Some Affine Invariants Revisited

Alina Stancu

Abstract We present several sharp inequalities for the $SL(n)$ invariant $\Omega_{2,n}(K)$ introduced in our earlier work on centro-affine invariants for smooth convex bodies containing the origin. A connection arose with the Paouris-Werner invariant Ω_K defined for convex bodies K whose centroid is at the origin. We offer two alternative definitions for Ω_K when $K \in C^2_+$. The technique employed prompts us to conjecture that any $SL(n)$ invariant of convex bodies with continuous and positive centro-affine curvature function can be obtained as a limit of normalized p-affine surface areas of the convex body.

Key words Affine surface area • Brunn-Minkowski-Firey theory • Centro-affine curvature • Centro-affine surface area • p-affine surface area

Mathematical Subject Classifications (2010): 52A40, 52A38, 52A20, 53A07

1 Introduction

Besides the intrinsic interest in affine invariants originating in Felix Klein's Erlangen Program, the extension to the Brunn-Minkowski-Firey theory [20, 21], and very recent connections between affine invariants and fields like stochastic geometry [3, 7] and information theory [17, 27, 30], led to an intense activity in this area of geometric analysis. The renewed interest in affine invariants has benefited also from a systematic approach classifying them, as for example in [8, 15, 16, 18], and

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from their use in affine and affine Sobolev inequalities [10, 11, 19, 23–26, 28, 39, 40] and problems arising in differential geometry [4–6, 9, 22, 35–38] which rely on isoperimetric-type functional inequalities. The study of such inequalities constitutes one of our primary goals of an on-going project.

The present paper spun as a follow-up of [34] in which we introduced new $SL(n)$-invariants for smooth convex bodies. We started by searching for sharp affine inequalities satisfied by one such invariant derived, in a certain sense, from the centro-affine surface area. The resulting inequalities are the subject of the next section. In the process, we encountered a connection to another $SL(n)$ invariant of convex bodies defined by Paouris and Werner who also related it to information theory [30]. In Sect. 3, we present two alternative definitions of this invariant. We noted that an additional $SL(n)$ invariant of convex bodies of class C^2_+ is defined with analogous techniques. This prompted us to conjecture that $SL(n)$ invariants for convex bodies with continuous and positive centro-affine curvature function can be obtained as limits of normalized p-affine surface areas of the convex body.

The setting for this paper is the Euclidean space $\mathbb{R}^n$, $n \geq 2$, in which we consider convex bodies containing the origin in their interior. Most of the time, we will also require that the convex bodies have smooth boundary, i.e. C^∞, with positive Gauss curvature. We will denote the set of such convex bodies by $\mathcal{K}_{\text{reg}}$. However, on several occasions, we will relax the regularity of the boundary to class C^2 with positive Gauss curvature and we will use the notation C^2_+ to indicate this latter class of convex bodies. The preferred parametrization of a convex body K will be with respect to the unit normal vector, $u \mapsto X_K(u)$, making many functions on the boundary ∂K to be considered as functions on the unit sphere $\mathbb{S}^{n-1}$.

We will denote the Gauss curvature of a convex body by $\mathcal{K}$ and its centro-affine curvature by $\mathcal{K}_0$. Geometrically, $\mathcal{K}_0^{-1/2}$ at a given point of ∂K is, up to a dimension dependent constant, the volume of the centered osculating ellipsoid at that point. Note that the centro-affine curvature is constant if and only if K is a centered ellipsoid. This can also be seen from a lemma due to Petty [31] since, analytically, as a function on the unit sphere, the centro-affine curvature is the ratio $\mathcal{K}_0(u) = \dfrac{\mathcal{K}(u)}{h^{n+1}(u)}$, $u \in \mathbb{S}^{n-1}$, where h is the support function of K: $h(u) = \max\{x \cdot u \mid \mathbf{x} \in K\}$ with $x \cdot u$ denoting the usual inner product in $\mathbb{R}^n$. Two additional notations are deemed necessary. First, $\mathcal{N}_0(u) := \mathcal{K}_0{}^{-\frac{1}{n+1}}(u)\,\mathcal{N}(u)$ is the centro-affine normal which is, pointwise, proportional to the (classical) affine normal $\mathcal{N}(u)$, [14]. Finally, we will use $d\mu_K$ to denote the cone measure of ∂K which, given that the Gauss curvature of K is positive, can be expressed by $d\mu_K(x) = h(\nu(x))\,\dfrac{1}{\mathcal{K}}(\nu(x))\,d\mu_{\mathbb{S}^{n-1}}(\nu(x))$, where $\nu : \partial K \to \mathbb{S}^{n-1}$ is the Gauss map of the boundary of K, hence the inverse of the parametrization X.

2 Inequalities for a Second Order Centro-Affine Invariant

In [34], we introduced a class of $SL(n)$ invariants for smooth convex bodies in $\mathbb{R}^n$. For a fixed convex body K, these invariants were the first, second, and, for an arbitrary integer k, the k-th variation of the volume of K while the boundary of the body was subject to a pointwise deformation in the direction of the centro-affine normal by a speed equal to a power of the centro-affine curvature at each specific point. The p-affine surface areas introduced by Lutwak [21] for p greater than one (later extended to the range $0 \le p < 1$ by Hug [13], to $-n \le p < 0$ by Meyer-Werner [29], and to $-\infty \le p < n$ by Schütt-Werner [33]) are, via this method, part of this class of invariants. To exemplify, and also bring the reader's attention to a particular such invariant which is one of the main objects of this paper, let us consider the following deformation of a convex body K with smooth boundary:

$$\begin{cases} \dfrac{\partial X(u,t)}{\partial t} = {\mathcal{K}_0}^{\frac{1}{2}}(u,t)\, \mathcal{N}_0(u,t) \\ X(u,0) = X_K(u). \end{cases} \tag{1}$$

Then, the first variation of $Vol(K)$ is the centro-affine surface area of K:

$$\frac{d}{dt}\,(Vol(K))_{t=0} = -\int_{\partial K} \mathcal{K}_0^{\frac{1}{2}}(\nu(x))\, d\mu_K(x) = -\Omega_n(K) =: \Omega_{1,n}(K), \tag{2}$$

see [34]. Recall that the centro-affine surface area of a convex body is the only one among the p-affine surface areas, $\Omega_p(K) = \int_{\partial K} \mathcal{K}_0^{\frac{p}{n+p}}\, d\mu_K$, invariant under $GL(n)$ transformations of the Euclidean space. Moreover, pursuing an additional variation, we obtain:

$$\begin{aligned} &\Omega_{2,n}(K) \\ &\quad := -\left(\frac{d^2\, Vol(K(t))}{dt^2}\right)_{|t=0} \\ &\quad = \frac{n(n-1)}{2}\, Vol(K^\circ) - \frac{n-1}{2} \int_{\mathbb{S}^{n-1}} h\sqrt{\mathcal{K}_0}\, s(h\sqrt{\mathcal{K}_0}, h, \ldots, h)\, d\mu_{\mathbb{S}^{n-1}}, \end{aligned} \tag{3}$$

where $s(f_1, f_2, \cdots, f_{n-1})$ is an extension of the mixed curvature function usually defined on C^2, here smooth, support functions to arbitrary smooth functions on the unit sphere $\mathbb{S}^{n-1}$, see [32] page 115 and also [34]. For the reader familiar with mixed determinants, the following can be taken as definition for the function $s(f_1, f_2, \cdots, f_{n-1})(u) := D(((f_1)_{ij} + \delta_{ij} f_1)(u), ((f_2)_{ij} + \delta_{ij} f_2)(u), \ldots, ((f_{n-1})_{ij} + \delta_{ij} f_{n-1})(u)), u \in \mathbb{S}^{n-1}$, where D is a mixed determinant and $(\,.\,)_i$ represents the covariant differentiation with respect to the i-th vector of a positively oriented orthonormal frame on the unit sphere $\mathbb{S}^{n-1}$.

We will show in Proposition 1 that, in a certain sense, $\Omega_{2,n}(K)$ measures how far K is from being a centered ellipsoid. In preparation, we call the Aleksandrov body, A_f, associated with a continuous positive function f on the unit sphere the convex body whose support function h_f is the maximal element of

$$\{h \le f \mid h : \mathbb{S}^{n-1} \to \mathbb{R} \text{ support function of a convex body}\}.$$

If f is itself a support function of a convex body L, then A_f is precisely the body L. Moreover, in general, $f = h_f$ almost everywhere with respect to the surface area measure of A_f. We could not find where this notion first surfaced in the literature, yet the work [9] gives an excellent background on this notion. We are now ready to state the following comparison result which we will use in analyzing $\Omega_{2,n}$:

Lemma 1 (Monotonicity Lemma). *Suppose that f is a strictly positive smooth function on the unit sphere $\mathbb{S}^{n-1}$ and that h is the support function of a convex body $K \subset \mathbb{R}^n$ which belongs to $\mathcal{K}_{reg}$. Then, denoting by $m := \min_{\mathbb{S}^{n-1}} \frac{f}{h}$, respectively, $M := \max_{\mathbb{S}^{n-1}} \frac{f}{h}$, we have*

$$m \cdot n\, Vol(K) \le \int_{\mathbb{S}^{n-1}} f s(h, h, \dots, h)\, d\mu_{\mathbb{S}^{n-1}} \le M \cdot n\, Vol(K) \tag{4}$$

and, if the Aleksandrov body associated with f has continuous positive curvature function, then

$$m^2 \cdot n\, Vol(K) \le \int_{\mathbb{S}^{n-1}} f s(f, h, \dots, h)\, d\mu_{\mathbb{S}^{n-1}} \le M^2 \cdot n\, Vol(K). \tag{5}$$

Proof. Since K belongs to $\mathcal{K}_{\text{reg}}$, $s(h, h, \dots, h) > 0$ on $\mathbb{S}^{n-1}$, thus $mh \le f \le Mh$ implies directly (4). In fact, we will show that we also have

$$m \cdot V(h, g, h, \dots, h) \le V(f, g, h, \dots, h) \le M \cdot V(h, g, h, \dots, h), \tag{6}$$

for any g support function of a convex body, denoted for later use by K_2. Indeed, if f itself would be a support function of a convex body, this claim is simply due to the monotonicity of mixed volumes. If f is not a support function, then there exists a large enough constant c so that $f + ch$ is a support function of a convex body, say L, with the Gauss parametrization. Moreover, $L \subseteq K_1$, where the latter is the dilation of K by the factor $M + c$. Then, from the monotonicity of mixed volumes, we have that $V(L, K_2, K, \dots, K) \le V(K_1, K_2, K, \dots, K)$. Choosing to represent these mixed volumes through the notation emphasizing the support functions of the two convex bodies, we have $V(f + ch, g, h, \dots, h) \le V((M + c)h, g, h, \dots, h)$. Finally, using the linearity of mixed volumes, we obtain $V(f, g, h, \dots, h) + cV(h, g, h, \dots., h) \le (M + c)V(h, g, h, \dots, h)$ which is, after a trivial simplification, the right-hand side inequality of Eq. (6). Similarly, by considering the dilation K of factor $(m + c)$, we obtain a convex body K_3 such that $K_3 \subseteq L$ and an argument analogous with the one above will imply $mV(h, g, h, \ldots, h) \leq V(f, g, h, \ldots, h)$.

We will now proceed to prove Eq. (5). Note again that if f would be a support function of a convex body, the claim follows from the monotonicity of mixed volumes. If f is not a support function, consider the Aleksandrov body associated to f, A_f, whose support function we denote by h_f. Thus $Mh \geq f \geq h_f \geq mh$ and, S_{A_f}-a.e., $f \circ \nu_{A_f}(x) = h_f(x)$, where ν_{A_f} is the Gauss map of ∂A_f. As, by hypothesis, A_f has a continuous positive curvature function, and by using Eq. (6), we have

$$\begin{aligned}
\int_{\mathbb{S}^{n-1}} fs(f,h,...,h)\, d\mu_{\mathbb{S}^{n-1}} &= \int_{\partial A_f} f(\nu_{A_f}^{-1}(x))s(f,h,\ldots,h)(\nu_{A_f}^{-1}(x))\, dS_{A_f}(x) \\
&= \int_{\partial A_f} h_f(\nu_{A_f}^{-1}(x))s(f,h,\ldots,h)(\nu_{A_f}^{-1}(x))\, dS_{A_f}(x) \\
&= \int_{\mathbb{S}^{n-1}} h_f s(f,h,...,h)\, d\mu_{\mathbb{S}^{n-1}} \\
&= nV(f,h_f,h,...,h) \\
&\geq m \cdot n\, V(h,h_f,h,\ldots,h) \\
&= m \int_{\mathbb{S}^{n-1}} h_f s(h,h,...,h)\, d\mu_{\mathbb{S}^{n-1}} \\
&\geq m \int_{\mathbb{S}^{n-1}} m\, hs(h,h,...,h)\, d\mu_{\mathbb{S}^{n-1}} \\
&= m^2 \cdot n\, Vol(K).
\end{aligned} \tag{7}$$

The second inequality can be proved similarly. □

Consequently, we obtain the following inequalities for $\Omega_{2,n}(K)$.

Proposition 1. *Let $K \in \mathcal{K}_{reg}$ with the usual notations of h and $\mathcal{K}_0$ for the support function, respectively, the centro-affine curvature of K as functions on the sphere $\mathbb{S}^{n-1}$. Then*

1. *$\Omega_{n,2}(K) \geq 0$ with equality if and only if K is a centered ellipsoid.*
2. *If, in addition, the Aleksandrov body associated with $f := h\sqrt{\mathcal{K}_0}$ has continuous positive curvature function, then $\Omega_{n,2}(K) \leq \frac{(n-1)n}{2}(M - m)Vol(K)$, where M, m are the maximum and minimum of the centro-affine curvature of K. The equality occurs if and only if K is a centered ellipsoid.*

Proof. 1. The first claim follows immediately from the Minkowski-type inequality we detailed in Lemma 4.3 of [34]

$$\left(\int_{\mathbb{S}^{n-1}} fs(f,h,\dots,h)\,d\mu_{\mathbb{S}^{n-1}}\right)\left(\int_{\mathbb{S}^{n-1}} hs(h,h,\dots,h)d\mu_{\mathbb{S}^{n-1}}\right)$$
$$\le \left(\int_{\mathbb{S}^{n-1}} fs(h,h,\dots,h)\,d\mu_{\mathbb{S}^{n-1}}\right)^2,$$

where f is an arbitrary smooth function on the sphere, while h is a smooth support function of a convex body. It suffices to apply this inequality to the second term of $\Omega_{n,2}(K)$ with $f := h\sqrt{\mathcal{K}_0}$ to obtain

$$\Omega_{2,n}(K) \ge \frac{n(n-1)}{2}\,Vol(K^\circ) - \frac{n-1}{2n}\,\frac{\Omega_n^2(K)}{Vol(K)}$$

from which the result follows by Hölder's inequality

$$Vol(K^\circ)\cdot Vol(K)$$
$$= \frac{1}{n^2}\left(\int_{\partial K}\mathcal{K}_0\,d\mu_K\right)\cdot\left(\int_{\partial K} d\mu_K\right) \ge \frac{1}{n^2}\left(\int_{\partial K}\sqrt{\mathcal{K}_0}\,d\mu_K\right)^2. \tag{8}$$

Note that the equality is attained if and only if $\mathcal{K}_0$ is constant on $\mathbb{S}^{n-1}$, hence if and only if K is a centered ellipsoid.

2. By taking $f = h\sqrt{\mathcal{K}_0}$ with $m \le \mathcal{K}_0 \le M$, we can apply Eq. (5),

$$\begin{aligned}\Omega_{2,n}(K) &= \frac{n(n-1)}{2}\,Vol(K^\circ)\\ &\quad -\frac{n-1}{2}\int_{\mathbb{S}^{n-1}} h\sqrt{\mathcal{K}_0}\,s(h\sqrt{\mathcal{K}_0},h,\dots,h)\,d\mu_{\mathbb{S}^{n-1}},\\ &\le \frac{n(n-1)}{2}\,\frac{1}{n}\int_{\partial K}\mathcal{K}_0\,d\mu_K - \frac{n(n-1)}{2}\,m\,Vol(K)\\ &\le \frac{n(n-1)}{2}\,(M-m)\,Vol(K).\end{aligned} \tag{9}$$

Equality is attained if and only if $M = m$ which implies, as before, that K is a centered ellipsoid. Note that we have only used the left-hand side inequality of Eq. (5). It so happens that the right-hand side inequality of Eq. (5) follows for this choice of function f from the positivity of $\Omega_{2,n}(K)$ for any $K \in \mathcal{K}_{\text{reg}}$. □

Further, the previous result implies additional isoperimetric-type inequalities.

Theorem 1. *If $K \in \mathcal{K}_{reg}$, the following $Gl(n)$-invariant inequality holds*

$$\frac{1}{n^2}\,\Omega_n^2(K) \le Vol(K)\cdot Vol(K^\circ)$$
$$\le \frac{2}{n(n-1)}\min\{Vol(K)\cdot\Omega_{n,2}(K), Vol(K^\circ)\cdot\Omega_{n,2}(K^\circ)\} + \frac{1}{n^2}\,\Omega_n^2(K),$$

and equality occurs if and only if K is a centered ellipsoid.

If, in addition, K is such that the Aleksandrov body associated with $f := h\sqrt{\mathcal{K}_0}$ has continuous positive curvature function and $\frac{M}{m} \le \frac{1+\sqrt{5}}{2}$, the golden ratio, then the following $Gl(n)$-invariant inequality holds:

$$\frac{1}{n^2}\,\Omega_n^2(K) \le Vol(K)\cdot Vol(K^\circ) \le \frac{1}{n^2}\,\Omega_n^2(K)\left[1-\frac{M-m}{\sqrt{Mm}}\right]^{-1}$$

with equality if and only if K is a centered ellipsoid.

Proof. The left-hand inequality follows immediately from Hölder's inequality. In fact, this easy remark motivated a search for an upper bound of the volume product $Vol(K)\cdot Vol(K^\circ)$ in terms of the centro-affine surface area or, in other words, a reverse isoperimetric-type inequality.

Toward this goal, note that the sign of $\Omega_{2,n}(K)$ translates into the following $Gl(n)$-invariant inequality:

$$\frac{1}{n^2}\,\Omega_n^2(K) \le Vol(K)\cdot Vol(K^\circ) \le \frac{2}{n(n-1)}Vol(K)\cdot\Omega_{n,2}(K) + \frac{1}{n^2}\,\Omega_n^2(K),$$

with equality if and only if K is a centered ellipsoid. Apply the same inequality with the roles of K and K° reversed and use the fact that $\Omega_n(K) = \Omega_n(K^\circ)$, [12,18,39]. Therefore,

$$\frac{1}{n^2}\,\Omega_n^2(K) \le Vol(K)\cdot Vol(K^\circ)$$
$$\le \frac{2}{n(n-1)}\min\{Vol(K)\cdot\Omega_{n,2}(K), Vol(K^\circ)\cdot\Omega_{n,2}(K^\circ)\} + \frac{1}{n^2}\,\Omega_n^2(K),$$

with equality if and only if K is a centered ellipsoid.

From Proposition 1,

$$\frac{2}{n(n-1)}\,Vol(K)\cdot\Omega_{n,2}(K) \le (M-m)\,Vol^2(K)$$

and

$$\frac{2}{n(n-1)}\,Vol(K^\circ)\cdot\Omega_{n,2}(K^\circ) \le (M^\circ-m^\circ)\,Vol^2(K^\circ),$$

thus

$$\frac{2}{n(n-1)}\min\{Vol(K)\cdot\Omega_{n,2}(K), Vol(K^\circ)\cdot\Omega_{n,2}(K^\circ)\}$$
$$\leq\sqrt{(M-m)(M^\circ-m^\circ)}\,Vol(K)\cdot Vol(K^\circ).$$

Here m° and M° are the minimum, respectively, the maximum of the centro-affine curvature of ∂K°.

For any point of ∂K, x, there exists a point y on ∂K° such that $\mathcal{K}_0(x)\cdot\mathcal{K}_0^\circ(y)=1$, see [12], thus $M\cdot m^\circ=1$ and $m\cdot M^\circ=1$ otherwise a contradiction with one of the definitions of m°, M° occurs. Hence

$$\sqrt{(M-m)(M^\circ-m^\circ)}=\sqrt{(M-m)\left(\frac{1}{m}-\frac{1}{M}\right)}=\frac{M-m}{\sqrt{Mm}},$$

which is less or equal to 1 if and only if M/m is less or equal to the golden ratio above.

Thus

$$Vol(K)\cdot Vol(K^\circ)\leq\frac{M-m}{\sqrt{Mm}}\cdot Vol(K)\cdot Vol(K^\circ)+\frac{1}{n^2}\,\Omega_n^2(K)$$

which implies the right-hand side inequality. The equalities follow as before from $M=m$ equivalent to constant centro-affine curvature along the boundary ∂K. □

Note that in the next proposition we drop the smoothness assumption on the boundary of K to class C^2.

Proposition 2. *For any $p>1$, and any $K\in C_+^2$ with the origin in its interior, we have*

$$\frac{\Omega_p^{n+p}(K)}{Vol^{n-p}(K)}\leq n^{p-1}\,\big(Vol(K)\cdot Vol(K^\circ)\big)^{p-1}\cdot\frac{\Omega^{n+1}(K)}{Vol^{n-1}(K)}. \tag{10}$$

The equality holds if and only if $p=1$ or K is a centered ellipsoid.

The opposite inequality holds for $p<1$, $p\neq -n$.

Proof. Note that, for any $p\neq -n$,

$$\Omega_p(K)=\int_{\partial K}\mathcal{K}_0^{\frac{p}{n+p}}\,d\mu_K=\int_{\partial K}\left(\mathcal{K}_0^{\frac{n}{n+1}}\right)^{\frac{p-1}{n+p}}\,d\sigma_K, \tag{11}$$

where $d\sigma_K$ is the affine surface area measure, in other words the Blaschke metric, of K. As the function $x\mapsto x^{\frac{p-1}{n+p}}$, $x>0$, is concave for $p\geq 1$ and convex for $p\leq 1$, we apply the appropriate Jensen's inequality for each range and the normalized measure $\dfrac{1}{\Omega(K)}\,d\sigma_K$. If $p\geq 1$, we obtain

$$\left(\frac{nVol(K^\circ)}{\Omega(K)}\right)^{\frac{p-1}{n+p}} \geq \frac{\Omega_p(K)}{\Omega(K)} \quad \Leftrightarrow \quad \Omega_p(K) \leq \left(n\, Vol(K^\circ)\right)^{\frac{p-1}{n+p}} \cdot \Omega^{\frac{n+1}{n+p}}(K) \quad (12)$$

with equality if and only if $p = 1$ or K is a centered ellipsoid. A re-arrangement of terms, gives Eq. (10). The proof of the reverse inequality in the case $p \leq 1$ is perfectly similar. □

Corollary 1. *For any convex body $K \in \mathcal{K}_{reg}$,*

$$n^n \left[\frac{2}{n-1} Vol(K) \cdot \Omega_{n,2}(K) + \frac{1}{n}\, \Omega_n^2(K)\right]$$
$$\geq \frac{\Omega^{n+1}(K)}{Vol^{n-1}(K)} \geq \frac{\Omega_n^{2n}(K)}{[(2/(n-1))\, \Omega_{n,2}(K) Vol(K) + \Omega_n^2(K)/n]^{n-1}},$$

with equality iff K is a centered ellipsoid.

Proof. Apply the previous result for $p = 0$ and, respectively, $p = n$, and use the bounds on $Vol(K) \cdot Vol(K^\circ)$ from Theorem 1. □

Corollary 2 (Isoperimetric-like Inequality). *For any $K \in C_+^2$ with the centroid at the origin, and any $T \in Sl(n)$,*

$$\frac{S^n(TK)}{Vol^{n-1}(K)} \geq \frac{n}{\omega_n^{2n-3}} \max\left\{\frac{\Omega_n^{2n}(K)}{\Omega^{n+1}(K)/Vol^{n-1}(K)}, \left(\frac{\Omega^{n+1}(K)}{Vol^{n-1}(K)}\right)^{n-1}\right\}, \quad (13)$$

where $S(TK)$ stands for the surface area of TK and ω_n is the volume of the unit ball $x_1^2 + \ldots + x_n^2 = 1$ in $\mathbb{R}^n$. Equality occurs if and only if K is a centered ellipsoid and T is the linear transformation of determinant one such that TK is a ball.

Hence

Proof. Consider $p = n$ in the inequality of Proposition 2 to obtain

$$\Omega_n^{2n}(K) \leq n^{n-1}[Vol(K) \cdot Vol(K^\circ)]^{n-1} \cdot \frac{\Omega^{n+1}(K)}{Vol^{n-1}(K)}. \quad (14)$$

From the classical isoperimetric inequality,

$$Vol^{n-1}(K) \leq \left(Vol^{n-1}(B)/S^n(B)\right) S^n(K),$$

where B is the unit ball as above. On the other hand, by Blaschke-Santaló inequality, $Vol(K) \cdot Vol(K^\circ) \leq (Vol(B))^2$.

Therefore

$$\Omega_n^{2n}(K) \leq n^{n-1} \frac{Vol^{3(n-1)}(B)}{S^n(B)} \frac{S^n(K)}{Vol^{n-1}(K)} \cdot \frac{\Omega^{n+1}(K)}{Vol^{n-1}(K)}, \quad (15)$$

where all quantities, except $S(K)$, are invariant under linear transformations of determinant one. Hence, the conclusion follows as $n^{n-1}\dfrac{Vol^{3(n-1)}(B)}{S^n(B)} = \dfrac{\omega_n^{2n-3}}{n}$. To analyze the equality case one needs to take T to be the linear transformation of determinant one minimizing the surface area of K and note that all other equalities hold if and only if K is a centered ellipsoid.

We will now use $p = 0$ in Proposition 2, to obtain

$$\frac{\Omega^{n+1}(K)}{Vol^{n-1}(K)} \leq n\, Vol(K)\cdot Vol(K^\circ) \leq n\,\frac{V(B)}{S(B)^{n/(n-1)}}\cdot S(K)^{n/(n-1)}\cdot Vol(K^\circ)$$
$$\leq n\,\frac{V(B)^3}{S(B)^{n/(n-1)}}\cdot\frac{S(K)^{n/(n-1)}}{Vol(K)} = n^{1-\frac{n}{n-1}}\,\omega_n^{3-\frac{n}{n-1}}\,\frac{S(K)^{n/(n-1)}}{Vol(K)}, \tag{16}$$

relying again on Blaschke-Santaló inequality.

From here,

$$\frac{S^n(TK)}{Vol^{n-1}(K)} \geq \frac{n}{\omega_n^{2n-3}}\left(\frac{\Omega^{n+1}(K)}{Vol^{n-1}(K)}\right)^{n-1}, \tag{17}$$

with the same condition for the equality case as above. □

One can use K. Ball's reverse isoperimetric ratio which gives an upper bound on $\dfrac{S^n(TK)}{Vol^{n-1}(K)}$ by the corresponding ratio for the regular solid simplex in $\mathbb{R}^n$ (or the solid cube in the centrally-symmetric case), [1,2], in the above corollary to get lower bounds on the affine isoperimetric ratio of bodies in C^2_+. However, these bounds will not be sharp.

As in Corollary 1, one can drop the requirement that the centroid of K is at the origin, consider $K \in \mathcal{K}_{\text{reg}}$, and use the upper bound on the volume product from Theorem 1 instead of Blaschke-Santaló inequality, to obtain $SL(n)$ invariant lower bounds on the isoperimetric ratio $S(TK)^n/Vol(K)^{n-1}$.

Finally, we include the next corollary, due to [30], which follows immediately from Proposition 2.

Corollary 3. *For any convex body K of class C^2_+ containing the origin in its interior,*

$$\Omega_K \leq \frac{\Omega^{n+1}(K)}{(nVol(K^\circ))^{n+1}}, \tag{18}$$

where $\Omega_K := \lim_{p\to\infty}\left(\dfrac{\Omega_p(K)}{nVol(K^\circ)}\right)^{n+p}$ is the affine invariant introduced by Paouris and Werner in [30]. The equality occurs if and only if K is a centered ellipsoid.

Note that in [30], for certain considerations, the invariant Ω_K has been defined for convex bodies whose centroid is at the origin, yet the above definition makes sense for any convex body K of class C^2_+ containing the origin in its interior for which one can show as in [30] that the limit exists.

3 More on the Paouris-Werner Invariant

Motivated by the earlier occurrence of Ω_K, we would like to give here a couple of other definitions of this invariant when K belongs to C^2_+. To do so, let us also recall that Paouris and Werner showed in [30] that Ω_K is related to the Kullback-Leibler divergence D_{KL} of two specific probability measures P, Q on ∂K via the relation $D_{KL}(P\|Q) = \ln\left(\frac{Vol(K)}{Vol(K^\circ)}\Omega_K^{-1/n}\right)$, where, in slightly different terms than in [30],

$$D_{KL}(P\|Q) := \frac{1}{nVol(K^\circ)}\int_{\partial K}\mathcal{K}_0 \ln\left(\mathcal{K}_0\,\frac{Vol(K)}{Vol(K^\circ)}\right) d\mu_K.$$

Hence, it is useful to note the identity

$$\ln(\Omega_K) = -\frac{1}{Vol(K^\circ)}\int_{\partial K}\mathcal{K}_0 \ln \mathcal{K}_0 \, d\mu_K, \tag{19}$$

and note that, in this paper, we assume only that the origin is contained in the interior of the convex body K.

Proposition 3. *For any K of class C^2_+ containing the origin in its interior, and any integer $p > 1$, the following $Gl(n)$-invariant inequalities hold*

$$\Omega_n^2(K) \ge \frac{\left(\Omega_{n/3}(K)\right)^4}{(nVol(K))^2} \ge \frac{\left(\Omega_{n/7}(K)\right)^8}{(nVol(K))^6} \ge \dots \ge \frac{\left(\Omega_{n/(2^p-1)}(K)\right)^{2^p}}{(nVol(K))^{2^p-2}} \ge \dots, \tag{20}$$

or, alternately,

$$\Omega_n^2(K) \ge \frac{(\Omega_{3n}(K^\circ))^4}{(nVol(K))^2} \ge \frac{(\Omega_{7n}(K^\circ))^8}{(nVol(K))^6} \ge \dots \ge \frac{\left(\Omega_{n(2^p-1)}(K^\circ)\right)^{2^p}}{(nVol(K))^{2^p-2}} \ge \dots, \tag{21}$$

$$\Omega_n^2(K) \ge \frac{(\Omega_{3n}(K))^4}{(nVol(K^\circ))^2} \ge \frac{(\Omega_{7n}(K))^8}{(nVol(K^\circ))^6} \ge \dots \ge \frac{\left(\Omega_{n(2^p-1)}(K)\right)^{2^p}}{(nVol(K^\circ))^{2^p-2}} \ge \dots. \tag{22}$$

In all sequences, all equalities hold if and only if K is a centered ellipsoid (which is the only reason why we did not include $p = 1$ in the statement).

Proof. Note that Eqs. (20) and (21) are equivalent through the equality $\Omega_q(K) = \Omega_{n^2/q}(K^\circ)$, [12, 18, 39]. The same goes for Eq. (22) due to $\Omega_n(K) = \Omega_n(K^\circ)$ and interchanging the roles of K and K° in the previous sequence of inequalities. Thus, it suffices to prove (20).

We will use the concavity of the function $x \mapsto \sqrt{x}$ on $(0, \infty)$ and Jensen's inequality as follows:

$$\left(\frac{\Omega_n(K)}{nVol(K)}\right)^{1/2} = \left(\int_{\partial K} \sqrt{\mathcal{K}_0}\, \frac{d\mu_K}{n\, Vol(K)}\right)^{1/2} \geq \int_{\partial K} \sqrt[4]{\mathcal{K}_0}\, \frac{1}{n\, Vol(K)}\, d\mu_K, \quad (23)$$

thus

$$\left(\frac{\Omega_n(K)}{nVol(K)}\right)^{1/2} \geq \frac{\Omega_{n/3}(K)}{nVol(K)},$$

which is, after raising both sides to power four, the first inequality of Eq. (20). Re-iterate now the same argument for $\Omega_{n/3}(K)$:

$$\left(\frac{\Omega_{n/3}(K)}{nVol(K)}\right)^{1/2} = \left(\int_{\partial K} \sqrt[4]{\mathcal{K}_0}\, \frac{d\mu_K}{n\, Vol(K)}\right)^{1/2} \geq \int_{\partial K} \sqrt[8]{\mathcal{K}_0}\, \frac{1}{n\, Vol(K)}\, d\mu_K, \quad (24)$$

which translates into

$$\left(\frac{\Omega_{n/3}(K)}{nVol(K)}\right)^{1/2} \geq \frac{\Omega_{n/7}(K)}{nVol(K)}.$$

Hence

$$\Omega_n(K) \geq \frac{\Omega^2_{n/3}(K)}{nVol(K)} \geq \frac{\Omega^4_{n/7}(K)}{(nVol(K))^3}$$

and so on, the sequence is obtained by iterating the argument. □

Theorem 2 (Alternative Definition of Ω_K). *For any K of class C^2_+ containing the origin in its interior, the scaling invariant sequence*

$$\left\{ \frac{\left(\Omega_{n(2^p-1)}(K)\right)^{2^p}}{(nVol(K^\circ))^{2^p}} \right\}_{p\in\mathbb{N},\ p\geq 1}$$

converges and

$$\lim_{p\to\infty} \left(\frac{\Omega_{n(2^p-1)}(K)}{nVol(K^\circ)}\right)^{2^p} = \Omega_K. \quad (25)$$

Proof. By Eq. (22), the positive sequence $\dfrac{\left(\Omega_{n(2^p-1)}(K)\right)^{2^p}}{(nVol(K^\circ))^{2^p-2}}$ is decreasing, thus converges. Therefore, so does the sequence above whose general term differs from general term of the former sequence by a factor of $(nVol(K^\circ))^{-2}$.

Let $q := n(2^p - 1)$, and, similarly with Proposition 3.6 in [30], consider

$$
\begin{aligned}
&\ln\left[\lim_{p\to\infty} \frac{\left(\Omega_{n(2^p-1)}(K)\right)^{2^p}}{(nVol(K^\circ))^{2^p}}\right] \\
&= \lim_{p\to\infty} 2^p \ln\left(\frac{\Omega_{n(2^p-1)}(K)}{nVol(K^\circ)}\right) \\
&= -\frac{2^p}{\ln 2} \frac{\frac{d}{dp}\left(\Omega_{n(2^p-1)}(K)\right)}{\Omega_{n(2^p-1)}(K)} \\
&= -\lim_{p\to\infty} \frac{2^p}{\ln 2} \frac{\frac{d}{dq}\left(\int_{\partial K} \exp\left(\ln \mathcal{K}_0^{\frac{q}{n+q}}\right) d\mu_K\right) \frac{dq}{dp}}{\Omega_{n(2^p-1)}(K)} \\
&= -\lim_{p\to\infty} 2^{2p} \frac{\frac{d}{dq}\left(\int_{\partial K} \exp\left(\ln \mathcal{K}_0^{\frac{q}{n+q}}\right) d\mu_K\right)}{\Omega_{n(2^p-1)}(K)} \\
&= -\lim_{p\to\infty} 2^{2p} \frac{\left(\int_{\partial K} \exp\left(\ln \mathcal{K}_0^{\frac{q}{n+q}}\right) \ln(\mathcal{K}_0) \frac{n}{(n+q)^2} d\mu_K\right)}{\Omega_{n(2^p-1)}(K)} \\
&= -n \lim_{p\to\infty} \frac{\int_{\partial K} \mathcal{K}_0^{\frac{2^p-1}{2^p}} \ln(\mathcal{K}_0)\, d\mu_K}{\Omega_{n(2^p-1)}(K)} \\
&= -n \frac{\int_{\partial K} \mathcal{K}_0 \ln(\mathcal{K}_0)\, d\mu_K}{n\, Vol(K^\circ)} = \ln(\Omega_K).
\end{aligned}
$$

The last equality, due to Eq. (19), completes the proof. □

Following from the monotonicity of the sequence (22), we have

Corollary 4. *For any K of class C^2_+ containing the origin in its interior, and any integer $p \geq 1$,*

$$\Omega_K \cdot (nVol(K^\circ))^2 \leq \frac{(\Omega_{n(2^p-1)}(K))^{2^p}}{(nVol(K^\circ))^{2^p-2}}, \tag{26}$$

in particular $\Omega_K \cdot (nVol(K^\circ))^2 \leq \Omega_n^2(K)$, with equalities everywhere if and only if K is a centered ellipsoid.

Corollary 5. *For any K of class C^2_+ containing the origin in its interior, and any integer $p \geq 1$,*

$$\Omega_K \cdot \Omega_{K^\circ} \leq \frac{(\Omega_{n(2^p-1)}(K) \cdot \Omega_{n(2^p-1)}(K^\circ))^{2^p}}{(n^2 Vol(K) \cdot Vol(K^\circ))^{2^p}}, \tag{27}$$

in particular $\Omega_K \cdot \Omega_{K^\circ} \leq \dfrac{\Omega_n^2(K)\cdot\Omega_n^2(K^\circ)}{(n^2 Vol(K)\cdot Vol(K^\circ))^2}$*, with equalities everywhere if and only if* K *is a centered ellipsoid in which case the right-hand sides of the two inequalities are equal to* 1.

The definition of Ω_K can be extended to affine surface areas of negative exponent using a similar result with Proposition 3:

Theorem 3 (Second Alternative Definition of Ω_K). *For any* K *of class* C_+^2 *containing the origin in its interior, the sequence*

$$\left\{\left(\frac{\Omega_{-(n+2^p)}(K^\circ)}{nVol(K)}\right)^{2^p}\right\}_{p\in\mathbb{N},\ p\geq 1}$$

converges and

$$\lim_{p\to\infty}\left(\frac{\Omega_{-(n+2^p)}(K^\circ)}{nVol(K)}\right)^{2^p} = \Omega_K^{-1}. \tag{28}$$

Proof. By applying again Jensen's inequality for the concave function $x \mapsto \sqrt{x}$, $x > 0$, we have, for any integer $p \geq 1$,

$$\begin{aligned}\int_{\partial K}\mathcal{K}_0^{-\frac{n}{2}}\,d\mu_K &\geq \frac{\left(\int_{\partial K}\mathcal{K}_0^{-\frac{n}{4}}\,d\mu_K\right)^2}{n\ Vol(K)} \geq \frac{\left(\int_{\partial K}\mathcal{K}_0^{-\frac{n}{8}}\,d\mu_K\right)^4}{(n\ Vol(K))^3}\\ &\geq \ldots \geq \frac{\left(\int_{\partial K}\mathcal{K}_0^{-\frac{n}{2^p}}\,d\mu_K\right)^{2^p}}{(n\ Vol(K))^{2^p-1}} \geq \ldots\end{aligned} \tag{29}$$

therefore the sequence of general term

$$\begin{aligned}\left(\frac{\Omega_{-(n+2^p)}(K^\circ)}{nVol(K)}\right)^{2^p} &= \left(\frac{\Omega_{-n^2/(n+2^p)}(K)}{nVol(K)}\right)^{2^p}\\ &= \left(\frac{1}{n\ Vol(K)}\cdot\frac{(\Omega_{-n^2/(n+2^p)}(K))^{2^p}}{(nVol(K))^{2^p-1}}\right)\end{aligned}$$

is monotone. Interchanging K with K°, we conclude that the sequence $\left\{\left(\frac{\Omega_{-(n+2^p)}(K)}{nVol(K^\circ)}\right)^{2^p}\right\}_{p\in\mathbb{N},\ p\geq 1}$ is monotone.

We now proceed as in the previous theorem with

$$\ln\left[\lim_{p\to\infty}\frac{\left(\Omega_{-(n+2^p)}(K)\right)^{2^p}}{(nVol(K^\circ))^{2^p}}\right] = \lim_{p\to\infty}2^p\ln\left(\frac{\Omega_{-(n+2^p)}(K)}{nVol(K^\circ)}\right)$$

$$= -\frac{2^p}{\ln 2} \frac{\frac{d}{dp}\left(\Omega_{-(n+2^p)}(K)\right)}{\Omega_{-(n+2^p)}(K)}$$

$$= - \lim_{p\to\infty} \frac{2^p}{\ln 2} \frac{\frac{d}{dp}\left(\int_{\partial K} \exp\left(\ln \mathcal{K}_0^{\frac{n}{2^p}+1}\right) d\mu_K\right)}{\Omega_{-(n+2^p)}(K)}$$

$$= n \lim_{p\to\infty} \frac{\left(\int_{\partial K} \exp\left(\ln \mathcal{K}_0^{\frac{n+2^p}{2^p}}\right) \ln(\mathcal{K}_0)\, d\mu_K\right)}{\Omega_{-(n+2^p)}(K)}$$

$$= n \frac{\int_{\partial K} \mathcal{K}_0 \ln(\mathcal{K}_0)\, d\mu_K}{n\, Vol(K^\circ)} = -\ln(\Omega_K),$$

and, using Eq. (19), we complete the proof of the theorem. □

While it is known that integrals of the form $\displaystyle\int_{\partial K} \phi(\mathcal{K}_0)\, d\mu_K$ are $SL(n)$-invariant, see also [16, 18], considering the results in [30], and others, including for example the next theorem, we conjecture that the set of p-affine surface areas, with algebraic operations, can generate, by taking the *closure*, all integrals of the above form.

Theorem 4. *For any K of class C^2_+ containing the origin in its interior, the $SL(n)$-invariant $\Lambda(K) := \exp\left[\frac{1}{nVol(K)} \int_{\partial K} \ln(\mathcal{K}_0)\, d\mu_K\right]$ is the limit, as $p \to +\infty$, of the sequence* $\left\{\left(\frac{\Omega_{-\frac{n}{2^p}}(K)}{n\, Vol(K)}\right)^{2^p}\right\}_{p\in\mathbb{N},\ p>1}$.

Proof. The claim follows directly from

$$\ln\left[\lim_{p\to\infty} \frac{\left(\Omega_{-n/2^p}(K)\right)^{2^p}}{(nVol(K^\circ))^{2^p}}\right]$$

$$= \lim_{p\to\infty} 2^p \ln\left(\frac{\Omega_{-n/2^p}(K)}{nVol(K^\circ)}\right)$$

$$= -\frac{2^p}{\ln 2} \frac{\frac{d}{dp}\left(\Omega_{-n/2^p}(K)\right)}{\Omega_{-n/2^p}(K)}$$

$$= - \lim_{p\to\infty} \frac{2^p}{\ln 2} \frac{\frac{d}{dp}\left(\int_{\partial K} \exp\left(\ln \mathcal{K}_0^{-\frac{1}{2^p-1}}\right) d\mu_K\right)}{\Omega_{-n/2^p}(K)}$$

$$= \lim_{p\to\infty} \frac{2^{2p}}{(2^p-1)^2} \frac{\left(\int_{\partial K} \exp\left(\ln \mathcal{K}_0^{-\frac{1}{2^p-1}}\right) \ln(\mathcal{K}_0)\, d\mu_K\right)}{\Omega_{-n/2^p}(K)}$$

$$= \lim_{p\to\infty} \frac{2^{2p}}{(2^p-1)^2} \frac{\left(\int_{\partial K} \mathcal{K}_0^{-\frac{1}{2^p-1}} \ln(\mathcal{K}_0)\, d\mu_K\right)}{\Omega_{-n/2^p}(K)}$$

$$= \frac{\int_{\partial K} \ln(\mathcal{K}_0)\, d\mu_K}{n\, Vol(K)} = \ln(\Lambda_K). \qquad \square$$

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References

1. K.M. Ball, in *Volumes of Sections of Cubes and Related Problems*, Israel Seminar on G.A.F.A., 1987–88, ed. by Lindenstrauss and Milman, Lecture Notes in Mathematics, Vol. 1376, (Springer, Berlin, 1989) pp. 251–260
2. K.M. Ball, Volume ratios and a reverse isoperimetric inequality. J. London Math. Soc. **44**, 351–359 (1991)
3. I. Bárány, Affine perimeter and limit shape. J. Reine Angew. Math. **484**, 71–84 (1997)
4. K. Böröczky, E. Lutwak, D. Yang, G. Zhang, The logarithmic Minkowski problem, to appear in J. Amer. Math. Soc.
5. K. Chou, X.-J. Wang, The L_p-Minkowski problem and the Minkowski problem in centroaffine geometry. Adv. Math. **205**, 33–83 (2006)
6. A. Cianchi, E. Lutwak, D. Yang, G. Zhang, Affine Moser-Trudinger and Morrey-Sobolev inequalities. Calc. Var. Partial Diff. Equations **36**, 419–436 (2009)
7. P.M. Gruber, Asymptotic estimates for best and stepwise approximation of convex bodies II. Forum Math. **5**, 521–538 (1993)
8. C. Haberl, M. Ludwig, A characterization of L_p intersection bodies. Int. Math. Res. Not. **2006**, 1–29 (2006). doi:10.1155/imrn/2006/10548.
9. C. Haberl, E. Lutwak, D. Yang, G. Zhang, The even Orlicz Minkowski problem. Adv. Math. **224**, 2485–2510 (2010)
10. C. Haberl, F. Schuster, Asymmetric affine L_p Sobolev inequalities. J. Funct. Anal. **257**, 641–658 (2009)
11. C. Haberl, F. Schuster, General L_p affine isoperimetric inequalities. J. Differential Geom. **83**, 1–26 (2009)
12. D. Hug, Curvature Relations and Affine Surface Area for a General Convex Body and its Polar. Results in Math. **29**, 233–248 (1996)
13. D. Hug, Contributions to affine surface area, Manuscripta Math. **91**, 283–301 (1996)
14. K. Leichtweiss, *Affine Geometry of Convex Bodies* (John Wiley and Sons, New York, 1999)
15. M. Ludwig, Minkowski valuations. Trans. Amer. Math. Soc. **357**, 4191–4213 (2005)
16. M. Ludwig, General affine surface areas. Adv. Math. **224**, 2346–2360 (2010)
17. M. Ludwig, Fisher information and matrix-valued valuations. Adv. Math. **226**, 2700–2711 (2011)
18. M. Ludwig, M. Reitzner, A classification of $SL(n)$ invariant valuations. Annals of Math. **172**, 1223–1271 (2010)
19. M. Ludwig, J. Xiao, G. Zhang, Sharp convex Lorentz-Sobolev inequalities. Math. Ann. **350**, 169–197 (2011)

20. E. Lutwak, The Brunn-Minkowski-Firey theory. I: Mixed volumes and the Minkowski problem. J. Differential Geom. **38**, 131–150 (1993)
21. E. Lutwak, The Brunn-Minkowski-Firey theory II: Affine and geominimal surface areas. Adv. Math. **118**, 244–294 (1996)
22. E. Lutwak, V. Oliker, On the regularity of solutions to a generalization of the Minkowski problem. J. Differential Geom. **41**, 227–246 (1995)
23. E. Lutwak, D. Yang, G. Zhang, L_p affine isoperimetric inequalities. J. Differential Geom. **56**, 111–132 (2000)
24. E. Lutwak, D. Yang, G. Zhang, Sharp affine L_p Sobolev inequalities. J. Differential Geom. **62**, 17–38 (2002)
25. E. Lutwak, D. Yang, G. Zhang, Volume inequalities for subspaces of L_p. J. Differential Geom. **68**, 159–184 (2004)
26. E. Lutwak, D. Yang, G. Zhang, Optimal Sobolev norms and the L_p Minkowski problem. Int. Math. Res. Not. IMRN 62987, 1–10 (2006)
27. E. Lutwak, D. Yang, G. Zhang, Cramér-Rao and moment-entropy inequalities for Renyi entropy and generalized Fisher information. IEEE Trans. Inform. Theory **51**, 473–478 (2005)
28. E. Lutwak, G. Zhang, Blaschke–Santaló inequalities. J. Differential Geom. **47**, 1–16 (1997)
29. M. Meyer, E. Werner, On the p-affine surface area. Adv. Math. **152**, 288–313 (2000)
30. G. Paouris, E. Werner, Relative entropy of cone measures and L_p centroid bodies. Proc. London Math. Soc. Vol. **104**, 253–286 (2012)
31. C.M. Petty, Affine isoperimetric problems, in *Discrete Geometry and Convexity* ed. by J.E. Goodman, E. Lutwak, J. Malkevitch, R. Pollack. Proceedings Conference New York 1982. Annals of the New York Academic of Sciences, vol. 440, 1985, pp. 113–127
32. R. Schneider, *Convex bodies: The Brunn-Minkowski theory* (Cambridge Univ. Press, New York, 1993)
33. C. Schütt, E. Werner, Surface bodies and p-affine surface area. Adv. Math. **187**, 98–145 (2004)
34. A. Stancu, Centro-Affine Invariants for Smooth Convex Bodies. Int. Math. Res. Notices **2012**, 2289–2320 (2012) doi: 10.1093/imrn/rnr110.
35. N.S. Trudinger, X.-J. Wang, The Bernstein problem for affine maximal hypersurfaces. Invent. Math. **140**, 399–422 (2000)
36. N.S. Trudinger, X.-J. Wang, Affine complete locally convex hypersurfaces. Invent. Math. **150**, 45–60 (2002)
37. N.S. Trudinger, X.-J. Wang, The Affine Plateau Problem. J. Amer. Math. Soc. **18**, 253–289 (2005)
38. N.S. Trudinger, X.-J. Wang, Boundary regularity for the Monge-Ampère and affine maximal surface equations. Ann. of Math. **167**, 993–1028 (2008)
39. E. Werner, D. Ye, New L_p affine isoperimetric inequalities. Adv. in Math. **218**, 762–780 (2008)
40. E. Werner, D. Ye, Inequalities for mixed p-affine surface area. Math. Annalen **347**, 703–737 (2010)