Chapter 2
q-Discrete Operators and Their Results

This chapter deals with the q-analogue of some discrete operators of exponential type. We study some approximation properties of the q-Bernstein polynomials, q-Szász–Mirakyan operators, q-Baskakov operators, and q-Bleimann, Butzer, and Hahn operators. Here, we present moment estimation, convergence behavior, and shape-preserving properties of these discrete operators.

2.1 q-Bernstein Operators

After the development of quantum calculus, A. Lupaş was the first who gave the q-analogue of the Bernstein polynomials. Let $f \in C[0,1]$. The linear operator $L_{n,q}: C[0,1] \to C[0,1]$, defined by

$$L_{n,q}(f;x) := \sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right) b_{n,k}^{q}(x), \tag{2.1}$$

where

$$b_{n,k}^{q}(x) = \frac{\begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-1)/2} x^k (1-x)^{n-k}}{\prod_{j=0}^{n-1}(1-x+q^j x)}$$

is called Lupaş q-analogue of Bernstein polynomials. He established some direct results for the operators $L_{n,q}$, which were later studied in details by Ostrovska [127].

In the year 1997 Phillips [133] introduced another q-analogue of Bernstein polynomials by using the q-binomial coefficients and the q-binomial theorem. Phillips and his colleagues have intensively studied these operators and many applications and generalizations have been investigated (see [134] and references therein). Also Ostrovska (see [125, 126, 129, 130]) established some interesting properties on such operators. In [128], she gave a systematic study on these

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operators on the completion of one of the decade on q-Bernstein polynomials. Also the other researcher who worked on q-Bernstein polynomials, we mention the work of Wang Heping and collaborators [96–98, 100]. Recently II'inski and Ostrovska [102, 125] obtained new results about convergence properties of the q-Bernstein polynomials. This section is based on the q-Bernstein polynomials by Phillips [133].

2.1.1 Introduction

We can verify by induction, using (1.2) or (1.3), that

$$(1+x)(1+qx)\dots(1+q^{k-1}x) = \sum_{r=0}^{k} q^{\frac{r(r-1)}{2}} \begin{bmatrix} k \\ r \end{bmatrix}_q x^r \tag{2.2}$$

which generalizes the binomial expansion.

For any real function f we define q-differences recursively from

$$\Delta_q^0 f_i = f_i$$

for $i = 0, 1, \dots, n$, where n is a fixed positive integer, and

$$\Delta_q^{k+1} f_i = \Delta_q^k f_{i+1} - q^k \Delta_q^k f_i \tag{2.3}$$

for $k = 0, 1, \dots, n-i-1$, where f_i denotes $f(\frac{[i]_q}{[n]_q})$. When $q = 1$, these reduce to ordinary forward differences. It is easily established by induction that the q-differences satisfy

$$\Delta_q^k f_i = \sum_{r=0}^{k} (-1)^r q^{\frac{r(r-1)}{2}} \begin{bmatrix} k \\ r \end{bmatrix}_q f_{i+k-r}. \tag{2.4}$$

See Schoenberg [140], Lee and Phillips [108] and [134, p. 46].

2.1.2 Bernstein Polynomials

Theorem 2.1. *For each positive integer n, we define*

$$B_n(f;x) = \sum_{r=0}^{n} f_r \begin{bmatrix} n \\ r \end{bmatrix}_q x^r \prod_{r=0}^{n-r-1} (1-q^s x), \tag{2.5}$$

where an empty product denotes 1 *and, as above,* $f_r = f(\frac{[r]_q}{[n]_q})$. *When* $q = 1$, *we obtain the classical Bernstein polynomial. We observe immediately from (2.5) that, independently of q,*

$$B_n(f;0) = f(0),\ B_n(f;1) = f(1) \tag{2.6}$$

for all functions f. We now state a generalization of the well-known forward difference form (see, e.g., Davis [46]) of the classical Bernstein polynomial.

Theorem 2.2. *The generalized Bernstein polynomial defined by (2.5) may be expressed in the q-difference form*

$$B_n(f;x) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix}_q \Delta_q^r f_0 x^r. \tag{2.7}$$

Proof. The coefficient of x^k in (2.5) is

$$\sum_{s=0}^{\infty} f_{k-s} \begin{bmatrix} n \\ k-s \end{bmatrix}_q (-1)^s q^{\frac{s(s-1)}{2}} \begin{bmatrix} n-k+s \\ s \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q \sum_{s=0}^{k} (-1)^s q^{\frac{s(s-1)}{2}} \begin{bmatrix} k \\ s \end{bmatrix}_q f_{k-s}.$$

We see immediately from the expansion of the q-difference (2.4) that the coefficient of x^k in (2.5) simplifies to give $\begin{bmatrix} n \\ k \end{bmatrix}_q \Delta_q^k f_0$, thus verifying (2.7). ■

We note in passing that (2.7) provides an efficient means of computing $B_n(f;x)$, using (1.2) or (1.3) to evaluate the q-binomial coefficient recursively and (2.3) to compute the q-difference recursively. Let us write the interpolating polynomial for f at the points $x_0, \ldots, x_n$ in the Newton divided difference form

$$p_n(x) = \sum_{r=0}^{n} (\prod_{s=0}^{r-1} (x - x_s)) f[x_0, \ldots, x_r],$$

where the empty product denotes 1. For the choice of points $x_r = \frac{[r]_q}{[n]_q}$, $0 \le r \le n$, we can express the divided differences in the form of q-differences. Specifically, we may verify by induction on k that

$$f[x_i, \ldots, x_{i+k}] = q^{\frac{-k(2i+k-1)}{2}} [n]_q^k \frac{\Delta_q^k f_i}{[k]!}. \tag{2.8}$$

(See Schoenberg [140], Lee and Phillips [108].) From the uniqueness of the interpolating polynomial it is clear that if f is a polynomial of degree m, then $\Delta_q^r f_0 = 0$ for $r > m$ and $\Delta_q^m f_0 \ne 0$. Thus it follows from (2.7) that, if f is a polynomial of degree m, then $B_n(f;x)$ is a polynomial of degree $\min(m,n)$. In particular, we will evaluate $B_n(f;x)$ explicitly for $f(x) = 1, x$ and x^2. First we obtain

$$B_n(1;x) = 1 \tag{2.9}$$

and with $f(x) = x$, we have $f_0 = 0$ and

$$\Delta_q f_0 = f_1 - f_0 = \frac{[1]_q}{[n]_q} - \frac{[0]_q}{[n]_q} = \frac{1}{[n]_q}.$$

Since $\begin{bmatrix} n \\ 1 \end{bmatrix}_q = [n]_q$, we deduce from (2.7) that

$$B_n(x;x) = x. \tag{2.10}$$

For $f(x) = x^2$, we compute $f_0 = 0$,

$$\Delta_q f_0 = \left(\frac{[1]_q}{[n]_q}\right)^2 - \left(\frac{[0]_q}{[n]_q}\right)^2 = \frac{1}{[n]_q^2}$$

and using (2.4),

$$\Delta_q^2 f_0 = \left(\frac{[2]_q}{[n]_q}\right)^2 - \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \left(\frac{[1]_q}{[n]_q}\right)^2 + q\left(\frac{[0]_q}{[n]_q}\right)^2 = \frac{(1+q)^2-(1+q)}{[n]_q^2} = \frac{q(1+q)}{[n]_q^2}.$$

Thus

$$B_n(x^2;x) = [n]_q \frac{1}{[n]_q^2} x + \frac{[n]_q [n-1]_q}{[2]_q} \frac{q(1+q)}{[n]_q^2} x^2$$

and, since $[2]_q = 1+q$ and $q\,[n-1]_q = [n]_q - 1$, we obtain

$$B_n(x^2;x) = x^2 + \frac{x(1-x)}{[n]_q}. \tag{2.11}$$

We note that the relations (2.9) and (2.10) are identical to those obtained for the classical Bernstein polynomials (corresponding to the case $q = 1$), while (2.11) differs only in having $[n]_q$ in place of n.

2.1.3 Convergence

In the classical case, the uniform convergence of $B_n(f;x)$ to $f(x)$ on $[0,1]$ for each $f \in C[0,1]$ is assured by the following two properties:

1. B_n is a positive operator.
2. $B_n(f;x)$ converges uniformly to $f \in C[0,1]$ for $f(x) = 1, x$ and x^2.

Recall that if a linear operator L maps an element $f \in C[0,1]$ to $Lf \in C[0,1]$, then L is said to be monotone if $f(x) \geq 0$ on $[0,1]$ implies that $Lf(x) \geq 0$ on $[0,1]$. We observe that the generalized Bernstein operator defined by (2.5) is monotone for $0 < q < 1$. On the other hand, for a fixed value of q with $0 < q < 1$, we see that

$$[n]_q \to \frac{1}{1-q} \text{ as } n \to \infty$$

and in the case it is clear from (2.11) that $B_n(x^2;x)$ does not converge to x^2. To obtain a sequence of generalized Bernstein polynomials with $q \neq 1$ which converges, we let $q = q_n$ depend on n. We then choose a sequence (q_n) such that

$$1 - \frac{1}{n} \leq q_n < 1.$$

Then we have

$$1 - \frac{r}{n} \leq q_n^r < 1 \text{ for } 1 \leq r \leq n-1$$

and thus

$$[n]_{q_n} = 1 + q_n + q_n^2 + \ldots + q_n^{n-1} \geq n - \frac{1}{2}(n-1) = \frac{1}{2}(n+1),$$

so that $[n]_{q_n} \to \infty$ as $n \to \infty$.

We now state formally our result on convergence.

Theorem 2.3. *Let $q = (q_n)$ satisfy $0 < q_n < 1$ and let $q_n \to 1$ as $n \to \infty$. Then, if $f \in C[0,1]$,*

$$B_n(f;x) = \sum_{r=0}^{n} f_r \begin{bmatrix} n \\ r \end{bmatrix}_{q_n} x^r \prod_{s=0}^{n-r-1} (1 - q_n^s x)$$

converges uniformly to f on $[0,1]$.

Proof. This is a special case of the Bohman–Korovkin theorem. (See, e.g., Cheney [42], Lorentz [114].) Alternatively, we may follow the proof given in Rivlin [138] for the convergence of the classical Bernstein polynomials, except that n must be replaced by $[n]_q$ when estimating how closely $B_n(x^2;x)$ approximates to x^2, as in (2.11) above. ■

Given a function f defined on $[0,1]$, let

$$w(\delta) = \sup_{|x_1 - x_2| < \delta} | f(x_1) - f(x_2) |,$$

the usual modulus of continuity, where the supremum is taken over all $x_1, x_2 \in [0,1]$ such that $| x_1 - x_2 | \leq \delta$. Then we have:

Theorem 2.4. *If f is bounded on $[0,1]$ and B_n denotes the generalized Bernstein operator defined by (2.5), then*

$$\| f - B_n f \|_\infty \leq \frac{3}{2} w\left(\frac{1}{[n]_q^{1/2}}\right). \tag{2.12}$$

Proof. Rivlin [138] states this theorem for the case where $q_n = 1$ for all n, and his proof is easily adapted to justify (2.12). ■

2.1.4 Voronovskaya's Theorem

In this section we will follow Davis [46], beginning with the sums

$$S_m(x) = \sum_{r=0}^{n} ([r]_q - [n]_q x)^m \begin{bmatrix} n \\ r \end{bmatrix}_q x^r \prod_{s=0}^{n-r-1} (1 - q^s x). \tag{2.13}$$

Let us write

$$([r]_q - [n]_q x)^m = [n]_q^m \sum_{s=0}^{m} \binom{m}{s} \left(\frac{[r]_q}{[n]_q}\right)^s (-x)^{m-s}$$

and so express $S_m(x)$ in the form

$$S_m(x) = [n]_q^m \sum_{s=0}^{m} \binom{m}{s} (-x)^{m-s} B_n(x^s;x). \tag{2.14}$$

We have already noted that $B_n(x^s;x)$ is a polynomial of degree $\min(s,n)$ in x. Thus $S_m(x)$ is a polynomial of degree at most m in x. Since B_n is a linear operator we also obtain from (2.14) that

$$S_m(1) = [n]_q^m B_n((x-1)^m;1) = 0, \tag{2.15}$$

using the property (2.6) that $B_n(f;x)$ interpolates $f(x)$ at $x=0$ and $x=1$. We deduce from (2.15) that $(1-x)$ is a factor of $S_m(x)$, for $m>0$. From (2.7) we find by direct calculation (using the symbolic language Maple) that S_6 has the form

$$S_6(x) = (1-x) \sum_{r=1}^{5} a_r x^r [n]_q^r, \tag{2.16}$$

where a_r is a polynomial in x and q. In the lemma below, we are concerned with the dependence of S_6 on $[n]_q$. The coefficients a_1, a_2 and a_3 are as follows:

$$a_1 = 1-(4+10q+10q^2+5q^3+q^4)x+ \\ (6+20q+35q^2+39q^3+29q^4+15q^5+5q^6+q^7)x^2- \\ (4+15q+31q^2+46q^3+51q^4+44q^5+29q^6+14q^7+5q^8+q^9)x^3+ \\ (1+4q+9q^2+15q^3+20q^4+22q^5+20q^6+15q^7+9q^8+4q^9+q^{10})x^4,$$

$$a_2 = (-1+10q+10q^2+5q^3+q^4)+ \\ (3-4q-36q^2-52q^3-40q^4-19q^5-6q^6-q^7)x \\ (-3-3q+12q^2+46q^3+72q^4+67q^5+43q^6+19q^7+6q^8+q^9)x^2+ \\ (1+2q-7q^3-20q^4-30q^5-32q^6-25q^7-13q^8-5q^9-q^{10})x^3,$$

and

$$a_3 = (1-16q+q^2+13q^3+11q^4+4q^5+q^6)+ \\ (-2+6q+30q^2+5q^3-22q^4-26q^5-15q^6-5q^7-q^8)x \\ (1-6q^2-14q^3-6q^4+9q^5+15q^6+11q^7+4q^8+q^9)x^2.$$

We have quoted the values of a_1, a_2 and a_3 for the sake of completeness, although we do not need to know their values. However, we do require the values of

$$a_4 = (1-q)^2(q(10+10q+5q^2+q^3)(1-qx)-1+x) \tag{2.17}$$

and

$$a_5 = (1-q)^4. \tag{2.18}$$

The presence of the factors $(1-q)^2$ and $(1-q)^4$ in (2.17) and (2.18), respectively, proves to be significant. For we observe that, with $0<q<1$, we have

$$0 < 1-q < \frac{1}{[n]_q} \tag{2.19}$$

for any positive integer n. Thus if $[n]_{q_n} \to \infty$ as $n \to \infty$, we see from (2.16) that $S_6(x)$ behaves like $[n]_{q_n}^3$ for large n and not like $[n]_{q_n}^5$. We now give a generalization of Lemma 6.3.5 of Davis [46] as a prelude to a generalization of Voronovskaya's theorem.

Lemma 2.1. *Let $q = q_n$ satisfy $0 < q_n < 1$ and let $q_n \to 1$ as $n \to \infty$. Then there exists a constant C independent of n such that, for all $x \in [0,1]$,*

$$\sum_{\left|\frac{[r]_{q_n}}{[n]_{q_n}}-x\right| \ge [n]_{q_n}^{-\frac{1}{4}}} \begin{bmatrix} n \\ r \end{bmatrix}_{q_n} x^r \prod_{s=0}^{n-r-1}(1-q^s x) \le \frac{C}{[n]_{q_n}^{\frac{3}{2}}}.$$

Proof. From the inequality (2.19), together with (2.16)–(2.18), we deduce that there exists a constant C independent of n such that

$$| S_6(x) | \leq C [n]_{q_n}^3$$

for all $x \in [0,1]$. Since

$$\left| \frac{[r]_{q_n}}{[n]_{q_n}} - x \right| \geq [n]_{q_n}^{-\frac{1}{4}} \Rightarrow \frac{([r] - [n]_{q_n} x)^6}{[n]_{q_n}^{\frac{9}{2}}} \geq 1,$$

it follows that

$$\sum_{\left|\frac{[r]_{q_n}}{[n]_{q_n}} - x\right| \geq [n]^{-\frac{1}{4}}} \begin{bmatrix} n \\ r \end{bmatrix}_{q_n} x^r \prod_{s=0}^{n-r-1} (1 - q_n^s x) \leq \frac{1}{[n]_{q_n}^{\frac{9}{2}}} S_6(x) \leq \frac{C}{[n]_{q_n}^{\frac{3}{2}}}.$$ ■

Theorem 2.5. *Let f be bounded on $[0,1]$ and let x_0 be a point of $[0,1]$ at which $f''(x_0)$ exists. Further, let $q = q_n$ satisfy $0 < q_n < 1$ and let $q_n \to 1$ as $n \to \infty$. Then the rate of convergence of the sequence of generalized Bernstein polynomials is governed by*

$$\lim_{n\to\infty} [n]_{q_n} (B_n(f;x_0) - f(x_0)) = \frac{1}{2} x_0 (1 - x_0) f''(x_0). \tag{2.20}$$

Proof. We replace Lemma 6.3.5 of Davis [46] by the lemma stated above and then the proof of Theorem 6.3.6 of Davis is readily extended to justify (2.20). Thus the error $B_n(f;x) - f(x)$ tends to zero like $\frac{1}{[n]_{q_n}}$. At best this is like $\frac{1}{n}$, for the classical Bernstein polynomials. However, through our choice of the sequence (q_n), we can achieve a rate of convergence which is slower than $\frac{1}{n}$ and indeed may be as slow as we please. Such a birthday gift! ■

2.2 q-Szász Operators

In this section, we give a generalization of Szász–Mirakyan operators based on q-integers that we call q-Szász–Mirakyan operators. Depending on the selection of q, these operators are more flexible than the classical Szász–Mirakyan operators while retaining their approximation properties. For these operators, we give a Voronovskaya-type theorem related to q-derivatives. Furthermore, we obtain convergence properties for functions belonging to particular subspaces of $C[0, \infty)$ and give some representation formulae of q-Szász–Mirakyan operators and their rth q-derivatives. This section is based on [25, 29].

2.2.1 Introduction

In this section, as Phillips has done for Bernstein operators, we introduce a similar modification of the Szász–Mirakyan operators [148] that we call q-Szász–Mirakyan operators and examine the main properties of this new approximation process. Recall that the Bernstein operators were defined with the aid of the functions defined on $[0,\ 1]$ as opposed to the classical Szász–Mirakyan operators which are defined on $\mathbb{R}_0 := [0,\ \infty)$ in order to analyze the approximation problems for the functions defined on the same interval. Although, from the structural point of view q-Szász–Mirakyan operators have some resemblances to classical Szász–Mirakyan operators, they have some similarities to Bernstein–Chlodowsky operators from the properties of convergence standpoint. That is, the interval of convergence grows as $n \to \infty$ as in Bernstein–Chlodowsky operators. Our new operators with this construction are sensitive or flexible to the rate of convergence to f. That is, the proposed estimate with rates in terms of modulus of continuity tells us that, depending on our selection of q, the rates of convergence in weighted norm of the new operators are better than the classical Bernstein–Chlodowsky operators.

2.2.2 Construction of Operators

For $0 < q < 1$, we now define new operators that we call the q-Szász–Mirakyan operators as follows:

$$\begin{aligned} S_n(f;q;x) &:= S_n^q(f;x) \\ &:= E_q\left(-[n]_q \frac{x}{b_n}\right) \sum_{k=0}^{\infty} f\left(\frac{[k]_q b_n}{[n]_q}\right) \frac{\left([n]_q x\right)^k}{[k]_q!\,(b_n)^k}, \end{aligned} \tag{2.21}$$

where $0 \le x < \alpha_q(n)$, $\alpha_q(n) := \frac{b_n}{(1-q)[n]_q}$, $f \in C(\mathbb{R}_0)$, and (b_n) is a sequence of positive numbers such that $\lim\limits_{n\to\infty} b_n = \infty$.

We observe that these operators are positive and linear. Furthermore, in the case of $q = 1$, the operators (2.21) are similar to the classical Szász–Mirakyan operators.

By the properties of the series in (1.7), the interval of domain of operators (2.21) is the interval $0 \le x < \alpha_q(n)$ for $0 < q < 1$; in the mean while the operators are interpolating the function f on $\mathbb{R}_0$. Note that the interval of convergence grows as $n \to \infty$. A similar situation arises for Bernstein–Chlodowsky operators (see [35, 43, 70, 113]).

We denote by e_m the test functions defined by $e_m(t) := t^m$ for every integer $m \ge 0$ and for each $x \ge 0$, $\varphi_x(t) := t - x$ such that $t - x \ge 0$.

2.2.3 Auxiliary Result

In the sequel, we need the following results:

Lemma 2.2. *For $0 \le x < \alpha_q(n)$, when $0 < q < 1$ and integer $m \ge 0$, we have*

$$S_n^q(e_m;x) = q^{\frac{m(m-1)}{2}} x^m + \sum_{j=1}^{m-1} \left(\frac{b_n}{[n]_q}\right)^{m-j} \mathbb{S}_q(m,j)\, q^{\frac{j(j-1)}{2}} x^j, \tag{2.22}$$

where $\mathbb{S}_q(m,j)$ are q-Stirling polynomials of the second kind.

Proof. Using (1.8) and the Cauchy rule for multiplication of two series [21, p. 376], from (2.21) we have the representation

$$\begin{aligned} S_n^q(f;x) &= \sum_{j=0}^{\infty}\sum_{i=0}^{j} (-1)^i q^{\frac{i(i-1)}{2}} f\left(\frac{[j-i]_q b_n}{[n]_q}\right) \frac{\left([n]_q x\right)^j}{[i]_q!\,[j-i]_q!\,b_n^j} \\ &= \sum_{j=0}^{\infty} \left(\frac{[n]_q}{b_n}\right)^j \Delta_q^j f_0 \frac{x^j}{[j]_q!}, \end{aligned} \tag{2.23}$$

where $\Delta_q^j f_0$ as in (2.4).

We can easily see from (2.4) for $f(x) = x^m$, $m = 0, 1, 2, \ldots$

$$\Delta_q^j(t^m)(0) = \sum_{i=0}^{j} (-1)^i q^{\frac{i(i-1)}{2}} \begin{bmatrix} j \\ i \end{bmatrix} \left(\frac{b_n}{[n]_q}\right)^m [j-i]_q^m$$

for $j \ge 0$.

Also we know that the connection with q-differences $\Delta_q^j f_0$ and jth derivative $f^{(j)}$ is the following:

$$\frac{\Delta_q^j f_0}{q^{j(j-1)}[j]_q!} = \frac{f^{(j)}(\xi)}{j!},$$

where $\xi \in (0,[j])$ (see [134, p. 268]). From this equality, it is obvious that q-differences of monomial t^m of order greater than m are zero. Thus, we have

$$\begin{aligned} S_n^q(e_m;x) &= \sum_{j=0}^{m} \left(\frac{b_n}{[n]_q}\right)^{m-j} \mathbb{S}_q(m,j)\, q^{\frac{j(j-1)}{2}} x^j \\ &= q^{\frac{m(m-1)}{2}} x^m + \sum_{j=1}^{m-1} \left(\frac{b_n}{[n]_q}\right)^{m-j} \mathbb{S}_q(m,j)\, q^{\frac{j(j-1)}{2}} x^j, \end{aligned}$$

where

$$\mathbb{S}_q(m, j) = \frac{1}{[j]_q! q^{\frac{j(j-1)}{2}}} \sum_{i=0}^{j} (-1)^i q^{\frac{i(i-1)}{2}} \begin{bmatrix} j \\ i \end{bmatrix} [j-i]_q^m$$

are the Stirling polynomials of the second kind satisfying the equality

$$\mathbb{S}_q(m+1, j) = \mathbb{S}_q(m, j-1) + [j]_q \mathbb{S}_q(m, j),$$

for $m \geq 0$ and $j \geq 1$ with $\mathbb{S}_q(0,0) = 1$, $\mathbb{S}_q(m,0) = 0$ for $m > 0$. Also $\mathbb{S}_q(m, j) = 0$ for $j > m$. Thus the proof is completed. ■

Lemma 2.2 gives the explicit expression of $S_n^q(e_m; x)$ for $m = 0, 1, 2$:

$$S_n^q(e_0; x) = 1 \tag{2.24}$$

$$S_n^q(e_1; x) = x \tag{2.25}$$

$$S_n^q(e_2; x) = qx^2 + \frac{b_n}{[n]_q} x. \tag{2.26}$$

The equality (2.24) can also be obtained from $(1.1, b)$.

Remark 2.1. Since for a fixed value of q with $0 < q < 1$,

$$\lim_{n\to\infty} [n]_q = \frac{1}{1-q},$$

to ensure the convergence properties of (2.21), we will assume $q = q_n$ as a sequence such that $q_n \to 1$ as $n \to \infty$ for $0 < q_n < 1$ and so that $[n]_{q_n} \to \infty$ as $n \to \infty$.

2.2.4 Convergence of $S_n^{q_n}(f)$

Proposition 2.1 (q-L'Hopital's Rule). *Suppose that throughout some interval containing a, each of f and g are q-differentiable and continuous functions and $(D_q g)(x) \neq 0$ for $q \in (0,1) \cup (1,\infty)$. If*

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$$

and there exists $\widehat{q} \in (0,1)$ such that for all $q \in (\widehat{q}, 1) \cup \left(1, \widehat{q}^{-1}\right)$

$$\lim_{x\to a} \frac{(D_q f)(x)}{(D_q g)(x)} = L,$$

then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=L.$$

Proof. Suppose that x is close enough to a so that throughout the interval between a and x, f and g are q-differentiable with $(D_q g)(x)\neq 0$. Then by q-Lagrange theorem (see [137]), there exists $\widehat{q}\in(0,1)$ such that for all $q\in(\widehat{q},1)\cup\left(1,\widehat{q}^{-1}\right)$

$$\frac{f(x)}{g(x)}=\frac{(D_q f)(\mu_1)}{(D_q g)(\mu_2)},$$

where $\mu_1,\mu_2\in(a,x)$.

Since μ_1 and μ_2 are between a and x, $x\to a$ implies that $\mu_1\to a$ and $\mu_2\to a$. Hence for all $q\in(\widehat{q},1)\cup\left(1,\widehat{q}^{-1}\right)$,

$$\begin{aligned}\lim_{x\to a}\frac{f(x)}{g(x)}&=\lim_{x\to a}\frac{(D_q f)(\mu_1)}{(D_q g)(\mu_2)}\\&=\lim_{x\to a}\frac{(D_q f)(x)}{(D_q g)(x)}.\end{aligned}$$

■

Now we give a Voronovskaya-type relation for the operator (2.21).

Theorem 2.6. *Let $f\in C(\mathbb{R}_0)$ be a bounded function and (q_n) denote a sequence such that $0<q_n<1$ and $q_n\to 1$ as $n\to\infty$. Suppose that the second derivative $D^2_{q_n}f(x)$ exists at a point $x\in[0,\alpha_{q_n}(n))$ for n large enough. If $\lim_{n\to\infty}\frac{b_n}{[n]_{q_n}}=0$, then*

$$\lim_{n\to\infty}\frac{[n]_{q_n}}{b_n}\left(S_n^{q_n}(f;x)-f(x)\right)=\frac{1}{2}x\lim_{q_n\to 1}D^2_{q_n}f(x).$$

Proof. By the q-Taylor formula [137] for f, we have

$$f(t)=f(x)+D_q f(x)(t-x)+\frac{1}{[2]_q}D_q^2 f(x)(t-x)_q^2+\Phi_q(x;t)(t-x)_q^2$$

for $0<q<1$ where $(t-x)_q^2=(t-x)(t-qx)$. By application of q-L'Hopital's Rule, there exists $\widehat{q_1}\in(0,1)$ such that for all $q\in(\widehat{q}_1,1)$

$$\lim_{t\to x}\Phi_q(x;t)=\lim_{t\to x}\frac{D_q f(t)-D_q f(x)-D_q^2 f(x)(t-x)}{[2]_q(t-x)}$$

where we use the equality

$$\left(D_q(t-x)_q^n\right)(t)=[n]_q(t-x)_q^{n-1},$$

where

$$(t-x)_q^n = \prod_{k=0}^{n-1}\left(t-q^k x\right)$$

(see [59]).

By applying again of q-L'Hopital's Rule, there exist $\widehat{q_2} \in (0,1)$ $(\widehat{q_1} < \widehat{q_2})$ such that for all $q \in (\widehat{q_2}, 1)$

$$\lim_{t\to x}\Phi_q(x;t) = \lim_{t\to x}\frac{D_q^2 f(t) - D_q^2 f(x)}{[2]_q} = 0. \tag{2.27}$$

By assumption the function $\Phi_q(t) := \Phi_q(t;x)$ is a bounded function for all $q \in (\widehat{q_2}, 1)$. Consequently, we can write

$$\begin{aligned}
&\frac{[n]_{q_n}}{b_n}\left(S_n^{q_n}(f;x) - f(x)\right) \\
&\quad = D_{q_n} f(x)\frac{[n]_{q_n}}{b_n}S_n^{q_n}(\varphi_x;x) + \frac{D_{q_n}^2 f(x)}{[2]_{q_n}}\frac{[n]_{q_n}}{b_n}S_n^{q_n}\left((t-x)_{q_n}^2;x\right) \\
&\qquad + \frac{[n]_{q_n}}{b_n}S_n^{q_n}\left(\Phi_{q_n}(t)(t-x)_{q_n}^2;x\right) \\
&\quad = D_{q_n} f(x)\frac{[n]_{q_n}}{b_n}S_n^{q_n}(\varphi_x;x) + \frac{D_{q_n}^2 f(x)}{[2]_{q_n}}\frac{[n]_{q_n}}{b_n}S_n^{q_n}\left(\varphi_x^2;x\right) \\
&\qquad + \frac{[n]_{q_n}}{b_n}\left(\frac{D_{q_n}^2 f(x)}{[2]_{q_n}}x(1-q_n)S_n^{q_n}(\varphi_x;x) + S_n^{q_n}\left(\Phi_{q_n}(t)(t-x)_{q_n}^2;x\right)\right)
\end{aligned}$$

By (2.24)–(2.26), we get

$$\lim_{n\to\infty}\frac{[n]_{q_n}}{b_n}S_n^{q_n}(\varphi_x;x) = 0 \tag{2.28}$$

$$\lim_{n\to\infty}\frac{[n]_{q_n}}{b_n}S_n^{q_n}\left(\varphi_x^2;x\right) = x \tag{2.29}$$

and thus

$$\begin{aligned}
&\lim_{n\to\infty}\frac{[n]_{q_n}}{b_n}\left(S_n^{q_n}(f;x) - f(x)\right) \\
&= \frac{1}{2}x\lim_{q_n\to 1}D_{q_n}^2 f(x) + \lim_{n\to\infty}\frac{[n]_{q_n}}{b_n}S_n^{q_n}\left(\Phi_{q_n}(t)(t-x)_{q_n}^2;x\right)
\end{aligned}$$

Now, the last term on the right-hand side can be estimated in the following way. Since $\lim_{t\to x}\Phi_{q_n}(t)=0$, then for all $\varepsilon>0$, there exists $\delta>0$ such that $|t-x|<\delta$ implies $\left|\Phi_{q_n}(t)\right|<\varepsilon$ for $x\in[0,\alpha_{q_n}(n))$ where n is large enough. While if $|t-x|\geq\delta$, then $\left|\Phi_{q_n}(t)\right|\leq\frac{M}{\delta^2}\varphi_x^2(t)$, where $M>0$ is a constant. Hence we can infer

$$\begin{aligned}
&\frac{[n]_{q_n}}{b_n}\left(S_n^{q_n}\left(\Phi_{q_n}(t)(t-x)_{q_n}^2;x\right)\right)\\
&=\frac{[n]_{q_n}}{b_n}\left(S_n^{q_n}\left(\Phi_{q_n}(t)\varphi_x^2;x\right)\right)\\
&\quad+\frac{[n]_{q_n}}{b_n}x(1-q_n)\left(S_n^{q_n}\left(\Phi_{q_n}(t)\varphi_x;x\right)\right)\\
&\leq\varepsilon\frac{[n]_{q_n}}{b_n}\left(S_n^{q_n}\left(\varphi_x^2;x\right)+x(1-q_n)S_n^{q_n}(\varphi_x;x)\right)\\
&\quad+\frac{M}{\delta^2}\frac{[n]_{q_n}}{b_n}\left(S_n^{q_n}\left(\varphi_x^4;x\right)+x(1-q_n)S_n^{q_n}\left(\varphi_x^3;x\right)\right).
\end{aligned}\tag{2.30}$$

If we calculate $S_n^{q_n}\left(\varphi_x^4;x\right)$ by using Lemma 2.2, we get

$$\begin{aligned}
S_n^{q_n}\left(\varphi_x^4;x\right)&=x^4\left(q_n^6-4q_n^3+6q_n-3\right)\\
&\quad+x^3\frac{\left(q_n^3\left(1+[2]_{q_n}+[3]_{q_n}\right)-4\left(1+[2]_{q_n}\right)q_n+6\right)b_n}{[n]_{q_n}}\\
&\quad+x^2\frac{\left(q_n\left(1+[2]_{q_n}+[2]_{q_n}^2\right)-4\right)b_n^2}{[n]_{q_n}^2}+x\frac{b_n^3}{[n]_{q_n}^3}.
\end{aligned}$$

Since $\lim_{n\to\infty}b_n=\infty$ then we have

$$\lim_{n\to\infty}\frac{[n]_{q_n}\left(q_n^6-4q_n^3+6q_n-3\right)}{b_n}=\lim_{n\to\infty}\frac{\left(1-q_n^n\right)\left(\frac{q_n^6-4q_n^3+6q_n-3}{1-q_n}\right)}{b_n}=0.$$

We thus obtain

$$\lim_{n\to\infty}\frac{[n]_{q_n}}{b_n}S_n^{q_n}\left(\varphi_x^4;x\right)=0\tag{2.31}$$

for fixed $x\in[0,\alpha_{q_n}(n))$ where n is large enough. Using Lemma 2.2 we can easily see that

$$\lim_{n\to\infty}(1-q_n)\frac{[n]_{q_n}}{b_n}\left(S_n^{q_n}\left(\varphi_x^3;x\right)\right)=0\tag{2.32}$$

and therefore by (2.31), (2.32), and (2.30), we conclude that

$$\lim_{n\to\infty}\frac{[n]_{q_n}}{b_n}\left(S_n^{q_n}\left(\Phi_{q_n}(t)(t-x)_{q_n}^2;x\right)\right)=0$$

for fixed $x\in[0,\alpha_{q_n}(n))$ where n is large enough and therefore, we have the desired result. ■

We know that if f is differentiable n times, then $\lim_{q\to1}D_q^n f(x)=f^{(n)}(x)$ (see [81, p. 22]). Using this property we have the following corollary.

Corollary 2.1. *Let $f\in C(\mathbb{R}_0)$ be a bounded function and (q_n) denote a sequence such that $0<q_n<1$ and $q_n\to1$ as $n\to\infty$. Suppose that the second derivative $f''(x)$ exists at a point $x\in[0,\alpha_n(q_n))$ for n large enough. If $\lim_{n\to\infty}\frac{b_n}{[n]_{q_n}}=0$, then*

$$\lim_{n\to\infty}\frac{[n]_{q_n}}{b_n}\left(S_n^{q_n}(f;x)-f(x)\right)=\frac{1}{2}xf''(x).$$

Recall that a continuous function on an interval, which does not include 0, is continuous q-differentiable. According to this, for every x in an interval not including 0, since q-derivatives of f become finite, we deduce the q-differentiable condition in Theorem 2.6. In other words, Voronovskaya-type theorem is valid only for continuous and bounded functions.

Corollary 2.2. *Let $f\in C(\mathbb{R}_0)$ be a bounded function and (q_n) denote a sequence such that $0<q_n<1$ and $q_n\to1$ as $n\to\infty$. If $\lim_{n\to\infty}\frac{b_n}{[n]_{q_n}}=0$, then*

$$\lim_{n\to\infty}\frac{[n]_{q_n}}{b_n}\left(S_n^{q_n}(f;x)-f(x)\right)=\frac{1}{2}x\lim_{q_n\to1}D_{q_n}^2f(x)$$

for every point $x\in(0,\alpha_n(q_n))$, where n is large enough.

Remark 2.2. If the assumption of Theorem 2.6 holds for the function f, then the pointwise convergence rate of the operators (2.21) to f is $\mathcal{O}\left(\frac{b_n}{[n]_{q_n}}\right)$. Also this convergence rate can be made better depending on the chosen q_n and is at least as fast as $\frac{b_n}{n}$ which is the convergence rate of the classical Bernstein–Chlodowsky operators (see [35, 133]).

2.2.5 Convergence Properties in Weighted Space

As we mentioned above, when $q_n\to1$ as $n\to\infty$, the interval $[0,\alpha_{q_n}(n))$ which is the domain of the operator $S_n^{q_n}(f)$ grows. In this case the uniform norm is not valid to compute the rate of convergence for these operators. So we will consider weighted function spaces and the weighted norm. In this section, we obtain a direct approximation theorem in weighted norm and an estimate in terms of the modulus of continuity. These types of theorems are given in [65, 66]. Now we recall this theorem.

Let φ be a continuous and monotonically increasing function on the positive real axis, such that $\lim\limits_{x\to\infty}\varphi(x)=\pm\infty$ and $\rho(x)=1+\varphi^2(x)$.

Let $B_\rho(\mathbb{R}_0)$ be the set of all functions f satisfying the condition $|f(x)|\leq M_f\rho(x), x\in\mathbb{R}_0$ with some constant M_f, depending only on f. We denote by $C_\rho(\mathbb{R}_0)$ the space of all continuous functions belongs $B_\rho(\mathbb{R}_0)$ with the norm

$$\|f\|_\rho := \sup_{x\in\mathbb{R}_0}\frac{|f(x)|}{\rho(x)}$$

and $C_\rho^0(\mathbb{R}_0)=\left\{f\in C_\rho(\mathbb{R}_0): \lim\limits_{x\to\infty}\frac{|f(x)|}{\rho(x)}<\infty\right\}$.

Theorem 2.7 ([65]). *Let $\{A_n\}$ be a sequence of positive linear operators acting from $C_\rho(\mathbb{R}_0)$ to $B_\rho(\mathbb{R}_0)$ satisfying the following three conditions:*

$$\lim_{n\to\infty}\|A_n(\varphi^\nu;x)-\varphi^\nu(x)\|_\rho=0,\quad \nu=0,1,2.$$

Then

$$\lim_{n\to\infty}\|A_n(f;x)-f(x)\|_\rho=0$$

for any function $f\in C_\rho^0(\mathbb{R}_0)$.

The definitions of the spaces $C_\rho(\mathbb{R}_0)$ and $C_\rho^0(\mathbb{R}_0)$ are the same as $C_m(\mathbb{R}_0)$ and $C_m^0(\mathbb{R}_0)$, respectively, if we take $\rho(x)=1+x^m$ $(m\geq 2)$ instead of $\rho(x)=1+\varphi^2(x)$.

Theorem 2.8. *Let (q_n) denote a sequence such that $0<q_n<1$ and $q_n\to 1$ as $n\to\infty$. For any function $f\in C_{2m}^0(\mathbb{R}_0)$, if $\lim\limits_{n\to\infty}\frac{b_n}{[n]_{q_n}}=0$, then*

$$\lim_{n\to\infty}\sup_{0\leq x\leq\alpha_{q_n}(n)}\frac{\left|S_n^{q_n}(f;x)-f(x)\right|}{1+x^{2m}}=0.$$

Moreover, for n large enough

$$\sup_{0\leq x\leq\alpha_{q_n}(n)}\frac{\left|S_n^{q_n}(f;x)-f(x)\right|}{1+x^{2m}}\leq\left(2+\sqrt{2}\right)\omega\left(f;\sqrt{\frac{b_n}{[n]_{q_n}}}\right)$$

where $\omega(f;\cdot)$ is the classical modulus of continuity.

Proof. Applying Theorem 2.7 with $\varphi(t)=e_m(t)$, $m\geq 1$, to the operators

$$A_n(f;x)=\begin{cases} S_n^{q_n}(f;x) & \text{if } 0\leq x\leq\alpha_{q_n}(n)\\ f(x) & \text{if } x>\alpha_{q_n}(n)\end{cases}$$

to complete the proof, it is sufficient to show that the conditions

$$\lim_{n\to\infty}\sup_{0\le x\le\alpha_{q_n}(n)}\frac{\left|S_n^{q_n}\left(e_m^{\nu}(t);x\right)-x^{m\nu}\right|}{1+x^{2m}}=0,\quad \nu=0,1,2 \tag{2.33}$$

are satisfied. As a consequence of Lemma 2.2, since $\left|S_n^{q_n}\left(1+t^{2m};x\right)\right|\le C\left(1+x^{2m}\right)$ for $x\in[0,\alpha_{q_n}(n))$ where n is large enough and C is a positive constant, $\left\{S_n^{q_n}\right\}$ is a sequence of linear positive operators acting from $C_{2m}(\mathbb{R}_0)$ to $C_{2m}(\mathbb{R}_0)$.

From (2.24)

$$\lim_{n\to\infty}\sup_{0\le x\le\alpha_{q_n}(n)}\frac{\left|S_n^{q_n}(e_0;x)-1\right|}{1+x^{2m}}=0$$

holds. Thus the condition (2.33) holds for $\nu=0$. Since $\lim\limits_{n\to\infty}\frac{b_n}{[n]_{q_n}}=0$, then there exists $n_0\in\mathbb{N}$ such that $\left(\frac{b_n}{[n]_{q_n}}\right)^{m-j}\le\frac{b_n}{[n]_{q_n}}$ for $n>n_0$ and $j=1,\ 2,\ldots,m-1$. By Lemma 2.2 we have, for $n>n_0$,

$$\begin{aligned}
&\sup_{x\in\left[0,\alpha_{q_n}(n)\right)}\frac{\left|S_n^{q_n}(e_m;x)-x^m\right|}{1+x^{2m}}\\
&\le\left(1-q_n^{\frac{m(m-1)}{2}}\right)+\sup_{x\in\left[0,\alpha_{q_n}(n)\right)}\frac{\sum\limits_{j=1}^{m-1}\left(\frac{b_n}{[n]_{q_n}}\right)^{m-j}\mathbb{S}_{q_n}(m,j)\,q_n^{\frac{j(j-1)}{2}}x^j}{1+x^{2m}}\\
&\le\left(1-q_n^{\frac{m(m-1)}{2}}\right)+\frac{b_n}{[n]_{q_n}}\left(\sum_{j=1}^{m-1}\mathbb{S}_{q_n}(m,j)\,q_n^{\frac{j(j-1)}{2}}\right).
\end{aligned}$$

Hence we obtain

$$\lim_{n\to\infty}\sup_{x\in\left[0,\alpha_{q_n}(n)\right)}\frac{\left|S_n^{q_n}(e_m;x)-x^m\right|}{1+x^{2m}}=0.$$

Thus the condition (2.33) holds for $\nu=1$.

Similarly, we have, for $n>n_0$

$$\begin{aligned}
&\sup_{x\in\left[0,\alpha_{q_n}(n)\right)}\frac{\left|S_n^{q_n}(e_{2m};x)-x^{2m}\right|}{1+x^{2m}}\\
&\le\left(1-q_n^{m(2m-1)}\right)+\sup_{x\in\left[0,\alpha_{q_n}(n)\right)}\frac{\sum\limits_{j=1}^{2m-1}\left(\frac{b_n}{[n]_{q_n}}\right)^{2m-j}\mathbb{S}_{q_n}(2m,j)\,q_n^{\frac{j(j-1)}{2}}x^j}{1+x^{2m}}\\
&\le\left(1-q_n^{m(2m-1)}\right)+\frac{b_n}{[n]_{q_n}}\left(\sum_{j=1}^{2m-1}\mathbb{S}_{q_n}(2m,j)\,q_n^{\frac{j(j-1)}{2}}\right).
\end{aligned}$$

That is, for $\nu = 2$, the condition (2.33) is satisfied. Therefore, the proof is completed from Theorem 2.7.

For the second part of the theorem, using the property of the modulus of continuity $\omega(f,\cdot)$ for every $\delta > 0$, $t \geq 0$ and $x \geq 0$,

$$|f(t) - f(x)| \leq \left(1 + \delta^{-2}(t-x)^2\right)\omega(f,\delta).$$

Using this inequality we can write

$$|S_n^{q_n}(f;x) - f(x)| \leq 2\omega\left(f, \sqrt{S_n^{q_n}(\varphi_x^2;x)}\right)$$

for $f \in C_{2m}^0(\mathbb{R}_0)$. Since

$$\begin{aligned}\sup_{x\in[0,\alpha_{q_n}(n))} \frac{S_n^{q_n}\left(\varphi_x^2;x\right)}{1+x^{2m}} &\leq (1-q_n) + \frac{b_n}{[n]_{q_n}} \\ &= (1-q_n) + (1-q_n)\,\alpha_{q_n}(n) \\ &\leq 2(1-q_n)\,\alpha_{q_n}(n) \\ &= 2\frac{b_n}{[n]_{q_n}}\end{aligned}$$

for n large enough, we have the desired result. ■

Remark 2.3. In [35, Theorem 2.1] it has been shown that for any function f satisfying Theorem 2.8, the weighted rate of convergence of classical Bernstein–Chlodowsky operators is $\mathcal{O}\left(\frac{b_n}{n}\right)$. As a consequence of Theorem 2.8 we say that the rate of convergence of $S_n^{q_n}(f)$ to f in the weighted norm is $\frac{b_n}{[n]_{q_n}}$, which is at least as fast as $\frac{b_n}{n}$.

2.2.6 Other Properties

In this section, we give two representations of the rth q-derivative of q-Szász–Mirakyan operators in terms of the q-differences and the divided difference and then obtain a representation of q-Szász–Mirakyan operators in terms of the divided differences which is the modified form of the representation of the classical Szász–Mirakyan operator given in [145, pp. 1183–1184]. Note that these representations are not obtained using classical derivatives and forward differences.

Proposition 2.2. *For each integer $r > 0$*

$$D_q^r\left(S_n^q(f;x)\right) = E_q\left(-[n]_q q^r \frac{x}{b_n}\right)\sum_{j=0}^{\infty}\left(\frac{[n]_q}{b_n}\right)^r \Delta_q^r f_j \frac{\left([n]_q x\right)^j}{[j]_q!\,(b_n)^j}. \tag{2.34}$$

Proof. The proof is by induction on r. According to (1.4) we set $D_q\left(E_q\left(-[n]\,\frac{x}{b_n}\right)\right) = -\frac{[n]}{b_n}E_q\left(-[n]\,q\frac{x}{b_n}\right)$. Applying the D_q-differential operator to (2.21) and using Lemma 1.1, (1.5), and (2.3) we find

$$\begin{aligned}
D_q\left(S_n^q(f;x)\right) &= -\frac{[n]_q}{b_n}E_q\left(-[n]_q q\frac{x}{b_n}\right)\sum_{j=0}^{\infty} f\left(\frac{[j]_q b_n}{[n]_q}\right)\frac{\left([n]_q x\right)^j}{[j]_q!\,(b_n)^j} \\
&\quad + \frac{[n]_q}{b_n}E_q\left(-[n]_q q\frac{x}{b_n}\right)\sum_{j=0}^{\infty} f\left(\frac{[j+1]_q b_n}{[n]_q}\right)\frac{\left([n]_q x\right)^j}{[j]_q!\,(b_n)^j} \\
&= E_q\left(-[n]_q q\frac{x}{b_n}\right)\sum_{j=0}^{\infty}\frac{[n]_q}{b_n}\left(f\left(\frac{[j+1]_q b_n}{[n]_q}\right) - f\left(\frac{[j]_q b_n}{[n]_q}\right)\right)\frac{\left([n]_q x\right)^j}{[j]_q!\,(b_n)^j} \\
&= E_q\left(-[n]_q q\frac{x}{b_n}\right)\sum_{j=0}^{\infty}\frac{[n]_q}{b_n}\Delta_q^1 f_j\frac{\left([n]_q x\right)^j}{[j]_q!\,(b_n)^j}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&D_q^2\left(S_n^q(f;x)\right) \\
&= D_q\left(D_q\left(S_n^q(f;x)\right)\right) \\
&= -q\left(\frac{[n]_q}{b_n}\right)^2 E_q\left(-[n]_q q^2\frac{x}{b_n}\right)\sum_{j=0}^{\infty}\left(f\left(\frac{[j+1]_q b_n}{[n]_q}\right) - \left(\frac{[j]_q b_n}{[n]_q}\right)\right)\frac{\left([n]_q x\right)^j}{[j]_q!\,(b_n)^j} \\
&\quad + \left(\frac{[n]_q}{b_n}\right)^2 E_q\left(-[n]_q q^2\frac{x}{b_n}\right)\sum_{j=0}^{\infty}\left(f\left(\frac{[j+2]_q b_n}{[n]_q}\right) - \left(\frac{[j+1]_q b_n}{[n]_q}\right)\right)\frac{\left([n]_q x\right)^j}{[j]_q!\,(b_n)^j} \\
&= E_q\left(-[n]_q q^2\frac{x}{b_n}\right)\sum_{j=0}^{\infty}\left(\frac{[n]_q}{b_n}\right)^2\Delta_q^2 f_j\frac{\left([n]_q x\right)^j}{[j]_q!\,(b_n)^j}.
\end{aligned}$$

Thus (2.34) holds for $r=1$ and $r=2$. Let us assume that it holds for some $r \geq 3$. Applying the q-differential operator to (2.34) we find

$$\begin{aligned}
&D_q\left(D_q^r\left(S_n^q(f;x)\right)\right)\\
&=D_q^{r+1}\left(S_n^q(f;x)\right)\\
&=-q^r E_q\left(-[n]_q q^{r+1}\frac{x}{b_n}\right)\sum_{j=0}^{\infty}\left(\frac{[n]_q}{b_n}\right)^{r+1}\Delta_q^r f_j \frac{\left([n]_q x\right)^j}{[j]_q!\,(b_n)^j}\\
&\quad +E_q\left(-[n]_q q^{r+1}\frac{x}{b_n}\right)\sum_{j=0}^{\infty}\left(\frac{[n]_q}{b_n}\right)^{r+1}\Delta_q^r f_{j+1} \frac{\left([n]_q x\right)^j}{[j]_q!\,(b_n)^j}\\
&=E_q\left(-[n]_q q^{r+1}\frac{x}{b_n}\right)\sum_{j=0}^{\infty}\left(\frac{[n]_q}{b_n}\right)^{r+1}\Delta_q^{r+1} f_j \frac{\left([n]_q x\right)^j}{[j]_q!\,(b_n)^j}
\end{aligned}$$

by using (2.3). This shows that (2.34) holds when r is replaced by $r+1$, which completes the proof. ■

Using the following connection between the divided differences and the q-differences given in [134, p. 44]

$$\Delta_q^r f_j=\left(\frac{b_n}{[n]}\right)^r [r]!\,q^{rj} q^{\frac{r(r-1)}{2}} f\left[\frac{b_n[j]}{[n]},\frac{b_n[j+1]}{[n]},\cdots,\frac{b_n[j+r]}{[n]}\right], \tag{2.35}$$

then we have the following representation formula.

Corollary 2.3. *For each integer* $r>0$

$$\begin{aligned}
&D_q^r\left(S_n^q(f;x)\right)\\
&=q^{\frac{r(r-1)}{2}}[r]!\,E_q\!\left(-[n]_q q^r\frac{x}{b_n}\right)\sum_{j=0}^{\infty} q^{rj} f\left[\frac{b_n[j]}{[n]},\frac{b_n[j+1]}{[n]},\cdots,\frac{b_n[j+r]}{[n]}\right]\frac{\left([n]_q x\right)^j}{[j]_q!\,(b_n)^j}.
\end{aligned}$$

Corollary 2.4. *The q-Szász–Mirakyan operator can be represented as*

$$S_n^q(f;x)=\sum_{j=0}^{\infty} q^{\frac{j(j-1)}{2}} f\left[0,\frac{b_n[1]}{[n]},\ldots,\frac{b_n[j]}{[n]}\right]x^j.$$

Proof. From the equalities (2.23) and (2.35), the proof is obvious. ■

2.3 q-Baskakov Operators

In this section we propose a generalization of the Baskakov operators, based on q-integers. We also estimate the rate of convergence in the weighted norm. We also study some shape-preserving and monotonicity properties of the q-Baskakov operators and also different generalizations of classical Baskakov operators based on q-integers defined in [30, 136].

First, we recall classical Baskakov operators [37], which for $f \in C[0, \infty)$ are defined as

$$B_n(f,x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k (1+x)^{-n-k} f\left(\frac{k}{n}\right)$$

This section is based on [32].

2.3.1 Construction of Operators and Some Properties of Them

For $f \in C[0, \infty)$, $q > 0$, and each positive integer n, a new q-Baskakov operators can be defined as

$$\begin{aligned} B_{n,q}(f,x) &= \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} x^k (-x,q)_{n+k}^{-1} f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right) \\ &= \sum_{k=0}^{\infty} \mathcal{P}_{n,k}^q(x) f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right). \end{aligned} \tag{2.36}$$

While for $q = 1$ these polynomials coincide with the classical ones.

Definition 2.1. Let f be a function defined on an interval (a,b) and h be a positive real number. The q-forward differences Δ_h^r of f are defined recursively as

$$\begin{aligned} \Delta_q^0 f(x_j) &:= f(x_j) \\ \Delta_q^{r+1} f(x_j) &:= q^r \Delta_q^r f(x_{j+1}) - \Delta_q^r f(x_j) \end{aligned}$$

for $r \geq 0$.

Note that the above definition is different from definition given in [134, p. 44].

As usual, we show divided differences with $f[x_0, x_1, \ldots, x_n]$ at the abscissas $x_0, x_1, \ldots, x_n$.

We now show the following general relation that connect the divided differences $f[x_0, x_1, \ldots, x_n]$ and q-forward differences.

Lemma 2.3. *For all $j,r \geq 0$, we have*

$$f\left[x_j,x_{j+1},\ldots,x_{j+r}\right] = q^{\frac{r(2j+r-1)}{2}} \frac{\Delta_q^r f(x_j)}{[r]_q!}, \tag{2.37}$$

where $x_j = \frac{[j]_q}{q^{j-1}}$.

Proof. Let us use induction on r. By Definition 2.1, the result is obvious for $r = 0$. Let us assume that the equality (2.37) is true for some $r \geq 0$ and all $j \geq 0$. Since

$$x_{j+r+1} - x_j = \frac{[r+1]_q}{q^{j+r}},$$

we have

$$\begin{aligned}
f\left[x_j,x_{j+1},\ldots,x_{j+r+1}\right] &= \frac{f\left[x_{j+1},\ldots,x_{j+r+1}\right] - f\left[x_j,\ldots,x_{j+r}\right]}{x_{j+r+1}-x_j} \\
&= \frac{q^{j+r}}{[r+1]_q}\left(q^{\frac{r(2j+r+1)}{2}} \frac{\Delta_q^r f\left(x_{j+1}\right)}{[r]_q!} - q^{\frac{r(2j+r-1)}{2}} \frac{\Delta_q^r f(x_j)}{[r]_q!}\right) \\
&= q^{\frac{r(2j+r-1)}{2}+j+r}\left(\frac{q^r \Delta_q^r f\left(x_{j+1}\right) - \Delta_q^r f(x_j)}{[r+1]_q!}\right) \\
&= q^{\frac{(r+1)(2j+r)}{2}} \frac{\Delta_q^{r+1} f(x_j)}{[r+1]_q!}.
\end{aligned}$$

■

Lemma 2.4. *For $n,\ k \geq 0$, we have*

$$\mathcal{D}_q\left[x^k(-x,q)_{n+k}^{-1}\right] = [k]_q x^{k-1}(-x,q)_{n+k}^{-1} - q^k x^k [n+k]_q (-x,q)_{n+k+1}^{-1} \tag{2.38}$$

Proof. First, we prove that $\mathcal{D}_q(-x,q)_n = [n]_q(-qx,q)_{n-1}$. Using q-derivative operator (1.5) we have

$$\begin{aligned}
\mathcal{D}_q(-x,q)_n &= \frac{1}{(q-1)x}\left(\prod_{j=0}^{n-1}\left(1+q^{j+1}x\right) - \prod_{j=0}^{n-1}\left(1+q^j x\right)\right) \\
&= \frac{1}{(q-1)x}\prod_{j=0}^{n-2}\left(1+q^{j+1}x\right)\left((1+q^n x)-(1+x)\right) \\
&= \frac{q^n-1}{q-1}\prod_{j=0}^{n-2}\left(1+q^{j+1}x\right) \\
&= [n]_q(-qx,q)_{n-1}.
\end{aligned}$$

The q-derivative formula for a quotient (1.6) imply that

$$\begin{aligned}\mathcal{D}_q(-x,q)_{n+k}^{-1} &= \frac{-[n+k]_q(-qx,q)_{n+k-1}}{(-x,q)_{n+k}(-qx,q)_{n+k}} \\ &= -[n+k]_q(-x,q)_{n+k+1}^{-1}. \qquad (2.39)\end{aligned}$$

Also it is obvious that

$$\mathcal{D}_q x^k = [k]_q x^{k-1}. \qquad (2.40)$$

Then using (2.40) and (2.39), the result follows by (1.5)

$$\mathcal{D}_q\left[x^k(-x,q)_{n+k}^{-1}\right] = [k]_q x^{k-1}(-x,q)_{n+k}^{-1} - q^k x^k [n+k]_q (-x,q)_{n+k+1}^{-1}. \qquad \blacksquare$$

We wish to calculate the moments. For this purpose we give q-derivative of $B_{n,q}$. Next theorem gives a representation of the rth derivative of $B_{n,q}$ in terms of q-forward differences.

Theorem 2.9. *Let $r \geq 0$. Then the rth derivative of q-Baskakov operator has the representation*

$$\mathcal{D}_q^r B_{n,q}(f,x) = \frac{[n+r-1]_q!}{[n-1]_q!}\sum_{k=0}^{\infty} q^{rk}\mathcal{P}_{n+r,k}^q(x)\Delta_q^r f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right) \qquad (2.41)$$

Proof. We use induction on r. Equality (2.38),

$$\begin{bmatrix} n+k \\ k+1 \end{bmatrix}_q [k+1]_q = [n]_q \begin{bmatrix} n+k \\ k \end{bmatrix}_q$$

and

$$\begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q [n+k]_q = [n]_q \begin{bmatrix} n+k \\ k \end{bmatrix}_q,$$

imply that

$$\begin{aligned}\mathcal{D}_q B_{n,q}(f,x) &= \sum_{k=1}^{\infty}\begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}}[k]_q x^{k-1}(-x,q)_{n+k}^{-1} f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right) \\ &\quad -\sum_{k=0}^{\infty}\begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} q^k x^k [n+k]_q (-x,q)_{n+k+1}^{-1} f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right) \\ &= [n]_q \sum_{k=0}^{\infty}\begin{bmatrix} n+k \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}+k} x^k (-x,q)_{n+k+1}^{-1}\left(f\left(\frac{[k+1]_q}{q^k[n]_q}\right) - f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right)\right) \\ &= [n]_q \sum_{k=0}^{\infty} q^k \mathcal{P}_{n+1,k}^q(x)\Delta_q^1 f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right).\end{aligned}$$

It is clear that (2.41) holds for $r=1$. Let us assume that (2.41) holds for some $r\geq 2$. Applying q-derivative operator to (2.41), we have

$$\begin{aligned}
\mathcal{D}_q^{r+1}\left(B_{n,q}(f,x)\right) &= \frac{[n+r-1]_q!}{[n-1]_q!}\sum_{k=1}^{\infty}\begin{bmatrix} n+k+r-1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}+rk}[k]_q \\
&\quad \times x^{k-1}(-x,q)_{n+k+r}^{-1}\Delta_q^r f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right) \\
&\quad -\frac{[n+r-1]_q!}{[n-1]_q!}\sum_{k=0}^{\infty}\begin{bmatrix} n+k+r-1 \\ k \end{bmatrix}_q [n+k+r]_q q^k q^{\frac{k(k-1)}{2}+rk} \\
&\quad \times x^k(-x,q)_{n+k+r+1}^{-1}\Delta_q^r f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right) \\
&= \frac{[n+r]_q!}{[n-1]_q!}\sum_{k=0}^{\infty}\begin{bmatrix} n+k+r \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}+(r+1)k}x^k(-x,q)_{n+k+r+1}^{-1} \\
&\quad \times\left(q^r\Delta_q^r f\left(\frac{[k+1]_q}{q^k[n]_q}\right)-\Delta_q^r f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right)\right) \\
&= \frac{[n+r]_q!}{[n-1]_q!}\sum_{k=0}^{\infty}q^{(r+1)k}\mathcal{P}_{n+r+1,k}^q(x)\,\Delta_q^{r+1}f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right)
\end{aligned}$$

This completes the proof of the theorem. ■

Corollary 2.5. *q-Baskakov operators can be represented as*

$$B_{n,q}(f,x)=\sum_{r=0}^{\infty}\frac{[n+r-1]_q!}{[n-1]_q!}\Delta_q^r f(0)\frac{x^r}{[r]_q!}.$$

Proof. By Theorem 2.9, we have

$$\begin{aligned}
\mathcal{D}_q^r\left(B_{n,q}(f,x)\right)\big|_{x=0} &= \frac{[n+r-1]_q!}{[n-1]_q!}\mathcal{P}_{n+r,0}^q(0)\,\Delta_q^r f(0) \\
&= \frac{[n+r-1]_q!}{[n-1]_q!}\Delta_q^r f(0)
\end{aligned}$$

for $r\geq 1$. By using the above equality in q-Taylor formula given in [137], we get

$$B_{n,q}(f,x)=\sum_{r=0}^{\infty}\frac{[n+r-1]_q!}{[n-1]_q!}\Delta_q^r f(0)\frac{x^r}{[r]_q!}. \tag{2.42}$$

■

From Lemma 2.3 and Corollary 2.5, we have the following corollary.

Corollary 2.6. *The q-Baskakov operators can be represented as*

$$B_{n,q}(f,x) = \sum_{r=0}^{\infty} \frac{[n+r-1]_q!}{[n-1]_q!} q^{-\frac{r(r-1)}{2}} f\left[0, \frac{1}{[n]_q}, \frac{[2]_q}{q[n]_q}, \ldots, \frac{[r]_q}{q^{r-1}[n]_q}\right] x^r.$$

We are now in a position to give the moments of the first and second orders of the operators $B_{n,q}$.

Lemma 2.5. *For* $B_{n,q}(t^m,x),\ m=0,1,2,$ *one has*

$$\begin{aligned} B_{n,q}(1,x) &= 1. \\ B_{n,q}(t,x) &= x, \\ B_{n,q}(t^2,x) &= x^2 + \frac{x}{[n]_q}\left(1+\frac{1}{q}x\right). \end{aligned}$$

Proof. It is well known [134, p. 10] that

$$f\left[x_0, x_1, \ldots, x_r\right] = \frac{f^{(r)}(\xi)}{r!}, \tag{2.43}$$

where $\xi \in (x_0,\ x_r)$. We also see from Lemma 2.3 and (2.43)

$$q^{\frac{r(r-1)}{2}} \frac{\Delta_q^r f(x_0)}{[n]_q^r [r]_q!} = \frac{f^{(r)}(\xi)}{r!}.$$

Thus it is observed that rth q-forward differences of $x^m, m>r$ are zero. From (2.42), we have

$$B_{n,q}(1,x) = 1. \tag{2.44}$$

For $f(x)=x$ we have $\Delta_q^0 f(0) = f(0) = 0$ and $\Delta_q^1 f(0) = f\left(\frac{1}{[n]_q}\right) - f(0) = \frac{1}{[n]_q}$ and it follows from (2.42)

$$B_{n,q}(t,x) = x \tag{2.45}$$

For $f(x)=x^2$ we have $\Delta_q^0 f(0) = f(0) = 0$ and $\Delta_q^1 f(0) = f\left(\frac{1}{[n]_q}\right) - f(0) = \frac{1}{[n]_q^2}$ and $\Delta_q^2 f(0) = qf\left(\frac{[2]_q}{q[n]_q}\right) - (1+q)f\left(\frac{1}{[n]_q}\right) - f(0)$

$$\begin{aligned} B_{n,q}(t^2,x) &= \frac{[n+1]_q}{[n]_q}\left(\frac{1}{q}[2]_q - 1\right)x^2 + \frac{x}{[n]_q} \\ &= \frac{q[n]_q+1}{[n]_q}\left(\frac{1}{q}(1+q) - 1\right)x^2 + \frac{x}{[n]_q} \end{aligned}$$

$$= x^2 + \frac{1}{q[n]_q}x^2 + \frac{x}{[n]_q}$$
$$= x^2 + \frac{x}{[n]_q}\left(1+\frac{1}{q}x\right). \tag{2.46}$$

■

The following proposition is another application of q-derivatives, which enables us to give the estimation of moments:

Proposition 2.3. *If we define*

$$U_{n,m}^q(x) := B_{n,q}(t^m,x) = \sum_{k=0}^{\infty} \mathcal{P}_{n,k}^q(x)\left(\frac{[k]_q}{q^{k-1}[n]_q}\right)^m,$$

then $U_{n,0}^q(x) = 1, U_{n,1}^q(x) = x$ and there holds the following recurrence relation:

$$[n]_q U_{n,m+1}^q(qx) = qx(1+x)D_q U_{n,m}^q(x) + qx[n]_q U_{n,m}^q(qx), m > 1.$$

Proof. Obviously $\sum_{k=0}^{\infty} \mathcal{P}_{n,k}^q(x) = 1$; thus, by this identity and (2.1), the values of $U_{n,0}^q(x)$ and $U_{n,1}^q(x)$ easily follow. From Lemma 2.4, it is obvious that $x(1+q^{n+k}x)D_q\mathcal{P}_{n,k}^q(x) = \left([k]_q - q^k[n]_q x\right)\mathcal{P}_{n,k}^q(x)$, which implies that

$$x(1+x)\mathcal{D}_q\mathcal{P}_{n,k}^q(x) = \left(\frac{[k]_q}{q^{k-1}[n]_q} - qx\right)\frac{[n]_q}{q}\mathcal{P}_{n,k}^q(qx)$$

Thus using this identity, we have

$$qx(1+x)D_q U_{n,m}^q(x) = \sum_{k=0}^{\infty} qx(1+x)D_q\mathcal{P}_{n,k}^q(x)\left(\frac{[k]_q}{q^{k-1}[n]_q}\right)^m$$
$$= [n]_q \sum_{k=0}^{\infty}\left(\frac{[k]_q}{q^{k-1}[n]_q} - qx\right)\mathcal{P}_{n,k}^q(qx)\left(\frac{[k]_q}{q^{k-1}[n]_q}\right)^m.$$
$$= [n]_q U_{n,m+1}^q(qx) - qx[n]_q U_{n,m}^q(qx).$$

This completes the proof of the recurrence relation. ■

2.3.2 Approximation Properties

We set

$$E_2(\mathbb{R}_+) := \left\{ f \in C(\mathbb{R}_+) : \lim_{x\to\infty}\frac{f(x)}{1+x^2} \text{ exist} \right\}$$

and

$$B_2(\mathbb{R}_+) := \left\{ f : \ |f(x)| \le B_f \left(1 + x^2\right) \right\}$$

where B_f is a constant depending on f, endowed with the norm $\|f\|_2 := \sup\limits_{x\ge 0} \frac{|f(x)|}{1+x^2}$. As a consequence of Lemma 2.5, the operators (2.36) map $E_2(\mathbb{R}_+)$ into $E_2(\mathbb{R}_+)$. Since for a fixed value of q with $q > 0$,

$$\lim_{n\to\infty} [n]_q = \frac{1}{1-q},$$

$B_{n,q}(t^2, x)$ does not converge to x^2 as $n \to \infty$. According to well-known Bohman–Korovkin theorem, relations (2.44), (2.45), and (2.46) don't guarantee that $\lim_{n\to\infty} B_{n,q_n} f = f$ uniformly on compact subset of $\mathbb{R}_+$ for every $f \in E_2(\mathbb{R}_+)$. To ensure this type of convergence properties of (2.36) we replace $q = q_n$ as a sequence such that $q_n \to 1$ as $n \to \infty$ for $q_n > 0$ and so that $[n]_{q_n} \to \infty$ as $n \to \infty$. Also, $B_{n,q_n} f$ are linear and positive operators for $q_n > 0$. In this situation, we can apply Bohman–Korovkin theorem to B_{n,q_n}. That is:

Theorem 2.10. *Let* (q_n) *be a sequence of real numbers such that* $q_n > 0$ *and* $\lim_{n\to\infty} q_n = 1$. *Then for every* $f \in E_2(\mathbb{R}_+)$

$$\lim_{n\to\infty} B_{n,q_n} f = f$$

uniformly on any compact subset of $\mathbb{R}_+$.

Theorem 2.11. *Let* $q = q_n$ *satisfies* $q_n > 0$ *and let* $q_n \to 1$ *as* $n \to \infty$. *For every* $f \in B_2(\mathbb{R}_+)$,

$$\lim_{n\to\infty} \sup_{x\ge 0} \frac{\left|B_{n,q_n}(f;x) - f(x)\right|}{\left(1+x^2\right)^3} = 0. \tag{2.47}$$

Proof. Since f is continuous, it is also uniformly continuous; on any closed interval, there exist a number $\delta > 0$, depending on ε and f; for $|t - x| < \delta$ we have

$$|f(t) - f(x)| < \varepsilon.$$

Since $f \in B_2(\mathbb{R}_+)$, we can write for $|t - x| \ge \delta$

$$|f(t) - f(x)| < A_f(\delta) \left\{ (t-x)^2 + \left(1+x^2\right)|t-x| \right\},$$

where $A_f(\delta)$ is a positive constant depending on f and δ.

On combining above results, we obtain

$$|f(t) - f(x)| < \varepsilon + A_f(\delta) \left\{ (t-x)^2 + \left(1+x^2\right)|t-x| \right\},$$

where $t,x\in\mathbb{R}_+$. Thus, we have

$$\left|B_{n,q_n}(f;x)-f(x)\right|<\varepsilon+A_f(\delta)\left\{B_{n,q_n}\left((t-x)^2;x\right)+\left(1+x^2\right)B_{n,q_n}(|t-x|;x)\right\}$$

and from Lemma 2.5

$$\sup_{x\geq 0}\frac{\left|B_{n,q_n}(f;x)-f(x)\right|}{1+x^2}<\varepsilon+A_f(\delta)\left\{\frac{1}{[n]_{q_n}}\left(1+\frac{1}{q_n}\right)+\sqrt{\frac{1}{[n]_{q_n}}\left(1+\frac{1}{q_n}\right)}\right\},$$

and this completes the proof. ■

Remark 2.4. Using the similar method given in [12, p. 301], we have

$$\left|B_{n,q_n}(f;x)-f(x)\right|\leq M\omega_2\left(f;\sqrt{\frac{x}{[n]_q}\left(1+\frac{1}{q}x\right)}\right),$$

where $\omega_2(f;\ \delta)$ is classical second modulus of smoothness of f and f is bounded uniformly continuous function on $\mathbb{R}_+$. Thus, we say that the rate of convergence of $B_{n,q_n}(f)$ to f in any closed subinterval of $\mathbb{R}_+$ is $\frac{1}{\sqrt{[n]_{q_n}}}$, which is at least as fast as $\frac{1}{\sqrt{n}}$ which is the rate of convergence of classical Baskakov operators.

2.3.3 Shape-Preserving Properties

Definition 2.2 ([115, 116, 131]). Let f be continuous and a nonnegative function such that $f(0)=0$. A function f is called star-shaped in $[0,a]$; a is a positive real number, if

$$f(\alpha x)\leq \alpha f(x)$$

for each $\alpha,\ \alpha\in[0,\ 1]$ and $x\in(0,a]$.

From the definition of q-derivative (1.5), the following lemma is obvious.

Lemma 2.6. *The function f is star-shaped if and only if $x\mathcal{D}_q(f)(x)\geq f(x)$ for each $q\in(0,1)$ and $x\in[0,a]$.*

Theorem 2.12. *If f is star-shaped, then $B_{n,q}(f)$ is star-shape.*

Proof. From Theorem 2.9, we can write

$$\begin{aligned}&\mathcal{D}_q\left(B_{n,q}(f,x)\right)-\frac{B_{n,q}(f,x)}{x}\\&=[n]_q\sum_{k=0}^{\infty}q^k\,\nabla_q^1 f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right)\begin{bmatrix}n+k\\k\end{bmatrix}_q q^{\frac{k(k-1)}{2}}x^k(-x,q)_{n+k+1}^{-1}\end{aligned}$$

$$-\sum_{k=1}^{\infty} f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right)\begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} x^{k-1}(-x,q)_{n+k}^{-1}$$

$$= [n]\sum_{k=0}^{\infty}\begin{bmatrix} n+k \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} q^k x^k (-x,q)_{n+k+1}^{-1}$$

$$\left(f\left(\frac{[k+1]_q}{q^k[n]_q}\right) - f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right) - \frac{1}{[k+1]_q} f\left(\frac{[k+1]_q}{q^k[n]_q}\right)\right).$$

Since

$$1 - \frac{1}{[k+1]_q} = \frac{q[k]_q}{[k+1]_q},$$

we have

$$\mathcal{D}_q\left(B_{n,q}(f,x)\right) - \frac{B_{n,q}(f,x)}{x}$$

$$= [n]\sum_{k=0}^{\infty} q^k \mathcal{P}_{n+1,k}^q\left(\frac{q[k]_q}{[k+1]_q} f\left(\frac{[k+1]_q}{q^k[n]_q}\right) - f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right)\right). \qquad (2.48)$$

Since f is star-shaped, we have

$$\frac{q[k]_q}{[k+1]_q} f\left(\frac{[k+1]_q}{q^k[n]_q}\right) \geq f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right).$$

From this inequality and (2.48), we have the desired result. ■

Now we give a certain monotonicity property of the q-Baskakov operators defined by (2.36). Similar results for the classical Baskakov operators were given in [41].

Theorem 2.13. *Suppose $f(x)$ is defined on $(0,\infty)$ and $f(x) \geq 0$ for $x \in (0,\infty)$. If $\frac{f(x)}{x}$ is decreasing for all $x \in (0,\infty)$, then $\mathcal{D}_q\left(\frac{B_{n,\,q}(f;\,x)}{x}\right) \leq 0$ for $x \in (0,\infty)$ and for all $q \in (0,\,\infty)$.*

Proof. From (2.36) we get

$$\frac{B_{n,\,q}(f;x)}{x} = \sum_{k=1}^{\infty} f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right)\begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} x^{k-1}(-x,q)_{n+k}^{-1} + \frac{f(0)}{x}(-x,q)_n^{-1}.$$

If we take q-derivative of above equality and using Lemma 2.4, then we have

$$\mathcal{D}_q\left(\frac{B_{n,\,q}(f;x)}{x}\right) = \sum_{k=2}^{\infty} f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right)\begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} [k-1]_q x^{k-2}(-x,q)_{n+k}^{-1}$$

$$-\sum_{k=1}^{\infty} f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right)\begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}}[n+k]_q q^{k-1}x^{k-1}(-x,q)_{n+k+1}^{-1}$$
$$+\mathcal{D}_q\left(\frac{f(0)}{x}(-x,q)_n^{-1}\right).$$

Also using (1.5) and (1.6), we get

$$\mathcal{D}_q\left(\frac{f(0)}{x}(-x,q)_n^{-1}\right) = -\frac{f(0)}{qx^2}(-x,q)_n^{-1} - [n]_q\frac{f(0)}{x}(-x,q)_{n+1}^{-1}$$

Therefore,

$$\mathcal{D}_q\left(\frac{B_{n,\,q}(f;x)}{x}\right)$$
$$= \sum_{k=1}^{\infty} f\left(\frac{[k+1]_q}{q^k[n]_q}\right)\begin{bmatrix} n+k \\ k+1 \end{bmatrix}_q q^{\frac{k(k-1)}{2}} q^k[k]_q x^{k-1}(-x,q)_{n+k+1}^{-1}$$
$$-\sum_{k=1}^{\infty} f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right)\begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}}[n+k]_q q^{k-1}x^{k-1}(-x,q)_{n+k+1}^{-1}$$
$$-\frac{f(0)}{qx^2}(-x,q)_n^{-1} - [n]_q\frac{f(0)}{x}(-x,q)_{n+1}^{-1}$$

Using the identities

$$\begin{bmatrix} n+k \\ k+1 \end{bmatrix}_q = \begin{bmatrix} n+k \\ k \end{bmatrix}_q \frac{[n]_q}{[k+1]_q}$$
$$\begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q [n+k]_q = \begin{bmatrix} n+k \\ k \end{bmatrix}_q [n]_q,$$

we have

$$\mathcal{D}_q\left(\frac{B_{n,\,q}(f;x)}{x}\right) = \sum_{k=1}^{\infty}\begin{bmatrix} n+k \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} x^{k-1}(-x,q)_{n+k+1}^{-1}$$
$$\left(f\left(\frac{[k+1]_q}{q^k[n]_q}\right)\frac{q^k[n]_q}{[k+1]_q} - f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right)\frac{q^{k-1}[n]_q}{[k]_q}\right)[k]_q$$
$$-\frac{f(0)}{qx^2}(-x,q)_n^{-1} - [n]_q\frac{f(0)}{x}(-x,q)_{n+1}^{-1}.$$

Since $f(x) \geq 0$ and $\frac{f(x)}{x}$ is nonincreasing for $x \in (0,\, \infty)$,

$$\mathcal{D}_q\left(\frac{B_{n,\,q}\left(f;x\right)}{x}\right)\leq 0$$

for all $q\in(0,\ \infty)$ and $x\in(0,\infty)$. ∎

2.3.4 Monotonicity Property

Now we give the following relation between two consecutive terms of the sequence $B_{n,q}\left(f\right)$. Note that similar result for classical Baskakov operators was given in [122].

Theorem 2.14. *If $f\in C\left(\mathbb{R}^{+}\right)$, then the following formula is valid*

$$B_{n+1,q}\left(f,x\right)-B_{n,q}\left(f,x\right)=-\frac{q^{n}}{[n]_q\,[n+1]_q}\sum_{k=0}^{\infty}q^{\frac{k(k+1)}{2}-2k}x^{k+1}\left(-x,q\right)_{n+k+1}^{-1}\begin{bmatrix}n+k\\k\end{bmatrix}_q$$
$$\times\frac{[n+k+1]_q}{[n+1]_q}f\left[\frac{[k]_q}{q^{k-1}[n+1]_q},\frac{[k+1]_q}{q^{k}[n+1]_q},\frac{[k+1]_q}{q^{k}[n]_q}\right]$$

Proof. Using the equality

$$1=1+q^{n+k}x-q^{n+k}x,$$

from (2.36) we can write

$$\begin{aligned}
B_{n+1,q}(f;x)&=\sum_{k=0}^{\infty}f\left(\frac{[k]_q}{q^{k-1}[n+1]_q}\right)\begin{bmatrix}n+k\\k\end{bmatrix}_q q^{\frac{k(k-1)}{2}}x^{k}\left(-x,q\right)_{n+k+1}^{-1}\\
&=\sum_{k=0}^{\infty}f\left(\frac{[k]_q}{q^{k-1}[n+1]_q}\right)\begin{bmatrix}n+k\\k\end{bmatrix}_q q^{\frac{k(k-1)}{2}}x^{k}\left(-x,q\right)_{n+k}^{-1}\\
&\quad-\sum_{k=0}^{\infty}f\left(\frac{[k]_q}{q^{k-1}[n+1]_q}\right)\begin{bmatrix}n+k\\k\end{bmatrix}_q q^{\frac{k(k-1)}{2}}q^{n+k}x^{k+1}\left(-x,q\right)_{n+k+1}^{-1}\\
&=f\left(0\right)\left(-x,q\right)_{n}^{-1}+\sum_{k=1}^{\infty}f\left(\frac{[k]_q}{q^{k-1}[n+1]_q}\right)\begin{bmatrix}n+k\\k\end{bmatrix}_q q^{\frac{k(k-1)}{2}}x^{k}\left(-x,q\right)_{n+k}^{-1}\\
&\quad-\sum_{k=0}^{\infty}f\left(\frac{[k]_q}{q^{k-1}[n+1]_q}\right)\begin{bmatrix}n+k\\k\end{bmatrix}_q q^{\frac{k(k-1)}{2}}q^{n+k}x^{k+1}\left(-x,q\right)_{n+k+1}^{-1}
\end{aligned}$$

Thus, we have

$$B_{n+1,q}(f;x) = f(0)(-x,q)_n^{-1} + \sum_{k=0}^{\infty} f\left(\frac{[k+1]_q}{q^k[n+1]_q}\right) \cdot \begin{bmatrix} n+k+1 \\ k+1 \end{bmatrix}_q q^{\frac{k(k-1)}{2}} q^k x^{k+1} (-x,q)_{n+k+1}^{-1} - \sum_{k=0}^{\infty} f\left(\frac{[k]_q}{q^{k-1}[n+1]_q}\right) \begin{bmatrix} n+k \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} q^{n+k} x^{k+1} (-x,q)_{n+k+1}^{-1}.$$

Since

$$\begin{aligned} B_{n,q}(f;x) &= f(0)(-x,q)_n^{-1} + \sum_{k=1}^{\infty} f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right) \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} x^k (-x,q)_{n+k}^{-1} \\ &= f(0)(-x,q)_n^{-1} + \sum_{k=0}^{\infty} f\left(\frac{[k+1]_q}{q^k[n]_q}\right) \begin{bmatrix} n+k \\ k+1 \end{bmatrix}_q q^{\frac{k(k-1)}{2}} q^k x^{k+1} (-x,q)_{n+k+1}^{-1}, \end{aligned}$$

we have

$$\begin{aligned} & B_{n+1,q}(f,x) - B_{n,q}(f,x) \\ &= \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} q^k x^{k+1} (-x,q)_{n+k+1}^{-1} \\ & \left(f\left(\frac{[k+1]_q}{q^k[n+1]_q}\right) \begin{bmatrix} n+k+1 \\ k+1 \end{bmatrix}_q - q^n f\left(\frac{[k]_q}{q^{k-1}[n+1]_q}\right) \begin{bmatrix} n+k \\ k \end{bmatrix}_q \right. \\ & \left. - f\left(\frac{[k+1]_q}{q^k[n]_q}\right) \begin{bmatrix} n+k \\ k+1 \end{bmatrix}_q \right) \end{aligned}$$

Using the equalities

$$\begin{bmatrix} n+k+1 \\ k+1 \end{bmatrix}_q = \frac{[n+k+1]_q}{[k+1]_q} \begin{bmatrix} n+k \\ k \end{bmatrix}_q$$

and

$$\begin{bmatrix} n+k \\ k+1 \end{bmatrix}_q = \frac{[n]_q}{[k+1]_q} \begin{bmatrix} n+k \\ k \end{bmatrix}_q,$$

we can write

$$
\begin{aligned}
&B_{n+1,q}(f,x) - B_{n,q}(f,x) \\
&= -\sum_{k=0}^{\infty} q^{\frac{k(k+1)}{2}} x^{k+1} (-x,q)_{n+k+1}^{-1} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \\
&\left(q^n f\left(\frac{[k]_q}{q^{k-1}[n+1]_q}\right) - \frac{[n+k+1]_q}{[k+1]_q} f\left(\frac{[k+1]_q}{q^k[n+1]_q}\right) + \frac{[n]_q}{[k+1]_q} f\left(\frac{[k+1]_q}{q^k[n]_q}\right)\right).
\end{aligned}
$$

Using the inequalities

$$
\frac{[k+1]_q}{q^k [n]_q} - \frac{[k]_q}{q^{k-1}[n+1]_q} = \frac{[n+k+1]_q}{q^k [n]_q [n+1]_q},
$$

$$
\frac{[k+1]_q}{q^k [n+1]_q} - \frac{[k]_q}{q^{k-1}[n+1]_q} = \frac{1}{q^k [n+1]_q},
$$

and

$$
\frac{[k+1]_q}{q^k [n]_q} - \frac{[k+1]_q}{q^k [n+1]_q} = \frac{q^n [k+1]_q}{q^k [n+1]_q [n]_q},
$$

we can easily see that

$$
\begin{aligned}
f\left[\frac{[k]_q}{q^{k-1}[n+1]_q}, \frac{[k+1]_q}{q^k[n+1]_q}, \frac{[k+1]_q}{q^k[n]_q}\right] &= \frac{q^{2k}[n]_q [n+1]_q^2}{q^n [n+k+1]_q} \left(q^n f\left(\frac{[k]_q}{q^{k-1}[n+1]_q}\right)\right. \\
&\left. - \frac{[n+k+1]_q}{[k+1]_q} f\left(\frac{[k+1]_q}{q^k[n+1]_q}\right) + \frac{[n]_q}{[k+1]_q} f\left(\frac{[k+1]_q}{q^k[n]_q}\right)\right).
\end{aligned}
$$

This proves the theorem. ■

We know that a function f is convex if and only if all second-order divided differences of f are nonnegative. Using this property and Theorem 2.14, we have the following result:

Corollary 2.7. *If $f(x)$ is a convex function defined on $\mathbb{R}^+$, then the q-Baskakov operator $B_{n,q}(f,x)$ defined by (2.36) is strictly monotonically nondecreasing in n, unless f is the linear function (in which case $B_{n,q}(f,x) = B_{n+1,q}(f,x)$ for all n).*

2.4 Approximation Properties of *q*-Baskakov Operators

We establish direct estimates for the q-Baskakov operator given by (2.36), using the second-order Ditzian–Totik modulus of smoothness. Furthermore, we define and study the limit q-Baskakov operator.

This section based on [63].

2.4.1 Introduction

We denote by $C_B[0,\infty)$ the space of all real valued, continuous, and bounded functions defined on $[0,\infty)$. This space equipped with the norm $\|f\| = \sup\{|f(x)| : x \in [0,\infty)\},\ f \in C_B[0,\infty)$ is a Banach space.

We know that from (2.36), q-analogue of Baskakov operators, which for $q \in (0,1)$, $n = 1,2,\ldots,$ $f \in C_B[0,\infty)$ and $x \in [0,\infty)$, is defined as

$$\begin{aligned}\mathcal{B}_{n,q}(f,x) &= \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} x^k (-x,q)^{-1}_{n+k} f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right) \\ &= \sum_{k=0}^{\infty} \mathcal{P}^q_{n,k}(x) f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right).\end{aligned}$$

For $q = 1$, we recover the well-known Baskakov operators [37].

Here, to obtain direct global estimates for the q-Baskakov operators, we use the second-order Ditzian–Totik modulus of smoothness, defined for $f \in C_B[0,\infty)$ by

$$\omega^2_\varphi(f;\delta) = \sup_{0<h\le\delta}\ \sup_{x\pm h\varphi(x)\in[0,\infty)} |f(x+h\varphi(x)) - 2f(x) + f(x-h\varphi(x))|, \qquad (2.49)$$

where $\varphi(x) = \sqrt{x(1+x)},\ x \in [0,\infty)$, let us consider the following K-function:

$$\bar{K}_{2,\varphi}(f;\delta) = \inf_{g' \in AC_{loc}[0,\infty)} \{\|f-g\| + \delta\|\varphi^2 g''\| + \delta^2\|g''\|\},$$

where $g' \in AC_{loc}[0,\infty)$ means that g is differentiable and g' is absolutely continuous in every closed finite interval $[a,b] \subset [0,\infty)$. In view of [51, pp. 24–25], we have known that $\omega^\varphi_2(f;\delta)$ and $\bar{K}_{2,\varphi}(f;\delta^2)$ are equivalent, i.e., there exists $C > 0$ such that

$$C^{-1}\omega^2_\varphi(f;\delta) \le \bar{K}_{2,\varphi}(f;\delta^2) \le C\omega^2_\varphi(f;\delta). \qquad (2.50)$$

Here we mention that C will denote throughout this paper an absolute positive constant which can be different at each occurrence. Analogously, for the K-function

$$K_{2,\varphi}(f;\delta) = \inf_{g' \in AC_{loc}[0,\infty)} \{\|f-g\| + \delta\|\varphi^2 g''\|\},$$

we have the equivalence of $\omega_2^{\varphi}(f;\delta)$ and $K_{2,\varphi}(f;\delta^2)$ (see [51, p. 11, Theorem 2.1.1], i.e., there exists $C > 0$ such that

$$C^{-1}\omega_2^{\varphi}(f;\delta) \le K_{2,\varphi}(f;\delta^2) \le C\omega_2^{\varphi}(f;\delta). \tag{2.51}$$

Furthermore, for $f \in C_B[0,\infty)$, $q \in (0,1)$, and $x \in [0,\infty)$, we define the limit q-Baskakov operator as

$$\mathcal{B}_{\infty,q}(f;x) \equiv (\mathcal{B}_{\infty,q}f)(x) = \sum_{k=0}^{\infty} v_{\infty,k}(q;x) f\left(\frac{1-q^k}{q^{k-1}}\right), \tag{2.52}$$

where

$$b_{\infty,k}(q;x) = q^{k(k-1)/2}(1-q)^{-1}(1-q^2)^{-1}\dots(1-q^k)^{-1} \times x^k \prod_{s=0}^{\infty}(1+xq^s)^{-1}. \tag{2.53}$$

By Euler's identity (see [20, Chap. 10, Corollary 10.2.2]), we have

$$\sum_{k=0}^{\infty} q^{k(k-1)/2} x^k (1-q)^{-1}(1-q^2)^{-1}\dots(1-q^k)^{-1} = \prod_{s=0}^{\infty}(1+xq^s),$$

where $x \in [0,\infty)$ and $q \in (0,1)$. Due to (2.53), the last identity implies that

$$\sum_{k=0}^{\infty} b_{\infty,k}(q;x) = 1 \tag{2.54}$$

for $q \in (0,1)$ and $x \in [0,\infty)$. Hence

$$|\mathcal{B}_{\infty,q}(f;x)| \le \|f\| \sum_{k=0}^{\infty} b_{\infty,k}(q;x) = \|f\|,$$

i.e., $\|\mathcal{B}_{\infty,q}f\| \le \|f\|$ for $f \in C_B[0,\infty)$. This means that the limit q-Baskakov operator is well defined.

In what follows we shall estimate the rate of approximation $\|\mathcal{B}_{n,q}f - \mathcal{B}_{\infty,q}f\|$ by the second-order Ditzian–Totik modulus of smoothness of f (see (2.49)).

2.4.2 Main Results

We introduce the space $\tilde{C}_B[0,\infty) = \{f \in C_B[0,\infty) : \text{there exists} \lim_{x\to\infty} f(x) = 0\}$. Obviously $\tilde{C}_B[0,\infty) \subset C_B[0,\infty)$ and $\tilde{C}_B[0,\infty)$ is also a Banach space. The following theorems were studied in [63].

Theorem 2.15. *Let $C_0 \in (0,1)$ be an absolute constant with the property that $q = q(n) \in (C_0^{1/n}, 1)$ for every $n = 1,2,\ldots$. Then there exists $C > 0$ such that*

$$\|\mathcal{B}_{n,q}f - f\| \le C\,\omega_\varphi^2(f;[n]_q^{-1/2}) \tag{2.55}$$

for all $f \in \tilde{C}_B[0,\infty)$ and $n = 3,4,\ldots$.

In the next theorem we estimate the rate of approximation $\|\mathcal{B}_{n,q}f - \mathcal{B}_{\infty,q}f\|$ for $f \in \tilde{C}_B[0,\infty)$, using the modulus of smoothness (2.49).

Theorem 2.16. *There exists $C > 0$ such that*

$$\|\mathcal{B}_{n,q}f - \mathcal{B}_{\infty,q}f\| \le C\,\omega_\varphi^2\left(f;\sqrt{q^{n-1}/(1-q^n)}\right), \tag{2.56}$$

for all $f \in \tilde{C}_B[0,\infty)$, $n = 1,2,\ldots$ and $q \in (0,1)$.

2.4.3 Proofs

The q-forward differences lead us to the moments of the first and second orders of $\mathcal{B}_{n,q}$.

Lemma 2.7. *We have*

$$\|\mathcal{B}_{n,q}f\| \le \|f\|$$

for all $f \in C_B[0,\infty)$, $n = 1,2,\ldots$ and $q \in (0,1)$.

Proof. For $x \in [0,\infty)$ one has $|\mathcal{B}_{n,q}(f;x)| \le \|f\|\,\mathcal{B}_{n,q}(1;x) = \|f\|$, taking into account Lemma 2.5. Thus $\|\mathcal{B}_{n,q}f\| \le \|f\|$, which completes the proof.
Proof of Theorem 2.15. Let $g \in C_B[0,\infty)$ with $g' \in AC_{loc}[0,\infty)$ be arbitrary. From Taylor's expansion

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t g''(u)(t-u)\,du, \quad t \in [0,\infty),$$

we have, by Lemma 2.5, that

$$\mathcal{B}_{n,q}(g;x) - g(x) = \mathcal{B}_{n,q}\left(\int_x^t g''(u)(t-u)\,du;x\right).$$

Hence, in view of [51, p. 140, Lemma 9.6.1], we obtain

$$
\begin{aligned}
|\mathcal{B}_{n,q}(g;x)-g(x)| &\leq \mathcal{B}_{n,q}\left(\left|\int_x^t |g''(u)||(t-u)|\,du\right|;x\right)\\
&\leq \mathcal{B}_{n,q}\left(\frac{(t-x)^2}{x}\left(\frac{1}{1+x}+\frac{1}{1+t}\right);x\right)\|\varphi^2 g''\|\\
&= \frac{1}{x(1+x)}\mathcal{B}_{n,q}((t-x)^2;x)\|\varphi^2 g''\|\\
&\quad +\frac{1}{x}\mathcal{B}_{n,q}\left(\frac{(t-x)^2}{1+t};x\right)\|\varphi^2 g''\|. \qquad (2.57)
\end{aligned}
$$

But, by Lemma 2.5,

$$
\begin{aligned}
\mathcal{B}_{n,q}((t-x)^2;x) &= \mathcal{B}_{n,q}(t^2;x)-2x\mathcal{B}_{n,q}(t;x)+x^2_{n,q}\mathcal{B}(1;x) \qquad (2.58)\\
&= \frac{1}{q[n]_q}x(q+x) \leq \frac{1}{q[n]_q}\varphi^2(x).
\end{aligned}
$$

Furthermore, by Hölder's inequality, we have

$$
\mathcal{B}_{n,q}\left(\frac{(t-x)^2}{1+t};x\right) \leq \left\{\mathcal{B}_{n,q}((1+t)^{-2};x)\right\}^{1/2}\left\{\mathcal{B}_{n,q}((t-x)^4;x)\right\}^{1/2}. \qquad (2.59)
$$

Using (2.36), we find that

$$
\begin{aligned}
\mathcal{B}_{n,q}((1+t)^{-2};x) = \sum_{k=0}^{\infty} & \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{k(k-1)/2}\\
& x^k(1+x)^{-1}(1+xq)^{-1}\dots(1+xq^{n+k-1})^{-1}\\
& \left(\frac{q^{k-1}[n]_q}{[k]_q+q^{k-1}[n]_q}\right)^2. \qquad (2.60)
\end{aligned}
$$

Because

$$
\begin{aligned}
&(1+xq^{n+k-2})(1+xq^{n+k-1})(1+q^{-2n-2k+3})\\
&= (1+q^{-2n-2k+3})+(1+q^{-2n-2k+3})(q^{n+k-2}+q^{n+k-1})x\\
&\quad +(1+q^{-2n-2k+3})q^{2n+2k-3}x^2\\
&\geq 1+2x+x^2 = (1+x)^2
\end{aligned}
$$

for $x \in [0,\infty)$, we get, by (2.60),

$$\begin{aligned}
&\mathcal{B}_{n,q}((1+t)^{-2};x) \\
&\leq \sum_{k=0}^{\infty} \begin{bmatrix} n+k-3 \\ k \end{bmatrix}_q \frac{[n+k-2]_q[n+k-1]_q}{[n-2]_q[n-1]_q} q^{k(k-1)/2} \\
&\qquad x^k(1+x)^{-1}(1+xq)^{-1}\ldots(1+xq^{n+k-3})^{-1} \\
&\qquad (1+q^{-2n-2k+3})(1+x)^{-2}\frac{q^{2k-2}[n]_q^2}{([k]_q+q^{k-1}[n]_q)^2}. \qquad (2.61)
\end{aligned}$$

Using the identities $[n+k-2]_q = [k]_q + q^k[n-2]_q$ and $[n+k-1]_q = [k]_q + q^k[n-1]_q$, we obtain

$$\begin{aligned}
[n+k-2]_q[n+k-1]_q &= [k]_q^2 + q^k[k]_q([n-2]_q+[n-1]_q) \\
&\quad + q^{2k}[n-2]_q[n-1]_q \\
&\leq [k]_q^2 + 2q^{k-1}[k]_q[n]_q + q^{2k-2}[n]_q^2 \\
&= ([k]_q + q^{k-1}[n]_q)^2.
\end{aligned}$$

Hence

$$\frac{[n+k-2]_q[n+k-1]_q}{([k]_q+q^{k-1}[n]_q)^2} \leq 1. \qquad (2.62)$$

Analogously, the identities $[n]_q = 1+q[n-1]_q$ and $[n]_q = 1+q+q^2[n-2]_q$ for $n = 3,4,\ldots$ imply that

$$\begin{aligned}
\frac{[n]_q^2}{[n-2]_q[n-1]_q} &= \frac{[n]_q}{[n-2]_q}\frac{[n]_q}{[n-1]_q} \\
&= \left(\frac{1+q}{[n-2]_q}+q^2\right)\left(\frac{1}{[n-1]_q}+q\right) \leq 6. \qquad (2.63)
\end{aligned}$$

The condition $q = q(n) \in (C_0^{1/n},1)$ implies

$$(1+q^{-2n-2k+3})q^{2k-2} \leq \frac{2}{q^{2n}} \leq \frac{2}{C_0^2}. \qquad (2.64)$$

Now combining (2.61)–(2.64), we obtain

$$\begin{aligned}
\mathcal{B}_{n,q}((1+t)^{-2};x) &\leq \frac{12}{C_0^2}\frac{1}{(1+x)^2}\mathcal{B}_{n-2,q}(1;x) \\
&= \frac{12}{C_0^2}\frac{1}{(1+x)^2} \qquad (2.65)
\end{aligned}$$

for $x \in [0,\infty)$ and $n = 3,4,\ldots$.

On the other hand

$$\mathcal{B}_{n,q}((t-x)^4;x) = \mathcal{B}_{n,q}(t^4;x) - 4x\mathcal{B}_{n,q}(t^3;x) + 6x^2_{n,q}\mathcal{B}(t^2;x) - 4x^3_{n,q}\mathcal{B}(t;x) + x^4_{n,q}\mathcal{B}(1;x). \tag{2.66}$$

To compute $\mathcal{B}_{n,q}(t^m;x),\ m=0,1,2,3,4$, we use Lemma 2.5 and the definition of the q-forward differences given above. Then, by direct computations, we get

$$\mathcal{B}_{n,q}(t^3;x) = \frac{1}{[n]_q^2}x + \frac{1+2q+1}{q^2[n]_q^2}x^2 + \frac{1}{q^3}\frac{[n+1]_q[n+2]_q}{[n]_q^2}x^3$$

and

$$\begin{aligned}
&\mathcal{B}_{n,q}(t^4;x)\\
&= \frac{1}{[n]_q^3}x + \frac{1}{q^3}(1+3q+3q^2)\frac{[n+1]_q}{[n]_q^3}x^2\\
&\quad + \frac{1}{q^5(1+q)}(1+3q+5q^2+3q^3)\frac{[n+1]_q[n+2]_q}{[n]_q^3}x^3\\
&\quad + \frac{1}{q^6(1+q)(1+q+q^2)(1+q+q^2+q^3)}(1+3q+5q^2+6q^3\\
&\quad + 5q^4+3q^5+q^6)\frac{[n+1][n+2]_q[n+3]_q}{[n]_q^3}x^4.
\end{aligned}$$

Hence, by (2.66),

$$\begin{aligned}
&B_{n,q}((t-x)^4;x)\\
&\quad = \frac{1}{[n]_q^3}x + \frac{1}{q^3[n]_q^3}\left\{q(1+3q-q^2)[n]_q + (1+3q+3q^2)\right\}x^2\\
&\qquad + \frac{1}{q^5[n]_q^3}\left\{q^3(1-q)^2[n]_q^2 + q(1+4q+3q^2-2q^3)[n]_q\right.\\
&\qquad \left. + (1+3q+5q^2+3q^3)\right\}x^3\\
&\qquad + \frac{1}{q^6[n]_q^3}\left\{q^3(1-q)^2[n]_q^2 + q(1+3q-q^3)[n]_q\right.\\
&\qquad \left. + (1+2q+2q^2+q^3)\right\}x^4.
\end{aligned}$$

Taking into account the condition $q \in (C_0^{1/n},1)$, we obtain that $q \in (C_0,1)$. Then, for $x \geq \frac{1}{[n]_q}$, we have

$$
\begin{aligned}
&\mathcal{B}_{n,q}((t-x)^4;x) \\
&\leq \frac{1}{[n]_q^2}x^2 + \frac{1}{C_0^3[n]_q^2}\left\{q(1+3q+q^2)+(1+3q+3q^2)\frac{1}{[n]_q}\right\}x^2 \\
&\quad + \frac{1}{C_0^5[n]_q^2}\left\{q^3(1-q^n)^2\frac{1}{[n]_q}+q(1+4q+3q^2+2q^3)\right. \\
&\quad \left. + (1+3q+5q^2+3q^3)\frac{1}{[n]_q}\right\}x^3 \\
&\quad + \frac{1}{C_0^6[n]_q^2}\left\{q^3(1-q^n)^2\frac{1}{[n]_q}+q(1+3q+q^3)\right. \\
&\quad \left. + (1+2q+2q^2+q^3)\frac{1}{[n]_q}\right\}x^4 \\
&\leq \frac{1}{C_0^6[n]_q^2}\left\{x^2+12x^2+23x^3+12x^4\right\} \\
&\leq \frac{C}{[n]_q^2}\varphi^4(x).
\end{aligned}
$$

Hence, in view of (2.57)–(2.60), we find for $x \geq \frac{1}{[n]}$ that

$$
\begin{aligned}
|\mathcal{B}_{n,q}(g;x)-g(x)| &\leq \frac{1}{x(1+x)}\frac{1}{C_0[n]}\varphi^2(x)\|\varphi^2 g''\| \\
&\quad + \frac{1}{x}\frac{C}{1+x}\frac{C}{[n]}\varphi^2(x)\|\varphi^2 g''\| \\
&\leq \frac{C}{[n]}\|\varphi^2 g''\|. \qquad (2.67)
\end{aligned}
$$

For $0 \leq x \leq \frac{1}{[n]}$ we have, by Taylor's expansion,

$$
g(t) = g(x) + g'(x)(t-x) + \int_x^t g''(u)(t-u)\,du, \quad t \in [0,\infty),
$$

and Lemma 2.5 and (2.58) that

$$
\begin{aligned}
|\mathcal{B}_{n,q}(g;x)-g(x)| &\leq \mathcal{B}_{n,q}\left(\left|\int_x^t |g''(u)||t-u|\,du\right|;x\right) \\
&\leq \mathcal{B}_{n,q}((t-x)^2;x)\|g''\| \leq \frac{1}{q[n]_q}x(1+x)\|g''\| \\
&\leq \frac{C}{[n]_q^2}\|g''\|. \qquad (2.68)
\end{aligned}
$$

Then (2.67) and (2.68) imply

$$|\mathcal{B}_{n,q}(g;x)-g(x)| \le C\left\{\frac{1}{[n]_q}\|\varphi^2 g''\| + \frac{1}{[n]_q^2}\|g''\|\right\}$$

for all $x \in [0,\infty)$. Hence, by Lemma 2.7, we obtain, for $f \in \tilde{C}_B[0,\infty)$,

$$\begin{aligned}\|\mathcal{B}_{n,q}f-f\| &\le \|\mathcal{B}_{n,q}f-\mathcal{B}_{n,q}g\| + \|\mathcal{B}_{n,q}g-g\| + \|g-f\| \\ &\le \|f-g\| + C\left\{\frac{1}{[n]_q}\|\varphi^2 g''\| + \frac{1}{[n]_q^2}\|g''\|\right\} + \|g-f\| \\ &\le C\left\{\|f-g\| + \frac{1}{[n]_q}\|\varphi^2 g''\| + \frac{1}{[n]_q^2}\|g''\|\right\}.\end{aligned}$$

Using the definition of the K-functional $\bar{K}_{2,\varphi}(f;1/[n]_q)$, we have $\|\mathcal{B}_{n,q}f-f\| \le C\bar{K}_{2,\varphi}(f;1/[n]_q)$. Then, in view of (2.50), we get (2.55), which was to be proved.

Proof of Theorem 2.16. Let $g \in C_B[0,\infty)$ with $g' \in AC_{loc}[0,\infty)$ be arbitrary. By Taylor's formula, we have

$$\begin{aligned}g\left(\frac{[k]_q}{q^{k-1}[n+1]_q}\right) &= g\left(\frac{[k+1]_q}{q^k[n+1]_q}\right) \\ &\quad + \left(\frac{[k]_q}{q^{k-1}[n+1]_q} - \frac{[k+1]_q}{q^k[n+1]_q}\right) g'\left(\frac{[k+1]_q}{q^k[n+1]_q}\right) \\ &\quad + \int_{[k+1]_q/q^k[n+1]_q}^{[k]_q/q^{k-1}[n+1]_q} \left(\frac{[k]_q}{q^{k-1}[n+1]_q} - u\right) g''(u)\,du\end{aligned}$$

and

$$\begin{aligned}g\left(\frac{[k+1]_q}{q^k[n]_q}\right) &= g\left(\frac{[k+1]_q}{q^k[n+1]_q}\right) \\ &\quad + \left(\frac{[k+1]_q}{q^k[n]_q} - \frac{[k+1]_q}{q^k[n+1]_q}\right) g'\left(\frac{[k+1]_q}{q^k[n+1]_q}\right) \\ &\quad + \int_{[k+1]_q/q^k[n+1]_q}^{[k+1]_q/q^k[n]_q} \left(\frac{[k+1]_q}{q^k[n]_q} - u\right) g''(u)\,du.\end{aligned}$$

Hence, by Theorem 2.14 and $[n+k+1]_q = [n]_q + q^n[k+1]_q$, we obtain

$$\begin{aligned}
&\mathcal{B}_{n,q}(g;x) - \mathcal{B}_{n+1,q}(g;x)\\
&= \sum_{k=0}^{\infty} q^{k(k+1)/2} x^{k+1} (1+x)^{-1}(1+xq)^{-1}\dots(1+xq^{n+k})^{-1}\\
&\quad \begin{bmatrix} n+k \\ k \end{bmatrix}_q \left\{ \left\{ q^n \left(\frac{[k]_q}{q^{k-1}[n+1]_q} - \frac{[k+1]_q}{q^k[n+1]_q} \right) \right.\right.\\
&\quad \left. + \frac{[n]_q}{[k+1]_q} \left(\frac{[k+1]_q}{q^k[n]_q} - \frac{[k+1]_q}{q^k[n+1]_q} \right) \right\} g'\left(\frac{[k+1]_q}{q^k[n+1]_q} \right)\\
&\quad + q^n \int_{[k+1]_q/q^k[n+1]_q}^{[k]_q/q^{k-1}[n+1]_q} \left(\frac{[k]}{q^{k-1}[n+1]} - u \right) g''(u)\,du\\
&\quad \left. + \frac{[n]_q}{[k+1]_q} \int_{[k+1]_q/q^k[n+1]_q}^{[k+1]_q/q^k[n]_q} \left(\frac{[k+1]_q}{q^k[n]_q} - u \right) g''(u)\,du \right\}. \qquad (2.69)
\end{aligned}$$

Because

$$\begin{aligned}
&q^n \left(\frac{[k]_q}{q^{k-1}[n+1]_q} - \frac{[k+1]_q}{q^k[n+1]_q} \right) + \frac{[n]_q}{[k+1]_q} \left(\frac{[k+1]_q}{q^k[n]_q} - \frac{[k+1]_q}{q^k[n+1]_q} \right)\\
&= -\frac{q^{n-k}}{[n+1]_q} + \frac{1}{q^k} \frac{[n+1]_q - [n]_q}{[n+1]_q}\\
&= -\frac{q^{n-k}}{[n+1]_q} + \frac{q^{n-k}}{[n+1]_q} = 0,
\end{aligned}$$

we get, by (2.69),

$$\begin{aligned}
&|\mathcal{B}_{n,q}(g;x) - \mathcal{B}_{n+1,q}(g;x)|\\
&\le \sum_{k=0}^{\infty} q^{k(k+1)/2} x^{k+1} (1+x)^{-1}(1+xq)^{-1}\dots(1+xq^{n+k})^{-1}\\
&\quad \begin{bmatrix} n+k \\ k \end{bmatrix}_q \left\{ q^n \left| \int_{[k+1]_q/q^k[n+1]_q}^{[k]_q/q^{k-1}[n+1]_q} \left| \frac{[k]_q}{q^{k-1}[n+1]_q} - u \right| |g''(u)|\,du \right| \right.\\
&\quad \left. + \frac{[n]_q}{[k+1]_q} \left| \int_{[k+1]_q/q^k[n+1]_q}^{[k+1]_q/q^k[n]_q} \left| \frac{[k+1]_q}{q^k[n]_q} - u \right| |g''(u)|\,du \right| \right\}. \qquad (2.70)
\end{aligned}$$

Taking into account the estimate

$$\left| \int_x^t (t-u)g''(u)\,du \right| \le (t-x)^2 \frac{1}{x}\left(\frac{1}{1+x}+\frac{1}{1+t}\right)\|\varphi^2 g''\|,$$

$t,x \in [0,\infty)$ (see [51, p. 140, Lemma 9.6.1]), we find that

$$\begin{aligned}
& q^n \left| \int_{[k+1]_q/q^k[n+1]_q}^{[k]_q/q^{k-1}[n+1]_q} \left| \frac{[k]_q}{q^{k-1}[n+1]_q} - u \right| |g''(u)|\,du \right| \\
& \le q^n \left(\frac{[k]_q}{q^{k-1}[n+1]_q} - \frac{[k+1]_q}{q^k[n+1]_q} \right)^2 \frac{q^k[n+1]_q}{[k+1]_q} \\
& \quad \left(\frac{q^k[n+1]_q}{[k+1]_q + q^k[n+1]_q} + \frac{q^{k-1}[n+1]_q}{[k]_q + q^{k-1}[n+1]_q} \right) \|\varphi^2 g''\| \\
& = q^{n-1} \left(\frac{q}{q^k + [n+k+1]_q} + \frac{1}{q^{k-1} + [n+k]_q} \right) \|\varphi^2 g''\| \qquad (2.71)
\end{aligned}$$

and

$$\begin{aligned}
& \frac{[n]_q}{[k+1]_q} \left| \int_{[k+1]_q/q^k[n+1]_q}^{[k+1]_q/q^k[n]_q} \left| \frac{[k+1]_q}{q^k[n]_q} - u \right| |g''(u)|\,du \right| \\
& \le \frac{[n]_q}{[k+1]_q} \left(\frac{[k+1]_q}{q^k[n]_q} - \frac{[k+1]_q}{q^k[n+1]_q} \right)^2 \frac{q^k[n+1]_q}{[k+1]_q} \\
& \quad \left(\frac{q^k[n+1]_q}{[k+1]_q + q^k[n+1]_q} + \frac{q^k[n]_q}{[k+1]_q + q^k[n]_q} \right) \|\varphi^2 g''\| \\
& = \frac{q^{2n}}{[n]_q[n+1]_q} \left(\frac{[n+1]_q}{q^k + [n+k+1]_q} + \frac{[n]_q}{q^k + [n+k]_q} \right) \|\varphi^2 g''\| \qquad (2.72)
\end{aligned}$$

Then (2.70)–(2.72) imply

$$\begin{aligned}
& |\mathcal{B}_{n,q}(g;x) - \mathcal{B}_{n+1,q}(g;x)| \\
& \le \sum_{k=0}^{\infty} q^{k(k+1)/2} x^{k+1} (1+x)^{-1}(1+xq)^{-1}\dots(1+xq^{n+k})^{-1} \\
& \quad \begin{bmatrix} n+k \\ k+1 \end{bmatrix}_q \frac{[k+1]_q}{[n]_q} \left\{ q^{n-1} \left(\frac{q}{q^k + [n+k+1]_q} + \frac{1}{q^{k-1} + [n+k]_q} \right) \right. \\
& \quad \left. + \frac{q^{2n}}{[n]_q[n+1]_q} \left(\frac{[n+1]_q}{q^k + [n+k+1]_q} + \frac{[n]}{q^k + [n+k]_q} \right) \right\} \|\varphi^2 g''\|
\end{aligned}$$

$$\leq \sum_{k=0}^{\infty} q^{k(k+1)/2} x^{k+1} (1+x)^{-1}(1+xq)^{-1} \dots (1+xq^{n+k})^{-1}$$
$$\begin{bmatrix} n+k \\ k+1 \end{bmatrix}_q \frac{1}{[n]_q} \left\{ q^{n-1} \left(\frac{[k+1]_q}{[n+k+1]_q} + \frac{[k+1]_q}{[n+k]_q} \right) \right.$$
$$\left. + q^{2n} \left(\frac{1}{[n]_q} \frac{[k+1]_q}{[n+k+1]_q} + \frac{1}{[n+1]} \frac{[k+1]_q}{[n+k]_q} \right) \right\} \|\varphi^2 g''\|$$
$$\leq \frac{4q^{n-1}}{[n]_q} \sum_{k=0}^{\infty} q^{k(k+1)/2} x^{k+1} (1+x)^{-1}(1+xq)^{-1} \dots (1+xq^{n+k})^{-1}$$
$$\begin{bmatrix} n+k \\ k+1 \end{bmatrix}_q \|\varphi^2 g''\|$$
$$\leq \frac{4q^{n-1}}{[n]_q} B_{n,q}(1;x) \|\varphi^2 g''\| = \frac{4q^{n-1}}{[n]_q} \|\varphi^2 g''\|$$

(see Lemma 2.5). Hence

$$\|\mathcal{B}_{n,q} g - \mathcal{B}_{n+1,q} g\| \leq \frac{4q^{n-1}}{[n]_q} \|\varphi^2 g''\|.$$

Now for $p = 1, 2, \dots$, we obtain

$$\begin{aligned} \|\mathcal{B}_{n,q} g - \mathcal{B}_{n+p,q} g\| &\leq \|\mathcal{B}_{n,q} g - \mathcal{B}_{n+1,q} g\| + \|\mathcal{B}_{n+1,q} g - \mathcal{B}_{n+2,q} g\| \\ &\quad + \dots + \|\mathcal{B}_{n+p-1,q} g - \mathcal{B}_{n+p,q} g\| \\ &\leq \frac{4q^{n-1}}{[n]_q} (1 + q + \dots + q^{n+p-2}) \|\varphi^2 g''\| \\ &\leq \frac{4q^{n-1}}{[n]_q (1-q)} \|\varphi^2 g''\| = \frac{4q^{n-1}}{1-q^n} \|\varphi^2 g''\|. \end{aligned}$$

Then, by Lemma 2.7 for $f \in C_B[0,\infty)$, we have

$$\begin{aligned} \|\mathcal{B}_{n,q} f - \mathcal{B}_{n+p,q} f\| &\leq \|\mathcal{B}_{n,q} f - \mathcal{B}_{n,q} g\| + \|\mathcal{B}_{n,q} g - \mathcal{B}_{n+p,q} g\| \\ &\quad + \|\mathcal{B}_{n+p,q} g - \mathcal{B}_{n+p,q} f\| \\ &\leq \|f - g\| + \frac{4q^{n-1}}{1-q^n} \|\varphi^2 g''\| + \|f - g\| \\ &\leq 4 \left\{ \|f - g\| + \frac{q^{n-1}}{1-q^n} \|\varphi^2 g''\| \right\}. \end{aligned}$$

Hence, in view of (2.51), we obtain

$$\|\mathcal{B}_{n,q}f - \mathcal{B}_{n+p,q}f\| \le 4K_{2,\varphi}(f; q^{n-1}/(1-q^n)) \\ \le C\,\omega_\varphi^2(f; \sqrt{q^{n-1}/(1-q^n)}) \tag{2.73}$$

for every $n, p = 1, 2, \ldots$.

On the other hand $\lim_{n\to\infty} \omega_\varphi^2(f; \sqrt{q^{n-1}/(1-q^n)}) = 0$, because of $f \in \tilde{C}_B[0,\infty)$ (see [51, pp. 36–37]). Then (2.73) implies that $\{V_{n,q}f\}$ is a Cauchy-sequence in the Banach space $C_B[0,\infty)$. Thus it converges in $C_B[0,\infty)$. In conclusion, by (2.73), there exists an operator $L : \tilde{C}_B[0,\infty) \to C_B[0,\infty)$ such that

$$\|\mathcal{B}_{n,q}f - Lf\| \le C\,\omega_\varphi^2(f; \sqrt{q^{n-1}/(1-q^n)}) \tag{2.74}$$

for all $f \in \tilde{C}_B[0,\infty)$ and $n = 1, 2, \ldots$.

Finally, we prove that $Lf \equiv \mathcal{B}_{\infty,q}f$ for $f \in \tilde{C}_B[0,\infty)$. Let $x \in [0,\infty)$ be arbitrary. Then

$$|L(f;x) - \mathcal{B}_{\infty,q}(f;x)| \le |L(f;x) - \mathcal{B}_{n,q}(f;x)| \\ + |\mathcal{B}_{n,q}(f;x) - \mathcal{B}_{\infty,q}(f;x)|.$$

By (2.36), (2.52), and (2.53), we have

$$\begin{aligned}
&|B_{n,q}(f;x) - B_{\infty,q}(f;x)| \\
&= \Bigg| \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{k(k-1)/2} x^k \prod_{s=0}^{n+k-1} (1+xq^s)^{-1} \\
&\quad (1 - \prod_{s=n+k}^{\infty} (1+xq^s)^{-1}) f\left(\frac{1-q^k}{q^{k-1}(1-q^n)}\right) \\
&\quad + \sum_{k=0}^{\infty} \left\{ \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q - (1-q)^{-1}(1-q^2)^{-1}\ldots(1-q^k)^{-1} \right\} \\
&\quad q^{k(k-1)/2} x^k \prod_{s=0}^{\infty} (1+xq^s)^{-1} f\left(\frac{1-q^k}{q^{k-1}(1-q^n)}\right) \\
&\quad + \sum_{k=0}^{\infty} q^{k(k-1)/2} (1-q)^{-1}(1-q^2)^{-1}\ldots(1-q^k)^{-1} \\
&\quad x^k \prod_{s=0}^{n+k-1} (1+xq^s)^{-1} \left\{ f\left(\frac{1-q^k}{q^{k-1}(1-q^n)}\right) - f\left(\frac{1-q^k}{q^{k-1}}\right) \right\} \Bigg|
\end{aligned}$$

$$\leq \|f\| \sum_{k=0}^{\infty} \left\{ \begin{bmatrix} n+k-1 \\ k \end{bmatrix} q^{k(k-1)/2} x^k \prod_{s=0}^{n+k-1} (1+xq^s)^{-1} \right.$$

$$(1 - \prod_{s=n+k}^{\infty} (1+xq^s)^{-1}) + \|f\| \sum_{k=0}^{\infty} \left| \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q - (1-q)^{-1} \right.$$

$$\left. (1-q^2)^{-1} \dots (1-q^k)^{-1} \right| q^{k(k-1)/2} x^k \prod_{s=0}^{\infty} (1+xq^s)^{-1}$$

$$+ \sum_{k=0}^{\infty} q^{k(k-1)/2} (1-q)^{-1} (1-q^2)^{-1} \dots (1-q^k)^{-1}$$

$$x^k \prod_{s=0}^{\infty} (1+xq^s)^{-1} \left| f\left(\frac{1-q^k}{q^{k-1}(1-q^n)} \right) - f\left(\frac{1-q^k}{q^{k-1}} \right) \right|$$

$$=: I_1 + I_2 + I_3. \tag{2.75}$$

Furthermore, the infinite product $\prod_{s=0}^{\infty} (1+xq^s)^{-1}$ is convergent; thus for every $\varepsilon > 0$ there exists n'_ε such that

$$0 < 1 - \prod_{s=n+k}^{\infty} (1+xq^s)^{-1} \leq 1 - \prod_{s=n}^{\infty} (1+xq^s)^{-1} \leq \frac{\varepsilon}{3\|f\|}$$

for $n > n'_\varepsilon$ and $k = 0, 1, 2, \dots$. Hence, by Lemma 2.5,

$$I_1 < \frac{\varepsilon}{3} \mathcal{B}_{n,q}(1;x) = \frac{\varepsilon}{3}. \tag{2.76}$$

In view of (2.54), we have $\mathcal{B}_{\infty,q}(1;x) = 1$. By [99, p. 156, (2.8)], we know that the inequality

$$1 - \prod_{s=j}^{\infty} (1-q^s) \leq \frac{q^j}{q(1-q)} \ln \frac{1}{1-q}$$

holds for $0 < q < 1$ and $j = 1, 2, \dots$. Then we obtain

$$I_2 = \|f\| \sum_{k=0}^{\infty} \left| (1-q)(1-q^2) \dots (1-q^k) \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q - 1 \right|$$

$$q^{k(k-1)/2} (1-q)^{-1} (1-q^2)^{-1} \dots (1-q^k)^{-1} x^k \prod_{s=0}^{\infty} (1+xq^s)^{-1}$$

$$= \|f\| \sum_{k=0}^{\infty} |(1-q^n)(1-q^{n+1}) \dots (1-q^{n+k-1}) - 1|$$

$$q^{k(k-1)/2}(1-q)^{-1}(1-q^2)^{-1}\dots(1-q^k)^{-1}x^k\prod_{s=0}^{\infty}(1+xq^s)^{-1}$$

$$\leq \|f\|\sum_{k=0}^{\infty}(1-\prod_{s=n}^{\infty}(1-q^s))$$

$$q^{k(k-1)/2}(1-q)^{-1}(1-q^2)^{-1}\dots(1-q^k)^{-1}x^k\prod_{s=0}^{\infty}(1+xq^s)^{-1}$$

$$\leq \|f\|\frac{q^n}{q(1-q)}\ln\frac{1}{1-q}V_{\infty,q}(1;x)$$

$$= \frac{q^n}{q(1-q)}\ln\frac{1}{1-q}\|f\|.$$

In conclusion, if $\varepsilon > 0$ is arbitrary, then there exists n''_ε such that $q^n < \varepsilon q(1-q)/(\|f\|\ln(1-q)^{-1})$ for every $n > n''_\varepsilon$. Thus

$$I_2 < \frac{\varepsilon}{3}. \tag{2.77}$$

Finally, because $f \in \tilde{C}_B[0,\infty)$ and $\varepsilon > 0$ is arbitrary, there exists $y_\varepsilon > 0$ such that $|f(y)| < \varepsilon/12$ for $y > y_\varepsilon$. Because $\frac{1-q^k}{q^{k-1}} \leq \frac{1-q^k}{q^{k-1}(1-q^n)}$ for all $k = 0,1,2,\dots$ and $n = 1,2,\dots$, there exists k_ε such that $\frac{1-q^k}{q^{k-1}} > y_\varepsilon$ for $k > k_\varepsilon$. Then

$$\left| f\left(\frac{1-q^k}{q^{k-1}}\right)\right| < \frac{\varepsilon}{12} \tag{2.78}$$

and

$$\left| f\left(\frac{1-q^k}{q^{k-1}(1-q^n)}\right)\right| < \frac{\varepsilon}{12} \tag{2.79}$$

for $k > k_\varepsilon$.

On the other hand

$$\frac{1-q^k}{q^{k-1}(1-q^n)} - \frac{1-q^k}{q^{k-1}} = \frac{(1-q^k)q^n}{q^{k-1}(1-q^n)};$$

therefore we obtain for $k = 0,1,\dots,k_\varepsilon$ that

$$\left| f\left(\frac{1-q^k}{q^{k-1}(1-q^n)}\right) - f\left(\frac{1-q^k}{q^{k-1}}\right)\right|$$

$$\leq \omega\left(f;\frac{(1-q^k)q^n}{q^{k-1}(1-q^n)}\right) \leq \left(1+\frac{1-q^k}{q^{k-1}}\right)\omega\left(f;\frac{q^n}{1-q^n}\right)$$
$$= (1-q+q^{-k+1})\,\omega\left(f;\frac{q^n}{1-q^n}\right)$$
$$\leq (1-q+q^{-k_\varepsilon+1})\,\omega\left(f;\frac{q^n}{1-q^n}\right), \tag{2.80}$$

where $\omega(f;\delta) = \sup\{|f(x)-f(y)| : x,y \in [0,\infty), |x-y| \leq \delta\}$ is the modulus of continuity of $f \in \tilde{C}_B[0,\infty)$. Then, for every $\varepsilon > 0$ there exists n_ε''' such that

$$(1-q+q^{-k_\varepsilon+1})\,\omega\left(f;\frac{q^n}{1-q^n}\right) < \frac{\varepsilon}{6}$$

for $n > n_\varepsilon'''$. Hence, by (2.78)–(2.80) and (2.54), we get

$$I_3 < \sum_{k=0}^{k_\varepsilon} q^{k(k-1)/2}(1-q)^{-1}(1-q^2)^{-1}\dots(1-q^k)^{-1}$$
$$x^k \prod_{s=0}^{\infty}(1+xq^s)^{-1}(1-q+q^{-k_\varepsilon+1})\,\omega\left(f;\frac{q^n}{1-q^n}\right)$$
$$+ \sum_{k=k_\varepsilon+1}^{\infty} q^{k(k-1)/2}(1-q)^{-1}(1-q^2)^{-1}\dots(1-q^k)^{-1}$$
$$x^k \prod_{s=0}^{\infty}(1+xq^s)^{-1}\left\{\left|f\left(\frac{1-q^k}{q^{k-1}(1-q^n)}\right)\right| + \left|f\left(\frac{1-q^k}{q^{k-1}}\right)\right|\right\}$$
$$< \frac{\varepsilon}{6}V_{\infty,q}(1;x) + \frac{\varepsilon}{6}V_{\infty,q}(1;x) = \frac{\varepsilon}{3}. \tag{2.81}$$

Now combining (2.74)–(2.77) and (2.81), we find that

$$|L(f;x) - \mathcal{B}_{\infty,q}(f;x)| \leq C\,\omega_\varphi^2(f;\sqrt{q^{n-1}/(1-q^n)}) + \varepsilon$$

for arbitrary $\varepsilon > 0$ and $n > \max\{n_\varepsilon', n_\varepsilon'', n_\varepsilon'''\}$. Thus $L(f;x) = \mathcal{B}_{\infty,q}(f;x)$, which was to be proved.

2.5 *q*-Bleimann–Butzer–Hahn Operators

There are several studies related to the approximation properties of the Bleimann, Butzer, and Hahn operators (or, briefly, BBH). There are many approximating operators that their Korovkin-type approximation properties and rates of convergence are investigated. The results involving Korovkin-type approximation properties can be found in [13] with details. In [68], A.D. Gadjiev and Ö. Çakar gave a Korovkin-type theorem using the test function $\left(\frac{t}{1+t}\right)^{\nu}$ for $\nu = 0, 1, 2$. Some generalization of the operators (2.82) were given in [6, 7, 52].

2.5.1 Introduction

In [39], Bleimann, Butzer, and Hahn introduced the following operators:

$$B_n(f)(x) = \frac{1}{(1+x)^n} \sum_{k=0}^{n} f\left(\frac{k}{n-k+1}\right) \binom{n}{k} x^k, \quad x > 0, n \in \mathbb{N}. \tag{2.82}$$

Here we derive a q-integers-type modification of BBH operators that we call q-BBH operators and investigate their Korovkin-type approximation properties by using the test function $\left(\frac{t}{1+t}\right)^{\nu}$ for $\nu = 0, 1, 2$. Also, we define a space of generalized Lipschitz-type maximal function and give a pointwise estimation. Then a Stancu-type formula of the remainder of q-BBH is given. We shall also give a generalization of these operators and study on the approximation properties of this generalization. We emphasize that while Bernstein and Meyer–König and Zeller operators based on q-integers depend on a function defined on a bounded interval, these operators defined on unbounded intervals. Also, these operators are more flexible than classical BBH operators. That is, depending on the selection of q, the rate of convergence of the q-BBH operators is better than the classical one.

We refer to readers for additional information on q-Bleimann, Butzer, and Hahn operators to [10, 60, 120]. This section is based on [27].

2.5.2 Construction of the Operators

Also, let us recall the following Euler identity (see [134, p. 293])

$$\prod_{k=0}^{n-1}(1+q^k x) = \sum_{k=0}^{n} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k. \tag{2.83}$$

It is clear that when $q = 1$, these q-binomial coefficients reduce to ordinary binomial coefficients.

According to these explanations, similarly in [53], we defined a new Bleimann, Butzer, and Hahn-type operators based on q-integers as follows:

$$L_n(f;x) = \frac{1}{\ell_n(x)} \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n-k+1]_q q^k}\right) q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k, \tag{2.84}$$

where

$$\ell_n(x) = \prod_{s=0}^{n-1}(1+q^s x)$$

and f defined on semiaxis $[0,\infty)$.

Note that taking $f\left(\frac{[k]_q}{[n-k+1]_q}\right)$ instead of $f\left(\frac{[k]_q}{[n-k+1]_q q^k}\right)$ in (2.84), then we obtain usual generalization of Bleimann, Butzer, and Hahn operators based on q-integers. But in this case it is impossible to obtain explicit expressions for the monomials t^ν and $(t/(1+t))^\nu$ for $\nu = 1,2$. If we define the Bleimann, Butzer, and Hahn-type operators as in (2.84), then we can obtain explicit formulas for the monomials $(t/(1+t))^\nu$ for $\nu = 0, 1, 2$.

By a simple calculation, we have

$$q^k[n-k+1]_q = [n+1]_q - [k]_q, q[k-1]_q = [k]_q - 1. \tag{2.85}$$

From (2.83) to (2.85), we have

$$L_n(1;x) = 1 \tag{2.86}$$

and

$$\begin{aligned} L_n(\frac{t}{1+t};x) &= \frac{1}{\ell_n(x)}\sum_{k=1}^{n}\frac{[k]_q}{[n+1]_q}q^{\frac{k(k-1)}{2}}\begin{bmatrix} n \\ k \end{bmatrix}_q x^k \\ &= \frac{1}{\ell_n(x)}\sum_{k=1}^{n}\frac{[n]_q}{[n+1]_q}q^{\frac{k(k-1)}{2}}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q x^k \\ &= \frac{[n]_q}{[n+1]_q}x\frac{1}{\ell_n(x)}\sum_{k=0}^{n-1}q^{\frac{k(k-1)}{2}}\begin{bmatrix} n-1 \\ k \end{bmatrix}_q (qx)^k \\ &= \frac{x}{x+1}\frac{[n]_q}{[n+1]_q}. \end{aligned} \tag{2.87}$$

We can also write

$$\begin{aligned} L_n(\frac{t^2}{(1+t)^2};x) &= \frac{1}{\ell_n(x)}\sum_{k=1}^{n}\frac{[k]_q^2}{[n+1]_q^2}q^{\frac{k(k-1)}{2}}\begin{bmatrix} n \\ k \end{bmatrix}_q x^k \\ &= \frac{1}{\ell_n(x)}\sum_{k=2}^{n}\frac{q[k]_q[k-1]_q}{[n+1]_q^2}q^{\frac{k(k-1)}{2}}\begin{bmatrix} n \\ k \end{bmatrix}_q x^k \\ &\quad + \frac{1}{\ell_n(x)}\sum_{k=1}^{n}\frac{[k]_q}{[n+1]_q^2}q^{\frac{k(k-1)}{2}}\begin{bmatrix} n \\ k \end{bmatrix}_q x^k \end{aligned}$$

$$= \frac{1}{\ell_n(x)} \sum_{k=0}^{n-2} \frac{[n]_q[n-1]_q}{[n+1]_q^2} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n-2 \\ k \end{bmatrix}_q (q^2x)^k q^2x^2$$

$$+ \frac{1}{\ell_n(x)} \sum_{k=0}^{n-1} \frac{[n]_q}{[n+1]_q^2} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q (qx)^k x$$

$$= \frac{[n]_q[n-1]_q}{[n+1]_q^2} q^2 \frac{x^2}{(1+x)(1+qx)} + \frac{[n]_q}{[n+1]_q^2} \frac{x}{x+1}. \tag{2.88}$$

Remark 2.5. Note that, if we choose $q = 1$ then L_n operators turn out into the classical Bleimann, Butzer, and Hahn operators given by (2.82). Also using the similar methods as in [53, 54, 133], to ensure the convergence properties of L_n, we will assume $q = q_n$ as a sequence such that $q_n \to 1$ as $n \to \infty$ for $0 < q_n < 1$.

2.5.3 Properties of the Operators

In this section we will give the theorems on uniform convergence and rate of convergence of the operators (2.82). As in [68], for this purpose we give a space of function ω of the type of modulus of continuity which satisfies the following condition:

(a) ω is a nonnegative increasing function on $[0, \infty)$.
(b) $\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$.
(c) $\lim_{\delta \to 0} \omega(\delta) = 0$.

And H_ω is the subspace of real-valued function and satisfies the following condition:
For any $x, y \in [0, \infty)$

$$|f(x) - f(y)| \leq \omega\left(\left|\frac{x}{1+x} - \frac{y}{1+y}\right|\right). \tag{2.89}$$

Also $H_\omega \subset C_B[0, \infty)$, where $C_B[0, \infty)$ is the space of functions f which is continuous and bounded on $[0, \infty)$ endowed with norm $\|f\|_{C_B} = \sup_{x \geq 0} |f(x)|$.

It is easy to show that from the condition (b), the function ω satisfies the inequality

$$\omega(n\delta) \leq n\omega(\delta) \quad n \in \mathbb{N},$$

and from condition (a) for $\lambda > 0$, we have

$$\begin{aligned} \omega(\lambda\delta) &\leq \omega((1 + [|\lambda|])\delta) \\ &\leq (1+\lambda)\omega(\delta) \end{aligned} \tag{2.90}$$

where $[|\lambda|]$ is the greatest integer of λ.

Remark 2.6. The operator L_n maps H_ω into $C_B[0,\infty)$ and it is continuous with respect to sup-norm.

The properties of linear positive operators acting from H_ω to $C_B[0,\infty)$ and the Korovkin-type theorems for them have been studied by Gadjiev and Çakar who have established the following theorem (see [68]).

Theorem 2.17. *If A_n is the sequence of positive linear operators from H_ω to $C_B[0,\infty)$ satisfy the following conditions for $U = 0, 1, 2$.*

$$\left\| \left(A_n \left(\frac{t}{1+t} \right)^{\upsilon} \right)(x) - \left(\frac{x}{1+x} \right)^{\upsilon} \right\|_{C_B} \to 0 \quad \textit{for } n \to \infty$$

then, for any function f in H_ω, one has

$$\|A_n f - f\|_{C_B} \to 0 \quad \textit{for } n \to \infty.$$

Theorem 2.18. *Let $q = q_n$ satisfies $0 < q_n < 1$ and let $q_n \to 1$ as $n \to \infty$. If L_n is defined by (2.84), then for any $f \in H_\omega$,*

$$\lim_{n\to\infty} \|L_n f - f\|_{C_B} = 0.$$

Proof. Using Theorem 2.17 we see that it is sufficient to verify the following three conditions:

$$\lim_{n\to\infty} \left\| L_n \left(\left(\frac{t}{1+t} \right)^{\upsilon} ; x \right) - \left(\frac{x}{1+x} \right)^{\upsilon} \right\|_{C_B} = 0, \ \ \upsilon = 0,1,2. \tag{2.91}$$

From (2.86), the first condition of (2.91) is fulfilled for $\upsilon = 0$. Now it is easy to see that from (2.87)

$$\begin{aligned} \left\| L_n \left(\left(\frac{t}{1+t} \right) ; x \right) - \frac{x}{1+x} \right\|_{C_B} &\leq \left| \frac{[n]_{q_n}}{[n+1]_{q_n}} - 1 \right| \\ &\leq \left| \frac{1}{q_n} - \frac{1}{q_n [n+1]_{q_n}} - 1 \right| \end{aligned}$$

and since $[n+1]_{q_n} \to \infty$, $q_n \to 1$ as $n \to \infty$, the condition (2.91) holds for $\upsilon = 1$. To verify this condition for $\upsilon = 2$, consider (2.88). We see that

$$\begin{aligned} \left\| L_n \left(\left(\frac{t}{1+t} \right)^2 ; x \right) - \left(\frac{x}{1+x} \right)^2 \right\|_{C_B} &= \sup_{x\geq 0} \left(\frac{x^2}{(1+x)^2} \left(\frac{[n]_{q_n}[n-1]_{q_n}}{[n+1]_{q_n}^2} q_n^2 \frac{1+x}{1+q_n x} - 1 \right) \right. \\ &\quad \left. + \frac{[n]_{q_n}}{[n+1]_{q_n}^2} \frac{x}{1+x} \right). \end{aligned}$$

A small calculation shows that

$$\frac{[n]_{q_n}[n-1]_{q_n}}{[n+1]_{q_n}^2}=\frac{1}{q_n^3}\left(1-\frac{2+q_n}{[n+1]_{q_n}}+\frac{1+q_n}{[n+1]_{q_n}^2}\right).$$

Thus, we have

$$\left\|L_n\left(\left(\frac{t}{1+t}\right)^2;x\right)-\left(\frac{x}{1+x}\right)^2\right\|_{C_B}\leq\frac{1}{q_n^2}\left(1-q_n^2-\frac{2}{[n+1]_{q_n}}+\frac{1}{[n+1]_{q_n}^2}\right).$$

This means that the condition (2.91) holds also for $\upsilon=2$ and the proof is completed by Theorem 2.17. ■

Theorem 2.19. *Let $q=q_n$ satisfies $0<q_n<1$ with $q_n\to 1$ as $n\to\infty$. If L_n is defined by (2.84), then for each $x\geq 0$ and for any $f\in H_\omega$, the following inequality holds*

$$|L_n(f;x)-f(x)|\leq 2\omega\left(\sqrt{\mu_n(x)}\right)$$

where

$$\mu_n(x)=\left(\tfrac{x}{1+x}\right)^2\left(1-2\tfrac{[n]_{q_n}}{[n+1]_{q_n}}+\tfrac{[n]_{q_n}[n-1]_{q_n}}{[n+1]_{q_n}^2}q_n^2\tfrac{(1+x)}{(1+q_nx)}\right)+\tfrac{[n]_{q_n}}{[n+1]_{q_n}^2}\tfrac{x}{1+x}. \qquad (2.92)$$

Proof. Since $L_n(1;x)=1$, we can write

$$|L_n(f;x)-f(x)|\leq L_n(|f(t)-f(x)|;x). \qquad (2.93)$$

On the other hand from (2.89) and (2.90)

$$\begin{aligned}|f(t)-f(x)|&\leq\omega\left(\left|\frac{t}{1+t}-\frac{x}{1+x}\right|\right)\\&\leq\left(1+\frac{\left|\frac{t}{1+t}-\frac{x}{1+x}\right|}{\delta}\right)\omega(\delta),\end{aligned}$$

where we choose $\lambda=\delta^{-1}\left|\frac{t}{1+t}-\frac{x}{1+x}\right|$. This inequality and (2.93) imply

$$|L_n(f;x)-f(x)|\leq\omega(\delta)\left(1+\frac{1}{\delta}L_n\left(\left|\frac{t}{1+t}-\frac{x}{1+x}\right|;x\right)\right).$$

According to the Cauchy–Schwarz inequality we have

$$|L_n(f;x)-f(x)|\leq\omega(\delta)\left(1+\frac{1}{\delta}L_n\left(\left|\frac{t}{1+t}-\frac{x}{1+x}\right|^2;x\right)^{\frac{1}{2}}\right).$$

By choosing $\delta = \mu_n(x) = L_n\left(\left|\frac{t}{1+t} - \frac{x}{1+x}\right|^2; x\right)$, we obtain the desired result. ■

Now we will give an estimate concerning the rate of convergence as given in [8, 52, 109]. We define the space of general Lipschitz-type maximal functions on $E \subset [0, \infty)$ by $W^{\sim}_{\alpha,E}$ as

$$W^{\sim}_{\alpha,E} = \left\{ f : \sup (1+x)^{\alpha} f_{\alpha}(x, y) \leq M \frac{1}{(1+y)^{a}}, x \geq 0 \text{ and } y \in E \right\},$$

where f is bounded and continuous on $[0, \infty)$, M is a positive constant, $0 < \alpha \leq 1$, and f_{α} is the following function:

$$f_{\alpha}(x, t) = \frac{|f(t) - f(x)|}{|x-t|^{\alpha}}.$$

Also, let $d(x, E)$ be the distance between x and E, that is

$$d(x, E) = \inf\{|x-y|; y \in E\}.$$

Theorem 2.20. *For all $f \in W^{\sim}_{\alpha,E}$ we have*

$$|L_n(f;x) - f(x)| \leq M\left(\mu_n^{\frac{\alpha}{2}}(x) + 2(d(x, E))^{\alpha}\right), \tag{2.94}$$

where $\mu_n(x)$ defined in (2.92).

Proof. Let $\overline{E}$ denote the closure of the set E. Then there exists a $x_0 \in \overline{E}$ such that $|x - x_0| = d(x, E)$, where $x \in [0, \infty)$. Thus we can write

$$|f - f(x)| \leq |f - f(x_0)| + |f(x_0) - f(x)|.$$

Since L_n is a positive and linear operator and $f \in W^{\sim}_{\alpha,E}$, by using the above inequality we have

$$\begin{aligned} |L_n(f;x) - f(x)| &\leq L_n(|f - f(x_0)|;x) + |f(x_0) - f(x)| \\ &\leq ML_n(\left|\frac{t}{1+t} - \frac{x_0}{1+x_0}\right|^{\alpha};x) + M\frac{|x-x_0|^{\alpha}}{(1+x)^{\alpha}(1+x_0)^{\alpha}}. \end{aligned} \tag{2.95}$$

If we use the classical inequality $(a+b)^{\alpha} \leq a^{\alpha} + b^{\alpha}$ for $a \geq 0, b \geq 0$, one can write

$$\left|\frac{t}{1+t} - \frac{x_0}{1+x_0}\right|^{\alpha} \leq \left|\frac{t}{1+t} - \frac{x}{1+x}\right|^{\alpha} + \left|\frac{x}{1+x} - \frac{x_0}{1+x_0}\right|^{\alpha}$$

for $0 < \alpha \leq 1$ and $t \in [0, \infty)$. Consequently we obtain

$$L_n(\left|\tfrac{t}{1+t}-\tfrac{x_0}{1+x_0}\right|^{\alpha};x)\le L_n(\left|\tfrac{t}{1+t}-\tfrac{x}{1+x}\right|^{\alpha};x)+\tfrac{|x-x_0|^{\alpha}}{(1+x)^{\alpha}(1+x_0)^{\alpha}}.$$

Since $L_n(1;x)=1$, applying Hölder inequality with $p=\frac{2}{\alpha}$ and $q=\frac{2}{2-\alpha}$, we have

$$L_n(\left|\tfrac{t}{1+t}-\tfrac{x_0}{1+x_0}\right|^{\alpha};x)\le L_n((\tfrac{t}{1+t}-\tfrac{x}{1+x})^2;x)^{\frac{\alpha}{2}}+\tfrac{|x-x_0|^{\alpha}}{(1+x)^{\alpha}(1+x_0)^{\alpha}}.$$

Thus in view of (2.95), we have (2.94). ■

As a particular case of Theorem 2.20, when $E=[0,\infty)$, the following is true:

Corollary 2.8. *If $f\in W^{\sim}_{\alpha,[0,\infty)}$ then we have*

$$|L_n(f;x)-f(x)|\le M\mu_n^{\frac{\alpha}{2}}(x),$$

where $\mu_n(x)$ defined in (2.92).

In the following theorem a Stancu-type formula for the remainder of q-BBH operators is obtained which reduces to the formula of the remainder of classical BBH operators (see [2, p. 151]). Similar formula is obtained for q-Szász–Mirakyan operators in [29].

Here, $[x_0,x_1\dots,x_n;f]=f[x_0,x_1\dots,x_n]$ denotes the divided difference of the function f given in Lemma 2.3.

Theorem 2.21. *If $x\in(0,\infty)\setminus\left\{\frac{[k]_q}{[n-k+1]_q q^k}\,\middle|\,k=0,1,2,\dots,n\right\}$, then the following identity holds:*

$$\begin{aligned}&L_n(f;x)-f\left(\frac{x}{q}\right)\\&=-\frac{x^{n+1}}{\ell_n(x)}\left[\frac{x}{q},\frac{[n]_q}{q^n};f\right]\\&+\frac{x}{\ell_n(x)}\sum_{k=0}^{n-1}\left[\frac{x}{q},\frac{[k]_q}{[n-k+1]_q q^k},\frac{[k+1]_q}{[n-k]_q q^{k+1}};f\right]\frac{q^{\frac{k(k+1)}{2}-2}}{[n-k]_q}\begin{bmatrix}n+1\\k\end{bmatrix}_q x^k.\end{aligned}\tag{2.96}$$

Proof. By using (2.84), we have

$$\begin{aligned}L_n(f;x)-f\left(\frac{x}{q}\right)&=\frac{1}{\ell_n(x)}\sum_{k=0}^{n}\left[f\left(\frac{[k]_q}{[n-k+1]_q q^k}\right)-f\left(\frac{x}{q}\right)\right]q^{\frac{k(k-1)}{2}}\begin{bmatrix}n\\k\end{bmatrix}_q x^k\\&=-\frac{1}{\ell_n(x)}\sum_{k=0}^{n}\left(\frac{x}{q}-\frac{[k]_q}{[n-k+1]_q q^k}\right)\\&\quad\times\left[\frac{x}{q},\frac{[k]_q}{[n-k+1]_q q^k};f\right]q^{\frac{k(k-1)}{2}}\begin{bmatrix}n\\k\end{bmatrix}_q x^k\end{aligned}$$

Since

$$\frac{[k]_q}{[n-k+1]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k-1 \end{bmatrix}_q,$$

we have

$$L_n(f;x) - f\left(\frac{x}{q}\right) = -\frac{1}{\ell_n(x)}\sum_{k=0}^{n}\left[\frac{x}{q}, \frac{[k]_q}{[n-k+1]_q q^k}; f\right] q^{\frac{k(k-1)}{2}-1}\begin{bmatrix} n \\ k \end{bmatrix}_q x^{k+1}$$
$$+\frac{1}{\ell_n(x)}\sum_{k=1}^{n}\left[\frac{x}{q}, \frac{[k]_q}{[n-k+1]_q q^k}; f\right] q^{\frac{k(k-1)}{2}-k}\begin{bmatrix} n \\ k-1 \end{bmatrix}_q x^k.$$

Rearranging the above equality, we can write

$$L_n(f;x) - f\left(\frac{x}{q}\right) = -\frac{x^{n+1}}{\ell_n(x)}\left[\frac{x}{q}, \frac{[n]_q}{q^n}; f\right] q^{\frac{n(n-1)}{2}-1}$$
$$+\frac{1}{\ell_n(x)}\sum_{k=0}^{n-1}\left(\left[\frac{x}{q}, \frac{[k+1]_q}{[n-k]_q q^{k+1}}; f\right]\right.$$
$$\left.-\left[\frac{x}{q}, \frac{[k]_q}{[n-k+1]_q q^k}; f\right]\right) q^{\frac{k(k-1)}{2}-1}\begin{bmatrix} n \\ k \end{bmatrix}_q x^{k+1}.$$

Using the equality

$$\frac{[k+1]_q}{[n-k]_q q^{k+1}} - \frac{[k]_q}{[n-k+1]_q q^k} = \frac{[n+1]_q}{[n-k]_q[n-k+1]_q q^{k+1}},$$

we have following formula for divided differences:

$$\left[\frac{x}{q}, \frac{[k]_q}{[n-k+1]_q q^k}, \frac{[k+1]_q}{[n-k]_q q^{k+1}}; f\right]\frac{[n+1]_q}{[n-k]_q[n-k+1]_q q^{k+1}}$$
$$= \left[\frac{x}{q}, \frac{[k+1]_q}{[n-k]_q q^{k+1}}; f\right] - \left[\frac{x}{q}, \frac{[k]_q}{[n-k+1]_q q^k}; f\right]$$

and therefore, we obtain that the remainder formula for q-BBH operators, which is expressible in the form (2.96). ∎

We know that a function is convex on an interval if and only if all second-order divided differences of f are nonnegative. From this property and Theorem 2.21 we have the following result:

Corollary 2.9. *If f is convex and nonincreasing, then*

$$f\left(\frac{x}{q}\right) \le L_n(f;x) \quad (n = 0, 1, \ldots)$$

2.5.4 Some Generalization of L_n

In this section, similarly as in [52], we shall define some generalization of the operators L_n.

We consider a sequence of linear positive operators as follows:

$$L_n^{\gamma}(f;x)=\frac{1}{\ell_n(x)}\sum_{k=0}^{n} f\left(\frac{[k]_q+\gamma}{b_{n,k}}\right) q^{\frac{k(k-1)}{2}}\begin{bmatrix} n \\ k \end{bmatrix}_q x^k, (\gamma\in\mathbb{R}) \tag{2.97}$$

where $b_{n,k}$ satisfies the following condition:

$$[k]_q+b_{n,k}=c_n \text{ and } \frac{[n]_q}{c_n}\to 1 \quad \text{for } n\to\infty.$$

It is easy to check that if $b_{n,k}=[n-k+1]q^k+\beta$ for any $n,\ k$ and $0<q<1$, then $c_n=[n+1]_q+\beta$ and these operators turn out into D.D. Stancu-type generalization of Bleimann, Butzer, and Hahn operators based on q-integers (see [145]). If we choose $\gamma=0$ and $q=1$, then the operators become the special case of the Balázs-type generalization of the operators (2.82) given in [52].

Theorem 2.22. *Let $q=q_n$ satisfies $0<q_n\leq 1$ and let $q_n\to 1$ as $n\to\infty$. If $f\in W^{\sim}_{\alpha,[0,\infty)}$, then the following inequality holds for a large n*

$$\begin{aligned}&\|L_n^{\gamma}(f;x)-f(x)\|_{C_B}\\ &\leq 3M\max\left\{\left(\frac{[n]_{q_n}}{c_n+\gamma}\right)^{\alpha}\left(\frac{\gamma}{[n]_{q_n}}\right)^{\alpha},\ \left|1-\frac{[n+1]_{q_n}}{c_n+\gamma}\right|^{\alpha}\right.\\ &\quad\left.\times\left(\frac{[n]_{q_n}}{[n+1]_{q_n}}\right)^{\alpha}, 1-2\frac{[n]_{q_n}}{[n+1]_{q_n}}+\frac{[n]_{q_n}[n-1]_{q_n}}{[n+1]^2_{q_n}}q_n\right\}.\end{aligned} \tag{2.98}$$

Proof. Using (2.84) and (2.97) we have

$$\begin{aligned}|L_n^{\gamma}(f;x)-f(x)| &\leq \frac{1}{\ell_n(x)}\sum_{k=0}^{n}\left|f\left(\frac{[k]_{q_n}+\gamma}{b_{n,k}}\right)-f\left(\frac{[k]_{q_n}}{\gamma+b_{n,k}}\right)\right| q_n^{\frac{k(k-1)}{2}}\begin{bmatrix} n \\ k \end{bmatrix}_{q_n} x^k\\ &\quad+\frac{1}{\ell_n(x)}\sum_{k=0}^{n}\left|f\left(\frac{[k]_{q_n}}{\gamma+b_{n,k}}\right)-f\left(\frac{[k]_{q_n}}{[n-k+1]q_n^k}\right)\right| q_n^{\frac{k(k-1)}{2}}\begin{bmatrix} n \\ k \end{bmatrix}_{q_n} x^k\\ &\quad+|L_n(f;x)-f(x)|.\end{aligned}$$

Since $f \in W^{\sim}_{\alpha,[0,\infty)}$ and by using Corollary 2.8, we can write

$$\begin{aligned}
\left|L_n^{\gamma}(f;x) - f(x)\right| \leq\ & \frac{M}{\ell_n(x)} \sum_{k=0}^{n} \left| \frac{[k]_{q_n} + \gamma}{[k]_{q_n} + \gamma + b_{n,k}} - \frac{[k]_{q_n}}{\gamma + [k]_{q_n} + b_{n,k}} \right|^{\alpha} q_n^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{q_n} x^k \\
& + \frac{M}{\ell_n(x)} \sum_{k=0}^{n} \left| \frac{[k]_{q_n}}{[k]_{q_n} + \gamma + b_{n,k}} - \frac{[k]_{q_n}}{[n+1]_{q_n}} \right|^{\alpha} q_n^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{q_n} x^k + \mu_n^{\frac{\alpha}{2}}(x) \\
\leq\ & \left(\frac{[n]}{c_n + \gamma} \right)^{\alpha} \left(\frac{\gamma}{[n]_{q_n}} \right)^{\alpha} + \left| 1 - \frac{[n+1]_{q_n}}{c_n + \gamma} \right|^{\alpha} \\
& \times \frac{1}{\ell_n(x)} \sum_{k=0}^{n} \left(\frac{[k]_{q_n}}{[n+1]_{q_n}} \right)^{\alpha} q_n^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{q_n} x^k + \mu_n^{\frac{\alpha}{2}}(x).
\end{aligned}$$

Using the Hölder inequality for $p = \frac{1}{\alpha}$, $q = \frac{1}{1-\alpha}$ and (2.87), we obtain

$$\begin{aligned}
\left|L_n^{\gamma}(f;x) - f(x)\right| \leq\ & M \left(\frac{[n]_{q_n}}{c_n + \gamma} \right)^{\alpha} \left(\frac{\gamma}{[n]_{q_n}} \right)^{\alpha} + M \left| 1 - \frac{[n+1]_{q_n}}{c_n + \gamma} \right|^{\alpha} \left(\frac{x}{x+1} \frac{[n]_{q_n}}{[n+1]_{q_n}} \right)^{\alpha} \\
& + \mu_n^{\frac{\alpha}{2}}(x).
\end{aligned}$$

Thus the inequality (2.98) holds for $x \in [0,\infty)$. ■