Chapter 5
q-Summation–Integral Operators

5.1 q-Baskakov–Durrmeyer Operators

Aral and Gupta [32], proposed a q-analogue of the Baskakov operators and investigated its approximation properties. In continuation of their work they introduced Durrmeyer-type modification of q-Baskakov operators. These operators, opposed to Bernstein–Durrmeyer operators, are defined to approximate a function f on $[0, \infty)$. The Durrmeyer-type modification of the q-Bernstein operators was first introduced in [48]. Some results on the approximation of functions by the q-Bernstein–Durrmeyer operators were recently studied in [94]. In [62], some direct local and global approximation theorems were given for the q-Bernstein–Durrmeyer operators. We may also mention that some article related to Baskakov–Durrmeyer operators and different generalizations of them given in [61, 83, 153].

The main motivation of this section is to present a local approximation theorem and a rate of convergence of these new operators as well as their weighted approximation properties. The resulting approximation processes turn out to have an order of approximation at least as good as the classical Baskakov–Durrmeyer operators in certain subspace of continuous functions.

Recently, in [32], we introduced the following q-generalization of the classical Baskakov operators. For $f \in C[0, \infty)$, $q > 0$ and each positive integer n, the q-Baskakov operators are defined as

$$\begin{aligned}\mathcal{B}_{n,q}(f,x) &= \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} \frac{x^k}{(1+x)_q^{n+k}} f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right) \\ &= \sum_{k=0}^{\infty} b_{n,k}^q(x) f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right). \qquad (5.1)\end{aligned}$$

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Lemma 5.1 ([32]). *For* $\mathcal{B}_{n,q}(t^m,x)$, $m=0,1,2$, *one has the following:*

$$\mathcal{B}_{n,q}(1,x) = 1.$$
$$\mathcal{B}_{n,q}(t,x) = x,$$
$$\mathcal{B}_{n,q}(t^2,x) = x^2 + \frac{x}{[n]_q}\left(1+\frac{1}{q}x\right).$$

5.1.1 Construction of Operators

For every $n \in \mathbb{N}$, $q \in (0,\ 1)$, the positive linear operator $\mathcal{D}_n^q$ is defined by

$$\mathcal{D}_n^q\left(f(t),x\right) := [n-1]_q \sum_{k=0}^{\infty} \mathcal{P}_{n,k}^q(x) \int_0^{\infty/A} \mathcal{P}_{n,k}^q(t) f(t)\, d_qt, \tag{5.2}$$

where

$$\mathcal{P}_{n,k}^q(x) := \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k^2}{2}} \frac{x^k}{(1+x)_q^{n+k}}$$

for $x \in [0,\ \infty)$ and for every real-valued continuous and bounded function f on $[0,\ \infty)$ (see [31]).
These operators satisfy linearity property. Also it can be observed that in case $q=1$ the above operators reduce to the Baskakov–Durrmeyer operators discussed in [139] and [142]. Also see [144] for similar type of operators.

Lemma 5.2. *The following equalities hold:*

(i) $\mathcal{D}_n^q(1,x) = 1.$

(ii) $\mathcal{D}_n^q(t,x) = \left(1+\frac{[2]_q}{q^2[n-2]_q}\right)x + \frac{1}{q[n-2]_q}$, *for* $n>2$.

(iii) $\mathcal{D}_n^q\left(t^2,x\right) = \left(1+\frac{[3]_q}{q^3[n-3]_q}+\frac{[2]_q}{q^2[n-2]_q}+\frac{q[2]_q[3]_q+[n]_q}{q^6[n-2]_q[n-3]_q}\right)x^2.$

$$+\frac{[n]_q+q(1+[2]_q)[n]_q}{q^5[n-2]_q[n-3]_q}x + \frac{[2]_q}{q^3[n-2]_q[n-3]_q}, \textit{for } n>3.$$

Proof. The operators $\mathcal{D}_n^q$ are well defined on the function $1,\ t,\ t^2$. Then for every $n>3$ and $x \in [0,\ \infty)$, we obtain

$$\begin{aligned}\mathcal{D}_n^q(1,x) &= [n-1]_q \sum_{k=1}^{\infty} \mathcal{P}_{n,k}^q(x) \int_0^{\infty/A} \mathcal{P}_{n,k}^q(t)\, d_qt \\ &= [n-1]_q \sum_{k=0}^{\infty} \mathcal{P}_{n,k}^q(x) \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k^2}{2}} \int_0^{\infty/A} \frac{t^k}{(1+t)_q^{n+k}} d_qt.\end{aligned}$$

Using (1.15) and (1.17), we can write

$$
\begin{aligned}
\mathcal{D}_n^q(1,x) &= [n-1]_q \sum_{k=0}^{\infty} \mathcal{P}_{n,k}^q(x) \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k^2}{2}} \frac{B_q(k+1,n-1)}{K(A,k+1)} \\
&= [n-1]_q \sum_{k=0}^{\infty} \mathcal{P}_{n,k}^q(x)\, q^{\frac{k^2}{2}} \frac{[n+k-1]_q!}{[n-1]_q!\,[k]_q!} \frac{[k]_q!\,[n-2]_q!}{[n+k-1]_q!\,q^{\frac{k(k+1)}{2}}} \\
&= [n-1]_q \sum_{k=0}^{\infty} \mathcal{P}_{n,k}^q(x) \frac{q^{-\frac{k}{2}}}{[n-1]_q} \\
&= \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} \frac{x^k}{(1+x)_q^{n+k}} \\
&= \mathcal{B}_n^q(1,x) = 1,
\end{aligned}
$$

where $\mathcal{B}_n^q(f,x)$ is the q-Baskakov operator defined by (5.1).

Similarly

$$
\begin{aligned}
\mathcal{D}_n^q(t,x) &= [n-1]_q \sum_{k=0}^{\infty} \mathcal{P}_{n,k}^q(x) \int_0^{\infty/A} \mathcal{P}_{n,k}^q(t)\, t\, d_q t \\
&= [n-1]_q \sum_{k=0}^{\infty} \mathcal{P}_{n,k}^q(x) \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k^2}{2}} \int_0^{\infty/A} \frac{t^{k+1}}{(1+t)_q^{n+k}} d_q t \\
&= [n-1]_q \sum_{k=0}^{\infty} \mathcal{P}_{n,k}^q(x) \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k^2}{2}} \frac{B_q(k+2,n-2)}{K(A,k+2)} \\
&= \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k^2-3k-2}{2}} \frac{[k+1]_q}{[n-2]_q} \frac{x^k}{(1+x)_q^{n+k}}.
\end{aligned}
$$

Using the equality $[k+1]_q = [k]_q + q^k$

$$
\begin{aligned}
\mathcal{D}_n^q(t,x) &= \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} q^{-k+1} q^{-2} \frac{[k]_q}{[n-2]_q} \frac{x^k}{(1+x)_q^{n+k}} \\
&\quad + \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} q^{-k+1} q^{-2} \frac{q^k}{[n-2]_q} \frac{x^k}{(1+x)_q^{n+k}} \\
&= \frac{[n]_q}{q^2[n-2]_q} \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} q^{-k+1} \frac{[k]_q}{[n]_q} \frac{x^k}{(1+x)_q^{n+k}} \\
&\quad + \frac{1}{q[n-2]_q} \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} \frac{x^k}{(1+x)_q^{n+k}} \\
&= \frac{[n]_q}{q^2[n-2]_q} \mathcal{B}_n^q(t,x) + \frac{1}{q[n-2]_q} \mathcal{B}_n^q(1,x).
\end{aligned}
$$

From Lemma 5.1, we can write

$$\mathcal{D}_n^q(t,x) = \frac{[n]_q}{q^2[n-2]_q}x + \frac{1}{q[n-2]_q}$$
$$= \left(1 + \frac{[2]_q}{q^2[n-2]_q}\right)x + \frac{1}{q[n-2]_q}.$$

Finally

$$\mathcal{D}_n^q\left(t^2,x\right) = [n-1]_q \sum_{k=0}^{\infty} \mathcal{P}_{n,k}^q(x) \int_0^{\infty/A} \mathcal{P}_{n,k}^q(t)\, t^2 d_q t$$
$$= [n-1]_q \sum_{k=0}^{\infty} \mathcal{P}_{n,k}^q(x) \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k^2}{2}} \int_0^{\infty/A} \frac{t^{k+2}}{(1+t)_q^{n+k}} d_q t$$
$$= [n-1]_q \sum_{k=0}^{\infty} \mathcal{P}_{n,k}^q(x) \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k^2}{2}} \frac{B_q(k+3,n-3)}{K(A,k+3)}$$
$$= \sum_{k=0}^{\infty} \mathcal{P}_{n,k}^q(x)\, q^{\frac{-5k}{2}-3} \frac{[k+2]_q [k+1]_q}{[n-2]_q [n-3]_q}.$$

Using $[k+2]_q = [k]_q + q^k[2]$ and $[k+1]_q = [k]_q + q^k$, we have

$$\mathcal{D}_n^q\left(t^2,x\right) = \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \frac{x^k}{(1+x)_q^{n+k}} q^{\frac{k^2-5k}{2}-3} \frac{\left([k]_q + q^k[2]_q\right)\left([k]_q + q^k\right)}{[n-2]_q [n-3]_q}$$
$$= \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \frac{x^k}{(1+x)_q^{n+k}} q^{\frac{k^2-5k}{2}-3} \frac{[k]_q^2}{[n-2]_q [n-3]_q}$$
$$+ \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \frac{x^k}{(1+x)_q^{n+k}} q^{\frac{k^2-3k}{2}-3} \frac{\left(1+[2]_q\right)[k]_q}{[n-2]_q [n-3]_q}$$
$$+ \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \frac{x^k}{(1+x)_q^{n+k}} q^{\frac{k^2-k}{2}-3} \frac{[2]_q}{[n-2]_q [n-3]_q}.$$

Again using (5.1) and Lemma 5.1, we have

$$\mathcal{D}_n^q\left(t^2,x\right) = \frac{q^{-5}[n]_q^2}{[n-2]_q [n-3]_q} \mathcal{B}_n^q\left(t^2,x\right) + \frac{q^{-4}\left(1+[2]_q\right)[n]_q}{[n-2]_q [n-3]_q} \mathcal{B}_n^q(t,x)$$

$$+\frac{q^{-3}[2]}{[n-2]_q[n-3]_q}\mathcal{B}_n^q(1,x)$$

$$=\frac{q[n]_q^2+[n]_q}{q^6[n-2]_q[n-3]_q}x^2+\frac{[n]_q+q(1+[2]_q)[n]_q}{q^5[n-2]_q[n-3]_q}x+\frac{[2]_q}{q^3[n-2]_q[n-3]_q}.$$

Since $[n]_q=[3]_q+q^3[n-3]_q$ and $[n]_q=[2]_q+q^2[n-2]_q$, we have the desired result. ∎

Remark 5.1. If we put $q=1$, we get the moments of Baskakov–Durrmeyer operators as

$$D_n^1(t-x,x)=\frac{1+2x}{n-2},n>2$$

$$D_n^1(t,x)=\frac{1+nx}{n-2},\ n>2$$

and

$$D_n^1((t-x)^2,x)=\frac{2[(n+3)x^2+(n+3)x+1]}{(n-2)(n-3)},\ n>3$$

$$D_n^1(t^2,x)=\frac{(n^2+n)x^2+4nx+2}{(n-2)(n-3)},\ n>3$$

Lemma 5.3. *Let $n>3$ be a given number. For every $q\in(0,1)$ we have*

$$\mathcal{D}_n^q\left((t-x)^2,x\right)\leq\frac{15}{q^6[n-2]_q}\left(\varphi^2(x)+\frac{1}{[n-3]_q}\right),$$

where $\varphi^2(x)=x(1+x),x\in[0,\infty)$.

Proof. By Lemma 5.2, we have

$$\mathcal{D}_n^q\left((t-x)^2,x\right)=\left(\frac{[3]_q}{q^3[n-3]_q}-\frac{[2]_q}{q^2[n-2]_q}+\frac{q[2]_q[3]_q+[n]_q}{q^6[n-2]_q[n-3]_q}\right)x^2$$

$$+\left(\frac{[n]_q+q(1+[2]_q)[n]_q}{q^5[n-2]_q[n-3]_q}-\frac{2}{q[n-2]_q}\right)x+\frac{[2]_q}{q^3[n-2]_q[n-3]_q}$$

$$=\left(\frac{q^3[3]_q[n-2]_q-q^4[n-3]_q[2]_q+q[2]_q[3]_q+[n]_q}{q^6[n-2]_q[n-3]_q}\right)x^2$$

$$+\left(\frac{q[n]_q+q^2(1+[2]_q)[n]_q-2q^5[n-3]_q}{q^6[n-2]_q[n-3]_q}\right)x+\frac{[2]_q}{q^3[n-2]_q[n-3]_q}$$

$$= x(1+x)\left(\frac{q^3[3]_q[n-2]_q - q^4[n-3]_q[2]_q + q[2]_q[3]_q + [n]_q}{q^6[n-2]_q[n-3]_q}\right)$$
$$+ x\left(\frac{q[n]_q + q^2(1+[2]_q)[n]_q - 2q^5[n-3]_q - q^3[3]_q[n-2]_q}{q^6[n-2]_q[n-3]_q}\right.$$
$$\left. + \frac{q^4[n-3]_q[2]_q - q[2]_q[3]_q - [n]_q}{q^6[n-2]_q[n-3]_q}\right) + \frac{[2]_q}{q^3[n-2]_q[n-3]_q}.$$

By direct computation, for $n > 3$, we have
$q^3[3]_q[n-2] - q^4[n-3]_q[2]_q + q[2]_q[3]_q + [n]_q$

$$= \left(q^3+q^4+q^5\right)\left([n-3]_q + q^{n-3}\right) - \left(q^4+q^5\right)[n-3]_q + \left(q+q^2\right)\left(1+q+q^2\right)$$
$$+ \left([n-3]_q + q^{n-3} + q^{n-2} + q^{n-1}\right)$$
$$= [n-3]_q\left(q^3+q^4+q^5 - \left(q^4+q^5\right) + 1\right) + q^n + q^{n+1} + q^{n+2}$$
$$+ \left(q+q^2\right)\left(1+q+q^2\right) + q^{n-3} + q^{n-2} + q^{n-1}$$
$$= [n-3]_q\left(q^3+1\right) + q^n + q^{n+1} + q^{n+2} + \left(q+q^2\right)\left(1+q+q^2\right) + q^{n-3} + q^{n-2} + q^{n-1} > 0$$

for every $q \in (0,1)$. Furthermore
$q[n]_q + q^2(1+[2]_q)[n]_q - 2q^5[n-3]_q$

$$= q\left(1+2q+q^2\right)\left(1+q+\ldots+q^{n-1}\right) - 2q^5\left(1+q+\ldots+q^{n-4}\right)$$
$$= q\left(1+q^2\right)\left(1+q+\ldots+q^{n-1}\right) + 2\left[\left(q^2+q^3+\ldots+q^{n+1}\right) - \left(q^5+q^6+\ldots+q^{n+1}\right)\right]$$
$$= \left(q+q^3\right)\left(1+q+\ldots+q^{n-1}\right) + 2\left[q^2+q^3+q^4\right]$$

and
$\left[q+q^2(1+[2]_q)\right][n]_q - 2q^5[n-3]_q - \left[q^3[3]_q[n-2]_q - q^4[n-3]_q[2]_q + q[2]_q[3]_q + [n]_q\right]$

$$= \left(q+q^3\right)[n]_q + 2\left[q^2+q^3+q^4\right] - \left(q+q^2+q^3\right)\left([n]_q - (1+q)\right)$$
$$+ \left(q+q^2\right)\left([n]_q - \left(1+q+q^2\right)\right) - q(1+q)\left(1+q+q^2\right) - [n]_q$$
$$= (q-1)[n]_q + q^4 - q < 0$$

for every $q \in (0,1)$.

Thus we have

$$\mathcal{D}_n^q\left((t-x)^2,x\right) \leq x(1+x)\left(\frac{q^3[3]_q[n-2]_q-q^4[n-3]_q[2]_q+q[2]_q[3]_q+[n]_q}{q^6[n-2]_q[n-3]_q}\right)$$

$$+\frac{[2]_q}{q^3[n-2]_q[n-3]_q}$$

$$\leq \frac{15[n-3]_q}{q^6[n-2]_q[n-3]_q}\varphi^2(x)+\frac{2}{q^3[n-2]_q[n-3]_q}$$

$$\leq \frac{15}{q^6[n-2]_q}\left(\varphi^2(x)+\frac{1}{[n-3]_q}\right)$$

for every $q\in(0,1)$ and $x\in[0,\infty)$. Thus the result holds. ∎

5.1.2 Local Approximation

In this section we establish direct and local approximation theorems in connection with the operators $\mathcal{D}_n^q$. Let $C_B[0,\infty)$ be the space of all real-valued continuous and bounded functions f on $[0,\infty)$ endowed with the norm $\|f\|=\sup\{|f(x)|:x\in[0,\infty)\}$. Further let us consider the following K-functional:

$$K_2(f,\delta) = \inf_{g\in W^2}\left\{ \|f-g\|+\delta\|g''\| \right\},$$

where $\delta>0$ and $W^2=\{g\in C_B[0,\infty):g',g''\in C_B[0,\infty)\}$. By [50, p. 177, Theorem 2.4] there exists an absolute constant $C>0$ such that

$$K_2(f,\delta) \leq C\,\omega_2\left(f,\sqrt{\delta}\right), \tag{5.3}$$

where

$$\omega_2\left(f,\sqrt{\delta}\right) = \sup_{0<h\leq\sqrt{\delta}}\ \sup_{x\in[0,\infty)} |f(x+2h)-2f(x+h)+f(x)|$$

is the second-order modulus of smoothness of $f\in C_B[0,\infty)$. By

$$\omega(f,\delta) = \sup_{0<h\leq\delta}\ \sup_{x\in[0,\infty)} |f(x+h)-f(x)|$$

we denote the usual modulus of continuity of $f\in C_B[0,\infty)$. In what follows we shall use the notations $\varphi(x)=\sqrt{x(1+x)}$ and $\delta_n^2(x)=\varphi^2(x)+\frac{1}{[n-3]_q}$, where $x\in[0,\infty)$ and $n\geq 4$.

Our first result is a direct local approximation theorem for the operators $\mathcal{D}_n^q$.

Theorem 5.1. *Let $q \in (0,1)$ and $n \geq 4$. We have*

$$|\mathcal{D}_n^q(f,x) - f(x)| \leq C\omega_2\left(f, \frac{\delta_n(x)}{\sqrt{q^6[n-2]_q}}\right) + \omega\left(f, \frac{q^{-2}[2]_q x + q^{-1}}{[n-2]_q}\right),$$

for every $x \in [0,\infty)$ and $f \in C_B[0,\infty)$, where C is a positive constant.

Proof. Let us introduce the auxiliary operators $\overline{\mathcal{D}}_n^q$ defined by

$$\overline{\mathcal{D}}_n^q(f,x) = \mathcal{D}_n^q(f,x) - f\left(x + \frac{q^{-2}[2]_q x + q^{-1}}{[n-2]_q}\right) + f(x), \tag{5.4}$$

$x \in [0,\infty)$. The operators $\overline{\mathcal{D}}_n^q$ are linear and preserve the linear functions:

$$\overline{\mathcal{D}}_n^q(t-x,x) = 0 \tag{5.5}$$

(see Lemma 5.2).
Let $g \in W^2$. From Taylor's expansion

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)\, g''(u)\, du, \quad t \in [0,\infty)$$

and (5.5), we get

$$\overline{\mathcal{D}}_n^q(g,x) = g(x) + \overline{\mathcal{D}}_n^q\left(\int_x^t (t-u)\, g''(u)\, du, x\right).$$

Hence, by (5.4) one has
$|\overline{\mathcal{D}}_n^q(g,x) - g(x)| \leq$

$$\leq \left| \mathcal{D}_n^q\left(\int_x^t (t-u)\, g''(u)\, du, x\right)\right| + \left| \int_x^{x+\frac{q^{-2}[2]_q x + q^{-1}}{[n-2]_q}} \left(x + \frac{q^{-2}[2]_q x + q^{-1}}{[n-2]_q} - u\right) g''(u)\, du \right|$$

$$\leq \mathcal{D}_n^q\left(\left|\int_x^t |t-u|\, |g''(u)|\, du\right|, x\right) + \int_x^{x+\frac{q^{-2}[2]_q x + q^{-1}}{[n-2]_q}} \left| x + \frac{q^{-2}[2]_q x + q^{-1}}{[n-2]_q} - u\right| |g''(u)| du$$

$$\leq \left[\mathcal{D}_n^q\left((t-x)^2, x\right) + \left(\frac{q^{-2}[2]_q x + q^{-1}}{[n-2]_q}\right)^2\right] \|g''\|. \tag{5.6}$$

Using Lemma 5.3 and $n \geq 4$, we obtain
$D_n^q\left((t-x)^2, x\right) + \left(\frac{q^{-2}[2]_q x + q^{-1}}{[n-2]_q}\right)^2 \leq$

$$\leq \frac{15}{q^6\,[n-2]_q}\left(\varphi^2(x) + \frac{1}{[n-3]_q}\right) + \left(\frac{q^{-2}[2]_q x + q^{-1}}{[n-2]_q}\right)^2.$$

Since

$$\left(\frac{q^{-2}[2]_q x+q^{-1}}{[n-2]_q}\right)^2 .\delta_n^{-2}(x)$$
$$= \frac{(1+q)^2x^2+2q(1+q)x+q^2}{q^4[n-2]_q^2}\cdot\frac{[n-3]_q}{[n-3]_q x(x+1)+1}$$
$$\leq \frac{1}{q^4[n-2]_q}\cdot\frac{[n-3]_q}{[n-2]_q}\cdot\frac{4x^2+4x+1}{[n-3]_q x(x+1)+1},$$

we have

$$D_n^q\left((t-x)^2,x\right)+\left(\frac{q^{-2}[2]_q x+q^{-1}}{[n-2]_q}\right)^2 \leq \frac{15}{q^6[n-2]_q}\delta_n^2(x).$$

Then, by (5.6), we get

$$|\overline{\mathcal{D}}_n^q(g,x)-g(x)| \leq \frac{15}{q^6[n-2]_q}\delta_n^2(x)\|g''\|. \tag{5.7}$$

On the other hand, by (5.4) and (5.2) and Lemma 5.2, we have

$$|\overline{\mathcal{D}}_n^q(f,x)| \leq |\mathcal{D}_n^q(f,x)|+2\|f\| \leq \|f\|\mathcal{D}_n^q(1,x)+2\|f\| \leq 3\|f\|. \tag{5.8}$$

Now (5.4), (5.7), and (5.8) imply

$$|\mathcal{D}_n^q(f,x)-f(x)| \leq |\overline{\mathcal{D}}_n^q(f-g,x)-(f-g)(x)|$$
$$+|\overline{\mathcal{D}}_n^q(g,x)-g(x)|+\left|f\left(x+\frac{q^{-2}[2]_q x+q^{-1}}{[n-2]_q}\right)-f(x)\right|$$
$$\leq 4\|f-g\|+\frac{15}{q^6[n-2]_q}\delta_n^2(x)\|g''\|$$
$$+\left|f\left(x+\frac{q^{-2}[2]_q x+q^{-1}}{[n-2]_q}\right)-f(x)\right|.$$

Hence taking infimum on the right-hand side over all $g\in W^2$, we get $|D_n^q(f,x)-f(x)| \leq$

$$\leq 15K_2\left(f,\frac{1}{q^6[n-2]_q}\delta_n^2(x)\right)+\omega\left(f,\frac{q^{-2}[2]_q x+q^{-1}}{[n-2]_q}\right).$$

In view of (5.3), for every $q \in (0,1)$ we get

$$|\mathcal{D}_n^q(f,x) - f(x)| \le C\omega_2\left(f, \frac{\delta_n(x)}{\sqrt{q^6[n-2]_q}}\right) + \omega\left(f, \frac{q^{-2}[2]_q x + q^{-1}}{[n-2]_q}\right).$$

This completes the proof of the theorem.

5.1.3 Rate of Convergence

Let $B_{x^2}[0, \infty)$ be the set of all functions f defined on $[0, \infty)$ satisfying the condition $|f(x)| \le M_f\left(1+x^2\right)$, where M_f is a constant depending only on f. By $C_{x^2}[0, \infty)$, we denote the subspace of all continuous functions belonging to $B_{x^2}[0, \infty)$. Also, let $C^*_{x^2}[0, \infty)$ be the subspace of all functions $f \in C_{x^2}[0, \infty)$, for which $\lim\limits_{x\to\infty} \frac{f(x)}{1+x^2}$ is finite. The norm on $C^*_{x^2}[0, \infty)$ is $\|f\|_{x^2} = \sup_{x\in[0, \infty)} \frac{|f(x)|}{1+x^2}$. For any positive a, by

$$\omega_a(f, \delta) = \sup_{|t-x|\le\delta}\ \sup_{x,t\in[0, a]} |f(t) - f(x)|$$

we denote the usual modulus of continuity of f on the closed interval $[0, a]$. We know that for a function $f \in C_{x^2}[0, \infty)$, the modulus of continuity $\omega_a(f, \delta)$ tends to zero.

Now we give a rate of convergence theorem for the operator $\mathcal{D}_n^q$.

Theorem 5.2. *Let $f \in C_{x^2}[0, \infty)$, $q = q_n \in (0, 1)$ such that $q_n \to 1$ as $n \to \infty$ and $\omega_{a+1}(f, \delta)$ be its modulus of continuity on the finite interval $[0, a+1] \subset [0, \infty)$, where $a > 0$. Then for every $n > 3$,*

$$\|\mathcal{D}_n^q(f) - f\|_{C[0, a]} \le \frac{K}{q^6[n-3]_q} + 2\omega_{a+1}\left(f, \sqrt{\frac{K}{q^6[n-3]_q}}\right),$$

where $K = 90M_f\left(1+a^2\right)\left(1+a+a^2\right)$.

Proof. For $x \in [0, a]$ and $t > a+1$, since $t - x > 1$, we have

$$\begin{aligned}|f(t) - f(x)| &\le M_f\left(2 + x^2 + t^2\right)\\ &\le M_f\left(2 + 3x^2 + 2(t-x)^2\right)\\ &\le 6M_f\left(1+a^2\right)(t-x)^2. \end{aligned} \tag{5.9}$$

For $x \in [0, a]$ and $t \le a+1$, we have

$$|f(t) - f(x)| \le \omega_{a+1}(f, |t-x|) \le \left(1 + \frac{|t-x|}{\delta}\right)\omega_{a+1}(f, \delta) \tag{5.10}$$

with $\delta > 0$.

From (5.9) and (5.10) we can write

$$|f(t)-f(x)| \leq 6M_f\left(1+a^2\right)(t-x)^2+\left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\ \delta) \tag{5.11}$$

for $x \in [0,\ a]$ and $t \geq 0$. Thus

$$\begin{aligned}|\mathcal{D}_n^q(f,x)-f(x)| &\leq \mathcal{D}_n^q(|f(t)-f(x)|,x)\\ &\leq 6M_f\left(1+a^2\right)\mathcal{D}_n^q\left((t-x)^2,x\right)\\ &\quad+\omega_{a+1}(f,\ \delta)\left(1+\frac{1}{\delta}\mathcal{D}_n^q\left((t-x)^2,x\right)^{\frac{1}{2}}\right).\end{aligned}$$

Hence, by Schwarz's inequality and Lemma 5.3, for every $q \in (0,1)$ and $x \in [0,\ a]$

$$\begin{aligned}|\mathcal{D}_n^q(f,x)-f(x)| &\leq \frac{90M_f\left(1+a^2\right)}{q^6[n-2]_q}\left(\varphi^2(x)+\frac{1}{[n-3]_q}\right)\\ &\quad+\omega_{a+1}(f,\ \delta)\left(1+\frac{1}{\delta}\sqrt{\frac{15}{q^6[n-2]_q}\left(\varphi^2(x)+\frac{1}{[n-3]_q}\right)}\right)\\ &\leq \frac{K}{q^6[n-3]_q}+\omega_{a+1}(f,\ \delta)\left(1+\frac{1}{\delta}\sqrt{\frac{K}{q^6[n-3]_q}}\right).\end{aligned}$$

By taking $\delta=\sqrt{\frac{K}{q^6[n-3]_q}}$, we get the assertion of our theorem. ■

Corollary 5.1. *If $f \in Lip_M\alpha$ on $[0,\ a+1]$, then for $n>3$*

$$\|\mathcal{D}_n^q(f)-f\|_{C[0,\ a]} \leq (1+2M)\sqrt{\frac{K}{q^6[n-3]_q}}.$$

Proof. For a sufficiently large n,

$$\frac{K}{q^6[n-3]_q} \leq \sqrt{\frac{K}{q^6[n-3]_q}},$$

because of $\lim_{n\to\infty}[n-3]_q=\infty$. Hence, by $f \in Lip_M\alpha$, we obtain the assertion of the corollary. ■

5.1.4 Weighted Approximation

Now we shall discuss the weighted approximation theorem, where the approximation formula holds true on the interval $[0,\ \infty)$.

Theorem 5.3. *Let $q = q_n$ satisfies $0 < q_n < 1$ and let $q_n \to 1$ as $n \to \infty$. For each $f \in C^*_{x^2}[0, \infty)$, we have*

$$\lim_{n\to\infty} \|\mathcal{D}_n^{q_n}(f) - f\|_{x^2} = 0.$$

Proof. Using the theorem in [65] we see that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty} \|\mathcal{D}_n^{q_n}(t^\nu, x) - x^\nu\|_{x^2} = 0, \quad \nu = 0, 1, 2. \tag{5.12}$$

Since $D_n^{q_n}(1, x) = 1$, the first condition of (5.12) is fulfilled for $\nu = 0$.

By Lemma 5.2 we have for $n > 2$

$$\begin{aligned} \|\mathcal{D}_n^{q_n}(t,x) - x\|_{x^2} &= \sup_{x\in[0,\,\infty)} \frac{\left|\mathcal{D}_n^{q_n}(t,x) - x\right|}{1+x^2} \\ &\le \frac{[2]_{q_n}}{q_n^2[n-2]_{q_n}} \sup_{x\in[0,\,\infty)} \frac{x}{1+x^2} + \frac{1}{q_n[n-2]_{q_n}} \\ &\le \frac{[2]_{q_n}}{q_n^2[n-2]_{q_n}} + \frac{1}{q_n[n-2]_{q_n}}, \end{aligned}$$

and the second condition of (5.12) holds for $\nu = 1$ as $n \to \infty$.

Similarly we can write for $n > 3$

$$\begin{aligned} \left\|\mathcal{D}_n^{q_n}\left(t^2,x\right) - x^2\right\|_{x^2} &\le \left(\frac{[3]_{q_n}}{q_n^3[n-3]_{q_n}} + \frac{[2]_{q_n}}{q_n^2[n-2]_{q_n}} + \frac{q_n[2]_{q_n}[3]_{q_n} + [n]_{q_n}}{q_n^6[n-2]_{q_n}[n-3]_{q_n}}\right) \sup_{x\in[0,\,\infty)} \frac{x^2}{1+x^2} \\ &\quad + \left(\frac{[n]_{q_n} + q_n\left(1+[2]_{q_n}\right)[n]_{q_n}}{q_n^5[n-2]_{q_n}[n-3]_{q_n}}\right) \sup_{x\in[0,\,\infty)} \frac{x}{1+x^2} + \frac{[2]}{q_n^3[n-2]_{q_n}[n-3]_{q_n}} \\ &\le \frac{[3]_{q_n}}{q_n^3[n-3]_{q_n}} + \frac{[2]_{q_n}}{q_n^2[n-2]_{q_n}} + \frac{q_n[2]_{q_n}[3]_{q_n} + [n]_{q_n}}{q_n^6[n-2]_{q_n}[n-3]_{q_n}} \\ &\quad + \frac{[n]_{q_n} + q_n\left(1+[2]_{q_n}\right)[n]_{q_n}}{q_n^5[n-2]_{q_n}[n-3]_{q_n}} + \frac{[2]_{q_n}}{q_n^3[n-2]_{q_n}[n-3]_{q_n}}, \end{aligned}$$

which implies that

$$\lim_{n\to\infty} \left\|\mathcal{D}_n^{q_n}\left(t^2,x\right) - x^2\right\|_{x^2} = 0.$$

Thus the proof is completed. ■

We give the following theorem to approximate all functions in $C_{x^2}[0, \infty)$. This type of results are given in [71] for locally integrable functions.

Theorem 5.4. *Let $q = q_n$ satisfies $0 < q_n < 1$ and let $q_n \to 1$ as $n \to \infty$. For each $f \in C_{x^2}[0, \infty)$ and $\alpha > 0$, we have*

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{\left|\mathcal{D}_n^{q_n}(f,x) - f(x)\right|}{\left(1+x^2\right)^{1+\alpha}} = 0.$$

Proof. For any fixed $x_0 > 0$,

$$\begin{aligned}\sup_{x\in[0,\infty)} \frac{\left|\mathcal{D}_n^{q_n}(f,x) - f(x)\right|}{\left(1+x^2\right)^{1+\alpha}} &\le \sup_{x\le x_0} \frac{\left|\mathcal{D}_n^{q_n}(f,x) - f(x)\right|}{\left(1+x^2\right)^{1+\alpha}} + \sup_{x\ge x_0} \frac{\left|\mathcal{D}_n^{q_n}(f,x) - f(x)\right|}{\left(1+x^2\right)^{1+\alpha}} \\ &\le \|\mathcal{D}_n^q(f) - f\|_{C[0,x_0]} + \|f\|_{x^2} \sup_{x\ge x_0} \frac{\left|\mathcal{D}_n^{q_n}\left(1+t^2,x\right)\right|}{\left(1+x^2\right)^{1+\alpha}} \\ &\quad + \sup_{x\ge x_0} \frac{|f(x)|}{\left(1+x^2\right)^{1+\alpha}}.\end{aligned}$$

The first term of the above inequality tends to zero from Theorem 5.2. By Lemma 5.2 for any fixed $x_0 > 0$, it is easily seen that $\sup_{x\ge x_0} \frac{\left|\mathcal{D}_n^{q_n}\left(1+t^2,x\right)\right|}{\left(1+x^2\right)^{1+\alpha}}$ tends to zero as $n \to \infty$. We can choose $x_0 > 0$ so large that the last part of above inequality can be made small enough.

Thus the proof is completed. ■

5.1.5 Recurrence Relation and Asymptotic Formula

The q-Baskakov–Durrmeyer operators $\mathcal{D}_n^q(f,x)$ can be defined in alternate form as

$$\mathcal{D}_n^q(f(t),x) := [n-1]_q \sum_{k=0}^{\infty} p_{n,k}^q(x) \int_0^{\infty/A} q^k p_{n,k}^q(t) f(t)\, d_qt, \tag{5.13}$$

where

$$p_{n,k}^q(x) := \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k^2-k}{2}} \frac{x^k}{(1+x)_q^{n+k}}$$

for $x \in [0, \infty)$ and for every real-valued continuous and bounded function f on $[0, \infty)$ (see [88]). Also

$$(1+x)_q^n = (-x;q)_n := \begin{cases} (1+x)(1+qx)\ldots\left(1+q^{n-1}x\right), & n = 1, 2, \ldots \\ 1, & n = 0. \end{cases}$$

Lemma 5.4. *If we define the central moments as*

$$T_{n,m}(x) = \mathcal{D}_n^q\left((t-x)_q^m, x\right) = [n-1]_q \sum_{k=0}^{\infty} p_{n,k}^q(x) \int_0^{\infty/A} q^k p_{n,k}^q(t)(t-x)_q^m d_q t,$$

then

$$T_{n,0}(x) = 1, T_{n,1}(x) = \frac{[2]_q}{q^2[n-2]} x + \frac{1}{q[n-2]},$$

and for $n > m+2$, we have the following recurrence relation:

$$\begin{aligned}
\Big([n]_q - [m+2]_q\Big) T_{n,m+1}(qx) &= qx(1+x)\Big[D_q T_{n,m}(x) + [m]_q T_{n,m-1}(qx)\Big] \\
&\quad + \Big([3]_q q^m x + q - x\Big)[m+1]_q T_{n,m}(qx) \\
&\quad + \Big[[2]_q q^m x\Big([3]_q q^m x + q - x\Big) - [3]_q q^{2m+1} x^2 - qx\Big][m]_q T_{n,m-1}(qx) \\
&\quad + \Big[q^m x\Big\{[2]_q q^m x\Big([3]_q q^m x + q - x\Big) - [3]_q q^{2m+1} x^2 - qx\Big\} \\
&\quad + q^{2m+1} x^2 \Big\{q^2 x - [3]_q q^m x - q + x\Big\}\Big][m-1]_q T_{n,m-2}(qx) \\
&\quad + x(1-q^{m+1})[n]_q T_{n,m}(qx) + qx(1-q^{m-1})[n]_q T_{n,m}(qx) \\
&\quad - qx^2(1-q^{m-1})(1-q^m)[n]_q T_{n,m-1}(qx),
\end{aligned}$$

and we consider $T_{n,-ve}(x) = 0$.

Proof. Using the identity

$$qx(1+x)D_q[p_{n,k}^q(x)] = \left(\frac{[k]_q}{q^{k-1}[n]_q} - qx\right)[n]_q p_{n,k}^q(qx)$$

and q-derivatives of product rule, we have

$$\begin{aligned}
qx(1+x)D_q[T_{n,m}(x)] &= [n-1]_q \sum_{k=0}^{\infty} qx(1+x) D_q[p_{n,k}^q(x)] \int_0^{\infty/A} q^k p_{n,k}^q(t)(t-x)_q^m d_q t \\
&\quad - [m]_q [n-1]_q \sum_{k=0}^{\infty} qx(1+x) p_{n,k}^q(qx) \int_0^{\infty/A} q^k p_{n,k}^q(t)(t-qx)_q^{m-1} d_q t.
\end{aligned}$$

Thus

$$E := qx(1+x)\left[D_q[T_{n,m}(x)]+[m]_q T_{n,m-1}(qx)\right]$$

$$= [n]_q[n-1]_q \sum_{k=0}^{\infty} p_{n,k}^q(qx)\left(\frac{[k]_q}{q^{k-1}[n]_q}-qx\right)\int_0^{\infty/A} q^k p_{n,k}^q(t)(t-x)_q^m d_qt$$

$$= [n]_q[n-1]_q \sum_{k=0}^{\infty} p_{n,k}^q(qx)\int_0^{\infty/A} q^k\left(\frac{[k]_q}{q^{k-1}[n]_q}-t+t-q^m x-qx+q^m x\right)p_{n,k}^q(t)(t-x)_q^m d_qt$$

$$= [n-1]_q \sum_{k=0}^{\infty} p_{n,k}^q(qx)\int_0^{\infty/A} q^k q^2\left[\frac{t}{q}\left(1+\frac{t}{q}\right)\right]D_q\left[p_{n,k}^q\left(\frac{t}{q}\right)\right](t-x)_q^m d_qt$$

$$+[n]_q[n-1]_q \sum_{k=0}^{\infty} p_{n,k}^q(qx)\int_0^{\infty/A} q^k p_{n,k}^q(t)(t-x)_q^{m+1} d_qt$$

$$+[n]_q[n-1]_q qx(q^{m-1}-1)\sum_{k=0}^{\infty} p_{n,k}^q(qx)\int_0^{\infty/A} q^k p_{n,k}^q(t)(t-x)_q^m d_qt$$

$$= [n-1]_q \sum_{k=0}^{\infty} p_{n,k}^q(qx)\int_0^{\infty/A} q^k(tq+t^2)D_q\left[p_{n,k}^q\left(\frac{t}{q}\right)\right](t-x)_q^m d_qt$$

$$+[n]_q[n-1]_q \sum_{k=0}^{\infty} p_{n,k}^q(qx)\int_0^{\infty/A} q^k p_{n,k}^q(t)(t-x)_q^{m+1} d_qt$$

$$+[n]_q[n-1]_q qx(q^{m-1}-1)\sum_{k=0}^{\infty} p_{n,k}^q(qx)\int_0^{\infty/A} q^k p_{n,k}^q(t)(t-x)_q^m d_qt.$$

Using the identities

$$(t-q^m x)(t-q^{m+1}x) = t^2-[2]_q q^m xt+q^{2m+1}x^2$$

and

$$(t-q^m x)(t-q^{m+1}x)(t-q^{m+2}x) = t^3-[3]_q q^m xt^2+[3]_q q^{2m+1}x^2 t-q^{3m+3}x^3,$$

we obtain the following identity after simple computation:

$$(qt+t^2)(t-x)_q^m = (qt+t^2)(t-x)(t-qx)_q^{m-1} = \left[t^3+(q-x)t^2-qxt\right](t-qx)_q^{m-1}$$

$$= (t-qx)_q^{m+2} + \left([3]_q q^m x + q - x\right)(t-qx)_q^{m+1}$$

$$+\left[[2]_q q^m x\left\{[3]_q q^m x+q-x\right\} - [3]_q q^{2m+1}x^2 - qx)\right](t-qx)_q^m$$

$$+\left[q^m x\left\{[2]_q q^m x\left([3]_q q^m x+q-x\right)-[3]_q q^{2m+1}x^2-qx\right\}\right.$$

$$\left.+q^{2m+1}x^2\left\{q^2x-[3]_q q^m x-q+x\right\}\right](t-qx)_q^{m-1}.$$

Using the above identity and q-integral by parts

$$\int_a^b u(t)D_q(v(t))d_qt = [u(t)v(t)]_a^b - \int_a^b v(qt)D_q[u(t)]d_qt,$$

we have

$$E = -[m+2]_q T_{n,m+1}(qx) - \left([3]_q q^m x+q-x\right)[m+1]_q T_{n,m}(qx)$$

$$-\left[[2]_q q^m x\left([3]_q q^m x+q-x\right)-[3]_q q^{2m+1}x^2-qx\right][m]_q T_{n,m-1}(qx)$$

$$-\left[q^m x\left\{[2]_q q^m x\left([3]_q q^m x+q-x\right)-[3]_q q^{2m+1}x^2-qx\right\}\right.$$

$$\left.+q^{2m+1}x^2\left\{q^2x-[3]_q q^m x-q+x\right\}\right][m-1]_q T_{n,m-2}(qx)$$

$$+[n]_q[n-1]_q \sum_{k=0}^{\infty} p_{n,k}^q(qx)\int_0^{\infty/A} q^k p_{n,k}^q(t)(t-x)_q^{m+1}d_qt$$

$$-[n]_q[n-1]_q qx(1-q^{m-1})\sum_{k=0}^{\infty} p_{n,k}^q(qx)\int_0^{\infty/A} q^k p_{n,k}^q(t)(t-x)_q^m d_qt.$$

Finally using

$$(t-x)_q^{m+1} = (t-x)(t-qx)_q^m = (t-qx)_q^{m+1} - x(1-q^{m+1})(t-qx)_q^m$$

and

$$(t-x)_q^m = (t-x)(t-qx)_q^{m-1} = (t-qx)_q^m - x(1-q^m)(t-qx)_q^{m-1},$$

we get

$$\begin{aligned}
E = & -[m+2]_q T_{n,m+1}(qx) - \left([3]_q q^m x + q - x\right)[m+1]_q T_{n,m}(qx) \\
& - \left[[2]_q q^m x\left([3]_q q^m x + q - x\right) - [3]_q q^{2m+1} x^2 - qx\right][m]_q T_{n,m-1}(qx) \\
& - \left[q^m x\left\{[2]_q q^m x\left([3]_q q^m x + q - x\right) - [3]_q q^{2m+1} x^2 - qx\right\}\right. \\
& \left. + q^{2m+1} x^2 \left\{q^2 x - [3]_q q^m x - q + x\right\}\right][m-1]_q T_{n,m-2}(qx) \\
& + [n]_q T_{n,m+1}(qx) - x(1-q^{m+1})[n] T_{n,m}(qx) \\
& - qx(1-q^{m-1})[n]_q T_{n,m}(qx) + qx^2(1-q^{m-1})(1-q^m)[n]_q T_{n,m-1}(qx).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\left([n]_q - [m+2]_q\right) T_{n,m+1}(qx) = & \; qx(1+x)\left[D_q T_{n,m}(x) + [m]_q T_{n,m-1}(qx)\right] \\
& + \left([3]_q q^m x + q - x\right)[m+1]_q T_{n,m}(qx) \\
& + \left[[2]_q q^m x\left([3] q^m x + q - x\right) - [3]_q q^{2m+1} x^2 - qx\right][m]_q T_{n,m-1}(qx) \\
& + \left[q^m x\left\{[2]_q q^m x\left([3]_q q^m x + q - x\right) - [3]_q q^{2m+1} x^2 - qx\right\}\right. \\
& \left. + q^{2m+1} x^2 \left\{q^2 x - [3]_q q^m x - q + x\right\}\right][m-1]_q T_{n,m-2}(qx) \\
& + x(1-q^{m+1})[n]_q T_{n,m}(qx) + qx(1-q^{m-1})[n]_q T_{n,m}(qx) \\
& - qx^2(1-q^{m-1})(1-q^m)[n]_q T_{n,m-1}(qx).
\end{aligned}$$

This completes the proof of recurrence relation. ■

Theorem 5.5 ([88]). *Let $f \in C[0,\infty)$ be a bounded function and (q_n) denote a sequence such that $0 < q_n < 1$ and $q_n \to 1$ as $n \to \infty$. Then we have for a point $x \in (0,\infty)$*

$$\lim_{n\to\infty} [n]_{q_n} \left(D_n^{q_n}(f,x) - f(x)\right) = (2x+1) \lim_{n\to\infty} D_{q_n} f(x) + x(1+x) \lim_{n\to\infty} D_{q_n}^2 f(x).$$

Proof. By q-Taylor formula [49] for f we have

$$f(t) = f(x) + D_q f(x)(t-x) + \frac{1}{[2]_q} D_q^2 f(x)(t-x)_q^2 + \Phi_q(x;t)(t-x)_q^2$$

for $0 < q < 1$ where

$$\Phi_q(x;t) = \begin{cases} \frac{f(t)-f(x)-D_q f(x)(t-x)-\frac{1}{[2]_q}D_q^2 f(x)(t-x)_q^2}{(t-x)_q^2}, & \text{if } x \neq y \\ 0, & \text{if } x = y. \end{cases} \tag{5.14}$$

We know that for n large enough

$$\lim_{t\to x} \Phi_{q_n}(x;t) = 0. \tag{5.15}$$

That is, for any $\varepsilon > 0$, $A > 0$, there exists a $\delta > 0$ such that

$$\left|\Phi_{q_n}(x;t)\right| < \varepsilon \tag{5.16}$$

for $|t-x| < \delta$ and n sufficiently large. Using (5.14) we can write

$$D_n^{q_n}(f,x) - f(x) = D_{q_n} f(x)\, T_{n,1}(x) + \frac{D_{q_n}^2 f(x)}{[2]_{q_n}} T_{n,2}(x) + E_n^{q_n}(x),$$

where

$$E_n^q(x) = [n-1]_q \sum_{k=0}^{\infty} p_{n,k}^q(x) \int_0^{\infty/A} q^k p_{n,k}^q(t)\, \Phi_q(x;t)(t-x)_q^2\, d_q t.$$

We can easily see that

$$\lim_{n\to\infty} [n]_{q_n} T_{n,1}(x) = 2x+1 \quad \text{and} \quad \lim_{n\to\infty} [n]_{q_n} T_{n,2}(x) = 2x(1+x).$$

In order to complete the proof of the theorem, it is sufficient to show that $lim_{n\to\infty} [n]_{q_n} E_n^{q_n}(x) = 0$. We proceed as follows:

Let

$$R_{n,1}^{q_n}(x) = [n]_{q_n} [n-1]_{q_n} \sum_{k=0}^{\infty} p_{n,k}^{q_n}(x) \int_0^{\infty/A} q_n^k p_{n,k}^{q_n}(t)\, \Phi_{q_n}(x;t)(t-x)_{q_n}^2 \chi_x(t)\, d_q t$$

and

$$R_{n,2}^{q_n}(x) = [n]_{q_n}[n-1]_{q_n}\sum_{k=0}^{\infty} p_{n,k}^{q_n}(x)\int_0^{\infty/A} q_n^k p_{n,k}^{q_n}(t)\,\Phi_{q_n}(x;t)\,(t-x)_{q_n}^2\,(1-\chi_x(t))\,d_qt,$$

so that

$$[n]_{q_n} E_n^{q_n}(x) = R_{n,1}^{q_n}(x) + R_{n,2}^{q_n}(x),$$

where $\chi_x(t)$ is the characteristic function of the interval $\{t : |t-x| < \delta\}$.

It follows from (5.14)

$$\left|R_{n,1}^{q_n}(x)\right| < \varepsilon 2x(x+1) \quad \text{as } n \to \infty.$$

If $|t-x| \geq \delta$, then $\left|\Phi_{q_n}(x;t)\right| \leq \frac{M}{\delta^2}(t-x)^2$, where $M > 0$ is a constant. Since

$$\begin{aligned}(t-x)^2 &= \left(t-q^2x+q^2x-x\right)\left(t-q^3x+q^3x-x\right)\\ &= \left(t-q^2x\right)\left(t-q^3x\right)+x\left(q^3-1\right)\left(t-q^2x\right)+x\left(q^2-1\right)\left(t-q^2x\right)\\ &\quad +x^2\left(q^2-1\right)\left(q^2-q^3\right)+x^2\left(q^2-1\right)\left(q^3-1\right),\end{aligned}$$

we have

$$\begin{aligned}\left|R_{n,2}^{q_n}(x)\right| &\leq \frac{M}{\delta^2}[n]_{q_n}[n-1]_{q_n}\sum_{k=0}^{\infty} p_{n,k}^{q_n}(x)\int_0^{\infty/A} q_n^k p_{n,k}^{q_n}(t)\,(t-x)_{q_n}^4\,d_qt\\ &\quad +\frac{M}{\delta^2}x\left(\left|\left(q_n^3-1\right)+\left(q_n^2-1\right)\right|\right)[n]_{q_n}[n-1]_{q_n}\sum_{k=0}^{\infty} p_{n,k}^{q_n}(x)\int_0^{\infty/A} q_n^k p_{n,k}^{q_n}(t)\,(t-x)_{q_n}^3\,d_qt\\ &\quad +\frac{M}{\delta^2}x^2\left(q_n^2-1\right)^2[n]_{q_n}[n-1]_{q_n}\sum_{k=0}^{\infty} p_{n,k}^{q_n}(x)\int_0^{\infty/A} q_n^k p_{n,k}^{q_n}(t)\,(t-x)_{q_n}^2\,d_qt\end{aligned}$$

and

$$\left|R_{n,2}^{q_n}(x)\right| \leq \frac{M}{\delta^2}\left\{[n]_{q_n}T_{n,4}(x)+x\left(2-q_n^2-q_n^3\right)[n]_{q_n}T_{n,3}(x)+x^2\left(q_n^2-1\right)^2[n]_{q_n}T_{n,2}(x)\right\}.$$

Using Lemma 5.4, we have

$$T_{n,4}(x) \leq \frac{C_m}{[n]_{q_n}^2},\quad T_{n,3}(x) \leq \frac{C_m}{[n]_{q_n}^2} \text{ and } T_{n,4}(x) \leq \frac{C_m}{[n]_{q_n}}.$$

We have the desired result. ■

Corollary 5.2. *Let $f \in C[0,\infty)$ be a bounded function and (q_n) denote a sequence such that $0 < q_n < 1$ and $q_n \to 1$ as $n \to \infty$. Suppose that the first and second derivatives $f'(x)$ and $f''(x)$ exist at a point $x \in (0,\infty)$, we have*

$$\lim_{n\to\infty}[n]_{q_n}\left(D_n^{q_n}(f,x)-f(x)\right) = f'(x)(2x+1)+x(1+x)f''(x).$$

5.2 q-Szász-Beta Operators

Very recently Radu [136] established the approximation properties of certain q-operators. She also proposed the q-analogue of well-known Szász–Mirakian operators, different from [29]. After the Durrmeyer variants of well-known exponential-type operators, namely, Bernstein, Baskakov, and Szász–Mirakian operators, several researchers proposed the hybrid operators. In this direction Gupta and Noor [90] introduced certain Szász-beta operators, which reproduce constant as well as linear functions. In approximation theory because of this property, the convergence becomes faster. Very recently Song et al. [143] observed that signals are often of random characters and random signals play an important role in signal processing, especially in the study of sampling results. For this purpose, one usually uses stochastic processes which are stationary in the wide sense as a model [141]. A wide-sense stationary process is only a kind of second-order moment processes. They obtained a Korovkin-type approximation theorem and mentioned the operators such as Bernstein, Baskakov, and Szász operators and their Kantorovich variants as applications. Here we extend the study and consider more complex operators by dealing with the q-summation–integral operators. In the present study, as an application of q-beta functions [49], we introduce the q- analogue of the Szász-beta operators and obtain its moments up to second order to study their convergence behaviors.

Radu [136] proposed q-generalization of the Szász operators as

$$\mathcal{S}_{n,q}(f,x) = \sum_{k=0}^{\infty} s_{n,k}^{q}(x)\, q^{k(k-1)} f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right), \tag{5.17}$$

where

$$s_{n,k}^{q}(x) = \frac{([n]_q x)^k}{[k]_q!} E_q\left(-[n]_q q^k x\right).$$

Lemma 5.5 ([136]). *We have the following:*

$$\mathcal{S}_{n,q}(1,x) = 1.$$

$$\mathcal{S}_{n,q}(t,x) = x.$$

$$\mathcal{S}_{n,q}(t^2,x) = x^2 + \frac{x}{[n]_q}.$$

5.2.1 Construction of Operators

For every $n \in \mathbb{N}$, $q \in (0,\ 1)$, the linear positive operators $\mathcal{D}_n^q$ are defined by

$$\mathcal{D}_n^q(f(t),x) := \sum_{k=1}^{\infty} q^{\frac{3k^2-3k}{2}} s_{n,k}^q(x) \int_0^{\infty/A} p_{n,k}^q(t) f(qt)\, d_q t + E_q(-[n]_q x) f(0) \quad (5.18)$$

where

$$p_{n,k}^q(t) := \frac{1}{B_q(n+1,k)} \frac{t^{k-1}}{(1+t)_q^{n+k+1}}$$

and

$$s_{n,k}^q(x) = \frac{([n]_q x)^k}{[k]_q!} E_q\left(-[n]_q q^k x\right)$$

for $x \in [0, \infty)$ and for every real-valued continuous and bounded function f on $[0, \infty)$ (see [87]). In case $q = 1$ the above operators reduce to the Szász-beta operators discussed in [90] .

Lemma 5.6 ([87]). *The following equalities hold:*

(i) $\mathcal{D}_n^q(1,x) = 1.$
(ii) $\mathcal{D}_n^q(t,x) = x.$
(iii) $\mathcal{D}_n^q\left(t^2,x\right) = \frac{[n]_q x^2 + [2]_q x}{q[n-1]_q}$*, for* $n > 1$.

Proof. For $x \in [0, \infty)$ by (5.18), we have

$$\begin{aligned}\mathcal{D}_n^q(1,x) &= \sum_{k=1}^{\infty} s_{n,k}^q(x) \frac{q^{\frac{3k^2-3k}{2}}}{B_q(n+1,k)} \int_0^{\infty/A} \frac{t^{k-1}}{(1+t)_q^{n+k+1}} d_q t + E_q(-[n]_q x) \\ &= \sum_{k=1}^{\infty} s_{n,k}^q(x) \frac{q^{\frac{3k^2-3k}{2}}}{B_q(n+1,k)} \int_0^{\infty/A} \frac{t^{k-1}}{(1+t)_q^{n+k+1}} d_q t + E_q(-[n]_q x).\end{aligned}$$

Using (1.15) and (1.17), we can write

$$\begin{aligned}\mathcal{D}_n^q(1,x) &= \sum_{k=1}^{\infty} s_{n,k}^q(x) \frac{q^{\frac{3k^2-3k}{2}}}{B_q(n+1,k)} \frac{B_q(n+1,k)}{K(A,k)} + E_q(-[n]_q x) \\ &= \sum_{k=1}^{\infty} s_{n,k}^q(x)\, q^{\frac{3k^2-3k}{2}} \frac{1}{q^{\frac{k(k-1)}{2}}} + E_q(-[n]_q x) \\ &= \sum_{k=1}^{\infty} s_{n,k}^q(x)\, q^{k(k-1)} + E_q(-[n]_q x) \\ &= \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{[k]_q!} q^{k(k-1)} E_q\left(-[n]_q q^k x\right) = \mathcal{S}_{n,q}(1,x) = 1,\end{aligned}$$

where $\mathcal{S}_n^q(f,x)$ is the q-Szász operator defined by (5.17).

Similarly

$$\begin{aligned}
\mathcal{D}_n^q(t,x) &= \sum_{k=1}^{\infty} s_{n,k}^q(x) \int_0^{\infty/A} p_{n,k}^q(t)\, qt\, d_q t \\
&= \sum_{k=1}^{\infty} s_{n,k}^q(x) \frac{q^{\frac{3k^2-3k}{2}}}{B_q(n+1,k)} \int_0^{\infty/A} \frac{qt^k}{(1+t)_q^{n+k+1}} d_q t \\
&= \sum_{k=1}^{\infty} s_{n,k}^q(x) \frac{q^{\frac{3k^2-3k}{2}}}{B_q(n+1,k)} \frac{qB_q(n,k+1)}{K(A,k+1)} \\
&= \sum_{k=1}^{\infty} s_{n,k}^q(x)\, q^{\frac{3k^2-3k+2}{2}} \frac{[n+k]_q!}{[n]_q!\,[k-1]_q!} \frac{[k]_q!\,[n-1]_q!}{[n+k]_q!\, q^{\frac{k(k+1)}{2}}} \\
&= \sum_{k=0}^{\infty} \frac{[k]_q}{[n]_q} s_{n,k}^q(x)\, q^{k^2-2k+1} = \sum_{k=0}^{\infty} \frac{[k]_q}{q^{k-1}[n]_q} s_{n,k}^q(x)\, q^{k^2-k} = \mathcal{S}_{n,q}(t,x) = x.
\end{aligned}$$

Finally for $n > 1$, we have

$$\begin{aligned}
\mathcal{D}_n^q\left(t^2,x\right) &= \sum_{k=1}^{\infty} q^{\frac{3k^2-3k}{2}} s_{n,k}^q(x) \int_0^{\infty/A} p_{n,k}^q(t)\, q^2 t^2 d_q t \\
&= \sum_{k=1}^{\infty} s_{n,k}^q(x) \frac{q^{\frac{3k^2-3k}{2}}}{B_q(n+1,k)} \int_0^{\infty/A} \frac{q^2 t^{k+1}}{(1+t)_q^{n+k+1}} d_q t \\
&= \sum_{k=1}^{\infty} s_{n,k}^q(x) \frac{q^{\frac{3k^2-3k}{2}}}{B_q(n+1,k)} \frac{q^2 B_q(n-1,k+2)}{K(A,k+2)\, q^{\frac{k^2+3k+2}{2}}} \\
&= \sum_{k=1}^{\infty} s_{n,k}^q(x)\, q^{\frac{3k^2-3k}{2}} \frac{[k+1]_q\,[k]_q}{[n]_q\,[n-1]_q} \frac{q^2}{q^{\frac{k^2+3k+2}{2}}}.
\end{aligned}$$

Using $[k+1]_q = [k]_q + q^k$, we have

$$\begin{aligned}
\mathcal{D}_n^q\left(t^2,x\right) &= \sum_{k=0}^{\infty} s_{n,k}^q(x)\, q^{k^2-3k+1} \frac{\left([k]_q + q^k\right)[k]_q}{[n]_q\,[n-1]_q} \\
&= \frac{[n]_q}{q\,[n-1]_q} \mathcal{S}_{n,q}(t^2,x) + \frac{1}{[n-1]_q} \mathcal{S}_{n,q}(t,x) \\
&= \frac{[n]_q}{q\,[n-1]_q} \left(x^2 + \frac{x}{[n]_q}\right) + \frac{x}{[n-1]_q} \\
&= \frac{[n]_q x^2 + [2]_q x}{q[n-1]_q}.
\end{aligned}$$

■

Remark 5.2. Let $n > 1$ and $x \in [0,\infty)$, and then for every $q \in (0,1)$, we have

$$\mathcal{D}_n^q((t-x),x) = 0$$

and

$$\mathcal{D}_n^q((t-x)^2,x) = \frac{x^2+[2]_q x}{q\,[n-1]_q}.$$

5.2.2 Direct Theorem

By $C_B[0,\infty)$, we denote the space of real-valued continuous and bounded functions f defined on the interval $[0,\infty)$. The norm-$||.||$ on the space $C_B[0,\infty)$ is given by

$$||f|| = \sup_{0\leq x<\infty} |f(x)|.$$

The Peetre K-functional is defined as

$$K_2(f,\delta) = \inf\{||f-g|| + \delta||g''|| : g \in W_\infty^2\},$$

where $W_\infty^2 = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. For $f \in C_B[0,\infty)$ the modulus of continuity of second order is defined by

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h\leq\sqrt{\delta}} \sup_{0\leq x<\infty} |f(x+2h) - 2f(x+h) + f(x)|.$$

By [50], there exists a positive constant $C > 0$ such that

$$K_2(f,\delta) \leq C\omega_2(f,\delta^{1/2}), \delta > 0.$$

Theorem 5.6. *Let $f \in C_B[0,\infty)$ and $0 < q < 1$. Then for all $x \in [0,\infty)$ and $n > 1$, there exists an absolute constant $C > 0$ such that*

$$|\mathcal{D}_n^q(f,x) - f(x)| \leq C\omega_2\left(f, \sqrt{\frac{x^2+[2]_q x}{2q\,[n-1]_q}}\right).$$

Proof. Let $g \in W_\infty^2$ and $x,t \in [0,\infty)$. By Taylor's expansion, we have

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du$$

Applying Remark 5.2, we obtain

$$\mathcal{D}_n^q(g,x) - g(x) = \mathcal{D}_n^q\left(\int_x^t (t-u)g''(u)du, x\right).$$

Obviously, we have $|\int_x^t (t-u)g''(u)du| \leq (t-x)^2||g''||$. Therefore

$$|\mathcal{D}_n^q(g,x)-g(x)| \leq \mathcal{D}_n^q((t-x)^2,x)||g''|| = \frac{x^2+[2]_q x}{q\,[n-1]_q}||g''||.$$

Using Lemma 5.6, we have

$$|\mathcal{D}_n^q(f,x)| \leq \sum_{k=1}^{\infty} q^{\frac{3k^2-3k}{2}} s_{n,k}^q(x) \int_0^{\infty/A} p_{n,k}^q(t)\,|f(qt)|\,d_q t + E_q(-[n]_q x)\,|f(0)| \leq ||f||.$$

Thus

$$|\mathcal{D}_n^q(f,x)-f(x)| \leq |\mathcal{D}_n^q(f-g,x)-(f-g)(x)| + |\mathcal{D}_n^q(g,x)-g(x)|$$

$$\leq 2||f-g|| + \frac{x^2+[2]_q x}{q\,[n-1]_q}||g''||.$$

Finally taking the infimum over all $g \in W_\infty^2$ and using the inequality $K_2(f,\delta) \leq C\omega_2(f,\delta^{1/2}), \delta > 0$, we get the required result. This completes the proof of Theorem 5.6. ■

We consider the following class of functions.

Let $H_{x^2}[0,\,\infty)$ be the set of all functions f defined on $[0,\,\infty)$ satisfying the condition $|f(x)| \leq M_f\left(1+x^2\right)$, where M_f is a constant depending only on f. By $C_{x^2}[0,\,\infty)$, we denote the subspace of all continuous functions belonging to $H_{x^2}[0,\,\infty)$. Also, let $C_{x^2}^*[0,\,\infty)$ be the subspace of all functions $f \in C_{x^2}[0,\,\infty)$, for which $\lim\limits_{|x|\to\infty} \frac{f(x)}{1+x^2}$ is finite. The norm on $C_{x^2}^*[0,\,\infty)$ is $\|f\|_{x^2} = \sup_{x\in[0,\,\infty)} \frac{|f(x)|}{1+x^2}$. We denote the modulus of continuity of f on closed interval $[0,\,a]\,, a>0$ as by

$$\omega_a(f,\,\delta) = \sup_{|t-x|\leq\delta}\ \sup_{x,t\in[0,\,a]} |f(t)-f(x)|.$$

We observe that for function $f \in C_{x^2}[0,\,\infty)$, the modulus of continuity $\omega_a(f,\,\delta)$ tends to zero.

Theorem 5.7. *Let $f \in C_{x^2}[0,\,\infty)$, $q \in (0,1)$ and $\omega_{a+1}(f,\,\delta)$ be its modulus of continuity on the finite interval $[0,\,a+1] \subset [0,\,\infty)$, where $a > 0$. Then for every $n > 1$,*

$$||\mathcal{D}_n^q(f)-f||_{C[0,\,a]} \leq \frac{6M_f a(1+a^2)(2+a)}{q[n-1]_q} + 2\omega\left(f, \sqrt{\frac{a(2+a)}{q[n-1]_q}}\right).$$

Proof. For $x \in [0,\,a]$ and $t > a+1$, since $t-x > 1$, we have

$$\begin{aligned}|f(t)-f(x)| &\leq M_f\left(2+x^2+t^2\right)\\ &\leq M_f\left(2+3x^2+2(t-x)^2\right)\\ &\leq 6M_f\left(1+a^2\right)(t-x)^2. \end{aligned} \tag{5.19}$$

For $x\in[0,\ a]$ and $t\leq a+1$, we have

$$|f(t)-f(x)|\leq \omega_{a+1}(f,|t-x|)\leq\left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\ \delta) \tag{5.20}$$

with $\delta>0$.

From (5.19) and (5.20), we can write

$$|f(t)-f(x)|\leq 6M_f\left(1+a^2\right)(t-x)^2+\left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\ \delta) \tag{5.21}$$

for $x\in[0,\ a]$ and $t\geq 0$. Thus

$$\begin{aligned}|\mathcal{D}_n^q(f,x)-f(x)| &\leq \mathcal{D}_n^q(|f(t)-f(x)|,x)\\ &\leq 6M_f\left(1+a^2\right)\mathcal{D}_n^q\left((t-x)^2,x\right)\\ &\quad+\omega_{a+1}(f,\ \delta)\left(1+\frac{1}{\delta}\mathcal{D}_n^q\left((t-x)^2,x\right)\right)^{\frac{1}{2}}.\end{aligned}$$

Hence, by using Schwarz inequality and Remark 5.2, for every $q\in(0,1)$ and $x\in[0,\ a]$,

$$\begin{aligned}|\mathcal{D}_n^q(f,x)-f(x)| &\leq \frac{6M_f\left(1+a^2\right)(x^2+[2]_qx)}{q[n-1]_q}\\ &\quad+\omega_{a+1}(f,\ \delta)\left(1+\frac{1}{\delta}\sqrt{\frac{x^2+[2]_qx}{q[n-1]_q}}\right)\\ &\leq \frac{6M_fa(1+a^2)(2+a)}{q[n-1]_q}+\omega_{a+1}(f,\ \delta)\left(1+\frac{1}{\delta}\sqrt{\frac{a(2+a)}{q[n-1]_q}}\right).\end{aligned}$$

By taking $\delta=\sqrt{\frac{a(2+a)}{q[n-1]_q}}$, we get the assertion of our theorem. This completes the proof of the theorem. ■

Remark 5.3. It is observed that under the assumptions of Theorem 5.7, the pointwise convergence rate of the operators (5.18) to f is $\frac{1}{\sqrt{q_n[n-1]_{q_n}}}$ for $0<q_n<1$ *and* $q_n\to 1$ as $n\to\infty$. Also this convergence rate can be made better depending on the choice of q_n and is at least as fast as than $\frac{1}{\sqrt{n-1}}$.

5.2.3 Weighted Approximation

Now, we shall discuss the weighted approximation theorem as follows:

Theorem 5.8. *Let $q = q_n$ satisfies $0 < q_n < 1$ and let $q_n \to 1$ as $n \to \infty$. For each $f \in C^*_{x^2}[0, \infty)$, we have*

$$\lim_{n\to\infty} \|\mathcal{D}_n^{q_n}(f) - f\|_{x^2} = 0.$$

Proof. Using Korovkin's theorem (see [65]), it is sufficient to verify the following three conditions:

$$\lim_{n\to\infty} \|\mathcal{D}_n^{q_n}(t^\nu, x) - x^\nu\|_{x^2} = 0, \quad \nu = 0, 1, 2. \tag{5.22}$$

Since $D_n^{q_n}(1, x) = 1$ and $D_n^{q_n}(t, x) = x$, (5.22) holds for $\nu = 0$ and $\nu = 1$.

Next for $n > 1$, we have

$$\begin{aligned}\left\|\mathcal{D}_n^{q_n}\left(t^2, x\right) - x^2\right\|_{x^2} &\le \left(\frac{[n]_q}{q_n[n-1]_{q_n}} - 1\right) \sup_{x\in[0,\,\infty)} \frac{x^2}{1+x^2} + \frac{[2]_q}{q_n[n-1]_{q_n}} \sup_{x\in[0,\,\infty)} \frac{x}{1+x^2} \\ &\le \frac{1}{q_n[n-1]_{q_n}} + \frac{[2]_q}{q_n[n-1]_{q_n}}\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \left\|\mathcal{D}_n^{q_n}\left(t^2, x\right) - x^2\right\|_{x^2} = 0.$$

Thus the proof is completed. ∎

Next we give the following theorem to approximate all functions in $C_{x^2}[0, \infty)$. This type of result is given in [70] for locally integrable functions.

Theorem 5.9. *Let $q = q_n$ satisfies $0 < q_n < 1$ and let $q_n \to 1$ as $n \to \infty$. For each $f \in C_{x^2}[0, \infty)$ and $\alpha > 0$, we have*

$$\lim_{n\to\infty} \sup_{x\in[0,\,\infty)} \frac{\left|\mathcal{D}_n^{q_n}(f, x) - f(x)\right|}{\left(1+x^2\right)^{1+\alpha}} = 0.$$

Proof. For any fixed $x_0 > 0$,

$$\begin{aligned}\sup_{x\in[0,\,\infty)} \frac{\left|\mathcal{D}_n^{q_n}(f, x) - f(x)\right|}{\left(1+x^2\right)^{1+\alpha}} &\le \sup_{x\le x_0} \frac{\left|\mathcal{D}_n^{q_n}(f, x) - f(x)\right|}{\left(1+x^2\right)^{1+\alpha}} + \sup_{x\ge x_0} \frac{\left|\mathcal{D}_n^{q_n}(f, x) - f(x)\right|}{\left(1+x^2\right)^{1+\alpha}} \\ &\le \|\mathcal{D}_n^q(f) - f\|_{C[0,\,x_0]} + \|f\|_{x^2} \sup_{x\ge x_0} \frac{\left|\mathcal{D}_n^{q_n}\left(1+t^2, x\right)\right|}{\left(1+x^2\right)^{1+\alpha}} \\ &\quad + \sup_{x\ge x_0} \frac{|f(x)|}{\left(1+x^2\right)^{1+\alpha}}.\end{aligned}$$

Obviously, the first term of the above inequality tends to zero, which is evident from Theorem 5.6. By Lemma 5.6 for any fixed $x_0 > 0$, it is easily seen that $\sup_{x\geq x_0} \frac{\left|\mathcal{D}_n^{q_n}\left(1+t^2,x\right)\right|}{\left(1+x^2\right)^{1+\alpha}}$ tends to zero as $n \to \infty$. Finally, we can choose $x_0 > 0$ so large that the last part of above inequality can be made small enough. ■

5.3 q-Szász–Durrmeyer Operators

In this section we present direct approximation result in weighted function space with the help of a weighted Korovkin-type theorem for new q-Szász–Durrmeyer operators (see [33]). Then we give the weighted approximation error of these operators in terms of weighted modulus of continuity. Finally, we establish an asymptotic formula.

Recently for $0 < q < 1$, Aral [25] (also see [29]) defined the q-Szász–Mirakian operators as

$$S_n^q(f,x) = E_q\left(-[n]_q \frac{x}{b_n}\right) \sum_{k=0}^{\infty} f\left(\frac{[k]_q b_n}{[n]_q}\right) \frac{([n]_q x)^k}{[k]_q!(b_n)^k}, \tag{5.23}$$

where $0 \leq x < \alpha_q(n), \alpha_q(n) := \frac{b_n}{(1-q)[n]_q}, f \in C([0,\infty)$ and (b_n) is a sequence of positive numbers such that $\lim_{n\to\infty} b_n = \infty$. Some approximation properties of these operators are studied in [29].

Based on this, we now propose the q-Szász–Durrmeyer operators for $0 < q < 1$ as

$$\mathcal{Z}_n^q\left(f\left(t\right),x\right) = \frac{[n]_q}{b_n} \sum_{k=0}^{\infty} s_{n,k}^q\left(x\right) \int_0^{\frac{qb_n}{1-q^n}} s_{n,k}^q\left(t\right) f\left(t\right) d_q t \ , \tag{5.24}$$

where

$$s_{n,k}^q\left(x\right) = \frac{\left([n]_q x\right)^k}{q^{\frac{k+1}{2}}\,[k]_q!\,(b_n)^k} E_q\left(-[n]_q \frac{x}{b_n}\right).$$

Remark 5.4. Note that the q-Szász–Durrmeyer operators can be rewritten via an improper integral by using Definition (1.13). We can easily see that $E_q\left(-\frac{q^n}{1-q}\right) = 0$ for $n \leq 0$. Thus for $0 < q < 1$ we can write

$$\mathcal{Z}_n^q\left(f\left(t\right),x\right) = \frac{[n]_q}{b_n} \sum_{k=0}^{\infty} s_{n,k}^q\left(x\right) \int_0^{\frac{\infty}{\frac{1-q^n}{qb_n}}} s_{n,k}^q\left(t\right) f\left(t\right) d_q t.$$

By [29], we have

$$S_n^q(1,x) = 1, \qquad S_n^q(t,x) = x, \qquad S_n^q(t^2,x) = qx^2 + \frac{b_n}{[n]_q}x,$$

$$S_n^q(t^3,x) = q^3x^3 + \left([2]_q + 1\right) q \frac{b_n}{[n]_q}x^2 + \frac{b_n}{[n]_q}x,$$

$$S_n^q(t^4,x) = q^6x^4 + a_1(q) S_n^q(t^3,x) + a_2(q) S_n^q(t^2,x) + [2]_q [3]_q \left(\frac{b_n}{[n]_q}\right)^2 x,$$

where

$$a_1(q) = \left(1 + [2]_q + [3]_q\right) \frac{b_n}{[n]_q} \text{ and } a_2(q) = -\left([2]_q [3]_q + [2]_q + [3]_q\right) \left(\frac{b_n}{[n]_q}\right)^2.$$

5.3.1 Auxiliary Results

In the sequel, we shall need the following auxiliary results.

Lemma 5.7. *We have*

$$\mathcal{Z}_n^q(1,x) = 1, \qquad \mathcal{Z}_n^q(t,x) = q^2x + \frac{qb_n}{[n]_q},$$

$$\mathcal{Z}_n^q\left(t^2,x\right) = q^6x^2 + \left(q^5 + 2q^4 + q^3\right) \frac{b_n}{[n]_q}x + q^2(1+q)\left(\frac{b_n}{[n]_q}\right)^2,$$

$$\mathcal{Z}_n^q\left(t^3,x\right) = q^9\mathcal{S}_n^q\left(t^3,x\right) + d_1(q)\mathcal{S}_n^q\left(t^2,x\right) + d_2(q)\mathcal{S}_n^q(t,x) + q^3\left(\frac{b_n}{[n]_q}\right)^3 [2]_q [3]_q,$$

$$\mathcal{Z}_n^q\left(t^4,x\right) = q^{14}\mathcal{S}_n^q\left(t^4,x\right) + d_3(q)\mathcal{S}_n^q\left(t^3,x\right) + d_4(q)\mathcal{S}_n^q\left(t^2,x\right) + d_5(q)\mathcal{S}_n^q(t,x) + q^4[2]_q[3]_q[4]_q\left(\frac{b_n}{[n]_q}\right)^4,$$

where

$$d_1(q) = q^6\left([3]_q + q[2]_q + q^2\right)\frac{b_n}{[n]_q}, \quad d_2(q) = q^4\left([2]_q[3]_q + q[3]_q + q^2[2]_q\right)\left(\frac{b_n}{[n]_q}\right)^2,$$

$$d_3(q) = q^{10}\left[[4]_q + \left(q[3]_q + q^2[2]_q + q^3\right)\right]\frac{b_n}{[n]_q},$$

$$d_4(q) = q^7 \left[[4]_q \left([3]_q + q[2]_q + q^2 \right) + q^2 \left([2]_q [3]_q + q[3]_q + q^2 [2]_q \right) \right] \left(\frac{b_n}{[n]_q} \right)^2, \text{ and}$$

$$d_5(q) = q^4 \left(q^4 [2]_q [3]_q + [4]_q \left(q[2]_q [3]_q + q^2 [3]_q + q^3 [2]_q \right) \right) \left(\frac{b_n}{[n]_q} \right)^3.$$

Proof. Using (5.24), we have

$$\begin{aligned} \mathcal{Z}_n^q(1,x) &= \frac{[n]_q}{b_n} \sum_{k=0}^{\infty} s_{n,k}^q(x) \int_0^{\frac{qb_n}{1-q^n}} s_{n,k}^q(t)\, d_q t \\ &= \frac{[n]_q}{b_n} \sum_{k=0}^{\infty} s_{n,k}^q(x) \frac{\left([n]_q\right)^k}{q^{\frac{k+1}{2}} [k]_q! (b_n)^k} \int_0^{\frac{qb_n}{1-q^n}} t^k E_q \left(-[n]_q \frac{t}{b_n} \right) d_q t. \end{aligned}$$

Using (5.19) and change of variable formula for q-integral with $t = q\frac{b_n}{[n]_q} y$, then we have

$$\mathcal{Z}_n^q(1,x) = \frac{[n]_q}{b_n} \sum_{k=0}^{\infty} s_{n,k}^q(x) \frac{q^{k+1} b_n}{q^{\frac{k+1}{2}} [n]_q [k]_q!} \int_0^{\frac{1}{1-q}} y^k E_q(-qy)\, d_q y.$$

From (5.24) and (5.23), it follows that

$$\begin{aligned} \mathcal{Z}_n^q(1,x) &= \frac{[n]_q}{b_n} \sum_{k=0}^{\infty} s_{n,k}^q(x) \frac{q^{\frac{k+1}{2}} b_n}{[n]_q [k]_q!} \Gamma_q(k+1) \\ &= E_q \left(-[n]_q \frac{x}{b_n} \right) \sum_{k=0}^{\infty} \frac{\left([n]_q x\right)^k}{[k]_q! (b_n)^k} \\ &= \mathcal{S}_n^q(1,x) = 1. \end{aligned}$$

Also, using a similar technique, from (5.24) with $t = q\frac{b_n}{[n]_q} y$, we have

$$\begin{aligned} \mathcal{Z}_n^q(t,x) &= \frac{[n]_q}{b_n} \sum_{k=0}^{\infty} s_{n,k}^q(x) \int_0^{\frac{qb_n}{1-q^n}} s_{n,k}^q(t)\, t\, d_q t \\ &= \frac{[n]_q}{b_n} \sum_{k=0}^{\infty} s_{n,k}^q(x) \frac{\left([n]_q\right)^k}{q^{\frac{k+1}{2}} [k]_q! (b_n)^k} \int_0^{\frac{qb_n}{1-q^n}} t^{k+1} E_q \left(-[n]_q \frac{t}{b_n} \right) d_q t \\ &= \frac{qb_n}{[n]_q} \sum_{k=0}^{\infty} s_{n,k}^q(x) \frac{q^{\frac{k+1}{2}}}{[k]_q!} \int_0^{\frac{1}{1-q}} y^{k+1} E_q(-qy)\, d_q y \\ &= \frac{qb_n}{[n]_q} \sum_{k=0}^{\infty} s_{n,k}^q(x) \frac{q^{\frac{k+1}{2}}}{[k]_q!} \Gamma_q(k+2). \end{aligned}$$

From (5.24) and (5.23), it follows that

$$\begin{aligned}
\mathcal{Z}_n^q(t,x) &= qE_q\left(-[n]_q\frac{x}{b_n}\right)\sum_{k=0}^{\infty}\frac{\left([n]_q x\right)^k}{[k]_q!\,(b_n)^k}b_n\frac{[k+1]_q}{[n]_q}\\
&= qE_q\left(-[n]_q\frac{x}{b_n}\right)\sum_{k=0}^{\infty}\frac{\left([n]_q x\right)^k}{[k]_q!\,(b_n)^k}b_n\frac{\left(q[k]_q+1\right)}{[n]_q}\\
&= q^2\mathcal{S}_n^q(t,x)+\frac{qb_n}{[n]_q}\mathcal{S}_n^q(1,x)=q^2x+\frac{qb_n}{[n]_q}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\mathcal{Z}_n^q\left(t^2,x\right) &= \frac{[n]_q}{b_n}\sum_{k=0}^{\infty}s_{n,k}^q(x)\frac{\left([n]_q\right)^k}{q^{\frac{k+1}{2}}[k]_q!\,(b_n)^k}\frac{q^{k+3}(b_n)^{k+3}}{\left([n]_q\right)^{k+3}}\int_0^{\frac{1}{1-q}}y^{k+2}E_q(-qy)\,d_qy\\
&= \frac{[n]_q}{b_n}\sum_{k=0}^{\infty}s_{n,k}^q(x)\frac{\left([n]_q\right)^k}{q^{\frac{k+1}{2}}[k]_q!\,(b_n)^k}\frac{q^{k+3}(b_n)^{k+3}}{\left([n]_q\right)^{k+3}}\Gamma_q(k+3)\\
&= q^2E_q\left(-[n]_q\frac{x}{b_n}\right)\sum_{k=0}^{\infty}\frac{\left([n]_q x\right)^k}{[k]_q!\,(b_n)^k}\frac{(b_n)^2[k+1]_q[k+2]_q}{\left([n]_q\right)^2}\\
&= q^2E_q\left(-[n]_q\frac{x}{b_n}\right)\sum_{k=0}^{\infty}\frac{\left([n]_q x\right)^k}{[k]_q!\,(b_n)^k}\frac{(b_n)^2\left(q[k]_q+1\right)\left(q^2[k]_q+[2]_q\right)}{\left([n]_q\right)^2}\\
&= q^5\mathcal{S}_n^q\left(t^2,x\right)+q^2\left(q[2]_q+q^2\right)\frac{b_n}{[n]_q}\mathcal{S}_n^q(t,x)+q^2\left(\frac{b_n}{[n]_q}\right)^2[2]_q\mathcal{S}_n^q(1,x)\\
&= q^6x^2+\left(q^5+2q^4+q^3\right)\frac{b_n}{[n]_q}x+q^2(1+q)\left(\frac{b_n}{[n]_q}\right)^2.
\end{aligned}$$

Other moments can be calculated similarly.

Lemma 5.8. *We have the following:*

1. $\mathcal{Z}_n^q(t-x,x)=\left(q^2-1\right)x+q\frac{b_n}{[n]_q}$.
2. $\mathcal{Z}_n^q\left((t-x)^2,x\right)=\left(\left(q^6-2q^2+1\right)x^2+q^2\left(q^3+2q^2+q-2\right)\frac{b_n}{[n]_q}x+q^2(1+q)\left(\frac{b_n}{[n]_q}\right)^2\right)$.
3. $\mathcal{Z}_n^q\left((t-x)^4,x\right)=x^4\left(q^{20}-4q^{12}+6q^6-4q^2+1\right)$

$$+\begin{pmatrix} q^{17}a_1(q)+d_3(q)q^3-4\left([2]_q+1\right)\frac{b_n}{[n]_q}q^{10}-4d_1(q) \\ +6\left(q^5+2q^4+q^3\right)\frac{b_n}{[n]_q}-4\frac{qb_n}{[n]_q} \end{pmatrix}x^3$$

$$+\begin{pmatrix} q^{15}a_1(q)\left([2]_q+1\right)\frac{b_n}{[n]_q}+q^{15}a_2(q)+d_3(q)q\left([2]_q+1\right)\frac{b_n}{[n]_q} \\ +d_4(q)q-4q^9\left(\frac{b_n}{[n]_q}\right)^2-4d_2(q)-4d_1(q)\frac{b_n}{[n]_q}+6(1+q)q^2\left(\frac{b_n}{[n]_q}\right)^2 \end{pmatrix}x^2$$

$$+\begin{pmatrix} q^{14}a_1(q)\left(\frac{b_n}{[n]_q}\right)^2+q^{14}a_2(q)\frac{b_n}{[n]_q}+d_3(q)\left(\frac{b_n}{[n]_q}\right)^2+d_4(q)\frac{b_n}{[n]_q} \\ -4q^3\left(\frac{b_n}{[n]_q}\right)^3[2]_q[3]_q \end{pmatrix}x$$

$$+q^{14}\left(\frac{b_n}{[n]_q}\right)^3[2]_q[3]_q+d_5(q)+q^4[2]_q[3]_q[4]_q\left(\frac{b_n}{[n]_q}\right)^2.$$

5.3.2 Approximation Properties

Let B_2 be the set of all functions f defined on $[0,\infty)$ satisfying the condition $|f(x)|\le M_f\left(1+x^2\right)$, where M_f is a constant depending only f. C_2 denotes the subspace of all continuous function in B_2, and C_2^* denotes the subspace of all functions $f\in C_2$ for which $\lim\limits_{x\to\infty}\frac{|f(x)|}{1+x^2}$ exists finitely.

Let (α_n) be a sequence of positive numbers, such that $\lim\limits_{n\to\infty}a_n=\infty$ and

$$\|f\|_{2,[0,a_n]}=\sup_{0\le x\le\alpha_n}\frac{|f(x)|}{1+x^2},$$

for $f\in B_2$. These type functions are mentioned in [71].

Theorem 5.10. *Let $f\in C_2^*$ and $q=q_n$ satisfies $0<q_n<1$ such that $q_n\to 1$ as $n\to\infty$. If $\lim_{n\to\infty}\frac{b_n}{[n]_q}=0$, we have*

$$\lim_{n\to\infty}\|\mathcal{Z}_n^q(f)-f\|_{2,\left[0,\alpha_{q_n}(n)\right]}=0.$$

Proof. On account of Theorem 1 in [71], it is enough to show the validity of the following:

$$\lim_{n\to\infty}\|\mathcal{Z}_n^{q_n}(t^\nu,x)-x^\nu\|_{2,\left[0,\alpha_{q_n}(n)\right]}=0,\quad \nu=0,1,2. \tag{5.25}$$

Since, $\mathcal{Z}_n^{q_n}(1,x)=1$, it obvious that

$$\lim_{n\to\infty}\|\mathcal{Z}_n^{q_n}(1,x)-1\|_{2,\left[0,\alpha_{q_n}(n)\right]}=0.$$

Using Lemma 5.7, we obtain

$$\lim_{n\to\infty} \|\mathcal{Z}_n^{q_n}(t,x)-x\|_{2,[0,\alpha_{q_n}(n)]} \leq \left(1-q_n^2\right) \sup_{0\leq x\leq \alpha_{q_n}(n)} \frac{x}{1+x^2} + \frac{q_n b_n}{[n]_{q_n}}$$
$$\leq \left(1-q_n^2\right) + \frac{q_n b_n}{[n]_{q_n}}$$

and

$$\lim_{n\to\infty} \left\|\mathcal{Z}_n^{q_n}\left(t^2,x\right)-x^2\right\|_{2,[0,\alpha_{q_n}(n)]}$$
$$= \lim_{n\to\infty} \sup_{0\leq x\leq \alpha_{q_n}(n)} \frac{\left|\mathcal{Z}_n^{q_n}\left(t^2,x\right)-x^2\right|}{1+x^2}$$
$$\leq \left(1-q_n^6\right) \sup_{0\leq x\leq \alpha_{q_n}(n)} \frac{x^2}{1+x^2} + \left(q_n^5+2q_n^4+q_n^3\right)\frac{b_n}{[n]_{q_n}} \sup_{0\leq x\leq \alpha_{q_n}(n)} \frac{x}{1+x^2}$$
$$+q_n^2(1+q_n)\left(\frac{b_n}{[n]_{q_n}}\right)^2 \sup_{0\leq x\leq \alpha_{q_n}(n)} \frac{1}{1+x^2}$$
$$\leq \left(1-q_n^6\right) + + \left(q_n^5+2q_n^4+q_n^3\right)\frac{b_n}{[n]_{q_n}} + \left(\frac{b_n}{[n]_{q_n}}\right)^2 q_n^2(1+q_n).$$

Since $\mathit{lim}_{n\to\infty} \frac{b_n}{[n]_{q_n}} = 0$ and $\mathit{lim}_{n\to\infty} q_n = 1$, we have $\lim_{n\to\infty} \left\|\mathcal{Z}_n^{q_n}(t,x)-x\right\|_{2,[0,\alpha_{q_n}(n)]} = 0$ and $\lim_{n\to\infty} \left\|\mathcal{D}_n^{q_n}\left(t^2,x\right)-x^2\right\|_{2,[0,\alpha_{q_n}(n)]} = 0$. Hence the conditions of (5.25) are fulfilled and we get $\lim_{n\to\infty} \left\|\mathcal{Z}_n^{q}(f)-f\right\|_{2,[0,\alpha_{q_n}(n)]} = 0$ for every $f \in C_2^*$. ■

Now, we find the order of approximation of the functions $f \in C_2^*$ by the operators $\mathcal{Z}_n^q$ with the help of following weighted modulus of continuity (see [153]).

Let

$$\Omega_2(f;\delta) = \sup_{0<h<\delta, x\in[0,a(n)]} \frac{|f(x+h)-f(x)|}{1+(x+h)^2}, \quad \text{for each } f \in C_2^*.$$

The weighted modulus of continuity has the following properties which are similar to usual first modulus of continuity.

Lemma 5.9. *Let $f \in C_2^*$. Then, we have the following:*

(i) $\Omega_2(f;\delta)$ *is a monotone increasing function of* δ.
(ii) *For each* $f \in C_2^*$, $\lim_{\delta\to 0^+} \Omega_2(f;\delta)$.
(iii) *For each* $\lambda > 0$, $\Omega_2(f;\lambda\delta) \leq (1+\lambda)\Omega_2(f;\delta)$.

Now we give the main theorem of this section.

Theorem 5.11. *Let $f \in C_2^*$ and $q = q_n$ satisfies $0 < q_n < 1$ such that $q_n \to 1$ as $n \to \infty$. If $\lim_{n\to\infty} \frac{b_n}{[n]_{q_n}} = 0$, then there exists a positive constant A such that the inequality*

$$\sup_{x\in\left[0,\alpha_{q_n}(n)\right]} \frac{\left|\mathcal{Z}_n^q(f,x) - f(x)\right|}{(1+x^2)^{\frac{3}{2}}} \le A\Omega_2\left(f;\sqrt{a_q(n)}\right)$$

holds, where $a_q(n) = \max\left\{1-q^3, \frac{b_n}{[n]_q}\right\}$ and A is a positive constant.

Proof. For $t \ge 0$, $x \in [0, \alpha_{q_n}(n)]$ and $\delta > 0$, using the definition of $\Omega_2(f;\delta)$ and Lemma 5.9 (iii), we get

$$\begin{aligned} |f(t) - f(x)| &\le \left(1 + (x + |t-x|)^2\right)\left(1 + \frac{|t-x|}{\delta}\right)\Omega_2(f;\delta) \\ &\le 2\left(1+x^2\right)\left(1+(t-x)^2\right)\left(1+\frac{|t-x|}{\delta}\right)\Omega_2(f;\delta). \end{aligned}$$

Since $\mathcal{Z}_n^q$ is linear and positive, we have

$$\begin{aligned} &\left|\mathcal{Z}_n^q(f,x) - f(x)\right| \\ &\le 2\left(1+x^2\right)\Omega_2(f;\delta) \\ &\quad \times \left\{1 + \mathcal{Z}_n^q((t-x)^2,x) + \mathcal{Z}_n^q((1+(t-x)^2)\frac{|t-x|}{\delta},x)\right\}. \end{aligned} \tag{5.26}$$

To estimate the first term of above inequality, using Lemma 5.7, we have

$$\begin{aligned} \mathcal{Z}_n^q\left((t-x)^2, x\right) &= \mathcal{Z}_n^q\left(t^2,x\right) - 2x\mathcal{Z}_n^q(t,x) + x^2\mathcal{Z}_n^q(1,x) \\ &= \left(q^6 - 2q^2 + 1\right)x^2 + \left(q^5 + 2q^4 + q^3 - 2q\right)\frac{b_n}{[n]_q}x \\ &\quad + q^2(1+q)\left(\frac{b_n}{[n]_q}\right)^2 \\ &\le \left(q^6 - 2q^2 + 1\right)x^2 + \frac{2b_n}{[n]_q}x + 2\left(\frac{b_n}{[n]_q}\right)^2 \\ &\le \left(1-q^3\right)^2x^2 + \frac{2b_n}{[n]_q}x + 2\left(\frac{b_n}{[n]_q}\right)^2 \\ &\le A_1\mathcal{O}\left(a_q(n)\right)\left(1+x+x^2\right), \end{aligned} \tag{5.27}$$

where $A_1 > 0$ and $a_q(n) = \max\left\{1-q^3, \frac{b_n}{[n]_q}\right\}$. Since $\lim\limits_{n\to\infty} \frac{b_n}{[n]_{q_n}} = 0$ and $\lim\limits_{n\to\infty} q_n = 1$, there exists a positive constant A_2 such that

$$\mathcal{Z}_n^q\left((t-x)^2, x\right) \le A_2\left(1+x^2\right).$$

To estimate the second term of (5.26), applying the Cauchy–Schwarz inequality, we have

$$\mathcal{Z}_n^q\left(\left(1+(t-x)^2\right)\frac{|t-x|}{\delta}, x\right) \le 2\left(\mathcal{Z}_n^q\left(1+(t-x)^4, x\right)\right)^{\frac{1}{2}}\left(\mathcal{Z}_n^q\left(\frac{(t-x)^2}{\delta^2}, x\right)\right)^{\frac{1}{2}}.$$

Using (5.27) and Lemma 5.8, by direct computation we get

$$\left(\mathcal{Z}_n^q\left(1+(t-x)^4, x\right)\right)^{\frac{1}{2}} \le A_3\left(1+x+x^2\right)$$

and

$$\left(\mathcal{Z}_n^q\left(\frac{(t-x)^2}{\delta^2}, x\right)\right)^{\frac{1}{2}} \le \frac{A_4}{\delta}\mathcal{O}\left(a_q(n)\right)^{\frac{1}{2}}\left(1+x+x^2\right)^{\frac{1}{2}}$$

for $A_3 > 0$ and $A_4 > 0$. If we take $\delta = a_q(n)^{\frac{1}{2}}$, $A = 2\left(1+A_2+2A_3A_4\right)$ and combine above estimates, we have the inequality of the theorem. ■

Now we give an asymptotic formula with respect to weighted norm. The symbol UC_2^2 will stand for the space of all twice-differentiable functions on $[0,\infty)$ with uniformly continuous and bounded second derivative.

Theorem 5.12. *Let $f \in UC_2^2$, $q = q_n$ satisfies $0 < q_n < 1$ such that $q_n \to 1$ as $n \to \infty$ and $\lim\limits_{n\to\infty} \frac{b_n}{[n]_{q_n}} = 0$ then*

$$\lim_{n\to\infty} \frac{1}{\left(1+x^2\right)^2}\left\{\frac{b_n}{[n]_{q_n}}\left(\mathcal{Z}_n^{q_n}(f,x) - f(x) - \left(xf'' + f'\right)\right)\right\} = 0$$

uniformly on $[0,\ \alpha_{q_n}(n)]$. Particularly

$$\lim_{n\to\infty} \frac{b_n}{[n]_{q_n}}\left(\mathcal{Z}_n^{q_n}(f,x) - f(x) - \left(xf'' + f'\right)\right) = 0$$

uniformly on compact subsets of $\alpha_{q_n}(n)$.

Proof. On account of Theorem 1 in [12], we need the show that:

1. $\lim_{n\to\infty} \frac{1}{\left(1+x^2\right)^2}\left(\frac{[n]_{q_n}}{b_n}\mathcal{Z}_n^{q_n}\left((t-x)^2, x\right) - 2x\right) = 0.$
2. $\lim_{n\to\infty} \frac{x^k}{\left(1+x^2\right)^2}\left(\frac{[n]_{q_n}}{b_n}\mathcal{Z}_n^{q_n}\left((t-x), x\right) - 1\right) = 0$, for $k = 0, 1$.

3. $\lim_{n\to\infty} \frac{1}{\left(1+x^2\right)^2} \frac{[n]_{q_n}}{b_n} \mathcal{Z}_n^{q_n} \left((t-x)^4, x\right) = 0.$
4. $\sup_{x\in[0,\alpha_n(q))} \sup_{n\geq 1} \frac{1}{\left(1+x^2\right)^2} \frac{[n]_{q_n}}{b_n} \mathcal{Z}_n^{q_n} \left((t-x)^2, x\right) < \infty.$

Since $\lim_{n\to\infty} \frac{[n]_{q_n}}{b_n} \left(q_n^6 - 2q_n^2 + 1\right) = \lim_{n\to\infty} \frac{1-q_n^n}{b_n} \left(\frac{q_n^6-2q_n^2+1}{1-q_n}\right) = 0$, we have

$$\lim_{n\to\infty} \frac{1}{\left(1+x^2\right)^2} \left(\frac{[n]_{q_n}}{b_n} \mathcal{Z}_n^{q_n} \left((t-x)^2, x\right) - 2x\right)$$

$$\begin{aligned} &= \frac{x^2}{(1+x^2)^2} \lim_{n\to\infty} \frac{[n]_{q_n}}{b_n} \left(q_n^6 - 2q_n^2 + 1\right) \\ &\quad + \frac{x}{(1+x^2)^2} \lim_{n\to\infty} \left(q_n^5 + 2q_n^4 + q_n^3 - 2q_n - 2\right) \\ &\quad + \frac{1}{(1+x^2)^2} \lim_{n\to\infty} q_n^2 (1+q_n) \frac{b_n}{[n]_{q_n}} = 0 \end{aligned}$$

uniformly on $[0,\ \alpha_{q_n}(n)]$. Also, for every $x \in [0,\ \alpha_{q_n}(n)]$

$$\lim_{n\to\infty} \frac{x^k}{\left(1+x^2\right)^2} \left(\frac{[n]_{q_n}}{b_n} \mathcal{Z}_n^{q_n} \left((t-x), x\right) - 1\right)$$

$$\begin{aligned} &= \frac{x^k}{(1+x^2)^2} \lim_{n\to\infty} \frac{[n]_{q_n}}{b_n} \left(q_n^2 - 1\right) x + q_n - 1 \\ &= \frac{x^k}{(1+x^2)^2} \lim_{n\to\infty} \frac{1-q_n^n}{b_n} (q_n - 1) x + q_n - 1 \\ &= 0 \end{aligned}$$

for $k = 0, 1$ uniformly on $[0,\ \alpha_{q_n}(n)]$. Since

$$\lim_{n\to\infty} \frac{[n]_{q_n}}{b_n} \left(q_n^{20} - 4q_n^{12} + 6q^{6_n} - 4q_n^2 + 1\right) = \lim_{n\to\infty} \frac{1-q_n^n}{b_n} \left(\frac{q_n^{20} - 4q_n^{12} + 6q_n^6 - 4q_n^2 + 1}{1-q_n}\right) = 0,$$

we have

$$\lim_{n\to\infty} \frac{1}{(1+x^2)^2} \frac{[n]_{q_n}}{b_n} \mathcal{Z}_n^{q_n} \left((t-x)^4, x\right) = 0.$$

Finally

$$\sup_{x\in[0,\alpha_n(q))}\sup_{n\geq 1}\frac{1}{\left(1+x^2\right)^2}\frac{[n]_{q_n}}{b_n}\mathcal{Z}_n^{q_n}\left((t-x)^2,x\right)$$

$$=\sup_{x\in[0,\alpha_n(q))}\sup_{n\geq 1}\frac{x^2}{\left(1+x^2\right)^2}\left[\frac{[n]_{q_n}}{b_n}\left(q_n^6-2q_n^2+1\right)\right.$$

$$\left.+\left(q_n^5+2q_n^4+q_n^3-2q_n\right)+q_n^2\left(1+q_n\right)\frac{b_n}{[n]_{q_n}}\right]$$

$$\leq\sup_{n\geq 1}\left[\frac{[n]_{q_n}}{b_n}\left(q_n^6-2q_n^2+1\right)\right.$$

$$\left.+\left(q_n^5+2q_n^4+q_n^3-2q_n\right)+q_n^2\left(1+q_n\right)\frac{b_n}{[n]_{q_n}}\right]<\infty,$$

and hence the result follows. ■

5.4 q-Phillips Operators

Phillips [135] defined the well-known linear positive operators

$$P_n(f;x)=n\sum_{k=1}^{\infty}e^{-nx}\frac{n^kx^k}{n!}\int_0^{\infty}e^{-nt}\frac{n^{k-1}t^{k-1}}{n!}f(t)dt+e^{-nx}f(0),$$

where $x\in[0,\infty)$. Some approximation properties of these operators were studied by Gupta and Srivastava [93] and by May [123]. Bézier variant of these Phillips operators was proposed and studied by Gupta [85], where the rate of convergence for the Bézier variant of the Phillips operators for bounded variation functions was discussed. Very recently, Mahmudov in [119] introduced the following q-Szász–Mirakian operator

$$\mathcal{S}_{n,q}(f;x)=\frac{1}{\prod_{j=0}^{\infty}\left(1+(1-q)q^j[n]_qx\right)}\sum_{k=0}^{\infty}f\left(\frac{[k]_q}{q^{k-2}[n]_q}\right)q^{\frac{k(k-1)}{2}}\frac{[n]_q^kx^k}{[k]_q!},$$

where $x\in[0,\infty)$, $0<q<1$, $f\in C[0,\infty)$, and investigated their approximation properties.

Definition 5.1 ([118]). For $f\in R^{[0,\infty)}$, we define the following q-parametric Phillips operators

$$\mathcal{P}_{n,q}(f;x) = [n]_q \sum_{k=1}^{\infty} q^{k-1} S_{n,k}(q;qx) \int_0^{\infty/(1-q)} S_{n,k-1}(q;t) f(t) d_q t + e_q\left(-[n]_q qx\right) f(0), \tag{5.28}$$

where $x \in [0,\infty)$ and $S_{n,k}(q;x) = e_q(-[n]x) q^{\frac{k(k-1)}{2}} \frac{[n]_q^k x^k}{[k]_q!}$.

These operators generalize the sequence of classical Phillips operators.

In this section we present the approximation properties of the q-Phillips operators defined by (5.28), establish some local approximation result for continuous functions in terms of modulus of continuity, and obtain inequalities for the weighted approximation error of q-Phillips operators. Furthermore, we study Voronovskaja-type asymptotic formula for the q-Phillips operators.

5.4.1 Moments

There are two q-analogues of the exponential function e^z; see [104]:

$$e_q(z) = \sum_{k=0}^{\infty} \frac{z^k}{[k]_q!} = \frac{1}{(1-(1-q)z)_q^{\infty}}, \quad |z| < \frac{1}{1-q}, \quad |q| < 1,$$

and

$$E_q(z) = \prod_{j=0}^{\infty} \left(1 + (1-q) q^j z\right) = \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{z^k}{[k]_q!} = (1 + (1-q)z)_q^{\infty}, \quad |q| < 1, \tag{5.29}$$

where $(1-x)_q^{\infty} = \prod_{j=0}^{\infty} \left(1 - q^j x\right)$.

We set

$$\begin{aligned} s_{n,k}(q;x) &= \frac{1}{E_q\left([n]_q x\right)} q^{\frac{k(k-1)}{2}} \frac{[n]_q^k x^k}{[k]_q!} \\ &= e_q\left(-[n]_q x\right) q^{\frac{k(k-1)}{2}} \frac{[n]_q^k x^k}{[k]_q!}, \quad n = 1, 2, \ldots. \end{aligned} \tag{5.30}$$

It is clear that $s_{n,k}(q;x) \geq 0$ for all $q \in (0,1)$ and $x \in [0,\infty)$ and moreover

$$\sum_{k=0}^{\infty} s_{n,k}(q;x) = e_q\left(-[n]_q x\right) \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{\left([n]_q x\right)^k}{[k]_q!} = 1.$$

The two q-gamma functions are defined as

$$\Gamma_q(x) = \int_0^{\frac{1}{1-q}} t^{x-1} E_q(-qt)\, d_q t, \quad \gamma_q^A(x) = \int_0^{\infty/A(1-q)} t^{x-1} e_q(-qt)\, d_q t.$$

For every $A, x > 0$ one has

$$\Gamma_q(x) = K(A;x)\, \gamma_q^A(x), 0$$

where $K(A;x) = \frac{1}{1+A} A^x \left(1 + \frac{1}{A}\right)_q^x (1+A)_q^{1-x}$. In particular for any positive integer n

$$K(A;n) = q^{\frac{n(n-1)}{2}} \quad \text{and} \quad \Gamma_q(n) = q^{\frac{n(n-1)}{2}} \gamma_q^A(n);$$

see [49].

In this section, we will calculate $\mathcal{P}_{n,q}(t^i;x)$ for $i = 0,1,2$. By the definition of q-gamma function γ_q^1, we have

$$\begin{aligned}
\int_0^{\infty/(1-q)} t^s S_{n,k}(q;t) d_q t &= \int_0^{\infty/(1-q)} t^s e_q(-[n]_q t) q^{\frac{k(k-1)}{2}} \frac{[n]_q^k t^k}{[k]_q!} d_q t \\
&= \frac{1}{[n]_q^{s+1}} \frac{1}{[k]_q!} q^{\frac{k(k-1)}{2}} \int_0^{\infty/(1-q)} \left([n]_q t\right)^{k+s} e_q(-[n]_q t) [n]_q d_q t \\
&= \frac{1}{[n]_q^{s+1}} \frac{1}{[k]_q!} q^{\frac{k(k-1)}{2}} \int_0^{\infty/(1-q)} (u)^{k+s} e_q(-u) d_q u \\
&= \frac{1}{[n]_q^{s+1}} \frac{1}{[k]_q!} q^{\frac{k(k-1)}{2}} \gamma_q^1(k+s+1) \\
&= \frac{1}{[n]_q^{s+1}} \frac{1}{[k]_q!} q^{\frac{k(k-1)}{2}} \frac{[k+s]_q!}{q^{(k+s+1)(k+s)/2}} \\
&= \frac{1}{[n]_q^{s+1}} \frac{[k+s]_q!}{[k]_q!} \frac{1}{q^{(2k+s)(s+1)/2}}.
\end{aligned}$$

Lemma 5.10. *We have*

$$\mathcal{P}_{n,q}(1;x) = 1, \ \ \mathcal{P}_{n,q}(t;x) = x,$$

$$\mathcal{P}_{n,q}(t^2;x) = \frac{1}{q^2} x^2 + \frac{(1+q)}{q^2 [n]_q} x,$$

$$\mathcal{P}_{n,q}((t-x)^2;x) = \left(\frac{1}{q^2} - 1\right) x^2 + \frac{(1+q)}{q^2 [n]_q} x.$$

Proof. For $f(t)=1$,

$$\begin{aligned}
\mathcal{P}_{n,q}(1;x) &= [n]_q \sum_{k=1}^{\infty} q^{k-1} S_{n,k}(q;qx) \int_0^{\infty/(1-q)} S_{n,k-1}(q;t)\,d_qt + e_q\left(-[n]_q qx\right) \\
&= [n] \sum_{k=1}^{\infty} q^{k-1} S_{n,k}(q;qx) \frac{1}{[n]_q} \frac{1}{q^{k-1}} + e_q\left(-[n]_q qx\right) \\
&= \sum_{k=1}^{\infty} S_{n,k}(q;qx) + e_q\left(-[n]_q qx\right) = \sum_{k=0}^{\infty} S_{n,k}(q;qx) = 1.
\end{aligned}$$

For $f(t)=t$

$$\begin{aligned}
\mathcal{P}_{n,q}(t;x) &= [n] \sum_{k=1}^{\infty} q^{k-1} S_{n,k}(q;qx) \int_0^{\infty/(1-q)} t S_{n,k-1}(q;t)\,d_qt \\
&= [n] \sum_{k=1}^{\infty} q^{k} S_{n,k}(q;qx) \frac{[k]}{[n]^2} \frac{1}{q^{2k-1}} = \sum_{k=0}^{\infty} S_{n,k}(q;qx) \frac{[k]}{[n]} \frac{1}{q^k} \\
&= \frac{1}{q^2} \sum_{k=0}^{\infty} S_{n,k}(q;qx) \frac{[k]}{[n]} \frac{1}{q^{k-2}} = \frac{1}{q^2} q^2 x = x.
\end{aligned}$$

For $f(t)=t^2$

$$\begin{aligned}
\mathcal{P}_{n,q}(t^2;x) &= [n]_q \sum_{k=1}^{\infty} q^{k-1} S_{n,k}(q;qx) \int_0^{\infty/(1-q)} t^2 S_{n,k-1}(q;t)\,d_qt \\
&= \sum_{k=1}^{\infty} S_{n,k}(q;qx) \frac{[k+1]_q [k]_q}{[n]_q^2} \frac{1}{q^{2k+1}} = \sum_{k=0}^{\infty} S_{n,k}(q;qx) \frac{[k+1]\,[k]}{[n]^2} \frac{1}{q^{2k+1}} \\
&= \sum_{k=0}^{\infty} S_{n,k}(q;qx) \frac{\left([k]_q + q^k\right)[k]_q}{[n]_q^2} \frac{1}{q^{2k+1}} = \sum_{k=0}^{\infty} S_{n,k}(q;qx) \frac{[k]_q^2}{[n]_q^2} \frac{1}{q^{2k+1}} \\
&\quad + \sum_{k=0}^{\infty} S_{n,k}(q;qx) \frac{q^k [k]_q}{[n]_q^2} \frac{1}{q^{2k+1}} \\
&= \frac{1}{q^5} \sum_{k=0}^{\infty} S_{n,k}(q;qx) \frac{[k]_q^2}{[n]_q^2} \frac{1}{q^{2k-4}} + \frac{1}{[n]_q q^3} \sum_{k=0}^{\infty} S_{n,k}(q;qx) \frac{[k]_q}{[n]_q} \frac{1}{q^{k-2}} \\
&= \frac{1}{q^5}\left(q^3x^2 + \frac{q^3}{[n]_q}x\right) + \frac{1}{[n]_q q}x = \frac{1}{q^2}x^2 + \frac{1}{q^2 [n]_q}x + \frac{1}{[n]_q q}x \\
&= \frac{1}{q^2}x^2 + \frac{(1+q)}{q^2 [n]_q}x.
\end{aligned}$$

■

Lemma 5.11. *For all* $0<q<1$ *the following identity holds:*

$$\mathcal{P}_{n,q}(t^m;x)=\frac{1}{[n]_q^m\,q^{(m^2-m)/2}}\sum_{s=1}^{m}C_s(m)\,[n]_q^s\sum_{k=0}^{\infty}\frac{[k]_q^s}{[n]_q^s}\frac{1}{q^{k(m+1)}}S_{n,k}(q;qx).$$

Proof. We have

$$\begin{aligned}\mathcal{P}_{n,q}(t^m;x)&=[n]_q\sum_{k=1}^{\infty}q^{k-1}S_{n,k}(q;qx)\int_0^{\infty/(1-q)}t^mS_{n,k-1}(q;t)d_qt\\&=[n]_q\sum_{k=1}^{\infty}q^{k-1}S_{n,k}(q;qx)\frac{1}{[n]_q^{m+1}}\frac{1}{[k-1]_q!}q^{\frac{(k-1)(k-2)}{2}}\frac{[k-1+m]_q!}{q^{(k+m)(k-1+m)/2}}\\&=\sum_{k=1}^{\infty}\frac{[k-1+m]_q\dots[k]_q}{[n]_q^m}\frac{1}{q^{\left(m^2+2mk+2k-m\right)/2}}S_{n,k}(q;qx)\\&=\sum_{k=0}^{\infty}\frac{[k-1+m]_q\dots[k]_q}{[n]_q^m\,q^{\left(m^2+2mk+2k-m\right)/2}}S_{n,k}(q;qx).\end{aligned}$$

Using $[k+s]_q=[s]_q+q^s\,[k]_q$, we obtain

$$[k]_q\,[k+1]_q\dots[k+m-1]_q=\prod_{s=0}^{m-1}\left([s]_q+q^s\,[k]_q\right)=\sum_{s=1}^{m}C_s\,(m)\,[k]_q^s$$

where $C_s\,(m)>0$, $s=1,2,\dots,m$ are the constants independent of k. Hence

$$\begin{aligned}\mathcal{P}_{n,q}(t^m;x)&=\frac{1}{[n]_q^m\,q^{(m^2-m)/2}}\sum_{k=0}^{\infty}\frac{1}{q^{k(m+1)}}\sum_{s=1}^{m}C_s(m)\,[k]_q^sS_{n,k}(q;qx)\\&=\frac{1}{[n]_q^m\,q^{(m^2-m)/2}}\sum_{k=0}^{\infty}\sum_{s=1}^{m}C_s(m)\,[k]_q^s\frac{1}{q^{k(m+1)}}S_{n,k}(q;qx)\\&=\frac{1}{[n]_q^m\,q^{(m^2-m)/2}}\sum_{s=1}^{m}C_s(m)\,[n]_q^s\sum_{k=0}^{\infty}\left(\frac{[k]_q}{[n]_q}\right)^s\frac{1}{q^{k(m+1)}}S_{n,k}(q;qx).\end{aligned}$$ ■

5.4.2 Direct Results

Let $C_B\,[0,\infty)$ be the space of all real-valued continuous and bounded functions f on $[0,\infty)$, endowed with the norm $\|f\|=\sup_{x\in[0,\infty)}|f(x)|$. The Peetre K-functional is defined by

$$K_2(f;\delta)=\inf_{g\in C^2[0,\infty)}\left\{\|f-g\|+\delta\left\|g''\right\|\right\},$$

where $C_B^2[0,\infty) := \{g \in C_B[0,\infty) : \ g', g'' \in C_B[0,\infty)\}$. By [50, Theorem 2.4] there exists an absolute constant $M > 0$ such that

$$K_2(f,\delta) \le M\omega_2(f;\sqrt{\delta}), \tag{5.31}$$

where $\delta > 0$ and the second-order modulus of smoothness is defined as

$$\omega_2(f;\sqrt{\delta}) = \sup_{0<h\le\delta} \sup_{x\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|,$$

where $f \in C_B[0,\infty)$ and $\delta > 0$. Also, we let

$$\omega(f;\delta) = \sup_{0<h\le\delta} \sup_{x\in[0,\infty)} |f(x+h) - f(x)|.$$

Lemma 5.12. *Let $f \in C_B[0,\infty)$. Then, for all $f \in C_B^2[0,\infty)$, we have*

$$\left|\mathcal{P}_{n,q}(f;x) - f(x)\right| \le \left\{\left(\frac{1-q^2}{q^2}\right)x^2 + \frac{(1+q)}{q^2[n]_q}x\right\}\|f''\|. \tag{5.32}$$

Proof. Let $x \in [0,\infty)$ and $f \in C_B^2[0,\infty)$. Using Taylor's formula

$$f(t) - f(x) = (t-x)f'(x) + \int_x^t (t-u)f''(u)du,$$

we can write

$$\mathcal{P}_{n,q}(f;x) - f(x) = \mathcal{P}_{n,q}((t-x)f'(x);x) + \mathcal{P}_{n,q}(\int_x^t (t-u)f''(u)du;x)$$

$$= f'(x)\mathcal{P}_{n,q}((t-x);x) + \mathcal{P}_{n,q}(\int_x^t (t-u)f''(u)du;x) - \int_x^x (x-u)f''(u)du$$

$$= \mathcal{P}_{n,q}(\int_x^t (t-u)f''(u)du;x).$$

On the other hand, since

$$\left|\int_x^t (t-u)f''(u)du\right| \le \int_x^t |t-u|\,|f''(u)|\,du \le \|f''\| \int_x^t |t-u|\,du \le (t-x)^2\,\|f''\|.$$

we conclude that

$$\begin{aligned}\left|\mathcal{P}_{n,q}(f;x)-f(x)\right| &= \left|\mathcal{P}_{n,q}(\int_x^t (t-u)g''(u)du;x)\right| \\ &\leq \mathcal{P}_{n,q}((t-x)^2 \left\|f''\right\|;x) \\ &= \left\{\left(\frac{1-q^2}{q^2}\right)x^2+\frac{(1+q)}{q^2[n]_q}x\right\}\left\|f''\right\|. \end{aligned}$$ ■

Lemma 5.13. *For $f \in C[0,\infty)$, we have*

$$\left\|\mathcal{P}_{n,q}f\right\| \leq \|f\|.$$

Theorem 5.13. *Let $f \in C_B[0,\infty)$. Then, for every $x \in [0,\infty)$, there exists a constant $M > 0$ such that*

$$\left|\mathcal{P}_{n,q}(f;x)-f(x)\right| \leq M\omega_2(f;\sqrt{\delta_n(x)}),$$

where

$$\delta_n(x)=\left(\frac{1-q^2}{q^2}\right)x^2+\frac{(1+q)}{q^2[n]_q}x.$$

Proof. Now, taking into account boundedness of $\mathcal{P}_{n,q}$, we get

$$\begin{aligned}\left|\mathcal{P}_{n,q}(f;x)-f(x)\right| &= \left|\mathcal{P}_{n,q}(f;x)-\mathcal{P}_{n,q}(g,x)-f(x)+g(x)+\mathcal{P}_{n,q}(g,x)-g(x)\right| \\ &\leq \left|\mathcal{P}_{n,q}(f-g;x)-(f-g)(x)\right|+\left|\mathcal{P}_{n,q}(g;x)-g(x)\right| \\ &\leq .\left|\mathcal{P}_{n,q}(f-g;x)+(f-g)(x)\right|+\left|\mathcal{P}_{n,q}(g;x)-g(x)\right| \\ &\leq 2\|f-g\|+\left\{\left(\frac{1-q^2}{q^2}\right)x^2+\frac{(1+q)}{q^2[n]_q}x\right\}\left\|g''\right\| \\ &\leq 2\left(\|f-g\|+\delta_n(x)\left\|g''\right\|\right). \end{aligned}$$

Now, taking infimum on the right-hand side over all $g \in C_B^2[0,\infty)$ and using (5.31), we get the following result

$$\left|\mathcal{P}_{n,q}(f;x)-f(x)\right| \leq 2K_2(f;\delta_n(x)) \leq 2M\omega_2(f;\sqrt{\delta_n(x)}).$$ ■

Theorem 5.14. *Let $0<\alpha\leq 1$ and E be any subset of the interval $[0,\infty)$. Then, if $f \in C_B[0,\infty)$ is locally Lip(α), i.e., the condition*

$$|f(y)-f(x)| \leq L|y-x|^\alpha,\ y \in E \text{ and } x \in [0,\infty) \tag{5.33}$$

holds, then, for each $x \in [0,\infty)$, we have

$$\left|\mathcal{P}_{n,q}(f;x)-f(x)\right|\leq L\left\{\delta_n^{\frac{\alpha}{2}}(x)+2\left(d\left(x,E\right)\right)^{\alpha}\right\},$$

where L is a constant depending on α and f ; and $d(x,E)$ is the distance between x and E defined as

$$d(x,E)=\inf\{|t-x|:t\in E\}.$$

Proof. Let $\overline{E}$ denote the closure of E in $[0,\infty)$. Then, there exists a point $x_0\in\overline{E}$ such that $|x-x_0|=d(x,E)$. Using the triangle inequality

$$|f(t)-f(x)|\leq|f(t)-f(x_0)|+|f(x)-f(x_0)|$$

we get, by (5.33)

$$\begin{aligned}\left|\mathcal{P}_{n,q}(f;x)-f(x)\right| &\leq \mathcal{P}_{n,q}(|f(y)-f(x_0)|;x)+\mathcal{P}_{n,q}(|f(x)-f(x_0)|;x)\\ &\leq L\left\{\mathcal{P}_{n,q}(|t-x_0|^{\alpha};x)+|x-x_0|^{\alpha}\right\}\\ &\leq L\left\{\mathcal{P}_{n,q}(|t-x|^{\alpha}+|x-x_0|^{\alpha};x)+|x-x_0|^{\alpha}\right\}\\ &= L\left\{\mathcal{P}_{n,q}(|t-x|^{\alpha};x)+2|x-x_0|^{\alpha}\right\}.\end{aligned}$$

Using the Hölder inequality with $p=\frac{2}{\alpha}$, $q=\frac{2}{2-\alpha}$, we find that

$$\begin{aligned}\left|\mathcal{P}_{n,q}(f;x)-f(x)\right| &\leq L\left\{\left[\mathcal{P}_{n,q}(|t-x|^{\alpha p};x)\right]^{\frac{1}{p}}\left[\mathcal{P}_{n,q}(1^q;x)\right]^{\frac{1}{q}}+2\left(d(x,E)\right)^{\alpha}\right\}\\ &= M\left\{\left[\mathcal{P}_{n,q}(|t-x|^2;x)\right]^{\frac{\alpha}{2}}+2\left(d(x,E)\right)^{\alpha}\right\}\\ &\leq M\left\{\left[\left(\frac{1-q^2}{q^2}\right)x^2+\frac{(1+q)}{q^2[n]_q}x\right]^{\frac{\alpha}{2}}+2\left(d(x,E)\right)^{\alpha}\right\}\\ &= M\left\{\delta_n^{\frac{\alpha}{2}}(x)+2\left(d(x,E)\right)^{\alpha}\right\}.\end{aligned}$$

■

We consider the following classes of functions:

$$C_m[0,\infty):=\left\{f\in C[0,\infty):\exists M_f>0\ |f(x)|<M_f(1+x^m)\ \text{ and }\ \|f\|_m:=\sup_{x\in[0,\infty)}\frac{|f(x)|}{1+x^m}\right\}.$$

$$C_m^*[0,\infty):=\left\{f\in C_m[0,\infty):\lim_{x\to\infty}\frac{|f(x)|}{1+x^m}<\infty\right\},\quad m\in\mathbb{N}.$$

Next, we obtain a direct approximation theorem in $C_1^*[0,\infty)$ and an estimation in terms of the weighted modulus of continuity. It is known that if f is not uniformly continuous on the interval $[0,\infty)$, then the usual first modulus of continuity $\omega(f,\delta)$

does not tend to zero, as $\delta \to 0$. For every $f \in C_m^*[0,\infty)$ the weighted modulus of continuity is defined as follows

$$\Omega_m(f,\delta) = \sup_{x\geq 0,\ 0<h\leq\delta} \frac{|f(x+h)-f(x)|}{1+(x+h)^m}.$$

See [112].

Lemma 5.14 ([112]). *Let $f \in C_m^*[0,\infty)$, $m \in \mathbb{N}$. Then, we have the following:*

1. *$\Omega_m(f,\delta)$ is a monotone increasing function of δ.*
2. $\lim_{\delta\to 0^+}\Omega_m(f,\delta) = 0$.
3. *For any $\alpha \in [0,\infty)$, $\Omega_m(f,\alpha\delta) \leq (1+\alpha)\Omega_m(f,\delta)$.*

In the next theorem we give an expression of the approximation error with the operators $S_{n,q}$ by means of Ω_1.

Theorem 5.15. *If $f \in C_1^*[0,\infty)$, then the inequality*

$$\left\|\mathcal{P}_{n,q}(f)-f\right\|_2 \leq k(q)\,\Omega_1\left(f;\frac{1}{\sqrt{[n]_q}}\right),$$

where k is a constant independent of f and n.

Proof. From the definition of $\Omega_1(f,\delta)$ and Lemma 5.14, we may write

$$\begin{aligned}|f(t)-f(x)| &\leq (1+x+|t-x|)\left(\frac{|t-x|}{\delta}+1\right)\Omega_1(f,\delta)\\ &\leq (1+2x+t)\left(\frac{|t-x|}{\delta}+1\right)\Omega_1(f,\delta).\end{aligned}$$

Then

$$\begin{aligned}\left|\mathcal{P}_{n,q}(f;x)-f(x)\right| \leq \mathcal{P}_{n,q}(|f(t)-f(x)|;x) &\leq \Omega_1(f,\delta)\,(\mathcal{P}_{n,q}((1+2x+t);x)\\ &+\mathcal{P}_{n,q}\left((1+2x+t)\frac{|t-x|}{\delta};x\right)\Big).\end{aligned}$$

Applying the Cauchy–Schwarz inequality to the second term, we get

$$\mathcal{P}_{n,q}\left((1+2x+t)\frac{|t-x|}{\delta};x\right) \leq \left(\mathcal{P}_{n,q}\left((1+2x+t)^2;x\right)\right)^{1/2}\left(\mathcal{P}_{n,q}\left(\frac{|t-x|^2}{\delta^2};x\right)\right)^{1/2}.$$

Consequently

$$\begin{aligned}\left|\mathcal{P}_{n,q}(f;x)-f(x)\right| \le \Omega_m(f,\delta)\,&(\mathcal{P}_{n,q}((1+2x+t);x)\\&+\left(\mathcal{P}_{n,q}\left((1+2x+t)^2;x\right)\right)^{1/2}\left(\mathcal{P}_{n,q}\left(\frac{|t-x|^2}{\delta^2};x\right)\right)^{1/2}\Bigg).\end{aligned} \tag{5.34}$$

On the other hand, there is a positive constant $K(q)$ such that

$$\mathcal{P}_{n,q}((1+2x+t);x) = 1+3x \le 3(1+x),$$

$$\begin{aligned}\left(\mathcal{P}_{n,q}\left((1+2x+t)^2;x\right)\right)^{1/2} &= \left(\left((1+2x)^2+(1+2x)x+\frac{1}{q^2}x^2+\frac{(1+q)}{q^2[n]_q}x;x\right)\right)^{1/2}\\&\le K(q)(1+x),\end{aligned} \tag{5.35}$$

and

$$\begin{aligned}\left(\mathcal{P}_{n,q}\left(\frac{|t-x|^2}{\delta^2};x\right)\right)^{1/2} &= \frac{1}{\delta q}\sqrt{(1-q^2)x^2+\frac{(1+q)}{[n]_q}x} \le \frac{1}{\delta q}\sqrt{\frac{(1-q^n)}{[n]_q}x^2+\frac{2x}{[n]_q}}\\&\le \frac{2}{\delta q\sqrt{[n]_q}}\sqrt{x^2+x} \le \frac{2}{\delta q\sqrt{[n]_q}}(1+x).\end{aligned} \tag{5.36}$$

Now from (5.34)–(5.36), we have

$$\begin{aligned}\left|\mathcal{P}_{n,q}(f;x)-f(x)\right| &\le \Omega_1(f,\delta)\left(3(1+x)+K(q)\frac{2(1+x)^2}{q\delta\sqrt{[n]_q}}\right)\\&\le \left(1+x^2\right)\Omega_1(f,\delta)\left(3K_1+K(q)\frac{4}{q\delta\sqrt{[n]_q}}\right),\end{aligned}$$

where

$$K_1 = \sup_{x\ge 0}\frac{1+x^m+x+x^{m+1}}{1+x^{m+1}}.$$

If we take $\delta = [n]^{q-\frac{1}{2}}$, then from the above inequality we obtain the desired result. ■

5.4.3 Voronovskaja-Type Theorem

In this section, we proceed to state and prove a Voronovskaja-type theorem for the q-Phillips operators. We first prove the following lemma:

Lemma 5.15. *Let* $0 < q < 1$. *We have*

$$\mathcal{P}_{n,q}(t^3;x) = \frac{1}{q^6}x^3 + \frac{[2]_q\,[3]_q}{[n]_q\,q^6}x^2 + \frac{[2]\,[3]}{[n]^2\,q^5}x$$

$$\mathcal{P}_{n,q}(t^4;x) = \frac{1}{q^{12}}x^4 + \frac{[2]_q\,[3]_q\,(1+q^2)}{[n]_q\,q^{12}}x^3 + \frac{[2]_q\,[3]_q^2\,(1+q^2)}{[n]_q^2\,q^{11}}x^2 + \frac{[2]_q^2\,[3]_q\,(1+q^2)}{[n]_q^3\,q^9}x.$$

Proof. Simple calculations show that

$$\begin{aligned}
\mathcal{P}_{n,q}(t^3;x) &= \frac{1}{[n]_q^3\,q^3}\sum_{k=0}^{\infty}\frac{[k+2]_q\,[k+1]_q\,[k]_q}{q^{3k}}S_{n,k}(q;qx)\\
&= \frac{1}{[n]_q^3\,q^3}\sum_{k=0}^{\infty}\frac{[k]_q^3 + q^k\,(2+q)\,[k]_q^2 + q^{2k}\,(1+q)\,[k]_q}{q^{3k}}S_{n,k}(q;qx)\\
&= \frac{1}{[n]_q^3\,q^3}\left\{\sum_{k=0}^{\infty}\frac{[k]_q^3}{q^{3k}}S_{n,k}(q;qx) + \sum_{k=0}^{\infty}\frac{(2+q)\,[k]_q^2}{q^{2k}}S_{n,k}(q;qx)\right.\\
&\quad\left. + \sum_{k=0}^{\infty}\frac{(1+q)\,[k]_q}{q^k}S_{n,k}(q;qx)\right\}\\
&= \frac{1}{q^3}\sum_{k=0}^{\infty}\frac{[k]_q^3}{[n]_q^3\,q^{3k}}S_{n,k}(q;qx) + \frac{(2+q)}{[n]_q\,q^3}\sum_{k=0}^{\infty}\frac{[k]_q^2}{[n]_q^2\,q^{2k}}S_{n,k}(q;qx)\\
&\quad + \frac{(1+q)}{[n]_q^2\,q^3}\sum_{k=0}^{\infty}\frac{[k]_q}{[n]_q\,q^k}S_{n,k}(q;qx)\\
&= \frac{1}{q^9}\sum_{k=0}^{\infty}\frac{[k]_q^3}{[n]_q^3\,q^{3k-6}}S_{n,k}(q;qx) + \frac{(2+q)}{[n]_q\,q^7}\sum_{k=0}^{\infty}\frac{[k]_q^2}{[n]_q^2\,q^{2k-4}}S_{n,k}(q;qx)\\
&\quad + \frac{(1+q)}{[n]_q^2\,q^5}\sum_{k=0}^{\infty}\frac{[k]_q}{[n]_q\,q^{k-2}}S_{n,k}(q;qx)
\end{aligned}$$

$$= \frac{1}{q^9}\sum_{k=0}^{\infty}\left(\frac{[k]_q}{[n]_q\,q^{k-2}}\right)^3 S_{n,k}(q;qx) + \frac{(2+q)}{[n]_q\,q^7}\sum_{k=0}^{\infty}\left(\frac{[k]_q}{[n]_q\,q^{k-2}}\right)^2 S_{n,k}(q;qx)$$

$$+\frac{(1+q)}{[n]_q^2 q^5}\sum_{k=0}^{\infty}\frac{[k]_q}{[n]_q q^{k-2}}S_{n,k}(q;qx)$$

$$=\frac{1}{q^9}\left(\frac{q^4}{[n]_q^2}x+\left(2q^4+q^3\right)\frac{x^2}{[n]_q}+q^3x^3\right)+\frac{(2+q)}{[n]_q q^7}\left(q^3x^2+\frac{q^3}{[n]_q}x\right)+\frac{(1+q)\,q^2}{[n]_q^2 q^5}x$$

$$=\frac{1}{q^5\,[n]_q^2}x+\frac{(2q+1)}{q^6\,[n]_q}x^2+\frac{1}{q^6}x^3+\frac{(2+q)}{[n]_q q^4}x^2+\frac{(2+q)}{[n]_q^2 q^4}x+\frac{(1+q)}{[n]_q^2 q^3}x$$

$$=\frac{1}{q^6}x^3+\frac{(1+2q+2q^2+q^3)}{q^6\,[n]_q}x^2+\frac{(1+2q+2q^2+q^3)}{q^5\,[n]_q^2}x$$

$$=\frac{1}{q^6}x^3+\frac{(1+q)(1+q+q^2)}{[n]_q q^6}x^2+\frac{(1+q)(1+q+q^2)}{[n]_q^2 q^5}x$$

$$\mathcal{P}_{n,q}(t^4;x)$$

$$=\frac{1}{[n]_q^4 q^6}\sum_{k=0}^{\infty}\frac{[k+3]_q\,[k+2]_q\,[k+1]_q\,[k]_q}{q^{4k}}S_{n,k}(q;qx)$$

$$=\frac{1}{q^{14}}\sum_{k=0}^{\infty}\left(\frac{[k]_q}{[n]_q q^{k-2}}\right)^4 S_{n,k}(q;qx)+\frac{\left(3+2q+q^2\right)}{[n]_q q^{12}}\sum_{k=0}^{\infty}\left(\frac{[k]_q}{[n]_q q^{k-2}}\right)^3 S_{n,k}(q;qx)$$

$$+\frac{\left(3+4q+3q^2+q^3\right)}{[n]_q^2 q^{10}}\sum_{k=0}^{\infty}\left(\frac{[k]_q}{[n]_q q^{k-2}}\right)^2 S_{n,k}(q;qx)$$

$$+\frac{\left(1+2q+2q^2+q^3\right)}{[n]_q^3 q^8}\sum_{k=0}^{\infty}\frac{[k]_q}{[n]_q q^{k-2}}S_{n,k}(q;qx)$$

$$=\frac{1}{q^{14}}\left(\frac{q^5}{[n]_q^3}x+\left(3q^3+3q^2+q\right)\frac{q^2}{[n]_q^2}x^2+\left(3q+2+\frac{1}{q}\right)\frac{q^3}{[n]_q}x^3+q^2x^4\right)$$

$$+\frac{\left(3+2q+q^2\right)}{[n]_q q^{12}}\left(\frac{q^4}{[n]_q^2}x+\left(2q^2+q\right)\frac{q^2}{[n]_q}x^2+q^3x^3\right)$$

$$+\frac{\left(3+4q+3q^2+q^3\right)}{[n]_q^2 q^{10}}\left(q^3x^2+\frac{q^3}{[n]_q}x\right)+\frac{\left(1+2q+2q^2+q^3\right)q^2}{[n]_q^3 q^8}x$$

$$=\frac{1}{q^{12}}x^4+\frac{1+2q+3q^2+(3+2q+q^2)q^3}{[n]_q q^{12}}x^3$$

$$+\frac{1+3q+3q^2+(3+2q+q^2)(2q+1)q^2+(3+4q+3q^2+q^3)q^4}{[n]_q^2 q^{11}}x^2$$

$$+\frac{1+(3+2q+q^2)q+(3+4q+3q^2+q^3)q^2+(1+2q+2q^2+q^3)q^3}{[n]_q^3 q^9}x$$

$$=\frac{1}{q^{12}}x^4+\frac{(1+q)(1+q^2)(1+q+q^2)}{[n]_q q^{12}}x^3+\frac{(1+q)(1+q^2)(1+q+q^2)^2}{[n]_q^2 q^{11}}x^2$$

$$+\frac{(1+q)^2(1+q^2)(1+q+q^2)}{[n]_q^3 q^9}x.$$ ■

Theorem 5.16. *Let $q_n \in (0,1)$. Then the sequence $\{\mathcal{P}_{n,q_n}(f)\}$ converges to f uniformly on $[0,A]$ for each $f \in C_2^*[0,\infty)$ if and only if $\lim_{n\to\infty} q_n = 1$.*

Proof. The proof is similar to that of Theorem 2 [86]. ■

Lemma 5.16. *Assume that $q_n \in (0,1)$, $q_n \to 1$, and $q_n^n \to a$ as $n \to \infty$. For every $x \in [0,\infty)$ there hold*

$$\lim_{n\to\infty}[n]_{q_n}\mathcal{P}_{n,q_n}((t-x)^2;x)=2(1-a)x^2+2x,$$

$$\lim_{n\to\infty}[n]_{q_n}^2\mathcal{P}_{n,q_n}((t-x)^4;x)=12x^2+24(1-a)x^3+12(1-a)^2x^4.$$

Proof. First, we have

$$\begin{aligned}\lim_{n\to\infty}[n]_{q_n}\mathcal{P}_{n,q_n}((t-x)^2;x)&=\lim_{n\to\infty}[n]_{q_n}\left\{\left(\frac{1}{q_n^2}-1\right)x^2+\frac{(1+q_n)}{q_n^2[n]_{q_n}}x\right\}\\&=\lim_{n\to\infty}\left(\frac{(1-q_n^n)(1+q_n)}{q_n^2}x^2+\frac{(1+q_n)}{q_n^2}x\right)\\&=2(1-a)x^2+2x.\end{aligned}$$

In order to calculate the second limit, we need expression for $\mathcal{P}_{n,q_n}((t-x)^4;x)$:

$$\begin{aligned}&\mathcal{P}_{n,q_n}((t-x)^4;x)\\&=\mathcal{P}_{n,q_n}(t^4;x)-4x\mathcal{P}_{n,q_n}(t^3;x)+6x^2\mathcal{P}_{n,q_n}(t^2;x)-4x^3\mathcal{P}_{n,q_n}(t;x)+x^4\\&=\frac{1}{q_n^{12}}x^4+\frac{[2]_{q_n}[3]_{q_n}(1+q_n^2)}{[n]_{q_n}q_n^{12}}x^3+\frac{[2]_{q_n}[3]_{q_n}^2(1+q_n^2)}{[n]_{q_n}^2q_n^{11}}x^2+\frac{[2]_{q_n}^2[3]_{q_n}(1+q_n^2)}{[n]_{q_n}^3q_n^9}x\end{aligned}$$

$$-4x\left\{\frac{1}{q_n^6}x^3+\frac{[2]_{q_n}[3]_{q_n}}{[n]_{q_n}q_n^6}x^2+\frac{[2]_{q_n}[3]_{q_n}}{[n]_{q_n}^2q_n^5}x\right\}+6x^2\left\{\frac{1}{q_n^2}x^2+\frac{[2]_{q_n}}{q_n^2[n]_{q_n}}x\right\}-3x^4$$

$$=\frac{(1-4q_n^6+6q_n^{10}-3q_n^{12})}{q_n^{12}}x^4+\left\{\frac{[2]_{q_n}[3]_{q_n}(1+q_n^2)-4[2]_{q_n}[3]_{q_n}q_n^6+6q_n^{10}[2]_{q_n}}{q_n^{12}[n]_{q_n}}\right\}x^3$$

$$+\left\{\frac{[2]_{q_n}[3]_{q_n}^2(1+q_n^2)-4q_n^6[2]_{q_n}[3]_{q_n}}{q_n^{11}[n]_{q_n}^2}\right\}x^2$$

$$+\frac{[2]_{q_n}^2[3]_{q_n}(1+q_n^2)}{[n]_{q_n}^3q_n^9}x$$

$$=\frac{\left(1+2q_n^2+3q_n^4-3q_n^8\right)(1-q_n^n)^2(q_n+1)^2}{q_n^{12}[n]_{q_n}^2}x^4$$

$$+\left\{\frac{(q_n^n-1)(q_n+1)\left(2q_n^7-4q_n^2-5q_n^3-6q_n^4-6q_n^5-2q_n^6-2q_n+6q_n^8+6q_n^9-1\right)}{q_n^{12}[n]_{q_n}^2}\right\}x^3$$

$$+\left\{\frac{[2]_{q_n}[3]_{q_n}^2(1+q_n^2)-4q_n^6[2]_{q_n}[3]_{q_n}}{q_n^{11}[n]_{q_n}^2}\right\}x^2+\frac{[2]_{q_n}^2[3]_{q_n}(1+q_n^2)}{[n]_{q_n}^3q_n^9}x.$$

Thus

$$\lim_{n\to\infty}[n]_{q_n}^2\mathcal{P}_{n,q_n}((t-x)^4;x)$$

$$=\lim_{n\to\infty}\frac{(1-q_n^n)^2}{(1-q_n)^2}\left\{\frac{\left(2q_n^2+3q_n^4-3q_n^8+1\right)(q_n-1)^2(q_n+1)^2}{q_n^{12}}x^4\right.$$

$$+\left(\frac{(q_n-1)(q_n+1)\left(2q_n^7-4q_n^2-5q_n^3-6q_n^4-6q_n^5-2q_n^6-2q_n+6q_n^8+6q_n^9-1\right)}{q_n^{12}[n]_{q_n}}\right)x^3$$

$$+\left(\frac{(q_n+1)\left(q_n+2q_n^2+q_n^3+q_n^4-4q_n^6+1\right)\left(q_n+q_n^2+1\right)}{q_n^{11}[n]_{q_n}^2}\right)x^2$$

$$\left.+\left(\frac{(1+q_n)^2(1+q_n^2)(1+q_n+q_n^2)}{q_n^9[n]_{q_n}^3}\right)x\right\}$$

$$=12(1-a)^2x^4+24(1-a)x^3+12x^2.$$ ■

Theorem 5.17. *Assume that* $q_n \in (0,1)$*,* $q_n \to 1$*, and* $q_n^n \to a$ *as* $n \to \infty$*. For any* $f \in C_2^*[0,\infty)$ *such that* $f', f'' \in C_2^*[0,\infty)$*, the following equality holds*

$$\lim_{n\to\infty} [n]_{q_n} \left(\mathcal{P}_{n,q_n}(f;x) - f(x)\right) = \left((1-a)x^2 + x\right) f''(x)$$

uniformly on any $[0,A]$*,* $A > 0$*.*

Proof. Let $f, f', f'' \in C_2^*[0,\infty)$ and $x \in [0,\infty)$ be fixed. By the Taylor formula we may write

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2} f''(x)(t-x)^2 + r(t;x)(t-x)^2, \tag{5.37}$$

where $r(t;x)$ is the Peano form of the remainder, $r(.;x) \in C_2^*[0,\infty)$, and $\lim_{t\to x} r(t;x) = 0$. Applying $\mathcal{P}_{n,q_n}$ to (5.37) we obtain

$$\begin{aligned}[n]_{q_n} \left(\mathcal{P}_{n,q_n}(f;x) - f(x)\right) &= \frac{1}{2} f''(x) [n]_{q_n} \mathcal{P}_{n,q_n}\left((t-x)^2;x\right) \\ &\quad + [n]_{q_n} \mathcal{P}_{n,q_n}\left(r(t;x)(t-x)^2;x\right).\end{aligned}$$

By the Cauchy–Schwarz inequality, we have

$$\mathcal{P}_{n,q_n}\left(r(t;x)(t-x)^2;x\right) \le \sqrt{\mathcal{P}_{n,q_n}\left(r^2(t;x);x\right)} \sqrt{\mathcal{P}_{n,q_n}\left((t-x)^4;x\right)}. \tag{5.38}$$

Observe that $r^2(x;x) = 0$ and $r^2(.;x) \in C_2^*[0,\infty)$. Then it follows from Theorem 5.16 that

$$\lim_{n\to\infty} \mathcal{P}_{n,q_n}\left(r^2(t;x);x\right) = r^2(x;x) = 0 \tag{5.39}$$

uniformly with respect to $x \in [0,A]$. Now from (5.38) and (5.39) and Lemma 5.16, we get immediately

$$\lim_{n\to\infty} [n]_{q_n} \mathcal{P}_{n,q_n}\left(r(t;x)(t-x)^2;x\right) = 0.$$

Then we get the following

$$\begin{aligned}&\lim_{n\to\infty} [n]_{q_n} \left(\mathcal{P}_{n,q_n}(f;x) - f(x)\right) \\ &= \lim_{n\to\infty} \left(\frac{1}{2} f''(x) [n]_{q_n} \mathcal{P}_{n,q_n}\left((t-x)^2;x\right) + [n]_{q_n} \mathcal{P}_{n,q_n}\left(r(t;x)(t-x)^2;x\right)\right) \\ &= \left((1-a)x^2 + x\right) f''(x).\end{aligned}$$

■