Chapter 5
Viability Theory

The results of this chapter deal with the existence of viable solutions for stochastic functional and backward inclusions. Weak compactness of sets of all viable weak solutions of stochastic functional inclusions is also considered.

1 Some Properties of Set-Valued Stochastic Functional Integrals Depending on Parameters

Let $F : [0,T] \times \mathbb{R}^d \to \mathrm{Cl}(\mathbb{R}^d)$ and $G : [0,T] \times \mathbb{R}^d \to \mathrm{Cl}(\mathbb{R}^{d\times m})$ be measurable and square integrably bounded set-valued mappings. Given a set-valued stochastic process $(K(t))_{0\le t\le T}$ with values in $\mathrm{Cl}(\mathbb{R}^d)$, we denote by $\overline{SFI}(F,G,K)$ the following viability problem:

$$\begin{cases} x_t - x_s \in \mathrm{cl}_{\mathbb{L}}\{J_{st}[S_{\mathbb{F}}(F\circ x)] + \mathcal{J}_{st}[S_{\mathbb{F}}(G\circ x)]\} & \text{for} \quad 0\le s\le t\le T, \\ x_t \in K(t) \;\; \text{a.s.} \;\; \text{for} \quad t\in[0,T], \end{cases} \tag{1.1}$$

associated with $\overline{SFI}(F,G)$. Similarly, we denote by $BSDI(F,K)$ the backward viability problem:

$$\begin{cases} x_s \in E[x_t + \int_s^t F(\tau,x_\tau)\mathrm{d}\tau|\mathcal{F}_s] & \text{a.s.} \;\; \text{for} \quad 0\le s\le t\le T, \\ x_t \in K(t) \quad \text{a.s.} \;\; \text{for} \quad t\in[0,T], \end{cases} \tag{1.2}$$

associated with $BSDI(F,K(T))$.

We precede the existence theorems for such problems by some properties of set-valued stochastic functional integrals depending on parameters. Given a Banach space $(X,\|\cdot\|)$, by $\mathrm{Cl}(X)$ we denote the space of all nonempty closed subsets

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of X. In particular, we shall consider X to be equal to $\mathbb{R}^d$, $\mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{R}^r)$, and $\mathbb{L}^2([0,T] \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^r$ with $r = d$ and $r = d \times m$, respectively. The Hausdorff metrics on these spaces will be denoted by h, D, and H, respectively.

Let $\mathcal{P}_{\mathbb{F}} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ satisfying the usual conditions. Similarly as above, for set-valued mappings F and G as given above and an $\mathbb{F}$-nonanticipative d-dimensional stochastic process $x = (x_t)_{0 \le t \le T}$, we shall denote by $S_{\mathbb{F}}(F \circ x)$ and $S_{\mathbb{F}}(G \circ x)$ the sets of all $\mathbb{F}$-nonanticipative stochastic processes $f = (f_t)_{0 \le t \le T}$ and $g = (g_t)_{0 \le t \le T}$, respectively, such that $f_t \in F(t, x_t)$ and $g_t \in G(t, x_t)$ a.s. for a.e. $t \in [0,T]$. It is clear that $S_{\mathbb{F}}(F \circ x)$ and $S_{\mathbb{F}}(G \circ x)$ are decomposable closed subsets of $\mathbb{L}^2([0,T] \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^d)$ and $\mathbb{L}^2([0,T] \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d \times m})$, respectively, where $\Sigma_{\mathbb{F}}$ denotes the σ-algebra of all $\mathbb{F}$-nonanticipative subsets of $[0,T] \times \Omega$. Therefore, by virtue of Theorem 3.2 of Chap. 2, there exist $\Sigma_{\mathbb{F}}$-measurable mappings Φ and Ψ such that $S_{\mathbb{F}}(F \circ x) = S_{\mathbb{F}}(\Phi)$ and $S_{\mathbb{F}}(G \circ x) = S_{\mathbb{F}}(\Psi)$, which by virtue of Corollary 3.1 of Chap. 2, implies that $\Phi = F \circ x$ and $\Psi = G \circ x$.

In what follows, we shall denote by $|\cdot|$ the norm of the Banach space $\mathcal{X}^r = \mathbb{L}^2([0,T] \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^r)$ with $r = d$ or $r = d \times m$. Similarly as above, $\mathbf{C}(\mathbb{F}, \mathbb{R}^d)$ denotes the space of all d-dimensional continuous $\mathbb{F}$-adapted stochastic processes $x = (x_t)_{0 \le t \le T}$ with norm $\|x\| = (E[\sup_{0 \le t \le T} |x_t|^2])^{1/2}$. Given a measurable and uniformly square integrably bounded set-valued mapping $K : [0,T] \times \Omega \to \mathrm{Cl}(\mathbb{R}^d)$, we shall assume that the set $\mathcal{K}(t) = \{u \in \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{R}^d) : u \in K(t, \cdot)\ a.s.\}$ is nonempty for every $0 \le t \le T$. It is clear that this requirement is satisfied for a square integrably bounded multifunction $K : [0,T] \to \mathrm{Cl}(\mathbb{R}^d)$. Recall that $K : [0,T] \times \Omega \to \mathrm{Cl}(\mathbb{R}^d)$ is said to be uniformly square integrably bounded if there exists $\lambda \in \mathbb{L}^2([0,T], \mathbb{R}^+)$ such that $\|K(t,\omega)\| \le \lambda(t)$ for a.e. $(t,\omega) \in [0,T] \times \Omega$, where $\|K(t,\omega)\| = h(K(t,\omega), \{0\})$. Let us observe that for the above multifunctions F and G and a d-dimensional $\mathcal{F}_t$-measurable random variable X, the set-valued processes $F \circ X$ and $G \circ X$ are $\beta_T \otimes \mathcal{F}_t$-measurable.

Assume that the above set-valued mappings F and G satisfy the following conditions $(\mathcal{H}_1)$:

(i) $F : [0,T] \times \mathbb{R}^d \to \mathrm{Cl}(\mathbb{R}^d)$ and $G : [0,T] \times \mathbb{R}^d \to \mathrm{Cl}(\mathbb{R}^{d \times m})$ are measurable and uniformly square integrably bounded, i.e., there exists $m \in \mathbb{L}^2([0,T], \mathbb{R}^+)$ such that $\max(\|F(t,x)\|, \|G(t,x)\|) \le m(t)$ for a.e. $t \in [0,T]$ and $x \in \mathbb{R}^d$, where $\|F(t,x)\| = \sup\{|z| : z \in F(t,x)\}$ and $\|G(t,x)\| = \sup\{|z| : z \in G(t,x)\}$;

(ii) $F(t,\cdot)$ and $G(t,\cdot)$ are Lipschitz continuous for a.e. fixed $t \in [0,T]$, i.e., there exists $k \in \mathbb{L}^2([0,T], \mathbb{R}^+)$ such that $H(F(t,x), F(t,z)) \le k(t)|x - z|$ and $H(G(t,x), G(t,z)) \le k(t)|x - z|$ for a.e. $t \in [0,T]$ and $x, z \in \mathbb{R}^d$.

Lemma 1.1. *If F and G satisfy conditions $(\mathcal{H}_1)$, then the set-valued mappings* $\mathbf{C}(\mathbb{F}, \mathbb{R}^d) \ni x \to S_{\mathbb{F}}(F \circ x) \in \mathrm{Cl}(\mathcal{X}^d)$ *and* $\mathbf{C}(\mathbb{F}, \mathbb{R}^d) \ni x \to S_{\mathbb{F}}(G \circ x) \in \mathrm{Cl}(\mathcal{X}^{d \times m})$ *are Lipschitz continuous with Lipschitz constant* $L = [\int_0^T k^2(t)\mathrm{d}t]^{1/2}$.

Proof. The proof is quite similar to the proof of Lemma 3.7 of Chap. 2. Let $x, z \in \mathbf{C}(\mathbb{F}, \mathbb{R}^d)$ and $f^x \in S_{\mathbb{F}}(F \circ x)$. By virtue of Theorem 3.1 of Chap. 2 applied to the $\Sigma_{\mathbb{F}}$-measurable set-valued mapping $F \circ z$, we get

$$\begin{aligned}\operatorname{dist}^2(f^x, S_{\mathbb{F}}(F \circ z)) &= \inf\left\{E\int_0^T |f^x_\tau - f_\tau|^2 \mathrm{d}\tau : f \in S_{\mathbb{F}}(F \circ z)\right\}\\ &= E\int_0^T \operatorname{dist}^2(f^x_\tau, F(\tau, z_\tau))\mathrm{d}\tau\\ &\leq E\int_0^T k^2(t)|x_t - z_t|^2 \mathrm{d}t \leq L^2\|x - z\|^2.\end{aligned}$$

Then $\overline{H}(S_{\mathbb{F}}(F \circ x), S_{\mathbb{F}}(F \circ z)) \leq L\|x - z\|$. In a similar way, we also get $\overline{H}(S_{\mathbb{F}}(F \circ z), S_{\mathbb{F}}(F \circ x)) \leq L||x - z||$. Therefore, $H(S_{\mathbb{F}}(F \circ x), S_{\mathbb{F}}(F \circ z)) \leq L\|x - z\|$. In a similar way, we obtain $H(S_{\mathbb{F}}(G \circ x), S_{\mathbb{F}}(G \circ z)) \leq L\|x - z\|$. □

Lemma 1.2. *Let* $K : [0, T] \times \Omega \to \mathrm{Cl}(\mathbb{R}^d)$ *be* $\mathbb{F}$*-adapted and square integrably bounded uniformly with respect to* $t \in [0, T]$. *If* $K(\cdot, \omega)$ *is continuous for a.e.* $\omega \in \Omega$*, then the set-valued mapping* $\mathcal{K} : [0, T] \to \mathrm{Cl}(\mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{R}^d))$ *is continuous.*

Proof. Let $t_0 \in [0, T]$ be fixed and let $(t_k)_{k=1}^{\infty}$ be a sequence of $[0, T]$ converging to t_0. By virtue of Theorem 3.1 of Chap. 2, for every $u \in \mathcal{K}(t_0)$ and $k \geq 1$, one has

$$\begin{aligned}\operatorname{dist}^2(u, \mathcal{K}(t_k)) &= \inf\left\{E|u - v|^2 : v \in \mathcal{K}(t_k)\right\}\\ &\leq E\left[\operatorname{dist}^2(u, K(t_k, \cdot)\right]\\ &\leq E\left[h^2(K(t_k, \cdot), K(t_0, \cdot))\right].\end{aligned}$$

Then $\overline{D}^2(\mathcal{K}(t_0), \mathcal{K}(t_k)) \leq E\left[h^2(K(t_k, \cdot), K(t_0, \cdot))\right]$. In a similar way, we also get $\overline{D}^2(\mathcal{K}(t_k), \mathcal{K}(t_0)) \leq E\left[h^2(K(t_k, \cdot), K(t_0, \cdot))\right]$. Therefore, for every $k \geq 1$, one has $D^2(\mathcal{K}(t_k), \mathcal{K}(t_0)) \leq E\left[h^2(K(t_k, \cdot), K(t_0, \cdot))\right]$. Hence, by the continuity of $K(\cdot, \omega)$ and its uniformly square integrable boundedness, it follows that $\lim_{k\to\infty} D(\mathcal{K}(t_k), \mathcal{K}(t_0)) = 0$. □

Lemma 1.3. *If* F *and* G *satisfy conditions* $(\mathcal{H}_1)$*, then the set-valued mappings* $\mathbf{C}(\mathbb{F}, \mathbb{R}^d) \ni x \to \mathrm{cl}_{\mathbb{L}}\{J_{st}[S_{\mathbb{F}}(F \circ x)]\} \subset \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{R}^d)$ *and* $\mathbf{C}(\mathbb{F}, \mathbb{R}^d) \ni x \to \mathrm{cl}_{\mathbb{L}}\{\mathcal{J}_{st}[S_{\mathbb{F}}(G \circ x)]\} \subset \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{R}^d)$ *are Lipschitz continuous uniformly with respect to* $0 \leq s < t \leq T$ *with Lipschitz constants equal to* $\sqrt{T}L$ *and* L*, respectively, where L is as in Lemma 1.1.*

Proof. Let $x, z \in \mathcal{C}(\mathbb{F}, \mathbb{R}^d)$ and $f^x \in S_{\mathbb{F}}(F \circ x)$. For fixed $0 \leq s < t \leq T$, we have $\operatorname{dist}^2(J_{st}(f^x), J_{st}[S_{\mathbb{F}}(F \circ z]) = \inf\left\{E|J_{st}(f^x - f^z)|^2 : f^z \in S_{\mathbb{F}}(F \circ z)\right\}$. But for every $0 \leq s < t \leq T$, one has

$$E\,|J_{st}(f^x - f^z)|^2 \leq TE\left[\int_0^T |f^x - f^z|^2 \mathrm{d}t\right].$$

Therefore, by Lemma 3.6 of Chap. 2, it follows that

$$\begin{aligned}\mathrm{dist}^2\left(J_{st}(f^x), J_{st}[S_{\mathbb{F}}(F\circ z)]\right) &\le T\inf\left\{E\int_0^T |f^x - f^z|^2 \mathrm{d}t : f^z \in S_{\mathbb{F}}(F\circ z)\right\}\\ &= T\,\mathrm{dist}^2\left(f^x, S_{\mathbb{F}}(F\circ z)\right)\\ &\le TH(S_{\mathbb{F}}(F\circ x), S_{\mathbb{F}}(F\circ z)) \le TL^2\|x-z\|^2.\end{aligned}$$

Then for every $0 \le s < t \le T$, one obtains

$$\overline{D}^2\left(J_{st}[S_{\mathbb{F}}(F\circ x)], J_{st}[S_{\mathbb{F}}(F\circ z)]\right) \le TL^2\|x-z\|^2.$$

Similarly, for every fixed $0 \le s < t \le T$, we also get

$$\overline{D}^2\left(J_{st}[S_{\mathbb{F}}(F\circ z)], J_{st}[S_{\mathbb{F}}(F\circ x)]\right) \le TL^2\|x-z\|^2.$$

Therefore, for every $0 \le s < t \le T$, one has

$$D\left(J_{st}[S_{\mathbb{F}}(F\circ x)], J_{st}[S_{\mathbb{F}}(F\circ z)]\right) \le \sqrt{T}L\|x-z\|.$$

In a similar way, for fixed $0 \le s < t \le T$, we obtain

$$D\left(\mathcal{J}_{st}[S_{\mathbb{F}}(G\circ x)], \mathcal{J}_{st}[S_{\mathbb{F}}(G\circ z)]\right) \le L\|x-z\|.$$

Hence it follows that

$$\sup_{0\le s<t\le T} D\left(\mathrm{cl}_{\mathbb{L}}\{J_{st}[S_{\mathbb{F}}(F\circ x)]\}, \mathrm{cl}_{\mathbb{L}}\{J_{st}[S_{\mathbb{F}}(F\circ z)]\}\right) \le \sqrt{T}L\|x-z\|$$

and

$$\sup_{0\le s<t\le T} D\left(\mathrm{cl}_{\mathbb{L}}\{\mathcal{J}_{st}[S_{\mathbb{F}}(G\circ x)]\}, \mathrm{cl}_{\mathbb{L}}\{\mathcal{J}_{st}[S_{\mathbb{F}}(G\circ z)]\}\right) \le L\|x-z\|.$$

□

Lemma 1.4. *Assume that F and G satisfy (i) of $(\mathcal{H}_1)$ and let $x_n, x \in \mathbf{C}(\mathbb{F}, \mathbb{R}^d)$ for $n = 1,2,\ldots$ be such that $\sup_{0\le t\le T}|x_n(t) - x(t)| \to 0$ a.s. as $n \to \infty$. If $F(t,\cdot)$ and $G(t,\cdot)$ are continuous for a.e. fixed $0 \le t \le T$, and $(\theta_n)_{n=1}^{\infty}$ is a sequence of functions $\theta_n : [0,T] \to [0,T]$ such that $\theta_n(t) \to t$ as $n \to \infty$ for every $t \in [0,T]$, then $\mathrm{cl}_{\mathbb{L}}\{J_{st}[S_{\mathbb{F}}(F\circ(x_n\circ\theta_n))] + \mathcal{J}_{st}[S_{\mathbb{F}}(G\circ(x_n\circ\theta_n))]\} \to \mathrm{cl}_{\mathbb{L}}\{J_{st}[S_{\mathbb{F}}(F\circ x)] + \mathcal{J}_{st}[S_{\mathbb{F}}(G\circ x)]\}$ in the D-metric topology of $\mathrm{Cl}(\mathbb{L}^2(\Omega,\mathcal{F},\mathbb{R}^d))$ as $n\to\infty$ for every $0 \le s \le t \le T$.*

Proof. Let $0 \le s \le t \le T$ be fixed and set $y^n = x_n\circ\theta_n$ for every $n = 1,2,\ldots$. One has

$$
\begin{aligned}
|y_t^n(\tau) - x(\tau)| &= |x_n(\theta_n(\tau)) - x(\tau)| \\
&\le |x_n(\theta_n(\tau)) - x(\theta_n(\tau))| + |x(\theta_n(\tau)) - x(\tau)| \\
&\le \sup_{0\le u\le T} |x_n(u) - x(u)| + |x(\theta_n(t)) - x(t)|
\end{aligned}
$$

for $n = 1, 2, \ldots$ and $0 \le \tau \le T$. Then $y_t^n(\tau) \to x(\tau)$ a.s. for every $0 \le \tau \le T$ as $n \to \infty$. Similarly as in the proof of Lemma 1.3, we can verify that the set-valued mappings $\mathbf{C}(\mathbb{F}, \mathbb{R}^d) \ni x \to \mathrm{cl}_{\mathbb{L}}\{J_{st}[S_{\mathbb{F}}(F \circ x)]\} \in \mathrm{Cl}(\mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{R}^d))$ and $\mathbf{C}(\mathbb{F}, \mathbb{R}^d) \ni x \to \mathrm{cl}_{\mathbb{L}}\{\mathcal{J}_{st}[S_{\mathbb{F}}(G \circ x)]\} \in \mathrm{Cl}(\mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{R}^d))$ are continuous. Therefore, $\mathrm{cl}_{\mathbb{L}}\{J_{st}[S_{\mathbb{F}}(F \circ (x_n \circ \theta_n))] + \mathcal{J}_{st}[S_{\mathbb{F}}(G \circ (x_n \circ \theta_n))]\} \to \mathrm{cl}_{\mathbb{L}}\{J_{st}[S_{\mathbb{F}}(F \circ x)] + \mathcal{J}_{st}[S_{\mathbb{F}}(G \circ x)]\}$ in the D-metric topology as $n \to \infty$. □

2 Viable Approximation Theorems

The existence of solutions of viability problems (1.1) and (1.2) will follow from some viable approximation theorems by applying the standard methods presented in the proofs of the existence of strong and weak solutions for stochastic functional inclusions. We shall now present such approximation theorems. In what follows, it will be convenient to denote by $d(x, A)$ the distance $\mathrm{dist}(x, A)$ of $x \in X$ to a nonempty set $A \subset X$. We shall also denote the set-valued functional integrals $J_{st}[S_{\mathbb{F}}\Phi)]$ and $\mathcal{J}_{st}[S_{\mathbb{F}}\Psi)]$ of $\mathbb{F}$-nonanticipative set-valued processes $\Phi \in \mathcal{L}^2_{\mathbb{F}}(T, \Omega, \mathbb{R}^d)$ and $\Psi \in \mathcal{L}^2_{\mathbb{F}}(T, \Omega, \mathbb{R}^{d\times m})$ by $\int_s^t \Phi_\tau \mathrm{d}\tau$ and $\int_s^t \Psi_\tau \mathrm{d}B_\tau$, respectively.

We shall prove the following approximation theorems.

Theorem 2.1. *Assume that F and G satisfy condition (i) of $(\mathcal{H}_1)$ and let $\mathcal{P}_{\mathbb{F}} = (\Omega, \mathcal{F}, \mathbb{F}, P)$ be a complete filtered probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ such that there exists an m-dimensional $\mathbb{F}$-Brownian motion $B = (B_t)_{0\le t\le T}$ defined on $\mathcal{P}_{\mathbb{F}}$. Let $K : [0, T] \to \mathrm{Cl}(\mathbb{R}^d)$ be such that a set-valued process $(\mathcal{K}(t))_{0\le t\le T}$ is continuous. If*

$$
\liminf_{h\to 0+} \frac{1}{h} \overline{D}\left[x + \mathrm{cl}_{\mathbb{L}}\left(\int_t^{t+h} F(\tau, x)\mathrm{d}\tau + \int_t^{t+h} G(\tau, x)\mathrm{d}B_\tau\right), \mathcal{K}(t + h)\right] = 0 \tag{2.1}
$$

for every $(t, x) \in Graph(\mathcal{K}^\varepsilon)$ and every $\varepsilon \in (0, 1)$, where $\mathcal{K}^\varepsilon(t) = \{u \in \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{R}^d) : d(u, \mathcal{K}(t)) \le \varepsilon\}$ for every $0 \le t \le T$, then for every $\varepsilon \in (0, 1)$ and $x_0 \in \mathcal{K}(0)$, there exist a step function $\theta_\varepsilon : [0, T] \to [0, T]$ and $\mathbb{F}$-nonanticipative stochastic processes $f^\varepsilon = (f_t^\varepsilon)_{0\le t\le T}$ and $g^\varepsilon = (g_t^\varepsilon)_{0\le t\le T}$ such that

(i) $f^\varepsilon \in S_{\mathbb{F}}(F \circ (x^\varepsilon \circ \theta_\varepsilon))$ and $g^\varepsilon \in S_{\mathbb{F}}(G \circ (x^\varepsilon \circ \theta_\varepsilon))$, where $x^\varepsilon(t) = x_0 + \int_0^t f_\tau^\varepsilon \mathrm{d}\tau + \int_0^t g_\tau^\varepsilon \mathrm{d}B_\tau$ for $0 \le t \le T$;
(ii) $E[\mathrm{dist}(x^\varepsilon(\theta_\varepsilon(t)), K(\theta_\varepsilon(t))] \le \varepsilon$ for $0 \le t \le T$;

(iii) $E\left[l(x^\varepsilon(s))\left(h(x^\varepsilon(t)) - h(x^\varepsilon(s)) - \int_s^t (\mathbb{L}^{x^\varepsilon}_{f^\varepsilon g^\varepsilon} h)_\tau \mathrm{d}\tau\right)\right] = 0$ *for every* $0 \le s \le t \le T$, $l \in C_b(\mathbb{R}^d, \mathbb{R})$ *and* $h \in C_b^2(\mathbb{R}^d, \mathbb{R})$.

Proof. Let $\varepsilon \in (0,1)$ and $x_0 \in \mathcal{K}(0)$ be fixed. Select $\delta \in (0,\varepsilon)$ such that $\int_t^{t+\delta} m^2(\tau)\mathrm{d}\tau \le \varepsilon^2/2^4$ and $D(\mathcal{K}(t+\delta), \mathcal{K}(t)) \le \varepsilon/2^2$ for $t \in [0,T]$. By virtue of (2.1), there exists $h_0 \in (0,\delta)$ such that

$$\overline{D}\left[x_0 + \mathrm{cl}_{\mathbb{L}}\left(\int_0^{h_0} F(\tau, x_0)\mathrm{d}\tau + \int_0^{h_0} G(\tau, x_0)\mathrm{d}B_\tau\right), \mathcal{K}(h_0)\right] \le \frac{\varepsilon h_0}{2^2}.$$

Then for every $u_0 \in x_0 + cl_{\mathbb{L}}\left(\int_0^{h_0} F(\tau, x_0)\mathrm{d}\tau + \int_0^{h_0} G(\tau, x_0)\mathrm{d}B_\tau\right)$, one has $d(u_0, \mathcal{K}(h_0)) \le \varepsilon h_0/2^2$. Let $t_0 = 0$ and $t_1 = h_0$. Select arbitrarily $\beta_T \otimes \mathcal{F}_0$-measurable selectors f^0 and g^0 of $F \circ x_0$ and $G \circ x_0$, respectively. It is clear that $f^0 \in S_{\mathbb{F}}(F \circ x_0))$ and $g^0 \in S_{\mathbb{F}}(G \circ x_0))$. Let $x^\varepsilon(t) = x_0 + \int_0^t f_\tau^0 \mathrm{d}\tau + \int_0^t g_\tau^0 \mathrm{d}B_\tau$ for $0 \le t \le t_1$. Put $\theta_\varepsilon(t) = 0$ for $0 \le t < t_1$ and $\theta_\varepsilon(t_1) = t_1$. We have

$$x^\varepsilon(h_0) \in x_0 + \mathrm{cl}_{\mathbb{L}^2}\left(\int_0^{h_0} F(\tau, x_0)\mathrm{d}\tau + \int_0^{h_0} G(\tau, x_0)\mathrm{d}B_\tau\right).$$

Therefore, $d(x^\varepsilon(h_0), \mathcal{K}(h_0)) \le \varepsilon h_0/2^2 \le \varepsilon/2^2$. Together with the properties of the number $\delta > 0$, it follows that

$$\begin{aligned} d(x^\varepsilon(t), \mathcal{K}(t)) &\le \|x^\varepsilon(t) - x^\varepsilon(h_0)\| + d(x^\varepsilon(h_0), \mathcal{K}(h_0)) \\ &\le \varepsilon/2 + \varepsilon h_0/2^2 + D(\mathcal{K}(h_0), \mathcal{K}(t)) \le \varepsilon \end{aligned}$$

for $0 \le t \le t_1$, because

$$\begin{aligned} &\|x^\varepsilon(t) - x^\varepsilon(h_0)\| \\ &\le \sqrt{h_0}\left[E\int_0^{h_0} |f_\tau^0|^2 \mathrm{d}\tau\right]^{1/2} + \left[E\int_0^{h_0} |g_\tau^0|^2 \mathrm{d}\tau\right]^{1/2} \le 2\varepsilon/2^2 = \varepsilon/2 \end{aligned}$$

for $0 \le t \le t_1$. Let $x_1 \in \mathcal{K}(t_1)$ be such that $\|x^\varepsilon(h_0) - x_1\| \le d(x^\varepsilon(h_0), \mathcal{K}(h_0)) + \varepsilon/2^2$. Hence, by Theorem 3.1 of Chap. 2, it follows that

$$E[\mathrm{dist}(x^\varepsilon(h_0), K(h_0))] = \inf\{E|x^\varepsilon(h_0) - u| : u \in \mathcal{K}(h_0)\} \le$$

$$E[|x^\varepsilon(h_0) - x_1|] \le (E[|x^\varepsilon(h_0) - x_1|^2])^{1/2} = ||x^\varepsilon(h_0) - x_1|| \le \varepsilon/2^2 + \varepsilon/2^2 \le \varepsilon.$$

By Itô's formula, for every $h \in C_b^2(\mathbb{R}^d, \mathbb{R})$ and $0 \le s \le t \le T$, we have

$$h(x^\varepsilon(t)) - h(x^\varepsilon(s)) - \int_s^t (\mathbb{L}^{x^\varepsilon}_{f^0 g^0} h)_\tau \mathrm{d}\tau = \sum_{i=1}^n \sum_{j=1}^m \int_s^t h''_{x_i x_j}(x^\varepsilon(\tau)) g_{ij}^0(\tau)\mathrm{d}B_\tau^j,$$

a.s., where $B = (B^1, \dots, B^m)$ and $g^0_\tau = (g^0_{ij}(\tau))_{d\times m}$. But $x^\varepsilon(s)$ is $\mathcal{F}_s$-measurable. Then for $0 \le s \le t \le t_2$, $i = 1, 2, \dots, n$, and $j = 1, 2, \dots, m$, we have

$$E\left[l(x^\varepsilon(s))\int_s^t h''_{x_i x_j}(x^\varepsilon(\tau))g^0_{ij}(\tau)\mathrm{d}B^j_\tau\right] = E\left[\int_s^t l(x^\varepsilon(s))h''_{x_i x_j}(x^\varepsilon(\tau))g^0_{ij}(\tau)\mathrm{d}B^j_\tau\right] = 0.$$

Therefore, for every $l \in C_b(\mathbb{R}^d, \mathbb{R})$, $h \in C^2_b(\mathbb{R}^d, \mathbb{R})$, and $0 \le s \le t \le t_1$, we get

$$E\left[l(x^\varepsilon(s))\left(h(x^\varepsilon(t)) - h(x^\varepsilon(s)) - \int_s^t (\mathbb{L}^{x^\varepsilon}_{f^0 g^0} h)_\tau \mathrm{d}\tau\right)\right] = 0.$$

Suppose $h_0 < T$. We have $(h_0, x^\varepsilon(h_0)) \in Graph(\mathcal{K}^\varepsilon)$ because $d(x^\varepsilon(h_0), \mathcal{K}(h_0)) \le \varepsilon$. Therefore, we can repeat the above procedure and select $h_1 \in (0, \delta)$ such that

$$\overline{D}\left[x^\varepsilon(h_0) + \mathrm{cl}_{\mathbb{L}}\left(\int_{t_1}^{t_1+h_1} F(\tau, x^\varepsilon(h_0))\mathrm{d}\tau + \int_{t_1}^{t_1+h_1} G(\tau, x^\varepsilon(h_0))\mathrm{d}B_\tau\right), \mathcal{K}(t_1+h_1)\right] \le \frac{\varepsilon h_1}{2^2}.$$

Similarly as above, we can select $f^1 \in S_{\mathbb{F}}(F \circ x^\varepsilon(h_0))$ and $g^1 \in S_{\mathbb{F}}(G \circ x^\varepsilon(h_0))$, and define $x^\varepsilon(t) = x^\varepsilon(t_1) + \int_{t_1}^t f^1_\tau \mathrm{d}\tau + \int_{t_1}^t g^1_\tau \mathrm{d}B_\tau$ for $t_1 \le t \le t_2$, where $t_2 = t_1 + h_1$. We can also extend the function θ on $[0, t_2]$ by taking $\theta(t) = t_1$ for $t_1 \le t < t_2$ and $\theta_\varepsilon(t_2) = t_2$. We have

$$x^\varepsilon(t_2) \in x^\varepsilon(t_1) + \mathrm{cl}_{\mathbb{L}}\left(\int_{t_1}^{t_2} F(\tau, x^\varepsilon(t_1))\mathrm{d}\tau + \int_{t_1}^{t_2} G(\tau, x^\varepsilon(t_1))\mathrm{d}B_\tau\right).$$

Therefore, for every $t_1 \le t \le t_2$, one has

$$d(x^\varepsilon(t), \mathcal{K}(t)) \le \|x^\varepsilon(t) - x^\varepsilon(t_2)\| + d(x^\varepsilon(t_2), \mathcal{K}(t_2)) + H(\mathcal{K}(t_2), \mathcal{K}(t)) \le \varepsilon,$$

because similarly as above, we get $\|x^\varepsilon(t) - x^\varepsilon(t_2)\| \le \varepsilon/2$ for every $t_1 \le t \le t_2$. Similarly as above, for every $l \in C_b(\mathbb{R}^d, \mathbb{R})$, $h \in C^2_b(\mathbb{R}^d, \mathbb{R})$, and $t_1 \le s \le t \le t_2$, we also get

$$E\left[l(x^\varepsilon(s))\left(h(x^\varepsilon(t)) - h(x^\varepsilon(s)) - \int_s^t (\mathbb{L}^{x^\varepsilon}_{f^\varepsilon g^\varepsilon} h)_\tau \mathrm{d}\tau\right)\right] = 0.$$

Let $x_2 \in \mathcal{K}(t_2)$ be such that $\|x^\varepsilon(t_2) - x_2\| \le d(x^\varepsilon(t_2), \mathcal{K}(t_2)) + \varepsilon/2^2$. Hence it follows that $E[\mathrm{dist}(x^\varepsilon(t_2), K(t_2))] \le \varepsilon$. Let us observe that the above relations can be written in the form presented in (i)–(iii) above with $T = t_2$, where $f^\varepsilon = \mathbb{1}_{[0,t_1)} f^0 + \mathbb{1}_{(t_1,t_2]} f^1$, $g^\varepsilon = \mathbb{1}_{[0,t_1)} g^0 + \mathbb{1}_{(t_1,t_2]} g^1$ and $x^\varepsilon(t) = x_0 + \int_0^t f^\varepsilon_\tau \mathrm{d}\tau + \int_0^t g^\varepsilon_\tau \mathrm{d}B_\tau$ for $0 \le t \le t_2$.

Continuing the above procedure, we can extend the function θ_ε and processes f^ε, g^ε, and x^ε on the whole interval $[0, T]$ such that the above conditions (i)–(iii) are satisfied. To see this, let us denote by Λ_ε the set of all extensions of the vector function $\Phi_\varepsilon = (\theta_\varepsilon, f^\varepsilon, g^\varepsilon, x^\varepsilon)$ on $[0, \alpha] \times \Omega$ with $\alpha \in (0, T]$ and $\theta_\varepsilon|_{[0,\alpha]}$ not depending on $\omega \in \Omega$. We have $\Lambda_\varepsilon \ne \emptyset$. Let us introduce in Λ_ε the partial order relation $\preceq$ by setting $\Phi^\alpha_\varepsilon \preceq \Phi^\beta_\varepsilon$ if and only if $\alpha \le \beta$ and $\Phi^\alpha_\varepsilon = \Phi^\beta_\varepsilon|_{[0,\alpha]}$, where Φ^α_ε

and Φ_ε^β denote extensions of Φ_ε to $[0,\alpha]$ and $[0,\beta]$, respectively. Let P_ε^α be a set containing an extension Φ_ε^α and all its restrictions $\Phi_\varepsilon^\alpha|_{[0,a]}$ for every $a \in (0,\alpha]$. It is clear that each completely ordered subset of Λ_ε is of the form P_ε^α determined by some extension Φ_ε^α. It is also clear that every set P_ε^α has Φ_ε^α as its upper bound. Then by the Kuratowski–Zorn lemma, there exists a maximal element Ψ_ε of Λ_ε defined on $[0,b]\times\Omega$ with $b \le T$. It has to be $b = T$. Indeed, if it were $b < T$, then we could repeat the above procedure and extend Ψ_ε to the vector function Γ_ε defined on $[0,\gamma]\times\Omega$ with $b \le \gamma$. It would be $\Psi_\varepsilon \preceq \Gamma_\varepsilon$, in contradiction to the assumption that Ψ_ε is a maximal element of Λ_ε. Then Φ_ε can be extended on $[0,T]\times\Omega$ in such a way that conditions (i)–(iii) are satisfied. □

Remark 2.1. Theorem 2.1 is also true if instead of (2.1), we assume that

$$\liminf_{h\to 0+}\frac{1}{h}\overline{D}\left[x + \mathrm{cl}_{\mathbb{L}}\left(\int_t^{t+h} F(\tau,x)\mathrm{d}\tau\right) + \int_t^{t+h} G(\tau,x)\mathrm{d}B_\tau, \mathcal{K}(t+h)\right] = 0 \tag{2.2}$$

for every $(t,x) \in$ Graph $(\mathcal{K}^\varepsilon)$. □

Theorem 2.2. *Assume that F and G satisfy conditions $(\mathcal{H}_1)$. Suppose $\mathcal{P}_{\mathbb{F}} = (\Omega,\mathcal{F},\mathbb{F},P)$ is a complete filtered probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ such that there exists an m-dimensional $\mathbb{F}$-Brownian motion $B = (B_t)_{0\le t\le T}$ defined on $\mathcal{P}_{\mathbb{F}}$. Let $K : [0,T]\times\Omega \to \mathrm{Cl}(\mathbb{R}^d)$ be $\mathbb{F}$-nonanticipative such that $\mathcal{K}(t) \ne \emptyset$ for every $0 \le t \le T$ and $(\mathcal{K}(t))_{0\le t\le T}$ is continuous. If (2.1) is satisfied for every $(t,x) \in Graph(\mathcal{K})$, then for every $\varepsilon \in (0,1)$, $a \in (0,T)$, $x_0 \in \mathcal{K}(0)$, and $\mathbb{F}$-nonanticipative processes $\phi = (\phi_t)_{0\le t\le T}$ and $\psi = (\psi_t)_{0\le t\le T}$ with $\phi_t \in \mathbb{L}^2(\Omega,\mathcal{F}_T,\mathbb{R}^d)$, $\psi_t \in \mathbb{L}^2(\Omega,\mathcal{F}_T,\mathbb{R}^{d\times m})$ for $0 \le t \le T$ and $(\phi_0,\psi_0) \in F(0,x_0)\times G(0,x_0)$ a.s., there exist a partition $0 = t_0 < t_1 < \cdots < t_p = a$ of the interval $[0,a]$, a step function $\theta_\varepsilon : [0,a] \to [0,a]$, $\mathbb{F}$-nonanticipative stochastic processes $f^\varepsilon = (f_t^\varepsilon)_{0\le t\le a}$ and $g^\varepsilon = (g_t^\varepsilon)_{0\le t\le a}$, and a step stochastic process $z^\varepsilon = (z^\varepsilon(t))_{0\le t\le a}$ such that*

- *(i) $t_{j+1}-t_j \le \delta$, where $\delta \in (0,\varepsilon)$ is such that $\max\left(\int_t^{t+\delta} k^2(\tau)\mathrm{d}\tau, \int_t^{t+\delta} m^2(\tau)\mathrm{d}\tau\right) \le \varepsilon^2/2^4$ and $D(\mathcal{K}(t+\delta),\mathcal{K}(t)) \le \varepsilon/2^2$ for $t \in [0,T]$;*
- *(ii) $\theta_\varepsilon(t) = t_j$ for $t_j \le t < t_{j+1}$ for $j = 0,1,\dots p-2$ and $\theta_\varepsilon(t) = t_{p-1}$ for $t_{p-1} \le t \le a$;*
- *(iii) $f^\varepsilon \in S_{\mathbb{F}}(F\circ(x^\varepsilon\circ\theta_\varepsilon))$, $g^\varepsilon \in S_{\mathbb{F}}(G\circ(x^\varepsilon\circ\theta_\varepsilon))$, $|\phi_t(\omega) - f_t^\varepsilon(\omega)| \le \mathrm{dist}(\phi_t, F(t,(x^\varepsilon\circ\theta_\varepsilon)(t)))$ and $|\psi_t(\omega) - g_t^\varepsilon(\omega)| \le \mathrm{dist}(\psi_t, G(t,(x^\varepsilon\circ\theta_\varepsilon)(t)))$ for $(t,\omega) \in [0,a]\times\Omega$, where $x^\varepsilon(t) = x_0 + \int_0^t (f_\tau^\varepsilon + z^\varepsilon(\tau))\mathrm{d}\tau + \int_0^t g_\tau^\varepsilon \mathrm{d}B_\tau$ a.s. for $0 \le t \le a$;*
- *(iv) $\|z^\varepsilon(t)\| \le \varepsilon/2^2$ for $0 \le t \le a$, where $\|z^\varepsilon(t)\|^2 = E|z(t)|^2$;*
- *(v) $d(x^\varepsilon(\theta_\varepsilon(t)),\mathcal{K}(\theta_\varepsilon(t))) = 0$ for $0 \le t \le a$;*
- *(vi) $d\left(x^\varepsilon(t) - x^\varepsilon(s), \mathrm{cl}_{\mathbb{L}}\left(\int_s^t F(\tau,(x^\varepsilon\circ\theta_\varepsilon)(\tau))\mathrm{d}\tau + \int_s^t G(\tau,(x^\varepsilon\circ\theta_\varepsilon)(\tau))\mathrm{d}B_\tau\right)\right) \le \varepsilon$ for every $0 \le s < t \le a$.*

Proof. Let $x_0 \in \mathcal{K}(0)$, $\varepsilon \in (0,1)$ and $a \in (0,T)$ be fixed. Without loss of generality, we can assume that $T = 1$. By virtue of (2.1), there exists $h_0 \in (0,\delta)$ such that

$$\overline{D}\left[x_0 + \mathrm{cl}_{\mathbb{L}}\left(\int_0^{h_0} F(\tau, x_0)\mathrm{d}\tau + \int_0^{h_0} G(\tau, x_0)\mathrm{d}B_\tau\right), \mathcal{K}(h_0)\right] \leq \frac{\varepsilon h_0}{2^2},$$

where $\delta > 0$ is such that condition (i) is satisfied. By virtue of Corollary 2.3 of Chap. 2 applied to $\Sigma_{\mathbb{F}}$-measurable multifunctions $F \circ x_0$ and $G \circ x_0$, and given the above processes ϕ and ψ, there exist $f^0 \in S_{\mathbb{F}}(F \circ x_0)$ and $g^0 \in S_{\mathbb{F}}(G \circ x_0)$ such that $|\phi_t(\omega) - f_t^0(\omega)| = \mathrm{dist}(\phi_t, F(t, x_0))$ and $|\psi_t(\omega) - g_t^0(\omega)| = \mathrm{dist}(\psi_t, G(t, x_0))$ for $(t,\omega) \in [0,a] \times \Omega$. Similarly as in the proof of Theorem 2.1, we define now the function θ_ε by taking $\theta_\varepsilon(t) = 0$ for $0 \leq t < t_1$ and $\theta_\varepsilon(t_1) = t_1$, where $t_1 = h_0$. Hence it follows that $f_t^0 \in F(t, \theta_\varepsilon(t))$ and $g_t^0 \in G(t, \theta_\varepsilon(t))$ a.s. for $0 \leq t < t_1$. Let $y_0 = x_0 + \int_0^{t_1} f_\tau^0 \mathrm{d}\tau + \int_0^{t_1} g_\tau^0 \mathrm{d}B_\tau$ a.s. We have

$$y_0 \in x_0 + \mathrm{cl}_{\mathbb{L}}\left(\int_0^{h_0} F(\tau, x_0)\mathrm{d}\tau + \int_0^{h_0} G(\tau, x_0)\mathrm{d}B_\tau\right).$$

Then $d(y_0, \mathcal{K}(h_0)) \leq \varepsilon h_0/2^2$, which by Theorem 3.1 of Chap. 2, implies that $d^2(y_0, \mathcal{K}(h_0)) = E[\mathrm{dist}(y_0, K(h_0, \cdot)]^2$. Therefore, by Corollary 2.3 of Chap. 2, there exists an $\mathcal{F}_{t_1}$-measurable random variable x_1 such that $x_1 \in K(h_0, \cdot)$ for $\omega \in \Omega$ and

$$\|y_0 - x_1\| = \left(E\left[dist^2(y_0, K(h_0, \cdot)\right]\right)^{1/2} = d(y_0, \mathcal{K}(h_0)) \leq \varepsilon h_0/2^2.$$

Define $z_t^\varepsilon = (1/h_0)(x_1 - y_0)$ a.s. for $0 \leq t \leq t_1$. We get $||z^\varepsilon(t)|| \leq (1/h_0)||x_1 - y_0|| \leq (1/h_0)(\varepsilon h_0/2^2) = \varepsilon/4$ for $0 \leq t \leq t_1$. We define now a process x^ε on $[0, t_1)$ by setting

$$x^\varepsilon(t) = x_0 + \int_0^t (f_\tau^0 + z^\varepsilon(\tau))\mathrm{d}\tau + \int_0^t g_\tau^0 \mathrm{d}B_\tau \quad \text{a.s.} \quad \text{for} \quad 0 \leq t \leq t_1.$$

We have $x^\varepsilon(0) = x_0 \in \mathcal{K}(0)$ and $x^\varepsilon(t_1) = y_0 + h_0(1/h_0)(x_1 - y_0) = x_1 \in \mathcal{K}(h_0)$, which is equivalent to $d(x^\varepsilon(\theta_\varepsilon(t), \mathcal{K}(\theta_\varepsilon(t))) = 0$ for $t \in [0, t_1]$. Similarly, for $0 \leq s \leq t < t_1$, one obtains

$$\begin{aligned}
&d\left[x^\varepsilon(t) - x^\varepsilon(s), \mathrm{cl}_{\mathbb{L}}\left(\int_s^t F(\tau, (x^\varepsilon \circ \theta_\varepsilon)(\tau))\mathrm{d}\tau + \int_s^t G(\tau, (x^\varepsilon \circ \theta_\varepsilon)(\tau))\mathrm{d}B_\tau\right)\right] \\
&\leq d\left[\int_s^t f_\tau^0 \mathrm{d}\tau + \int_s^t g_\tau^0 \mathrm{d}B_\tau, \mathrm{cl}_{\mathbb{L}}\left(\int_s^t F(\tau, x^0)\mathrm{d}\tau + \int_s^t G(\tau, x^0)\mathrm{d}B_\tau\right)\right] \\
&\quad + (t-s) \sup_{0 \leq \tau \leq t_1} ||z^\varepsilon(\tau)|| \leq \frac{\varepsilon}{4} < \varepsilon.
\end{aligned}$$

If $h_0 < a$, we can repeat the above procedure. Applying (2.1) to $(t_1, x_1) \in Graph(\mathcal{K})$, we can select $h_1 \in (0, \delta)$ such that

$$\overline{D}\left[x_1 + \mathrm{cl}_{\mathbb{L}}\left(\int_{t_1}^{t_1+h_1} F(\tau, x_1)\mathrm{d}\tau + \int_{t_1}^{t_1+h_1} G(\tau, x_1)\mathrm{d}B_\tau\right), \mathcal{K}(t_1 + h_1)\right] \le \frac{\varepsilon h_1}{2^2}.$$

Similarly as above, we can select $x_2 \in \mathcal{K}(t_1 + h_1)$, $f^1 \in S_{\mathbb{F}}(F \circ x_1)$, and $g^1 \in S_{\mathbb{F}}(G \circ x_1)$ such that $|\phi_t(\omega) - f_t^1(\omega)| = \mathrm{dist}(\phi_t, F(t, x^1))$ and $|\psi_t(\omega) - g_t^1(\omega)| = \mathrm{dist}(\psi_t, G(t, x^1))$ for $(t, \omega) \in [0, a] \times \Omega$ and $\|y_1 - x_2\| \le \varepsilon h_1/2^2$, where $y_1 = x_1 + \int_{t_1}^{t_1+h_1} f_\tau^1 \mathrm{d}\tau + \int_{t_1}^{t_1+h_1} g_\tau^1 \mathrm{d}B_\tau$ a.s. We can extend the function θ_ε and the process z^ε on the interval $[0, t_2]$ by setting $\theta_\varepsilon(t) = t_1$ for $t_1 \le t < t_2$, $\theta(t_2) = t_2$, and $z^\varepsilon(t) = (1/h_1)(x_2 - y_1)$ for $t_1 < t \le t_2$, where $t_2 = t_1 + h_2$. Define on the interval $[0, t_2]$ the process x^ε by setting

$$x^\varepsilon(t) = x_0 + \int_0^t (f_\tau^\varepsilon + z^\varepsilon(\tau))\mathrm{d}\tau + \int_0^t g_\tau^\varepsilon \mathrm{d}B_\tau \quad \text{a.s. for } 0 \le t \le t_2,$$

where $f^\varepsilon = \mathbb{1}_{[0,t_1)} f^0 + \mathbb{1}_{[t_1,t_2)} f^1$ and $g^\varepsilon = \mathbb{1}_{[0,t_1)} g^0 + \mathbb{1}_{[t_1,t_2)} g^1$. Similarly as above, we obtain $d(x^\varepsilon(\theta_\varepsilon(t), \mathcal{K}(\theta_\varepsilon(t))) = 0$ for $0 \le t < t_2$ and $d(x^\varepsilon(\theta_\varepsilon(t_2), \mathcal{K}(\theta_\varepsilon(t_2))) = 0$, because $x^\varepsilon(t_2) = x_2$. Then $d(x^\varepsilon(\theta_\varepsilon(t), \mathcal{K}(\theta_\varepsilon(t))) = 0$ for $0 \le t \le t_2$. It is clear that $\|z_t^\varepsilon\| \le \varepsilon/4 \le \varepsilon$ for every $0 \le t \le t_2$. Then for every $0 \le s \le t \le t_2$, we get

$$\begin{aligned} &d\left[x^\varepsilon(t) - x^\varepsilon(s), \mathrm{cl}_{\mathbb{L}}\left(\int_s^t F(\tau, (x^\varepsilon \circ \theta_\varepsilon)(\tau))\mathrm{d}\tau + \int_s^t G(\tau, (x^\varepsilon \circ \theta_\varepsilon)(\tau))\mathrm{d}B_\tau\right)\right] \\ &\le d\left[\int_s^t f_\tau^\varepsilon \mathrm{d}\tau + \int_s^t g_\tau^\varepsilon \mathrm{d}B_\tau, \mathrm{cl}_{\mathbb{L}}\left(\int_s^t F(\tau, (x^\varepsilon \circ \theta_\varepsilon)(\tau))\mathrm{d}\tau\right.\right. \\ &\quad \left.\left. + \int_s^t G(\tau, (x^\varepsilon \circ \theta_\varepsilon)(\tau))\mathrm{d}B_\tau\right)\right] + (t - s) \sup_{0 \le \tau \le t_2} ||z^\varepsilon(\tau|| \le \frac{\varepsilon}{4} < \varepsilon. \end{aligned}$$

Suppose that for some $i \ge 1$, the inductive procedure is realized on $[0, t_i) \subset [0, a]$ and the above step function θ_ε, and stochastic processes z^ε f^ε, g^ε, and x^ε are extended to $[0, t_i]$ and $[0, t_i)$, respectively, with the above properties on this interval. Denote by S_i the set of all positive numbers h such that $h \in (0, \min(\delta, a - t_i))$ and

$$\overline{D}\left[x_i + \mathrm{cl}_{\mathbb{L}}\left(\int_{t_i}^{t_i+h} F(\tau, x_i)\mathrm{d}\tau + \int_{t_i}^{t_i+h} G(\tau, x_i)\mathrm{d}B_\tau\right), \mathcal{K}(t_i + h)\right] \le \frac{\varepsilon h}{2^3},$$

where $x_i = x^\varepsilon(t_i)$. We have $S_i \ne \emptyset$ and $\sup S_i > 0$. Choose $h_i \in S_i$ such that $\sup S_i - (1/2)\sup S_i \le h_i$. Put $t_{i+1} = t_i + h_i$ and let $f^i \in S_{\mathbb{F}}(F \circ x_i)$ and $g^i \in S_{\mathbb{F}}(G \circ x_i)$ be such that $|\phi_t(\omega) - f_t^i(\omega)| = \mathrm{dist}(\phi_t, F(t, x_i))$ and $|\psi_t(\omega) - g_t^i(\omega)| = \mathrm{dist}(\psi_t, G(t, x_i))$. We can now extend θ_ε, f^ε, and g^ε to the interval $[0, t_{i+1}]$ by taking $\theta_\varepsilon(t) = t_i$ for $t_i \le t < t_{i+1}$ and $\theta_\varepsilon(t_{i+1}) = t_{i+1}$, $f_t^\varepsilon = f_t^i$, $g_t^\varepsilon = g_t^i$ for $t_i \le t < t_{i+1}$. Then $|\phi_t(\omega) - f_t^\varepsilon(\omega)| \le \mathrm{dist}(\phi_t, F(t, (x^\varepsilon \circ \theta_\varepsilon)(t)))$ and $|\psi_t(\omega) - g_t^\varepsilon(\omega)| \le \mathrm{dist}(\psi_t, G(t, (x^\varepsilon \circ \theta_\varepsilon)(t)))$ for $(t, \omega) \in [0, t_{i+1}) \times \Omega$, where

$$x^\varepsilon(t) = x_0 + \int_0^t (f_\tau^\varepsilon + z^\varepsilon(\tau))\mathrm{d}\tau + \int_0^t g_\tau^\varepsilon \mathrm{d}B_\tau$$

a.s. for $0 \le t \le t_{i+1}$ with

$$z^\varepsilon(t) = (1/h_i)\left(x_{i+1} - x_i - \int_{t_i}^{t_{i+1}} f_\tau^\varepsilon \mathrm{d}\tau - \int_{t_i}^{t_{i+1}} g_\tau^\varepsilon \mathrm{d}B_\tau\right)$$

a.s for $t_i < t \le t_{i+1}$, where $x_{i+1} \in \mathcal{K}(t_{i+1})$ is such that

$$\left\| x_i + \int_{t_i}^{t_{i+1}} f_\tau^\varepsilon \mathrm{d}\tau + \int_{t_i}^{t_{i+1}} g_\tau^\varepsilon \mathrm{d}B_\tau - x_{i+1} \right\| \le \varepsilon h_i/4.$$

Similarly as above, we obtain $||z^\varepsilon(t)|| \le \varepsilon/4$ for $t_i < t \le t_{i+1}$. Hence it follows that

$$\begin{aligned} &d\left[x^\varepsilon(t) - x^\varepsilon(s), \mathrm{cl}_{\mathbb{L}}\left(\int_s^t F(\tau, (x^\varepsilon \circ \theta_\varepsilon)(\tau))\mathrm{d}\tau + \int_s^t G(\tau, (x^\varepsilon \circ \theta_\varepsilon)(\tau))\mathrm{d}B_\tau\right)\right] \\ &\le d\left[\int_s^t f_\tau^\varepsilon \mathrm{d}\tau + \int_s^t g_\tau^\varepsilon \mathrm{d}B_\tau, \mathrm{cl}_{\mathbb{L}}\left(\int_s^t F(\tau, (x^\varepsilon \circ \theta_\varepsilon)(\tau))\mathrm{d}\tau + \int_s^t G(\tau, (x^\varepsilon \circ \theta_\varepsilon)(\tau))\mathrm{d}B_\tau\right)\right] \\ &\quad + (t-s) \sup_{0 \le \tau \le t_2} ||z^\varepsilon(\tau|| \le \frac{\varepsilon}{4} < \varepsilon \end{aligned}$$

for $0 \le s < t < t_{i+1}$ and $d(x^\varepsilon(\theta_\varepsilon(t), \mathcal{K}(\theta_\varepsilon(t))) = 0$ for $0 \le t \le t_2$.

We can continue the above procedure up to $n > 1$ such that $t_n \in [a, 1]$. Suppose to the contrary that such $n > 1$ does not exist, i.e., that for every $n > 1$, one has $0 < t_n < a$. Then we obtain a sequence $(t_i)_{i=1}^\infty$ converging to $t^* \le a$ such that for every $0 \le j < k \le i + 1$ and $i \ge 0$, we have

$$\begin{aligned} ||x^\varepsilon(t_k) - x^\varepsilon(t_j)|| &\le \left\|\int_{t_j}^{t_k} f_\tau^\varepsilon \mathrm{d}\tau\right\| + \left\|\int_{t_j}^{t_k} g_\tau^\varepsilon \mathrm{d}B_\tau\right\| + \left\|\int_{t_j}^{t_k} z^\varepsilon(\tau)\mathrm{d}\tau\right\| \\ &\le 2\left(\int_{t_j}^{t_k} m^2(\tau)\mathrm{d}\tau\right)^{1/2} + \varepsilon \cdot (t_k - t_j)/4. \end{aligned}$$

Let $x_j = x^\varepsilon(t_j)$ and $x_k = x^\varepsilon(t_k)$ for $0 \le j < k < \infty$. For every $0 \le j < k < \infty$, one gets

$$\|x_k - x_j\| \le 2\left(\int_{t_j}^{t_k} m^2(\tau)\mathrm{d}\tau\right)^{1/2} + \varepsilon \cdot (t_k - t_j)/4.$$

Then $(x_i)_{i=1}^{\infty}$ is a Cauchy sequence of $\mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{R}^d)$. Therefore, there exists $x^* \in \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{R}^d)$ such that $\|x_i - x^*\| \to 0$ as $i \to \infty$. By the continuity of the set-valued mapping $\mathcal{K}$, we get $(t^*, x^*) \in Graph(\mathcal{K})$. Then by (2.1), there exists $h^* \in (0, \min(\delta, 1-a))$ such that

$$\overline{D}\left[x^* + \mathrm{cl}_{\mathbb{L}}\left(\int_{t^*}^{t^*+h^*} F(\tau, x^*)\mathrm{d}\tau + \int_{t^*}^{t^*+h^*} G(\tau, x^*)\mathrm{d}B_\tau\right), \mathcal{K}(t^* + h^*)\right] \leq \frac{\varepsilon h^*}{2^3}.$$

Let $N > 1$ be such that for every $i \geq N$, one has $0 < t^* - t_i < \min(h^*, a, \eta_\varepsilon)$, $\|x_i - x^*\| \leq \varepsilon h^*/(2^6 A)$, and $D(\mathcal{K}(t_i), \mathcal{K}(t^*)) \leq \varepsilon h^*/2^6$, where $A = 1 + 2\left(\int_0^1 k^2(t)\mathrm{d}t\right)^{1/2}$ and $\eta_\varepsilon \in (0, 1-a)$ is such that $\left(\int_t^{t+\eta_\varepsilon} m^2(\tau)\mathrm{d}\tau\right)^{1/2} \leq \varepsilon h^*/2^7$ for every $0 \leq t \leq a$. For every $i \geq N$ and arbitrarily taken $\phi^i \in S_{\mathbb{F}}(F \circ x_i)$ and $\psi^i \in S_{\mathbb{F}}(G \circ x_i)$, we can select $f^* \in S_{\mathbb{F}}(F \circ x^*)$ and $g^* \in S_{\mathbb{F}}(G \circ x^*)$ such that $|\phi_t^i(\omega) - f_t^*(\omega)| = \mathrm{dist}(\phi^i, F(t, x^*))$ and $|\psi_t^i(\omega) - g_t^*(\omega)| = \mathrm{dist}(\psi^i, G(t, x^*))$ for $(t, \omega) \in [t_i, t^* + h^*] \times \Omega$. In particular, this implies

$$\|\phi^i - f^*\|_*^2 \leq E\int_{t_i}^{t^*+h^*} [h(F(t, x_i), F(t, x^*))]^2\mathrm{d}t \leq \int_{t_i}^{t^*+h^*} k^2(t)\|x_i - x^*\|^2\mathrm{d}t$$

and

$$\|\psi^i - f^*\|_*^2 \leq E\int_{t_i}^{t^*+h^*} [h(G(t, x_i), G(t, x^*))]^2\mathrm{d}t \leq \int_{t_i}^{t^*+h^*} k^2(t)\|x_i - x^*\|^2\mathrm{d}t$$

for $i \geq 1$. Therefore, for every $i \geq N$, we get

$$\begin{aligned}
& d\left[x_i + \int_{t_i}^{t_i+h^*} \phi_\tau^i \mathrm{d}\tau + \int_{t_i}^{t_i+h^*} \psi_\tau^i \mathrm{d}B_\tau, \mathcal{K}(t_i + h^*)\right] \\
&\leq \left\| \left[x_i + \int_{t_i}^{t_i+h^*} \phi_\tau^i \mathrm{d}\tau + \int_{t_i}^{t_i+h^*} \psi_\tau^i \mathrm{d}B_\tau\right] - \left[x^* + \int_{t^*}^{t^*+h^*} f_\tau^* \mathrm{d}\tau + \int_{t^*}^{t^*+h^*} g_\tau^* \mathrm{d}B_\tau\right] \right\| \\
&\quad + d\left[x^* + \int_{t^*}^{t^*+h^*} f_\tau^* \mathrm{d}\tau + \int_{t^*}^{t^*+h^*} g_\tau^* \mathrm{d}B_\tau, \mathcal{K}(t^* + h^*)\right] + D(\mathcal{K}(t^* + h^*), \mathcal{K}(t_i + h^*)) \\
&\leq ||x_i - x^*|| + \left\| \int_{t_i}^{t^*+h^*} (\phi_\tau^i - f_\tau^*)\mathrm{d}\tau \right\| + \left\| \int_{t_i}^{t^*+h^*} (\psi_\tau^i - g_\tau^*)\mathrm{d}B_\tau \right\| \\
&\quad + \left\| \int_{t_i+h^*}^{t^*+h^*} \phi_\tau^i \mathrm{d}\tau \right\| + \left\| \int_{t_i+h^*}^{t^*+h^*} \psi_\tau^i \mathrm{d}B_\tau \right\| + \left\| \int_{t_i}^{t^*} f_\tau^* \mathrm{d}\tau \right\| + \left\| \int_{t_i}^{t^*} g_\tau^* \mathrm{d}B_\tau \right\| \\
&\quad + d\left[x^* + \int_{t^*}^{t^*+h^*} f_\tau^* \mathrm{d}\tau + \int_{t^*}^{t^*+h^*} g_\tau^* \mathrm{d}B_\tau, \mathcal{K}(t^* + h^*)\right] + D(\mathcal{K}(t^* + h^*), \mathcal{K}(t_i + h^*)) \\
&\leq \|x_i - x^*\| + 2\sqrt{(t^* - t_i) + h^*}\,\|x_i - x^*\| \left(\int_{t_n}^{t^*+h^*} k^2(\tau)\mathrm{d}\tau\right)^{1/2}
\end{aligned}$$

$$+\left(1+\sqrt{(t^*-t_i)}\right)\left(\int_{t_i+h^*}^{t^*+h^*} m^2(\tau)\mathrm{d}\tau\right)^{1/2}+\left(1+\sqrt{(t^*-t_i)}\right)\left(\int_{t_i}^{t^*} m^2(\tau)\mathrm{d}\tau\right)^{1/2}$$

$$+2\frac{\varepsilon h^*}{2^6}+\frac{\varepsilon h^*}{2^6}\leq\left[1+2\sqrt{(t^*-t_i)+h^*}\left(\int_{t_i}^{t^*+h^*} k^2(\tau)\mathrm{d}\tau\right)^{1/2}\right]\|x_i-x^*\|$$

$$+\left(1+\sqrt{(t^*-t_i)}\right)\max\left\{\left[\int_{t_i+h^*}^{t^*+h^*} m^2(\tau)\mathrm{d}\tau\right]^{1/2},\left[\int_{t_i}^{t^*} m^2(\tau)\mathrm{d}\tau\right]^{1/2}\right\}+2\frac{\varepsilon h^*}{2^6}$$

$$+\frac{\varepsilon h^*}{2^6}\leq\left[1+2\left(\int_0^1 k^2(\tau)\mathrm{d}\tau\right)^{1/2}\right]\|x_i-x^*\|$$

$$+\left(1+\sqrt{(t^*-t_i)}\right)\max\left\{\left[\int_{t_i+h^*}^{t^*+h^*} m^2(\tau)\mathrm{d}\tau\right]^{1/2},\left[\int_{t_i}^{t^*} m^2(\tau)\mathrm{d}\tau\right]^{1/2}\right\}$$

$$+2\frac{\varepsilon h^*}{2^6}+\frac{\varepsilon h^*}{2^6}\leq\frac{\varepsilon h^*}{2^6\cdot A}\cdot A+2\cdot\frac{\varepsilon h^*}{2\cdot 2^6}+2\cdot\frac{\varepsilon h^*}{2^6}+\frac{\varepsilon h^*}{2^6}$$

$$=5\cdot\frac{\varepsilon h^*}{2^6}=\frac{5}{8}\cdot\frac{\varepsilon h^*}{2^3}<\frac{\varepsilon h^*}{2^3}.$$

Then for every $i\geq N$, we have

$$D\left[x_i+\mathrm{cl}_{\mathbb{L}}\left(\int_{t_i}^{t_i+h^*} F(\tau,x_i)\mathrm{d}\tau+\int_{t_i}^{t_i+h^*} G(\tau,x_i)\mathrm{d}B_\tau\right),\mathcal{K}(t_i+h^*)\right]\leq\frac{\varepsilon h^*}{2^3}$$

and $h^*\in(0,\min(\delta,1-a))$. But $t_i<a$ for every $i\geq 1$. Then $1-a<1-t_i$ for every $i\geq 1$. Therefore, for every $i\geq N$, we have $h^*\in(0,\min(\delta,1-t_i))$. Hence it follows that $h^*\in S_i$ for every $i\geq N$. Then for every $i\geq N$, one has $(1/2)h^*\leq(1/2)\sup S_i\leq h_i=t_{i+1}-t_i$, which contradicts the convergence of the sequence $(t_i)_{i=1}^\infty$. Therefore, there exists $p\geq 1$ such that $0=t_0<t_1<\cdots<t_p=a$. □

Remark 2.2. Theorem 2.2 is also true if instead of (2.1), we assume that (2.2) is satisfied for every $(t,x)\in Graph(\mathcal{K})$. □

Theorem 2.3. *Assume that F satisfies conditions $(\mathcal{H}_1)$, and let $\mathcal{P}_{\mathbb{F}}=(\Omega,\mathcal{F},\mathbb{F},P)$ be a complete filtered probability space with a continuous filtration $\mathbb{F}=(\mathcal{F}_t)_{0\leq t\leq T}$ such that $\mathcal{F}_T=\mathcal{F}$. Suppose $K:[0,T]\times\Omega\to\mathrm{Cl}(\mathbb{R}^d)$ is an $\mathbb{F}$-adapted set-valued process such that $\mathcal{K}(t)\neq\emptyset$ for every $0\leq t\leq T$ and such that the set-valued mapping $\mathcal{K}:[0,T]\to\mathrm{Cl}(\mathbb{L}(\Omega,\mathcal{F}_T,\mathbb{R}^d)$ is continuous. If*

$$\liminf_{h\to 0+}\frac{1}{h}\overline{D}\left[S\left(E\left[x+\int_{t-h}^{t} F(\tau,x)\mathrm{d}\tau|\mathcal{F}_{t-h}\right]\right),\mathcal{K}(t-h)\right]=0 \qquad (2.3)$$

is satisfied for every $(t,x)\in Graph(\mathcal{K})$, where $S(E[x+\int_{t-h}^t F(\tau,x)\mathrm{d}\tau|\mathcal{F}_{t-h}])=\{E[x+\int_{t-h}^t f_\tau\mathrm{d}\tau|\mathcal{F}_{t-h}]:f\in S(coF\circ x)\}$, then for every $\varepsilon\in(0,1)$, $x_T\in\mathcal{K}(x_T)$, $a\in(0,T)$ and measurable process $\phi=(\phi)_{0\leq t\leq T}$ such that $\phi_t\in\mathbb{L}(\Omega,\mathcal{F}_T,\mathbb{R}^d)$ for $0\leq t\leq T$ and $\phi_T\in F(T,x_T)$ a.s., there exist a partition $a=t_p<t_{p-1}<$

$\cdots < t_1 < t_0 = T$ of the interval $[a,T]$, a step function $\theta_\varepsilon : [a,T] \to [a,T]$, a step stochastic process $z^\varepsilon = (z_t^\varepsilon)_{a\le t\le T}$, and a measurable process $f^\varepsilon = (f_t^\varepsilon)_{a\le t\le T}$ on $\mathcal{P}_{\mathbb{F}}$ such that

(i) $t_j - t_{j+1} \le \delta$, *where* $\delta \in (0,\varepsilon)$ *is such that* $\max\{\int_t^{t+\delta} k(\tau)\mathrm{d}\tau, \int_t^{t+\delta} m(\tau)\mathrm{d}\tau\} \le \varepsilon^2/2^4$ *and* $D(\mathcal{K}(t+\delta),\mathcal{K}(t)) \le \varepsilon/2$ *for* $t \in [0,T]$;
(ii) $\|z_t^\varepsilon\| \le \varepsilon/2$ *for every* $a \le t \le T$, *where* $\|z_t^\varepsilon\| = E|z_t^\varepsilon|$;
(iii) $\theta_\varepsilon(t) = t_{j-1}$ *for* $t_j < t \le t_{j-1}$ *and* $\theta_\varepsilon(t_j) = t_j$ *with* $j = 1,\dots,p-1$ *and* $\theta_\varepsilon(t) = t_{p-1}$ *for* $a \le t \le t_{p-1}$;
(iv) $f^\varepsilon \in S(\mathrm{co}\,F \circ (x^\varepsilon \circ \theta_\varepsilon))$, $|\phi_t(\omega) - f_t^\varepsilon(\omega)| = \mathrm{dist}(\phi_t, \mathrm{co}\,F(t,(x^\varepsilon\circ\theta_\varepsilon)(t)))$ *for* $(t,\omega) \in [a,T]\times\Omega$, *where* $x^\varepsilon(t) = E[x_T + \int_t^T f_\tau^\varepsilon \mathrm{d}\tau|\mathcal{F}_t] + \int_t^T z_\tau^\varepsilon \mathrm{d}\tau$ *a.s. for* $a \le t \le T$ *and* $S(\mathrm{co}\,F\circ(x^\varepsilon\circ\theta_\varepsilon))=\{f \in \mathbb{L}([a,T]\times\Omega, \beta_T\otimes\mathcal{F}_T,\mathbb{R}^d) : f_t \in \mathrm{co}\,F(t,x^\varepsilon(\theta_\varepsilon(t)))$ a.s. for *a.e.* $a \le t \le T\}$;
(v) $E[\mathrm{dist}(x^\varepsilon(s), E[x^\varepsilon(t)+\int_s^t F(\tau,(x^\varepsilon\circ\theta_\varepsilon)(\tau)\mathrm{d}\tau|\mathcal{F}_s])] \le \varepsilon$ *for* $a \le s \le t \le T$,
(vi) $d(x^\varepsilon(\theta_\varepsilon(t)),\mathcal{K}(\theta_\varepsilon(t))) = 0$ *for* $a \le t \le T$.

Proof. Let $\varepsilon \in (0,1)$, $a \in (0,T)$, $x_T \in \mathcal{K}(T)$, and a measurable process $\phi = (\phi)_{0\le t\le T}$ be given. By virtue of (2.3), there exists $h_0 \in (0,\min(\delta,T))$ such that

$$\overline{D}\left[S\left(E\left[x_T + \int_{T-h_0}^T F(\tau,x_T)\mathrm{d}\tau|\mathcal{F}_{T-h_0}\right]\right),\mathcal{K}(T-h_0)\right] \le \varepsilon h_0/2.$$

Let $t_1 = T-h_0$. By virtue of Corollary 2.3 of Chap. 2, there exists $f^0 \in S(\mathrm{co}\,F\circ x_T)$ such that $|\phi_t(\omega)-f_t^0(\omega)| = \mathrm{dist}(\phi_t(\omega),\mathrm{co}\,F(t,x_T(\omega))$ for $(t,\omega)\in[t_1,T]\times\Omega$. Let $y_0 = E[x_T + \int_{t_1}^T f_\tau^0\mathrm{d}\tau|\mathcal{F}_{t_1}]$ a.s. We have $y_0 \in E[x_T + \int_{t_1}^T F(\tau,x_T)\mathrm{d}\tau|\mathcal{F}_{t_1}]$ a.s., i.e., $y_0 \in S(E[x_T + \int_{t_1}^T F(\tau,x_T)\mathrm{d}\tau|\mathcal{F}_{t_1}])$. Therefore, $d(y_0,\mathcal{K}(t_1)) \le \varepsilon h_0/2$. Similarly as above, we can see that there exists $x_1 \in \mathcal{K}(t_1)$ such that $E|y_0-x_1| = E[\mathrm{dist}(y_0,K(t_1\cdot))] = d(y_0,\mathcal{K}(t_1)) \le \varepsilon h_0/2$. Then $\|y_0-x_1\| \le \varepsilon h_0/2$. Let $z_t^\varepsilon = 1/h_0(x_1-y_0)$ a.s. for $t_1 \le t \le T$. We have $\|z_t^\varepsilon\| \le (1/h_0)\|y_0-x_1\| \le \varepsilon/2$. Furthermore, by the definition of z_t^ε, it follows that $\int_t^T z_\tau^\varepsilon\mathrm{d}\tau$ is $\mathcal{F}_{t_1}$-measurable. Define $\theta_\varepsilon(t) = T$ for $t_1 < t \le T$ and $\theta(t_1) = t_1$. One has $f_t^0 \in \mathrm{co}F(t,x_T)$ a.s. for $t_1 \le t \le T$. Let

$$x^\varepsilon(t) = E\left[x_T + \int_t^T f_\tau^0\mathrm{d}\tau|\mathcal{F}_t\right] + \int_t^T z_\tau^\varepsilon\mathrm{d}\tau$$

for $t_1 \le t \le T$. We have $x^\varepsilon(T) = x_T$ and $x^\varepsilon(t_1) = y_0 + h_0(1/h_0)(x_1-y_0) = x_1$. Therefore, $d(x^\varepsilon(\theta(t)),\mathcal{K}(\theta(t))) = 0$ for $t_1 \le t \le T$ and $|\phi_t(\omega) - f_t^0(\omega)| = \mathrm{dist}(\phi_t(\omega),\mathrm{co}\,F(t,x^\varepsilon(\theta_\varepsilon(t)(\omega)))$ for $(t,\omega)\in[t_1,T]\times\Omega$. By the definition of x^ε, it follows that it is $\mathbb{F}$-adapted. By properties of f^0 and x^ε, it follows that

$$E\left[\mathrm{dist}\left(x^\varepsilon(s), E[x_T + \int_s^t F(\tau,x^\varepsilon(\theta(\tau)))\mathrm{d}\tau|\mathcal{F}_s]\right)\right] \le \varepsilon/2 \quad \text{for} \quad t_1 \le s \le t \le T.$$

If $t_1 > a$, we can repeat the above procedure starting with $(t_1, x_1) \in Graph(\mathcal{K})$. Immediately from (2.3), it follows that there exists an $h_1 \in (0, \delta)$ such that

$$\overline{D}\left[S(E[x_1 + \int_{t_1-h_1}^{t_1} F(\tau, x_1)\mathrm{d}\tau|\mathcal{F}_{t_1-h_1}]), \mathcal{K}(t_1 - h_1)\right] \leq \varepsilon h_1/2.$$

Similarly as above, we can select $f^1 \in S(\mathrm{co}F \circ x_1)$ and $x_2 \in \mathcal{K}(t_1 - h_1)$ such that $|\phi_t(\omega) - f_t^1(\omega))| = \mathrm{dist}(\phi_t(\omega), \mathrm{co}(F \circ x_1)(t, \omega)$ for $(t, \omega) \in [t_1 - h_1, t_1] \times \Omega$ and $\|y_1 - x_2\| \leq \varepsilon h_1/2^2$, where $y_1 = E[x_1 + \int_{t_1-h_1}^{t_1} f_\tau^1 \mathrm{d}\tau|\mathcal{F}_{t_1-h_1}]$ and $t_2 = t_1 - h_1$. We can now extend the step function θ_ε and step process z^ε on the interval $[t_2, T]$ by taking $\theta_\varepsilon(t_2) = t_2$, $\theta_\varepsilon(t) = t_1$ for $t_2 < t \leq t_1$ and $z_t^\varepsilon = (1/h_1)(x_2 - y_1)$ for $t_2 \leq t \leq t_1$. We have $f_t^1 \in \mathrm{co}F(t, \theta_\varepsilon(t))$ a.s. for $t_2 \leq t \leq t_1$. We can also extend the process x^ε to the interval $[t_2, T]$ by taking

$$x^\varepsilon(t) = E\left[x_1 + \int_t^{t_1} f_\tau^1 \mathrm{d}\tau|\mathcal{F}_t\right] + \int_t^{t_1} z_\tau^\varepsilon \mathrm{d}\tau$$

a.s. for $t_2 \leq t \leq t_1$. We have $d(x^\varepsilon(\theta_\varepsilon(t)), \mathcal{K}(\theta(t))) = 0$ for $t_2 \leq t \leq T$, because $x^\varepsilon(t_2) = x_2$. Let $f^\varepsilon = \mathbb{1}_{(t_2,t_1]} f^1 + \mathbb{1}_{(t_1,T]} f^0$. We have $x^\varepsilon(t) = E[x_T + \int_t^T f_\tau^\varepsilon \mathrm{d}\tau|\mathcal{F}_t] + \int_t^T z_\tau^\varepsilon \mathrm{d}\tau$ a.s. for $t_2 < t \leq T$. Similarly as above, we can verify that $f_t^\varepsilon \in \mathrm{co}F(t, x^\varepsilon(\theta_\varepsilon(t)))$ a.s. for $t_2 < t \leq T$ and $|\phi_t - f_t^\varepsilon)| \leq \mathrm{dist}(\phi_t, \mathrm{co}\, F(t, x^\varepsilon(\theta(t)))$ a.s. for $t_2 < t \leq T$. Furthermore, $d(x^\varepsilon(\theta_\varepsilon(t)), \mathcal{K}(\theta_\varepsilon(t))) = 0$ and $E[\mathrm{dist}(x^\varepsilon(s), E[\int_s^t F(\tau, x^\varepsilon(\theta(\tau)))\mathrm{d}\tau|\mathcal{F}_s] \leq \varepsilon/2$ for $t_2 \leq t \leq T$ and $t_2 \leq s \leq t \leq T$, respectively.

Suppose that for some $i \geq 1$, the inductive procedure is realized. Then there exist $t_{i-1} \in [a, T)$ and $x_{i-1} \in \mathcal{K}(t_{i-1})$ such that we can extend the step function θ_ε, step process z^ε, and process f^ε to the whole interval $[t_{i-1}, T]$ such that $f_t^\varepsilon \in \mathrm{co}\, F(t, x^\varepsilon(\theta_\varepsilon(t))$ and $|\phi_t - f_t^\varepsilon| = \mathrm{dist}(\phi_t, \mathrm{co}\, F(t, x^\varepsilon(\theta_\varepsilon(t)))$ for $t_{i-1} \leq t \leq T$. Define

$$x^\varepsilon(t) = E\left[x_T + \int_t^T f_\tau^\varepsilon \mathrm{d}\tau|\mathcal{F}_t\right] + \int_t^T z_\tau^\varepsilon \mathrm{d}\tau$$

a.s. for $t_{i-1} \leq t \leq T$. We have $x^\varepsilon(t_{i-1}) = x_{i-1}$, $d(x^\varepsilon(\theta_\varepsilon(t)), \mathcal{K}(\theta_\varepsilon(t))) = 0$, and

$$E\left[\mathrm{dist}\left(x^\varepsilon(s), E[x^\varepsilon(t) + \int_s^t F(\tau, (x_{i-1}^\varepsilon \circ \theta_\varepsilon)(\tau))\mathrm{d}\tau|\mathcal{F}_s]\right)\right] \leq \varepsilon/2$$

for $t_{i-1} \leq s \leq t \leq T$.

Denote by S_i the set of all positive numbers $h \in (0, \min(\delta, t_{i-1}))$ such that

$$\overline{D}\left[S(E[x^\varepsilon(t_{i-1}) + \int_{t_{i-1}-h}^{t_{i-1}} F(\tau, x_{i-1})\mathrm{d}\tau|\mathcal{F}_{t_{i-1}-h}]), \mathcal{K}(t_{i-1})\right] \leq \varepsilon h/2.$$

By the properties of x^ε, we have $x^\varepsilon(t_{i-1}) = x_{i-1}$ and $(t_{i-1}, x^\varepsilon(t_{i-1})) \in Graph(\mathcal{K})$. Therefore, by virtue of (2.3), we have $S_i \neq \emptyset$ and $\sup S_i > 0$. Choose $h_{i-1} \in S_i$ such that $(1/2)\sup S_i \leq h_{i-1}$. Put $t_i = t_{i-1} - h_{i-1}$. We can extend again the step function θ_ε, step process z^ε, and processes f^ε and x^ε to the interval $[t_i, T]$ such that $d(x^\varepsilon(\theta_\varepsilon(t)), \mathcal{K}(\theta(t)))) = 0$ for $t_i \le t \le T$, and $f_t^\varepsilon \in \mathrm{co}F(t, x^\varepsilon(\theta_\varepsilon(t))$ and $|\phi_t - f_t^\varepsilon| = \mathrm{dist}(\phi_t, \mathrm{co}F(t, x^\varepsilon(\theta_\varepsilon(t))$ a.s. for $t_i \le t \le T$. Furthermore,

$$E[\mathrm{dist}(x^\varepsilon(s), E[x^\varepsilon(t) + \int_s^t F(\tau, (x_{i-1}^\varepsilon \circ \theta_\varepsilon)(\tau))\mathrm{d}\tau|\mathcal{F}_s])] \le \varepsilon/2$$

for $t_i \le s \le t \le T$. We can continue the above procedure up to $n \ge 1$ such that $0 < t_n \le a < t_{n-1}$. Suppose to the contrary that there does not exist such $n \ge 1$, i.e., that for every $n \ge 1$, one has $a < t_n < T$. Then we can extend the step function θ_ε, the step process z^ε, and the stochastic processes f^ε and x^ε to the interval $[t_n, T]$ for every $n \ge 1$ such that $x^\varepsilon(t_n) \in \mathcal{K}(t_n)$ a.s. for every $n \ge 1$ and so that the above properties are satisfied on $[t_n, T]$ for every $n \ge 1$. By the boundedness of the sequence $(t_n)_{n=1}^\infty$, we can select a decreasing subsequence $(t_i)_{i=1}^\infty$ converging to $t^* \in [a, T]$. Let $(x_i)_{i=1}^\infty$ be a sequence defined by $x_i = x^\varepsilon(t_i)$ a.s. for every $i \ge 0$. In particular, we have $x_i \in \mathcal{K}(t_i)$ a.s. for every $i \ge 1$. For every $j > k \ge 0$, we obtain

$$\begin{aligned} E|x_k - x_j| &\le E|E[x_T|\mathcal{F}_{t_k}] - E[x_T|\mathcal{F}_{t_j}]| + \int_{t^*}^{t_k} m(t)\mathrm{d}t + \int_{t^*}^{t_j} m(t)\mathrm{d}t \\ &+ (t_k - t_j)E|z_t^\varepsilon| + E\left|E\left[\int_{t^*}^T f_t^\varepsilon \mathrm{d}t|\mathcal{F}_{t_k}\right] - E\left[\int_{t^*}^T f_t^\varepsilon \mathrm{d}t|\mathcal{F}_{t^*}\right]\right| \\ &+ E\left|E\left[\int_{t^*}^T f_t^\varepsilon \mathrm{d}t|\mathcal{F}_{t_j}\right] - E\left[\int_{t^*}^T f_t^\varepsilon \mathrm{d}t|\mathcal{F}_{t^*}\right]\right|. \end{aligned}$$

By the continuity of the filtration $\mathbb{F}$, it follows that $\lim_{j,k\to\infty} E|x_k - x_j| = 0$. Then $(x_i)_{i=1}^\infty$ is a Cauchy sequence of $\mathbb{L}(\Omega, \mathcal{F}_T, \mathbb{R}^d)$. Therefore, there is $x^* \in \mathbb{L}(\Omega, \mathcal{F}_T, \mathbb{R}^d$ such that $\|x_i - x^*\| \to 0$ as $i \to \infty$. But $x_i \in \mathcal{K}(t_i))$ for every $i \ge 1$ and $\mathcal{K}$ is continuous. Then $(t^*, x^*) \in Graph(\mathcal{K})$, which by virtue of (3), implies that we can select $h^* \in (0, \min(\delta, t^*))$ such that

$$\overline{D}\left[S(E[x^* + \int_{t^*-h^*}^{t^*} F(\tau, x^*)\mathrm{d}\tau|\mathcal{F}_{t^*-h^*}]), \mathcal{K}(t^* - h^*)\right] \le \varepsilon h^*/2^5.$$

Similarly as above, for every $i \ge 1$ and $\phi_i \in S(\mathrm{co}F \circ x_i)$, we can select $f^* \in S(\mathrm{co}F \circ x^*)$ such that $|\phi_t^i - f_t^*)| = \mathrm{dist}(\phi_t^i, F(t, x^*))$ a.s. for every $t^* - h^* \le t \le t^*$. By the continuity of the filtration $\mathbb{F}$, we obtain $\|E[x^*|\mathcal{F}_{t_i-h^*}] - E[x^*|\mathcal{F}_{t^*-h^*}]\| \to 0$ and

$$E\left|E\left[\int_{t^*-h^*}^{t^*} f_\tau^* \mathrm{d}\tau|\mathcal{F}_{t_i-h^*}\right] - E\left[\int_{t^*-h^*}^{t^*} f_\tau^* \mathrm{d}\tau|\mathcal{F}_{t^*-h*}\right]\right| \to 0$$

as $i \to \infty$. Let $N \ge 1$ be such that for every $i \ge N$, we have $0 < t_i - t^* < \min(h^*, \delta)$, $\|x_i - x^*\| < \varepsilon h^*/(2^5 \cdot A)$, $D(\mathcal{K}(t_i - h^*), \mathcal{K}(t^* - h^*)) \le$

$\varepsilon h^*/2^5$, $\|E[x^*|\mathcal{F}_{t_i-h^*}] - E[x^*|\mathcal{F}_{t^*-h^*}]\| \le \varepsilon h^*/2^5$, $E\int_{t_i-h^*}^{t^*-h^*}|\phi_\tau^i|\mathrm{d}\tau \le \varepsilon h^*/2^5$, $E\int_{t^*}^{t_i}|\phi_\tau^i|\mathrm{d}t \le \varepsilon h^*/2^5$, and $E|E[\int_{t^*-h^*}^{t^*} f_\tau^*\mathrm{d}\tau|\mathcal{F}_{t_i-h^*}] - E[\int_{t^*-h^*}^{t^*} f_\tau^*\mathrm{d}\tau|\mathcal{F}_{t^*-h*}]| \le \varepsilon h^*/2^5$, where $A = 1 + \int_0^T k(t)\mathrm{d}t$. By the properties of the multifunction $F(t,\cdot)$ and selector f^* of $F \circ x^*$, it follows that

$$\begin{aligned}\|\mathbb{1}_{[t^*-h^*,t^*]}(\phi^i - f^*)\| &= E\int_{t^*-h^*}^{t^*}|\phi_\tau^i - f_\tau^*|\mathrm{d}\tau \\ &\le E\int_{t^*-h^*}^{t^*} h((F(t,x_i), F(t,x^*))]\mathrm{d}t \\ &\le \|x_i - x^*\|\int_{t^*-h^*}^{t^*} k(t)\mathrm{d}t.\end{aligned}$$

For every $i \ge N$, one gets

$$\begin{aligned}&d\left(E[x_i + \int_{t_i-h^*}^{t_i}\phi_\tau^i\mathrm{d}\tau|\mathcal{F}_{t_i-h^*}], \mathcal{K}(t_i - h^*)\right) \\ &\le E\left|E[x_i + \int_{t_i-h^*}^{t_i}\phi_\tau^i\mathrm{d}\tau|\mathcal{F}_{t_i-h^*}] - E[x^* + \int_{t^*-h^*}^{t^*} f_\tau^*\mathrm{d}\tau|\mathcal{F}_{t^*-h*}]\right| \\ &\quad + d\left(E[x^* + \int_{t^*-h^*}^{t^*} f_\tau^*\mathrm{d}\tau|\mathcal{F}_{t^*-h*}], \mathcal{K}(t^*-h^*)\right) + D(\mathcal{K}(t^*-h^*), \mathcal{K}(t_i - h^*)).\end{aligned}$$

But for every $i \ge N$, we have

$$\begin{aligned}&E\left|E\left[x_i + \int_{t_i-h^*}^{t_i}\phi_\tau^i\mathrm{d}\tau|\mathcal{F}_{t_i-h^*}\right] - E\left[x^* + \int_{t^*-h^*}^{t^*} f_\tau^*\mathrm{d}\tau|\mathcal{F}_{t^*-h*}\right]\right| \\ &\le E|E[(x_i - x^*)|\mathcal{F}_{t_i-h*}]| + E|E[x^*|\mathcal{F}_{t_i-h^*}] - E[x^*|\mathcal{F}_{t^*-h^*}]| \\ &\quad + E\left|E\left[\int_{t^*}^{t^*-h^*}(\phi_\tau^i - f_\tau^*)\mathrm{d}\tau|\mathcal{F}_{t_i-h^*}\right]\right| + E\int_{t_i-h^*}^{t^*-h^*}|\phi_\tau^i|\mathrm{d}\tau + E\int_{t^*}^{t_i}|\phi_\tau^i|\mathrm{d}t \\ &\quad + E\left|E\left[\int_{t^*}^{t^*-h^*} f_\tau^*\mathrm{d}\tau|\mathcal{F}_{t_i-h^*}\right] - E\left[\int_{t^*-h^*}^{t^*} f_\tau^*\mathrm{d}\tau|\mathcal{F}_{t^*-h*}\right]\right| \le 6\varepsilon h^*/2^5.\end{aligned}$$

Therefore, for every $i \ge N$, one gets

$$d\left[E\left[x_i + \int_{t_i-h^*}^{t_i}\phi_\tau^i\mathrm{d}\tau|\mathcal{F}_{t_i-h^*}\right], \mathcal{K}(t_i)\right] \le 8\varepsilon h^*/2^5 = \varepsilon h^*/2^2,$$

which implies that

$$\overline{D}(S(E[x_i + \int_{t_i-h^*}^{t_i} F(\tau, x_i)\mathrm{d}\tau|\mathcal{F}_{t_i-h^*}], \mathcal{K}(t_i)) \leq \varepsilon h^*/2^2.$$

But $t^* \leq t_i$ for $i \geq 1$. Therefore, for every $i \geq N$, one has $h^* \in S_{i+1}$ and $(1/2)h^* \leq \sup S_{i+1} \leq h_i = t_i - t_{i+1}$, which contradicts the convergence of the sequence $(t_i)_{i=1}^{\infty}$. Then there is a $p > 1$ such that $a = t_p < t_{p-1}, \ldots, t_1 < t_0 = T$. Taking $f^{\varepsilon} = \mathbb{1}_{[a,t_{p-1}]} f^p + \sum_{i=p-2}^{0} \mathbb{1}_{(t_{i+1},t_i]} f^i$, we obtain the desired selector of $\mathrm{co}F \circ (x^{\varepsilon} \circ \theta_{\varepsilon})$. □

Remark 2.3. The above results are also true if instead of continuity of the set-valued mapping $\mathcal{K}$, we assume that it is uniformly upper semicontinuous on $[0, T]$, i.e., that $\lim_{\delta\to 0} \sup_{0\leq t\leq T} \overline{D}(\mathcal{K}(t + \delta), \mathcal{K}(t)) = 0$. □

Conditions (2.1) and (2.3) can be expressed by certain types of stochastic tangent sets. To see this, let $(t, x) \in Graph(\mathcal{K})$ and denote by $\mathcal{T}_K(t, x)$ the set of all pairs $(f, g) \in \mathbb{L}^2([t, T] \times \Omega, \Sigma_{\mathbb{F}}^t, \mathbb{R}^d) \times \mathbb{L}^2([t, T] \times \Omega, \Sigma_{\mathbb{F}}^t, \mathbb{R}^{d\times m})$ such that

$$\liminf_{h\to 0+}(1/h)d\left[x + \int_t^{t+h} f_\tau \mathrm{d}\tau + \int_t^{t+h} g_\tau \mathrm{d}B_\tau, \mathcal{K}(t + h)\right] = 0,$$

where $\Sigma_{\mathbb{F}}^t$ denotes the σ-algebra of all $\mathbb{F}$-nonanticipative subsets of $[t, T] \times \Omega$. In a similar way, for $(t, x) \in Graph(\mathcal{K}^{\varepsilon})$ and $\varepsilon \in (0, 1)$, we can define a backward stochastic tangent set $\mathcal{T}_K^b(t, x)$ with respect to a filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ as the set of all measurable processes $f \in \mathbb{L}([0, T] \times \Omega, \mathcal{F}_T, \mathbb{R}^d)$ such that

$$\liminf_{h\to 0+}(1/h)d\left(E\left[x + \int_{t-h}^{t} f_\tau \mathrm{d}\tau|\mathcal{F}_{t-h}\right], \mathcal{K}(t - h)\right) = 0.$$

Lemma 2.1. *Let $\mathcal{P}_{\mathbb{F}}$ be a complete filtered probability space. Assume that F and G satisfy condition (i) of $(\mathcal{H}_1)$ and let $K : [0, T] \times \Omega \to \mathrm{Cl}(\mathbb{R}^d)$ be $\mathbb{F}$-adapted and such that $\mathcal{K}(t) \neq \emptyset$ for every $0 \leq t \leq T$. The condition (2.1) is satisfied for every $(t, x) \in Graph(\mathcal{K})$ if and only if $S_{\mathbb{F}}^t(F \circ x) \times S_{\mathbb{F}}^t(G \circ x) \subset \mathcal{T}_K(t, x)$ for every $(t, x) \in Graph(\mathcal{K})$, where $S_{\mathbb{F}}^t(F \circ x)$ and $S_{\mathbb{F}}^t(G \circ x)$ denote the sets of all restrictions of all elements of $S_{\mathbb{F}}(F \circ x)$ and $S_{\mathbb{F}}(G \circ x)$, respectively, to the set $[t, T] \times \Omega$.*

Proof. It is clear that if (2.1) is satisfied for every $(t, x) \in Graph(\mathcal{K})$, then $S_{\mathbb{F}}^t (F \circ x) \times S_{\mathbb{F}}^t(G \circ x) \subset \mathcal{T}_K(t, x)$ for every $(t, x) \in Graph(\mathcal{K})$. Let $S_{\mathbb{F}}^t(F \circ x) \times S_{\mathbb{F}}^t(G \circ x) \subset \mathcal{T}_K(t, x)$ for fixed $(t, x) \in Graph(\mathcal{K})$. Then for every $(f, g) \in S_{\mathbb{F}}^t(F \circ x) \times S_{\mathbb{F}}^t(G \circ x)$, one has

$$\liminf_{h\to 0+}(1/h)d\left[x + \int_t^{t+h} f_\tau \mathrm{d}\tau + \int_t^{t+h} g_\tau \mathrm{d}B_\tau, \mathcal{K}(t + h)\right] = 0.$$

Thus for every $(t, x) \in Graph(\mathcal{K})$ and $(f, g) \in S^t_{\mathbb{F}}(F \circ x) \times S^t_{\mathbb{F}}(G \circ x)$ and every $\varepsilon \in (0, 1)$, there exists $h^{f,g}_\varepsilon(t) \in (0, \varepsilon)$ such that

$$d\left[x + \int_t^{t+h} f_\tau \mathrm{d}\tau + \int_t^{t+h} g_\tau \mathrm{d}B_\tau, \mathcal{K}(t+h)\right] \le h^{f,g}_\varepsilon(t) \cdot \varepsilon.$$

Let $h_\varepsilon = \sup\{h^{f,g}_\varepsilon(t) : (f, g) \in S^t_{\mathbb{F}}(F \circ x) \times S^t_{\mathbb{F}}(G \circ x)),\ 0 \le t \le T\}$. We have

$$d\left[x + \int_t^{t+h} f_\tau \mathrm{d}\tau + \int_t^{t+h} g_\tau \mathrm{d}B_\tau, \mathcal{K}(t+h)\right] \le h_\varepsilon \cdot \varepsilon$$

for every $(t, x) \in Graph(\mathcal{K})$ and $(f, g) \in S_{\mathbb{F}}(F \circ x) \times S_{\mathbb{F}}(G \circ x)$. Then

$$\overline{D}\left[x + \int_t^{t+h} F(\tau, x)\mathrm{d}\tau + \int_t^{t+h} G(\tau, x)\mathrm{d}B_\tau, \mathcal{K}(t+h)\right] \le h_\varepsilon \varepsilon,$$

which implies that

$$\liminf_{h\to 0+}(1/h)\overline{D}\left(x + \int_t^{t+h} F(\tau, x)\mathrm{d}\tau + \int_t^{t+h} G(\tau, x)\mathrm{d}B_\tau, \mathcal{K}(t+h)\right) = 0$$

for every $(t, x) \in Graph(\mathcal{K})$. □

Remark 2.4. The results of Theorems 2.1 and 2.2 also hold if instead of condition (2.1), we assume that $[S^t_{\mathbb{F}}(F \circ x) \times S^t_{\mathbb{F}}(G \circ x)] \cap \mathcal{T}_K(t, x) \ne \emptyset$ for every $\varepsilon \in (0, 1)$ and $(t, x) \in Graph(\mathcal{K}^\varepsilon)$. □

There are another types of stochastic tangent sets. For a given $\mathbb{F}$-adapted set-valued stochastic process $K : [0, T] \times \Omega \to \mathrm{Cl}(\mathbb{R}^d)$ and $(t, x) \in Graph(\mathcal{K})$, by $\mathcal{S}_K(t, x)$ we denote the stochastic "tangent set" to K at (t, x) with respect to the filtration $\mathbb{F}$ defined as the set of all pairs $(f, g) \in \mathbb{L}^2([t, T] \times \Omega, \Sigma^t_{\mathbb{F}}, \mathbb{R}^d) \times \mathbb{L}^2([t, T] \times \Omega, \Sigma^t_{\mathbb{F}}, \mathbb{R}^{d\times m})$ such that for every $(f, g) \in \mathcal{S}_K(t, x)$, there exist a sequence $(h_n)_{n=1}^\infty$ of positive numbers converging to 0 and sequences $(a^n)_{n=1}^\infty$ and $(b^n)_{n=1}^\infty$ of d- and $d \times m$-dimensional $\mathbb{F}$-adapted stochastic processes $a^n = (a^n_t)_{0\le t\le T}$ and $b^n = (b^n_t)_{0\le t\le T}$, respectively, such that

$$\sup_{n\ge 1} d\left[x + \int_t^{t+h_n} (f_\tau + a^n_s)\mathrm{d}\tau + \int_t^{t+h_n} (g_\tau + b^n_s)\mathrm{d}B_\tau, \mathcal{K}(t+h_n)\right] = 0$$

and

$$\lim_{n\to\infty} (1/h_n)E\left[\left|\int_t^{t+h_n} a^n_\tau \mathrm{d}\tau + \int_t^{t+h_n} b^n_\tau \mathrm{d}B_\tau\right|^2\right]^{1/2} = 0.$$

We shall show that such stochastic tangent sets are smaller then $\mathcal{T}_K(t,x)$, i.e., that $\mathcal{S}_K(t,x) \subset \mathcal{T}_K(t,x)$ for every $(t,x) \in Graph(\mathcal{K})$.

Lemma 2.2. *Let* $K : [0,T] \times \Omega \to \mathrm{Cl}(\mathbb{R}^d)$ *be an* $\mathbb{F}$*-adapted set-valued process such that* $\mathcal{K}(t) \neq \emptyset$ *for every* $0 \leq t \leq T$*. For every* $(t,x) \in Graph(\mathcal{K})$*, one has* $\mathcal{S}_K(t,x) \subset \mathcal{T}_K(t,x)$*.*

Proof. Let $(t,x) \in Graph(\mathcal{K})$ be fixed and $(f,g) \in \mathcal{S}_K(t,x)$. There exist a sequence $(h_n)_{n=1}^{\infty}$ of positive numbers converging to 0 and sequences $(a^n)_{n=1}^{\infty}$ and $(b^n)_{n=1}^{\infty}$ of d- and $d \times m$-dimensional $\mathbb{F}$-adapted stochastic processes $a^n = (a_t^n)_{0 \leq t \leq T}$ and $b^n = (b_t^n)_{0 \leq t \leq T}$, respectively, such that the above conditions are satisfied. For every $n \geq 1$, one has

$$d^2\left[x + \int_t^{t+h_n} f_\tau \mathrm{d}\tau + \int_t^{t+h_n} g_\tau \mathrm{d}B_\tau, \mathcal{K}(t+h)\right] \leq 2E\left[\left|\int_t^{t+h_n} a_\tau^n \mathrm{d}\tau + \int_t^{t+h_n} b_\tau^n \mathrm{d}B_\tau\right|^2\right].$$

Hence, by the properties of sequences $(a^n)_{n=1}^{\infty}$ and $(b^n)_{n=1}^{\infty}$, it follows that

$$\lim_{n\to\infty}(1/h_n)d\left[x + \int_t^{t+h_n} f_\tau \mathrm{d}\tau + \int_t^{t+h_n} g_\tau \mathrm{d}B_\tau, \mathcal{K}(t+h)\right] = 0,$$

which implies

$$\liminf_{h\to 0+}(1/h)d\left(x + \int_t^{t+h} f_\tau \mathrm{d}\tau + \int_t^{t+h} g_\tau \mathrm{d}B_\tau, \mathcal{K}(t-h)\right) = 0.$$

Then $(f,g) \in \mathcal{T}_K(t,x)$ for every $(f,g) \in \mathcal{S}_K(t,x)$. □

Denote by $\tau_K(t,x)$ that stochastic "contingent set" to K at (t,x) with respect to $\mathbb{F}$, defined as the set of all pairs $(f,g) \in \mathbb{L}^2([t,T] \times \Omega, \Sigma_{\mathbb{F}}^t, \mathbb{R}^d) \times \mathbb{L}^2([t,T] \times \Omega, \Sigma_{\mathbb{F}}^t, \mathbb{R}^{d\times m})$ such that for every such pair (f,g), there exist a sequence $(h_n)_{n=1}^{\infty}$ of positive numbers converging to 0 and sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ of d- and $d \times m$-dimensional $\mathcal{F}_t$-measurable random variables a_n and b_n, respectively, such that $x + \int_t^{t+h_n} f_s ds + \int_t^{t+h_n} g_s \mathrm{d}B_s + h_n a_n + \sqrt{h_n} b_n \in \mathcal{K}(t+h_n)$ for every $n \geq 1$ and $\max\left\{E|a_n|^2, (1/h_n)E|b_n|^2\right\} \to 0$ as $n \to \infty$. Similarly as above, we obtain the following result.

Lemma 2.3. *Let* $K : [0,T] \times \Omega \to \mathrm{Cl}(\mathbb{R}^d)$ *be an* $\mathbb{F}$*-adapted set-valued process such that* $\mathcal{K}(t) \neq \emptyset$ *for every* $0 \leq t \leq T$*. For every* $(t,x) \in Graph(\mathcal{K})$*, one has* $\tau_K(t,x) \subset \mathcal{S}_K(t,x)$*.*

Proof. Let $(t,x) \in Graph(\mathcal{K})$ be fixed and $(f,g) \in \tau_K(t,x)$. There are a sequence $(h_n)_{n=1}^{\infty}$ of positive numbers converging to zero and sequences $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ of $\mathcal{F}_t$-measurable random variables $a_n : \Omega \to \mathbb{R}^d$ and $b_n : \Omega \to \mathbb{R}^{d\times m}$ such that the above conditions are satisfied. For every $n \geq 1$, one gets

$$\sup_{n\geq 1} d\left[x + \int_t^{t+h_n}(f_s + a_n)\mathrm{d}s + \int_t^{t+h_n}(g_s + b_n)\mathrm{d}B_s, \mathcal{K}(t+h)\right] = 0$$

and

$$\left[E\left|\int_t^{t+h_n} a_n \mathrm{d}s + \int_t^{t+h_n} b_n \mathrm{d}B_s\right|^2\right]^{1/2} \leq h_n \left[E|a_n|^2\right]^{1/2} + \sqrt{h_n}\left[E|b_n|^2\right]^{1/2}.$$

Hence, for $n \geq 1$ sufficiently large, it follows that

$$(1/h_n)\left[E\left|\int_t^{t+h_n} a_n \mathrm{d}s + \int_t^{t+h_n} b_n \mathrm{d}B_s\right|^2\right]^{1/2} \leq \left[E|a_n|^2\right]^{1/2} + \left[1/h_n E|b_n|^2\right]^{1/2},$$

which implies that

$$(1/h_n)\left[E\left|\int_t^{t+h_n} a_n \mathrm{d}s + \int_t^{t+h_n} b_n \mathrm{d}B_s\right|^2\right]^{1/2} \to 0 \quad \text{as} \quad n \to \infty.$$

Then $(f, g) \in \mathcal{S}_K(t, x)$. □

Remark 2.5. The results of Theorems 2.1 and 2.2 are also true if instead of condition (2.1), we assume that $[S_{\mathbb{F}}^t(F \circ x) \times S_{\mathbb{F}}^t(G \circ x)] \cap \tau_K(t, x) \neq \emptyset$ for every $\varepsilon \in (0, 1)$ and $(t, x) \in Graph(\mathcal{K}^\varepsilon)$. □

3 Existence of Viable Solutions

We shall prove now that if F and G satisfy conditions $(\mathcal{H}_1)$, then for every continuous set-valued $\mathbb{F}$-adapted process $K : [0, T] \times \Omega \to \mathrm{Cl}(\mathbb{R}^d)$, the viability problems $\overline{SFI}(F, G, K)$ and $BSDI(F, K)$ possess viable strong solutions. Furthermore, the existence of viable weak solutions of $\overline{SFI}(F, G, K)$ is considered. Similarly as above, we define $\mathcal{K}(t)$ and $\mathcal{K}^\varepsilon(t)$ by setting $\mathcal{K}(t) = \{u \in \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{R}^d) : d(u, \mathcal{K}(t)) = 0\}$ and $\mathcal{K}^\varepsilon(t) = \{u \in \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{R}^d) : d(u, \mathcal{K}(t)) \leq \varepsilon\}$.

Theorem 3.1. *Let $\mathcal{P}_{\mathbb{F}} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space and $B = (B_t)_{0 \leq t \leq T}$ an m-dimensional $\mathbb{F}$-Brownian motion on $\mathcal{P}_{\mathbb{F}}$. Assume that F and G satisfy conditions $(\mathcal{H}_1)$ and let $K : [0, T] \times \Omega \to \mathrm{Cl}(\mathbb{R}^d)$ be an $\mathbb{F}$-adapted set-valued process such that $\mathcal{K}(t) \neq \emptyset$ for every $0 \leq t \leq T$ and such that the mapping $\mathcal{K} : [0, T] \ni t \to \mathcal{K}(t) \in \mathrm{Cl}(\mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{R}^d))$ is continuous. If $\mathcal{P}_{\mathbb{F}}$, B, F, G, and K are such that (2.1) is satisfied for every $(t, x) \in Graph(\mathcal{K})$, then the problem $\overline{SFI}(F, G, K)$ possesses on $\mathcal{P}_{\mathbb{F}}$ a strong viable solution.*

Proof. Let $a \in (0, T)$ and select arbitrarily $x_0 \in \mathcal{K}(0)$. Let $u_0 \in \mathbb{L}^2(\Omega, \mathcal{F}_0, \mathbb{R}^d)$ and $v_0 \in \mathbb{L}^2(\Omega, \mathcal{F}_0, \mathbb{R}^{d \times m})$ be such that $u_0 \in F(0, x_0)$ and $v_0 \in G(0, x_0)$ a.s. By virtue of Theorem 2.2, for $\varepsilon_1 = 1/2^{3/2}$ and stochastic processes $\phi^1 = (\phi_t^1)_{a \leq t \leq T}$

and $\psi^1 = (\psi_t^1)_{a\le t\le T}$ defined by $\phi_t^1 = u_0$ and $\psi_t^1 = v_0$ a.s. for every $a \le t \le T$, there exist a partition $0 = t_0^1 < t_1^1 < \cdots < t_{p_1-1}^1 < t_{p_1}^1 = a$, a step function θ_1, and stochastic processes f^1, g^1, and z^1 such that conditions (i)–(v) of Theorem 2.2 are satisfied with

$$x^1(t) = x_0 + \int_0^t (f_\tau^1 + z_\tau^1)\mathrm{d}\tau + \int_0^t g_\tau^1 \mathrm{d}B_\tau$$

a.s. for $a \le t \le T$. Similarly, for $\varepsilon_2 = 1/2$ and $\phi^2 = f^1$ and $\psi^2 = g^1$, we can select a partition $0 = t_0^2 < t_1^2 < \cdots < t_{p_2-1}^2 < t_{p_1}^2 = a$, a step function θ_2, and stochastic processes f^2, g^2, and z^2 such that conditions (i)–(v) of Theorem 2.2 are satisfied with

$$x^2(t) = x_0 + \int_0^t (f_\tau^2 + z_\tau^2)\mathrm{d}\tau + \int_0^t g_\tau^2 \mathrm{d}B_\tau$$

a.s. for $a \le t \le T$. Continuing this procedure for $\varepsilon_k = 1/2^{3k/2}$ and $\phi^k = f^{k-1}$ and $\psi^k = g^{k-1}$, we obtain a partition $0 = t_0^k < t_1^k < \cdots < t_{p_k-1}^k < t_{p_k}^k = a$, a step function θ_k, and stochastic processes f^k, g^k, and z^k such that conditions (i)–(v) of Theorem 2.2 are satisfied for every $k \ge 1$ with

$$x^k(t) = x_0 + \int_0^t (f_\tau^k + z_\tau^k)\mathrm{d}\tau + \int_0^t g_\tau^k \mathrm{d}B_\tau$$

a.s. for $a \le t \le T$ such that $d(x^k(\theta_k(t)), \mathcal{K}(\theta_k(t)) = 0$. For every $k \ge 1$, one has $f^k \in S_{\mathbb{F}}(F \circ (x^{k-1} \circ \theta_{k-1}))$, $g^k \in S_{\mathbb{F}}(G \circ (x^{k-1} \circ \theta_{k-1}))$, $|f^k - f^{k-1}| \le \mathrm{dist}(f_t^{k-1}, F(t, (x^{k-1}(\theta_{k-1}(t)))$, $|g^k - g^{k-1}| \le \mathrm{dist}(g_t^{k-1}, G(t, (x^{k-1}(\theta_{k-1}(t)))$, $\|z^k(t)\| \le \varepsilon_k$, and

$$d\left(x^k(t) - x^k(s), \mathrm{cl}_{\mathbb{L}}\left(\int_s^t F(\tau, (x^k \circ \theta_k)(\tau))\mathrm{d}\tau + \int_s^t G(\tau, (x^k \circ \theta_k)(\tau))\mathrm{d}B_\tau\right)\right) \le \varepsilon_k$$

for $0 \le s \le t \le a$. Furthermore, one has $|\theta_k(t) - \theta_{k-1}(t)| \le \delta_{k-1}$,

$$\int_{\theta_{k-1}(t)}^{\theta_k(t)} |f_\tau^{k-1}|^2 \mathrm{d}\tau \le \varepsilon_{k-1}^2/2^4 \quad \text{and} \quad \int_{\theta_{k-1}(t)}^{\theta_k(t)} |g_\tau^{k-1}|^2 \mathrm{d}\tau \le \varepsilon_{k-1}^2/2^4$$

for $0 \le t \le a$, because by (i) of Theorem 2.2, $\delta_k \in (0, \varepsilon_k)$ is such that

$$\max\left[\sup_{0\le s<t\le T} \int_t^{t+\delta_k} k^2(\tau)\mathrm{d}\tau, \sup_{0\le s<t\le T} \int_t^{t+\delta_k} m^2(\tau)\mathrm{d}\tau\right] \le \varepsilon_k^2/2^4.$$

We shall now evaluate $E[\sup_{0\le\tau\le t} |x^{k+1}(\tau) - x^k(\tau)|^2]$ for $k = 1, 2, \ldots$ and $0 \le t \le a$. Let us observe first that $E[\sup_{0\le\tau\le t} |x^k(\theta_{k+1}(\tau)) - x^k(\theta_k(\tau))|^2] \to 0$ as $k \to \infty$, because $|\theta_{k+1}(t) - \theta_k(t)| \le \delta_k$ and

$$E[\sup_{0\le\tau\le t} |x^k(\theta_{k+1}(\tau)) - x^k(\theta_k(\tau))|^2] \le 3(\delta_k + 1) \int_{\theta_k(t)}^{\theta_{k+1}(t)} m^2(\tau)\mathrm{d}\tau + \varepsilon_k^2$$

for $k = 2, 3, \ldots$ and $0 \le t \le a$. Hence it follows that

$$E[\sup_{0\le\tau\le t} |x^{k+1}(\tau) - x^k(\tau)|^2] \le \alpha\varepsilon_k^2 + \beta \int_0^t k^2(\tau) E[\sup_{0\le u\le\tau} |x^k(u) - x^{k-1}(u)|^2] \mathrm{d}\tau$$

for every $k = 1, 2, \ldots$ and $0 \le t \le a$, where $x_t^0 = x_0$, $\alpha = (4T)^2$ and $\beta = 2^2(T+1)$.

Now, by the definition of the processes x^1 and x^0, one gets $E[\sup_{0\le\tau\le t} |x^1(\tau) - x^0(\tau)|^2] \le \gamma$ with $\gamma = 2^2[(T+1)\int_0^T m^2(t)\mathrm{d}t + T^2]$. Therefore,

$$E[\sup_{0\le\tau\le t} |x^2(\tau) - x^1(\tau)|^2] \le \alpha\varepsilon_1^2 + \beta\gamma \int_0^t k^2(\tau)\mathrm{d}\tau$$

for $0 \le t \le a$. From this and (3), it follows that

$$E[\sup_{0\le\tau\le t} |x^3(\tau) - x^2(\tau)|^2] \le \alpha\varepsilon_2^2 + \alpha\beta\varepsilon_1^2 \int_0^t k^2(\tau)\mathrm{d}\tau + \frac{\beta^2\gamma}{2!}\left(\int_0^t k^2(\tau)\mathrm{d}\tau\right)^2.$$

Similarly, we get

$$E[\sup_{0\le\tau\le t} |x^4(\tau) - x^3(\tau)|^2]$$
$$\le \alpha\varepsilon_3^2 + \alpha\beta\varepsilon_2^2 \int_0^t k^2(\tau)\mathrm{d}\tau + \alpha\frac{\beta^2\varepsilon_1^2}{2!}\left(\int_0^t k^2(\tau)\mathrm{d}\tau\right)^2 + \gamma\frac{\beta^3}{3!}\left(\int_0^t k^2(\tau)\mathrm{d}\tau\right)^3$$

for $0 \le t \le a$. By the inductive procedure, we obtain

$$E[\sup_{0\le\tau\le t} |x^{n+1}(\tau) - x^n(\tau)|^2]$$
$$\le \alpha\varepsilon_n^2 + \alpha\beta\varepsilon_{n-1}^2 \int_0^t k^2(\tau)\mathrm{d}\tau + \alpha\varepsilon_{n-2}^2\frac{\beta^2}{2!}\left(\int_0^t k^2(\tau)\mathrm{d}\tau\right)^2 + \cdots + \gamma\frac{\beta^n}{n!}\left(\int_0^t k^2(\tau)\mathrm{d}\tau\right)^n$$
$$\le M\varepsilon_n^2\left[1 + 8\beta\int_0^t k^2(\tau)\mathrm{d}\tau + \frac{(8\beta)^2}{2!}\left(\int_0^t k^2(\tau)\mathrm{d}\tau\right)^2 + \cdots + \frac{(8\beta)^n}{n!}\left(\int_0^t k^2(\tau)\mathrm{d}\tau\right)^n\right]$$
$$\le M\varepsilon_n^2 \exp\left[8\beta\int_0^t k^2(\tau)\mathrm{d}\tau\right]$$

for $n \ge 1$ with $M = \max(\alpha, \gamma)$. By Chebyshev's inequality, we obtain

$$P\left[\sup_{0\le\tau\le a} |x^{n+1}(\tau) - x^n(\tau)| > 2^{-n}\right]$$
$$\le 2^{2n} E[\sup_{0\le\tau\le T} |x^{n+1}(\tau) - x^n(\tau)|^2] \le 2^{2n}\varepsilon_n^2 M \exp\left[8\beta\int_0^t k^2(\tau)\mathrm{d}\tau\right]$$
$$= 2^{-n} M \exp\left[8\beta\int_0^t k^2(\tau)\mathrm{d}\tau\right].$$

Therefore, by the Borel–Cantelli lemma, one gets

$$P\left[\sup_{0\le\tau\le a}|x^{n+1}(\tau)-x^n(\tau)|>2^{-n}\quad\text{for infinitely many } n\right]=0.$$

Thus for a.e. $\omega\in\Omega$, there exists $n_0=n_0(\omega)$ such that $\sup_{0\le\tau\le a}|x^{n+1}(\tau)-x^n(\tau)|\le 2^{-n}$ for $n\ge n_0(\omega)$. Therefore, the sequence $\{x^n(\cdot)(\omega)\}_{n=1}^{\infty}$ is uniformly convergent on $[0,a]$ for a.a. $\omega\in\Omega$, because $x^n(t)(\omega)=x^1(t)(\omega)+\sum_{k=1}^{n-1}[x^{k+1}(t)(\omega)-x^k(t)(\omega)]$ for every $0\le t\le T$ and a.a. $\omega\in\Omega$. Denote the limit of the above sequence by $x_t(\omega)$ for $0\le t\le a$ and a.a. $\omega\in\Omega$. By virtue of (3), it follows that $E[\sup_{0\le\tau\le t}|x^{n+1}(\tau)-x^n(\tau)|^2]\to 0$ as $n\to\infty$. On the other hand, by the properties of sequences $(f^k)_{k=1}^{\infty}$ and $(f^k)_{k=1}^{\infty}$, we get

$$\begin{aligned}\int_0^a E[|f_\tau^{k+1}-f_\tau^k|^2]\mathrm{d}\tau &\le \int_0^a E[h^2(F(\tau,(x^k\circ\theta_k)(\tau))),F(\tau,(x^{k-1}\circ\theta_{k-1})(\tau))))]\mathrm{d}\tau\\ &\le \int_0^a k^2(\tau)E[\sup_{0\le u\le\tau}|x^k(u)-x^{k-1}(u)|^2]\mathrm{d}\tau\end{aligned}$$

and

$$\begin{aligned}\int_0^a E[|g_\tau^{k+1}-g_\tau^k|^2]\mathrm{d}\tau &\le \int_0^a E[H^2(G(\tau,(x^k\circ\theta_k)(\tau))),G(\tau,(x^{k-1}\circ\theta_{k-1})(\tau))))]\mathrm{d}\tau\\ &\le \int_0^a k^2(\tau)E[\sup_{0\le u\le\tau}|x^k(u)-x^{k-1}(u)|^2]\mathrm{d}\tau\end{aligned}$$

for every $k=0,1,\ldots$. Hence it follows that $(f^k)_{k=1}^{\infty}$ and $(g^k)_{k=1}^{\infty}$ are Cauchy sequences of Banach spaces $(\mathbb{L}^2([0,a]\times\Omega,\Sigma_{\mathbb{F}},\mathbb{R}^d),|\cdot|)$ and $(\mathbb{L}^2([0,a]\times\Omega,\Sigma_{\mathbb{F}},\mathbb{R}^{d\times m}),|\cdot|)$, respectively. Then there exist $f\in\mathbb{L}^2([0,a]\times\Omega,\Sigma_{\mathbb{F}},\mathbb{R}^d)$ and $g\in\mathbb{L}^2([0,a]\times\Omega,\Sigma_{\mathbb{F}},\mathbb{R}^{d\times m})$ such that $|f^n-f|\to 0$ and $|g^n-g|\to 0$ as $n\to\infty$. Let $y_t=x_0+\int_0^t f_\tau\mathrm{d}\tau+\int_0^t g_\tau\mathrm{d}B_\tau$ for $0\le t\le a$. For every $n\ge 1$, one gets

$$\begin{aligned}&E[\sup_{0\le t\le a}|x^n(t)-y_t|^2]\\ &\le E[\sup_{0\le t\le a}\left|\int_0^t(f_\tau^n-f_\tau)\mathrm{d}\tau+\int_0^t(g_\tau^n-g_\tau)\mathrm{d}B_\tau+\int_0^t z^n(\tau)\mathrm{d}\tau\right|^2\\ &\le 3T|f^n-f|^2+3|g^n-g||^2+3T^2\varepsilon_n.\end{aligned}$$

Therefore, we have $E[\sup_{0\le t\le a}|x^n(t)-y_t|^2]\to 0$ and $E[\sup_{0\le t\le a}|x^n(t)-x(t)|^2]\to 0$ as $n\to\infty$, which implies that $x(t)=y_t$ a.s. for every $0\le t\le a$. Then $x(t)=x_0+\int_0^t f_\tau\mathrm{d}\tau+\int_0^t g_\tau\mathrm{d}B_\tau$ a.s. for $0\le t\le a$. Now, by Lemma 1.3 and Theorem 2.2, we obtain

$$\begin{aligned}
0 &\leq d\left(x(t)-x(s), \mathrm{cl}_{\mathbb{L}}\left(\int_s^t F(\tau, x(\tau))\mathrm{d}\tau + \int_s^t G(\tau, x(\tau))\mathrm{d}B_\tau\right)\right)\\
&\leq ||(x(t)-x(s)) - (x^n(t)-x^n(s))||\\
&\quad + d\left(x^n(t)-x^n(s), \mathrm{cl}_{\mathbb{L}}\left(\int_s^t F(\tau, (x^n\circ\theta_n)(\tau))\mathrm{d}\tau + \int_s^t G(\tau, (x^n\circ\theta_n)(\tau))\mathrm{d}B_\tau\right)\right)\\
&\quad + H\left(\mathrm{cl}_{\mathbb{L}}\left(\int_s^t F(\tau, (x^n\circ\theta_n)(\tau))\mathrm{d}\tau + \int_s^t G(\tau, (x^n\circ\theta_n)(\tau))\mathrm{d}B_\tau\right),\right.\\
&\qquad \left.\mathrm{cl}_{\mathbb{L}}\left(\int_s^t F(\tau, x(\tau))\mathrm{d}\tau + \int_s^t G(\tau, x(\tau))\mathrm{d}B_\tau\right)\right)\\
&\leq 2||x^n - x|| + \varepsilon_n + (1+\sqrt{T})\left(\int_0^T k^2(t)\mathrm{d}t\right)^{1/2}\|x^n\circ\theta_n - x\|
\end{aligned}$$

for every $0 \leq s \leq t \leq a$. But

$$\begin{aligned}
\|x^n\circ\theta_n - x\|^2 &= E[|x^n(\theta_n(t)) - x(t)|]\\
&\leq E[\sup_{0\leq u\leq a}|x^n(u)-x(u)|^2] + E[\sup_{0\leq t\leq a}|x(\theta_n(t))-x(t)|].
\end{aligned}$$

Then $\lim_{n\to\infty}\|x^n\circ\theta_n - x\| = 0$. Therefore, for every $0 \leq s \leq t \leq a$, we get

$$d\left(x(t)-x(s), \mathrm{cl}_{\mathbb{L}}\left(\int_s^t F(\tau, x(\tau))\mathrm{d}\tau + \int_s^t G(\tau, x(\tau))\mathrm{d}B_\tau\right)\right) = 0.$$

Thus

$$x(t)-x(s) \in \mathrm{cl}_{\mathbb{L}}\left(\int_s^t F(\tau, x(\tau))\mathrm{d}\tau + \int_s^t G(\tau, x(\tau))\mathrm{d}B_\tau\right)$$

for every $0 \leq s \leq t \leq a$. In a similar way, we get $d(x(t), \mathcal{K}(t)) \leq \|x^n - x\| + d(x^n(t), \mathcal{K}(t)) \leq \|x^n - x\| + \varepsilon_n$ for every $n \geq 1$ and $0 \leq s \leq t \leq a$. Therefore, $d(x(t), \mathcal{K}(t)) = 0$ for every $0 \leq t \leq a$, which by Theorem 3.1 of Chap. 2, implies that $x(t) \in K(t, \cdot)$ a.s. for $0 \leq t \leq a$.

We can now extend our solution to the whole interval $[0, T]$. Let us denote by Λ_x the set of all extensions of the viable solution x of $\overline{SFI}(F, G, K)$ obtained above. We have $\Lambda_x \neq \emptyset$, because we can repeat the above procedure for every interval $[a, \alpha]$ with $\alpha \in (a, T)$. Let us introduce in Λ_x the partial order relation $\preceq$ by setting $x \preceq z$ if and only if $a_x \leq a_z$ and $x = z|_{[0,a_x]}$, where $a_z \in (0, T)$ is such that z is a strong viable solution for $\overline{SFI}(F, G, K)$ on $[0, a_z]$, and $z|_{[0,a_x]}$ denotes the restriction of the solution z to the interval $[0, a_x]$. Let $\psi : [0, \alpha] \to \mathbb{R}^d$ be an extension of x to $[0, \alpha]$ with $\alpha \in (a, T)$ and denote by $P_x^\psi \subset \Lambda_x$ the set containing ψ and all its restrictions $\psi|_{[0,\beta]}$ for every $\beta \in [a, \alpha)$. It is clear that each completely ordered subset of Λ_x is of the form P_x^ψ, determined by some extension ψ of x. It is also clear that every P_x^ψ has ψ as its upper bound. Then by the Kuratowski–Zorn lemma, there exists a maximal element γ of Λ_x. It has to be $a_\gamma = T$. Indeed, if we had $a_\gamma < T$, then we could repeat the above procedure and extend γ as a viable strong solution $\xi \in \Lambda_x$ of $\overline{SFI}(F, G, K)$ to the interval $[0, b]$ with $a_\gamma < b$, which would imply that $\gamma \preceq \xi$, a contradiction to the assumption that γ is a maximal element of Λ_x. Then x can be extended to the whole interval $[0, T]$. □

In a similar way, by virtue of Remark 2.2, we can prove the following existence theorem for $SFI(\overline{F}, G)$.

Theorem 3.2. *Let $\mathcal{P}_{\mathbb{F}} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered separable probability space and $B = (B_t)_{0 \le t \le T}$ an m-dimensional $\mathbb{F}$-Brownian motion on $\mathcal{P}_{\mathbb{F}}$. Assume that F and G satisfy conditions $(\mathcal{H}_1)$ and that $K : [0, T] \times \Omega \to \mathrm{Cl}(\mathbb{R}^d)$ is $\mathbb{F}$-adapted such that $\mathcal{K}(t) \neq \emptyset$ for every $0 \le t \le T$ and such that the mapping $\mathcal{K} : [0, T] \to \mathrm{Cl}(\mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{R}^d))$ is continuous. If $\mathcal{P}_{\mathbb{F}}$, B, F, G, and K are such that (2.2) is satisfied for every $(t, x) \in Graph(\mathcal{K})$, then the problem*

$$\begin{cases} x_t - x_s \in \mathrm{cl}_{\mathbb{L}}\{\int_s^t F(\tau, x_\tau)\mathrm{d}\tau\} + \int_s^t G(\tau, x_\tau)\mathrm{d}B_\tau & \text{for } 0 \le s \le t \le T, \\ x_t \in K(t) \ \text{ a.s. for } t \in [0, T], \end{cases}$$

possesses on $\mathcal{P}_{\mathbb{F}}$ a strong viable solution. □

We shall now prove the existence of weak viable solutions for stochastic functional inclusions. The proof of such an existence theorem is based on the first viable approximation theorem presented above.

Theorem 3.3. *Assume that $F : [0, T] \times \mathbb{R}^d \to \mathrm{Cl}(\mathbb{R}^d)$ and $G : [0, T] \times \mathbb{R}^d \to \mathrm{Cl}(\mathbb{R}^{d \times m})$ are measurable, bounded, convex-valued and are such that $F(t, \cdot)$ and $G(t, \cdot)$ are continuous for a.e. fixed $t \in [0, T]$. Let G be diagonally convex and let $K : [0, T] \to \mathrm{Cl}(\mathbb{R}^d)$ be continuous. If there exist a complete filtered probability space $\mathcal{P}_{\mathbb{F}} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and a d-dimensional $\mathbb{F}$-Brownian motion on $\mathcal{P}_{\mathbb{F}}$ such that (2.1) is satisfied for every $\varepsilon \in (0, 1)$ and $(t, x) \in Graph(\mathcal{K}^\varepsilon)$, then $\overline{SFI}(F, G, K)$ possesses a weak viable solution.*

Proof. Let $x_0 \in \mathcal{K}(0)$ be fixed and let $\varepsilon_n = 1/2^n$. By virtue of Theorem 2.1, we can define on $[0, T]$ a step function $\theta_n = \theta_{\varepsilon_n}$ and $\mathbb{F}$-nonanticipative stochastic processes $f^n = f^{\varepsilon_n}$, $g^n = g^{\varepsilon_n}$, and $x_t^n = x_0 + \int_0^t f_\tau^n \mathrm{d}\tau + \int_0^t g_\tau^n \mathrm{d}B_\tau$ for $0 \le t \le T$ such that conditions (i)–(iii) of Theorem 2.1 are satisfied. In particular, for every $m \ge 1$, $n \ge 1$, and $0 \le s \le t \le T$, we obtain

$$\begin{aligned} E|x^n(t) - x^n(s)|^{2m} &\le C_m^1 E\left[\left|\int_s^t f_\tau^n \mathrm{d}\tau\right|^{2m}\right] + C_m^2 E\left[\left|\int_s^t g_\tau^n \mathrm{d}B_\tau\right|^{2m}\right] \\ &\le C_m^1 T^m E\left(\int_s^t |f_\tau^n|^2 \mathrm{d}\tau\right)^m + C_m^2 E\left[\left|\int_s^t g_\tau^n \mathrm{d}B_\tau\right|^{2m}\right], \end{aligned}$$

where C_m^1 and C_m^2 are positive integers depending on $m \geq 1$. Let us observe that

$$E\left(\int_s^t |f_\tau^n|^2 \mathrm{d}\tau\right)^m \leq M^{2m}|t-s|^m \text{ and } E\left[\left|\int_s^t g_\tau^n \mathrm{d}B_\tau\right|^{2m}\right] \leq M^{2m}(2m-1)!!|t-s|^m.$$

Therefore,

$$E|x^n(t) - x^n(s)|^{2m} \leq \left[C_m^1 T^m + C_m^2(2m-1)!!\right] M^{2m} |t-s|^m$$

for every $0 \leq s \leq t \leq T$ and $n, m \geq 1$. In a similar way, we can verify that there exist positive numbers K and γ such that $E|x_0^n|^\gamma \leq K$. Then the sequence $(x^n)_{n=1}^\infty$ of continuous processes $x^n = (x_t^n)_{0 \leq t \leq T}$ satisfies on the probability space $(\Omega, \mathcal{F}, P)$ the assumptions of Theorem 3.5 of Chap. 1. Furthermore, immediately from Theorem 2.1, it follows that $E[\mathrm{dist}(x^n(\theta_n(t)), K(\theta_n(t)))] \leq \varepsilon_n$ and

$$E\left[l(x^n(s))\left(h(x^n(t)) - h(x^n(s)) - \int_s^t (\mathbb{L}_{f^n g^n}^{x^n} h)_\tau \mathrm{d}\tau\right)\right] = 0$$

for every $0 \leq s \leq t \leq T$, $l \in C_b(\mathbb{R}^d, \mathbb{R})$, and $h \in C_b^2(\mathbb{R}^d, \mathbb{R})$.

By virtue of Theorems 3.5 and 2.4 of Chap. 1, there exist an increasing subsequence $(n_k)_{k=1}^\infty$ of $(n)_{n=1}^\infty$, a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, and d-dimensional continuous stochastic processes $\tilde{x}^{n_k}$ and $\tilde{x}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ for $k = 1, 2, \ldots$ such that $P(x^{n_k})^{-1} = P(\tilde{x}^{n_k})^{-1}$ for $k = 1, 2, \ldots$ and $\sup_{0 \leq t \leq T} |\tilde{x}^{n_k} - \tilde{x}| \to 0$ as $k \to \infty$. Let $\tilde{\mathcal{F}}_t^{n_k} = \bigcap_{\varepsilon > 0} \sigma(\tilde{x}_u^{n_k} : u \leq t + \varepsilon)$ for $0 \leq t \leq T$ and let $\tilde{\mathbb{F}}_{n_k} = (\tilde{\mathcal{F}}_t^{n_k})_{0 \leq t \leq T}$. For every $k \geq 1$, x^{n_k} and $\tilde{x}^{n_k}$ are continuous $\mathbb{F}$- and $\tilde{\mathbb{F}}_{n_k}$-adapted. Furthermore, immediately from (3), it follows that $\mathcal{M}_{FG}^{x^{n_k}} \neq \emptyset$ for every $k \geq 1$, which by Lemma 1.3 of Chap. 4, implies that $\mathcal{M}_{FG}^{\tilde{x}} \neq \emptyset$. This, by Theorem 1.3 of Chap. 4, implies the existence of an $\tilde{\mathbb{F}}$-Brownian motion $\hat{B}$ on the standard extension $\hat{\mathcal{P}}_{\hat{\mathbb{F}}} = (\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{P})$ of the filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{P})$, with $\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{0 \leq t \leq T}$ defined by $\tilde{\mathcal{F}}_t = \bigcap_{\varepsilon > 0} \sigma(\tilde{x}(u) : u \leq t + \varepsilon)$, such that $(\hat{\mathcal{P}}_{\hat{\mathbb{F}}}, \hat{x}, \hat{B})$ is a weak solution of $\overline{SFI}(F, G)$ with $\hat{x}_t(\hat{\omega}) = \tilde{x}_t(\pi(\hat{\omega}))$ satisfying the initial condition $P\hat{x}_0^{-1} = P\tilde{x}_0^{-1}$, where $\pi : \hat{\Omega} \to \tilde{\Omega}$ is the $(\hat{\mathcal{F}}, \tilde{\mathcal{F}})$-measurable mapping described in the definition of the extension of $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{P})$, because the standard extension $\hat{\mathcal{P}}_{\hat{\mathbb{F}}}$ is also an extension. Similarly as in the proof of Corollary 3.2 of Chap. 1, we obtain $P(e_s \circ x^{n_k})^{-1} = P(e_s \circ \tilde{x}^{n_k})^{-1}$ with $s = \theta_{n_k}(t)$ for $0 \leq t \leq T$. This, together with the inequality $E[\mathrm{dist}(x^{n_k}(\theta_{n_k}(t)), K(\theta_{n_k}(t)))] \leq 1/2^{n_k}$ for $k \geq 1$ and properties of the sequence $(\tilde{x}^{n_k})_{k=1}^\infty$, implies that $E[\mathrm{dist}(\tilde{x}_t, K(t))] = 0$. Similarly as in the proof of Theorem 1.3 of Chap. 4, by the definition of the process $\hat{x}$, it follows that $P\hat{x}^{-1} = P\tilde{x}^{-1}$, which implies that $P(e_t \circ \hat{x})^{-1} = P(e_t \circ \tilde{x})^{-1}$ for every $0 \leq t \leq T$. Therefore, $E[\mathrm{dist}(\hat{x}_t, K(t))] = 0$ for every $0 \leq t \leq T$. Thus $\hat{x}_t \in K(t)$, $\hat{P}$-a.s. for $0 \leq t \leq T$. □

Remark 3.1. The results of Theorem 3.3 again hold if instead of (2.1), we assume that (2.2) is satisfied. It is also true if instead of (2.1), we assume that $[S^t_{\mathbb{F}}(F \circ x) \times S^t_{\mathbb{F}}(G \circ x)] \cap \mathcal{T}_K(t,x) \neq \emptyset$ for every $(t,x) \in Graph(\mathcal{K}^\varepsilon)$ and $\varepsilon \in (0,1)$. □

In a similar way as above, we obtain immediately from Theorem 2.3 the following existence theorem.

Theorem 3.4. *Let $\mathcal{P}_{\mathbb{F}} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space with a continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ such that $\mathcal{F}_T = \mathcal{F}$. Assume that F satisfies conditions $(\mathcal{H}_1)$ and let $K : [0,T] \times \Omega \to \mathrm{Cl}(\mathbb{R}^d)$ be an $\mathbb{F}$-adapted set-valued process such that $\mathcal{K}(t) \neq \emptyset$ for every $0 \le t \le T$ and that the mapping $\mathcal{K} : [0,T] \ni t \to \mathcal{K}(t) \in \mathrm{Cl}(\mathbb{L}(\Omega, \mathcal{F}_T, \mathbb{R}^d))$ is continuous. If $\mathcal{P}_{\mathbb{F}}$, F, and K are such that (2.3) is satisfied for every $(t,x) \in Graph(\mathcal{K})$, then $BSDI(F,K)$ possesses a strong viable solution.*

Proof. Let $x_T \in \mathcal{K}(T)$ and $a \in (0,T)$ be fixed. Put $x^0_t = E[x_T|\mathcal{F}_t]$ a.s. for $a \le t \le T$ and let $f^0 = (f^0_t)_{a \le t \le T}$ be a measurable process on $\mathcal{P}_{\mathbb{F}}$ such that $f^0_t \in \mathrm{co}F(t,(x^0 \circ \theta_0)(t))$ a.s. for a.e. $a \le t \le T$, where $\theta_0(t) = T$ for $a \le t \le T$. Let $\phi_t = f^0_t$ a.s. for a.e. $a \le t \le T$. By virtue of Theorem 2.3, for $\varepsilon_1 = 1/2^{3/2}$ and the above process $\phi = (\phi_t)_{a \le t \le T}$, there exist a partition $a = t^1_{p_1} < t^1_{p_1-1} < \cdots < t^1_1 < t^1_0 = T$, a step function $\theta_1 : [a,T] \to [a,T]$, a step process $z^1 = (z^1_t)_{a \le t \le T}$, and a measurable process $f^1 = (f^1_t)_{a \le t \le T}$ on $\mathcal{P}_{\mathbb{F}}$ such that conditions (i)–(vi) of Theorem 2.3 are satisfied. In particular, $f^1_t \in \mathrm{co}F(t,(x^1 \circ \theta_1)(t))$, $|f^1_t - f^0_t| = \mathrm{dist}(f^0_t, \mathrm{co}F(t,(x^1 \circ \theta_1)(t)))$ a.s. for a.e. $a \le t \le T$ and $d(x^1(t), \mathcal{K}(t)) \le \varepsilon_1$ for $a \le t \le T$, because $d(x^1(t),\mathcal{K}(t)) \le |x^1(t) - x^1(\theta(t))| + d(x^1(\theta(t)), \mathcal{K}(\theta(t))) + D(\mathcal{K}(\theta(t)), \mathcal{K}(t)) \le \varepsilon_1$, where $x^1_t = E[x_T + \int_t^T f^0_\tau \mathrm{d}\tau|\mathcal{F}_t] + \int_t^T z^1_\tau \mathrm{d}\tau$ a.s. for $a \le t \le T$. In a similar way, for $\phi = (f^1_t)_{a \le t \le T}$ and $\varepsilon_2 = 1/2^3$, we can define a partition $a = t^2_{p_2} < t^2_{p_2-1} < \cdots < t^2_1 < t^2_0 = T$, a step function $\theta_2 : [a,T] \to [a,T]$, a step process $z^2 = (z^2_t)_{a \le t \le T}$, and a measurable process $f^2 = (f^2_t)_{a \le t \le T}$ such that $f^2_t \in \mathrm{co}F(t,(x^2 \circ \theta_2)(t))$, $|f^2_t - f^1_t| = \mathrm{dist}(f^1_t, \mathrm{co}F(t,(x^2 \circ \theta_2)(t)))$ a.s. for a.e. $a \le t \le T$ and $d(x^2(t), \mathcal{K}(t)) \le \varepsilon_2$ for $a \le t \le T$, where $x^2_t = E[x_T + \int_t^T f^1_\tau \mathrm{d}\tau|\mathcal{F}_t] + \int_t^T z^2_\tau \mathrm{d}\tau$ a.s. for $a \le t \le T$. Furthermore, for $i = 1,2$, we have

$$E\left[\mathrm{dist}\left(x^i(s), E\left[x^i(t) + \int_s^t F(\tau,(x^i \circ \theta_i)(\tau))\mathrm{d}\tau|\mathcal{F}_s\right]\right)\right] \le \varepsilon_i$$

a.s. for $a \le s \le t \le T$. By the inductive procedure, for $\varepsilon_k = 1/2^{3k/2}$ and $\phi^k = (f^k_t)_{a \le t \le T}$, we can select for every $k \ge 1$, a partition $a = t^k_{p_k} < t^k_{p_k-1} < \cdots < t^k_1 < t^k_0 = T$, a step function $\theta_k : [a,T] \to [a,T]$, a step process $z^k = (z^k_t)_{a \le t \le T}$, and a measurable process $f^k = (f^k_t)_{a \le t \le T}$ such that $f^k_t \in \mathrm{co}F(t,(x^k \circ \theta_k)(t))$, $|f^k_t - f^{k-1}_t| = \mathrm{dist}(f^k_t, \mathrm{co}F(t,(x^k \circ \theta_k)(t)))$ a.s. for a.e. $a \le t \le T$ and $d(x^k(t), \mathcal{K}(t)) \le \varepsilon_k$ for $a \le t \le T$, where

$$x^k_t = E[x_T + \int_t^T f^{k-1}_\tau \mathrm{d}\tau|\mathcal{F}_t] + \int_t^T z^k_\tau \mathrm{d}\tau$$

a.s. for $a \le t \le T$. Furthermore,

$$E\left[\operatorname{dist}\left(x^k(s), E\left[x^k(t) + \int_s^t F(\tau, (x^k \circ \theta_k)(\tau))\mathrm{d}\tau | \mathcal{F}_s\right]\right)\right] \le \varepsilon_k$$

for $a \le s \le t \le T$. Of course, $x^k \in \mathcal{S}(\mathbb{F}, \mathbb{R}^m)$ for $k \ge 1$. By Corollary 3.2 of Chap. 3 and the continuity of the filtration $\mathbb{F}$, the process x^k is continuous for every $k \ge 1$. Furthermore, by the properties of the sequence $(f^k)_{k=1}^\infty$, one gets

$$\begin{aligned} |x^{k+1}(t) - x^k(t)| &\le E\left[\int_t^T |f_\tau^k - f_\tau^{k-1}|^2 \mathrm{d}\tau | \mathcal{F}_t\right] + \int_t^T E|z_\tau^{k+1} - z_\tau^k|\mathrm{d}\tau \\ &\le E\left[\int_t^T \operatorname{dist}^2(f_\tau^{k-1} \text{ço}\, F(\tau, (x^k \circ \theta_k)(\tau)))\mathrm{d}\tau | \mathcal{F}_t\right] + 8T^2\varepsilon_k \\ &\le \alpha\varepsilon_k + E\left[\int_t^T k(\tau) \sup_{\tau \le s \le T} |x^k(s) - x^{k-1}(s)|\mathrm{d}\tau | \mathcal{F}_t\right], \end{aligned}$$

a.s. for $a \le t \le T$, where $\alpha = 8T^2$. Therefore,

$$\begin{aligned} \sup_{t \le u \le T} |x^{k+1}(u) - x^k(u)| &\le \alpha\varepsilon_k + \sup_{t \le u \le T} E\left[\int_u^T k(\tau) \sup_{\tau \le s \le T} |x^k(s) - x^{k-1}(s)|\mathrm{d}\tau | \mathcal{F}_u\right] \\ &\le \alpha\varepsilon_k + \sup_{t \le u \le T} E\left[\int_t^T k(\tau) \sup_{\tau \le s \le T} |x^k(s) - x^{k-1}(s)|\mathrm{d}\tau | \mathcal{F}_u\right] \end{aligned}$$

a.s. for $a \le t \le T$ and $k = 1, 2, \ldots$. By Doob's inequality, we get

$$\begin{aligned} &E\left[\sup_{t \le u \le T} E[\int_t^T k(\tau) \sup_{\tau \le s \le T} |x^k(s) - x^{k-1}(s)|\mathrm{d}\tau | \mathcal{F}_u]\right]^2 \\ &\le 4E\left[\int_t^T k(\tau) \sup_{\tau \le s \le T} |x^k(s) - x^{k-1}(s)|\mathrm{d}\tau]\right]^2 \end{aligned}$$

for $a \le t \le T$. Therefore, for every $a \le t \le T$ and $k = 1, 2, \ldots$, we have

$$E[\sup_{t \le u \le T} |x^{k+1}(u) - x^k(u)|^2] \le \alpha^2\varepsilon_k^2 + \beta \int_t^T k^2(\tau) E[\sup_{\tau \le s \le T} |x^k(s) - x^{k-1}(s)|^2]\mathrm{d}\tau,$$

where $\beta = 8T$. By the definitions of x^1 and x^0, we obtain $E[\sup_{t \le u \le T} |x^1(u) - x^0(u)|^2] \le L$, where $L = T \int_0^T m^2(t)\mathrm{d}t$. Therefore,

$$E[\sup_{t \le u \le T} |x^2(u) - x^1(u)|^2] \le 2\alpha^2\varepsilon_1^2 + L\beta \int_t^T k^2(\tau)\mathrm{d}\tau$$

for $a \leq t \leq T$. Hence it follows that

$$E[\sup_{t\leq u\leq T}|x^3(u)-x^2(u)|^2] \leq 2\alpha\varepsilon_2^2+\alpha\beta\varepsilon_1^2\int_t^T k^2(\tau)\mathrm{d}\tau + L\beta^2\int_t^T k^2(\tau)\left(\int_\tau^T k^2(u)\mathrm{d}u\right)\mathrm{d}\tau$$
$$\leq 2\alpha^2\varepsilon_2^2+\alpha^2\beta\varepsilon_1^2\int_t^T k^2(\tau)\mathrm{d}\tau + L\frac{\beta^2}{2!}\left(\int_t^T k^2(\tau)\mathrm{d}\tau\right)^2$$

for $a \leq t \leq T$. By the inductive procedure, for every $k = 1, 2, \ldots$ and $a \leq t \leq T$, we obtain

$$E[\sup_{t\leq u\leq T}|x^{n+1}(u)-x^n(u)|^2]$$
$$\leq M\varepsilon_2^2\left[1+(8\beta)\int_t^T k^2(\tau)\mathrm{d}\tau+\frac{(8\beta)^2}{2!}\left(\int_t^T k^2(\tau)\mathrm{d}\tau\right)^2+\cdots+\frac{(8\beta)^n}{n!}\left(\int_t^T k^2(\tau)\mathrm{d}\tau\right)^n\right]$$
$$\leq M\varepsilon_n^2 exp\left[8\beta\int_t^T k^2(\tau)\mathrm{d}\tau\right],$$

where $M = \max\{2\alpha^2, L\}$. Hence, by Chebyshev's inequality and the Borel–Cantelli lemma, it follows that the sequence $(x^k)_{k=1}^\infty$ of stochastic processes $(x^k(t))_{a\leq t\leq T}$ is for a.e. $\omega \in \Omega$ uniformly convergent in $[a, T]$ to a continuous process $(x(t))_{a\leq t\leq T}$. We can verify that the sequence $(f^k)_{k=1}^\infty$ is a Cauchy sequence of $\mathbb{L}([a, T] \times \Omega, \beta_T \otimes \mathcal{F}_T, \mathbb{R}^m)$. Indeed, for every $k = 0, 1, 2, \ldots$, one has

$$\int_0^a E[|f_\tau^{k+1}-f_\tau^k|]\mathrm{d}\tau$$
$$\leq \int_0^a E[H(F(\tau,(x^k\circ\theta_k)(\tau))), F(\tau,(x^{k-1}\circ\theta_{k-1})(\tau))))]\mathrm{d}\tau$$
$$\leq \int_0^a k(\tau)E[\sup_{0\leq u\leq\tau}|x^k(u)-x^{k-1}(u)|]\mathrm{d}\tau,$$

which by the properties of the sequence $(x^k)_{k=1}^\infty$, implies that $(f^k)_{k=1}^\infty$ is a Cauchy sequence. Then there is an $f \in \mathbb{L}([a, T]\times\Omega, \beta_T\otimes\mathcal{F}_T, \mathbb{R}^m)$ such that $|f^k - f| \to 0$ as $k \to \infty$. Let $y_t = E[x_T + \int_t^T f_\tau \mathrm{d}\tau|\mathcal{F}_t]$ a.s. for $a \leq t \leq T$. For every $k \geq 1$, we have

$$E[\sup_{a\leq t\leq T}|x(t)-y_t|] \leq E[\sup_{a\leq t\leq T}|x(t)-x_t^k|]+E[\sup_{a\leq t\leq T}|x^k(t)-y_t|]$$
$$\leq E[\sup_{a\leq t\leq T}|x(t)-x_t^k|]+E\left[\sup_{a\leq t\leq T}E[\int_t^T|f_\tau^k-f_\tau|\mathrm{d}\tau|\mathcal{F}_t]\right]+\int_t^T E|z_\tau^k|\mathrm{d}\tau$$
$$\leq E[\sup_{a\leq t\leq T}|x(t)-x_t^k|]+E\left[E[\int_0^T|f_\tau^k-f_\tau|\mathrm{d}\tau|\mathcal{F}_t]\right]+T\varepsilon_k^2$$
$$\leq E[\sup_{a\leq t\leq T}|x(t)-x_t^k|]+E\int_0^T|f_\tau^k-f_\tau|\mathrm{d}\tau+T\varepsilon_k^2,$$

which implies that $E[\sup_{a\le t\le T}|x(t)-y_t|]=0$. Then $x(t)=E[x_T+\int_t^T f_\tau \mathrm{d}\tau|\mathcal{F}_t]$ a.s. for $a\le t\le T$. Now, for every $a\le s\le t\le T$, we get

$$\begin{aligned}
&E\left[\operatorname{dist}\left(x(s),E\left[x(t)+\int_s^t F(\tau,x(\tau))\mathrm{d}\tau|\mathcal{F}_s\right]\right)\right]\\
&\le E\left[|x(s)-x^k(s)|\right]+E\left[\operatorname{dist}\left(x^k(s),E\left[x^k(t)+\int_s^t F(\tau,x^k(\theta_k(\tau)))\mathrm{d}\tau|\mathcal{F}_s\right]\right)\right]\\
&\quad+E\left[H\left(E\left[\int_s^t F(\tau,x^k(\theta_k(\tau)))\mathrm{d}\tau|\mathcal{F}_s\right],E\left[\int_s^t F(\tau,x^k(\tau))\mathrm{d}\tau|\mathcal{F}_s\right]\right)\right]\\
&\quad+E\left[H\left(E\left[\int_s^t F(\tau,x^k(\tau))\mathrm{d}\tau|\mathcal{F}_s\right],E\left[\int_s^t F(\tau,x(\tau))\mathrm{d}\tau|\mathcal{F}_s\right]\right)\right]\\
&\le ||x^k-x||+\varepsilon_k+E\int_a^T k(t)|x^k(\theta_k(t))-x^k(t)|\mathrm{d}t+E\int_a^T k(t)|x^k(t)-x(t)|\mathrm{d}t.
\end{aligned}$$

But

$$E[|x^k(\theta_k(t))-x^k(t)|]\le ||x^k-x||+E[\sup_{a\le t\le T}|x(\theta_k(t))-x^k(t)|]$$

for every $k\ge 1$ and $a\le t\le T$. Then

$$\begin{aligned}
&E\left[\operatorname{dist}\left(x(s),E\left[x(t)+\int_s^t F(\tau,x(\tau))\mathrm{d}\tau|\mathcal{F}_s\right]\right)\right]\\
&\le\left(\int_0^T k(t)\mathrm{d}t\right)\left\{E[\sup_{a\le t\le T}|x(\theta_k(t))-x^k(t)|]+E[\sup_{a\le t\le T}|x(t)-x_t^k|]\right\}\\
&\quad+\|x^k-x\|+\varepsilon_k\le||x^k-x||\left(1+\int_0^T k(t)\mathrm{d}t\right)+\varepsilon_k
\end{aligned}$$

for every $k\ge 1$ and $a\le s\le t\le T$. Thus

$$E\left[\operatorname{dist}\left(x(s),E\left[x(t)+\int_s^t F(\tau,x(\tau))\mathrm{d}\tau|\mathcal{F}_s\right]\right)\right]=0$$

for every $a\le s\le t\le T$. In a similar way, we also get that $d(x(t),\mathcal{K}(t))=0$ for every $a\le t\le T$. Then x is a strong solution of $BSDI(F,K)$ on the interval $[a,T]$ for every $a\in(0,T)$.

We can now extend the above solution to the whole interval $[0,T]$. Let us denote by Λ_x the set of all extensions of the above-obtained viable solution x of $BSDI(F,K)$. We have $\Lambda_x\ne\emptyset$, because we can repeat the above procedure for every interval $[\alpha,T]$ with $\alpha\in(0,a]$ and get a solution x^α of $BSDI(F,K)$ on the interval $[\alpha,T]$. The process $z=\mathbb{1}_{[\alpha,a]}x^\alpha+\mathbb{1}_{(a,T]}x$ is an extension of x to the interval

$[\alpha, T]$. Let us introduce in Λ_x the partial order relation $\preceq$ by setting $x \preceq z$ if and only if $a_z \leq a_x$ and $x = z|_{[a_x,T]}$, where $a_x, a_z \in (0, a)$ are such that x and z are strong viable solutions for $BSDI(F, K)$ on $[a_x, T]$ and $[a_z, T]$, respectively, and $z|_{[a_x,T]}$ denotes the restriction of the solution z to the interval $[a_x, T]$. Let $\psi : [\alpha, T] \to \mathbb{R}^d$ be an extension of x to $[\alpha, T]$ with $\alpha \in (0, a]$ and denote by $P_x^\psi \subset \Lambda_x$ the set containing ψ and all its restrictions $\psi|_{[\beta,T]}$ for every $\beta \in (\alpha, a)$. It is clear that each completely ordered subset of Λ_x is of the form P_x^ψ determined by some extension ψ of x and contains its upper bound ψ. Then by the Kuratowski–Zorn lemma, there exists a maximal element γ of Λ_x. It has to be $a_\gamma = 0$, where $a_\gamma \in [0, T)$ is such that γ is a strong viable solution of $BSDI(F, K)$ on the interval $[a_\gamma, T]$. Indeed, if we had $a_\gamma > 0$, then we could repeat the above procedure and extend γ as a viable strong solution $\xi \in \Lambda_x$ of $BSFI(F, K)$ to the interval $[b, T]$ with $0 \leq b < a_\gamma$. This would imply that $\gamma \preceq \xi$, a contradiction to the assumption that γ is a maximal element of Λ_x. Then x can be extended to the whole interval $[0, T]$. □

Remark 3.2. Theorem 3.4 is also true if $\mathcal{K}(t) = \{u \in \mathbb{L}(\Omega, \mathcal{F}_0, \mathbb{R}^d) : u \in K(t)\}$. In such a case, instead of (2.3), we can assume that $\liminf_{h\to 0+} \overline{D}(x + \int_{t-h}^t F(\tau, x)\mathrm{d}\tau, \mathcal{K}(t)) = 0$ for every $(t, x) \in Graph(\mathcal{K})$.

Proof. For every $(t, x) \in Graph(\mathcal{K})$, $f \in S(coF \circ x)$, and $u \in \mathcal{K}(t)$, we have

$$\begin{aligned} E\left(\left|E[x + \int_{t-h}^t f_\tau \mathrm{d}\tau|\mathcal{F}_{t-h}] - u\right|\right) &= E\left(\left|E[x + \int_{t-h}^t f_\tau \mathrm{d}\tau|\mathcal{F}_{t-h}] - E[u|\mathcal{F}_{t-h}]\right|\right) \\ &\leq E\left(E\left[\left|x + \int_{t-h}^t f_\tau \mathrm{d}\tau - u\right| \middle| \mathcal{F}_{t-h}\right]\right) \\ &= E\left|x + \int_{t-h}^t f_\tau \mathrm{d}\tau - u\right|. \end{aligned}$$

Therefore, $d(E[x + \int_{t-h}^t f_\tau \mathrm{d}\tau|\mathcal{F}_{t-h}], \mathcal{K}(t)) \leq d(x + \int_{t-h}^t f_\tau \mathrm{d}\tau, \mathcal{K}(t))$ for every $f \in S(coF \circ x)$. Then

$$\overline{D}\left[S(E[x + \int_{t-h}^t F(\tau, x)\mathrm{d}\tau|\mathcal{F}_{t-h}]), \mathcal{K}(t-h)\right] \leq \overline{D}\left[x + \int_{t-h}^t F(\tau, x)\mathrm{d}\tau, \mathcal{K}(t-h)\right]$$

for every $(t, x) \in Graph(\mathcal{K})$. Thus, $\liminf_{h\to 0+} \overline{D}(x + \int_{t-h}^t F(\tau, x)\mathrm{d}\tau, \mathcal{K}(t-h)) = 0$ implies that (2.3) is satisfied. □

Remark 3.3. The results of the above existence theorems are also true if instead of continuity of the set-valued mapping $\mathcal{K}$, we assume that it is uniformly upper semicontinuous on $[0, T]$, i.e., that $\lim_{\delta\to 0} \sup_{0\leq t\leq T} \overline{D}(\mathcal{K}(t + \delta), \mathcal{K}(t)) = 0$. □

It can be verified that the requirement $X_t \in K(t)$ a.s. for $0 \leq t \leq T$ in the above viability problems is too strong to be satisfied for some stochastic differential equations. For example, the stochastic differential equation $dX_t = f(X_t) + dB_t$ with Lipschitz continuous and bounded function $f : \mathbb{R} \to \mathbb{R}$ does not have any solution $X = (X_t)_{0\leq t\leq T}$ with X_t belonging to a compact set $K \subset \mathbb{R}$ a.s. for every $0 \leq t \leq T$. This is a consequences of the following theorem.

Theorem 3.5. *Let $\mathcal{P}_{\mathbb{F}} = (\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space and $B = (B_t)_{t\geq 0}$ a real-valued $\mathbb{F}$-Brownian motion on $\mathcal{P}_{\mathbb{F}}$. Assume that $\xi = (\xi_t)_{0\leq t\leq T}$ is an Itô diffusion such that $d\xi_t = \alpha_t(\xi)dt + dB_t$, $\xi_0 = 0$ for $0 \leq t \leq T$. Then $P(\{\int_0^T \alpha_t^2(\xi)\mathrm{d}t < \infty\}) = 1$ and $P(\{\int_0^T \alpha_t^2(B)\mathrm{d}t < \infty\}) = 1$ if and only if ξ and B have the same distributions as C_T-random variables on $\mathcal{P}_{\mathbb{F}}$, where $C_T = C([0,T], \mathbb{R})$.*

Example 3.1. Let $f : \mathbb{R} \to \mathbb{R}$ be bounded and Lipschitz continuous. Let $\mathcal{P}_{\mathbb{F}}$ and B be as in Theorem 3.5. Put $\alpha_t(x) = f(e_t(x))$ for $x \in C_T$, where $C_T = C([0,T], \mathbb{R})$ and e_t is the evaluation mapping on C_T, i.e., $e_t(x) = x(t)$ for $x \in C_T$ and $0 \leq t \leq T$. Assume that K is a nonempty compact subset of $\mathbb{R}$ such that $0 \in K$ and consider the viable problem

$$\begin{cases} dX_t = f(X_t)dt + dB_t \quad a.s.\ for\ \ 0 \leq t \leq T, \\ X_t \in K \quad a.s.\ for\ \ t \in [0,T]. \end{cases}$$

Suppose there is a solution X, an Itô diffusion, of the above viability problem such that $X_0 = 0$. By the properties of f, we have $\int_0^T f^2(X_t)\mathrm{d}t < \infty$ and $\int_0^T f^2(B_t)\mathrm{d}t < \infty$ a.s. Therefore, by virtue of Theorem 3.4, for every $A \in \beta(C_T)$ with $PX^{-1}(A) = 1$, one has $PX^{-1}(A) = PB^{-1}(A)$. By the properties of the process X, one has $P(\{X_t \in K\}) = 1$. But $P(\{X_t \in K\}) = P(\{e_t(X) \in K\}) = PX^{-1}(e_t^{-1}(K))$, where e_t is the evolution mapping. Hence it follows that $1 = PX^{-1}(e_t^{-1}(K)) = PB^{-1}(e_t^{-1}(K)) = P(\{B_t \in K\}) < 1$, a contradiction. Then the problem (3) does not have any K-viable strong solution.

Remark 3.4. We can consider viability problems with weaker viable requirements of the form $P(\{X_t \in K(t)\}) \in (\varepsilon, 1)$ for $0 \leq t \leq T$ and $\varepsilon \in (0,1)$ sufficiently large. Solutions to such problems can be regarded as a type of approximations to viable solutions. □

4 Weak Compactness of Viable Solution Sets

Let us denote by $\mathcal{X}(F,G,K)$ the set of (equivalence classes of) all weak viable solutions of $SFI(F,G,K)$. We shall show that for every F, G, and K satisfying the assumptions of Theorem 3.3, the set $\mathcal{X}(F,G,K)$ is weakly compact, i.e., the set $\mathcal{X}^P(F,G,K)$ of distributions of all weak solutions of $SFI(F,G,K)$ is weakly compact subsets of the space $\mathcal{M}(C_T)$ of all probability measures on the Borel σ-algebra $\beta(C_T)$, where $C_T =: C([0,T], \mathbb{R}^d)$.

Theorem 4.1. *Assume that F and G are measurable, bounded, and convex-valued such that $F(t,\cdot)$ and $G(t,\cdot)$ are continuous for a.e. fixed $t \in [0,T]$. Let G be diagonally convex and $K : [0,T] \to \mathrm{Cl}(\mathbb{R}^d)$ continuous. If there exist a complete filtered probability space $\mathcal{P}_{\mathbb{F}} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with a filtration $\mathbb{F}$ satisfying the usual* *conditions and an m-dimensional* $\mathbb{F}$*-Brownian motion on* $\mathcal{P}_{\mathbb{F}}$ *such that (2.1) is satisfied for every* $\varepsilon \in (0,1)$ *and* $(t,x) \in Graph(\mathcal{K}^\varepsilon)$*, then the set* $\mathcal{X}(F,G,K)$ *of all weak viable solutions* $(\mathcal{P}_{\mathbb{F}}, x, B)$ *of* $\overline{SFI}(F,G,K)$ *is weakly compact.*

Proof. By virtue of Theorem 3.3, the set $\mathcal{X}(F,G,K)$ is nonempty. Similarly as in the proof of Theorem 4.1 of Chap. 4, we can verify that $\mathcal{X}(F,G,K)$ is relatively weakly compact. We shall prove that it is a weakly closed subset of the space $\mathcal{M}(C_T)$. Let $(x^r)_{r=1}^\infty$ be a sequence of $\mathcal{X}(F,G,K)$ convergent in distribution. Then there exists a probability measure $\mathcal{P}$ on $\beta(C_T)$ such that $P(x^r)^{-1} \Rightarrow \mathcal{P}$ as $r \to \infty$. By virtue of Theorem 2.3 of Chap. 1, there are a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and random variables $\tilde{x}^r : \tilde{\Omega} \to C_T$ and $\tilde{x} : \tilde{\Omega} \to C_T$ for $r = 1,2,\dots$ such that $P(x^r)^{-1} = P(\tilde{x}^r)^{-1}$ for $r = 1,2,\dots$, $\tilde{P}(\tilde{x})^{-1} = \mathcal{P}$ and $\lim_{r\to\infty} \sup_{0\le t\le T} |\tilde{x}_t^r - \tilde{x}_t| = 0$ with $(\tilde{P}.1)$. By Theorem 1.3 of Chap. 4, we have $\mathcal{M}_{FG}^{x_r} \neq \emptyset$ for every $r \ge 1$, which by Theorem 1.5 of Chap. 4, implies that $\mathcal{M}_{FG}^{\tilde{x}} \neq \emptyset$. Therefore, by Theorem 1.3 of Chap. 4, there exist a standard extension $\hat{\mathcal{P}}_{\hat{\mathbb{F}}} = (\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{P})$ of $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{P})$ and an m-dimensional Brownian motion $\hat{\mathcal{P}}_{\hat{\mathbb{F}}}$ such that $(\hat{\mathcal{P}}_{\hat{\mathbb{F}}}, \hat{x}, \hat{B})$ is a weak solution of $\overline{SFI}(F,G,\mu)$ with an initial distribution μ equal to the probability distribution $P\tilde{x}_0^{-1}$. Similarly as in the proof of Theorem 3.3, this solution is defined by $\hat{x}(\hat{\omega}) = \tilde{x}(\pi(\hat{x}))$ for $\hat{\omega} \in \hat{\Omega}$. Similarly as in the proof of Theorem 4.1 of Chap. 4, we obtain $P(x^r)^{-1} \Rightarrow P(\hat{x})^{-1}$ as $r \to \infty$, which by the properties of the sequence $(\tilde{x}^r)_{n=1}^\infty$ implies that $P(\tilde{x}^r)^{-1} \Rightarrow P(\hat{x})^{-1}$ as $r \to \infty$. By the properties of the sequence $(x^r)_{r=1}^\infty$, we have $E^r[\mathrm{dist}(x^r(t), K(t))] = 0$ for every $r \ge 1$, which implies that $\tilde{E}[\mathrm{dist}(\tilde{x}^r(t), K(t))] = 0$ for every $r \ge 1$. Hence, by the continuity of the mapping $\mathrm{dist}(\cdot, K(t))$ and properties of the sequence $(\tilde{x}^r)_{n=1}^\infty$, it follows that $\hat{E}[\mathrm{dist}(\hat{x}_t, K(t))] = 0$. Thus $(\hat{\mathcal{P}}_{\hat{\mathbb{F}}}, \hat{x}, \hat{B})$ is a weak solution of $\overline{SFI}(F,G,\mu)$, with an appropriately chosen initial distribution μ, such that $x^r \Rightarrow \hat{x}$ and $\hat{x}_t \in K(t)$ with $(\hat{P}.1)$ for every $t \in [0,T]$. Then $(\hat{\mathcal{P}}_{\hat{\mathbb{F}}}, \hat{x}, \hat{B}) \in \mathcal{X}(F,G,K)$, and $\mathcal{X}(F,G,K)$ is weakly closed. □

Remark 4.1. The results of Theorem 4.1 continue to hold if instead of (2.1), we assume that $[S_{\mathbb{F}}^t(F \circ x) \times S_{\mathbb{F}}^t(G \circ x)] \cap \mathcal{T}_K(t,x) \neq \emptyset$ for every $(t,x) \in \mathcal{K}^\varepsilon$ and $\varepsilon \in (0,1)$. □

5 Notes and Remarks

The viability approach to optimal control problems is especially useful for problems with state constraints. There is a great number of papers dealing with viability problems for differential inclusions. The first results dealing with viability problems for differential inclusions were given by Aubin and Cellina in [5]. The first result extending to the stochastic case of Nagumo's viability theorem due to Aubin and Da Prato [7]. Most of the results concerning this topic have now been collected in the excellent book by Aubin [6]. Interesting generalizations of viability and invariance problems were given by Plaskacz [88]. A new approach to viability problems for stochastic differential equations was initiated by Aubin and Da Prato in [8] and [9] and by Millian in [79]. Later on, these results were extended by Aubin, Da Prato, and Frankowska [10, 12] in the case of stochastic inclusions written in differential form. Independently, viability problems for stochastic inclusions were also considered by Kisielewicz in [54] and Motyl in [85]. Viability theory provides geometric conditions that are equivalent to viability or invariance properties. Illustrations of viability approach to some issues in control theory and dynamical games with the problem of dynamic valuation and management of a portfolio, can be found in Aubin et al. [13]. The stochastic viability condition presented in Example 3.1 was constructed by M. Michta. The results contained in the present chapter are mainly based on methods applied in Aitalioubrahim and Sajid [3], Van Benoit and Ha [18], and Aubin and Da Prato [9]. The main results of this chapter dealing with the existence of viable strong and weak solutions of stochastic and backward stochastic inclusions and weak compactness with respect to convergence in the sense of distributions of viable weak solution sets are due to the author of this book.