

# Two Differences Between Interval Type-2 and Type-1 Fuzzy Logic Controllers: Adaptiveness and Novelty

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**Abstract** Interval type-2 fuzzy logic controllers (IT2 FLCs) have been attracting great research interests recently. Many reported results have shown that IT2 FLCs are better able to handle uncertainties than their type-1 (T1) counterparts. A challenging question is: *What are the fundamental differences between IT2 and T1 FLCs?* Once the fundamental differences are clear, we can better understand the advantages of IT2 FLCs and hence better make use of them. This chapter explains two fundamental differences between IT2 and T1 FLCs: (1) *Adaptiveness*, meaning that the embedded T1 fuzzy sets used to compute the bounds of the type-reduced interval change as input changes; and, (2) *Novelty*, meaning that the upper and lower membership functions of the same IT2 fuzzy set may be used simultaneously in computing each bound of the type-reduced interval. T1 FLCs do not have these properties; thus, a T1 FLC cannot implement the complex control surface of an IT2 FLC given the same rulebase.

## 1 Introduction

Interval type-2 fuzzy logic controllers (IT2 FLCs) have been attracting great research interests recently. Many reported results have shown that IT2 FLCs are better able to handle uncertainties than their type-1 (T1) counterparts [1, 5, 10, 22, 23]. A challenging question is: *What are the fundamental differences between IT2 and T1 FLCs?* Once the fundamental differences are clear, we can better understand the advantages of IT2 FLCs and hence better make use of them. In the literature, there has been considerable effort on answering this challenging and fundamental question. Some important arguments are [17]:

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1. An IT2 fuzzy set (FS) can better model intra-personal and inter-personal uncertainties,<sup>1</sup> which are intrinsic to natural language, because the membership grade of an IT2 FS is an interval instead of a crisp number in a T1 FS. Mendel [11] also showed that IT2 FS is a scientifically correct model for modeling linguistic uncertainties, whereas T1 FS is not.
2. Using IT2 FSs to represent the FLC inputs and outputs will result in the reduction of the rulebase when compared to using T1 FSs [5, 10], as the ability of the footprint of uncertainty (FOU) to represent more uncertainties enables one to cover the input/output domains with fewer FSs. This makes it easier to construct the rulebase using expert knowledge and also increases robustness [20, 22, 23].
3. An IT2 FLC can give a smoother control surface than its T1 counterpart, especially in the region around the steady state (i.e., when both the error and the change of error approach 0) [6, 20, 22, 23]. Wu and Tan [24] showed that when the baseline T1 FLC implements a linear PI control law and the IT2 FSs of an IT2 FLC are obtained from symmetrical perturbations of the T1 FSs, the resulting IT2 FLC implements a variable gain PI controller around the steady state. These gains are smaller than the PI gains of the baseline T1 FLC, which result in a smoother control surface around the steady state. The PI gains of the IT2 FLC also change with the inputs, which cannot be achieved by the baseline T1 FLC.
4. IT2 FLCs are more adaptive and they can realize more complex input–output relationships which cannot be achieved by T1 FLCs. Karnik and Mendel [8] pointed out that an IT2 fuzzy logic system can be thought of as a collection of many different embedded T1 fuzzy logic systems. Wu and Tan [21] proposed a systematic method to identify the *equivalent generalized T1 FSs* that can be used to replace the FOU. They showed that the equivalent generalized T1 FSs are significantly different from traditional T1 FSs, and there are different equivalent generalized T1 FSs for different inputs. Du and Ying [3], and Nie and Tan [14], also showed that a symmetrical IT2 fuzzy-PI (or the corresponding PD) controller, obtained from a baseline T1 PI FLC, partitions the input domain into many small regions, and in each region it is equivalent to a nonlinear PI controller with variable gains. The control law of the IT2 FLC in each small region is much more complex than that of the baseline T1 FLC, and

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<sup>1</sup> According to Mendel [11], intra-personal uncertainty describes “*the uncertainty a person has about the word.*” It is also explicitly pointed out by psychologists Wallsten and Budescu [15] as “*except in very special cases, all representations are vague to some degree in the minds of the originators and in the minds of the receivers,*” and they suggest to model it by a T1 FS. According to Mendel [11], inter-personal uncertainty describes “*the uncertainty that a group of people have about the word,*” i.e., “*words mean different things to different people.*” It is also explicitly pointed out by psychologists Wallsten and Budescu [15] as “*different individuals use diverse expressions to describe identical situations and understand the same phrases differently when hearing or reading them.*”

hence it can realize more complex input–output relationship that cannot be achieved by a T1 FLC using the same rulebase.

5. *IT2 FLCs have a novelty that does not exist in traditional T1 FLCs.* Wu [16, 17] showed that in an IT2 FLC *different* membership grades from the same IT2 FS can be used in different rules, whereas for traditional T1 FLC the *same* membership grade from the same T1 FS is always used in different rules. This again implies that an IT2 FLC is more complex than a T1 FLC and it cannot be implemented by a T1 FLC using the same rulebase.

This chapter explains why *adaptiveness* and *novelty* are two fundamental differences between IT2 and T1 FLCs. Methods for visualizing the effects of these two differences can be found in [17].

## 2 Interval Type-2 Fuzzy Sets and Controllers

This section introduces background materials on IT2 FSs and FLCs, and shows two numerical examples on IT2 FLCs.

### 2.1 Interval Type-2 Fuzzy Sets (IT2 FSs)

T1 FS theory was first introduced by Zadeh [25] in 1965 and has been successfully applied in many areas.

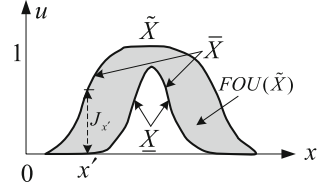
**Definition 1** A T1 FS  $X$  is comprised of a *domain*  $D_X$  of real numbers (also called the *universe of discourse* of  $X$ ) together with a *membership function* (MF)  $\mu_x : D_X \rightarrow [0, 1]$ , i.e.,

$$X = \int_{D_X} \mu_x(x)/x. \quad (1)$$

Here  $\int$  denotes the collection of all points  $x \in D_X$  with associated *membership grade*  $\mu_x(x)$ . □

Despite having a name which carries the connotation of uncertainty, research has shown that there are limitations in the ability of T1 FSs to model and minimize the effect of uncertainties [4, 5, 10, 22]. This is because a T1 FS is certain in the sense that its membership grades are crisp values. Recently, type-2 FSs [26], characterized by MFs that are themselves fuzzy, have been attracting great interests. IT2 FSs [10], a special case of type-2 FSs, are currently the most widely used for their reduced computational cost.

**Fig. 1** An IT2 FS,  $\tilde{X}$  (the LMF) and  $\bar{X}$  (the UMF) are two embedded T1 FSs



**Definition 2** [10, 12] An IT2  $\tilde{X}$  is characterized by its MF  $\mu_{\tilde{X}}(x, u)$ , i.e.,

$$\begin{aligned} \tilde{X} &= \int_{x \in D_{\tilde{X}}} \int_{u \in J_x \subseteq [0,1]} \mu_{\tilde{X}}(x, u) / (x, u) = \int_{x \in D_{\tilde{X}}} \int_{u \in J_x \subseteq [0,1]} 1 / (x, u) \\ &= \int_{x \in D_{\tilde{X}}} \left[ \int_{u \in J_x \subseteq [0,1]} 1 / u \right] / x \end{aligned} \quad (2)$$

where  $x$ , called the *primary variable*, has domain  $D_{\tilde{X}}$ ;  $u \in [0, 1]$ , called the *secondary variable*, has domain  $J_x \subseteq [0, 1]$  at each  $x \in D_{\tilde{X}}$ ;  $J_x$  is also called the *support of the secondary MF*; and the amplitude of  $\mu_{\tilde{X}}(x, u)$ , called a *secondary grade* of  $\tilde{X}$ , equals 1 for  $\forall x \in D_{\tilde{X}}$  and  $\forall u \in J_x \subseteq [0, 1]$ .  $\square$

An example of an IT2 FS,  $\tilde{X}$ , is shown in Fig. 1. Observe that unlike a T1 FS, whose membership grade for each  $x$  is a number, the membership of an IT2 FS is an interval. Observe also that an IT2 FS is bounded from above and below by two T1 FSs,  $\bar{X}$  and  $\underline{X}$ , which are called *upper membership function* (UMF) and *lower membership function* (LMF), respectively. The area between  $\bar{X}$  and  $\underline{X}$  is the *footprint of uncertainty* (FOU). An *embedded T1 FS* is any T1 FS within the FOU.  $\underline{X}$  and  $\bar{X}$  are two such sets.

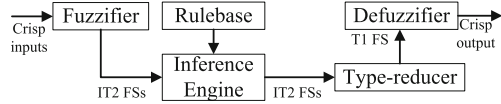
## 2.2 Interval Type-2 Fuzzy Logic Controllers (IT2 FLCs)

Figure 2 shows the schematic diagram of an IT2 FLC. It is similar to its T1 counterpart, the major difference being that at least one of the FSs in the rulebase is an IT2 FS. Hence, the outputs of the inference engine are IT2 FSs, and a type-reducer [8, 10] is needed to convert them into a T1 FS before defuzzification can be carried out.

In practice the computations in an IT2 FLC can be significantly simplified. Consider the rulebase of an IT2 FLC consisting of  $N$  rules assuming the following form:

$$\tilde{R}^n : \text{IF } x_1 \text{ is } \tilde{X}_1^n \text{ and } \cdots \text{ and } x_l \text{ is } \tilde{X}_l^n, \text{ THEN } y \text{ is } Y^n. n = 1, 2, \dots, N$$

**Fig. 2** The schematic diagram of an IT2 FLC



where  $\tilde{X}_i^n$  ( $i = 1, \dots, I$ ) are IT2 FSs, and  $Y^n = [\underline{y}^n, \bar{y}^n]$  is an interval, which can be understood as the centroid [7, 10] of a consequent IT2 FS,<sup>2</sup> or the simplest TSK model. In many applications [20, 22, 23] we use  $\underline{y}^n = \bar{y}^n$ , i.e., each rule consequent is represented by a crisp number.

For an input vector  $\mathbf{x}' = (x'_1, x'_2, \dots, x'_J)$ , typical computations in an IT2 FLC involve the following steps:

1. Compute the membership interval of  $x'_i$  on each  $X_i^n$ ,  $[\mu_{\underline{X}_i^n}(x'_i), \mu_{\bar{X}_i^n}(x'_i)]$ ,  $i = 1, 2, \dots, I$ ,  $n = 1, 2, \dots, N$ .
2. Compute the firing interval of the  $n^{\text{th}}$  rule,  $F^n(\mathbf{x}')$ :

$$F^n(\mathbf{x}') = [\mu_{\underline{X}_1^n}(x'_1) \times \dots \times \mu_{\underline{X}_I^n}(x'_I), \mu_{\bar{X}_1^n}(x'_1) \times \dots \times \mu_{\bar{X}_I^n}(x'_I)] \equiv [f^n, \bar{f}^n] \quad (3)$$

Note that the *minimum*  $t$ -norm may also be used in (3). However, this chapter focuses only on the *product*  $t$ -norm.

3. Perform type-reduction to combine  $F^n(\mathbf{x}')$  and the corresponding rule consequents. There are many such methods [10]. The most commonly used one is the center-of-sets type-reducer [10], derived from the Extension Principle [25]:

$$Y_{\text{cos}}(\mathbf{x}') = \bigcup_{\substack{f^n \in F^n(\mathbf{x}') \\ y^n \in Y^n}} \frac{\sum_{n=1}^N f^n y^n}{\sum_{n=1}^N f^n} = [y_l, y_r] \quad (4)$$

It has been shown that [10, 13, 18]:

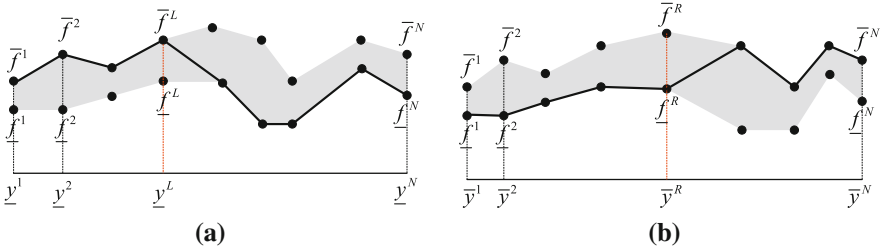
$$y_l = \min_{k \in [1, N-1]} \frac{\sum_{n=1}^k \bar{f}^n \underline{y}^n + \sum_{n=k+1}^N f^n \underline{y}^n}{\sum_{n=1}^k \bar{f}^n + \sum_{n=k+1}^N f^n} \equiv \frac{\sum_{n=1}^L \bar{f}^n \underline{y}^n + \sum_{n=L+1}^N f^n \underline{y}^n}{\sum_{n=1}^L \bar{f}^n + \sum_{n=L+1}^N f^n} \quad (5)$$

$$y_r = \max_{k \in [1, N-1]} \frac{\sum_{n=1}^k f^n \bar{y}^n + \sum_{n=k+1}^N \bar{f}^n \bar{y}^n}{\sum_{n=1}^k f^n + \sum_{n=k+1}^N \bar{f}^n} \equiv \frac{\sum_{n=1}^R f^n \bar{y}^n + \sum_{n=R+1}^N \bar{f}^n \bar{y}^n}{\sum_{n=1}^R f^n + \sum_{n=R+1}^N \bar{f}^n} \quad (6)$$

where the *switch points*  $L$  and  $R$  are determined by

$$\underline{y}^L \leq y_l \leq \underline{y}^{L+1} \quad (7)$$

<sup>2</sup> The rule consequents can be IT2 FSs; however, when the popular center-of-sets type-reduction method [10] is used, these consequent IT2 FSs are replaced by their centroids in the computation; so, it is more convenient to represent the rule consequents as intervals directly.



**Fig. 3** Illustration of the switch points in computing  $y_l$  and  $y_r$ . **(a)** Computing  $y_l$  switch from the *upper* bounds of the firing intervals to the *lower* bounds. **(b)** Computing  $y_r$  switch from the *lower* bounds of the firing intervals to the *upper* bounds

$$\bar{y}^R \leq y_r \leq \bar{y}^{R+1} \quad (8)$$

and  $\{\underline{y}^n\}_{n=1,\dots,N}$  and  $\{\bar{y}^n\}_{n=1,\dots,N}$  have been sorted in ascending order, respectively.  $y_l$  and  $y_r$  can be computed by the KM algorithms [8, 10] or their many variants [18, 19]. The main idea of the KM algorithms is to find the switch points for  $y_l$  and  $y_r$ . Take  $y_l$  as an example.  $y_l$  is the minimum of  $Y_{\cos}(\mathbf{x}')$ . Since  $\underline{y}^n$  increases from the left to the right along the horizontal axis of Fig. 3a, we should choose a large weight (upper bound of the firing interval) for  $\underline{y}^n$  on the left and a small weight (lower bound of the firing interval) for  $\underline{y}^n$  on the right. The KM algorithm for  $y_l$  finds the switch point  $L$ . For  $n \leq L$ , the upper bounds of the firing intervals are used to calculate  $y_l$ ; for  $n > L$ , the lower bounds are used. This ensures  $y_l$  is the minimum.

4. Compute the defuzzified output as:

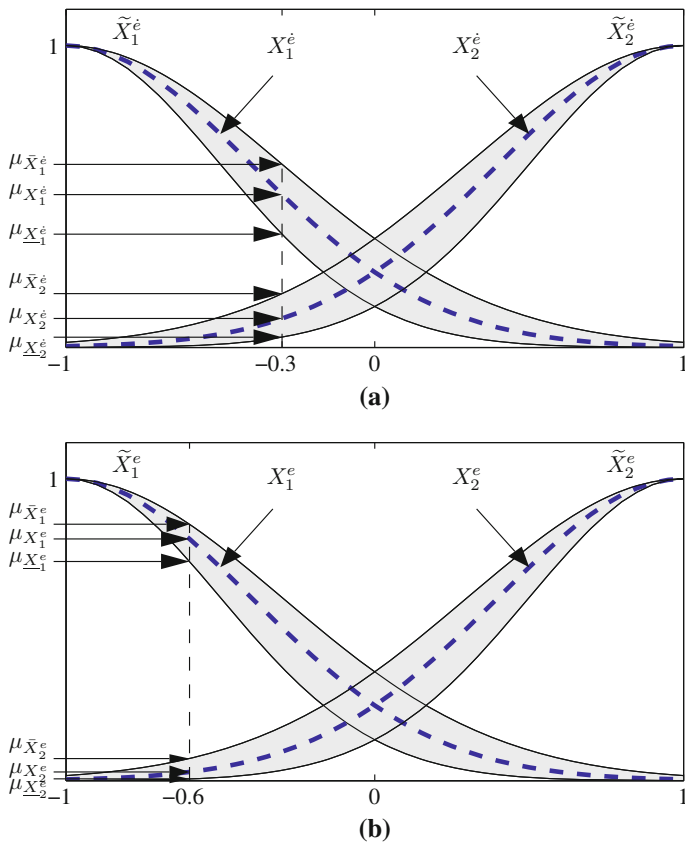
$$y = \frac{y_l + y_r}{2}. \quad (9)$$

### 3 Examples of IT2 FLC

A pair of T1 and IT2 PI FLCs are used in this section to illustrate the differences between them.

#### 3.1 The T1 and IT2 PI FLCs

The MFs of the T1 PI FLC are shown in Fig. 4 as the bold dashed lines, where the standard deviation of all Gaussian MFs is 0.6. Its four rules are:



**Fig. 4** Firing levels of the T1 FLC, and firing intervals of the IT2 FLC, when  $\mathbf{x}' = (\dot{e}', e') = (-0.3, -0.6)$

- $R^1$  : IF  $\dot{e}$  is  $X_1^{\dot{e}}$  and  $e$  is  $X_1^e$ , THEN  $\dot{u}$  is  $y^1$ .
- $R^2$  : IF  $\dot{e}$  is  $X_1^{\dot{e}}$  and  $e$  is  $X_2^e$ , THEN  $\dot{u}$  is  $y^2$ .
- $R^3$  : IF  $\dot{e}$  is  $X_2^{\dot{e}}$  and  $e$  is  $X_1^e$ , THEN  $\dot{u}$  is  $y^3$ .
- $R^4$  : IF  $\dot{e}$  is  $X_2^{\dot{e}}$  and  $e$  is  $X_2^e$ , THEN  $\dot{u}$  is  $y^4$ .

where  $\dot{u}$  is the change of the control signal,  $e$  is the feedback error, and  $\dot{e}$  is the change of error.  $y^1 - y^4$  are given in Table 1. For simplicity the rule consequents are crisp numbers instead of intervals. However, they can be arbitrary numbers or intervals and do not affect the conclusions of this chapter.

**Table 1** The rule consequents of the T1 and IT2 FLCs

	$X_1^e (\tilde{X}_1^e)$	$X_2^e (\tilde{X}_2^e)$
$X_1^e (\tilde{X}_1^e)$	$y^1 = -1$	$y^2 = -0.5$
$X_2^e (\tilde{X}_2^e)$	$y^3 = 0.5$	$y^4 = 1$

An IT2 fuzzy PI controller may be constructed by blurring the T1 FSs of the T1 FLC to IT2 FSs.<sup>3</sup> In this chapter, we blur the standard deviation of the Gaussian MFs from 0.6 to an interval [0.5, 0.7], as shown in Fig. 4. The rulebase of the IT2 FLC is

$$\begin{aligned} \tilde{R}^1 : & \text{ IF } \dot{e} \text{ is } \tilde{X}_1^e \text{ and } e \text{ is } \tilde{X}_1^e, \text{ THEN } \dot{u} \text{ is } y^1. \\ \tilde{R}^2 : & \text{ IF } \dot{e} \text{ is } \tilde{X}_1^e \text{ and } e \text{ is } \tilde{X}_2^e, \text{ THEN } \dot{u} \text{ is } y^2. \\ \tilde{R}^3 : & \text{ IF } \dot{e} \text{ is } \tilde{X}_2^e \text{ and } e \text{ is } \tilde{X}_1^e, \text{ THEN } \dot{u} \text{ is } y^3. \\ \tilde{R}^4 : & \text{ IF } \dot{e} \text{ is } \tilde{X}_2^e \text{ and } e \text{ is } \tilde{X}_2^e, \text{ THEN } \dot{u} \text{ is } y^4. \end{aligned}$$

Next, the mathematical operations in an IT2 FLC, introduced in Sect. 2.2, are illustrated using two numerical examples, which will be revisited in Sect. 4.

### 3.2 Example 1

Consider an input vector  $\mathbf{x}' = (\dot{e}', e') = (-0.3, -0.6)$ , as shown in Fig. 4. The firing levels of the four T1 FSs are:

$$\mu_{X_1^e}(\dot{e}') = 0.5063, \quad \mu_{X_2^e}(\dot{e}') = 0.0956, \quad \mu_{X_1^e}(e') = 0.8007, \quad \mu_{X_2^e}(e') = 0.0286$$

The firing levels of their four rules are shown in Table 2. The output of the T1 FLC is

$$\dot{u} = \frac{f^1 y^1 + f^2 y^2 + f^3 y^3 + f^4 y^4}{f^1 + f^2 + f^3 + f^4} = -0.3886.$$

For the IT2 FLC, the firing intervals of the four IT2 FSs are:

$$\begin{aligned} [\mu_{\underline{X}_1^e}(\dot{e}'), \mu_{\overline{X}_1^e}(\dot{e}')] &= [0.3753, 0.6065], & [\mu_{\underline{X}_2^e}(\dot{e}'), \mu_{\overline{X}_2^e}(\dot{e}')] &= [0.0340, 0.1783] \\ [\mu_{\underline{X}_1^e}(e'), \mu_{\overline{X}_1^e}(e')] &= [0.7261, 0.8494], & [\mu_{\underline{X}_2^e}(e'), \mu_{\overline{X}_2^e}(e')] &= [0.0060, 0.0734] \end{aligned}$$

The firing intervals of the four rules are shown in Table 3. From the KM algorithms, we find that  $L = 1$  and  $R = 2$ . So,

<sup>3</sup> An IT2 FLC can also be constructed from scratch without using a baseline T1 FLC [22, 23]. This chapter uses a baseline T1 FLC for comparison purposes.



**Table 2** Firing levels of the four rules of the T1 FLC in Example 1

Rule no.:	Firing level	→	Rule consequent
$R^1$ :	$f^1 = \mu_{X_1^e}(\dot{e}') \cdot \mu_{X_1^e}(e') = 0.5063 \times 0.8007 = 0.4054$	→	$y^1 = -1$
$R^2$ :	$f^2 = \mu_{X_1^e}(\dot{e}') \cdot \mu_{X_2^e}(e') = 0.5063 \times 0.0286 = 0.0484$	→	$y^2 = -0.5$
$R^3$ :	$f^3 = \mu_{X_2^e}(\dot{e}') \cdot \mu_{X_1^e}(e') = 0.0956 \times 0.8007 = 0.0766$	→	$y^3 = 0.5$
$R^4$ :	$f^4 = \mu_{X_2^e}(\dot{e}') \cdot \mu_{X_2^e}(e') = 0.0956 \times 0.0286 = 0.0027$	→	$y^4 = 1$

**Table 3** Firing intervals of the four rules of the IT2 FLC in Example 1

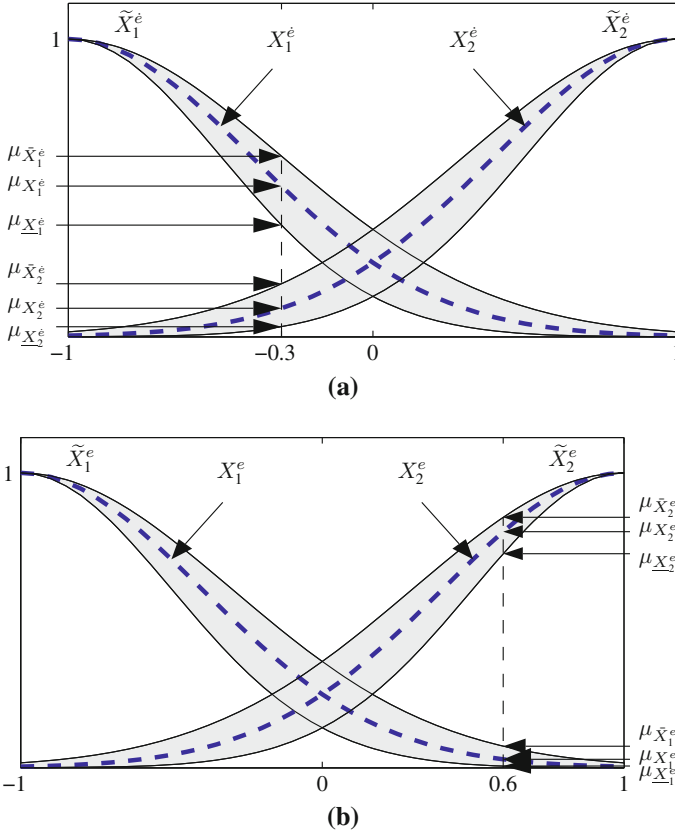
Rule no.:	Firing interval	→	Rule consequent
$\tilde{R}^1$ :	$[\underline{f}^1, \bar{f}^1] = [\mu_{X_1^e}(\dot{e}') \cdot \mu_{X_1^e}(e'), \mu_{X_1^e}(\dot{e}') \cdot \mu_{X_1^e}(e')]$ $= [0.3753 \times 0.7261, 0.6065 \times 0.8494] = [0.2725, 0.5152]$	→	$y^1 = -1$
$\tilde{R}^2$ :	$[\underline{f}^2, \bar{f}^2] = [\mu_{X_1^e}(\dot{e}') \cdot \mu_{X_2^e}(e'), \mu_{X_1^e}(\dot{e}') \cdot \mu_{X_2^e}(e')]$ $= [0.3753 \times 0.0060, 0.6065 \times 0.0734] = [0.0022, 0.0445]$	→	$y^2 = -0.5$
$\tilde{R}^3$ :	$[\underline{f}^3, \bar{f}^3] = [\mu_{X_2^e}(\dot{e}') \cdot \mu_{X_1^e}(e'), \mu_{X_2^e}(\dot{e}') \cdot \mu_{X_1^e}(e')]$ $= [0.0340 \times 0.7261, 0.1783 \times 0.8494] = [0.0247, 0.1514]$	→	$y^3 = 0.5$
$\tilde{R}^4$ :	$[\underline{f}^4, \bar{f}^4] = [\mu_{X_2^e}(\dot{e}') \cdot \mu_{X_2^e}(e'), \mu_{X_2^e}(\dot{e}') \cdot \mu_{X_2^e}(e')]$ $= [0.0340 \times 0.0060, 0.1783 \times 0.0734] = [0.0002, 0.0131]$	→	$y^4 = 1$

$$\begin{aligned}
 y_l &= \frac{\bar{f}^1 y^1 + \bar{f}^2 y^2 + \bar{f}^3 y^3 + \bar{f}^4 y^4}{\bar{f}^1 + \bar{f}^2 + \bar{f}^3 + \bar{f}^4} \\
 &= \frac{0.5152 \times (-1) + 0.0022 \times (-0.5) + 0.0247 \times 0.5 + 0.0002 \times 1}{0.5152 + 0.0022 + 0.0247 + 0.0002} \\
 &= -0.9288 \\
 y_r &= \frac{\underline{f}^1 y^1 + \underline{f}^2 y^2 + \underline{f}^3 y^3 + \underline{f}^4 y^4}{\underline{f}^1 + \underline{f}^2 + \underline{f}^3 + \underline{f}^4} \\
 &= \frac{0.2725 \times (-1) + 0.0022 \times (-0.5) + 0.1514 \times 0.5 + 0.0131 \times 1}{0.2725 + 0.0022 + 0.1514 + 0.0131} \\
 &= -0.4209
 \end{aligned} \tag{10}$$

Finally, the crisp output of the IT2 FLC is:

$$\dot{u} = \frac{y_l + y_r}{2} = \frac{-0.9288 - 0.4209}{2} = -0.6748.$$

Observe from the above example that for the same input, IT2 and T1 FLCs give quite different outputs.



**Fig. 5** Firing levels of the T1 FLC, and firing intervals of the IT2 FLC, when  $\mathbf{x}' = (e', e') = (-0.3, 0.6)$

### 3.3 Example 2

Consider another input vector  $\mathbf{x}' = (e', e') = (-0.3, 0.6)$ , as shown in Fig. 5. The firing levels of the four T1 FSs are:

$$\mu_{X_1^e}(e') = 0.5063, \quad \mu_{X_2^e}(e') = 0.0956, \quad \mu_{\tilde{X}_1^e}(e') = 0.0286, \quad \mu_{\tilde{X}_2^e}(e') = 0.8007$$

The firing levels of its four rules are shown in Table 4. The output of the T1 FLC is

$$\dot{u} = \frac{f^1 y^1 + f^2 y^2 + f^3 y^3 + f^4 y^4}{f^1 + f^2 + f^3 + f^4} = 0.0393.$$

**Table 4** Firing levels of the four rules of the T1 FLC in Example 2

Rule no.:	Firing level	→	Rule consequent
$R^1$ :	$f^1 = \mu_{X_1^e}(\dot{e}') \cdot \mu_{X_1^e}(e') = 0.5063 \times 0.0286 = 0.0145$	→	$y^1 = -1$
$R^2$ :	$f^2 = \mu_{X_1^e}(\dot{e}') \cdot \mu_{X_2^e}(e') = 0.5063 \times 0.8007 = 0.0484$	→	$y^2 = -0.5$
$R^3$ :	$f^3 = \mu_{X_2^e}(\dot{e}') \cdot \mu_{X_1^e}(e') = 0.0956 \times 0.0286 = 0.0027$	→	$y^3 = 0.5$
$R^4$ :	$f^4 = \mu_{X_2^e}(\dot{e}') \cdot \mu_{X_2^e}(e') = 0.0956 \times 0.8007 = 0.0766$	→	$y^4 = 1$

**Table 5** Firing intervals of the four rules of the IT2 FLC in Example 2

Rule no.:	Firing interval	→	Rule consequent
$\tilde{R}^1$ :	$[\underline{f}^1, \bar{f}^1] = [\mu_{X_1^e}(\dot{e}') \cdot \mu_{X_1^e}(e'), \mu_{X_1^e}(\dot{e}') \cdot \mu_{X_1^e}(e')]$ $= [0.3753 \times 0.0060, 0.6065 \times 0.0734] = [0.0022, 0.0445]$	→	$y^1 = -1$
$\tilde{R}^2$ :	$[\underline{f}^2, \bar{f}^2] = [\mu_{X_1^e}(\dot{e}') \cdot \mu_{X_2^e}(e'), \mu_{X_1^e}(\dot{e}') \cdot \mu_{X_2^e}(e')]$ $= [0.3753 \times 0.7261, 0.6065 \times 0.8494] = [0.2725, 0.5152]$	→	$y^2 = -0.5$
$\tilde{R}^3$ :	$[\underline{f}^3, \bar{f}^3] = [\mu_{X_2^e}(\dot{e}') \cdot \mu_{X_1^e}(e'), \mu_{X_2^e}(\dot{e}') \cdot \mu_{X_1^e}(e')]$ $= [0.0340 \times 0.0060, 0.1783 \times 0.0734] = [0.0002, 0.0131]$	→	$y^3 = 0.5$
$\tilde{R}^4$ :	$[\underline{f}^4, \bar{f}^4] = [\mu_{X_2^e}(\dot{e}') \cdot \mu_{X_2^e}(e'), \mu_{X_2^e}(\dot{e}') \cdot \mu_{X_2^e}(e')]$ $= [0.0340 \times 0.7261, 0.1783 \times 0.8494] = [0.0247, 0.1514]$	→	$y^4 = 1$

For the IT2 FLC, the firing intervals of the four IT2 FSs are:

$$[\mu_{X_1^e}(\dot{e}'), \mu_{X_1^e}(e')] = [0.3753, 0.6065], \quad [\mu_{X_2^e}(\dot{e}'), \mu_{X_2^e}(e')] = [0.0340, 0.1783]$$

$$[\mu_{X_1^e}(e'), \mu_{X_1^e}(\dot{e}')] = [0.0060, 0.0734], \quad [\mu_{X_2^e}(e'), \mu_{X_2^e}(\dot{e}')] = [0.7261, 0.8494]$$

The firing intervals of the four rules are show in Table 5. From the KM algorithms, we find that  $L = 2$  and  $R = 2$ . So,

$$y_l = \frac{\bar{f}^1 y^1 + \bar{f}^2 y^2 + \bar{f}^3 y^3 + \bar{f}^4 y^4}{\bar{f}^1 + \bar{f}^2 + \bar{f}^3 + \bar{f}^4}$$

$$= \frac{0.0445 \times (-1) + 0.5152 \times (-0.5) + 0.0002 \times 0.5 + 0.0247 \times 1}{0.0445 + 0.5152 + 0.0002 + 0.0247}$$

$$= -0.4743 \tag{11}$$

$$y_r = \frac{\underline{f}^1 y^1 + \underline{f}^2 y^2 + \underline{f}^3 y^3 + \underline{f}^4 y^4}{\underline{f}^1 + \underline{f}^2 + \underline{f}^3 + \underline{f}^4}$$

$$= \frac{0.0022 \times (-1) + 0.2725 \times (-0.5) + 0.0131 \times 0.5 + 0.1514 \times 1}{0.0022 + 0.2725 + 0.0131 + 0.1514}$$

$$= 0.0443$$

Finally, the crisp output of the IT2 FLC is:

$$\dot{u} = \frac{y_l + y_r}{2} = \frac{-0.4743 + 0.0443}{2} = -0.2150.$$

Observe again from the above example that for the same input, IT2 and T1 FLCs give quite different outputs. The next section explains the fundamental reasons behind this difference.

## 4 Fundamental Differences Between IT2 and T1 FLCs

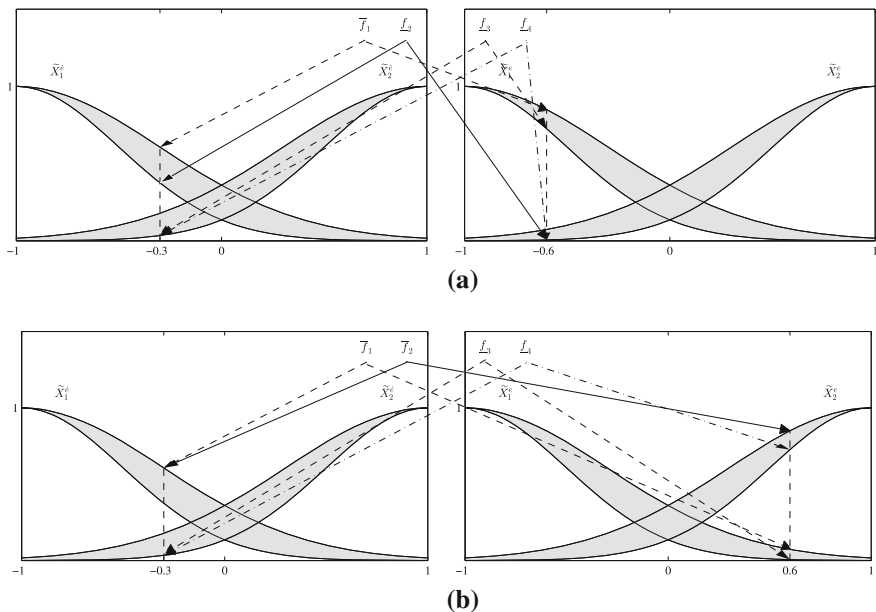
Observe from (9), and also Examples 1 and 2, that the output of an IT2 FLC is the average of two ‘‘T1 FLCs’’. However, these two ‘‘T1 FLCs’’ are *fundamentally* different from traditional T1 FLCs, for the following reasons [16, 17]:

1. **Adaptiveness**, meaning that the embedded T1 FSs used to compute the bounds of the type-reduced interval change as input changes. Take  $y_l$  in (10) and (11) as an example. The firing levels of the four rules in (10) are  $\bar{f}^1$ ,  $f_-^2$ ,  $f_-^3$ , and  $f_-^4$ , respectively, which are computed from different lower and upper MFs, as shown in the first part of Table 6 and Fig. 6a. The firing levels of the four rules in (11) are shown in the second part of Table 6 and Fig. 6b. Comparing the two parts of Table 6, and the two sub-figures in Fig. 6, we can observe that when the input  $(\dot{e}', e')$  changes from  $(-0.3, -0.6)$  to  $(-0.3, 0.6)$ , different embedded

**Table 6** The embedded T1 FSs from which the four firing levels in (10) and (11) are obtained.

	$\tilde{X}_1^e$		$\tilde{X}_2^e$		$\tilde{X}_1^e$		$\tilde{X}_2^e$	
	UMF	LMF	UMF	LMF	UMF	LMF	UMF	LMF
Equation (10) $(\dot{e}', e') = (-0.3, -0.6)$	$\bar{f}^1$	✓					✓	
	$f_-^2$		✓					✓
	$f_-^3$			✓		✓		
	$f_-^4$			✓				✓
Equation (11) $(\dot{e}', e') = (-0.3, 0.6)$	$\bar{f}^1$	✓					✓	
	$\bar{f}^2$	✓						✓
	$f_-^3$			✓		✓		
	$f_-^4$			✓				✓

Observe that  $f_-^2$  is used when  $(\dot{e}', e') = (-0.3, -0.6)$  and  $\bar{f}^2$  is used when  $(\dot{e}', e') = (-0.3, 0.6)$ ; as a result, different embedded T1 FSs are used in Rule  $\tilde{R}^2$  when the input changes. Observe also that when  $(\dot{e}', e') = (-0.3, -0.6)$  both the UMFs and LMFs of  $\tilde{X}_1^e$  and  $\tilde{X}_1^e$  are used in computing  $y_l$ , and when  $(\dot{e}', e') = (-0.3, 0.6)$  both the UMFs and LMFs of  $\tilde{X}_1^e$  and  $\tilde{X}_2^e$  are used in computing  $y_l$



**Fig. 6** The embedded T1 FSs used in (a) Eq. (10) for computing  $y_l$ , where  $(\tilde{e}', e') = (-0.3, -0.6)$  and the LMFs of  $\tilde{X}_1^e$  and  $\tilde{X}_2^e$  are used in computing the firing level  $f_2$  of Rule  $\tilde{R}^2$ ; and, (b) Eq. (11) for computing  $y_l$ , where  $(\tilde{e}', e') = (-0.3, 0.6)$  and the UMFs of  $\tilde{X}_1^e$  and  $\tilde{X}_2^e$  are used in computing the firing level  $\tilde{f}_2$  of Rule  $\tilde{R}^2$ . Observe that in (a) both the UMFs and LMFs of  $\tilde{X}_1^e$  and  $\tilde{X}_2^e$  are used in computing  $y_l$ , and in (b) both the UMFs and LMFs of  $\tilde{X}_1^e$  and  $\tilde{X}_2^e$  are used in computing  $y_l$

T1 FSs of  $\tilde{X}_1^e$  and  $\tilde{X}_2^e$  are used in computing the firing levels for Rule  $\tilde{R}^2$  and hence  $y_l$ . This adaptiveness is impossible for a T1 FLC since it does not have such embedded T1 FSs.

2. **Novelty**, meaning that the UMF and LMF of the same IT2 FS may be used simultaneously in computing each bound of the type-reduced interval. Observe from the first part of Table 6, and also Fig. 6a, that both the upper and lower MFs of  $\tilde{X}_1^e$  are used in computing  $y_l$ , and they are used in different rules: The UMF of  $\tilde{X}_1^e$  is used in computing  $\tilde{f}^1$ , the firing level of Rule  $\tilde{R}^1$ , whereas the LMF of  $\tilde{X}_1^e$  is used in computing  $\tilde{f}^2$ , the firing level of Rule  $\tilde{R}^2$ . Similarly, the upper and lower MFs of  $\tilde{X}_2^e$  are used simultaneously in different rules for computing  $y_l$ . Observe also from the second part of Table 6 and Fig. 6b that the upper and lower MFs of  $\tilde{X}_1^e$  and  $\tilde{X}_2^e$  are used simultaneously in different rules for computing  $y_l$ . This novelty is again impossible for a T1 FLC because it does not have embedded T1 FSs and the same MFs are always used in computing the firing levels of all rules.

We consider adaptiveness and novelty as two fundamental differences between IT2 and T1 FLCs. Though they are illustrated by specific numerical examples, they are fundamental to an arbitrary IT2 FLC. Furthermore, [17] presents several methods for visualizing and analyzing the effects of these two fundamental differences, including the control surface, the P-map, the equivalent generalized T1 fuzzy sets, and the equivalent PI gains. It also examines five alternative type-reducers for IT2 FLCs and explain why they do not capture the fundamentals of IT2 FLCs.

**Theorem 1** [17]  $y_l$  in (5) cannot be implemented by a T1 FLC using the same rulebase.

*Proof* In this proof we make use of the following two facts:

- *Fact 1:* The rule firing levels used in the KM algorithms are the bounds of the firing intervals. For an upper bound, all involved embedded T1 FSs must be UMFs, and for a lower bound, all involved embedded T1 FSs must be LMFs. There is no mixture of UMFs and LMFs in computing the firing level of any rule.
- *Fact 2:*  $\bar{f}^1$  and  $\underline{f}^N$  are always used for computing  $y_l$  in (5), though we are not sure about whether the upper or lower firing levels should be used for the rest of the rules. For  $\bar{f}^1$ , all involved embedded T1 FSs must be UMFs. For  $\underline{f}^N$ , all involved embedded T1 FSs must be LMFs.

We consider two cases separately:

1. Rules  $\tilde{R}^1$  and  $\tilde{R}^N$  share at least one IT2 FS  $\tilde{X}_i$ . In this case, according to Fact 2, for Rule  $\tilde{R}^1$   $\bar{X}_i$  must be used, whereas for Rule  $\tilde{R}^N$   $\underline{X}_i$  must be used. This novelty cannot be implemented by a T1 FLC using the same rulebase.
2. Rules  $\tilde{R}^1$  and  $\tilde{R}^N$  do not have any IT2 FS in common, (e.g., for  $y_l$  in (11),  $\tilde{R}^1$  involves  $\tilde{X}_1^e$  and  $\tilde{X}_1^e$ , whereas  $\tilde{R}^4$  involves  $\tilde{X}_2^e$  and  $\tilde{X}_2^e$ ). This case is more complicated than the previous one. We prove it by contradiction. Assume  $y_l$  in (5) can be implemented by a T1 FLC, where the same T1 MFs are used in computing all firing levels, e.g., if the UMF of  $\tilde{X}_1^e$  is used in computing the firing level of Rule  $\tilde{R}^1$ , it must also be used in computing the firing levels of all other rules involving  $\tilde{X}_1^e$ . In the second case, it is always possible to find a Rule  $\tilde{R}^k$  such that Rules  $\tilde{R}^1$  and  $\tilde{R}^k$  share at least one common IT2 FS  $\tilde{X}_i$ , and Rules  $\tilde{R}^k$  and  $\tilde{R}^N$  share at least one common IT2 FS  $\tilde{X}_j$  (e.g., for  $y_l$  in (11), Rules  $\tilde{R}^1$  and  $\tilde{R}^2$  share  $\tilde{X}_1^e$ , and Rules  $\tilde{R}^2$  and  $\tilde{R}^4$  share  $\tilde{X}_2^e$ ). According to Fact 2,  $\bar{X}_i$  must be used in Rule  $\tilde{R}^1$  for computing  $\bar{f}^1$ . If  $y_l$  can be implemented by a T1 FLC using the same rulebase, then  $\bar{X}_i$  must also be used in Rule  $\tilde{R}^k$ . According to Fact 1,  $\bar{X}_j$  must also be used for Rule  $\tilde{R}^k$ . For a T1 FLC, this means  $\bar{X}_j$  must also be used in Rule  $\tilde{R}^N$ , which is a contradiction with Fact 2. So, again  $y_l$  in (5) cannot be implemented by a T1 FLC using the same rulebase.  $\square$

**Theorem 2** [17]  $y_r$  in (6) cannot be implemented by a T1 FLC using the same rulebase.

The proof is very similar to that for Theorem 1, so it is omitted.

Based on Theorems 1 and 2, we can easily reach the following conclusion:

**Theorem 3** [17] An IT2 FLC using the KM type-reducer cannot be implemented by a T1 FLC using the same rulebase.

Theorem 3 is very helpful in understanding why IT2 FLCs may outperform T1 FLCs. It suggests that an IT2 FLC can implement a more complex control surface than a T1 FLC: when there is no FOU, an IT2 FLC collapses to a T1 FLC; with FOU, an IT2 FLC can implement a control surface that cannot be obtained from a T1 FLC using the same rulebase. Note that Theorem 3 does not conflict with the fact that T1 fuzzy logic systems are universal approximators [2, 9]: Being a universal approximator requires a T1 fuzzy logic system to have an arbitrarily large number of MFs, whereas in this chapter we only consider IT2 and T1 FLCs with the same rulebase and a fixed (small) number of MFs.

## 5 Conclusions

IT2 FLCs have been widely used and demonstrated better ability to handle uncertainties than their T1 counterparts. A challenging question is what the fundamental differences are between IT2 and T1 FLCs. Once the fundamental differences are clear, we can better understand the advantages of IT2 FLCs and hence better make use of them. In this chapter, we have explained two fundamental differences between IT2 and T1 FLCs: 1) *Adaptiveness*, meaning that the embedded T1 FSs used to compute the bounds of the type-reduced interval change as input changes; and, 2) *Novelty*, meaning that the UMF and LMF of the same IT2 FS may be used simultaneously in computing each bound of the type-reduced interval. As a result, an IT2 FLC can implement a complex control surface that cannot be achieved by a T1 FLC using the same rulebase.

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