Chapter 4
q-Bernstein-Type Integral Operators

4.1 Introduction

In order to approximate integrable functions on the interval $[0,1]$, Kantorovich gave modified Bernstein polynomials. Later in the year 1967 Durrmeyer [58] considered a more general integral modification of the classical Bernstein polynomials, which were studied first by Derriennic [47]. Also some other generalizations of the Bernstein polynomials are available in the literature. The other most popular generalization as considered by Goodman and Sharma [82], namely, genuine Bernstein–Durrmeyer operators. In this chapter we discuss the q analogues of various integral modifications of Bernstein polynomials. The results were discussed in recent papers [45, 62, 86, 89, 92, 94, 121], etc.

4.2 *q*-Bernstein–Kantorovich Operators

Recently, Dalmanoglu [45] proposed the q-Kantorovich–Bernstein operators as

$$K_{n,q}(f,x) = [n+1]_q \sum_{k=0}^{n} p_{n,k}(q;x) \int_{[k]_q/[n+1]_q}^{[k+1]_q/[n+1]_q} f(t)d_qt, \;\; x \in [0,1] \tag{4.1}$$

where

$$p_{n,k}(q;x) := \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1} (1-q^s x).$$

In case $q = 1$, the operators (4.1) reduce to well-known Bernstein–Kantorovich operators

$$K_n(f,x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt, \;\; x \in [0,1]$$

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where $p_{n,k}(x)$ is the Bernstein basis function given by

$$p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}.$$

4.2.1 Direct Results

For the operators (4.1), Dalmanoglu [45] obtained the following theorems:

Theorem 4.1. *If the sequence* (q_n) *satisfies the conditions* $\lim_{n\to\infty} q_n = 1$ *and* $\lim_{n\to\infty} \frac{1}{[n]_{q_n}} = 0$ *and* $0 < q_n < 1$*, then*

$$||K_{n,q}(f,x) - f|| \to 0, n \to \infty,$$

for every $f \in C[0,a]$, $0 < a < 1$.

Proof. First, we have

$$K_{n,q}(1,x) = [n+1]_q \sum_{k=0}^{n} q^{-k} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1} (1-q^s x) \int_{[k]_q/[n+1]_q}^{[k+1]_q/[n+1]_q} d_q t.$$

Also by definition of q-integral

$$\int_{[k]_q/[n+1]_q}^{[k+1]_q/[n+1]_q} d_q t = \int_0^{[k+1]_q/[n+1]_q} d_q t - \int_0^{[k]_q/[n+1]_q} d_q t$$

$$= (1-q)\frac{[k+1]_q}{[n+1]_q} \sum_{j=0}^{\infty} q^j - (1-q)\frac{[k]_q}{[n+1]_q} \sum_{j=0}^{\infty} q^j$$

$$= \frac{1-q}{[n+1]_q}([k+1]_q - [k]_q) \sum_{j=0}^{\infty} q^j = \frac{q^k}{[n+1]_q}.$$

Thus $K_{n,q}(1,x) = 1$. Next

$$K_{n,q}(t,x) = [n+1]_q \sum_{k=0}^{n} q^{-k} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1} (1-q^s x) \int_{[k]_q/[n+1]_q}^{[k+1]_q/[n+1]_q} t\, d_q t.$$

Again by definition of q-integral

$$\int_{[k]_q/[n+1]_q}^{[k+1]_q/[n+1]_q} t d_q t = \int_0^{[k+1]_q/[n+1]_q} t d_q t - \int_0^{[k]_q/[n+1]_q} t d_q t$$

$$= (1-q)\frac{[k+1]_q}{[n+1]_q}\sum_{j=0}^{\infty} q^{2j}\frac{[k+1]_q}{[n+1]_q} - (1-q)\frac{[k]_q}{[n+1]_q}\sum_{j=0}^{\infty} q^{2j}\frac{[k]_q}{[n+1]_q}$$

$$= \frac{1-q}{[n+1]_q^2}([k+1]_q^2 - [k]_q^2)\sum_{j=0}^{\infty} q^{2j} = \frac{q^k}{[n+1]_q^2}\frac{1}{1+q}([k]_q(1+q)+1).$$

Therefore

$$K_{n,q}(t,x) = [n+1]_q \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1}(1-q^s x)\frac{1}{[n+1]_q^2}\frac{1}{1+q}([k]_q(1+q)+1)$$

$$\frac{[n]_q}{[n+1]_q}x + \frac{1}{1+q}\frac{1}{[n+1]_q}.$$

To estimate $K_{n,q}(t^2,x)$, we have

$$\int_{[k]_q/[n+1]_q}^{[k+1]_q/[n+1]_q} t^2 d_q t = \int_0^{[k+1]_q/[n+1]_q} t^2 d_q t - \int_0^{[k]_q/[n+1]_q} t^2 d_q t$$

$$= \frac{1}{[n+1]_q^3}\frac{1}{1+q+q^2}(q^k[k+1]_q^2 + [k]_q[k+1]_q + [k]_q^2).$$

Therefore using $[k+1]_q = q[k]_q + 1$ and using the similar methods as above, we have

$$K_{n,q}(t^2,x) = \frac{[n]_q[n-1]_q}{[n+1]_q^2}\frac{q^3+q^2+q}{1+q+q^2}x^2 + \frac{[n]_q}{[n+1]_q^2}\frac{q^2+3q+2}{1+q+q^2}x + \frac{1}{[n+1]_q^2}\frac{1}{1+q+q^2}.$$

Replacing q by a sequence $\{q_n\}$ such that $\lim_{n\to\infty} q_n = 1$, it is easily seen that $K_{n,q}(t^i,x), i=0,1,2$ converges uniformly to t^i. Thus the result follows by Korovkin's theorem. ■

Theorem 4.2. *If the sequence* (q_n) *satisfies the conditions* $\lim_{n\to\infty} q_n = 1$ *and* $\lim_{n\to\infty}\frac{1}{[n]_{q_n}} = 0$ *and* $0 < q_n < 1$, *then*

$$|K_{n,q}(f,x) - f(x)| \le 2\omega(f,\sqrt{\delta_n}),$$

for all $f \in C[0,a]$ *and* $\delta_n = K_{n,q}((t-x)^2,x)$.

Proof. Let $f \in C[0,a]$. From the linearity and monotonicity of $K_{n,q}(f,x)$, we can write

$$|K_{n,q}(f,x)-f(x)| \leq K_{n,q}(|f(t)-f(x)|,x)$$
$$= [n+1]_q \sum_{k=0}^{n} q^{-k} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1}(1-q^s x) \int_{[k]_q/[n+1]_q}^{[k+1]_q/[n+1]_q} |f(t)-f(x)| d_q t.$$

On the other hand

$$|f(t)-f(x)| \leq \omega(f,|t-x|).$$

If $|t-x| < \delta$, it is obvious that

$$|f(t)-f(x)| \leq \left(1+\frac{(t-x)^2}{\delta^2}\right)\omega(f,\delta) \tag{4.2}$$

If $|t-x| > \delta$, we use the property of modulus of continuity

$$\omega(f,\lambda\delta) \leq (1+\lambda)\omega(f,\delta) \leq (1+\lambda^2)\omega(f,\delta), \lambda \in R^+$$

as $\lambda = \frac{|t-x|}{\delta}$. Therefore, we have

$$|f(t)-f(x)| \leq \left(1+\frac{(t-x)^2}{\delta^2}\right)\omega(f,\delta) \tag{4.3}$$

for $|t-x| > \delta$. Consequently by (4.2) and (4.3), we get

$$|K_{n,q}(f,x)-f(x)| \leq [n+1]_q \sum_{k=0}^{n} q^{-k} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k$$
$$\prod_{s=0}^{n-k-1}(1-q^s x) \int_{[k]_q/[n+1]_q}^{[k+1]_q/[n+1]_q} \left(1+\frac{(t-x)^2}{\delta^2}\right)\omega(f,\delta) d_q t$$
$$= \left(K_{n,q}(1,x)+\frac{1}{\delta^2}K_{n,q}((t-x)^2,x)\right)\omega(f,\delta).$$

Taking $q=(q_n)$ satisfies the conditions $\lim_{n\to\infty} q_n = 1$, $\lim_{n\to\infty} \frac{1}{[n]_{q_n}} = 0$, and $0 < q_n < 1$, using the methods of Theorem 4.1, that

$$\lim_{n\to\infty} K_{n,q_n}((t-x)^2,x) = 0,$$

letting $\delta_n = K_{n,q_n}((t-x)^2,x)$ and taking $\delta = \sqrt{\delta_n}$, we finally get the desired result. This completes the proof of theorem. ∎

4.3 q-Bernstein–Durrmeyer Operators

For $f \in C[0,1], x \in [0,1], n = 1,2,,,,; 0 < q < 1$, very recently Gupta [86] defined the q-Durrmeyer-type operators as

$$D_{n,q}(f,x) \equiv (D_{n,q}f)(x) = [n+1]_q \sum_{k=0}^{n} q^{-k} p_{n,k}(q;x) \int_0^1 f(t) p_{n,k}(q;qt) d_q t \tag{4.4}$$

where

$$p_{n,k}(q;x) := \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1} (1 - q^s x).$$

It can be easily verified that in the case $q = 1$, the operators defined by (4.4) reduce to the well-known Bernstein–Durrmeyer operators

$$D_n(f,x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 f(t) p_{n,k}(t) dt,$$

where

$$p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}.$$

4.3.1 Auxiliary Results

In the sequel, we shall need the following auxiliary results:

Lemma 4.1. *For $n, k \geq 0$, we have*

$$D_q(1-x)_q^{n-k} = -[n-k]_q (1-qx)_q^{n+k-1}, \tag{4.5}$$

Proof. Using the q-derivative operator, we can write

$$\begin{aligned} D_q(1-x)_q^{n-k} &= \frac{1}{(q-1)x} \left(\prod_{j=0}^{n-k-1} (1 - q^{j+1}x) - \prod_{j=0}^{n-k-1} (1 - q^j x) \right) \\ &= -\frac{(q^{n-k}-1)}{(q-1)} \prod_{j=0}^{n-k-2} (1 + q^{j+1}x) \\ &= -[n-k]_q (1-qx)_q^{n-k-1}. \end{aligned}$$

■

Remark 4.1. By using (4.5) and $D_q x^k = [k]_q x^{k-1}$, we get

$$\begin{aligned} D_q(x^k(1-x)_q^{n-k}) &= [k]_q x^{k-1}(1-x)_q^{n-k} - q^k x^k [n-k]_q (1-qx)_q^{n-k-1} \\ &= x^{k-1}(1-qx)_q^{n-k-1}((1-x)[k]_q - q^k x[n-k]_q) \\ &= x^{k-1}(1-qx)_q^{n-k-1}([k]_q - [n]_q x). \end{aligned}$$

Hence, we obtain

$$x(1-x)D_q\left(x^k(1-x)_q^{n-k}\right) = x^k(1-x)_q^{n-k}[n]_q\left(\frac{[k]_q}{[n]_q} - x\right). \tag{4.6}$$

Lemma 4.2. *We have the following equalities:*

$$x(1-x)D_q(p_{n,k}(q;x)) = [n]_q p_{n,k}(q;x)\left(\frac{[k]_q}{[n]_q} - x\right), \tag{4.7}$$

$$t(1-qt)D_q(p_{n,k}(q;qt)) = [n]_q p_{n,k}(q;qt)\left(\frac{[k]_q}{[n]_q} - qt\right). \tag{4.8}$$

Proof. Above equalities can be obtained by direct computations using definition of operator and (4.6). ■

Theorem 4.3 ([92]). *If m-th $(m > 0, m \in \mathbb{N})$ order moments of operator (4.4) is defined as*

$$D_{n,m}^q(x) := D_{n,q}(t^m, x) = [n+1]_q \sum_{k=0}^{n} q^{-k} p_{n,k}(q;x) \int_0^1 p_{n,k}(q;qt)t^m d_q t, x \in [0,1],$$

then $D_{n,0}^q(x) = 1$ and for $n > m+2$, we have the following recurrence relation:
$[n+m+2]D_{n,m+1}^q(x)$

$$= ([m+1]_q + q^{m+1}x[n]_q)D_{n,m}^q(x) + x(1-x)q^{m+1}D_q(D_{n,m}^q(x)). \tag{4.9}$$

Proof. By (4.7), we have
$x(1-x)D_q(D_{n,m}^q(x))$

$$\begin{aligned} &= [n+1]_q \sum_{k=0}^{n} q^{-k} x(1-x) D_q(p_{n,k}(q;x)) \int_0^1 p_{n,k}(q;qt)t^m d_q t \\ &= [n+1]_q [n]_q \sum_{k=0}^{n} q^{-k} p_{n,k}(q;x) \int_0^1 \left(\frac{[k]_q}{[n]_q} - qt\right) p_{n,k}(q;qt)t^m d_q t \\ &+ q[n+1]_q [n]_q \sum_{k=0}^{n} q^{-k} x(1-x) D_q(p_{n,k}(q;x)) \int_0^1 p_{n,k}(q;qt)t^{m+1} d_q t \end{aligned}$$

$$- x[n+1]_q[n]_q \sum_{k=0}^{n} q^{-k} x(1-x) D_q(p_{n,k}(q;x)) \int_0^1 p_{n,k}(q;qt)t^m d_qt$$
$$= I + [n]_q D^q_{n,m+1}(x) - x[n]_q D^q_{n,m}(x),$$

Set

$$u(t) = \frac{t^{m+1}}{q^{m+1}} - \frac{t^{m+2}}{q^{m+1}},$$

by q-integral by parts, we get
$\int_0^1 u(qt) D_q(p_{n,k}(q;qt))d_qt$

$$= [u(t)p_{n,k}(q;qt)]_0^1 - \frac{1}{q^{m+1}} \int_0^1 p_{n,k}(q;qt)([m+1]_q t^m - [m+2]_q t^{m+1})d_qt$$
$$= -\frac{1}{q^{m+1}} \int_0^1 p_{n,k}(q;qt)([m+1]_q t^m - [m+2]_q t^{m+1})d_qt,$$

therefore

$$I = -\frac{1}{q^{m+1}} \left([m+1]_q D^q_{n,m}(x) - [m+2]_q D^q_{n,m+1}(x) \right)$$

by combining the above two equations, we can write

$$q^{m+1} x(1-x) D_q(D^q_{n,m}(x)) = -\left([m+1]_q D^q_{n,m}(x) - [m+2]_q D^q_{n,m+1}(x) \right)$$
$$+ q^{m+1} \left([n]_q D^q_{n,m+1}(x) - x[n]_q D^q_{n,m}(x) \right).$$

Hence we get the result. ∎

Corollary 4.1. *We have*

$$D^q_{n,1}(x) = \frac{(1+qx[n]_q)}{[n+2]_q}, \tag{4.10}$$

$$D^q_{n,2}(x) = \frac{q^3x^2[n]_q([n]_q-1) + (1+q)^2 qx[n]_q + 1 + q}{[n+2]_q[n+3]_q}. \tag{4.11}$$

The corollary follows from (4.9).

Lemma 4.3. *For $f \in C[0,1]$, we have $||D_{n,q}f|| \le ||f||$.*

Proof. By definition (4.4) and using Theorem 4.3, we have

$$|D_{n,q}(f;x)| \le [n+1]_q \sum_{k=0}^{n} q^{-k} p_{n,k}(q;x) \int_0^1 |f(t)| p_{n,k}(q;qt) d_qt$$
$$\le ||f|| D_{n,q}(1;x) = ||f||.$$

∎

Lemma 4.4. *Let $n > 3$ be a given natural number and let $q_0 = q_0(n) \in (0,1)$ be the least number such that $q^{n+2} - q^{n+1} - 2q^n - 2q^{n-1} - \cdots - 2q^3 - q^2 + q + 2 < 0$ for every $q \in (q_0, 1)$. Then*

$$D_{n,q}((t-x)^2, x) \;\le\; \frac{2}{[n+2]_q}\left(\varphi^2(x) + \frac{1}{[n+3]_q}\right),$$

where $\varphi^2(x) = x(1-x)$, $x \in [0,1]$.

Proof. In view of Theorem 4.3, we obtain

$$D_{n,q}((t-x)^2, x) = x^2 \cdot \frac{q^3[n]_q([n]_q - 1) - 2q[n]_q[n+3]_q + [n+2]_q[n+3]_q}{[n+2]_q[n+3]_q}$$
$$+ x \cdot \frac{q(1+q)^2[n]_q - 2[n+3]_q}{[n+2]_q[n+3]_q} + \frac{1+q}{[n+2]_q[n+3]_q}$$

By direct computations, using the definition of the q-integers, we get

$$\begin{aligned} q(1+q)^2[n]_q - 2[n+3]_q &= q(1+q)^2(1+q+\cdots+q^{n-1}) - 2(1+q+\cdots+q^{n+2}) \\ &= -q^{n+2} + q^{n+1} + 2q^n + 2q^{n-1} + \cdots + 2q^3 + q^2 - q - 2 > 0, \end{aligned}$$

for every $q \in (q_0, 1)$. Furthermore

$$\begin{aligned} q(1+q)^2[n]_q - 2[n+3]_q &\le 4[n] - q - 2[n+3]_q \\ &= 4([n+3]_q - q^n - q^{n+1} - q^{n+2}) - 2[n+3]_q \\ &\le 4[n+3]_q - 2[n+3]_q \;=\; 2[n+3]_q \end{aligned}$$

and

$$\begin{aligned} &q(1+q)^2[n]_q - 2[n+3]_q + q^3[n]_q([n]_q - 1) - 2q[n]_q[n+3]_q + [n+2]_q[n+3]_q \\ &= q(1+q)^2[n]_q - 2(1+q+q^2+q^3[n]_q) + q^3[n]_q^2 - q^3[n]_q \\ &\quad - 2q[n]_q(1+q+q^2+q^3[n]_q) + (1+q+q^2[n]_q)(1+q+q^2+q^3[n]_q) \\ &= q^3(1-q)^2[n]_q^2 - (q - q^2 + 2q^3 - 2q^4)[n]_q - (1-q^3) \\ &= q^3(1-q)^2 \cdot \left(\frac{1-q^n}{1-q}\right)^2 - q(1-q)(1+2q^2) \cdot \frac{1-q^n}{1-q} - (1-q^3) \\ &= q^{2n+3} + q^{n+1} - q - 1 \le 0. \end{aligned}$$

In conclusion, for $x \in [0,1]$, we have
$D_{n,q}((t-x)^2,x)$

$$= \frac{q(1+q)^2[n]_q - 2[n+3]_q}{[n+2]_q[n+3]_q} \cdot x(1-x) + \left(\frac{q(1+q)^2[n]_q - 2[n+3]_q}{[n+2]_q[n+3]_q}\right.$$
$$\left. + \frac{q^3[n]_q([n]_q-1) - 2q[n]_q[n+3]_q + [n+2]_q[n+3]_q}{[n+2]_q[n+3]_q}\right) \cdot x^2 + \frac{1+q}{[n+2]_q[n+3]_q}$$
$$\leq \frac{2[n+3]_q}{[n+2]_q[n+3]_q} \cdot \varphi^2(x) + \frac{2}{[n+2]_q[n+3]_q} \leq \frac{2}{[n+2]_q} \cdot \left(\varphi^2(x) + \frac{1}{[n+3]_q}\right),$$

which was to be proved. ■

For $\delta > 0$ and $W^2 = \{g \in C[0,1] : g', g'' \in C[0,1]\}$, the K-functional are defined as

$$K_2(f,\delta) = \inf\{||f-g|| + \eta||g''|| : g \in W^2\},$$

where norm-$||.||$ is the uniform norm on $C[0,1]$. Following [50], there exists a positive constant $C > 0$ such that

$$K_2(f,\delta) \leq C\omega_2(f,\sqrt{\delta}), \tag{4.12}$$

where the second-order modulus of smoothness for $f \in C[0,1]$ is defined as

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h\leq\sqrt{\delta}} \sup_{x,x+h\in[0,1]} |f(x+h) - f(x)|.$$

We define the usual modulus of continuity for $f \in C[0,1]$ as

$$\omega(f,\delta) = \sup_{0<h\leq\delta} \sup_{x,x+h\in[0,1]} |f(x+h) - f(x)|.$$

4.3.2 Direct Results

Our first main result is the following local theorem:

Theorem 4.4. *Let $n > 3$ be a natural number and let $q_0 = q_0(n) \in (0,1)$ be defined as in Lemma 4.4. Then there exists an absolute constant $C > 0$ such that*

$$|D_{n,q}(f,x) - f(x)| \leq C\,\omega_2\left(f, [n+2]_q^{-1/2}\delta_n(x)\right) + \omega\left(f, \frac{1-x}{[n+2]_q}\right),$$

where $f \in C[0,1]$, $\delta_n^2(x) = \varphi^2(x) + \frac{1}{[n+3]_q}$, $x \in [0,1]$, and $q \in (q_0,1)$.

Proof. For $f \in C[0,1]$ we define

$$\tilde{D}_{n,q}(f,x) \;=\; D_{n,q}(f,x) + f(x) - f\left(\frac{1+q[n]_q x}{[n+2]_q}\right).$$

Then, by Corollary 4.1, we find

$$\tilde{D}_{n,q}(1,x) = D_{n,q}(1,x) = 1 \tag{4.13}$$

and

$$\tilde{D}_{n,q}(t,x) = D_{n,q}(t,x) + x - \frac{1+q[n]_q x}{[n+2]_q} = x. \tag{4.14}$$

Using Taylor's formula

$$g(t) = g(x) + (t-x)\, g'(x) + \int_x^t (t-u)\, g''2,$$

we obtain

$$\begin{aligned}
\tilde{D}_{n,q}(g,x) \;&=\; g(x) + \tilde{D}_{n,q}\left(\int_x^t (t-u)\, g''(u)\, du, x\right) \\
&= g(x) + D_{n,q}\left(\int_x^t (t-u)\, g''(u)\, du, x\right) \\
&\quad - \int_x^{\frac{1+q[n]_q x}{[n+2]_q}} \left(\frac{1+q[n]_q x}{[n+2]_q} - u\right) g''(u)\, du
\end{aligned}$$

Hence $|\tilde{D}_{n,q}(g,x) - g(x)| \;\le$

$$\begin{aligned}
&\le D_{n,q}\left(\left|\int_x^t |t-u| \cdot |g''(u)|\, du\right|, x\right) + \left|\int_x^{\frac{1+q[n]_q x}{[n+2]_q}} \left|\frac{1+q[n]_q x}{[n+2]_q} - u\right| \cdot |g''(u)|\, du\right| \\
&\le D_{n,q}((t-x)^2, x) \cdot \|g''\| + \left(\frac{1+q[n]_q x}{[n+2]_q} - x\right)^2 \cdot \|g''\| \qquad (4.15)
\end{aligned}$$

On the other hand

$$D_{n,q}((t-x)^2, x) + \left(\frac{1+q[n]_q x}{[n+2]_q} - x\right)^2 \;\le$$

$$\le\; \frac{2}{[n+2]_q}\left(\varphi^2(x) + \frac{1}{[n+3]_q}\right) + \left(\frac{1-([n+2]_q - q[n]_q)x}{[n+2]_q}\right)^2, \tag{4.16}$$

by Lemma 4.4. Because $[n+2]_q - q[n]_q = (1+q+\ldots+q^{n+1}) - q(1+q+\ldots+q^{n-1}) = 1+q^{n+1}$, we have

$$1 \leq [n+2]_q - q[n]_q \leq 2 \tag{4.17}$$

Then using (4.17), we have

$$\left(\frac{1-([n+2]_q - q[n]_q)x}{[n+2]_q}\right)^2 \cdot \delta_n^{-2}(x) \ \leq$$

$$= \frac{1-2([n+2]_q - q[n]_q)x + ([n+2]_q - q[n]_q)^2 x^2}{[n+2]_q^2} \cdot \frac{[n]_q}{[n]_q x(1-x)+1}$$

$$\leq \frac{1-2x+4x^2}{[n+2]_q} \cdot \frac{[n]_q}{[n+2]_q} \cdot \frac{1}{[n]_q x(1-x)+1} \leq \frac{3}{[n+2]_q}, \tag{4.18}$$

for $n = 1,2,\ldots$ and $0 < q < 1$. In conclusion, by (4.16) and (4.18), we get

$$D_{n,q}((t-x)^2,x) + \left(\frac{1+q[n]_q x}{[n+2]_q} - x\right)^2 \leq \frac{5}{[n+2]_q} \cdot \delta_n^2(x), \tag{4.19}$$

where $x \in [0,1]$. Hence, by (4.15),

$$|\tilde{D}_{n,q}(g,x) - g(x)| \leq \frac{5}{[n+2]_q} \cdot \delta_n^2(x) \cdot \|g''\|, \tag{4.20}$$

where $n > 3$ and $x \in [0,1]$. Furthermore, by Theorem 4.3, we have

$$|\tilde{D}_{n,q}(f,x)| \leq |D_{n,q}(f,x)| + |f(x)| + \left| f\left(\frac{1+q[n]_q x}{[n+2]_q}\right) \right| \leq 3\|f\|.$$

Thus

$$\|\tilde{D}_{n,q}(f,x)\| \leq 3\ \|f\|, \tag{4.21}$$

for all $f \in C[0,1]$.

Now, for $f \in C[0,1]$ and $g \in W^2$, we obtain

$$|D_{n,q}(f,x) - f(x)| \ \leq$$

$$= \left| \tilde{D}_{n,q}(f,x) - f(x) + f\left(\frac{1+q[n]_q x}{[n+2]_q}\right) - f(x) \right|$$

$$\leq |\tilde{D}_{n,q}(f-g,x)| + |\tilde{D}_{n,q}(g,x) - g(x)| + |g(x) - f(x)| + \left| f\left(\frac{1+q[n]_q x}{[n+2]_q}\right) - f(x) \right|$$

$$\leq 4\,\|f-g\| + \frac{5}{[n+2]} \cdot \delta_n^2(x) \cdot \|g''\| + \omega\left(f, \left| \frac{1-([n+2]_q - q[n]_q)x}{[n+2]_q} \right|\right)$$

$$\leq 5\left(\|f-g\| + \frac{1}{[n+2]_q} \cdot \delta_n^2(x) \cdot \|g''\|\right) + \omega\left(f, \frac{1-x}{[n+2]_q}\right),$$

where we used (4.20) and (4.21). Taking the infimum on the right hand side over all $g \in W^2$, we obtain

$$|D_{n,q}(f,x) - f(x)| \leq 5\, K_2\left(f, \frac{1}{[n+2]_q}\delta_n^2(x)\right) + \omega\left(f, \frac{1-x}{[n+2]_q}\right).$$

In view of (4.12), we find

$$|D_{n,q}(f,x) - f(x)| \;\leq\; C\, \omega_2\left(f, [n+2]_q^{-1/2}\delta_n(x)\right) + \omega\left(f, \frac{1-x}{[n+2]_q}\right),$$

this completes the proof of the theorem. ■

For the next theorem we shall use some notations: for $f \in C[0,1]$ and $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$, let

$$\omega_2^{\varphi}(f, \sqrt{\delta}) \;=\; \sup_{0<h\leq\sqrt{\delta}} \; \sup_{x\pm h\varphi\in[0,1]} |f(x+h\varphi(x)) - 2f(x) + f(x-h\varphi(x))|$$

be the second-order Ditzian–Totik modulus of smoothness, and let

$$\overline{K}_{2,\varphi}(f,\delta) \;=\; \inf\{\|f-g\| + \delta\|\varphi^2 g''^2\| g''^2(\varphi)\}$$

be the corresponding K-functional, where

$$W^2(\varphi) = \{g \in C[0,1] : g' \in AC_{loc}[0,1], \varphi^2 g'' \in C[0,1]\}$$

and $g' \in AC_{loc}[0,1]$ means that g is differentiable and g' is absolutely continuous on every closed interval $[a,b] \subset [0,1]$. It is well known (see [51, p. 24, Theorem 1.3.1]) that

$$\overline{K}_{2,\varphi}(f,\delta) \;\leq\; C\, \omega_2^{\varphi}(f, \sqrt{\delta}) \tag{4.22}$$

for some absolute constant $C > 0$. Moreover, the Ditzian–Totik moduli of first order is given by

$$\omega_{\psi}(f,\delta) \;=\; \sup_{0<h\leq\delta} \; \sup_{x,x\pm h\psi(x)\in[0,1]} |f(x+h\psi(x)) - f(x)|,$$

where ψ is an admissible step-weight function on $[0,1]$.
Now we state our next main result.

Theorem 4.5. *Let $n > 3$ be a natural number and let $q_0 = q_0(n) \in (0,1)$ be defined as in Lemma 4.3. Then there exists an absolute constant $C > 0$ such that*

$$\|D_{n,q}f - f\| \le C\, \omega_2^{\varphi}(f, [n+2]_q^{-1/2}) + \omega_{\psi}(f, [n+2]_q^{-1}),$$

where $f \in C[0,1]$, $q \in (q_0, 1)$, and $\psi(x) = 1 - x$, $x \in [0,1]$.

Proof. Again, let

$$\tilde{D}_{n,q}(f,x) = D_{n,q}(f,x) + f(x) - f\left(\frac{1+q[n]_q x}{[n+2]_q}\right),$$

where $f \in C[0,1]$. Using Taylor's formula:

$$g(t) = g(x) + (t-x)\, g'(x) + \int_x^t (t-u)\, g''^2(\varphi),$$

the formulas (4.13) and (4.14), we obtain

$$\tilde{D}_{n,q}(g,x) = g(x) + D_{n,q}\left(\int_x^t (t-u)\, g''(u)\, du, x\right) - \int_x^{\frac{1+q[n]_q x}{[n+2]-q}} \left(\frac{1+q[n]_q x}{[n+2]_q} - u\right) g''(u)\, du$$

Hence

$$|\tilde{D}_{n,q}(g,x) - g(x)|$$

$$\le D_{n,q}\left(\left|\int_x^t |t-u| \cdot |g''(u)|\, du\right|, x\right) + \left|\int_x^{\frac{1+q[n]_q x}{[n+2]}} \left|\frac{1+q[n]_q x}{[n+2]_q} - u\right| \cdot |g''(u)|\, du\right| \tag{4.23}$$

Because the function δ_n^2 is concave on $[0,1]$, we have for $u = t + \tau(x-t)$, $\tau \in [0,1]$, the estimate

$$\frac{|t-u|}{\delta_n^2(u)} = \frac{\tau|x-t|}{\delta_n^2(t+\tau(x-t))} \le \frac{\tau|x-t|}{\delta_n^2(t) + \tau(\delta_n^2(x) - \delta_n^2(t))} \le \frac{|t-x|}{\delta_n^2(x)}.$$

Hence, by (4.23), we find

$$|\tilde{D}_{n,q}(g,x) - g(x)| \le$$

$$\le D_{n,q}\left(\left|\int_x^t \frac{|t-u|}{\delta_n^2(u)}\, du\right|, x\right) \cdot \|\delta_n^2 g''\| + \left|\int_x^{\frac{1+q[n]_q x}{[n+2]_q}} \frac{\left|\frac{1+q[n]_q x}{[n+2]_q} - u\right|}{\delta_n^2(u)}\, du\right| \cdot \|\delta_n^2 g''\|$$

$$\le \frac{1}{\delta_n^2(x)} \cdot D_{n,q}((t-x)^2, x) \cdot \|\delta_n^2 g''\| + \frac{1}{\delta_n^2(x)} \cdot \left(\frac{1+q[n]_q x}{[n+2]_q} - x\right)^2 \cdot \|\delta_n^2 g''\|$$

In view of (4.19) and

$$\delta_n^2(x)\cdot|g''^2(x)g''(x)|+\frac{1}{[n+3]_q}\cdot|g''^2g''\|+\frac{1}{[n+3]_q}\cdot\|g''\|,$$

where $x\in[0,1]$, we get

$$|\tilde{D}_{n,q}(g,x)-g(x)|\leq\frac{5}{[n+2]_q}\cdot\left(\|\varphi^2g''\|+\frac{1}{[n+3]_q}\cdot\|g''\|\right) \tag{4.24}$$

Using $[n]_q\leq[n+2]_q$, (4.21), and (4.24), we find for $f\in C[0,1]$,

$$|D_{n,q}(f,x)-f(x)|\ \leq$$

$$\leq|\tilde{D}_{n,q}(f-g,x)|+|\tilde{D}_{n,q}(g,x)-g(x)|+|g(x)-f(x)|+\left|f\left(\frac{1+q[n]_qx}{[n+2]_q}\right)-f(x)\right|$$

$$\leq 4\,\|f-g\|+\frac{5}{[n+2]_q}\cdot\|\varphi^2g''\|+\frac{5}{[n+2]_q}\cdot\|g''\|+\left|f\left(\frac{1+q[n]_qx}{[n+2]_q}\right)-f(x)\right|$$

Taking the infimum on the right hand side over all $g\in W^2(\varphi)$, we obtain

$$|D_{n,q}(f,x)-f(x)|\leq 5\overline{K}_{2,\varphi}\left(f,\frac{1}{[n+2]_q}\right)+\left|f\left(\frac{1+q[n]_qx}{[n+2]_q}\right)-f(x)\right| \tag{4.25}$$

On the other hand

$$\left|f\left(\frac{1+q[n]x}{[n+2]}\right)-f(x)\right|=$$

$$=\left|f\left(x+\psi(x)\cdot\frac{1-([n+2]_q-q[n]_q)x}{[n+2]_q\psi(x)}\right)-f(x)\right|$$

$$\leq\sup_{t,t+\psi(t)\cdot(1-([n+2]_q-q[n]_q)x)/[n+2]_q\in[0,1]}\left|f\left(t+\psi(t)\cdot\frac{1-([n+2]_q-q[n]_q)x}{[n+2]_q\psi(x)}\right)-f(t)\right|$$

$$\leq\omega_\psi\left(f,\frac{|1-([n+2]_q-q[n]_q)x|}{[n+2]_q\psi(x)}\right)\leq\omega_\psi\left(f,\frac{1-x}{[n+2]_q\psi(x)}\right)=\omega_\psi\left(f,\frac{1}{[n+2]_q}\right).$$

Hence, by (4.25) and (4.22), we get

$$\|D_{n,q}f-f\|\ \leq\ C\ \omega_2^\varphi(f,[n+2]_q^{-1/2})\ +\ \omega_\psi(f,[n+2]_q^{-1}),$$

$x\in[0,1]$, which completes the proof of the theorem. ■

Remark 4.2. In [86] it is proved for $q = q(n) \to 1$ as $n \to \infty$ that the sequence $\{D_{n,q}f\}$ converges to f uniformly on $[0,1]$ for each $f \in C[0,1]$. The same result follows from Theorem 4.5, because

$$\lim_{n\to\infty}[n+2]_{q_n} = \lim_{n\to\infty}\frac{1-(q(n))^{n+2}}{1-q(n)} = \infty,$$

if $\lim_{n\to\infty} q(n) = 1$.

4.3.3 Applications to Random and Fuzzy Approximation

Let $(X, ||.||)$ be a normed space over K, where $K = R$ or $K = C$. Similar to the case of real-valued functions can be introduced the following concepts.

Definition 4.1 (Gal [74]).

(i) For $f : [0,1] \to X$, the first-order Ditzian–Totik modulus of continuity $\omega_\psi(f,\delta)$ and the second-order Ditzian–Totik modulus of smoothness $\omega_2^\varphi(f,\delta)$ are respectively defined as

$$\omega_\psi(f,\delta) \;=\; \sup_{0<h\le\delta}\ \sup_{x,x\pm h\psi(x)\in[0,1]} \|f(x+h\psi(x))-f(x)\|,$$

and

$$\omega_2^\varphi(f,\delta) =$$
$$\sup\{\sup\{||f(x+h\varphi(x))-2f(x)+f(x-h\varphi(x))||, x\in I_{2,h}\}, h\in[0,\delta]\}$$

where $I_{2,h} = \left[-\frac{1-h^2}{1+h^2}, \frac{1-h^2}{1+h^2}\right]$, $\varphi(x) = \sqrt{x(1-x)}$, $\psi(x) = 1-x, 0<\delta\le 1$.

(ii) $f : [0,a] \to X$ is called q-integrable $(0<q<1)$ on $[0,a]$ if there exists $I \in X$ denoted by $I := \int_0^a f(u)d_q u$ with the property

$$\lim_{n\to\infty}\|I-(1-q)\sum_{k=1}^{n} q^k f(aq^k)\| = 0.$$

Remark 4.3. Let $(X, ||.||)$ be a Banach space. If $f : [0,a] \to X$ is continuous on $[0,a]$, then it is q-integrable. Indeed, denoting $S_n(f) = (1-q)\sum_{k=1}^{n} q^k f(aq^k)$, we get $S_{n+p}(f) - S_n(f) = (1-q)\sum_{k=n}^{n+p} q^k f(aq^k)$ and since $\|f(x)\|$ is bounded (by continuity) by a positive constant denoted by M, for all $n, p \in \mathbb{N}$ it follows

$$\|S_{n+p}(f)-S_n(f)\| \le M(1-q)\sum_{k=n}^{n+p} q^k \le M(1-q)q^n\sum_{j=0}^{\infty} q^j = Mq^n,$$

which shows that $(S_n(f))_{n\in\mathbb{N}}$ is a Cauchy sequence. Since X is a Banach space, it follows that this sequence is convergent and therefore f is q-integrable.

Definition 4.2 (see Gupta [86] for real-valued functions). For $f:[0,1]\to X, 0<q<1$, q-integrable on $[0,1]$, the q-Durrmeyer operators attached to f can be defined as

$$D_{n,q}(f,x)\equiv(D_{n,q}f)(x)=[n+1]\sum_{k=0}^{n}q^{-k}p_{n,k}(q;x)\int_0^1 f(u)p_{n,k}(q;qu)d_qu \quad (4.26)$$

where

$$p_{n,k}(q;x):=\begin{bmatrix} n \\ k \end{bmatrix} x^k(x;q)_{n-k}.$$

Theorem 4.6 (see, e.g., [124], p. 183). *Let $(X,||.||)$ be a normed space over K, where $K=R$ or $K=C$ and denote by $X^*=\{x^*:X\to K, x^*$ is linear and continuous$\}$. Then*

$$||x||=\sup\{|x^*(x)|:x^*\in X^*,\|x^*\|<1\}.$$

Gal and Gupta [77] established the following theorem:

Theorem 4.7. *Let $(X,\|\cdot\|)$ be a Banach space and suppose that $f:[0,1]\to X$ is continuous on $[0,1]$. Then under the conditions on q as given in Lemma 4.4, we have*

$$||D_{n,q}f-f||_u\le C\,\omega_2^{\varphi}(f,[n+2]^{-1/2})\;+\;\omega_\psi(f,[n+2]^{-1}),$$

where $||f||_u=\sup\{||f(x)||:x\in[0,1]\}$.

Proof. Let $x^*\in X^*, 0<|||x^*|||\le 1$ and define $g:[0,1]\to\mathbb{R}, g(x)=x^*(f(x))$. Obviously g is continuous on $[0,1]$. First, we have

$$\begin{aligned}\omega_\psi(g,\frac{1}{[n+2]}) &= \sup_{0<h\le 1/[n+2]}\sup_{x,x\pm h\psi(x)\in[0,1]}|x^*[f(x+h\psi(x))-f(x)]|\\ &\le \sup_{0<h\le 1/[n+2]}\sup_{x,x\pm h\psi(x)\in[0,1]}|||x^*|||\cdot\|[f(x+h\psi(x))-f(x)]\|\\ &\le \sup_{0<h\le 1/[n+2]}\sup_{x,x\pm h\psi(x)\in[0,1]}\|[f(x+h\psi(x))-f(x)]\|\\ &=\omega_\psi(f,\frac{1}{[n+2]}),\end{aligned}$$

and

$$\begin{aligned}&\omega_2^{\varphi}(g,[n+2]^{-1/2})\\&=\sup\{\sup\{|x^*[f(x+h\varphi(x))-2f(x)+f(x-h\varphi(x))]|,\ x\in I_{2,h}\},h\in[0,[n+2]^{-1/2}]\}\\&\le\sup\{\sup\{|||x^*|||\cdot\|f(x+h\varphi(x))-2f(x)+f(x-h\varphi(x))\|,x\in I_{2,h}\},h\in[0,[n+2]^{-1/2}]\}\\&\le\omega_2^{\varphi}(f,[n+2]^{-1/2}).\end{aligned}$$

Now, by Theorem 4.5, for all $x\in[0,1]$ and $n\in\mathbb{N}$, we have

$$|D_{n,q}g(x)-g(x)|\le C[\ \omega_2^{\varphi}(g,[n+2]^{-1/2})\ +\ \omega_{\psi}(g,[n+2]^{-1})].$$

But by the linearity and the continuity of x^* (the continuity allows to x^* to commutes with the integral), we easily get $D_{n,q}g(x)-g(x)=x^*[D_{n,q}f(x)-f(x)]$, which combined with the above inequalities lead to

$$|x^*[D_{n,q}f(x)-f(x)]|\le C[\ \omega_2^{\varphi}(f,[n+2]^{-1/2})\ +\ \omega_{\psi}(f,[n+2]^{-1})],$$

for all $x\in[0,1]$. Passing to supremum with $|||x^*|||\le 1$ and taking into account Theorem 4.6, it follows

$$\|D_{n,q}f(x)-f(x)]\|\le C[\ \omega_2^{\varphi}(f,[n+2]^{-1/2})\ +\ \omega_{\psi}(f,[n+2]^{-1})],$$

for all $x\in[0,1]$, which proves the theorem. ■

Some applications to the approximation of random functions by q-Durrmeyer random polynomials and of fuzzy-number-valued functions by q-Durrmeyer fuzzy polynomials were discussed in [77] as

If (S,B,P) is a probability space (P is the probability), then the set of almost sure (a.s.) finite real random variables is denoted by $L(S,B,P)$ and it is a Banach space with respect to the norm $||g||=\int_S|g(t)|dP(t)$. Here, for $g_1,g_2\in L(S,B,P)$, we consider $g_1=g_2$ if $g_1(t)=g_2(t)$, a.s. $t\in S$.

A random function defined on $[0,1]$ is a mapping $f:[0,1]\to L(S,B,P)$ and we denote $f(x)(t)\in\mathbb{R}$ by $f(x,t)$. For this kind of f, the q-Durrmeyer random polynomials are defined by

$$(D_{n,q}f)(x,t)=[n+1]\sum_{k=0}^{n}q^{-k}p_{n,k}(q;x)\int_0^1 f(u,t)p_{n,k}(q;qu)d_qu.$$

Corollary 4.2. *If* $f:[0,1]\to L(S,B,P)$ *is continuous on* $[0,1]$*, then*

$$||D_{n,q}f-f||_u\le C\ \omega_2^{\varphi}(f,[n+2]^{-1/2})\ +\ \omega_{\psi}(f,[n+2]^{-1}),$$

where $||f||_u=\sup\{||f(x)||;x\in[0,1]\}=\sup\{\int_S|f(x,t)|dP(t);x\in[0,1]\}$.

Given a set $X \neq \emptyset$, a fuzzy subset of X is a mapping $u : X \to [0,1]$, and obviously any classical subset A of X can be considered as a fuzzy subset of X defined by $\chi_A : X \to [0,1]$, $\chi_A(x) = 1$, if $x \in A$, $\chi_A(x) = 0$ if $x \in X \setminus A$. (see, e.g., Zadeh [154]).

Let us denote by $\mathbb{R}_{\mathcal{F}}$ the class of fuzzy subsets of real axis $\mathbb{R}$ (i.e., $u : \mathbb{R} \to [0,1]$), satisfying the following properties:

(i) $\forall u \in \mathbb{R}_{\mathcal{F}}$, u is normal, i.e., $\exists x_u \in \mathbb{R}$ with $u(x_u) = 1$.
(ii) $\forall u \in \mathbb{R}_{\mathcal{F}}$, u is convex fuzzy set (i.e., $u(tx + (1-t)y) \geq \min\{u(x), u(y)\}$, $\forall t \in [0,1], x, y \in \mathbb{R}$).
(iii) $\forall u \in \mathbb{R}_{\mathcal{F}}$, u is upper semicontinuous on $\mathbb{R}$.
(iv) $\overline{\{x \in \mathbb{R} : u(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of A.

Then $\mathbb{R}_{\mathcal{F}}$ is called the space of fuzzy real numbers (see, e.g., Dubois–Prade [56]).

Remark 4.4. Obviously $\mathbb{R} \subset \mathbb{R}_{\mathcal{F}}$, because any real number $x_0 \in \mathbb{R}$ can be described as the fuzzy number whose value is 1 for $x = x_0$ and 0 otherwise.

For $0 < r \leq 1$ and $u \in \mathbb{R}_{\mathcal{F}}$, define $[u]^r = \{x \in \mathbb{R}; u(x) \geq r\}$ and $[u]^0 = \overline{\{x \in \mathbb{R}; u(x) > 0\}}$. Then it is well known that for each $r \in [0,1]$, $[u]^r$ is a bounded closed interval. For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we have the sum $u \oplus v$ and the product $\lambda \odot u$ defined by $[u \oplus v]^r = [u]^r + [v]^r$, $[\lambda \odot u]^r = \lambda [u]^r$, $\forall r \in [0,1]$, where $[u]^r + [v]^r$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and $\lambda [u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see, e.g., Dubois–Prade [56], Congxin–Zengtai [44]).

Let $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+ \cup \{0\}$ by

$$D(u,v) = \sup_{r \in [0,1]} \max\left\{\left|u_-^r - v_-^r\right|, \left|u_+^r - v_+^r\right|\right\},$$

where $[u]^r = \left[u_-^r, u_+^r\right]$, $[v]^r = \left[v_-^r, v_+^r\right]$. The following properties are known (Dubois–Prade [56]):

$D(u \oplus w, v \oplus w) = D(u,v)$, $\forall u, v, w \in \mathbb{R}_{\mathcal{F}}$

$D(k \odot u, k \odot v) = |k| D(u,v)$, $\forall u, v \in \mathbb{R}_{\mathcal{F}}, \forall k \in \mathbb{R}$;

$D(u \oplus v, w \oplus e) \leq D(u,w) + D(v,e)$, $\forall u, v, w, e \in \mathbb{R}_{\mathcal{F}}$ and $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space.

Also, we need the following concept of q-integral. A function $f : [0,a] \to \mathbb{R}_{\mathcal{F}}$, $[0,a] \subset \mathbb{R}$ will be called q-integrable on $[0,a]$, if there exists $I \in \mathbb{R}_{\mathcal{F}}$, denoted by $I = \int_0^a f(u) d_q u$ with the property

$$\lim_{n\to\infty} D[I, (1-q) \odot \Sigma^{*n}_{k=1} q^k \odot f(aq^k)]\| = 0.$$

Here the sum $\sum^*$ is considered with respect to the operation $\oplus$.

Remark 4.5. If $f : [0,a] \to \mathbb{R}_{\mathcal{F}}$ is continuous on $[0,a]$, then it is q-integrable. Indeed, denoting $S_n(f) = (1-q) \odot \Sigma^{*n}_{k=1} q^k \odot f(aq^k)$, from the above properties of the metric D, we can write

$$D[S_n(f), S_{n+p}(f)] = (1-q)D[0_{\mathbb{R}_\mathcal{F}}, \Sigma^{*\,n+p}_{k=n} q^k \odot f(aq^k)] \leq$$

$$(1-q)\sum_{k=n}^{n+p} q^k D[0_{\mathbb{R}_\mathcal{F}}, f(aq^k)] \leq M(1-q)\sum_{k=n}^{n+p} q^k,$$

where the continuity implies that f is bounded and that there exists $M > 0$ such that $D[0_{\mathbb{R}_\mathcal{F}}, f(x)] \leq M$ for all $x \in [0,a]$. In continuation, taking into account that $(\mathbb{R}_\mathcal{F}, D)$ is a complete metric space, the reasonings are similar to those in the Remark 4.3.

Theorem 4.8 (see [44]). *$\mathbb{R}_\mathcal{F}$ can be embedded in $\mathbb{B} = \bar{C}[0,1] \times \bar{C}[0,1]$, where $\bar{C}[0,1]$ is the class of all real-valued bounded functions $f : [0,1] \to \mathbb{R}$ such that f is left continuous for any $x \in (0,1]$, f has right limit for any $x \in [0,1)$, and f is right continuous at 0. With the norm $\|\cdot\| = \sup_{x\in[0,1]} |f(x)|$, $\bar{C}[0,1]$ is a Banach space. Denote $\|\cdot\|_\mathbb{B}$ the usual product norm, i.e., $\|(f,g)\|_\mathcal{B} = \max\{\|f\|, \|g\|\}$. Let us denote the embedding by $j : \mathbb{R}_\mathcal{F} \to \mathbb{B}$, $j(u) = (u_-, u_+)$. Then $j(\mathbb{R}_\mathcal{F})$ is a closed convex cone in $\mathbb{B}$ and j satisfies the following properties:*

(i) $j(s \odot u \oplus t \odot v) = s \cdot j(u) + t \cdot j(v)$ for all $u, v \in \mathbb{R}_\mathcal{F}$ and $s, t \geq 0$ (here "$\cdot$" and "$+$" denote the scalar multiplication and addition in $\mathbb{B}$)

(ii) $D(u,v) = \|j(u) - j(v)\|_\mathbb{B}$ (i.e., j embeds $\mathbb{R}_\mathcal{F}$ in $\mathbb{B}$ isometrically)

Let $f : [0,1] \to \mathbb{R}_\mathcal{F}$ be a continuous fuzzy-number-valued function. The fuzzy q-Durrmeyer polynomials attached to f can be defined by

$$(D_{n,q}f)(x) = [n+1]\sum_{k=0}^{n} q^{-k} p_{n,k}(q;x) \odot \int_0^1 p_{n,k}(q;qu) \odot f(u) d_q u.$$

Also, let us define the following moduli of continuity and smoothness of f :

$$\omega_\psi(f,\delta) = \sup_{0<h\leq\delta} \sup_{x, x\pm h\psi(x)\in[0,1]} D[f(x+h\psi(x)), f(x)],$$

$$\begin{aligned}\omega_2^\phi(f;\delta) = \sup\{&D[f(x+h\phi(x)) \oplus f(x-h\phi(x)), 2\odot f(x)];\\ &x, x+h\phi(x), x-h\phi(x) \in [0,1], 0 \leq h \leq \delta\}.\end{aligned}$$

Here $\phi^2(x) = x(1-x)$, $\psi(x) = 1-x$.

Theorem 4.9. *Let $f : [0,1] \to \mathbb{R}_\mathcal{F}$ be continuous on $[0,1]$. There exist the absolute constant C, such that for all $n \in \mathbb{N}$ we have*

$$\sup\{D[(D_{n,q}f)(x), f(x)]; x \in [0,1]\} \leq C\, \omega_2^\varphi(f, [n+2]^{-1/2}) + \omega_\psi(f, [n+2]^{-1}).$$

4.4 Discretely Defined q-Durrmeyer Operators

For $f \in C[0,1]$, Gupta and Wang [94] proposed the following q-Durrmeyer operators as

$$M_{n,q}(f;x) = [n+1]_q \sum_{k=1}^{n} q^{1-k} p_{n,k}(q;x) \int_0^1 f(t) p_{n,k-1}(q;qt) d_q t + f(0) p_{n,0}(q;x) \tag{4.27}$$

It can be easily verified that in the case $q = 1$, the operators defined by (4.27) reduce to the Durrmeyer-type operators recently introduced and studied in [3].

4.4.1 Moment Estimation

By the definition of q-Beta function, we have

$$\int_0^1 t^s p_{n,k}(q;qt) d_q t = \begin{bmatrix} n \\ k \end{bmatrix} q^k \int_0^1 t^{k+s} (1-qt)_q^{n-k} d_q t$$

$$= \frac{q^k [n]_q!}{[k]_q! [n-k]_q!} \frac{[k+s]_q! [n-k]_q!}{[k+s+n-k+1]_q!} = \frac{q^k [n]_q! [k+s]_q!}{[n+s+1]_q! [k]_q!} \tag{4.28}$$

and

$$\int_0^1 t^s p_{\infty,k}(q;qt) d_q t = \frac{q^k}{(1-q)^k [k]_q!} \int_0^1 t^{k+s} (1-qt)_q^{\infty} d_q t$$

$$= \frac{q^k}{(1-q)^k [k]_q!} [k+s]_q! (1-q)^{k+s+1} = (1-q)^{s+1} \frac{q^k [k+s]_q!}{[k]_q!}. \tag{4.29}$$

Lemma 4.5. *We have*

$$M_{n,q}(1;x) = 1, \quad M_{n,q}(t;x) = x \frac{[n]_q}{[n+2]_q}$$

and

$$M_{n,q}(t^2;x) = \frac{(1+q)x[n]_q}{[n+3]_q [n+2]_q} + x^2 \frac{q[n]_q([n]_q - 1)}{[n+3]_q [n+2]_q}.$$

Proof. In order to prove the theorem we shall use the following identities:

$$\sum_{k=0}^{n} p_{n,k}(q;x) = 1, \quad \sum_{k=0}^{n} \frac{[k]_q}{[n]_q} p_{n,k}(q;x) = x,$$
$$\sum_{k=0}^{n} \left(\frac{[k]_q}{[n]_q}\right)^2 p_{nk}(q;x) = x^2 + \frac{x(1-x)}{[n]_q}.$$

By (4.28) and (4.29), it can easily be verified that $M_{n,q}(1;x) = 1$. Next, using the above, we have

$$\begin{aligned} M_{n,q}(t;x) &= [n+1]_q \sum_{k=1}^{n} q^{1-k} p_{n,k}(q;x) \frac{q^{k-1}[n]_q![k]_q}{[n+2]_q!} \\ &= \frac{1}{[n+2]_q} \sum_{k=1}^{n} [k]_q p_{n,k}(q;x) = x\frac{[n]_q}{[n+2]_q}. \end{aligned}$$

Finally, using $[a+1]_q = 1 + q[a]_q$, we have

$$\begin{aligned} M_{n,q}(t^2;x) &= \frac{1}{[n+3]_q [n+2]_q} \sum_{k=1}^{n} p_{n,k}(q;x) [k+1]_q [k]_q \\ &= \frac{1}{[n+3]_q [n+2]_q} \left\{ \sum_{k=1}^{n} p_{n,k}(q;x)(1+q[k]_q)[k]_q \right\} \\ &= \frac{1}{[n+3]_q [n+2]_q} \left\{ \sum_{k=1}^{n} p_{n,k}(q;x)[k]_q + q \sum_{k=1}^{n} p_{n,k}(q;x)[k]_q^2 \right\} \\ &= \frac{1}{[n+3]_q [n+2]_q} \left\{ x[n]_q + q(x^2[n]_q^2 + x(1-x)[n]_q) \right\} \\ &= \frac{x[n]_q(1+q)}{[n+3]_q [n+2]_q} + \frac{q^2x^2}{[n+3]_q [n+2]_q} \left[\frac{[n]_q^2 - [n]_q}{q} \right]. \end{aligned}$$

Thus,

$$M_{n,q}(t^2;x) = \frac{x[n]_q(1+q)}{[n+3]_q [n+2]_q} + \frac{qx^2[n]_q([n]_q - 1)}{[n+3]_q [n+2]_q}.$$

This completes the proof of the lemma. ∎

Remark 4.6. By simple computation, it can easily be verified that

$$M_{n,q}(t^r;x) = \frac{[n+1]_q!}{[n+r+1]_q!} \sum_{k=1}^{n} [k]_q[k+1]_q \cdots [k+r-1]_q p_{n,k}(q;x), \quad r \geq 1.$$

Using $[k+s]_q = [s]_q + q^s[k]_q$, we get

$$[k]_q[k+1]_q \cdots [k+r-1]_q = \prod_{s=0}^{r-1}([s]_q + q^s[k]_q) = \sum_{s=1}^{r} c_s(r)[k]_q^s,$$

where $c_s(r) > 0,\ s = 1,2,\ldots,r$ are the constants independent of k. Hence

$$M_{n,q}(t^r;x) = \frac{[n+1]_q!}{[n+r+1]_q!}\sum_{s=1}^{r} c_s(r)\sum_{k=1}^{n}[k]_q^s p_{n,k}(q;x) = \frac{[n+1]_q!}{[n+r+1]_q!}\sum_{s=1}^{r} c_s(r)[n]_q^s B_{n,q}(t^s;x).$$

Since $c_s(r) > 0$ for $s = 1,2,\ldots,r$ and $B_{n,q}(t^s;x)$ is a polynomial of degree $\le \min(s,n)$ (see [7]), we get $M_{n,q}(t^r;x)$ is a polynomial of degree $\le \min(r,n)$.

4.4.2 Rate of Approximation

Theorem 4.10. *Let* $q_n \in (0,1)$. *Then the sequence* $\{M_{n,q_n}(f)\}$ *converges to* f *uniformly on* $[0,1]$ *for each* $f \in C[0,1]$ *if and only if* $\lim_{n\to\infty} q_n = 1$.

Proof. Since the operators M_{n,q_n} are positive linear operators on $C[0,1]$ and preserve constant functions, the well-known Korovkin theorem [113] implies that $M_{n,q_n}(f;x)$ converges to $f(x)$ uniformly on $[0,1]$ as $n\to\infty$ for any $f \in C[0,1]$ if and only if

$$M_{n,q_n}(t^i;x) \to x^i \ \ (i = 1,2), \tag{4.30}$$

uniformly on $[0,1]$ as $n\to\infty$. If $q_n \to 1$, then $[n]_{q_n} \to \infty$ (see [151]) and for $s = 1,2,3$, $\lim_{n\to\infty}\frac{[n+s]_{q_n}}{[n]_{q_n}} = 1$, hence (4.30) follows from Lemma 4.5.

On the other hand, if we assume that for any $f \in C[0,1]$, $M_{n,q_n}(f,x)$ converges to $f(x)$ uniformly on $[0,1]$ as $n\to\infty$, then $q_n \to 1$. In fact, if the sequence (q_n) does not tend to 1, then it must contain a subsequence (q_{n_k}) such that $q_{n_k} \in (0,1),\ q_{n_k} \to q_0 \in [0,1)$ as $k\to\infty$. Thus, $\frac{1}{[n_k+s]_{q_{n_k}}} = \frac{1-q_{n_k}}{1-(q_{n_k})^{n_k+s}} \to (1-q_0)$ as $k\to\infty,\ s = 0,1,2,3$. Taking $n = n_k,\ q = q_{n_k}$ in $M_{n,q}(t^2;x)$, by Lemma 4.5, we get

$$M_{n_k,q_{n_k}}(t^2;x) \to x(1-q_0^2) + x^2q_0^2 \not\to x^2 \quad (k\to\infty)\ ,$$

which leads to a contradiction. Hence, $q_n \to 1$.
This completes the proof of Theorem 4.10. ■

Let $q \in (0,1)$ be fixed. We define $M_{\infty,q}(f,1) = f(1)$ and for $x \in [0,1)$

$$M_{\infty,q}(f,x) := \frac{1}{1-q}\sum_{k=1}^{\infty} p_{\infty,k}(q;x)q^{1-k}\int_0^1 f(t)p_{\infty,k-1}(q;qt)d_qt + f(0)p_{\infty,0}(q;x)$$

$$=: \sum_{k=0}^{\infty} A_{\infty k}(f)p_{\infty,k}(q;x). \tag{4.31}$$

Using (4.29), (4.31), and the fact that (see [125])

$$\sum_{k=0}^{\infty} p_{\infty,k}(q;x) = 1, \quad \sum_{k=0}^{\infty} (1-q^k) p_{\infty,k}(q;x) = x$$

and

$$\sum_{k=0}^{\infty} (1-q^k)^2 p_{\infty,k}(q;x) = x^2 + (1-q)x(1-x),$$

it is easy to prove that

$$M_{\infty,q}(1;x) = 1, \qquad M_{\infty,q}(t;x) = x,$$

and

$$M_{\infty,q}(t^2;x) = \sum_{k=0}^{\infty} (1-q^k)(1-q^{k+1}) p_{\infty,k}(q;x)$$
$$= (1-q)x + q(x^2 + (1-q)x(1-x)) = (1-q^2)x + q^2x^2.$$

For $f \in C[0,1]$, $t > 0$, we define the modulus of continuity $\omega(f,t)$ as follows:

$$\omega(f,t) := \sup_{\substack{|x-y| \le t \\ x,y \in [0,1]}} |f(x) - f(y)|.$$

Lemma 4.6. *Let $f \in C[0,1]$ and $f(1) = 0$. Then we have*

$$|A_{nk}(f)| \le A_{nk}(|f|) \le \omega(f,q^n)(1+q^{k-n})$$

and

$$|A_{\infty k}(f)| \le A_{\infty k}(|f|) \le \omega(f,q^n)(1+q^{k-n}).$$

Proof. By the well-known property of modulus of continuity (see [4], pp. 20)

$$\omega(f,\lambda t) \le (1+\lambda)\omega(f,t), \ \ \lambda > 0,$$

we get

$$|f(t)| = |f(t) - f(1)| \le \omega(f,1-t) \le \omega(f,q^n)(1+(1-t)/q^n).$$

Thus,

$$|A_{nk}(f)| \le A_{nk}(|f|) := [n+1]_q \int_0^1 q^{1-k} |f(t)| p_{n,k-1}(q;qt) d_q t$$

$$\le [n+1]_q \int_0^1 q^{1-k} \omega(f,q^n)(1+(1-t)/q^n) p_{n,k-1}(q;qt) d_q t$$

$$= \omega(f,q^n)(1+q^{-n}(1-\frac{[k]_q}{[n+2]_q}))$$

$$= \omega(f,q^n)\left(1+\frac{q^k(1-q^{n+2-k})}{q^n(1-q^{n+2})}\right) \le \omega(f,q^n)(1+q^{k-n}).$$

Similarly,

$$|A_{\infty k}(f)| \le A_{\infty k}(|f|) := \frac{q^{1-k}}{1-q} \int_0^1 |f(t)| p_{\infty,k-1}(q;qt) d_q t$$

$$\le \omega(f,q^n) \frac{q^{1-k}}{1-q} \int_0^1 (1+(1-t)/q^n) p_{\infty,k-1}(q;qt) d_q t$$

$$= \omega(f,q^n)(1+(1-(1-q^k))/q^n) = \omega(f,q^n)(1+q^{k-n}).$$

Lemma 4.6 is proved. ■

Theorem 4.11. *Let* $0 < q < 1$. *Then for each* $f \in C[0,1]$ *the sequence* $\{M_{n,q}(f;x)\}$ *converges to* $M_{\infty,q}(f;x)$ *uniformly on* $[0,1]$. *Furthermore,*

$$\|M_{n,q}(f) - M_{\infty,q}(f)\| \le C_q\ \omega(f,q^n). \tag{4.32}$$

Remark 4.7. When $f(x) = x^2$, we have

$$\|M_{n,q}(f) - M_{\infty,q}(f)\| \ge c_1 q^n \ge c_2\ \omega(f,q^n),$$

where $c_1, c_2 > 0$ are the constants independent of n. Hence, the estimate (4.32) is sharp in the following sense: The sequence q^n in (4.32) cannot be replaced by any other sequence decreasing to zero more rapidly as $n \to \infty$.

Proof. The operators $M_{n,q}$ and $M_{\infty,q}$ preserve constant functions, that is,

$$M_{n,q}(1,x) = M_{\infty,q}(1,x) = 1.$$

Without loss of generality, we assume that $f(1) = 0$. If $x = 1$, then by Lemma 4.1, we have

$$|M_{n,q}(f;1) - M_{\infty,q}(f;1)| = |A_{nn}(f) - f(1)| = |A_{nn}(f)| \le 2\omega(f,q^n).$$

For $x \in [0,1)$, by the definitions of $M_{n,q}(f;x)$ and $M_{\infty,q}(f;x)$, we know that

$$|M_{n,q}(f;x) - M_{\infty,q}(f;x)| = \Big| \sum_{k=0}^{n} A_{nk}(f) p_{n,k}(q;x) - \sum_{k=0}^{\infty} A_{\infty k}(f) p_{\infty,k}(q;x) \Big|$$

$$\leq \sum_{k=0}^{n} |A_{nk}(f) - A_{\infty k}(f)| p_{n,k}(q;x) + \sum_{k=0}^{n} |A_{\infty k}(f)| |p_{n,k}(q;x) - p_{\infty,k}(q;x)|$$

$$+ \sum_{k=n+1}^{\infty} |A_{\infty k}(f))| p_{\infty,k}(q;x) =: I_1 + I_2 + I_3.$$

First we have

$$|p_{n,k}(q;x) - p_{\infty,k}(q;x)| = \Big| \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1} (1 - q^s x) - \frac{x^k}{(1-q)^k [k]_q!} \prod_{s=0}^{\infty} (1 - q^s x) \Big|$$

$$= \Big| \begin{bmatrix} n \\ k \end{bmatrix}_q x^k (\prod_{s=0}^{n-k-1} (1 - q^s x) - \prod_{s=0}^{\infty} (1 - q^s x))$$

$$+ x^k \prod_{s=0}^{\infty} (1 - q^s x) (\begin{bmatrix} n \\ k \end{bmatrix}_q - \frac{1}{(1-q)^k [k]_q!}) \Big|$$

$$\leq p_{n,k}(q;x) \Big| 1 - \prod \lim_{s=n-k}^{\infty} (1 - q^s x) \Big|$$

$$+ p_{\infty k}(q;x) \Big| \prod_{s=n-k+1}^{n} (1 - q^s) - 1 \Big|$$

$$\leq \frac{q^{n-k}}{1-q} (p_{n,k}(q;x) + p_{\infty k}(q;x)),$$

where in the last formula, we use the following inequality, which can be easily proved by the induction on n (see [100]):

$$1 - \prod_{s=1}^{n} (1 - a_s) \leq \sum_{s=1}^{n} a_s, \quad (a_1, \ldots, a_n \in (0,1),\ n = 1, 2, \ldots, \infty).$$

Using the above inequality we get

$$|A_{nk}(f) - A_{\infty k}(f)| \leq \int_0^1 q^{1-k} |f(t)| |[n+1]_q p_{n,k-1}(q;qt) - \frac{1}{1-q} p_{\infty,k-1}(q;qt)| d_q t$$

$$\leq \int_0^1 q^{1-k} |f(t)| \Big| [n+1]_q - \frac{1}{1-q} \Big| p_{\infty,k-1}(q;qt) d_q t$$

$$+\int_0^1 q^{1-k}|f(t)|[n+1]_q\Big|p_{n,k-1}(q;qt)-p_{\infty,k-1}(q;qt)\Big|d_qt$$

$$\leq \frac{q^{n+1}}{1-q}\int_0^1 q^{1-k}|f(t)|p_{\infty,k-1}(q;qt)d_qt$$

$$+\frac{q^{n-k}}{1-q}\int_0^1 q^{1-k}|f(t)|[n+1](p_{n,k-1}(q;qt)+p_{\infty,k-1}(q;qt))d_qt$$

$$=q^{n+1}A_{\infty k}(|f|)+\frac{q^{n-k}}{1-q}A_{nk}(|f|)+q^{n-k}[n+1]_qA_{\infty k}(|f|)$$

$$\leq q^{n+1}\omega(f,q^n)(1+q^{k-n})+2\frac{q^{n-k}}{1-q}\omega(f,q^n)(1+q^{k-n})\leq \frac{5\omega(f,q^n)}{1-q}.$$

Now we estimate I_1 and I_3. We have

$$I_1\leq \frac{5\omega(f,q^n)}{1-q}\sum_{k=0}^{n}p_{n,k}(q;x)=\frac{5\omega(f,q^n)}{1-q}.$$

and

$$I_3\leq \omega(f,q^n)\sum_{k=n+1}^{\infty}(1+q^{k-n})p_{\infty,k}(q;x)\leq 2\omega(f,q^n)\sum_{k=n+1}^{\infty}p_{\infty,k}(q;x)\leq 2\omega(f,q^n).$$

Finally we estimate I_2 as follows:

$$I_2\leq \sum_{k=0}^{n}\omega(f,q^n)(1+q^{k-n})\frac{q^{n-k}}{1-q}(p_{n,k}(q;x)+p_{\infty,k}(q;x))$$

$$\leq \frac{2\omega(f,q^n)}{1-q}\sum_{k=0}^{n}(p_{n,k}(q;x)+p_{\infty,k}(q;x))\leq \frac{4\omega(f,q^n)}{1-q}.$$

We conclude that for $x\in[0,1)$,

$$|M_{n,q}(f;x)-M_{\infty,q}(f;x)|\leq C_q\omega(f,q^n),$$

where $C_q=2+\frac{9}{1-q}$. This completes the proof of Theorem 4.11. ∎

Since $M_{\infty,q}(t^2,x)=(1-q^2)x+q^2x^2>x^2$ for $0<q<1$, as a consequence of Lemma 3.10, we have the following:

Theorem 4.12. *Let $0<q<1$ be fixed and let $f\in C[0,1]$. Then $M_{\infty,q}(f;x)=f(x)$ for all $x\in[0,1]$ if and only if f is linear.*

Remark 4.8. Let $0<q<1$ be fixed and let $f\in C[0,1]$. Then by Theorem 4.11 and Theorem 4.12, it can easily be verified that the sequence $\{M_{n,q}(f;x)\}$ does not approximate $f(x)$ unless f is linear. This is completely in contrast to the classical Bernstein polynomials, by which $\{B_{n,1}(f;x)\}$ approximates $f(x)$ for any $f \in C[0,1]$.

At last, we discuss approximating property of the operators $M_{\infty,q}$.

Theorem 4.13. *For any $f \in C[0,1]$, $\{M_{\infty,q}(f)\}$ converges to f uniformly on $[0,1]$ as $q \to 1-$.*

Proof. The proof is standard. We know that the operators $M_{\infty,q}$ are positive linear operators on $C[0,1]$ and reproduce linear functions. Also,

$$M_{\infty,q}(t^2;x) = (1-q^2)x + q^2x^2 \to x^2$$

uniformly on [0,1] as $q \to 1-$. Theorem 4.5 follows from the Korovkin theorem. ■

4.5 Genuine q-Bernstein–Durrmeyer Operators

For $f \in C[0,1]$, Mahmudov and Sabancigil [121] defined the following genuine q-Bernstein–Durrmeyer operators as

$$\begin{aligned} U_{n,q}(f;x) &= [n-1]_q \sum_{k=1}^{n-1} q^{1-k} p_{n,k}(q;x) \int_0^1 f(t) p_{n-2,k-1}(q;qt) d_q t \\ &\quad + f(0) p_{n,0}(q;x) + f(1) p_{n,n}(q;x) \\ &=: \sum_{k=0}^{n} A_{nk}(f) p_{n,k}(q;x), \quad 0 \le x \le 1. \end{aligned} \tag{4.33}$$

It can be easily verified that in the case $q = 1$, the operators defined by (4.33) reduce to the genuine Bernstein–Durrmeyer operators [82].

4.5.1 Moments

Lemma 4.7 ([121]). *We have*

$$U_{n,q}(1;x) = 1, U_{n,q}(t;x) = x$$

$$U_{n,q}(t^2;x) = \frac{(1+q)x(1-x)}{[n+1]_q} + x^2$$

and

$$U_{n,q}((t-x)^2;x) = \frac{(1+q)x(1-x)}{[n+1]_q} \le \frac{2}{[n+1]_q} x(1-x).$$

Lemma 4.8 ([121]). *$U_{n,q}(t;x)$ is a polynomial of degree less than or equal to $\min\{m,n\}$.*

Proof. By simple computation,

$$U_{n,q}(t^m;x) = [n-1]_q \sum_{k=1}^{n-1} q^{1-k} p_{n,k}(q;x) \int_0^1 f(t) p_{n-2,k-1}(q;qt) t^m d_q t + p_{n,n}(q;x)$$

$$= [n-1]_q \sum_{k=1}^{n-1} p_{n,k}(q;x) \frac{[n-2]_q![k+m-1]_q!}{[k-1]_q![n+m-1]_q!} + p_{n,n}(q;x)$$

$$= \frac{[n-1]_q!}{[n+m-1]_q!} \sum_{k=1}^{n-1} p_{n,k}(q;x) \frac{[k+m-1]_q!}{[k-1]_q!} + p_{n,n}(q;x)$$

$$= \frac{[n-1]_q!}{[n+m-1]_q!} \sum_{k=1}^{n} p_{n,k}(q;x)[k]_q[k+1]_q \cdots [k+m-1]_q + p_{n,n}(q;x).$$

Next using

$$[k]_q[k+1]_q \cdots [k+m-1]_q = \prod_{s=0}^{m-1} (q^s[k]_q + [s]_q) = \sum_{s=1}^{m} c_c(m)[k]_q^s,$$

where $c_s(m) > 0, s = 1,2,3,\cdots,m$ are the constants independent of k, we get

$$U_{n,q}(t^m;x) = \frac{[n-1]_q!}{[n+m-1]_q!} \sum_{k=1}^{n} \sum_{s=1}^{m} c_s(m)[n]_q^s B_{n,q}(t^s;x),$$

where $B_{n,q}$ is the q Bernstein operator. Since $B_{n,q}(t^s;x)$ is a polynomial of degree less than or equal to $\min\{s,n\}$ and $c_s(m) > 0, s = 1,2,3,\ldots,m$, it follows that $U_{n,q}(t^m;x)$ is a polynomial of degree less than or equal to $\min\{m,n\}$. ■

4.5.2 Direct Results

The following theorems were established by [121]:

Theorem 4.14. *Let $0 < q_n < 1$. Then the sequence $\{U_{n,q}(f;x)\}$ converges to f uniformly on $[0,1]$ for each $f \in C[0,1]$, if and only if $\lim_{n\to\infty} q_n = 1$.*

Theorem 4.15. *Let $0 < q < 1$ and $n > 3$. Then for each $f \in C[0,1]$ the sequence $\{U_{n,q}(f;x)\}$ converges to $f(x)$ uniformly on $[0,1]$. Furthermore*

$$||U_{n,q}(f;.) - U_{\infty,q}(f;.)|| \le c_q \omega(f, q^{n-2}),$$

where $c_q = \frac{10}{1-q} + 4$ and $||.||$ is the uniform norm on $[0,1]$.

Theorem 4.16. *There exists an absolute constant* $C > 0$ *such that*

$$|U_{n,q}(f;x) - f(x)| \leq C\ \omega_2\left(f, \sqrt{\frac{x(1-x)}{[n+1]_q}}\right),$$

where $f \in C[0,1]$, $0 < q < 1$, *and* $x \in [0,1]$.

Proof. Using Taylor's formula

$$g(t) = g(x) + (t-x)\ g'(x) + \int_x^t (t-u)\ g''^2[0,1],$$

we obtain

$$U_{n,q}(g;x) = g(x) + U_{n,q}\left(\int_x^t (t-u)\ g''(u)\ du;x\right), g \in C^2[0,1]$$

Hence

$$|U_{n,q}(g;x) - g(x)| \leq U_{n,q}\left(\left|\int_x^t |t-u| \cdot |g''(u)|\ du\right|, x\right)$$

$$\leq U_{n,q}((t-x)^2;x) \cdot \|g''\| \leq \|g''\| \frac{2}{[n+1]_q} x(1-x).$$

Now for $f \in C[0,1]$ and $g \in C^2[0,1]$ and with the fact $||U_{n,q}(f,;.)|| \leq ||f||$, we obtain

$$|U_{n,q}(f;x) - g(x)| \leq |U_{n,q}(f-g;x)| + |U_{n,q}(g;x) - g(x)| + \|f(x) - g(x)\|$$

$$\leq 2\ \|f-g\| + \|g''\| \frac{2}{[n+1]_q} x(1-x).$$

Taking the infimum on the right hand side over all $g \in C^2[0,1]$, we obtain

$$|U_{n,q}(f;x) - f(x)| \leq 2K_2\left(f, \frac{1}{[n+1]_q} x(1-x)\right). \tag{4.34}$$

The desired results follow from (4.12), (4.34). This completes the proof of the theorem. ∎

4.6 q-Bernstein Jacobi Operators

In the year 2005, Derriennic [48] introduced the generalization of modified Bernstein polynomials for q-Jacobi weights using the q-Bernstein basis functions. For $q \in (0,1)$ and $\alpha, \beta > -1$

$$L_{n,q}^{\alpha,\beta}(f;x) = \sum_{k=0}^{n} f_{n,k,q}^{\alpha,\beta} p_{n,k}(q;x) \tag{4.35}$$

where

$$p_{n,k}(q;x) := \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1} (1-q^s x)$$

and

$$f_{n,k,q}^{\alpha,\beta} = \frac{\int_0^1 t^{k+\alpha}(1-qt)_q^{n-k+\beta} f(q^{\beta+1}t)d_qt}{\int_0^1 t^{k+\alpha}(1-qt)_q^{n-k+\beta} d_qt}.$$

It is observed in [48] that for any $n \in N$, $L_{n,q}^{\alpha,\beta}(f;x)$ is linear and positive and preserves the constant functions.
It is self adjoint. It preserves the degree of polynomials of degree $\leq n$.

The polynomial $L_{n,q}^{\alpha,\beta}(f;x)$ is well defined if there exists $\gamma \geq 0$ such that $x^\gamma f(x)$ is bounded on $(0,A]$ for some $A \in 90,1]$ and $\alpha > \gamma - 1$. Indeed $x^\alpha f(x)$ is then q-integrable for the weight $w_q^{\alpha,\beta}(x) = x^\alpha (1-qx)_q^\beta$. Thus we call that f is said to satisfy the condition $C(\alpha)$. Also $< f,g >_q^{\alpha,\beta}$ is well defined if the product fg satisfies $C(\alpha)$, particularly if f^2 and g^2 do it, where

$$< f,g >_q^{\alpha,\beta} = \int_0^{q^{\beta+1}} t^\alpha (1-q^{-\beta}t)_q^\beta f(t)g(t)d_qt$$

and

$$< f,g >_q^{\alpha,\beta} = q^{(\alpha+1)(\beta+1)} \int_0^1 t^\alpha (1-qt)_q^\beta f(q^{\beta+1}t)g(q^{\beta+1}t)d_qt.$$

4.6.1 Basic Results

Proposition 4.1. *If f verifies the condition $C(\alpha)$, we have*

$$D_q L_{n,q}^{\alpha,\beta}(f;x) = \frac{[n]_q}{[n+\alpha+\beta+2]_q} q^{\alpha+\beta+2} L_{n-1,q}^{\alpha+1,\beta+1} D_q\left(f\left(\frac{\cdot}{q}\right);qx\right), x \in [0,1]$$

Proposition 4.2. *For any $m,n \in N, x \in [0,1]$ and $q \in [1/2,1]$ if*

$$T_{n,m,q}(x) = \sum_{k=0}^n p_{n,k}(q;x) \frac{\int_0^1 t^{k+\alpha}(1-qt)_q^{n-k+\beta}(x-t)^m d_qt}{\int_0^1 t^{k+\alpha}(1-qt)_q^{n-k+\beta} d_qt}.$$

Lemma 4.9. *For any $m,n \in N, x \in [0,1]$ and $q \in [1/2,1]$ if*

$$T_{n,m,q}^1(x) = \sum_{k=0}^n p_{n,k}(q;x) \frac{\int_0^1 t^{k+\alpha}(1-qt)_q^{n-k+\beta}(x-t)_q^m d_qt}{\int_0^1 t^{k+\alpha}(1-qt)_q^{n-k+\beta} d_qt}.$$

Then for $m \geq 2$, the following recurrence formula holds

$$\begin{aligned}
&[n+m+\alpha+\beta+2]_q q^{-\alpha-2m-1} T^1_{n,m+1,q}(x) \\
&= (-x(1-x)D_q T^1_{n,m,q}(x) + T^1_{n,m,q}(x)(p_{1,m}(x) + x(1-q)[n+\alpha+\beta]_q[m+1]_q q^{1-\alpha-m}) \\
&= +T^1_{n,m-1,q}(x) p_{2,m}(x) + T^1_{n,m-2,q}(x) p_{3,m}(x)(1-q),
\end{aligned}$$

where the polynomials $p_{i,m}(x), i = 1,2,3$ are uniformly bounded with regard to n and q.

Lemma 4.10. *For any $m \in N, x \in [0,1]$ and $q \in [1/2,1]$, the expansion of $(x-t)^m$ on the Newton basis at the points $x/q^i, i = 0,1,2,\ldots .m-1$ is*

$$(x-t)^m = \sum_{k=1}^{m} d_{m,k}(1-q)^{m-k}(x-t)^k_q, \tag{4.36}$$

where the coefficient $d_{m,k}$ verify $|d_{m,k}| \leq d_m, k = 1,2,\ldots,m$ and d_m does not depend on x,t,q.

Remark 4.9. From Lemmas 4.9 and 4.10, we have for any m there exists a constant $K_m > 0$ independent of n and q, such that

$$\sup_{x\in[0,1]} |T_{n,m,q}(x)| \leq \begin{cases} \frac{K_m}{[n]_q^{m/2}}, & \text{if } m \text{ is even} \\ \frac{K_m}{[n]_q^{(m+1)/2}}, & \text{if } m \text{ is odd}. \end{cases}$$

Remark 4.10. The sequence (q_n) has the property S if and only if there exists $n \in N$ and $c > 0$ such that for any $n > N, 1 - q_n < c/n$.

4.6.2 Convergence

Theorem 4.17. *If f is continuous at the point $x \in (0,1)$, then*

$$\lim_{n\to\infty} L^{\alpha,\beta}_{n,q_n}(f;x) = f(x)$$

in the following cases:

1. *If f is bounded on $[0,1]$ and the sequence (q_n) is such that $\lim_{n\to\infty} q_n = 1$*
2. *If there exist real numbers $\alpha',\beta' \geq 0$ and a real $k' > 0$ such that, for any $x \in (0,1), |x^{\alpha'}(1-x)^{\beta'} f(x)| \leq k', \alpha' < \alpha+1, \beta' < \beta+1$ and the sequence (q_n) owns the property S*

Theorem 4.18. *If the function f admits a second derivative at the point $x \in [0,1]$, then as in cases 1 and 2 of Theorem 4.17, we have*

$$\lim_{n\to\infty}[n]_{q_n}[L_{n,q}^{\alpha,\beta}(f;x)-f(x)] = \frac{d}{dx}\frac{\left(x^{\alpha+1}(1-x)^{\beta+1}f'x)\right)}{x^{\alpha}(1-x)^{\beta}} \tag{4.37}$$

Proof. By Taylor's formula, we have

$$f(t) = f(x)+(t-x)f'(x)+\frac{(t-x)^2}{2!}f''(x)+(t-x)^2\varepsilon(t-x),$$

where $\lim_{u\to 0}\varepsilon(u)=0$. Thus

$$L_{n,q_n}^{\alpha,\beta}(f;x)-f(x) = -f'(x)T_{n,1,q_n}(x)+\frac{f''(x)}{2!}T_{n,2,q_n}(x)+R_n(x),$$

where $R_n(x)=L_{n,q_n}^{\alpha,\beta}((t-x)^2\varepsilon(t-x);x)$. Using $\lim_{q\to 1}[a]_q = a$ for any $a\in R$. Using Lemmas 4.9 and 4.10, we have $\lim_{[n]_{q_n}\to\infty}[n]_{q_n}T_{n,1,q_n}(x)=(\alpha+\beta+2)x-\alpha-1$ and $\lim_{[n]_{q_n}\to\infty}[n]_{q_n}T_{n,2,q_n}(x)=2x(1-x)$. The result follows immediately if we show that $\lim_{[n]_{q_n}\to\infty}[n]_{q_n}R_n(x)=0$. Proceeding along the same manner as in Theorem 4.17. For any $\eta>0$ we can find a $\delta>0$ such that for n large enough $\varepsilon(t-x)<\eta$ if $|x-q_n^{\beta+1}t|<\delta$.

We obtain the inequality $|(t-x)^2\varepsilon(t-x)|\le\eta(x-t)^2+(\rho_x+|f(t)|)I_{x,\delta}(q^{-(\beta+1)}t)$ for any $t\in(0,1)$ where ρ_x is independent of t and δ. We deduce

$$[n]_{q_n}|R_n(x)| \le \begin{cases} [n]_{q_n}\left(\eta T_{n,2,q_n}(x)+(\rho_x+k)\,T_{n,4,q_n}(x)/\delta^4\right), & \text{in case 1}\\ [n]_{q_n}\left(\eta T_{n,2,q_n}(x)+\rho_x T_{n,4,q_n}(x)/\delta^4\right)+k'nE_n(x,\delta), & \text{in case 1}\end{cases}$$

The right hand side tends to $2\eta x(1-x)$ when n (hence $[n]_{q_n}$) tends to infinity is as small as wanted. ■