Chapter 35
Spectral Element Method for Cable Harnessed Structure

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Abstract This paper presents a predictive model of a cable-harnessed structure through using the Spectral Element Method (SEM) and this is compared to a finite element approach. SEM is an element-based method that combines the generality of the finite element method with the accuracy of spectral techniques. The exact dynamic stiffness matrices are used as the element matrices in the Finite Element Method (FEM). Thus it is possible to generate the meshes on geometric domains of concern. The spectral element can be assembled in the same terminology of the FEM. After assembly and application of the boundary conditions, the global matrix can be solved for response of model, repeatedly at all discrete frequencies because the dynamic stiffness matrices are computed at each frequency. Here we model a cable-harnessed structure as a double beam system. The presented SEM model can define location and number of connections very conveniently. Comparison is conducted between the FEM and SEM for several cases. The results show that the proposed SEM approach can be used as the exact solution of cable harnessed structures.

Keywords FEM • SEM • Spectral element method • Double beam system • Cable harnessed structure

35.1 Introduction

In satellite applications modeling is extremely important in order to predict behavior during launch and in orbit. Today detailed finite element models of satellites are accurate and agree well with experimental data. However once the satellites are hung with cable harnesses the ability to model the system dynamics has eluded modeling engineers. Here we investigate in a simple way the effects of adding cables to a simple structure with the goal of developing an understanding of the physics of cables interacting with a structure. The goal of this research is to obtain the predictive and precise model of the cable harnessed structure.

To obtain a simplified model of cable harnessed structure, the total system can be considered as a double beam system with both beams connected by spring connection at specific locations to emulate the effect of attaching cables to the structure.

The response of single Euler-Bernoulli beam is a classical example of a distributed system. And exact solution of this beam can be obtained in various ways [1]. However, when secondary beam is attached to the main beam by means of several spring connections, obtaining the solution of system can be more complicated. Several authors have investigated double-beam systems elastically combined by a distributed spring in parallel. Seelig and Hoppman II [2], worked out the solution of differential equation of elastically connect parallel beams. Rao [3] considered the free response of several Timoshenko beam systems. Gürgöze [4, 5] dealt with the derivation of the frequency equation of a clamped-free Euler Bernoulli beam with several spring-mass systems attached in mid of span by means of the Lagrange multipliers method. Vu et al. [6] presented an exact method for the vibration of a double-beam system subject to harmonic excitation. The system consists of a main beam

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with an applied force, and an auxiliary beam, with a distributed spring k and dashpot c in parallel between the two beams. Wu and Chou [7] considered the beam connected with two degrees-of-freedom systems at specific location.

Doyle [8] and Lee [9] describe the spectral element method very well. Doyle introduced the basic formulation of spectral element matrix by Euler-Bernoulli beam. And Lee summarized the various ways to derive the spectral element matrix for structural elements such as Euler beam, Timoshenko beam and plate. Lee also presented several practical applications of SEM. Many authors used the dynamic stiffness approach very closely related to the SEM. Banerjee [10] and Chen [11] used dynamic stiffness matrix for the beam with attached two degree-of-freedoms system. Li and Hua [12, 13] considered elastically connected two and three parallel beams by dynamic stiffness analysis. Jiao et al. [14] investigated the Euler beam with an arbitrary cross section. However, most of papers only considered the clamped-free or fixed-fixed or simply supported boundary conditions. Thus the free-free boundary condition is considered and tried for both single beam and double beam in this paper. And the procedure of modeling the cable-harnessed system with the SEM is presented. Through the SEM approach, we can obtain the extremely high accurate result with a minimum number of DOFs. And it can be considered as a numerical exact solution because the SEM is based on the exact solutions of the governing differential equation of the element. Thus the SEM results are compared with FEM results and exact solution in single Euler Bernoulli beam model to validate the accuracy. Then the SEM approach has extended to double beam system. Furthermore, various connections will be considered by presented SEM model.

35.2 The SEM for Euler-Bernoulli Beam with a Free-Free Boundary Condition

A single beam can be considered as Euler-Bernoulli beam or a long slender beam if width and thickness is 10 times less than length [1]. For Euler beam, the shear deformation can be neglected. Thus we can only consider the transverse displacement. The bending moment of beam and shear force can be defined by

$$M(x,t) = EI(x)\frac{\partial^2 w(x,t)}{\partial x^2},\ V(x,t) = -\frac{\partial M(x,t)}{\partial x} = -EI(x)\frac{\partial^3 w(x,t)}{\partial x^3} \tag{35.1}$$

If EI(x) and A are assumed to be constant and the no external force exists, the free vibration of beam is governed by

$$EI\frac{\partial^4 w(x,t)}{\partial x^4} + \rho A\frac{\partial^2 w(x,t)}{\partial t^2} = 0 \tag{35.2}$$

Assume the solution of Eq. (35.2) in spectral form to be

$$w(x,t) = \frac{1}{N}\sum_{n=0}^{N-1} W_n(x)\, e^{i\omega_n t} \tag{35.3}$$

Substitute Eq. (35.3) into (35.2), then we get the eigenvalue problems at the specific frequency $\omega = \omega_n$ such as

$$EI\frac{\partial^4 W(x)}{\partial x^4} - \omega^2 \rho A W(x) = 0 \tag{35.4}$$

Assume the general solution of Eq. (35.4) as

$$W\ (x) = A\cos\beta x + B\sin\beta x + C\cosh\beta x + D\sinh\beta x \tag{35.5}$$

Substituting Eq. (35.5) into (35.4) yields the following dispersion relation equation by

$$\beta^4 - \frac{\rho A\omega^2}{EI} = 0 \tag{35.6}$$

The nodal displacement and slope at both ends can be expressed by

$$\{d\} = \begin{pmatrix} W(0) \\ \frac{\partial W}{\partial x}(0) \\ W(L) \\ \frac{\partial W}{\partial x}(L) \end{pmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \beta & 0 & \beta \\ \cos\beta L & \sin\beta L & \cosh\beta L & \sinh\beta L \\ -\beta\sin\beta L & \beta\cos\beta L & \beta\sinh\beta L & \beta\cosh\beta L \end{bmatrix}\{a\} \tag{35.7}$$

where $\{a\} = \begin{bmatrix} A & B & C & D \end{bmatrix}^T$ and

$$[D(\omega)] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \beta & 0 & \beta \\ \cos\beta L & \sin\beta L & \cosh\beta L & \sinh\beta L \\ -\beta\sin\beta L & \beta\cos\beta L & \beta\sinh\beta L & \beta\cosh\beta L \end{bmatrix} \tag{35.8}$$

And the transverse shear force and bending moments at the nodal points are given by

$$\{f\} = \begin{Bmatrix} V(0) \\ M(0) \\ -V(L) \\ -M(L) \end{Bmatrix} = EI \begin{bmatrix} 0 & -\beta^3 & 0 & \beta^3 \\ \beta^2 & 0 & -\beta^2 & 0 \\ -\beta^3\sin\beta L & \beta^3\cos\beta L & -\beta^3\sinh\beta L & -\beta^3\cosh\beta L \\ -\beta^2\cos\beta L & -\beta^2\sin\beta L & \beta^2\cosh\beta L & \beta^2\sinh\beta L \end{bmatrix} \{a\} \tag{35.9}$$

where

$$[F(\omega)] = EI \begin{bmatrix} 0 & -\beta^3 & 0 & \beta^3 \\ \beta^2 & 0 & -\beta^2 & 0 \\ -\beta^3\sin\beta L & \beta^3\cos\beta L & -\beta^3\sinh\beta L & -\beta^3\cosh\beta L \\ -\beta^2\cos\beta L & -\beta^2\sin\beta L & \beta^2\cosh\beta L & \beta^2\sinh\beta L \end{bmatrix} \tag{35.10}$$

From the Eqs. (35.8), (35.9) and (35.10), the relation f and d can be obtained such as

$$[S]\{d\} = \{f\}, \quad [S] = [F][D]^{-1} \tag{35.11}$$

Finally, the spectral element matrix of Euler Bernoulli Beam is given by

$$[S] = EI \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{12} & S_{22} & S_{23} & S_{24} \\ S_{13} & S_{23} & S_{33} & S_{34} \\ S_{14} & S_{24} & S_{34} & S_{44} \end{bmatrix} \tag{35.12}$$

where

$$\begin{aligned} \eta &= \frac{1}{1 - \cos(\beta L)\cosh(\beta L)} \\ S_{11} &= S_{33} = \eta(\beta L)^3 (\cos(\beta L)\sinh(\beta L) + \sin(\beta L)\cosh(\beta L)) \\ S_{22} &= S_{44} = \eta\beta L^3 (-\cos(\beta L)\sinh(\beta L) + \sin(\beta L)\cosh(\beta L)) \\ S_{12} &= -S_{34} = \eta\beta^2 L^3 \sin(\beta L)\sinh(\beta L) \\ S_{13} &= -\eta(\beta L)^3 (\sin(\beta L) + \sinh(\beta L)) \\ S_{14} &= -S_{23} = \eta\beta^2 L^3 (-\cos(\beta L) + \cosh(\beta L)) \\ S_{24} &= \eta\beta L^3 (-\sin(\beta L) + \sinh(\beta L)) \end{aligned} \tag{35.13}$$

After obtaining the spectral element matrix, we can assemble elements to generate the global spectral matrix. And we apply the boundary conditions to the global system. The global system can be expressed by

$$[S_g]\{d_g\} = \{f_g\} \tag{35.14}$$

Owing to usage of the exact dynamic stiffness matrix to formulate the spectral element matrix, we can solve exactly the system characteristics with minimum number of element matrix. Especially, single Euler-Bernoulli beam with free-free boundary condition case, we can obtain the exact solution with only one element and no application of boundary condition. Now, we can calculate the natural frequencies by solving the eigenvalue problem for spectral element model given by

$$\left[S_g\right]\left\{d_g\right\} = 0 \tag{35.15}$$

Table 35.1 Natural freqecies (Hz) of exact solution, SEM and FEM

Mode	ω_{exact}	$\omega_{spec.}$	ω_{FEM} n = 10	ω_{FEM} n = 30	ω_{FEM} n = 50
1	0	0	0.00001	0.00009	0.0002
2	11.1794	11.1794	11.1798	11.1794	11.1794
3	30.8165	30.8165	30.8242	30.8166	30.8165
4	60.4127	60.4127	60.4683	60.4135	60.4128
5	99.8654	99.8654	100.1052	99.8687	99.8658
6	149.5272	149.1817	149.9387	149.1928	149.1831
7	208.3612	208.3612	210.2857	208.3915	208.3652
8	277.4040	277.4040	281.5237	277.4749	277.4134

Similar to the regular eigenvalue problem, we need to consider that determinant of spectral element matrix $[S_g]$ is zero, $\det(S(\omega_{NAT})) = 0$, to find the natural frequencies. However, the spectral element matrix consists of transcendental functions such as sine, cosine, hyperbolic cosine (cosh), and hyperbolic sine (sinh). We cannot use the linear eigensolver such as 'eig' in MATLAB. The several approaches to find the eigenvalues are summarized by Lee [9]. The determinant of global stiffness matrix $[S_g]$ can be simplified as

$$\det\left[S_g\right] = \frac{E^4 I^4 \beta^8 \left(\cos\left(2\beta L\right) - \cos\left(\beta L\right)\cosh\left(\beta L\right)\right)\sin^2\left(\beta L\right)\sinh^2\left(\beta L\right)}{4\left(-1+\cos\left(\beta L\right)\cosh\left(\beta L\right)\right)^3} \tag{35.16}$$

The conventional and spectral beam elements will be compared using a simple example. A free vibration of free-free beam will be analyzed using both types of elements. The material used in the example is aluminum with elastic modulus of $7 \times 10^{10}\,\mathrm{N/m^2}$ and density of $2,700\,\mathrm{kg/m^3}$. The dimensions of beam are $1.2192 \times 0.0254 \times 0.003175$ m. From the characteristic equation of free-free beam [1], $\cos(\beta L)\cosh(\beta L) = 1$, the exact natural frequencies are calculated. And the root-finding algorithms [15] are used to find the ω_n in SEM.

Table 35.1 shows the natural frequencies of free-free beam. The SEM results are identical to exact solution. And the FEM results go closer to SEM and exact solution as the number of elements is increased. This tendency also can be notified by means of the receptance versus frequency graph in Fig. 35.1. In Fig. 35.1a, FEM gives 3 good natural frequencies. The number of elements is increased to 10 in Fig 35.1b. And the 5 natural frequencies are almost identical with SEM frequencies. Considering the high frequency range, the more elements are necessary to obtain the good results.

35.3 The SEM for Double Beam with a Free-Free Boundary Condition

Most of papers considered the double beam such that two beams are connected with distributed spring connections, not in specific location. In this paper, we can define the exact number and locations of connections. Through this approach, the effect of number and locations of connection can also be identified.

Considering the double beam system in Fig. 35.2 each beam is $2N$ d.o.f system. And the spring connection exists between ith nodes of beam 1 and beam 2. For beam 1 and beam 2, we can formulate the global stiffness matrix by means of Eq. (35.12). From the relation between nodal displacement and force, the combined global stiffness matrix without spring connection can be shown as

$$\begin{bmatrix} S_{g,1} & 0 \\ 0 & S_{g,2} \end{bmatrix} \begin{Bmatrix} \{d_{g,1}\} \\ \{d_{g,2}\} \end{Bmatrix} = \begin{Bmatrix} \{f_{g,1}\} \\ \{f_{g,2}\} \end{Bmatrix} \tag{35.17}$$

where

$$\begin{aligned} \{d_{g,1}\} &= \left[\, W_{1,1}\ \Theta_{1,1} \cdots W_{i,1}\ \Theta_{i,1} \cdots W_{N,1}\ \Theta_{N,1} \,\right]^T \\ \{d_{g,2}\} &= \left[\, W_{1,2}\ \Theta_{1,2} \cdots W_{i,2}\ \Theta_{i,2} \cdots W_{N,2}\ \Theta_{N,2} \,\right]^T \\ \{f_{g,1}\} &= \left[\, V_{1,1}\ M_{1,1} \cdots V_{i,1}\ M_{i,1} \cdots V_{N,1}\ M_{N,1} \,\right]^T \\ \{f_{g,2}\} &= \left[\, V_{1,2}\ M_{1,2} \cdots V_{i,2}\ M_{i,2} \cdots V_{N,2}\ M_{N,2} \,\right]^T \end{aligned} \tag{35.18}$$

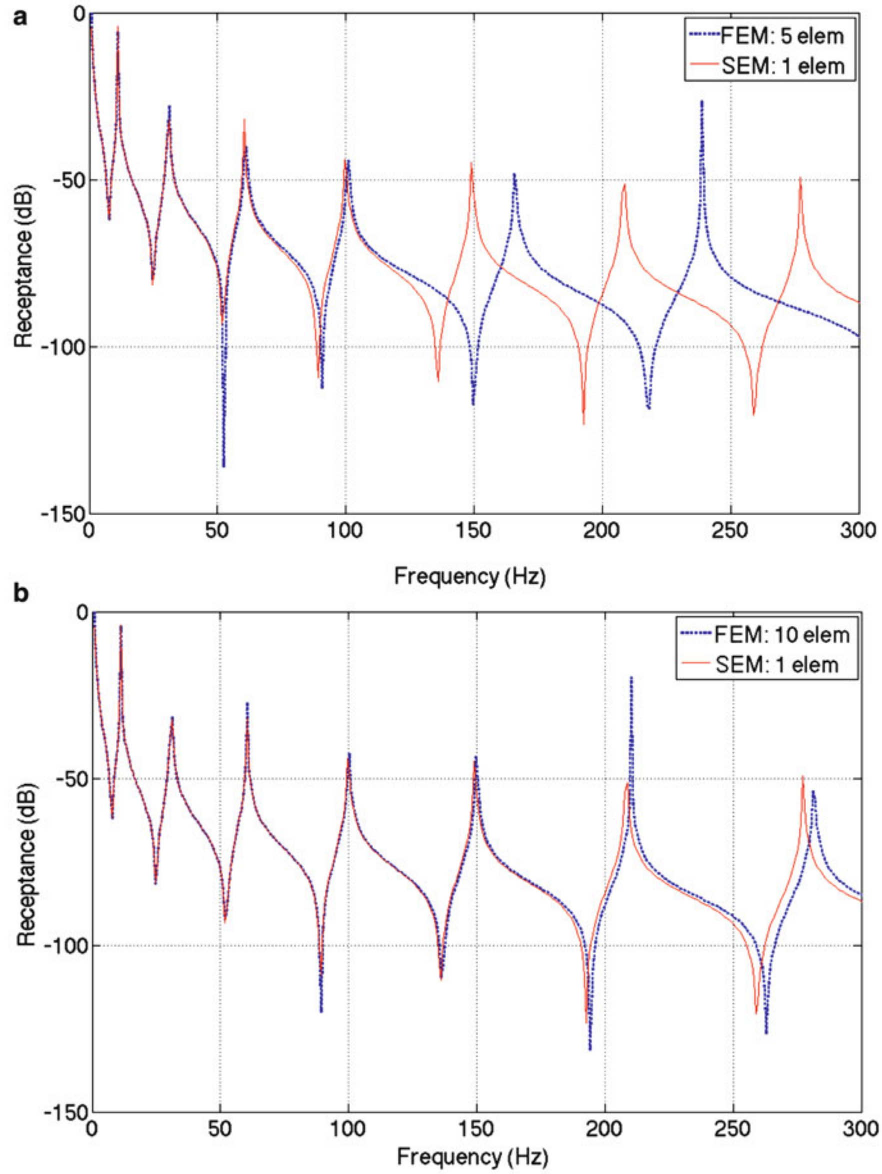

Fig. 35.1 Comparision of SEM and FEM by means of Receptance graph. (**a**) Receptance of SEM (1 element) and FEM (5 elements). (**b**) Receptance of SEM (1 element) and FEM (10 elements)

The global stiffness matrix $S_{g,1}$ and $S_{g,2}$ are $2N \times 2N$ matrices and the global displacement and force vector $d_{g,1}$, $d_{g,2}$, $f_{g,1}$ and $f_{g,2}$ are $2N \times 1$ vectors. If the force f_k is working on between ith nodes of beam 1 and beam 2 due to the spring connection, f_k can be given by

$$f_k = k(W_{i,1} - W_{i,2}) \tag{35.19}$$

The total combined system can be expressed as

$$\begin{aligned} [S_{g,1}]\{d_{g,1}\} + f_k &= \{f_{g,1}\} \\ [S_{g,2}]\{d_{g,2}\} - f_k &= \{f_{g,2}\} \end{aligned} \tag{35.20}$$

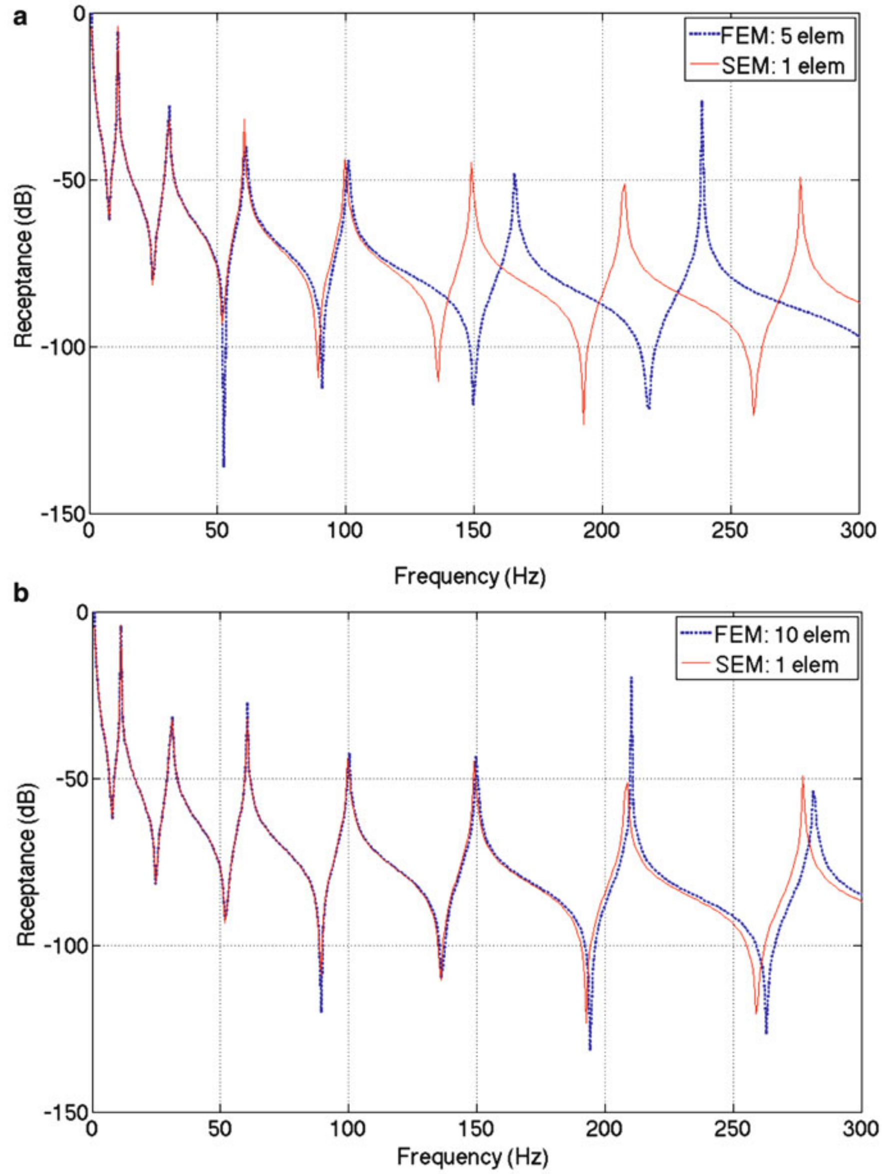

Fig. 35.2 Double beam model

Table 35.2 Material properties of two beams

–	E(N/m^3)	I(m^4)	A(m^2)	ρ(kg/m^3)	L (m)
Beam1	7×10^{10}	6.7746×10^{-11}	8.0645×10^{-5}	2,700	1.2192
Beam2	2.333×10^{10}	2.7098×10^{-12}	4.03225×10^{-5}	540	1.2192

Let's introduce the locator vector $\{L_i\}$ The vector $\{L_i\}$ is $4N \times 1$ vector correspondent with $[d_{g,1}\, d_{g,2}]^T$. The $2i-1$th component of $\{L_i\}$ is 1 and the $2N+2i-1$th component of $\{L_i\}$ is -1 and the remainders are zero.

$$\begin{bmatrix} S_{g,1} & 0 \\ 0 & S_{g,2} \end{bmatrix} \begin{Bmatrix} d_{g,1} \\ d_{g,2} \end{Bmatrix} + \sum_{i=1}^{p} k_i \{L_i\}\{L_i\}^T \begin{Bmatrix} d_{g,1} \\ d_{g,2} \end{Bmatrix} = \begin{Bmatrix} f_{g,1} \\ f_{g,2} \end{Bmatrix} \tag{35.21}$$

where p is the total number of connections. The FEM and SEM will be compared using a double beam example. A free vibration of free-free double beam will be analyzed for several connection cases. The material properties of both beams are summarized in Table 35.2. And the spring constant k is 10^5 N/m. And the root-finding algorithms [15] are utilized to find the ω_n in the SEM.

Table 35.3 Natural frequencies (Hz) for double beam with several connections

	1 spring connection		2 spring connections		3 spring connections		5 spring connections	
Mode	FEM 128	SEM 2	FEM 120	SEM 3	FEM 128	SEM 4	FEM 120	SEM 6
1	0.0024	0.0213	0.0027	0.0124	0.0025	0.0119	0.0006	0.0053
2	2.6341	2.6341	3.3250	3.3250	6.8085	6.8085	10.5753	10.5753
3	10.7977	10.7977	4.6189	4.6189	7.1844	7.1844	15.7602	15.7603
4	11.2526	11.2526	10.9921	10.9921	11.1687	11.1687	16.2981	16.2982
5	16.5676	16.5679	21.1613	21.1613	29.5254	29.5254	30.1176	30.1176
6	30.8166	30.8166	27.9176	27.9176	34.6464	34.6464	58.0927	58.0927
7	36.4657	36.4648	32.4038	32.4037	44.8028	44.8028	69.8780	69.8780
8	44.9628	44.9938	37.1127	37.1016	58.4510	58.4506	84.7031	84.7031
9	59.9995	59.9995	60.3813	60.3813	58.8538	58.8545	94.2957	94.2947
10	76.0828	76.0828	73.9227	73.9226	66.7777	66.7782	106.1145	106.1145
11	88.7920	88.8003	90.6776	90.6776	99.6240	99.6240	121.6623	121.6622

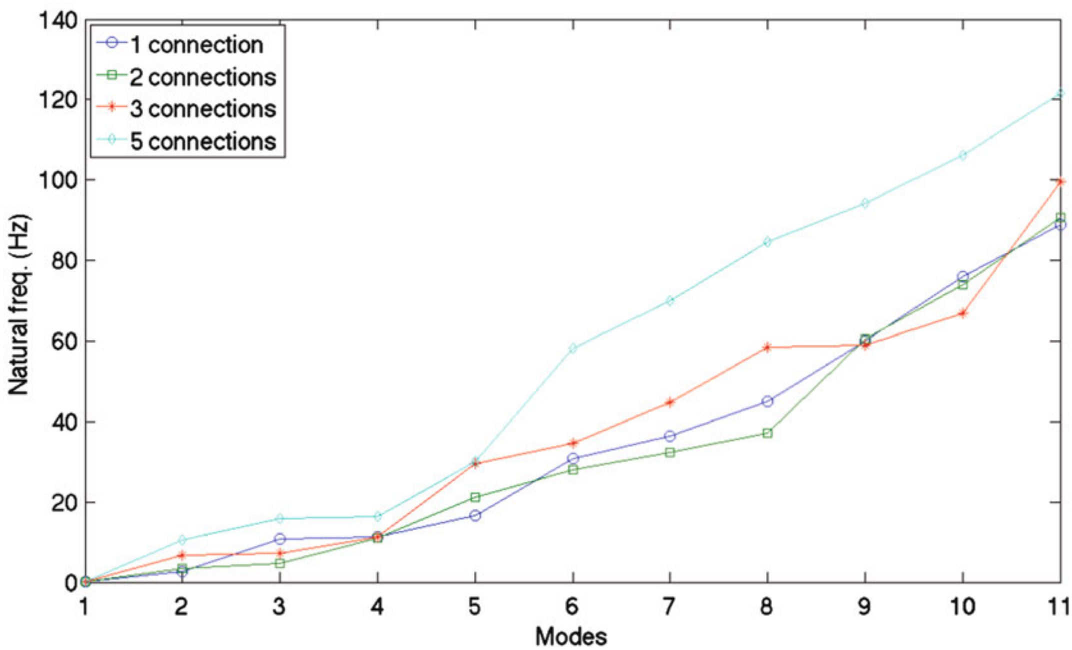

Fig. 35.3 Natural frequencies of 11 modes for each connection case

The double beam system with 1, 2, 3, and 5 connections has simulated by means of material properties in Table 35.2. The result from 1st ∼ 11th modes are summarized in Table 35.3. The SEM uses minimum number of element to obtain the exact natural frequencies. As the connections are added, the natural frequencies increase as in Fig. 35.3.

In (a) of Fig. 35.4 ∼ 35.7, the small number of elements are used in the FEM, the difference between the FEM and the SEM goes bigger in higher frequency range. About 10 times more number of elements are used in (b) of Fig. 35.4 ∼ 35.7. Frequencies by the FEM are very close with the SEM results.

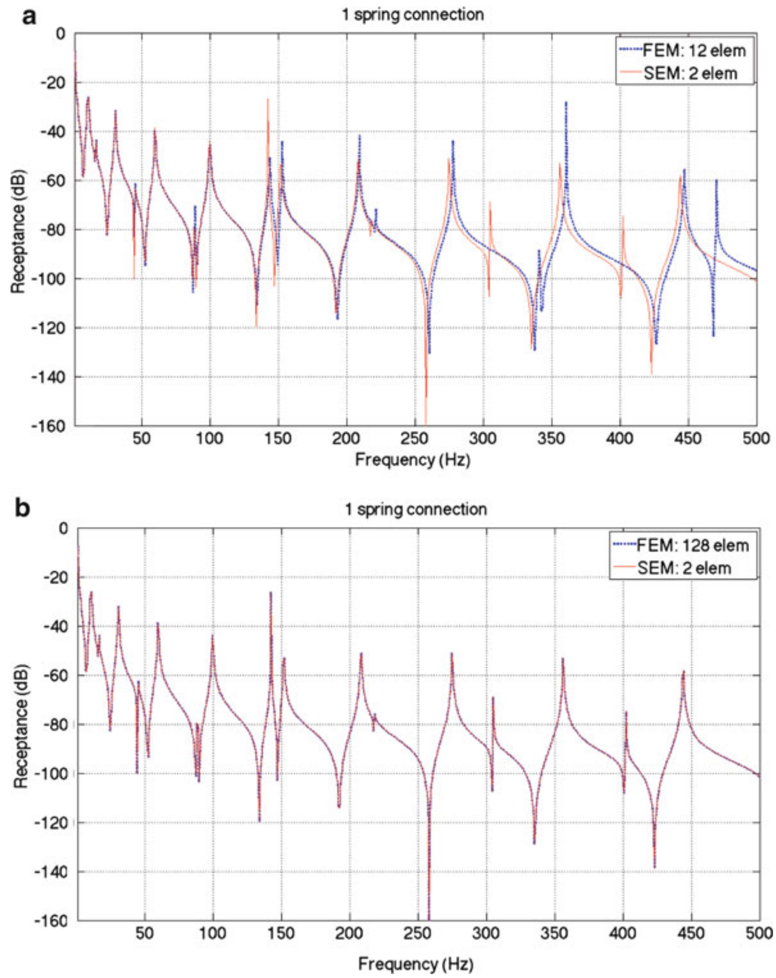

Fig. 35.4 Receptance versus frequency graph of double beam with 1 connections. (**a**) Receptance of SEM (2 element) and FEM (12 elements) (**b**) Receptance of SEM (2 element) and FEM (128 elements)

35.4 Conclusion

The SEM is presented as the numerical exact solution to predict the response of a cable harnessed structure. The single and double beam cases are considered to validate the exactness of the SEM. In all cases, the comparisons are performed by means of calculated natural frequencies and the receptance graphs. The SEM shows the exact solution, requiring fewer elements than the FEM, with the minimum number of degree-of-freedoms. Moreover, the SEM allows us to define the locations very conveniently. Through this convenience, the effect of number of connections is also investigated. As the number of connections increases, the natural frequencies increase. The effectiveness and accuracy of the proposed SEM are demonstrated with several numerical examples.

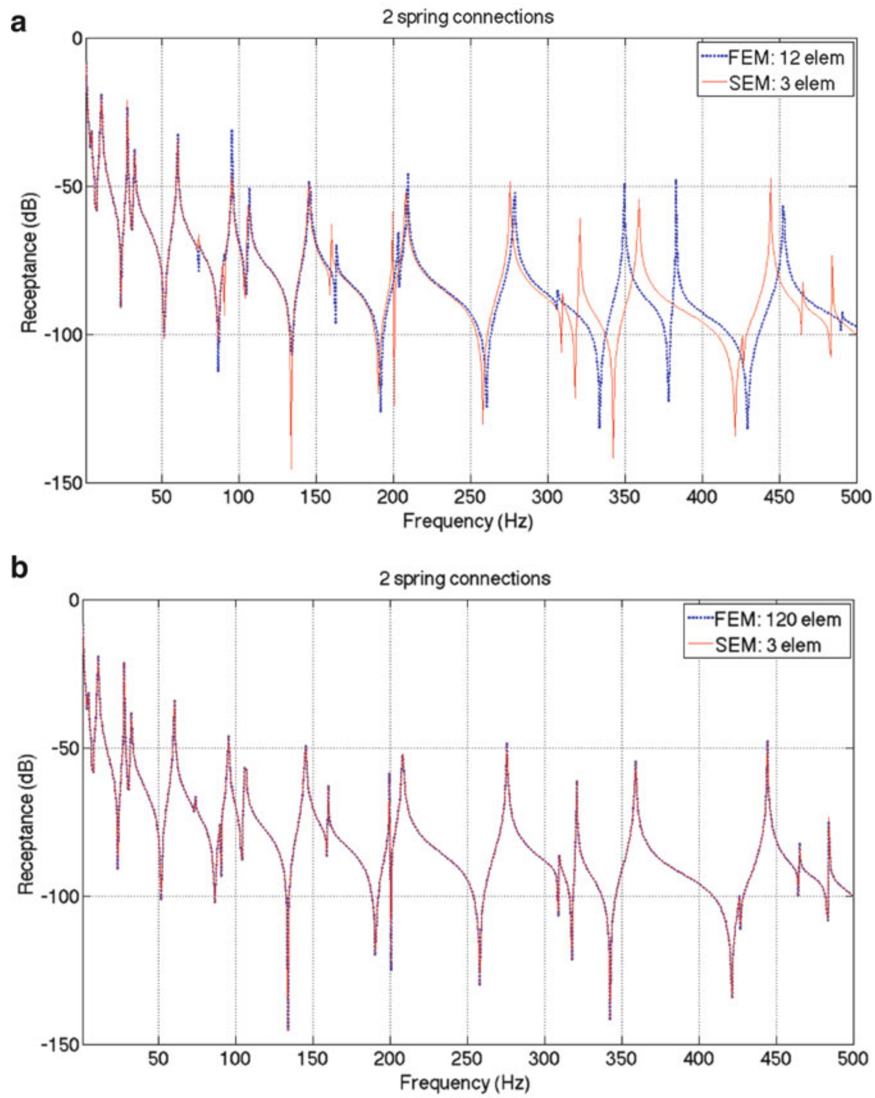

Fig. 35.5 Receptance versus frequency graph of double beam with 3 connections. (**a**) Receptance of SEM (3 element) and FEM (12 elements). (**b**) Receptance of SEM (3 element) and FEM (120 elements)

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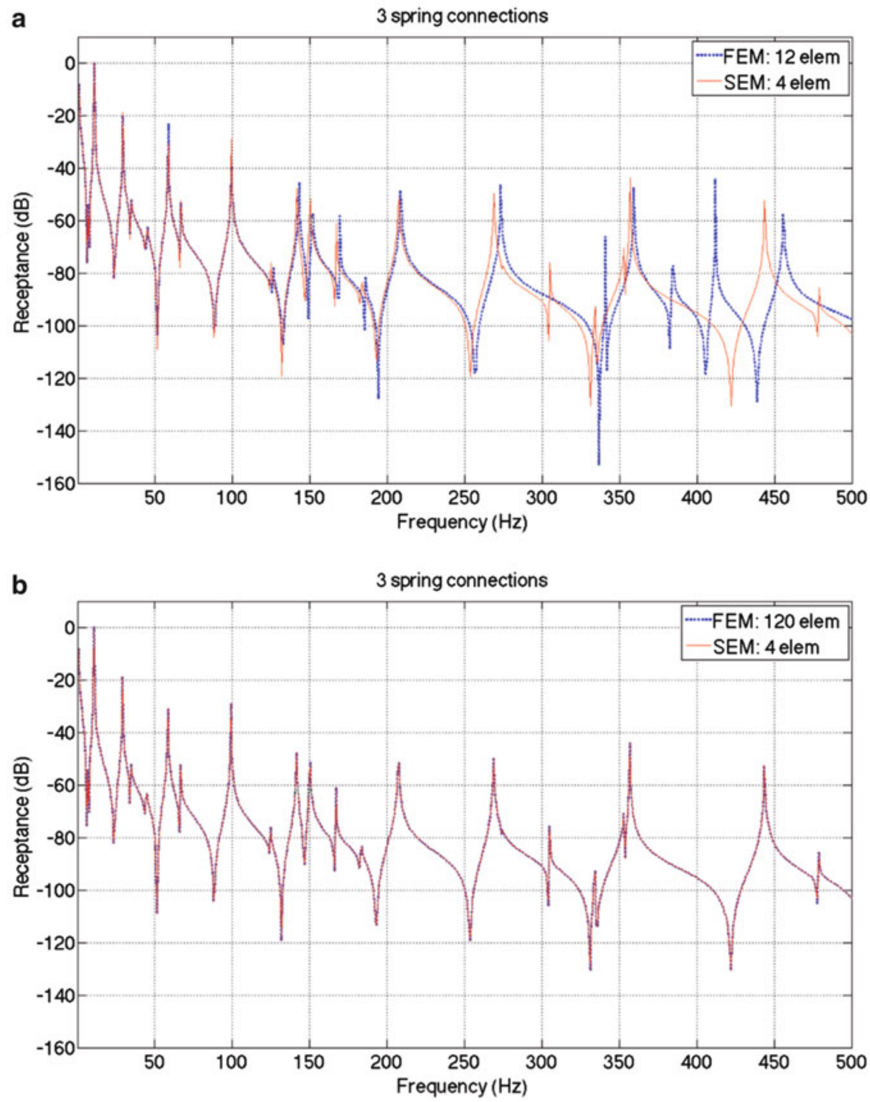

Fig. 35.6 Receptance versus frequency graph of double beam with 3 connections. (**a**) Receptance of SEM (4 element) and FEM (12 elements). (**b**) Receptance of SEM (4 element) and FEM (120 elements)

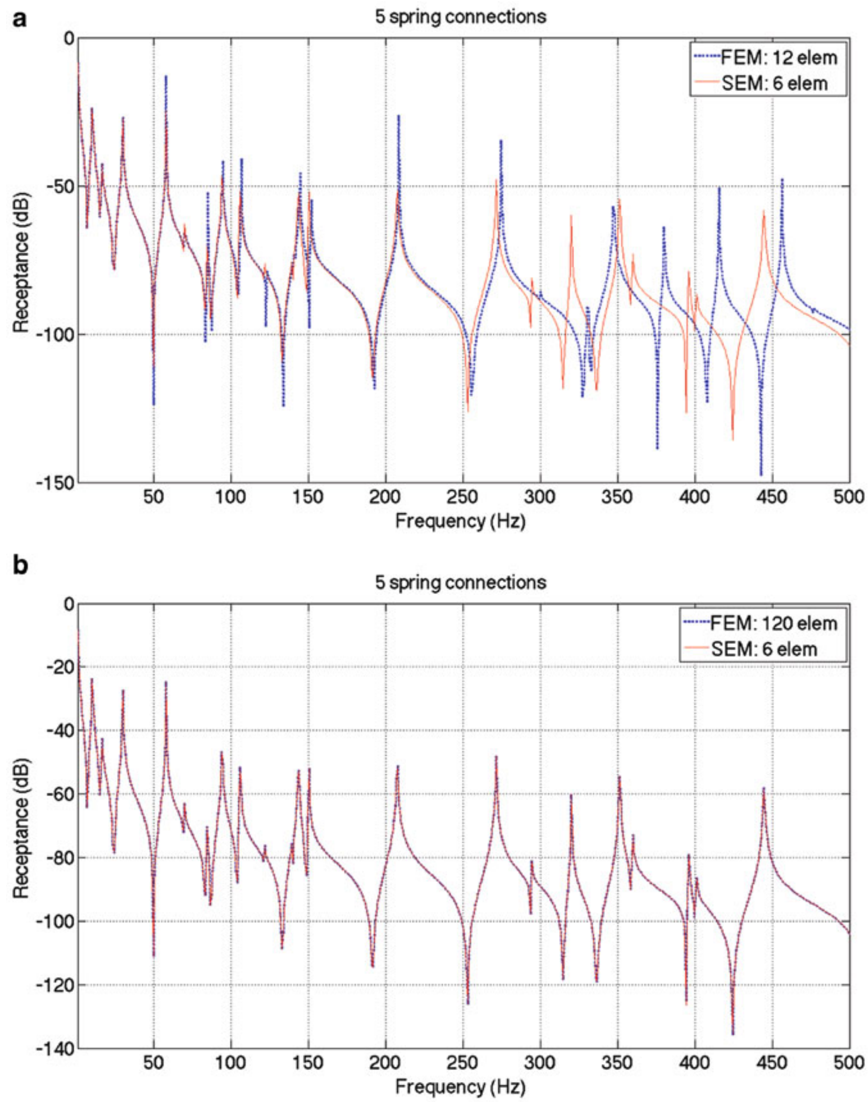

Fig. 35.7 Receptance versus frequency graph of double beam with 5 connections. (**a**) Receptance of SEM (6 element) and FEM (12 elements). (**b**) Receptance of SEM (6 element) and FEM (120 elements)