Construction of Interval Type-2 Fuzzy Sets From Fuzzy Sets: Methods and Applications

Miguel Pagola, Edurne Barrenechea, Javier Fernández, Aranzazu Jurio, Mikel Galar, Jose Antonio Sanz, Daniel Paternain, Carlos Lopez-Molina, Juan Cerrón and Humberto Bustince

Abstract In this chapter, we present some methods to construct interval type-2 membership functions from fuzzy membership functions and their applications in image processing, classification, and decision making. First, we review some basic concepts of interval type-2 fuzzy sets (IT2FSs). Next, we analyze three different approaches to construct IT2FSs starting from fuzzy sets and their applications in different fields.

1 Interval Type-2 Fuzzy Sets

From the beginning, it was clear that fuzzy set theory [30] was an extraordinary tool for representing human knowledge. The use of linguistic labels enables the acquisition of interpretable knowledge systems, and in this manner the choice of the membership function plays an essential role in their success. The punctual value set as membership degree is usually defined either by means of expert knowledge or homogeneously over the input space. Nevertheless, Zadeh himself established (see [31]) that sometimes, in decision-making processes, knowledge is better represented by means of some generalizations of fuzzy sets.

Extensions of fuzzy sets are not as specific as their counter-parts of fuzzy sets, but this lack of specificity makes them more realistic for some applications. Their advantage is that they allow us to express our uncertainty in identifying a particular membership function. This uncertainty is involved when extensions of fuzzy sets are processed, making results of the processing less specific but more reliable.

The concept of *type-2 fuzzy set* was suggested by Zadeh in 1975 [31] as a generalization of an ordinary fuzzy set. Type-2 fuzzy sets are characterized by a fuzzy membership function, that is, the membership value for each element of the

M. Pagola (✉) · E. Barrenechea · J. Fernández · A. Jurio · M. Galar · J. A. Sanz · D. Paternain · C. Lopez-Molina · J. Cerrón · H. Bustince
Universidad Pública de Navarra, Campus Arrosadia sn, 31006 Pamplona, Spain
e-mail: miguel.pagola@unavarra.es

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set is given by a fuzzy set defined in the reference set $[0, 1]$. These sets were first studied and analyzed in [20].

There is still some discussion about the notation for type-2 fuzzy sets. We shall follow the standard mathematical notation in the following definitions (for equivalences with other notations, see [1]). A good review of these sets can be found in [18].

Sometimes, it is appropriate to represent the membership degree of each element to the fuzzy set by means of an interval. Hence, not only vagueness (lack of sharp class boundaries), but also a feature of uncertainty (lack of information) can be addressed intuitively.

A particular case of type-2 fuzzy sets called interval type-2 fuzzy sets (see [19]). In May 1975 Sambuc (see [24]) presented in his doctoral thesis, the concept of an interval-valued fuzzy set named a Φ-fuzzy set. That same year, Zadeh [31] discussed the representation of type 2 fuzzy sets and its potential in approximate reasoning. One year later, Grattan-Guinness [13] established a definition of an interval-valued membership function. In that decade, interval-valued fuzzy sets appeared in the literature in various guises and it was not until the 1980s, that the importance of these sets, as well as their name, was definitely established. In [10, 16, 18], it is proved that interval-valued fuzzy sets are a particular case of IT2FSs. It turns out that interval type-2 fuzzy sets are isomorphic to *interval-valued fuzzy set* [24].

In this chapter, we work with finite, nonempty reference sets. We denote by $L([0, 1])$ the set of all closed subintervals of the unit interval $[0, 1]$ in the following way:

$$L([0,1]) = \{\mathbf{x} = [\underline{x}, \overline{x}] | (\underline{x}, \overline{x}) \in [0,1]^2 \text{ and } \underline{x} \leq \overline{x}\}. \tag{1}$$

We use bold letters to refer the elements $\mathbf{x} \in L([0, 1])$ and we denote with W the length of an interval, that is, $W(\mathbf{x}) = \overline{x} - \underline{x}$.

$L([0, 1])$ is a partially ordered set with respect to the relation $\leq_L$ defined in the following way: given $\mathbf{x}, \mathbf{y} \in L([0, 1])$,

$$\mathbf{x} \leq_L \mathbf{y} \text{ if andonly if } \underline{x} \leq \underline{y} \text{ and} \overline{x} \leq \overline{y}. \tag{2}$$

With this order relation, $(L([0, 1]), \leq_L)$ is a complete lattice, where the smallest element is $0_L = [0, 0]$ and the largest is $1_L = [1, 1]$.

An interval type 2 fuzzy set $\tilde{A}$ on U is defined by

$$\tilde{A} = \{(u, A(u), \mu_u(x)) | u \in U, A(u) \in L([0, 1])\},$$

where $A(u) = [\underline{A}(u), \overline{A}(u)]$ is a closed subinterval of $[0, 1]$, and the function $\mu_u(x)$ represents the fuzzy set associated with the element $u \in U$ obtained when x covers the interval $[0, 1]$; $\mu_u(x)$ is given in the following way:

$$(x) = \begin{cases} a & \text{if } \underline{A}(u) \leq x \leq \overline{A}(u) \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

where $0 \leq a \leq 1$. As we have said previously with $a = 1$ an interval type 2 fuzzy set is the same as an interval valued fuzzy set.

Mendel and others [17] defined IT2FSs using the footprint of uncertainty (FOU). An IT2FS $\tilde{A}$ for a primary variable $(x \in X)$ is characterized by its footprint of uncertainty, $FOU(\tilde{A})$, which in turn is completely described by its lower membership function, $LMF(\tilde{A})$, also denoted by $\underline{\mu}_{\tilde{A}}(x)$, and upper membership function $UMF(\tilde{A})$, also denoted by $\overline{\mu}_{\tilde{A}}(x)$, i.e., the lower and upper bounding functions of $FOU(\tilde{A})$ respectively.

Through the chapter we denote by $\mathcal{IT}2\mathcal{FS}s(U)$ the set of all the interval type-2 fuzzy sets defined on U, and $\mathcal{FS}s(U)$ all the fuzzy sets on U.

2 Construction Methods of IT2FSs

When we will develop an application using IT2FSs, the first step is to define the membership functions that will represent these sets. For example, if we use an IT2FS system then we must define the rules and the lower and upper membership functions of the linguistic labels. It is known that a key problem of the fuzzy systems is the definition of the membership functions, as we have previously stated.

Usually, IT2FS are defined manually or from data extracted [17]. Other typical method to obtain a good definition of the membership functions is to optimize their shape using genetic algorithms [14].

When we are working with IT2FSs, we must take into account that the FOU of the IT2FSs represents the uncertainty in the membership degree. Therefore the FOUs must represent the uncertainty that exists in the model.

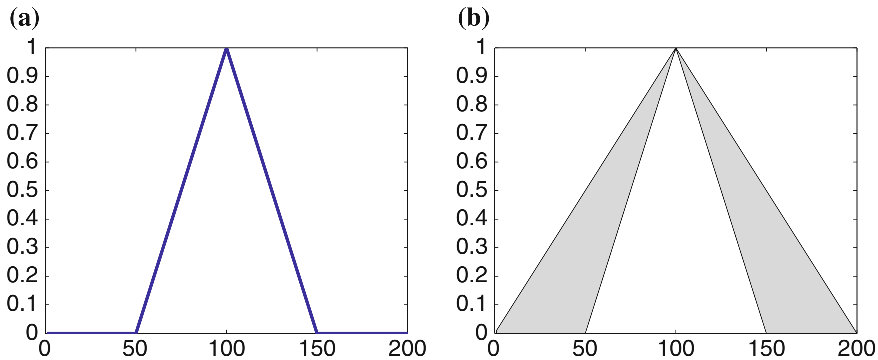

Fig. 1 **a** Triangular fuzzy membership function. **b** Triangular interval type-2 fuzzy membership function

In this work, we present three different methods to construct IT2FS from fuzzy sets that try to generate FOUs adapted to the model's uncertainty. We have studied three different cases:

- Using several fuzzy membership functions.
- Using two fuzzy membership functions that represent opposite objects or concepts.
- Using only one fuzzy membership function.

2.1 Construction of an IT2FS from Several Fuzzy Membership Functions

When we define a fuzzy system one key problem is the definition of the membership functions. In the context of fuzzy rule-based systems, sometimes the expert can choose between different functions (triangular, gaussian, etc.) and different parameters. Therefore, the expert is not sure about which is the best membership function, he can choose several adequate membership functions. If we want to construct an IT2FS from different membership functions, the IT2FS should be such that the lengths of the intervals represent the uncertainty that the expert has in the selection of these fuzzy sets. That is, if the expert is absolutely sure of the membership degree of an element, then the length of the interval associated to such element is zero (a fuzzy set). On the other hand, if the expert does not know the membership degree of an element at all, then the length of the interval associated to this element should be the maximum possible.

$$\begin{aligned} &\Phi : \mathcal{FS}s(U)\overbrace{\times\cdots\times}^{k\text{ times}}\mathcal{FS}s(U) \longrightarrow \mathcal{IT2FS}s(U) \text{ given by} \\ &\Phi(A^1,\cdots,A^k) = \{(u,\Phi(A^1,\cdots,A^k)(u))|u\in U\} \quad \text{such that} \\ &\Phi(A^1,\cdots,A^k)(u) = [T(\mu_{A^1}(u),\cdots,\mu_{A^k}(u)) \quad S(\mu_{A^1}(u),\cdots,\mu_{A^k}(u))], \end{aligned} \tag{4}$$

where T and S are a t-norm and a t-conorm, respectively, in $[0,1]$.

Remark The associativity of triangular norms and triangular t-conorms allows us to extend these mappings to an arbitrary finite number of arguments in a unique way, by means of a recursive definition. For example, n-ary triangular norms are defined as follows. Let $(x_i,\cdots,x_n)$ be a finite family in $[0,1]^n$. Then $T(x_1,\cdots,x_n) = T(T(x_1,\cdots,x_{n-1}),x_n)$.

We denote by W_{TS} the length of an interval constructed by the above method, where T is a t-norm and S is a t-conorm.

As U is discrete, our method can be seen as a construction of the footprint of uncertainty from several fuzzy sets. Suppose that the expert gives two different opinions (or two different experts each giving a single opinion). We can use a t-norm and a t-conorm to construct an IT2FS. Figure 2 depicted different IT2FSs

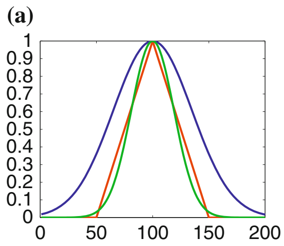

Fig. 2 (**a**) Three different fuzzy membership functions. (**b**) Interval type-2 fuzzy membership function generated using the t-norm minimum and the t-conorm maximum. (**c**) Interval type-2 fuzzy membership function generated using the t-norm product and the t-conorm probabilistic sum

constructed using different t-norms and t-conorms. We can also observe in Fig. 2 that the FOU generated with the t-norm product and the t-conorm probabilistic sum is wider than the one generated with the t-norm minimum and the t-conorm maximum.

Corollary 1 *Under the conditions of the construction method described in Eq.* (4), *the following statement is true:*

If T and S are any t-norm and t-conorm in [0,1], then

$$W_{TS}(\Phi(Q^1,\cdots,Q^k)(u)) \geq W_{\wedge\vee}(\Phi(A^1,\cdots,A^k)(u)) \forall u \in U.$$

Proof It is enough to take into account the fact that $\wedge$ is the largest t-norm and $\vee$ is the smallest t-conorm. □

This corollary proves that the FOU constructed with the t-norm minimum and t-conorm maximum is the smallest one. If other combination of t-norm and t-conorm is used the FOU will be greater.

2.2 Construction of an IT2FS from Two Fuzzy Membership Functions

Next we introduce the concept of *Ignorance function* and the way we use it to construct IT2FS from two related fuzzy sets. In this case, the fuzzy sets must be related with each other; they must represent opposite concepts. For example, one set represents the concept *near* and the other the concept *far*, or the concepts *small* and *big*. We have proposed this method in the context of image segmentation where we have two sets, one to represent the object and another to represent the background; but it can be used in any environment in which we have two different sets that represent opposite concepts.

The concept of *ignorance function* [6] tries to model the lack of knowledge that sometimes experts suffer when determining the membership degrees of some pixels of an image Q to the fuzzy set representing the background (B) of the image and to the fuzzy set representing the object (A) in the image.

For us, $\mu_B(x)$ $(\mu_A(x))$ is the quantification of the expert knowledge that the pixel with intensity x belongs to the background (object). In this sense, if $\mu_B(x) = 1(\mu_A(x) = 1)$, then the expert has total knowledge (total sureness) that the pixel belongs to the background (object). When $\mu_B(x) = 0.5$ $(\mu_A(x) = 0.5)$, we say that the expert is totally ignorant of whether the pixel belongs to the background (object) (total doubt). If the expert is totally sure that the pixel belongs to the background (object), then he should take $\mu_B(x) = 1$ and in this case the membership to the object (background) should be close to 0, ($\mu_A(x) \approx 0$). In spite of this, the simultaneous ignorance of a pixel's membership to the background and to the object will be given when the two membership functions are close to 0.5.

Evidently, there are pixels of the image for which the expert is absolutely sure that the chosen representation is the correct one. Nevertheless, there are also pixels for which the expert does not know if the representation taken is the best. We will represent the expert's ignorance in terms of μ_B and μ_A by means of what we denote as ignorance functions.

Under this interpretation, the following conditions must be fulfilled by these functions:

1. The ignorance function depends only on $\mu_B(x)$ and $\mu_A(x)$.
2. The ignorance does not depend on whether we first consider the membership to the background and then the membership to the object or we first consider the membership to the object and then the membership to the background.
3. (Representation of total knowledge) The ignorance of the expert in the choice of the membership of a pixel must be zero if and only if he is certain that the pixel belongs to the object or the background.
4. (Representation of total doubt) If $\mu_B(x) = 0.5$ and $\mu_A(x) = 0.5$; that is if the expert is not capable of distinguishing whether a pixel belongs to the background or to the object, then we will say that the expert's ignorance of the membership of this pixel to the background or to the object is one.
5. If the membership of the pixel to the background and its membership to the object are greater than 0.5, then the greater both memberships are, the smaller the ignorance should be.
6. If the membership of the pixel to the background and its membership to the object are smaller than 0.5, then the greater both memberships are, the greater the ignorance should be.

We just recall that these properties are equivalent for any problem in which there are two objects that represent opposite things, and therefore the mathematical definition is valid in those environments. The considerations above have led us to present the following definition.

Definition 1 A function $G_i : [0,1]^2 \rightarrow [0,1]$ is called an *ignorance function*, if it satisfies the following conditions:

(G_i1) $G_i(x,y) = G_i(y,x)$ for all $x, y \in [0,1]$;
(G_i2) $G_i(x,y) = 0$ if and only if $x = 1$ or $y = 1$;
(G_i3) If $x = 0.5$ and $y = 0.5$, then $G_i(x,y) = 1$;
(G_i4) G_i is decreasing in $[0.5,1]^2$;
(G_i5) G_i is increasing in $[0,0.5]^2$.

In some cases, it is advisable to require ignorance functionss to be continuous, since the ignorance must not present a chaotic reaction to small changes in the degree of knowledge that the experts possess regarding to the membership of the pixel in question to the background or to object. If this is the case, we will say that the ignorance functionss are continuous.

In the following theorem, we show a construction method of continuous ignorance functions from t-norms.

Theorem 1 [6] *Let T be a continuous t-norm such that*
$$T(x,y) = 0 \text{ if and only if } x \cdot y = 0.$$
Under these conditions, the function

$$G_i(x,y) = \begin{cases} \dfrac{T(1-x,1-y)}{T(0.5,0.5)} & \text{if } T(1-x,1-y) \leq T(0.5,0.5) \\ \dfrac{T(0.5,0.5)}{T(1-x,1-y)} & \text{otherwise} \end{cases}$$

is a continuous ignorance function.

Example 1

(1) The t-norm minimum satisfies the conditions in Theorem 1, so

$$G_i(x,y) = \begin{cases} 2 \cdot \min(1-x,1-y) & \text{if } \min(1-x,1-y) \leq 0.5 \\ \dfrac{1}{2 \cdot \min(1-x,1-y)} & \text{otherwise} \end{cases}$$

is a continuous ignorance function.

(2) The t-norm product satisfies the conditions in Theorem 1., so

$$G_i(x,y) = \begin{cases} 4 \cdot (1-x) \cdot (1-y) & \text{if } (1-x) \cdot (1-y) \leq 0.25 \\ \dfrac{1}{4 \cdot (1-x) \cdot (1-y)} & \text{otherwise} \end{cases}$$

is a continuous ignorance function.

In [6], we developed a method to construct ignorance functions from functions different than the t-norms. Next, we show an example.

Example 2 If we take $\varphi(x) = \sqrt{x}$ for all $x \in [0, 1]$ we recover the following ignorance function:

$$G_i(x, y) = \begin{cases} 2\sqrt{(1-x)\cdot(1-y)} & \text{if } (1-x)\cdot(1-y) \leq 0.25 \\ \dfrac{1}{2\sqrt{(1-x)\cdot(1-y)}} & \text{otherwise} \end{cases}$$

Taking into account, the value of ignorance and the original fuzzy set we can construct the IT2FS. First, we assign the value of the ignorance function to the length W of the interval. Such a way the ignorance calculated represent the FOU:

$$W(x) = G_i(\mu_A(x), \mu_B(x)).$$

The main problem is that the lower membership function must always be greater than zero and the upper membership function must be lower than one. Therefore in [21], we propose the following method to construct IT2FS for two opposite fuzzy sets A and B:

$$\tilde{A}(u) = [S(0, \mu_A(u) + \lambda \times W(u)/2) \quad T(1, \mu_A(u) + \lambda \times W(u)/2)], \tag{5}$$

where T and S are a t-norm and a t-conorm, respectively, in $[0, 1]$ *and* $\lambda > 0$.

With this method the interval generated is always within $[0, 1]$. Also the parameter λ modifies the length of the intervals. If $\lambda = 1$ then the length of the interval is the same as the value of the ignorance function G_i calculated.

One of the advantages of this method is that the shape of the FOU is related with the shape of the membership functions, as we can see in Fig. 3.

2.3 Construction of an IT2FS from One Fuzzy Membership Function

If we have a membership function that represents the fuzzy set that modelizes certain concept, sometimes we know that there exist uncertainty in this membership. There exist several works that try to obtain from the proper membership function a value of the uncertainty and from this value to construct an IT2FS. Mainly two different approaches have been proposed. The first one intervals are generated using one or two parameters, we denote this method as interval generators. The second approach intervals are constructed by means of a function that only depends on the value of the membership function. This function gives an ignorance value, related with the membership degree, allowing us to construct an IT2FS.

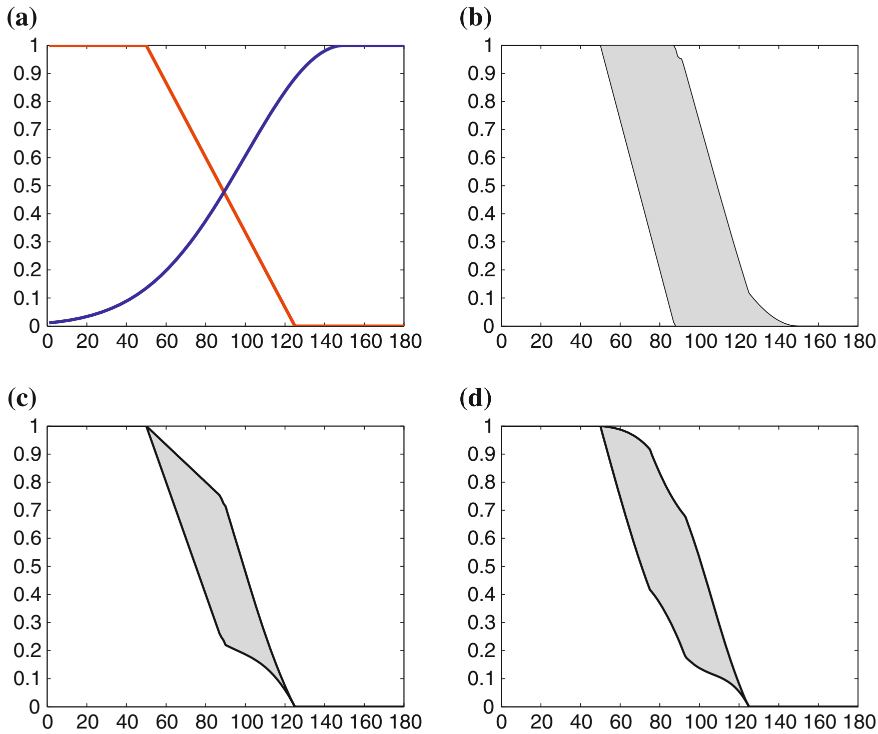

Fig. 3 **a** Two different fuzzy membership functions. **b** IT2FS generated from the ignorance function of example 1.1 and $\lambda = 1$. **c** IT2FS generated from the ignorance function of example 1.1 and $\lambda = 0.5$. **d** IT2FS generated from the ignorance function of example 1.2 and $\lambda = 0.5$

2.3.1 Interval Generators

If the uncertainty presented in the problem is due to a known cause, we can modelize it with some functions [3], called generators, and construct an IT2FS from the former fuzzy set.

In the following example we present an interval generator with two parameters.

Let $A \in \mathcal{F}Ss(U)$ and let the functions:

$$\begin{cases} f : [0,1] \to [0,1] \text{ given by} \\ \qquad f(x) = x^{\alpha} \text{ with } \alpha \geq 1. \\ g : [0,1] \to [0,1] \text{ given by} \\ \qquad g(x) = x^{\frac{1}{\beta}} \text{ with } \beta \geq 1. \end{cases}$$

Under these conditions

$$\tilde{A}_{\alpha,\beta} = \{(u, [\mu_A^{\alpha}(u)\ \mu_A^{\frac{1}{\beta}}(u))]|u \in U\} \in \mathcal{IT}2\mathcal{F}Ss(U).$$

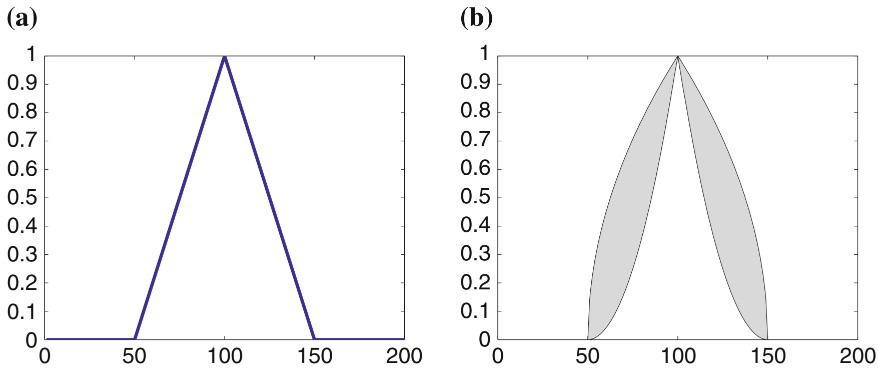

Fig. 4 **a** Original fuzzy set. **b** IT2FS generated from an interval generator with $\alpha = 2$ and $\beta = 2$.

The verification that $\tilde{A}_{\alpha,\beta} \in \mathcal{IT2FSs}(U)$ is evident: $0 \leq \mu_A^{\alpha}(u) \leq \mu_A^{\frac{1}{\beta}}(u) \leq 1$. The parameters α and β can be related with the ignorance of the expert in the membership function selection. An specific case with only one parameter α is:

$$\tilde{A}_{\alpha} = \{(u, [\mu_A^{\alpha}(u)\ \mu_A^{\frac{1}{\alpha}}(u))] | u \in U\} \in \mathcal{IT2FSs}(U). \tag{6}$$

Figure 4 depicted a fuzzy set and an IT2FS generated with values of $\alpha = 2$ and $\beta = 2$.

2.3.2 Weak Ignorance Function

The length of the IT2FSs can be seen as a representation of the ignorance when assigning punctual values as membership degrees. In order to measure the ignorance degree, we define the concept of weak ignorance functions [26], which are a particular case of ignorance functions depending on a single variable and demanding a less number of properties.

Definition 2 [26] A weak ignorance function is a mapping $g : [0, 1] \rightarrow [0, 1]$ that satisfies:

- $(g1)$ $g(x) = g(1 - x)$ for all $x \in [0, 1]$;
- $(g2)$ $g(x) = 0$ if and only if $x = 0$ or $x = 1$;
- $(g3)$ $g(0.5) = 1$.

Example 3 $g(x) = 2 \cdot \min(x, 1 - x)$ is a weak ignorance function.

We also present in [26] the following construction method of IT2FSs. First, we assign the length of the interval the value of ignorance of the membership degree of the fuzzy set A, i.e., $W(u) = g(\mu_A(u))$ and then we construct the IT2FSs $\tilde{A}$ in the following way:

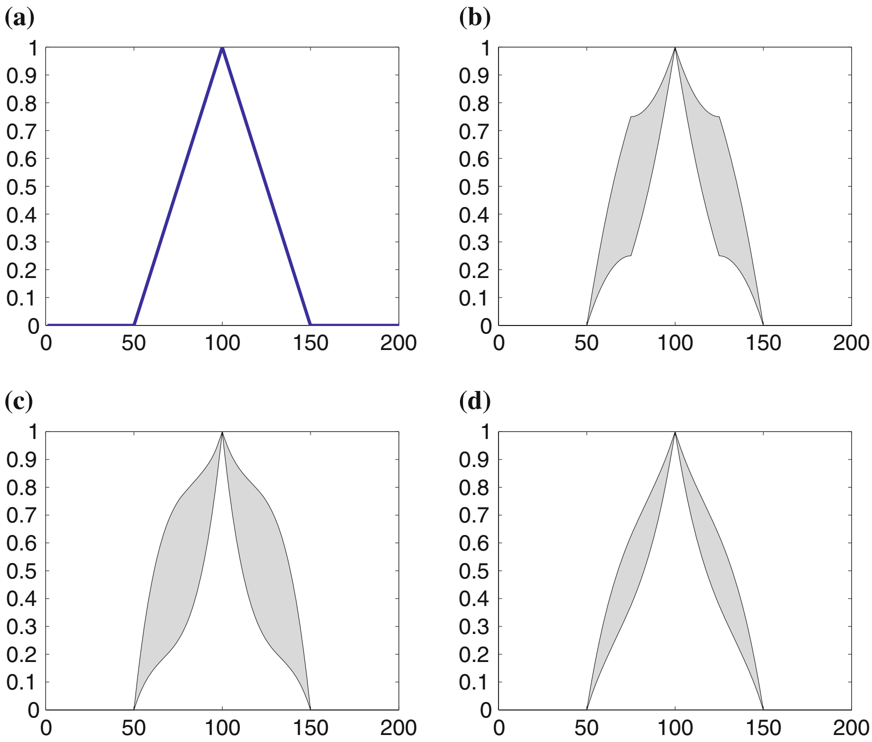

Fig. 5 **a** Original fuzzy set. **b** IT2FS generated with weak ignorance function of equation $g(x) = 2 \cdot \min(x, 1-x)$ and $\lambda = 1$. **c** IT2FS generated with weak ignorance function of equation $g(x) = 4 \cdot (x \cdot (1-x))$ and $\lambda = 1$. **d** IT2FS generated with weak ignorance function of equation $g(x) = 4 \cdot (x \cdot (1-x))$ and $\lambda = 0.5$

$$\tilde{A} = \{(u, [\mu_A(u)(1 - \lambda \times W(u)) \quad \mu_A(u)(1 - \lambda \times W(u)) + \lambda \times W(u)]|u \in U\}. \tag{7}$$

Also the parameter λ modifies the length of the intervals. If $\lambda = 1$ then the length of the interval is the same as the value of the ignorance function g calculated.

Figure 5 depicted three different IT2FSs generated from different weak ignorance functions.

3 Applications

Next we present three different applications where we use IT2FSs constructed with the methods presented in the previous section.

3.1 Classification

Fuzzy rule-based classification systems (FRBCSs) are widely employed in classification tasks since they allow us to deal with noisy, imprecise or incomplete information which is often present in many real world problems. They provide a good trade-off between the empirical precision of traditional engineering techniques and the interpretability achieved through the use of linguistic labels whose semantic is close to the natural language.

However, FRBCSs can suffer a lack of system accuracy as a result of the uncertainty related to the definition of the membership functions.

In [25], we propose a methodology in which we use IT2FSs to model the linguistic labels of the classification system. To do so, we define a new parametrized IT2FSs construction method using triangular shaped membership IT2FSs. Specifically, the amplitude of the support of the upper bound of the IT2FSs is determined by the value of the parameter W, which establishes the relationship between the length of the lower and the upper bounds of each IT2FS. In this manner, we can build an IT2FSs model using the initial knowledge base generated by any fuzzy rule learning algorithm. Furthermore, the representation of the linguistic labels by means of IT2FSs leads to a natural extension of the classical fuzzy reasoning method (FRM) [8]. Specifically, we modified the two first steps out of the four, which compose the original FRM, in the following way:

- *Matching degree*: we apply a t-norm to the lower and upper bounds of the interval membership degrees of the elements to the IT2FSs composing the antecedent of the rules.
- *Association degree*: we take the mean between the product of the matching degree by the rule weight associated with the lower bound and the product of the matching degree by the rule weight associated with the upper bound.

In addition, we defined an evolutionary tuning in which we modified the value of the parameter W for each IT2FS used in the system. In this way, we tried to improve the system's performance by looking for the best amount of uncertainty that the FOU of each IT2FS represents.

In the experimental study, we used two well-recognized fuzzy rule learning methods, i.e., the algorithm proposed by Chi et al. [9] and the fuzzy hybrid genetics-based machine learning (FH-GBML) defined by Ishibuchi and Yamamoto [11]. In both cases, the application of our methodology (to the knowledge base generated by each algorithm) allowed to notably enhance the results provided by the initial nonIT2 fuzzy methods.

In [26], using the concept of weak ignorance function, we formalize the IT2FSs construction method introduced in [25] by establishing the relationship between the uncertainty represented by the FOUs of the IT2FSs and the ignorance degree. Specifically, we achieve that the length of the intervals, which are assigned as the membership degree of the elements to the set, are proportional to the weak ignorance degree computed by $g(x)$.

The experimental study supported the suitability of our method, since we outperformed the results of: (1) the original FH-GBML method; (2) the tuning approach based on the linguistic 3-tuples representation applied to the original fuzzy knowledge base, and (3) the lateral tuning applied to both the nonIT2 and the IT2 fuzzy versions of the knowledge base.

3.2 Image Segmentation

In 2005, Tizhoosh [28] presented an image thresholding approach using interval type 2 fuzzy sets (we must point out that he tries to use type 2 fuzzy sets, however he only uses interval type 2 fuzzy sets [7]). His study is based on the modification of the classical fuzzy algorithm of Huang and Wang [15], so that he applies an α factor as an interval generator to the membership function. Starting from a membership function, Tizhoosh obtains an interval type-2 fuzzy set that "contains" different membership functions and is useful for finding the threshold of an image. Tizhooh's algorithm is applied directly to color segmentation using RGB in [27] and it is also used to segment color image skin lesions [29]. Starting form the idea of obtaining the uncertainty from the information given by the user, we have proposed an approximation using interval type-2 fuzzy sets generated from interval generators [3] (where the key point is to choose the correct parameters). Also we have used interval type-2 membership functions within an algorithm of stereo matching [12] (in this case we use the terminology of interval-valued fuzzy sets). In said paper, we were interested in eliminating the sensitivity to the radiometric gain, bias, and noise using IT2FSs to represent the images. In this way, we managed the cited problems by splitting the image into two different areas (background and objects), where the membership degree of each pixel to an object or to the background is represented with an interval.We proposed a thresholding-based segmentation to build these interval type-2 fuzzy sets. These works led us to introduce the concept of ignorance function to try to model the lack of knowledge from which experts may suffer when determining the membership degrees of some pixels of a given image. This concept was presented in [5] and [6] where we modified the classical fuzzy thresholding algorithm such way the user should pick two functions, one to represent the background and another one to represent the object, instead of using one membership function to represent the whole image; that is, we proposed by means of *ignorance functions* to modelize the user's ignorance for choosing these two membership functions. From this value of ignorance we constructed the IT2FSs. The rest of the algorithm remained similar to the algorithm using IT2FS constructed from interval generators.

We evaluated the performance of the algorithm that uses ignorance functions in natural images and prostate ultrasound images. We must take into account that, since ultrasound images depend on the particular settings of the machine is very important that our algorithm gives good solutions even if some membership functions that do not represent accurately the background and the prostate are chosen. The IT2FS algorithm performance was compared with the classical fuzzy algorithm and we can conclude that *for the pairs of membership functions such that the fuzzy algorithm solution is good (small error), the IT2FS algorithm does not provide better results but if the error we get with the fuzzy algorithm begins to be high (i.e., if we have used bad-chosen membership functions), then the result of the IT2FS algorithm improves the other algorithm's result.*

3.3 Decision Making

Fuzzy preference relations have been widely used to model preferences for decision-making problems due to their high expressiveness and their effectiveness as a tool for modeling decision processes. In the fuzzy case, the experts express their opinions using a difference scale [0,1]. In [2] we presented a generalization of the nondominance criterion proposed by Orlovsky using interval preferences.

Our method starts from fuzzy preferences and by means of weak ignorance functions we construct an interval type-2 fuzzy preference matrix (in the paper we use the notation of interval valued fuzzy preference relation).

Let $R^* \in FR(X \times X)$ be a fuzzy preference relation over a set of alternatives $X = \{x_1, \ldots, x_n\}$; for each pair of alternatives x_i and x_j, $R^*_{ij} = R^*(x_i, x_j)$ represents a degree of (weak) preference of x_i over x_j, namely the degree to which x_i is considered as least as good as x_j.

Given $R^* \in FR(X \times X)$ we normalize it to $[0, 1]$ in such a way that for each element of the new relation, denoted by $R \in FR(X \times X)$, holds that $R_{ij} = 1 - R_{ji}$.

Next, from R we must extract a set of nondominated alternatives as the solution of the decision-making problem. Specifically, the maximal nondominated elements of R are calculated extending the nondominance criterion proposed by Orlovsky in [22] to intervals.

The *Non-dominance Interval Algorithm* that we proposed [2] is the following: Given a fuzzy preference relation R^* (without defined elements in the main diagonal) and a *weak fuzzy ignorance function* g,

1. Construct R normalizing R^*
2. Compute the fuzzy strict preference relation R^s in Orlovsky's sense
3. Build the interval type-2 fuzzy relation $\mathbf{r}$:

$$\mathbf{r_{ij}} = \begin{cases} [R^s_{ij} \cdot (1 - g(R_{ij})), R^s_{ij} \cdot (1 - g(R_{ij})) + g(R_{ij})] & \text{if } R_{ij} > R_{ji} \\ [0, g(R_{ij})] & \text{otherwise} \end{cases} \tag{8}$$

4. Build the interval type-2 fuzzy set:

$$ND_{IV} = \{(x_j, ND_{IV}(x_j))|x_j \in X\} \quad \text{where}$$
$$ND_{IV}(x_j) = \mathbf{S}(\mathbf{r}_{ij}) = \left[\bigvee_{i=1}^{n}(\underline{r}_{ij}), \bigvee_{i=1}^{n}(\overline{r}_{ij})\right] \tag{9}$$

5. Build the interval type-2 fuzzy set:

$$N_{IV}(ND_{IV}) = \{(x_j, N_{IV}(ND_{IV}(x_j)))|x_j \in X\} \quad \text{where} \tag{10}$$

$$N_{IV}(ND_{IV})(x_j) = \left[1 - \bigvee_{i=1}^{n}(\overline{r}_{ij}), 1 - \bigvee_{i=1}^{n}(\underline{r}_{ij})\right] \tag{11}$$

6. Order the elements of $N_{IV}(ND_{IV})$ in a decreasing way in terms of accuracy and score functions.
7. If there exist several alternatives occupying the firstplace, take as solution the alternative with the biggest upper bound of its interval associated.

We must remark that if for a majority of the elements $\mathbf{r_{ij}}$ we have that $g(R_{ij}) \to 0$, then the resulting intervals have a very small length and it is reasonable to assume that the result obtained with the algorithm is the same than the result obtained with the nondominance algorithm.

If for a majority of the elements $\mathbf{r_{ij}}$ we have that $g(R_{ij}) \to 1$, then the algorithm allows us to distinguish better than the nondominance algorithm the alternative or alternatives that we must take as solution.

4 Conclusions

A key problem of fuzzy systems and algorithms is the accurate election of the membership function. In this chapter, we have presented three different methods to generate interval type-2 fuzzy sets from fuzzy sets, such that they are very goodtools to represent the uncertainty existing in the problem or specifically in the election of the correct membership function. We have presented three different applications in which these methods have been applied successfully. In some cases, the IT2FS systems or algorithms achieved an improvement in the results of the original fuzzy cases.

As future research we plan to study different methods to construct IT2FS from data. Another interesting study is how to construct a general type-2 fuzzy set from a fuzzy set.

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