For teacher education programs to effectively teach future teachers to recognize, for example, how objectives are addressed in specific lessons and how topics are developed across a textbook, it would require having a curriculum to teach people to use in the first place, one that would be used wherever a teacher is hired.

Related to this need for building a system for supplying skilled teachers to every school—teachers who are prepared to teach the school's curriculum—mathematics teacher educators need a concept of "safe to practice." It is irresponsible to put untrained people into classrooms to teach. Students are real—not the manikins used by nurses and doctors for learning practice. Other professionals responsible for people do not just watch for some time and then jump in to see how it goes, but this is common for teachers. With regard to teacher preparation, the expression "sink or swim" is more often used for what it means to learn professional practice than are analogies from professional domains more closely resembling teaching. Novices should not be put in classrooms to teach without supervision. No other occupation functions in such a way. There are performance standards for cutting hair and for plumbing a building, and there ought to be at least comparable levels of standards for teaching children.

In addition, we argue that licensure needs to be associated with performance. Current forms of teacher training are not stellar, but if agreement were established on some core practices that teachers should be able to perform, then a common licensure system could be built, which would position teacher educators to build a better system to train teachers. Mathematics teacher educators need to establish some minimal threshold—agreeing that no one should be in the classroom unless they know X and can do Y. Such a step would help to build multiple pathways into teaching. The country needs a diverse teaching workforce, and a large one. Having different ways for people to enter teaching would be good, but multiple pathways are not helpful when there is no agreement about what qualifies someone to teach children. Less worry needs to be given to which pathways, or how many pathways, and more worry needs to be given to establishing a standard that says whether someone is safe to practice or not. Then, a standard for continued professional education could be set so that once people meet the initial licensure standard, a second license would be required in order to remain or to specialize.

At present, people are leaving teaching in droves. Two points are worth making about this problem. One is that the relationship between teacher age and their leaving follows a U-shaped curve (Ingersoll, 2001). In any occupation for which people enter in their twenties, a high proportion of them leave by the time they are thirty. People in their twenties are often still seeking their niche, and some change course. That should be expected and it would be good to accept this reality and to design for it. A path could be planned for people who are willing to teach for a few years in ways that ensured they did a good job for their time teaching in the classroom before moving on to other roles and occupations. Indeed, it would be good to have more adults in U.S. society who had taught for a few years, who understand something about schools from the perspective of having actually taught. This might help to counteract the apprenticeship of observation that Lortie (1975) argued profoundly distorts people's view of classroom teaching, creating a society ill prepared to engage in informed public discourse about teaching and its improvement. In addition, these journeymen and -women could then carry knowledge about children and teaching into other roles, for instance as parents, managers, or voters. Likewise, it would be good to have business leaders who are oriented toward the teaching and learning of people in their corporation, and connected to and invested in schools. Teaching and learning have applicability broader than the classroom and having more people in our society who are familiar with classroom teaching and learning would be helpful.

At present, people often leave teaching because they do a bad job (Ingersoll, 2001; Smith & Ingersoll, 2004). On the one hand, we might want such teachers to leave, but on the other hand, it is frustrating to have so many leave because they were not adequately prepared and feel badly about the role they have played. Leaving because while they like the profession, they haven't been prepared to be successful, is bound to leave a bad taste for schools and for education, further eroding public sentiment toward mathematics education.

At present, there is no evidence that teacher training or certification actually produces people who are more skilled in teaching than anyone else (National Mathematics Panel, 2008). This is not a popular view in the education community, but admitting it is the first step toward building training systems and assessments that are effective. Otherwise, the message is, "anyone can teach," even though that is not what anyone in the field thinks. It is not the case that anyone who knows mathematics well is prepared to teach. Teaching mathematics is a mathematically specialized endeavor. The current system does not equip people with specialized training. The fact that training mechanisms do not work now does not mean that good training and assessments cannot be developed. They just do not exist now. It would be better to acknowledge that the current system is inadequate and to start building training and assessment that is adequate. This would be a better investment of time and energy than defending the status quo.

A fourth strategic idea for designing a strong instructional system would be to focus mathematics teacher licensure and teacher training on practice. When we say practice here, we mean to include content as it relates to practice. Included here is the work of our own research group at the University of Michigan on mathematical knowledge for teaching, that is, a practical use of mathematics in teaching (Ball, Hill, & Bass, 2005). This has been written about extensively elsewhere, but the point here is to be clear that a focus on practice includes attention to practice-based content knowledge. There is evidence that such professional knowledge and skill are positively related to student achievement (Hill, Rowan, & Ball, 2005; Rockoff, Jacob, Kane, & Staiger, 2008) and that it can be effectively taught (Hill & Ball, 2004). What we still need, however, is research that further builds this knowledge base and builds systems to scaffold people's capacity to know this mathematics and to know how to use it to teach students.

A more direct focus on practice, and a much-needed focus, concerns finding ways to distill teaching practice into a strategic and manageable set of high-leverage practices and designing ways of integrating these practices into teacher development. Several scholars have begun exploring issues of closely modeling, training, and coaching teaching practice (Ball, Sleep, Boerst, & Bass, 2009; Grossman et al., 2009; Grossman & McDonald, 2008; Lampert, 2010; Lampert & Graziani, 2009). There is increasing evidence that teaching practice can be taught and scaffolded and that doing so addresses the unpredictability of learning from experience and the problem of building capacity at scale, especially as it reshapes the culture around teaching in schools.

The final key target area for designing a healthy instructional system, one discussed earlier, is the need for a system to support the early years of teaching so that people go from an initial, basic, safe-to-practice stage to full membership in a professionally skillful staff. That would require different licensure levels for the same school (and differentiated staffing) and that would require that teachers be able to continue to develop their skill as they become more accomplished. For example, it might be good if first-year teachers were able to identify some of the most frequent difficulties children have when learning a specific topic, and then, as beginning teachers become more accomplished, they could learn about some of the more subtle difficulties that students have. They do not need to grasp all of the nuances in the first year. It is important to recognize that beginning teachers who have two or three things that they know about frequent student difficulties for each topic are in a noticeably better place to teach than those who do not. A teacher who does not know the most frequent things that come up and has to puzzle at every turn should probably not be teaching. Qualified beginning teachers need to know the most common and central student difficulties and, as they become more advanced, they need to be exposed to more difficult and complex issues that students face as they learn specific content. In addition, they should be supported in developing the more complex practices of teaching. In short, teacher learning of practice needs to be more staged across time than it currently is.

Early career support is assumed in most other occupations in our country, from unskilled or blue-collar occupations to professional fields, such as architecture, nursing, or social work. It is quite astounding that early-career support is not built into teacher induction and development. Instead, the work of beginners is undifferentiated from the work of more experienced experts. One might ask why. As we mentioned before, teaching is by far the largest occupation in this country. There are roughly 3.75 million teachers, and the number is rising (Keigher, 2010). This is not surprising because children are the largest demographic group. Hence, the scale of the problem is significant. The improvement of mathematics education will require preparing a large number of ordinary people to do important and skilled work. That is why our point about systemic-ness is important. Little progress will result from recruiting a few talented people. It is great to have some retired engineers, and others, turning to teaching as a second career. It is great that Math for America recruits mathematically well-trained people and that Teach for America recruits able, elite, college students. Adding these people to the teaching workforce is a boon, but these additions do not come close to obtaining the millions of quality teachers that are needed. The United States must add to the teaching workforce without putting teachers in the classroom who do not yet know what they are doing and do not yet know the mathematics in ways that would allow them to teach it effectively. Mathematics teacher educators in the United States are responsible for supplying the training and licensure components of the system and they have a vital role to play in advancing national agendas that address key features of a revamped instructional system.

Moving Beyond Myths

Returning to the pretest at the beginning of this chapter (see Fig. 2.1), the issues raised are interpretative and are meant to be engaging and provocative. It may be surprising to know, for example, that the teacher education curriculum that is taught in most institutions of higher education is the same as it was in 1940. That is telling when one thinks about how much the situations that teachers have to deal with in schools have changed. The third statement is particularly important, and worth emphasizing. As people engage in the profession of education, it is important that mathematics teacher educators be spokespeople for the fact that there is extensive evidence that individual teachers have an enormous impact on student learning and that one good teacher can make a big difference in children's gains in a school year. Studies consistently indicate that, adjusting for student characteristics, roughly one-tenth of the variance in student achievement gains is associated with teachers, that cumulative effects are even larger, and that effective teaching substantially lessens differences in achievement predictable by student characteristics, in particular differences predictable by race/ethnicity and social class (Goldhaber, 1999; Nye, Konstantopoulos, & Hedges, 2004; Rivkin, Hanushek, & Kain, 2005; Rockoff, 2004; Sanders & Rivers, 1996). These effects are as significant within a single school building as they are between schools. For instance, two teachers can work side by side with the same population of students in the same context, with one teacher producing, year after year, more than a year's worth of growth in students while the next-door teacher does not. The problem, of course, is that no one has yet completely figured out what differentiates the skill of those two teachers. Education has an effect, even at the level of the individual teacher. And this is the positive note, one about the profound importance of teaching, which we have argued is central to making progress on the problem of mathematics education in the United States.

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Chapter 3
The Constantly Underestimated Challenge of Improving Mathematics Instruction

James Hiebert

Abstract Many U.S. efforts to improve mathematics learning opportunities for students have not made it through the classroom door. Teaching is the single common pathway along which improvements reach students. But mathematics teaching has not changed much over the past century. Why is teaching so hard to change? The reasons seem to lie in an incomplete understanding of why mathematics teaching looks like it does, what aspects of teaching should change, and how teachers learn to teach mathematics in different ways. Examining these issues reveals a number of challenging research agendas.

Regardless of how difficult you think it is to improve classroom mathematics teaching on a wide scale, it is more difficult than that. The evidence suggests that changing the core of teaching—the way in which a teacher and students interact about the content—is so difficult that U.S. classrooms have changed little over the past century (Cuban, 1993; Fey, 1979; Hoetker & Ahlbrand, 1969). Your grandfather would likely recognize the math class your children attend. Of course, there have been changes at the margins, such as more colorful textbooks, and even the types of mathematics problems presented. But the basic nature of teaching—presenting definitions and rules, demonstrating solution procedures on sample problems, and then asking students to practice the procedures on similar problems—has remained remarkably consistent over the years.

The persistence of the way mathematics is taught in the face of numerous efforts to change it poses a serious and urgent problem for mathematics educators. If teaching didn't matter much, changing it still would be a challenge but it wouldn't be a

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serious problem. But, of all the school factors that affect students' academic learning, teaching matters the most (Hiebert & Grouws, 2007; Nye, Konstantopoulos, & Hedges, 2004; Pianta & Hamre, 2009; Raudenbush, 2008; Sanders & Rivers, 1996). Teaching is the common pathway along which all efforts to improve schooling reach the students. Better teacher preparation, more effective professional development, clearer standards, new curricula, school-wide improvement programs, and mandated accountability policies, all reach students through the way teachers interact with students about the content. If the teaching doesn't change, students will hardly notice these other differences. Even new curricula and new textbooks can be so transformed by the teaching that the learning opportunities for students remain unchanged (Stigler & Hiebert, 2009).

If students' mathematics learning in the United States was acceptable, then improving teaching would not be an urgent problem. But the major national (National Center for Education Statistics, 2009) and international (Gonzales et al., 2008; Lemke et al., 2004) assessments of students' mathematics learning suggest that U.S. students are not learning nearly as much as they could. Many factors account for the nature and level of students' learning. These include factors outside the control of educators such as home support, neighborhood and peer pressures, and state resources provided to schools. But they also include factors directly under the control of classroom teachers. And teaching is the prime example. Because teaching sets the learning opportunities for students, one might expect the U.S. students' limited learning to be associated with limited classroom opportunities to learn, that is, limited teaching. That is exactly what the video study associated with TIMSS revealed. The U.S. eighth-grade mathematics teaching was characterized by "frequent reviews of unchallenging, procedurally oriented mathematics during lessons that were unnecessarily fragmented" (Hiebert et al., 2005, p. 116). So, although teaching cannot completely explain students' learning, it surely contributes to the relatively average levels of learning exhibited by the U.S. students.

That teaching is under teachers' control places it in the spotlight as a feature of the educational system that could be improved. As argued above, changing teaching is not easy. But this doesn't mean changing teaching is impossible. And changing teaching is educators' best hope for improving students' learning. Consequently, I believe improving classroom teaching is one of the most serious and urgent problems facing mathematics educators. This chapter is devoted to understanding the problem of improving teaching more fully in order to place the profession in a better position to create and adopt improvement strategies that will work better than those often tried in the past.

Why Teaching Is So Hard to Change: Three Underappreciated Explanations

A first step in creating effective strategies for improving teaching is understanding more fully why teaching has resisted past efforts to change. What makes teaching so immune to reforms? I'll suggest three reasons. None of these are new; they have been the subject of previous research and policy efforts. But I don't think educators have appreciated their significance and, in some cases, I think the research evidence is insufficient to shape and guide actions that could be effective.

Explanation One: The Profession Does Not Agree on Well-Defined Learning Goals for Students

The United States has experienced a continuing debate on the most important mathematics learning goals for K-12 students (Resnick & Ford, 1981). The debate is often characterized, in simple terms, as a tug-of-war between procedural skill and conceptual understanding. Although it has become generally accepted in the research community that both skill and understanding are essential and can feed off each other as they develop (Baroody, Feil, & Johnson, 2007; Hiebert, 1986; National Research Council, 2001; Star, 2005, 2007), there remain disagreements among the public and these disagreements influence local decisions about learning goal priorities (e.g., Klein, 2007; see also the Mathematically Correct website http://www.mathematicallycorrect.com/).

In addition to public debates about priorities, the learning goal confusion is amplified by the strategies of textbook publishers to include the union of material requested by multiple users. This strategy results in oversized textbooks filled with much more material that can be taught in a single year. Teachers are left to pick and choose from the smorgasbord of material offered in the curriculum. This approach can lead to a surface-level exposure to many topics but the deeper development of none (Schmidt, McKnight, & Raizen, 1997).

The reason that a lack of agreement on well-defined learning goals for students prevents improvements in teaching is that teaching is not, in general, effective or ineffective. Teaching is effective (or not) for helping students achieve particular learning goals. It doesn't make sense to ask whether a particular approach to mathematics teaching is effective until you know what learning goal it is trying to help students achieve. Some approaches are more effective for some learning goals than others. Research recommends, for example, a different set of teaching features if the learning goal is rapid, error-free execution of a procedural skill than if the learning goal is deep understanding of a concept (Hiebert & Grouws, 2007).

That different methods of teaching can be differentially effective for different learning goals means that improvements in teaching are linked to particular learning goals. Teaching improves with respect to particular learning goals. Without stable and well-defined learning goals, efforts to improve teaching keep shooting at different targets, and the targets keep changing. What is learned about more effective teaching is left behind when the goals change. And, with different teachers teaching toward different goals, there is no way to share and accumulate what they are learning.

Rarely is the absence of agreement on learning goals viewed as an obstacle to improving teaching. But that is likely its most significant consequence. There is little reason to expect steady and cumulating improvements in teaching without wider agreement on specific learning goals for students.

Toward Agreement on Learning Goals for Students

The absence of a consensus, well-defined set of learning goals for students in the United States is not due to lack of trying. Recent efforts can be traced to the work of the National Council of Teachers of Mathematics [NCTM] in the 1980s. With the release of the *Curriculum and Evaluation Standards for School Mathematics* in 1989, NCTM brought new attention to the value of professional agreement on learning goals for students. Following hotly contested discussions of the priorities indicated in the 1989 document, NCTM released an updated version of the document titled *Principles and Standards for School Mathematics* in 2000 and a follow-up *Curriculum Focal Points* in 2006. Adoption of a country-wide shared set of learning goals took a major step with the release of the *Common Core State Standards* (Common Core State Standards Initiative, 2010). It is too early to tell what affect these core standards will have on the extent to which teachers actually share learning goals across classrooms, schools, and districts, and it is too early to tell whether wider acceptance of shared learning goals will be used to build cumulating knowledge of how to help students achieve these standards, but the release of the *Common Core State Standards* indicates that shared learning goals, essential for improving teaching, might eventually be realized.

It is useful to note that amidst the political debate that has surrounded the development of nationally acceptable standards, a 2001 document, *Adding It Up*, produced by a committee of the National Research Council, attempted to bridge the ideological divides by creating a general statement of student learning goals acceptable to all stakeholders. Called "mathematical proficiency," the learning goals consisted of five interrelated strands: adaptive reasoning, strategic competence, conceptual understanding, productive disposition, and procedural fluency. Although this statement gained widespread acceptance and was endorsed in 2008 by the National Mathematics Advisory Panel, the goals were stated at such a general level that they do not easily guide efforts to study and improve teaching. Each of the five strands can be interpreted in many different ways by different teachers as different mathematical topics are taught through daily lessons.

Due also to the recency of the construct, there is no research of which I am aware that has attempted to study what kind of teaching is effective for helping students achieve mathematical proficiency. In a review of research, Hiebert and Grouws (2007) conjectured that two features of teaching might eventually be implicated in teaching for mathematical proficiency: allowing students a chance to struggle or grapple with important mathematical ideas, and making the key mathematical relationships related to those ideas explicit in some way during the lesson. These two descriptions might guide further research but, like the goal of mathematical proficiency itself, they are at too general a level to lead directly to improvements in teaching. There are too many ways to interpret and implement these general features. It is difficult to accumulate evidence on effectiveness because different teachers likely implement the features in very different ways. Whether the more specific *Common Core State Standards* sets the stage for the kind of work on improving teaching described in the next section has yet to be determined.

Improving Teaching When Specific Learning Goals Are Shared

When teachers agree on specific, lesson-level learning goals, they can study the effectiveness of their teaching with respect to these goals and share what they are learning. The knowledge often comes in the form of small bits of information about particular aspects of the lesson that could be improved. Although each improvement might be small, they can accumulate over time to yield steady and lasting change.

One example from teaching preservice teachers provides an image of how the process can work. Berk and Hiebert (2009) described a multiyear effort to improve a lesson for K-8 preservice teachers on subtracting fractions. The lesson was part of a mathematics course for preservice teachers that is taught every semester in multiple sections with multiple instructors. The critical fact for this example is that all the instructors agreed on a learning goal for this lesson: Preservice teachers should learn to write a realistic story problem for a subtraction number sentence such as 5/8 – 1/4 = ? Based on prior experience, the instructors who created the first lesson anticipated that many preservice teachers would write a version of the following incorrect story: “Kathy has 5/8 lb of coffee. She uses 1/4 of it. How much does she have left?” They decided to expose this error early in the lesson and discuss as a class the difference between this story and an appropriate one (e.g., “Kathy has 5/8 lb of coffee. She uses 1/4 lb. How much does she have left?”).

The instructors met after the lesson and shared information regarding its effectiveness by reviewing preservice teachers’ performance on similar number sentences near the end of the lesson. Although most preservice teachers corrected their earlier errors, the instructors were not satisfied. Too many preservice teachers still did not appreciate the importance of being clear about the referent for the number 1/4. So, they created a second version of the lesson, to be taught by another set of instructors the following semester, which began the same as the first version but then introduced the reverse activity. Preservice teachers were asked to write number sentences for the following stories: (1) Kathy has 1/2 lb of chocolate. She eats 1/2 lb. How much does she have left? (2) Kathy has 1/2 lb of chocolate. She eats 1/2 of it. How much does she have left? The easy number 1/2 was chosen so that preservice teachers could focus their attention on the referent for the number. Notes were included in the lesson to explain its history and the rationale for the current version.

After the second version was taught, the instructors met to share information on its effectiveness. They still were not satisfied. A third version of the lesson was created to be taught by the instructors the following semester. The third version kept the two activities and added a new homework assignment. The assignment asked preservice teachers to draw diagrams and write story problems for the following two number sentences: (1) 8/9 – 1/4 = ? (2) 8/9 – (1/4 of 8/9) = ? The second number sentence matches the incorrect story that some preservice teachers continued to write.

The instructors of the third version judged this lesson to be quite effective and added notes to this effect. Although the changes to the lesson were small, and they were directed toward a single learning goal, the teaching toward this goal improved in a steady and lasting way. It is possible to imagine small improvements to other lessons accumulating in much the same way until significant improvements in teaching the entire course are clearly visible. These improvements are enabled by an agreement by the instructors on the learning goals. This allows multiple instructors, over multiple semesters, to contribute to *building* improvements, one on top of the other.

Explanation Two: Confusing Teachers with Teaching

Educators and policy makers often think that they are addressing teaching when they are actually focusing on teachers. What does it mean to focus on teachers rather than teaching, and why is this a problem? What it means is illustrated in a striking way by *Education Week*'*s* 2008 annual report on the quality of the U.S. schools ("Quality Counts," 2008). The report is titled "Tapping Into Teaching" and one of the lead articles is titled "Taking Teaching Quality Seriously." Based on the titles, one would expect a discussion of the nature of teaching in the U.S. classrooms, the methods used to teach, the classroom practices that usually are labeled "instruction," the ways in which teachers and students interact about the content, and so on. But there are no analyses of classroom teaching in the entire report. Instead, the report is about the qualifications, characteristics, recruitment, and retention of teachers. It is as if knowing the characteristics of teachers tells you about the methods of teaching they will use. But that just isn't true.

Education Week is not the only source that confuses teaching with teachers. Most press reports and policy statements that critique education and call for better teaching point to characteristics of teachers, not methods of instruction. Malcolm Gladwell's (2008) popular essay in the *New Yorker* is a prime example. Influential policy documents often make the same mistake. In the first two sentences of the 2004 annual report of the U.S. Secretary of Education, "teacher" and "teaching" are equated and then used interchangeably with improved teaching, apparently assumed to result from changing teacher characteristics (Paige, 2004). The policy document *Before It*'*s Too Late* produced by the highly visible commission chaired by U.S. Senator John Glenn includes four key points in the foreword about improving mathematics and science teaching in the twenty-first century (National Commission on Mathematics and Science Teaching for the 21st Century, 2000). The third of the four points actually focuses on teaching, and suggests that improving teaching is a key to improving students' learning. But, consistent with Secretary Paige's report, the point is developed by recommending upgrading the quality of teachers as measured by professional and personal characteristics, not by recommending upgrading the quality of teaching methods.

The problem caused by confusing teachers with teaching is that improving teaching by increasing the qualifications of teachers will not work. Research shows little correlation between the qualifications of teachers, as measured by usual certification markers, and students' learning (Buddin & Zamarro, 2009; Goe, 2007; Solmon, Bigler, Hanushek, Shulman, & Walberg, 2004). If better "qualified" teachers taught in more effective ways, one would expect their students to learn more. But this connection isn't there. Furthermore, some evidence shows that even teachers with the highest qualifications do not routinely display high-quality teaching methods (Silver, Mesa, Morris, Star, & Benken, 2009). Teach*ing* quality, even crudely measured, is more highly related to student achievement than are traditional teach*er* qualifications (Rivkin, Hanushek, & Kain, 2005). Branson and Grow (1987) summarized the situation by noting that improving teaching by improving the qualifications of teachers has been tried for at least 200 years and its effects reached asymptotic levels years ago. "Trying to increase the quality of the educational system by somehow improving the quality of teachers without, at the same time, implementing an improved model of operations [teaching] is an approach doomed to fail" (Branson & Grow, 1987, p. 422).

Despite the evidence that changing the qualifications of teachers does not significantly improve teaching, efforts to do so persist. A striking case, as reported by the *New York Times* (Gootman, 2009), is the "Equity Project," a new school founded by Zeke Vanderhoek, an entrepreneur, with a mission of assembling the best teachers and thereby offering the best instruction for students. Mr. Vanderhoek recruits these teachers by, among other things, paying them salaries of $125,000. He has interviewed 100 people from 600 applicants to choose the 7–10 teachers for the first year. Mr. Vanderhoek acknowledges that it is difficult to divine teaching effectiveness from reading resumes. Whether the teachers he chooses will use more effective teaching methods than teachers in neighboring schools getting one-fourth the salary is an open question. The evidence is not on Mr. Vanderhoek's side.

Two reasons might explain why the public as well as many educators persist in believing that higher teacher qualifications produce better teaching. One is that it seems intuitively obvious. If teachers are better trained or have more qualifications, shouldn't they teach more effectively? This has an appealing logic. Why this cause–effect logic doesn't work so easily is taken up in the third explanation (coming shortly) for why teaching is so hard to change. The second reason the myth of higher qualifications producing better teaching persists might be that teacher qualifications are easy to measure whereas teaching quality is hard to measure. Researchers and policy makers need to measure the things they study, so teacher qualifications have become accepted as a measurable proxy for high-quality teaching. The problems with substituting teacher characteristics for teaching should now be apparent.

Parenthetically, confusing teachers with teaching is a notable example of a research failure producing a policy failure. If researchers could develop direct measures of teaching quality, it is likely that policy makers would be less inclined to keep using teacher qualifications. Measuring teaching quality is not achieved by just assessing students' learning gains because numerous factors, besides the nature of teaching, contribute to students' learning (see Fenstermacher and Richardson's (2005) for distinction between good teaching and successful teaching). Student learning gains could, of course, be part of the measure of teaching quality, but independent measures of teaching quality must also be developed.

Explanation Three: Under-Appreciation of the Cultural Nature of Teaching

Teaching, in general, and mathematics teaching, in particular, is a cultural activity (Stigler & Hiebert, 2009). Cultural activities evolve over time to satisfy society's needs within the constraints imposed by surrounding forces. For teaching, these forces include keeping large number of students in schools for 5 days per week, organizing schools into classrooms with one teacher per classroom, defining a teacher's job as caring for children as well as helping them achieve academic goals, handing teachers a curriculum determined by others, expecting students to graduate with particular competencies (for most of U.S. history, the mathematical competencies have been arithmetic skills), and measuring students' competencies with standardized tests and ranking their performance. Methods of teaching have evolved to fit within these various constraints and to do so in a sustainable way—with as little conflict and confusion as possible.

Because multiple forces constrain the nature of teaching, it is overdetermined (as are all cultural activities). There is more than one force holding teaching in place; loosening one constraint will not change things much. Many other forces will fill the vacuum and keep the long-standing methods of teaching operating as before. Attempts to change teaching meet many forms of resistance.

A consequence of seeing teaching as a cultural activity is seeing that the methods of teaching are not invented new by each teacher. Methods of teaching are handed down from one generation to the next. Cultural activities are learned by growing up in a culture, watching how others do things, and following their lead. Eight-year-olds already know how to play school. They know what teachers typically do. Beginning teachers have spent 16 years in classrooms, watching their teachers teach. They acquire their training by observing what their teachers do. In spite of graduating from teacher preparation programs, most teachers employ methods that are more like those of their K-12 teachers than those they might have been formally taught as preservice teachers (Borko & Putnam, 1996; Hiebert, Morris, & Glass, 2003; Lortie, 1975).

It is now possible to see why a teacher's qualifications or formal training do not predict well how effective they will be in the classroom. The methods they use to teach—the ways in which they interact with students around content—are likely to be determined by their own experiences as students in K-12 classrooms, and students within the same culture have similar experiences. Beginning teachers with impressive credentials are likely to use methods similar to those of their less "qualified" colleague down the hall because they had similar classroom experiences as students.

How Does Mathematics Teaching Change?

Based on the previous analysis, it is appropriate to ask whether teaching can change. Is it even possible? The historic record doesn't offer much hope. But reconsidering each of the three reasons for the persistence of teaching opens a window onto some possible paths toward change.

Asking how cultural activities change is a start. Cultural activities do *not* change by writing and reading reports. It is unlikely that producing written documents that prescribe changes to teaching will have much effect. It is more likely that changes will begin to occur when there is a widespread consensus on the learning goals for students. Learning goals provide targets for change. If all members of the teaching culture buy into a new set of learning goals, they might begin to align their methods of teaching more closely to those that best help students achieve these goals.

Realigning methods of teaching to match students' learning goals will be enabled by changes in some deep-seated beliefs about teaching. Some people still believe that teaching is an innate gift (Green, 2010). Either you have the gift or you don't—teachers are born, not made. This belief places the hope for improved teaching on finding the right people. But, as I argued earlier, this belief ignores the cultural nature of teaching.

A more productive belief is that teaching can be learned. Just as any complex skill, learning teaching takes time and requires lots of on-the-job practice, but it can be learned. And all teachers can learn to teach more effectively toward well-defined learning goals over time. David Moore (1995), on receiving the Mathematical Association of America's award for distinguished teaching, made this simple but still controversial claim: "Good teaching is based on the teacher's learning" (p. 5). Believing that teaching can be learned and that such learning takes time and practice would dramatically alter the nature of schools. They would become places where teachers, not only students, would learn (Schaefer, 1967). The culture of schools would necessarily change. And the changes would be fundamental. Time would be set aside for teacher learning. Collaborations among teachers sharing the same learning goals for students would emerge naturally. Observations of teaching would be centered on learning rather than evaluation. Evidence of improvements in teaching would be shared among teachers within and across schools. Schools would become intellectually invigorating and demanding places for teachers to work.

What teachers would actually do in these new environments would be simply to turn teaching into an object of study. As a cultural activity, much of what occurs as teaching has become so common that it is invisible. Making teaching an object of study brings the routines of teaching to conscious awareness. Studying teaching means slowing it down, taking apart its details. Is it best to ask the question this way or this way? How would each version of the question support or undermine students' efforts to achieve the learning goal(s) for this lesson?

In addition to unpacking the details of teaching, studying teaching means seeing the cause–effect relationships between teaching and learning that infuse an ordinary lesson. Many teachers do not appreciate that slight changes in lessons—in the ways they interact with students around content—influence directly what students learn. When teachers see the effects of the changes they make on what and how well students learn, they can begin to appreciate the powerful impact of studying the details of teaching (Gallimore, Ermeling, Saunders, & Goldenberg, 2009).

To summarize, there are good explanations for the persistence of traditional methods of teaching mathematics. In fact, the explanations are powerful enough that they prompt one to question whether change is possible. But, there also are reasons to think that understanding these explanations helps one see how change could occur. By developing a shared consensus on students' learning goals, by changing beliefs about how teaching improves, and by implementing a set of practices focused on teacher learning, improvements in teaching could begin to emerge and even accumulate over time.

An important caveat for the United States, however, is that Americans are addicted to quick fixes. Arne Duncan, U.S. Secretary of Education, when commenting on the state of teacher preparation in the country, said that Americans need "revolutionary change—not evolutionary tinkering" (Duncan, 2009). Such statements are typical of American impatience. If changes to teaching take the path sketched out here, a path more attuned to evolutionary tinkering than revolutionary reform, this impatience could easily scuttle these efforts before they have a chance to work. On the other hand, maybe the past record of failed quick fixes will be motivation enough to give another strategy a little time.

In this spirit, the final words will be those of John Wooden, one of the greatest basketball coaches of all time. Coach Wooden died at age 99 the day I was writing this chapter. Because he saw himself as a teacher, and the words I quote are his thoughts about improving his teaching skills, they are a fitting conclusion:

> When you improve a little each day, eventually big things occur …. Not tomorrow, not the next day, but eventually a big gain is made. Don't look for the big, quick improvement. Seek the small improvement one day at a time. That's the only way it happens—and when it happens, it lasts. (Wooden & Jamison, 1997, p. 143)

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Chapter 4
In the Absence of Meaning…

Patrick W. Thompson

Abstract There are many diagnoses of the bad state of U.S. mathematics education, ranging from incoherent curricula to low-quality teaching. In this chapter I will address a foundational reason for the many manifestations of failure—a systemic, cultural inattention to mathematical meaning and coherence. The result is teachers' inability to teach for understanding and students' inability to develop personal mathematical meanings that support interest, curiosity, and future learning. In developing this argument I discuss the subtle ways in which actual meanings with which teachers currently teach and actual meanings students currently develop in interaction with instruction contribute to dysfunctional mathematics education. I end by proposing a long-term strategy to address this situation.

I hope to address the issue of meaning in mathematics education in a way that conveys its nature and importance and also that conveys ramifications of addressing this issue for teaching, learning, and research in mathematics education. One ramification is to become aware of how deeply meaningless mathematics teaching and learning are in the United States. We must be aware of the depth of the problem as a prelude to devising solutions for it.

In this chapter I discuss meanings of "meaning," the creation of meaning through teaching, and difficulties that students have in creating mathematical meanings. I hasten to note that an incoherent meaning is a meaning, so please do not read "creating meaning through teaching" as pointing only to rosy outcomes.

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I also share some consequences of systemic inattention to mathematical meaning in the United States and a positive outcome of one teacher's attempt to help students build coherent meanings in algebra. My hope is that inspecting these examples will clarify ways that attending to issues of meaning allows us to see problems of mathematics learning as emergent from fundamental cultural orientations as much as from epistemological problems of learning sophisticated ideas. I end with a proposed agenda for how to move forward so that a focus on meaning is central to improving mathematics teaching and learning in the United States.

Meanings of "Meaning"

I have yet to find anyone who finds the phrase *the meaning of* "*meaning*" odd. They might ask, "What do you mean?" but they do not act as if I've spoken nonsense. What this points to is something that is innately human. Any time that we invoke the idea of meaning we invoke the idea of meaning. The idea of meaning is so deeply recursive that when we talk about issues of meaning we are talking about an intellectual capacity that is unique to humans. The recursive nature of attempts to examine the nature of meaning suggests, in line with Dewey (1910, 1933), that reflection and abstraction are at its core.

Philosophical disputes about the nature of meaning have centered historically around the referential relationship between language and reality. Ogden and Richards (1923/1989) offered their well-known semantic triangle (Fig. 4.1), which places referents in the world, but the relationship between a symbol and a referent exists only by way of a person making the association. Putnam (1973, 1975) argued strongly that meanings cannot be characterized by individuals' psychological states. "Meanings just ain't in the head," he famously said (Putnam, 1975, p. 227).

A second perspective on meaning focuses on what people intend to convey via an utterance, and what people imagine being conveyed as they hear an utterance. Grice (1957) presented an entertaining analysis of this perspective. He first distinguished between natural meanings and nonnatural meanings. A natural meaning, in

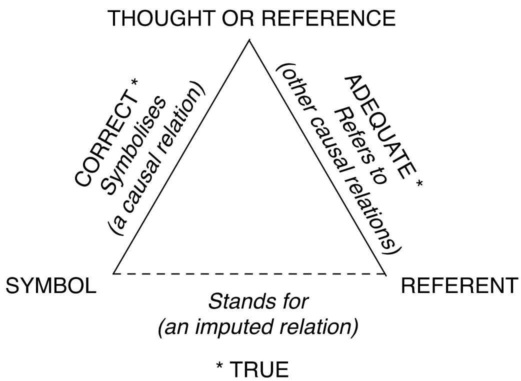


Fig. 4.1 Ogden and Richards (1923/1989) semiotic triangle

Grice's usage, is an inference one makes from observing something in the world (e.g., "red spots mean measles"). He was not much interested in this type of meaning, focusing instead on what he called nonnatural meanings—"meaning$_{NN}$"—ideas and ways of thinking that someone intends to convey to someone else and uses signs or symbols to do so. Grice distinguished among three ways that meaning$_{NN}$ can be seen at play in typical uses of "to mean":

> (1) "*A* meant$_{NN}$ something by *x*" is (roughly) equivalent to "*A* intended the utterance of *x* to produce some effect in an audience by means of the recognition of this intention"; and we may add that to ask what *A* meant is to ask for a specification of the intended effect (though, of course, it may not always be possible to get a straight answer involving a "that" clause, for example, "a belief that …").
>
> (2) "*x* meant something" is (roughly) equivalent to "Somebody meant$_{NN}$ something by *x*." Here again there will be cases where this will not quite work. I feel inclined to say that (as regards traffic lights) the change to red meant$_{NN}$ that the traffic was to stop; but it would be very unnatural to say, "Somebody (e.g., the Corporation) meant$_{NN}$ by the red-light change that the traffic was to stop." Nevertheless, there seems to be some sort of reference to somebody's intentions.
>
> (3) "*x* means$_{NN}$ (timeless) that so-and-so" might as a first shot be equated with some statement or disjunction of statements about what "people" (vague) intend (with qualifications about "recognition") to effect by *x*. (Grice, 1957, p. 385)

The significance of Grice's position for mathematics education is that the "mathematics on the page" cannot be the conveyor of meaning. Meanings reside in the minds of the person producing it and the person interpreting it.

Walker Percy, in his famous *Delta Factor* (Percy, 1975a, 1975b), expressed the result of his years-long puzzlement over the nature of man through an analysis of Helen Keller learning the word "water." He began from the perspective of Ogden and Richards' triangle, but later abandoned that approach because of the difficulty he had arguing that Keller had access to a real-world material called "water." Percy realized that Keller's connections were not between water and a sign. Rather, the connections were between experiences that she was having. Instead of the causal relations that Ogden and Richards posited between symbol and thought, Percy insisted that there was no causation at all—that the triangle was irreducible, and that the links in the triangle were all made by Helen. Percy called Keller's irreducible construction an instance of *The Delta Phenomenon*. He reflected upon his attempt to use Ogden and Richards' triangle to capture "what happened" when Keller learned the word "water":

> The longer one thought about the irreducible triangle and its elements and relations, the queerer they got.
>
> Compare Delta Δ phenomenon with the pseudo triangle of Ogden and Richards: buzzer → dog → food. The latter is a pseudo triangle because one needn't think of it as a triangle at all but can conceive the event quite easily as a series of energy exchanges beginning with buzzer and ending in the dog's salivation and approaching food.
>
> But consider the Delta phenomenon in its simplest form. A boy has just come into the naming stage of language acquisition and one day points to a balloon and looks questioningly at his father. The father says, "That's a balloon," or perhaps just, "Balloon."
>
> Here the Delta phenomenon is as simple as Helen's breakthrough in the well-house, the main difference being that the boy is stretching out over months what Helen took by storm in a few hours.

> But consider.
>
> Unlike the buzzer-dog-salivation sequence, one runs immediately into difficulty when one tries to locate and specify the Delta elements—balloon (thing), balloon (word), boy (organism).
>
> In a word, my next discovery was bad news. It was the discovery of three mystifying negatives. In the Delta phenomenon it seems: The balloon is not the balloon out there. The word balloon is not the sound in the air. The boy is not the organism boy. (Percy, 1975a, Kindle Locations 661–673).

In other words, Percy saw Keller's construction of meaning as an epiphenomenon, an emergent unification created by Keller's association of her tactile experience of what an observer would call *water pouring over her hand* and the tactile experience of what Anne Sullivan would have called *signing into her hand*. Keller made her experiences whole through an act of *naming*. It was Keller's connecting these experiences that made Sullivan's sign have a referent, and the meaning *within* Keller was irreducible. It had all three components simultaneously.

Dewey (1910, 1933) considered meaning and understanding to be synonymous, and either to be the product of thinking. His idea of thinking was very special, however. His interest was in what he sometimes called *reflective* thinking. To Dewey, coherence is a characteristic outcome of thinking—thinking leads to "the organization of facts and conditions which, just as they stand, are isolated, fragmentary, and discrepant, the organization being effected through the introduction of connecting links, or middle terms" (Dewey, 1910, p. 79). Dewey also considered thinking to be the primary mechanism for the construction and refinement of meaning: "That thinking both employs and expands notions, conceptions, is then simply saying that in inference and judgment we use meanings, and that this use also corrects and widens them" (Dewey, 1910, p. 125). He also emphasized the role of meaning in human communication:

> It is significant that one meaning of the term *understood* is something so thoroughly mastered, so completely agreed upon, as to be *assumed*; that is to say, taken as a matter of course without explicit statement. The familiar "goes without saying" means "it is understood." If two persons can converse intelligently with each other, it is because common experience supplies a background of mutual understanding upon which their respective remarks are projected. To dig up and to formulate this common background would be imbecile; it is "understood," that is, it is silently sup-plied and im-plied as the taken-for-granted.
>
> If, however, the two persons find themselves at cross purposes, it is necessary to dig up and compare the presuppositions, the implied context, on the basis of which each is speaking. The im-plicit is ex-plicit; what was unconsciously assumed is exposed to the light of conscious day. (Dewey, 1910, p. 214).

Meaning and understanding were synonymous to Piaget, also. But he put it differently than Dewey. Though I know of no place where Piaget said this directly, I agree with Skemp (1961, 1962, 1979) that, to Piaget, "to understand" was synonymous with "to assimilate to a scheme." Of course, this is entirely unhelpful if we do not know what Piaget meant by a scheme.

My understanding of what Piaget meant by "scheme" differs from that proposed by Cobb and Glaersfeld (1983) and by Glasersfeld (1995, 1998). They proposed that, to Piaget, a scheme was a three-part mental structure: a condition that would trigger a scheme, an action or a system of actions, and an anticipation of what the action should produce. I believe what Cobb and Glasersfeld described fits better with Piaget called a *schema* of action (Piaget, 1968, p. 11; Piaget & Inhelder, 1969, p. 4). Piaget spoke of a child's sucking schema, for example. I believe Piaget had larger organizations in mind when he spoke of schemes—organizations of operations, images, schemata, and schemes—that did not have easily identified entry points that might trigger action.[1] I have spoken, for example, of a *rate of change scheme* (Thompson, 1994a, 1994c; Thompson & Thompson, 1992, 1996) that entails a complex coordination of understandings of quantity, variation, relative change, accumulation, and proportionality. Thompson and Saldanha (2003) wrote about a coordination among understandings of quantity, measure, proportionality, multiplication, and division as comprising a *fraction scheme*. So, in Piaget's system, *to understand* means *to assimilate to a scheme*, but this is still somewhat unsatisfactory because we need to understand Piaget's meaning of assimilation.

Standard meanings of "assimilate" all entail some sense of something being absorbed by something else. As Piaget famously said, "A rabbit that eats a cabbage doesn't become cabbage; it is the cabbage that becomes rabbit—that's assimilation. It's the same thing at the psychological level. Whatever a stimulus is, it is integrated with internal structures" (Bringuier, 1980, p. 42). Piaget's use of "assimilate" is in a cognitive sense. It does not entail energy transfer. Rather, it emphasized absorption of information. A physical stimulation on a retina creates information that is processed by the nervous system. What looks like absorption is actually imbuement. Montangero and Maurice-Naville (1997, p. 72) quoted Piaget as saying,

> Assimilating an object to a scheme involves giving one or several meanings to this object, and it is this conferring of meanings that implies a more or less complete system of inferences, even when it is simply a question of verifying a fact. In short, we could say that an assimilation is an association accompanied by inference. (Johnckheere, Mandelbrot, & Piaget, 1958, p. 59)

So, to understand is to assimilate to a scheme. But whence schemes? From assimilation. From a Piagetian viewpoint, to construct a meaning is to construct an understanding—a scheme—and to construct a scheme requires applying the same operations of thought repeatedly to understand situations being made meaningful by that scheme. "Assimilation … is the source of schemes …. Assimilation is the operation of integration of which the scheme is the result" (Piaget, 1977, p. 70). Put another way, we construct stable understandings by repeatedly constructing them anew.

Hiebert and Carpenter (Carpenter, 1986; Hiebert & Carpenter, 1992; Hiebert & Lefevre, 1986) characterized mathematical understandings similarly to Piaget. They spoke of desirable understandings as rich networks of connections among concepts

[1]The concept of scheme is recursive. A scheme can entail other schemes.

and procedures. Their characterization of concepts, however, is largely noncognitive. They did not address how someone thinks to have them and their notion of meaning is static. I find Piaget's ideas on understanding (meaning) to do more work for thinking about teaching and learning. As Piaget and Garcia (1991) made clear, their notion of meaning is implicative—meaning comes from an assimilation's implications for further action. Moreover, Piaget's genetic epistemology entails a rich conception of ways that understandings can be made and how they work in reasoning.

Why Attending to Meaning Matters

In one sense, the issue of meaning is irrelevant to mathematics education—if we accept the current state of mathematics education. It is rare for a mathematics teacher, at least one in the United States, to be concerned with meaning, either intended or conveyed. If we believe the results of TIMSS classroom studies (e.g., Hiebert et al., 2005; Schmidt, Houang, & Cogan, 2002; Schmidt, Wang, & McKnight, 2005; Stigler & Hiebert, 1999), the main goal of most U.S. mathematics teachers is that students learn to perform prescribed procedures. Issues of meaning are largely irrelevant. But if we intend that students develop mathematical understandings that will serve them as creative and spontaneous thinkers outside of school, then issues of meaning are paramount.

I am therefore speaking to educators who are concerned about the conveyance of mathematical meaning. To convey meaning is one of the most important goals towards which teachers can strive. As we think about teaching and the conveyance of mathematical meaning, it will be productive to look for useful ways to imagine how "conveyance" happens. Is meaning on a printed page? Written on a whiteboard? Does it appear on a computer screen? Is meaning conveyed to students by directing their attention to "real-world" referents? Each of these stances puts meaning in the world so that there are "correct" meanings to be had and any meanings that depart from them are incorrect. Simon, Tzur, Heinz, Kinzel, and Smith (2000) characterized this image of conveying mathematical meaning "perception-based" mathematics. They claimed that this is the predominant view of mathematics schoolteachers and that teachers expect students to see in mathematical statements what they see. It is "there" for students to take up.

I concur with Cobb (2007) in taking the stance that we should adopt theoretical perspectives only to the extent that they help us do our work. I maintain that any stance that puts meaning outside of individuals is less helpful for purposes of instructional and curricular design, teacher preparation, and professional development than a stance that puts meaning within individuals. This is because most of our efforts in working with students occur at a time when they do not possess mathematical meanings that we hope they will have eventually. Ogden and Richards' triangle is of little use in this case. Their triangle offers no guidance. The meanings that matter at the moment of interacting with students are the meanings that students have, for it is their current meanings that constitute the framework within which they operate and it is their personal meanings that we hope students will transform. Our making explicit the meanings we intend that they have eventually is important, because they comprise our instructional and curricular goals. But those targeted meanings must come to exist within individual students—in the sense that Percy (1975a) described Helen Keller's acquisition of "water"—for us to have succeeded.

There is empirical evidence that the mathematical meanings a teacher possesses matter in regard to what students learn. For example, Branca (1980) studied the communication of semantic structure from teacher to student regarding the content of a unit on operational systems. He studied six high school mathematics teachers over the duration of the instructional unit. At the beginning of the unit there was little resemblance between what teachers meant and what students meant by key terms and phrases. By the end of the study, students' meanings regarding operational systems very much resembled their teacher's—including inappropriate meanings. Teachers' and students' meanings became aligned even about ideas that were not taught. So, even tacit meanings that teachers carry can be conveyed to students. (I address how this might happen in the next section.) Branca's study, however, was tightly focused on definitions and theorems having to do with systems defined by operations on sets, so issues of meaning were central to the subject matter being taught. Examples given later in this chapter show that students develop understandings and ways of thinking about the mathematics they learn even when meaning is not central to the teachers' subject matter. But the understandings that students develop in those settings are not propitious for later learning.

Conveying Meaning Through Teaching

If we maintain the stance that meanings are entirely within individuals, we face the immediate question of how people can appear to learn a meaning from someone else. How shall we explain the seemingly evident fact that teachers *can* convey meanings to students? I find two sources immensely helpful in conceptualizing human communication so that we can speak sensibly about the conveyance of meaning without violating our self-imposed stance that all meanings lay within individuals. The first is Piaget's notion of *intersubjectivity* (Glasersfeld, 1995; Piaget, 1995; Thompson, 2000) and the other is Pask's *conversation theory* (Pask, 1975, 1976; Scott, 2009). Piaget placed great emphasis on the idea of *decentering*, or attempting to adopt a viewpoint that differs from your own. He used the term *intersubjective operations* to describe thoughts that are directed at another. As Glasersfeld (1995) put it, once a child starts to think that another person "thinks like me," he or she can then also notice occasions where the person seems not to think like her. This is at the root of what Glasersfeld called "the construction of others" (Glasersfeld, 1995, Chap. 6).

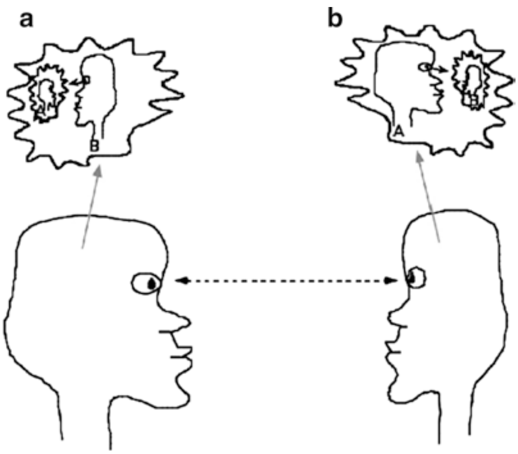


Fig. 4.2 Summary of intersubjective operations involved in the communication of meaning

Pask's conversation theory attempted to explain how social interaction can lead to participants' construction of knowledge. His theory was rather technical, but the important part for this chapter is his concept of a conversation. To Pask (1975, 1976), a conversation is more than face-to-face verbal exchanges confined to a specific place and time. Rather, a conversation involves all the actions entailed in conversants' attempts to convey and discern meaning. So, a classroom conversation, in Pask's sense, could include an exchange that involved the teacher introducing an idea, handing out a worksheet, and discussing how he or she expects students to use it. The teacher's soliloquy can be considered part of an ongoing conversation, as can students asking questions about a worksheet or about what the teacher expects. It goes without saying that conversations are most productive when each participant is oriented to understand what others have in mind and is oriented to have others understand what he or she intends.

What follows is an amalgam of Piaget's notion of intersubjective operations and Pask's conversation theory. The amalgam is necessary simply because neither Piaget nor Pask focused squarely on the construction of mathematical meaning. Pask's theory was quite technical, and it was more interested in conversations than in participants. His interest in teaching expressed itself largely in the form of adaptive teaching machines that made decisions about problems that a student should work given the student's performance on prior problems. He paid little attention to imagery and its role in meaning, and he did not consider specific mathematical meanings, such as what it might mean for someone to understand the idea of quantity. On the other hand, Piaget's theory embraced imagery as a key component of cognitive development (see Thompson, 1994a, 1996), but he was not interested in specific mathematical ideas.

Figure 4.2 shows Persons A and B attempting to have a meaningful conversation. Person A intends to convey something to Person B. The intention is constituted by a thought that A holds that he wishes B to hold as well. The figure shows A not just considering how to express his thought, but considering how B might interpret A's utterances and actions. It is worthwhile noting that A's action towards B is not really towards B. A's action towards B is towards A's image of B. In a sophisticated conversation A's action towards B is not just towards B, but it's towards B with some understanding of how B might hear A. Likewise, B is doing the same thing. He assimilates A's utterances, imbuing them with meanings that he would have were he to say the same thing. But B then colors those understandings with what he knows about A's meanings and according to the extent to which A said something differently than B would have said it to mean what B thinks A means. B then formulates a response to A with the intent of conveying to A what B now has in mind, but B colors his intention with his model of how he thinks A might hear him, where the model is updated by anything he has just learned from attempting to understand A's utterance. And so on.

The process of mutual interpretation and accommodation described above, which Steffe and Thompson (2000) also called *reciprocal assimilations*, is what I understand Piaget to have meant by the negotiation of meaning. The negotiation is not sitting down and developing a contract, like negotiation of meaning is often portrayed. The negotiations that happen are rarely negotiated explicitly. The negotiations that happen involve each person monitoring the other's responses, comparing them to the responses he anticipated, and then adjusting his model of the other to make better decisions about how to act and what to expect in the future. This, I believe, is what Bauersfeld (1980, 1988) meant by communication as interactions among mutually reflexive systems. Both A and B adjust their understandings of the other's understanding, and possibly adjust their personal understandings in the process. In Piaget's and Glasersfeld's usage, A's and B's conversation enters a state of intersubjectivity when neither of them has a reason to believe that he has misunderstood the other. They may in fact have completely misunderstood each other, but they have not discerned any evidence of such. As Glasersfeld (1995) makes clear, a conversation being in a state of intersubjectivity has no implication for whether the participants' meanings align. Rather, the nature of a conversation that is in a state of intersubjectivity is that neither participant has any reason to believe that he has misunderstood the other. It is important to note that it is a conversation that is in a state of intersubjectivity. It is a category error to say that the participants are in a state of intersubjectivity.

The above description of conversation assumes that all participants really are participants—that they care to understand other participants. If, for some reason, B were to not care what A meant, then there is no conversation. This observation has important implications for teachers' management of classroom conversations: students must intend to discern meaning in order to construct meaning from a conversation. The teacher's guidance in creating an atmosphere where making meaning is valued and expected is central to students' construction of meaning through conversations (Cobb, Boufi, McClain, & Whitenack, 1997). It is also important to distinguish between a conversation of equals and a classroom conversation. The teacher is a very special participant in classroom conversations. A teacher has power and trust in a conversation that students do not have. Teachers can manage conversations; students are rarely positioned to manage a conversation. Teachers can manage a conversation so that students *do* have power and trust, but it is teachers who allow and nurture those opportunities.

Many people take the case of intersubjectivity in a two-participant conversation as being completely unlike a conversation that involves many participants. "It is impossible for each participant to have a model of every other participant's understanding," they might say. This stance, however, misses the essential character of a conversation that is in a state of intersubjectivity. A conversation is in a state of intersubjectivity when it is in a state of equilibrium—when each participant takes for granted that no one has misunderstood anyone else's understanding. Disagreements do not necessarily puncture a conversation's equilibrium. Two people can disagree with what they discern the other to mean, but if neither person feels that he or she has misunderstood the other, then the equilibrium persists. In a sense, each person has created an epistemic "other" to which he or she can attach a variety of ways of thinking about the conversation's subject.

We can return now to Branca's (1980) study and the question of how teachers conveyed meaning to students. The teachers needn't have said, "No, no, no, that meaning is wrong, I want you to have this meaning." The teachers probably rarely said anything like this. Rather, by focusing their attention on their meaning of an operational system, students adapted their understandings to fit what they discerned the teacher to have in mind. This would account for students ending with meanings that were compatible with their teachers' meanings even when those meanings were either normatively inappropriate or never discussed explicitly.

The notion of intersubjectivity, as described above, can also give us insight into how *mis*communication happens. The example given below illustrates a conversation that was in a state of intersubjectivity for a relatively long period of time even though the participants had misinterpreted each other quite severely.

Mindi is a ninth-grader enrolled in Algebra I. Her teacher, Sheila, is a participant in a professional development project that emphasizes student-oriented instruction that focuses on supporting students' creation of meaningful, coherent mathematics. Sheila's review of arithmetic at the beginning of the year emphasized meanings and ways of thinking that underlie arithmetical operations (e.g., division as measuring or partitioning, multiplication as multiple copies or as dilation, fractions as a reciprocal relation of relative size, order of operations as a system of conventions that imposes structure on arithmetic and symbolic expressions, and so on). Prior to where this example starts, Sheila had drawn on these meanings of addition, subtraction, multiplication, division, and order of operations to help students build ways to think about expressions and equations. For example, instead of teaching "do the same to both sides" Sheila emphasized inferences one could make about numerical relationships that would allow you to see numerical relationships that were not directly stated. For instance, when discussing what value or values of x makes $x/5+15=30$ true, she guided students to reason about what the equation is saying:

> If $x/5+15$ is 30, then $x/5$ must be 15, because $15+15$ is 30. Now $x/5$ means "one-fifth of the number represented by x." If one-fifth of the number x is 15, and since there are five 1/5ths of x in x, then x must contain five fifteens, and therefore must be five times as large as 15.

Sheila used diagrams and illustrations to accompany her statements, and she consciously designed her instruction so that students would generate equation-solving methods as abstractions from their experiences of repeated reasoning.

Many of Sheila's students seemed to thrive in the context of her instruction; some did not. Sheila asked that I interview Mindi, a hardworking, bright girl who did well in class until the section on solving linear equations. Several excerpts of that interview follow.

Excerpt 1: Meaning of Equations. P: Pat; M: Mindy

1. P: Before we start, could you tell me what pops into your head when you see an equation?
2. M: Well, you are supposed to isolate the variable so that it equals a number or another expression. If it equals another expression, then you try again to isolate the variable so that it equals a number.

In a sense, Mindi's description of her meaning for equations seems quite standard. When you see an equation you think to solve it. Excerpt 2, however, reveals that Mindi did not interpret equations as stating numerical relationships. It reveals the depth of her procedural perspective.

Excerpt 2: w/3 = *11. P: Pat; M: Mindy*

1. P: I'm not asking you to solve this equation. Instead, just tell me what it says.
2. M: It says that when you divide some number by 3, you get 11.
3. P: Okay. Now, can you tell me what this expression stands for (circles w/3)?
4. M: It stands for a number.
5. P: Any idea what that number is?
6. M: No. I'd have to solve for w. Then I could tell you what w slash three stands for.[2]

Mindi revealed the same way of thinking when discussing $8m-4=8$ (I circled $8m-4$) and $a/5+15=30$ (I circled $a/5+15$). Thus, it seems safe to say that the meaning of an equation, for Mindi, was that it was a symbolic form that she was expected to act on to end with another form $x=number$. This was confirmed again when she actually solved the equation $8m-4=8$, getting $m=12/8$ ("12 over 8,

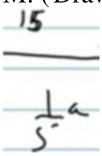
[2]Sheila's expectation was that students would understand that, in the context of the equation $w/3=11$, $w/3$ stood for the number 11. Though she expected this interpretation, she never stated it.

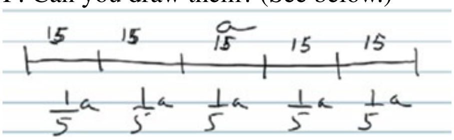
which reduces to 3 over 2”). I asked her whether the answer is “12 over 8” or “12 eighths.” She stated that she preferred saying “12 over 8” because it made more sense to her than “12 eighths.”

I then asked Mindi to revisit the intermediate result $8m=12$ that she had written, and which appeared immediately above “$m=3/2$.” I said, “This says that 12 is some number of times as large as 8. Twelve is how many times as large as 8?” No answer. “Is it 25 % larger, 100 % larger?” Mindi thought that 100 % was too much. I then asked, “What is 150 % of 8?” Mindi replied that she had difficulty finding 150 % of a number. In other words, Mindi’s meaning for equations did not entail seeing them as an expression of numerical relationships. She did not see that her answer $m=3/2$ meant that 12 is 3/2 times as large as 8. Her numerical reasoning did not support seeing equations that way. She could not imbue an equation with meanings drawn from numerical relationships. Her scheme for equations entailed actions for operating on them and little else.

Excerpt 3, below, was about the equation $a/5+15=30$. It reveals more about Mindi’s schemes—not just her scheme for equations, but her schemes for engaging with classroom mathematical instruction. It picks up after I had led Mindi to the conclusion that $a/5$ had to be 15 because $15+15$ is 30.

Excerpt 3: a/5+15=30. P: Pat; M: Mindy

1. P: You said that *a* divided by 5 is 15. Can you interpret *a*/5 as a fraction?
2. M: It is 1/5 of *a*.
3. P: Can you draw a diagram to show 1/5 of *a*?
4. M: (Draws the upper part of this diagram. Pat writes “1/5 *a*” below it.)
5. 

6. P: Does this tell you anything about what *a* is?
7. M: No. Not really.
8. P: How many fifths of *a* are in *a*?
9. M: Five.
10. P: Can you draw them? (See below.)
11. 

12. […]
13. P: You figured out that 1/5 of *a* is 15, and therefore that 5 of those things make *a*.
14. M: Yeah.

15. P: Does that make sense?
16. M: Yeah, kind of.
17. P: What's the "kind of" part?
18. M: Well, I just would probably never do it that way, 'cause // it's kind of confusing. Like it kinda makes sense, but it's still kinda confusing.
19. P: Well, notice here (on a quiz she took the prior week) what you wrote. (Mindi had written $a/5=45$, and $a=45/5$). So, even doing it your way there was something confusing about it.
20. M: Yeah.
21. P: So, let me ask you a question. You seem to be reluctant to figure things out, without a rule. Is that right?
22. M: Yeah.
23. P: Why is that?
24. M: I don't know. Well, whenever I do it this way (reasoning with numerical relationships) I feel like I'm doing it wrong. You know, like I mean, with a rule, I can be sure. Because a rule says to do this, I know what I'm supposed to do and I know I'm doing it right, but with this way there is too much room for error. I think.
25. P: Okay. What about all these places where you used rules (pointing to errors she made on her quiz) and …
26. M: Most of them are stupid mistakes. Like here I added 15 instead of subtracting 15. With that one (another error) // a stupid mistake.
27. P: So, how can you avoid stupid mistakes?
28. M: Just by practicing more. Studying more.
29. P: Do you practice a lot?
30. M: No, not really. But I never needed to! Because I'd always just got // done it, like perfect.
31. P: So it's a little more complicated now?
32. M: Yeah, but I'll just practice more and I'm sure I'll do okay.
33. P: Okay. So you have more faith in practicing the rules than you do in practicing reasoning it through?
34. M: Yes.
35. […]
36. P: Well, this has been very useful for me. I hope it's been useful for you.
37. M: Yeah.
38. P: Is there anything you would like to ask me?
39. M: (Pause.) Ummm. Well, maybe just like // why do you want us to do it this way (reasoning) so much?

It is clear from Excerpt 3 that Mindi had little faith that reasoning was a reliable problem-solving technique and that she had much greater trust in using procedures that she had memorized.

In answering Mindi's last question I explained the benefits of reasoning—making fewer mistakes and catching mistakes when you make them. I also explained that by practicing reasoning students often found learning new ideas to be easier. I then asked Mindi about what she did when Sheila tried to teach her to reason about problems. It turned out that Mindi understood the reasoning that Sheila thought she was teaching as just the steps that Sheila wanted Mindi to remember. Mindi found the steps to be confusing—she couldn't remember them. So Mindi waited until Sheila taught "the rule." "The rule" from Sheila's perspective was a generalization of the reasoning she thought she had taught, but from Mindi's perspective it was the meat of the lesson, and Mindi could not understand why Sheila waited so long to tell it.

To summarize, Mindi's scheme for equations (apply procedures to isolate a variable) existed within a larger scheme that anticipated what she should get from instruction (rules) and how she should participate in lessons (remember the rule). Mindi saw her role as remembering steps; the teacher's role was to provide steps. Successful participation, for Mindi, was that she had a clear idea of the rules she was supposed to use. This is the way Mindi assimilated Sheila's actions and utterances—even though Sheila intended her actions and utterances to help students construct meanings. Mindi assimilated Sheila's reasoning steps as "new rules"—rules that were harder for her to remember than the bottom-line rules that she eventually discerned. Despite its dysfunctional nature, the conversation constituted by Sheila–Mindi interactions was in a state of intersubjectivity until Sheila discerned that something was amiss with her understanding of Mindi's understanding.

Sheila was aghast as she listened to Mindi's interview. She had no idea that Mindi was hearing her as she was. Sheila decided to find out whether other students had understood her instruction as Mindi did—as just providing steps they should memorize. She opened up classroom discussions to include several questions I'd asked in Mindi's interview. As a result, she found that Mindi's perspective was common, even among students who Sheila thought had been solving problems from a basis of meanings of equations and meanings of operations.

I suspect that Mindi's predominant experience in mathematics classrooms prior to entering Sheila's had been such that understanding was equated with correct performance, and that classroom conversations, even when everyone thought they were about understanding, were actually about procedures. Once Mindi developed her way of thinking about what mathematics *is*, she then heard her teachers projecting that same way of thinking, and she found no occasion to believe otherwise. Similarly, her teachers thought that Mindi understood what they intended because Mindi performed successfully, and the conversations in which Mindi participated were such that her teachers saw no occasion to believe otherwise.

The final comment to be made here is about teachers' expectations for students' understanding. Branca's (1980) study, mentioned earlier, suggests that teachers' meanings create a space for students' meanings. If a teacher's image of what students are to learn entails weak meanings, or no meanings, then intersubjectivity can be attained with students collectively possessing a wide variety of meanings that fit the discourse, many of which we would identify as problematic. When a teacher's image of what students should learn entails a strong system of meanings, then the space for possible student meanings is much smaller, assuming that teacher and students mutually adapt their understanding of the other. The teacher will find more occasions to discern that students' meanings differ from what he or she intends, and students will find more occasions to discern that what the teacher has in mind differs from what they understand. If the classroom culture is such that participants expect that noticed differences in meaning should be resolved, then it is more likely that students will develop coherent systems of meaning that guide their mathematical performance.

The discerning reader might object to the previous paragraph—Sheila had strong meanings and yet Mindi's thinking seemed unconstrained by them. How is this possible? It is possible because the conversations that Sheila managed had strong overtones of "what should we *do*," not "what should we *mean*." Sheila told students what they should mean, and then too quickly moved the discussion to how to answer questions in the worksheets "meaningfully." She demonstrated ways to answer questions using the meanings she had told them, but the conversation allowed students to think that she was simply showing them how to answer the questions. Students could safely ignore those occasions when Sheila asked them questions they couldn't answer (e.g., "How does the meaning of division tell us to multiply both sides by 5?"). The conversation's bottom line, in the students' eyes, was that you should multiply both sides by 5.

What Happens in the Absence of Meaning?

The case of Mindi illustrates how a well-meaning teacher who has a fairly strong system of meanings can nevertheless fail to influence a student in the way he or she intends. Another case, though, is when a teacher does not have a strong system of meanings regarding a particular body of ideas. A teacher with a weak system of meanings for an idea cannot help being vague or confusing when he or she speaks about ideas, and naturally avoids issues of meaning. However, even if he or she avoids speaking about ideas explicitly, his or her actions will be unconstrained by a strong system of meanings, and a conversation's meaning-spaces will have a high probability of entailing many inappropriate possibilities. As Dewey (1910) said, vagueness of meaning is a source of *mis*understanding, *mis*apprehension, and *mis*-taking. Confused meanings (i.e., undifferentiated, vague, confounded) are "too gelatinous" to support students' productive analysis and reflection: "Vagueness disguises the unconscious mixing of different meanings, and facilitates the substitution of one for the other" (Dewey, 1910, p. 129ff).

Dewey's point is illustrated by a group of high school teachers who were working together in weekly Professional Learning Community (PLC) meetings that they used to discuss material that they taught in common. At the time of this meeting, January 20, 2006, they were in the midst of teaching a unit on trigonometry to tenth-graders. The current topics were angle and angle measure.

An outside facilitator met with the group in the role of a consultant. The teachers and facilitator were arranged in a semicircle. The camera was about 20 ft away, directly in front of them. Excerpt 4 begins with the facilitator asking, offhandedly, "What is an angle measure?"

Excerpt 4: Teachers Discuss the Meaning of Angle Measure. F: Facilitator; T: Teacher

1. F: So, what is angle measure? You might raise this issue …
2. T1: (Interrupting) What is angle measure? I think that is a good question.
3. F: What is angle measure?
4. T2: It is very different from measuring the length of a side // I had a couple of students who thought they could be the same thing.
5. F: What did you say to them?
6. T2: You can't do that! They're not the same thing!
7. T1: So, how would you define it [angle measure]?
8. T3: How do you // how do you define angle measure?
9. T1: The ray sweeps // isn't the angle created when the terminal ray sweeps from the initial side to the terminal side // so angle measure is defined as what?
10. T3: Are you talking about, then you start getting into that thing of are you talking about arc length?
11. T1: Well, I don't know. How do you define angle measure?
12. T4: The curvature.
13. T3: (To T1) You mean your initial ray?
14. F: How do you [say], "Angle measure means this."
15. T3: (Reading from a textbook.) "The measure of angle A is denoted by // The measure of an angle can be approximated with a protractor using units called degrees. For instance" // they don't ever get into what is a degree.
16. T1: (Reading) An angle consists of two different rays.
17. T3: That's just defining an angle.
18. T1: It's the portion of a complete rotation that you take out as the terminal side sweeps (stops).

The teachers' only meaning for angle measure was to lay a protractor down and read off a number. They realized, however, that reading off a number from a protractor does not explain what an angle measure *is*. The teachers' meaning for angle measure (or lack thereof) had consequences for students' learning.

We asked the teachers to give this question to their students: "What are you measuring when we measure an angle?" Students' responses are summarized in Table 4.1, which shows that 93 % of the students thought that an angle measure was measuring something between the sides, either a distance from one side to the other or an area bounded by the angle's sides. Only one student said anything related to an arc, and even this answer seems to be oriented towards a distance. We should note that teachers never discussed with students what an angle measure is, or what one measures when measuring an angle. The answers students gave might have been preformed, in the sense that these are meanings that they created prior to their geometry class. However, even if their meanings existed prior to taking geometry, it seems that there was nothing in their experiences within their geometry class to alert them that their particular ways of thinking might be problematic. We must also

Table 4.1 Students' responses to "What are you measuring when you measure an angle?" ($n=110$)

Student response	Percentage (%)
Distance between sides	51
Distance between labeled points	2
Shape of the angle (directions of the rays)	3
Diameter of the angle	2
"Arc of the angle (how wide it is)"	1
Area of the angle	42

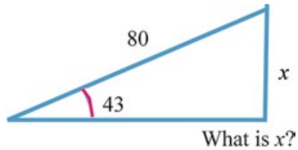


Fig. 4.3 Problem included in teachers' student interview protocol

entertain the possibility that students had never thought about what they were measuring when finding an angle measure. This interpretation seems sensible if their only experience with measuring angles was simply to follow a procedure that employed a protractor.

Independently of our question in Table 4.1, teachers created a set of geometry questions and an interview protocol as part of their PLC work, and they each interviewed three students from their respective class. One of the interview questions is given in Fig. 4.3. The teachers were to ask students to solve the problem and then were to discuss the students' solutions.

The interviews took place in mid-March, 2006, at the end of this particular instructional unit. In their March 26 PLC meeting they discussed students' responses to the interview questions. Excerpt 5 presents the portion of that meeting in which they discussed students' answers to the problem shown in Fig. 4.3.

Excerpt 5: Teachers Discuss Results of Student Interviews

1. T4: I was really surprised at the interviews. Two of the three students I interviewed really mixed information. They mixed 180° in a triangle // They confused 180 with a side length. They subtracted 180 – 43 and got 137. Then they subtracted 80 from 137 to get 57 for the other side length.
2. T3: Triangles have to add up to 180.
3. T2: My kids make no distinction between angles and sides.
4. T5: My honors kids today were going to take 360 and subtract a length, and I told them you are mixing angles and lengths! You can't do that!!

I find it remarkable in Excerpt 5 that none of the teachers considered the possibility that the students' confusions were rooted in the teachers' teaching. Why should students *not* confuse (what we take as) angle measures and (what we take as) side lengths when, to the students, numbers rarely have any meaning? By the teachers' own admission in Excerpt 4, they paid no attention to the meaning of an angle measure. Moreover, it is ironic that in the context of complaining that students cannot differentiate between angle measures and side lengths that T3 uttered, "Triangles have to add up to 180." I do not know what it means to add up triangles. If this is the level of precision T3 used in class, then it is no wonder that her students cannot distinguish between angle measures and side lengths.

I suspect that, in the context of classroom instruction, the teachers' students could easily succeed in the moment without paying any attention to the meanings of the numbers that appeared in problems. Within the context of the problems they were working in a particular section, students simply applied the procedure that was being taught at that moment. A number was a number was a number. When different numbers mean different things within the context of one situation, to distinguish between numbers that are side lengths and numbers that are angle measures students must have a system of meanings that keep them separate.

An Example of a Teacher Attending to Meaning

To construct a meaning requires repeatedly constructing and using the operations (ways of thinking) whose organization constitutes that meaning. But constitutes that meaning. The most usable meanings are those that are richly connected with imagery action and that tie into other meanings.

In 2006–2007, I had the pleasure of working with a ninth-grade Algebra I teacher, Augusta, who took seriously the matter of students learning mathematics meaningfully and coherently. She structured the subject over the year so that students would build ways of thinking that would constitute an understanding of algebra that had a clear trajectory for supporting their future learning of calculus. I share a sample drawn from her unit on polynomial functions to illustrate what I have said about constructing a meaning by repeatedly constructing and using the operations (ways of thinking) whose organization constitutes that meaning. But to make this sample understandable, I must first describe how she prepared students to participate in the conversations about the idea of polynomial function that I will share.

Augusta was conscientious about helping students build meanings that would lend coherence to their algebraic thinking and provide a foundation for later learning. She

- Began with building variation as a way of thinking about quantities changing. Students could *imagine* a quantity changing continuously. Variables varied. Always.
- Built covariation as a way of thinking about two quantities varying simultaneously. Time on a clock varies while a runner runs. The clock doesn't *cause* a

runner to run. We simply keep track of how far she has run in relation to how much time has elapsed on the clock.

- Built the idea of function as an invariant relationship between the values of covarying quantities. The perimeter of a circle is always 2π times the length of its radius no matter how we change either (assuming that it remains a circle).
- Built the idea of linear function as a function that has a constant rate of change.
- Built the idea of a graph as having points, where the coordinates of each point tell us the value that each quantity has in relation to the other. Each point provides a "snapshot" of the quantities' covariation.
- Built an understanding of constant rate of change as a relationship between two quantities that are changing simultaneously such that all changes in the value of one quantity are proportional to changes in the value of the other.
- Built an understanding of average rate of change. First, two quantities, A and B, need to change simultaneously, and each has a total change. The average rate of change of Quantity A with respect to Quantity B is that constant rate of change of A with respect to B would produce the same change in A in relation to the change in B that actually happened.

Augusta also had an agenda with regard to symbolic facility and representational equivalence. To explain what she did in regard to symbol sense is not important for this example, though I will say more about it later.

Augusta intended that students understand a polynomial function as a function that is the sum of monomial functions (Dugdale, Wagner, & Kibbey, 1992). That is, she wanted them to think of $f(x)=2x^3-x^2+5x+2$ as the sum of $f_1(x)=2x^3$, $f_2(x)=-x^2$, $f_3(x)=5x$, and $f_4(x)=2$, and hence that $f(x)=f_1(x)+f_2(x)+f_3(x)+f_4(x)$. She aimed for this understanding so that students could anticipate the behavior of a polynomial function, expressed in standard form, by examining its addends. To develop this way of thinking about polynomial functions, Augusta needed to help students understand the meaning of a sum of two functions. This is the focus of the example I share below—Augusta is introducing the idea of a function that is the sum of two functions.

The sum of two functions f and g is often defined as $(f+g)(x)=f(x)+g(x)$ [3] which emphasizes how you would calculate the value of a sum for a given value of x. Augusta's aim was that students could also *imagine* the sum of two functions in a way that was nonsymbolic, yet true to the definition. She wanted her students to have a *way of thinking* about making a function that is a sum.

The example enters a lesson at the time that Augusta is displaying the graphs of two functions within the same coordinate system. A special feature of her display is that she has not included any numbers and she designed the functions so that their graphs were unlike anything the students might recognize and be able to name. Her reason for doing this is that she had discovered in the past that when she placed

[3] The "+" in "$(f+g)$" does not mean the same thing as "+" in "$f(x)+g(x)$". The first instance of "+" is part of the function's name; the second instance is the arithmetic operation of addition.

numbers on the axes, students tried to estimate points' coordinates and add them numerically to get a value of the sum function. They then used that number to plot a point, again with great concern for accuracy. When she included numbered axes, students became bogged down trying to be highly accurate and they also often made addition errors. In the process of all this focus on accuracy, they lost the image of combining the values of two functions to get the value of a third.

To draw students' attention away from numbers and accurate placement of points, Augusta gave students blank straightedges (rulers with no markings). She showed them how to use the rulers to estimate the functions' values simply as magnitudes, and to imagine the value of the sum as putting one magnitude on top of the other. Excerpt 6 picks up Augusta's lesson after she has estimated the value of the sum function at several places along the horizontal axis. Students have a copy of the displayed graph and are attempting to replicate Augusta's placement of points on the sum's graph.

Excerpt 6: Augusta Attempts to Convey Meaning of "Sum of Two Functions." A: Augusta; S: Student

1. A: Let's go forward some more. I don't know how much more, but go forward some more (see Fig. 4.4a). Uhhhhhm. Again, you can use your ruler to help you estimate. How positive is function A?
2. Ss: It's positive.
3. A: It's positive. Is it very positive?
4. Ss: A little bit.
5. A: Yeah maybe. It depends on how you scale it. But, it's about // we can pinch it that much positive (see Fig. 4.4b). So you guys, on your scales, on your graphs can pinch off just how positive the A value, the A function is. (She waits for students to "pinch off" the value of A.) What about the B function?
6. S: It's positive.

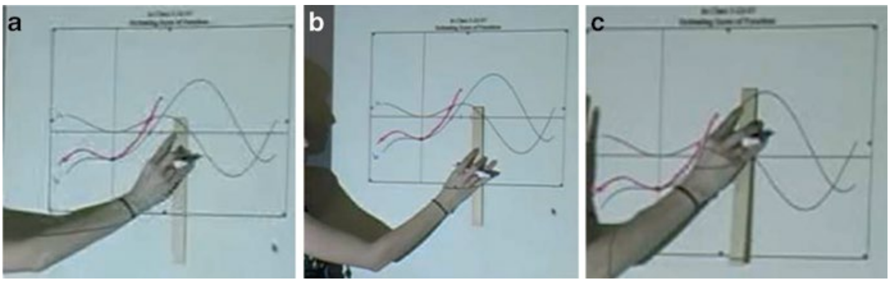


Fig. 4.4 Augusta indicates (**a**) a value in the domain of both functions, (**b**) the value of one function, and (**c**) that value added to the value of the other function

7. A: It's also positive. How can I show with my ruler how I'm going to add this to the value of the B function?
8. A: So that's how positive A is. Now I need to add that to the value of B. So how so? What do I want to do with this length? To show the sum? // How do you show adding with two lengths?
9. S: Mark another one.
10. A: How, next to it?
11. S: No, down farther.
12. A: Yeah! Right on top of it! So, if this is how much positive my A function is, and that's how positive the B function is, I'm going to take this and … add it! You can literally think of stacking it. So here, that's maybe how positive the value of A is. That's perhaps how positive the value of B is. So their sum? How would you show it?
13. S: It's bigger.
14. A: How much bigger?
15. S: Add ′em.
16. A: That on top of that. Exactly! It's that much bigger. So you're *stacking* these magnitudes now, because they're positive. You're literally stacking the lengths that you're estimating, because they're positive. So you can still use your ruler to help you pinch, so that's how positive A is, and that's how positive B is. So *stack* it, and you are actually up … about here (see Fig. 4.4c).

In Excerpt 6 we see Augusta employing covariation ("Let's move forward a little bit," Line 1) and thinking in magnitudes ("Pinch off just how much positive it is," Line 5) and thinking of combining magnitudes ("you're literally stacking the lengths," Line 16).

After Excerpt 6, Augusta turned responsibility over to the students to complete sketching the sum function's graph. Her motive for asking students to complete the sketch was that they create the value of the sum function as the result of an action of combining. She wanted students to develop what Dubinsky and Harel (1992) called an *action conception* of a sum function—the image of actually combining the function's values. Students' action conception of a sum function prepared them to develop later what Dubinsky and Harel called a *process conception* of a sum function—the ability to envision the action of summing immediately, focusing on the outcome of the action.

Excerpt 7 captures an interaction between Augusta and a student as he attempts to complete the sketch. Prior to this excerpt, Augusta and the student had a somewhat rambling conversation in which the student expressed his confusion about where to look for the functions' values ("there aren't any numbers") and how to think about adding them.[4]

[4]Part of Augusta's management of this conversation was to anticipate the difficulties students would experience making sense of what she demonstrated during the whole-class discussion of combining function's magnitudes. She anticipated that they would find it odd not to have numbers. Thus, she was not surprised at the student's comment.

Excerpt 7: Augusta Discusses Worksheet with Student Who Is Having Difficulty

1. A: Where are you looking? Maybe around here somewhere?
2. S: Yeah, down here.
3. A: Pinch off how negative the negative function is.
4. S: That right there.
5. A: That much. How much will the positive lift it?
6. S: It will get lower, won't it? (Appearing to look at the positive function.) Because … less.
7. A: It will get less negative (looking at the negative function). Yeah, exactly. Right now it is this negative. But since you are adding a positive to it, it will be less negative. How much less negative?
8. S: This much (see Fig. 4.5a).
9. A: That much. So keep your finger where how negative it is. And then, lift with me, keep your finger on it, it will get lifted … that much. That much. I mean, this is estimation.
10. S: So this (the value on the negative graph) goes up higher!
11. A: Yeah! It used to be *that* negative, but it will get lifted that much. So take that negative value, and lift it … that much (see Fig. 4.5b).
12. S: Oh, I get it.

Augusta's language and actions while speaking with this student emphasized building an image of "stacking" function values, the same way of thinking she attempted to convey during the immediately prior whole-class discussion. The payoff of Augusta's emphasis on having students solidify the action of combining two functions came in subsequent lessons. She asked students to imagine the location of points on the graph of a sum of two functions whose graphs were displayed simultaneously as she steadily moved her finger along the horizontal axis. She asked

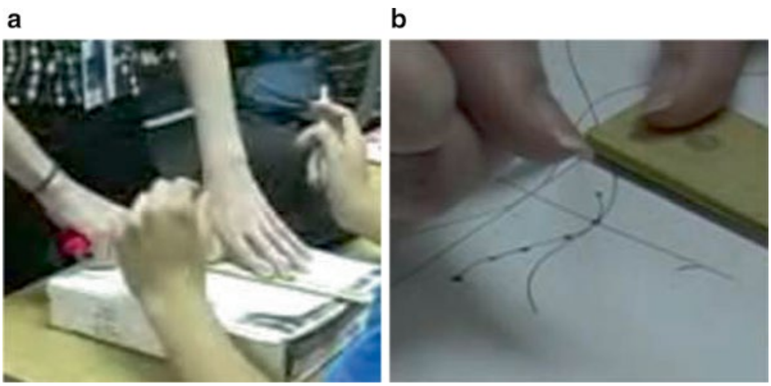


Fig. 4.5 Augusta and student discuss how to interpret "sum of functions" at a value of the domain

students to imagine the sum's graph "evolving" simultaneously with running through values in the addend functions' domain. Students' eventual ability to rapidly anticipate a visual estimation of the sum functions' values were expressions of their process conception of a sum function and gave them opportunities to solidify an understanding of a function that is the sum of two functions. It paid off further when Augusta came to polynomial functions, where she asked students to envision the behavior of the sum of two or more monomial functions given their prior knowledge of the monomials' graphs.

This example from Augusta's class focused on her attempt to create a meaning for a function that is the sum of two functions. I should point out that Augusta's lesson, which emphasized imagistic meaning, also reflects her year-long struggle with de-emphasizing talk about "what to do." We often discussed the value of stepping back and talking with students about what she intended that they create and, once created, what they had created and what it meant.

I would be remiss not to point out that the meaning of function sum was just one part of a larger scheme that Augusta intended that students build. That scheme entailed their prior work on understanding functions defined as a product of factors and an understanding of producing an equivalent representation by using the distributive property of multiplication over addition. Put more broadly, Augusta's intent was that students see a function's graph as invariant across representations of the function, and to build meaning within each representation by focusing on schemes for imagining a function's behavior. Her long-term instructional design was attentive to what Lehrer, Schauble, Carpenter, and Penner (2000) described as the inseparable, interrelated development of inscription and meaning. In the case of polynomial functions, she aimed to develop a scheme of meanings that entailed students' abilities to transform one symbolic representation of a polynomial function into other symbolic representations, and that they take for granted that there was something called "the function" (a relationship expressed as a graph) that remained the same. Augusta's approach to having one meaning be invariant across representations of a polynomial goes beyond the issue I raised in Thompson (1994b), where I questioned what was then called the "multiple representations" movement.

> I believe that the idea of multiple representations, as currently construed, has not been carefully thought out, and the primary construct needing explication is the very idea of representation. Tables, graphs, and expressions might be multiple representations of functions to us, but I have seen no evidence that they are multiple representations of anything to students. In fact, I am now unconvinced that they are multiple representations even to us, but instead may be, as Moschkovich, Schoenfeld, and Arcavi (1993) have said, areas of representational activity among which we have built rich and varied connections. It could well be a fiction that there is any interior to our network of connections, that our sense of "common referent" among tables, expressions, and graphs is just an expression of our sense, developed over many experiences, that we can move from one type of representational activity to another, keeping a current situation somehow intact. Put another way, the core concept of "function" is not represented by any of what are commonly called the multiple representations of function, but instead our making connections among representational activities produces a subjective sense of invariance.

> I do not make these statements idly, as I was one to jump on the multiple- representations bandwagon early on (Thompson, 1987, 1989), and I am now saying that I was mistaken. I agree with Kaput (1993) that it may be wrongheaded to focus on graphs, expressions, or tables as representations of function, but instead focus on them as representations of something that, from the students' perspective, is *representable*, such as some aspect of a specific situation. The key issue then becomes twofold: (1) To find situations that are sufficiently propitious for engendering multitudes of representational activity and (2) Orient students to draw connections among their representational activities in regard to the situation that engendered them. (Thompson, 1994b, pp. 39–40)

Augusta went beyond the concern I raised in 1994 by first addressing it squarely (developing students' meanings for each form of expression in terms of ways to read it for information about joint variation) and then raising the question of how you could change to another form of expression while retaining the information students discerned originally. In this way, she helped students develop a "subjective sense of invariance" while moving from one representation of polynomial function to another, taking the graph of a function as the "most basic" representation of it.

Lastly, Augusta supported her class conversations with specially designed didactic objects (Thompson, 2002)—displays, diagrams, graphs, mathematical expressions, or class activities that she designed conscientiously to support specific reflective conversations she intended to have with students. For example, the graphs that she used during the function-stacking activity had blank axes and unfamiliar shapes. This design feature enabled Augusta to focus students' attention on function's magnitudes at a common value of their domains instead of on points' coordinates.

Absence of Meaning in Mathematics Education

The preponderance of research on learning mathematics in the United States suggests that my examples of meaningless learning and teaching are far from uncommon and that meaningful instruction is rare. One study in particular stands out—the TIMSS eighth-grade video study (Stigler, Gonzales, Kawanaka, Knoll, & Serrano, 1999; Stigler & Hiebert, 1999). They formed nationally representative samples consisting of 81 U.S. classrooms, 50 Japanese classrooms, and 100 German classrooms. As part of this study a team of U.S. mathematicians and mathematics educators examined the lessons (which were blinded for national identity) with regard to the quality of the lessons:

> They based their judgments on a detailed written description of the content that was altered for each lesson to disguise the country of origin (deleting, for example, references to currency). They completed a number of in-depth analyses, the simplest of which involved making global judgments of the quality of each lesson's content on a three-point scale (Low, Medium, High). Quality was judged according to several criteria, including the coherence of the mathematical concepts across different parts of the lesson, and the degree to which deductive reasoning was included. Whereas 39 percent of the Japanese lessons and 28 percent of the German ones received the highest rating, none of the U.S. lessons received the highest rating. Eighty-nine percent of U.S. lessons received the lowest rating, compared with 11 percent of Japanese lessons. (Stigler et al., 1999, p. iv)

The rarity of meaningful, coherent mathematics instruction in the United States—instruction that aims to develop students' mathematical thinking in the sense of Dewey—is very troubling. The rarity with which popular textbooks, both elementary and secondary, and both traditional and reform, attempt to develop mathematics as a coherent system of meanings is also troubling.

What I find more troubling is the rarity of research in mathematics education that takes the issue of mathematical meaning seriously. Research that is ostensibly on knowing or understanding, whether the context is teaching or learning, too often examines performance instead of clarifying the meanings students or teachers have when they perform correctly or the meanings they are working from when they fail to perform correctly. Neither correct performance nor incorrect performance says anything about the nature of a person's system of meanings that expresses itself therein. This is not to say that *no* research considers students' or teachers' meanings. Rather, it is too rare.

Some publications fail to address the issue of meaning even when their titles say it is about meaning. The chapters in Kilpatrick, Hoyles, Skovsmose, and Valero (2005) discuss the many ways that "meaning" is used in mathematics education, but they do not explicate a meaning of "meaning" that does work for designing curriculum or instruction that will improve mathematics learning. Kieran's (2007) review of research on learning and teaching algebra is a case in point. Its subtitle is, "Building meaning for symbols and their manipulation." The article is an astonishing piece of scholarship in the scope of the research it reviews, but by the criteria I've set in this chapter, it fails to say what Kieran or any of the articles she reviews take "meaning" to mean, and the article gives few examples of anyone's thinking that might constitute a meaning for symbols or their manipulation. Moreover, the article is devoid of references to research on quantitative reasoning as a source of meaning for arithmetic and algebra, and its review of research on function completely misses the research on ways of thinking that might constitute various understandings of function. Instead, it focuses on evidence that students find the concept of function, whatever it is, difficult.

Research on calculus learning is another case in point. Research on students' understanding of the derivative (e.g., Clark et al., 1997; Ferrini-Mundy & Gauadard, 1992; Ferrini-Mundy & Graham, 1994; Heid, 1988; Machín, Rivero, & Santos-Trigo, 2010; Orton, 1983; Sofronos & DeFranco, 2010; White & Mitchelmore, 1996) takes "slope of secant" as a primary meaning of average rate of change (the other is the computation $\Delta y/\Delta x$) and takes "slope of tangent" as a primary meaning of instantaneous rate of change (the other is the limit of average rates of change, where average rate of change is defined as slope of a secant). I am puzzled by the approach of taking "slope of secant" and "slope of tangent" as fundamental meanings for average rate of change and instantaneous rate of change, respectively. Secants and tangents are lines. They are geometric objects. I can easily imagine a thoughtful student asking, for example, "What do lines have to do with speed?" Clearly, there is a complex system of meanings behind thinking of a secant as somehow embodying an average rate of change, and there is an even more complex system of meanings behind taking a tangent as somehow embodying an instantaneous rate of change. I outlined part of that system earlier, when I spoke of a rate of change scheme. However, none of these studies explicates such a system of meanings. Hence they do not investigate them.

Unfortunately, when researchers treat meanings for slope (whose computation students often take as an index of "slantiness"), secant (which students often think of as a piece of wire that is laid across a graph), and tangent (which students often think of as a line that "just touches" a curve) as *primary* meanings, not as emergent meanings, they cannot understand the sources of students' success or failure to learn. Hackworth (1994) drove this point home. She studied 90 first-semester calculus students' understandings of rate of change. Her question was, "What have calculus students, after studying differentiation and derivatives, learned about rate of change?" By her measures, they learned nothing about rate of change. In some instances students understood more about rate of change before receiving instruction than they did after the course.

Carla Stroud (2010), in a follow-up to Hackworth's (1994) study, interviewed 15 students in Calculus 2 and Calculus 3 about their meaning of instantaneous speed. One question was this:

> When the Discovery space shuttle is launched, its speed increases continually until its booster engines separate from the shuttle. During the time it is continually speeding up, the shuttle is *never* moving at a constant speed. What, then, would it mean to say that at precisely 2.15823 s after launch the shuttle is traveling at precisely 183.8964 miles per hour? (Hackworth, 1994, p. 108)

Consistent with Hackworth's (1994) findings, the primary meaning held by students in Stroud's study was that of a speedometer. The space shuttle's instantaneous speed 2.15823 s after launch is whatever number its speedometer points at. There are two problems with this way of thinking: (1) the space shuttle doesn't have a speedometer, and (2) even if it did, what about the speedometer's design guarantees that it is pointing at the correct number? Some students had a backup way of thinking—you would take the limit of the space shuttle's average speed over smaller and smaller intervals or you would simply differentiate the shuttle's position function. Carla asked, "And how would you do that?" The students presumed that there was some function they could act upon symbolically—and the shuttle's speed would pop out of that.

The area of mathematics education research that is most wanting today regarding attention to meaning is research on teachers' mathematical knowledge for teaching (MKT). First, the verb "to know" is used in this research as a primitive, undefined term. The question of what "to know" means in regard to knowing mathematics is unaddressed. Second, this area is quite taken with the idea that teachers' knowledge, whatever that means, can be categorized (Ferrini-Mundy, Floden, McCrory, Burrill, & Sandow, 2005; Hill, 2010; Hill et al., 2008). I suspect that the desire to create instruments to assess teachers' knowledge is the driving force behind this focus. Item specifications need categories. When you categorize a teacher's knowledge based on an answer to an item, however, your attention is necessarily drawn away from the system of meanings by which the teacher was operating. Assessments that do not address teachers' meanings can be summative, but they cannot be diagnostic. I'll illustrate this point with an example.

Before I can share this example I must review the idea of continuous variation. Castillo-Garsow (2010) identified two ways, in principle, that one can think about continuous variation, what he called "chunky" and "smooth." A conception of continuous variation that is *chunky* is one where someone thinks of a variable varying in discretely continuous amounts. By "discretely continuous" I mean that they imagine that the variable varies, but they imagine "next" values and mentally connect the values. The variation comes in one chunk between current and next values. The value of x goes directly from initial to end without passing through the values in between. The values in between current and next values are "there," but the person imagining the variation does not imagine passing through them. A conception of a variable varying smoothly is recursive. One might imagine a "next" value, but does so with the anticipation that the variable varies smoothly between current and next value by varying smoothly between values that exist between current and next (Thompson, 2011).

A ninth-grade algebra teacher, Sandra, was in the midst of teaching a lesson on the point–slope and point–point formulas. She was attempting to use a method that she had just learned which takes a rate-of-change approach to the point–slope formula. The method works like this: Suppose a function has a constant rate of change r. You start by assuring that students have an appropriate meaning of constant rate of change, such as "r is the constant rate of change of y with respect to x" means that however much x changes, y changes r times as much.[5] With this meaning in hand, if you know that a function with a constant rate of change of 1.7 passes through the point (3, 9), then if you decrease the value of x by 3 (i.e., increase it by −3), the function's value will change by 1.7 times −3. Thus, the value of the function at $x=0$ is $9+(1.7)(-3)$. The function's definition is therefore $y=1.7x+\left(9+\left(1.7\right)\left(-3\right)\right)$. The two-point method follows as a corollary by determining the function's average rate of change between two points and realizing that you now have a situation where a function has a known constant rate of change and its graph passes through a known point. Sandra was excited to try this method with her class.

Sandra worked through several examples using this method to find a function definition when given one point and a rate of change. Things fell apart, though, when she moved to the case of having two points that the function's graph passes through. Excerpt 8 picks up as she discusses the two-point case.

Excerpt 8: Sandra Discusses the Two-Point Case

1. S: (Plots the points (3,1) and (7,4) in a coordinate system on the board.) Now we'll look at something that is a little bit different. Now all we're given is two

[5]The phrase "assure that students have" can be misleading. It does not mean "teach this idea in the 5 min before the point–slope lesson." Rather, it means to assure that this meaning of constant rate of change has been the target of instruction over a long period of time, long enough so that students have this meaning and all its entailments.

points, and we're supposed to find the equation for the line that goes through them. Any ideas?
2. (Silence)
3. S: Well, let's notice something. This function goes over 4 and up 3 (sketches segments). So if we do the same thing as before and move x back to 0 we'll know what the y intercept is! So if we go 4 to the left (draws a horizontal segment of length 4 to the left from (3,1); see Fig. 4.6).
4. S: (Long pause) We'll pick this up tomorrow. (Pause) Here are some practice problems. Do just the ones with one point.

Though Sandra's difficulty actually began in Line 3, where she described the change as "over 4 and up 3," her entire difficulty resided in her schemes for variation, slope, division, and rate of change.[6] First, she saw the change in x as a chunk. This was unproblematic in the case of one point. However, her chunk in this problem did not place her at $x=0$ as she wished. Second, her meaning for slope was "rise over run," where rise and run were both chunks. Third, her computation of slope, not evident in this excerpt but made clear later, was of a procedure that produced a number that is an index of a line's "slantiness." Division did not produce a quotient that has the meaning that the dividend is so many times as large as the divisor—3/4 as a slope was not a number that gave a rate of change. It gave a "slantiness." Fourth, her meaning for rate of change entailed neither smooth variation nor proportionality. It was more akin to her meaning of slope—two things changing in chunks. These meanings not only failed to provide Sandra a connection between her current setting (two points) and prior method, but they also led her down the dead-end path

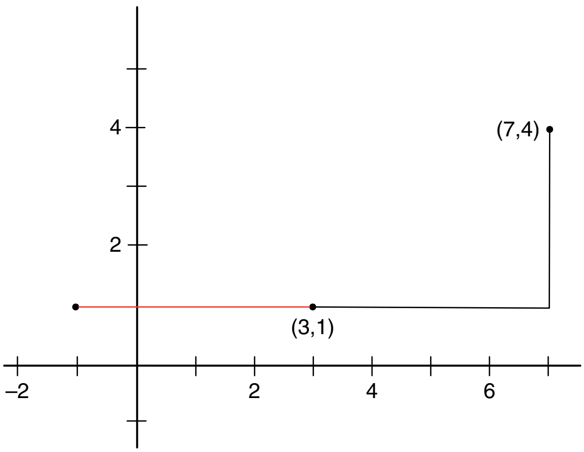


Fig. 4.6 Sandra's boardwork while working the two-point problem

[6]It is important to notice that I said *schemes*. Sandra's meanings for variation, slope, division, and rate of change did not exist within a single scheme. They were unrelated.

she followed. Had Sandra reasoned proportionally and with smooth continuous variation, she might have said "… over 3/4 of 4 and up 3/4 of 3." That would have given her the graph's *y*-intercept.

I fail to see how categorizing Sandra's knowledge would enhance our understanding of why her lesson fell apart. With our above understanding of what Sandra knew (i.e., the meanings from which she operated) we are positioned to help her improve. Putting her knowledge in categories like "curricular knowledge," "common mathematical knowledge," or "specialized mathematical knowledge" serves no practical purpose except to see whether her score on a test meets a benchmark. I feel strongly that assessments of MKT must be rooted in developmental theories of MKT. Otherwise, despite being ostensibly rooted in the work teachers do, the assessments will have little explanatory power with regard to *why* teachers do what they do and will have little usefulness in helping them improve what they do.

I propose that we develop a new type of assessment, aimed at assessing teachers' mathematical meanings for teaching (MMT). The enterprise of developing such assessments might redirect the field's attention to the subtle, yet foundational, role that meanings play in what teachers and students do. It might also redirect the field's attention towards an emphasis on explicating desirable, powerful systems of meanings that we feel students should develop. Lest I be misinterpreted, I hasten to add that the issue of skill would still be paramount. But our conception of skilled performance would change. Our descriptions of students' skilled performance would necessarily entail our intention that it be evidence that they have built powerful, rich, integrated systems of mathematical meanings.

A focus on MMT would also foster the field's conceptualization of bridges among what teachers know (as a system of meanings), how they teach (their orientation to high-quality conversations), what they teach (the meanings that an observer can reasonably imagine that students might construct, over time, from teachers' actions), and what students learn (the meanings they construct).

Assessments of MMT would be more diagnostic than current assessments of MKT. Information from them would be useful for teachers' professional development. I imagine that such instruments would also be useful in designing professional development aimed at improving teachers' ability to help their students learn. Sample items from an assessment of MMT might alert teachers to ways of understanding the ideas they teach that are expected of them and of their students. For example, can teachers explain a meaning of division that gives a meaning of quotient? Are they *inclined* to teach a meaning of division that gives a meaning of quotient? Can they explain that 8 divided by 5 equaling 1.6 means that 8 is 1.6 times as large as 5? Do they think it is important for students to know this? Do they have a coherent system of meanings of multiplication, division, and fractions that allows them to explain that $43 \times 18 = 774$ means, at once, that 774 is 18 times as large as 43, that 774 is 43 times as large as 18, that 43 is 1/18 of 774, and that 18 is 1/43 of 774? Are they *inclined* to explain those meanings? Do they think these are important meanings for students to have? Are they inclined to ask students questions that force those connections?

An Agenda for Change

My intention in this chapter was to convey the nature of meaning as it relates to mathematics education and the importance of taking meaning as a foundational consideration in mathematics learning, teaching, and instructional design.[7] How, though, might we as a nation bring about changes that resolve the lack of meaningful mathematics in schools and colleges? I draw inspiration from Tucker (2011) to answer how we might move forward with such an agenda for change. Tucker examined the educational policies of Ontario, Finland, Japan, Shanghai, and Singapore to see what policies they either have in place to sustain an excellent educational system or put in place to pull themselves to a level of internationally elite educational systems. He pointed to five areas of policy that are central to elite systems attaining and sustaining excellence, and he turned each into a set of recommendations to be implemented at the state level. His recommendations are as follows: Benchmark the best, design for quality, design for equity, design for productivity, design for coherence. These categories serve well as organizers for thinking about how to make meaning central to mathematics education.

Benchmark the Best

Tucker pointed out that prior to World War II, the United States borrowed ideas and practices from other countries at a rapid rate, but after World War II we seemed to think that no one had anything to offer. Recently, there have been several efforts to benchmark international standards (e.g., National Mathematics Advisory Panel, 2008). However, benchmarking standards is like surveying a landscape from 50,000 ft. You might see broad outlines, but you have no sense of the details by which things are made to happen. Elsewhere (Thompson, 2008a), I stated that the National Mathematics Panel Report recommendations read like a table of contents. The Panel did not attend to what it might mean to understand the things in their lists. By that I meant that they paid little heed to how other countries actually implemented their standards and that the Panel ignored the idea that attending carefully to issues of meaning was one way that elite countries attained excellence in mathematics education.

[7] I did not emphasize issues of curriculum. I agree wholeheartedly with Marilyn Carlson et al. (2010), who has argued convincingly that a well-designed curriculum will play a central role in supporting teachers' reconceptualization of the mathematics they teach and will be an essential component in efforts to make meaning central to teaching and learning. It is my experience, however, that teachers' meanings trump curriculum, so I have emphasized teachers' meanings for the purposes of this chapter. On the other hand, there is a large intersection between issues of instructional design and issues of curriculum, so I have not ignored curricular issues entirely.

Funding agencies should commission studies to benchmark the systems of mathematical meanings towards which elite systems strive. They should also document how those meanings are achieved and the consequences of achieving them. For example,

- Singapore elementary education targets a deep understanding of speed as rate of change in grades 1–5 (as does Russia). This deep understanding entails ideas of variation, covariation, and proportionality. The Singapore curriculum outline does not state this specifically, but if you examine their texts and instructional guides it leaps at you. Their early attention to speed is later leveraged in developing students' understanding of variable and linear function. What other meanings does Singapore target, how does it build them, and how does it leverage those meanings in students' later learning?
- Japanese elementary education emphasizes whole-number numeration as a systematic way to represent numerical value—to a far great extent than in the United States. In Japan, numerical algorithms arise out of a system of meanings that constitute an understanding of place value. They are not taught as meaningless, memorized notational procedures. What other meanings does Japan target, how does it build them, and how does it leverage those meanings in students' later learning?
- Russian elementary education emphasizes quantity and measurement (as, to a lesser extent, do Singapore and Japan). A deep understanding of measurement entails understanding ratio and proportion. Russians leverage this early learning by intermingling it with the idea of generalization, which necessitates ideas of representation and representational equivalence. What other meanings does Russia target, how does it build them, and how does it leverage those meanings in students' later learning?

Design for Quality

Tucker's (2011) first bullet in this section is, "Get your goals clear, and get public and professional consensus on them" (p. 5). This feat will not be easily accomplished with regard to targeted systems of mathematical meanings, but it is essential. National funding agencies will play an essential role in the effort to clarify systems of meanings, and how they might be expressed skillfully, that mathematics education should take as its primary goals.

The clarification of goals will also address the matter of coherence in the mathematics curriculum. A number of studies have stated boldly that the typical judgment of U.S. mathematics curricula at all levels is that they are conceptually incoherent (Cai, 2010; Oehrtman, Carlson, & Thompson, 2008; Schmidt et al., 2002, 2005; Thompson, 2008b; Thompson, Carlson, & Silverman, 2007). A focus on developing coherent meanings will not guarantee coherent curricula, but it surely will increase the likelihood that a curriculum designed to support students' development of a coherent system of meanings will be coherent.

Another aspect of designing for quality is that targeted meanings must be worth having. We must be able to argue that having them will pay off in important ways, either in preparation for life or in preparation for future learning. The arguments must be specific in regard to *how* having them will be important. Research will play an essential role in the quest to design for quality, because we often realize the intricacies of a targeted meaning's "payoff," or lack thereof, only in attempting to help students develop it. Research will also play an essential role in identifying and characterizing important meanings that students and teachers should have, and conveying those meanings to parties who can use that information. (Please understand that I use "convey" in the sense that I've described repeatedly in this chapter.)

Design for Equity

Tucker's (2011) emphasis on equity is largely in regard to allocation of resources. He argues that school systems and students should get resources according to their need. Hardest-to-educate students should receive sufficient resources necessary to enable them to attain the high standards set in the quest for quality already described.

With regard to issues of meaning, hardest-to-educate students will be those who are farthest from developing the meanings that we decide are essential. Designing for equity with regard to meaning requires that we identify ways of thinking and systems of meaning that are highly obstructive for constructing the meanings we intend, and then investigating means of support to effect change most efficiently. Just as we do not give steak to someone who is malnourished, we cannot expect someone who reasons additively to participate productively in instruction on making multiplicative comparisons as a foundation for reasoning proportionally.

Design for Productivity

Tucker (2011) characterized designing for productivity in terms of making frequent and timely checks for quality. With regard to a focus on meanings, this translates into helping teachers attend to the meanings that students are actually constructing and adjust instruction appropriately. Research will play a central role in identifying effective ways that teachers can do this and effective ways to help teachers do this. I am not speaking of grand assessment strategies or high-stakes tests, though I anticipate that we will continue to have them—and that their character will change. Rather, the road to students building powerful, coherent, and useful systems of mathematical meaning will be built upon teachers' ability to conduct constant, formative assessments of students' learning.

Design for Coherence

The coherence Tucker (2011) had in mind was that efforts to address the prior four areas cohere, and that they complement and draw from each other. The same is surely true for mathematics education. In regard to the issue of moving the field towards making meaning a primary concern, I see several additional ways in which we must design for coherence. The obvious one is designing coherent systems of meanings that we target for student learning. We also must coordinate efforts in instructional design, teacher preparation, and professional development around those meanings. We not only want school students to develop coherent meanings for arithmetic operations, but we also want future teachers to be prepared to convey and assess them and professional development programs that support continued teacher growth.

It is on the criterion of coherence that Tucker found the greatest strength in elite systems and the greatest fault in the U.S. system. Culture surely plays a large role in both cases. Cultures change over time, but they rarely change abruptly. They continually regenerate themselves through intersubjective operations among their participants. An educational system is slow to change for the same reasons—entering teachers have images of mathematical teaching and learning that they formed as students. Lortie (1975) noted this when explaining why U.S. instruction seemed to change so little—adults who choose to enter teaching developed a deep resonance with their experience of schooling as students. We can leverage Lortie's observation to gain insight into different educational systems' clear differences in teachers' and textbooks' orientations to meaning. Ma (1999) observed Chinese elementary school teachers with the equivalent of high school education displaying what she called profound understandings of the mathematics they taught. The U.S. elementary teachers in her study, with college degrees, displayed superficial and fragmented understandings of the mathematics they taught. Why the difference? Because, I suspect, the Chinese teachers developed meanings while school students that they later refined as prospective and practicing teachers, while the U.S. teachers developed unproductive meanings while students—meanings that served them poorly as a foundation for conveying coherent mathematical understandings to their students. Coherence, and incoherence, is inherited as much as it is produced.

Tucker (2011) ended his report by noting that each of these elite educational systems took from 30 to 100 years to transform themselves and that the United States should not expect less. Also, he noted that each reform effort involved a high degree of experimentation within its policy frameworks without losing sight of its goals. We must take Tucker's observation to heart—expect that it will take 30–100 years of concerted, purposeful effort to transform the U.S. educational system so that coherent meanings are at the core of mathematics teaching, learning, and curriculum. At the same time, we cannot expect to succeed without striving to develop a clear vision of what we mean that meaning be at the core of mathematics teaching, learning, and curriculum.

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Chapter 5
The Need for Theories of Conceptual Learning and Teaching of Mathematics

Martin A. Simon

Abstract One of the important challenges in mathematics education is the development of pedagogical theory that can guide mathematics instruction, instructional design, teacher education, and research in each of these areas. In this chapter, I begin with a brief review of the current state of the field with respect to pedagogical theory. I then offer a vision of what might constitute needed pedagogical theory. Such theory would include useful ways of describing the mechanisms of mathematics conceptual learning and, derived from the characterization of these mechanisms, design and instructional principles for fostering mathematics conceptual learning. Using the research program I am involved in as an example, I describe some of the methodological challenges in producing an empirical basis for such theory development.

One of the most important challenges in mathematics education today is the challenge to produce better theories of mathematics pedagogy. I use the term *mathematics pedagogy* rather than mathematics *teaching* to be inclusive of all efforts to promote mathematics conceptual learning, including teaching, curriculum development, and the development of tools for learning mathematics. In this chapter, I discuss a vision for theories of conceptual learning and teaching of mathematics (theories of CLTM), theories that are each made up of a theory of mathematics conceptual learning *and* a theory of instruction that builds on and is integrated with the theory of learning. A theory of CLTM should be based on an understanding of students' conceptual learning processes and provide a framework for the design and

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implementation of instruction that harnesses students' learning processes to promote the learning of particular mathematical concepts. These ideas will be elaborated throughout the chapter.

I begin this discussion with a look at the current situation in the United States, a situation that is shared to a large extent by many other countries. This is followed by brief comments about some of the Asian countries that score highest on international comparisons. I then consider two theoretical frameworks that are more developed than others, the French Theory of Didactical Situations and the Dutch Realistic Mathematics Education. Finally, I go into greater depth on a vision of a theory of CLTM and describe the direction of our current work in this area.

The Need for Pedagogical Theory in the United States and Beyond[1]

I will discuss the current situation from both the perspective of practice and the perspective of research. Traditionally, teaching and curriculum development were based on showing and telling students what they were to learn. The assumption, often implicit, was that motivated students would take in the knowledge shared by the teacher and incorporate it into their mathematical knowledge, perhaps not after one teaching session, but certainly with sufficient repetition and practice. Not only was the show-and-tell approach clear, it was in alignment with people's natural instincts about how to "share" knowledge. The model was fundamentally unproblematic. Whereas some teachers and some curricula were judged to be more effective in carrying out this approach, and some lessons were more successful than others, the approach itself was not broadly challenged until relatively recently. Although this approach is still probably the most frequently used, it has been discredited as a primary approach to mathematics instruction, because the results have not been good. With the loss of confidence in direct instruction came a loss of clarity about the teacher's role in promoting student learning.

In Teaching

Through recent reform efforts, teachers have internalized a message of "minimize showing and telling." However, they are uncertain about what should replace this traditional approach to instruction. Following are descriptions of some common ways teachers are meeting the mandates of reform. These approaches are not mutually exclusive.

[1] The chapter is written from my perspective as a U.S. mathematics educator. I try to accurately represent the contributions from other countries, although my perspective is as an outsider, sometimes quite distant from the work going on in these countries.

An Initial Attempt at Reform: The Telling–Asking Continuum

Many teachers in the United States are attempting to participate in current reforms by moving away from a direct teaching model. However, often the change is in teacher behavior and not in underlying perspectives (Cohen & Ball, 1990). Rather than telling or showing a new mathematical idea, these teachers often ask their students for the idea. This approach engenders conceptual development occasionally for individual students, if the students are at the point of formulating the idea asked for, and they can do so in response to the teacher's question. Of course, many students in the class are not yet at that point and are therefore unable to answer the teacher's questions based on their current understandings. The limited value of this teaching strategy is often masked by the presence in the classroom of students who already have the knowledge asked for by the teacher. These students take on the telling role that would be filled by a teacher using a direct-instruction approach.

I consider this initial attempt at creating an alternative to direct instruction as existing along a continuum bounded on one end by telling students the mathematics they are to learn and at the other end by asking students for the mathematics they are to learn. The teacher endeavors to keep their teaching towards the "asking" end of the continuum.[2] When none of the students can answer the teacher's question, we often see the teacher move back towards the middle of the continuum, asking leading questions and supplying hints. This becomes the compromise position. The teacher has avoided telling yet accomplished the results of telling—getting the students to say what the teacher is refraining from saying. However, neither the teacher's use of leading questions and hints nor the reliance on an advanced student telling the other students the mathematics to be learned represents a theoretical alternative to teacher telling.

Two factors combined have tended to encourage teachers to move away from telling and to gravitate towards asking students for the mathematics: the discrediting of direct instruction (identified earlier) and an interpretation of constructivism as "let students construct (come up with) their own ideas." The latter interpretation has led to misconceptions characterized by Cobb, Yackel, and Wood (1992) as "romanticism," an unrealistic view in which "to construct [would mean] to learn by spontaneously apprehending fixed mathematical relationships without the teacher's [or curriculum's] guidance" (p. 28). Constructivism points to the active role of the learner in learning, but does not imply that students be left totally to their own devices to learn mathematics.

[2]The continuum is our conception and not how the teachers would describe what they do.

Focus on Classroom Structures and Teaching Strategies

A second way that teachers have dealt with the mandate for reform has been to focus on particular tools of instruction that have become popular in the last 20 years. These tools include the use of whole-class discussion, collaborative small groups, math journals, manipulatives, software environments, calculators, nonroutine problems, and probing questions. Whereas all of these tools can be used effectively to promote learning, and even unskilled use of these tools sometimes is advantageous for student learning, the use of tools without a theoretical framework for their use is of limited value. Teachers can regularly make use of each of these tools and not have any idea how to help students, who do not understand a concept, come to understand it.

A Perception-Based Perspective

Earlier I implied that the movement away from direct instruction created a void with respect to articulated theories of mathematics teaching. However, many teachers participating in reform (and who have moved beyond the telling–asking continuum described above) do not seem to experience such a void. This was puzzling for us until research that we were doing on teacher development provided a way to understand this observation (see Simon, Tzur, Heinz, Kinzel, & Smith, 2000, for a detailed description of this work).

In our investigation of the perspectives underlying the practice of elementary teachers engaged in the mathematics reform, we came to postulate a perspective we refer to as a "perception-based perspective." In describing this perspective, we characterized major organizing, but *implicit*, conceptions of these teachers. (It is not a description of how *they* would describe mathematics and mathematics learning.)

A perception-based perspective is based on three ideas:

1. Mathematics is an understandable, logical, and connected body of knowledge.
2. Optimal learning with understanding involves learners' first-hand perception of the mathematics and the connections among different aspects of the mathematics.
3. Mathematics exists independent of human activity and is available for all to perceive.

Teaching, based on a perception-based perspective, involves creating situations that provide the students an opportunity for first-hand perception of the mathematics. This is frequently accomplished by using tools of the reform (discussed earlier), including nonroutine problems, manipulatives, computer environments, and identification of patterns. From this perspective, these "tools" provide particular ways to give students access to the mathematics that they need to learn (perceive). Thus, teaching mathematics involves choosing representations and/or problem situations in which the mathematics is "transparent," so students can perceive the mathematics first hand. Because mathematics is seen as existing independent of human activity and universally available to be seen, teachers consider the mathematical ideas to be *transparent* when they perceive them to be clear.

Although limited, teaching based on a perception-based perspective offers several advantages over traditional direct instruction. First, the students have more freedom to think and communicate (in contrast with direct instruction), and thus, more chance to engage their prior knowledge. Second, with the introduction of computer environments and manipulatives, students engage with a richer set of mathematical representations on a regular basis. Third, intermittently, and somewhat by chance, instructional tasks that are aimed at perception of *transparent* relationships lead to cognitive processes that actually lead to new conceptualizations.

The principal limitation of a perception-based perspective is its inability to inform a response to the situation in which learners do not perceive what the teacher anticipates they will perceive. It is common for students not to see what the teacher sees. For example, they may not see place value in a set of base-ten blocks or the structure of multiplication and division in an array. Students' inability to see the mathematics in question can be explained using Piaget's (1970) construct of *assimilation*. The mathematical relationships that are perceived are those that learners currently understand (can assimilate). Thus, learners who have a concept of place value organize the visual input from base-ten blocks according to the base-ten relationships they understand. Learners who have no base-ten conceptions (in particular those who lack a conception of composite units) see blocks of different sizes, but may impose no base-ten relationships on them. The important question that drives from the construct of assimilation is the following: How can a teacher promote the construction of a new (for the learners) and more advanced conception given that the learners cannot observe the new concept prior to having that concept? It is this question that makes mathematics teaching problematic (and has been fundamental to our work, which I describe in a later section).

In Curriculum Development

Curriculum development in the United States generally suffers from a lack of theories of mathematics pedagogy. In the early 1990s the U.S. National Science Foundation supported several major curriculum development projects at the elementary, middle school, and high school levels. Many of the new curricula that resulted from this initiative, while offering some opportunities for learning not found in many traditional curricula, reflect the particular experience, knowledge, and intuitions of their various authors, rather than a coherent approach to building on students' learning processes. There has been no theory of mathematics teaching and learning to structure the design process. Curriculum developers refer to general theories of learning resulting in very general ideas with respect to mathematics instruction. For example,

> We are in general agreement with constructivist explanations of the ways that knowledge is developed, especially the social constructivist ideas about influence of discourse on learning. This position is reflected in the authors' decision to write materials that support student-centered investigation of mathematical problems and in our attempt to design problem content and formats that encourage student-student and student-teacher dialogue about the work. (Connected Mathematics Project, 2010, "Social Constructivism")

Similarly, the "design principles" identified in these curriculum projects tend to be general statements that represent stances taken by the designers rather than a framework that can inform the design of tasks, lessons, and units. For example, Fey and Hirsch (2007) reported, "Core-Plus Mathematics curriculum materials were based on an explicit intention to support problem-based, student-centered classroom activity shaped by current theory and research on teaching and learning" (p. 132). However, the principles they list are similar to those listed by the other curriculum projects and do not involve pedagogical theory as I have defined it:

- School mathematics is best learned and understood as an active science of patterns involving quantity and change, shape and motion, data and chance, and enumeration and algorithms (Steen, 1990).
- Any introduction of new mathematics will be most effective if the ideas and techniques appear in problem contexts that students can relate to and that connect to their prior knowledge. Authentic applied problems are especially useful contexts for learning, but significant pure mathematical problems are often useful also (Hiebert et al., 1996).
- Effective mathematics instruction frequently engages students in collaborative small-group investigations of problem situations that encourage student-to-student dialogue, followed by teacher-led whole-group summarizing activities that lead to analysis, abstraction, and further application of underlying mathematical ideas (Cobb, 1994; Davidson & Kroll, 1991). There is also some evidence that small-group collaborative learning encourages a range of social skills conducive to the learning styles of groups that are currently underrepresented in mathematics (Oakes, 1990).
- Students should be regularly involved in mathematical activities like searching for patterns, making and verifying conjectures, generalizing, applying, proving, and reflecting on the process (Fey & Hirsch, 2007, pp. 132–133; Freudenthal, 1983).

In addition to their general nature, some of these design principles point to aspects of theory that require further elaboration. For example, one of the "criteria for a mathematical task" in *Connected Mathematics* is "Investigating the problem should contribute to the conceptual development of important mathematical ideas" (Lappan & Phillips, 2009, p. 8). Specifically how problems can contribute to conceptual development is one of the key theoretical issues that must be addressed. One of the principles on which *Everyday Mathematics* was based raises similar questions:

> The K-6 curriculum should help children make transitions from intuition and concrete operations to abstractions and symbol processing skills while at the same time building new intuitions that will mature in the years beyond sixth grade. (Bell & Isaacs, 2007, p. 10)

Understanding the transitions and mechanisms that lead to abstractions and formal mathematics is key to ongoing efforts to develop effective curricula.

Russell (2007), in her discussion of *Investigations in Number*, *Data*, *and Space*, indicated a kind of concrete-to-abstract or informal-to-formal characterization of learning common to several of these curriculum projects. However such characterizations fall short of the pedagogical theory needed to guide curriculum development:

> Using representations and contexts to visualize mathematical relationships is an essential principle of Investigations. Students may first use representations or contexts concretely, drawing or modeling with materials. Later, they incorporate these representations and contexts into mental models that they can call on to visualize the structure of problems and their solutions. (p. 28)

Isaacs, Carroll, and Bell (2001) discussed Vygotskian theory as one of the foundations of *Everyday Mathematics*. However, their application of the theory was limited to encouraging conversations between adults and children:

> Early learning appears to be greatly enhanced by ongoing interactions between children and their world, including adults in that world. Talking about ideas, with informal error corrections by adults and peers, is often as important as thinking about ideas, and conversations can gradually become internal dialogues that guide the child's progress through a problem. (p. 2)

In some cases the developers of these curricula identified models of teaching that guide their work. These models are similarly very general. For example, at least two of the projects describe their model of teaching as launch–explore–summarize (c.f., Fey & Hirsch, 2007; Lappan & Phillips, 2009).

The one curriculum project of this group that differed with respect to the pedagogical theory on which it was based is *Mathematics in Context* (*MiC*). *MiC* was a collaboration with Dutch mathematics educators who use the Realistic Mathematics Education (RME) approach. RME is discussed later in this chapter.

In Mathematics Teacher Education

Mathematics teacher education and research in mathematics teacher education have been limited by a lack of pedagogical theory. Without a specification of the teacher's role in promoting the learning of mathematical concepts, teaching is merely categorized into broad responsibilities (e.g., monitoring students' progress, facilitating classroom discussions, introducing tasks). Consequently, teacher education aims at fostering particular skills (e.g., listening to students, asking probing questions); developing particular dispositions (e.g., inclination towards inquiry and reflection); and teaching about resources (e.g., textbooks, software, manipulatives), classroom strategies (e.g., collaborative small groups, whole-class discussions), and aspects of working with students in particular mathematical areas.

The lack of a theory of pedagogy[3] means that teachers receive no *specific* instruction on how to promote mathematical concepts. Engaging students in small-group problem solving and later having a class discussion are classroom structures, not a theory-based approach to fostering conceptual learning. Whereas teachers may learn to create conditions that are supportive of learning, there remains a lack of knowledge about how to meet the challenge of promoting the learning of particular concepts for students who may not readily learn them. For teacher education, this means that specific goals for teacher learning are lacking. Research on teacher education similarly lacks a sufficient theoretical basis for the design and analysis of teacher education situations.

To investigate informally my hypothesis that research on teacher education generally lacks an underling theory of pedagogy, I perused the titles and abstracts of all articles published in the last 2.5 years in the *Journal of Mathematics Teacher Education* (2008–2010). Based on the titles and abstracts, I selected seven articles that I judged would most likely, given their subject matter, have explicit description of a theory of pedagogy. (This should be construed as demonstrative and not as research.) The results of my reading of these papers confirmed my hypothesis. None of the papers reviewed discussed a specific approach to fostering mathematical concept learning. I discuss some of these articles in more detail.

Two articles were concerned with *inquiry-based* instruction. Both considered inquiry-based instruction to be a set of general processes. Wilkins (2008) wrote,

> Inquiry-based mathematics instruction is characterized by students' active engagement in meaningful mathematical problems and activities that involve conjecturing, investigating, collecting and analyzing data, reasoning, making conclusions, and communicating … Further, inquiry-based classrooms tend to reflect the notion that mathematics is a social activity in which discussion, justification, argumentation, and negotiation are central to the mathematical discourse among students, and between students and teachers. (pp. 140–141)

The only specification of the teacher's role was, "Teachers play an important role in developing the 'sociomathematical' norms of the classroom that promote a community of inquiry in which students feel comfortable sharing their ideas, challenging others' ideas, and justifying their own views" (p. 141).

Towers (2010) pointed out, "Inquiry-based practice is a slippery concept" (p. 246). And, "inquiry-oriented teaching rests upon a particular set of teacher competencies and dispositions, though it is not easy to discern a coherent or agreed set of such capacities from the emerging literature" (p. 246). She went on to provide a list of "practices and dispositions typically attributed to inquiry-oriented teachers" (p. 246). These practices and dispositions were at a general level and did not reflect a particular conceptualization of fostering concept learning (e.g., "a level of comfort

[3] My use of the term *theory* is not meant to refer to something that only the most sophisticated theoreticians can think about. Rather, it suggests teachers' need for a clear, coherent, and explicit approach to instruction.

with ambiguity and uncertainty, … a commitment to exploring student thinking as well as skill in probing and making sense of students' ideas" p. 247). Perhaps most intriguing was the statement, "knowing how to 'teach for understanding', including fluency in teaching with manipulatives, guiding small-group work, capitalising on students' multiple solution strategies, and so on" (p. 247). The crux of my argument rests on what was unspecified in this statement. What *is* involved in teaching for understanding and what knowledge *is* needed to do so? How does one capitalize on students' solutions? The lack of specification of pedagogical theory in Towers and Wilkins' articles is particularly striking given their interest in looking at teachers' pedagogical content knowledge.

Koirala, Davis, and Johnson (2008) worked to develop "a performance-based assessment designed to measure preservice teachers' pedagogical content knowledge and skills" (p. 129). They specified, "This performance assessment measures teacher candidates' ability to analyze student work and use the results in developing lesson plans" (p. 129). However, a conceptualization beyond that was not developed. What is the theoretical basis for using the analyses of student work to develop lesson plans? Nipper and Sztajn (2008) wrote about their conceptual frame for working to improve professional development. They cited several leading researchers to establish that "mathematics instruction can be understood as the contextualized interactions among the teacher, the students, and the mathematics" (p. 334). The discussion, however, stays at this level of generality.

Baxter and Williams (2010) discuss the "dilemma of telling." In their article, they cite Windschitl (2002), who also suggests the inadequacy of pedagogical theory development:

> Windschitl (2002) points out that "principles of instruction that derive from constructivist explanations for learning have not cohered into any comprehensible, widely accepted models" (p. 138).[4] In the absence of established models, Windschitl suggests that educators often create their own version of this form of teaching through a sort of conceptual metonymy—letting isolated pieces of a presumed constructivist approach stand in for the whole. (p. 7)

The lack of "comprehensible, widely accepted models" led to Baxter and Williams' (2010) central problem: "the dilemma of telling: how to facilitate students coming to certain understandings, without directly telling them what they need to know or to do" (p. 8). Interestingly, Baxter and Williams took an empirical approach to the problem, analyzing what two teachers did in terms of this dilemma. Although teaching is filled with dilemmas of judgment (e.g., choice of actions when each supports one of the teacher's goals over the other), I consider the "dilemma of telling" to be the result of inadequate theory development and, therefore, not for the most part an inherent aspect of teaching. That is, the alternative to telling, as a means of fostering understanding of a specific concept, has not been sufficiently developed theoretically.

[4] I discuss this point in the section on the basis of our recent work on pedagogical theory.

Pedagogical Theory and the Countries at the Top of International Comparisons

In international mathematics comparisons (e.g., TIMMS, 2007), five Asian countries, Taiwan, South Korea, Singapore, Hong Kong, and Japan, have clearly demonstrated their superiority. It is common for mathematics educators from other countries to look to these countries for ideas about improving mathematics education in their own countries. A perusal of literature available in English suggests that none of these countries have made a major contribution to pedagogical theory. Their students' success in mathematics seems to be attributable to several factors, including greater mathematical knowledge of the teachers (e.g., elementary mathematics teacher specialists), cultural differences (e.g., the value the family places on school learning and success, the lack of discipline problems in school), a host of useful educational practices (problem focus, more is less, coherent curricula, solicitation of students' ideas, considerable professional development time, and support structures), and smaller percentages of the student population that are not competent in the language of instruction.

An example of the importance of culture is Hong Kong, which finished fourth in the TIMMS 2007 comparison despite a lack of cutting-edge instruction. The Hong Kong curriculum has been focused towards results on written examinations:

> In order to achieve this objective, students have been exposed to constant drills on skills and content… Primary and junior secondary students seem to associate mathematics with its terminology and content, and doing mathematics is often perceived to be applying a set of rules rather than a thinking process. (Lam, 2002, p. 204)

Those responsible for mathematics education in Hong Kong are engaged in a mathematics education reform that has similar goals to the reforms in a number of other countries. The intention is "to develop students' abilities in inquiring, reasoning, conceptualizing, problem-solving, and communicating" (Lam, 2002, p. 205). The emerging curricula are organized to promote learning from the concrete to the abstract using real-world problems and technology. However, there is no explication of an underlying pedagogical theory beyond these practical principles.

Much international attention has been focused on the Singapore national curriculum (Kho, Yeo, & Lim, 2009). Descriptions of the Singapore method do not focus on instructional theory. The curriculum uses a consistent set of diagram models, the *Model Method* (Kho, 2007), as the basis for concept development, symbolic mathematics, and problem solving. The method seems to hinge on the notion that these diagram models are accessible representations of informal reasoning by students and serve as a foundation for formal mathematics (e.g., algebra equation writing and solving).[5]

[5]This approach is related to a principle of RME discussed later.

Two Developing Pedagogical Theories

The review of design principles underlying U.S. and Asian curricula revealed a lack of well-developed pedagogical theory. Several practices and principles were shared by a number of these curricular efforts including building on informal knowledge, using concrete models and diagrams, working from more concrete to more abstract representations, using real-world problems, using collaborative groups and classroom discussions, and engaging students more actively in classroom activities. However, these practices and general principles fall short of offering a framework for instructional and curricular decisions.

Over the last four decades, two research and development programs have made significant contributions to pedagogical theory, *Realistic Mathematics Education* in the Netherlands and the *Theory of Didactical Situations* in France. I will discuss each briefly, leaving out much, but trying to highlight the aspects of each that are most relevant to the focus of this chapter.[6]

Theory of Didactical Situations

Theory of didactical situations (TDS) is the principal theoretical framework for mathematics education research in France. It incorporates a number of constructs (including some already discussed) into an overall framework. The theoretical framework, which was grounded in the Piaget's (1970) notion of learning as adaptation and Bachelard's (1938) theory of epistemological obstacles, is built particularly on the work of Brousseau (1997) and Chevallard (1985) and focuses on both cultural and cognitive factors. As the name suggests, the focus is on didactical situations. The framework represents a systemic approach to classroom instruction—*not* a focus on students' learning processes. In describing situations for the learning and teaching of mathematics, contributors to TDS have attempted to capture dynamics of the classroom as well as the planning of instruction.

Design of lessons begins with finding *fundamental situations*, situations that are accessible to the student, represent the mathematical concept, and allow students to develop appropriate conceptions through adapting to those situations. Instruction begins with the creation of an *adidactical situation* and the design of the *milieu*. The adidactical situation is a problem that students can attack using their prior knowledge. Ruthven, Laborde, Leach, and Tiberghien (2009) elaborate:

> Although an adidactical situation is designed to condition the construction of some specific new knowledge by students, it must be experienced by students not as a matter of learning some ready-made result, but rather as one of resolving a genuinely problematic state of affairs with whatever knowledge they already have available. In particular, an adidactical

[6] It should be understood that as an outsider to both programs my perspectives may be limited.

> situation depends on the problem being such that some starting strategy is available for students, but one that turns out to be unsatisfactory in some way. The ideal is that students, as a result of observing the inadequacy of their strategy, will be motivated to look for others and that this will lead them to devise solution strategies that provide a basis for constructing the intended new knowledge. (p. 332).

This latter aspect of the adidactical situation is reminiscent of Piaget's (1970) notion of disequilibrium.

The initial lesson design also involves the design of the *milieu*. *Milieu* is a complex and comprehensive construct that refers broadly to the physical and social context of the work on the problem. It includes anticipation of the organization of work on the problem and interactions among the various classroom components. Much of the development of constructs within TDS is related to the design of the milieu: "The notion of 'milieu' has been developed within TDS to refer to that component of the situation that offers possibilities of interaction to students, providing means of gaining feedback to validate or invalidate their solution strategies" (Ruthven et al., 2009, p. 332).

The design is also informed by a set of didactical variables. These variables specify factors that, although often overlooked, have significant impact on the success of the didactical situation. The final step in an instructional cycle is the situation of *institutionalization*, in which the new concept that emerged through the students' solutions is made explicit and given a status in the class. This situation often involves the introduction of vocabulary and symbolization, connecting the new concept with the larger domain of mathematics and with prior concepts.

TDS has proven to be a useful theoretical framework for the design of instruction and for research on classroom instruction. It provides some powerful ways to conceptualize mathematics teaching. However, for the purposes of this chapter, I will focus on a particular aspect of TDS. To do this, let us examine excerpts from Ruthven et al.'s (2009) description of the design of an adidactical situation. This is done mostly through example using Brousseau's (Brousseau, Brousseau, & Warfield, 2008) classical work on teaching decimals. Ruthven et al. (2009) claimed: "This situation was created expressly with the intention of addressing a crucial epistemological obstacle. It seeks to invalidate an additive model of the scaling operation through providing students with strong feedback that convinces them that their solution is wrong" (p. 333). They went on to explain:

> At this stage, however, the pragmatic feedback offered by the material milieu is no longer sufficient. It becomes necessary for the teacher to offer *intellectual* feedback drawing on the knowledge available to the students. The original square needs to be transformed into a larger square. By focusing on the process of applying the doubling-and-subtracting rule to the differing partitions of the sides of the original square and adding the results, it becomes clear that squareness is not preserved. It is usually only after students have dismissed computationally simple formulations of this type that they find a strategy based on the linearity of the transformation for sums of lengths. (p. 333)

What we see from this description of the example is first the dependence on disequilibrium. As my colleagues and I have argued in other venues (c.f., Simon, Tzur, Heinz, & Kinzel, 2004), disequilibrium may trigger a need for a new conception; however the construct does not offer any theoretical frame for provoking the development of any particular conception. The example in question hinges on a confidence that the students, once dissuaded from certain solutions, will find the appropriate one. Brousseau (1997) emphasized, “The teacher must imagine and present to the students situations within which they can live and within which the knowledge will appear as the optimal and discoverable solution to the problems posed” (p. 22). This description of the responsibility of the teacher raises questions about how that discovery process is planned for. Part of the answer for TDS is the consideration of the historical evolution of the concept to trigger ideas for problem sequences. TDS also makes use of specific research in the teaching and learning of the concepts in question. However, in this chapter, I argue for additional theory development with respect to the process by which new conceptions are developed and how such development can be fostered.

Realistic Mathematics Education

The pedagogical approach of RME begins with experientially real situations for the students in which they can use their current knowledge to solve problems informally. Learning is described as involving horizontal and vertical *mathematization* (Treffers, 1987). *Horizontal mathematization* involves finding ways to represent and work with the mathematical relationships in a realistic problem. Vertical mathematization involves progressively refining the mathematics used towards more formal solutions. Thus, the learning process proceeds from students' spontaneous productions in response to realistic problems towards formal mathematics. A key construct in RME is *guided reinvention* (Freudenthal, 1973), an active process of students inventing ideas for themselves supported by carefully planned and orchestrated lessons. Guided reinvention involves students building on their prior mathematical and real-world knowledge.

For me, the most sophisticated pedagogical construct in RME is the notion of *model of becoming model for* (Gravemeijer, 1997). RME instructional designers study students' spontaneous productions in response to realistic problems. They then develop a physical or paper-and-pencil model that is consistent with the models that students develop spontaneously *and* that affords vertical mathematization. This model can be used first as a *model of* a situation (e.g., the arithmetic rack as a model of passengers on a double-decker bus) and later as a *model for* reasoning about mathematical ideas (e.g., the arithmetic rack for reasoning about addition and subtraction of numbers and the development of non-counting strategies). I claim that this is a sophisticated pedagogical construct, because it is one of the few constructs that addresses conceptual transitions, the *processes* by which learning takes place.

So Where Are We with Respect to Pedagogical Theory?

As we have seen, instructional design in many countries is based on instruction beginning with realistic problems, the importance of encouraging an active role for students in developing mathematical ideas, and the use of particular pedagogical tools (e.g., collaborative group work, class discussions, software environments). However, in most of these cases the specific design of lessons and units is insufficiently guided by pedagogical theory.

Two theories, TDS and RME, offer theoretical frameworks based on several decades of research and development. TDS offers an integrated way of characterizing and focusing on many aspects of the instructional enterprise. It explains the critical roles of particular situations and the importance of careful design of the milieu. RME offers some significant principles for instructional design and notably describes the model-of-model-for construct, which provides a basis for planning a type of conceptual transition.

However, none of these efforts entail detailed understanding of students' mathematics conceptual learning processes and pedagogical theory derived from it.[7] In the remainder of this chapter I endeavor to promote a vision of such work and its potential.

Grand Theories and Intermediate Frameworks for Design

Ruthven et al. (2009) made a distinction between *grand theories* (e.g., constructivism, sociocultural theory, enactivism) and *intermediate frameworks* (pedagogical frameworks based on grand theories). They explained, "Intermediate frameworks and design tools serve to organize the contribution of grand theories to the process of designing and evaluating teaching sequences by extracting relevant components of the theories and coordinating and contextualizing their application" (p. 340). TDS and RME are examples of intermediate frameworks.

In contrast to TDS and RME is the El'konin–Davydov (E–D) elementary mathematics program (Davydov, Gorbov, Mikulina, & Savel'eva, 1995). The E–D curriculum was developed in Russia based on Russian activity theory (Leontyev, 1979), an outgrowth of Vygotskian sociocultural theory. The curriculum is a coherent, logical approach to teaching elementary students arithmetic concepts. Aspects of activity theory seem to have undergirded the curriculum effort, particularly inquiry into the historical development of particular mathematical concepts and the serious attention to the use of symbols. However, the developers of this curriculum (to my knowledge) have not contributed an intermediate framework, a pedagogical theory

[7] Steffe (2003) has contributed greatly to identifying hierarchies of students' schemes. However, in his seminal teaching experiments, the focus has remained on the students and not on the pedagogical principles behind the researchers' actions. In Simon et al. (2010) we argued that his focus has been on the schemes and less on the transition between these schemes.

for mathematics instruction and design. Rather their work has been focused on how specific mathematical concepts should be sequenced and on articulating broad constructs of activity theory. Thus, their work on learning, thinking, and generalization (c.f., Davydov, 1988) tends not to be specific to mathematics education. Curriculum developers who wish to build on the efforts of El'konin, Davydov, and their colleagues must find ways to adapt activity theory. A key point is that theories of knowing and learning (grand theories) do not in themselves provide frameworks for instructional design and implementation. Cobb (1994) explained that such theories "do not constitute axiomatic foundations from which to deduce pedagogical principles" (p. 4).

In building pedagogical theory on the basis of these grand theories, several questions must be addressed, including the following:

1. What does each learning (grand) theory contribute to understanding classroom learning of mathematics?
2. What aspects of the theory (theories) are most relevant?
3. What is insufficiently understood and theorized about mathematics learning?
4. What type of pedagogical approach might be effective given particular understandings of mathematics learning?

In the remainder of the chapter, I attempt to clarify a vision of a theory of CLTM by describing our program of research and how it aims to address these four questions.

Towards a Theory of Conceptual Learning and Teaching of Mathematics

The development of any empirically based theory depends in part on researchers' ideas about what is possible. Progress in any field is not only afforded by recent increases in knowledge and tools but also by a concomitant idea about what might be accomplished in the foreseeable future. The vision of a theory of CLTM that I describe in this chapter is not one that is widely shared in the mathematics education research community.

As I have indicated, a theory of CLTM is a theory of design and instruction based on an understanding of how mathematical concepts are learned. It is specific to the learning of mathematical concepts and general enough to be relevant in many different mathematical content domains. The basic idea is that the more we understand about the mechanism(s) by which learners come to understand a new concept, the greater the opportunity to design instruction to support and make use of these learning mechanisms. By analogy, the more that is understood about the physics, biomechanics, and physiology involved in hitting a golf ball, the more that can be done in the design of clubs and the coaching and teaching of the golf swing. Menchinskaya (1969) wrote, "It is of great theoretical and practical interest to establish the specific regularities of that inner processing to which the child's concepts are subject" (p. 78).

The vision I describe is based on the assumption that there is commonality in the way humans come to know mathematical concepts. Also, I assume that no intervention or set of interventions based on pedagogical theory can be deterministic of learning. Pedagogical theory is intended to provide more powerful ways to think about fostering mathematics learning and therefore to increase the probability of producing positive learning outcomes in a reasonable time frame. Part of having a more powerful framework is the ability to successfully modify unsuccessful lessons. *The goal therefore is to develop characterizations of the mechanisms by which people learn mathematical concepts* (*the transition process*) *at an appropriate level of detail to afford the construction of a set of mathematics instructional design principles*.

The Benefits of a Theory of a CLTM

I made a case earlier for the need for pedagogical theory in mathematics teaching and curriculum design. Such theory could also be central to defining important goals for teacher education. What could be more important than teaching teachers about how students learn mathematical concepts and how they can promote that learning? Advances in pedagogical theory would likewise be of critical importance in research. For example, currently important research on students' conceptions is often done using teaching experiment methodology. Whereas researchers, such as Steffe and his colleagues (e.g., Steffe, 2003), have contributed substantially to our understanding of students' schemes in the areas of early number, fractions, etc., the contribution to pedagogical theory in mathematics has not been nearly as great. Researchers who wish to conduct research in different mathematical areas have only the most general principles on which to begin to design instructional sequences.

Enhanced pedagogical theory would also be important in research on teaching and teacher education. Advances in pedagogical theory would provide theoretical frameworks for specifying key foci for careful study of teaching and the progress and impact of teacher education.

Impediments to Realizing this Vision

The greatest impediment to realizing this vision is the difficulty in studying learning. There are researchers who would say that they study learning. In some cases they are studying conceptual steps through which students pass as they learn particular content in particular contexts (reviewed in detail in Simon et al., 2010). In other cases, researchers study the success of particular pedagogical interventions. In neither case is there an analysis of the mechanism by which students learn a new concept. It is very difficult to study the transition between steps, as opposed to just the steps themselves. The vision being described is based on the assumption that study of these mechanisms is possible, though difficult, and needs to be prioritized. Siegler (1995) asserted,

> How change occurs is perhaps the single, fundamental issue in the study of cognitive development. Progress in understanding the issue has been slow … Part of the reason is the inherent conceptual complexity of the subject. Understanding changes in children's thinking presents all of the demands of understanding their thinking at any one time, plus the added demands of understanding what is changing and how the change is being accomplished. (p. 225)

He suggested,

> Focusing on change … will require reformulation of our basic assumptions about children's thinking, the kinds of questions we ask about it, our methods for studying it, the mechanisms we propose to explain it, and the basic metaphors that underlie our thinking about it. (Siegler, 1996, p. 218)

Some researchers argue that learning takes place over extended periods of time and to some extent when learners are not engaged in mathematics learning activities. Whereas this is undoubtedly true, it does not eliminate the possibility of well-designed studies that create opportunities to study learning in the context of particular instructional interventions (i.e., during teaching experiments). This approach allows study of only a subset of mathematics learning situations. However, such studies might yield important insights into mathematics learning that are relevant to a wider set of learning situations.

One issue that complicates the discussion of the study of mathematics learning is the widespread use of general constructs for describing learning (e.g., reflective abstraction, working in the zone of proximal development) as if these mechanisms are well understood. Siegler (1996) argued,

> The standard labels for hypothesized transition processes: assimilation, accommodation, and equilibration; change in M-space; conceptual restructuring; differentiation and hierarchic integration; are more promissory notes, telling us that we really should work on this some time, than serious mechanistic accounts. (p. 223)

These broad constructs provide us with orienting frameworks for looking at learning. However, they do not provide sufficient insight into the mechanisms by which conceptual transitions take place to allow careful design of instruction based on these mechanisms.

A Program of Research and Theory Development

Origins of the Research

This research effort derived[8] from the confluence of two prior studies—Tzur's (1996) dissertation study and Simon's (1995) postulation of the *hypothetical learning trajectory* (*HLT*) and the *mathematics teaching cycle* of which the HLT is part. The

[8]Discussing the origins of a research program can involve a chicken and egg problem. One can talk about the reasons for the initial choice of theory. However, it is likely that the thinking and observations that led to the theory choice were influenced by a prior theoretical stance. Please read the following with that caveat in mind.

collaborative work was based on two shared observations. First, students' activity, whether young children or advanced mathematics students, seems to have a major effect on their learning. "Activity" includes mental activity that goes on even when students are sitting listening. Second, learning is not just a gradual increase in knowledge, but rather is characterized by significant steps that represent conceptual breakthroughs (e.g., the student who comes to understand conservation of number, or the student who constructs a notion of rate). We thought about these breakthroughs as *abstractions*. Menchinskaya (1969) considered the process of abstraction to be central to investigation of internal conceptual learning processes: "When investigating the mastery of concepts, we constantly encounter the fact that precisely that aspect of scientific knowledge which is the result of 'the human mind's work of abstraction' also constitutes the greatest problem for students" (p. 80).

We were drawn to Piaget's (2001) construct of *reflective abstraction*, because it focused on explaining the production of new abstractions, it took the learners' activity as the raw material for learning, and it postulated inherent and often not conscious mental processes (reflection) as critical in realizing abstractions from activity. However, *the construct of reflective abstraction was not sufficiently elaborated to guide mathematics pedagogy*, *and*, *because of that*, *had yet to have significant impact on mathematics instructional design*.

In addition, our thinking was grounded in Piaget's (1970) construct of *assimilation*. We took as given that students cannot perceive mathematical relationships they do not already conceptualize. For example, students who have no concept of multiplication cannot see multiplicative situations as having a common structure and cannot see those situations in the world distinct from non-multiplicative situations. Therefore the challenge is to explain how they can build up a concept of multiplication through their activity based on their extant knowledge. This statement does not imply the absence of a teacher, well-structured tasks, or any other aspect of an optimum situation for learning a mathematical concept. Rather, these aspects were backgrounded as we focused our initial inquiry on the learning processes of the students. One of the unique aspects of our ongoing work is our interest in studying and elucidating the making of *single* abstractions (as opposed to a set of abstractions related to a particular topic).

The Simon–Tzur collaboration was productive. Simon et al. (2004) offered further elaboration of reflective abstraction of mathematical concepts using a framework we called *reflection on activity–effect relationships* (RAER). The framework offered a way to understand how conceptual transitions derive from the learners' activity and reflection. We also identified implications of the framework for the design of instructional task sequences (Simon & Tzur, 2004).

Direction of Recent Work

Following this period of collaboration, Tzur initiated research projects using the RAER framework (e.g., Tzur, 2007). I have taken a somewhat different tack. To explain the choices I made for continuing this work, I discuss the affordances and limitation of the RAER framework from my perspective.[9] The RAER framework demonstrated what it might mean to articulate a mechanism for conceptual learning. In this way it contributed to concretizing the vision discussed. It demonstrated the usefulness of the constructs of reflection, activity, abstraction, and assimilation in doing so. It also demonstrated the possibility of generating a framework that can explain conceptual learning both in and outside of instructional settings.[10]

However, the RAER framework had three significant limitations. First, generation of the framework was based on a small number of learning examples. For the most part, it was a hypothesis that was worked out to explain learning of a pair of fraction concepts in a particular teaching experiment. Second, if one used the rigorous definition of activity–effect relationship articulated in Simon et al. (2004), the framework proved difficult (from my perspective) to use to generate task sequences. The framework seemed to be more useful for explaining learning that had occurred than for informing instructional design. On the other hand, the general notions of starting with an activity available to the students and creating mathematical tasks from which they would abstract important relationships continued to be useful and productive. That is, the instructional design implications we had identified were still useful. Third, the data we used were collected using teaching experiment methodology that was not sufficiently adapted to the purpose of uncovering the mechanisms by which students learn through their mathematical activity. This limitation resulted in data that were sometimes not sufficiently revealing the issues under study.

To take the work further, we decided not to assume the RAER framework, but do extensive empirical work in order to build up the framework very rigorously. We set out to begin generating multiple examples of conceptual learning across different mathematical domains and age groups, examples that could be studied in terms of mechanisms of learning.[11] We were working from the same initial constructs (activity, reflection, abstraction, and assimilation); however this time we were developing specific adaptations of teaching experiment methodology in order to generate useful sets of data for focusing on the transitions by which conceptual learning takes place. In particular, we wanted to generate continuous evidence of students' relevant thinking and understanding from a point where they did not have a particular understanding, through the transition, to the point where they had the understanding.[12] To optimize the collection of data, we aimed to create the following conditions (Simon et al., 2010):

1. Conceptual learning goals that might result in students' learning *during* the instructional intervention: Learning that takes place when the students are not with us creates no data for us to analyze.

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[9] I make no attempt here to report similarities and differences of Tzur's and my perspectives on the RAER framework, only to articulate my perspective.

[10] We worked from an assumption that the same human mechanisms for learning mathematical concepts should be at play whether or not learners are in instructional settings.

[11] The ultimate impact of this work depends on many well-researched examples. We hope to enlist colleagues interested in contributing to this effort.

[12] Understanding is relative and complex. We are referring here to a single idea before and after it is abstracted.

2. Teaching experiments with one student: When students watch or listen to another student, there is no trace of their activity for us to follow. Thus, when two or more students work together, it creates holes in our data.
3. A limited role for the researcher/teacher consisting of posing problems and probing the students' thinking: When the student is listening to the teacher's hints, suggestions, or leading questions, the continuity of the students' thinking is interrupted and the data we have to analyze are not as reliable an indication of their process of abstraction.
4. Timely and frequent assessments: Knowing which data to analyze to explain learning is based on knowing at what point the conceptual transition took place. If we observe a student solving a problem, but do not know that the student was incapable of solving that problem in that way at an earlier point in the instructional sequence, we have no basis for claiming that a conceptual transition occurred during that instructional session.

This brief review of the main points of the adaptation of the methodology is intended to give a sense of the methodological challenge of studying learning and one way that this challenge is being met. Even with these adaptations, the generation of a useful data set is an uncertain process.

One frequent critique of this methodology is that teaching someone in a one-on-one situation with a limited teacher's role is very different from teaching in classroom situations. This is true. However, this work rests on the idea that if we can understand the process of abstraction in these limited situations, it could provide an important basis for understanding learning in classroom settings and inform the design of task sequences for classroom learning. How might classroom learning be enhanced by task sequences that consistently foster important abstractions in individuals? How might the discourse be changed and the participation of certain students be increased through the use of such task sequences?

We take as an open question whether there are different categories of mathematical concepts (e.g., development of a new mathematical object versus the development of an algorithm) that might be characterized by somewhat different learning processes. The diversity of the set of learning examples that are accumulated through this work will contribute to attempts to answer this question.

The first teaching experiment we conducted involved preservice teachers learning concepts of division of fractions. The adapted research methodology allowed us to infer a consistent trace of the student's evolving thought process as the learning transition occurred. Because of space limitations, I cannot report the research finding here. The reader is referred to Simon et al. (2010) for a detailed analysis. The product of the study was an account of a student's learning. A rich set of such accounts would provide the data for an analysis of mechanisms. Our current teaching experiments are with elementary students learning fraction and ratio concepts grounded in measurement models of quantities and quantitative operations.[13]

[13]National Science Foundation supported "Measurement Approach to Rational Number (MARN) Project" 2010–2015.

Conclusion

In traditional lecture-only mathematics instruction, the teaching approach was explicit while the theory of learning underlying the approach remained implicit. Over the last 30 years mathematics education has imported constructivist and socio-cultural theories affording a more explicit (albeit general) characterization of learning. The instructional implications of such theories, however, remain underspecified and under-theorized. Advances in pedagogical theory are needed.

Historically, there has often been a dichotomy between research on mathematics learning[14] and research on mathematics teaching. Studies of mathematics learning (e.g., Confrey & Smith, 1995; Steffe, 2003; Thompson, 1994) have not tended to focus on the theoretical basis of the teaching interventions promoting learning, whereas studies of mathematics teaching (e.g., Hill et al., 2008; Lampert, 2001; Silver & Stein, 1996) have not tended to be grounded in theories of learning. As described, there are programs of research (e.g., TDS, RME, E–D) that have moved in the direction of more integrated attention to teaching and learning. In this chapter, I have argued for programs of research that focus on the conceptual transitions that make up learning, at a level of detail that can provide a basis for theorizing instruction and curriculum design. Of course mathematics instruction is complex and rests on understanding more than conceptual transitions and how to foster them. Absent from this discussion are issues of classroom norms, diversity, social justice, institutional context, and many others that have significant effect on what is learned and by whom. However, progress in the area of pedagogical theory would go a long way towards better task design and sequencing, more informed use of tools and symbol systems in instruction, and greater direction for managing discourse in the instructional setting.[15] Progress of this type rests on both belief in the potential to characterize learning mechanisms and willingness to develop methodologies and simplified systems for studying learning transitions.

Studying how mathematical concepts are learned depends on progress in a different but related research program—the specification of mathematical concepts. In order to investigate how a particular concept is developed, we must be able to articulate what that concept is. Often in mathematics education the concept to be learned is inadequately articulated. Instead, a proxy is used, such as the following:

- Focusing on the ability to solve a particular type of problem (e.g., “learn to solve maximum–minimum problems”).
- Specifying behavioral goals (e.g., “appropriate use of the distributive property”).

[14] Here I am using the term broadly to include work on students’ schemes, even though I have argued that such work does not generally focus on the transitions between schemes.

[15] Although not the focus of this article, useful theories of pedagogy that support greater success in teaching new concepts have a significant potential to increase equity in mathematics learning.

- Putting the word "understand" in front of a mathematical topic (e.g., "understand function," "understand place value").

Note that none of these ways of specifying learning goals offer any information as to what the understanding (concept) is.

The vision presented in this chapter could have profound effects. Just as understanding a particular anatomical system opens up potential in medicine, greater understanding of the processes of mathematical conceptual learning has the potential to impact mathematics instruction, instructional design, teacher education, and research in each of these areas. Conceptual learning processes have long been a black box. Examining the contents of that box will not be easy and success will be uncertain. However, I have argued that it is possible and its potential impact merits the effort and risk of failure.

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Chapter 6
Intellectual Need

Guershon Harel

Abstract Most students, even those who desire to succeed in school, are intellectually aimless in mathematics classes because often they do not realize an intellectual need for what we intend to teach them. The notion of intellectual need is inextricably linked to the notion of epistemological justification: the learners' discernment of how and why a particular piece of knowledge came to be. This chapter addresses historical and philosophical aspects of these two notions, as well as ways teachers can be aware of students' intellectual need and address it directly in the mathematics classroom.

Years of experience with schools have left me with a strong impression that most students, even those who are eager to succeed in school, feel intellectually aimless in mathematics classes because we (teachers) fail to help them realize an *intellectual need* for what we intend to teach them. The main goal of this chapter is to define *intellectual need*, discuss its manifestations in mathematical practice, and demonstrate its absence and potential presence in mathematics instruction.

Intellectual need is inextricably linked to problem solving. Problem solving is usually defined as engagement in a problem "for which the solution method is not known in advance" (NCTM, 2000, p. 52). Alas, many of the situations students encounter in school satisfy this definition and yet do not constitute "true" problem solving because, from the students' perspective, these problems are often devoid of any intellectual purpose. Thus, another goal of this chapter is to advance the perspective, articulated by many other scholars (e.g., Brownell, 1946; Davis, 1992;

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Hiebert, 1997; Schoenfeld, 1985; Thompson, 1985), that problem solving is not just a goal, but also the means—the only means—for learning mathematics.

The chapter is organized around five sections. The first section briefly outlines a set of underlying premises in which the concept of intellectual need resides. The second section defines the concept of *intellectual need* on the basis of these premises. The third section defines five categories of intellectual needs, describes their functions in mathematical practices, and offers concrete curricular implications. The fourth section introduces several common fundamental characteristics to these needs. They are fundamental because without them the concept of intellectual need is both pedagogically and epistemologically incoherent. The last section abstracts the themes of the paper into a definition of learning and a consequent instructional principle.

Underlying Premises

The perspective put forth in this paper is oriented within the Piagetian theory of equilibration and is part of a conceptual framework called *DNR-based instruction in mathematics* (*DNR*). *DNR* can be thought of as a system consisting of three categories of constructs: *premises*—explicit assumptions underlying the *DNR* concepts and claims; *concepts* oriented within these premises; and *instructional principles*—claims about the potential effect of teaching actions on student learning justifiable in terms of these premises and empirical observations. The initials *D*, *N*, and *R* stand for the three foundational instructional principles of the framework: *Duality*, *Necessity*, *and Repeated reasoning*. Here we only discuss the four *DNR* premises that are needed for our definition (see Fig. 6.1) of *intellectual need*: the *knowledge*

Premise	
Knowledge of Mathematics	Knowledge of mathematics consists of two related but different categories of knowledge: all the ways of understanding and ways of thinking that have been institutionalized throughout history.
Knowing	Knowing is a developmental process that proceeds through a continual tension between assimilation and accommodation, directed toward a (temporary) equilibrium.
Knowledge - Knowing Linkage	Any piece of knowledge humans know is an outcome of their resolution of a problematic situation.
Subjectivity	Any observations humans claim tohave made are due to what their mental structure attributes to their environment.

Fig. 6.1 Four DNR premises

(*of mathematics*) *premise*, the *knowing premise*, the *knowledge-knowing linkage premise*, and the *subjectivity premise*.[1]

Antecedent to the concepts of *way of understanding* and *way of thinking* referred to in the *Knowledge of Mathematics Premise* is the primary concept of *mental act*. Examples of mental acts include the acts of interpreting, conjecturing, inferring, proving, explaining, structuring, generalizing, applying, predicting, classifying, searching, and problem solving. When one carries out a mental act, one produces a particular outcome. For example, when reading a string of symbols, a statement, or a problem, one of the mental acts a person carries out is the interpreting act, which, in turn, results in a particular meaning for it. Similarly, upon encountering an assertion, one may carry out the justification act and produce, accordingly, a particular justification. Such a product of a mental act is called a *way of understanding* associated with that act. Different individuals are likely to produce different ways of understanding associated with the same mental act. For example, students engaged in a dynamic geometry software activity may carry out conjecturing and justifying acts and, accordingly, produce different conjectures and justifications. Each conjecture and justification is a way of understanding—a product of the conjecturing act and justification act, respectively.

A common cognitive characteristic of a person's (or a community's) ways of understanding associated with a particular mental act is referred to as that person's *way of thinking* associated with that act. For example, a teacher or a researcher may infer (from a multitude of observations) one or more of the following characteristics: that a student's interpretations of arithmetic operations are characteristically inflexible, devoid of quantitative referents, or, alternatively, flexible and connected to other concepts; that a student's justifications of mathematical assertions are typically based on empirical evidence or, alternatively, based on rules of deduction. Each of these characteristics is a way of thinking. It is important to emphasize that in *DNR*, ways of understanding and ways of thinking are distinguished from their qualities. Namely, one's way of understanding or way of thinking can be judged as correct or wrong, useful or impractical in a given context. Of course, the goal is to help students gradually advance their ways of understanding and ways of thinking toward those that have been institutionalized in the mathematics community.

The *Knowing Premise* is after Piaget and is about the mechanism of knowing: that the means—the only means—of knowing is a process of assimilation and accommodation. Disequilibrium, or perturbation, is a state that results when one encounters an obstacle or fails to assimilate. It leads the mental system to seek equilibrium, that is, to reach a balance between the structure of the mind and the environment. Its cognitive effect in suitable emotional conditions is that the subject feels compelled "to go beyond his current state and strike out in new directions" (Piaget, 1985, p. 10). Equilibrium, on the other hand, is a state in which one perceives success in removing such an obstacle. In Piaget's terms, it occurs when one modifies

[1]These are four of the eight *DNR* premises (see Harel, 1998, 2008a, 2008b, 2008c).

his or her viewpoint (accommodation) and is able, as a result, to integrate new ideas toward the solution of the problem (assimilation).

The *Knowledge-Knowing Linkage Premise*, too, is inferable from Piaget, and is consistent with Brousseau's claim that "for every piece of knowledge there exists a fundamental situation to give it an appropriate meaning" (Brousseau, 1997, p. 42). The *Subjectivity Premise* orients our interpretations of the actions and views of the learner. Many scholars (e.g., Confrey, 1991; Dubinsky, 1991; Steffe & Thompson, 2000; Steffe, Cobb, & von Glasersfeld, 1988) have articulated essential implications of the *Subjectivity Premise* to mathematics curriculum and instruction.

These and the rest of the *DNR* premises (see Harel, 2008b, 2008c) were not conceived a priori, but emerged in the process of reflection on and exploration of justifications for the *DNR* concepts and claims.

Definition

With these premises at hand, we can now define the concept of *intellectual need* and its associated concept, *epistemological justification*. If K is a piece of knowledge possessed by an individual or a community, then, by the *Knowing-Knowledge Linkage Premise*, there exists a problematic situation S out of which K arose. S (as well as K) is subjective, by the *Subjectivity Premise*, in the sense that it is a perturbational state resulting from an individual's encounter with a situation that is incompatible with, or presents a problem that is unsolvable by, his or her current knowledge. Such a problematic situation S, prior to the construction of K, is referred to as an individual's *intellectual need*: S is the need to reach equilibrium by learning a new piece of knowledge. Thus, intellectual need has to do with disciplinary knowledge being created out of people's current knowledge through engagement in problematic situations conceived as such by them. One may experience S without succeeding to construct K. That is, intellectual need is only a necessary condition for constructing an intended piece of knowledge, and, as discussed below, other motivational conditions are also necessary. Methodologically, however, intellectual need is best observed when we see that (a) one's engagement in the problematic situation S has led one to construct the intended piece of knowledge K and (b) one sees how K resolves S. The latter relation between S and K is crucial, in that it constitutes the geneses of mathematical knowledge—the perceived reasons for its birth in the eyes of the learner. We call this relation *epistemological justification*. An individual's or the institutionalized epistemological justification may not (and often does not) coincide with the historical epistemological justification. For example, many central concepts of real analysis—and some argue the entire field of real analysis (Bressoud, 1994)—were intellectually necessitated from Fourier's solution to Laplace's equation, $\frac{\partial^2 z}{\partial w^2} + \frac{\partial^2 z}{\partial x^2} = 0$. In particular, his solution as an infinite cosine series led, after major objections from the leading mathematician of the time, to a reconceptualization of the concept of function. Specifically, the expansion $f(x)=\frac{\pi}{4}\left(\cos\frac{\pi x}{2}-\frac{1}{3}\cos\frac{3\pi x}{2}+\cdots\right)$, whose graph oscillates between the −1 and 1, was not conceived as a function, because

> functions were polynomials; roots, powers, and logarithms; trigonometric functions and their inverses; and whatever could be built up by addition, subtraction, multiplication, division, or composition of these functions. Functions had graphs with unbroken curves. Functions had derivatives and Taylor series. Fourier's cosine series flew in the face of everything that was *known* about the behavior of functions. (Bressoud, 1994, p. 7).[2]

Thus, the historical epistemological justification for the concept of function is not necessarily that currently held by most mathematicians.

Categories of Intellectual Need

Although laying claim to neither completeness nor uniqueness, I offer five categories of intellectual needs: (1) *need for certainty*, (2) *need for causality*, (3) *need for computation*, (4) *need for communication*, and (5) *need for structure*. In modern mathematical practices these categories of needs are inextricably linked, which makes it difficult to discuss them in isolation. Despite this difficulty, they will be discussed in separate sections in an effort to demonstrate the existence of each need and to better elucidate their distinctions. Each of these sections is divided into two parts. The first part (a) defines the respective need, (b) discusses its cognitive primitives (preconceptualizations whose function is to orient us to the intellectual needs we experience when we learn mathematics[3]), and (c) illustrates its occurrence in the history of mathematics. The second part of each section discusses pedagogical considerations of the respective need.[4] None of these discussions intends to provide a comprehensive epistemological, historical, cognitive, or instructional account for any of these needs; rather, the goal is to describe the intended meaning for each need and illustrate its function in mathematical practice and its possible application in the teaching of mathematics. Nor are these discussions of equal length. The need for computation, for example, occupies the largest space due to its ubiquity in mathematical practice, on the one hand, and its special role in mathematics curricula, on the other hand.

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[2] See also Lakatos (1976, Footnote 3, pp. 19–20, Footnote 2, pp. 22–23, and Appendix 2, pp. 151–152) for an interesting discussion on a similar resistance "monstrous" conceptualization of function.

[3] Here and elsewhere in this chapter it is essential to understand the phrase "learn mathematics" in the sense described earlier, that is, in accordance with the *Knowledge of Mathematics Premise* and the definition of learning presented earlier.

[4] Since the discussion of pedagogical considerations follow the discussion of historical phenomena, it is important to state our belief that the intellectual necessity for a learner need not—and in most cases cannot—be the one that occurred in the history of mathematics.

Need for Certainty

Definition and function. When an individual (or a community) considers an assertion, he or she conceives it either as a *fact* or as a *conjecture*—an assertion made by a person who has doubts about its truth. The assertion ceases to be a conjecture and becomes a fact in his or her view once the person becomes certain of its truth. The *need for certainty* is the natural human desire to know whether a conjecture is true—whether it is a fact. When the person fulfills this need, through whatever means deemed appropriate by him or her, the person gains new knowledge about the conjecture.

We reserve the term *proving* for the mental act one carries out to achieve certainty about a conjecture, and *explaining* (to be discussed in the next section) for the mental act one carries out to understand the cause for a conjecture to be true or false. A person is said to have proved an assertion if the person has produced an argument that convinced him or her that the assertion is true. Such an argument is called *proof*. The proof someone produces may not be one that is acceptable by the mathematics community, but it is a proof for the person who has produced it. Hence, a proof is a way of understanding; it is a cognitive product of one's mental act of proving. A *proof scheme*, on the other hand, is a way of thinking; it is a collective cognitive characteristic of the proofs one produces. Proof schemes can be thought of as the means by which one obtains certainty. For example, a proof scheme may be empirical, where conviction is reached through perceptual or inductive observations (e.g., drawings, measurements, a series of examples, etc.), or deductive, where conviction is reached through application of rules of logic (see Harel (2008a) for a more thorough discussion).

Humans' instinctual desire to seek certainty is a cognitive primitive to the mathematical certainty reached through deductive proof schemes. Throughout history, proof schemes have not been static but varied from civilization to civilization, generation to generation within the same civilization, and community to community within the same generation (Kleiner, 1991). For example, the Babylonians merely prescribed specific solutions to specific problems, and so their proof schemes were mainly empirical. The deductive proof scheme—that is, the approach of establishing mathematical certainty by deducing facts from accepted principles—was first conceived by the Greeks and continues to dominate the mathematics discipline today.

Pedagogical considerations. Our subjectivity toward the meaning of proof does not imply ambiguous goals in the teaching of this concept. Ultimately, the goal is to help students learn to produce mathematical proofs and acquire mathematical proof schemes. A proof or a proof scheme is mathematical if it is consistent with those shared and practiced in contemporary mathematics. It is due to these schemes and practices that mathematicians trust the validation process of proofs established by the mathematics community. Clearly a mathematician is certain of a result when he or she proved it or read its proof. However, mathematicians are certain of numerous results, especially those outside their mathematical specialty, whose proofs they have not read. They accept a result if it has been validated by a mathematician they trust or has gone through a certification process by the community (e.g., published in a reputable journal). Auslander (2008) points out that this process of validation and certification "is an indication that we are part of a community whose members trust one another," and that "mathematics could not be a coherent discipline, as opposed to a random collection of techniques and results, without [this process]" (p. 64).

These socio-mathematical norms for conviction are fundamentally different from the norms prevalent in the mathematics classroom. Strong evidence exists that students at all grade levels, and even school teachers, draw certainty from undesirable proof schemes, such as verification on the basis of specific examples (the *inductive proof scheme*), appearances in drawings (the *perceptual proof scheme*), forms in which a proof is conveyed (the *ritualistic proof scheme*), and teacher's authority (the *authoritative proof scheme*) (Harel & Sowder, 1998, 2007). These behaviors are not surprising, given common teaching practices. Harel and Rabin (2010) identified a series of teaching practices that might account for the strong presence of these proof schemes among students. These practices include the following: the teacher's answers to students' questions mainly tell them how to perform a task and whether an action is correct or incorrect; the justification of the need for content taught is social rather than intellectual; and the teacher's justifications are mainly authoritative, and those that are not authoritative are mainly empirical rather than deductive.

Beyond such detrimental teaching practices, other intuitively sound teaching practices aimed at changing students' undesirable proof scheme have turned out to be largely ineffective. In particular, raising skepticism as to whether an assertion is true beyond the cases evaluated, and showing the limitations inherent in the use of examples through situations where an assertion is true for a very large number n of cases but untrue for the $n+1$ case, does not, in most cases, alter students' proof-related behaviors. This observation was made repeatedly in my teaching experiments with undergraduate math and engineering students as well as in-service teachers. An explanation for this phenomenon rests on the recognition that doubts and conviction—and more generally disequilibrium and equilibrium—are interdependent. A person's doubts about an observation cannot be defined independently of what constitutes certainty for him or her, and, conversely, a person's certainty cannot be defined independently of what doubt is for that person. The presence of doubts necessarily implies the presence of conditions for their removal, and, conversely, a fulfillment of these conditions is necessary for attaining certainty. Thus, since the students viewed their actions of verifying an assertion in a finite number of cases as sufficient for removing their doubts about the truth of the assertion, the question of whether the assertion is true beyond the cases evaluated is unlikely to generate intellectual perturbation with the students. Moreover, since in most cases the teacher's verification actions confirm what the students have already concluded, these actions add little or nothing to the students' conviction about the truth or falsity of the assertion. The counterexample cases students (rarely) encounter, where assertions are true for a large number of cases but untrue for all cases, do not shake students' confidence in their empirical methods of proving. This is so because students' conditions for gaining certainty have not been fulfilled; the attempt to bring students to doubt their empirical proving methods is done by a method those students do not accept in the first place.

The experience of disequilibrium cannot be described independently of its corresponding experience of equilibrium, and, therefore, as a form of perturbational experience, intellectual need cannot be determined independently of what satisfies it. An important implication of this observation is that curriculum developers and teachers must think hard as to what constitutes perturbation and equilibrium for students and how to enculturate them into a milieu of *mathematical* perturbations and equilibriums. The rest of this chapter is an attempt to make a contribution toward defining the content of this milieu.

Need for Causality

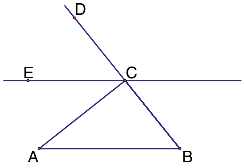
Definition and function. Certainty is achieved when an individual determines (by whatever means he or she deems appropriate) that an assertion is true. Truth alone, however, may not be the only aim for the individual, and he or she may desire to know *why* the assertion is true—the cause that makes it true. Thus, the *need for causality* is one's desire to *explain*, to determine a cause of a phenomenon. "Mathematicians routinely distinguish proofs that merely demonstrate from proofs which explain" (Steiner, 1978, p. 135). For many, the role of mathematical proofs goes beyond achieving certainty—to show that something is true; rather, "they're there to show … why [an assertion] is true," as Gleason, one of the solvers of Hilbert's Fifth Problem (Yandell, 2002, p. 150), points out. Two millennia before him, Aristotle, in his *Posterior Analytic*, asserted,

> We suppose ourselves to possess unqualified scientific knowledge of a thing, as opposed to knowing it in the accidental way in which the sophist knows, when we think that we know the cause on which the fact depends as the cause of the fact and of no other. (p. 4)

Like with certainty, humans' instinctual desire to explain phenomena in their environments serves as a cognitive primitive to mathematical justification. The distinction between achieving certainty and finding causality in mathematics was the focus of a debate during the sixteenth and seventeenth centuries. Some philosophers of this period argued that mathematics is not a perfect science because mathematics is concerned with mere certainty rather than cause: Mathematicians are satisfied when they arrive at a conclusion by logical implications but do not require the demonstration of the *cause* of their conclusion (Mancosu, 1996). These philosophers point, for example, to Euclid's proof of Proposition 1.32 (the sum of the three interior angles of any triangle ABC is equal to 180°). Consider Euclid's proof of this proposition (Fig. 6.2).

In this proof, these philosophers argue, the cause of the property that is proved is absent. The two facts to which the proof appeals—the one about the auxiliary segment *CE* and the one about the external angle *ACD*—cannot be the true cause of the property, for the property holds whether or not the segment *CE* is produced and the angle *ACD* considered. A causal proof, according to these philosophers, gives not just evidence of the truth of the theorem but of the *cause* for the proposition's truth.

Proof by contradiction was another example of a noncausal proof in the eyes of these philosophers. When a statement "*A* implies *B*" is proved by showing how not



Construct CE parallel to AB. Then the alternate angles BAC and ACE are congruent and the corresponding angles ABC and ECD are congruent. Hence, $m(\sphericalangle BAC) + m(\sphericalangle ABC) + m(\sphericalangle ACB) = 180°$.

Fig. 6.2 Euclid's proof for the triangle angle sum theorem

B (and *A*) leads logically to an absurdity, we do not learn anything about the causal relationship between *A* and *B*. Nor, continued these philosophers to argue, do we gain any insight into how the result was obtained. Proofs by contradiction continued to be controversial until the late nineteenth and early twentieth centuries. In 1888, for example, Hilbert astonished the mathematical community of the time when he proved the Gordan conjecture: There is a finite "basis" from which all algebraic invariants of a given polynomial form could be constructed by applying a specified set of additions and multiplications. It was more the form of Hilbert's solution than the shear success in solving an open problem that was controversial. Hilbert didn't *find* a basis that everyone had been searching for; he merely proved that if we accept Aristotle's law of the excluded middle ("Any statement is either true or its negation is true") then such a basis had to exist, whether we could produce it or not.

> At first this result was greeted with disbelief. Gordan said, "Das ist nicht Mathematic. Das ist Theology." Cayley at first failed to grasp the proof. Lindemann thought the proof unheimlich ("uncomfortable, sinister, weird"). Only Klein got it right away: "Wholly simple and, therefore, logically compelling." Within the next five years organized opposition disappeared, and this was the result that initially made Hilbert's reputation. (Yandell, 2002, p. 12)

Why was Hilbert's use of proof by contradiction so controversial? After all, he was not the first to use this method of argument? According to Yandell, previous uses had not dealt with a subject of such obvious calculational complexity. A pure existence proof does not produce a specific object that can be checked—one had to trust the logical consistency of the growing body of mathematics to trust the proof. The presence of an actual object that can be evaluated provides more than mere certainty; it constitutes a cause (in the Aristotelian sense) for the observed phenomenon.

The philosophical stance about the scientific nature of understanding and its implication that mathematical proofs must conform to the Aristotelian definition of science seems to have played a role, perhaps implicitly, in Grassmann's (1809–1877) work. According to Lewis (2004), when Grassmann published his theory of extension (*Ausdehnungslehre*) in 1844, and again, in a modified version, in 1862, it went unnoticed, partly due to its novel and large-scale discoveries, and partly due to its novel method of presentation. The latter is of particular relevance to our discussion about the need for certainty versus the need for causality. Grassmann insisted on a presentation that met the highest standards of rigor, on the one hand, and provided the reader with a clear understanding of the epistemological justifications behind his concepts and proofs, on the other. Grassmann's insistence on such a presentation, according to Lewis, goes beyond pedagogical considerations to help the reader grasp his new concepts and techniques; rather, Grassmann "appears to regard the pedagogical involvement as an essential part of the justification of mathematics as a science" (p. 19).

Recall that *proving* and *explaining* are two different, yet related, mental acts: the first is carried out to remove doubts, and the second to determine cause. Accordingly, a *proof* is a way of understanding associated with the mental act of proving, and an *explanation* is a way of understanding associated with the mental act of explaining. Often, when facing a particular assertion or arriving at a conjecture, one may carry out the two mental acts of proving and explaining together, resulting in a single product that is both a proof and an explanation—it removes doubts about the truth of the assertion and provides a reason, or a cause, for its truth. The issue of what makes a proof a causal proof (i.e., proof *and* explanation) was addressed by Steiner (1978). He distinguishes between proofs that prove and proofs that explain, but his distinction is a priori, independent of the individual's conceptions. This distinction and its corresponding ontological position are adopted by Hanna (1990), who argues that proofs by mathematical induction, for example, are proofs that prove but do not explain. Our position is different. We hold that it is the individual's scheme of doubts, truths, and convictions in a given context that determines whether an argument is a proof or an explanation.

Pedagogical considerations. This historical analysis, together with the findings discussed earlier about the ineffectiveness of some intuitively sound teaching practices, led to a pedagogical lesson regarding the transition from undesirable proof schemes, especially the empirical proof schemes, to deductive proof schemes. The idea is to shift students' attention from *certainty* to *cause*. Rather than justifying the need for deductive proofs by raising questions about the logical legitimacy of empirical proofs—which, as indicated earlier, turned out to have little or no perturbational effect—we turned students' attention to the cause (or causes) that makes an assertion true or false. By repeatedly attending to explanations as well as to proofs, we aimed at enculturating students into the habit of seeking to understand cause, not only attaining certainty. To illustrate how this can be done, consider the following episode: A group of in-service secondary teachers participating in a professional development summer institute were given the Quilt Problem (Fig. 6.3).

A company makes square quilts. Each quilt is made out of small congruent squares, where the squares on the main diagonals are black and the rest are white. The cost of a quilt is calculated as follows: Materials: \$1.00 for each black square and \$0.50 for each white square; Labor: \$0.25 for each square. To order a quilt, one must specify the number of black squares, or the number of white squares, or the total number of squares on the following order form:

Number of Black Squares	**Number of White Squares**	**Total of Squares**

April, Bonnie, and Chad ordered three identical quilts. Each of the three filled out a different order form. April entered the number of black squares in the Black Cell. The other two entered the same number as April's, but accidentally Bonnie entered her number in the Whites Cell, and Chad entered his number in the Total Cell. April was charged \$139.25. How much money were Bonnie and Chad charged?

Fig. 6.3 The Quilt Problem

Fig. 6.4 Nina's equations

$$\text{Price} = \#\text{Black} + \frac{(\#\text{ White})}{2} + \frac{(\#\text{Black} + \#\text{ White})}{4}.$$

$$139.25 = 2x - 1 + \frac{(x-1)^2}{2} + \frac{\left((2x-1)+(x-1)^2\right)}{4}.$$

Fig. 6.5 Nina's pattern for the number of white squares

Size	# White Squares
1	0
3	4 (4·1)
5	16 $(3\cdot 1)+(1\cdot 4) = 4(4) = 4(2\cdot 2) = 4\cdot\left(\frac{5-1}{2}\right)^2$
7	36 $4(9) = 4(3\cdot 3) = 4\cdot\left(\frac{7-1}{2}\right)^2$
…	
x	$4\cdot\left(\frac{x-1}{2}\right)^2 = 4\cdot\frac{(x-1)^2}{4} = (x-1)^2$

The teacher participants worked in small groups on the problem for some time, and then each group presented its solution (whether complete or partial). Nina,[5] a teacher participant in the institute, presented her group's solution. The solution considers two cases: an even-sized quilt and an odd-sized quilt. Our discussion here pertains to the odd-sized case, but for the sake of completeness the even-sized case is also presented.

Nina noted that viewing each partial cost in terms of units of \$0.25 excludes the possibility that the quilt is of an even-sized dimension, for if the dimension were even then each partial cost, and therefore the total cost as well, would comprise an even number of \$0.25 – units. But the total cost (\$139.25) comprises an odd number of these units. For the odd-sized case, Nina first wrote the two lines in Fig. 6.4: where x is the size of the quilt (the number of squares on each side). Following this, she proceeded to solve the algebraic equation. When asked by one of the teacher participants in the class why the number of whites is $(x-1)^2$, Nina responded by presenting a table (see Fig. 6.5).

Nina indicated that this table was the result of an effort by her group to express the number of white squares as a function of the quilt's size. Based on these special cases, the group concluded that for an x-sized quilt, the number of white squares is $4\left(\frac{x-1}{2}\right)^2$, or $(x-1)^2$.

In the discussion that followed this presentation, it was clear that the class as a whole was impressed by Nina's solution, and was convinced that the generalization was valid. Rather than dwelling on the question of how we know the pattern

[5]Pseudonyms.

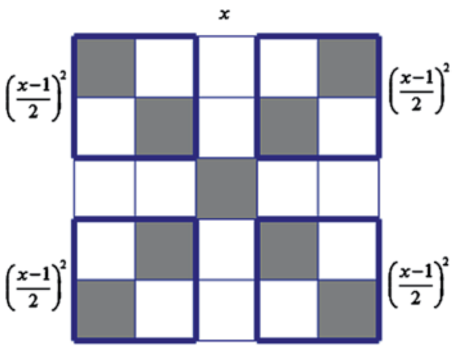


Fig. 6.6 The drawing that accompanied John's solution

continues to be valid for all positive odd integers, the instructor presented an alternative solution offered by John, one of the teacher participants in an earlier summer institute. In John's solution, x is an odd number representing the size of the quilt (the number of square on its side). Removing the middle row and middle column of squares (those containing the black square shared by the main diagonals) leaves four "subsquares" having $\left(\frac{x-1}{2}\right)^2$ squares, including black squares from the diagonals (Fig. 6.6). So, excluding the row and column previously removed, there would be a total of $4\left(\frac{x-1}{2}\right)^2$, or $(x-1)^2$, squares in the four subsquares. Since each black square along the diagonal corresponds to one square that had been removed by eliminating the row and column containing the center black square, the number of white squares in an x-sized quilt remains $(x-1)^2$.

The teacher participants had been impressed by Nina's solution and they were equally impressed by John's solution. The general consensus among the teachers was that both solutions are convincing, but John's solution has an added value; it reveals the *reason* (i.e., the *cause*) for *why* the number of white squares is $(x-1)^2$.[6]

Our experience from these professional development institutes and other teaching experiments is that through repeated experiences such as the one described here—of comparing empirical solutions (such as Nina's) with causal solutions (such as John's)—learners gradually come to the realization that one type of reasoning is of more intellectual value than the other. Whereas empirical reasoning provides them with certainty (because of their robust *empirical proof scheme*), causal reasoning provides them with both certainty and enlightenment (understanding of cause). We observed a change in the teacher participants' argumentation for

[6]Other solutions were offered by the class. For example, one solution examined all the possible cases for the size of the quilt, and another solution simply calculated the number of white squares by subtracting the number of black squares from the total number of squares.

ascertainment and persuasion after having gone through this experience for an extended period of time. Thus, shifting the focus from certainty to causality seems to have effected the teacher participants' schemes of doubts and, in turn, their proof schemes. Though they continued to produce empirical proofs, they also sought casual justifications.[7]

Need for Computation

Definition and function. After Piaget, quantifying is the act of transforming a sensation (i.e., a perceptual action scheme—visual, auditory, tactile, etc.) into a quantity—a measurable sensation. For example, the sensation *fastness* is transformed into *speed*; *heaviness* into *weight*; *extent* into *length*, *area*, or *volume*; *pushing or pulling* into *force*; *rotational twist* into *torque*; *hotness* into *heat* (i.e., *thermal energy*), etc. Some sensations might be difficult to quantify; "texture," "taste," "pain," "happiness," and "instructional quality" are examples. The quantification process involves assigning a unit of measure to a quantity; for example, "mph," "gram," "Newton meter," and "square meter" are unit measures assigned to the quantities "speed," "weight," "torque," and "area," respectively. As can be seen from these examples, often quantification is a nested act: one quantity is constructed from previously formed quantities.

Sensations such as *fastness* and *heaviness* constitute cognitive primitives to the need to quantify, which is one expression of the *need to compute*. Another expression is the act of determining a missing quantity from a set of quantitative constraints, as when, for example, one seeks to determine the dimensions of a right triangle from its area and the ratio of two of its sides. Collectively, these two expressions of the need to compute manifest humans' desire to accurately compare different sensations, determine their interrelationships, and, in turn, better understand and control their own physical and social environment.

The need to compute is not the invention of modern mathematics. The Babylonians (around 2000 B.C.) engaged in problems that required determining the value of a quantity from other given quantities. For example, they invented procedures for solving what we now view as quadratic equations (e.g., how to find the side of a square when the difference between the area and the side is given). This practice of computing continued to develop in different cultures throughout history, and it led gradually to the development of symbolic algebra, and, in turn, to new mathematical concepts (such as complex numbers, equations, and polynomials) and a system of symbols to represent these concepts. These invented symbols necessitated the creation of new concepts. For example, the Babylonian numerical system

[7]This transition involved interesting cognitive disequilibria, which are not discussed in this paper.

is a positional notation system (i.e., utilizes the principle of "place value") but denotes all the multiples $k60^{\pm n}$ by the same sign (e.g., the string of symbols, ٢ ٢   ٢ ٢, might mean $2(60)+2$, $2(60)^2+2(60)$, or $2(60)^3+2(60)^2$). The computational ambiguity of this system necessitated the conception of zero as a number and the introduction of this number into calculation. This conception, in turn, led to the creation of the numerical system of our present time, which removed the ambiguity and advanced the computational effectiveness of the place value system used by Babylonians. Another example of how the need for computation led to the creation of symbols, and, in turn, to the creation of new concepts, is from a later period. The Leibnitzian notation Df, $D^2 f$, etc. was needed to display the number of successive differentiations, but it also suggested the possibility of extending the meaning of $D^{\alpha} f$ for negative and fractional α. Davis and Hersh (1981) point out that this invention contributed powerfully to the development of abstract algebra in the mid-nineteenth century. In addition, this notation may have necessitated, or at least helped to advance, the object conception of function (in the sense of Dubinsky, 1991), namely, that f, in addition to being a process that assigns to a given input-number a single output-number, is itself an operand (an input) for another process, D. Overall, the nature of computing evolved rather slowly. As late as the fifteenth century, mathematicians lacked the ability to compute with symbols independent of their spatial referents—and encountered major difficulties as a result. For example, a major obstacle in justifying the formula for the roots of the cubic equation was the inability to figure out the identity $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$, whose proof required dissection of a cube in three-dimensional space (Tignol, 1988). Only later, with the work of Cardono (1501–1576), was the formula for the cubic equation justified by means of *symbolic algebra*—specifically, by transforming different forms of cubic equations into systems of equations.

To *compute by means of symbolic algebra* reflects two inseparable abilities: (a) the ability to represent a situation symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their reference, and (b) the ability to pause at will during the manipulation process in order to probe into the referential meanings for the symbols involved in the manipulation. The attempt to form a referential meaning need not always succeed or even occur. What matters is that the person who carries out the manipulation has the ability to investigate, when needed, the referential meaning of any symbol and transformation involved. In this paper, the *need to compute* is in the sense of this definition, in that it refers to one's desire to quantify, determine a missing object or construct an object (e.g., a number, geometric figure, function, or matrix), determine the property of an object or relations among objects, etc. by means of symbolic algebra. It also includes the need to find more efficient computational methods, such as one might need to extend computations to larger numbers in a reasonable "running time."

Historically, the practice of manipulating symbols without *necessarily* examining their meanings played a significant role in the development of mathematics. For example, during the nineteenth century a significant work was done in differential and difference calculus using a technique called "operational method," a method whose results are obtained by symbol manipulations without understanding their meaning, and in many cases in violation of well-established mathematical rules (see, for example, the derivation of the Euler–MacLaurin summation formula for approximating integrals by sums, in Friedman (1991)). Mathematicians sought meaning for the operational method, and with the aid of functional analysis, which emerged early in the twentieth century, they were able to justify many of its techniques.

Computing by means of symbolic algebra marked a revolutionary change in the history of mathematics. In particular, it provided a conceptual foundation for the critical shift from "results of operations" as the object of study to the operations themselves as the object of study. While the Greeks restricted their attention to attributes of spatial configurations and paid no attention to the operations underlying them, nineteenth-century mathematics investigated the operations, their algebraic representations, and their structures. For example, Euclidean constructions using only a compass and straightedge were translated into statements about the constructability of real numbers, which, in turn, led to observations about the structure of constructible numbers. A deeper investigation into the theory of fields led to the understanding of why certain constructions are possible whereas others are not. The Greeks had no means to build such an understanding, since they did not attend to the nature of the operations underlying Euclidean construction. Thus, by means of symbolic algebra and analytic geometry, mathematicians realized that all Euclidean geometry problems can be solved by a single approach, that of reducing the problems into equations and applying algebraic techniques to solve them. Euclidean straightedge-and-compass constructions were understood to be equivalent to equations, and hence the solvability of a Euclidean problem became equivalent to the solvability of the corresponding equation(s) in the constructible field.

Pedagogical considerations. The need for computation, perhaps the most powerful need in the context of school mathematics, is rarely utilized adequately. For example, after learning how to multiply polynomials, secondary-school students typically learn techniques for factoring polynomials, and then how to apply these techniques to simplify rational expressions. Judging from the students' perspective, the tasks of multiplying and factoring polynomials and simplifying rational expressions are intellectually purposeless. They learn to transform one form of expression into another without a clear understanding of the mathematical purpose such transformations serve and the circumstances under which one form of expression is more advantageous than another. A case in point is the way the quadratic formula is taught. Some algebra textbooks present the quadratic formula before the method of completing the square. Seldom do students see an intellectual purpose for the latter method (i.e., to solve quadratic equations and to derive a general formula for their solutions), rendering completing the square problems intellectually purposeless to most students. An alternative approach that would intellectually necessitate such problems builds on what the students know: Assuming that the students have already learned how to solve equations of the form $(x+T)^2 = L$, the teacher's action would be geared toward helping them manipulate the quadratic equation $ax^2 + bx + c = 0$ with a goal in mind—that of transforming the latter equation form into the former known equation form but maintaining the solution set unchanged. The intellectual gain is that students learn that algebraic expressions are reformed for a reason.

Often problems used to introduce a new concept do not demonstrate the intellectual benefit of the concept at the time of its introduction. For example, some high-school mathematics texts introduce the idea of using equations to solve word problems through trivial, one-step addition or multiplication word problems (see Harel, 2009). This approach is contrived, and is unlikely to intellectually necessitate this idea since students can easily solve such problems with tools already available to them. To make this point clearer, it is worth presenting an alternative approach—one that is more likely to intellectually necessitate algebraic tools to solve word problems. In this alternative approach, students first learn to solve nontrivial word problems with their current arithmetic tools. For example, they can reason about problems of the following kind directly, without any explicit use of variables.

> Towns A and B are 280 miles apart. At 12:00 PM, a car leaves A toward B, and a truck leaves B toward A. The car drives at 80 m/h and the truck at 60 m/h. When will they meet?

Students can do so by, for example, reasoning as follows:

> After 1 h, the car drives 80 miles and the truck 60 miles. Together they drive 140 miles. In 2 h, the car drives 160 miles and the truck 120 miles. Together they drive 280 miles. Therefore, they will meet at 2:00 PM.

Through this kind of reasoning, students develop the habit of building coherent images for the problems—a habit they often lack.

These problems can then be gradually modified (in context, as well as in quantities) so as to make them harder to solve with arithmetic tools alone, whereby necessitating the use of algebraic tools. For example, varying the distance between the two towns through the sequence of numbers, 420, 350, 245, and 309, results in a new sequence of problems with increasing degree of difficulty. Students still can solve these problems with their arithmetic tools but the problems become harder as the relationship between the given distance and the quantity 140 (the sum of the two given speeds) becomes less obvious. For example, for the case where the distance is 245 miles, the time it takes until the two vehicles meet must be between 1 and 2 h, and so one might search through the values 1 h and 15 min $\left(80\frac{75}{60}+60\frac{75}{60}\overset{?}{=}245\right)$, 1 h and 30 min $\left(80\frac{90}{60}+60\frac{90}{60}\overset{?}{=}245\right)$, and 1 h and 45 min $\left(80\frac{105}{60}+60\frac{105}{60}\overset{?}{=}245\right)$, and find that the last value is the time sought for. This activity of varying the time needed can give rise to the concept of variable (or unknown) and, in turn, to the equation, $80x+60x=245$. Granted, this is not the only approach to intellectually necessitate the use of algebraic tools for solving word problems. However, whatever approach is used, it is critical to give students ample opportunities to repeatedly reason about problems with their current arithmetic tools and to gradually lead them to incorporate new, algebraic tools. The goal is for students to learn to build coherent mental representations for the quantities involved in the problem and to intellectually necessitate the use of equations to represent these relationships. An added value of this approach is the development of computational fluency with numbers (especially fractions).

The inadequate use of the need for computation is prevalent in undergraduate mathematics as well. For example, typically, linear algebra textbooks introduce the pivotal concepts of "eigenvalue," "eigenvector," and "matrix diagonalization" with statements such as the following:

> The concepts of "eigenvalue" and "eigenvector" are needed to deal with the problem of factoring an $n \times n$ matrix A into a product of the form XDX^{-1}, where D is diagonal. The latter factorization would provide important information about A, such as its rank and determinant.
>
> The concepts of "eigenvalue" and "eigenvector" are needed to deal with the problem of computing a higher order of power of a given matrix, to study the long-term behavior of linear systems.
>
> The concepts of "eigenvalue" and "eigenvector" are needed to deal with a problem that arises frequently in application of linear algebra—that of finding values of a scalar parameter λ for which there exists $x \neq 0$ satisfying $Ax = \lambda x$, where A is a square matrix.

Each of these introductory statements aims at pointing out to the student an important problem. While the problem is intellectually intrinsic to its poser (a university instructor), it is most likely to be alien to the students, since a student in an elementary linear algebra course is unlikely to realize from such statements the true nature of the problem, its mathematical importance, and the role the concepts to be taught ("eigenvalue," "eigenvector," and "diagonalization") play in determining its solution.

An alternative approach, based particularly on students' intellectual need for computation, is through linear systems of differential equations. In what follows, I briefly outline part of a unit in a linear algebra course I have taught numerous times, some of which as teaching experiments. The goal of the unit is to necessitate fundamental ideas of the Eigen Theory, from the basic concepts of eigenvalue, eigenvector, diagonalization, and their related theorems up to the Jordan Theorem (i.e., "Every vector is a linear combination of generalized eigenvectors.") and its related Jordan Canonical Form. The unit begins with an investigation of the linear system of differential equations:

$$\begin{cases} AY(t) = Y'(t) \\ Y(0) = C \end{cases} \tag{*}$$

(Here A is a square matrix, and the matrix A and the vector C, the initial condition vector, are over the complex field.) Obviously, this system and its representation in a matrix form do not emerge in a vacuum, but out of a context established in previous units. The investigation consists of a series of stages. Here I focus on the first several stages that lead up to the concept of diagonalization.

In the first stage of the investigation, we help students analogize system (*) to the scalar case:

$$\begin{cases} ay(t) = y'(t) \\ y(0) = c \end{cases} \tag{**}$$

(Here a and c are real numbers.) The students are familiar with this equation and its (unique) solution, $y(t) = c\mathrm{e}^{at}$, from their calculus classes. To refresh their memory of this topic, we assign them (prior to the start of the unit on Eigen Theory) a few problems involving this equation and the exponential function power series. Students notice the similarity in form between (*) and (**), and accordingly offer the analogous expression, $Y(t) = C\mathrm{e}^{At}$, as a solution to system (*). It takes some prompting from the instructor for the students to attend to the meaning of the objects and operations involved in this expression. After some discussion, the students offer to rewrite the product At as tA, and ask about the meaning of the phrase "e to the power of a matrix." At this point, students' attention is centered on this phrase, and so questions concerning the dimension of the matrix and whether the product $C\mathrm{e}^{tA}$ is meaningful are not raised. Often, but not always, students suggest that e^{B} is the matrix with the entries $(\mathrm{e}^{B})_{i,j} = \mathrm{e}^{B_{i,j}}$. With this definition at hand,[8] the instructor provides a special case of system (*) and asks the students to verify whether the expression $Y(t) = C\mathrm{e}^{At}$ is a solution to the system, as they have conjectured. In this process, the students first realize the need to reverse the order of the product $C\mathrm{e}^{At}$ into $\mathrm{e}^{At}C$, and then conclude that the revised expression $Y(t) = \mathrm{e}^{tA}C$, under their definition of the matrix-valued exponential function, is not a solution to system (*). Consequently, students conclude that the solution to system (*) must be of different form from the one they offered; it does not occur to them to seek a different definition for the matrix-valued exponential function.

The second phase of the investigation commences with the instructor suggesting a different approach for defining this function. He reminds the students of the definition of the real-valued function e^{b} as a power series (a topic they reviewed in the preceding unit). Some students suggest analogizing e^{B} to e^{b}; namely, that analogous to $\mathrm{e}^{b} = \sum_{i=0}^{\infty}(1/i!)b^{i}$, we define $\mathrm{e}^{B} = \sum_{i=0}^{\infty}(1/i!)B^{i}$. Again, despite the use of the term "define," students do not view the latter equality as a definition but as a formula, perhaps because the former equality was derived from a Taylor expansion rather stated as a definition. Nor do they raise any concern about the convergence of the series. Furthermore, only when the instructor asks the class to compute e^{B} for a particular simple 2×2 matrix B do the students realize that e^{B} is meaningless unless B is a squared matrix, and consequently they observe that e^{B} too is a squared matrix. With this new definition at hand, the instructor leads the class in the process of verifying that $Y(t) = \mathrm{e}^{tA}C = \sum_{i=0}^{\infty}(t^{i}/i!)A^{i}C$ is a solution to system (*). As with the question of convergence, the question of uniqueness too is never addressed in this class.

In the third phase of the investigation, the instructor returns to the above solution in its expansion form ($Y(t) = \mathrm{e}^{tA}C = \sum_{i=0}^{\infty}(t^{i}/i!)A^{i}C$) and points out the following critical observation: If $AC = \lambda C$ for some scalar λ, then the solution to system (*) is easily computable. Specifically, it is $Y(t) = \mathrm{e}^{\lambda t}C$, for under this condition $Y(t) = \sum_{i=0}^{\infty}(t^{i}/i!)A^{i}C = \sum_{i=0}^{\infty}(t^{i}/i!)\lambda^{i}C = \sum_{i=0}^{\infty}((\lambda t)^{i}/i!)C = \mathrm{e}^{\lambda t}C$. This

[8]The use of the term "definition" here should not imply that the students' intention was to *define*—in the mathematical sense of the term—the concept "e to the power of a matrix" (see the discussion on *definitional reasoning*).

observation necessitates attention to the relationship $AC = \lambda C$, and therefore a name: C is called an eigenvector of A and λ its corresponding eigenvalue. Following a few examples of solving system (*), the instructor (and in a few cases a student) raises the question about the computability of the solution in cases where the condition vector is not an eigenvector of the coefficient matrix. The instructor suggests looking at the case where C is not an eigenvector of A but it is a linear combination of eigenvectors of A. We proceed to show that in this case too the solution to system (*) is easily computable. Specifically $Y(t) = \sum_{i=1}^{n} a_i e^{\lambda_i t} C_i$, where $C = \sum_{i=1}^{n} a_i C_i$ and $AC_i = \lambda_i C_i$. This result is then used to conclude that if the coefficient matrix has a basis of eigenvectors then for any condition vector the solution to system (*) is easily computable. Such a matrix, therefore, is of a computational significance, and hence it warrants attention. This concludes the third phase of the investigation.

The content of the next phases depends on the level of the course. For an elementary linear algebra course, the proceeding phases deal with the factorization of matrices with a basis of eigenvectors (i.e., *diagonalization*) and change of basis. For the more advanced linear algebra course, the proceeding phases continue the investigation of the computability of the solution to system (*). The investigation leads up to the Jordan Theorem (and its related Canonical Form), which yields the interesting results that the solution to system (*) is always easily computable.

All the alternative approaches discussed here demonstrate how both conditions (a) and (b) in our definition of computing by means of symbolic algebra are implemented. It is never the case that every single symbol in the manipulation process is referential. Rather, it is only in critical stages (viewed as such by the person who carries the symbol manipulations) that one forms, or attempts to form, referential meanings. One does not usually attend to interpretation in the middle of symbol manipulations unless one encounters a barrier or recognizes a symbolic form that is of interest to the problem at hand. Thus, for most of the process the symbols are treated as if they have a life of their own. It is in this sense that symbol manipulation skills should be understood and, accordingly, be taught.

Needs for Communication

Definition and function. In mathematics, the *need for communication* refers collectively to two reflexive acts: *formulating* and *formalizing*. Formulating is the act of transforming strings of spoken language into algebraic expressions (i.e., expression amenable to computation by means of symbolic algebra as discussed in the preceding section). Formalization is the act of externalizing the exact intended meaning of an idea or a concept or the logical basis underlying an argument. A cognitive primitive of these two acts is the act of conveying and exchanging ideas by means of a spoken language and gestures, which are defining features of humans.

In modern mathematics the acts of formulation and formalizations are reflexive in that as one formalizes a mathematical idea it is often necessary to formulate it, and, conversely, as one formulates an idea one often encounters a need to formalize it.

Historically, however, the need for formulation seems to have emerged well after the need for formalization. At least in the Western world the need for formalization began with Greeks, whereas that of formulating with Viete (1540–1603) and Stevin (1548–1620). These two scholars are viewed by historians as milestones in the evolution of the need for formulation, and, in turn, in the evolution of the need for formalization beyond the Greeks. Until then, the exchange of mathematical ideas was largely colloquial (i.e., idiomatic and conversational). The Babylonians (around 2000 B.C.), for example, used only text to exchange problems and procedures for their solutions, as can be seen in one of their tablets:

> I have subtracted from the area the side of my square: 14.30 [meaning, the result is 14.30]. [To solve], divide 1 into two parts: 30. Multiply 30 and 30: 15. You add to 14.30, and 14.30.15 has the root 29.30. You add to 29.30 the 30 which you have multiplied by itself: 30, and this is the side of the square. (Tignol, 1988, p. 7).

The arithmetic here is in base 60, so, for example, 14.30 in base 10 is $14\times 60+30=870$. Tignol points out that the "Babylonians had no symbol to indicate the absence of a number or to indicate that certain numbers are intended as fractions. For instance, when 1 is divided by 2, the result which is indicated as 30 really means 30×60^{-1}, i.e., 0.5" (p. 7).

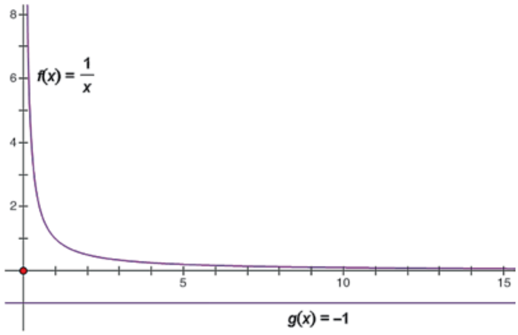
Three and a half millennia later, Cardano began to *formulate* the notation of equations. For example, the equation $x^2+2x=48$ is written by Cardano as, "*1. quad. P : 2 pos. aeq. 48* (*quad*. for 'quadratum'; *pos*. for 'positiones' and *aeq*. for 'aequatur')" (Tignol, 1988, p. 36). An essential characteristic of this type of notation is that *its syntax is in the form of a spoken language*. Of course, it is both simplistic and wrong to bundle a span of over three millennia of development of mathematical notation into a single characteristic. This is not our intention here. Rather, we merely aim at pointing to one of the features of the notational conventions of the time: the use of immediate, natural tools of a spoken language to exchange ideas. Remarkably, this level of notation was sufficient to attain major achievements, the most notable of which is the solution of the cubic equation.

Pedagogical considerations. Spoken knowledge is an essential means for the development of the need for (mathematical) communication. Consider the following example: Students may be satisfied with their intuitive explanation of why $\lim_{x\to\infty} 1/x=0$, which typically is communicated through a statement such as the following: "$\lim_{x\to\infty} 1/x=0$ because the larger x gets the closer $1/x$ is to 0." A teacher whose goal is to help students see a need to formulate and formalize their concept of *limit* might proceed, upon hearing this explanation, by writing it on the board along with the graphs of $f(x)=1/x$ and $g(x)=-1$ (see Fig. 6.7).

Then the teacher may point out to the students that based on their own statement one can rightly argue $\lim_{x\to\infty} 1/x=-1$, because, by their own words, "the larger x gets the closer $1/x$ is to -1." This exchange may, as our experience confirms, result in a conflict for the students, whereby they see a need to formulate and formalize their idea of limit.

The reflexive nature between the need for formulation and the need for formalization is best captured by Thompson (1992), in analyzing the use of concrete

Fig. 6.7 The graphs of $f(x) = 1/x$ and $g(x) = -1$



materials in elementary mathematics instruction: "When students are aware of reciprocal relationships between notation and reasoning they may be more inclined to concentrate on their reasoning when experiencing difficulty and concentrate less on performing correct notational actions" (p. 124). Thompson places students' use of concrete materials in the context of their development of the use of notation to express their reasoning. He points out that authoritative need deprives students from the opportunity to see a (intellectual) need for formalization:

> Students' reenactment of a prescribed procedure does not give them opportunities to construct constraints in their meanings and reasoning—they meet constraints only because they are obliged to adhere to prescription, and it matters little that the prescriptions entail use of concrete materials. In reenacting prescribed procedures, students do not experience constraints as arising from tensions between their attempts to say what they have in mind and their attempts to be systematic in their expressions of it. (p. 124)

Conversely, it is the need for formalization that compels students to formulate (or reformulate) their symbolic system:

> As students come to be systematic in their expressions of reasoning and make a commitment to express their reasoning within their system, that same systematicity places constraints on the reasoning they wish to express. When students are aware of the constraining influence exerted by their arbitrary use of notation, they may feel freer to modify their standard uses of notation to express better what they have in mind. (p. 124)

Repeated application of the need for formulation and formalization is necessary to advance students' conception of the notion of *mathematical definition*. This conception is associated with *definitional reasoning*—a way of thinking by which one defines objects and proves assertions in terms of mathematical definitions. A mathematical definition is a description that applies to all objects to be defined and only to them. A crucial feature of this way of thinking is that with it one is compelled to conclude logically that there can be only one mathematical definition for a concept within a given theory; namely, if D_1 and D_2 are such definitions for a concept C, then D_1 is a logical consequence of D_2, or vice versa; otherwise, C is not well defined. Typically, students' definitions of concepts are not mathematical, even if the concepts were defined to them mathematically. Understanding the notion of mathematical definition and appreciating the role and value of mathematical definitions in proving is a developmental process, which is not achieved for most students until adulthood (if at all). Many students even in advanced grades do not possess definitional reasoning. For instance, in Van Hiele's (1980) model, only in the highest stage of geometric reasoning are students' definitions of Euclidean objects mathematical (see Burger & Shaughnessy, 1986). Definitional reasoning is largely absent among college students as well, even among undergraduate mathematics and engineering majors (Harel, 1999). For example, when asked to define "invertible matrix," many linear algebra students stated a series of equivalent properties (e.g., "a square matrix with a non-zero determinant," "a square matrix with full rank," etc.) rather than a definition. The fact that they provided more than one such property is an indication that they were not thinking in terms of mathematical definition.

Need for Structure

Definition and function. The *need for structure* is the need to reorganize the knowledge one has learned into a logical structure. A critical element in this definition is the verb "to reorganize," and, by implication, its source verb "to organize." The verb "to organize" implies an action on something that already exists, and the verb "to reorganize" implies that something has already been organized. Accordingly, the need for structure is not a forward need; that is, one does not feel intellectually compelled to learn new knowledge in a particular order and from that fit a predetermined structure; rather, one assimilates knowledge into one's existing structure, and reorganizes it if and when one perceives a need to do so. The nature of the structure into which one organizes one's own knowledge is idiosyncratic and depends entirely on one's past experience. Such a structure is unlikely to be logically hierarchical, and even mathematicians are unlikely to involuntarily organize their knowledge into a systematic logical structure. Thus, the term "reorganize" in the above definition recognizes that individual learners or communities of learners first organize the mathematical knowledge they learn in a form determined by their existing cognitive structures; later they may meet the need to *reorganize* what they have learned into a logical structure. The history of Euclidian geometry illustrates this point. Perhaps the most recognized mathematical structure is *Euclid's Elements*, a geometrical edifice organized in a logical structure where each assertion depends on the previous ones. Relevant to our discussion here are two historical observations. First, the development of the theorems in the *Euclid's Elements* did not follow a systematic logical progression, as it is laid out in this treatise, but evolved largely unsystematically over several centuries. Second, it was the need to organize this accumulated body of knowledge that led to the production of the logical structure of axioms, definitions, and propositions, as we know it; it was the need to perfect this structure that, in turn, led to the two-millennium-long attempt to prove the parallel postulate.

The need for structure often leads to the discovery of unifying principles (e.g., *associativity*), common elements to different systems (e.g., the *identity element*), invariants (e.g., for the quadratic form, $ax^2 + bx + cy^2$, the form $b^2 - 4ac$ remains unchanged under rotations and scalar changes of the axes), and similarities or analogies of form, which may, in turn, lead to recognizing isomorphism between different systems. It also often leads to a unification of scattered ideas into a single concept. "Convergence," as was formalized by Cauchy in 1827, is an example of knowledge reorganization. The particularities of convergence were well known and widely used prior to this time, but Cauchy's formalization reorganized and unified this knowledge into a single concept: "convergence."

Sometimes the need for structure compels us to define objects in a particular way. For example, we define $x^0 = 1$ for $x \neq 0$ in order for the familiar law of exponents to hold for nonzero bases, x. Specifically $1 = \frac{x}{x} = \frac{x^1}{x^1} = x^{1-1} = x^0$. On the other hand, 0^0 is excluded in this definition because it leads to the ill-founded statement, $0^0 = \frac{0}{0}$. The proposed definition $0^0 = 1$ is not forced by any demands of consistency with laws of exponents. That being said, mathematicians frequently adopt the convention that $0^0 = 1$ anyway, in order, for example, to make the binomial theorem and Taylor's theorem valid for zero values of a variable.

Another important aspect of the need for structure is the need to make connections—for example, the need to analogize structures, problems, and solutions to problems. H. Bass (personal communication, May 15, 2012) calls these aspects *theory building*. Our earlier discussion concerning Eigen Theory provides an example for the need to analogize structures. We have seen how students successfully analogized between two structures: from a scalar differential equation to a matrix differential equation, and from a real-valued exponential function to a matrix-valued exponential function.

As to the role of analogy in mathematical practice, this topic has been debated widely in the literature in cognitive psychology and mathematics education (see, for example, English, 1997; Simon & Hayes, 1976). For the sake of completeness, however, I briefly discuss here one example. Consider the problem, "In how many ways can 8 identical chocolate bars be distributed into three groups, where none of the bars are to be broken and each group must contain at least one bar?" A tenth-grade student solved the problem by analogizing it to what was to him a simpler problem. He began by saying something to the effect that when the eight bars are placed in a row, seven spaces (one space between two bars) are created. Each choice of two spaces among the seven will determine one possible distribution. For example, if the second and seventh spaces are selected, the corresponding distribution is as follows: one group consists of two bars, the second group of five bars, and the third group of one bar. Thus, the student reduced the original problem into a different, familiar problem—in how many ways can two objects (spaces in our case) be chosen among seven objects? The student then easily determined the answer to be $\binom{7}{2} = \frac{7!}{(7-2)!2!} = 21$.

Pedagogical considerations. As is evident throughout the history of mathematics, the rigor of a logical structure—that is, the level of scrupulousness in which a mathematical argument is examined—is not absolute, but a process of continual development. Intellectual need applies here too. It is a vital guide in determining the level of rigor suitable for a particular group of students. The question is always whether students, given their current knowledge and mathematical maturity, can see a need for an idea we intend to teach them. Often students are asked to provide justifications to claims they view as self-evident. This is particularly true for certain properties of the real numbers and geometrical objects. We observed, for example, a ninth-grade teacher, teaching algebra and geometry, who requires his students to accompany each assertion written on the left-hand side of two-column proofs by a reason on the right-hand column. Students in his geometry class were required to justify the assertion "$AB \cong AB$" by the phrase "reflexive property" and the assertion "If $\angle ABC = 30^\circ$ and $\angle CBD = 45^\circ$, then $\angle ABD = 75^\circ$" by "additive property." Similarly, students in his class were required to justify the assertion "$a + b = b + a$" by the phrase "commutative property," "$(a+b)+c = a+(b+c)$" by "associative property," and "$(-1)b = -b$" by "multiplying by -1 property." It turned out that both the teacher and his students viewed these assertion as obvious (ones that require no justification) but all felt compelled to follow rules; the students had to follow rules imposed by their teacher, and the teacher those imposed by the textbook. Thus, the task to justify was alien to the teacher and to his students, and the tasks added no understanding of logical structure or rigor.

The requirement to justify operations on real numbers in terms of basic properties such as "commutativity," "associativity," and "identity" is not exclusive to secondary school mathematics; it is also common in elementary mathematics. Here, too, the task to justify is commonly *alien* to both the teachers and students. For example, a fifth-grade teacher assigned the problem: "Use properties to find n in the following equations: (1) $55 + 8 = n + 55$, (2) $8 + (2 + 3) = (n + 2) + 3$, and (3) $17 + 0 = n$." The properties referred to in this assignment are the commutative, associative, and identity properties. Students were expected to solve the three problems by resorting to these three properties, respectively. The attention of many of these fifth graders was focused solely on the teacher's demand to use these properties rather than on the quantitative meaning of the equations. There were students who solved each of these problems directly (e.g., in Problem 1, some students first added 55 and 8 to get 63, and then looked for and found a number whose sum with 55 is 63), and then accompanied their answer by the property they guessed to be the one expected by the teacher ("commutative property," in Problem 1). From the students' point of view, the task to use the properties to find the unknown n is likely to have been intellectually alien (merely to satisfy the teacher's will) rather than intellectually intrinsic (to solve a problem they find intellectually puzzling). The teacher's justification for the task she assigned, too, was intellectually alien: "So that students will do well when tested on these properties."

Geometry is perhaps the only place in high-school mathematics where a relatively complete and rigorous mathematical structure can be necessitated. Deductive geometry can be treated in numerous ways and in different levels of rigor. Deciding what constitutes an "adequate level of rigor" is crucial, of course. Earlier works, especially the work by Van Hiele (1980), suggest that subtle concepts and axioms, such as those related to "betweenness" and "separation," must be dealt with intuitively. However, the progression from definitions and intuitive axioms to theorems and from one theorem to the next must be coherent, be logical, and exhibit a clear mathematical structure. In passing, I speculate that a program that sequences its instructional unit so that *neutral geometry* (a geometry without the parallel postulate) precedes *Euclidean geometry* (a geometry with the parallel postulate) would enhance students' understanding of the concept of logical structure.

Unfortunately, some current high-school geometry textbooks amount to empirical observations of geometric facts; they have little or nothing to do with deductive geometry (for an extended detailed review, see Harel (2009)). There is definitely a need for intuitive treatment of geometry in any textbook, especially one intended for high-school students. But the experiential geometry presented in these texts is hardly utilized to develop geometry as a deductive system. In one of the texts reviewed, most assertions appear in the form of conjectures and most of the conjectures are not proved deductively. It is difficult, if not impossible, to systematically differentiate which of the conjectures are postulates and which are theorems. It is difficult to learn from these texts what a mathematical definition is or to distinguish between a necessary condition and sufficient condition. Another text presents the entire mathematical content through problems (an approach we support wholeheartedly) but fails to convey a clear mathematical structure. It is not clear which assertions are to be proved and which are not, and which are needed for the deductive progressions and which are not. Only one who knows the development in advance is likely to identify a deductive structure for the material from the set of problems in a given lesson. And to identify such a structure, it is necessary to go over the entire set of problems, including the homework problems. If, for example, one skips certain problems on uniqueness of perpendicularity, an important piece of the structure would be missing. Similarly, other problems appear as homework problems and yet they are needed for the development of a logical progression. Furthermore, even if all the problems are assigned and solved correctly, without a guide as to how these problems, together with some problems from the lesson, form a logical structure, it is difficult, if not impossible, to organize the material within a deductive structure.

Summary

We have identified five categories of intellectual need: (1) the *need for certainty* is the need to prove, to remove doubts. One's certainty is achieved when one determines, by whatever means one deems appropriate, that an assertion is true. Truth alone, however, may not be the only need of an individual, who may also strive to explain *why* the assertion is true. (2) The *need for causality* is the need to explain—to determine a cause of a phenomenon, to understand what makes a phenomenon the way it is. This need does not refer to physical causality in some real-world situation being mathematically modeled, but to logical explanation within the mathematics itself. (3) The *need for computation* includes the need to quantify and to calculate values of quantities and relations among them by means of symbolic algebra. (4) The *need for communication* consists of two reflexive needs: *the need for formulation*—the need to transform strings of spoken language into algebraic expressions—and the *need for formalization*—the need to externalize the exact meaning of ideas and concepts and the logical justification for arguments. (5) The *need for structure* includes the need to reorganize knowledge learned into a logical structure.

As was indicated earlier, in modern mathematical practices these five needs are inextricably linked, and the reason for discussing them in different sections was merely to demonstrate the existence of each and to explicate their distinctions. The need for computation, in particular, is strongly connected to other needs. For example, the need to compute the roots of the cubic equations led to advances in exponential notation, which, in turn, has helped to abolish the psychological barrier of dealing with the third degree "by placing all the powers of the unknown on an equal footing" (Tignol, 1988, p. 38).

Collectively, these five needs are ingrained in all aspects of mathematical practice—in forming hypotheses, proving and explaining proofs, establishing common interpretations, definitions, notations, and conventions, describing mathematical ideas unambiguously, etc. They have driven the historical development of mathematics and characterized the organization and practice of the subject today. In modern mathematical practice different needs often occur concurrently. DNR-based instruction is structured, so these same needs drive student learning of specific topics, *and* by realizing the different needs that drive mathematical practice, students are likely to construct a global understanding of the epistemology of mathematics as a discipline. The notion of intellectual need is related to the Realistic Mathematics Education (Gravemeijer, 1994) dictum that students must engage in mathematical activities that are real to them, for which they see a purpose. Initially, this may mean problems arising in the "real" (nonmathematical) world, but as students progress, mathematics becomes part of their world and "self-contained" or "abstract" mathematical problems become equally real. Thus, again, what stimulates intellectual need depends on the learner at any given time.

Fundamental Characteristics of Intellectual Need

This discussion of intellectual need is unfinished without addressing its fundamental characteristics. Without these characteristics, the concept of intellectual need is devoid of instructional value and lacks sufficient epistemological basis. The decision to postpone the presentation of these characteristics to the end, after an extensive discussion of the definitions, functions, and pedagogical implications of the five categories of intellectual needs, was purely pedagogical (to first allow for the formation of a solid concept image for the concept definition of intellectual need).

Subjectivity

Intellectual needs are subjective. When we talk about intellectual need we always refer to the need of the learner, not the need of a teacher or an observer. There should be no ambiguity about the sources of intellectual need—it is a learner's conception, not a teacher's conception. And since intellectual need depends on the learner's background and knowledge, what constitutes an intellectual need for one particular population of students may not be so for another population of students. This view is rooted in the Subjectivity Premise and entailed from the very definition of intellectual need. Without it, the concept of intellectual need, as well as other central concepts of *DNR* such as ways of understanding and ways of thinking, loses its substance. In particular, the pedagogical discussions discussed previously would be devoid of instructional value should one lose sight of where intellectual needs reside.

Innateness and Cognitive Primitives

The five needs discussed here are not claimed to be exhaustive or final; additional or different categories might be found. Further, and more important, these categories are not static constructs; rather, they have developed over millennia of mathematics practice and are likely to continue to develop in the future. This historical fact leads to the hypothesis that intellectual needs are learned, not innate. If accepted, as we do, this hypothesis has two consequences. The first consequence is pedagogical. Intellectual needs cannot be taken for granted in mathematics teaching. A continual and sustained instructional effort is necessary for students' mathematical behaviors to become oriented within and driven by these needs.

The second consequence is epistemological. If intellectual needs are learned, not innate, then by the *Knowledge-Knowing Linkage Premise*, they evolve out of resolutions of problematic situations. But then one is compelled to conclude that the learning of an intellectual need A requires the occurrence of an intellectual need B, which in turn requires the occurrence of an intellectual need C and so on, ad infinitum. To resolve this puzzle, we need a second conjecture: intellectual needs have cognitive primitives, whose role is to orient us to the intellectual needs we experience when we learn mathematics. In this respect, they are like subitizing (Kaufman et al., 1949), the ability to recognize the number of briefly presented items without actually counting, whose function is to orient us to recognize numerosity as a property that can be measured (English & Halford, 1995). For example, as we have discussed earlier, the *need for communication* occurs in mathematical practice when one is compelled to express ideas in a form and syntax that is amenable to computation by means of symbolic algebra, or when one is compelled to externalize the exact intended meaning of a concept and its logical basis (as when we ensure that a concept is well defined). A cognitive primitive to this need is the act of conveying and exchanging ideas by means of a spoken language and gestures, which is a defining feature of humans.

Interdependency

Intellectual need cannot be determined independently of what satisfies it. Human's experience of disequilibrium cannot be described independently of its corresponding experience of equilibrium, and, therefore, as a form of perturbational experience, intellectual need cannot be determined independently of what satisfies it.[9] For example, to understand the nature of one's doubts about a particular assertion, it is necessary to understand what evidence would be sufficient for that person to remove these doubts. And, conversely, to understand why a person is certain about an assertion, it is necessary to understand what caused him or her to doubt the assertion before he or she became certain of its truth.

Intellectual Need Versus Affective Need

Often there is confusion between intellectual need and application. Cognitively and pedagogically, the term "application" refers to those problematic situations that aim at helping students solidify mathematical knowledge they have already learned. Intellectual need problems, on the other hand, aim at eliciting knowledge students are yet to learn. This does not mean that problems from other fields cannot serve as intellectual need problems. As we know from history, many mathematical concepts emerged from the need to solve problems in fields outside mathematics.

One's engagement in a problem can be purely affective (e.g., self-interest) or social (e.g., to cure diseases, clean the environment, develop forensic tools to achieve justice, etc.). Affective need is different from intellectual need. While intellectual need has to do with *the epistemology of a discipline*, affective need has to do with *people's desire*, *volition*, *interest*, *self-determination*, *and the like*. Affective need is the drive to initially engage in a problem and pursue its solution. As such, it is strongly linked to social and cultural values and conventions. For example, by and large, students accept the obligation to attend school to learn, an obligation rooted in the cultural values and social conventions of the society in which we live. This need may manifest itself in different but interrelated ways. First, there is the need that originates from external expectation, explicit or implicit, by authoritative figures, such as teachers, parents, and society in general. This need is particularly dominant in current teaching practices and is utilized through a complex system of rewards and punishments (e.g., grades, contests, etc.). Second, there is the need driven by causes of self-advancement, such as a desire to advance one's social stature or improve

[9] More precisely, intellectual need cannot be determined independently of what *hypothetically* satisfies it. The added qualification ("hypothetically") is needed, for otherwise this claim would mean that the experience of disequilibrium over famous unsolved problems such as the Riemann hypothesis would not be describable.

one's economic conditions. Third, there is the need that stems from a desire to advance societal causes, such as technological, political, environmental, and social justice causes. Such causes might be less global, as when one might go into medicine because a sibling has some complex medical condition. Common to these types of need is a sense of a social obligation, to an authority, to oneself within a community, or to the society in general. Affective needs thus belong to the field of motivation, which addresses conditions that activate and boost (or, alternatively, halt and inhibit) learning in general. Undoubtedly questions about the fulfillment of such conditions are of paramount importance, but these are beyond the scope of this chapter.

Local Intellectual Need Versus Global Intellectual Need

By the *Knowledge-Knowing Linkage Premise* any piece of mathematical knowledge is an outcome of a resolution to a problematic situation. These situations, however, do not usually occur haphazardly, but emerge along paths toward a resolution of a major problem. Such a problematic situation, understood as such by an individual, is referred to as a *global intellectual necessity*. A problem that emerges along the way to solve a major problem is referred to as a *local intellectual need*. This is a rough characterization, of course, since it is not uncommon that some of these intermediate problems become themselves major milestones, or global necessities. The pedagogical goal is that students develop a general image of the overall problem toward which all activities relate. I illustrate this point with two examples.

Linear algebra. A curriculum in elementary linear algebra can be developed in numerous ways. What is said here is not to advocate one way over another. Rather, the goal is to illustrate the application of global necessity in teaching elementary linear algebra. If, for example, one decides to teach this topic from a matrix theory perspective, one might start with systems of equations, both linear and nonlinear. Systems of equations, if understood by the students as quantitative constraints on a set of unknowns, constitute a need for computation—the need to determine the value of the unknowns by means of symbolic algebra. Students entering their first course in linear algebra are familiar with systems of equations and understand their importance (in solving word problems, for example). Once this need is in place—and our experience suggests that undergraduate students do realize this need—students can be brought to appreciate the importance of a special kind of systems of equations, those whose equations are linear. This can be done in different ways, for example by showing how the solution of certain nonlinear systems cannot be found accurately but can be approximated by suitable linear systems, or by showing how many application problems can be modeled by linear systems. The leading questions would constitute global need. Such questions include the following: Given a linear system, how do we solve it? Are there ways to solve linear systems systematically—algorithmically, that is? Can we determine, without necessarily solving the system, if the system has a solution? If the system is solvable, how many solutions does it have? Can the system have a finite number of solutions? If yes, what are the necessary and sufficient conditions for this to happen? When the system has infinitely many solutions (a situation students should observe early on), can all the solutions be listed? The need for formulation then is applied to translate these questions in formal terms involving central concepts, such as “linear combination,” “linear independence,” “basis,” etc. What is crucial here is that students come to understand that any new concept is formed to advance investigation of these questions (see Harel, 1998). Once the scalar case (i.e., systems whose unknown are numbers) is completed, one can turn to systems of differential equations. As we discussed earlier, we introduced the global need for Eigen Theory through the question whether it is always the case that the solution of a linear system of differential equations with an initial condition is easily computable. Through it we necessitated fundamental concepts of the Eigen Theory, from the basic concepts of eigenvalue, eigenvector, diagonalization, and their related theorems up to the Jordan Theorem and its related Jordan Canonical Form.

Rate of change. The concept of rate of change can be necessitated around the *need to model reality*. When seeking a function to model a natural phenomenon, the data typically available consist of how the phenomenon changes. Thus, one of the main purposes of examining rates of change is to use some information about the rate to gain information about a function, a purpose which is often masked in traditional calculus courses. We (Harel, Fuller, Rabin, & Stevens (n.d.)) have designed a sequence of problems consistent with this purpose as a global necessity. We began with a set of problems on functions—in particular, problems in which the objective is to describe a physical situation (e.g., At any time, what is the population?). One of our primary goals was that students understand functions as models of reality. In these problems, attending to rate of change is necessary for determining a model. The need to determine a model, in turn, necessitates an in-depth study of rates of change—in particular, an exploration of average rate of change, which leads naturally to an intuitive notion of instantaneous rate of change. The *need for communication*—in this case the need to communicate to others a precise definition of “approaches” and “arbitrarily close”—demands the formalization of our intuitive notion (i.e., the definition of the derivative). With the definition of the derivative in hand, we prove properties of functions that follow from properties of their derivatives. Many of these properties are intuitive, but the *need for certainty* (to know that something is true) demands formal proof. Truth alone, however, is not our only aim; we desire students to know *why* something is true, and thus appeal to the *need for causality*.

Concluding Remark

In its current form, *DNR* is primarily concerned with the intellectual components, not with the motivational components, of perturbation, though its definition of *learning* incorporates intellectual needs and affective needs, as well as the ways of understanding and ways of thinking currently held by the learner. Specifically,

> *Learning* is a continuum of disequilibrium-equilibrium phases manifested by (a) *intellectual needs* and *affective needs* that instigate or result from these phases and (b) *ways of understanding* or *ways of thinking* that are utilized and newly constructed during these phases. (Harel, 2008b, p. 897)

Learning in *DNR*, thus, is driven by exposure to problematic situations that result in a learner experiencing perturbation, or disequilibrium in the Piagetian sense. The drive to resolve these perturbations has both psychological and intellectual components. The psychological components pertain to the learner's *motivation*, whereas the intellectual components pertain to *epistemology*—the structure of the knowledge domain in question, both for the learner as an individual and as the domain developed historically and is viewed by experts today.

In essence, this chapter deals with the question of how instruction can help students experience the need to construct an epistemological justification for the knowledge we intend to teach them. The basis for this question is the stipulation, rooted in the *DNR* premises, that the responsibility of curriculum developers and teachers is to intellectually necessitate the mathematical knowledge intended for students to learn. Elsewhere I formulated this stipulation as an instructional principle, called the *necessity principle*: "For students to learn the mathematics we intend to teach them, they must see a need for it, where 'need' means *intellectual need*, not *social* or *cultural need*" (Harel, 2008b, p. 900). The pedagogical considerations of the different intellectual needs are rooted in this fundamental principle. In all, this principle translates into the following four concrete instructional steps:

1. Recognize what constitutes a global intellectual need for a particular population of students, relative to a particular subject (e.g., in linear algebra such a need might be solving systems of equations).
2. Translate this need into a set of general questions formulated in terms that students can understand and appreciate.
3. Structure the subject around a sequence of problems whose solutions contribute to the investigation of these questions. These problems, in turn, serve as local necessities for the emergence of particular concepts needed to advance the investigation at hand.
4. Help students elicit the concepts from solutions to these problems.

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Chapter 7
The False Dichotomy in Mathematics Education Between Conceptual Understanding and Procedural Skills: An Example from Algebra

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Abstract The history of mathematics education provides ample evidence of the dichotomous distinction that has been made over the years between concepts and procedures, between concepts and skills, and between "knowing that" and "knowing how." In no field of school mathematics learning has this dichotomy been so damaging as in algebra. While reform efforts of the past decade have attempted to imbue algebra learning with meaning by focusing on "real-life" problems and their various representations, these efforts have missed the main point with respect to the literal-symbolic: that is, that conceptual aspects of algebra abound within the literal-symbolic and that these are integral to most of the so-called procedures of algebra. Both theoretical and empirical arguments will be used to make the point for adopting a different vision of the literal-symbolic domain, in which the procedural is so permeated with the conceptual as to render obsolete a primarily procedure-based view of algebra in school mathematics.

> Of course, at certain moments a technique can take the form of a manipulative skill. This is particularly the case when a certain routinization is necessary. … But techniques must not be considered only in their routinized form. The work of constituting techniques in response to tasks, and of theoretical elaboration on the problems posed by these techniques, remains

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> fundamental to learning.[1] … Technique plays an epistemic role by contributing to an understanding of the objects that it handles, particularly during its elaboration. It also serves as an object for a conceptual reflection when compared with other techniques and when discussed with regard to consistency.[2] (Jean-Baptiste Lagrange)

> Much of the elaborate architecture of a mature mind is made of hierarchies of automatized skills that are constructed in, and constantly revised by, consciousness. … Conscious processing is needed to establish and maintain our own internal cognitive habits. It is also needed to alter them. And it is needed as well to use them in any complex situation. … New knowledge – ideas, facts, words, concepts – can be used to update and to be integrated within that which has been automatized. … But even that which is automatized in long-term memory is constantly being updated and revised.[3] (Merlin Donald)

The traditional dichotomy in mathematics education between conceptual understanding and procedural skills masks two important aspects about procedures. The first, which is reflected in the opening words of the French didactician Jean-Baptiste Lagrange, is that during their period of elaboration, procedures are conceptual in nature. The second aspect, suggested by the quotation drawn from the Canadian cognitive neuroscientist Merlin Donald, is that procedures, even when they function as automatized skills, are regularly being updated, revised, and extended by means of conceptual elements. Thus, within the procedural domain, the distinction between concepts and procedures is indeed a fuzzy one. Although the falseness of the dichotomy between procedural skills and conceptual understanding within mathematics education has been argued in the past (e.g., Wu, 1999), the two specific aspects reflected in the words of Lagrange and Donald have not been elaborated within these discussions. And while arithmetic has often been the topic of these discussions regarding the priority of conceptual understanding or procedural skills, algebra has not. Algebra has traditionally been viewed as a domain of school mathematics that is dominated by procedures of symbol manipulation and where the presence of the conceptual has been considered all but an oxymoron.

The ultimate aim of this chapter is to propose a rethinking of algebraic symbol activity in terms of the conceptual, by looking both at examples of research-based tasks, as well as the student discourse elicited by them. But first, we embark on a brief historical voyage through the relevant mathematics education landscape, which not only illustrates the ways in which the dichotomy between conceptual understanding and procedural skills has permeated the field but also indicates how some researchers have been reconsidering the relationship between the two. Then, the chapter describes the development of recent theoretical perspectives that emphasize the importance of the technical for the conceptual, but which also point to the conceptual components of procedures. This is followed by presentation of the viewpoint from neuroscience that even skilled procedural performance is constantly being updated by the conceptual. An example of a task activity, along with empirical data drawn from the research of my team on the learning of algebra within technological environments, is included to support the theoretical ideas that are being discussed.

[1] Lagrange, J.-B. (2000, p. 16).

[2] Lagrange, J.-B. (2003, p. 271).

[3] Donald, M. (2001, pp. 52–57).

Historical Voyage Through the Dichotomous Land of Procedural Skills and Conceptual Understanding

From its earliest years at the beginning of the twentieth century in North America, the mathematics education community has been both fostering (e.g., Thorndike, 1921) and arguing against (e.g., Brownell, 1935) the dichotomy between conceptual understanding and procedural skills. Signs of this duality could still be seen in the 1960s and 1970s, at the same time the research community was experiencing its first major growth spurt (Kilpatrick, 1992). For example, in recognition of the emergence of the new community of researchers within the National Council of Teachers of Mathematics (NCTM), the NCTM commissioned in the 1970s the publication of a major research volume. The volume, which was published in 1980 (Shumway), included within its 14 chapters one titled *Skill Learning* and another titled *Concept and Principle Learning*. These titles suggest the dichotomous relation between conceptual understanding and procedural skills that would be elaborated within. In addition, the NCTM also published in 1978 a yearbook titled *Developing Computational Skills*. This was the first time that any of the NCTM yearbooks, which had begun publication in 1926, included either the term *concepts* or that of *skills* within the title. One earlier yearbook, however, with the title *The Learning of Mathematics*: *Its Theory and Practice* (Fehr, 1953) contained a chapter on concept formation and several dealing with skill proficiency. We now take a closer look at that 1980 NCTM research volume, as well as other more recent publications, to uncover some of the details of the expression of this dichotomy during the last century.

Research in Mathematics Education (Shumway, 1980)

Suydam and Dessart (1980) began their chapter *Skill Learning* with the following remarks:

> One of the most frequently stated goals of mathematics instruction involves the development of skills. Skills are comparatively easy to describe or specify, to teach, and to evaluate. A skill is what a learner should be able to *do*. Skills arise from concepts and principles and provide a foundation for the development of other concepts and principles. Conceptual thought is derived in part from the understanding attained as skills are developed.… Practice is obviously one component of learning a skill. It is simply not efficient to perform most skills in other than a routine way, and practice aids in their mastery for routine use. Sometimes this is interpreted to mean that skills should be taught by rote procedures emphasizing drill. But understanding what makes a procedure work—including the application of concepts and principles—is a necessary concomitant to skill learning. (pp. 207–208)

Their position seems at once both nuanced and reasonable: Conceptual understanding serves procedural skills, which in turn provide a foundation for further conceptual development. Although this position threads through much of the later literature of the period, earlier work of the century emphasized, by and large, the learning of procedural skills per se. As disclosed by their review of the literature on skill learning, drill theory predominated in nearly all elementary school teaching until the mid-1930s, when Brownell proposed meaningful approaches involving materials and discussion. Nevertheless, much research continued its focus on drill, attempting to determine which drill and practice procedures were most effective. For example, Suydam and Dessart pointed to the 1978 NCTM yearbook on developing computational skills, in which Davis (1978), in summarizing his own research and the evidence from psychological studies, argued that the principles for drill include, for instance, "During drill sessions, emphasize remembering—don't explain!" (p. 54).

During the 1970s, the mathematics education community was also witnessing "Back to Basics," widely considered now as a reaction to the excesses of the New Math movement of the 1960s. Questions such as, "Should learning the meaning of skills precede practice for mastery or should mastery be attained and then meaning developed?" and "Should some skills be taught only rotely and others with meaning?" were raised by Suydam and Dessart (1980) as they discussed the myriad issues involved in research on skill learning. They concluded that the predominant need was for studies that would inform the community as to how skills are learned.

In bringing their chapter on skill learning to a close, Suydam and Dessart (1980) propose a four-stage research model for developing a skill theory in mathematical learning. They walk the reader through this model, using as an example the skill of factoring the difference of two squares. Even if Suydam and Dessart began their chapter with the professed belief that "understanding what makes a procedure work—including the application of concepts and principles—is a necessary concomitant to skill learning" (p. 208), their exemplification of factoring the difference of squares provides little evidence of the role the conceptual might play in the development of procedural skill. By the same token, a reading of the chapter *Concept and Principle Learning* by Sowder (1980) in the same NCTM research volume yields no discussion regarding the role that procedural skills might play in the development of conceptual knowledge. These two chapters in the 1980 NCTM research volume thus suggest that the dichotomy between conceptual understanding and procedural skills was alive and well within the mathematics education community in the 1970s.

Conceptual and Procedural Knowledge: The Case of Mathematics (Hiebert, 1986)

Within 6 years of the publication of the NCTM research volume (Shumway, 1980), the beginnings of a shift could be discerned. The year 1986 marked the publication of *Conceptual and Procedural Knowledge*: *The Case of Mathematics* (Hiebert). The movement in perspective is reflected first in the title: from *procedural skill* to *procedural knowledge*. Nevertheless, Hiebert and Lefevre (1986) lament in their introductory chapter that

> Over the past century, considerations of these two kinds of mathematical knowledge [i.e., conceptual and procedural] have taken different forms using different labels. Probably the most widely recognized distinction has been that between skill and understanding.… But, regardless of the labels, the division between types of knowledge lies in approximately the same place today as it has in the past. (p. 2)

Then, on a more optimistic note, they also point out that "current discussions treat the two forms of knowledge as distinct, but linked in critical, mutually beneficial ways" (p. 2).

Nonetheless, the linkages that Hiebert and Lefevre (1986) profess between the two forms of knowledge are difficult to find within the definitions they offer. They characterize conceptual knowledge, on the one hand, as knowledge that is rich in relationships. Procedural knowledge, on the other hand, is defined as having two distinct parts: (1) the formal language or symbol representation system and (2) the algorithms or rules for completing mathematical tasks. No integration of conceptual knowledge within procedural knowledge is even hinted at here. They add

> Perhaps the biggest difference between procedural knowledge and conceptual knowledge is that the primary relationship in procedural knowledge is "after," which is used to sequence subprocedures and superprocedures linearly; in contrast, conceptual knowledge is saturated with relationships of many kinds (Hiebert & Lefevre, 1986, p. 8).

Just as did Suydam and Dessart (1980), Hiebert and Lefevre (1986) argue that "procedures that are learned with meaning are procedures that are linked to conceptual knowledge" (p. 8).

More explicitly, Carpenter (1986) suggests an order to these linkages with his stance that meaning for procedures cannot be developed unless a rich conceptual knowledge base is in place. While Hiebert and Lefevre (1986) do acknowledge that, in theory, procedural knowledge can occasionally take the lead and spur the development of new concepts, they are reticent to admit that such occurs in general practice. Although the position of Hiebert and Lefevre might allow for the mutual co-support of the procedural and the conceptual, the nature of the interactions remains tenuous at best. In fact, the basic definitions of conceptual and procedural knowledge presented by Hiebert and Lefevre remain so far apart as to seem mutually exclusive—dichotomously so. Nevertheless, Silver (1986), within the same volume, while recognizing that discussions of procedural–conceptual linkages often maintain the dependence of procedural knowledge on conceptual knowledge, offers an example from his own problem-solving research that illustrates the apparent dependence of conceptual knowledge on procedural knowledge.

Adding It Up: Helping Children Learn Mathematics (National Research Council, 2001)

Adding It Up (NRC, 2001), which was written by the Mathematics Learning Study Committee consisting of mathematicians, mathematics education researchers, and mathematics practitioners, succeeds in blurring the dichotomy between procedural skills and conceptual understanding, and in fact opens up the definitions of both. At the heart of the volume is a theoretical frame for mathematical proficiency that intertwines five strands: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. Conceptual understanding is described as "comprehension of mathematical concepts, operations, and relations" (p. 116) and procedural fluency as "knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently" (p. 121). One notices first that conceptual understanding is deemed to include "operations"—not seen in the literature surveyed up to now, but not elaborated either within the volume's description of conceptual understanding. It is in the discussion of procedural fluency that one finds commentary related to the conceptual role that can be played by procedures:

> In the domain of number, procedural fluency is especially needed to support conceptual understanding of place value and the meanings of rational numbers. It also supports the analysis of similarities and differences between methods of calculating…. In addition to providing tools for computing, some algorithms are important as concepts in their own right.… Procedural fluency and conceptual understanding are often seen as competing for attention in school mathematics. But pitting skill against understanding creates a false dichotomy. As we noted earlier, the two are interwoven. Understanding makes learning skills easier, less susceptible to common errors, and less prone to forgetting. By the same token, a certain level of skill is required to learn many mathematical concepts with understanding, and using procedures can help strengthen and develop that understanding. (pp. 121–122)

What is striking about this quotation is its unequivocal statement that the development of concepts needs procedural fluency. While it also postulates, and importantly so, that some procedures are conceptual in their own right, this idea is not expanded upon. Nevertheless, the statement that procedures can be concepts and that concepts can have a procedural face represents a significant advance in tearing down the wall that has traditionally separated conceptual understanding and procedural skill. However, the authors of *Adding It Up* might have gone further; they might have argued that the very process of elaborating a procedure is a conceptually oriented activity. For this notion we turn elsewhere. Although the learning of procedures with a conceptual orientation has been a central feature of the Vygotskian-inspired approach elaborated by Davydov (Schmittau, 2004), this approach has focused by and large on the primary and early middle school levels of mathematics. Thus, for a perspective more directly applicable to the secondary level and beyond, we call upon the French *didactique* community of mathematics education.

A Theoretical Frame for Interpreting the Constitution of Procedures in a Conceptual Light

In the mid-1990s, when Computer Algebra System (CAS) technology started to make its appearance in secondary school mathematics classes in France, researchers (Artigue, Defouad, Duperier, Juge, & Lagrange, 1998) noticed that teachers were emphasizing the conceptual dimensions while neglecting the role of technical work in algebra learning. However, this emphasis on conceptual activity was producing neither a clear lightening of the technical aspects of the work nor a definite enhancement of students' conceptual reflection (Lagrange, 1996). From their observations, the research team of Artigue and her collaborators came to think of techniques as a link between tasks and conceptual reflection—in other words, that the learning of techniques was vital to related conceptual thinking.

Chevallard (1999) describes, within his anthropological theory of didactics, four components of practice by which mathematical objects are brought into play in didactic institutions: task, technique, technology, and theory. He states that *tasks* are normally expressed in terms of verbs, for example, "multiply the given algebraic expression" (p. 225). He defines *technique* as "a way of accomplishing, of carrying out tasks" (p. 225). In his theory, Chevallard separates *technique* from the discourse that justifies, explains or produces it, which he refers to as *technology*. But he also admits that this type of discourse is often integrated into technique, and points out that such technique can be characterized in terms of theoretical progress. According to Chevallard, *theory* takes the form of abstract speculation, a distancing from the empirical. Thus, within the anthropological approach, techniques are a bridge from tasks to theoretical discourse.

Artigue (2002) and her research collaborators adapted Chevallard's anthropological theory by collapsing *technology* and *theory* into the one term, *theory*. In this way, the theoretical component was accorded a wider interpretation than is usual in the anthropological approach; it also reserved the use of the term *technology* for digital devices. Furthermore, and more importantly for this chapter, Artigue also gave *technique* a wider meaning than is usual in educational discourse: "A technique is a manner of solving a task and, as soon as one goes beyond the body of routine tasks for a given institution, each technique is a complex assembly of reasoning and routine work" (p. 248).

Lagrange (2002), one of Artigue's collaborators, expressed the interrelationship of task, technique, and theory as follows:

> Within this dynamic, tasks are first of all problems. Techniques become elaborated relative to tasks, then become hierarchically differentiated. Official techniques emerge and tasks lose their problematic character: tasks become routinized, the means to perfect techniques. The theoretical environment takes techniques into account—their functioning and their limits. Then the techniques themselves become routinized to ensure the production of results useful to mathematical activity.… Thus, technique has a pragmatic role that permits the production of results; but it also plays an epistemic role (Rabardel & Samurçay, 2001) in that it constitutes understanding of objects and is the source of new questions. (p. 163, personal translation)

Lagrange (2003) further extended this latter idea: "Technique plays an epistemic role by contributing to an understanding of the objects that it handles, particularly during its elaboration. It also serves as an object for a conceptual reflection when compared with other techniques and when discussed with regard to consistency" (p. 271).

Lagrange contrasted his vision of the epistemic role played by technique during the period of its elaboration with the more traditional view that espouses a dichotomy between conceptual understanding and procedural skills, by using an example drawn from Heid's (1988) CAS-supported, applied calculus study. Lagrange (2000) points out that the technical domain in that study was defined more narrowly in terms of routine manipulations, computational procedures, and algorithmic skills. Heid (1988) designed her study in such a way that the conceptual was emphasized during the first 12 weeks and the teaching of skills in the latter three. According to Lagrange (2000), Heid's (1988) traditional vision of both concepts and skills led her to miss the broader role that was being played by technique during the first 12 weeks. Lagrange (2000) argues that the so-called conceptual activity during the first part of the study was laced with technical work, in fact, with several new techniques that were constituted in response to a broader range of tasks, as well as the theoretical elaboration of these techniques. Further, Lagrange has claimed that, for this level of mathematics course, a concept cannot exist without the techniques that are associated with it, and that ultimately, “the opposition between concepts and manipulative skills masks an essential point; there exists a technical dimension to the mathematical activity of students that does not reduce to skills” (p. 14). In other words, procedures have a dual face: one where the focus is skills and the other where the focus is clearly the elaboration of techniques, which of necessity includes conceptual activity. Ruthven (2002), in an analysis of the French view on technique, has reiterated that, “from this perspective, technique—whether mediated by technology or not—fulfills not only a pragmatic function in accomplishing mathematical tasks, but an epistemic function in building mathematical concepts” (p. 283).

This perspective strikes a crushing blow to the traditional dichotomy between conceptual understanding and procedural skills. No longer can the two be viewed as separate entities. Nor is it sufficient to argue that conceptual understanding can lead to the meaningful development of procedural knowledge. Rather, the elaboration of procedures has within itself a conceptual component. In other words, the technical activity of students contains, during the period of elaboration of technique, an epistemic, that is, conceptual, element that is so intertwined with it that one codevelops with the other.[4] At these moments, the procedural is conceptual. But the conceptual aspects of the procedural do not end here.

The Updating of Procedural Skills by New Conceptual Knowledge

In the previous section, the emphasis was the *constitution* of procedures—a process whereby conceptual aspects of techniques are elaborated. This epistemic role played by technique gradually gives way to the pragmatic, the more routine, whereby techniques are viewed under the guise of manipulative skills. But the conceptual component of procedures does not disappear once the procedures have evolved to function pragmatically as skills. Procedural skills continue to be updated and revised by means of new technically related conceptual knowledge—as was suggested in this chapter’s opening quote from Donald (2001). The interaction between the conceptual and the procedural is an ongoing recursive process. The procedural, in which its initial elaboration constitutes new conceptual knowledge related to the given technique, acquires pragmatic, skill-related characteristics, which are in turn updated and revised by means of further technical activity of an epistemic nature.

[4]Please note that, henceforth in this chapter, the terms *procedure* and *technique* will be used synonymously, except in those cases where the context clearly indicates otherwise.

In the past, in mathematics education, procedural skills have tended to be viewed as quite static entities, existing within "mindless," well-practiced activity. But that view fails to take into consideration the neurologically based argument that procedural skills adapt over time as the conceptual domain to which they are applied is broadened. For example, equation-solving skills that are perfectly honed to the solving of first-degree equations must be revised as the set of mathematical objects being treated is widened to encompass second-degree equations. One of the revisions that will be required is the integration of factoring as a solving technique, accompanied by its justification in terms of the zero-product property and the notion of the zeros of a function, along with an understanding of what all of this means with respect to equation solving. Such updating of automatized skills is far from mindless. In this regard, Donald (2001) has argued that

> According to a long line of scientific research on human skill, one of the primary functions of conscious processing is the systematic refinement and automatization of action.… Automaticity is not the antithesis of consciousness. Conscious processing is needed for most kinds of learning.… The unconscious mind may passively register certain very elementary impressions of the world, such as color, brightness, or movement, but it will neither identify an object nor locate it in the world without active participation of attention, which, in this context, amounts to conscious awareness. Conscious capacity is also needed for acquiring and automatizing complex skills and representations, including, of course, more elaborate symbolic skills, such as mathematics, music, writing, speaking, and computer programming.… Conscious processing is needed to establish and maintain our own internal cognitive habits. It is also needed to alter them. And it is needed as well to use them in any complex situation.… New knowledge—ideas, facts, words, concepts—can be used to update and to be integrated within that which has been automatized. (pp. 57–58)

In the mathematics education world, procedural automaticity has often been considered as necessary in order to free up the mind to do other things: "The automaticity in putting a skill to use frees up mental energy to focus on the more rigorous demands of a complicated problem" (Wu, 1999, p. 2). While we do not contest this point, neither Wu's argument that "precision and fluency in the execution of the skills are the requisite vehicles to convey conceptual understanding," nor that "conceptual advances are invariably built on the bedrock of technique" (p. 1), the issue is broader than his arguments suggest. It is not so much that skills with certain procedures allow one to do things of a conceptual nature, which they clearly do, it is rather that even procedures that have become automatized are regularly being updated by the constitution of new techniques that have been elaborated conceptually.

The second blow has now been delivered to the traditional dichotomy between conceptual understanding and procedural skills. It is not just the case that procedures are conceptual in nature during the period in which they are being constituted and contribute to a theoretical understanding of both the objects being treated as well as to the techniques themselves. It is also the case that procedural skills have a significant conceptual component during processes of updating and extension. Automatized skills are not static in nature, but rather are regularly being revised by new techniques with their conceptual elaborations.

The Theoretical Elaboration of New Techniques and the Updating of Existing Technical Skills: An Example from an Algebra Project

Our research team[5] adopted the task–technique–theory framework (Artigue, 2002) as a vehicle for investigating issues surrounding the relation between the technical and the theoretical within the domain of school algebra—a domain where the traditional cleavage between the procedural and the conceptual has tended to persist and where algebraic procedures continue to be taught, by and large, as if concept free. In thinking about both the conceptual elaboration and consequent updating of techniques within algebraic activity, one of our first objectives related to rendering explicit how we ourselves conceptualized the theoretical aspects of algebraic technique.

A Conceptual Understanding of Algebraic Technique

We decided a *conceptual understanding of algebraic technique* should include at least the following understandings:

- Being able to see a certain *form* in algebraic expressions and equations, such as a linear or quadratic form. For example, this can involve seeing x^6-1 as $((x^3)^2-1)$ and as $((x^2)^3-1)$, and so being able to factor it in two ways, or seeing that x^2+5x+6 and x^4+7x^2+10 are both of the form ax^2+bx+c.
- Being able to see *relationships*, such as the equivalence between factored and expanded expressions. For example, this can include the awareness that the same numerical substitution (not a restricted value) in each step of the rewriting process of expanding will yield the same value and so substituting, say 3, into all four of the following expressions is seen to yield 20 for each expression:

  $(x+1)(x+2)$—factored form
  $=x(x+2)+1(x+2)$
  $=x^2+2x+x+2$
  $=x^2+3x+2$—expanded form.

- Being able to *see through algebraic transformations* to the underlying changes in form of the algebraic object and being able to explain or justify these changes. For example, this can involve seeing within the following equation sequence that the equation has been rewritten so as to make it equal to zero, and knowing that this not only maintains the solutions but also makes it easier to find them; this example also involves seeing that the left side has been reexpressed so as to make

[5] Our research team has included over various periods of this program of research André Boileau, Caroline Damboise, Paul Drijvers, José Guzmán, Fernando Hitt, Ana Isabel Sacristán, Luis Saldanha, Armando Solares, and Denis Tanguay—as well as our project consultant, Michèle Artigue.

evident the common factor of $(y-2)$, which will be followed by factoring the rest of the left side so as to be able to use the zero-product property:

$$(y-2)^3-10(y-2)=y(y-2)$$
$$(y-2)^3-10(y-2)-y(y-2)=0$$
$$(y-2)[(y-2)^2-10-y]=0$$

The Task Activity on Factoring x^n-1

In light of the perspective described in the preceding sections, we designed a study on the theoretical elaboration and consequent extension of technique and thus on the epistemic role played by technique. An essential component of the design phase was the nature of the task activities to be created. For example, for one task activity, we adapted a problem situation that had originally been set by Mounier and Aldon (1996). Their version involved conjecturing and proving general factorizations of x^n-1. Our task activity had three parts (for more details, see Kieran & Drijvers, 2006). The first part, which involved CAS technology as well as paper and pencil, aimed at promoting an awareness of the presence of the factor $(x-1)$ in the given factored forms of the expressions x^2-1, x^3-1, and x^4-1 (see Fig. 7.1), as well as leading to the *generalized* form $x^n-1=(x-1)(x^{n-1}+x^{n-2}+\cdots+x+1)$ for integral values of *n*.

The next part of the activity involved students *confronting* the paper-and-pencil factorizations they produced for x^n-1, for integer values of *n* from 2 to 6 (and then from 7 to 13), with the completely factored forms produced by the CAS and *reconciling* these two factorizations (see Fig. 7.2). An important aspect of this part of the activity involved reflecting and *forming conjectures* (see Fig. 7.3) on the relations between particular expressions of the x^n-1 family and their completely factored forms. The final part of the activity (see Fig. 7.4) focused on students *proving* one of the conjectures they had generated during the previous part of the task activity.

1. Perform the indicated operations: $(x-1)(x+1)$; $(x-1)(x^2+x+1)$.
2. Without doing any algebraic manipulation, anticipate the result of the following product $(x-1)\left(x^3+x^2+x+1\right)=$
3. Verify the above result using paper and pencil, and then using the calculator.
4. What do the following three expressions have in common? And, also, how do they differ? $(x-1)(x+1)$, $(x-1)(x^2+x+1)$, and $(x-1)\left(x^3+x^2+x+1\right)$.
5. How do you explain the fact that when you multiply: i) the two binomials above, ii) the binomial with the trinomial above, and iii) the binomial with the quadrinomial above, you always obtain a binomial as the product?
6. On the basis of the expressions we have found so far, predict a factorization of the expression x^5-1.

Fig. 7.1 Some of the initial questions of the task activity

In this activity each line of the table below must be filled in completely (all three cells), one row at a time. Start from the top row (the cells of the three columns) and work your way down. If, for a given row, the results in the left and middle columns differ, reconcile the two by using algebraic manipulations in the right hand column.

Factorization using paper and pencil	Result produced by the FACTOR command	Calculation to reconcile the two, if necessary
$x^2 - 1 =$		
$x^3 - 1 =$		
$x^4 - 1 =$		
$x^5 - 1 =$		
$x^6 - 1 =$		

Fig. 7.2 Factorization Task where students confront completely factored forms produced by CAS

Conjecture, in general, for what numbers n will the factorization of $x^n - 1$:
i) contain exactly two factors?
ii) contain more than two factors?
iii) include $(x + 1)$ as a factor?
Please explain.

Fig. 7.3 Conjecturing Task where students examine more closely the nature of the factors produced by CAS

Prove that $(x+1)$ is always a factor of $x^n - 1$ for even values of n.

Fig. 7.4 The Proving Task

In short, the aim of the task activity was to develop new techniques for factoring a certain family of polynomial expressions and to enable the students to view their existing techniques for factoring differences of squares and cubes within a new, and much broader and more general, perspective—and thereby extend them. In designing a task activity in which technical work was intertwined with conceptual reflection, we aimed to cast light upon the nature of the conceptual elaboration of algebraic technique.

The Conceptual Elaboration of New Techniques and the Updating of Older Techniques

The Grade 10 students who were participating in our study had already learned, during Grade 9, how to factor the difference of squares and factorable trinomials of the form ax^2+bx+c, as well as how to reexpress in factored form certain polynomials whose terms contained a common factor. In addition, an earlier task activity during their 10th grade participation in our project had involved factoring the sum and difference of cubes. But they had not yet learned to see, for example, potential relationships between the factoring techniques for the difference of squares and the difference of cubes, nor more global factoring methods that would allow them to place these two former techniques into a wider and more structured perspective.

To illustrate the ways in which the task activity on factoring x^n-1 led to learning new techniques with conceptual components such as these, I draw on episodes from the unfolding of this task activity within one of the participating classes. As will be seen, the inconsistencies between students' paper-and-pencil results and the factorizations produced by the CAS for different cases of x^n-1 led to questions of both a conceptual and a technical nature. Their resolution, both technically and conceptually, fostered the co-emergence of both technique and theory in the students.

The students began the Factorization Task (see Fig. 7.2) having just experienced the general form of factorization for x^n-1, for integral values of n. Thus, they were thinking about factoring the various examples of x^n-1 according to this general form. However, when they entered "Factor (x^4-1)" into their CAS, it yielded $(x-1)(x+1)(x^2+1)$ and not $(x-1)(x^3+x^2+x+1)$, which they had expected. It did not take long before students could be heard commenting, "It can be factored further," "It's not completely factored," and "It gives you all the factors"—comments that reflected their rethinking about factoring, more specifically that not all techniques provide a complete factorization.

During the class discussion that followed the completion of the first set of examples for n from 2 to 6 in the factoring of x^n-1, some clarification of the notion of complete factorization took place. In addition, even if some students were already quite skilled with factoring differences of squares, the idea that expressions with even exponents greater than 2 could also be regarded as a difference of squares seemed unfamiliar to others. Furthermore, while it was mentioned by a few students that x^6-1 could be treated either as a difference of squares or as a difference of cubes, the forthcoming task which was to involve the factoring of x^9-1 was to provide evidence that *seeing* a difference of cubes was even more difficult for some students than seeing a difference of squares.

Before tackling the next part of the task activity on factoring x^n-1, for values of n from 7 to 13, the students were asked whether they had, thus far, observed any new patterns emerging from their factoring. For example, *Were there some exponents for which the general rule was providing a complete factorization and others for which this was not the case*? Based on their limited set of examples for n from

2 to 6, it was inevitable that most students would generate the conjecture that, for odd values of n, the general rule seemed to be holding. In other words, they thought the complete factorization of $x^n - 1$ had exactly two factors for odd ns, while for even values of n it contained more than two factors, one of which was $(x+1)$. The counterexample would occur with the factoring of the expression $x^9 - 1$ (for a case study related to this part of the task activity, see Hitt & Kieran, 2009).

In fact, the expression $x^9 - 1$ pushed a significant number of students to the limits of their current thinking on factoring. A few erroneously handled the expression as if it were a difference of squares, $(x^3+1)(x^3-1)$, or as a "sort of difference of squares," $(x^3-1)(x^6+1)$. Others used the general rule. When they compared their paper-and-pencil factors with the CAS factors, they came to the realization that the CAS had produced a factored form that most were unable to obtain themselves. Even those who had used the general rule for $x^n - 1$ and who could reconcile their factorization with the factors produced by the CAS (by multiplying all the CAS factors except $(x-1)$ to produce their second paper-and-pencil factor) were still not satisfied. They insisted on knowing how to factor $x^9 - 1$ themselves and explicitly requested such help from the teacher: "How do you get those factors?" The teacher suggested they to try to "see" x^9 as $(x^3)^3$, and thus $x^9 - 1$ as ($(x^3)^3 - 1$), which could then be treated as a difference of cubes, which they supposedly knew how to factor.

By the time the students had completed the Factorization Task for integral ns from 2 to 13 and the Conjecturing Task, they had not only developed new techniques for factoring the $x^n - 1$ family of polynomials but, in the process, they had also elaborated new theoretical ideas related to factoring. This spontaneously led to revising and updating some of their old techniques and theories about factoring. In trying to make sense of the factors the CAS had produced, the students came to extend their skills with the difference of squares technique. They also came to see that expressions of the form $x^n - 1$ whose exponents have several divisors can generally be factored in more than one way. They began to look at expressions in terms of multiple possible structures. Their understanding of the notion of complete factorization evolved. Finally, some students were even able to detect new patterns, and with the aid of the CAS (and provoked by the complete factorization of $x^{10} - 1$), developed another general rule, this time for the factoring of $x^n + 1$ with odd ns as exponent (see Kieran & Drijvers, 2006).

Further Discussion

The above brief description of student classroom work involving one of the task activities that was designed for the study illustrates the conceptual nature of the elaboration of new techniques, as well as the way in which such activity can serve to update previously learned skills. The episodes provide evidence for the theoretical claim that not only is the conceptual an integral component of new technical knowledge as the latter is being constituted, but also that this same technical–conceptual activity is central to the revising and extending of existing technical skills.

In the above episodes, this revising and extending occurred simultaneously with the theoretical elaboration of new techniques. Similar updating of procedural skills was also seen in other task activities, such as one involving the learning of techniques for solving equations containing radical expressions. There, students came to add constraints to their previous skills for solving first- and second-degree equations within the context of issues related to extraneous and missing solutions for equations such as $5\left(\sqrt{x-4}\right)^3 + 11\sqrt{x-4} = \left(2x+1\right)\sqrt{x-4}$.

Our observations, analyses, and reflections within this study provide support for the notion that the conceptual in literal-symbolic algebra is an integral part of the procedural, whether it be in the constitution of new procedures or in the updating of procedures that have already become more-or-less skilled. Properties, characteristics, form, structure, and relationships are as much a part of the procedural as they are of the conceptual in algebra. In the above classroom episodes, students' technical talk about their new factoring procedures always included reference to conceptual aspects, such as the nature of the exponents and their relation to the factored form, and whether an expression was fully factored or not. The procedural could not escape the conceptual. But not all researchers view the procedural in this way.

Star (2005), in his *JRME* Research Commentary on *Reconceptualizing Procedural Knowledge*, proposed that conceptual knowledge be defined as "knowledge of concepts or principles" (p. 407) and procedural knowledge as "knowledge of procedures" (p. 407). However, just as conceptual knowledge could be superficial or deep, he argued that this too is the case for procedural knowledge. He defined *deep procedural knowledge* as

> knowledge of procedures that is associated with comprehension, flexibility, and critical judgment and that is distinct from (but possibly related to) knowledge of concepts; separating these independent characteristics of knowledge (type versus quality) allows for the reconceptualization of procedural knowledge as potentially deep. (p. 408)

But these definitions continue the same old dichotomy that suggests that procedural and conceptual can be viewed separately, in isolation of each other, and furthermore maintain that knowledge of procedures does not involve knowledge of conceptual relations, connections, or principles.

Star (2005) insists that the procedural and the conceptual constitute two different types of knowledge and that one could have a deep understanding of the procedural alone. However, that which he describes as deep procedural knowledge would appear to have much in common with what I have been describing as the technical–theoretical interplay in letter-symbolic mathematics, even if Star seems unwilling to call such activity conceptual. Indeed, Baroody, Feil, and Johnson (2007) in their reaction to the Star commentary suggest that "although conceptual knowledge is not necessary for the former [i.e., superficial procedural knowledge], it is unclear how substantially deep comprehension of a procedure can exist without understanding its rationale (e.g., the conceptual basis for each of its steps)" (p. 119).

Within a study involving the comparison of various methods for solving first-degree equations, which Star conducted with Rittle-Johnson (Rittle-Johnson & Star, 2009), the two cognitive-science researchers constructed assessment tasks for each of the following three components of mathematical competence: procedural

Conceptual Knowledge	Procedural Flexibility
Here are two equations: $98 = 21x$ $98 + 2(x+1) = 21x + 2(x+1)$ (a) Look at this pair of equations. Without solving the equations, decide if these equations are equivalent (have the same answer), (b) Explain your reasoning.	$5(x+3) + 6 = 5(x+3) + 2x$ $6 = 2x$ (a) What step did the student use to get from the first line to the second line? (b) Do you think that this is a good way to start this problem? (c) Explain your reasoning.

Fig. 7.5 Sample assessment items from Rittle-Johnson and Star (2009)

knowledge, procedural flexibility, and conceptual knowledge. If we look closely at these tasks, we notice immediately a certain similarity between the sample item for conceptual knowledge and the sample item for procedural flexibility (see Fig. 7.5). The equation pairs for each question call upon the same knowledge, that is, recognizing that the same algebraic object (referred to as *composite variables* by the researchers) has been subtracted from (or added to) both sides of the equation and that this is a valid equation-solving transformation. This similarity leads us to ask the following question: If comparable questions tap into both conceptual knowledge and procedural flexibility, does this not suggest that these two aspects are so closely related as to be nearly one and the same?

If even more evidence were needed to make a case for the conceptual character of techniques, Rittle-Johnson and Star (2005) may have unwittingly supplied it with their pretest results as well as their main findings: “Procedural knowledge correlated with both conceptual knowledge… and flexibility knowledge, and flexibility and conceptual knowledge were also related” (p. 536). They then conclude,

> In the case of equation solving, comparing solution methods was more effective for supporting conceptual knowledge and procedural flexibility than comparing equivalent equations or comparing problem types.… [and] all three types of comparison were equally effective for supporting procedural knowledge. (p. 541)

Certainly, the findings of this study were of a nature to warrant Rittle-Johnson and Star’s reconsidering their original position regarding the distinctness of conceptual and procedural types of knowledge, in particular conceptual knowledge and procedural flexibility. However, in the discussion of their results, they did not return to the issue of the conceptual–procedural distinctions that were fundamental to the design of their study. This return was all the more compelling in that Rittle-Johnson and Star used the theoretical frame of *Adding It Up* (NRC, 2001) as a basis for the construction of their tasks—a frame in which procedural fluency and conceptual understanding are deemed to be interwoven.

Concluding Remarks

I have argued in this chapter against the false dichotomy in mathematics education between conceptual understanding and procedural skills. Others before me have taken the position either that conceptual understanding must precede the meaningful learning of procedures (e.g., Carpenter, 1986) or that precision and fluency in the execution of procedural skills are required in order to acquire and to communicate conceptual understanding (Wu, 1999), thereby maintaining a discourse of dichotomy. However, my arguments against this dichotomy have focused more directly on the intrinsically conceptual nature of the learning of procedures and of their updating and revising. Inspired by both Jean-Baptiste Lagrange and Merlin Donald, scientists from quite disparate domains, I have discussed two ideas that are central to this argument: One that the initial learning, elaboration, and constitution of technique has a conceptual component that contributes to the understanding of the mathematical objects being treated and of the technique itself; the other that this technical–conceptual activity is an integral part of the process of updating procedures, even procedural skills that have become automatized.

Using episodic extracts drawn from our recent project on the learning of algebra, I illustrated the co-emergence of the technical and the conceptual and their joint interplay. For the task activity that was designed around the factoring of the $x^n - 1$ family of polynomials for integral values of *n*, the classes of 10th grade students who participated in our project constituted various factoring techniques that involved the elaboration of conceptual ideas such as the following: (1) there are multiple possible structures to expressions whose exponents are composite numbers (e.g., seeing and expressing $x^6 - 1$ as either $(x^3)^2 - 1$ or $(x^2)^3 - 1$) and thus more than one way to factor such expressions; (2) factoring an expression does not necessarily mean that it is completely factored nor does factoring an expression necessarily produce a unique set of factors; (3) expressions of the form $x^n + 1$ can be factored for odd values of *n*, as per the factoring patterns for the expressions $x^3 + 1$ and $x^5 + 1$.

At the same time that they were elaborating the conceptual ideas that constituted their new techniques, they were integrating these new technical concepts into their existing set of factoring skills, as when they insisted on knowing how they might themselves factor with paper and pencil $x^9 - 1$ so as to obtain the same factors that the CAS had produced. In short, they had developed techniques for factoring any expression of the $x^n - 1$ family of polynomials with integral values of the exponent, along with conceptual elaboration of these techniques.

The implications of this perspective, which considers the procedural activity of literal-symbolic algebra from a conceptual point of view, are potentially far-reaching. The dichotomy of conceptual understanding and procedural skills, which has been with us in mathematics education for years, has permitted algebra to be relegated to the strictly procedural arena within school mathematics. The teaching of algebra as a set of concept free, manipulative procedures has led to the failure of countless numbers of students in their high school algebra classes (Kieran, 1992). Although recent reform efforts have partially shifted the focus in algebra, at least during the earlier years of high school, from procedural work to real-world problem solving and multiple representations for these problems (NCTM, 2000), the issue remains, especially during the later years of high school. When students are eventually faced with the literal-symbolic, transformational activity of algebra, the cleavage between the procedural and the conceptual reappears; the teaching of algebraic procedures is approached by and large as a skills-based endeavor where the conceptual is generally absent. This way of thinking about algebra must change.

The perspective that has been advanced in this chapter, supported by both theory and research, suggests both a feasible and appropriate direction for such change.

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Chapter 8
Needed: Critical Foxes

Jeremy Kilpatrick

Abstract Beyond issues of the research we need are issues arising from the research we already have. Research in mathematics education lacks critical friends, but that phrase implies someone who is on the outside. We in the community especially need insiders who can help us see our work whole. These insiders should have a synoptic view. Isaiah Berlin once drew an important distinction between the hedgehog (who knows one big thing) and the fox (who knows many things). Drawing primarily on my own experience in the field, I argue that more of us in mathematics education ought to become critical foxes.

The question I address in this chapter is not a problem calling for empirical investigation by researchers; instead, it is a developmental problem for the entire field: How can we—*we* meaning people in mathematics education—become more productively self-critical? Reflecting on my work with doctoral students over many years, I have realized that self-criticism is one of the toughest qualities for them to develop. It is extremely easy for beginning graduate students to become cynical about studying mathematics education and say, "All this research is worthless. It has nothing to do with real mathematics teaching. It's not helpful." Students typically find it hard to strike a balance between being critical, which we all need to be, and being overly critical, which none of us needs to be. Mathematics education does not need more people saying, "This is all worthless." That is much too easy a way to approach research in the field. If you say it is worthless, then you do not have to read any research reports. It is not helpful to have prospective scholars discount the

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whole enterprise. Therefore, we need mathematics educators who take research seriously but also can be appropriately critical about it. I see that as more a people problem than a substantive research problem, and in that sense this chapter is somewhat different from the others.

In what follows, I first consider the title "Needed: Critical Foxes." What do I mean by *critical*? And why do I mention foxes? What does that have to do with anything? Next, I discuss some ideas that can (but maybe should not) dominate thinking about research in mathematics education. To illustrate the question of being (or not being) critical of research, I survey contributions to the Research Commentary Section of the *Journal for Research in Mathematics Education* (JRME). In a more personal vein and to illustrate influences on my own critical stance, I then discuss my doctoral committee of 40 years ago and say something about its three members. And finally, I consider further how we as a field need to go critical and consequently how, in my view, being critical means first and foremost being self-critical.

Critical?

Critical Friend?

Judith Jacobs is one of the cofounders of the Association of Mathematics Teacher Educators (AMTE) and the person who gave the first Judith Jacobs lecture in 2003, when that group met in Atlanta. In January 2009, I had the honor of giving the Judith Jacobs lecture in Orlando at the annual AMTE meeting. In her introduction, Judith called me a "critical friend," which was a thoughtful compliment. She and I have known each other for a long time, and it was very generous of her to put me in that category.

The more I thought about it, however, the more I said to myself, "I know Judith meant that phrase in the nicest possible way. I don't mean to disparage her gracious comment, but 'critical friend' sounds as though I'm not actually part of the group." I can be a critical friend of the BYU Department of Mathematics Education because I am not a member of it; it is easy for me to be, or to aspire to be, the department's critical friend. But to call me a critical friend of mathematics educators, or of researchers in mathematics education, or of mathematics teacher educators, is to imply that I am not one of them. It sounds as though I am looking in from outside—that I am not a colleague but only a friend. So I decided I did not want to use the word *friend* in this chapter. Instead, I want to discuss critical foxes.

What Does It Mean to Be Critical?

First, however, let me say a little more about what it means to be critical. In an editorial I wrote in the November 1988 JRME, I noted that critics have an unsavory reputation. I quoted Mark Twain, who said, “The trade of critic, in literature, music, and drama, is the most degraded of all the trades.” I liked even better Kenneth Tynan’s characterization: “A critic is someone who knows the way but can’t drive the car.” Those quotations may well capture how people think about critics, but we need criticism. We need it in our field, even though we tend to look at critics as people who really cannot do the job—which is why they are critics. I think it is especially important, given the work that we are trying to do in our field, that we have critics. Here is what I said in my editorial in 1988, which was a long time ago: “Scholarly work, and especially scientific scholarly work, requires a community of committed critics—people who will subject one another’s ideas to continual scrutiny” (Kilpatrick, 1988, p. 370). We certainly need such people. If we do not scrutinize each other’s ideas, we are not in a scholarly field. Part of being a scholar is learning how to criticize without being ugly, as we say in the South—without being vicious, mean, or destructive. That is a role that we have to take on. As I said in 1988, “If research in mathematics education is to prosper, it must cultivate the give and take of serious scholarship” (p. 370). We have come a long way since I wrote that editorial, but we still have further to go, which is what I want to explore in this chapter.

Foxes?

Isaiah Berlin

The great scholar, Isaiah Berlin, who was born just a century ago, in 1909, once wrote a very famous essay that appeared in book form, *The Hedgehog and the Fox* (Berlin, 1953). Berlin was the first person of Jewish descent to be given a position at Oxford University. He had been born in what was at the time Russia and is now Latvia, but he left there early in life and went to England. People tend to think of Berlin as a native-born Englishman, but he actually was an immigrant.

Hedgehogs and Foxes

In his essay, Berlin quoted the Greek historian, poet, and soldier, Archilochus, who said, “The fox knows many things, but the hedgehog knows one big thing.” People have differed in how they interpret that image, but Berlin used it to argue that there are two categories of writers and thinkers. I want to argue that there are two categories of people in mathematics education, including researchers in mathematics education. I see these not as discrete categories, however, but as extremes of a continuum.

People who are hedgehogs look at the world through the lens of a single defining idea. They have a big idea that determines how they interpret the world, whereas foxes are fascinated by the world’s variety. Foxes do not have one big idea to cover everything but are eclectic instead. They are very pragmatic.

You may have heard the expression, "There are two kinds of people in the world: those who think there are two kinds of people in the world, and those who don't." Isaiah Berlin apparently thought there were two kinds of people in the world. In his essay about Leo Tolstoy's view of history, Berlin says that Tolstoy's talents were those of a fox but that Tolstoy thought a scholar ought to be a hedgehog. He misleadingly, according to Berlin, assessed his own work as having a unitary vision. A fox who wanted to be a hedgehog was how Berlin finally summed up Tolstoy.

Expert Political Judgment

I wanted to use this fox–hedgehog contrast in part because other people have taken it a little further. In particular, Philip Tetlock (2005), who is at the University of California at Berkeley, wrote a book on expert political judgment that made use of the contrast. Tetlock studied those so-called experts in the media who make forecasts about what is likely to happen. He looked at people who had a track record for predicting, for example, whether South Africa would achieve a democratic government without a big revolution. When Quebec was on the verge of seceding from Canada, would Canada then fall apart? What would happen when the Soviet Union disintegrated; would there be a violent reaction or not? What would the reaction be? Given various events around the world, the question was: Were the experts able to anticipate them correctly?

Tetlock (2005) found that among so-called experts, the hedgehogs were worse at prediction than the foxes. The hedgehogs had a big idea of what would happen, and, when it did not happen, had reasons why. The big idea of the hedgehogs dictated the outcome they foresaw, but that idea did not always pan out. In contrast, the foxes, who knew a lot of different things, knew tricks of the trade. They were skeptical of big schemes that covered all eventualities and consequently had a better track record at forecasting.

Of course, hedgehogs are not always wrong. At times, they can be right. And when they are right, they really are right. As has often been pointed out, many great scientists have been hedgehogs. In some ways, you have to be something of a hedgehog to be a scientist so that you can focus your attention and concentrate on something for a lifetime. And journalists are partial to hedgehogs. They love people who can interpret everything in terms of one big idea. In fact, the only thing journalists like better than a hedgehog is two hedgehogs that have different ideas. They do not particularly like foxes, however, because foxes put too many conditions on their comments and predictions. That is the world in which we live: It has hedgehogs; it has foxes.

One of the interesting things that Tetlock (2005) also found was that it is difficult to teach people to look at alternative scenarios so as to broaden their outlook. Tetlock was not able to broaden the outlook of hedgehogs, and with foxes, his training actually made their performance worse. The foxes began to think of reasons why other events might happen, and so their predictive power went down. Political scientists have not figured out how to help either hedgehogs or foxes improve their performance as predictors.

Hedgehog Ideas

One reason I want to make this contrast between hedgehogs and foxes is that we in mathematics education have an oversupply of hedgehogs, or at least of hedgehog ideas. We have many people who have one idea that governs everything they do. Therefore, I think it would be good if a few of us were more fox-like, and that is what I argue for: more foxes. Let me give you some examples of hedgehog ideas in our field. See if you recognize any of them.

U.S. Students Know Less Mathematics than Their Parents or Grandparents

Consider the assertion that U.S. students know less mathematics than their parents or grandparents did. I am not sure how many researchers in mathematics education would buy that hedgehog idea, but there are plenty of people outside of mathematics education who look at how U.S. students do on international tests and say, "Well, yeah. Our students don't know as much math as they used to."

The issue is not whether the statement is true or false—although in this case, there is much evidence to suggest that it is false. Whether it is true or false, however, is separate from whether people believe it to be the case and whether it determines how they think about schools and students. I think there are many people who use that hedgehog idea to interpret what they see going on in schools and what they think is happening when students learn mathematics. The idea seems to dominate their thinking.

Constructivism Explains the Learning and Teaching of Mathematics

Another hedgehog idea is that constructivism explains the learning and teaching of mathematics. I come from the University of Georgia, which is ground zero for radical constructivism, so I am continually confronted by colleagues and students with this idea. Many mathematics educators are hedgehogs when it comes to constructivism. They use it as a grand idea to explain everything. And again, it is not a question of whether the idea is true or false. The question is whether you are using the idea as a lens through which you look at everything.

What is constructivism? Strictly speaking, it is a theory of epistemology. It concerns how people know things. In the simplest terms, constructivism claims that we know something because we construct it mentally and not because we take it in from outside. We build our own ideas inside our heads; ideas do not come to us from outside ourselves—that is the simplest interpretation I can give.

The reason I have trouble with the idea is that, with respect to learning, I think we learn both by constructing our own ideas and by taking in ideas from outside. But people called radical constructivists think that we should not assume that we can know anything outside ourselves; everything we know is constructed in our own heads—that is the radical view in oversimplified terms. My radical constructivist colleagues would probably be unhappy with the definition I have given you, but that is my short answer to what constructivism is. At base, it is a theory about how we know things: We know them because we make them.

An additional problem is that some people go on to claim that therefore there is such a thing as constructivist teaching. For me, it does not follow at all that if you think all knowledge is constructed, you will teach a particular way. Teaching is much more complicated than that. I do not see any necessary connection between how you think people know things and how you think people do or could teach effectively. That is a different story that I do not elaborate here.

When I was becoming a teacher, we used the terminology of discovery rather than construction. Discovery is a metaphor implying that the thing to be learned is out there, and I am finding it. Constructivism uses a different metaphor: What I am learning is here in my head, and I'm building it. Constructivism is not all that new. Depending on how you look at it, Plato was a constructivist.

We can ask the question: How do we know anything? My answer is that we know it because we have thought about it. We know some things because we have put them together ourselves and other things because they have come to us from elsewhere. I think it is both, but that is just me. I am arguing here only that constructivism is a big hedgehog idea in our field.

Randomized Controlled Trials Are the Gold Standard of Education Research

Let me take another idea: Randomized controlled trials are the gold standard of education research. If you are familiar at all with the report of the National Mathematics Advisory Panel (2008), you will recognize that guiding idea: If we are going to have high-quality research in our field, it needs to consist of randomized controlled trials. The panel took a very narrow view of research. Again, that is a hedgehog idea because if you believe the idea, it dominates how you think about research. If the only good research is the small number of research studies that meet the gold standard, however, what does that do to the rest of the research in the field?

Design Experiments Are the Gold Standard of Education Research

In contrast, another idea is that design experiments are the gold standard of education research. That idea is more congenial for many mathematics education researchers I know, who would say, "Well, yes. This is the sort of research we ought to have more of." This view, however, has the same quality of "hedgehogness" as the previous one. Both are ideas that are used to interpret and judge everything. I think that once you do that, you are bound to get into trouble.

Standardized Tests Are the Best Measures of the Mathematics Worth Knowing

A final idea: Standardized tests are the best measures of the mathematics worth knowing. Whether you agree with that view or not, it is a hedgehog idea about assessment that many people have—not everybody, but many.

These are a few examples of what I call hedgehog ideas. As I say, I think we have a lot of hedgehogs out there in our field. It would be nice if we had a few more foxes. Research commentaries can provide one venue for developing fox-like critical faculties.

Research Commentary

In the fall of 2004, Steve Williams recruited me to be the editor of the new Research Commentary section of the JRME. In what follows, I give a little of the history of the section so as to address the way we have been criticizing our own work.

In May 1970, the third issue of the JRME, the editorial board announced that they were going to have a "Forum for Researchers" (Kilpatrick, 2007). At first, they said forum articles would be three pages, but they soon increased the limit to six pages. They got very few manuscripts in response, which was in a way one sign of the field's immaturity. The big exception, which I cannot explain, occurred when Jim Wilson was the editor. He averaged about one forum article each issue from 1977 to 1982. Not all these forum articles were critical discussions—some were just expositions—but I still do not know how Jim did it. I think he must have solicited manuscripts to get that many. But that was the only time when such articles were very frequent.

In preparing a guest editorial on the new Research Commentary section (Kilpatrick, 2007), I surveyed forum articles over the years. From 1972 to 1974, for example, there was only one forum article in those three volumes. And from 2000 to 2004, another three volumes, there were only four forum articles. Our field did not have a very good track record when it came to publishing these more expository, and sometimes analytic, articles in the Forum.

Just before Steve took over as editor of the JRME, there was a survey of the readers. One of their common comments was that the journal needed to publish more commentaries on research. In response, the editorial panel decided to launch a commentary section in which they would allow longer manuscripts (usually from 8 to

12 pages and no longer than 20 pages). The panel proposed several types of topics for the section:

- *Commentaries on research*, which turns out to be the bulk of what we have received and published.
- *Discussions of the connections between research, policy, and practice*, which we have not seen.
- *Scholarly analyses of policy trends related to mathematics education*, which we have not seen.
- *Commentaries on the relationship between research and evaluation*, a few of which have been submitted.
- *Extended reviews of books with critical commentary*, which we have not seen.
- *Scholarly debates among proponents of different views*, a few of which we have published.

Now let us consider one by one the Research Commentaries that JRME published up to May 2009 (see Table 8.1). They have appeared in nine issues. I take you through them because I think it is interesting to look at the kinds of commentary that people in our field have been putting out there for the rest of us to read. I discuss the last column after going through the titles and authors.

The first commentary published was authored by Joan Ferrini-Mundy and Bill Schmidt. It is about international comparative studies in mathematics. I was working on a project with Joan in which we were writing about the mathematics results of the 2003 Trends in International Mathematics and Science Study and the 2003 Program for International Student Assessment. As part of that work, Joan was talking about writing a more general article, and I said, "Great. Write a manuscript for the Research Commentary, this new section we have in the JRME. I would appreciate it if you would submit to us a manuscript on international comparisons." So she did. We had it reviewed, and although we accepted it right away, when I sent it back to Joan with some suggestions for revision, she decided to recruit Bill Schmidt as coauthor.

After soliciting that first article, I happened to be at Michigan State talking with Jon Star. He told me that he had an idea for a commentary on procedural knowledge and wanted to check whether it would be appropriate. He thought that researchers in mathematics education were not thinking about procedural knowledge in quite the right way. So we talked a little about what he might write. His manuscript came in and was published in November 2005.

The next article did not appear until May 2006. Kathryn Chval from Missouri submitted a manuscript about school-based research and the difficulty of conducting research in schools these days because of the various pressures on school people. After we had accepted her manuscript and asked for some changes, she decided to add four coauthors from Missouri. I think she thought that even though she had drafted the manuscript herself, she should bring her coauthors on board because they were all working on the same project together, so the article was really a project production for which others should receive some credit.

After Jon Star's commentary appeared, Arthur Baroody submitted a manuscript in which he challenged Jon's ideas, taking a different view of procedural and

Table 8.1 Research Commentaries in the *Journal for Research in Mathematics Education* from May 2005 to March 2009

Issue	Title	Author(s)	Topic
May 2005	International Comparative Studies in Mathematics Education: Opportunities for Collaboration and Challenges for Researchers	Joan Ferrini-Mundy and William H. Schmidt	Type (Intl)
November 2005	Reconceptualizing Procedural Knowledge	Jon R. Star	Construct (PK)
May 2006	Pressures to Improve Student Performance: A Context That Both Urges and Impedes School-Based Research	Kathryn B. Chval, Robert Reys, Barbara J. Reys, James E. Tarr, and Óscar Chávez	Context (Schools)
March 2007	An Alternative Reconceptualization of Procedural and Conceptual Knowledge	Arthur J. Baroody, Yingying Feil, and Amanda R. Johnson	Construct (PK & CK)
	Foregrounding Procedural Knowledge	Jon R. Star	
November 2007	Toward a Conceptualization of Statistical Knowledge for Teaching	Randall E. Groth	Construct (SKT)
July 2008	On "Gap Gazing" in Mathematics Education: The Need for Gaps Analyses	Sarah Theule Lubienski	Type (Gap Gazing)
	A "Gap-Gazing" Fetish in Mathematics Education? Problematizing Research on the Achievement Gap	Rochelle Gutiérrez	
	Bridging the Gaps in Perspectives on Equity in Mathematics Education	Sarah Theule Lubienski and Rochelle Gutiérrez	
January 2009	The Effects of Spacing and Mixing Practice Problems	Doug Rohrer	Construct (Practice)
March 2009	Transfer, Abstraction, and Context	Matthew G. Jones	Construct (Transfer)
	Concrete Instantiations of Mathematics: A Double-Edged Sword	Jennifer A. Kaminski, Vladimir M. Sloutsky, and Andrew F. Heckler	
	Examining Surface Features in Context	Matthew G. Jones	
May 2009	Examining the Quality of Statistical Mathematics Education Research	Heather C. Hill and Jeffrey Shih	Quality (Stat)

conceptual knowledge. Asked to revise his accepted manuscript, Art added two coauthors and came back with the longest article we have published in the section. We gave Jon an opportunity to reply and published the two articles together in March 2007.

In November 2007, we published an article by Randy Groth on statistical knowledge for teaching. Randy asked the interesting related question: What knowledge of statistics do mathematics teachers need? That is a good question, and Randy's article gave readers something to think about.

In July 2008, we published a series of three articles by Sarah Lubienski and Rochelle Gutiérrez at the University of Illinois. They had submitted the manuscripts as a package, which is how they were reviewed and accepted. After a long phone conversation in which we discussed putting them together as a single article, they convinced me that the manuscripts would work better as separate pieces. In the published articles, Sarah leads off with her argument, Rochelle states hers, and then the two engage in a kind of dialogue. They present two very different views about the so-called gap gazing: Do we need more research on the gaps in performance between different groups, minority and majority? Sarah says yes, we need it. Rochelle says no, we do not; we need other kinds of research. In the third article, they engage in debate and moderate some of their views. It is a nice exchange.

In January 2009, we published an article in which Doug Rohrer, a psychologist from Florida who has done research on spaced and mixed practice, discussed questions of how to orchestrate practice in mathematics.

The trio of articles in March 2009 came about because of a controversial article on the advantage of abstract examples in learning mathematics that Jennifer Kaminski and her colleagues at Ohio State had published in *Science* the previous year (Kaminski, Sloutsky, & Heckler, 2008). Matt Jones submitted a critique of the article that we accepted. According to the media, the *Science* article argued that abstract examples are better than concrete examples in teaching mathematics. That overly simple conclusion turns out to be, according to Matt, not the right conclusion. It took us a while to get a response from Jen and her colleagues to what Matt wrote, but we got it. Then I gave Matt the opportunity to have the last word, so we have an exchange of a somewhat different kind in the issue.

A 14th article appeared in May 2009. It is by Heather Hill and Jeff Shih and concerns the quality of research studies published in the JRME during the previous decade that used statistics in some way or another. Heather and Jeff are very critical of what they found. Those of us who think that JRME is the best journal in the field may be somewhat dismayed to discover how many flaws there are in the statistical analyses in that journal.

Now let us go back and look at the last column in Table 8.1. Although the categorization of the articles it offers is very crude, it does provide some idea of the types of articles the journal has been publishing, which in turn reflect the kind of manuscripts received. The first article, by Ferrini-Mundy and Schmidt, is about a type of research: international comparative research. They advocate more of it—that more people should do it and that it presents many opportunities. In contrast, Star's article is more about a construct in our field. What do we mean by *procedural knowledge* (PK)? How are we interpreting that term? Star is not so much calling for more research on procedural knowledge as he is saying that we need to think differently about the construct.

The Chval et al. paper is the lone research commentary that deals with the context in which we do research, namely, schools. It is about the logistical difficulties of getting research done out there in schools when they are under such pressure to improve their performance. What time do schools have to give to researchers? That is a big issue for the field, and I was pleased to have the article.

With Baroody and his colleagues, we are back to the constructs of procedural and conceptual knowledge—another article about constructs. Moreover, Groth's statistical knowledge for teaching (SKT) is yet another construct. The articles on "gap gazing," in contrast, are about a type of research. Should we have more of this kind of research on equity or should we have less of it?

A number of the commentaries in the series have been psychological in their orientation. Rohrer's is about the construct of practice and what psychological research says about how we ought to orchestrate it in mathematics instruction. Similarly, Jones's article concerns transfer, which is what that whole debate regarding the *Science* article was about—the construct of transfer. Only the article by Hill and Shih, the last one in the group, gets into the question of the quality of the research that we do.

The nine issues in Table 8.1 contain the articles I accepted and published as the first editor of the Research Commentary section (the second editor is Ed Silver). As I survey the articles in the table, I am both proud and happy to have had a hand in getting them into print, and I think that collectively they have added much to our field. But I must say that they are not critical in quite the sense I would like to see us be critical. For me, only the last commentary explores deeply the quality of the work we do. You can point out that it looks only at the statistical side of things. That's true, but the authors do try to take an evaluative look.

My Doctoral Committee

Now let me shift gears rather dramatically—to my doctoral committee and how they helped me learn to be a critic. I had a committee of three wonderfully talented people. Sometimes you just get lucky because the stars are in alignment. My major professor was Ed Begle, who was at Yale University when I started at Stanford but then moved to Stanford and became my major professor. George Pólya was the reason I went to Stanford—to work with him. I was in an academic year institute with him when Begle came. And Lee Cronbach, an educational psychologist, turned out to be influential in my program, too.

E.G. Begle (1914–1978)

Ed Begle, as I say, was my major professor. At the first International Congress on Mathematics Education in France in 1969, he gave a famous talk, later published in *Educational Studies in Mathematics* (Begle, 1969), on the role of research in the improvement of mathematics education. In the talk, he said he thought we ought to turn mathematics education into an experimental science. That was his hedgehog idea: that we needed to make our field experimental like physics and the natural sciences. I never agreed with him about that, but I learned a lot from him about research. I would call him a critical hedgehog. He wrote a book entitled *Critical Variables in Mathematics Education* (Begle, 1979) that showed that he was a good critic. But he was also a hedgehog.

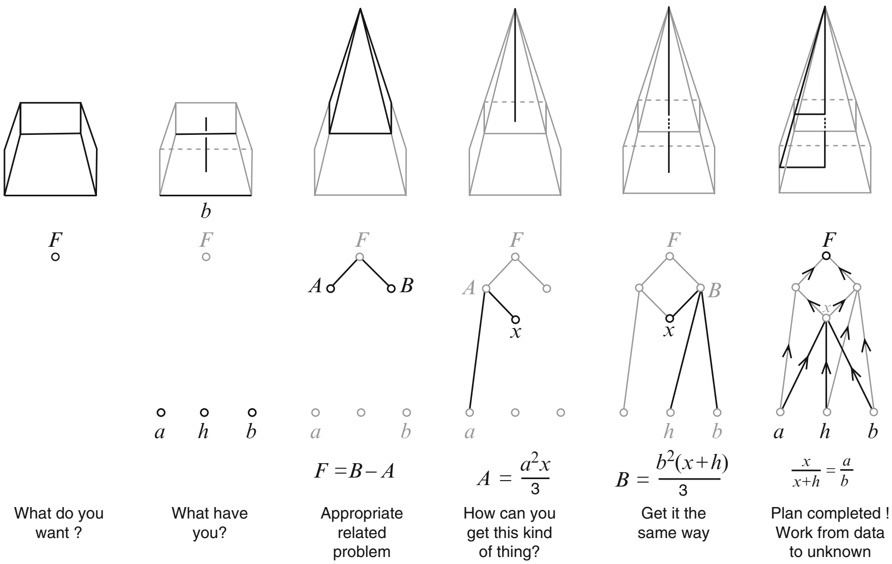

Fig. 8.1 Simultaneous progress on four levels (Pólya, 1962, 1965/1981, Vol. 2, Fig. 7.8, p. 9)

George Pólya (1887–1985)

George Pólya was in a way a kind of mathematics education researcher—one who never wrote up his ideas in a research report but who reflected on his experience and had a lot to say about mathematics teaching and learning. He tells the story (Pólya, 1962, 1965/1981, Vol. 2, p. 1) that when he was a postgraduate student in Vienna, he was tutoring a boy and was trying to explain something in solid geometry. Pólya lost the thread of what he wanted to say and got stuck. He sat down and worked through what it was he was trying to explain to the boy, and did that thoroughly so that he would not forget it again. He arrived at a geometric representation of the problem-solving process (see Fig. 8.1).

The problem illustrated in Fig. 8.1 is not the same problem that Pólya was doing with the boy, but it is similar. The problem is to find the formula for the volume of a frustum of a square pyramid. The solver begins by asking: What do you want? You are looking for a formula, *F*, for a volume. Then you ask: What do you have? What are you given? Well, you have the sides of the lower and upper bases, and you have the height. And so on. The main idea that Pólya was coming up with was that if you look at what you are given, and you look at where you are headed, you can work forward and backward from there. In *Mathematical Discovery* (Pólya, 1962, 1965/1981), Volume 2, Pólya works through the problem showing the four levels in Fig. 8.1: (a) the *image* level at the top, which suggests the geometric figure as it evolves in the solver's mind; (b) below it, the *relational* level, which symbolizes the objects being considered (unknown, data, etc.) as points being connected by lines; (c) the *mathematical* level, consisting of formulas; and finally (d) the *heuristic* level, containing the questions or the suggestions that move the solution process forward.

Pólya's metaphor for problem solving is essentially that of building a bridge. Begin with what you want and what you have at hand and see if you cannot connect the two by working from both sides. Here is where I am; here is where I want to go. Sometimes I can work backward from where I want to go; sometimes I can move forward from what I have to develop that further. Eventually, if I am successful in reaching a solution, I connect where I have started from with where I need to go. I think that metaphor is a very powerful way to think about problem solving in our field. More teachers of mathematics should be asking their students the questions Pólya suggests. Questions like: What do you want? What do you have? What is a related problem? For more examples of problems that illustrate how such questions might be used, see Pólya and Kilpatrick (1974/2009).

I see Pólya's ideas as emerging from a kind of introspective psychological study. Pólya was interested in teaching from the very beginning. He appreciated research in mathematics education. He thought it should support mathematics teaching. For example, he thought I should do my dissertation study on the effectiveness of the geoboard. That was the kind of study he thought would be appropriate. I would call him a sympathetic fox.

Lee J. Cronbach (1916–2001)

In a way, Lee Cronbach taught me more than anybody else about how to be a critic. You may have heard, perhaps, of Cronbach's alpha, which is a method for measuring reliability. He worked on many topics in testing and measurement, including generalizability theory, construct validity, aptitude–treatment interactions, and program evaluation. He made important contributions in all those areas, and I consider him a first-rate critical fox.

For my dissertation, I studied eighth graders thinking aloud while solving mathematical problems, trying to see how they came up with solutions and whether they used the sort of heuristic procedures that Pólya proposed. When Cronbach read the first draft of my dissertation, he first made me take out all of the items in my coding scheme that I was not able to code reliably. So I had to go back and revise the coding scheme, which was a good lesson for me. The second lesson was more painful. I had written a chapter containing four case studies of students. Cronbach managed to convince me that those case studies had nothing to do with the rest of my study, and so I took that chapter out.

At the time, I thought—and I still think—that removing that chapter was like learning to perform surgery on one's own child. Scrapping a whole chapter of one's dissertation is a major feat. One thing I have learned as an editor is that most people have serious trouble performing surgery on their manuscripts. You return the manuscript to them saying, "You need to fix this and this and this," and they send it back to you having done the least possible work on it. They cannot perform radical surgery. They love what they have written so much, it has meant so much to them, and they have worked so hard to write it that the idea that it needs major changes is really tough for them to handle. I think it is important for all of us early in our careers to learn to do this surgery on our own children, so we can then do it on other people's children.

I think we can develop the critic in people. The main way that Cronbach helped me develop my inner critic was by taking my work seriously. Not just my dissertation but all the papers I wrote for him were returned with extensive comments of all kinds. In fact, when Cronbach died, some of his former students remarked that he had often written more on their papers than they had. My own students know that I take their work seriously by the comments I make. That is a way, it seems to me, to help a writer reflect: "What am I trying to say in this passage? Here's what someone else has thought about what I've said. How does that help me think about saying it better?"

I think that we in the profession, especially those of us who train teachers, need to take our students' work seriously so that they can take their students' work seriously. That's how we develop this self-critical attitude. If you have not had someone else tear one of your essays apart, you do not really know how to do your own criticism. I think that's how you learn it. You learn it from other people doing it with you, and then you decide:

> OK, I can do this myself. I'm strong. I survived this tearing apart of my beautiful work. I can learn to be more critical with my own work, and in doing that I can help other people become more critical.

I think there's nothing like the experience of being an editor to help you become more self-critical and therefore more critical, in a good way, of what other people are doing.

Going (Self-)Critical

To conclude, in 1988 I wrote the following:

> Although an individual scholar's reputation is built on her or his own original work, the status of a field rests ultimately on the level of critical analyses that reviewers, editors, and readers do of each other's efforts. Such analyses are never given much weight in a curriculum vitae, yet they are essential to the collective enterprise. (Kilpatrick, 1988, p. 370)

I want to underline this point: You cannot critique the work of others without first becoming self-critical. Here is the King James version of what Jesus said in his Sermon on the Mount: "And why beholdest thou the mote that is in thy brother's eye, but considerest not the beam that is in thine own eye?" (Matthew 7:3). We have to become self-critical before we can become critical of the work of others.

A Dilemma

The hard thing for us as researchers is that we need both faith and skepticism. We need a bit of hedgehog in us so that we believe enough in what we are doing to move forward. At the same time, we have to be fox-like and doubt our idea enough to put it to the test:

> As researchers, we must maintain enough faith in our theory to test it with confidence while simultaneously doubting it enough to test it with skepticism. As George Pólya often said about solving problems, what we need is courage—both the courage to guess and the courage to doubt our guess. Young children and most adults may need the security of unquestioned belief as a platform from which to make cognitive progress. The risky business of science, however, demands researchers who know their theory is faulty and actively seek to refute it. If we hold our theory in the rigid grip of certainty, we cannot use it effectively as a tool for scientific inquiry. (Kilpatrick, 1987, p. 330)

I like to use the image of the hammer here because if you think about it, in order to use a hammer, you have to be flexible. You cannot hold a hammer rigidly and use it to pound at the same time. And that is true of any tool: You need to use the tool flexibly, not rigidly.

So what does it mean to be critical? Henry James said, "To criticize is to appreciate, to appropriate, to take intellectual possession." And that is, I think, a good way for all of us to think of it.

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Chapter 9
Where Are the Foxes in Mathematics Education?

Jo Boaler, Sarah Kate Selling, and Kathy Sun

Abstract Mathematics education researchers have produced many important research insights into how students learn mathematics, but relatively few of these insights have influenced the practice of classrooms. This chapter takes up Kilpatrick's call for more "foxes" in mathematics education and considers the ways the field may move to work in more "foxy" ways. This includes reviewing some examples of fox-like research, and offering suggestions for ways in which researchers of mathematics education may turn their research results into useful and useable aspects of practice.

We thoroughly enjoyed reading the different chapters in this book, all of them setting out the authors' ideas for ways to improve mathematics teaching and learning through research. The authors are well qualified to offer their thoughts on the research topics that, if studied, could change the landscape of mathematics education. All of the ideas put forward are good candidates, ranging from focused pleas to wider ranging calls. Thompson, for example, makes a focused call for change within a particular part of teaching. He calls for the need to develop *mathematical meaning* in classrooms—meaning that supports students' interest, curiosity, and future learning. He then makes a plea to address this need by developing a new assessment tool to capture the mathematical meanings necessary for teaching. Simon also focuses on learning environments, citing the lack of a coherent *pedagogical theory* for the teaching and learning of mathematics. He argues for developing a detailed characterization of the ways people learn mathematical concepts in order to build a set of design principles for mathematics instruction that closely matches this learning process. Harel raises concerns about the lack of *intellectual aim* in most mathematics classrooms, and proposes four concrete instructional steps to address this concern.

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By contrast, Hiebert asks the broader question: Why is teaching so hard to change? He argues for a need to address this systemic issue by (1) developing a shared consensus regarding student learning goals, (2) shifting the focus from the "teacher" to "teaching," and (3) working to implement practices to support learning to teach. Kieran also asks a broader question—one that is very important for mathematics education—Why has there been a persistent dichotomy in the treatment of procedures and concepts? She considers the relationship between conceptual and procedural understanding and the important role played by the conceptual in different aspects of procedural development, extension, and application. Thames and Ball also take a broad perspective centered on the improvement of teaching, calling for a shift in focus from "teacher quality" to "teaching quality." Like Hiebert, they propose a changed system that (1) builds a common mathematics curriculum, (2) develops valid and reliable assessments that are linked to the curriculum, (3) builds a system to supply all schools with skilled teachers to teach the curriculum, (4) centers teacher training and credentialing on teaching practice, and (5) organizes schools to support beginning teachers.

These different chapters are illustrative of an issue that Kilpatrick raises and that we will focus on in our response to this thought-provoking book. Kilpatrick draws from Isaiah Berlin's (1953) famous essay *The Hedgehog and the Fox*, which sets out two types of writers and thinkers—hedgehogs and foxes. Hedgehogs see the world through the lens of a single idea, whereas foxes are fascinated by variety and tend to look across different ideas in order to make recommendations for change. Kilpatrick argues that, in mathematics education, we have many hedgehogs, who dedicate their research lives to studying an idea or a set of related ideas. This is not surprising. Kilpatrick acknowledges that most scientists need to be hedgehogs in order to focus on one idea and conduct detailed study. But Kilpatrick also makes a contentious and interesting claim—he proposes that mathematics education as a field has an oversupply of hedgehogs and that it would be good for the mathematics teaching and learning landscape if there were many more foxes in our field.

Do We Really Need More Foxes in Mathematics Education?

The idea that we need more foxes in mathematics education is one that intrigued us and sparked many animated conversations among our group. What exactly is a fox in mathematics education? Are Thames and Ball and Hiebert's chapters examples of foxy analyses, whereas Simon, Thompson, and Harel's are more hedgehog like? What about Kieran's interesting chapter? Does that fall between foxiness and hedgehoginess? Who are the foxes in mathematics education? Do we have enough of them? Do other fields have more foxes? We thought they did. If so, why is that?

In considering the foxiness of the field, we also took some time to discuss the characteristics of a fox. Kilpatrick argues that we need more "critical foxes"—people who consider research ideas and critique them. One of the analyses Kilpatrick nominates as being foxy is Hill and Shih's (2009) critique of statistical research in mathematics education, which appeared as a research commentary in JRME. This article casts a critical eye over statistical research in our field. Using the criteria set forth by AERA, APA, and NCME in 1999, the authors propose that the majority of statistical studies published in JRME between 1997 and 2006 are methodologically lacking in one or more respects. Based on their review, Hill and Shih make a number of recommendations for statistical research, including disentangling treatment, classroom, and instructor effects. The authors emphasize the importance of methodological rigor in order to gain credibility with policymakers and practitioners.

In their article Hill and Shih (2009) also highlight an issue that seemed to us to pertain to the role of foxes in mathematics education—that of making the results of research available to a wider audience, including policymakers and practitioners. For although the production of research ideas is extremely worthy, if research ideas in mathematics education do not get taken up and used, by teachers, parents, and other educators, then their worth diminishes significantly. Boaler has argued that there is a huge gap between what we know works from research and what happens in most classrooms and homes—in the United States (Boaler, 2009b), the UK (Boaler, 2010), and elsewhere. This gap means that many mathematics teaching and learning interactions take place that are oppositional to those that would encourage students' interest, enjoyment, and achievement. This is despite the fact that we now know a lot about productive learning environments. A number of the chapters in this book describe features of learning environments that the authors regard to be lacking. Thompson, Simon, and Harel all propose new research to help improve the learning environments in mathematics classrooms. Such research is important, but unless we find more ways to make research results accessible to practitioners and credible to policymakers, it seems likely that the results of the research will lie dormant, as have the results of thousands of studies before them, having no impact on the teaching and learning interactions that take place in classrooms across the country. Many wonderful hedgehogs have produced important research insights into the ways students should learn mathematics, but these insights have had little impact on practices within classrooms. Perhaps Kilpatrick has put his finger on the problem—there are just not enough foxes in mathematics education.

Foxy Research

If our field is to produce important research that is also taken up and used, then we agree that researchers need to be more fox-like in the way they work. The role that Kilpatrick highlights—that of looking across research to critique it—is clearly important. Another foxy role that we wish to highlight is that of taking research ideas and making them more accessible to the audiences who might use them. Some researchers make this move when they publish their research results in books and articles for teachers, parents, and others. I (JB) was prompted to do this when presenting the results of my research on different teaching approaches and their effect on student learning to an audience of senior academics, none of whom worked in education.

At the conclusion of my talk, a number of them rushed up to me expressing shock and dismay, saying "You must get these results out to the general public." Some of them also told me that they felt they had been misled in the past, and some had even campaigned against moves towards new teaching environments because of the misinformation they had been given. They urged me to write a book for the general public, which I did, and the book was published by the "trade press" publisher Penguin in the United States (Boaler, 2009b), with versions for the UK (Boaler, 2010) and for Sweden (Boaler, 2011). The writing of that book presented many occasions for meetings with policymakers and numerous media opportunities, such as newspaper coverage and radio interviews, all of which helped to communicate mathematics education research results (mine and others) widely. But at the same time some academics frowned on my writing of a book they regarded as less than *objective*, as I had turned research results into ideas for teachers.

In addition to publishing research in a variety of traditional print formats, it is important in this technological era for researchers to begin using alternative forms of media. The use of multimedia communication has the potential to make our research more accessible to a broader audience. The creation of Web sites to convey information could play an instrumental role in disseminating important research findings. Likewise, the use of video has powerful effects for communicating ideas. Both researchers and practitioners may benefit from the use of video case studies, which illustrate key findings and offer examples of best practice. One possibility for making this dissemination happen is to take one or more of the key findings of a research study and ask, "How might we turn this idea into a multi-media case?" Such cases could serve as artifacts of practice for teacher (and other) learning as Ball and Cohen argued so well in 1999.

Foxy Collaborations

Another key strategy for making research more accessible to a broader audience is to form partnerships between various invested parties. Greeno et al. (1999) argued that we should remove the boundaries between knowledge and "domains of practical activity" (p. 303) by engaging teachers, researchers, and students in collaborative participation structures to generate new research knowledge. Foxes would play an instrumental role in orchestrating such collaborative activity. In the past decade, two particular groups comprising mathematics education scholars, teachers, mathematicians, and policymakers have demonstrated the potential of such collaboration. The first was the National Research Council's (NRC) Mathematics Learning Study committee in the United States, chaired by Kilpatrick himself, which worked to synthesize research on mathematics learning. The document produced by this committee—*Adding it up*: *Helping children learn mathematics* (NRC, 2001)—is highly cited and used frequently in teacher education programs. It also played a key role in informing the Common Core State Standards in the United States, in particular the section on *Mathematical Practices*, a document that is poised to significantly impact classrooms through curricular material and student achievement assessment.

A second significant collaboration between various stakeholders occurred in 2003 with the RAND Mathematics Study Panel in the United States. Their report (Ball, 2003)—*Mathematical proficiency for all students*: *Toward a strategic research and development program in mathematics education*—was intended to be relevant to policymakers and funding agencies as well as to mathematics education researchers. Not only was the panel fox-like in trying to impact practice, but it also made a foxy move by calling for the strategic accumulation of research and for collaborations between different groups.

In other countries we see examples of collaborative institutions that play a fox-like role. Singapore has such an institution, called the Centre for Research in Pedagogy and Practice which was established to conduct research to support education. The centre describes itself in this way:

> The Centre for Research in Pedagogy and Practice is the largest educational research centre in the Asia Pacific. It was established in 2002 by the National Institute of Education, Singapore's sole teacher education institution, and funded by the Singapore Ministry of Education. The Centre brings together researchers, educators and administrators to research, develop and implement new and innovative ways of teaching and learning. The Centre's research will provide the basis for educational policy and decision making in Singapore, to help our schools and students address the complex challenges of new economies, cultures and technologies. (National Institute of Education, 2011)

The claim that "the Centre's research will provide the basis for educational policy and decision making in Singapore" is supported by a Ministry of Education that seems highly receptive to the research conducted in the institute. As an example, the Centre conducted a study of classrooms in 2004–2005 during which they coded over 200 lessons in different subject areas, including mathematics. One of the results of the study was that mathematics lessons were found to be largely procedural (Yeo & Zhu, 2005). This finding prompted the ministry to implement a new approach to teaching, learning, and assessment in schools. The policy initiative, called PETALS (Pedagogy, Experience of learning, Tone of environment, Assessment, Learning) was a direct response to the findings of the research. This direct link between a research-conducting institution and government policy is unusual and illustrates a system that few countries seem willing or able to emulate.

In the Netherlands a different type of institution exists that is also extremely influential in the practice of teachers worldwide. The Freudenthal Institute is devoted to mathematics education and conducts research, designs classroom materials, and implements and evaluates research findings in classrooms. It was founded by and named after the mathematician Hans Freudenthal, who designed a teaching approach based on realistic mathematics education (RME). The influence of research that emanates from the Freudenthal Institute is widespread in and beyond the Netherlands; and the institute has links with universities across the world. Research and ideas emanating from the institute have changed practice in countries as diverse as the United States (e.g., the Freudenthal Institute, USA, at Wisconsin) and Indonesia (see Zulkardi, Nieveen, van den Akker, & de Lange, 2002). Indeed it is hard to overestimate the impact this energetic and generative center, devoted to work in mathematics education, has had upon practices in classrooms across the world. In some ways it seems that a center such as the Freudenthal Institute—comprising a group of people working under the same roof, inspired by similar ideals and grounded in careful research and theory—is the ideal way for research to have an impact on practice, as it is the place to which policymakers and practitioners turn for ideas on ways to improve classrooms. The reality for many researchers of mathematics education is quite different. Many work as the single mathematics educator in an institution, with fewer opportunities to collaborate on research, to take research findings into schools, to integrate research ideas into curricula and other materials, and to be inspired by the presence of colleagues with similar interests and concerns. The isolated nature of much research that does not build upon what has gone before it is one of the criticisms commonly given about research in education (e.g., Hargreaves, 1996).

Collaborations between the various stakeholders in mathematics education are important and valuable, but individual researchers can also work in ways that would encourage their research results to have an impact on practice. In 2008 I (JB) was asked to deliver a plenary address on the subject of research impacting practice in Rome at the centennial celebration of 100 years of ICMI. In preparation for this talk, I asked key figures in mathematics education from seven different countries on four continents to nominate studies that had impacted practice (Boaler, 2009a). This proved to be an interesting exercise. First, it showed that almost all of the nominated research had taken place in elementary mathematics education. The author of one of the nominated studies, Bob Wright, the designer of the "mathematics recovery" program in Australia, reflected by saying, "In my view, early years teachers have both a major need to learn more about young children's number development and a significant unrealised capacity for such learning" (Boaler, 2009a, p. 6). This "significant unrealised capacity" raises an important issue around the choice of research topics and the openness of teachers and others to learn from them. Wright suggests that elementary teachers need to learn about children's number development (which he describes as a "major need") and that they are willing and able to learn, which he describes as an "unrealised capacity."

In secondary mathematics education the issues are different, as teachers are content specialists and may not be quite as willing to learn from research unless they themselves have identified an aspect of their practice as problematic. Presenting secondary mathematics teachers with research findings on the importance of students developing meaning or even on progressions of student thinking may be met with resistance if the teachers have not identified a need for mathematical meaning or knowledge of student thinking. If teachers are less willing to take up research that they have not identified as important, then collaborative research that involves teachers at the design stage is likely to have a much greater impact.

A highly significant aspect of the research studies that had impacted practice in different countries (Boaler, 2009a) was the work that the researchers had purposefully done to weave the results of their research into practice, usually through teacher learning opportunities. Perhaps the best example of this interesting move comes from the Cognitively Guided Instruction program in the United States.

Cognitively Guided Instruction: An Example of Foxy Research

One mathematics education research program that has been quite successful at influencing practice comes from the work of Thomas Carpenter and colleagues in the development of Cognitively Guided Instruction (CGI). CGI is a research-based professional development program that emerged from research findings about the development of children's mathematical thinking (Carpenter, 1985; Carpenter & Moser, 1984). The CGI program took student thinking to include understanding teachers' knowledge of students' mathematical thinking, and teachers' use of such knowledge in instructional decision-making. Carpenter, Fennema, Franke, Levi, and Empson (2000) found that

> although teachers had a great deal of intuitive knowledge about children's mathematical thinking, it was fragmented and, as a consequence, generally did not play an important role in most teachers' decision-making (Carpenter et al., 1988). If teachers were to be expected to plan instruction based on their knowledge of students' thinking, they needed some coherent basis for making instructional decisions. To address this problem, we designed CGI to help teachers construct conceptual maps of the development of children's mathematical thinking in specific content domains. (p. 1)

The CGI team intentionally addressed the gap between what is perceived to be important to implement (based on research) and what actually occurs in classrooms (practice). CGI's early work was heavily based on research on student thinking about addition and subtraction. The CGI professional development program was designed to be relevant to teachers by helping them to develop a framework to better understand student thinking. The CGI researchers did not just try to teach teachers about students' mathematical strategies; instead they introduced teachers to their research findings and then set them upon their own paths of discovery. The researchers organized professional development workshops in which teachers became familiar with the strategies and frameworks. Teachers were then encouraged to watch and listen to their own students in their own classrooms, in order to develop their own frameworks for student thinking. In the first CGI study, researchers found that those teachers who participated in the CGI training subsequently listened to their students more, encouraged a variety of solution strategies, and focused more on problem solving and less on computational skills than the control teachers (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989). Subsequent studies have also found similar positive effects of CGI on classroom practices. A follow-up study (Franke, Carpenter, Levi, & Fennema, 2001) found that, four years after participation in the CGI training, all CGI teachers continued to make some use of student thinking in their classrooms. Additionally, approximately half of the teachers had changed more profoundly, demonstrating generative growth as they continued to develop and elaborate upon their knowledge of student thinking in the years following the original intervention. These teachers viewed student thinking as central and perceived themselves as creating their own knowledge about student thinking.

The CGI researchers did something very interesting and highly effective. They took research findings on student thinking and they designed opportunities for teachers to learn from the research, encouraging teachers to become active inquirers. Ultimately, the CGI team took a research idea and found a way to make such an idea relevant and influential to teacher practices while at the same time creating a program of research, which evaluated this integration between past research and teacher practices (Carpenter et al., 1989; Carpenter, Franke, Jacobs, & Fennema, 1998; Fennema et al., 1996; Fennema, Franke, Carpenter, & Carey, 1993; Franke et al., 2001). CGI professional development workshops now take place across the United States and beyond. These programs have been well researched, and there is evidence that they not only change teachers' practices in the short term but also substantively reorient teachers' attention towards students' mathematical thinking in ways that have a long-lasting impact (Franke et al., 2001). In this regard, the CGI program serves as a model for how we might integrate important research findings and understandings into the practitioner's world while still maintaining a research perspective.

There are other examples of foxy research, such as Paul Cobb and Kay McClain's recent work with school districts (Cobb & McClain, 2011). Rather than attempting to "reform" mathematics teaching by working only with teachers, Cobb and McClain have situated their efforts within the broader sphere of the school district and worked to understand the interconnections and boundary crossing that need to take place in order to promote instructional innovations (Cobb & McClain, 2011). But foxy research remains rare in mathematics education, as most researchers work in more hedgehoggy and focused ways and report the results of their research in scientific journals, which teachers rarely read. This is not surprising, as the publication of journal articles is the "gold standard" by which most researchers who work in universities are judged. Indeed, while research journal articles are rewarded, publications for practitioners (even articles in highly regarded practitioner journals) are sometimes frowned upon and deemed too "popular" to be considered serious academic work. It is our contention that if researchers were to convert their findings into opportunities for teacher learning—substantive opportunities such as having teachers ask similar questions *of their own students*—then research would have a much greater impact upon practice.

Conclusion

The different chapters in this book all nominate important areas for further research but they are quite different. The chapters from Harel, Simon, and Thompson each highlight something in classrooms that the authors believe needs to change and to which research could provide important answers. Hiebert asks a broader question about why it has proved so difficult to change teaching over the years, and Kieran asks a question of a similar grain size—Why has there been so much confusion and insufficient awareness of the role of conceptual thinking in mathematics learning? Thames and Ball set out various aspects of the system that need to change. Taken together these different chapters highlight most aspects of mathematics education that are in need of attention. We have chosen not to add to the list in our response, although we could have made an impassioned plea for more research on the inequities that persist in mathematics learning, in the United States and beyond (Martin, 2009). Instead we have chosen to consider the way research studies may be designed and the opportunities that are built in for teachers or others to conduct their own learning. The United States does not have a central organization such as those that exist in the Netherlands and Singapore that serves the purpose of reviewing research studies and disseminating them, making particular opportunities for teachers (and others) to learn from them (although there are regional versions of these). This void is one of the reasons that the studies that have had the most impact on practice in the United States are those that have built teacher learning into the design of the study. When this dimension is possible, it seems ideal, but when it is not, then other methods for creating teacher learning opportunities need to be considered, including the creation of multimedia cases and Web sites that are more accessible to multiple audiences than journal articles. We have spent some time in this chapter supporting Kilpatrick's call for more foxes in mathematics education and highlighting one important aspect of foxiness—that of working with research ideas to make them accessible to educators and to the public. This role can be taken on with one's own research, as evidenced by the CGI work, by researchers communicating a range of research ideas, and by collaborations and institutions that bring together a range of people to conduct research and turn research ideas into practical resources for teachers and others. It seems that it is important to be a fox, but when we look at those in mathematics education who work in foxy ways—for example, three of the authors who feature in this book, namely, Deborah Ball, James Hiebert, and Jeremy Kilpatrick—we note that they are all seniors in the field. This may not be a coincidence; it is difficult for new researchers to head research programs such as CGI or to be invited into high-level policy discussions. Nevertheless, new technologies offer even the most hedgehoggy and inexperienced researchers opportunities to raise their heads above the ground that hedgehogs so faithfully traverse to consider ways to convert their carefully produced findings into ideas for practice that are accessible and exciting for teachers and others to use.

It is also important to recognize that if those of us within mathematics education do not take on the role of a fox, communicating research ideas widely, then we leave ourselves vulnerable to predators who work in less than ethical ways. Kilpatrick describes the need for scholars to be critical without being "vicious, mean, or destructive." In recent years some of us have suffered extremely vicious attacks from campaigners for traditional education attempting to miscommunicate and distort research ideas and destroy the reputations of researchers (see Boaler, 2009a). If researchers in mathematics education do not step up to work as foxes, then policies and practices in classrooms could well be determined by "weasels."

When we survey the landscape of mathematics teaching and learning in the United States, it is clear that many problems remain—of "savage inequalities" (Kozol, 1991), serious underachievement, and low student interest (Boaler, 2009b). Most students still receive mathematics instruction that comprises a teacher delivering content that students then practice, devoid of opportunities to engage actively or to develop meaning (Thompson). This is despite the fact that our field has produced thousands of important research studies on the ways that students learn productively. If we are going to change this situation we would do well to heed the advice of Kilpatrick, as well as many of the other authors in this book, and to conduct research with the rigor of a hedgehog but then to work as a fox, turning research results into useful and useable aspects of practice.

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