Duality on Convex Sets in Generalized Regions

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Abstract Recently, the duality relation on several families of convex sets was shown to be completely characterized by the simple property of reversing order. The families discussed in aforementioned results were convex sets in $\mathbb{R}^n$. Our goal in this note is to generalize this type of results to regions in $\mathbb{R}^n$ bounded between two convex sets.

Key words Duality of convex bodies • Fractional linear transformations • Order isomorphism

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1 Introduction

Let A and B be two convex sets in $\mathbb{R}^n$ such that $A \subset B$. Denote by $\mathcal{K}^n(A, B)$ the class of all closed convex sets containing A and contained in B:

$$\mathcal{K}^n(A, B) = \{K \in \mathcal{K}^n : A \subseteq K \subseteq B\}.$$

Note that we may consider simply all sets contained in a given set B, or alternatively all sets containing A, if we allow for $A = \phi$ and $B = \mathbb{R}^n$, so that for example $\mathcal{K}^n = \mathcal{K}^n(\emptyset, \mathbb{R}^n)$. Our goal is to determine all order-preserving isomorphisms and as a result order-reversing isomorphisms on this class. Note that characterizations for the special cases $\mathcal{K}^n(0, \mathbb{R}^n)$ and $\mathcal{K}^n(\emptyset, \mathbb{R}^n)$ were done by Artstein–Milman in [1, 2]. In addition, the family of convex bodies with zero in the interior was considered by Böröczky and Schneider in [4]. A special case of the aforementioned

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family where the bodies are origin symmetric was considered by Gruber in [5]. A somewhat different family of closed convex cones was also treated by Schneider in [9]. We remark that in the aforementioned papers [4, 5, 9] the authors actually consider a more general setting, namely the classification of the endomorphisms of corresponding lattices. In this paper we generalize these theorems to the class $\mathcal{K}^n(A, B)$ for any convex A, B such that A is compact. To state our first result, we introduce the class of n-dimensional fractional linear maps, which will play a central role in this paper. In the sequel we assume we have some fixed Euclidean structure $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$.

Definition 1. A map $F : \mathbb{R}^n \to \mathbb{R}^n$ will be called fractional linear if

$$F(x) = \frac{Ax + b}{\langle c, x\rangle + d},$$

where A is a $n \times n$ matrix, $b, c \in \mathbb{R}^n$ and $d \in \mathbb{R}$, such that the matrix

$$\hat{A} = \begin{pmatrix} A & b \\ c & d \end{pmatrix}$$

is invertible.

We denote the family of fractional linear maps by $F.L(n)$. Note that a fractional linear map is actually the restriction of a projective map. For further information of such maps, we refer the reader to [3, 10]. We will use a result by Shiffman, roughly stating that any injective map, that preserves, in some sense, most of the intervals, on some open set must be fractional linear. More precisely, given an open set U, denote by $\mathcal{L}(U)$ the set of lines in $\mathbb{R}^n$ intersecting U. Then:

Theorem 1 (Shiffman). *Let $n \geq 2$. Let $U \subset \mathbb{R}^n$ be an open connected set and let $\mathcal{L}_0$ be an open subset of $\mathcal{L}(U)$ that covers U. Assume that $F : U \to \mathbb{R}^n$ is an injective continuous map and that $F(l \cap U)$ is contained in a line for every $l \in \mathcal{L}_0$. Then F is a fractional linear map.*

Let us state our main result.

Theorem 2. *Let $n \geq 2$ and $A_1, B_1, A_2, B_2 \in \mathcal{K}^n$ such that A_1, A_2 are compact and $A_i \subset B_i$ for $i = 1, 2$. Also, assume that $int(B_1) \neq \emptyset$. Let $T : \mathcal{K}^n(A_1, B_1) \to \mathcal{K}^n(A_2, B_2)$ be a bijective mapping satisfying for all $K, L \in \mathcal{K}^n(A_1, B_1)$:*

$$K \subset L \Leftrightarrow T(K) \subset T(K).$$

Then there exists $F \in F.L(n)$ such that $T(K) = F(K)$ for all $K \in \mathbb{K}$. Moreover, $A_2 = F(A_1)$ and $B_2 = F(B_1)$.

As a corollary of Theorem 2 we get a characterization of duality on the class $\mathcal{K}^n(A, B)$ in case where $0 \in A$. The result is presented in Sect. 3.6.

Remark 1. A theorem in the spirit of Theorem 2 was proved for the case of convex functions by Artstein–Florentin–Milman in [3]. For the sake of completeness, we state the result here; given two compact sets $K \subset T$ define the class

$$Cvx_T(K) = \{f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} : f \text{ is convex }, 1_T^\infty \leq f \leq 1_K^\infty\},$$

where $1_K^\infty = -\log 1_K$ is the convex indicator function.

Theorem 3 (Artstein–Florentin–Milman). *Let $n \geq 2$. Let $A_1 \subset B_1$ and $A_2 \subset B_2$ be compact convex sets. Let $T : Cvx_{A_1}(B_1) \to Cvx_{A_2}(B_2)$ be an order-preserving bijection. Then, there exists a fractional linear map $F : B_1 \times \mathbb{R}^+ \to B_2 \times \mathbb{R}^+$ such that for every $f \in Cvx_{A_1}(B_1)$ we have*

$$epi(Tf) = F(epif),$$

where $epi(f)$ is the epigraph of f.

We remark that in the heart of all previous results (where $B = \mathbb{R}^n$) lies the use of extremal sets of the classes in hand that satisfy some useful properties. Using these extremal sets leads to a construction of a point map, inducing the orderisomorphism. Then the use of the fundamental theorem of affine geometry (see [8]) essentially completes the proof. In our setting, the use of extremal families is also central and, in fact, an extremal property which holds in general for all of our cases is described and proved in Sect. 2. Using Theorem 1 that generalizes the classical fundamental theorem of projective geometry, we will conclude that the inducing map is a fractional linear map.

2 Extremal Families

We will denote by $A \vee B$ the closed convex hull of A and B, i.e. $\overline{\text{conv}\{A, B\}}$.

Definition 2. Let K be a closed convex set. A closed convex subset $L \subset K$ is said to be *extremal* in K if for every $x \in L, a, b \in K$,

$$x \in (a, b) \implies [a, b] \subset L.$$

Definition 3. Let F be a family of closed convex sets. A set $K \in F$ is said to be *extremal* in F if for every $A, B \in F$,

$$K = A \vee B \implies A = K \quad \text{or} \quad B = K.$$

Characterization of extremal sets of $\mathcal{K}^n(A, B)$ is given in Sect. 3.2.

Recall Klee's theorem [6, 7], which states that if a closed convex set in $\mathbb{R}^n$ does not contain a full line, it is the convex hull of its extremal points and extremal rays (extremal subsets, which are a point/ray respectively).

Definition 4. Consider the set $\mathcal{K}^n(A, B)$. Given two points $x, y \in B \setminus A$, we will say that they are comparable if $x \in A \vee \{y\}$ or $y \in A \vee \{x\}$.

3 Proof of Theorem 2

The plan of the proof is as follows. First, we will describe all the possibilities for extremal elements in $\mathcal{K}^n(A, B)$. Then, we will show that they are preserved under T. This will provide us with a point map F that induces the map T. After that, we will show that the point map preserves, in some sense, most of the intervals and apply a stronger version of the fundamental theorem of affine geometry to conclude that F is a fractional linear map.

3.1 Largest and Smallest Elements

Since $A_1 \subset K$ for all $K \in \mathcal{K}^n(A_1, B_1)$ it must hold that $T(A_1) \subset T(K)$. Since T is onto we get that $T(A_1)$ is a subset of every element of $\mathcal{K}^n(A_2, B_2)$ and thus $T(A_1) = A_2$. In the same way, $T(B_1) = B_2$.

3.2 Extremal Elements

Remark 2. Several proofs in this section are based on separation of convex sets by hyperplanes. Unless stated otherwise, separation is assumed to be strict, i.e. if a hyperplane H separates sets X, Y then it is assumed that H, X and Y are disjoint.

Lemma 1. *Let $A \in \mathcal{K}^n$ be a compact convex set and $B \in \mathcal{K}^n$. Then, if $K \in \mathcal{K}^n(A, B)$ is extremal, then either $K = A \vee \{x\}$ for some point x or $K = A \vee R$ for some ray R.*

Proof. Case 1: Assume that K contains some ray R. If $K \neq A \vee R$, then there exists some point $x \in K \setminus (A \vee R)$. Due to compactness of A there exists a hyperplane H that separates x from $A \vee R$ and contains no translate of R. Denote by H^+ the closed half-space that contains x and by H^- the half-space that contains $A \vee R$. Denote by $K^+ = A \vee (K \cap H^+)$ and $K^- = A \vee (K \cap H^-)$. Then, by the choice of H we know that K^+ does not contain R, K^- does not contain x and $K = K^+ \vee K^-$, which is a contradiction to extremality of K.

Case 2: Assume that K contains no rays. In this case it is not hard to check that K is a compact set (see e.g. [6]). Thus, K can be written as $A \vee E$ where E is the set of extremal exposed points in K which are not in A. Take a point $x_1 \in E$. If $E \setminus \{x_1\}$ is empty, we are done. Otherwise, there exists a point $x_2 \in E \setminus \{x_1\}$. Choose a hyperplane H that separates x_2 from $A \vee \{x_1\}$. Define H^+ the closed half-space containing x_2 and H^- the closed half-space containing $A \vee \{x_1\}$. Consider $K^+ = A \vee (K \cap H^+)$ and $K^- = A \vee (K \cap H^-)$. Obviously, $K = K^+ \vee K^-$. It is also clear that K^+ does not contain x_1. Indeed, since A and $K \cap H^+$ are both compact, their convex hull is the union of all convex combinations of $a \in A$ and $k \in K \cap H^+$. Since x_1 is an extremal point of K it cannot be written as convex combination of points in K. Thus $x_1 \notin K^+$. Obviously $x_2 \notin K^-$; hence we get a contradiction to extremality of K. $\square$

3.3 Segments Are Mapped to Segments

Let $T : \mathcal{K}^n(A_1, B_1) \to \mathcal{K}^n(A_2, B_2)$ be an order-preserving isomorphism. As we already noticed, $T(A_1) = A_2$ and $T(B_1) = B_2$. Since T is order isomorphism, it is easy to see that the following holds:

1. $T(K_1 \vee K_2) = T(K_1) \vee T(K_2)$.
2. $T(K_1 \cap K_2) = T(K_1) \cap T(K_2)$.

Thus, we conclude that extremal elements of $\mathcal{K}^n(A_1, B_1)$ are mapped to extremal elements of $\mathcal{K}^n(A_2, B_2)$.

Denote by E_i the set $B_i \setminus A_i$, for $i = 1, 2$. Before we define our point map, let us check that an extremal element of the form $A_1 \vee \{x\}$ cannot be mapped to $A_2 \vee R$ for some ray:

Lemma 2. *For every point $x_1 \in E_1$ there exists $x_2 \in E_2$ such that $T(A_1 \vee \{x_1\}) = A_2 \vee \{x_2\}$.*

Proof. Assume the contrary: There exists a point $x \in E_1$ and a ray R such that $T(A_1 \vee \{x\}) = A_2 \vee \{R\}$. Since E_1 is open, there exists $y \in E_1$ such that $A_1 \vee \{x\} \subsetneq A_1 \vee \{y\}$. The image of $A_1 \vee \{y\}$ must be an extremal element that contains $A_2 \vee R$, but this is impossible, unless $T(A_1 \vee \{y\}) = A_2 \vee R$ and thus $y = x$, which is a contradiction. $\square$

Now, since T is bijective, we get a well-defined point map $F : B_1 \setminus A_1 \to B_2 \setminus A_2$ as follows:

$$T(A_1 \vee \{x\}) = A_2 \vee \{F(x)\}.$$

Clearly F is also bijective. Our main goal now is to show that F is a fractional linear map.

Lemma 3. *Let $a, b \in E_1$ two points such that the line passing through them does not intersect A_1. Then, the line passing through $F(a), F(b)$ does not intersect A_2.*

Proof. First notice that $F(a)$ and $F(b)$ are not comparable since T^{-1} preserves order. Assume the claim does not hold. Then, there exist two points $a, b \in E_1$, such that the line passing through a, b does not intersect A_1, but the line through $F(a), F(b)$ does intersect A_2. Since $F(a), F(b)$ are not comparable it must hold that the segment $[F(a), F(b)]$ intersects A_2. Let us check that $A_2 \vee F(a) \vee F(b) = (A_2 \vee F(a)) \cup (A_2 \vee F(b))$. Obviously, $[F(a), F(b)] \subset (A_2 \vee F(a)) \cup (A_2 \vee F(b))$. Indeed, denote by z_0 the point where $[F(a), F(b)]$ intersects the boundary of A_2. Then, $[F(a), z_0] \subset A_2 \vee F(a)$ and $[z_0, F(b) \subset A_2 \vee F(b)$. Consider a point z_1 on the boundary of $A_2 \vee F(b)$ and denote by z_2 the point where the line through $F(b)$ and z_1 intersects the boundary of A_2. Clearly, $[F(b), z_2] \subset A_2 \vee F(b)$ and $[z_2, F(a)] \subset A_2 \vee F(a)$. Thus, the triangle created by $F(a), F(b), z_2$ is contained in $(A_2 \vee F(a)) \cup (A_2 \vee F(b))$, which implies that $[F(a), z_1] \subset (A_2 \vee F(a)) \cup (A_2 \vee F(b))$. Since A_2 is compact we get that the convex hull $A_2 \vee F(a) \vee F(b) = (A_2 \vee F(a)) \cup (A_2 \vee F(b))$, which is a contradiction since $A_1 \vee a \vee b \neq (A_1 \vee a) \cup (A_1 \vee b)$. □

Lemma 4. *Let $a, b \in E_1$ be two incomparable points. Then $F([a, b]) = [F(a), F(b)]$.*

Proof. By the previous argument $[F(a), F(b)]$ does not intersect A_2. Define the set $K = A_1 \vee \{a\} \vee \{b\}$ and consider some point $z \in (a, b)$. Since T is an order isomorphism, we have that $T(K) = T(A_1 \vee \{a\}) \vee T(A_1 \vee \{b\})$ and since $K = K \vee (A \vee \{z\})$ we have that $T(K) = T(K) \vee (A_2 \vee F(z))$. Thus $F(z) \in T(K)$. If $F(z) \notin [F(a), F(b)]$, we could find some point $z' \in [F(a), F(b)]$ such that $A_2 \vee F(z) \subset A_2 \vee z'$. Since T^{-1} preserves order, we know that the pre-image of $A_2 \vee z'$ contains $A_1 \vee z$. On the other hand, $A_2 \vee z' \subset T(K)$ and thus its pre-image is contained in K. Since $T^{-1}(A_2 \vee z')$ is an extremal element, we conclude that $T^{-1}(A_2 \vee z') = A_1 \vee z$, which is a contradiction to injectivity. □

At this stage we have a point map that induces T and sends, in some sense, a large set of intervals to intervals. In order to show that F is a fractional linear map we must show that it is continuous. Since F is defined on some subset of $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ with the standard Euclidean structure, we get a naturally induced metric. The continuity we discuss is with respect to this metric.

3.4 Continuity of F

Lemma 5. *Let $K_n \in \mathcal{K}^n(A_1, B_1)$ be a decreasing sequence such that $K_n \searrow K$. Then, $T(K_n) \searrow T(K)$.*

Proof. Obviously, since T is order preserving it holds that $T(K_n) \searrow M$ for some $M \in \mathcal{K}^n(A_2, B_2)$. Since $K \subset K_n$ we have that $T(K) \subset T(K_n)$ which implies that

$T(K) \subset M$. On the other hand, since $T(K_n) \supset M$ and T^{-1} is order-preserving isomorphism, we know that $K_n \supset T^{-1}(M)$. Thus $T^{-1}(M) \subset \cap_1^\infty K_n$ and $M \subset T(\cap_1^\infty K_n)$, which implies that $M = T(\cap_1^\infty K_n) = T(K)$. □

Lemma 6. *Let* $\{K_n\} \subset \mathcal{K}^n(A_1, B_1)$ *be an increasing sequence such that* $K_n \nearrow K$. *Then,* $T(K_n) \nearrow T(K)$.

Proof. The proof is similar to the proof of Lemma 5. □

Lemma 7. *Let* $\{x_n\} \subset E_1$ *be a sequence that converges to some point* x. *For a given* n*, consider the set*

$$K_n = A_1 \vee \{x_n, x_{n+1}, \ldots\}.$$

Then,

$$A_1 \vee x = \bigcap_{n=1}^{\infty} K_n.$$

Proof. Denote by $K_{n,m} = A_1 \vee x_n \ldots \vee x_m$. Then, $K_n = \bigcup_{m \geq n+1} K_{n,m}$. Assume we have a point $z \notin A_1 \vee x$ and $z \in \bigcap_{n=1}^\infty \bigcup_{m=n+1}^\infty K_{n,m}$. This implies that for each n there exists p such that $z \in K_{n,p}$. Since $z \notin A_1 \vee x$, there exists $\epsilon_0 > 0$ such that $d(z, A_1 \vee x) > \epsilon_0$. On the other hand, since $\{x_n\}$ converges to x, there exists n_0 such that for all $n > n_0$ $|x - x_n| < \epsilon$ and thus $d_{\mathrm{H}}(K_{n,p}, A_1 \vee x) < \epsilon_0$ for all $p > n$ (where d_{H} stands for the Hausdorff distance). Since $z \in K_{n,p}$ for some p, we get a contradiction.

Now assume that $x \notin \bigcap_{n=1}^\infty \bigcup_{m=n+1}^\infty K_{n,m}$. Thus, there exists n_0 such that $x \notin \bigcup_{m=n_0+1}^\infty K_{n_0,m}$. This implies that $d(x, K_{n_0,m}) > \epsilon_0$ for some $\epsilon_0 > 0$ and for all $m > n_0$. In particular, for all $m > n_0$ we have that $d(x, x_m) > \epsilon_0$, but this cannot hold since the sequence $\{x_n\}$ converges to x. Thus, since $A_1 \subset \bigcap_{n=1}^\infty \bigcup_{m=n+1}^\infty K_{n,m}$ and $x \in \bigcap_{n=1}^\infty \bigcup_{m=n+1}^\infty K_{n,m}$ we have that $A_1 \vee x \subset \bigcap_{n=1}^\infty \bigcup_{m=n+1}^\infty K_{n,m}$. This completes the proof. □

Lemma 8. *The map* F *is continuous on* E_1.

Proof. We would like to show that for any sequence $\{x_n\} \subset E_1$ such that $x_n \to x$ we have that $F(x_n) \to F(x)$. For a given n, we may consider an increasing sequence (with respect to m) of convex sets defined in Lemma 7:

$$K_{n,m} := A_1 \vee x_n \vee x_{n+1} \ldots \vee x_m$$

and a decreasing sequence

$$K_n = \lim_{m \to \infty} K_{n,m}.$$

By Lemma 7 we have that $K_n \searrow K$, where $K = A_1 \vee x$. By Lemma 5 $T(K_n) \searrow T(K) = A_2 \vee F(x)$. Assume that $F(x_n)$ converges to some point y. Since T is an order isomorphism we have that $T(K_{n,m}) = A_2 \vee F(x_n) \ldots \vee F(x_m)$, $T(K_{n,m})$ is a sequence increasing to $T(K_n)$ while $T(K_n)$ is a sequence decreasing to $T(K) = A_2 \vee F(x)$. On the other hand, using Lemma 7 we get that $T(K_n)$ converges to $A_2 \vee y$. This implies that $A_2 \vee F(x) = T(K) = A_2 \vee y$ which means that $y = F(x)$. $\square$

3.5 Completing the Proof

Now we know that the map $F : E_1 \to E_2$ is continuous and maps intervals connecting non-comparable points to intervals. Since we do not know that any interval is mapped to interval under F we cannot apply the classical fundamental theorem of projective geometry. Thus, we would like to apply Theorem 1. Since we are considering open sets in a family of lines in $\mathbb{R}^n$, we must discuss the relevant topology. A line $l \in \mathcal{L}(\mathbb{R}^n)$ is defined uniquely by its direction (up to a sign) and the distance from the origin. Hence a line l can be determined by a non-negative number d_l and a vector (u_l) on the sphere S^{n-1}. The metric on $\mathcal{L}(\mathbb{R}^d)$ is inherited from the Grassmannian. A neighbourhood of the line defined by (d, u) is given by a small perturbation of both d and u.

Consider the set $\tilde{E}_1 = \mathrm{int}(E_1)$ and $\mathcal{L}_1 := \mathcal{L}(\tilde{E}_1) \setminus \mathcal{L}(A_1)$. Notice that $\mathcal{L}_1$ is the set of lines that intersect the interior of E_1 but do not intersect A_1. Now, we will show that the interior of $\mathcal{L}_1$ is an open set that covers the interior of E_1.

Remark 3. The open set $\mathcal{L}_1' = \mathrm{int}\mathcal{L}_1$ satisfies:

$$\tilde{E}_1 \subset \bigcup_{l_0 \in \mathcal{L}_1'} l.$$

Indeed, take some point $x \in \tilde{E}_1$. Since $\tilde{E}_1$ is open, there exists a point y such that the line l_0 passing through x and y does not intersect A_1. Since both $x, y \in \tilde{E}_1$ it is clear that $l_0 \in \mathcal{L}_1'$.

Clearly, two points $x, y \in \tilde{E}_1$ are comparable if and only if the line $l_{x,y}$ passing through them is not in $\mathcal{L}_1'$. Thus, applying Lemma 4 we get that for any $l \in \mathcal{L}_1'$, $F(l \cap \tilde{E}_1)$ is contained in a line. Therefore, we have shown that on the interior of E_1, all the conditions of Theorem 1 are satisfied. Hence, we conclude that $F|_{\tilde{E}_1}$ is a fractional linear map. If the defining hyperplane of $F|_{\tilde{E}_1}$ has no common points with E_1, then it is obvious that F is fractional linear on E_1 (two continuous functions that coincide on a dense subset are equal). If the defining hyperplane of $F|_{\tilde{E}_1}$ has some common point x with E_1, then for any sequence $\{x_n\} \subset \tilde{E}_1$ that converges to x it must hold that $F(x_n)$ converges to ∞ which is a contradiction to continuity. To see that $T(K) = F(K)$ for $K \in \mathcal{K}^n(A_1, B_1)$ notice that every $K \in \mathcal{K}^n(A_1, B_1)$ can be written as the convex hull of all the extremal elements contained in K.

3.6 Duality

After we have a characterization of all known order-preserving isomorphism on the classes mentioned above, we may easily write characterizations of duality on such classes. Recall that given a convex set K in $\mathbb{R}^n$, the polar set is defined as follows:

$$K^\circ = \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1, \quad \forall y \in K\}.$$

Theorem 4. *Let $n \geq 2$, A_1, A_2, B_1, B_2 convex sets in $\mathbb{R}^n$ such that $0 \in A_1$. Let $T : \mathcal{K}^n(A_1, B_1) \to \mathcal{K}^n(A_2, B_2)$ be a bijection satisfying for all $K, L \in \mathcal{K}^n(A_1, B_1)$*

$$K \subset L \Leftrightarrow T(K) \supset T(L).$$

Then, there exists a fractional linear map $F : A_1^\circ \setminus B_1^\circ \to B_2 \setminus A_2$ such that for every $K \in \mathcal{K}^n(A_1, B_1)$ we have $T(K) = F(K^\circ)$.

Proof. Consider the map $T_1(K) = T(K^\circ)$. Obviously, the domain of T_1 is $\mathcal{K}^n(B_1^\circ, A_1^\circ)$ and T is an order-preserving isomorphism. Thus, by Theorem 2, we know that there exists a fractional linear map $F : A_1^\circ \setminus B_1^\circ \to B_2 \setminus A_2$ such that $T_1(K) = F(K)$ for all $K \in \mathcal{K}^n(B_1^\circ, A_1^\circ)$. Hence $T(K) = F(K^\circ)$. This means that $T(A_1) = B_2$ and $T(B_1) = A_2$. □

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