Measurement Error Analysis from Independent to Longitudinal Setup

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Abstract In a generalized linear models (GLMs) setup, when scalar responses along with multidimensional covariates are collected from a selected sample of independent individuals, there are situations where it is realized that the observed covariates differ from the corresponding true covariates by some measurement error, but it is of interest to find the regression effects of the true covariates on the scalar responses. Further it may happen that the true covariates may be fixed but unknown or they may be random. It is understandable that when observed covariates are used for either likelihood or quasi-likelihood-based inferences, the naive regression estimates would be biased and hence inconsistent for the true regression parameters. Over the last three decades there have been a significant number of studies dealing with this bias correction problem for the regression estimation due to the presence of measurement error. In general these bias correction inferences are relatively easier for the linear and count response models, whereas the inferences are complex for the logistic binary models. In the first part of the paper, we review some of the widely used bias correction inferences in the GLMs setup and highlight their advantages and drawbacks where appropriate. As opposed to the independent setup, the bias correction inferences for clustered (longitudinal) data are, however, not adequately addressed in the literature. To be a bit more specific, some attention has been given to deal with bias correction in linear longitudinal setup (also called panel data setup) only. Bias corrected generalized method of moments (BCGMM) and bias corrected generalized quasi-likelihood (BCGQL) approaches are introduced and discussed. In the second part of this paper, we review these BCGMM and BCGQL approaches along with their advantages and drawbacks. The bias correction inferences for count and binary data are, however, more complex, because of the fact that apart from the mean functions, the variance and covariance functions of the clustered responses also involve time-dependent covariates. This makes the bias correction

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B.C. Sutradhar (ed.), *ISS-2012 Proceedings Volume On Longitudinal Data Analysis Subject to Measurement Errors, Missing Values, and/or Outliers*, Lecture Notes in Statistics 211,
DOI 10.1007/978-1-4614-6871-4_1,

difficult. However, following some recent works, in the second part of the paper, we also discuss a BCGQL approach for longitudinal models for count data subject to measurement error in covariates. Developing a similar bias correction approach for longitudinal binary data appears to be difficult and it requires further in-depth investigations.

1 Introduction

When responses along with covariates are collected from a group of independent individuals in a generalized linear model (GLM) setup, in some practical situations the observed covariates may be subject to measurement errors differing from the true covariates values. These imprecise observed covariates, when used directly, the standard statistical methods such as likelihood and quasi-likelihood methods yield biased and hence inconsistent regression estimates. Bias corrected estimation for the regression effects involved in generalized linear measurement-error models (GLMEMs) with normal measurement errors in covariates has been studied extensively in the literature. See, for example, Fuller (1987), Carroll et al. (2006), and Buonaccorsi (2010), and the references therein. In general, this type of bias correction is studied under two scenarios. First, if for a sample of observed responses and covariates, namely, $\{(y_i, x_i)(i = 1, \ldots, K)\}$, the true covariates $\{z_i\}$ are independent and identically distributed random vectors from some unknown distribution, *a structural error-in-variables model* is obtained; second if $\{z_i\}$ are unknown constants, *a functional error-in-variables model* is obtained (Kendall and Stuart 1979, Chap. 29; Stefanski and Carroll 1987; also known as Berkson error model). Note that the second scenario is more challenging technically because unknown fixed $\{z_i\}$ makes a large set of parameters and direct estimation or prediction of each of them may be impossible, specially when K is large.

For discussions on structural models, especially for inferences, in addition to the aforementioned references, namely, Fuller (1987), Carroll et al. (2006), and Buonaccorsi (2010), one may consult, for example, an instrumental variable technique to obtain bias corrected estimates for regression parameters in GLMs, studied by Buzas and Stefanski (1996a) (see also Stefanski and Buzas 1995; Buzas and Stefanski 1996b; Amemiya 1990; Carroll and Stefanski 1990), among others. In this paper, we, however, mainly deal with functional models, and among many existing studies based on such functional models, we, for example, refer to Stefanski and Carroll (1985), Stefanski (1985), Armstrong (1985), Stefanski and Carroll (1987), Carroll and Stefanski (1990), Nakamura (1990), Carroll and Wand (1991), Liang and Liu (1991), Stefanski (2000), and Carroll et al. (2006). Some of these studies address measurement error problems in various complicated situations such as when the data also contain outliers, and regression function is partly specified. But, they are confined to the independent setup.

As opposed to the independent setup, not much attention is given to the measurement error models for longitudinal count and binary data. Sutradhar and Rao (1996) have developed a bias correction approach as an extension of Stefanski (1985) for the longitudinal binary data with covariates subject to measurement errors. To be specific, these authors have used a small measurement error variance asymptotic approach to achieve the bias correction, which works well if the measurement error variance is small or moderately large. Wang et al. (1996) considered a measurement error model in a generalized linear regression setup where covariates are replicated and the measurement errors for replicated covariates are assumed to be correlated with a stationary correlation structure such as Gaussian auto-regressive of order 1 (AR(1)) structure. As far as the responses are concerned, they were assumed to be independent, collected at a cross-sectional level from a large number of independent individuals. Thus this study does not address the measurement error problems in the longitudinal setup where responses are collected repeatedly from a large number of independent individuals. With regard to the correlations for the repeated responses, there, however, exit some studies for continuous responses subject to measurement error, in time series setup. For example, we refer to the study by Staudenmayer and Buonaccorsi (2005), where time series responses are assumed to follow the Gaussian auto-regressive order 1 (AR(1)) correlation process subject to measurement error. But, these studies are not applicable to the longitudinal setup, especially for discrete longitudinal data such as for repeated count and binary data with covariates subject to measurement error.

In this paper, first we review some of the widely used inference approaches in the GLMs setup for independent responses, for the estimation of the regression effects on such responses when associated covariates are subject to mainly functional measurement error. The structural measurement error models are discussed in Sect. 2.1.2. The advantages and drawbacks of each approach are highlighted.

As pointed out above, the measurement error analysis is not so developed in the longitudinal setup specially for binary and count data. For linear longitudinal measurement error models, there exist some studies with concentration on econometric data analysis. For example, Wansbeek (2001) (see also Wansbeek and Meijer 2000) considered a measurement error model for linear panel data, where on top of the fixed true covariates z_i, some of the other covariates are strictly exogenous. To be more specific, Wansbeek (2001) developed necessary moment conditions to form bias corrected method of moments (BCMM) estimating equations in order to obtain consistent generalized method of moments (GMM) estimates for the regression parameters involved including the effect of the exogenous covariates. More recently, Xiao et al. (2007) studied the efficiency properties of the BCGMM (bias corrected generalized method of moments) approach considered by Wansbeek (2001). Note that the derivation of the efficient BCGMM estimators by Xiao et al. (2007) may be considered as the generalization of the GMM approach of Hansen (1982) to the measurement error models. In studying the efficiency of the BCGMM approach, Xiao et al. (2007), however, assumed that the model errors $\varepsilon_{i1},\ldots,\varepsilon_{iT_i}$ are independent to each other. Also they assume that the measurement errors collected over times are serially correlated. Recently Fan et al. (2012) have developed a bias corrected generalized quasi-likelihood

(BCGQL) approach that produces more efficient estimates than the BCGMM approach.

As far as the measurement error models for longitudinal count and binary data are concerned, in developing a bias correction method, one has to accommodate both longitudinal correlations of the repeated responses and the measurement errors in covariates. Recently, Sutradhar et al. (2012) have developed a BCGQL approach so that the BCGQL estimating function is unbiased for the GQL estimating function involving the true covariates. They then solved the BCGQL estimating equation to obtain bias corrected regression estimates. These estimates are also efficient. We describe this BCGQL approach in brief from Sutradhar et al. (2012). As opposed to the small measurement error variance-based estimating equation approach (Sutradhar and Rao 1996), developing a similar BCGQL estimating equation for regression effects involved in longitudinal binary data models does not appear to be easy. This would require further in-depth investigations.

2 Measurement Error Analysis in Independent Setup

For $i = 1, \ldots, K$, let Y_i denote the binary or count response variable for the ith individual and $x_i = (x_{i1}, \ldots, x_{ip})'$ be the associated p-dimensional covariate vector subject to normal measurement errors. Let $z_i = (z_{i1}, \ldots, z_{ip})'$ be the unobserved true covariate vector which may be fixed constant or random and β be the regression effect of z_i on y_i. For discrete responses, such as for count and binary data, by using exponential family density for y_i given z_i, the GLMEM is written as

$$f(y_i; z_i) = \exp[\{y_i \theta_i(z_i) - a(\theta_i(z_i))\} + b(y_i)] \tag{1}$$

$$x_i = z_i + \delta v_i \text{ with } v_i \sim N_p(0, \Lambda = \text{diag}[\sigma_1^2, \ldots, \sigma_p^2]), \tag{2}$$

where $\theta_i(z_i) = h(z_i'\beta)$, with $a(\cdot), b(\cdot)$, and $h(\cdot)$ being known functional form, yielding the first and second derivatives, $a'(\theta_i(z_i))$ and $a''(\theta_i(z_i))$, as the mean and variance of y_i, respectively; v_i is a random measurement error vector and δ^2 is a scalar parameter. Note that if for a sample $(y_i, x_i)(i = 1, \ldots, K)$ the covariates $\{z_i\}$ are unknown constants, a functional error-in-variables model (also known as Berkson error model) is obtained; if $\{z_i\}$ are independent and identically distributed random vectors from some unknown distribution, a structural error-in-variables model is obtained (Kendall and Stuart 1979, Chap. 29; Stefanski and Carroll 1987).

Under the functional model, Nakamura (1990) has proposed a corrected score (CS) estimation approach, where for given z_i, the log likelihood function for β is written by (1) as

$$\ell(\beta; y, z) = \sum_{i=1}^{K} [\{y_i \theta_i(z_i) - a(\theta_i(z_i))\} + b(y_i)],$$

and observed covariates x_i-based corrected log likelihood function $\ell^*(\beta;y,x)$ is written such that $E_x[\ell^*(\beta;y,x)] = \ell(\beta;y,z)$. The corrected score estimate of β, say $\hat{\beta}_{\mathrm{CS}}$, is then obtained by solving the corrected score equation

$$U^*(\beta;y,x) = \frac{\partial \ell^*(\beta;y,x)}{\partial \beta} = 0. \tag{3}$$

This corrected score approach provides closed form estimating equation for β for the Poisson regression model, but, the binary logistic regression model does not yield a corrected score function which is a limitation to this approach.

Stefanski and Carroll (1987) proposed a method based on conditional scores (CNS). In this approach, unbiased score equations are obtained by conditioning on certain parameter-dependent sufficient statistics for the true covariates z, and the authors have developed the approach in both functional and structural setups. The conditional score equations have a closed form for GLMs such as for normal, Poisson, and binary logistic models. Obtaining a closed form unbiased equation for logistic regression parameter by this conditional approach is an advantage over the direct corrected score approach (Nakamura 1990) which does not yield corrected score function. To elaborate a little more on the conditional score approach, consider, for example, the functional version of the logistic measurement error model with scalar predictor z_i so that the measurement error v_i in (2) follows $N_1(0,\sigma_1^2)$ (Stefanski 2000, Sect. 4.1). For convenience, consider $\delta = 1$ in (2). In this case, the density of $(y_i;x_i)$ is given by

$$f(y_i,x_i;\beta,z_i) = [\frac{\exp(z_i'\beta)}{1+\exp(z_i'\beta)}]^{y_i}[\frac{1}{1+\exp(z_i'\beta)}]^{1-y_i}\frac{1}{\sigma_1}\phi(\frac{x_i-z_i}{\sigma_1}),$$

where $\phi(.)$ is the standard normal density function. The estimation of β also requires the estimation of the nuisance parameters z_i or some functions of z_i's for $i = 1,\ldots,K$. However, Stefanski and Carroll (1987) have demonstrated that the parameter-dependent statistic $\lambda_i = x_i + y_i\sigma_1^2\beta$ is sufficient for unknown z_i in the sense that the conditional distribution of (y_i,x_i) given λ_i does not depend on the nuisance parameter z_i. This fact was exploited to obtain unbiased estimating equation for β using either conditional likelihood method or mean variance function models (based on conditional density of y_i given λ_i) and quasi-likelihood methods. For the scalar regression parameter β, the unbiased estimating equation has the form (Sutradhar and Rao 1996, Eq. (2.10))

$$\sum_{i=1}^{K}(\lambda_i - \sigma_1^2\beta)(y_i - \tilde{p}_i) = 0, \tag{4}$$

where $\tilde{p}_i = F[\{\lambda_i - (\sigma_1^2/2)\beta\}\beta]$ with $F(t) = 1/[1+\exp(-t)]$. Let $\hat{\beta}_{\mathrm{CNS}}$ denote the solution of (4) for β.

In structural error-in-variables setup, there exists an instrumental variable technique to obtain bias corrected estimates for regression parameters in GLMs. For this, for example, we refer to Buzas and Stefanski (1996a) (see also Stefanski and Buzas 1995; Buzas and Stefanski 1996a; Amemiya 1990; Fuller 1987). We do not discuss about this technique any further in this paper as our purpose is to deal with functional models as opposed to the structural models.

Note that as in the absence of measurement errors, regression parameters involved in GLMs such as for count and binary models, may be estimated consistently and efficiently by using the first two moments-based quasi-likelihood (QL) approach (Wedderburn 1974), there has been a considerable attention to modify the naive QL (NQL) approach (that directly uses observed covariates ignoring measurement errors) in order to accommodate measurement errors in covariates and obtain bias corrected QL (BCQL) estimates. Some of these BCQL approaches are developed for both structural and functional models, some are developed for the functional models and others are more appropriate for structural models only. Stefanski (1985) proposed a small measurement error variance-based BCQL approach for structural models, Carroll and Stefanski (1990) have used a similar small measurement error variance-based QL approach which is developed to accommodate either of the structural or functional models or both. Liang and Liu (1991) have discussed a BCQL approach for structural model, which was later on generalized by Wang et al. (1996) to accommodate correlated replicates in covariates. Sutradhar and Rao (1996) have used Stefanski's (1985) small measurement error-based BCQL approach for the longitudinal binary data, independent setup being a special case, under functional model only. In the next section, we provide a brief review of some of these existing simpler BCQL approaches which are suitable for functional models.

In Sect. 2.1, we provide an alternative BCQL approach which yields the same corrected regression estimates as the corrected score estimates (Nakamura 1990) for the Poisson model in functional setup. In the binary case, the proposed alternative approach provides a first order approximate BCQL regression estimates.

2.1 BCQL Estimation

Note that if z_i were known, then one would have obtained a consistent estimator of β by solving the so-called quasi-likelihood (QL) estimating equation

$$\sum_{i=1}^{K}\left[\frac{\partial a'(\theta_i(z_i))}{\partial\beta}\frac{(y_i - a'(\theta_i(z_i)))}{a''(\theta_i(z_i))}\right] = \sum_{i=1}^{K}\psi_i(y_i, z_i, \beta) = 0 \tag{5}$$

(Wedderburn 1974), where for $\theta_i(z_i) = h(z_i'\beta)$, both $a'(\theta_i(z_i))$ and $a''(\theta_i(z_i))$ are functions of β. For example, for the Poisson and binary data $h(\cdot) = 1$, and $a'(\theta_i(z_i)) = \exp(z_i'\beta)$ for the Poisson data, and $a'(\theta_i(z_i)) = \frac{\exp(z_i'\beta)}{1+\exp(z_i'\beta)}$ for the binary data. Thus, for both Poisson and binary models, the QL estimating (5) reduces to $\sum_{i=1}^{K} z_i(y_i - a'(z_i'\beta)) = 0$, where $a'(z_i'\beta) = \mu_{iz}$ is the mean of y_i. Note that this QL estimating equation is also a likelihood estimating equation. However, because the true covariate z_i is not observed, one cannot use the estimating (5) for the estimation of β.

2.1.1 Small Measurement Error Variance-Based QL (SVQL) Approach

Suppose that by replacing z_i with x_i in (5), one constructs a NQL estimating equation, namely

$$\begin{aligned}\sum_{i=1}^{K} \psi_i(y_i, x_i, \beta) &= \sum_{i=1}^{K} w_i[y_i - a'(h(x_i'\beta))]h'(x_i'\beta)x_i \\ &= \sum_{i=1}^{K} g_i(x_i'\beta)x_i = 0, \end{aligned} \tag{6}$$

which is the naive version of the Eq. (10) in Stefanski (1985, 588), where $w_i x_i = \frac{\partial a'(h(x_i'\beta))/\partial\beta}{a''(h(x_i'\beta))}$. Let $\hat{\beta}$ be the solution of this NQL estimating (6). But, because the NQL estimating function in the left-hand side of (6) is a function of $x_i x_i'$ and because $x_i = z_i + \delta v_i$ with $E[x_i x_i'] = z_i z_i' + \delta^2 \Lambda$ in the functional setup, $\hat{\beta}$ obtained from (6) cannot converge to β, it rather converges to a different parameter say $\beta(\delta\Lambda)$. Thus, the naive estimator $\hat{\beta}$ is biased and hence inconsistent for β. As a remedy, assuming that δ is small, by expanding the expected function

$$E_x \sum_{i=1}^{K} \psi_i(y_i, x_i, \beta) = E_x \sum_{i=1}^{K} g_i(x_i'\beta)x_i = \sum_{i=1}^{K} \psi_i^*(y_i, z_i, \beta(\delta\Lambda)), \text{ (say)}, \tag{7}$$

about $\delta = 0$, and then equating the expanded function to zero followed by replacing z_i with x_i and β with $\hat{\beta}$, Stefanski (1985) obtained a SVQL estimator of β as a function of δ as

$$\begin{aligned}\hat{\beta}_{\text{SVQL}}(\delta) = \hat{\beta} + \frac{1}{2}\delta^2 &\left[\sum_{i=1}^{K} g_i'(x_i'\hat{\beta})x_i x_i'\right]^{-1} \\ &\times \left[\sum_{i=1}^{K} g_i''(x_i'\hat{\beta})\hat{\beta}'\Lambda\hat{\beta}x_i + 2g_i'(x_i'\hat{\beta})\Lambda\hat{\beta}\right], \end{aligned} \tag{8}$$

where $g_i'(\eta_i) = \frac{\partial g_i(\eta_i)}{\partial \eta_i}$, and similarly $g_i''(\eta_i) = \frac{\partial^2 g_i(\eta_i)}{\partial \eta_i^2}$. Note that because in the present independent setup, the mean and variance functions-based QL estimating (5) is the same as the likelihood estimating equation based on GLM (1), Stefanski's

(1985) small variance-based bias correction to naive likelihood estimates is quite flexible. See also Whittemore and Keller (1988) for a similar QL-based modification to the NQL or likelihood estimates. Armstrong (1985) (see also Schafer 1987) also has used QL approach but solved for bias corrected estimates numerically as opposed to obtaining SVQL estimates. Based on small δ^2 approach, Carroll and Stefanski (1990) have developed an approximate SVQL approach in a general framework which can accommodate either structural or functional model or both. In this paper, we, however, concentrate on the functional model only.

Note that as in the count data case $g_i(x_i'\beta) = y_i - \mu_{ix} = y_i - \exp(x_i'\beta)$, the SVQL estimator of β by (8) has the formula

$$\text{Poisson case:} \quad \hat{\beta}_{\text{SVQL}}(\delta) = \hat{\beta} + \frac{1}{2}\delta^2 \left[-\sum_{i=1}^{K} \hat{\mu}_{ix} x_i x_i' \right]^{-1} \times \left[\sum_{i=1}^{K} (-1)\hat{\mu}_{ix}\hat{\beta}'\Lambda\hat{\beta}x_i - 2\hat{\mu}_{ix}\Lambda\hat{\beta} \right], \tag{9}$$

where $\hat{\mu}_{ix} = \exp(x_i'\hat{\beta})$. Similarly, for the binary data case with $\hat{\mu}_{ix} = \hat{p}_{ix} = \exp(x_i'\hat{\beta})/[1+\exp(x_i'\hat{\beta})]$, the SVQL estimator of β has the formula

$$\text{Binary case:} \quad \hat{\beta}_{\text{SVQL}}(\delta) = \hat{\beta} + \frac{1}{2}\delta^2 \left[-\sum_{i=1}^{K} \hat{p}_{ix} x_i x_i' \right]^{-1} \times \left[\sum_{i=1}^{K} \hat{p}_{ix}\hat{q}_{ix}\{1-\hat{q}_{ix}\}\hat{\beta}'\Lambda\hat{\beta}x_i - 2\hat{p}_{ix}\hat{q}_{ix}\Lambda\hat{\beta} \right], \tag{10}$$

(see also Sutradhar and Rao 1996, Eq. (2.2), p. 181), where $\hat{q}_{ix} = 1 - \hat{p}_{ix}$.

2.1.2 Conditional QL (CNQL) Estimation

In structural setup, there exists a QL approach, developed conditional on x_i. Let the true covariate vector z_i be a stochastic variable, distributed as

$$z_i \sim N_p(m, V).$$

Next because $x_i = z_i + \delta v_i$ by (2), it then follows that conditional on z_i, x_i has the conditional normal distribution

$$x_i | z_i \sim N_p(z_i, \delta^2\Lambda).$$

Unconditionally x_i has the normal distribution given by

$$
\begin{aligned}
x_i &\sim N_p[E(z_i),\, E(\delta^2\Lambda) + \text{var}(z_i)] \\
&\equiv N_p[m, \delta^2\Lambda + V].
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\text{cov}(x_i, z_i) &= E_z[\text{cov}((x_i, z_i)|z_i)] + \text{cov}_z[E(x_i|z_i), E(z_i|z_i)] \\
&= \text{cov}_z[z_i, z_i] = V.
\end{aligned}
$$

It then follows that z_i and x_i have the $2p$-dimensional joint normal distribution given as

$$
\begin{pmatrix} z_i \\ x_i \end{pmatrix} \sim N_{2p}\left[\begin{pmatrix} m \\ m \end{pmatrix}, \begin{pmatrix} V & V \\ V & \delta^2\Lambda + V \end{pmatrix}\right], \tag{11}
$$

yielding the conditional distribution of z_i given x_i as

$$
\begin{aligned}
z_i|x_i &\sim N_p[m + V(\delta^2\Lambda + V)^{-1}(x_i - m),\, V - V(\delta^2\Lambda + V)^{-1}V] \\
&\equiv N_p[\{I_p - V(\delta^2\Lambda + V)^{-1}\}m \\
&\qquad + V(\delta^2\Lambda + V)^{-1}x_i,\, \{I_p - V(\delta^2\Lambda + V)^{-1}\}V] \\
&\equiv N_p[\eta_{z|x},\, V_{11.2}].
\end{aligned} \tag{12}
$$

The CNQL estimate of β, say $\hat{\beta}_{\text{CNQL}}$ is then obtained by solving the QL estimating equation

$$
\sum_{i=1}^{K} \frac{\partial\{E[Y_i|x_i]\}}{\partial\beta}[\text{var}(Y_i|x_i)]^{-1}(y_i - E[Y_i|x_i]) = 0, \tag{13}
$$

(Liang and Liu 1991, Eq. (4.11), p. 51), where by applying (12), the conditional expectation and covariance matrix may be computed by using the formulas

$$
\begin{aligned}
E[Y_i|x_i] &= E_{z_i|x_i}[Y_i|z_i] = E_{z_i|x_i}[a'(z_i'\beta)], \\
\text{var}[Y_i|x_i] &= E_{z_i|x_i}[\text{var}(Y_i|z_i)] + \text{var}_{z_i|x_i}[E(Y_i|z_i)] \\
&= E_{z_i|x_i}[a''(z_i'\beta)] + \text{var}_{z_i|x_i}[a'(z_i'\beta)].
\end{aligned}
$$

Note that in this structural setup, Wang et al. (1996) have used a naive mean and variance-based QL approach where QL estimating equation for β is constructed by replacing the observed covariate vector x_i with its mean obtained from a repeated sampling. In fact this type of repeated samples is usually employed to estimate the measurement error variances. Their approximate QL estimating equation has the form

$$\sum_{i=1}^{K}\left[\frac{\partial\{a'(x_i'\beta)\}}{\partial\beta}[a''(x_i'\beta)]^{-1}(y_i-a'(x_i'\beta))\right]_{|x_i=\tilde{x}_i}=0,$$

where $\tilde{x}_i$ is the mean computed from the replicates of x_i. The relative performance of this approximate QL approach with other existing approaches is, however, not known.

Turning back to the functional setup, the CNQL estimating (12) may be modified by using fixed z_i and its relationship to x_i given in (2), that is, $x_i = z_i + \delta v_i$. It follows in this case that one may still solve the CNQL (12) for β, but the conditional expectation and variance are computed as

$$\begin{aligned} E[Y_i|x_i] &= E_{v_i}[\{a'(z_i'\beta)\}_{|z_i=x_i-\delta v_i}] \\ \mathrm{var}[Y_i|x_i] &= E_{v_i}[\{a''(z_i'\beta)\}_{|z_i=x_i-\delta v_i}] + \mathrm{var}_{v_i}[\{a'(z_i'\beta)\}_{|z_i=x_i-\delta v_i}], \end{aligned} \tag{14}$$

where $v_i \sim N_p[0, \Lambda = \mathrm{diag}(\sigma_1^2, \ldots, \sigma_u^2, \ldots, \sigma_p^2)]$.

2.1.3 An Approximate BCQL Approach Using Corrected Estimating Function

We propose a bias correction approach along the lines of Nakamura (1990). The difference between Nakamura's and our approach is that Nakamura (1990) developed a corrected score function $\ell^*(\beta; y, x)$ such that its expectation is the true but unknown score function, that is, $E_x[\ell^*(\beta; y, x)] = \ell(\beta; y, z)$, and then solved the corrected score (3) for β, whereas in our approach we develop a corrected quasi-likelihood function, say $Q^*(y, x, \beta)$, such that

$$E_x[Q^*(y,x,\beta)] = \psi(y,z,\beta), \tag{15}$$

where by (5), $\psi(y,z,\beta) = \sum_{i=1}^{K}\psi_i(y_i,z_i,\beta)$ is the true QL function in unknown covariates z_i, and solve the corrected QL equation, that is, $Q^*(\beta,y,x) = 0$ for β. Also, this bias correction approach is different than the SVQL approach of Stefanski (1985) as it does not require any small variance assumption to hold.

Poisson Regression Model

If the true covariates z_i were known, then for the Poisson regression model it follows from (5) that the QL estimating equation would have the form

$$\psi(y,z,\beta) = \sum_{i=1}^{K}\psi_i(y_i,z_i,\beta) = \sum_{i=1}^{K} z_i(y_i-\mu_{iz}) = 0, \tag{16}$$

with $\mu_{iz} = \exp(z_i'\beta)$. For the purpose of developing a corrected QL function $Q^*(\beta, y, x)$, by replacing z_i with x_i, we first write the NQL estimating equation as

$$Q(y,x,\beta) = \sum_{i=1}^{K} Q_i(y_i,x_i,\beta) = \sum_{i=1}^{K} x_i(y_i - \mu_{ix}) = 0, \tag{17}$$

where $\mu_{ix} = \exp(x_i'\beta)$. Under the measurement error model (2), that is, when $x_i = z_i + \delta v_i$, it is clear that NQL function $Q(y,x,\beta)$ is not unbiased for the true QL function $\psi(y,z,\beta)$. That is,

$$E_x[Q(y,x,\beta)] = E_x \sum_{i=1}^{K} x_i(y_i - \mu_{ix}) \neq \psi(y,z,\beta) = \sum_{i=1}^{K} z_i(y_i - \mu_{iz}).$$

Note, however, that under the Gaussian measurement error model (2), that is when $x_i \sim N_p(z_i, \delta^2 \Lambda)$, one obtains $E_x[\exp(x_i'\beta)|z_i] = \exp(z_i'\beta + \xi) = \mu_{iz}\exp(\xi)$, where $\xi = \frac{\delta^2}{2}\beta'\Lambda\beta$, yielding

$$E_x \mu_{ix} \exp(-\xi) = \mu_{iz}. \tag{18}$$

Further it may be shown that $E_x[x_i \exp(x_i'\beta)|z_i] = [z_i + \delta^2\Lambda\beta]\mu_{iz}\exp(\xi)$ (Nakamura 1990), yielding

$$E_x[x_i \mu_{ix} \exp(-\xi)] = z_i \mu_{iz} + \delta^2 \Lambda \beta \mu_{iz}. \tag{19}$$

Now by using (18), it follows from (19) that

$$E_x[\{x_i - \delta^2 \Lambda \beta\} \mu_{ix} \exp(-\xi)] = z_i \mu_{iz}. \tag{20}$$

Consequently, one obtains the BCQL function

$$Q^*(y,x,\beta) = \sum_{i=1}^{K} [x_i y_i - \{(x_i - \delta^2 \Lambda \beta)\mu_{ix} \exp(-\xi)\}] \tag{21}$$

which satisfies

$$E_x[Q^*(y,x,\beta)] = \sum_{i=1}^{K} z_i(y_i - \mu_i z), \tag{22}$$

yielding the BCQL estimating equation for β in the Poisson model as

$$\sum_{i=1}^{K} [x_i y_i - \{(x_i - \delta^2 \Lambda \beta)\mu_{ix} \exp(-\xi)\}] = 0. \tag{23}$$

We denote the solution of (23) by $\hat{\beta}_{\text{BCQL}}$. This estimator is consistent for β. Remark that this BCQL estimating (23) is the same as the corrected score equation derived by Nakamura (1990, Sect. 4.3). Thus, in the Poisson measurement model setup, the BCQL approach provides the same regression estimate as the bias corrected likelihood approach.

Binary Regression Model

In the binary regression case, the true but unknown mean function is given by $\mu_{iz} = \exp(z_i'\beta)/[1+\exp(z_i'\beta)]$, whereas in the Poisson case $\mu_{iz} = \exp(z_i'\beta)$. This makes it difficult to find a corrected QL function $\tilde{Q}(y,x,\beta)$ such that

$$E_x[\tilde{Q}(y,x,\beta)] = \sum_{i=1}^{K} z_i[y_i - \frac{\exp(z_i'\beta)}{1+\exp(z_i'\beta)}] = \tilde{\psi}(y,z,\beta) \tag{24}$$

in the binary case. However, a softer, that is, a first order approximate BCQL (SBCQL) estimating function may be developed as follows. We denote this SBCQL function as $\tilde{Q}_S(y,x,\beta)$ which will be approximately unbiased for $\tilde{\psi}(y,z,\beta)$, that is,

$$E_x[\tilde{Q}_S(y,x,\beta)] \simeq \tilde{\psi}(y,z,\beta).$$

Recall from (18) and (20) that

$$E_x[\exp(x_i'\beta - \xi)] = \exp(z_i'\beta), \tag{25}$$

$$E_x[\{x_i - \delta^2\Lambda\beta\}\exp(x_i'\beta - \xi)] = z_i\exp(z_i'\beta), \tag{26}$$

where $\xi = \frac{\delta^2}{2}\beta'\Lambda\beta$. It then follows that

$$E_x\left[\frac{\{x_i - \delta^2\Lambda\beta\}\exp(x_i'\beta - \xi)}{1+\exp(x_i'\beta - \xi)}\right] \simeq \frac{z_i\exp(z_i'\beta)}{1+\exp(z_i'\beta)}. \tag{27}$$

Next because the true QL function has the form

$$\tilde{\psi}(y,z,\beta) = \sum_{i=1}^{K} z_i y_i - \sum_{i=1}^{K}[\frac{z_i\exp(z_i'\beta)}{1+\exp(z_i'\beta)}],$$

by using (27), one may write a softer BCQL (SBCQL) estimating equation as

$$\sum_{i=1}^{K}\left[x_i y_i - \frac{\{x_i - \delta^2\Lambda\beta\}\exp(x_i'\beta - \xi)}{1+\exp(x_i'\beta - \xi)}\right] = 0. \tag{28}$$

We denote the solution of the SBCQL estimating (28) by $\hat{\beta}_{\text{SBCQL}}$. Note that this estimator may still be biased and on a more serious note it may not even converge to β. This is because the expectation shown in (27) may differ to a great extent from the actual expectation. However exploiting a better approximation for the expectation as follows may remove the convergence problem and also may yield estimates with smaller bias.

For the purpose, rewrite the expectation in (27) as

$$E_x\left[\frac{\{x_i - \delta^2\Lambda\beta\}\exp(x_i'\beta - \xi)}{1+\exp(x_i'\beta - \xi)}\right] \simeq \frac{z_i\exp(z_i'\beta)}{1+\exp(z_i'\beta)} = \frac{\mu_{W_{z,N}}}{\mu_{W_{z,D}}}, \tag{29}$$

and improve the expectation as follows. To be specific, we first compute an improved expectation as

$$\begin{aligned} E_x\left[\frac{\{x_i - \delta^2\Lambda\beta\}\exp(x_i'\beta - \xi)}{1+\exp(x_i'\beta - \xi)}\right] &= E_x\left[\frac{W_{x,N}}{W_{x,D}}\right] \\ \simeq \frac{\mu_{W_{z,N}}}{\mu_{W_{z,D}}} - \frac{\hat{\text{cov}}[W_{x,N}, W_{x,D}]}{\hat{\mu}^2_{W_{z,D}}} &+ \frac{\hat{\mu}_{W_{z,N}}}{\hat{\mu}^3_{W_{z,D}}}\hat{\text{var}}[W_{x,D}], \end{aligned} \tag{30}$$

where we use

$$\begin{aligned} \hat{\mu}_{W_{z,N}} &= \frac{1}{K}\sum_{i=1}^{K}[\{x_i - \delta^2\Lambda\beta\}\exp(x_i'\beta - \xi)] \\ \hat{\mu}_{W_{z,D}} &= \frac{1}{K}\sum_{i=1}^{K}[1+\exp(x_i'\beta - \xi)] \\ \hat{\text{var}}[W_{x,D}] &= \frac{1}{K}\sum_{i=1}^{K}[1+\exp(x_i'\beta - \xi)]^2 - \hat{\mu}^2_{W_{z,D}} \\ \hat{\text{cov}}[W_{x,N}, W_{x,D}] &= \frac{1}{K}\sum_{i=1}^{K}\left[\{(x_i - \delta^2\Lambda\beta)\exp(x_i'\beta - \xi)\}\{1+\exp(x_i'\beta - \xi)\}\right] \\ &\quad - \hat{\mu}_{W_{z,N}}\hat{\mu}_{W_{z,D}} \end{aligned} \tag{31}$$

We then rewrite (30) as

$$E_x\left[\frac{\{x_i - \delta^2\Lambda\beta\}\exp(x_i'\beta - \xi)}{1+\exp(x_i'\beta - \xi)} + t_c\right] = \frac{\mu_{W_{z,N}}}{\mu_{W_{z,D}}}, \tag{32}$$

where

$$t_c = \frac{\hat{\text{cov}}[W_{x,N}, W_{x,D}]}{\hat{\mu}^2_{W_{z,D}}} - \frac{\hat{\mu}_{W_{z,N}}}{\hat{\mu}^3_{W_{z,D}}}\hat{\text{var}}[W_{x,D}].$$

Thus, instead of (28), we now solve the improved SBCQL estimating equation given by

$$\sum_{i=1}^{K}\left[x_i y_i - \frac{\{x_i - \delta^2 \Lambda \beta\}\exp(x_i'\beta - \xi)}{1+\exp(x_i'\beta - \xi)} - t_c\right] = 0. \tag{33}$$

3 Measurement Error Analysis in Longitudinal Setup

With regard to the correlations for the repeated responses, not much attention is paid to model such correlations, where the associated covariates are subject to measurement error. However, in time series setup, there exist some studies for continuous responses subject to measurement error. For example, we refer to the study by Staudenmayer and Buonaccorsi (2005), where time series responses are assumed to follow the Gaussian auto-regressive order 1 (AR(1)) correlation process subject to measurement errors. But, these studies are not applicable to the longitudinal setup, especially for discrete longitudinal data such as for repeated count data with covariates subject to measurement error.

In longitudinal setup, both repeated responses and measurement errors in covariates are likely to be correlated. Because the repeated measurement errors usually share a common instrument/machine/individual effect, in this study we assume that this type of errors follow a familial correlation structure such as mixed model-based equi-correlation structure. As far as the repeated responses are concerned, it is likely that they will follow a dynamic relationship causing certain auto-correlations among them as time effects. Thus, similar to Sutradhar (2011), in this study we assume that the repeated responses will follow a general class of auto-correlation structures. It is, however, known that the repeated linear, count, and binary data exhibit similar but different auto-correlation structures especially when the covariates are time dependent (nonstationary). For this reason, in this section, we deal with the measurement error models for these three types of response data separately and discuss them in sequence in the following three subsections.

3.1 Linear Auto-correlation Models with Measurement Error in Covariates

In this section, we consider functional error-in-variables models for continuous (linear) panel data. Let

$$\begin{aligned} y_{it} &= z_{it}'\beta + w_i\gamma_i^* + \varepsilon_{it},\ \text{for } t = 1,\ldots,T_i, \\ x_{it} &= z_{it} + v_{it}, \end{aligned} \tag{34}$$

represent such a measurement error model, where y_{it} denotes a continuous response for the ith $(i = 1, \ldots, K)$ individual recorded at time t $(t = 1, \ldots, T_i)$ with $2 \leq T_i \leq T$, $z_{it} = (z_{it1} \ldots, z_{itu}, \ldots, z_{itp})'$ be the $p \times 1$ true but unobserved time-dependent covariate vector, $\beta = (\beta_1, \ldots, \beta_u, \ldots, \beta_p)'$ be the $p \times 1$ vector of regression parameters, γ_i^* is the ith individual random effect with $\gamma_i^* \overset{iid}{\sim} (0, \sigma_\gamma^2)$, and w_i is a known additional covariate for the ith individual on top of the fixed covariates z_{it}. Furthermore, ε_{it} in (34) is the model error such that marginally $\varepsilon_{it} \sim (0, \sigma_\varepsilon^2)$, but jointly $\varepsilon_{i1}, \ldots, \varepsilon_{it}, \ldots, \varepsilon_{iT_i}$ follow a serially correlated such as AR(1) (auto-regressive order 1) or MA(1) (moving average order 1) process. Furthermore, in (34),

$$x_{it} = (x_{it1} \ldots, x_{itu}, \ldots, x_{itp})', \text{ and } v_{it} = (v_{it1} \ldots, v_{itu}, \ldots, v_{itp})',$$

with

$$v_{itu} \sim (0, \sigma_u^2), \text{ for } u = 1, \ldots, p$$

at any time point $t = 1, \ldots, T_i$. Here, as in Sect. 1, σ_u^2 is known as the measurement error variance for the uth covariate. Because the measurement errors $v_{i1u}, \ldots, v_{itu}, \ldots, v_{iT_iu}$ for measuring the same uth covariate values at different times are likely to be correlated due to a common instrumental random effect $m_{i|u}$, (say), we consider

$$v_{itu} = m_{i|u} + a_{itu}, \text{ for } t = 1, \ldots, T_i \tag{35}$$

and assume that $m_{i|u} \overset{iid}{\sim} (0, \tilde{\sigma}_u^2)$ and $a_{itu} \overset{iid}{\sim} (0, \sigma_a^2)$, and $m_{i|u}$ and a_{itu} are independent. It is then clear from (35) that the variance of v_{itu} and the correlation between v_{isu} and v_{itu} are given by

$$\text{var}(v_{itu}) = \sigma_u^2 = \tilde{\sigma}_u^2 + \sigma_a^2, \text{ and } \text{corr}(v_{isu}, v_{itu}) = \phi_u = \frac{\tilde{\sigma}_u^2}{\tilde{\sigma}_u^2 + \sigma_a^2}, \tag{36}$$

for all $s \neq t, s, t = 1, \ldots, T_i$.

By writing $Z_i = [z_{i(1)}, \ldots, z_{i(u)}, \ldots, z_{i(p)}] : T_i \times p$, with $z_{i(u)} = (z_{i1u}, \ldots, z_{itu}, \ldots, z_{iT_iu})'$; $X_i = [x_{i(1)}, \ldots, x_{i(u)}, \ldots, x_{i(p)}] : T_i \times p$, with $x_{i(u)} = (x_{i1u}, \ldots, x_{itu}, \ldots, x_{iT_iu})'$; and $V_i = [v_{i(1)}, \ldots, v_{i(u)}, \ldots, v_{i(p)}] : T_i \times p$, with $v_{i(u)} = (v_{i1u}, \ldots, v_{itu}, \ldots, v_{iT_iu})'$, and expressing the measurement error model (34) in matrix notation as

$$y_i = Z_i\beta + 1_{T_i} w_i \gamma_i^* + \varepsilon_i \tag{37}$$

$$X_i = Z_i + V_i \tag{38}$$

with $y_i = (y_{i1}, \ldots, y_{iT_i})'$, $\varepsilon_i = (\varepsilon_{i1}, \ldots, \varepsilon_{iT_i})'$, and 1_{T_i} as the T_i-dimensional unit vector, one can first write the so-called naive MM (NMM) estimating equation for β as

$$\psi^* = \sum_{i=1}^{K} X_i'(y_i - X_i\beta) = 0, \tag{39}$$

but its solution would produce biased and hence inconsistent estimate for β, because $E_{x|y}[\sum_{i=1}^{K} X_i'(y_i - X_i\beta)] \neq \sum_{i=1}^{K} Z_i'(y_i - Z_i\beta)$, due to the fact that in the present measurement error setup $E[V_i'V_i] \neq 0$ even though $E[V_i] = 0$. As a remedy, by exploiting

$$\begin{aligned} E[V_i'V_i] &= T_i\mathrm{diag}[\tilde{\sigma}_1^2 + \sigma_a^2, \ldots, \tilde{\sigma}_u^2 + \sigma_a^2, \ldots, \tilde{\sigma}_p^2 + \sigma_a^2] \\ &= T_i\mathrm{diag}[\sigma_1^2, \ldots, \sigma_u^2, \ldots, \sigma_p^2] \\ &= T_i\Lambda(\sigma_1^2, \ldots, \sigma_u^2, \ldots, \sigma_p^2), \text{(say)} \end{aligned} \tag{40}$$

that is,

$$E[X_i'X_i] = Z_i'Z_i + E[V_i'V_i] = Z_i'Z_i + T_i\Lambda(\sigma_1^2, \ldots, \sigma_u^2, \ldots, \sigma_p^2),$$

one may obtain a BCMM estimator for β by solving the BCMM estimating equation

$$\begin{aligned} \psi(x, y; \beta, \sigma_1^2, \ldots, \sigma_p^2) &= \sum_{i=1}^{K} X_i'y_i - [\sum_{i=1}^{K} \{X_i'X_i - T_i\Lambda(\sigma_1^2, \ldots, \sigma_u^2, \ldots, \sigma_p^2)\}]\beta \\ &= \sum_{i=1}^{K} \psi_i(x_i, y_i; \beta, \sigma_1^2, \ldots, \sigma_p^2) \end{aligned} \tag{41}$$

(Griliches and Hausman 1986) yielding the BCMM estimator as

$$\hat{\beta}_{BCMM} = \left[\sum_{i=1}^{K} \{X_i'X_i - T_i\Lambda(\sigma_1^2, \ldots, \sigma_u^2, \ldots, \sigma_p^2)\}\right]^{-1} \sum_{i=1}^{K} X_i'y_i. \tag{42}$$

This BCMM estimator is consistent for β but can be inefficient.

Recently, some authors such as Wansbeek (2001) (see also Wansbeek and Meijer 2000) considered a slightly different model than (37)–(38) by also involving certain strictly exogenous explanatory variables (in addition to Z_i) and by absorbing the random effects γ_i^* into the error vector ε_i that avoids the estimation of the variance component of the random effects σ_γ^2. Wansbeek (2001) developed necessary moment conditions to form BCMM estimating equations in order to obtain consistent GMM estimates for the regression parameters involved including the effect of the exogenous covariates. More recently, Xiao et al. (2007) studied the efficiency properties of the BCGMM approach considered by Wansbeek (2001). Note that the derivation of the efficient BCGMM estimators by Xiao et al. (2007) may be considered as the generalization of the GMM approach of Hansen (1982) to the measurement error models. In studying the efficiency of the BCGMM approach, Xiao et al. (2007), however, assumed that the model errors $\varepsilon_{i1},\ldots,\varepsilon_{iT_i}$ are independent to each other. Also they assume that the measurement errors $v_{i1u},\ldots,v_{iT_iu}$ in (38) (see also (34)) are serially correlated.

More recently, by treating the model errors $\varepsilon_{i1},\ldots,\varepsilon_{iT_i}$ as serially correlated with a general auto-correlation structure

$$C_i(\rho) = \begin{bmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{T_i-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{T_i-2} \\ \vdots & \vdots & \vdots & & \vdots \\ \rho_{T_i-1} & \rho_{T_i-2} & \rho_{T_i-3} & \cdots & 1 \end{bmatrix}, \tag{43}$$

(Sutradhar 2003) and by considering a more practical familial type equi-correlation structure (36) for the measurement errors, that is,

$$E[v_{i(u)}v'_{i(u)}] = \sigma_u^2[\phi_u 1_{T_i}1'_{T_i} + (1-\phi)I_{T_i}], \tag{44}$$

Fan et al. (2012) compared the efficiency of the BCGMM estimator with a new BCGQL (also referred to as BCGLS) approach, the latter being more efficient. These two approaches are briefly described in the following two sub-sections.

3.1.1 BCGMM Estimation for Regression Effects

Note that the BCMM estimating (41) is an unbiased estimating equation because of the fact that

$$E_yE_{x|y}\psi(x,y;\beta,\sigma_1^2,\ldots,\sigma_p^2) = \sum_{i=1}^{K} E_{y_i}E_{x_i|y_i}\psi_i(x_i,y_i;\beta,\sigma_1^2,\ldots,\sigma_p^2) = 0.$$

Consequently, the BCMM estimator for β in (42) was obtained by solving

$$\psi(x,y;\beta,\sigma_1^2,\ldots,\sigma_p^2) = 0,$$

but this estimator can be inefficient. As a remedy, following Hansen (1982) (see also Xiao et al. 2007, Eq. (2.4)), Fan et al. (2012) discuss a BCGMM approach, where one estimates β by minimizing the quadratic form

$$Q = \psi' C \psi \tag{45}$$

for a suitable $p\times p$, positive definite matrix C, with $C = [\mathrm{cov}(\psi)]^{-1}$ as an optimal choice. In (45), ψ is an unbiased moment function given by (41). Note that since the computation of the $\mathrm{cov}(\psi)$ matrix requires the formulas for the third and fourth order moments of $\{x_{itu}\}$ as well, one cannot compute such a covariance matrix provided the measurement error distributions for the model (34) are known. However, as argued in the independent setup, it is reasonable for many practical situations that measurement errors are normally distributed. As far as their covariance structure is concerned we assume that they follow the structure in (36). Based on this normality assumption for the measurement error, we reexpress the C matrix in (45) as C_N and obtain the BCGMM estimator for β by solving the estimating equation

$$\frac{\partial \psi'}{\partial \beta} C_N \psi = 0, \tag{46}$$

where by (41)

$$\frac{\partial \psi'}{\partial \beta} = [\sum_{i=1}^{K} \{X_i' X_i - T_i \Lambda(\sigma_1^2, \ldots, \sigma_u^2, \ldots, \sigma_p^2)\}].$$

It then follows that the solution of (46), i.e., the BCGMM estimator of β is given by

$$\hat{\beta}_{BCGMM} = \left[\frac{\partial \psi'}{\partial \beta} C_N \frac{\partial \psi}{\partial \beta'}\right]^{-1} \left[\frac{\partial \psi'}{\partial \beta} C_N \sum_{i=1}^{K} X_i' y_i\right], \tag{47}$$

with its variance as

$$\begin{aligned} \mathrm{var}(\hat{\beta}_{BCGMM}) &= \left[\frac{\partial \psi'}{\partial \beta} C_N \frac{\partial \psi}{\partial \beta'}\right]^{-1} \\ &\times \left[\frac{\partial \psi'}{\partial \beta} C_N \sum_{i=1}^{K} \mathrm{var}(X_i' y_i) C_N \frac{\partial \psi}{\partial \beta}\right] \left[\frac{\partial \psi'}{\partial \beta} C_N \frac{\partial \psi}{\partial \beta'}\right]^{-1}. \end{aligned} \tag{48}$$

Construction of C_N Matrix

Note that $C_N = [\mathrm{var}(\psi)]^{-1}$ under the assumption that the measurement errors $\{v_{itu}\}$ and hence observed covariates $\{x_{itu}\}$ are normally distributed. For the purpose, we first compute $\mathrm{var}(\psi)$ as follows where ψ is given as in (41):

$$\begin{aligned} \mathrm{var}(\psi) &= \mathrm{var}[\sum_{i=1}^{K} X_i' y_i - \{\sum_{i=1}^{K} X_i' X_i\}\beta] \\ &= \sum_{i=1}^{K} [\mathrm{var}\{X_i' y_i - X_i' X_i \beta\}] \\ &= \sum_{i=1}^{K} [\mathrm{var}\{X_i' y_i\} + \mathrm{var}\{X_i' X_i \beta\} - 2\mathrm{cov}\{X_i' y_i, X_i' X_i \beta\}], \end{aligned} \tag{49}$$

which, in addition to the formulas for the covariance matrix of y_i, requires the formulas for all possible second, third, and fourth order moments of $\{x_{itu}\}$. The following two lemmas will be useful in computing the covariance matrices in (49).

Lemma 3.1. *Under the measurement error model (34)–(37), let* $var(Y_i) = \Sigma_i = w_i^2\sigma_\gamma^2 J_{T_i} + \sigma_\varepsilon^2 R_i = (\sigma_{i\ell m})$ *denote the* $T_i \times T_i$ *covariance matrix of the response vector* y_i*, where* J_{T_i} *is the* $T_i \times T_i$ *unit matrix and* $R_i = (\rho_{i\ell m})$ *is the* $T_i \times T_i$ *correlation matrix for the components of* ε_i *such as for AR(1) process* $\rho_{i\ell m} = \rho^{|\ell - m|}$*,* ρ *being the correlation index parameter. It then follows that*

$$\sigma_{i\ell m} = cov[Y_{i\ell}, Y_{im}] = \begin{cases} \sigma_i^{*2} & for\ \ell = m = 1,\ldots,T_i \\ \sigma_i^{*2}[\theta_i + (1-\theta_i)\rho_{i\ell m}] & for\ \ell \neq m, \end{cases} \tag{50}$$

where $\sigma_i^{*2} = w_i^2\sigma_\gamma^2 + \sigma_\varepsilon^2$*, and* $\theta_i = \frac{w_i^2\sigma_\gamma^2}{w_i^2\sigma_\gamma^2 + \sigma_\varepsilon^2}$.

Lemma 3.2. *Let* $\Delta_{i(u)} = (\delta_{i(uu)\ell m})$ *denote the* $T_i \times T_i$ *covariance matrix of* $x_{i(u)} = (x_{i1u}, \ldots, x_{itu}, \ldots, x_{iT_iu})'$*, where by (36)*

$$cov[x_{i\ell u}, x_{imu}] = \delta_{i(uu)\ell m} = \begin{cases} \sigma_u^2 = \tilde{\sigma}_u^2 + \sigma_a^2 & for\ \ell = m = 1,\ldots,T_i \\ \tilde{\sigma}_u^2 = \phi_u \sigma_u^2 & for\ \ell \neq m. \end{cases} \tag{51}$$

Under the assumption that $v_{i(u)}$ or $x_{i(u)}$ in (38) follows the T_i-dimensional normal distribution with covariance matrix $\Delta_{i(u)}$ as in Lemma 3.2, the third and fourth order corrected product moments for the components of $x_{i(u)}$ are given by

$$\eta_{i\ell mt} = E\left[(x_{i\ell u} - z_{i\ell u})(x_{imu} - z_{imu})(x_{itu} - z_{itu})\right] = 0, \tag{52}$$

and

$$\begin{aligned} \xi_{i\ell mst} &= E\left[(x_{i\ell u} - z_{i\ell u})(x_{imu} - z_{imu})(x_{isu} - z_{isu})(x_{itu} - z_{itu})\right] \\ &= \delta_{i(uu)\ell m}\delta_{i(uu)st} + \delta_{i(uu)\ell s}\delta_{i(uu)mt} + \delta_{i(uu)\ell t}\delta_{i(uu)ms}, \end{aligned} \tag{53}$$

respectively.

By applying the Lemmas 3.1 and 3.2, one may compute the covariance matrices in (49). For example, by writing the $p \times 1$ vector $X_i'y_i$ as $X_i'y_i = [\sum_{t=1}^{T_i} x_{it1}y_{it}, \ldots, \sum_{t=1}^{T_i} x_{itu}y_{it}, \ldots, \sum_{t=1}^{T_i} x_{itp}y_{it}]'$, one may compute its $p \times p$ covariance matrix as

$$\mathrm{var}[X_i'y_i] = \begin{cases} \mathrm{var}[\sum_{t=1}^{T_i} x_{itu}y_{it}] & \text{for } u = 1,\ldots,p \\ \mathrm{cov}[\sum_{t=1}^{T_i} x_{itu}y_{it}, \sum_{t=1}^{T_i} x_{itr}y_{it}] & \text{for } u \neq r,\ u,r = 1,\ldots,p, \end{cases} \tag{54}$$

where

$$\begin{aligned}
\mathrm{var}[\sum_{t=1}^{T_i} x_{itu}y_{it}] &= \mathrm{var}_y E_x[\sum_{t=1}^{T_i} x_{itu}y_{it}|y] + E_y \mathrm{var}_x[\sum_{t=1}^{T_i} x_{itu}y_{it}|y] \\
&= \mathrm{var}_y[\sum_{t=1}^{T_i} z_{itu}y_{it}] + E_y[\sum_{t=1}^{T_i}\sum_{m=1}^{T_i} \delta_{i(uu)tm}y_{it}y_{im}] \\
&= \sum_{t=1}^{T_i}\sum_{m=1}^{T_i} z_{itu}z_{imu}\sigma_{itm} \\
&+ \sum_{t=1}^{T_i}\sum_{m=1}^{T_i} \delta_{i(uu)tm}[\sigma_{itm} + \beta' z_{it}z'_{im}\beta]
\end{aligned} \tag{55}$$

and

$$\begin{aligned}
\mathrm{cov}[\sum_{t=1}^{T_i} x_{itu}y_{it}, \sum_{t=1}^{T_i} x_{itr}y_{it}] &= \mathrm{cov}_y[E_x\{\sum_{t=1}^{T_i} x_{itu}y_{it}|y\}, E_x\{\sum_{t=1}^{T_i} x_{itr}y_{it}|y\}] \\
&+ E_y \mathrm{cov}_x[\{\sum_{t=1}^{T_i} x_{itu}y_{it}, \sum_{t=1}^{T_i} x_{itr}y_{it}\}|y] \\
&= \mathrm{cov}_y[\sum_{t=1}^{T_i} z_{itu}y_{it}, \sum_{t=1}^{T_i} z_{itr}y_{it}] + E_y[\sum_{t=1}^{T_i}\sum_{m=1}^{T_i} y_{it}y_{im}\delta_{i(ur)tm}|y] \\
&= \sum_{t=1}^{T_i}\sum_{m=1}^{T_i} z_{itu}z_{imu}\sigma_{itm},
\end{aligned} \tag{56}$$

because two covariates ($u \neq r$) are always independent, i.e., $\delta_{i(ur)tm} = 0$ irrespective of the time points of their measurements. The remaining two covariance matrices in (49) may be computed similarly.

3.1.2 BCGQL Estimation for Regression Effects

In this approach, by pretending that the model (37)–(38) does not contain any measurement error, we first write the naive generalized quasi-likelihood (NGQL) estimating equation

$$\Psi^* = \sum_{i=1}^{K} X_i' \Sigma_i^{-1}[y_i - X_i\beta] = 0, \tag{57}$$

where Σ_i is the covariance matrix of y_i. Note that the estimating function Ψ^* is similar but different than the MM estimating function ψ^* given in (39). The solution of (57) yields an NGQL estimator for β as

$$\hat{\beta}_{NGQL} = \left[\sum_{i=1}^{K} X_i' \Sigma_i^{-1} X_i\right]^{-1} \sum_{i=1}^{K} X_i' \Sigma_i^{-1} y_i, \tag{58}$$

which is also familiar as the generalized least squares (GLS) estimator for β. Note that this NGQL estimator $\hat{\beta}_{NGQL} = \hat{\beta}_{GLS}$ is not consistent for β. This is because, Ψ^* in (57) is not an unbiased function under the true model (37)–(38), that is, $E(\Psi^*) \neq 0$.

Now to obtain an unbiased and hence consistent estimator for β, it is necessary to consider an unbiased GQL function under the present model. This would be a generalization of finding the moment conditions for MM studied by Wansbeek (2001) to the actual correlation setup for the panel data.

In order to obtain an unbiased function from the Ψ_i^* function in (57), we first note that in probability $(\rightarrow_p)$, $X_i' \Sigma_i^{-1} X_i$ converges as

$$X_i' \Sigma_i^{-1} X_i \rightarrow_p [Z_i' \Sigma_i^{-1} Z_i + \text{diag}\{tr(\Sigma_i^{-1} \Delta_{i(1)}), \ldots, tr(\Sigma_i^{-1} \Delta_{i(p)})\}], \tag{59}$$

where for $u = 1, \ldots, p$, $\Delta_{i(u)}$ is given in Lemma 3.2 (see also (36)). Now by using (59), we may modify (57) to obtain an unbiased estimating function given by

$$\begin{aligned} \Psi = & \sum_{i=1}^{K} X_i' \Sigma_i^{-1} y_i - [\sum_{i=1}^{K} \{X_i' \Sigma_i^{-1} X_i \\ & - \text{diag}[tr(\Sigma_i^{-1} \Delta_{i(1)}), \ldots, \ldots, tr(\Sigma_i^{-1} \Delta_{i(p)})]\}]\beta, \end{aligned} \tag{60}$$

that is, $E[\Psi] = 0$ under the model (37)–(38). Consequently, for known measurement error variances, it is now clear from (60) that one may obtain the BCGQL estimator given by

$$\begin{aligned} \hat{\beta}_{BCGQL} = & \left[\sum_{i=1}^{K} \{X_i' \Sigma_i^{-1} X_i \right. \\ & \left. - \text{diag}[tr(\Sigma_i^{-1} \Delta_{i(1)}), \ldots, tr(\Sigma_i^{-1} \Delta_{i(p)})]\}\right]^{-1} \sum_{i=1}^{K} X_i' \Sigma_i^{-1} y_i, \end{aligned} \tag{61}$$

which is consistent for β. Also, this BCGQL estimator would be more efficient than the BCMM estimator given in (42). This is because, unlike the BCMM estimator, the BCGQL estimator is constructed by using the covariance matrix Σ_i of y_i as the weight matrix in the estimating equation. In fact, in view of the comparative results for GQL and GMM estimators in the linear panel data setup (Rao et al. 2012), this BCGQL estimator (61) may also be more efficient than the BCGMM estimator obtained in (47).

Note that the asymptotic variance of $\hat{\beta}_{BCGQL}$ may be estimated as follows. By writing

$$P_{i,x} = X_i'\Sigma_i^{-1}X_i - \text{diag}[tr(\Sigma_i^{-1}\Delta_{i(1)}),\ldots,tr(\Sigma_i^{-1}\Delta_{i(p)})],$$

and because

$$E_yE_{x|y}[X_i'\Sigma_i^{-1}y_i] = [Z_i'\Sigma_i^{-1}Z_i]\beta$$

is estimated by $P_{i,x}\beta$, one obtains a moment estimator of var$[\hat{\beta}_{BCGQL}]$ as

$$\hat{\text{var}}[\hat{\beta}_{BCGQL}] = [\sum_{i=1}^{K} P_{i,x}]^{-1}\sum_{i=1}^{K}[X_i'\Sigma_i^{-1}y_i - P_{i,x}\hat{\beta}_{BCGQL}] \times [X_i'\Sigma_i^{-1}y_i - P_{i,x}\hat{\beta}_{BCGQL}]'[\sum_{i=1}^{K} P_{i,x}]^{-1}. \quad (62)$$

A Two-Stage BCGQL (BCGQL2) Estimation of β

Instead of solving the first stage estimating (60) for BCGQL estimator, similar to the BCGMM estimation (46), Fan et al. (2012) have solved the second stage estimating equation

$$\sum_{i=1}^{K}\left[\frac{\partial \Psi_i'}{\partial \beta}D_{iN}^{-1}\Psi_i\right] = 0, \quad (63)$$

where, for $\Psi = \sum_{i=1}^{K}\Psi_i$ (60), with $\Psi_i = X_i'\Sigma_i^{-1}y_i - P_{i,x}\beta$,

$$D_{iN} = \text{cov}[\Psi_i]$$

under the assumption of multivariate normality for the random covariates $x_{i(u)} = [x_{i1u},\ldots,x_{itu},\ldots,x_{iT_iu}]'$. It then follows that the solution of (63), i.e., the two stage BCGQL BCGQL2 estimator of β is given by

$$\hat{\beta}_{BCGQL2} = \left[\sum_{i=1}^{K}\frac{\partial \Psi_i'}{\partial \beta}D_{iN}^{-1}\frac{\partial \Psi_i}{\partial \beta'}\right]^{-1}\sum_{i=1}^{K}\left[\frac{\partial \Psi_i'}{\partial \beta}D_{iN}^{-1}X_i'\Sigma_i^{-1}y_i\right], \quad (64)$$

with its variance as

$$\text{var}[\hat{\beta}_{BCGQL2}] = \left[\sum_{i=1}^{K}\frac{\partial \Psi_i'}{\partial \beta}D_{iN}^{-1}\frac{\partial \Psi_i}{\partial \beta'}\right]^{-1}\sum_{i=1}^{K}\left[\frac{\partial \Psi_i'}{\partial \beta}D_{iN}^{-1}\text{var}(X_i'\Sigma_i^{-1}y_i)D_{iN}^{-1}\frac{\partial \Psi_i}{\partial \beta'}\right] \times \left[\sum_{i=1}^{K}\frac{\partial \Psi_i'}{\partial \beta}D_{iN}^{-1}\frac{\partial \Psi_i}{\partial \beta'}\right]^{-1}, \quad (65)$$

where $\text{var}(X_i' \Sigma_i^{-1} y_i)$ may be computed similar to that of $\text{var}(X_i' y_i)$ in (54). Further, the covariance matrix D_{iN} can be computed in the fashion similar to that of C_N in Sect. 3.1.1.

3.2 *Longitudinal Count Data Models with Measurement Error in Covariates*

When compared to the linear measurement error model for correlated data (34), in the present case, one has to deal with a correlation model for repeated count data $y_{i1}, \ldots, y_{it}, \ldots, y_{iT}$, where y_{it} marginally, as in Sect. 2.1.3, follows a count data distribution such as Poisson distribution with mean $\mu_{iz} = \exp(z_{it}' \beta)$. However, as far as the measurement errors are concerned, they arise through the same relationship $x_{it} = z_{it} + v_{it}$, as in the correlated linear model setup.

For the correlation structure for count data, we consider a practically important AR(1) model following Sutradhar (2010) (see also Sutradhar 2011). The model is written such that conditional on the true covariate vector z_{it}, the marginal means and variances satisfy the Poisson distribution-based relationship

$$E(Y_{it}|z_{it}) = \text{var}(Y_{it}|z_{it}) = \mu_{iz,t} = \exp(z_{it}' \beta), \tag{66}$$

for all $t = 1, \ldots, T$. Note that these two moments are nonstationary as they depend on the time-dependent covariates z_{it}. As far as the AR(1) correlations among repeated counts are concerned, they arise from the following dynamic relationships:

$$\begin{aligned} y_{i1} &\sim Poi(\mu_{iz,1}) \\ y_{it} &= \rho * y_{i,t-1} + d_{it} = \sum_{j=1}^{y_{i,t-1}} b_j(\rho) + d_{it},\ t = 2, \ldots, T, \end{aligned} \tag{67}$$

where for given counts $y_{i,t-1}$ at time point $t-1$, $\sum_{j=1}^{y_{i,t-1}} b_j(\rho)$ denotes the sum of $y_{i,t-1}$ independent binary values with $Pr[b_j(\rho) = 1] = \rho$ and $Pr[b_j(\rho) = 0] = 1 - \rho$, ρ being the longitudinal correlation index parameter. Now under the assumptions that $y_{i,t-1} \sim Poi(\mu_{iz,t-1})$, $d_{it} \sim Poi(\mu_{iz,t} - \rho \mu_{iz,t-1})$, for $t = 2, \ldots, T$, and d_{it} and $y_{i,t-1}$ are independent, it follows from (67) that y_{ir} and y_{it} have nonstationary lag $t - r$ correlations given by

$$\text{corr}(Y_{ir}, Y_{it}) = c_{iz,rt} = \begin{cases} \rho^{t-r} [\mu_{iz,r} \mu_{iz,t}^{-1}]^{\frac{1}{2}}, \text{ for } r < t \\ \\ \rho^{r-t} [\mu_{iz,t} \mu_{ir,t}^{-1}]^{\frac{1}{2}}, \text{ for } r > t. \end{cases} \tag{68}$$

Note that the lag correlations given by (68) are nonstationary by nature as they depend on the time-dependent variances through the covariates z_{it} and z_{iu}, whereas in the stationary case when $z_{it} = z_{iu}$ for all $u \neq t$, they reduce to ρ^{t-u}, a Gaussian-type $AR(1)$ correlation structure satisfying (43). Further note that because $E[Y_{it}] = \mu_{iz,t} = \exp(z'_{it}\beta)$ by (67), the regression parameters vector β measures the effects of z_{it} on y_{it} for all $t = 1,\ldots,T$. But in the present setup, z_{it}'s are unobservable, and hence they cannot be used to estimate β. Instead, one must use the observed covariates x_{it}, which are, however, subject to measurement error explained through the relationship

$$x_{it} = z_{it} + v_{it},$$

with $v_{it} = (v_{it1},\ldots,v_{itu},\ldots,v_{itp})'$ satisfying the following assumptions:

1. $v_{it} \sim N(0, \Lambda = \text{diag}[\sigma_1^2,\ldots,\sigma_u^2,\ldots,\sigma_p^2])$ for all $t = 1,\ldots,T$.
2. Also,

$$\text{corr}[v_{iru}, v_{itm}] = \begin{cases} \phi_u, \text{ for } m = u; r \neq t, r,t = 1,\ldots,T \\ 0, \quad \text{for } m \neq u; r,t = 1,\ldots,T. \end{cases}$$

These two assumptions imply that the uth covariate has the measurement error variance σ_u^2 for $u = 1,\ldots,p$, at a given time t for all $t = 1,\ldots,T$. Also, the covariate values for the same uth covariate recorded at two different times r and t are equally correlated with correlation ϕ_u for all $r \neq t$. This correlation assumption is similar to that of the time-dependent covariates considered by Wang et al. (1996). One may also consider other correlation structures such as AR(1) among the repeated values for the same covariate. More specifically, the above assumptions is equivalent to writing

$$\begin{pmatrix} x_{ir} \\ x_{it} \end{pmatrix} \sim N_{2p}\left[\begin{pmatrix} z_{ir} \\ z_{it} \end{pmatrix}, \begin{pmatrix} \Lambda & \Lambda_\phi \\ \Lambda_\phi & \Lambda \end{pmatrix}\right], \tag{69}$$

where $\Lambda_\phi = \text{cov}(v_{ir}, v'_{it}) = \text{diag}[\phi_1\sigma_1^2,\ldots,\phi_u\sigma_u^2,\ldots,\phi_p\sigma_p^2]$.

3.2.1 Bias Corrected GQL Estimation

Suppose that by using the observed covariates one writes a NGQL estimating equation given by

$$\sum_{i=1}^{K} \frac{\partial \mu'_{ix}}{\partial \beta} \Sigma_{ix}^{-1}(y_i - \mu_{ix}) = 0, \tag{70}$$

where

$$\mu_{ix,t} = \mu_{iz,t}|_{z_{it}=x_{it}}, \text{ and } \Sigma_{ix} = (\sigma_{ix,rt}) = \Sigma_{iz}|_{z=x} = ([c_{iz,rt}\sqrt{\mu_{iz,r}\mu_{iz,t}}]|_{z_{it}=x_{it}}).$$

But, this NGQL estimating (70) will yield biased and hence inconsistent estimate for β. This is because the NGQL estimating function in the left-hand side of the (70) is not unbiased for the true covariates-based GQL estimating function. That is,

$$E_x\left[\sum_{i=1}^{K}\frac{\partial\mu'_{ix}}{\partial\beta}\Sigma_{ix}^{-1}(y_i-\mu_{ix})\right]\neq\sum_{i=1}^{K}\frac{\partial\mu'_{iz}}{\partial\beta}\Sigma_{iz}^{-1}(y_i-\mu_{iz}). \tag{71}$$

Recently, Sutradhar et al. (2012) have proposed a bias correction to the NGQL estimating function and developed a BCGQL estimating function which is unbiased for the true covariates-based estimating function $\sum_{i=1}^{K}\frac{\partial\mu'_{iz}}{\partial\beta}\Sigma_{iz}^{-1}(y_i-\mu_{iz})$. This provides the BCGQL estimating equation as

$$\begin{aligned}g_x(x,\beta,\rho,\Lambda,\phi_1,\ldots,\phi_p|y)=\sum_{i=1}^{K}&\big[\{M_{1\phi}X'_i-M_{1\phi}B_{1\phi}(\beta\otimes 1'_T)\}\\&\times\{A_{ix}^{\frac{1}{2}}\tilde{Q}_{ix}(\rho)A_{ix}^{-\frac{1}{2}}\}y_i-\{M_{2\phi}X'_i-M_{2\phi}B_{2\phi}(\beta\otimes 1'_T)\}\\&\times\{A_{ix}^{\frac{1}{2}}\tilde{Q}_{ix}(\rho)A_{ix}^{-\frac{1}{2}}\}\mu_{ix}\Big]=0,\end{aligned} \tag{72}$$

where $y_i=(y_{i1},\ldots,y_{it},\ldots,y_{iT})'$ is the $T\times 1$ vector of repeated count responses, with its mean $\mu_{ix}=\exp(x'_{it}\beta)$ in observed covariates; $X'_i=(x_{i1},\ldots,x_{it},\ldots,x_{iT})$ is the $p\times T$ observed covariates matrix; $A_{ix}=\text{diag}[\mu_{ix,1},\ldots,\mu_{ix,t},\ldots,\mu_{ix,T}]$; $1'_T=(1\ldots,1)$ is the $1\times T$ vector of unity, $\otimes$ denotes the well-known Kronecker or direct product, so that $\beta\otimes 1'_T$ is the $p\times T$ matrix containing $\beta=(\beta_1\ldots,\beta_p)'$ in each column of the matrix; and

$$\begin{aligned}B_{1\phi}&=\frac{1}{2}(\Lambda-\Lambda_\phi),B_{2\phi}=\frac{1}{2}(\Lambda+\Lambda_\phi),\\M_{1\phi}&=\text{diag}[m_1,\ldots,m_1]:p\times p;M_{2\phi}=\text{diag}[m_2,\ldots,m_2]:p\times p,\end{aligned}$$

with

$$m_1=\exp\{-\frac{1}{4}\beta'(\Lambda-\Lambda_\phi)\beta\},\text{ and }m_2=\exp\{-\frac{1}{4}\beta'(\Lambda+\Lambda_\phi)\beta\}.$$

Furthermore, in (72), $\tilde{Q}_{ix}(\rho)=\tilde{C}_{ix}^{-1}(\rho)$, with $C_{ix}(\rho)=(\tilde{c}_{ix,rt})$ as an unbiased correlation matrix for the AR(1) correlation matrix in true covariates, namely $C_{iz}(\rho)=(c_{iz,rt})$. The formula for the (r,t)-th element of the unbiased correlation matrix is given by

$$\tilde{c}_{ix,rt}=\rho^{t-r}[\exp(x_{ir}-x_{it})'\frac{\beta}{2}-\frac{1}{4}\beta'(\Lambda-\Lambda_\phi)\beta] \tag{73}$$

satisfying

$$E_x[\tilde{c}_{ix,rt}] = c_{iz,rt} = \rho^{t-r}\{\mu_{iz,r}/\mu_{iz,t}\}^{\frac{1}{2}}. \tag{74}$$

We reexpress the BCGQL estimating (72) as

$$g_x(x,\beta,\rho,\Lambda,\phi_1,\ldots,\phi_p|y) = \sum_{i=1}^{K}[D_{i1}(x)y_i - D_{i2}(x)\mu_{ix}] = 0, \tag{75}$$

where $D_{i1}(x)$ and $D_{i2}(x)$ are $p \times T$ matrix functions of observed covariates. Let $\hat{\beta}_{BCGQL}$ be the solution of (73). Now conditional on the observed covariates x_i, solving this equation for β is equivalent to use the iterative equation,

$$\begin{aligned}\hat{\beta}_{BCGQL}(r+1) \;=\; \hat{\beta}_{BCGQL}(r) \;-\; & \left[\{\hat{E}_y(\frac{\partial g_x(x,\beta,\rho,\Lambda,\phi_1,\ldots,\phi_p|y)}{\partial \beta'})\}^{-1}\right.\\ & \left.\times \sum_{i=1}^{K}\{D_{i1}(x)y_i - D_{i2}(x)\mu_{ix}\}\right]_{\hat{\beta}_{BCGQL}(r)}, \end{aligned}\tag{76}$$

where $\hat{\beta}_{BCGQL}(r)$ denote the β estimate at the r-th iteration. Note that under the true model involving covariates z_i,

$$y_i \sim [\mu_{iz}, \Sigma_{iz} = (\sigma_{itm})],$$

where for $t < m$, $\sigma_{itm} = \text{cov}(y_{it}, y_{im}) = (\rho^{m-t}\mu_{iz,t})$, and $y_1,\ldots,y_i,\ldots,y_K$ are T-dimensional independent vectors. Thus, under some mild moment conditions, by using Lindeberg-Feller central limit theorem (Amemiya 1985, Theorem 3.3.6, p. 92), it follows from (76) that as $K \to \infty$, $\hat{\beta}_{BCGQL} \sim N_p(\beta, V^*)$, where

$$\begin{aligned} V^* = & \left[\hat{E}_y(\frac{\partial g_x(x,\beta,\rho,\Lambda,\phi_1,\ldots,\phi_p|y)}{\partial \beta'})\right]^{-1}\sum_{i=1}^{K} D_{i1}(x)\Sigma_{iz}D'_{i1}(x)\\ & \times \left[\hat{E}_y(\frac{\partial g_x(x,\beta,\rho,\Lambda,\phi_1,\ldots,\phi_p|y)}{\partial \beta'})\right]^{-1}, \end{aligned}\tag{77}$$

which may be consistently estimated by using the moment estimate for Σ_{iz} in (77). For this moment estimate, when ρ is known, one estimates the (t,m)th element $(t < m)$ of this matrix by using

$$\hat{\sigma}_{itm} = \rho^{m-t}\hat{\mu}_{iz,t} = \rho^{m-t}\mu_{ix,t} = \rho^{m-t}[exp\,(x'_{it}\beta - \frac{1}{2}\beta'\Lambda\beta)]_{|\beta=\hat{\beta}_{BCGQL}}.$$

3.2.2 A Simulation-Based Numerical Illustration

We consider two $(p = 2)$ covariates with measurement error variances σ_1^2 and σ_2^2, respectively. It is expected that these measurement error variances are small in practice. We, however, consider them ranging from 0.1 to 0.3 for σ_1^2; and from 0.1 to 0.8 for σ_2^2. Note that these ranges are quite large, whereas in the independence ($\rho = 0.0$) setup and for one covariate case ($p = 1$), Nakamura (1990) examined the performance of the bias corrected score estimator for σ_1^2 up to 0.1. We have also included the independence case but for larger measurement error variances as compared to that of Nakamura (1990).

The main purpose of this section is to illustrate the performance of the proposed BCGQL estimator obtained from (72) (see also (76)) when AR(1) count responses are generated with some positive correlation index, where the covariates are subject to measurement error with variances σ_1^2 and σ_2^2 for the two covariate case. We consider $\rho = 0.5$. As mentioned above we also include the independence case ($\rho = 0.0$). In all these cases, we first show that if measurement errors are not adjusted, the so-called NGQL approach (70) produces highly biased estimates and the correction by using BCGQL approach performs well.
We consider 500 simulations and generate correlated count data following the AR(1) Poisson model (67)–(68) for $K = 100$ individuals over a period of $T = 4$ time points. The true covariates z_{it1} and z_{it2} were generated as

$$z_{it1} \overset{iid}{\sim} N(0,1), \text{ and } z_{it2} \overset{iid}{\sim} \frac{\chi_4^2 - 4}{\sqrt{8}}$$

with their effects $\beta_1 = 0.3$ and $\beta_2 = 0.1$, respectively, on the repeated response y_{it}. Note that even though the true covariates z_{itu} are generated following the standard normal and standardized χ^2 distribution, these values are treated as fixed under all simulations. Further note that these true covariates are unobserved in the present setup, instead x_{it1} and x_{it2} are observed. We generate the observed covariates following the relationship

$$x_{itu} = z_{itu} + v_{itu},\ u = 1, \ldots, p,$$

where v_{itu}'s are generated by using a random effect model given by

$$v_{itu} = k_u + e_{itu}, \text{ with } k_u \overset{iid}{\sim} N(0, \sigma_u^{*2}) \text{ and } e_{itu} \overset{iid}{\sim} N(0, \sigma_e^{*2}), \tag{78}$$

yielding

$$\begin{aligned} \text{var}(v_{itu}) &= \sigma_u^{*2} + \sigma_e^{*2} = \sigma_u^2 \\ \text{corr}(v_{itu}, v_{iru}) &= \frac{\sigma_u^{*2}}{\sigma_u^{*2} + \sigma_e^{*2}} = \phi_u, \end{aligned} \tag{79}$$

where σ_u^2 is the measurement error variance for the uth ($u = 1, 2$). Notice that ϕ_u represents the equi-correlations among the repeated values of the same covariate. Thus, $\phi_u = 1$ would represent the situation where covariate values are same over

Table 1 Simulated regression estimates, and their standard errors (SSEs), with true regression parameters $\beta_1 = 0.3,\ \beta_2 = 0.1$, under AR(1) count data model for selected response correlation ρ, measurement error variances $\sigma_1^2,\ \sigma_2^2$, with $K = 100$; $T = 4$; and measurement error correlations ϕ_1 and ϕ_2; and true covariate values $Z_1 \sim N(0,1)$ and $Z_2 \sim \frac{\chi_4^2 - 4}{\sqrt{8}}$

					Estimates			
					NGQL		BCGQL	
ρ	ϕ_1	ϕ_2	σ_1^2	σ_2^2	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_1$	$\hat{\beta}_2$
0.0	1.0	1.0	0.1	0.3	0.2683	0.0849	0.3025	0.1026
					(0.0501)	(0.0380)	(0.0583)	(0.0448)
			0.3	0.3	0.2274	0.0800	0.3052	0.1033
					(0.0462)	(0.0383)	(0.0680)	(0.0468)
			0.3	0.8	0.2221	0.0652	0.3068	0.1052
					(0.0450)	(0.0338)	(0.0701)	(0.0525)
0.5	1.0	1.0	0.1	0.3	0.2688	0.0900	0.3036	0.1085
					(0.0689)	(0.0535)	(0.0803)	(0.0634)
			0.3	0.3	0.2286	0.0854	0.3069	0.1099
					(0.0640)	(0.0536)	(0.0920)	(0.0662)
			0.3	0.8	0.2232	0.0707	0.3100	0.1138
					(0.0623)	(0.0493)	(0.0979)	(0.0786)
0.0	0.25	0.50	0.1	0.3	0.2680	0.0842	0.2772	0.0914
					(0.0502)	(0.0372)	(0.0523)	(0.0401)
			0.3	0.3	0.2270	0.0793	0.2435	0.0871
					(0.0461)	(0.0372)	(0.0499)	(0.0402)
			0.3	0.8	0.2218	0.0643	0.2404	0.0779
					(0.0450)	(0.0329)	(0.0495)	(0.0392)
0.5	0.25	0.50	0.1	0.3	0.2338	0.0751	0.2800	0.1050
					(0.0617)	(0.0484)	(0.0764)	(0.0652)
			0.3	0.3	0.1687	0.0683	0.2525	0.1148
					(0.0521)	(0.0476)	(0.0943)	(0.1065)
			0.3	0.8	0.1642	0.0492	0.2488	0.1097
					(0.0506)	(0.0406)	(0.1314)	(0.1579)

time and in this case we consider $x_{itu} = k_u$, which yields $\text{corr}(x_{itu}, x_{iru}) = \phi_u = 1.0$. But it does not mean though responses are same, rather responses follow the AR(1) correlation structure. In the simulation study, we, however, consider both situations where $\phi_u = 1.0$ for $u = 1,2$, in one situation; and in the other situation $\phi_1 = 0.25$ and $\phi_2 = 0.5$.

The simulated estimates along with their standard errors are presented in Table 1 for all selected values of the parameters. As expected, the NGQL estimates appear to be highly biased. For example, when $\phi_1 = \phi_2 = 1.0$, the response correlation index is 0.5, and measure error variances are $\sigma_1^2 = 0.3,\ \sigma_2^2 = 0.8$, the NGQL approach produces the estimates of $\beta_1 = 0.3$ and $\beta_2 = 0.1$ as 0.22 and 0.07, whereas the BCGQL approach yields almost unbiased estimates as 0.31 and 0.11, respectively. When $\phi_1 = 0.25$ and $\phi_2 = 0.5$, for this set of large measurement error variances, the NGQL approach produces useless estimates, 0.16 for $\beta_1 = 0.30$, and 0.05 for $\beta_2 = 0.10$. In this case, BCGQL approach still appears to produce reasonably good estimates, 0.25 for $\beta_1 = 0.30$, and 0.11 for $\beta_2 = 0.10$. The BCGQL estimates for β_2 appears to be unbiased in all selected situations. As far as the independence case $\rho = 0.0$ is concerned, the BCGQL approach works similarly to the correlation case with $\rho = 0.5$.

Acknowledgment The author fondly acknowledges the stimulating discussion by the audience of the symposium and wishes to thank for their comments and suggestions.

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