On a Stability Property of the Generalized Spherical Radon Transform

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Abstract In this note, we study the operator norm of the generalized spherical Radon transform, defined by a smooth measure on the underlying incidence variety. In particular, we prove that for small perturbations of the measure, the spherical Radon transform remains an isomorphism between the corresponding Sobolev spaces.

Key words Radon transform • Sobolev spaces • Pseudodifferential operators • Integral geometry

Mathematical Subject Classifications (2010): 44A12, 53C65

1 Introduction and Background

Throughout the note, we fix a Euclidean space $V = \mathbb{R}^{d+1}$, and consider the Euclidean spheres $X = S^d \subset V$, and $Y = S^d \subset V^*$. For $p \in Y$, $C_p \subset X$ will denote the copy of $S^{d-1} \subset X$ given by $C_p = \{q \in X : \langle q, p \rangle = 0\}$. Let $\sigma_{d-1}(q)$ denote the $SO(d)$-invariant probability measure on C_p. The set $C_q \subset Y$ and the measure $\sigma_{d-1}(p)$ on it are defined similarly. Then the spherical Radon transform is defined as follows:

$$\mathcal{R} : C^\infty(X) \to C^\infty(Y)$$
$$\mathcal{R}f(p) = \int_{C_p} f(q) d\sigma_{d-1}(q).$$

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Let σ be the unique $SO(d+1)$-invariant probability measure on the incidence variety $Z = \{(q, p) \in X \times Y : \langle q, p\rangle = 0\}$. Assume one is given a smooth, not necessarily positive measure $d\mu$ on Z, given by $\mu(q, p)\sigma$ where $\mu \in C^\infty(Z)$, and which satisfies $\mu(\pm q, \pm p) = \mu(q, p)$ (call such μ symmetric). Introduce $\mathcal{R}_\mu : C^\infty(X) \to C^\infty(Y)$ by

$$(\mathcal{R}_\mu f)(p) = \int_{C_p} f(q)\mu(q, p)d\sigma_{d-1}(q).$$

Introduce also the dual Radon transform $\mathcal{R}_\mu^T : C^\infty(Y) \to C^\infty(X)$ which is formally adjoint to $\mathcal{R}_\mu$ and given by

$$(\mathcal{R}_\mu^T g)(q) = \int_{C_q} g(p)\mu(q, p)d\sigma_{d-1}(p).$$

Let $L_s^2(\mathbb{P}X)$ and $L_s^2(\mathbb{P}Y)$ denote the Sobolev space of even functions on X and Y, respectively. It is well known (see [2]) that the spherical Radon transform extends to an isomorphism of Sobolev spaces:

$$\mathcal{R} : L_s^2(\mathbb{P}X) \to L^2_{s+\frac{d-1}{2}}(\mathbb{P}Y)$$

for every $s \in \mathbb{R}$. For general μ as above, $\mathcal{R}_\mu$ is a Fourier integral operator of order $\frac{d-1}{2}$ (see [3, 5]), and so extends to a bounded map $\mathcal{R}_\mu : L_s^2(\mathbb{P}X) \to L^2_{s+\frac{d-1}{2}}(\mathbb{P}Y)$. We look for conditions on μ so that this is again an isomorphism.

It follows from Guillemin's theorem on general Radon transforms associated to double fibrations [1], that $\mathcal{R}_\mu^T\mathcal{R}_\mu : C^\infty(\mathbb{P}X) \to C^\infty(\mathbb{P}X)$ is an elliptic pseudo-differential operator of order $d-1$ for all smooth, positive, symmetric measures μ on Z (for completeness, this is verified in the Appendix). The dependence of the principal symbol of $\mathcal{R}_\mu^T\mathcal{R}_\mu$ on μ was investigated in [5]. In this note, we analyze the dependence on μ of the operator norm of $\mathcal{R}_\mu^T\mathcal{R}_\mu : L_s^2(\mathbb{P}X) \to L^2_{s+(d-1)}(\mathbb{P}X)$. We then give a sufficient condition on a perturbation μ of μ_0 so that $\mathcal{R}_\mu : C^\infty(\mathbb{P}X) \to C^\infty(\mathbb{P}Y)$ remains an isomorphism. Namely, we prove the following

Theorem. *The set of C^∞ measures μ on Z for which $\mathcal{R}_\mu : C^\infty(\mathbb{P}X) \to C^\infty(\mathbb{P}Y)$ is an isomorphism, is open in the $C^{2d+1}(Z)$ topology.*

2 Bounding the Norm of $\mathcal{R}_\nu^T\mathcal{R}_\mu$

We start by recalling an equivalent description of the Radon transform. Consider the double fibration

$$X \xleftarrow{\;\pi\;} Z \xrightarrow{\;\rho\;} Y$$

Let $\sigma_X = \pi_* \sigma$ and $\sigma_Y = \rho_* \sigma$ be the rotation-invariant probability measures on X and Y, respectively. Then for $f \in C^\infty(X)$, $(\mathcal{R} f)\sigma_Y = \rho_*(\sigma(\pi^* f))$. For smooth symmetric measures $d\mu$, $d\nu$ on Z, given by $\mu(q,p)\sigma$ and $\nu(q,p)\sigma$ we can define $\mathcal{R}_\mu : C^\infty(X) \to C^\infty(Y)$, $\mathcal{R}_\nu^T : C^\infty(Y) \to C^\infty(X)$ respectively by

$$(\mathcal{R}_\mu f)\sigma_Y = \rho_*(\mu\sigma(\pi^* f))$$

and

$$(\mathcal{R}_\nu^T g)\sigma_X = \pi_*(\nu\sigma(\rho^* g)).$$

It follows from [1] that both $\mathcal{R}_\mu$ and $\mathcal{R}_\nu^T$ are Fourier integral operators of order $\frac{d-1}{2}$. Thus we restrict to even functions and consider $\mathcal{R}_\mu : L^2_s(\mathbb{P}X) \to L^2_{s+\frac{d-1}{2}}(\mathbb{P}Y)$ and $\mathcal{R}_\nu^T : L^2_{s+\frac{d-1}{2}}(\mathbb{P}Y) \to L^2_{s+(d-1)}(\mathbb{P}X)$.

As before, q will denote a point in X and p a point in Y. We will often write $q \in p$ instead of $\langle q, p\rangle = 0 \iff q \in C_p \iff p \in C_q$. In the following, the functions f, g are even. We also assume $d \geq 2$.

Proposition 1. *The Schwartz kernel of* $\mathcal{R}_\nu^T \mathcal{R}_\mu : L^2_s(\mathbb{P}X) \to L^2_{s+(d-1)}(\mathbb{P}X)$ *is*

$$K(q',q) = \frac{c_d}{\sin \operatorname{dist}(q',q)} \alpha(q,q')$$

that is,

$$\mathcal{R}_\nu^T \mathcal{R}_\mu f(q') = \int_X f(q) K(q',q)\, d\sigma_X(q).$$

Here c_d *is a constant, and* $\alpha(q,q')$ *is the average over all* $p \in Y$ *s.t.* $q, q' \in p$ *of* $\mu(q,p)\nu(q',p)$. *More precisely,*

$$\alpha(q,q') = \int_{SO(d-1)} \mu(q, Mp_0)\nu(q', Mp_0)\, dM$$

where $SO(d-1) = \{g \in SO(d+1) : gq = q, gq' = q'\}$, C_{p_0} *is any fixed copy of* S^d *through* q, q', *and* dM *is the Haar probability measure on* $SO(d-1)$.

Proof. Fix some $q' \in X$, and $p_0 \in Y$ s.t. $q' \in p_0$. Let $SO(d) \subset SO(d+1)$ be the stabilizer of $q' \in X$. For $g \in C^\infty(Y)$ we may write

$$\mathcal{R}_\nu^T g(q') = \int_{p \ni q'} g(p)\nu(q',p)\, d\sigma_{d-1}(p) = \int_{SO(d)} g(Mp_0)\nu(q', Mp_0)\, dM$$

where dM is the Haar probability measure on $SO(d)$. Then taking

$$g(p) = \mathcal{R}_\mu f(p) = \int_{\tilde{q}\in p} f(\tilde{q})\mu(\tilde{q}, p)d\sigma_{d-1}(\tilde{q})$$

we get

$$\begin{aligned}
\mathcal{R}_\nu^T \mathcal{R}_\mu f(q') &= \int_{SO(d)} \Big(\int_{\tilde{q}\in Mp_0} f(\tilde{q})\mu(\tilde{q}, Mp_0)d\sigma_{d-1}(\tilde{q})\Big)\nu(q', Mp_0)dM \\
&= \int_{SO(d)} \Big(\int_{\tilde{q}\in p_0} f(M\tilde{q})\mu(M\tilde{q}, Mp_0)d\sigma_{d-1}(\tilde{q})\Big)\nu(q', Mp_0)dM \\
&= \int_{\tilde{q}\in p_0} d\sigma_{d-1}(\tilde{q}) \int_{SO(d)} f(M\tilde{q})\mu(M\tilde{q}, Mp_0)\nu(q', Mp_0)dM.
\end{aligned}$$

Denote $\theta = \mathrm{dist}(\tilde{q}, q')$, and $S_\theta^{d-1} = \{q : \mathrm{dist}(q', q) = \theta\}$. Let $d\sigma_{d-1}^\theta(q)$ denote the rotationally invariant probability measure on S_θ^{d-1}. The inner integral may be written as

$$\int_{S_\theta^{d-1}} f(q)\alpha(q, q')d\sigma_{d-1}^\theta(q).$$

Here $\alpha(q, q') = \int_{SO(d-1)} \mu(q, Mp_0)\nu(q', Mp_0)dM$ with $SO(d-1) = Stab(q) \cap Stab(q_0)$ is just the average of $\mu(q, p)\nu(q', p)$ over all $(d-1)$-dimensional spheres C_p containing both q and q'. Then

$$\mathcal{R}_\nu^T \mathcal{R}_\mu f(q') = \int_{\tilde{q}\in p_0} d\sigma_{d-1}(\tilde{q}) \int_{S_\theta^{d-1}} f(q)\alpha(q, q')d\sigma_{d-1}^\theta(q)$$

and since the inner integral only depends on $\theta = \mathrm{dist}(\tilde{q}, q')$, this may be rewritten as

$$c_d \int_0^{\pi/2} d\theta \sin^{d-2}\theta \int_{S_\theta^{d-1}} f(q)\alpha(q, q')d\sigma_{d-1}^\theta(q).$$

Finally, $d\sigma_d = c_d \sin^{d-1}\theta d\theta d\sigma_{d-1}^\theta$, and so

$$\mathcal{R}_\nu^T \mathcal{R}_\mu f(q') = c_d \int_X \frac{1}{\sin\theta} f(q)\alpha(q, q')d\sigma_d(q).$$

We conclude that the Schwartz kernel is

$$K(q', q) = \frac{c_d}{\sin \mathrm{dist}(q', q)}\alpha(q, q').$$

□

We proceed to estimate the norm of $\mathcal{R}_\nu^T \mathcal{R}_\mu$. Our main tool will be the following proposition proved in Sect. C

Proposition. *Consider a pseudodifferential operator P of order m*

$$P : L^2_{s+m}(\mathbb{R}^n) \to L^2_s(\mathbb{R}^n)$$

between Sobolev spaces with $x-$compactly supported symbol $p(x,\xi)$ in $K \subset \mathbb{R}^n$ s.t.

$$|D^\alpha_x p(x,\xi)| \le C_{\alpha 0}(1+|\xi|)^\alpha.$$

There exists a constant $C(n,s)$ such that

$$\|P\|_{L^2_{s+m}(\mathbb{R}^n)\to L^2_s(\mathbb{R}^n)} \le C(n,s) \sup_{|\alpha|\le n+\lfloor |s|\rfloor+1} C_{\alpha 0}|K|.$$

Proposition 2. *The norm of $\mathcal{R}^T_\nu \mathcal{R}_\mu : L^2_{-(d-1)}(\mathbb{P}X) \to L^2_0(\mathbb{P}X)$ is bounded from above by*

$$\|\mathcal{R}^T_\nu \mathcal{R}_\mu\| \le C \sum_{j+k=0}^{2d+1} \|D^j\mu\|_\infty \|D^k\nu\|_\infty$$

for some constant C dependent on the double fibration.

Proof. First introduce coordinate charts. Choose a partition of unity $\chi_i(q')$ corresponding to a covering of X by charts U_i, and a function $\rho : [0,\infty) \to \mathbb{R}_+$ with support in [0, 1] s.t. $\rho(r) = 1$ for $r \le \frac{1}{2}$. Write

$$K(q',q) = \sum_i K_i(q',q) + L_i(q',q)$$

$$K_i(q',q) = \chi_i(q')\rho(\sin \operatorname{dist}(q',q))K(q',q)$$

and

$$L_i(q',q) = \chi_i(q')(1-\rho(\sin \operatorname{dist}(q',q)))K(q',q).$$

Let $\mathcal{R}^T_\nu \mathcal{R}_\mu = \sum_i T_{K_i} + T_{L_i}$ be the corresponding decomposition for the operators.

First we will bound the norm of the diagonal terms, i.e., the operators defined by K_i. Fix i, and choose some point $q' \in U_i$. Introduce polar coordinates (r,ψ) around q' so that $\psi \in S^{d-1}_1(q')$ and $r = \sin\theta$ for $r \le \frac{1}{2}$, $\theta = \operatorname{dist}(q,q')$. Note that $\alpha(q',(r,\psi)) = \alpha(q',(r,-\psi))$. By Proposition 5, the corresponding symbol is

$$p_1(q',\xi) = \chi_i(q') \int_0^1 \int_{S^{d-1}} \frac{\alpha(q',r\psi)}{r} e^{-i\langle \xi,\psi\rangle} \rho(r) r^{d-1} dr d\psi.$$

For a given q', introduce spherical coordinates $\psi = (\phi,\phi_1,\dots,\phi_{d-2})$ $0 \le \phi \le \pi$, on $S^{d-1}_1(q')$ in such a way that $\cos\phi = \psi_1$; Take $\xi_0 = (1,0,\dots,0)$. Then

$$p_1(q',T\xi_0) = C_d \int_0^1 \rho(r) r^{d-2} dr \int_0^\pi m(q',r,\phi) e^{-iTr\cos\phi} \cos^{d-2}\phi d\phi$$

where $m(q', r, \phi) = \chi_i(q') \int_0^\pi \dots \int_0^\pi \int_0^{2\pi} d\phi_1 \dots d\phi_{d-2} \alpha(q', r, \phi, \phi_1, \dots, \phi_{d-2})$. We then have $m(q', r, \pi/2 + \phi) = m(q', r, \pi/2 - \phi)$. Take $t = \cos\phi$ and $M(q', r, t) = m(q', r, \arccos t)$. Then $M(q', r, t) = M(q', r, -t)$ and

$$p_1(q', T\xi_0) = C_d \int_0^1 \rho(r) r^{d-2} dr \int_{-1}^1 M(q', r, t) e^{-iTrt} (1 - t^2)^{\frac{d-3}{2}} dt.$$

Since M is even, we may write

$$p_1(q', T\xi_0) = 2 \int_0^1 \rho(r) r^{d-2} dr \int_0^1 M(q', r, t) \cos(Trt) (1 - t^2)^{\frac{d-3}{2}} dt$$

and so for all multi-indices α

$$D_{q'}^\alpha p_1(q', T\xi_0) = 2 \int_0^1 \rho(r) r^{d-2} dr \int_0^1 D_{q'}^\alpha M(q', r, t) \cos(Trt) (1 - t^2)^{\frac{d-3}{2}} dt.$$

Define as in Appendix A

$$I(d, \rho, m) = \int_0^1 \rho(r) r^{d-2} dr \int_0^1 M(q', r, t) \cos(Trt) (1 - t^2)^{\frac{d-3}{2}} dt$$

then we can write

$$D_{q'}^\alpha p_1(q', T\xi_0) = 2I(d, \rho, D_{q'}^\alpha m(q', r, \phi))$$

and conclude by Proposition 3 that

$$\begin{aligned} |p_1(q', T\xi_0)| &\le \frac{C}{T^{d-1}} \sum_{\substack{a + b \le d \\ a \le d/2, b \le d - 1}} \sup \left| \frac{\partial^{a+b}}{\partial r^a \partial \phi^b} m \right| \\ &\le \frac{C}{T^{d-1}} \sum_{j+k \le d} \|D^j \mu\|_\infty \|D^k \nu\|_\infty \end{aligned}$$

and similarly for all multi-indices α

$$|D_{q'}^\alpha p_1(q', T\xi_0)| \le \frac{C}{T^{d-1}} \sum_{j+k \le |\alpha| + d} \|D^j \mu\|_\infty \|D^k \nu\|_\infty.$$

It is also immediate that

$$|p_1(q', T\xi_0)| \le C_d \int_0^1 \rho(r) r^{d-2} dr \int_0^\pi |m(q', r, \phi)| d\phi \le C \|\mu\|_\infty \|\nu\|_\infty$$

and similarly

$$|D^{\alpha}_{q'} p_1(q', T\xi_0)| \leq C \sum_{j+k\leq|\alpha|} \|D^j\mu\|_\infty \|D^k\nu\|_\infty.$$

So we can write

$$|D^{\alpha}_{q'} p_1(q', \xi)| \leq \frac{C}{(1+|\xi|)^{d-1}} \sum_{j+k\leq|\alpha|+d} \|D^j\mu\|_\infty \|D^k\nu\|_\infty$$

for some universal constant $C = C(d)$. Then choosing $s = 0$ and $|\alpha| = d+1$ in Proposition 4 we get that

$$\|T_{K_i}\|_{L^2_{-(d-1)}(\mathbb{P}X)\to L^2_0(\mathbb{P}X)} \leq C \sum_{j+k=0}^{2d+1} \|D^j\mu\|_\infty \|D^k\nu\|_\infty.$$

Now we bound the norm of the off-diagonal term, namely the sum of operators corresponding to L_i. They constitute a smoothing operator $T_L(\mu, \nu)$; its Schwartz kernel $k(q', q) = (1 - \rho(\mathrm{dist}(q', q)))K(q', q)$ is a smooth function in both arguments. Denoting by $\nabla^j : C^\infty(X) \to (T^*X)^{\otimes j}$ the j-th derivative obtained from the Levi-Civita connection,

$$\begin{aligned}
&\left\| \int_X d\sigma_X(q) f(q) k(q', q) \right\|^2_{L^2_{d-1}(\mathbb{P}X)} \\
&= \int_X \left| \nabla^{d-1}_{q'} \int_X d\sigma_X(q) f(q) k(q', q) \right|^2 d\sigma_X(q') \\
&= \int_X \left| \int_X d\sigma_X(q) f(q) \nabla^{d-1}_{q'} k(q', q) \right|^2 d\sigma_X(q') \\
&\leq \int_X d\sigma_X(q') \left(\int_X |f(q)^2| d\sigma_X(q) \int_X |\nabla^{d-1}_{q'} k(q', q)|^2 d\sigma_X(q) \right) \\
&= \|f\|^2_{L^2(X)} \int_X |\nabla^{d-1}_{q'} k(q', q)|^2 d\sigma_X(q)
\end{aligned}$$

and

$$\sup_{q'} \sqrt{\int_X |\nabla^{d-1}_{q'} k(q', q)|^2 d\sigma_X(q)} \leq C \sum_{j+k\leq d-1} \|D^j\mu\|_\infty \|D^k\nu\|_\infty.$$

So

$$\|T_L(\mu, \nu)\|_{L^2_0(\mathbb{P}X)\to L^2_{d-1}(\mathbb{P}X)} \leq C \sum_{j+k\leq d-1} \|D^j\mu\|_\infty \|D^k\nu\|_\infty.$$

It is easy to see that the adjoint operator $T_L(\mu,\nu)^* : L^2_{(d-1)}(\mathbb{P}X)^* \to L^2_0(\mathbb{P}X)^*$ equals $T_L(\nu,\mu)$ after the isomorphic identification $L^2_s(\mathbb{P}X)^* \simeq L^2_{-s}(\mathbb{P}X)$ for $s = 0, d-1$. Since the bound above is symmetric in μ,ν we conclude

$$\|T_L(\mu,\nu)\|_{L^2_{-(d-1)}(\mathbb{P}X)\to L^2_0(\mathbb{P}X)} \leq C \sum_{j+k\leq d-1} \|D^j\mu\|_\infty \|D^k\nu\|_\infty.$$

Finally

$$\|\mathcal{R}^T_\nu \mathcal{R}_\mu\| \leq \|\sum_i T_{K_i}\| + \|T_L\| \leq C \sum_{j+k=0}^{2d+1} \|D^j\mu\|_\infty \|D^k\nu\|_\infty.$$

□

Theorem 1. *Assume $d \geq 2$, and let $\mu_0 \in C^\infty(Z)$ be such that $\mathcal{R}_{\mu_0} : C^\infty(\mathbb{P}X) \to C^\infty(\mathbb{P}Y)$ is an isomorphism. Then there exists $\epsilon_0 > 0$ (depending on the double fibration), such that if $\|\mu-\mu_0\|_{C^{2d+1}(Z)} < \epsilon_0$ then $\mathcal{R}_\mu : C^\infty(\mathbb{P}X) \to C^\infty(\mathbb{P}Y)$ is an isomorphism (for all s).*

Proof. Since $\mathcal{R}^T_{\mu_0}\mathcal{R}_{\mu_0} : L^2_{-(d-1)}(\mathbb{P}X) \to L^2_0(\mathbb{P}X)$ (and likewise for Y) is elliptic, it is an isomorphism. Let us verify that both of the maps $\mathcal{R}^T_\mu\mathcal{R}_\mu : L^2_{-(d-1)}(\mathbb{P}X) \to L^2_0(\mathbb{P}X)$ and $\mathcal{R}_\mu\mathcal{R}^T_\mu : L^2_{-(d-1)}(\mathbb{P}Y) \to L^2_0(\mathbb{P}Y)$ remain an isomorphism for small perturbations μ of μ_0 in the $C^{2d+1}(Z)$ norm:

$$\begin{aligned}\|\mathcal{R}^T_\mu\mathcal{R}_\mu - \mathcal{R}^T_{\mu_0}\mathcal{R}_{\mu_0}\| &= \|(\mathcal{R}^T_{\mu_0} + \mathcal{R}^T_{\mu-\mu_0})(\mathcal{R}_{\mu_0} + \mathcal{R}_{\mu-\mu_0}) - \mathcal{R}^T_{\mu_0}\mathcal{R}_{\mu_0}\| \\ &\leq \|\mathcal{R}^T_{\mu_0}\mathcal{R}_{\mu-\mu_0}\| + \|\mathcal{R}^T_{\mu-\mu_0}\mathcal{R}_{\mu_0}\| + \|\mathcal{R}^T_{\mu-\mu_0}\mathcal{R}_{\mu-\mu_0}\|\end{aligned}$$

so by Corollary 2, there is an $\epsilon_0 > 0$ s.t. all norms are indeed small when $\|\mu-\mu_0\|_{C^{2d+1}(Z)} < \epsilon_0$. The operator $\mathcal{R}_\mu\mathcal{R}^T_\mu$ is treated identically, and we take the minimal of the two ϵ_0.

Since both $\mathcal{R}^T_\mu\mathcal{R}_\mu$ and $\mathcal{R}_\mu\mathcal{R}^T_\mu$ are elliptic operators, the dimension of the kernel and cokernel are independent of s. It follows that $\mathcal{R}^T_\mu\mathcal{R}_\mu : L^2_s(\mathbb{P}X) \to L^2_{s+d-1}(\mathbb{P}X)$ and $\mathcal{R}_\mu\mathcal{R}^T_\mu : L^2_s(\mathbb{P}Y) \to L^2_{s+d-1}(\mathbb{P}Y)$ are isomorphisms for all s. In particular, $\mathrm{Ker}(\mathcal{R}_\mu : L^2_s(\mathbb{P}X) \to L^2_{s+\frac{d-1}{2}}(\mathbb{P}Y)) = 0$ and $\mathrm{Coker}(\mathcal{R}_\mu : L^2_s(\mathbb{P}X) \to L^2_{s+\frac{d-1}{2}}(\mathbb{P}Y)) = 0$, implying by the open mapping theorem that $\mathcal{R}_\mu : L^2_s(\mathbb{P}X) \to L^2_{s+\frac{d-1}{2}}(\mathbb{P}Y)$ is an isomorphism for all s. The result follows. □

Remark 1. It is unlikely that the result is sharp. For instance, in the case of $d = 1$ one only needs $\|\mu - 1\|_{C^0}$ to be small to conclude that $\mathcal{R}_\mu$ is an isomorphism, while the statement (although non-applicable for $d = 1$) would suggest bounding the C^3-norm.

Appendix

A Some Integral Estimates

Fix some real $T > 0$. For an integer $d \geq 2$, a smooth function $\rho : [0, \infty) \to \mathbb{R}$ compactly supported in $[0, 1)$, and smooth functions $m(r, \phi), n(r, \phi) : [0, \infty) \times [0, \pi]\mathbb{R}$ we define the integrals

$$I(d, \rho, m) = \int_0^1 \rho(r) r^{d-2} dr \int_0^1 M(r, t) \cos(Trt)(1 - t^2)^{\frac{d-3}{2}} dt$$

and

$$J(d, \rho, m) = \int_0^1 \rho(r) r^{d-2} dr \int_0^1 N(r, t) \sin(Trt)(1 - t^2)^{\frac{d-3}{2}} dt$$

where $M(r, t) = m(r, \phi)$ and $N(r, t) = n(r, \phi)$ for $t = \cos \phi$. We assume m is even w.r.t. $\frac{\pi}{2}$, namely $m(r, \frac{\pi}{2} + \phi) = m(r, \frac{\pi}{2} - \phi) \iff M(r, t) = M(r, -t)$; while n is odd, i.e. $n(r, \frac{\pi}{2} + \phi) = -n(r, \frac{\pi}{2} - \phi) \iff N(r, t) = -N(r, -t)$.

Proposition 3. *There exists a constant $C = C(d, \rho)$ such that for $d \geq 2$ and all even functions m*

$$|I(d, \rho, m)| \leq \frac{C}{T^{d-1}} \sum_{\substack{a+b \leq d \\ a \leq d/2, b \leq d-1}} \sup \left| \frac{\partial^{a+b}}{\partial r^a \partial \phi^b} m \right|$$

and for odd functions n

$$|J(d, \rho, n)| \leq \frac{C}{T^{d-1}} \sum_{\substack{a+b \leq d \\ a \leq d/2, b \leq d-1}} \sup \left| \frac{\partial^{a+b}}{\partial r^a \partial \phi^b} n \right|.$$

Proof. Induction on d. Start by verifying the bounds for $d = 2$. To bound

$$I(2, \rho, m) = \int_0^1 (1 - t^2)^{-\frac{1}{2}} \int_0^1 \rho(r) M(r, t) \cos(Trt) dr$$

we first integrate the inner integral by parts:

$$\int_0^1 M(r, t) \rho(r) \cos(Trt) dr = -\frac{1}{Tt} \int_0^1 \sin(Trt)(M(r, t)\rho'(r) + \frac{\partial M}{\partial r}(r, t)\rho(r)) dr$$

Let us bound separately

$$\int_0^1 \frac{dt}{t\sqrt{1-t^2}} \int_0^1 \sin(Trt)M(r,t)\rho'(r)dr$$

and

$$\int_0^1 \frac{dt}{t\sqrt{1-t^2}} \int_0^1 \sin(Trt)\frac{\partial M}{\partial r}(r,t)\rho(r)dr.$$

Now

$$\left| \int_{\frac{1}{2}}^1 \frac{\sin(Trt)M(r,t)dt}{t\sqrt{1-t^2}} \right| \leq C \sup|m|$$

and since $\frac{\partial M}{\partial t} = -\frac{\partial m}{\partial \phi}\frac{1}{\sqrt{1-t^2}}$,

$$\begin{aligned}
&\int_0^{\frac{1}{2}} \frac{\sin(Trt)M(r,t)dt}{t\sqrt{1-t^2}} \\
&= Si(Trt)\frac{M(r,t)}{\sqrt{1-t^2}}\bigg|_0^{\frac{1}{2}} \\
&\quad + \int_0^{\frac{1}{2}} Si(Trt)\left(\frac{\partial m}{\partial \phi}(r,t)\frac{1}{\sqrt{1-t^2}^2} + M(r,t)\frac{t}{(1-t^2)^{3/2}}\right)dt.
\end{aligned}$$

Now since Si is bounded, it follows that

$$\left| \int_0^{\frac{1}{2}} \frac{\sin(Trt)M(r,t)dt}{t\sqrt{1-t^2}} \right| \leq C\left(\sup|m| + \sup\left|\frac{\partial m}{\partial \phi}\right|\right)$$

Thus

$$\begin{aligned}
&\left| \int_0^1 \frac{dt}{t\sqrt{1-t^2}} \int_0^1 \sin(Trt)M(r,t)\rho'(r)dr \right| \\
&\quad \leq C\left(\sup|m| + \sup\left|\frac{\partial m}{\partial \phi}\right|\right)\int_0^1 dr|\rho'(r)| \leq C\left(\sup|m| + \sup\left|\frac{\partial m}{\partial \phi}\right|\right).
\end{aligned}$$

Similarly,

$$\left| \int_0^1 \frac{dt}{t\sqrt{1-t^2}} \int_0^1 \sin(Trt)\frac{\partial M}{\partial r}(r,t)\rho(r)dr \right| \leq C\left(\sup\left|\frac{\partial m}{\partial r}\right| + \sup\left|\frac{\partial^2 m}{\partial r \partial \phi}\right|\right).$$

Tracing back,

$$|I(2,\rho,m)| \leq \frac{C}{T}\left(\sup|m| + \sup\left|\frac{\partial m}{\partial r}\right| + \sup\left|\frac{\partial m}{\partial \phi}\right| + \sup\left|\frac{\partial^2 m}{\partial r \partial \phi}\right|\right).$$

Next we bound $|J(2,\rho,n)|$

$$J(2,\rho,n) = \int_0^1 (1-t^2)^{-1/2} dt \int_0^1 N(r,t)\rho(r)\sin(Trt)dr.$$

Integrate the inner integral by parts:

$$\int_0^1 N(r,t)\rho(r)\sin(Trt)dr = -\frac{1}{Tt}\int_0^1 (1-\cos(Trt))(N(r,t)\rho'(r) + \frac{\partial N}{\partial r}(r,t)\rho(r))dr.$$

Let us bound separately

$$\int_0^1 \frac{dt}{t\sqrt{1-t^2}} \int_0^1 (1-\cos(Trt))N(r,t)\rho'(r)dr$$

and

$$\int_0^1 \frac{dt}{t\sqrt{1-t^2}} \int_0^1 (1-\cos(Trt))\frac{\partial N}{\partial r}(r,t)\rho(r)dr.$$

Now

$$\left|\int_{\frac{1}{2}}^1 \frac{(1-\cos(Trt))N(r,t)dt}{t\sqrt{1-t^2}}\right| \leq C \sup|n|$$

and since $\frac{\partial N}{\partial t} = -\frac{\partial n}{\partial \phi}\frac{1}{\sqrt{1-t^2}}$ and $N(r,0) = 0$ we get that $|\frac{\partial}{\partial t}N(r,t)| \leq C \sup|\frac{\partial n}{\partial \phi}|$ for $0 \leq t \leq \frac{1}{2}$ so $|N(r,t)| \leq C \sup|\frac{\partial n}{\partial \phi}|t$ and

$$\left|\int_0^{\frac{1}{2}} \frac{(1-\cos(Trt))N(r,t)dt}{t\sqrt{1-t^2}}\right| \leq C \sup\left|\frac{\partial n}{\partial \phi}\right| \int_0^1 \frac{dt}{\sqrt{1-t^2}} = C \sup\left|\frac{\partial n}{\partial \phi}\right|.$$

Thus

$$\left|\int_0^1 \frac{dt}{t\sqrt{1-t^2}} \int_0^1 (1-\cos(Trt))N(r,t)\rho'(r)dr\right| \leq C\left(\sup|n| + \sup\left|\frac{\partial n}{\partial \phi}\right|\right)\int_0^1 dr|\rho'(r)| \leq C\left(\sup|n| + \sup\left|\frac{\partial n}{\partial \phi}\right|\right).$$

Similarly,

$$\left|\int_0^1 \frac{dt}{t\sqrt{1-t^2}} \int_0^1 (1-\cos(Trt)) \frac{\partial N}{\partial r}(r,t)\rho(r)dr\right| \le C\left(\sup\left|\frac{\partial n}{\partial r}\right| + \sup\left|\frac{\partial^2 n}{\partial r \partial \phi}\right|\right)$$

and putting all together,

$$|J(2,\rho,n)| \le \frac{C}{T}\left(\sup|n| + \sup\left|\frac{\partial n}{\partial r}\right| + \sup\left|\frac{\partial n}{\partial \phi}\right| + \sup\left|\frac{\partial^2 n}{\partial r \partial \phi}\right|\right)$$

as required.

Next consider the case $d = 3$. We bound I, J simultaneously. Apply integration by parts to the inner integrals:

$$\begin{aligned} I(3,\rho,m) &= \int_0^1 \rho(r) r dr \int_0^1 M(r,t)\cos(Trt)dt \\ &= \frac{1}{T}\int_0^1 r\rho(r)dr \int_0^1 \frac{\partial m}{\partial \phi}(r,t)\frac{1}{\sqrt{1-t^2}}\frac{\sin(Trt)}{r}dt \\ &\quad + \frac{1}{T}\int_0^1 r\rho(r)M(r,1)\frac{\sin(Tr)}{r}dr \end{aligned}$$

and similarly

$$\begin{aligned} J(3,\rho,n) &= -\frac{1}{T}\int_0^1 r\rho(r)dr \int_0^1 \frac{\partial n}{\partial \phi}(r,t)\frac{1}{\sqrt{1-t^2}}\frac{\cos(Trt)}{r}dt \\ &\quad - \frac{1}{T}\int_0^1 r\rho(r)\left(N(r,1)\frac{\cos(Tr)}{r} - \frac{N(r,0)}{r}\right)dr. \end{aligned}$$

These first summands are

$$\frac{1}{T}\int_0^1 \rho(r)dr \int_0^1 \frac{\partial m}{\partial \phi}(r,t)\frac{1}{\sqrt{1-t^2}}\sin(Trt)dt = \frac{1}{T}J\left(2,\rho,\frac{\partial m}{\partial \phi}\right)$$

and

$$-\frac{1}{T}\int_0^1 \rho(r)dr \int_0^1 \frac{\partial n}{\partial \phi}(r,t)\frac{1}{\sqrt{1-t^2}}\cos(Trt)dt = -\frac{1}{T}I\left(2,\rho,\frac{\partial n}{\partial \phi}\right)$$

the second summand for I

$$\begin{aligned} &\frac{1}{T}\int_0^1 r\rho(r)M(r,1)\frac{\sin(Tr)}{r}dr \\ &= \frac{1}{T}\int_0^1 \rho(r)M(r,1)\sin(Tr)dr \end{aligned}$$

$$= -\frac{1}{T}\rho(r)M(r,1)\frac{\cos(Tr)}{T}\Big|_0^1 + \frac{1}{T^2}\int_0^1 (\rho'(r)M(r,1)$$
$$+\rho(r)\frac{\partial}{\partial r}M(r,1))\cos(Tr)dr$$

is bounded by

$$\frac{C}{T^2}\left(\sup|m| + \sup\left|\frac{\partial m}{\partial r}\right|\right).$$

Similarly since $N(r,0) = 0$, also the second summand for J

$$\left|\frac{1}{T}\int_0^1 r\rho(r)N(r,1)\frac{\cos(Tr)}{r}dr\right| \leq \frac{C}{T^2}\left(\sup|n| + \sup\left|\frac{\partial n}{\partial r}\right|\right)$$

thus we showed that

$$|I(3,\rho,m)| \leq \frac{C(3,\rho)}{T}J(2,\rho,\frac{\partial m}{\partial\phi}) + \frac{C(3,\rho)}{T^2}\left(\sup|m| + \sup\left|\frac{\partial m}{\partial r}\right|\right)$$

and

$$|J(3,\rho,n)| \leq \frac{C(3,\rho)}{T}I(2,\rho,\frac{\partial n}{\partial\phi}) + \frac{C(3,\rho)}{T^2}\left(\sup|n| + \sup\left|\frac{\partial n}{\partial r}\right|\right)$$

and plugging the already proved estimates for $d = 2$ concludes the case $d = 3$.

Finally, for $d > 3$ we will apply induction. Again consider both integrals simultaneously. Start by integrating by parts the inner integral: the boundary term is zero (for J since n is odd), so

$$\int_0^1 M(r,t)\cos(Trt)(1-t^2)^{\frac{d-3}{2}}dt$$
$$= \frac{1}{Tr}\int_0^1 \sin(Trt)\left(\frac{\partial m}{\partial\phi}(1-t^2)^{\frac{d-4}{2}} + (d-3)M(r,t)t(1-t^2)^{\frac{d-5}{2}}\right)dt$$

$$\int_0^1 N(r,t)\sin(Trt)(1-t^2)^{\frac{d-3}{2}}dt$$
$$= -\frac{1}{Tr}\int_0^1 \cos(Trt)\left(\frac{\partial n}{\partial\phi}(1-t^2)^{\frac{d-4}{2}} + (d-3)N(r,t)t(1-t^2)^{\frac{d-5}{2}}\right)dt.$$

Thus

$$\begin{aligned} I(d,\rho,m) &= \int_0^1 \rho(r) r^{d-2} dr \int_0^1 M(r,t)\cos(Trt)(1-t^2)^{\frac{d-3}{2}} dt \\ &= \frac{1}{T}\int_0^1 \rho(r) r^{d-3} dr \int_0^1 \sin(Trt)\frac{\partial m}{\partial \phi}(1-t^2)^{\frac{d-4}{2}} dt \\ &\quad + \frac{C_d}{T}\int_0^1 \rho(r) r^{d-3} \int_0^1 M(r,t) t (1-t^2)^{\frac{d-5}{2}} \sin(Trt) dt \end{aligned}$$

and

$$\begin{aligned} J(d,\rho,n) &= \int_0^1 \rho(r) r^{d-2} dr \int_0^1 N(r,t)\sin(Trt)(1-t^2)^{\frac{d-3}{2}} dt \\ &= -\frac{1}{T}\int_0^1 \rho(r) r^{d-3} dr \int_0^1 \cos(Trt)\frac{\partial n}{\partial \phi}(1-t^2)^{\frac{d-4}{2}} dt \\ &\quad - \frac{C_d}{T}\int_0^1 \rho(r) r^{d-3} \int_0^1 N(r,t) t (1-t^2)^{\frac{d-5}{2}} \cos(Trt) dt. \end{aligned}$$

The first terms are $\frac{1}{T} J(d-1, \rho, \frac{\partial m}{\partial \phi})$ and $-\frac{1}{T} I(d-1, \rho, \frac{\partial n}{\partial \phi})$, respectively.

In the second term, first change the order of integration:

$$\begin{aligned} &\frac{C_d}{T}\int_0^1 \rho(r) r^{d-3} \int_0^1 M(r,t) t (1-t^2)^{\frac{d-5}{2}} \sin(Trt) dt \\ &= \frac{C_d}{T}\int_0^1 t(1-t^2)^{\frac{d-5}{2}} dt \int_0^1 M(r,t)\sin(Trt) r^{d-3}\rho(r) dr. \end{aligned}$$

Now apply integration by parts to the inner integral. Since $d-3>0$ and $\rho(1)=0$, again there is no boundary term:

$$\begin{aligned} &\int_0^1 M(r,t)\sin(Trt) r^{d-3}\rho(r) dr \\ &= \int_0^1 dr \frac{\cos(Trt)}{Tt}\Big(r^{d-3}\rho(r)\frac{\partial M}{\partial r}(r,t) + (r^{d-3}\rho'(r) \\ &\quad + (d-3) r^{d-4}\rho(r)) M(r,t)\Big) \end{aligned}$$

Thus

$$\begin{aligned} &\frac{C_d}{T}\int_0^1 \rho(r) r^{d-3} \int_0^1 M(r,t) t (1-t^2)^{\frac{d-5}{2}} \sin(Trt) dt \\ &= \frac{C_d}{T^2}\int_0^1 (1-t^2)^{\frac{d-5}{2}} dt \int_0^1 dr \cos(Trt) r^{d-3}\rho(r)\frac{\partial M}{\partial r}(r,t) \end{aligned}$$

$$+\frac{C_d}{T^2}\int_0^1 (1-t^2)^{\frac{d-5}{2}}dt \int_0^1 dr \cos(Trt)(r^{d-3}\rho'(r)$$
$$+(d-3)r^{d-4}\rho(r))M(r,t)$$
$$= \frac{C_d}{T^2} I\left(d-2, r\rho(r), \frac{\partial m}{\partial r}\right) + \frac{C_d}{T^2} I(d-2, \rho(r) + r\rho'(r), m)$$

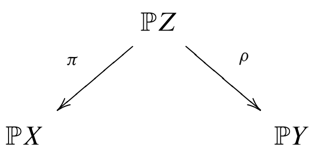
and the corresponding term for J:

$$-\frac{C_d}{T}\int_0^1 \rho(r)r^{d-3}\int_0^1 N(r,t)t(1-t^2)^{\frac{d-5}{2}}\cos(Trt)dt$$
$$= -\frac{C_d}{T}\int_0^1 t(1-t^2)^{\frac{d-5}{2}}dt \int_0^1 N(r,t)\cos(Trt)r^{d-3}\rho(r)dr$$
$$= -\frac{C_d}{T}\int_0^1 t(1-t^2)^{\frac{d-5}{2}}dt \int_0^1 dr \frac{\sin(Trt)}{Tt}$$
$$\times\left(r^{d-3}\rho(r)\frac{\partial N}{\partial r}(r,t) + (r^{d-3}\rho'(r) + (d-3)r^{d-4}\rho(r))N(r,t)\right)$$
$$= -\frac{C_d}{T^2} J(d-2, r\rho(r), \frac{\partial n}{\partial r}) + \frac{C_d}{T^2} J(d-2, \rho(r) + r\rho'(r), n)$$

and we conclude by induction. □

B Guillemin's Condition

For $q \in X$, we denote by $\bar{q} \in X$ the unique point proportional to q and distinct from it. We will consider the projective space $\mathbb{P}X = \mathbb{RP}^d$, $\mathbb{P}Y = \mathbb{RP}^d$ and the projectivized incidence variety $\mathbb{P}Z = \{(q, p) \in \mathbb{P}X \times \mathbb{P}Y : \langle q, p\rangle = 0\}$.
Consider the projectivized double fibration



Then any two fibers $F_p(\mathbb{P}X)$ intersect transversally (since before projectivization, the only non-transversal intersection was between fibers over antipodal points). Denote $N_W \subset T^*(\mathbb{P}X \times \mathbb{P}Y)$, $N_E \subset T^*(X \times Y)$ the conormal bundles of W, E respectively. Since $\dim E = 2d - 1$ and $\dim(X \times Y) = 2d$, the fibers of N_E, N_W are one-dimensional. Recall that $T_{(q,p)}E = \{(\xi, \eta) \in T_qX \times T_pY :$

$\langle q, \eta \rangle + \langle \xi, p \rangle = 0\}$. Therefore, N_E over $(q, p) \in E$ has its fiber spanned by $(p, q) \in T_q^* X \times T_p^* Y$. One thus has $N_E \backslash 0 \subset (T^* X \backslash 0) \times (T^* Y \backslash 0)$, and $\rho : N_E \backslash 0 \to T^* Y \backslash 0$ given by $((q, p), t(p, q)) \mapsto (p, tq)$ is an immersion, which is two-to-one since $\rho((q, p), t(p, q)) = \rho((\bar{q}, p), (-t)(p, \bar{q}))$. The corresponding map $\rho : N_W \backslash 0 \to T^* \mathbb{P} Y \backslash 0$ is already an injective immersion. Thus Guillemin's condition is satisfied, and we conclude

Corollary 1. *For any smooth positive measure $\mu \in \mathcal{M}^\infty(\mathbb{P}Z)$, $\mathcal{R}_\mu^T \mathcal{R}_\mu : C^\infty(\mathbb{P}X) \to C^\infty(\mathbb{P}X)$ is an elliptic pseudodifferential operator.*

C Pseudo-Differential Operators

For a survey of the subject, see for instance [4].
We will study the norm of a pseudodifferential linear operator $P : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ which is given by its symbol $p(x, \xi)$

$$Pf(x) = \int d\xi e^{i\langle x, \xi \rangle} p(x, \xi) \hat{f}(\xi)$$

where $p \in Sym^m(K)$, i.e.,

1. $p \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$
2. p has compact $x-$support $K \subset \mathbb{R}^n$
3. $|D_\xi^\beta D_x^\alpha p(x, \xi)| \leq C_{\alpha\beta}(1 + |\xi|)^{m - |\beta|}$

It is well known that for all $s \in \mathbb{R}$, P extends to a bounded operator between Sobolev spaces

$$P : L_{s+m}^2(\mathbb{R}^n) \to L_s^2(\mathbb{R}^n)$$

We will trace the proof of this fact to understand the dependence on p of the operator norm $\|P\|$.

Proposition 4. *There exists a constant $C(n, s)$ such that*

$$\|P\|_{L_{s+m}^2 \to L_s^2} \leq C(n, s) \sup_{|\alpha| \leq n + \lfloor |s| \rfloor + 1} C_{\alpha 0} |K|$$

Proof. All the integrals in the following are over $\mathbb{R}^n$. Start by integrating by parts:

$$\left| \int dx D_x^\alpha p(x, \xi) e^{i\langle x, \zeta \rangle} \right| = \left| \zeta^\alpha \right| \left| \int dx p(x, \xi) e^{i\langle x, \zeta \rangle} \right|.$$

So

$$\left|\int dxp(x,\xi)e^{i\langle x,\zeta\rangle}\right| \leq \min(|\zeta|^{-|\alpha|}C_{\alpha 0}(1+|\xi|)^m|K|, C_{00}(1+|\xi|)^m|K|)$$
$$\leq 2^{|\alpha|}(C_{00}+C_{\alpha 0})|K|(1+|\xi|)^m(1+|\zeta|)^{-|\alpha|}$$
$$= C_\alpha|K|(1+|\xi|)^m(1+|\zeta|)^{-|\alpha|}$$

where $C_\alpha = 2^{|\alpha|}(C_{00}+C_{\alpha 0})$. We want to bound

$$Pu(x) = \int d\xi e^{i\langle x,\xi\rangle}p(x,\xi)\hat{u}(\xi).$$

Take $v \in L^2_{-s}(\mathbb{R}^n)$, then

$$(Pu,v) = \int d\zeta\hat{v}(\zeta)\hat{Pu}(\zeta) = \int d\zeta\hat{v}(\zeta)\int dxPu(x)e^{-i\langle x,\zeta\rangle}$$
$$= \int\int d\zeta dx\hat{v}(\zeta)e^{-i\langle x,\zeta\rangle}\int d\xi\hat{u}(\xi)p(x,\xi)e^{i\langle x,\xi\rangle}$$
$$= \int\int d\zeta d\xi\hat{u}(\xi)\hat{v}(\zeta)\int dxp(x,\xi)e^{i\langle x,\xi-\zeta\rangle}$$

so denoting

$$\Phi(\xi,\zeta) = (1+|\xi|)^{-m-s}(1+|\zeta|)^s\left|\int dxp(x,\xi)e^{i\langle x,\xi-\zeta\rangle}\right|$$

we have

$$|(Pu,v)| \leq \int\int d\xi d\zeta\hat{u}(\xi)\hat{v}(\zeta)\Phi(\xi,\zeta)(1+|\xi|)^{m+s}(1+|\zeta|)^{-s}$$
$$\leq \Big(\int d\xi|\hat{u}(\xi)|^2(1+|\xi|)^{2(m+s)}\int d\zeta\Phi(\xi,\zeta)\Big)^{1/2}$$
$$\times\Big(\int\int d\zeta|\hat{v}(\zeta)|^2(1+|\zeta|)^{-2s}\int d\xi\Phi(\xi,\zeta)\Big)^{1/2}.$$

Now

$$\Phi(\xi,\zeta) \leq C_\alpha|K|(1+|\xi|)^{-m-s}(1+|\zeta|)^s(1+|\xi|)^m(1+|\xi-\zeta|)^{-|\alpha|}$$
$$\leq C_\alpha|K|(1+|\xi-\zeta|)^{|s|-|\alpha|}.$$

Therefore,

$$\int d\xi \Phi(\xi,\zeta) \leq A(n,|s|-|\alpha|)C_\alpha|K|$$

where

$$A(n,l) = \int d\xi(1+|\xi|)^l$$

similarly

$$\int d\zeta \Phi(\xi,\zeta) \leq C(n,|s|-|\alpha|)C_\alpha|K|$$

implying

$$|(Pu,v)| \leq A(n,|s|-|\alpha|)C_\alpha|K|\|u\|_{m+s}\|v\|_{-s}$$
$$\Rightarrow \|P\|_{L^2_{s+m}\to L^2_s} \leq A(n,|s|-|\alpha|)C_\alpha|K|$$

and this holds for all α s.t. $A(n,|s|-|\alpha|) = \int d\xi(1+|\xi|)^{|s|-|\alpha|} < \infty$, i.e. $|s|-|\alpha| < -n \iff |\alpha| > n+|s|$. We thus choose α s.t. $|\alpha| = \lfloor |s| \rfloor + n + 1$, and recall that $C_\alpha = 2^{|\alpha|}(C_{00}+C_{\alpha 0})$ to obtain the stated estimate. □

We will also need the relation between the Schwartz kernel and the symbol.

Proposition 5. *Suppose the Schwartz kernel of P is given by $K(x,y)$, namely*

$$\langle Pf(x), g(x)\rangle = \int dxdyK(x,y)f(y)g(x).$$

Then the symbol $p(x,\xi)$ of P is given by

$$p(x,\xi) = \int e^{-i\langle y,\xi\rangle}K(x,x-y)dy.$$

Proof. Write for smooth compactly supported f, g

$$\langle Pf(x), g(x)\rangle = \int dxd\xi e^{i\langle x,\xi\rangle}p(x,\xi)\hat{f}(\xi)g(x)$$
$$= \int dxdyd\xi e^{i\langle x-y,\xi\rangle}p(x,\xi)f(y)g(x).$$

That is, $K(x,y) = \int d\xi e^{i\langle x-y,\xi\rangle}p(x,\xi)$, and $\langle Pf, g\rangle = \int f(y)g(x)K(x,y)dydx$. Denoting by $\check{h}(x) = \int d\xi h(\xi)e^{i\langle x,\xi\rangle}$ the inverse Fourier transform, we can also write $K(x,y) = \check{p}(x,\bullet)(x-y) \iff K(x,x-y) = \check{p}(x,\bullet)(y)$, so

$$p(x,\xi) = \int e^{-i\langle y,\xi\rangle}K(x,x-y)dy$$

as claimed. □

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