Chapter 1
Introduction of q-Calculus

In the field of approximation theory, the applications of q-calculus are new area in last 25 years. The first q-analogue of the well-known Bernstein polynomials was introduced by Lupas in the year 1987. In 1997 Phillips considered another q-analogue of the classical Bernstein polynomials. Later several other researchers have proposed the q-extension of the well-known exponential-type operators which includes Baskakov operators, Szász–Mirakyan operators, Meyer–König–Zeller operators, Bleiman, Butzer and Hahn operators (abbreviated as BBH), Picard operators, and Weierstrass operators. Also, the q-analogue of some standard integral operators of Kantorovich and Durrmeyer type was introduced, and their approximation properties were discussed. This chapter is introductory in nature; here we mention some important definitions and notations of q-calculus. We give outlines of q-integers, q-factorials, q-binomial coefficients, q-differentiations, q-integrals, q-beta and q-gamma functions. We also mention some important q-basis functions and their generating functions.

1.1 Notations and Definitions in q-Calculus

In this section we mention some basic definitions of q-calculus, which would be used throughout the book.

Definition 1.1. Given value of $q > 0$, we define the q-integer $[n]_q$ by

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & q \neq 1 \\ n, & q = 1 \end{cases},$$

for $n \in \mathbb{N}$.

We can give this definition for any real number λ. In this case we call $[\lambda]_q$ a q-real.

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Definition 1.2. Given the value of $q > 0$, we define the q-factorial $[n]_q!$ by

$$[n]_q! = \begin{cases} [n]_q [n-1]_q \cdots [1]_q\,, & n = 1, 2, \ldots \\ 1 & n = 0. \end{cases},$$

for $n \in \mathbb{N}$.

Definition 1.3. We define the q-binomial coefficients by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!}, \quad 0 \le k \le n, \tag{1.1}$$

for $n, k \in \mathbb{N}$.

The q-binomial coefficient satisfies the recurrence equations

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \tag{1.2}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k \end{bmatrix}_q. \tag{1.3}$$

Definition 1.4. The q-analogue of $(1+x)_q^n$ is the polynomial

$$(1+x)_q^n := \begin{cases} (1+x)(1+qx)\ldots\left(1+q^{n-1}x\right) & n = 1, 2, \ldots \\ 1 & n = 0. \end{cases}$$

A q-analogue of the common Pochhammer symbol also called a q-shifted factorial is defined as

$$(x;q)_0 = 1, (x;q)_n = \prod_{i=0}^{n-1}(1-q^i x), (x;q)_\infty = \prod_{i=0}^{\infty}(1-q^i x).$$

Definition 1.5. The Gauss binomial formula:

$$(x+a)_q^n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{j(j-1)/2} a^j x^{n-j}.$$

Definition 1.6. The Heine's binomial formula:

$$\frac{1}{(1-x)_q^n} = 1 + \sum_{j=1}^{\infty} \frac{[n]_q[n+1]_q \ldots [n+j-1]_q}{[j]_q!} x^j.$$

Also, we have the following important property:

$$x^n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q (x-1)_q^j.$$

1.2 *q*-Derivative

Definition 1.7. The q-derivative $D_q f$ of a function f is given by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad \text{if } x \neq 0, \tag{1.4}$$

and $(D_q f)(0) = f'(0)$ provided $f'(0)$ exists.

Note that

$$\lim_{q \to 1} D_q f(x) = \lim_{q \to 1} \frac{f(qx) - f(x)}{(q-1)x} = \frac{df(x)}{dx}$$

if f is differentiable.

It is obvious that the q-derivative of a function is a linear operator. That is, for any constants a and b, we have

$$D_q \{af(x) + bg(x)\} = aD_q \{f(x)\} + bD_q \{g(x)\}.$$

Now we calculate the q-derivative of a product at $x \neq 0$, using Definition 1.7, as

$$\begin{aligned} D_q \{f(x) g(x)\} &= \frac{f(qx) g(qx) - f(x) g(x)}{(q-1)x} \\ &= \frac{f(qx) g(qx) - f(qx) g(x) + f(qx) g(x) - f(x) g(x)}{(q-1)x} \\ &= \frac{f(qx)(g(qx) - g(x))}{(q-1)x} + \frac{(f(qx) - f(x)) g(x)}{(q-1)x} \\ &= f(qx) D_q g(x) + D_q f(x) g(x). \end{aligned}$$

We interchange f and g and obtain

$$D_q \{f(x) g(x)\} = f(x) D_q g(x) + D_q f(x) g(qx). \tag{1.5}$$

The Leibniz rule for the q-derivative operator is defined as

$$D_q^{(n)}(fg)(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q D_q^{(k)} f(xq^{n-k}) D_q^{(n-k)} g(x).$$

If we apply Definition 1.7 to the quotient $f(x)$ and $g(x)$, we obtain

$$\begin{aligned} D_q\left\{\frac{f(x)}{g(x)}\right\} &= \frac{1}{(q-1)x}\left\{\frac{f(qx)}{g(qx)} - \frac{f(x)}{g(qx)} + \frac{f(x)}{g(qx)} - \frac{f(x)}{g(x)}\right\} \\ &= \frac{1}{g(qx)}\left\{\frac{f(qx)-f(x)}{(q-1)x}\right\} + \frac{1}{(q-1)x}\left\{\frac{f(x)g(x)-f(x)g(qx)}{g(qx)g(x)}\right\} \\ &= \frac{1}{g(qx)}D_q f(x) + \frac{f(x)}{g(qx)g(x)}\left\{\frac{g(x)-g(qx)}{(q-1)x}\right\} \\ &= \frac{g(x)D_q f(x) - f(x)D_q g(x)}{g(qx)g(x)}. \end{aligned} \tag{1.6}$$

The above formula can also be written as

$$D_q\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(qx)D_q f(x) - f(qx)D_q g(x)}{g(qx)g(x)}.$$

Note that there does not exist a general chain rule for q-derivative. We can give a chain rule for function of the form $f(u(x))$, where $u = u(x) = \alpha x^{\beta}$ with α, β being constant. For this chain rule, we can write

$$\begin{aligned} D_q\{f(u(x))\} &= D_q\left\{f\left(\alpha x^{\beta}\right)\right\} \\ &= \frac{f\left(\alpha q^{\beta}x^{\beta}\right) - f\left(\alpha x^{\beta}\right)}{(q-1)x} \\ &= \frac{f\left(\alpha q^{\beta}x^{\beta}\right) - f\left(\alpha x^{\beta}\right)}{\alpha q^{\beta}x^{\beta} - \alpha x^{\beta}} \cdot \frac{\alpha q^{\beta}x^{\beta} - \alpha x^{\beta}}{(q-1)x} \\ &= \frac{f\left(q^{\beta}u\right) - f(u)}{q^{\beta}u - u} \cdot \frac{u(qx) - u(x)}{(q-1)x} \end{aligned}$$

and, hence,

$$D_q\{f(u(x))\} = \left(D_{q^{\beta}} f\right)(u(x))\, D_q(u(x)).$$

Proposition 1.1. *For* $n \geq 1$,

$$D_q(1+x)_q^n = [n]_q (1+qx)_q^{n-1}$$

$$D_q\left\{\frac{1}{(1+x)_q^n}\right\} = -\frac{[n]_q}{(1+x)_q^{n+1}}.$$

Proof. According to the definition of q-derivative we have

$$D_q(1+x)_q^n = \frac{(1+qx)_q^n - (1+x)_q^n}{(q-1)x}$$
$$= (1+qx)_q^{n-1}\frac{\{(1+q^n x-(1+x)\}}{(q-1)x}$$
$$= [n]_q\,(1+qx)_q^{n-1}.$$

According to (1.6), we have

$$D_q\left\{\frac{1}{(1+x)_q^n}\right\} = -\frac{D_q(1+x)_q^n}{(1+qx)_q^n(1+x)_q^n}$$
$$= -\frac{[n]_q}{(1+q^n x)\,(1+x)_q^n}$$
$$= -\frac{[n]_q}{(1+x)_q^{n+1}}.$$

■

Remark 1.1. Suppose $n \geq 1$ and $a,b,r,s \in \Re$, then by simple computation, we immediately have the following:

$$D_q(a+bx)_q^n = [n]_q b(a+bqx)_q^{n-1},$$

$$D_q(ax+b)_q^n = [n]_q a(ax+b)_q^{n-1},$$

and

$$D_q\frac{(1+ax)_q^r}{(1+bx)_q^s} = [r]_q a\frac{(1+aqx)_q^{r-1}}{(1+bqx)_q^s} - b[s]_q\frac{(1+ax)_q^r}{(1+bx)_q^{s+1}}.$$

1.3 *q*-Series Expansions

Theorem 1.1. *For* $|x|<1$, $|q|<1$,

$$\sum_{k=0}^{\infty}\frac{(1-a)_q^k}{(1-q)_q^k}x^k = \frac{(1-ax)_q^\infty}{(1-x)_q^\infty},$$

where $(1-x)_q^\infty = \prod_{k=0}^{\infty}\left(1-q^k x\right)$.

Proof. Let

$$f_a(x) = \sum_{k=0}^{\infty} \frac{(1-a)_q^k}{(1-q)_q^k} x^k.$$

Clearly

$$\begin{aligned}
\frac{f_a(x) - f_a(qx)}{x} &= \sum_{k=0}^{\infty} \frac{(1-a)_q^k}{(1-q)_q^k} \left(1-q^k\right) x^{k-1} \\
&= (1-a) \sum_{k=1}^{\infty} \frac{(1-aq)_q^{k-1}}{(1-q)_q^{k-1}} x^{k-1} \\
&= (1-a) \sum_{k=0}^{\infty} \frac{(1-aq)_q^k}{(1-q)_q^k} x^k = (1-a) f_a(qx)
\end{aligned}$$

or

$$f_a(x) - f_a(qx) = (1-a) x f_a(qx).$$

Also

$$\begin{aligned}
f_a(x) - f_a(qx) &= \sum_{k=0}^{\infty} \frac{(1-qa)_q^{k-1}}{(1-q)_q^k} \left(1-a-1+aq^k\right) x^k \\
&= -ax f_{aq}(x)
\end{aligned}$$

or

$$f_a(x) = (1-ax) f_{aq}(x).$$

Combining the above two equations, we get

$$f_a(x) = \frac{1-ax}{1-x} f_a(qx).$$

Iterating this relation n times and letting $n \to \infty$ we have

$$f_a(x) = \frac{(1-ax)_q^n}{(1-x)_q^n} f_a(q^n x) = \frac{(1-ax)_q^\infty}{(1-x)_q^\infty}.$$

Thus we have the desired result. ■

Corollary 1.1. *(a) Taking $a = 0$ in Theorem 1.1, we have*

$$\sum_{k=0}^{\infty} \frac{x^k}{(1-q)_q^k} = \frac{1}{(1-x)_q^\infty}, \qquad |x| < 1, |q| < 1.$$

(b) Replacing a with $\frac{1}{a}$, and x with ax, and then taking $a = 0$ in Theorem 1.1, we have

$$\sum_{k=0}^{\infty} \frac{(-1)^n q^{\frac{k(k-1)}{2}} x^k}{(1-q)_q^k} = (1-x)_q^{\infty}, \qquad |q| < 1.$$

(c) Taking $a = q^N$ in Theorem 1.1, we have

$$\sum_{k=0}^{\infty} \begin{bmatrix} N-k-1 \\ k \end{bmatrix}_q x^k = \frac{1}{(1-x)_q^N}, \qquad |x| < 1.$$

We consider Corollary 1.1(a). We can write

$$\sum_{k=0}^{\infty} \frac{x^k}{(1-q)_q^k} = \sum_{k=0}^{\infty} \frac{\left(\frac{x}{1-q}\right)^k}{\left(\frac{1-q^2}{1-q}\right)\left(\frac{1-q^3}{1-q}\right)\cdots\left(\frac{1-q^k}{1-q}\right)}$$
$$= \sum_{k=0}^{\infty} \frac{\left(\frac{x}{1-q}\right)^k}{[k]_q!}$$

which resembles Taylor's expansion of classical exponential function e^x.

Definition 1.8. A q-analogue of classical exponential function e^x is

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!}.$$

Using Corollary 1.1, (a) and (b), we see that

$$e_q\left(\frac{x}{1-q}\right) = \frac{1}{(1-x)_q^{\infty}}$$

and

$$e_q(x) = \frac{1}{(1-(1-q)x)_q^{\infty}}. \tag{1.7}$$

Definition 1.9. Another q-analogue of classical exponential function is

$$E_q(x) = \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{x^k}{[k]_q!} = (1+(1-q)x)_q^{\infty}. \tag{1.8}$$

The q-exponential functions satisfy following properties:

Lemma 1.1. *(a)* $D_q e_q(x) = e_q(x), \qquad D_q E_q(x) = E_q(qx).$
(b) $e_q(x) E_q(-x) = E_q(x) e_q(-x) = 1.$

Note that for $q \in (0,1)$ the series expansion of $e_q(x)$ has radius of convergence $\frac{1}{1-q}$. On the contrary, the series expansion of $E_q(x)$ converges for every real x.

1.4 Generating Functions

In this section we present the generating functions for some of the important q-basis functions, namely, q-Bernstein basis function, q-MKZ basis function, and q-beta basis functions (see [95]).

We can consider the q-exponential function in the following form:

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{(1-x)_q^n} &= \lim_{n\to\infty}\sum_{k=0}^{\infty}\begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q x^k \\ &= \lim_{n\to\infty}\sum_{k=0}^{\infty}\frac{(1-q^{n+k-1})\ldots.(1-q^n)}{(1-q)(1-q^2)\ldots(1-q^k)}x^k \\ &= \sum_{k=0}^{\infty}\frac{x^k}{(1-q)(1-q^2)\ldots(1-q^k)} = e_q(x).\end{aligned}$$

Another form of q-exponential function is given as follows:

$$\lim_{n\to\infty}(1+x)_q^n = \sum_{k=0}^{\infty}\frac{q^{k(k-1)/2}x^k}{(1-q)(1-q^2)\ldots(1-q^k)} = E_q(x).$$

Based on the q-integers Phillips [132] introduced the q-analogue of the well-known Bernstein polynomials. For $f \in C[0,1]$ and $0 < q < 1$, the q-Bernstein polynomials are defined as

$$\mathcal{B}_{n,q}(f,x) = \sum_{k=0}^{n} b_{k,n}^q(x) f\left(\frac{[k]_q}{[n]_q}\right), \tag{1.9}$$

where the q-Bernstein basis function is given by

$$b_{k,n}^q(x) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k (1-x)_q^{n-k}, x \in [0,1]$$

and $(a-b)_q^n = \prod_{s=0}^{n-1}(a-q^s b), \quad a,b \in \mathbf{R}$.

Also Trif [150] proposed the q-analogue of well-known Meyer–König–Zeller operators. For $f \in C[0,1]$ and $0 < q < 1$, the q-Meyer–König–Zeller operators are defined as

$$\mathcal{M}_{n,q}(f,x) = \sum_{k=0}^{\infty} m_{k,n}^q(x) f\left(\frac{[k]_q}{[n]_q}\right), \tag{1.10}$$

where the q-MKZ basis function is given by

$$m_{k,n}^q(x) = \begin{bmatrix} n+k+1 \\ k \end{bmatrix}_q x^k(1-x)_q^n, x \in [0,1].$$

For $f \in C[0,\infty)$ and $0 < q < 1$, the q-beta operators are defined as

$$\mathcal{V}_n(f,x) = \sum_{k=0}^{\infty} v_{k,n}^q(x) f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right), \tag{1.11}$$

where the q-beta basis function is given by

$$v_{k,n}^q(x) = \frac{q^{k(k-1)/2}}{B_q(k+1,n)} \frac{x^k}{(1+x)_q^{n+k+1}}, x \in [0,\infty)$$

and $B_q(m,n)$ is q-beta function.

Now we give the generating functions for q-Bernstein, q-Meyer–König–Zeller, and q-beta basis functions.

1.4.1 Generating Function for q-Bernstein Basis

Theorem 1.2. *$b_{k,n}^q(x)$ is the coefficient of $\frac{t^n}{[n]_q!}$ in the expansion of*

$$\frac{x^k t^k}{[k]_q!} e_q((1-q)(1-x)_q t).$$

Proof. First consider

$$\begin{aligned}
\frac{x^k t^k}{[k]_q!} e_q((1-q)(1-x)_q t) &= \frac{x^k t^k}{[k]_q!} \sum_{n=0}^{\infty} \frac{(1-x)_q^n t^n}{[n]_q!} \\
&= \frac{1}{[k]_q!} \sum_{n=0}^{\infty} \frac{x^k (1-x)_q^n t^{n+k}}{[n]_q!} \\
&= \sum_{n=0}^{\infty} \frac{[n+1]_q [n+2]_q \ldots . . [n+k]_q x^k (1-x)_q^n t^{n+k}}{[n+k]_q! [k]_q!} \\
&= \sum_{n=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix}_q \frac{x^k (1-x)_q^n t^{n+k}}{[n+k]_q!} \\
&= \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{x^k (1-x)_q^{n-k} t^n}{[n]_q!} = \sum_{n=0}^{\infty} b_{k,n}^q(x) \frac{t^n}{[n]_q!}.
\end{aligned}$$

This completes the proof of generating function for $b_{k,n}^q(x)$. ∎

1.4.2 Generating Function for q-MKZ

Theorem 1.3. *$m_{k,n}^q(x)$ is the coefficient of t^k in the expansion of* $\frac{(1-x)_q^n}{(1-xt)_q^{n+2}}$.

Proof. It is easily seen that

$$\frac{(1-x)_q^n}{(1-xt)_q^{n+2}} = \sum_{k=0}^{\infty} \begin{bmatrix} n+k+1 \\ k \end{bmatrix}_q (1-x)_q^n x^k t^k = \sum_{k=0}^{\infty} m_{k,n}^q(x) t^k.$$

This completes the proof. ■

1.4.3 Generating Function for q-Beta Basis

Theorem 1.4. *It is observed by us that $v_{k,n}^q(x)$ is the coefficient of $\frac{t^k}{[n+k]_q!}$ in the expansion of* $\frac{1}{(1+x)_q^{n+1}} E_q\left(\frac{(1-q)xt}{(1+q^{n+1}x)_q}\right)$.

Proof. First using the definition of q-exponential $E_q(x)$, we have

$$\begin{aligned}
\frac{1}{(1+x)_q^{n+1}} E_q\left(\frac{(1-q)xt}{(1+q^{n+1}x)_q}\right) &= \frac{1}{(1+x)_q^{n+1}} \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{x^k}{(1+q^{n+1}x)_q^k} \frac{t^k}{[k]_q!} \\
&= \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{x^k}{(1+x)_q^{n+k+1}} \frac{t^k}{[k]_q!} \\
&= \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{x^k t^k}{(1+x)_q^{n+k+1}} \frac{[k+1]_q [k+2]_q \dots [n+k]_q}{[n+k]_q!} \\
&= \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{x^k}{(1+x)_q^{n+k+1}} \begin{bmatrix} n+k \\ n \end{bmatrix}_q \frac{[n]_q t^k}{[n+k]_q!} \\
&= \sum_{k=0}^{\infty} \frac{1}{B_q(k+1,n)} q^{k(k-1)/2} \frac{x^k}{(1+x)_q^{n+k+1}} \frac{t^k}{[n+k]_q!} \\
&= \sum_{k=0}^{\infty} v_{k,n}^q(x) \frac{t^k}{[n+k]_q!}.
\end{aligned}$$

This completes the proof of generating function. ■

1.5 q-Integral

The Jackson definite integral of the function f is defined by (see [103], [149]):

$$\int_0^a f(x)\,d_qx = (1-q)\,a\sum_{n=0}^{\infty} f(aq^n)\,q^n,\ a\in\mathbb{R}. \tag{1.12}$$

Notice that the series on the right-hand side is guaranteed to be convergent as soon as the function f is such that for some $C>0$, $\alpha>-1$, $|f(x)|<Cx^{\alpha}$ in a right neighborhood of $x=0$.

One defines the Jackson integral in a generic interval $[a,b]$:

$$\int_a^b f(x)\,d_qx = \int_0^b f(x)\,d_qx - \int_0^a f(x)\,d_qx.$$

Now we give the fundamental theorem of quantum calculus.

Theorem 1.5. *(a) If F is any anti q-derivative of the function f, namely, $D_qF=f$, continuous at $x=0$, then*

$$\int_0^a f(x)\,d_qx = F(a)-F(0).$$

(b) For any function f one has

$$D_q\int_0^x f(t)\,d_qt = f(x).$$

Remark 1.2. (a) The q-analogue of the rule of integration by parts is

$$\int_0^a g(x)\,D_qf(x)\,d_qx = f(x)\,g(x)|_a^b - \int_0^a f(qx)\,D_qg(x)\,d_qx.$$

(b) If $u(x)=\alpha x^{\beta}$, change of variable formula is

$$\int_{u(a)}^{u(b)} f(u)\,d_qu = \int_a^b f(u(x))\,D_{q^{1/\beta}}u(x)\,d_{q^{1/\beta}}x.$$

Definition 1.10. For $m,n>0$ the q-beta function [104] is defined as

$$B_q(m,n) = \int_0^1 t^{m-1}(1-qt)_q^{n-1}d_qt.$$

It can be easily seen that for $m>1, n>0$ after integrating by parts:

$$B_q(m,n) = \frac{[m-1]_q}{[n]_q}B_q(m-1,n+1).$$

Also from Definition 1.10, we have

$$\begin{aligned} B_q(m,n+1) &= \int_0^1 t^{m-1}(1-qt)_q^{n-1}(1-q^n t)d_q t \\ &= \int_0^1 t^{m-1}(1-qt)_q^{n-1}d_q t - q^n \int_0^1 t^m (1-qt)_q^{n-1} d_q t \\ &= B_q(m,n) - q^n B_q(m+1,n). \end{aligned}$$

The improper integral of function f is defined by [49, 107]:

$$\int_0^{\infty/A} f(x)\, d_q x = (1-q)\sum_{n=-\infty}^{\infty} f\left(\frac{q^n}{A}\right)\frac{q^n}{A},\ A \in \mathbb{R}. \tag{1.13}$$

Remark 1.3. If the function f satisfies the conditions $|f(x)| < Cx^{\alpha}$, $\forall x \in [0,\varepsilon)$, for some $C > 0$, $\alpha > -1$, $\varepsilon > 0$ and $|f(x)| < Dx^{\beta}$, $\forall x \in [N,\infty)$, for some $D > 0$, $\beta < -1$, $N > 0$, then the series on the right hand side is convergent. In general even though when these conditions are satisfied, the value of sum in the right side of (1.13) will be dependent on the constant A. In order to get the integral independent of A, in the anti q-derivative, we have to take the limits as $x \to 0$ and $x \to 1$, respectively.

Definition 1.11. The q-gamma function defined by

$$\Gamma_q(t) = \int_0^{1/1-q} x^{t-1} E_q(-qx)\, d_q x, \quad t > 0 \tag{1.14}$$

satisfies the following functional equation:

$$\Gamma_q(t+1) = [t]_q \Gamma_q(t),$$

where $[t]_q = \frac{1-q^t}{1-q}$ and $\Gamma_q(1) = 1$.

Remark 1.4. Note that the q-gamma integral given by (1.14) can be rewritten via an improper integral by using definition (1.13). From (1.8) we can easily see that $E_q\left(-\frac{q^n}{1-q}\right) = 0$ for $n \leq 0$. Thus, we can write

$$\Gamma_q(t) = \int_0^{\infty/1-q} x^{t-1} E_q(-qx)\, d_q x, \quad t > 0.$$

Definition 1.12. The q-beta function is defined as

$$B_q(t,s) = K(A,t)\int_0^{\infty/A} \frac{x^{t-1}}{(1+x)_q^{t+s}} d_q x, \tag{1.15}$$

and the q-gamma function is defined as

$$\Gamma_q(t) = K(A,t)\int_0^{\infty/A(1-q)} x^{t-1} e_q(-x)d_qx, \tag{1.16}$$

where $K(x,t) = \frac{1}{x+1}x^t\left(1+\frac{1}{x}\right)_q^t (1+x)_q^{1-t}$.

Remark 1.5. The q-gamma and q-beta functions are related to each other by the following identities:

$$B_q(t,s) = \frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t+s)} \tag{1.17}$$

and

$$\Gamma_q(t) = \frac{B_q(t,\infty)}{(1-q)^t}.$$

The function $K(x,t)$ is a q-constant, i.e., $K(qx,t) = K(x,t)$. In particular, for any positive integer n

$$K(x,n) = q^{\frac{n(n-1)}{2}}, \; K(x,0) = 1.$$

Also

$$\lim_{q\to 1} K(x,t) = 1, \forall x,t \in \Re$$

and

$$\lim_{q\to 0} K(x,t) = x^t + x^{t-1}, \forall t \in (0,1), \qquad x \in \Re.$$

It also satisfies $K(x,t+1) = q^t K(x,t)$ (see [49]).