Chapter 7
q-Complex Operators

In the recent years applications of q-calculus in the area of approximation theory and number theory are an active area of research. Several researchers have proposed the q-analogue of exponential, Kantorovich- and Durrmeyer-type operators. Also Kim [106] and [105] used q-calculus in the area of number theory. Recently, Gupta and Wang [94] proposed certain q-Durrmeyer operators in the case of real variables. The aim of this present chapter is to present the recent results [5] on q-Durrmeyer operators to the complex case. The main contributions for the complex operators are due to Sorin G. Gal; in fact, several important results have been complied in his recent monograph [76]. Also very recently, Gal and Gupta [78, 79], and [80] have studied some other complex Durrmeyer-type operators, which are different from the operators considered in the present article.

7.1 Summation-Integral-Type Operators in Compact Disks

In this section we shall study approximation results for the complex q-Durrmeyer operators (introduced and studied in the case of real variable by Gupta–Wang [94]), defined by

$$M_{n,q}(f;z) = [n+1]_q \sum_{k=1}^{n} q^{1-k} p_{n,k}(q;z) \int_0^1 f(t) p_{n,k-1}(q;qt) d_q t + f(0) p_{n,0}(q;z), \tag{7.1}$$

where $z \in \mathbb{C}, n = 1,2,\dots; q \in (0,1)$ and $(a-b)_q^m = \Pi_{j=0}^{m-1}(a - q^j b)$, q-Bernstein basis functions are defined as

$$p_{n,k}(q;z) := \begin{bmatrix} n \\ k \end{bmatrix}_q z^k (1-z)_q^{n-k}$$

A. Aral et al., *Applications of q-Calculus in Operator Theory*,
DOI 10.1007/978-1-4614-6946-9_7,

and also in the above q-beta functions [104] are given as

$$B_q(m,n) = \int_0^1 t^{m-1}(1-qt)_q^{n-1} d_q t, \ m,n > 0.$$

This section is based on [94]. Throughout the present section we use the notation $D_R = \{z \in \mathbb{C} : |z| < R\}$, and by $H(D_R)$, we mean the set of all analytic functions on $f : D_R \to \mathbb{C}$ with $f(z) = \sum_{k=0}^{\infty} a_k z^k$ for all $z \in D_R$. The norm $\|f\|_r = \max\{|f(z)| : |z| \leq r\}$. We denote $\pi_{p,n}(q;z) = M_{n,q}(e_p;z)$ for all $e_p = t^p, p \in \mathbb{N} \cup \{0\}$.

7.1.1 Basic Results

To prove the results of next subsections, we need the following basic results.

Lemma 7.1. *Let $q \in (0,1)$. Then, $\pi_{m,n}(q;z)$ is a polynomial of degree $\leq \min(m,n)$, and*

$$\pi_{m,n}(q;z) = \frac{[n+1]_q!}{[n+m+1]_q!} \sum_{s=1}^{m} c_s(m) \, [n]_q^s B_{n,q}(e_s;z),$$

where $c_s(m) \geq 0$ are constants depending on m and q, and $B_{n,q}(f;z)$ is the q Bernstein polynomials given by $B_{n,q}(f;z) = \sum_{k=0}^{n} p_{n,k}(q;z) f([k]_q/[n]_q)$.

Proof. By definition of q-beta function, with $B_q(m,n) = \frac{[m-1]_q![n-1]_q!}{[m+n-1]_q!}$, we have

$$\pi_{m,n}(q;z) = [n+1]_q \sum_{k=1}^{n} q^{1-k} p_{n,k}(q;z) \int_0^1 p_{n,k-1}(q;qt) t^m d_q t$$

$$= [n+1]_q \sum_{k=1}^{n} q^{1-k} p_{n,k}(q;z) \int_0^1 \begin{bmatrix} n \\ k-1 \end{bmatrix}_q (qt)^{k-1} (1-qt)_q^{n-k+1} t^m d_q t$$

$$= [n+1]_q \sum_{k=1}^{n} p_{n,k}(q;z) \frac{[n]_q!}{[k-1]_q![n-k+1]_q!} B_q(k+m, n-k+2)$$

$$= \frac{[n+1]_q!}{[n+m+1]_q!} \sum_{k=1}^{n} p_{n,k}(q;z) \frac{[k+m-1]_q!}{[k-1]_q!}.$$

For $m = 1$, we find

$$\pi_{1,n}(q;z) = \frac{[n+1]_q!}{[n+2]_q!}\sum_{k=1}^{n} p_{n,k}(q;z)[k]_q = \frac{1}{[n+2]_q}\sum_{k=0}^{n} p_{n,k}(q;z)[n]_q\frac{[k]_q}{[n]_q}$$

$$= \frac{1}{[n+2]_q}\sum_{s=1}^{1}[n]_q^s B_{n,q}(e_s;z);$$

thus, the result is true for $m = 1$ with $c_1(1) = 1 > 0$.

Next for $m = 2$, with $[k+1]_q = 1 + q[k]_q$, we get

$$\begin{aligned}\pi_{2,n}(q;z) &= \frac{[n+1]_q!}{[n+3]_q!}\sum_{k=0}^{n} p_{n,k}(q;z)(1+q[k]_q)[k]_q \\ &= \frac{[n+1]_q!}{[n+3]_q!}\left[[n]_q B_{n,q}(e_1;z) + q[n]_q^2 B_{n,q}(e_2;z)\right] \\ &= \frac{[n+1]_q!}{[n+3]_q!}\sum_{s=1}^{2} c_s(2)[n]_q^s B_{n,q}(e_s;z);\end{aligned}$$

thus the result is true for $m = 2$ with $c_1(2) = 1 > 0$, $c_2(2) = q > 0$.

Similarly for $m = 3$, using $[k+2]_q = [2]_q + q^2[k]_q$ and $[k+1]_q = 1 + q[k]_q$, we have

$$\pi_{3,n}(q;z) = \frac{[n+1]_q!}{[n+4]_q!}\sum_{s=1}^{3} c_s(3)[n]_q^s B_{n,q}(e_s;z),$$

where $c_1(3) = [2]_q > 0$, $c_2(3) = 2q^2 + q > 0$, and $c_3(3) = q^3 > 0$.

Continuing in this way the result follows immediately for all $m \in N$. ■

Lemma 7.2. *Let $q \in (0,1)$. Then, for all $m, n \in \mathbb{N}$, we have the inequality*

$$\frac{[n+1]_q!}{[n+m+1]_q!}\sum_{s=1}^{m} c_s(m)\,[n]_q^s \le 1.$$

Proof. By Lemma 7.1, with $e_m = t^m$, we have

$$\pi_{m,n}(q;1) = \frac{[n+1]_q!}{[n+m+1]_q!}\sum_{s=1}^{m} c_s(m)\,[n]_q^s B_{n,q}(e_s;1) = \frac{[n+1]_q!}{[n+m+1]_q!}\sum_{s=1}^{m} c_s(m)\,[n]_q^s.$$

Also

$$p_{n,k}(q;z) = \begin{bmatrix} n \\ k \end{bmatrix}_q z^k(1-z)(1-qz)(1-q^2z)\ldots(1-q^{n-k-1}z).$$

It immediately follows that $p_{n,k}(q;1)=0, \quad k=0,1,2,\ldots,n-1$, and $p_{n,n}(q;1)=1$. Thus, we obtain

$$\begin{aligned}\pi_{m,n}(q;1) &= [n+1]_q\, p_{n,n}(q;1)\, q^{1-n}\int_0^1 p_{n,n-1}(q;qt)\, t^m d_qt \\ &= [n+1]_q \int_0^1 [n]_q t^{n+m-1}(1-qt)d_qt \\ &= [n+1]_q[n]_q\left[\frac{t^{n+m}}{[n+m]_q} - q\frac{t^{n+m+1}}{[n+m+1]_q}\right]_0^1 \\ &= \frac{[n+1]_q\,[n]_q}{[n+m]_q\,[n+m+1]_q} \le 1.\end{aligned}$$

■

Corollary 7.1. *Let $r\ge 1$ and $q\in(0,1)$. Then, for all $m,n\in\mathbb{N}\cup\{0\}$ and $|z|\le r$, we have $|\pi_{m,n}(q;z)|\le r^m$.*

Proof. By using the methods [76], p. 61, proof of Theorem 1.5.6, we have $\left|B_{n,q}(e_s;z)\right|\le r^s$. By Lemma 7.2 and for all $m\in\mathbb{N}$ and $|z|\le r$,

$$\begin{aligned}|\pi_{m,n}(q;z)| &\le \frac{[n+1]_q!}{[n+m+1]_q!}\sum_{s=1}^m c_s(m)\,[n]_q^s\left|B_{n,q}(e_s;z)\right| \\ &\le \frac{[n+1]_q!}{[n+m+1]_q!}\sum_{s=1}^m c_s(m)\,[n]_q^s r^s \le r^m.\end{aligned}$$

■

Lemma 7.3. *Let $q\in(0,1)$; then for $z\in\mathbb{C}$, we have the following recurrence relation:*

$$\pi_{p+1,n}(q;z) = \frac{q^p z(1-z)}{[n+p+2]_q} D_q\pi_{p,n}(q;z) + \frac{q^p[n]_q z + [p]_q}{[n+p+2]_q}\pi_{p,n}(q;z).$$

Proof. By simple computation, we have

$$z(1-z)D_q\left(p_{n,k}(q;z)\right) = \left([k]_q - [n]_q z\right)p_{n,k}(q;z)$$

and

$$t(1-qt)D_q\left(p_{n,k-1}(q;qt)\right) = \left([k-1]_q - [n]_q qt\right)p_{n,k-1}(q;qt).$$

Using these identities, it follows that

$$z(1-z)D_q\left(\pi_{p,n}(q;z)\right)=[n+1]_q\sum_{k=1}^{n}q^{1-k}\left([k]_q-[n]_qz\right)p_{n,k}(q;z)\int_0^1 p_{n,k-1}(q;qt)t^pd_qt$$

$$=[n+1]_q\sum_{k=1}^{n}q^{1-k}p_{n,k}(q;z)\int_0^1\left(1+q[k-1]_q-[n]_qq^2t+[n]_qq^2t\right)p_{n,k-1}(q;qt)t^pd_qt$$

$$-z[n]_q[n+1]_q\sum_{k=1}^{n}q^{1-k}p_{n,k}(q;z)\int_0^1 p_{n,k-1}(q;qt)t^pd_qt$$

$$=q[n+1]_q\sum_{k=1}^{n}q^{1-k}p_{n,k}(q;z)\int_0^1\left(D_qp_{n,k-1}(q;qt)\right)t(1-qt)t^pd_qt$$

$$+\pi_{p,n}(q;z)+[n]_qq^2\pi_{p+1,n}(q;z)-z[n]_q\pi_{p,n}(q;z).$$

Let us denote $\delta(t)=\frac{t}{q}(1-t)\left(\frac{t}{q}\right)^p=\frac{1}{q^{p+1}}\left(t^{p+1}-t^{p+2}\right)$. Then, the last q-integral becomes

$$\int_0^1 D_q\left(p_{n,k-1}(q;qt)\right)t(1-qt)t^pd_qt=\int_0^1 D_q\left(p_{n,k-1}(q;qt)\right)\delta(qt)d_qt$$

$$=\delta(t)p_{n,k-1}(q;qt)\,|_0^1-\int_0^1 p_{n,k-1}(q;qt)D_q\delta(t)d_qt$$

$$=-q^{-p-1}\int_0^1 p_{n,k-1}(q;qt)D_q\left(t^{p+1}-t^{p+2}\right)d_qt$$

$$=-q^{-p-1}[p+1]_q\int_0^1 p_{n,k-1}(q;qt)t^pd_qt$$

$$+q^{-p-1}[p+2]_q\int_0^1 p_{n,k-1}(q;qt)t^{p+1}d_qt,$$

and hence,

$$z(1-z)D_q\pi_{p,n}(q;z)=-q^{-p}[p+1]_q\pi_{p,n}(q;z)+q^{-p}[p+2]_q\pi_{p+1,n}(q;z)$$

$$+\pi_{p,n}(q;z)+[n]_qq^2\pi_{p+1,n}(q;z)-z[n]_q\pi_{p,n}(q;z).$$

Therefore,

$$\pi_{p+1,n}(q;z)=\frac{z(1-z)}{q^{-p}[p+2]_q+[n]_qq^2}D_q\pi_{p,n}(q;z)+\frac{[n]_qz+q^{-p}[p+1]_q-1}{q^{-p}[p+2]_q+[n]_qq^2}\pi_{p,n}(q;z)$$

$$=\frac{q^pz(1-z)}{[p+2]_q+[n]_qq^{p+2}}D_q\pi_{p,n}(q;z)+\frac{q^p[n]_qz+[p]_q}{[p+2]_q+[n]_qq^{p+2}}\pi_{p,n}(q;z).$$

Finally, using the identity $[p+2]_q + [n]_q q^{p+2} = [n+p+2]_q$, we get the required recurrence relation. ■

7.1.2 Upper Bound

If $P_m(z)$ is a polynomial of degree m, then by the Bernstein inequality and the complex mean value theorem, we have

$$\left|D_q P_m(z)\right| \le \left\|P_m'\right\|_r \le \frac{m}{r}\|P_m\|_r \quad \text{for all } |z| \le r.$$

The following theorem gives the upper bound for the operators (7.1):

Theorem 7.1. *Let $f(z) = \sum_{p=0}^{\infty} a_p z^p$ for all $|z| < R$ and let $1 \le r \le R$; then for all $|z| \le r$, $q \in (0,1)$ and $n \in \mathbb{N}$,*

$$\left|M_{n,q}(f;z) - f(z)\right| \le \frac{K_r(f)}{[n+2]_q},$$

where $K_r(f) = (1+r)\sum_{p=1}^{\infty} |a_p| p(p+1) r^{p-1} < \infty$.

Proof. First we shall show that $M_{n,q}(f;z) = \sum_{p=0}^{\infty} a_p \pi_{p,n}(q;z)$. If we denote $f_m(z) = \sum_{j=0}^{m} a_j z^j$, $|z| \le r$ with $m \in \mathbb{N}$, then by the linearity of $M_{n,q}$, we have

$$M_{n,q}(f_m;z) = \sum_{p=0}^{m} a_p \pi_{p,n}(q;z).$$

Thus, it suffice to show that for any fixed $n \in \mathbb{N}$ and $|z| \le r$ with $r \ge 1$, $\lim_{m\to\infty} M_{n,q}(f_m,z) = M_{n,q}(f;z)$. But this is immediate from $\lim_{m\to\infty} ||f_m - f||_r = 0$ and by the inequality

$$\begin{aligned} &|M_{n,q}(f_m;z) - M_{n,q}(f;z)| \\ &\le |f_m(0) - f(0)| \cdot |(1-z)^n| + [n+1]_q \sum_{k=1}^{n} |p_{n,k}(q;z)| q^{1-k} \int_0^1 p_{n,k-1}(q,qt) |f_m(t) - f(t)| d_q t \\ &\le C_{r,n} ||f_m - f||_r, \end{aligned}$$

where

$$C_{r,n} = (1+r)^n + [n+1]_q \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (1+r)^{n-k} r^k \int_0^1 p_{n,k-1}(q;qt) d_q t.$$

Since, $\pi_{0,n}(q;z) = 1$, we have

$$|M_{n,q}(f;z) - f(z)| \leq \sum_{p=1}^{\infty} |a_p| \cdot |\pi_{p,n}(q;z) - e_p(z)|.$$

Now using Lemma 7.3, for all $p \geq 1$, we find

$$\begin{aligned}
\pi_{p,n}(q;z) - e_p(z) &= \frac{q^{p-1} z(1-z)}{[n+p+1]_q} D_q(\pi_{p-1,n}(q;z)) \\
&+ \frac{q^{p-1}[n]_q z + [p-1]_q}{[n+p+1]_q} (\pi_{p-1,n}(q;z) - e_{p-1}(z)) \\
&+ \frac{q^{p-1}[n]_q z + [p-1]_q}{[n+p+1]_q} z^{p-1} - z^p \\
&= \frac{q^{p-1} z(1-z)}{[n+p+1]_q} D_q(\pi_{p-1,n}(q;z)) \\
&+ \frac{q^{p-1}[n]_q z + [p-1]_q}{[n+p+1]_q} (\pi_{p-1,n}(q;z) - e_{p-1}(z)) \\
&+ \frac{[p-1]_q}{[n+p+1]_q} z^{p-1} + \frac{q^{p-1}[n]_q - [n+p+1]_q}{[n+p+1]_q} z^p.
\end{aligned}$$

However,

$$\begin{aligned}
\left| \frac{q^{p-1}[n]_q - [n+p+1]_q}{[n+p+1]_q} z^p \right| &= \left| \frac{q^{p-1}[n]_q - [p-1]_q - q^{p-1}[n]_q - q^{n+p-1} - q^{n+p}}{[n+p+1]_q} z^p \right| \\
&\leq \frac{[p+1]_q}{[n+p+1]_q} r^p.
\end{aligned}$$

Combining the above relations and inequalities, we find

$$\begin{aligned}
|\pi_{p,n}(q;z) - e_p(z)| &\leq \frac{r(1+r)}{[n+2]_q} \cdot \frac{p-1}{r} \|\pi_{p-1,n}(q;z)\|_r \\
&+ r|\pi_{p-1,n}(q;z) - e_{p-1}(z)| + \frac{[p+1]_q}{[n+2]_q} r^{p-1}(1+r) \\
&\leq \frac{(1+r)(p-1)}{[n+2]_q} r^{p-1} + r|\pi_{p-1,n}(q;z) - e_{p-1}(z)|
\end{aligned}$$

$$+\frac{[p+1]_q}{[n+2]_q}r^{p-1}(1+r)$$
$$\leq 2p\frac{(1+r)}{[n+2]_q}r^{p-1}+r\left|\pi_{p-1,n}(q;z)-e_{p-1}(z)\right|.$$

From the last inequality, inductively it follows that

$$\left|\pi_{p,n}(q;z)-e_p(z)\right| \leq r\left(r\left|\pi_{p-2,n}(q;z)-e_{p-2}(z)\right|+\frac{2(p-1)}{[n+2]_q}(1+r)r^{p-2}\right)$$
$$+2p\frac{(1+r)}{[n+2]_q}r^{p-1}$$
$$=r^2\left|\pi_{p-2,n}(q;z)-e_{p-2}(z)\right|+2\frac{(1+r)}{[n+2]_q}r^{p-1}(p-1+p)$$
$$\leq \ldots \leq \frac{(1+r)}{[n+2]_q}p(p+1)r^{p-1}.$$

Thus, we obtain

$$\left|M_{n,q}(f;z)-f(z)\right| \leq \sum_{p=1}^{\infty}\left|a_p\right|\cdot\left|\pi_{p,n}(q;z)-e_p(z)\right| \leq \frac{1+r}{[n+2]_q}\sum_{p=1}^{\infty}\left|a_p\right|p(p+1)r^{p-1},$$

which proves the theorem. ■

Remark 7.1. Let $q \in (0,1)$ be fixed. As, $\lim_{n\to\infty}\frac{1}{[n+2]_q}=1-q$, Theorem 7.1 is not a convergence result. To obtain the convergence, one can choose $0<q_n<1$ with $q_n \nearrow 1$ as $n\to\infty$. In that case, $\frac{1}{[n+2]_{q_n}}\to 0$ as $n\to\infty$ (see Videnskii [152], formula (2.7)); from Theorem 7.1 we get $M_{n,q_n}(f;z)\to f(z)$, uniformly for $|z|\leq r$ and for any $1\leq r<R$.

7.1.3 Asymptotic Formula and Exact Order

The following result is the quantitative Voronovskaja-type asymptotic result:

Theorem 7.2. *Suppose that $f\in H(D_R), R>1$. Then, for any fixed $r\in[1,R]$ and for all $n\in\mathbb{N}, |z|\leq r$ and $q\in(0,1)$, we have*

$$\left|M_{n,q}(f;z)-f(z)-\frac{z(1-z)f''(z)-2zf'(z)}{[n]_q}\right| \leq \frac{M_r(f)}{[n]_q^2}+2(1-q)\sum_{k=1}^{\infty}|a_k|kr^k,$$

where $M_r(f) = \sum_{k=1}^{\infty} |a_k| k B_{k,r} r^k < \infty$, *and*

$$B_{k,r} = (k-1)(k-2)(2k-3) + 8k(k-1)^2 + 6(k-1)k^2 + +4k(k-1)^2(1+r).$$

Proof. In view of the proof of Theorem 7.1, we can write $M_{n,q}(f;z) = \sum_{k=0}^{\infty} a_k \pi_{k,n}(q;z)$. Thus,

$$\left| M_{n,q}(f;z) - f(z) - \frac{z(1-z)f''(z) - 2zf'(z)}{[n]_q} \right|$$

$$\leq \sum_{k=1}^{\infty} |a_k| \left| \pi_{k,n}(q;z) - e_k(z) - \frac{(k(k-1) - k(k+1)z)z^{k-1}}{[n]_q} \right|,$$

for all $z \in D_R, n \in \mathbb{N}$. If we denote

$$E_{k,n}(q;z) = \pi_{k,n}(q;z) - e_k(z) - \frac{(k(k-1) - k(k+1)z)z^{k-1}}{[n]_q},$$

then $E_{k,n}(q;z)$ is a polynomial of degree $\leq k$, and by simple calculation and using Lemma 7.3, we have

$$E_{k,n}(q;z) = \frac{q^{k-1}z(1-z)}{[n+k+1]_q} D_q E_{k-1,n}(q;z) + \frac{q^{k-1}[n]_q z + [k-1]_q}{[n+k+1]_q} E_{k-1,n}(q;z) + X_{k,n}(q;z),$$

where

$$\begin{aligned}
X_{k,n}(q;z) = & \frac{z^{k-2}}{[n]_q[n+k+1]_q} \Big[q^{k-1}(k-1)(k-2)[k-2]_q + [k-1]_q(k-1)(k-2) \\
& + z\Big(q^{k-1}[n]_q[k-1]_q - q^{k-1}(k-1)(k-2)[k-2]_q - q^{k-1}k(k-1)[k-1]_q \\
& + q^{k-1}[n]_q(k-1)(k-2) + [k-1]_q[n]_q - [k-1]_q k(k-1) - k(k-1)[n+k+1]_q \Big) \\
& z^2\Big(k(k+1)[n+k+1]_q - [n]_q[n+k+1]_q - q^{k-1}[n]_q k(k-1) \\
& + q^{k-1}[n]_q^2 + q^{k-1}k(k-1)[k-1]_q - q^{k-1}[n]_q[k-1]_q \Big) \Big] \\
=: & \frac{z^{k-2}}{[n]_q[n+k+1]_q} \left(X_{1,q,n}(k) + zX_{2,q,n}(k) + z^2 X_{3,q,n}(k) \right).
\end{aligned}$$

Obviously as $0 < q < 1$, it follows that

$$\left| X_{1,q,n}(k) \right| \leq (k-1)(k-2)(2k-3).$$

Next with $[n+k+1]_q = [k-1]_q + q^{k-1}[n]_q + q^{n+k-1} + q^{n+k}$, we have

$$\begin{aligned}X_{2,q,n}(k) &= [n]_q \left(q^{k-1}[k-1]_q + [k-1]_q - 2q^{k-1}(k-1)\right)\\&- q^{k-1}(k-1)(k-2)[k-2]_q - q^{k-1}k(k-1)[k-1]_q\\&- [k-1]_q k(k-1) - k(k-1)[k-1]_q - k(k-1)q^{n+k-1} - k(k-1)q^{n+k}\end{aligned}$$

and

$$\begin{aligned}&[n]_q \left(q^{k-1}[k-1]_q + [k-1]_q - 2q^{k-1}(k-1)\right)\\&= [n]_q q^{k-1}\left([k-1]_q - (k-1)\right) + [n]_q \left([k-1]_q - q^{k-1}(k-1)\right)\\&= [n]_q q^{k-1}(q-1)\sum_{j=0}^{k-2}[j]_q + [n]_q (1-q)\sum_{j=1}^{k-1}[j]_q q^{k-1-j}\\&= q^{k-1}(q^n-1)\sum_{j=0}^{k-2}[j]_q + (1-q^n)\sum_{j=1}^{k-1}[j]_q q^{k-1-j}.\end{aligned}$$

Thus,

$$\begin{aligned}\left|X_{2,q,n}(k)\right| &\le (k-1)[k-2]_q + (k-1)[k-1]_q\\&+ (k-1)(k-2)[k-2]_q + k(k-1)[k-1]_q + [k-1]_q k(k-1)\\&+ k(k-1)[k-1]_q + k(k-1) + k(k-1)\\&\le 8k(k-1)^2.\end{aligned}$$

Now we will estimate $X_{3,q,n}(k)$:

$$\begin{aligned}X_{3,q,n}(k) &= k(k+1)[n+k+1]_q - [n]_q[n+k+1]_q - q^{k-1}[n]_q k(k-1)\\&+ q^{k-1}[n]_q^2 + q^{k-1}k(k-1)[k-1]_q - q^{k-1}[n]_q[k-1]_q\\&= k(k+1)\left([k-1]_q + q^{k-1}[n]_q + q^{n+k-1} + q^{n+k}\right)\\&- [n]_q\left([k-1]_q + q^{k-1}[n]_q + q^{n+k-1} + q^{n+k}\right) - q^{k-1}[n]_q k(k-1)\\&+ q^{k-1}[n]_q^2 + q^{k-1}k(k-1)[k-1]_q - q^{k-1}[n]_q[k-1]_q\\&= k(k+1)[k-1]_q + k(k+1)\left(q^{n+k-1} + q^{n+k}\right) - [n]_q[k-1]_q\\&- [n]_q\left(q^{n+k-1} + q^{n+k}\right) + 2kq^{k-1}[n]_q\\&+ q^{k-1}k(k-1)[k-1]_q - q^{k-1}[n]_q[k-1]_q\end{aligned}$$

$$= [n]_q \left(-q^{k-1}[k-1]_q - [k-1]_q + q^{k-1}(2k) - q^{n+k-1} - q^{n+k}\right)$$

$$+ k(k+1)[k-1]_q + k(k+1)\left(q^{n+k-1} + q^{n+k}\right) + q^{k-1}k(k-1)[k-1]_q$$

$$= -[n]_q q^{k-1}([k-1]_q - (k-1)) + [n]_q q^{k-1}(1-q^n) + [n]_q \left(kq^{k-1} - [k-1]_q - q^{n+k}\right)$$

$$+ k(k+1)[k-1]_q + k(k+1)\left(q^{n+k-1} + q^{n+k}\right) + q^{k-1}k(k-1)[k-1]_q$$

$$= -[n]_q q^{k-1}([k-1]_q - (k-1)) + [n]_q q^{k-1}(1-q^n) - [n]_q \left([k-1]_q - (k-1)q^{k-1}\right)$$

$$- [n]_q \left(q^{n+k} - q^{k-1}\right) + k(k+1)[k-1]_q + k(k+1)\left(q^{n+k-1} + q^{n+k}\right) + q^{k-1}k(k-1)[k-1]_q$$

$$= -q^{k-1}(q^n - 1)\sum_{j=0}^{k-2}[j]_q - (1-q^n)\sum_{j=1}^{k-1}[j]_q q^{k-1-j} + q^{k-1}(1-q^n)[n]_q$$

$$- [n]_q \left(q^{n+k} - q^{k-1}\right) + k(k+1)[k-1]_q + k(k+1)\left(q^{n+k-1} + q^{n+k}\right) + q^{k-1}k(k-1)[k-1]_q.$$

Hence, it follows that

$$\left|X_{3,q,n}(k)\right| \leq (k-1)[k-2]_q + (k-1)[k-1]_q + (1-q^n)[n]_q$$

$$+ \left(1-q^{n+1}\right)[n]_q + k(k+1)[k-1]_q + 2k(k+1) + k(k-1)[k-1]_q$$

$$\leq 6(k-1)k^2 + (1-q^n)[n]_q + \left(1-q^{n+1}\right)[n]_q.$$

Thus,

$$|X_{k,n}(q;z)| \leq \frac{r^{k-2}}{[n]_q^2}\left((k-1)(k-2)(2k-3) + r8k(k-1)^2 + r^2 6(k-1)k^2\right)$$

$$+ \frac{r^k}{[n]_q}(1-q^n) + \frac{r^k}{[n+1]_q}\left(1-q^{n+1}\right)$$

$$= \frac{r^{k-2}}{[n]_q^2}\left((k-1)(k-2)(2k-3) + r8k(k-1)^2 + r^2 6(k-1)k^2\right) + 2r^k(1-q)$$

for all $k \geq 1, n \in \mathbb{N}$ and $|z| \leq r$.

Next, using the estimate in the proof of Theorem 7.1, we have

$$|\pi_{k,n}(q;z) - e_k(z)| \leq \frac{(1+r)k(k+1)r^{k-1}}{[n+2]_q},$$

for all $k, n \in \mathbb{N}, |z| \leq r$, with $1 \leq r$.

Hence, for all $k,n \in \mathbb{N}, k \geq 1$ and $|z| \leq r$, we have

$$|E_{k,n}(q;z)| \leq \frac{q^{k-1}r(1+r)}{[n+k+1]_q}|E'_{k-1,n}(q;z)| + \frac{q^{k-1}[n]_q r + [k-1]_q}{[n+k+1]_q}|E_{k-1,n}(q;z)| + |X_{k,n}(q;z)|.$$

However, since $\frac{q^{k-1}r(1+r)}{[n+k+1]_q} \leq \frac{r(1+r)}{[n+k+1]_q}$ and $\frac{q^{k-1}[n]_q r+[k-1]_q}{[n+k+1]_q} \leq r$, it follows that

$$|E_{k,n}(q;z)| \leq \frac{r(1+r)}{[n+k+1]_q}|E'_{k-1,n}(q;z)| + r|E_{k-1,n}(q;z)| + |X_{k,n}(q;z)|.$$

Now we shall compute an estimate for $|E'_{k-1,n}(q;z)|$, $k \geq 1$. For this, taking into account the fact that $E_{k-1,n}(q;z)$ is a polynomial of degree $\leq k-1$, we have

$$|E'_{k-1,n}(q;z)| \leq \frac{k-1}{r}||E_{k-1,n}||_r$$

$$\leq \frac{k-1}{r}\left[||\pi_{k-1,n} - e_{k-1}||_r + \left|\left|\frac{\{(k-1)(k-2) - k(k-1)e_1\}\, e_{k-2}]}{[n]_q}\right|\right|_r\right]$$

$$\leq \frac{k(k-1)}{r}\left[\frac{(1+r)(k-1)kr^{k-2}}{[n+2]_q} + \frac{r^{k-2}k(k-1)(1+r)}{[n]_q}\right]$$

$$\leq \frac{k(k-1)^2}{[n]_q}\left[2r^{k-2} + 2r^{k-2}\right] = \frac{4k(k-1)^2 r^{k-2}}{[n]_q}.$$

Thus,

$$\frac{r(1+r)}{[n+k+1]_q}|E'_{k-1,n}(q;z)| \leq \frac{4k(k-1)^2(1+r)r^{k-1}}{[n]_q^2}$$

and

$$|E_{k,n}(q;z)| \leq \frac{4k(k-1)^2(1+r)r^k}{[n]_q^2} + r|E_{k-1,n}(q;z)| + |X_{k,n}(q;z)|,$$

where

$$|X_{k,n}(q;z)| \leq \frac{r^k}{[n]_q^2}A_k + 2r^k\,(1-q)\,,$$

for all $|z| \leq r, k \geq 1, n \in \mathbb{N}$, where

$$A_k = (k-1)(k-2)\,(2k-3) + 8k(k-1)^2 + 6\,(k-1)\,k^2.$$

Hence, for all $|z| \leq r, k \geq 1, n \in \mathbb{N}$,

$$|E_{k,n}(q;z)| \leq r|E_{k-1,n}(q;z)| + \frac{r^k}{[n]_q^2}B_{k,r} + 2r^k\,(1-q)\,,$$

where $B_{k,r}$ is a polynomial of degree 3 in k defined as

$$B_{k,r} = A_k + 4k(k-1)^2(1+r).$$

But $E_{0,n}(q;z) = 0$, for any $z \in C$, and therefore by writing the last inequality for $k = 1,2,\ldots$, we easily obtain step by step the following:

$$|E_{k,n}(q;z)| \leq \frac{r^k}{[n]_q^2} \sum_{j=1}^{k} B_{j,r} + 2r^k(1-q) \leq \frac{kr^k}{[n]_q^2} B_{k,r} + 2r^k k(1-q).$$

Therefore, we can conclude that

$$\left| M_{n,q}(f;z) - f(z) - \frac{z(1-z)f''(z) - 2zf'(z)}{[n]_q} \right| \leq \sum_{k=1}^{\infty} |a_k||E_{k,n}(q;z)|$$

$$\leq \frac{1}{[n]_q^2} \sum_{k=1}^{\infty} |a_k| k B_{k,r} r^k + 2(1-q) \sum_{k=1}^{\infty} |a_k| k r^k.$$

As $f^{(4)}(z) = \sum_{k=4}^{\infty} a_k k(k-1)(k-2)(k-3) z^{k-4}$ and the series is absolutely convergent in $|z| \leq r$, it easily follows that $\sum_{k=4}^{\infty} |a_k| k(k-1)(k-2)(k-3) r^{k-4} < \infty$, which implies that $\sum_{k=1}^{\infty} |a_k| k B_{k,r} r^k < \infty$. This completes the proof of theorem. ■

Remark 7.2. For $q \in (0,1)$ fixed, we have $\frac{1}{[n]_q} \to 1-q$ as $n \to \infty$; thus Theorem 7.2 does not provide convergence. But this can be improved by choosing $1 - \frac{1}{n^2} \leq q_n < 1$ with $q_n \nearrow 1$ as $n \to \infty$. Indeed, since in this case $\frac{1}{[n]_{q_n}} \to 0$ as $n \to \infty$ and $1 - q_n \leq \frac{1}{n^2} \leq \frac{1}{[n]_{q_n}^2}$ from Theorem 7.2, we get

$$\left| M_{n,q_n}(f;z) - f(z) - \frac{z(1-z)f''(z) - 2zf'(z)}{[n]_{q_n}} \right| \leq \frac{M_r(f)}{[n]_{q_n}^2} + \frac{2}{[n]_{q_n}^2} \sum_{k=1}^{\infty} |a_k| k r^k.$$

Our next main result is the exact order of approximation for the operator (7.1).

Theorem 7.3. *Let* $1 - \frac{1}{n^2} \leq q_n < 1$, $n \in \mathbb{N}$, $R > 1$, *and let* $f \in H(D_R), R > 1$. *If* f *is not a polynomial of degree* 0, *then for any* $r \in [1,R)$, *we have*

$$\|M_{n,q_n}(f;\cdot) - f\|_r \geq \frac{C_r(f)}{[n]_{q_n}}, \quad n \in \mathbb{N},$$

where the constant $C_r(f) > 0$ *depends on* f, r *and on the sequence* $(q_n)_{n\in\mathbb{N}}$, *but it is independent of* n.

Proof. For all $z \in \mathbb{D}_R$ and $n \in \mathbb{N}$, we have

$$M_{n,q_n}(f;z) - f(z) = \frac{1}{[n]_{q_n}} \Bigg[z(1-z)f''(z) - 2zf'(z)$$

$$+ \frac{1}{[n]_{q_n}} \left\{ [n]_{q_n}^2 \left(M_{n,q_n}(f;z) - f(z) - \frac{z(1-z)f''(z) - 2zf'(z)}{[n]_{q_n}} \right) \right\} \Bigg].$$

We use the following property:

$$\|F+G\|_r \geq |\, \|F\|_r - \|G\|_r \,| \geq \|F\|_r - \|G\|_r$$

to obtain

$$||M_{n,q_n}(f;\cdot) - f||_r$$

$$\geq \frac{1}{[n]_{q_n}} \Bigg[||e_1(1-e_1)f'' - 2e_1 f'||_r$$

$$- \frac{1}{[n]_{q_n}} \left\{ [n]_{q_n}^2 \left|\left| M_{n,q_n}(f;\cdot) - f - \frac{e_1(1-e_1)f'' - 2e_1 f'}{[n]_{q_n}} \right|\right|_r \right\} \Bigg].$$

By the hypothesis, f is not a polynomial of degree 0 in D_R; we get $||e_1(1-e_1)f'' - 2e_1 f'||_r > 0$. Supposing the contrary, it follows that $z(1-z)f''(z) - 2zf'(z) = 0$ for all $|z| \leq r$, that is, $(1-z)f''(z) - 2f'(z) = 0$ for all $|z| \leq r$ with $z \neq 0$. The last equality is equivalent to $[(1-z)f'(z)]' - f'(z) = 0$, for all $|z| \leq r$ with $z \neq 0$. Therefore, $(1-z)f'(z) - f(z) = C$, where C is a constant, that is, $f(z) = \frac{Cz}{1-z}$, for all $|z| \leq r$ with $z \neq 0$. But since f is analytic in $\overline{D_r}$ and $r \geq 1$, we necessarily have $C = 0$, a contradiction to the hypothesis.

But by Remark 7.2, we have

$$[n]_{q_n}^2 \left|\left| M_{n,q_n}(f;\cdot) - f - \frac{e_1(1-e_1)f'' - 2e_1 f'}{[n]_{q_n}} \right|\right|_r \leq M_r(f) + 2\sum_{k=1}^{\infty} |a_k| k r^k,$$

with $\frac{1}{[n]_{q_n}} \to 0$ as $n \to \infty$. Therefore, it follows that there exists an index n_0 depending only on f, r and on the sequence $(q_n)_n$, such that for all $n \geq n_0$, we have

$$||e_1(1-e_1)f'' - 2e_1 f'||_r$$

$$- \frac{1}{[n]_{q_n}} \left\{ [n]_{q_n}^2 \left|\left| M_{n,q_n}(f;\cdot) - f - \frac{e_1(1-e_1)f'' - 2e_1 f'}{[n]_{q_n}} \right|\right|_r \right\}$$

$$\geq \frac{1}{2} ||e_1(1-e_1)f'' - 2e_1 f'||_r,$$

which implies that

$$||M_{n,q_n}(f;\cdot)-f||_r \geq \frac{1}{2[n]_{q_n}}||e_1(1-e_1)f''-2e_1f'||_r, \forall n \geq n_0.$$

For $1 \leq n \leq n_0 - 1$, we clearly have

$$\|M_{n,q_n}(f;\cdot)-f\|_r \geq \frac{c_{r,n}(f)}{[n]_{q_n}},$$

where $c_{r,n}(f) = [n]_{q_n} \cdot \|M_{n,q_n}(f;\cdot)-f\|_r > 0$, which finally implies

$$\|M_{n,q_n}(f;\cdot)-f\|_r \geq \frac{C_r(f)}{[n]_{q_n}}, \text{ for all } n \in \mathbb{N},$$

where

$$C_r(f) = \min\{c_{r,1}(f), c_{r,2}(f)\ldots, c_{r,n_0-1}(f), \frac{1}{2}||e_1(1-e_1)f''-2e_1f'||_r\}. \qquad \blacksquare$$

Combining Theorem 7.3 with Theorem 7.1, we get the following.

Corollary 7.2. *Let* $1-\frac{1}{n^2} < q_n < 1$ *for all* $n \in \mathbb{N}$*,* $R > 1$ *and suppose that* $f \in H(D_R)$*. If* f *is not a polynomial of degree* 0*, then for any* $r \in [1,R)$*, we have*

$$\|M_{n,q_n}(f;\cdot)-f\|_r \sim \frac{1}{[n]_{q_n}}, \quad n \in \mathbb{N},$$

where the constants in the above equivalence depend on $f, r, (q_n)_n$*, but are independent of* n*.*

The proof follows along the lines of [80].

Remark 7.3. For $0 \leq \alpha \leq \beta$, we can define the Stancu-type generalization of the operators (7.1) as

$$M_{n,q}^{\alpha,\beta}(f;z) = [n+1]_q \sum_{k=1}^{n} q^{1-k} p_{n,k}(q;z) \int_0^1 f\left(\frac{[n]_q t+\alpha}{[n]_q+\beta}\right) p_{n,k-1}(q;qt)\, d_q t$$
$$+ f\left(\frac{\alpha}{[n]_q+\beta}\right) p_{n,0}(q;z).$$

The analogous results can be obtained for such operators. As analysis is different, it may be considered elsewhere.

7.2 q-Gauss–Weierstrass Operator

In this section we study a complex q-Gauss–Weierstrass integral operators taking into consideration the operators introduced by Anastassiou and Aral in [17]. We show that these operators are an approximation process in some subclasses of analytic functions giving Jackson-type estimates in approximation. Furthermore, we give q-calculus analogues of some shape-preserving properties for these operators satisfied by classical complex Gauss–Weierstrass integral operators. The results of this section were discussed in [36].

7.2.1 *Introduction*

In a recent study, Anastassiou and Aral [17] introduced a new q-analogue of Gauss–Weierstrass operators, which for $n \in \mathbb{N}$, $q \in (0,\ 1)$, $x \in \mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$ be a function, defined as

$$\mathcal{W}_n\left(f;q,x\right) := \frac{\sqrt{[n]_q}\,(q+1)}{2\Gamma_{q^2}\left(\frac{1}{2}\right)} \int_0^{\frac{2}{\sqrt{[n]_q}\sqrt{1-q^2}}} f\left(x+t\right) E_{q^2}\left(-q^2\,[n]_q\,\frac{t^2}{4}\right) d_q t. \quad (7.2)$$

The goal of the present section is to introduce complex q-Gauss–Weierstrass operators and to obtain Jackson-type estimates in approximation by these operators. Also, we prove shape-preserving properties and some geometric properties of the operators using q-derivative.
Note that geometric and approximation properties of some complex convolution polynomials, complex singular integrals, and complex variant of well known operators were studied in detail in [76]. Also shape-preserving approximation by real or complex polynomials in one or several variables was given in [75].

Definition 7.1. Let $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$ be the open unit disk and $A\left(\overline{\mathbb{D}}\right) = \left\{f : \overline{\mathbb{D}} \to \mathbb{C}\ ;\ \ f \text{ is analytic on } \mathbb{D}, \text{ continuous on } \overline{\mathbb{D}},\ f(0) = 0,\ D_q f(0) = 1\right\}$. For $\xi > 0$, $q \in (0,\ 1)$, the complex q-Gauss–Weierstrass integral of $f \in A\left(\overline{\mathbb{D}}\right)$ is defined as

$$\mathcal{W}_\xi\left(f;q,z\right) := \frac{(q+1)}{\sqrt{\xi}\,\Gamma_{q^2}\left(\frac{1}{2}\right)} \int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}} f\left(ze^{-it}\right) E_{q^2}\left(-q^2\frac{t^2}{\xi}\right) d_q t \quad (7.3)$$

for $z \in \overline{\mathbb{D}}$.

Remark 7.4. Noting that the complex q-Gauss–Weierstrass operators $\mathcal{W}_\xi\left(f\right)(z)$ given by (7.3) can be rewritten via an improper integral, we can easily see that

$E_q\left(-\frac{q^n}{1-q}\right) = 0$ for $n \le 0$. Thus, we may write

$$\mathcal{W}_\xi(f;q,z) = \frac{(q+1)}{\sqrt{\xi}\Gamma_{q^2}\left(\frac{1}{2}\right)} \int_0^{\frac{\infty}{\sqrt{(1-q^2)/\xi}}} f\left(ze^{-it}\right) E_{q^2}\left(-q^2\frac{t^2}{\xi}\right) d_qt$$

.

7.2.2 Approximation Properties

In this section, we obtain Jackson-type rate in approximation by complex operators given (7.3) and global smoothness preservation properties of them.

Lemma 7.4. *We have*

$$\mathcal{W}_\xi(1;q,z) = 1.$$

Proof. We can write the q-derivative of the equality $t = \sqrt{\xi}\sqrt{u}$ as

$$\begin{aligned} D_{q^2}(t) &= \sqrt{\xi}\frac{\sqrt{u}-\sqrt{q^2u}}{(1-q^2)u} \\ &= \sqrt{\xi}\frac{1}{(q+1)\sqrt{u}}. \end{aligned} \tag{7.4}$$

Also, using the change of variable formula for q-integral with $\beta = \frac{1}{2}$, we have

$$\begin{aligned} \int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}} E_{q^2}\left(-q^2\frac{t^2}{\xi}\right) d_qt &= \frac{\sqrt{\xi}}{(q+1)} \int_0^{\frac{1}{1-q^2}} u^{-\frac{1}{2}} E_{q^2}\left(-q^2u\right) d_{q^2}u \\ &= \frac{\sqrt{\xi}}{(q+1)}\Gamma_{q^2}\left(\frac{1}{2}\right), \end{aligned}$$

which proves $\mathcal{W}_\xi(1;q,z) = 1$. ■

Theorem 7.4. *Let* $f \in A\left(\overline{\mathbb{D}}\right)$.

(*i*) *For* $z \in \overline{\mathbb{D}}$, $\xi \in (0,1]$, *we have*

$$\left|\mathcal{W}_\xi(f;q,z) - f(z)\right| \le \omega_1\left(f;\sqrt{\xi}\right)_{\partial\mathbb{D}}\left(1 + \frac{1}{\Gamma_{q^2}\left(\frac{1}{2}\right)}\right),$$

where

$$\omega_1(f;\xi)_{\partial\mathbb{D}} = \sup\left\{\left|f\left(e^{i(x-t)}\right) - f\left(e^{-it}\right)\right|;\ x \in \mathbb{R},\ 0 \le t \le \xi\right\}.$$

(ii) *We have*

$$\omega_1\left(\mathcal{W}_\xi\left(f;q,z\right);\delta\right)_{\overline{\mathbb{D}}}\le C\omega_1\left(f;\delta\right)_{\overline{\mathbb{D}}},\forall\delta>0,\ \xi>0,$$

where

$$\omega_1\left(f;\delta\right)_{\overline{\mathbb{D}}}=\sup\left\{\left|f\left(z_1\right)-f\left(z_2\right)\right|;\ z_1,z_2\in\overline{\mathbb{D}},\ \left|z_1-z_2\right|\le\delta\right\}.$$

Proof.

(i) Since $\mathcal{W}_\xi\left(1;q,z\right)=1$, for $z\in\overline{\mathbb{D}}$, we get

$$\left|\mathcal{W}_\xi\left(f;q,z\right)-f\left(z\right)\right|\le\frac{(q+1)}{\sqrt{\xi}\Gamma_{q^2}\left(\frac{1}{2}\right)}\int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}}\left|f\left(ze^{-it}\right)-f\left(z\right)\right|E_{q^2}\left(-q^2\frac{t^2}{\xi}\right)d_qt.$$

By the maximum modulus principle we can restrict our considerations to $|z|=1$, and we can write

$$\begin{aligned}&\left|\mathcal{W}_\xi\left(f;q,z\right)-f\left(z\right)\right|\\&\quad\le\frac{(q+1)}{\sqrt{\xi}\Gamma_{q^2}\left(\frac{1}{2}\right)}\int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}}\omega_1\left(f;|z|\left|1-e^{-it}\right|\right)_{\partial\mathbb{D}}E_{q^2}\left(-q^2\frac{t^2}{\xi}\right)d_qt\end{aligned}$$

Combined this with the inequality

$$|z|\left|1-e^{-it}\right|\le2\left|\sin\frac{t}{2}\right|\le t,\ \ \forall t>0,$$

it follows that

$$\begin{aligned}&\left|\mathcal{W}_\xi\left(f;q,z\right)-f\left(z\right)\right|\\&\quad=\frac{(q+1)}{\sqrt{\xi}\Gamma_{q^2}\left(\frac{1}{2}\right)}\int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}}\omega_1\left(f;2\left|\sin\frac{t}{2}\right|\right)_{\partial\mathbb{D}}E_{q^2}\left(-q^2\frac{t^2}{\xi}\right)d_qt\\&\quad\le\frac{(q+1)}{\sqrt{\xi}\Gamma_{q^2}\left(\frac{1}{2}\right)}\int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}}\omega_1\left(f;t\right)_{\partial\mathbb{D}}E_{q^2}\left(-q^2\frac{t^2}{\xi}\right)d_qt\\&\quad\le\frac{(q+1)}{\sqrt{\xi}\Gamma_{q^2}\left(\frac{1}{2}\right)}\int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}}\left(1+\frac{t}{\sqrt{\xi}}\right)\omega_1\left(f;\sqrt{\xi}\right)_{\partial\mathbb{D}}E_{q^2}\left(-q^2\frac{t^2}{\xi}\right)d_qt\end{aligned}$$

$$= \omega_1\left(f;\sqrt{\xi}\right)_{\partial\mathbb{D}}\left(1+\frac{(q+1)}{\xi\Gamma_{q^2}\left(\frac{1}{2}\right)}\int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}} tE_{q^2}\left(-q^2\frac{t^2}{\xi}\right)d_qt\right).$$

Also, using the change of variable formula for q-integral with $\beta=\frac{1}{2}$, we have

$$\int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}} tE_{q^2}\left(-q^2\frac{t^2}{\xi}\right)d_qt = \frac{\xi}{(q+1)}\int_0^{\frac{1}{1-q^2}} E_{q^2}\left(-q^2u\right)d_{q^2}u$$
$$= \frac{\xi}{(q+1)}\Gamma_{q^2}(1) = \frac{\xi}{(q+1)}.$$

Thus, we have

$$\left|\mathcal{W}_\xi(f;q,z)-f(z)\right| \le \omega_1\left(f;\sqrt{\xi}\right)_{\partial\mathbb{D}}\left(1+\frac{1}{\Gamma_{q^2}\left(\frac{1}{2}\right)}\right).$$

(ii) For $z_1,z_2\in\overline{\mathbb{D}}$, $|z_1-z_2|\le\delta$, we have following:

$$\left|\mathcal{W}_\xi(f;q,z_1)-\mathcal{W}_\xi(f;q,z_2)\right|$$
$$\le \frac{(q+1)}{\sqrt{\xi}\Gamma_{q^2}\left(\frac{1}{2}\right)}\int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}}\left|f\left(z_1e^{-it}\right)-f\left(z_2e^{-it}\right)\right|E_{q^2}\left(-q^2\frac{t^2}{\xi}\right)d_qt$$
$$\le \omega_1(f;|z_1-z_2|)_{\overline{\mathbb{D}}}\mathcal{W}_\xi(1;q,z)$$
$$\le \omega_1(f;\delta)_{\overline{\mathbb{D}}}.$$

From which, we derive by passing supremum over $|z_1-z_2|\le\delta$

$$\omega_1\left(\mathcal{W}_\xi(f;q,z);\delta\right)_{\overline{\mathbb{D}}} \le \omega_1(f;\delta)_{\overline{\mathbb{D}}}$$ ■

7.2.3 Shape-Preserving Properties

In this section, we deal with some properties of the complex operators given in Definition 7.1. Firstly we present following function classes:

$$S_2=\left\{f \text{ is analytic on } \mathbb{D},\ f(z)=\sum_{k=1}^{\infty}a_kz^k,\ z\in\mathbb{D},\ |a_1|\ge\sum_{k=2}^{\infty}|a_k|\right\},$$
$$S_3^q=\left\{f\in A\left(\overline{\mathbb{D}}\right);\ \left|D_q^2f(z)\right|\le 1,\ \text{for all } z\in\mathbb{D}\right\}$$

and

$$\mathfrak{P}=\left\{f:\overline{\mathbb{D}}\to\mathbb{C};\, f \text{ is analytic on } \mathbb{D},\ f(0)=1,\ Re\left[f(z)\right]>0,\ \forall z\in\mathbb{D}\right\}.$$

Theorem 7.5. *If $f(z)=\sum_{k=0}^{\infty}a_kz^k$ is analytic in $\mathbb{D}$, then for $\xi>0$, $\mathcal{W}_\xi(f)(z)$ is analytic in $\mathbb{D}$, and we have*

$$\mathcal{W}_\xi(f;q,z)=\sum_{k=0}^{\infty}a_kd_k(\xi,q)z^k, \qquad \forall z\in\mathbb{D}$$

where

$$d_k(\xi,q)=\frac{(q+1)}{\sqrt{\xi}\Gamma_{q^2}\left(\frac{1}{2}\right)}\int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}}e^{-ikt}E_{q^2}\left(-q^2\frac{t^2}{\xi}\right)d_qt. \tag{7.5}$$

Also, if f is continuous on $\overline{\mathbb{D}}$, then $\mathcal{W}_\xi(f)$ is continuous on $\overline{\mathbb{D}}$.

Proof. For the continuity at $z_0\in\overline{\mathbb{D}}$, let $z_n\in\overline{\mathbb{D}}$ be with $z_n\to z_0$ as $n\to\infty$. From (7.3), we can write

$$\begin{aligned}
&\left|\mathcal{W}_\xi(f;q,z_n)-\mathcal{W}_\xi(f;q,z_0)\right|\\
&\quad\le\frac{(q+1)}{\sqrt{\xi}\Gamma_{q^2}\left(\frac{1}{2}\right)}\int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}}\left|f\left(z_ne^{-it}\right)-f\left(z_0e^{-it}\right)\right|E_{q^2}\left(-q^2\frac{t^2}{\xi}\right)d_qt\\
&\quad\le\frac{(q+1)}{\sqrt{\xi}\Gamma_{q^2}\left(\frac{1}{2}\right)}\int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}}\omega_1\left(f;\left|z_ne^{-it}-z_0e^{-it}\right|\right)_{\overline{\mathbb{D}}}E_{q^2}\left(-q^2\frac{t^2}{\xi}\right)d_qt\\
&\quad=\frac{(q+1)}{\sqrt{\xi}\Gamma_{q^2}\left(\frac{1}{2}\right)}\int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}}\omega_1\left(f;|z_n-z_0|\right)_{\overline{\mathbb{D}}}E_{q^2}\left(-q^2\frac{t^2}{\xi}\right)d_qt\\
&\quad=\omega_1\left(f;|z_n-z_0|\right)_{\overline{\mathbb{D}}},
\end{aligned}$$

from which the continuity of f at $z_0\in\overline{\mathbb{D}}$ immediately implies the continuity of $\mathcal{W}_\xi(f)$ too at z_0.

Since $f(z)=\sum_{k=0}^{\infty}a_kz^k$, $z\in\mathbb{D}$, we get

$$\begin{aligned}
\mathcal{W}_\xi(f;q,z)&=\frac{(q+1)}{\sqrt{\xi}\Gamma_{q^2}\left(\frac{1}{2}\right)}\int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}}\sum_{k=0}^{\infty}a_kz^ke^{-ikt}E_{q^2}\left(-q^2\frac{t^2}{\xi}\right)d_qt\\
&=\frac{\sqrt{1-q^2}}{\Gamma_{q^2}\left(\frac{1}{2}\right)}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_kz^ke^{-ik\frac{\sqrt{\xi}}{\sqrt{1-q^2}}q^n}E_{q^2}\left(-q^2\frac{q^{2n}}{1-q^2}\right)q^n.
\end{aligned} \tag{7.6}$$

If $g_{n,k}$ is absolutely summable, that is, if $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left|g_{n,k}\right|<\infty$, then we know from *Fubini's theorem*:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}g_{n,k}=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}g_{n,k}.$$

Since

$$\left|a_k e^{-ik\frac{\sqrt{\xi}}{\sqrt{1-q^2}}q^n}\right|=\left|a_k\right|,$$

for all $n\in\mathbb{N}$, the series $\sum_{k=0}^{\infty}a_kz^k$ is convergent, and it follows that the series $\sum_{k=0}^{\infty}a_kz^ke^{-ik\frac{\sqrt{\xi}}{\sqrt{1-q^2}}q^n}$ is uniformly convergent with respect to n. Also, we can write

$$\begin{aligned}\frac{\sqrt{1-q^2}}{\Gamma_{q^2}\left(\frac{1}{2}\right)}\sum_{n=0}^{\infty}E_{q^2}\left(-q^2\frac{q^{2n}}{1-q^2}\right)q^n&=\frac{1}{\Gamma_{q^2}\left(\frac{1}{2}\right)}\int_0^{\frac{1}{1-q^2}}t^{-\frac{1}{2}}E_{q^2}\left(-q^2t\right)d_{q^2}t\\&=1.\end{aligned}$$

These immediately imply that the series in (7.6) can be interchangeable by Fubini's theorem, that is,

$$\begin{aligned}\mathcal{W}_{\xi}\left(f;q,z\right)&=\frac{\sqrt{1-q^2}}{\Gamma_{q^2}\left(\frac{1}{2}\right)}\sum_{k=0}^{\infty}a_kz^k\sum_{n=0}^{\infty}e^{-ik\frac{\sqrt{\xi}}{\sqrt{1-q^2}}q^n}E_{q^2}\left(-q^2\frac{q^{2n}}{1-q^2}\right)q^n\\&=\sum_{k=0}^{\infty}a_kd_k\left(\xi,q\right)z^k,\end{aligned}$$

where

$$\begin{aligned}d_k\left(\xi,q\right)&=\frac{\sqrt{1-q^2}}{\Gamma_{q^2}\left(\frac{1}{2}\right)}\sum_{n=0}^{\infty}e^{-ik\frac{\sqrt{\xi}}{\sqrt{1-q^2}}q^n}E_{q^2}\left(-q^2\frac{q^{2n}}{1-q^2}\right)q^n\\&=\frac{(q+1)}{\sqrt{\xi}\Gamma_{q^2}\left(\frac{1}{2}\right)}\int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}}e^{-ikt}E_{q^2}\left(-q^2\frac{t^2}{\xi}\right)d_qt.\end{aligned}$$

■

Theorem 7.6. *For $\xi>0$, it holds that*

$$\mathcal{W}_{\xi}\left(S_2\right)\subset S_2\quad\textit{and}\quad\mathcal{W}_{\xi}\left(\mathfrak{P}\right)\subset\mathfrak{P}.$$

Proof. By Theorem 7.5, we get

$$\mathcal{W}_\xi\left(f;q,z\right)=\sum_{k=0}^{\infty}a_kd_k\left(\xi,q\right)z^k,$$

and

$$\begin{aligned}\left|d_k\left(\xi,q\right)\right| &\le \frac{(q+1)}{\sqrt{\xi}\Gamma_{q^2}\left(\frac{1}{2}\right)}\int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}}\left|e^{-ikt}\right|E_{q^2}\left(-q^2\frac{t^2}{\xi}\right)d_qt\\ &\le \frac{(q+1)}{\sqrt{\xi}\Gamma_{q^2}\left(\frac{1}{2}\right)}\int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}}E_{q^2}\left(-q^2\frac{t^2}{\xi}\right)d_qt\\ &=1.\end{aligned}$$

Since $f\in S_2$, it follows that

$$\sum_{k=2}^{\infty}\left|a_kd_k\left(\xi,q\right)\right|\le\sum_{k=2}^{\infty}\left|a_k\right|\le a_1.$$

Thus we have,

$$\mathcal{W}_\xi\left(f\right)\in S_2.$$

Let $f(z)=\sum_{k=0}^{\infty}a_kz^k\in\mathfrak{P}$, that is, $a_0=f(0)=1$ and if $f(z)=U(x,y)+iV(x,y)$, $z=x+iy\in\mathbb{D}$, then $U(x,y)>0$, for all $z=x+iy\in\mathbb{D}$.

We have

$$\mathcal{W}_\xi\left(f\right)(0)=a_0=1$$

with the condition $a_0=f(0)=1$ and for $\forall z=re^{it}$,

$$\begin{aligned}&\mathcal{W}_\xi\left(f;q,z\right)\\ &=\frac{(q+1)}{\sqrt{\xi}\Gamma_{q^2}\left(\frac{1}{2}\right)}\int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}}U\left(r\cos\left(t-u\right),r\sin\left(t-u\right)\right)E_{q^2}\left(-q^2\frac{u^2}{\xi}\right)d_qu\\ &\quad+i\frac{(q+1)}{\sqrt{\xi}\Gamma_{q^2}\left(\frac{1}{2}\right)}\int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}}V\left(r\cos\left(t-u\right),r\sin\left(t-u\right)\right)E_{q^2}\left(-q^2\frac{u^2}{\xi}\right)d_qu,\end{aligned}$$

which implies that

$$\begin{aligned}&Re\left[\mathcal{W}_\xi\left(f;q,z\right)\right]\\ &=\frac{(q+1)}{\sqrt{\xi}\Gamma_{q^2}\left(\frac{1}{2}\right)}\int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}}U\left(r\cos\left(t-u\right),r\sin\left(t-u\right)\right)E_{q^2}\left(-q^2\frac{u^2}{\xi}\right)d_qu>0,\end{aligned}$$

that is, $\mathcal{W}_\xi\left(f;q,z\right)\in\mathfrak{P}$. ∎

Remark 7.5. By [11], if $f \in S_2$, then f is starlike (and univalent) on $\mathbb{D}$. According to Theorem 7.6, the operators $\mathcal{W}_\xi$ possess this property.

7.2.4 Applications of q-Derivative to Operators

In this section, we present some properties of the complex operators $\mathcal{W}_\xi f(z)$, $\xi > 0$ via q-derivative.

Lemma 7.5. *The $d_k(\xi,q)$ is defined as (7.5). We have*

$$\lim_{\xi\to 0} d_k(\xi,q) = 1.$$

Proof. We can write

$$\begin{aligned}\lim_{\xi\to 0} d_k(\xi,q) &= \lim_{\xi\to 0} \frac{(q+1)}{\sqrt{\xi}\,\Gamma_{q^2}\left(\frac{1}{2}\right)} \int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}} e^{-ikt} E_{q^2}\left(-q^2\frac{t^2}{\xi}\right) d_q t \\ &= \lim_{\xi\to 0} \frac{\sqrt{1-q^2}}{\Gamma_{q^2}\left(\frac{1}{2}\right)} \sum_{n=0}^{\infty} e^{-ik\frac{\sqrt{\xi}}{\sqrt{1-q^2}}q^n} E_{q^2}\left(-\frac{q^2}{1-q^2}q^{2n}\right) q^n.\end{aligned}$$

Since the series of above equality is uniform convergent, it follows that the series can be interchangeable with limit, that is,

$$\begin{aligned}\lim_{\xi\to 0} d_k(\xi,q) &= \frac{\sqrt{1-q^2}}{\Gamma_{q^2}\left(\frac{1}{2}\right)} \sum_{n=0}^{\infty} \lim_{\xi\to 0} e^{-i\frac{\sqrt{\xi}}{\sqrt{1-q^2}}q^n} E_{q^2}\left(-\frac{q^2}{1-q^2}q^{2n}\right) q^n \\ &= \frac{\sqrt{1-q^2}}{\Gamma_{q^2}\left(\frac{1}{2}\right)} \sum_{n=0}^{\infty} E_{q^2}\left(-\frac{q^2}{1-q^2}q^{2n}\right) q^n \\ &= \frac{1}{\Gamma_{q^2}\left(\frac{1}{2}\right)} \int_0^{\frac{1}{1-q^2}} u^{-\frac{1}{2}} E_{q^2}\left(-q^2 u\right) d_{q^2} u \\ &= 1.\end{aligned}$$

■

Theorem 7.7. *For all $\xi > 0$,*

$$\frac{1}{d_1(\xi,q)} \mathcal{W}_\xi\left(S^q_{3,d_1(\xi,q)}\right) \subset S^q_3, \quad \frac{1}{d_1(\xi,q)} \mathcal{W}_\xi\left(S^q_M\right) \subset S^q_{\frac{M}{d_1(\xi,q)}},$$

where

$$S^q_{3,d_1(\xi,q)} = \left\{ f \in S^q_3;\, \left|D^2_q f(z)\right| \le d_1(\xi,q) \right\}$$

and

$$S^q_{\frac{M}{d_1(\xi,q)}} = \left\{ f \in S^q_M ; \left| D_q f(z) \right| \leq \frac{M}{d_1(\xi,q)} \right\}.$$

Proof. Let $f \in S^q_{3,d_1(\xi,q)}$. Since $f \in A\left(\overline{\mathbb{D}}\right)$, we know that $f(0) = a_0 = 0$, $D_q f(0) = a_1 = 1$. Also since $\mathcal{W}_\xi(f;q,z)$ is continuous from Theorem 7.5, we can take q-derivative of it. Thus, we have

$$\frac{1}{d_1(\xi,q)} \mathcal{W}_\xi(f;q,0) = 0, \quad \frac{1}{d_1(\xi,q)} D_q \mathcal{W}_\xi(f;q,0) = a_1 = 1.$$

Also, since

$$D_q^2 \mathcal{W}_\xi(f;q,z) = \frac{(q+1)}{\sqrt{\xi}\,\Gamma_{q^2}\left(\frac{1}{2}\right)} \int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}} D_q^2 f\left(ze^{-it}\right) e^{-2it} E_{q^2}\left(-q^2 \frac{t^2}{\xi}\right) d_q t,$$

and $\left|D_q^2 f(z)\right| \leq |d_1(\xi,q)|$, it follows that

$$\begin{aligned} &\left| \frac{1}{d_1(\xi,q)} D_q^2 \mathcal{W}_\xi(f;q,z) \right| \\ &\quad \leq \frac{(q+1)}{|d_1(\xi,q)|\sqrt{\xi}\,\Gamma_{q^2}\left(\frac{1}{2}\right)} \int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}} \left|D_q^2 f\left(ze^{-it}\right)\right| \left|e^{-2it}\right| E_{q^2}\left(-q^2 \frac{t^2}{\xi}\right) d_q t \\ &\quad \leq \frac{(q+1)}{\sqrt{\xi}\,\Gamma_{q^2}\left(\frac{1}{2}\right)} \int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}} E_{q^2}\left(-q^2 \frac{t^2}{\xi}\right) d_q t = 1, \end{aligned}$$

that is, $\frac{1}{d_1(\xi,q)} \mathcal{W}_\xi(f) \in S^q_3$.

Now, let $f \in S^q_M$, that is, $\left|D_q f(z)\right| \leq M$. It follows that

$$\begin{aligned} &\left| \frac{1}{d_1(\xi,q)} D_q \mathcal{W}_\xi(f;q,z) \right| \\ &\quad \leq \frac{(q+1)}{|d_1(\xi,q)|\sqrt{\xi}\,\Gamma_{q^2}\left(\frac{1}{2}\right)} \int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}} \left|D_q f\left(ze^{-it}\right)\right| \left|e^{-it}\right| E_{q^2}\left(-q^2 \frac{t^2}{\xi}\right) d_q t \\ &\quad \leq \frac{M}{|d_1(\xi,q)|} \frac{(q+1)}{\sqrt{\xi}\,\Gamma_{q^2}\left(\frac{1}{2}\right)} \int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}} E_{q^2}\left(-q^2 \frac{t^2}{\xi}\right) d_q t = \frac{M}{|d_1(\xi,q)|}, \end{aligned}$$

which implies that $\frac{1}{d_1(\xi,q)} \mathcal{W}_\xi(f) \in S^q_{\frac{M}{d_1(\xi,q)}}$. ■

7.2.5 Exact Order of Approximation

For exact order of approximation, we give a modification of the operator (7.3).

For $\xi > 0$, $q \in (0,\ 1)$, the complex q-Gauss–Weierstrass integral of $f \in A\left(\overline{\mathbb{D}}\right)$ is defined as

$$\mathcal{W}_{\xi}^{*}(f;q,z) := \frac{(q+1)}{2\sqrt{\xi}\Gamma_{q^2}\left(\frac{1}{2}\right)} \int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}} \left(f\left(ze^{-it}\right) + f\left(ze^{it}\right)\right) E_{q^2}\left(-q^2\frac{t^2}{\xi}\right) d_q t$$

for $z \in \overline{\mathbb{D}}$. The approximation properties of the $W_{\xi}^{*}(f;q,z)$ are expressed by the following theorem.

Theorem 7.8. (i) *Let* $f \in A\left(\overline{\mathbb{D}}\right)$. *For all* $\xi \in (0,1]$ *and* $z \in \overline{\mathbb{D}}$, *it follows*

$$\left|\mathcal{W}_{\xi}^{*}(f;q,z) - f(z)\right| \leq C\omega_2\left(f;\sqrt{\xi}\right)_{\partial\mathbb{D}}$$

.

(ii) *Let us suppose that* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *for all* $z \in \mathbb{D}_R$, $R > 1$. *If* f *is not constant for* $\mathbf{s} = 0$ *and not a polynomial of degree* $\leq s-1$ *for* $s \in \mathbb{N}$, *then for all* $1 \leq r < r_1 < R$, $\xi \in (0;1]$, *and* $s \in \mathbb{N} \cup \{0\}$

$$\left\|\left(\mathcal{W}_{\xi}^{*}\right)^{(s)}(f) - f^{(s)}\right\|_r \sim \xi$$

where the constants in the equivalence depend only on f, q, p, r, r_1.

Proof. (i) We get

$$\begin{aligned}
&\mathcal{W}_{\xi}^{*}(f;q,z) - f(z) \\
&= \frac{(q+1)}{2\sqrt{\xi}\Gamma_{q^2}\left(\frac{1}{2}\right)} \int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}} \left(f\left(ze^{-it}\right) - 2f(z) + f\left(ze^{it}\right)\right) E_{q^2}\left(-q^2\frac{t^2}{\xi}\right) d_q t.
\end{aligned}$$

For $|z| = 1$, we can write

$$\begin{aligned}
&\left|\mathcal{W}_{\xi}^{*}(f;q,z) - f(z)\right| \\
&\leq \frac{(q+1)}{2\sqrt{\xi}\Gamma_{q^2}\left(\frac{1}{2}\right)} \int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}} \left|f\left(ze^{-it}\right) - 2f(z) + f\left(ze^{it}\right)\right| E_{q^2}\left(-q^2\frac{t^2}{\xi}\right) d_q t \\
&\leq \frac{(q+1)}{2\sqrt{\xi}\Gamma_{q^2}\left(\frac{1}{2}\right)} \int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}} \omega_2(f;t)_{\partial\mathbb{D}} E_{q^2}\left(-q^2\frac{t^2}{\xi}\right) d_q t
\end{aligned}$$

$$\leq \omega_2\left(f;\sqrt{\xi}\right)_{\partial\mathbb{D}} \frac{(q+1)}{2\sqrt{\xi}\Gamma_{q^2}\left(\frac{1}{2}\right)} \int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}} \left(\frac{t}{\sqrt{\xi}}+1\right)^2 E_{q^2}\left(-q^2\frac{t^2}{\xi}\right) d_q t.$$

We can write the q-derivative of the equality $t=\sqrt{\xi}\sqrt{u}$ as

$$\begin{aligned} D_{q^2}(t) &= \sqrt{\xi}\,\frac{\sqrt{u}-\sqrt{q^2u}}{(1-q^2)\,u} \\ &= \sqrt{\xi}\,\frac{1}{(q+1)\sqrt{u}}. \end{aligned}$$

Also, using the change of variable formula for q-integral with $\beta=\frac{1}{2}$, we have

$$\begin{aligned} \frac{(q+1)}{2\xi\sqrt{\xi}\Gamma_{q^2}\left(\frac{1}{2}\right)} \int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}} t^2 E_{q^2}\left(-q^2\frac{t^2}{\xi}\right) d_q t &= \frac{1}{2\Gamma_{q^2}\left(\frac{1}{2}\right)} \int_0^{\frac{1}{1-q^2}} u^{\frac{1}{2}} E_{q^2}\left(-q^2u\right) d_{q^2}u \\ &= \frac{\Gamma_{q^2}\left(\frac{3}{2}\right)}{2\Gamma_{q^2}\left(\frac{1}{2}\right)} < \infty \end{aligned}$$

and

$$\begin{aligned} \frac{(q+1)}{\xi\Gamma_{q^2}\left(\frac{1}{2}\right)} \int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}} t E_{q^2}\left(-q^2\frac{t^2}{\xi}\right) d_q t &= \frac{1}{\Gamma_{q^2}\left(\frac{1}{2}\right)} \int_0^{\frac{1}{1-q^2}} E_{q^2}\left(-q^2u\right) d_{q^2}u \\ &= \frac{1}{\Gamma_{q^2}\left(\frac{1}{2}\right)} < \infty. \end{aligned}$$

Thus, we have desired result.

(ii) We follow here the ideas in the proof of [76, pp. 269–272]. We can easily see that for $r\geq 1$,

$$\omega_2\left(f;\sqrt{\xi}\right) \leq C_{r,q}(f)\,\xi,$$

where

$$\omega_2\left(f;\sqrt{\xi}\right)_{\partial\mathbb{D}_r} = \sup\left\{\Delta_u^2 f\left(re^{it}\right) : |u| < \sqrt{\xi}\right\}.$$

From (i) we have

$$\left\|\mathcal{W}_\xi^*(f)-f\right\|_r \leq C_{r,q}(f)\,\xi$$

for all $\xi\in(0,1]$ and $z\in\overline{\mathbb{D}_r}$ (see [76]).

Now, we find the upper estimate in (ii) by using the Cauchy's formulas. Let γ be a circle of radius $r_1>1$ and center 0. For $u\in\gamma$, we get

$$\left|f^{(s)}(z)-\mathcal{W}_{\xi}^{*(s)}(f)(z)\right|=\frac{s!}{2\pi}\left|\int_{\gamma}\frac{f(u)-\left(\mathcal{W}_{\xi}^{*}\right)(f)(u)}{(u-z)^{s+1}}du\right|.$$

This equality implies that

$$\begin{aligned}\left\|D_q^{(s)}f-D_q^{(s)}\mathcal{W}_{\xi}^{*}(f)\right\|_r &\leq \left\|f^{(s)}-\mathcal{W}_{\xi}^{*(s)}(f)\right\|_r\\ &\leq C_{r_1,q}(f)\,\xi\,\frac{s!r_1}{(r_1-r)^{s+1}}.\end{aligned}$$

For the lower estimate in (ii), firstly, let us show the $\mathcal{W}_{\xi}^{*}$ operator as series. Using (i), for the $\mathcal{W}_{\xi}^{*}(f)$ operator, we get

$$\mathcal{W}_{\xi}^{*}(f)(z)=\sum_{k=0}^{\infty}a_k d_k^{*}(\xi,q)\,z^k,$$

where

$$d_k^{*}(\xi,q)=\frac{(q+1)}{\sqrt{\xi}\Gamma_{q^2}\left(\frac{1}{2}\right)}\int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}}\cos(kt)\,E_{q^2}\left(-q^2\frac{t^2}{\xi}\right)d_qt.$$

By the mean value theorem applied to $h(t)=\cos kt$ on $[0,t]$, we get

$$\begin{aligned}\left|d_k^{*}(\xi,q)\right| &\leq \frac{(q+1)}{\sqrt{\xi}\Gamma_{q^2}\left(\frac{1}{2}\right)}\int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}}\left|\cos kt\right|E_{q^2}\left(-q^2\frac{t^2}{\xi}\right)d_qt\\ &\leq \frac{(q+1)}{\sqrt{\xi}\Gamma_{q^2}\left(\frac{1}{2}\right)}\int_0^{\frac{\sqrt{\xi}}{\sqrt{1-q^2}}}(1+kt)\,E_{q^2}\left(-q^2\frac{t^2}{\xi}\right)d_qt\\ &= 1+k\frac{\sqrt{\xi}}{\Gamma_{q^2}\left(\frac{1}{2}\right)}.\end{aligned}\qquad(7.7)$$

Using q-derivative and taking $z=re^{i\varphi}$, we have

$$\begin{aligned}&\left[D_q^{(s)}f(z)-D_q^{(s)}\left(\mathcal{W}_{\xi}^{*}\right)(f)(z)\right]e^{-ip\varphi}\\ &=\sum_{k=s}^{\infty}a_k[k]_q[k-1]_q\ldots[k-s+1]_q\,r^{k-s}e^{i(k-s-p)\varphi}\left[1-d_k^{*}(\xi,q)\right].\end{aligned}$$

Integrating from $-\pi$ to π, we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[D_q^{(s)}f(z)-D_q^{(s)}\left(\mathcal{W}_\xi^*\right)(f)(z)\right]e^{-ip\varphi}d\varphi$$
$$=a_{s+p}[s+p]_q[s+p-1]_q\dots[p+1]_q r^p\left[1-d_{s+p}^*(\xi,q)\right].$$

Then, passing to absolute value and using (7.7), we easily obtain for $\xi\in(0,1]$

$$\begin{aligned}&\left\|D_q^{(s)}f-D_q^{(s)}\left(\mathcal{W}_\xi^*\right)(f)\right\|_r\\ &\geq\left|a_{s+p}\right|[s+p]_q[s+p-1]_q\dots[p+1]_q r^p\left|1-d_{s+p}^*(\xi,q)\right|\\ &\geq\left|a_{s+p}\right|[s+p]_q[s+p-1]_q\dots[p+1]_q r^p\left|1-\left|d_{s+p}^*(\xi,q)\right|\right|\\ &\geq\left|a_{s+p}\right|[s+p]_q[s+p-1]_q\dots[p+1]_q r^p(s+p)\frac{\sqrt{\xi}}{\Gamma_{q^2}\left(\frac{1}{2}\right)}\\ &\geq\left|a_{s+p}\right|[s+p]_q[s+p-1]_q\dots[p+1]_q r^p(s+p)\frac{\xi}{\Gamma_{q^2}\left(\frac{1}{2}\right)}.\end{aligned}$$

Using this inequality, we have for $p\geq 1$ and $\xi\in(0,1]$

$$\left\|f-\mathcal{W}_\xi^*(f)\right\|_r\geq\left|a_p\right|r^p\frac{\xi}{\Gamma_{q^2}\left(\frac{1}{2}\right)}.$$

Thus, we can say that if there exists a subsequence $(\xi_k)_k$ in $(0,1]$ with $\lim_{k\to\infty}\xi_k=0$ and such that $\lim_{k\to\infty}\frac{\left\|f-\mathcal{W}_{\xi_k}^*(f)\right\|_r}{\xi_k}=0$, then $a_p=0$ for all $p\geq 1$, that is, f is constant on $\overline{\mathbb{D}_r}$.

Therefore, if f is not constant, then for $\xi\in(0,1]$, there exists a constant $C_{r,q}(f)>0$ such that $\left\|f-\mathcal{W}_\xi^*(f)\right\|_r\geq\xi C_{r,q}(f)$.

Now, we consider $s\geq 1$. We can write

$$\left\|D_q^{(s)}f-D_q^{(s)}\left(\mathcal{W}_\xi^*\right)(f)\right\|_r\geq\left|a_{s+p}\right|[s+p]_q[s+p-1]_q\dots[p+1]_q r^p(s+p)\frac{\xi}{\Gamma_{q^2}\left(\frac{1}{2}\right)}$$

for $\xi\in(0,1]$ and for all $p\geq 0$. Similarly, if there exists a subsequence $(\xi_k)_k$ in $(0,1]$ with $\lim_{k\to\infty}\xi_k=0$ and such that $\lim_{k\to\infty}\frac{\left\|D_q^{(s)}f-D_q^{(s)}\left(\mathcal{W}_{\xi_k}^*\right)(f)\right\|_r}{\xi_k}=0$, then $a_{s+p}=0$ for all $p\geq 0$, that is, f is a polynomial degree $\leq s-1$ on $\overline{\mathbb{D}_r}$.

Therefore, if f is not a polynomial of degree $\leq s-1$, then for $\xi\in(0,1]$, there exists a constant $C_{r,q}(f)>0$ such that

$$\left\|D_q^{(s)}f-D_q^{(s)}\left(\mathcal{W}_\xi^*\right)(f)\right\|_r\geq\xi C_{r,q}(f).$$ ■