Chapter 7
Stochastic Optimal Control Problems

This chapter contains some optimal control problems for systems described by stochastic functional and partial differential inclusions. The existence of optimal controls and optimal solutions for such systems is a consequence of the weak compactness of the set $\mathcal{X}_{sx}(F, G)$ of all weak solutions of (equivalence classes of) $SFI(F, G)$ satisfying an initial condition $x_s = x$, measurable selection theorems, and stochastic representation theorems for solutions of partial differential inclusions presented in Chap. 6. We begin with introductory remarks dealing with optimal control problems of systems described by stochastic differential equations.

1 Optimal Control Problems for Systems Described by Stochastic Differential Equations

Assume that the state of a dynamical system starting from a point $(s, x) \in \mathbb{R}^+ \times \mathbb{R}^d$ is described at time $t \geq s$ by a weak solution of the stochastic differential equation

$$\begin{cases} \mathrm{d}x_t = f(t, x_t, u_t)\mathrm{d}t + g(t, x_t, u_t)\mathrm{d}B_t \quad \text{a.s. for } t \geq s\,, \\ x_s = x \ \text{ a.s.}, \end{cases} \tag{1.1}$$

depending on a control process $u = (u_t)_{t\geq 0}$, where $f : \mathbb{R}^+ \times \mathbb{R}^d \times U \to \mathbb{R}^d$ and $g : \mathbb{R}^+ \times \mathbb{R}^d \times U \to \mathbb{R}^{d\times m}$ are given functions with $U \subset \mathbb{R}^k$. Given a domain $D \subset \mathbb{R}^d$ and an initial point $(s, x) \in \mathbb{R}^+ \times D$, a system $(\mathcal{P}_{\mathbb{F}}, u, X_{s,x}, B)$ consisting of a filtered probability space $\mathcal{P}_{\mathbb{F}} = (\Omega, \mathcal{F}, \mathbb{F}, P)$, $\mathbb{F}$-nonanticipative processes u and $X_{s,x}$, and an m-dimensional $\mathbb{F}$-Brownian motion $B = (B_t)_{t\geq 0}$ defined on $\mathcal{P}_{\mathbb{F}}$ satisfying (1.1) and such that $\tau_D^X < \infty$ a.s. is called an admissible system for the stochastic control system described by (1.1). As usual, τ_D^X denotes the first exit time of $X_{s,x}$ from the set D. For every $(s, x) \in \mathbb{R}^+ \times D$, we are also given a performance

M. Kisielewicz, *Stochastic Differential Inclusions and Applications*,
Springer Optimization and Its Applications 80, DOI 10.1007/978-1-4614-6756-4_7,

functional $J_D^{u,X}(s,x)$ defined for given functions $\Phi : \mathbb{R}^+ \times \mathbb{R}^d \times U \to \mathbb{R}$ and $K : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ and an admissible system $(\mathcal{P}_{\mathbb{F}}, u, X_{s,x}, B)$ by the formula

$$J_D^{u,X}(s,x) = E^{s,x}\left[\int_s^{\tau_D^X} \Phi(t, X_{s,x}(t), u(t))\mathrm{d}t + K\left(\tau_D^X, X_{s,x}\left(\tau_D^X\right)\right)\right], \tag{1.2}$$

where $E^{s,x}$ denotes the mean value operator with respect to the law $Q^{s,x}$ of $X_{s,x}$. For every admissible system $(\mathcal{P}_{\mathbb{F}}, u, X_{s,x}, B)$, a pair $(u, X_{s,x})$ is said to be an admissible pair for (1.1). The set of all admissible pairs for the control system (1.1) is denoted by $\Lambda_{fg}(s,x)$. For every $(u, X_{s,x}) \in \Lambda_{fg}(s,x)$, a process $X_{s,x}$ is called an admissible trajectory corresponding to an admissible control u. The performance functional $J_D^{u,X}(s,x)$ can be regarded as a functional defined on the set $\Lambda_{fg}(s,x)$.

An admissible pair $(\bar{u}, \bar{X}_{s,x}) \in \Lambda_{fg}(s,x)$ is said to be optimal for an optimal control problem (1.1) and (1.2) if $J_D^{\bar{u},\bar{X}}(s,x) = \sup\{J_D^{u,X}(s,x) : (u, X_{s,x}) \in \Lambda_{fg}(s,x)\}$ for every $(s,x) \in \mathbb{R}^+ \times D$. If $(\bar{u}, \bar{X}_{s,x})$ is the optimal pair for (1.1) and (1.2), then $\bar{u}$ is called the optimal control, and $\bar{X}_{s,x}$ the optimal trajectory for the optimal control problem described by (1.1) and (1.2). The function $v : \mathbb{R}^+ \times D \to \mathbb{R}$ defined by $v(s,x) = \sup\{J_D^{u,X}(s,x) : (u, X_{s,x}) \in \Lambda_{fg}(s,x)\}$ for every $(s,x) \in \mathbb{R}^+ \times D$ is said to be the value function associated to the optimal control problem (1.1) and (1.2). An admissible pair $(\bar{u}, \bar{X}_{s,x})$ is optimal if $v(s,x) = J_D^{\bar{u},\bar{X}}(s,x)$ for every initial condition $(s,x) \in \mathbb{R}^+ \times D$. The problem consisting in finding for each $(s,x) \in \mathbb{R}^+ \times D$ the number $v(s,x)$ for the optimal control problem (1.1) and (1.2) will be denoted by

$$\begin{cases} \mathrm{d}x_t = f(t, x_t, u_t)\mathrm{d}t + g(t, x_t, u_t)\mathrm{d}B_t \quad \text{a.s. for } t \geq s\,, \\ x_s = x \quad \text{a.s.}\,, \\ J_D^{u,X}(s,x) \xrightarrow{\Lambda_{fg}} \max\,. \end{cases} \tag{1.3}$$

Let us observe that if the optimal pair $(\bar{u}, \bar{X}_{s,x}) \in \Lambda_{fg}(s,x)$ exists and $(f(\cdot,\cdot,z), g(\cdot,\cdot,z))$ is such that $SDF(f(\cdot,\cdot,z), g(\cdot,\cdot,z))$ possesses for every fixed $z \in U$ a unique in law weak solution $X_{s,x}^z$ satisfying initial condition $X_{s,x}^z(s) = x$ a.s. for $(s,x) \in \mathbb{R}^+ \times \mathbb{R}^d$, then the standard approach to determine an optimal pair is to solve the Hamilton–Jacobi–Bellman (HJB) equation

$$\begin{cases} \sup_{z\in U}\{\Phi(s,x,z) + (\mathcal{A}_{fg}^z v)(s,x)\} = 0 \quad \text{for} \quad (s,x) \in \mathbb{R}^+ \times D\,, \\ v(s,x) = K(s,x) \quad \text{for} \quad (s,x) \in \mathbb{R}^+ \times \partial D, \end{cases}$$

where $\mathcal{A}_{fg}^z$ is the infinitesimal generator of a $(d+1)$-dimensional Itô diffusion defined, similarly as in Sect. 11 of Chap. 1, by $X_{s,x}^z$ for every fixed $z \in U$. If the above supremum is attained, i.e., if there exists an optimal control $\bar{u}(s,x)$, then

$$\begin{cases} \Phi(s,x,\bar{u}(s,x)) + (\mathcal{A}_{\bar{f}\bar{g}}v)(s,x) = 0 \quad \text{for} \quad (s,x) \in \mathbb{R}^+ \times D\,, \\ v(\bar{\tau}_D, \bar{X}_{s,x}) = K(\bar{\tau}_D, \bar{X}_{s,x}) \quad \text{for} \quad (s,x) \in \mathbb{R}^+ \times \partial D, \end{cases}$$

where $\bar{f}(s,x) = f(s,x,\bar{u}(s,x))$, $\bar{g}(s,x) = g(s,x,\bar{u}(s,x))$ for $(s,x) \in \mathbb{R}^+ \times D$, $\mathcal{A}_{\bar{f}\bar{g}}$ is an infinitesimal generator defined by a unique in law weak solution $\bar{X}_{s,x}$ of $SDE(\bar{f},\bar{g})$ satisfying an initial condition $\bar{X}_{s,x}(s) = x$ a.s. for $(s,x) \in \mathbb{R}^+ \times \mathbb{R}^d$, and $\bar{\tau}_D$ denotes the first exit time of $\bar{X}_{s,x}$ from the set D. Immediately from Theorem 5.5 of Chap. 6, it follows that if $v \in C_0^{1,2}([0,T] \times D, \mathbb{R})$ and $\bar{X}_{s,x}$ is such that $\bar{E}^{s,x}[\int_0^{\bar{\tau}_D} \Phi(t, \bar{X}_{s,x}(t))\mathrm{d}t] < \infty$ and there exists a number $C > 0$ such that $|v(t,x)| \le C(1 + \bar{E}^{s,x}[\int_0^{\bar{\tau}_D} \Phi(t, \bar{X}_{s,x}(t))\mathrm{d}t])$ for every $(s,x) \in (0,T) \times \mathbb{R}^d$, then

$$v(s,x) = \bar{E}^{s,x}[K(\bar{\tau}_D, \bar{X}_{s,x})] + \bar{E}^{s,x}\left[\int_0^{\bar{\tau}_D} \Phi(t, \bar{X}_{s,x}(t))\mathrm{d}t\right],$$

where $\bar{E}^{s,x}$ is a mean value operator taken with respect to a distribution of $\bar{X}_{s,x}$.

We shall consider now the optimal control problem (1.3) with continuous deterministic control parameters with values in a closed set $U \subset \mathbb{R}^k$ and a strong solution $X_{s,x}$ of (1.1) defined for a given m-dimensional $\mathbb{F}$-Brownian motion $B = (B_t)_{t\ge 0}$ on a given complete filtered probability space $\mathcal{P}_{\mathbb{F}} = (\Omega, \mathcal{F}, \mathbb{F}, P)$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\ge 0}$ satisfying the usual conditions. We consider a control system (1.1) with measurable functions $f : \mathbb{R}^+ \times \mathbb{R}^d \times U \to \mathbb{R}^d$ and $g : \mathbb{R}^+ \times \mathbb{R}^d \times U \to \mathbb{R}^{d\times m}$ satisfying the following conditions (H).

(H): There exist $k, m \in \mathbb{L}(\mathbb{R}^+, \mathbb{R}^+)$ such that

(i) $\max(|f(t,x,z)|, \|g(t,x,z)\|) \le m(t)$ for every $(t,x,z) \in \mathbb{R}^+ \times \mathbb{R}^d \times U$.
(ii) $\max(|f(t,x,z) - f(t,\bar{x},\bar{z})|^2, \|g(t,x,z) - g(t,\bar{x},\bar{z})\|^2) \le k(t)(|x-\bar{x}|^2 + |z-\bar{z}|^2)$ for every $t \ge 0$, $x, \bar{x} \in \mathbb{R}^d$, and $z, \bar{z} \in U$.
(iii) $g(t,x,z) \cdot g(t,x,z)^*$ is positive definite on $\mathbb{R}^+ \times \mathbb{R}^d$ for every fixed $z \in U$.

In what follows, by $\mathcal{U}_T$ we denote a nonempty compact subset of the Banach space $(C([0,T], \mathbb{R}^k), \|\cdot\|_T)$ with the supremum norm $\|\cdot\|_T$ such that $u_t \in U$ for every $u \in \mathcal{U}_T$ and $t \in [0,T]$.

Remark 1.1. Similarly as in the proof of Theorem 1.1 of Chap. 4, by an appropriate changing of the norm of the space $\mathcal{X}$ defined in the proof of Theorem 1.1 of Chap. 4, we can verify that if conditions (i) and (ii) of (H) are satisfied, then for every $(s,x) \in \mathbb{R}^+ \times \mathbb{R}^d$, $T > s$, a filtered probability space $\mathcal{P}_{\mathbb{F}} = (\Omega, \mathcal{F}, \mathbb{F}, P)$, an m-dimensional $\mathbb{F}$-Brownian motion $B = (B_t)_{t\ge 0}$, and $u \in \mathcal{U}_T$, there exists a unique strong solution $X_{s,x}^u$ of (1.1) defined on $[s,T] \times \Omega$.

Proof. Let $(s,x) \in \mathbb{R}^+ \times \mathbb{R}^d$, $T > s$, a filtered probability space $\mathcal{P}_{\mathbb{F}} = (\Omega, \mathcal{F}, \mathbb{F}, P)$, and an m-dimensional $\mathbb{F}$-Brownian motion $B = (B_t)_{t\ge 0}$ be given. Define, for fixed $u \in \mathcal{U}_T$, set-valued mappings F and G by taking $F(t,x) = \{f(t,x,u_t)\}$ and $G(t,x) = \{g(t,x,u_t)\}$ for $(t,x) \in [0,T] \times \mathbb{R}^d$. Let $X_{s,x}^{\alpha\beta}(t)$ be defined by $X^{\alpha\beta}_{s,x}(t) = x + \int_s^t \alpha_\tau \mathrm{d}\tau + \int_s^t \beta_\tau \mathrm{d}B_\tau$ for every $t \in [s, T]$ and $(\alpha, \beta) \in \mathcal{X}$. Similarly as in the proof of Theorem 1.1 of Chap. 4, we define on $\mathcal{X}$ an operator Q, which in the case of the above-defined multifunctions F and G, has the form $Q(\alpha, \beta) = \{(f(\cdot, X^{\alpha\beta}_{s,x}, u), g(\cdot, X^{\alpha\beta}_{s,x}, u))\}$ for every $(\alpha, \beta) \in \mathcal{X}$.

Let us define on $\mathbb{L}^2([0, T] \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^d)$ a family $\{\|\cdot\|_\lambda\}_{\lambda>0}$ of norms $\|\cdot\|_\lambda$ equivalent to the norm $|\cdot|$ of this space by setting $\|w\|^2_\lambda = \int_0^T \exp[-lK(t)]E|w_t|^2\mathrm{d}t$ for $w \in \mathbb{L}^2([0, T] \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^d)$, where $l = 1/\lambda^2$ and $K(t) = \int_0^t k(\tau)\mathrm{d}\tau$ with $k \in \mathbb{L}(\mathbb{R}^+, \mathbb{R}^+)$ satisfying conditions (H). For every $(\alpha, \beta), (\tilde{\alpha}, \tilde{\beta}) \in \mathcal{X}$, one gets

$$\begin{aligned}
&\|f(\cdot, X^{\alpha\beta}_{s,x}, u) - f(\cdot, X^{\tilde{\alpha}\tilde{\beta}}_{s,x}, u)\|^2_\lambda \\
&\quad = \int_0^T \exp[-lK(t)]E|f(t, X^{\alpha\beta}_{s,x}(t), u_t) - f(t, X^{\tilde{\alpha}\tilde{\beta}}_{s,x}(t), u_t)|^2\mathrm{d}t \\
&\quad \le \int_0^T k(t)\exp[-lK(t)]E|X^{\alpha\beta}_{s,x}(t) - X^{\tilde{\alpha}\tilde{\beta}}_{s,x}(t)|^2\mathrm{d}t.
\end{aligned}$$

Similarly as in the proof of Theorem 1.1 of Chap. 4, we get

$$\begin{aligned}
E[|X^{\alpha\beta}_{s,x}(t) - X^{\tilde{\alpha}\tilde{\beta}}_{s,x}(t)|^2] &= E\left|\int_s^t (\alpha_\tau - \tilde{\alpha}_\tau)\mathrm{d}\tau + \int_s^t (\beta_\tau - \tilde{\beta}_\tau)\mathrm{d}B\tau\right|^2 \\
&\le 2T\int_0^t E|\alpha_\tau - \tilde{\alpha}_\tau|^2\mathrm{d}\tau + 2\int_0^t E|\beta_\tau - \tilde{\beta}_\tau|^2\mathrm{d}\tau.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|f(\cdot, X^{\alpha\beta}_{s,x}, u) - f(\cdot, X^{\tilde{\alpha}\tilde{\beta}}_{s,x}, u)\|^2_\lambda \\
&\quad \le 2T\int_0^T\int_0^t k(t)\exp[-lK(t)]E|\alpha_\tau - \tilde{\alpha}_\tau|^2\mathrm{d}\tau\mathrm{d}t \\
&\quad\quad + 2\int_0^T\int_0^t k(t)\exp[-lK(t)]E|\beta_\tau - \tilde{\beta}_\tau|^2\mathrm{d}\tau\mathrm{d}t.
\end{aligned}$$

By interchanging the order of integration, we obtain

$$\begin{aligned}
&\int_0^T\int_0^t k(t)\exp[-lK(t)]E|\alpha_\tau - \tilde{\alpha}_\tau|^2\mathrm{d}\tau\mathrm{d}t = \int_0^T\int_\tau^T E|\alpha_\tau - \tilde{\alpha}_\tau|^2 k(t)\exp[-lK(t)]\mathrm{d}t\mathrm{d}\tau \\
&\quad = -\lambda^2 e^{-lK(T)}\int_0^T E|\alpha_\tau - \tilde{\alpha}_\tau|^2\mathrm{d}\tau + \lambda^2\int_0^T k(\tau)\exp[-lK(\tau)]E|\alpha_\tau - \tilde{\alpha}_\tau|^2\mathrm{d}\tau \\
&\quad \le \lambda^2\|\alpha - \tilde{\alpha}\|^2_\lambda.
\end{aligned}$$

In a similar way, we obtain

$$\int_0^T\int_0^t k(t)\exp[-lK(t)]E|\beta_\tau - \tilde{\beta}_\tau|^2\mathrm{d}\tau\mathrm{d}t \le \lambda^2\|\beta - \tilde{\beta}\|_\lambda.$$

Therefore,

$$\|f(\cdot, X_{s,x}^{\alpha\beta}, u) - f(\cdot, X_{s,x}^{\tilde{\alpha}\tilde{\beta}}, u)\|_\lambda^2 \leq 2\,\lambda^2(1+T)\,\|(\alpha,\beta) - (\tilde{\alpha} - \tilde{\beta}\|_\lambda^2\,,$$

where $\|(\alpha,\beta) - (\tilde{\alpha},\tilde{\beta})\|_\lambda = \max(\|\alpha - \tilde{\alpha}\|_\lambda, \|\beta - \tilde{\beta}\|_\lambda)$. In a similar way, for every $\lambda > 0$, we can define on the space $\mathbb{L}^2([0,T] \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ an equivalent norm, denoted again by $\|\cdot\|_\lambda$, and get

$$\|g(\cdot, X_{s,x}^{\alpha\beta}, u) - g(\cdot, X_{s,x}^{\tilde{\alpha}\tilde{\beta}}, u)\|_\lambda \leq 2\,\lambda^2(1+T)\;\|(\alpha,\beta) - (\tilde{\alpha},\tilde{\beta})\|_\lambda.$$

Therefore, for every $\lambda > 0$ and $(\alpha,\beta), (\tilde{\alpha},\tilde{\beta}) \in \mathcal{X}$, one has

$$d_\lambda(Q(\alpha,\beta), Q(\tilde{\alpha},\tilde{\beta})) \leq \lambda\,\sqrt{2(1+T)}\,\|(\alpha,\beta) - (\tilde{\alpha},\tilde{\beta})\|_\lambda,$$

where

$$\begin{aligned} &d_\lambda(Q(\alpha,\beta), Q(\tilde{\alpha},\tilde{\beta})) \\ &\quad = \max\{\|f(\cdot, X_{s,x}^{\alpha\beta}, u) - f(\cdot, X_{s,x}^{\tilde{\alpha}\tilde{\beta}}, u)\|_\lambda\,,\ \|g(\cdot, X_{s,x}^{\alpha\beta}, u) - g(\cdot, X_{s,x}^{\tilde{\alpha}\tilde{\beta}}, u)\|_\lambda\}. \end{aligned}$$

Taking in particular $\lambda \in (0, 1/\sqrt{2(1+T)})$, we obtain a contraction mapping Q defined on the complete metric space $(\mathcal{X}, d_\lambda)$. Then there exists a unique fixed point $(\alpha,\beta) \in \mathcal{X}$ of Q, which generates exactly one strong solution $X_{s,x}^{\alpha\beta}$ of (1.1) defined on $[s,T] \times \Omega$. □

Let $X_{s,x}^u$ be the unique strong solution of (1.1) defined for given $(s,x) \in \mathbb{R}^+ \times \mathbb{R}^d$, $T > s$, and $u \in \mathcal{U}_T$ on the interval $[s,T]$. We can extend such a solution to the whole interval $[0,T]$ by taking $X_{s,x}^u(t) = x$ a.s. for $0 \leq t < s$ and define on $\mathcal{U}_T$ an operator $\lambda_{s,x}$ with values in $C_{\mathbb{F}}^T$ by setting $\lambda_{s,x}(u) = \tilde{X}_{s,x}^u$, where $\tilde{X}_{s,x}^u = \mathbb{I}_{[0,s)}x + \mathbb{I}_{[s,T]}X_{s,x}^u$ and $(C_{\mathbb{F}}^T, \|\cdot\|)$ denotes the space of all $\mathbb{F}$-adapted d-dimensional continuous square integrable stochastic processes $X = (X_t)_{0\leq t\leq T}$ with norm $\|X\| = \{E[\sup_{0\leq t\leq T}|X_t|^2]\}^{1/2}$.

Lemma 1.1. *Let $B = (B_t)_{t\geq 0}$ be an m-dimensional $\mathbb{F}$-Brownian motion on a filtered probability space $\mathcal{P}_{\mathbb{F}}$, $(s,x) \in \mathbb{R}^+ \times \mathbb{R}^d$, and $T > s$. If f and g are measurable and satisfy (i) and (ii) of conditions (H), then $\lambda_{s,x}$ is a continuous mapping on $\mathcal{U}_T$ depending continuously on $(s,x) \in \mathbb{R}^+ \times \mathbb{R}^d$.*

Proof. By virtue of Remark 1.1, for every $u \in \mathcal{U}_T$, there exists a unique strong solution of (1.1) defined on $[s,T] \times \Omega$. Let $u \in \mathcal{U}_T$, and let $(u_n)_{n=1}^\infty$ be a sequence of $\mathcal{U}_T$ such that $\|u_n - u\|_T \to 0$ as $n \to \infty$. By the definition of the mapping $\lambda_{s,x}$, we have $\lambda_{s,x}(u) = \tilde{X}_{s,x}$ and $\lambda_{s,x}(u_n) = \tilde{X}_{s,x}^n$ for $n = 1, 2, \ldots$. By Corollary 4.4 of Chap. 1, for every $n \geq 1$ and $s \leq t \leq T$, we get

$$
\begin{aligned}
E\left[\sup_{0\le z\le t}|\tilde{X}^n_{s,x}(z)-\tilde{X}_{s,x}(z)|^2\right] &= E\left[\sup_{s\le z\le t}|X^n_{s,x}(z)-X_{s,x}(z)|^2\right]\\
&\le 2E\left(\sup_{s\le z\le t}\left|\int_s^z [f(\tau,X^n_{s,x}(\tau),u^n_\tau)-f(\tau,X_{s,x}(\tau),u_\tau)]\mathrm{d}\tau\right|^2\right)\\
&\quad + 2E\left(\sup_{s\le z\le t}\left|\int_s^z [g(\tau,X^n_{s,x}(\tau),u^n_\tau)-g(\tau,X_{s,x}(\tau),u_\tau)]\mathrm{d}B_\tau\right|^2\right)\\
&\le 2TE\int_s^t |f(\tau,X^n_{s,x}(\tau),u^n_\tau)-f(\tau,X_{s,x}(\tau),u_\tau)|^2\mathrm{d}\tau\\
&\quad + 8E\int_s^t |g(\tau,X^n_{s,x}(\tau),u^n_\tau)-g(\tau,X_{s,x}(\tau),u_\tau)|^2\mathrm{d}\tau\\
&\le 2(T+4)\,\|u^n-u\|_T^2\int_0^T k(t)\mathrm{d}t\\
&\quad + 2(T+4)\int_0^t k(\tau)\mathrm{d}\tau\, E\left[\sup_{s\le z\le \tau}|X^n_{s,x}(z)-X_{s,x}(z)|^2\right]\mathrm{d}\tau,
\end{aligned}
$$

which by Gronwall's inequality (see [49], p. 22) implies that

$$
\begin{aligned}
\|\tilde{X}^n_{s,x}-\tilde{X}_{s,x}\|^2 &= E\left[\sup_{0\le t\le T}|\tilde{X}^n_{s,x}(t)-\tilde{X}_{s,x}(t)|^2\right]\\
&\le 2(T+4)\left(\int_0^T k(t)\mathrm{d}t\right)\exp\left[2(T+4)\int_0^T k(t)\mathrm{d}t\right]\|u^n-u\|_T^2.
\end{aligned}
$$

Therefore, $\lim_{n\to\infty}\|\lambda_{s,x}(u_n)-\lambda_{s,x}(u)\|_T=0$ for every $u\in\mathcal{U}_T$ and every sequence $(u_n)_{n=1}^\infty$ of $\mathcal{U}_T$ converging to $u\in\mathcal{U}_T$. Finally, immediately from the definition of $\lambda_{s,x}$, for every $(s,x),(\bar{s},\bar{x})\in\mathbb{R}^+\times\mathbb{R}^d$ with $s<\bar{s}$, one gets

$$
\sup\{|\lambda_{s,x}(u)-\lambda_{\bar{s},\bar{x}}(u)|:u\in\mathcal{U}_T\}\le 2\left[|x-\bar{x}|+(\sqrt{T}+1)\sqrt{\int_s^{\bar{s}} m^2(t)\mathrm{d}t}\right],
$$

which implies that the mapping $\mathbb{R}^+\times\mathbb{R}^d\ni(s,x)\to\lambda_{s,x}(u)\in\mathbb{R}^d$ is uniformly continuous with respect to $u\in\mathcal{U}_T$. Similarly, this is true for the case $\bar{s}<s$. □

Now we can prove the following existence theorem.

Theorem 1.1. *Let f and g be measurable and satisfy conditions (H). If $K:\mathbb{R}^+\times\mathbb{R}^d\to\mathbb{R}$ and $\Phi:\mathbb{R}^+\times\mathbb{R}^d\times U\to\mathbb{R}$ are continuous and bounded, then for every bounded domain D, filtered probability space $\mathcal{P}_{\mathbb{F}}$, m-dimensional $\mathbb{F}$-Brownian motion $B=(B_t)_{t\ge 0}$ defined on $\mathcal{P}_{\mathbb{F}}$, and $(s,x)\in\mathbb{R}^+\times\mathbb{R}^d$, there* *exists* $\bar{u} \in \mathcal{U}_T$ *such that* $I^D_{s,x}(\bar{u}, \bar{X}^{\bar{u}}_{s,x}) = \sup\{I^D_{s,x}(u, X^u_{s,x}) : u \in \mathcal{U}_T\}$*, where* $I^D_{s,x}(u, X^u_{s,x}) = J^{u,X}_D(s, x)$ *and* $X^u_{s,x}$ *is the unique strong solution of (1.1) on the filtered probability space* $\mathcal{P}_{\mathbb{F}}$ *corresponding to the Brownian motion* B *and* $u \in \mathcal{U}_T$*.*

Proof. Similarly as above, by virtue of Remark 1.1, for every $u \in \mathcal{U}_T$, there exists a unique strong solution of (1.1) defined on $[s, T] \times \Omega$. Observe that $\sup\{I^D_{s,x}(u, X^u_{s,x}) : u \in \mathcal{U}_T\} = \sup\{I^D_{s,x}(u, \lambda_{s,x}(u)) : u \in \mathcal{U}_T\}$. Let $\alpha = \sup\{I^D_{s,x}(u, \lambda_{s,x}(u)) : u \in \mathcal{U}_T\}$, and let $(u_n)^\infty_{n=1}$ be a sequence of $\mathcal{U}_T$ such that $\alpha = \lim_{n\to\infty} I^D_{s,x}(u_n, \lambda_{s,x}(u_n))$. By the compactness of $\mathcal{U}_T$, there exist an increasing subsequence $(n_k)^\infty_{k=1}$ of $(n)^\infty_{n=1}$ and $\bar{u} \in \mathcal{U}_T$ such that $\|u_{n_k} - \bar{u}\|_T \to 0$ as $k \to \infty$. By virtue of Lemma 1.1, it follows that $\|\lambda_{s,x}(u_{n_k}) - \lambda_{s,x}(u)\|_T \to 0$ as $k \to \infty$. By the definitions of the operator $\lambda_{s,x}$ and the norm $\|\cdot\|$, it follows that there exists a subsequence, still denoted by $(X^{n_k}_{s,x})^\infty_{k=1}$, of the sequence $(X^{n_k}_{s,x})^\infty_{k=1}$ such that $\sup_{0\le t\le T} |\tilde{X}^{n_k}_{s,x} - \bar{X}_{s,x}| \to 0$ a.s. as $k \to \infty$, where $\bar{X}_{s,x} = \lambda_{s,x}(\bar{u})$. By virtue of Lemma 10.1 of Chap. 1 and Theorem 5.1 of Chap. 4, we have $\tilde{\tau}^{n_k}_D \to \bar{\tau}_D$ a.s. as $k \to \infty$, where $\tilde{\tau}^{n_k}_D$ and $\bar{\tau}_D$ denote the first exit times of $\tilde{X}^{n_k}_{s,x}$ and $\bar{X}_{s,x}$, respectively, from the domain D. Hence, by the continuity of Φ and K, it follows that $\alpha = \lim_{k\to\infty} I^D_{s,x}(u_{n_k}, \lambda_{s,x}(u_{n_k})) = I^D_{s,x}(\bar{u}, \lambda_{s,x}(\bar{u})) = I^D_{s,x}(\bar{u}, \bar{X}_{s,x})$. Thus $(\bar{u}, \bar{X}_{s,x}|_{[s,T]})$ is an optimal pair for (1.3).□

We can consider the above optimal control problem with a special type of controls $u = (u_t)_{t\ge0}$ of the form $u_t = \varphi(t, X_t)$ a.s. for $t \ge 0$ and a measurable function $\varphi : \mathbb{R}^+ \times \mathbb{R}^d \to U \subset \mathbb{R}^k$. Such controls are called Markov controls, because with such u, the corresponding process $X = (X_t)_{t\ge0}$ becomes an Itô diffusion, in particular a Markov process. In what follows, the above Markov control will be identified with a measurable function φ, and this function will be simply called a Markov control. The set of all such Markov controls will be denoted by $\mathcal{M}(U)$. The set of all restrictions of all $\varphi \in \mathcal{M}(U)$ to the set $[0, T] \times \mathbb{R}^d$ is denoted by $\mathcal{M}_T(U)$. Immediately from Theorem 1.1, it follows that for all measurable functions f and g satisfying conditions (H), there exists an optimal control for (1.3) in the set $\mathcal{S}_T$ consisting of all bounded and uniformly Lipschitz continuous Markov controls $\varphi \in \mathcal{M}_T(U)$, i.e., with the property that there exists a number $L > 0$ such that $|\varphi(t, z) - \varphi(s, v)| \le L(|t - s| + |z - v|)$ for every $\varphi \in \mathcal{S}_T$, $t, s \in [0, T]$, and $z, v \in \mathbb{R}^d$. Indeed, for functions f, g, and $\varphi, \psi \in \mathcal{S}_T \subset \mathcal{M}_T(U)$ as given above, we have

$$|f(t,x,\varphi(t,x)) - f(t,z,\psi(t,z))|^2 \le 2|f(t,x,\varphi(t,x)) - f(t,z,\varphi(t,z))|^2 + 2|f(t,z,\varphi(t,z)) - f(t,z,\psi(t,z))|^2 \le 2k(t)\left[(1+L^2)|x-z| + 2L^2\|\varphi-\psi\|^2_T\right]$$

and

$$\|g(t,x,\varphi(t,x)) - g(t,z,\psi(t,z))\|^2 \le 2k(t)\left[(1+L^2)|x-z| + 2L^2\|\varphi-\psi\|^2_T\right]$$

for every $t \in [0, T]$ and $x, z \in \mathbb{R}^d$, where $\|\cdot\|_T$ denotes the supremum norm of the space $\mathbf{C}([0, T] \times \mathbb{R}^d, \mathbb{R}^k)$ of all continuous bounded functions $v : [0, T] \times \mathbb{R}^d \to \mathbb{R}^k$.

Hence, similarly as in the proof of Lemma 1.1, it follows that the mapping $\lambda_{s,x}$: $\mathcal{S}_T \ni u \to \tilde{X}^{\varphi}_{s,x} \in C^T_{\mathbb{F}}$ with $\tilde{X}^{\varphi}_{s,x}$ and $C^T_{\mathbb{F}}$ as above is continuous. Here $X^{\varphi}_{s,x}$ is a strong solution of (1.1) corresponding to the Markov control $\varphi \in \mathcal{S}_T$. Therefore, immediately from Theorem 1.1, we obtain the existence in $\mathcal{S}_T$ of the optimal control for (1.3).

2 Optimal Control Problems for Systems Described by Stochastic Functional Inclusions

We shall now extend the above optimal control problem (1.3) on the case in which the dynamics of a control system is described by stochastic functional inclusions $SFI(F, G)$ of the form

$$\begin{cases} X_t - X_s \in \int_s^t F(\tau, X_\tau)\mathrm{d}\tau + \int_s^t G(\tau, X_\tau)\mathrm{d}B_\tau \; for \quad t \geq s\,, \\ X_s = x \quad a.s. \end{cases} \tag{2.1}$$

with the performance functional depending only on the weak solution $(\mathcal{P}_{\mathbb{F}}, X, B)$ of $SFI(F, G)$, i.e., with the performance functional $J^X_D(s, x)$ of the form

$$J^X_D(s, x) = E^{s,x}\left[\int_s^{\tau_D} \Psi(t, X_t)\mathrm{d}t + K(\tau_D, X_{\tau_D})\right], \tag{2.2}$$

where D is a bounded subset of $\mathbb{R}^d$, and $\Psi : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ and $K : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ are given continuous functions. By a solution of such a stochastic optimal control problem we mean a weak solution $(\tilde{\mathcal{P}}_{\tilde{\mathbb{F}}}, \tilde{X}, \tilde{B})$ of (2.1) such that $J^{\tilde{X}}_D(s, x) = \sup\{J^X_D(s, x) : X \in \mathcal{X}^D_{s,x}\}$, where $\mathcal{X}^D_{s,x}$ denotes the set of all weak solutions of (equivalence classes of) the stochastic functional inclusion $SFI(F, G)$ satisfying an initial condition $X(s) = x$ and such that $\tau_D = \inf\{t > s : X_{s,x}(t) \notin D\} < \infty$. Such an optimal control problem will be denoted by

$$\begin{cases} X_t - X_s \in \int_s^t F(\tau, X_\tau)\mathrm{d}\tau + \int_s^t G(\tau, X_\tau)\mathrm{d}B_t\tau \text{ for } t \geq s\,, \\ X_s = x \;\; a.s.\,, \\ J^X_D(s, x) \xrightarrow{\mathcal{X}^D_{s,x}} \max, \end{cases} \tag{2.3}$$

and called an optimal control problem for the control system described by the stochastic functional inclusion $SFI(F, G)$. In this case, the set $\mathcal{X}^D_{s,x}$ is said to be an admissible set for the optimal control problem (2.3). If there is $(\tilde{\mathcal{P}}_{\tilde{\mathbb{F}}}, \tilde{X}, \tilde{P}) \in \mathcal{X}^D_{s,x}$ such that $J^{\tilde{X}}_D(s, x) = \sup\{J^X_D(s, x) : X \in \mathcal{X}^D_{s,x}\}$, then $(\tilde{\mathcal{P}}_{\tilde{\mathbb{F}}}, \tilde{X}, \tilde{P})$ is called the optimal solution of the optimal control problem (2.3). Similarly as above, it will be simply denoted by $\tilde{X}$. We shall consider the optimal control problems of the form (2.3) with set-valued mappings $F : \mathbb{R}^+ \times \mathbb{R}^d \to \mathrm{Cl}(\mathbb{R}^d)$ and $G : \mathbb{R}^+ \times \mathbb{R}^d \to \mathrm{Cl}(\mathbb{R}^{d\times m})$ such that the set $\mathcal{X}_{s,x}(F,G)$ of all weak solutions of $SFD(F,G)$ satisfying the initial condition $X(s) = x$ is weakly compact in distribution and such that $\mathcal{X}^D_{s,x}(F,G) \neq \emptyset$. Hence, immediately from Theorem 5.1 of Chap. 4, it will follow that $\mathcal{X}^D_{s,x}$ is also weakly compact. We apply the result obtained to the case of F and G defined by $F(t,x) = \{f(t,x,z) : z \in U\}$ and $G(t,x) = \{g(t,x,z) : z \in U\}$. Hence in particular, the existence of optimal pairs for the optimal control problems of the system described by (1.1) and performance functionals of the form

$$J^X_D(s,x) = E^{s,x}\left[\int_s^{\tau_D} \sup_{u\in U} \Phi(t,X_t,u))\mathrm{d}t + K(\tau_D, X_{\tau_D})\right] \tag{2.4}$$

and

$$J^X_D(s,x) = E^{s,x}\left[\int_s^{\tau_D} \sup_{n\geq 1} \Phi(t,X_t,\varphi^n(t,X_t))\mathrm{d}t + K(\tau_D, X_{\tau_D})\right] \tag{2.5}$$

will follow, where $(\varphi^n)_{n=1}^{\infty}$ is a dense sequence of a bounded set $\mathcal{U} \subset C(\mathbb{R}^+\times\mathbb{R}^d, U)$. In what follows, we shall still denote by $(\mathcal{P})$ and $(\mathcal{A})$ the assumptions defined in Sect. 1 of Chap. 6.

Theorem 2.1. *Let $F : \mathbb{R}^+ \times \mathbb{R}^d \to \mathrm{Cl}(\mathbb{R}^d)$ and $G : \mathbb{R}^+ \times \mathbb{R}^d \to \mathrm{Cl}(\mathbb{R}^{d\times m})$ be convex-valued, continuous, and bounded, and let $\Psi : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ be a uniformly integrally bounded Carathéodory function. Assume that G is diagonally convex and satisfies item (iv′) of conditions $(\mathcal{A})$. Let D be a bounded domain in $\mathbb{R}^d$. If $K : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ is continuous and bounded, then for every $(s,x) \in \mathbb{R}^+ \times D$, the optimal control problem (2.3) possesses an optimal solution.*

Proof. Let us observe that $\mathcal{X}^D_{s,x}$ is nonempty and weakly compact in distribution. Indeed, similarly as in the proof of Theorem 4.1 of Chap. 4, we can verify that $\mathcal{X}_{s,x}(F,G)$ is weakly compact in distribution for every $(s,x) \in \mathbb{R}^+ \times \mathbb{R}^d$. By property $(\mathcal{P})$ of G, for every $(f,g) \in \mathcal{C}(F) \times \mathcal{C}(G)$, there exists a unique in law solution $(\tilde{\mathcal{P}}_{\tilde{\mathbb{F}}}, \tilde{x}, \tilde{B})$ of $SDE(f,g)$ with initial condition $\tilde{x}_s = x$ a.s., which by the properties of functions f and g, implies that $(\tilde{\mathcal{P}}_{\tilde{\mathbb{F}}}, \tilde{x}, \tilde{B}) \in \mathcal{X}_{s,x}(F,G)$. By Remark 10.4 of Chap. 1, we have $\tilde{\tau}_D < \infty$ a.s., where $\tilde{\tau}_D$ is the first exit time of $\tilde{x}$ from the set D. Then $(\tilde{\mathcal{P}}_{\tilde{\mathbb{F}}}, \tilde{x}, \tilde{B}) \in \mathcal{X}^D_{s,x}$. To verify that $\mathcal{X}^D_{s,x}$ is weakly compact, let us observe that by the weak compactness of $\mathcal{X}_{s,x}(F,G)$ and the relation $\mathcal{X}^D_{s,x} \subset \mathcal{X}_{s,x}(F,G)$, it is enough to verify that $\mathcal{X}^D_{s,x}$ is weakly closed.

Let $(x^r)_{r=1}^{\infty}$ be a sequence of $\mathcal{X}^D_{s,x}$ convergent in distributions. Then there exists a probability measure $\mathcal{P}$ on $\beta(C(\mathbb{R}^+,\mathbb{R}^d))$ such that $P(x^r)^{-1} \Rightarrow \mathcal{P}$ as $r \to \infty$. By virtue of Theorem 2.3 of Chap. 1, there exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and random variables $\tilde{x}^r : \tilde{\Omega} \to C(\mathbb{R}^+,\mathbb{R}^d)$ and $\tilde{x} : \tilde{\Omega} \to C(\mathbb{R}^+,\mathbb{R}^d)$ for $r = 1,2,\ldots$ such that $P(x^r)^{-1} = P(\tilde{x}^r)^{-1}$ for $r = 1,2,\ldots$, $\tilde{P}(\tilde{x})^{-1} = \mathcal{P}$ and $\lim_{r\to\infty} \rho(\tilde{x}^r, \tilde{x}) = 0$ with $(\tilde{P}.1)$, where ρ is the metric defined in $C(\mathbb{R}^+,\mathbb{R}^d)$ as in Theorem 2.4 of Chap. 1. For every $r \geq 1$, one has $\tau_D^r < \infty$ a.s., where τ_D^r is the first exit time of x^r from the set D, which by Theorem 5.2 of Chap. 4, implies that $\tilde{\tau}_D^r < \infty$ a.s., where $\tilde{\tau}_D^r$ denotes the first exit time of $\tilde{x}^r$ from the set D. Hence, by the properties of the sequence $(\tilde{x}^r)_{r=1}^{\infty}$, it follows that $\tilde{\tau}_D < \infty$ a.s., where $\tilde{\tau}_D$, is the first exit time of $\tilde{x}$ from D. Similarly as in the proof of Theorem 4.1 of Chap. 4, we can verify now that by virtue of Theorem 1.3 of Chap. 4, there exist a standard extension $\hat{\mathcal{P}}_{\hat{\mathbb{F}}} = (\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{P})$ of $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{P})$ and an m-dimensional Brownian motion $\hat{B}$ such that $(\hat{\mathcal{P}}_{\hat{\mathbb{F}}}, \hat{x}, \hat{B})$ is a weak solution of $SFI(F, G, \mu)$, with $\mu = P\tilde{x}_s^{-1}$ and such that $x^r \Rightarrow \hat{x}$. Furthermore, we have $P\hat{x}^{-1} = P\tilde{x}^{-1}$, which by Theorem 5.2 of Chap. 4, implies that $P\hat{\tau}_D^{-1} = P\tilde{\tau}_D^{-1}$. Hence in particular, it follows that $\hat{\tau}_D < \infty$. Thus $\mathcal{X}_{\mu}^D$ is weakly closed with respect to weak convergence in the sense of distributions.

By (2.2) and the properties of the functions Ψ and K, one has $\alpha := \sup\{J_D^X(s, x) : X \in \mathcal{X}_{s,x}^D\} < \infty$, because

$$\int_s^{\tau_D^X} \Psi(t, X(t))\mathrm{d}t \leq \int_s^{\tau_D^X} |\Psi(t, X(t))|\mathrm{d}t \leq \int_0^{\tau_D^X} m(t)\mathrm{d}t \leq \int_0^{\infty} m(t)\mathrm{d}t < \infty,$$

where $m \in \mathbb{L}(\mathbb{R}^+, \mathbb{R}^+)$ is such that $|\Psi(t, x)| \leq m(t)$ and there is $M > 0$ such that $|K(t, x)| \leq M$ for $x \in \mathbb{R}^d$ and $t \geq 0$. Let $(\mathcal{P}_{\mathbb{F}_n}^n, X^n, B^n) \in \mathcal{X}_{s,x}^D$ be for $n = 1, 2, \ldots$ such that $\alpha = \lim_{n\to\infty} J_D^n(s, x)$, with

$$J_D^n(s, x) = E^{s,x}\left[\int_s^{\tau_D^n} \Psi(t, X^n(t))\mathrm{d}t + K(\tau_D^n, X^n(\tau_D^n)),\right]$$

where $E_n^{s,x}$ denotes the mean value operator with respect to the probability law $Q_n^{s,x}$ of X^n and $\tau_D^n = \inf\{r > s : X^n(r) \notin D\}$ for $n = 1, 2, \ldots$. By the weak compactness of $\mathcal{X}_{s,x}^D$ and Theorem 2.3 of Chap. 1, there are an increasing subsequence $(n_k)_{k=1}^{\infty}$ of the sequence $(n)_{n=1}^{\infty}$, a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, and continuous processes $\tilde{X}^{n_k}$ and $\tilde{X}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ such that $P(X^{n_k})^{-1} = P(\tilde{X}^{n_k})^{-1}$ for $k = 1, 2, \ldots$ and $\rho(\tilde{X}^{n_k}, \tilde{X}) \to 0$, $\tilde{P}$-a.s. as $k \to \infty$, which by Corollary 3.3 of Chap. 1, implies that $P(X_s^{n_k})^{-1} \Rightarrow P\tilde{X}_s^{-1}$ as $k \to \infty$. Let $\tilde{\mathcal{F}}_t = \bigcap_{\varepsilon>0} \sigma(\{\tilde{X}(u) : s \leq u \leq t + \varepsilon\})$ for $t \geq s$ and put $\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{t\geq s}$. It is clear that $\tilde{X}$ is $\tilde{\mathbb{F}}$-adapted. By virtue of Lemma 1.3 of Chap. 4, we have $\mathcal{M}_{FG}^{\tilde{X}} \neq \emptyset$, and therefore, there exist $\tilde{f} \in S_{\tilde{\mathbb{F}}}(F \circ \tilde{X})$ and $\tilde{g} \in S_{\tilde{\mathbb{F}}}(G \circ \tilde{X})$ such that for every $h \in C_0^2(\mathbb{R}^d)$, a stochastic process $\varphi_h^{\tilde{X}} = ((\varphi_h^{\tilde{X}})_t)_{t\geq s}$ with $(\varphi_h^{\tilde{X}})_t = h(\tilde{X}_t) - h(\tilde{X}_s) - \int_s^t (\mathbb{L}_{\tilde{f}\tilde{g}}^{\tilde{X}} h)_\tau d\tau$ for $t \geq s$ is a continuous local $\tilde{\mathbb{F}}$-martingale on the filtered probability space $\tilde{\mathcal{P}}_{\tilde{\mathbb{F}}} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{P})$. Hence, by Theorem 1.3 of Chap. 4, it follows that there exists a standard extension of $\tilde{\mathcal{P}}_{\tilde{\mathbb{F}}}$, still denoted by $\tilde{\mathcal{P}}_{\tilde{\mathbb{F}}}$, and an m-dimensional $\tilde{\mathbb{F}}$-Brownian motion $\tilde{B} = (\tilde{B}_t)_{t\geq 0}$ such that $(\tilde{\mathcal{P}}_{\tilde{\mathbb{F}}}, \tilde{X}, \tilde{B})$ is a weak solution of $SFI(F, G, \mu)$ with an initial distribution $\mu = P\tilde{X}_s^{-1}$. Immediately from the properties of the stochastic processes $\tilde{X}^{n_k}$ and $\tilde{X}$, it follows that $\widetilde{X}_s^{n_k} = x$, P-a.s., and $P(\widetilde{X}_s^{n_k})^{-1} \Rightarrow P(\widetilde{X}_s)^{-1}$ as $k \to \infty$, which implies that $\widetilde{X}_s = x$, P̃-a.s. Therefore, $(\tilde{\mathcal{P}}_{\tilde{\mathbb{F}}}, \tilde{X}, \tilde{B}) \in \mathcal{X}_{s,x}(F, G)$. Similarly as above, we can verify that $(\tilde{\mathcal{P}}_{\tilde{\mathbb{F}}}, \tilde{X}, \tilde{B}) \in \mathcal{X}_{s,x}^D$. On the other hand, By (2.2) and the properties of processes X^{n_k} and $\tilde{X}^{n_k}$ and Theorem 5.2 of Chap. 4, it follows that $P(\tau_D^k)^{-1} = P(\tilde{\tau}_D^k)^{-1}$ for $k = 1, 2, \ldots$. Then $J_D^{n_k}(s,x) = \tilde{J}_D^{n_k}(s,x)$ for every $k = 1, 2, \ldots$, where

$$\tilde{J}_D^{n_k}(s,x) = \tilde{E}^{s,x}\left[\int_s^{\tilde{\tau}_D^k} \Psi(t, \tilde{X}^{n_k}(t))\mathrm{d}t + K(\tilde{\tau}_D^k, \tilde{X}^{n_k}(\tilde{\tau}_D^k))\right]$$

for $k = 1, 2, \ldots$ with $\tilde{\tau}_D^k$ and τ_D^k defined as above with $\widetilde{X}_t^{n_k} = \widetilde{X}^{n_k}(t)$. Hence, by Theorem 5.1 of Chap. 4, it follows that

$$\lim_{k\to\infty} \tilde{J}_D^{n_k}(s,x) = \tilde{E}^{s,x}\left[\int_s^{\tilde{\tau}_D} \Psi(t, \tilde{X}(t))\mathrm{d}t + K(\tilde{\tau}_D, \tilde{X}(\tilde{\tau}_D))\right],$$

where $\tilde{\tau}_D = \inf\{r > s : \tilde{X} \notin D\}$. But $\alpha = \lim_{k\to\infty} J_D^{n_k}(s,x) = \lim_{k\to\infty} \tilde{J}_D^{n_k}(s,x)$. Therefore,

$$\alpha = \tilde{E}^{s,x}\left[\int_s^{\tilde{\tau}_D} \Psi(t, \tilde{X}(t))\mathrm{d}t + K(\tilde{\tau}_D, \tilde{X}(\tilde{\tau}_D))\right].$$

□

Remark 2.1. Similarly as above, we can consider the following viable optimal control problem:

$$\begin{cases} X_t - X_s \in \int_s^t F(\tau, X_\tau)\mathrm{d}\tau + \int_s^t G(\tau, X_\tau)\mathrm{d}B_t\tau, \text{ for } t \geq s, \\ X_t \in \Gamma(t) \text{ a.s. for } t \geq s, \\ J(X) \xrightarrow{\mathcal{X}_D^\Gamma} \max, \end{cases}$$

where Γ is a given target set mapping and $\mathcal{X}_D^\Gamma$ denotes the set of all weak Γ-viable solutions $(\mathcal{P}_{\mathbb{F}}, X, B)$ of the stochastic functional inclusion $SFI(F,G)$ such that $\tau_D^X = \inf\{t > s : X(t) \notin D\} < \infty$. □

We shall consider now the existence of the optimal control problem (1.3) with a performance functional $J_D^X(s,x)$ defined by (2.4) and (2.5) above. Let us recall that for a given nonempty set $U \subset \mathbb{R}^k$, a bounded domain D, an initial point $(s,x) \in \mathbb{R}^+ \times D$, and functions $f : \mathbb{R} + \times\mathbb{R}^d \times U \to \mathbb{R}^d$, $g : \mathbb{R} + \times\mathbb{R}^d \times U \to \mathbb{R}^{d\times m}$, $\Psi : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$, and $K : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$, we are interested in the existence of an admissible pair $(\bar{u}, \bar{X}^{\bar{u}}) \in \Lambda_{fg}(s,x)$ such that $J_D^{\bar{X}}(s,x) = \sup\{J_D^X(s,x) : (u, X^u) \in \Lambda_{fg}(s,x)\}$. We shall show that such an optimal pair $(\bar{u}, \bar{X}) \in \Lambda_{fg}(s,x)$ exists if $f : \mathbb{R}^+ \times \mathbb{R}^d \times U \to \mathbb{R}^d$ and $g : \mathbb{R}^+ \times \mathbb{R}^d \times U \to \mathbb{R}^{d\times m}$ satisfy the following conditions (C):

(i) f and g are continuous and bounded such that $f(t,x,\cdot)$, $g(t,x,\cdot)$, and $(g \cdot g^*)(t,x,\cdot)$ are affine for every fixed $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$ on the compact convex set $U \subset \mathbb{R}^k$.

(ii) g is such that $g \cdot g^*$ is uniformly positive definite.
(iii) $\Phi : \mathbb{R}^+ \times \mathbb{R}^d \times U \to \mathbb{R}$ is a uniformly integrally bounded Carathéodory function and $K : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ is continuous and bounded.

Lemma 2.1. *If f and g satisfy conditions (C), then for every nonempty compact convex set $U \subset \mathbb{R}^k$, the set-valued mappings F and G defined by $F(t,x) = \{f(t,x,z) : z \in U\}$ and $G(t,x) = \{g(t,x,z) : z \in U\}$ satisfy $(\mathcal{P})$ and conditions (i), (iii), (iv′), and (v) of $(\mathcal{A})$.*

Proof. Immediately from (ii) of conditions (C), it follows that G satisfies the condition $(\mathcal{P})$. Let $\mathcal{T}_U$ be the induced topology in U. Then $(U, \mathcal{T}_U)$ is a compact topological space. Let $(\bar{t}, \bar{x}) \in \mathbb{R}^+ \times \mathbb{R}^d$ and $\bar{z} \in U$ be fixed and V an open set in $\mathbb{R}^d$. Suppose $(\bar{t}, \bar{x}, \bar{z})$ is such that $f(\bar{t}, \bar{x}, \bar{z}) \in V$. By the continuity of $f(\cdot,\cdot,\bar{z})$ at $(\bar{t}, \bar{x})$, there is a neighborhood $\mathcal{N}$ of $(\bar{t}, \bar{x})$ such that $f(t,x,\bar{z}) \in V$ for every $(t,x) \in \mathcal{N}$. Therefore, for every $(t,x) \in \mathcal{N}$, one has $F(t,x) \cap V \neq \emptyset$. Then F is l.s.c. In a similar way, we can also verify that G is l.s.c. By the compactness of the set U and continuity of $f(t,x,\cdot)$ and $g(t,x,\cdot)$, it follows that $F(t,x)$ and $G(t,x)$ are compact subsets of $\mathbb{R}^d$ and $\mathbb{R}^{d\times m}$, respectively, for every $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$. Similarly, by the convexity of U and affineness of $f(t,x,\cdot)$, $g(t,x,\cdot)$, and $(g \cdot g^*)(t,x,\cdot)$, it follows that F and G are convex-valued and G is diagonally convex. We shall verify that F and G are also u.s.c. Indeed, similarly as above, let $(\bar{t}, \bar{x}) \in \mathbb{R}^+ \times \mathbb{R}^d$ be arbitrarily fixed and suppose V is an open neighborhood of $F(\bar{t}, \bar{x})$. By the continuity of f, for every fixed $z \in U$ there exist neighborhoods W^z and $\mathcal{O}^z$ of $(\bar{t}, \bar{x}) \in \mathbb{R}^+ \times \mathbb{R}^d$ and $\bar{z} \in U$, respectively, such that $f(W^z \times \mathcal{O}^z) \subset V$. By the compactness of the topological space $(U, \mathcal{T}_U)$, there are $z_1, \ldots, z_n \in U$ such that $\bigcup_{i=1}^n \mathcal{O}^{z_i} = U$. For every $i = 1, 2, \ldots, n$, we have $f(W^{z_i} \times \mathcal{O}^{z_i}) \subset V$. Therefore, $\bigcup_{i=1}^n f(W^{z_i} \times \mathcal{O}^{z_i}) \subset V$. But

$$\bigcup_{i=1}^n f\left(\left[\bigcap_{i=1}^n W^{z_i}\right] \times \mathcal{O}^{z_i}\right) = f\left(\left[\bigcap_{i=1}^n W^{z_i}\right] \times \left[\bigcup_{i=1}^n \mathcal{O}^{z_i}\right]\right)$$
$$= f\left(\left[\bigcap_{i=1}^n W^{z_i}\right] \times U\right) \subset \bigcup_{i=1}^n f\left(W^{z_i} \times \mathcal{O}^{z_i}\right) \subset V.$$

Therefore, $F(\bigcap_{i=1}^n W^{z_i}) = f([\bigcap_{i=1}^n W^{z_i}] \times U) \subset V$. Then F is u.s.c. at $(\bar{t}, \bar{x}) \in \mathbb{R}^+ \times \mathbb{R}^d$. In a similar way, we can verify that also G is u.s.c.

Let $\sigma \in \mathcal{C}(l(G))$ be a continuous selector of $D(G) = l(G)$, where $l(u) = u \cdot u^*$ for every $u \in \mathbb{R}^{d\times m}$, and let $\lambda(t,x,z) = l(g(t,x,z)) = g(t,x,z) \cdot g^*(t,x,z)$ for $(t,x,z) \in \mathbb{R}^+ \times \mathbb{R}^d \times U$. We have $\sigma(t,x) \in \lambda(t,x,U)$ for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$. Therefore, by virtue of Theorem 2.2 of Chap. 2, there exists a sequence $(z_n)_{n=1}^\infty$ of continuous functions $z_n : \mathbb{R}^+ \times \mathbb{R}^d \to U$ such that $\sup_{(t,x)} |\sigma(t,x) - l(g_n(t,x))| \to 0$ as $n \to \infty$, where $g_n(t,x) = g(t,x,z_n(t,x)) \in G(t,x)$ for $n = 1, 2, \ldots$ and $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$. Then there exists a sequence $(g_n)_{n=1}^\infty$ of continuous selectors of G such that $l(g_n) \to \sigma$ uniformly in $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$ as $n \to \infty$. Thus (iv′) of conditions $(\mathcal{A})$ is also satisfied. □

We can now prove the existence of an optimal pair for the optimal control problem (1.3) with the performance functionals defined by (2.4) and (2.5).

Theorem 2.2. *Let D be a bounded domain in $\mathbb{R}^d$ and assume that conditions (C) are satisfied. There exists an optimal pair of the optimal control problem (1.3) with a performance functional defined by* (2.4).

Proof. Let F and G be defined as above. By virtue of Lemma 2.1, the multifunctions F and G satisfy the conditions of Theorem 2.1. Therefore, for every $(s,x) \in \mathbb{R}^+ \times \mathbb{R}^d$, there exists a weak solution $(\tilde{\mathcal{P}}_{\tilde{\mathbb{F}}}, \tilde{X}, \tilde{B})$ of $SFI(F,G)$ satisfying the initial condition $\tilde{X}(s) = x$, $\tilde{P}$-a.s., with $\tilde{\mathcal{P}}_{\tilde{\mathbb{F}}} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{P})$ such that $J_D^{\tilde{X}}(s,x) = \sup\{J_D^X(s,x) : X \in \mathcal{X}_{s,x}\}$, where

$$J_D^{\tilde{X}}(s,x) = \tilde{E}^{s,x}\left[\int_s^{\tilde{\tau}_D} \Psi(t,\tilde{X}(t))\mathrm{d}t + K(\tilde{\tau}_D, \tilde{X}(\tilde{\tau}_D))\right]$$

with $\Psi(t,x) = \sup\{\Phi(t,x,u) : u \in U\}$ and $\tilde{\tau}_D = \inf\{r > s : \tilde{X}(r) \notin D\}$. By virtue of Theorem 1.5 of Chap. 3, there are $\tilde{f} \in S_{\tilde{\mathbb{F}}}(F \circ \tilde{X})$ and $\tilde{g} \in S_{\tilde{\mathbb{F}}}(G \circ \tilde{X})$ such that $\tilde{X}(t) = x + \int_s^t \tilde{f}_\tau \mathrm{d}\tau + \int_s^t \tilde{g}_\tau \mathrm{d}\tilde{B}_\tau$, $\tilde{P}$-a.s. for $t \geq s$. Let $\Gamma(t,x) = \{(f(t,x,z), g(t,x,z)) : z \in U\}$ for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$. Similarly as in the proof of Lemma 2.1, we can verify that Γ is a continuous and bounded set-valued mapping with compact values in $\mathbb{R}^d \times \mathbb{R}^{d\times m}$. Therefore, the set-valued process $\tilde{\Gamma} = (\tilde{\Gamma}_t)_{t\geq s}$ defined by $\tilde{\Gamma}_t = \Gamma(t,\tilde{X}(t))$ is $\tilde{\mathbb{F}}$-nonanticipative and such that $(\tilde{f}_t, \tilde{g}_t) \in \tilde{\Gamma}_t$, $\tilde{P}$-a.s. for $t \geq s$. By virtue of Theorem 2.5 of Chap. 2, there exists an $\tilde{\mathbb{F}}$-nonanticipative process $\tilde{u} = (\tilde{u}_t)_{t\geq s}$ with values in the set U such that $(\tilde{f}_t, \tilde{g}_t) = (f(t,\tilde{X}(t),\tilde{u}_t), g(t,\tilde{X}(t),\tilde{u}_t))$, $\tilde{P}$-a.s. for $t \geq s$. Then an optimal solution $\tilde{X}$ of the optimal control problem (1.3) with the performance functional (2.4) can be expressed by the formula

$$\tilde{X}(t) = x + \int_s^t f(\tau,\tilde{X}(\tau),\tilde{u}_\tau)\mathrm{d}\tau + \int_s^t g(\tau,\tilde{X}(\tau),\tilde{u}_\tau)\mathrm{d}\tilde{B}_\tau$$

$\tilde{P}$-a.s. for $t \geq s$. Therefore, $(\tilde{u},\tilde{X}) \in \Lambda_{fg}(s,x)$. In a similar way, we deduce that for every weak solution $(\mathcal{P}_{\mathbb{F}}, X, B)$ of $SFI(F,G)$ satisfying the initial condition $X(s) = x$ a.s. with the above-defined set-valued mappings F and G, there exists an $\mathbb{F}$-nonanticipative stochastic process $u = (u_t)_{t\geq s}$ with values in U such that $(u,X) \in \Lambda_{fg}(s,x)$. By the properties of the performance functional $J_D^X(s,x)$ defined by (2.4), one has

$$J_D^{\tilde{X}}(s,x) = \sup\{J_D^X(s,x) : X \in \mathcal{C}_{s,x}^D\} = \sup\{J_D^X(s,x) : (u,X) \in \Lambda_{f,g}(s,x)\}$$

with $\mathcal{C}_{s,x}^D = \pi(\Lambda_{f,g}(s,x))$, where $\pi(u,X) = X$ for $(u,X) \in \Lambda_{fg}(s,x)$. Then $(\tilde{u},\tilde{X})$ is the optimal pair for the optimal control problem (1.3) with the performance functional defined by (2.4). □

In a similar way, we can prove the following existence theorem.

Theorem 2.3. *Let D be a bounded domain in $\mathbb{R}^d$, $\mathcal{U}$ a bounded subset of $C(\mathbb{R}^+ \times \mathbb{R}^d, U)$, and $(\varphi^n)_{n=1}^\infty$ a dense sequence of $\mathcal{U}$. Assume that conditions $(\mathcal{C})$ are satisfied and that f and g are such that $f(t,x,\cdot)$ and $g(t,x,\cdot)$ are linear. There exists an optimal pair $(\tilde{u}, \tilde{X})$ for the optimal control problem (1.3) with the performance functional $J_D^X(s,x)$ defined by (2.5) and $\tilde{u} = \lim^w_{j\to\infty} \sum_{k=1}^{m_j} \mathbb{I}_{C_k^j} \varphi^{n_j^k}(\cdot, \tilde{X}.)$, where $\lim^w$ denotes the weak limit of sequences in the space $\mathbb{L}(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^k)$, $\{C_1^j, \dots, C_{m_j}^j\}$ is a finite $\Sigma_{\mathbb{F}}$-partition of $\mathbb{R}^+ \times \Omega$, and $\{\varphi^{n_j^1}, \dots, \varphi^{n_j^{m_j}}\} \subset \{\varphi^n : n \geq 1\}$ for every $j \geq 1$.*

Proof. Let F and G be defined by $F(t,x) = \{f(t,x,\varphi(t,x)) : \varphi \in \mathcal{U}\}$ and $G(t,x) = \{g(t,x,\varphi(t,x)) : \varphi \in \mathcal{U}\}$ for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$. By virtue of Lemma 2.1, F and G satisfy the conditions of Theorem 2.1. Therefore, for every $(s,x) \in \mathbb{R}^+ \times \mathbb{R}^d$, there exists a weak solution $(\tilde{\mathcal{P}}_{\tilde{\mathbb{F}}}, \tilde{X}, \tilde{B})$ of $SFI(F,G)$ satisfying the initial condition $\tilde{X}(s) = x$, $\tilde{P}$-a.s., with $\tilde{\mathcal{P}}_{\tilde{\mathbb{F}}} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{P})$ such that $J_D^{\tilde{X}}(s,x) = \sup\{J_D^X(s,x) : X \in \mathcal{X}_{s,x}\}$. By virtue of Theorem 1.5 of Chap. 3, there are $\tilde{f} \in S_{\tilde{\mathbb{F}}}(F \circ \tilde{X})$ and $\tilde{g} \in S_{\tilde{\mathbb{F}}}(G \circ \tilde{X})$ such that $\tilde{X}_t = x + \int_s^t \tilde{f}_\tau \mathrm{d}\tau + \int_s^t \tilde{g}_\tau \mathrm{d}\tilde{B}_\tau$, $\tilde{P}$-a.s. for $t \geq s$. By the properties of the sequence $(\varphi^n)_{n=1}^\infty$, it follows that $F(t,x) = \mathrm{cl}\{f(t,x,\varphi^n(t,x)) : n \geq 1\}$ and $G(t,x) = \mathrm{cl}\{g(t,x,\varphi^n(t,x)) : n \geq 1\}$ for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$. Therefore, by virtue of Lemma 4.1 of Chap. 2, it follows that $S_{\tilde{\mathbb{F}}}(F \circ \tilde{X}) = \overline{\mathrm{dec}}\{f(\cdot, \tilde{X}., \varphi^n(\cdot, \tilde{X}.)) : n \geq 1\}$ and $S_{\tilde{\mathbb{F}}}(G \circ \tilde{X}) = \overline{\mathrm{dec}}\{g(\cdot, \tilde{X}., \varphi^n(\cdot, \tilde{X}.)) : n \geq 1\}$. Hence it follows that $(\tilde{f}, \tilde{g}) \in \overline{\mathrm{dec}}\{(f,g)(\cdot, \tilde{X}., \varphi^n(\cdot, \tilde{X}.)) : n \geq 1\}$. Thus there exists a sequence $(\alpha_j)_{j=1}^\infty$ of $\mathrm{dec}\{f,g)(\cdot, \tilde{X}., \varphi^n(\cdot, \tilde{X}.)) : n \geq 1\}$ converging to $(\tilde{f}, \tilde{g})$ in the metric topology of $\mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^d \times \mathbb{R}^{d\times m})$. But $\mathrm{dec}\{(f,g)(\cdot, \tilde{X}., \varphi^n(\cdot, \tilde{X}.)) : n \geq 1\} = (f,g)(\cdot, \tilde{X}., \mathrm{dec}\{\varphi^n(\cdot, \tilde{X}.) : n \geq 1\})$. Therefore, for every $j \geq 1$, there exist a finite $\Sigma_{\mathbb{F}}$-partition $\{C_1^j, \dots, C_{m_j}^j\}$ of $\mathbb{R}^+ \times \Omega$ and a family of $\{\varphi^{n_j^1}, \dots, \varphi^{n_j^{m_j}}\} \subset \{\varphi^n : n \geq 1\}$ such that $\alpha_j = (f,g)(\cdot, \tilde{X}., \sum_{k=1}^{m_j} \mathbb{I}_{C_k^j} \varphi^{n_j^k}(\cdot, \tilde{X}.))$ for $j \geq 1$. By the boundedness of the set $\mathcal{U}$, it follows that the sequence $(\sum_{k=1}^{m_j} \mathbb{I}_{C_k^j} \varphi^{n_j^k}(\cdot, \tilde{X}.))_{j=1}^\infty$ is relatively sequentially weakly compact. Then there exist $\tilde{u} \in \mathbb{L}(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^k)$ and a subsequence, still denoted by $(\sum_{k=1}^{m_j} \mathbb{I}_{C_k^j} \varphi^{n_j^k}(\cdot, \tilde{X}.))_{j=1}^\infty$, weakly converging to $\tilde{u}$. Hence, by the properties of the functions f and g, it follows that $(\tilde{f}, \tilde{g}) = (f(\cdot, \tilde{X}., \tilde{u}), g(\cdot, \tilde{X}., \tilde{u}))$. Similarly as in the proof of Theorem 2.3, it follows that $(\tilde{u}, \tilde{X})$, with the optimal control $\tilde{u}$ described above, is the optimal pair for the optimal control problem (1.3) with the performance functional $J_D^X(s,x)$ defined by (2.5). □

3 Optimal Problems for Systems Described by Partial Differential Inclusions

Let $F : \mathbb{R}^+ \times \mathbb{R}^d \to \mathrm{Cl}(\mathbb{R}^d)$ and $G : \mathbb{R}^+ \times \mathbb{R}^d \to \mathrm{Cl}(\mathbb{R}^{d\times m})$ be such that the following conditions $(\mathcal{D})$ are satisfied:

(i) F and G are bounded, continuous, and convex-valued, and for every $g \in \mathcal{C}(G)$, the matrix-valued mapping $l(g) = g \cdot g^*$ is uniformly positive definite.
(ii) G is diagonally convex, i.e., for every $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$, the set $D(G)(t,x) = \{v \cdot v^* : v \in G(t,x)\}$ is convex.
(iii) For every $\sigma \in \mathcal{C}(D(G))$, there exists a sequence $(g^n)_{n=1}^{\infty}$ of $\mathcal{C}(G)$ such that $\sup_{(t,x)\in\mathbb{R}^+\times\mathbb{R}^d} |\sigma_n(t,x) - \sigma(t,x)| \to 0$ as $n \to \infty$, where $\sigma_n = l(g_n)$ for $n \geq 1$.

For a bounded domain $D \subset \mathbb{R}^d$, $T > 0$, $(s,x) \in \mathbb{R}^+ \times \mathbb{R}^d$, $\tilde{h} \in C_0^2(\mathbb{R}^{d+1})$, $u \in C([0,T] \times D, \mathbb{R})$, and a continuous function $\Phi : (0,T) \times \partial D \to \mathbb{R}$, we shall consider the initial and boundary values problems (6.3) and (6.4) of Chap. 6 of the form:

$$\begin{cases} v_t'(t,s,x) - v_s'(t,s,x) \in (\mathbb{L}_{FG} v(t,\cdot))\,(s,x) - c(s,x)v(t,s,x) \\ \text{for } (s,x) \in [0,T) \times \mathbb{R}^d \text{ and } t \in [0, T-s], \\ v(0,s,x) = \tilde{h}(s,x) \text{ for } (s,x) \in [0,T) \times \mathbb{R}^d\,, \end{cases}$$

and

$$\begin{cases} u(t,x) - v_t'(t,x) \in (\mathbb{L}_{FG} v)\,(t,x) - c(t,x)v(t,x) \text{ for } (t,x) \in (0,T) \times D, \\ \lim_{D\ni x\to y} v(t,x) = \Phi(t,y) \text{ for } (t,y) \in (0,T] \times \partial D. \end{cases}$$

Let $H : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be measurable and uniformly integrably bounded and let $\Lambda_{FG}(c,\tilde{h})$ and $\Gamma_{FG}(c,u,\Phi)$ denote the sets of all solutions of the above initial and boundary value problems, respectively. For every $(s,x) \in [0,T) \times \mathbb{R}^d$, let $\mathcal{H}_{s,x}$ and $\mathcal{Z}_x$ denote the mappings defined on $\Lambda_{FG}(c,\tilde{h})$ and $\Gamma_{FG}(c,u,\Phi)$, respectively, by setting

$$\mathcal{H}_{s,x}(v) = \int_0^T H(t, v(t,s,x))\mathrm{d}t \quad \text{for} \quad v \in \Lambda_{FG}(c,\tilde{h})$$

and

$$\mathcal{Z}_x(w) = \int_0^T H(t, w(t,x))\mathrm{d}t \quad \text{for} \quad w \in \Gamma_{FG}(c,u,\Phi).$$

For every fixed $(s,x) \in [0,T) \times \mathbb{R}^d$, we shall look for $\tilde{v} \in \Lambda_{FG}^{\mathcal{C}}(c,\tilde{h})$ and $\tilde{v} \in \Gamma_{FG}^{\mathcal{C}}(c,u,\Phi)$ such that $\mathcal{H}_{sx}(\tilde{v}) = \inf\{\mathcal{H}_{s,x}(v) : v \in \Lambda_{FG}^{\mathcal{C}}(c,\tilde{h})\}$ and $\mathcal{Z}_x(\tilde{v}) = \inf\{\mathcal{Z}_x(u) : u \in \Gamma_{FG}^{\mathcal{C}}(c,u,\Phi)\}$, where $\Lambda_{FG}^{\mathcal{C}}(c,\tilde{h}) = \Lambda_{FG}(c,\tilde{h}) \cap C_b^{1,1,2}(\mathbb{R}^{d+2})$ and $\Gamma_{FG}^{\mathcal{C}}(c,u,\Phi) = \Gamma_{FG}(c,u,\Phi) \cap C^{1,2}(\mathbb{R}^{d+1})$.

Theorem 3.1. *Assume that conditions* $(\mathcal{D})$ *are satisfied. Let* $c \in C([0,T]\times\mathbb{R}^d,\mathbb{R})$ *be bounded,* $\tilde{h} \in C^{1,2}(\mathbb{R}^{d+1})$*, and let* $H : [0,T]\times\mathbb{R} \to \mathbb{R}$ *be measurable and uniformly integrally bounded such that* $H(t,\cdot)$ *is continuous. If* F *and* G *are furthermore such that for* $\tilde{h}$ *and* c *as given above, the set* $\Lambda^{\mathcal{C}}_{FG}(c,\tilde{h})$ *is nonempty, then there is* $\tilde{X} \in \mathcal{X}_{s,x}(F,G)$ *such that the function* $\tilde{v} \in \Lambda^{\mathcal{C}}_{FG}(c,\tilde{h})$ *defined by*

$$\tilde{v}(t,s,x) = \tilde{E}\left[\exp\left(-\int_s^{s+t} c(\tau,\tilde{X}(\tau))\mathrm{d}\tau\right)\tilde{h}(s+t,\tilde{X}(s+t))\right]$$

for every $(s,x) \in [0,T)\times\mathbb{R}^d$ *and* $t \in [0,T-s]$ *satisfies* $\mathcal{H}_{s,x}(\tilde{v}) = \inf\{\mathcal{H}_{s,x}(v) : v \in \Lambda^{\mathcal{C}}_{FG}(c,\tilde{h})\}$.

Proof. Let $(s,x) \in [0,T)\times\mathbb{R}^d$ be fixed. The set $\{\mathcal{H}_{s,x}(v) : v \in \Lambda^{\mathcal{C}}_{FG}(c,\tilde{h})\}$ is nonempty and bounded, because there is $k \in \mathbb{L}([0,T],\mathbb{R}_+)$ such that $|\mathcal{H}_{s,x}(v)| \leq \int_0^T k(t)\mathrm{d}t$ for every $v \in \Lambda^{\mathcal{C}}_{FG}(c,\tilde{h})$. Therefore, there exists a sequence $(v^n)_{n=1}^{\infty}$ of $\Lambda^{\mathcal{C}}_{FG}(c,\tilde{h})$ such that $\alpha =: \inf\{\mathcal{H}_{s,x}(v) : v \in \Lambda^{\mathcal{C}}_{FG}(c,\tilde{h})\} = \lim_{n\to\infty}\mathcal{H}_{s,x}(v^n)$. By virtue of Theorem 6.4 of Chap. 6, for every $n = 1,2,\ldots$ and $(s,x) \in [0,T)\times\mathbb{R}^d$, there is $X^n_{s,x} \in \mathcal{X}_{s,x}(F,G)$ such that

$$v^n(t,s,x) = E^{s,x}\left[\exp\left(-\int_s^{s+t} c(\tau,X^n_{s,x}(\tau))\mathrm{d}\tau\right)\tilde{h}(s+t,X^n_{s,x}(s+t))\right]$$

for $(t,x) \in [0,T-s]\times\mathbb{R}^d$. By the weak compactness of $\mathcal{X}_{s,x}(F,G)$ and Theorem 2.3 of Chap. 1, there are an increasing subsequence $(n_k)_{k=1}^{\infty}$ of $(n)_{n=1}^{\infty}$, a probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P})$, and stochastic processes $\tilde{X}^{n_k}$ and $\tilde{X}$ on $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P})$ such that $P(X^{n_k}_{s,x})^{-1} = P(\tilde{X}^{n_k})^{-1}$ for $k = 1,2,\ldots$ and $\sup_{0\leq t\leq T}|\tilde{X}^{n_k}(t) - \tilde{X}(t)| \to 0$ a.s. Hence in particular, it follows that

$$\begin{aligned} v^{n_k}(t,s,x) &= E^{s,x}\left[\exp\left(-\int_s^{s+t} c(\tau,X^{n_k}_{s,x}(\tau))\mathrm{d}\tau\right)\tilde{h}(s+t,X^{n_k}_{s,x}(s+t))\right] \\ &= \tilde{E}\left[\exp\left(-\int_s^{s+t} c(\tau,\tilde{X}^{n_k}(\tau))\mathrm{d}\tau\right)\tilde{h}(s+t,\tilde{X}^{n_k}(s+t))\right], \end{aligned}$$

where $\tilde{E}$ is the mean value operator taken with respect to the probability measure $\tilde{P}$. By the properties of processes $\tilde{X}^{n_k}$, $\tilde{X}$ and functions c and $\tilde{h}$, it follows that

$$\lim_{k\to\infty} v^{n_k}(t,s,x) = \tilde{E}\left[\exp\left(-\int_s^{s+t} c(\tau,\tilde{X}(\tau))\mathrm{d}\tau\right)\tilde{h}(s+t,\tilde{X}(s+t))\right].$$

By virtue of Theorem 6.3 of Chap. 6, it follows that the function $\tilde{v}(t,s,x) =: \lim_{k\to\infty} v^{n_k}(t,s,x)$ belongs to $\Lambda^{\mathcal{C}}(F,G,\tilde{h},c)$, because $(\mathcal{A}_{FG}v(t\cdot))(s,x) \subset v'_s(t,s,x) + (\mathbb{L}_{FG}v(t\cdot))(s,x)$ for $(s,x) \in [0,T]\times\mathbb{R}^d$ and $t \in [0,T-s]$. Hence, by the properties of the function H, we get $\alpha = \lim_{k\to\infty}\mathcal{H}_{s,x}(v^{n_k}) = \mathcal{H}_{s,x}(\tilde{v})$. □

Theorem 3.2. *Assume that conditions* $(\mathcal{D})$ *are satisfied,* $T > 0$*, and* D *is a bounded domain in* $\mathbb{R}^d$*. Let* $c \in C([0,T]\times\mathbb{R}^d,\mathbb{R})$*,* $u \in C((0,T)\times D,\mathbb{R})$*, and* $\Phi \in C([0,T]\times \partial D,\mathbb{R})$ *be bounded. Assume that* $H : [0,T]\times\mathbb{R}\to\mathbb{R}$ *is measurable and uniformly integrally bounded such that* $H(t,\cdot)$ *is continuous. If* F *and* G *are furthermore such that for* Φ*,* u*, and* c *given above, the set* $\Gamma^{\mathcal{C}}_{FG}(c,u,\Phi)$ *belongs to* $[0,T)\times\mathbb{R}^d$ *for every* (s,x)*, then there is* $\tilde{X}_{s,x} \in \mathcal{X}_{s,x}(F,G)$ *such that the function* $\tilde{v} \in \Gamma^{\mathcal{C}}_{FG}(c,u,,\Phi)$ *defined by*

$$\tilde{v}(s,x) = E^{s,x}\left[\Phi(\tau_D,\tilde{X}_{s,x}(\tau_D))\exp\left(-\int_s^{\tau_D} c(t,\tilde{X}_{s,x}(t))\mathrm{d}t\right)\right] - E^{s,x}\left\{\int_s^{\tau_D}\left[u(t,\tilde{X}_x(t))\exp\left(-\int_s^{s+t} c(z,\tilde{X}_{s,x}(z))\mathrm{d}z\right)\right]\mathrm{d}t\right\}$$

with $\tau_D = \inf\{r \in (s,T] : \tilde{X}_{s,x}(r) \notin D\}$ *satisfies* $\mathcal{Z}_x(\tilde{v}) = \inf\{\mathcal{Z}_x(v) : v \in \Gamma^{\mathcal{C}}_{FG}(c,u,\Phi)\}$*.*

Proof. Similarly as above, we can select a sequence $(v_n)_{n=1}^\infty$ of $\Gamma^{\mathcal{C}}_{FG}(c,u,,\Phi)$ such that $\alpha = \sup\{\mathcal{Z}_x(v) : v \in \Gamma^{\mathcal{C}}_{FG}(c,u,,\Phi)\} = \lim_{n\to\infty}\mathcal{Z}_x(v_n)$ for fixed $x \in \mathbb{R}^d$. By virtue of Theorem 6.6 of Chap. 6, for every $(s,x) \in [0,T)\times\mathbb{R}^d$, there exists a sequence $(X^n_{s,x})_{n=1}^\infty$ of $\mathcal{X}_{s,x}(F,G)$ such that

$$v_n(s,x) = E_n^{s,x}\left[\Phi(\tau_D^n,X^n_{s,x}(\tau_D))\exp\left(-\int_s^{\tau_D^n} c(t,X^n_{s,x}(t))\mathrm{d}t\right)\right] - E_n\left\{\int_s^{\tau_D^n}\left[u(t,X^n_{s,x}(t))\exp\left(-\int_s^{s+t} c(z,X^n_{s,x}(z))\mathrm{d}z\right)\right]\mathrm{d}t\right\}$$

for $n \ge 1$, where $\tau_D^n = \inf\{r \in (s,T] : X^n_{s,x}(r) \notin D\}$. By virtue of Theorem 4.1 of Chap. 4 and Theorem 2.3 of Chap. 1, there are an increasing subsequence $(n_k)_{k=1}^\infty$ of $(n)_{n=1}^\infty$, a probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P})$, and stochastic processes $\tilde{X}^{n_k}_{s,x}$ and $\tilde{X}_{s,x}$ on $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P})$ such that $P(X^{n_k}_{s,x})^{-1} = P(\tilde{X}^{n_k}_{s,x})^{-1}$ for $k = 1,2,\dots$ and $\sup_{s\le t\le T}|\tilde{X}^{n_k}_{s,x}(t) - \tilde{X}_{s,x}(t)| \to 0$ a.s. Hence by Theorem 5.2 of Chap. 4, it follows that

$$\begin{aligned} v_{n_k}(s,x) &= E^{s,x}_{n_k}\left[\Phi(\tau_D^{n_k},X^{n_k}_{s,x}(\tau_D^{n_k}))\exp\left(-\int_s^{\tau_D^{n_k}} c(t,X^{n_k}_x(t))\mathrm{d}t\right)\right] \\ &\quad - E^{s,x}_{n_k}\left\{\int_s^{\tau_D^{n_k}}\left[u(t,X^{n_k}_{s,x}x(t))\exp\left(-\int_s^{s+t} c(z,X^{n_k}_{s,x}(z))\mathrm{d}z\right)\right]\mathrm{d}t\right\} \\ &= \tilde{E}\left[\Phi(\tilde{\tau}_D^{n_k},\tilde{X}^{n_k}_{s,x}(\tilde{\tau}_D^{n_k}))\exp\left(-\int_s^{\tilde{\tau}_D^{n_k}} c(t,\tilde{X}^{n_k}_{s,x}(t))\mathrm{d}t\right)\right] \\ &\quad - \tilde{E}\left\{\int_s^{\tilde{\tau}_D^{n_k}}\left[u(t,\tilde{X}^{n_k}_{s,x}(t))\exp\left(-\int_s^{s+t} c(z,\tilde{X}^{n_k}_{s,x}(z))\mathrm{d}z\right)\right]\mathrm{d}t\right\} = \tilde{v}_{n_k}(s,x) \end{aligned}$$

for $(s,x) \in [0,T) \times D$ and $k \geq 1$, where $\tilde{\tau}_D^{n_k} = \inf\{r \in (s,T] : \tilde{X}_{s,x}^{n_k}(r) \notin D\}$. Therefore, by Lemma 10.1 of Chap. 1, Theorem 5.1 of Chap. 4, and the properties of the sequence $(\tilde{X}_{s,x}^{n_k})_{k=1}^{\infty}$, one obtains

$$\lim_{k\to\infty} \tilde{v}_{n_k}(s,x) = \tilde{E}\left[\Phi(\tilde{\tau}_D, \tilde{X}_{s,x}(\tilde{\tau}_D)) \exp\left(-\int_s^{\tilde{\tau}_D} c(t, \tilde{X}_x(t))\mathrm{d}t\right)\right] - \tilde{E}\left\{\int_s^{\tilde{\tau}_D}\left[u(t, \tilde{X}_{s,x}(t)) \exp\left(-\int_s^{s+t} c(z, \tilde{X}_{s,x}(z))\mathrm{d}z\right)\right]\mathrm{d}t\right\},$$

where $\tilde{\tau}_D = \inf\{r \in (s,T] : \tilde{X}_{s,x}(r) \notin D\}$. Immediately from Theorem 6.6 of Chap. 6, it follows that the function $\tilde{v}$ defined by

$$\tilde{v}(s,x) = \tilde{E}\left[\Phi(\tilde{\tau}_D, \tilde{X}_{s,x}(\tilde{\tau}_D)) \exp\left(-\int_s^{\tilde{\tau}_D} c(t, \tilde{X}_x(t))\mathrm{d}t\right)\right] - \tilde{E}\left\{\int_s^{\tilde{\tau}_D}\left[u(t, \tilde{X}_{s,x}(t)) \exp\left(-\int_s^{s+t} c(z, \tilde{X}_{s,x}(z))\mathrm{d}z\right)\right]\mathrm{d}t\right\}$$

belongs to $\Gamma_{FG}^{\mathcal{C}}(c,u,\Phi)$. Finally, similarly as above, we get $\alpha = \lim_{k\to\infty} \mathcal{Z}_x(\tilde{v}_{n_k}) = \mathcal{Z}_x(\tilde{v})$. □

In a similar way, we can also prove similar theorems for control systems described by set-valued stochastic Dirichlet, Poisson, and Dirichlet–Poisson problems. To formulate them, let us recall the basic notation dealing with such problems. Let $T > 0$ and let $D \subset \mathbb{R}^d$ be a nonempty bounded domain. Assume that $F : \mathbb{R}^+ \times \mathbb{R}^d \to \mathrm{Cl}(\mathbb{R}^d)$ and $G : \mathbb{R}^+ \times \mathbb{R}^d \to \mathrm{Cl}(\mathbb{R}^{d\times m})$ are measurable and bounded, and let $\Phi : (0,T) \times \partial D \to \mathbb{R}$, $\varphi : (0,T) \times D \to \mathbb{R}$ and $\psi : (0,T) \times D \to \mathbb{R}$ be continuous and bounded. Let $\mathcal{D}_{FG}(\Phi)$, $\mathcal{P}_{FG}(\varphi)$ and $\mathcal{R}_{FG}(\Phi,\psi)$ be defined by

$$\mathcal{D}_{FG}(\Phi) = \{u(s,x) = E^{s,x}[\Phi(\tau_D, X_{s,x}(\tau_D))] : X_{s,x} \in \mathcal{X}_{s,x}(F,G)\},$$

$$\mathcal{P}_{FG}(\varphi) = \left\{v(s,x) = E^{s,x}\left[\int_0^{\tau_D} \varphi(\tau_D, X_{s,x}(\tau_D))\right] : X_{s,x} \in \mathcal{X}_{s,x}(F,G)\right\},$$

and

$$\mathcal{R}_{FG}(\Phi,\psi) = \left\{w : w(s,x) = E^{s,x}[\Phi(\tau_D, X_{s,x}(\tau_D))] + E^{s,x}\left[\int_0^{\tau_D} \varphi(\tau_D, X_{s,x}(\tau_D))\right] : X_{s,x} \in \mathcal{X}_{s,x}(F,G)\right\}.$$

Immediately from Theorem 6.7, Theorem 6.8, and Theorem 6.9 of Chap. 6, it follows that $\mathcal{D}_{FG}(\Phi)$, $\mathcal{P}_{FG}(\varphi)$, and $\mathcal{R}_{FG}(\Phi,\psi)$ are subsets of the sets of all solutions of the following stochastic set-valued boundary value problems:

$$\begin{cases} 0 \in (\mathcal{L}_{FG}u)\,(t,x) \text{ for } (t,x) \in [0,T) \times D, \\ \lim_{t\to\tau_D} u(t, X_{s,x}(t) = \Phi(\tau_D, X_{s,x}(\tau_D)) \text{ for } (s,x) \in (0,T) \times D \quad a.s., \end{cases}$$

$$\begin{cases} -\varphi(s,x) \in (\mathcal{L}_{FG}v)\,(t,x) \text{ for } (t,x) \in [0,T) \times D, \\ \lim_{t\to\tau_D} v(t, X_{s,x}(t)) = 0)) \text{ for } (s,x) \in (0,T) \times D \quad a.s., \end{cases}$$

and

$$\begin{cases} -\varphi(s,x) \in (\mathcal{L}_{FG}w)\,(t,x) \text{ for } (t,x) \in [0,T) \times D, \\ \lim_{t\to\tau_D} w(t, X_{s,x}(t)) = \Phi(\tau_D, X_{s,x}(\tau_D)) \text{ for } (s,x) \in (0,T) \times D \quad a.s., \end{cases}$$

respectively. Similarly as above, we obtain the following results.

Theorem 3.3. *Assume that conditions* $(\mathcal{D})$ *are satisfied,* $T > 0$*, and* D *is a bounded domain in* $\mathbb{R}^d$*. Let* $\Phi \in C([0,T] \times \partial D, \mathbb{R})$ *be continuous and bounded. Assume that* $H : [0,T] \times \mathbb{R} \to \mathbb{R}$ *is measurable and uniformly integrally bounded such that* $H(t,\cdot)$ *is continuous. For every* $(s,x) \in (0,T) \times D$*, there is* $\tilde{X}_{s,x} \in \mathcal{X}_{s,x}(F,G)$ *such that the function* $\tilde{u}(s,x) = E^{s,x}[\Phi(\tilde{\tau}_D, \tilde{X}_{s,x}(\tilde{\tau}_D))]$ *satisfies* $\mathcal{Z}_x(\tilde{u}) = \sup\{\mathcal{Z}_x(u) : u \in \mathcal{D}_{FG}(\Phi)\}$*, where* $\tilde{\tau}_D = \inf\{r \in (0,T] : \tilde{X}_{s,x}(r) \notin D\}$.

Proof. Similarly as above, we can select a sequence $(u_n)_{n=1}^{\infty}$ of $\mathcal{D}_{FG}(\Phi)$ such that $\alpha = \sup\{\mathcal{Z}_x(u) : u \in \mathcal{D}_{FG}(\Phi)\} = \lim_{n\to\infty} \mathcal{Z}_x(u_n)$. By the definition of $\mathcal{D}_{FG}(\Phi)$, there exists a sequence $(X_x^n)_{n=1}^{\infty}$ of $\mathcal{X}_{s,x}(F,G)$ such that $u_n(s,x) = E^{s,x}[\Phi(\tau_D^n, X_{s,x}^n(\tau_D^n))]$, where $\tau_D^n = \inf\{r \in (0,T] : X_x^n(r) \notin D\}$. By virtue of Theorem 4.1 of Chap. 4 and Theorem 2.3 of Chap. 1, there are an increasing subsequence $(n_k)_{k=1}^{\infty}$ of $(n)_{n=1}^{\infty}$, a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, and stochastic processes $\tilde{X}^{n_k}$ and $\tilde{X}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ such that $P(X_{s,x}^{n_k})^{-1} = P(\tilde{X}^{n_k})^{-1}$ for $k = 1,2,\ldots$ and $\sup_{0\le t\le T} |\tilde{X}^{n_k}(t) - \tilde{X}(t)| \to 0$ a.s. Hence, similarly as in the proof of Theorem 3.2, it follows that $\alpha = \lim_{n\to\infty} \mathcal{Z}_x(u_{n_k}) = \mathcal{Z}_x(\tilde{u})$, where $\tilde{u}(s,x) = E^{s,x}[\Phi(\tilde{\tau}_D, \tilde{X}_{s,x}(\tilde{\tau}_D))]$.□

Theorem 3.4. *Assume that conditions* $(\mathcal{D})$ *are satisfied,* $T > 0$*, and* D *is a bounded domain in* $\mathbb{R}^d$*. Let* $\varphi : (0,T) \times D \to \mathbb{R}$ *be continuous and bounded, and let* $H : [0,T] \times \mathbb{R} \to \mathbb{R}$ *be measurable and uniformly integrally bounded such that* $H(t,\cdot)$ *is continuous. For every* $(s,x) \in (0,T) \times D$*, there is* $\tilde{X}_{s,x} \in \mathcal{X}_{s,x}(F,G)$ *such that the function* $\tilde{u}(s,x) = E^{s,x}[\Phi(\tilde{\tau}_D, \tilde{X}_{s,x}(\tilde{\tau}_D))]$ *satisfies* $\mathcal{Z}_x(\tilde{u}) = \sup\{\mathcal{Z}_x(u) : u \in \mathcal{P}_{FG}(\varphi)\}$*, where* $\tilde{\tau}_D = \inf\{r \in (0,T] : \tilde{X}_{s,x}(r) \notin D\}$. □

Theorem 3.5. *Assume that conditions* $(\mathcal{D})$ *are satisfied,* $T > 0$*, and* D *is a bounded domain in* $\mathbb{R}^d$*. Let* $\Phi \in C((0,T) \times \partial D, \mathbb{R})$ *and* $\psi : (0,T) \times D \to \mathbb{R}$ *be continuous and bounded, and let* $H : [0,T] \times \mathbb{R} \to \mathbb{R}$ *be measurable and uniformly integrally* *bounded such that* $H(t,\cdot)$ *is continuous. For every* $(s,x)\in(0,T)\times D$*, there exists* $\tilde{X}_{s,x}\in\mathcal{X}_{s,x}(F,G)$ *such that the function* $\tilde{u}(s,x)=E^{s,x}[\Phi(\tilde{\tau}_D,\tilde{X}_{s,x}(\tilde{\tau}_D))]$ *satisfies* $\mathcal{Z}_x(\tilde{u})=\sup\{\mathcal{Z}_x(u):u\in\mathcal{R}_{FG}(\Phi,\psi)\}$*, where* $\tilde{\tau}_D=\inf\{r\in(0,T]:\tilde{X}_{s,x}(r)\notin D\}$*.* □

4 Notes and Remarks

The results of this chapter are consequences of the properties of the set $\mathcal{X}_{s,x}(F,G)$ of all (equivalence classes of) weak solutions for $SFI(F,G)$ and the representation theorems presented in Chap. 6. It is possible to consider problems with weaker assumptions. It is important to observe that such an approach reduces the optimal control problems described by stochastic functional and partial differential inclusions to the existence of optimal problems of functionals defined on weakly compact subsets of the space $\mathcal{M}(\mathcal{X})$ of probability measures defined on a Borel σ-algebra $\beta(\mathcal{X})$ of a complete metric space $\mathcal{X}$. Furthermore, this approach, together with representation theorems, leads to the representation of optimal solutions of the above type of optimal control problems by weak solutions of stochastic functional inclusions. This allows us in some special cases to determine explicit solutions of such optimal control problems. Some applications of weak solutions of multivalued stochastic equations to optimal control problems are given by A. Zălinescu in [97]. Some optimal control problems described by stochastic differential equations depending on control parameters can be solved explicitly by solving appropriate HJB equations. As pointed out (see B. Øksendal [86]) at the beginning of this chapter, some solutions of these equations can also be represented by weak solutions of stochastic differential equations. More information dealing with such problems can be found in B. Øksendal [86] and J. Yong and X.Y. Zhou [96].

Let us observe (see [45]) that there are three major approaches to stochastic optimal control: dynamic programming, duality, and the maximum principle. Dynamic programming obtains, by means of the optimality principle of Bellman, the Hamilton–Jacobi–Bellman equation, which characterizes the value function (see [28, 29, 37, 64, 98]). Under some smoothness and regularity assumptions on the solution, it is possible to obtain, at least implicitly, the optimal control. This is the content of the so-called verification theorem, which appears in W.H. Fleming and R.M. Rishel [28] or W.H. Fleming and H.M. Soner [29]. However, the problem of recovering the optimal control from the gradient of the value function by means of solving a static optimization remains, and this can be difficult to do. Duality methods, also known in stochastic control theory as the martingale approach, have become very popular in recent years, because they provide powerful tools for studying some classes of stochastic control problems, usually connected with some approximative procedures (see [73]). Martingale methods are particularly useful for problems appearing in finance (see [26]), such as the model of R.C. Merton [74]. Duality reduces the original problem to one of finite dimension. The approach is based on the martingale representation theorem and the Girsanov transformation.

The stochastic maximum principle has been developed completely in recent years in S. Peng [87]. It is a counterpart of the maximum principle for deterministic problems. The distinctive feature is the use of the concept of forward–backward stochastic differential equations, which arise naturally, governing the evolution of the state variables. See H.J. Kushner [67], J.M. Bismut [19,20], or U.G. Haussmann [36].

Control problems and optimal control problems for systems described by stochastic and partial differential equations have been considered by many authors. The classical optimal control problems for systems described by stochastic differential equations and inclusions were considered by, among others, N.A. Ahmed [1], A. Friedman [30], W.H. Fleming and M. Nisio [27], and M. Michta [75]. Optimal control problems for partial differential equations were considered by, for example, W. Huckbusch in [34]