Overgroups of the Automorphism Group of the Rado Graph

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Abstract We are interested in overgroups of the automorphism group of the Rado graph. One class of such overgroups is completely understood; this is the class of reducts. In this article we tie recent work on various other natural overgroups, in particular establishing group connections between them and the reducts.

Key words Rado graph • Automorphism group • Overgroup • Homogeneous structures

Mathematical Subject Classifications (2010): 05C25, 05C80, 05C55, 05C63, 20F28

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M. Ludwig et al. (eds.), *Asymptotic Geometric Analysis*, Fields Institute Communications 68, DOI 10.1007/978-1-4614-6406-8_4,

1 Introduction

The Rado Graph $\mathcal{R}$ is the countable universal homogeneous graph: it is the unique (up to isomorphism) countable graph with the defining property that for every finite disjoint subsets of vertices A and B there is a vertex adjacent to all vertices in A and not adjacent to any of the vertices in B.

We are interested in overgroups of its automorphism group Aut($\mathcal{R}$) in Sym($\mathcal{R}$), the symmetric group on the vertex set of $\mathcal{R}$. One class of overgroups of Aut($\mathcal{R}$) is completely understood; this is the class of reducts, or automorphism groups of relational structures definable from $\mathcal{R}$ without parameters. Equivalently, this is the class of subgroups of $Sym(\mathcal{R})$ containing $Aut(\mathcal{R})$ which are closed with respect to the product topology. According to a theorem of Thomas [5], there are just five reducts of Aut($\mathcal{R}$):

- Aut($\mathcal{R}$)
- $\mathcal{D}(\mathcal{R})$, the group of dualities (automorphisms and anti-automorphisms) of $\mathcal{R}$
- $\mathcal{S}(\mathcal{R})$, the group of switching automorphisms of $\mathcal{R}$ (see below)
- $\mathcal{B}(\mathcal{R}) = \mathcal{D}(\mathcal{R}).\mathcal{S}(\mathcal{R})$ (the big group)
- $Sym(\mathcal{R})$, the full symmetric group

Given a set X of vertices in a graph G, we denote by $\sigma_X(G)$ the *switching* operation of changing all adjacencies between X and its complement in G, leaving those within or outside X unchanged, thus yielding a new graph. Now a switching automorphism of G is an isomorphism which maps G to $\sigma_X(G)$ for some X, and $\mathcal{S}(G)$ is the group of switching automorphisms. Thus the interesting question is often for which subset X is $\sigma_X(G)$ isomorphic G, and for this reason will sometimes abuse terminology and may call $\sigma_X(G)$ a switching automorphism.

Thomas also showed (see [6], and also the work of Bodirsky and Pinsker [1]) that the group $\mathcal{S}(\mathcal{R})$ can also be understood as the automorphism group of the 3-regular hypergraph whose edges are those 3-element subsets containing an odd number of edges. Similarly, $\mathcal{D}(\mathcal{R})$ is the automorphism group of the 4-regular hypergraph whose edges are those 4-element subsets containing an odd number of edges, and $\mathcal{B}(\mathcal{R})$ is the automorphism group of the 5-regular hypergraph whose edges are those 5-element subsets containing an odd number of edges.

One can see that $\mathcal{G}$ is any subgroup of Sym($\mathcal{R}$), then $\mathcal{G}$.FSym($\mathcal{R}$) (the group generated by the union of $\mathcal{G}$ and FSym($\mathcal{R}$), the group of all finitary permutations on $\mathcal{R}$) is a subgroup of Sym($\mathcal{R}$) containing $\mathcal{G}$ and highly transitive. The reducts $\mathcal{D}(\mathcal{R})$ and $\mathcal{S}(\mathcal{R})$ however are 2-transitive but not 3-transitive, while $\mathcal{B}(\mathcal{R})$ is 3-transitive but not 4-transitive. On the other hand we have the following.

Lemma 1. *Any overgroup of* Aut($\mathcal{R}$) *which is not contained in* $\mathcal{B}(\mathcal{R})$ *is highly transitive.*

Proof. Let G with Aut($\mathcal{R}$) $\leq G \not\leq \mathcal{B}(\mathcal{R})$, and let $\overline{G}$ be the closure of G in Sym($\mathcal{R}$). Since $\overline{G} \not\leq \mathcal{B}(\mathcal{R})$, we have $\overline{G} =$ Sym($\mathcal{R}$) by Thomas' theorem. Since G and $\overline{G}$ have the same orbits of n-uples, G is highly transitive. □

Now for a bit of notation. With the understanding that $\mathcal{R}$ is the only graph under consideration here, we write $v \sim w$ when v and w are adjacent (in $\mathcal{R}$), $\mathcal{R}(v)$ for the set of vertices adjacent to v (the neighbourhood of v), and will use $\mathcal{R}^c(v)$ for $\mathcal{R} \backslash \mathcal{R}(v)$ (note that $v \in \mathcal{R}^c(v)$). We say that a permutation g changes the adjacency of v and w if $(v \sim w) \Leftrightarrow (v^g \not\sim w^g)$. We say that g changes finitely many adjacencies at v if there are only finitely many points w for which g changes the adjacency of v and w.

Given two groups G_1, G_2 contained in a group H, we write $G_1.G_2$ for the subgroup of H generated by their union.

In Sect. 2, we present various other natural overgroups and tie recent work and in particular establish group connections between them and the reducts.

2 Other Overgroups of Aut($\mathcal{R}$)

Cameron and Tarzi in [2] have studied the following overgroups of $\mathcal{R}$.

(a) $\mathrm{Aut}_1(\mathcal{R})$, the group of permutations which change only a finite number of adjacencies.
(b) $\mathrm{Aut}_2(\mathcal{R})$, the group of permutations which change only a finite number of adjacencies at each vertex.
(c) $\mathrm{Aut}_3(\mathcal{R})$, the group of permutations which change only a finite number of adjacencies at all but finitely many vertices.
(d) $\mathrm{Aut}(\mathcal{F}_{\mathcal{R}})$, where $\mathcal{F}_{\mathcal{R}}$ is the neighbourhood filter of $\mathcal{R}$, the filter generated by the neighbourhoods of vertices of $\mathcal{R}$.

One shows that all these sets of permutations really are groups, as claimed. For $\mathrm{Aut}_i(\mathcal{G})$, this is because if $C(g)$ denotes the set of pairs $\{v, w\}$ whose adjacency is changed by g, then one verifies that $C(g^{-1}) = C(g)^{g^{-1}}$ and $C(gh) \subseteq C(g) \cup C(h)^{g^{-1}}$.

The main facts known about these groups are:

Proposition 1 ([2]).

(a) $\mathrm{Aut}(\mathcal{R}) < \mathrm{Aut}_1(\mathcal{R}) < \mathrm{Aut}_2(\mathcal{R}) < \mathrm{Aut}_3(\mathcal{R})$.
(b) $\mathrm{Aut}_2(\mathcal{R}) \leq \mathrm{Aut}(\mathcal{F}_{\mathcal{R}})$*, but* $\mathrm{Aut}_3(\mathcal{R})$ *and* $\mathrm{Aut}(\mathcal{F}_{\mathcal{R}})$ *are* incomparable.
(c) $\mathrm{FSym}(\mathcal{R}) < \mathrm{Aut}_3(\mathcal{R}) \cap \mathrm{Aut}(\mathcal{F}_{\mathcal{R}})$*, but* $\mathrm{FSym}(\mathcal{R}) \cap \mathrm{Aut}_2(\mathcal{R}) = 1$.
(d) $\mathcal{S}(\mathcal{R}) \not\leq \mathrm{Aut}(\mathcal{F}_{\mathcal{R}})$*, and* $\mathrm{Aut}(\mathcal{F}_{\mathcal{R}}) \cap \mathcal{D}(\mathcal{R}) = \mathrm{Aut}(\mathcal{F}_{\mathcal{R}}) \cap \mathcal{S}(\mathcal{R}) = \mathrm{Aut}(\mathcal{R})$.

Proof. (a) is clear.
(b) For the first part, let $g \in \mathrm{Aut}_2(\mathcal{R})$. It suffices to show that, for any vertex v, we have $\mathcal{R}(v)^g \in \mathcal{F}(\mathcal{R})$. Now by assumption, $\mathcal{R}(v)^g$ differs only finitely from $\mathcal{R}(v^g)$; let $\mathcal{R}(v)^g \setminus \mathcal{R}(v^g) = \{w_1, \ldots, w_n\}$. If we choose w such that $w_i \notin \mathcal{R}(w)$ for each i, then we have

$$\mathcal{R}(v^g) \cap \mathcal{R}(w) \subseteq \mathcal{R}(v)^g,$$

and we are done.

For the second part, choose a vertex v, and consider the graph $\mathcal{R}'$ obtained by changing all adjacencies at v. Then $\mathcal{R}' \cong \mathcal{R}$. Choose an isomorphism g from $\mathcal{R}$ to $\mathcal{R}'$; since $\mathcal{R}'$ is vertex-transitive, we can assume that g fixes v. So g maps $\mathcal{R}(v)$ to $\mathcal{R}_1(v) = \mathcal{R}^c(v) \setminus \{v\}$. Clearly $g \in \mathrm{Aut}_3(\mathcal{R})$, since it changes only one adjacency at any point different from v. But if $g \in \mathrm{Aut}(\mathcal{F}_{\mathcal{R}})$, then we would have $\mathcal{R}_1(v) \in \mathcal{F}_{\mathcal{R}}$, a contradiction since $\mathcal{R}(v) \cap \mathcal{R}_1(v) = \emptyset$.

In the reverse direction, let $\mathcal{R}''$ be the graph obtained by changing all adjacencies between non-neighbours of v. Again $\mathcal{R}'' \cong \mathcal{R}$, and we can pick an isomorphism g from $\mathcal{R}$ to $\mathcal{R}''$ which fixes v. Now g changes infinitely many adjacencies at all non-neighbours of v (and none at v or its neighbours), so $g \notin \mathrm{Aut}_3(\mathcal{R})$. Also, if w is a non-neighbour of v, then $\mathcal{R}(v) \cap \mathcal{R}(w)^g = \mathcal{R}(v) \cap \mathcal{R}(w^g)$, so $g \in \mathrm{Aut}(\mathcal{F}_{\mathcal{R}})$.

(c) Note that any non-identity finitary permutation belongs to $\mathrm{Aut}_3(\mathcal{R}) \setminus \mathrm{Aut}_2(\mathcal{R})$. For if g moves v, then g changes infinitely many adjacencies at v (namely, all v and w, where w is adjacent to v but not v^g and is not in the support of g). On the other hand, if g fixes v, then g changes the adjacency of v and w only if g moves w, and there are only finitely many such w.

Finally, if $g \in \mathrm{FSym}(\mathcal{R})$, then $\mathcal{R}(v)^g$ differs only finitely from $\mathcal{R}(v)$, for any vertex $v \in V$; so $g \in \mathrm{Aut}(\mathcal{F}_{\mathcal{R}})$.

Thus the left inclusion is proper: $\mathrm{Aut}_2(\mathcal{R})$ is contained in the right-hand side but intersects $\mathrm{FSym}(\mathcal{R})$ in $\{1\}$.

(d) The graph $\mathcal{R}'$ in the proof of (b) is obtained from $\mathcal{R}$ by switching with respect to the set $\{v\}$; so the permutation g belongs to the group $\mathcal{S}(\mathcal{R})$ of switching automorphisms. Thus $S(R) \not\leq \mathrm{Aut}(\mathcal{F}_{\mathcal{R}})$.

Now any anti-automorphism g of $\mathcal{R}$ maps $\mathcal{R}(v)$ to a set disjoint from $\mathcal{R}(v^g)$; so no anti-automorphism can belong to $\mathrm{Aut}(\mathcal{F}_{\mathcal{R}})$. Suppose that $g \in \mathrm{Aut}(\mathcal{F}_{\mathcal{R}})$ is an isomorphism from $\mathcal{R}$ to $\sigma_X(\mathcal{R})$. We may suppose that σ_X is not the identity, that is, $X \neq \emptyset$ and $Y = V \setminus X \neq \emptyset$. Choose x and y so that $x^g \in X$ and $y^g \in Y$. Then $\mathcal{R}(x)^g \triangle Y = \mathcal{R}(x^g)$ and $\mathcal{R}(y)^g \triangle X = \mathcal{R}(y^g)$. Hence $\mathcal{R}(x^g) \cap \mathcal{R}(x)^g \subseteq X$ and $\mathcal{R}(y^g) \cap \mathcal{R}(y^g) \subseteq Y$. Hence

$$\mathcal{R}(x^g) \cap \mathcal{R}(x)^g \cap \mathcal{R}(y^g) \cap \mathcal{R}(y)^g = \emptyset,$$

a contradiction. □

On the other hand, results of Laflamme, Pouzet and Sauer in [4] concern the hypergraph $\mathcal{H}$ on the vertex set of $\mathcal{R}$ whose edges are those sets of vertices which induce a copy of $\mathcal{R}$. Note that a cofinite subset of an edge is an edge. There are three interesting groups here:

(a) $\mathrm{Aut}(\mathcal{H})$.
(b) $\mathrm{FAut}(\mathcal{H})$, the set of permutations g with the property that there is a finite subset S of $\mathcal{R}$ such that for every edge E, both $(E \setminus S)g$ and $(E \setminus S)g^{-1}$ are edges.
(c) $\mathrm{Aut}^*(\mathcal{H})$, the set of permutations g with the property that, for every edge E, there is a finite subset S of E such that $(E \setminus S)g$ and $(E \setminus S)g^{-1}$ are edges.

Clearly $\mathrm{Aut}(\mathcal{H}) \leq \mathrm{FAut}(\mathcal{H}) \leq \mathrm{Aut}^*(\mathcal{H})$, and a little thought shows that all three are indeed groups. Moreover one will note that for $\mathrm{Aut}^*(\mathcal{H})$ and $\mathrm{FAut}(\mathcal{H})$ to be groups, both conditions on g and g^{-1} in their definitions are necessary. To see this, choose an infinite clique $C \subset \mathcal{R}$, and also partition $\mathcal{R}$ into two homogeneous edges E_1 and E_2: for every finite disjoint subsets of vertices A and B of $\mathcal{R}$ there is a vertex in E_2 adjacent to all vertices in A and not adjacent to any of the vertices in B. Then it is shown in [4] that there exists $g \in \mathrm{Sym}(\mathcal{R})$ such that $Cg = A$, and Eg is an edge for any edge E. But clearly $(A \setminus S)g^{-1}$ is not an edge for any (finite) S.

As a further remark let $\mathcal{H}^*$ be the hypergraph on the vertex set of $\mathcal{R}$ whose edges are subset of the form $E \cup F$ where E induces a copy of $\mathcal{R}$ and F is a finite subset of $\mathcal{R}$. Equivalently these are the subsets of $\mathcal{R}$ of the form $E \Delta F$ where E induces a copy of $\mathcal{R}$ and F is a finite subset of $\mathcal{R}$ (this follows from the fact that for every copy E and finite set F, $E \setminus F$ is a copy). Then observe that $Aut(\mathcal{H}^*) = Aut^*(\mathcal{H})$.

We now provide some relationships between these LPS groups and the CT groups.

Proposition 2. *(a)* $\mathrm{Aut}(\mathcal{H}) < \mathrm{FAut}(\mathcal{H})$.
(b) $\mathrm{Aut}_2(\mathcal{R}) \leq \mathrm{Aut}(\mathcal{H})$ *and* $\mathrm{Aut}_3(\mathcal{R}) \leq \mathrm{FAut}(\mathcal{H})$.
(c) $\mathrm{FSym}(\mathcal{R}) \leq \mathrm{FAut}(\mathcal{H})$ *but* $\mathrm{FSym}(\mathcal{R}) \cap \mathrm{Aut}(\mathcal{H}) = 1$.

Proof. (a) This follows from part (c).

(b) If we alter a finite number of adjacencies at any point of $\mathcal{R}$, the result is still isomorphic to $\mathcal{R}$. So induced copies of $\mathcal{R}$ are preserved by $\mathrm{Aut}_2(\mathcal{R})$. Similarly, given an element of $\mathrm{Aut}_3(\mathcal{R})$, if we throw away the vertices where infinitely many adjacencies are changed, we are in the situation of $\mathrm{Aut}_2(\mathcal{R})$.

(c) The first part follows from Proposition 1 part (c) and part (b) above. For the second part, choose a vertex v and let E be the set of neighbours of v in $\mathcal{R}$ (this set is an edge of $\mathcal{H}$). Now, for any finitary permutation, there is a conjugate of it whose support contains v and is contained in $\{v\} \cup E$. Then $Eg = E \cup \{v\} \setminus \{w\}$ for some w. But the induced subgraph on this set is not isomorphic to $\mathcal{R}$, since v is joined to all other vertices. □

We shall see later that $\mathrm{FAut}(\mathcal{H}) < \mathrm{Aut}^*(\mathcal{H})$, but we present a bit more information before doing so. In particular we now show that an arbitrary switching is almost a switching isomorphism.

Lemma 2. *Let $X \subseteq \mathcal{R}$ arbitrary and $\sigma = \sigma_X$ be the operation of switching $\mathcal{R}$ with respect to X. Then there is a finite set S such that $\sigma(\mathcal{R} \setminus S)$ is an edge of $\mathcal{H}$, namely isomorphic to the Rado graph.*

Proof. For $E \subseteq \mathcal{R}$ and disjoint $U, V \subseteq E$, denote by $W_E(U, V)$ the collection of all witnesses for (U, V) in E. Note that if E is an edge, then $W_E(U, V)$ is an edge for any such sets U and V. Now for $C \subseteq \mathcal{R}$, denote for convenience by C_X the set $C \cap X$, and by C_X^c the set $C \setminus X$.

Thus if $\sigma(\mathcal{R})$ is **not** already an edge of $\mathcal{H}$, then the Rado graph criteria regarding switching yields finite disjoint $U, V \subseteq \mathcal{R}$ such that both:

- $W_{\mathcal{R}}(U_X^c \cup V_X, U_X \cup V_X^c) \subseteq X$
- $W_{\mathcal{R}}(U_X \cup V_X^c, U_X^c \cup V_X) \cap X = \emptyset$

Define $S = U \cup V$ and $E = \mathcal{R} \setminus S$, we show that $\sigma(E)$ is an edge of $\mathcal{H}$.

For this let $\bar{U}, \bar{V} \subseteq E$. But now we have:

$$\begin{aligned} &W_{\mathcal{R}}(\bar{U}_X \cup \bar{V}_X^c \cup U_X^c \cup V_X, \bar{U}_X^c \cup \bar{V}_X \cup U_X \cup V_X^c) \\ &= W_E(\bar{U}_X \cup \bar{V}_X^c, \bar{U}_X^c \cup \bar{V}_X) \cap W_{\mathcal{R}}(U_X^c \cup V_X, U_X \cup V_X^c) \\ &\subseteq X \end{aligned}$$

In virtue of the Rado graph, the above first set contains infinitely many witnesses, and thus $W_E(\bar{U}_X \cup \bar{V}_X^c, \bar{U}_X^c \cup \bar{V}_X)$ is non empty. Hence $\sigma(E)$ contains a witness for $(\bar{U}, \bar{V})$, and we conclude that $\sigma(E)$ is an edge. □

The last item above shows that any graph obtained from $\mathcal{R}$ by switching has a cofinite subset inducing a copy of $\mathcal{R}$. This can be formulated as follows: Let G be a graph on the same vertex set as $\mathcal{R}$ and having the same parity of the number of edges in any 3-set as $\mathcal{R}$. Then G has a cofinite subset inducing $\mathcal{R}$.

The next result is about the relation between the LPS-groups and the reducts.

Proposition 3. *(a)* $\mathcal{D}(\mathcal{R}) < \mathrm{Aut}(\mathcal{H})$.
(b) $\mathcal{S}(\mathcal{R}) \not\leq \mathrm{Aut}(\mathcal{H})$.
(c) $\mathcal{S}(\mathcal{R}) \leq \mathrm{Aut}^*(\mathcal{H})$.

Proof. (a) Clearly $\mathcal{D}(\mathcal{R}) \leq \mathrm{Aut}(\mathcal{H})$ since $\mathcal{R}$ is self-complementary. We get a strict inequality since $\mathrm{Aut}(\mathcal{H})$ is highly transitive (since $\mathrm{Aut}_2(\mathcal{R}) \leq \mathrm{Aut}(\mathcal{H})$) while $\mathcal{D}(\mathcal{R})$ is not.

(b) We show that $\mathcal{R}$ can be switched into a graph isomorphic to $\mathcal{R}$ in such a way that some induced copy E of $\mathcal{R}$ has an isolated vertex after switching. Then the isomorphism is a switching-automorphism but not an automorphism of $\mathcal{H}$.

Let p, q be two vertices of $\mathcal{R}$. The graph we work with will be $\mathcal{R}_1 = \mathcal{R} \setminus \{p\}$, which is of course isomorphic to $\mathcal{R}$. Let A, B, C, D be the sets of vertices joined to p and q, p but not q, q but not p, and neither p nor q, respectively. Let σ be the operation of switching $\mathcal{R}_1$ with respect to C, and let $E = \{q\} \cup B \cup C$. It is clear that, after the switching σ, the vertex q is isolated in E. So we have to prove two things:

Claim. E induces a copy of $\mathcal{R}$.

Proof. Take U, V to be finite disjoint subsets of E. We may assume without loss of generality that $q \in U \cup V$.

Case 1: $q \in U$. Choose a witness z for $(U, V \cup \{p\})$ in $\mathcal{R}$. Then $z \not\sim p$ and $z \sim q$, so $z \in C$; thus z is a witness for (U, V) in E.
Case 2: $q \in V$. Now choose a witness for $(U \cup \{p\}, V)$ in $\mathcal{R}$; the argument is similar. □

Claim. $\sigma(\mathcal{R}_1)$ is isomorphic to $\mathcal{R}$.

Proof. Choose U, V finite disjoint subsets of $\mathcal{R} \setminus \{p\}$. Again, without loss, $q \in U \cup V$. Set $U_1 = U \cap C$, $U_2 = U \setminus U_1$, and $V_1 = V \cap C$, $V_2 = V \setminus V_1$.

Case 1: $q \in U$, so $q \in U_2$. Take z to be a witness for $(U_2 \cup V_1 \cup \{p\}, U_1 \cup V_2)$ in $\mathcal{R}$. Then $z \sim p, q$, so $z \in A$. The switching σ changes its adjacencies to U_1 and V_1, so in $\sigma(\mathcal{R}_1)$ it is a witness for $(U_1 \cup U_2, V_1 \cup V_2)$.
Case 2: $q \in V$, so $q \in V_2$. Now take z to be a witness for $(U_1 \cup V_2, U_2 \cup V_1 \cup \{p\})$ in $\mathcal{R}$. Then $z \sim q$, $z \not\sim p$, so $z \in C$, and σ changes its adjacencies to U_2 and V_2, making it a witness for $(U_1 \cup U_2, V_1 \cup V_2)$. □

(c) Let $X \subseteq \mathcal{R}$, σ be the operation of switching $\mathcal{R}$ with respect to X, and $g : \mathcal{R} \to \sigma(\mathcal{R})$ an isomorphism. In order to show that $g \in Aut(\mathcal{H}^*)$ we need to show that if E is an edge of $\mathcal{H}$ there is some finite S such that $(E \setminus S)g$ and $(E \setminus S)g^{-1}$ are edges of $\mathcal{H}$.

However the graph Eg (in $\mathcal{R}$) is obtained from switching the graph induced by $\sigma(\mathcal{R})$ on Eg. Since the latter is a copy of $\mathcal{R}$, Lemma 2 yields a finite $S_0 \subset \mathcal{R}$ such that $Eg \setminus S_0$ is an edge of $\mathcal{H}$. If $S_1 = S_0 g^{-1}$, then $(E \setminus S_1)g$ is an edge of $\mathcal{H}$.

Finally notice that g^{-1} is an isomorphism from $\mathcal{R}$ to $\sigma_{Xg^{-1}}(\mathcal{R})$, the above argument shows that there is a finite S_2 such that $(E \setminus S_2)g^{-1}$ is an edge of $\mathcal{H}$. Since cofinite subsets of edges are edges $S := S_1 \cup S_2$ has the required property. □

Corollary 1. $\mathcal{B}(\mathcal{R}) < \mathrm{Aut}^*(\mathcal{H})$

Proof. That $\mathcal{B}(\mathcal{R}) \leq \mathrm{Aut}^*(\mathcal{H})$ follows from parts (a) and (c) of Proposition 3.

We get a strict inequality since $\mathrm{Aut}^*(\mathcal{H})$ is highly transitive (since $\mathrm{Aut}_2(\mathcal{R}) \leq \mathrm{Aut}^*(\mathcal{H})$) while $\mathcal{D}(\mathcal{R})$ is not. □

Proposition 4. $\mathcal{S}(\mathcal{R}) \not\leq \mathrm{FAut}(\mathcal{H})$.

In view of $\mathcal{S}(\mathcal{R}) \leq \mathrm{Aut}^*(\mathcal{H})$ (by Proposition 3), this yields the following immediate Corollary.

Corollary 2. $\mathrm{FAut}(\mathcal{H}) < \mathrm{Aut}^*(\mathcal{H})$.

Clearly we have the following immediate observation:

Note 1.

$$\mathrm{Aut}(\mathcal{H}).\mathrm{FSym}(\mathcal{R}) \leq \mathrm{FAut}(\mathcal{H})$$

Hence, the Corollary yields yet that $\mathrm{Aut}(\mathcal{H}).\mathrm{FSym}(\mathcal{R}) < \mathrm{Aut}^*(\mathcal{H})$.

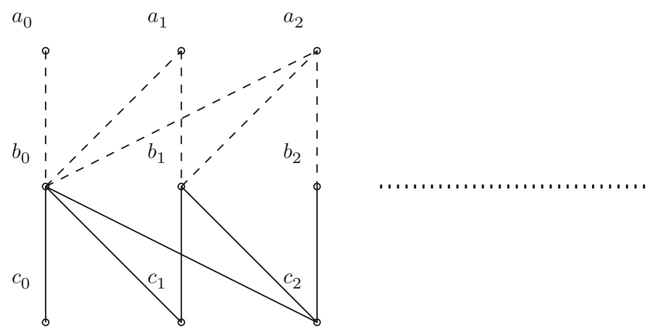
Proof (of Proposition 4). The argument can be thought of as an infinite version of the one given in part (b) of Proposition 3.

We shall recursively define subsets of $\mathcal{R}$:

- $A = \langle a_n : n \in \mathbb{N} \rangle$
- $B = \langle b_n : n \in \mathbb{N} \rangle$
- $C = \langle c_n : n \in \mathbb{N} \rangle$

and set D so that:

1. $\forall n \forall k \leq n\ a_n \not\sim b_k$ and $c_n \sim b_k$.
2. $\forall n\ E_n := \{a_k : k \geq n\} \cup \{b_n\} \cup \{c_k : k \geq n\}$ is an edge.
3. If σ is the operation of switching $\mathcal{R}$ with respect to C, then $\sigma(\mathcal{R})$ is isomorphic to $\mathcal{R}$.
4. $D = \mathcal{R} \setminus (A \cup B \cup C)$ is infinite.



The construction is as follows. First list all pairs (U, V) of disjoint finite subsets of $\mathcal{R}$ so that each one reoccurs infinitely often. Start with $A = B = C = D = \emptyset$ and at stage n, assume we have constructed $A_n = \{a_k : k \leq n\}$, $B_n = \{b_k : k \leq n\}$, and $C_n = \{c_k : k \leq n\}$ satisfying condition (1) above, together with a finite set D disjoint from A_n, B_n and C_n. Then given (U, V), proceed following one of the following cases:

(a) Suppose $U \cup V \subseteq A_n \cup B_n \cup C_n$ and contains at most one b_i (i.e. (U, V) is a type candidate for the eventual E_i). Then, choosing from $\mathcal{R} \setminus (A_n \cup B_n \cup C_n \cup D)$, add a_{n+1} or c_{n+1} as a witness for (U, V) depending as to whether b_i is in V or U (add a_{n+1} if there is no such b_i at all). Then choose two more elements from $\mathcal{R} \setminus D$ to complete the addition of elements a_{n+1}, b_{n+1}, and c_{n+1} as required by condition (1). Also throw a new point in D just to ensure it will become infinite.

(b) Else add the elements of $U \cup V \setminus A_n \cup B_n \cup C_n$ to D, and select an element of $\mathcal{R} \setminus (A_n \cup B_n \cup C_n \cup D)$ as witness to $(U \setminus C_n \cup V \cap C_n, V \setminus C_n \cup U \cap C_n)$.

The construction in part (b) will ensure that $\sigma(\mathcal{R})$ is isomorphic to $\mathcal{R}$. Indeed let U and V be disjoint finite subsets of $\mathcal{R}$, and without loss of generality $U \cap D \neq \emptyset$. Thus when the pair (U, V) is handled at some stage n, part (b) will add a witness d in D to $(U \setminus C_n \cup V \cap C_n, V \setminus C_n \cup U \cap C_n)$. But then d is a witness to (U, V) in $\sigma(\mathcal{R})$.

Let g be the isomorphism from $\sigma(\mathcal{R})$ to $\mathcal{R}$.

Finally the construction in part (a) clearly ensures condition (2). However note that in $\sigma(\mathcal{R})$, b_n is isolated in E_n, and therefore $\sigma(E_n)$ is not an edge.

Finally for any finite set $S \subset \mathcal{R}$, choose n so that $S \cap E_n = \emptyset$. Then $(E_n \setminus S)g$ is not an edge. Thus $g \in \mathcal{S}(\mathcal{R}) \setminus \mathrm{FAut}(\mathcal{H})$. ☐

We now go back to $\mathrm{Aut}(\mathcal{F}_{\mathcal{R}})$. One can readily verify that the automorphism g produced in the reverse direction of Proposition 1 is in fact not in $\mathrm{Aut}(\mathcal{H})$, thus $\mathrm{Aut}(\mathcal{F}_{\mathcal{R}}) \not\leq \mathrm{Aut}(\mathcal{H})$. However we have the following.

Proposition 5. $\mathrm{Aut}(\mathcal{F}_{\mathcal{R}}) \not\leq \mathrm{Aut}^*(\mathcal{H})$.

Proof. Fix a vertex $v \in \mathcal{R}$. Now partition $\mathcal{R}^c(v) = E \cup D$, where E is an edge, D is an infinite independent set. This is easily feasible since $\mathcal{R}^c(v)$ is an edge. Now define $g \in \mathrm{Sym}(\mathcal{R})$ such that:

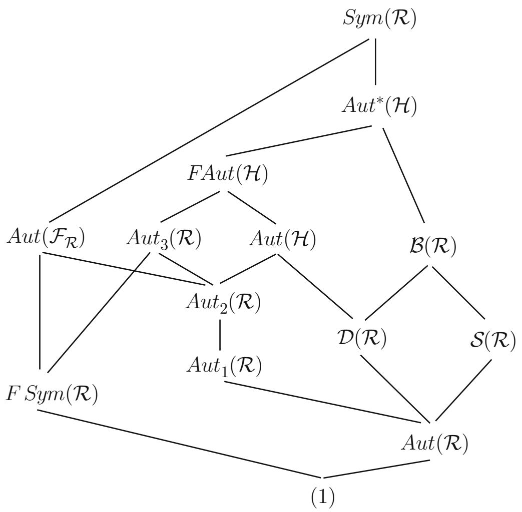
(a) $g \restriction \mathcal{R}(v)$ is the identity.
(b) $g \restriction E$ is a bijection to D.
(c) $g \restriction D$ is a bijection to $\mathcal{R}^c(v)$.

Now for any vertex w, $\mathcal{R}(w)^g \supseteq \mathcal{R}(v) \cap \mathcal{R}(w)$ so $g \in \mathrm{Aut}(\mathcal{F}_{\mathcal{R}})$. However, for any finite set S of E, then $(E \setminus S)g$ is again an independent set, and thus certainly not an edge.

Hence $g \notin \mathrm{Aut}^*(\mathcal{H})$ and the proof is complete. ☐

3 Conclusion

The following diagram summarizes the subgroup relationship between the various groups under discussion.



We do not know if the inclusion is strict in Observation 1.

Acknowledgements Claude Laflamme's work was supported by NSERC of Canada Grant # 690404.

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