The Fields Institute for Research in Mathematical Sciences

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Editors

Asymptotic Geometric Analysis

Proceedings of the Fall 2010 Fields Institute Thematic Program

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Asymptotic Geometric Analysis

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Thematic Program

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Preface

Asymptotic geometric analysis is concerned with geometric and linear properties of finite-dimensional objects, normed spaces, and convex bodies, especially with asymptotics of their various quantitative parameters as the dimension tends to infinity. Deep geometric, probabilistic, and combinatorial methods developed here are used outside the field in many areas, related to the subject of the program.

The main tools of the theory belong to the concentration phenomenon and large deviation inequalities. Concentration of measure is, in fact, an isomorphic form of isoperimetric problems, which was first developed inside asymptotic geometric analysis and then became pertinent to many branches of mathematics as an efficient tool and useful concept. Some new techniques of the theory are connected with measure transportation methods and with related PDEs. The concentration phenomenon is well known to be closely linked with combinatorics (Ramsey theory), and such links have been recently better understood in the setting of infinite-dimensional transformation groups, by means of the so-called fixed point on compacta property: on the one hand, every classical Ramsey-type result is equivalent to the fixed point on compacta property of the group of automorphisms of a suitable structure and on the other hand, the fixed point on compacta property is often established by using concentration of measure in subgroups.

The last few years also witnessed the development of small ball probability estimates and their applications, especially for quantitative results on random matrices. A deep understanding of classical convexity and its analytic methods is necessary to advance new types of "isomorphic" results. It is now difficult to draw a borderline between asymptotic and classical convexity theories; and results of each are used in the other and also have numerous applications. Among the recent important ones are results of a geometric-probabilistic flavor on the volume distribution in convex bodies and central limit theorems for convex bodies and others, with close links with geometric inequalities and optimal transport.

More recently, similar results have been established for a larger category of log-concave distributions on Euclidean space, replacing uniform distributions on convex bodies. This is a remarkable extension of the whole theory which could be called "Geometrization of Probability" because it extends to the class of (log-concave)

probability measures many typical geometric notions and results. For example, the notion of polarity, the Blaschke–Santaló inequality and its inverse (by Bourgain–Milman), the Brunn–Minkowski inequality, the Urysohn inequality, and many others are formulated and proved now on this larger category.

The achievements of asymptotic geometric analysis demonstrate new and unexpected phenomena characteristic for high dimensions. These phenomena appear in a number of domains of mathematics and adjacent domains of science dealing with functions of a large number of variables. Besides the main subject of our program, asymptotic theory of normed spaces and convex bodies, it includes the branch of discrete mathematics known as asymptotic combinatorics, including problems of complexity and graph theory; a considerable part of probability dealing with large numbers of correlated random components, including large deviation and the theory of random matrices; and many others. The theory of computational complexity studies the inherent computational difficulty of various computational problems, mostly originated in combinatorial optimization. Complexity theory is, actually, a purely asymptotic field, as is the notion of complexity classes; the most basic notion here is formulated and perceived as an asymptotic notion, where the growing parameter is the size of the computational problem under investigation. The famous "P versus NP" problem asks in fact to compare two asymptotic complexity classes. In recent years several important breakthroughs in complexity theory were obtained by applying tools from asymptotic geometric analysis such as the concentration of measure phenomenon, spectral theory, and discrete harmonic analysis.

The thematic program on asymptotic geometric analysis took place at the Fields Institute for Research in Mathematical Sciences in July–December 2010. The main directions of research covered by the program included:

- Asymptotic theory of convexity and normed spaces
- Concentration of measure and isoperimetric inequalities with an optimal transportation approach
- Applications of the concept of concentration
- Connections with transformation groups and Ramsey theory
- Geometrization of probability
- Random matrices
- Connection with asymptotic combinatorics and complexity theory

Avi Wigderson (Institute for Advanced Study) delivered the Distinguished Lecture Series "Randomness, pseudorandomness and derandomization" and Shiri Artstein-Avidan (Tel-Aviv University) delivered the Coxeter Lecture Series "Abstract duality, the Legendre transform and a new duality transform," "Order isomorphisms and the fundamental theorem of affine geometry," and "Multiplicative transforms and characterization of the Fourier transform."

Three workshops were held during the program:

1. Asymptotic Geometric Analysis and Convexity (September 13–17, 2010), organized by Monika Ludwig, Vitali Milman, and Nicole Tomczak-Jaegermann, preceded by a concentration period on convexity (September 8–10,

2010) and followed by a concentration period on asymptotic geometric analysis (September 20–22, 2010)
2. Concentration Phenomenon, Transformation Groups and Ramsey Theory (October 12–16, 2010), organized by Eli Glasner, Vladimir Pestov, and Stevo Todorcevic
3. Geometric Probability and Optimal Transportation (November 1–5, 2010), organized by Bo'az Klartag and Robert McCann, preceded by a concentration period on partial differential equations and geometric analysis (October 25–29) and followed by a concentration period on nonlinear dynamics and applications (November 8–10)

To give an idea of program's scale, there were 426 participants, including 85 graduate students and 17 postdocs and 49 long-term participants (those who came fore one month or more). The total number of participant days in the program was an impressive 5162.

The program organizers and editors of this volume are grateful to the Fields Institute for an excellent working environment and generous financial support. They also thank the US National Science Foundation for its support.

This volume contains a selection of papers by the participants of the thematic program, which reflects some of the main directions of the scientific activities.

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Contents

The Variance Conjecture on Some Polytopes

David Alonso–Gutiérrez and Jesús Bastero

Abstract We show that any random vector uniformly distributed on any hyperplane projection of B_1^n or B_∞^n verifies the variance conjecture

$$\mathrm{Var}\, |X|^2 \le C \sup_{\xi \in S^{n-1}} \mathbb{E}\langle X, \xi \rangle^2 \mathbb{E}|X|^2.$$

Furthermore, a random vector uniformly distributed on a hyperplane projection of B_∞^n verifies a negative square correlation property and consequently any of its linear images verifies the variance conjecture.

Key words Variance conjecture • Polytopes

Mathematical Subject Classifications (2010): 46B06, 52B12

1 Introduction and Notation

Let X be a random vector in $\mathbb{R}^n$ with a log-concave density, i.e. X is distributed on $\mathbb{R}^n$ according to a probability measure, μ_X, whose density with respect to the Lebesgue measure is $\exp(-V)$ for some convex function $V : \mathbb{R}^n \to (-\infty, \infty]$. For instance, vectors uniformly distributed on convex bodies and Gaussian random vectors are log-concave.

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A random vector X is said to be isotropic if:

1. The barycenter is at the origin, i.e., $\mathbb{E}X = 0$, and
2. The covariance matrix M_X is the identity I_n, i.e. $\mathbb{E}\langle X, e_i \rangle \langle X, e_j \rangle = \delta_{i,j}$, $1 \leq i, j \leq n$,

where $\{e_i\}_{i=1}^n$ denotes the canonical basis in $\mathbb{R}^n$ and $\delta_{i,j}$ denotes the Kronecker delta. It is well known that every centered random vector with full dimensional support has a unique, up to orthogonal transformations, linear image which is isotropic.

Given a log-concave random vector X, we will denote by λ_X^2 the highest eigenvalue of the covariance matrix M_X and by σ_X its "thin shell width", i.e.

$$\lambda_X^2 = \|M_X\|_{\ell_2^n \to \ell_2^n} = \sup_{\xi \in S^{n-1}} \mathbb{E}\langle X, \xi \rangle^2,$$

$$\sigma_X = \sqrt{\mathbb{E}\left||X| - (\mathbb{E}|X|^2)^{\frac{1}{2}}\right|^2}.$$

(S^{n-1} represents the Euclidean unit sphere in $\mathbb{R}^n$).

In asymptotic geometric analysis, the *variance* conjecture states the following:

Conjecture 1. There exists an absolute constant C such that for every isotropic log-concave vector X, if we denote by $|X|$ its Euclidean norm,

$$\mathrm{Var}\,|X|^2 \leq C\,\mathbb{E}|X|^2 = Cn.$$

This conjecture was considered by Bobkov and Koldobsky in the context of the Central Limit Problem for isotropic convex bodies (see [7]). It was conjectured before by Antilla, Ball and Perissinaki, (see [1]) that for an isotropic log-concave vector X, $|X|$ is highly concentrated in a "thin shell" more than the trivial bound $\mathrm{Var}|X| \leq \mathbb{E}|X|^2$ suggested. Actually, it is known that the variance conjecture is equivalent to the *thin shell width* conjecture:

Conjecture 2. There exists an absolute constant C such that for every isotropic log-concave vector X

$$\sigma_X = \sqrt{\mathbb{E}\left||X| - \sqrt{n}\right|^2} \leq C.$$

It is also known (see [3, 10]) that these two equivalent conjectures are stronger than the hyperplane conjecture, which states that every convex body of volume 1 has a hyperplane section of volume greater than some absolute constant.

The variance conjecture is a particular case of a stronger conjecture, due to Kannan, Lovász and Simonovits (see [15]) concerning the spectral gap of log-concave probabilities. This conjecture can be stated in the following way due to the work of Cheeger, Maz'ya and Ledoux:

Conjecture 3. There exists an absolute constant C such that for any locally Lipschitz function, $g : \mathbb{R}^n \to \mathbb{R}$, and any centered log-concave random vector X in $\mathbb{R}^n$

$$\text{Var } g(X) \leq C \lambda_X^2 \mathbb{E} |\nabla g(X)|^2.$$

Note that Conjecture 1 is the particular case of Conjecture 3, when we consider only isotropic vectors and $g(X) = |X|^2$. Our purpose in this paper is to study the particular case of Conjecture 3 in which $g(X) = |X|^2$ but X is not necessarily isotropic. Thus, we will study the following *general variance* conjecture:

Conjecture 4. There exists an absolute constant C such that for every centered log-concave vector X

$$\text{Var } |X|^2 \leq C \lambda_X^2 \mathbb{E} |X|^2.$$

In the same way that Conjecture 1 is equivalent to Conjecture 2, Conjecture 4 can be shown (see Sect. 2) to be equivalent to the following *general thin shell width* conjecture:

Conjecture 5. There exists an absolute constant C such that for every centered log-concave vector X

$$\sigma_X \leq C \lambda_X$$

Notice that since Conjecture 4 and Conjecture 5 are not invariant under linear maps, these two conjectures cannot easily be reduced to Conjecture 1 and Conjecture 2. We will study how these conjectures behave under linear transformations and we will also prove that random vectors uniformly distributed on a certain family of polytopes verify Conjecture 4 but, before stating our results, let us recall the results known, up to now, concerning the aforementioned conjectures.

Besides the Gaussian vectors only a few examples are known to satisfy Conjecture 3. For instance, the vectors uniformly distributed on ℓ_p^n-balls, $1 \leq p \leq \infty$, the simplex and some revolution convex bodies [5, 14, 20, 25]. In [18], Klartag proved Conjecture 3 with an extra $\log n$ factor for vectors uniformly distributed on unconditional convex bodies and recently Barthe and Cordero extended this result for log-concave vectors with many symmetries (see [4]). Kannan, Lovász and Simonovits proved Conjecture 3 with the factor $(\mathbb{E}|X|)^2$ instead of λ_X^2 (see [15]), improved by Bobkov to $(\text{Var}|X|^2)^{1/2} \simeq \sigma_X \mathbb{E}|X|$ (see [6]).

In [18], Klartag proved Conjecture 1 for random vectors uniformly distributed on isotropic unconditional convex bodies. The best known (dimension dependent) bound for general log-concave isotropic random vectors in Conjecture 2 was proved by Guédon and Milman with a factor $n^{1/3}$ instead of C, improving down to $n^{1/4}$ when X is ψ_2 (see, [12]). This results give better estimates than previous ones by Klartag (see [17]) and Fleury (see [11]). Given the relation between Conjecture 1 and Conjecture 2 we have that Conjecture 1 is known to be true with an extra factor $n^{2/3}$.

Very recently Eldan, [9] obtained a breakthrough showing that Conjecture 2 implies Conjecture 3 with an extra logarithmic factor. By using the result of Guédon–Milman, Conjecture 3 is obtained with an extra factor $n^{2/3}(\log n)^2$.

Since the variance conjecture is not linearly invariant, in Sect. 2 we will study its behavior under linear transformations i.e., given a centered log-concave random vector X, we will study the variance conjecture of the random vector TX, $T \in$

$GL(n)$. We will prove that if X is an isotropic random vector verifying Conjecture 1, then the non-isotropic $T \circ U(X)$ verifies the variance Conjecture 4 for a typical $U \in O(n)$. We will also show the equivalence between Conjecture 4 and Conjecture 5. As a consequence of Guédon and Milman's result we obtain that every centered log-concave random vector verifies the variance conjecture with constant $Cn^{\frac{2}{3}}$ rather than the $Cn^{\frac{2}{3}}(\log n)^2$ obtained from the best general known result in Conjecture 3.

The main results in this paper will be included in Sect. 3, where we will show that random vectors uniformly distributed on hyperplane projections of B_1^n or B_∞^n (the unit balls of ℓ_1^n and ℓ_∞^n respectively) verify Conjecture 4. Furthermore, in the case of the hyperplane projections of B_∞^n we will see that they verify a negative square correlation property with respect to any orthonormal basis, which will allow us to deduce that also a random vector uniformly distributed on any linear image of any hyperplane projection of B_∞^n will verify Conjecture 4.

In order to compute some quantities on the hyperplane projections of B_1^n and B_∞^n we will use Cauchy's formula which, in the case of polytopes can be stated like this:

Let K_0 be a polytope with facets $\{F_i : i \in I\}$ and $K = P_H K_0$ be the projection of K_0 onto a hyperplane. If X is a random vector uniformly distributed on K, for any integrable function $f : K \to \mathbb{R}$ we have

$$\mathbb{E}f(X) = \sum_{i \in I} \frac{\text{Vol}(P_H(F_i))}{\text{Vol}(K)} \mathbb{E}f(P_H Y^i),$$

where Y^i is a random vector uniformly distributed on the facet F_i and Vol denotes the volume or Lebesgue measure.

Let us now introduce some notation. Given a convex body K, we will denote by $\tilde{K}$ its homothetic image of volume 1 ($\text{Vol}(\tilde{K}) = 1$). $\tilde{K} = \frac{K}{\text{Vol}^{\frac{1}{n}}(K)}$. We recall that a convex body $K \subset \mathbb{R}^n$ is *isotropic* if it has volume $\text{Vol}(K) = 1$, the barycenter of K is at the origin and its inertia matrix is a multiple of the identity. Equivalently, there exists a constant $L_K > 0$ called isotropy constant of K such that $L_K^2 = \int_K \langle x, \theta \rangle^2 \, dx, \forall \theta \in S^{n-1}$. In this case if X denotes a random vector uniformly distributed on K, $\lambda_X = L_K$. Thus, K is isotropic if the random vector X, distributed on $L_K^{-1}K$ with density $L_K^n \chi_{L_K^{-1}K}$ is isotropic.

When we write $a \sim b$, for $a, b > 0$, it means that the quotient of a and b is bounded from above and from bellow by absolute constants. $O(n)$ will always denote the orthogonal group on $\mathbb{R}^n$.

2   General Results

In this section we are going to consider the variance conjecture for linear [16], transformations of a fixed centered log-concave random vector in $\mathbb{R}^n$. Our first result shows that if such random vector is not far from being isotropic and verifies the variance conjecture, then the average perturbation (in the sense we will state in the proposition) will also verify the variance conjecture.

Proposition 1. *Let X be a centered isotropic, log-concave random vector in $\mathbb{R}^n$ verifying the variance conjecture with constant C_1. Let $T \in GL(n)$ be any linear transformation. If U is a random map uniformly distributed in $O(n)$ then*

$$\mathbb{E}_U \, Var \, |T \circ U(X)|^2 \leq CC_1 \|T\|_{op}^2 \|T\|_{HS}^2 = CC_1 \lambda_{T \circ u(X)}^2 \mathbb{E}|T \circ u(X)|^2$$

for any $u \in O(n)$, where C is an absolute constant.

Proof. The non singular linear map T can be expressed by $T = V \Lambda U_1$ where $V, U_1 \in O(n)$ and $\Lambda = [\lambda_1, \dots, \lambda_n]$ $(\lambda_i > 0)$ a diagonal map.

Given $\{e_i\}_{i=1}^n$ the canonical basis in $\mathbb{R}^n$, we will identify every $U \in O(n)$ with the orthonormal basis $\{\eta_i\}_{i=1}^n$ such that $U_1 U \eta_i = e_i$ for all i. Thus, by uniqueness of the Haar measure invariant under the action of $O(n)$ we have that, for any integrable function F

$$\mathbb{E}_U \, F(U)$$
$$= \mathbb{E}_U \, F(\eta_1, \dots, \eta_n)$$
$$= \int_{S^{n-1}} \int_{S^{n-1} \cap \eta_1^\perp} \cdots \int_{S^{n-1} \cap \eta_1^\perp \cap \cdots \cap \eta_{n-1}^\perp} F(\eta_1, \dots, \eta_n) d\nu(\eta_n) \dots d\nu(\eta_1),$$

where $d\nu(\eta_i)$ is the Haar probability measure on $S^{n-1} \cap \eta_1^\perp \cap \cdots \cap \eta_{i-1}^\perp$. Then, since

$$|T \circ U(X)|^2 = |\Lambda U_1 UX|^2 = \sum_{i=1}^n \langle \Lambda U_1 UX, e_i \rangle^2 = \sum_{i=1}^n \lambda_i^2 \langle X, \eta_i \rangle^2$$

$$\mathbb{E}_U \, Var \, |T \circ U(X)|^2 = \sum_{i=1}^n \lambda_i^4 \mathbb{E}_U \left(\mathbb{E}\langle X, \eta_i \rangle^4 - (\mathbb{E}\langle X, \eta_i \rangle^2)^2 \right)$$
$$+ \sum_{i \neq j} \lambda_i^2 \lambda_j^2 \mathbb{E}_U \left(\mathbb{E}\langle X, \eta_i \rangle^2 \langle X, \eta_j \rangle^2 - \mathbb{E}\langle X, \eta_i \rangle^2 \mathbb{E}\langle X, \eta_j \rangle^2 \right).$$

Since for every i

$$\mathbb{E}_U \left(\mathbb{E}\langle X, \eta_i \rangle^4 - (\mathbb{E}\langle X, \eta_i \rangle^2)^2 \right) \leq \mathbb{E}_U \mathbb{E}\langle X, \eta_i \rangle^4 = \mathbb{E}|X|^4 \int_{S^{n-1}} \langle e_1, \eta_1 \rangle^4 d\nu(\eta_1)$$
$$= \frac{3}{n(n+2)} \mathbb{E}|X|^4,$$

and for every $i \neq j$, denoting by Y an independent copy of X,

$$\mathbb{E}_U \left(\mathbb{E}\langle X, \eta_i \rangle^2 \langle X, \eta_j \rangle^2 - \mathbb{E}\langle X, \eta_i \rangle^2 \mathbb{E}\langle X, \eta_j \rangle^2 \right)$$
$$= \mathbb{E}|X|^4 \int_{S^{n-1}} \left\langle \frac{X}{|X|}, \eta_1 \right\rangle^2 \int_{S^{n-1} \cap \eta_1^\perp} \left\langle \frac{X}{|X|}, \eta_2 \right\rangle^2 d\nu(\eta_2) d\nu(\eta_1)$$
$$- \mathbb{E}|X|^2 |Y|^2 \int_{S^{n-1}} \left\langle \frac{X}{|X|}, \eta_1 \right\rangle^2 \int_{S^{n-1} \cap \eta_1^\perp} \left\langle \frac{Y}{|Y|}, \eta_2 \right\rangle^2 d\nu(\eta_2) d\nu(\eta_1)$$

$$= \frac{\mathbb{E}|X|^4}{n-1} \left(\frac{1}{n} - \int_{S^{n-1}} \langle e_1, \eta_1 \rangle^4 d\nu(\eta_1) \right)$$

$$- \frac{\mathbb{E}|X|^2|Y|^2}{n-1} \left(\frac{1}{n} - \int_{S^{n-1}} \left\langle \frac{X}{|X|}, \eta_1 \right\rangle^2 \left\langle \frac{Y}{|Y|}, \eta_1 \right\rangle^2 d\nu(\eta_1) \right)$$

$$= \frac{\mathbb{E}|X|^4}{n-1} \left(\frac{1}{n} - \frac{3}{n(n+2)} \right)$$

$$- \frac{\mathbb{E}|X|^2|Y|^2}{n-1} \left(\frac{1}{n} - \frac{1}{n(n+2)} - \frac{2}{n(n+2)} \left\langle \frac{X}{|X|}, \frac{Y}{|Y|} \right\rangle^2 \right)$$

we have that

$$\mathbb{E}_U \operatorname{Var} |T \circ U(X)|^2 \leq \frac{3}{n(n+2)} \mathbb{E}|X|^4 \sum_{i=1}^{n} \lambda_i^4$$

$$+ \left(\frac{\mathbb{E}|X|^4 - (\mathbb{E}|X|^2)^2}{n(n+2)} - \frac{2\mathbb{E}|X|^2|Y|^2}{(n-1)n(n+2)} \left(1 - \left\langle \frac{X}{|X|}, \frac{Y}{|Y|} \right\rangle^2 \right) \right) \sum_{i \neq j} \lambda_i^2 \lambda_j^2$$

$$\leq \frac{3}{n(n+2)} \mathbb{E}|X|^4 \sum_{i=1}^{n} \lambda_i^4 + \frac{\operatorname{Var} |X|^2}{n(n+2)} \sum_{i \neq j} \lambda_i^2 \lambda_j^2.$$

Now, since for every $\theta \in S^{n-1}$, $\mathbb{E}\langle X, \theta \rangle^2 = 1$ and X satisfies the variance conjecture with constant C_1, we have

$$\mathbb{E}|X|^4 = Var|X|^2 + (\mathbb{E}|X|^2)^2 \leq C_1 n + n^2 \leq C C_1 n^2.$$

and

$$\mathbb{E}_U \operatorname{Var} |T \circ U(X)|^2 \leq C C_1 \sum_{i=1}^{n} \lambda_i^4 + \frac{C_1}{n} \sum_{i \neq j} \lambda_i^2 \lambda_j^2$$

Hence, given any $u \in O(n)$, let $\{v_i\}_{i=1}^{n}$ be the orthonormal basis defined by $v_i = U_1 \circ u(e_1)$, for all i. Then we have

$$\lambda_{T \circ u(X)}^2 = \sup_{\theta \in S^{n-1}} \mathbb{E}\langle T \circ u(X), \theta \rangle^2 = \sup_{\theta \in S^{n-1}} \mathbb{E}\langle \Lambda U_1 u X, \theta \rangle^2$$

$$= \sup_{\theta \in S^{n-1}} \mathbb{E} \left(\sum_{i=1}^{n} \lambda_i \langle X, v_i \rangle \langle e_i, \theta \rangle \right)^2$$

$$= \sup_{\theta \in S^{n-1}} \sum_{i=1}^{n} \lambda_i^2 \langle e_i, \theta \rangle^2 = \max_{1 \leq i \leq n} \lambda_i^2 = \|T\|_{op}^2$$

and

$$\mathbb{E}|T \circ u(X)|^2 = \sum_{i=1}^n \lambda_i^2 \mathbb{E}\langle X, v_i \rangle^2 = \sum_{i=1}^n \lambda_i^2 = \|T\|_{HS}^2$$

Thus

$$
\begin{aligned}
\mathbb{E}_U \operatorname{Var}|T \circ U(X)|^2 &\le CC_1\|T\|_{op}^2\|T\|_{HS}^2 + \frac{C_1}{n}\|T\|_{op}^2 \sum_{i \neq j} \lambda_j^2 \\
&\le CC_1\|T\|_{op}^2\|T\|_{HS}^2 + C_1\|T\|_{op}^2\|T\|_{HS}^2 \\
&= CC_1\|T\|_{op}^2\|T\|_{HS}^2 = CC_1\lambda_{T\circ u(X)}^2 \mathbb{E}|T \circ u(X)|^2
\end{aligned}
$$

$\square$

Remark 1. The same proof as before can be applied when X is not necessarily isotropic. In this case

$$\mathbb{E}_U \operatorname{Var}|T \circ U(X)|^2 \le CC_1 B^4 \lambda_{T\circ u(X)}^2 \mathbb{E}|T \circ u(X)|^2$$

for any $u \in O(n)$, where B is the spectral condition number of its covariance matrix i.e.

$$B^2 = \frac{\max_{\theta \in S^{n-1}} \mathbb{E}\langle X, \theta \rangle^2}{\min_{\theta \in S^{n-1}} \mathbb{E}\langle X, \theta \rangle^2}.$$

As a consequence of Markov's inequality we obtain the following

Corollary 1. *Let X be an isotropic, log-concave random vector in $\mathbb{R}^n$ verifying the variance conjecture with constant C_1. There exists an absolute constant C such that the measure of the set of orthogonal operators U for which the random vector $T \circ U(X)$ verifies the variance conjecture with constant CC_1 is greater than $\frac{1}{2}$.*

In [12] it was shown that every log-concave isotropic random vector verifies the thin-shell width conjecture with constant $C_1 = Cn^{\frac{1}{3}}$. Also, an estimate for σ_X was given when X is not isotropic.

The following proposition is well known for the experts. However we include it here for the sake of completeness. As a consequence and, by using the result in [12], we will obtain that every centered log-concave vector, isotropic or not, verifies the variance conjecture with constant $Cn^{\frac{2}{3}}$ rather than $Cn^{\frac{2}{3}}(\log n)^2$.

Proposition 2. *Let X be an isotropic log-concave random vector, T a linear map and σ_{TX} the thin-shell width of the random vector TX i.e.*

$$\sigma_{TX}^2 = \mathbb{E}\left| |TX| - (\mathbb{E}|TX|^2)^{\frac{1}{2}} \right|^2.$$

Then

$$\sigma_{TX}^2 \leq \frac{\text{Var } |TX|^2}{\mathbb{E}|TX|^2} \leq C_1 \sigma_{TX}^2 + C_2 \frac{\|T\|_{op}^2}{\|T\|_{HS}^2} \lambda_{TX}^2.$$

Proof. The first inequality is clear, since

$$\sigma_{TX}^2 = \mathbb{E} \left| |TX| - (\mathbb{E}|TX|^2)^{\frac{1}{2}} \right|^2$$

$$\leq \mathbb{E} \left| |TX| - (\mathbb{E}|TX|^2)^{\frac{1}{2}} \right|^2 \frac{\left| |TX| + (\mathbb{E}|TX|^2)^{\frac{1}{2}} \right|^2}{\mathbb{E}|TX|^2}$$

$$= \frac{\text{Var } |TX|^2}{\mathbb{E}|TX|^2}.$$

Let us now show the second inequality. Let $B > 0$ to be chosen later.

$$\text{Var } |TX|^2 = \mathbb{E} \left| |TX|^2 - \mathbb{E}|TX|^2 \right|^2 \chi_{\left\{ |TX| \leq B(\mathbb{E}|TX|^2)^{\frac{1}{2}} \right\}}$$
$$+ \mathbb{E} \left| |TX|^2 - \mathbb{E}|TX|^2 \right|^2 \chi_{\left\{ |TX| > B(\mathbb{E}|TX|^2)^{\frac{1}{2}} \right\}}.$$

The first term equals

$$\mathbb{E} \left| |TX| + (\mathbb{E}|TX|^2)^{\frac{1}{2}} \right|^2 \left| |TX| - (\mathbb{E}|TX|^2)^{\frac{1}{2}} \right|^2 \chi_{\left\{ |TX| \leq B(\mathbb{E}|TX|^2)^{\frac{1}{2}} \right\}}$$
$$\leq (1 + B)^2 \sigma_{TX}^2 \mathbb{E}|TX|^2.$$

If $B \geq \frac{1}{\sqrt{2}}$, the second term verifies

$$\mathbb{E} \left| |TX|^2 - \mathbb{E}|TX|^2 \right|^2 \chi_{\left\{ |TX| > B(\mathbb{E}|TX|^2)^{\frac{1}{2}} \right\}} \leq \mathbb{E}|TX|^4 \chi_{\left\{ |TX| > B(\mathbb{E}|TX|^2)^{\frac{1}{2}} \right\}}$$

By Paouris' strong estimate for log-concave isotropic probabilities (see [23]) there exists an absolute constant c such that

$$\mathbb{P}\{|TX| > ct(\mathbb{E}|TX|^2)^{\frac{1}{2}}\} \leq e^{-t\frac{\|T\|_{HS}}{\|T\|_{op}}} \qquad \forall t \geq 1.$$

Choosing $B = \max\left\{c, \frac{1}{\sqrt{2}}\right\}$ we have that the second term is bounded from above by

$$\mathbb{E}|TX|^4 \chi_{\{|TX| > B(\mathbb{E}|TX|^2)^{\frac{1}{2}}\}}$$
$$= B^4(\mathbb{E}|TX|^2)^2 \mathbb{P}\{|TX| > B(\mathbb{E}|TX|^2)^{\frac{1}{2}}\}$$
$$+ B^4(\mathbb{E}|TX|^2)^2 \int_1^\infty 4t^3 \mathbb{P}\{|TX| > Bt(\mathbb{E}|TX|^2)^{\frac{1}{2}}\} dt$$

$$\leq B^4 \|T\|_{op}^4 \frac{\|T\|_{HS}^4}{\|T\|_{op}^4} e^{-\frac{\|T\|_{HS}}{\|T\|_{op}}} + B^4 \|T\|_{HS}^4 \int_1^\infty 4t^3 e^{-t\frac{\|T\|_{HS}}{\|T\|_{op}}} \, dt$$

$$\leq C_2 \|T\|_{op}^4.$$

Hence, we achieve

$$\frac{\text{Var } |TX|^2}{\mathbb{E}|TX|^2} \leq \sigma_{TX}^2 (1+B)^2 + C_2 \frac{\|T\|_{op}^4}{\|T\|_{HS}^2}$$

$$\leq C_1 \sigma_{TX}^2 + C_2 \frac{\|T\|_{op}^4}{\|T\|_{HS}^2}$$

$$= C_1 \sigma_{TX}^2 + C_2 \frac{\|T\|_{op}^2}{\|T\|_{HS}^2} \lambda_{TX}^2.$$

$\square$

As a consequence of this proposition we obtain that Conjecture 4 and Conjecture 5 are equivalent. Combining it with the estimate of σ_{TX} given in [12] we obtain the following

Corollary 2. *There exists an absolute constant C such that for every log-concave isotropic random vector X and any linear map $T \in GL(n)$ we have*

$$\sigma_{TX} \leq C_1 \lambda_{TX} \qquad \implies \quad Var \, |TX|^2 \leq C_1 \mathbb{E}|TX|^2$$

and

$$Var \, |TX|^2 \leq C_2 \mathbb{E}|TX|^2 \qquad \implies \quad \sigma_{TX} \leq C \, C_2 \lambda_{TX}.$$

Moreover, both inequalities are true with $C_2 = C n^{2/3}$.

Proof. The two implications are a direct consequence of the previous proposition and the fact that $\|T\|_{op} \leq \|T\|_{HS}$. It was proved in [12] that

$$\sigma_{TX} \leq C \|T\|_{op}^{\frac{1}{3}} \|T\|_{HS}^{\frac{2}{3}}.$$

Thus, by the previous proposition

$$\frac{\text{Var } |TX|^2}{\mathbb{E}|TX|^2} \leq C_1 \sigma_{TX}^2 + C_2 \frac{\|T\|_{op}^2}{\|T\|_{HS}^2} \lambda_{TX}^2$$

$$\leq C \lambda_{TX}^2 \left(\frac{\|T\|_{HS}^{\frac{4}{3}}}{\|T\|_{op}^{\frac{4}{3}}} + \frac{\|T\|_{op}^2}{\|T\|_{HS}^2} \right)$$

$$\leq C \lambda_{TX}^2 \frac{\|T\|_{HS}^{\frac{4}{3}}}{\|T\|_{op}^{\frac{4}{3}}} \leq C n^{\frac{2}{3}} \lambda_{TX}^2,$$

since $\|T\|_{op} \leq \|T\|_{HS} \leq \sqrt{n} \|T\|_{op}$. $\square$

The square negative correlation property appeared in [1] in the context of the central limit problem for convex bodies.

Definition 1. Let X be a centered log-concave random vector in $\mathbb{R}^n$ and $\{\eta_i\}_{i=1}^n$ an orthonormal basis of $\mathbb{R}^n$. We say that X satisfies the square negative correlation property with respect to $\{\eta_i\}_{i=1}^n$ if for every $i \neq j$

$$\mathbb{E}\langle X, \eta_i\rangle^2 \langle X, \eta_j\rangle^2 \leq \mathbb{E}\langle X, \eta_i\rangle^2 \mathbb{E}\langle X, \eta_j\rangle^2.$$

In [1], the authors showed that a random vector uniformly distributed on B_p^n satisfies the square negative correlation property with respect to the canonical basis of $\mathbb{R}^n$. The same property was proved for random vectors uniformly distributed on generalized Orlicz balls in [26], where it was also shown that this property does not hold in general, even in the class of random vectors uniformly distributed on 1-symmetric convex bodies.

It is easy to see that if a random centered log-concave vector X satisfies the square negative correlation property with respect to some orthonormal basis, then it also satisfies the Conjecture 4. Furthermore, the following proposition shows that, in such case, also some class of linear perturbations of X verify the Conjecture 4.

Proposition 3. *Let X be a centered log-concave random vector in $\mathbb{R}^n$ satisfying the square negative correlation property with respect to any orthonormal basis, then the Conjecture 4 holds for any linear image $T \in GL(n)$.*

Proof. Let $T = V\Lambda U$, with $U, V \in O(n)$ and $\Lambda = [\lambda_1, \ldots, \lambda_n]$ ($\lambda_i > 0$) a diagonal map. Let $\{\eta_i\}_i$ the orthonormal basis defined by $U\eta_i = e_i$ for all i. By the square negative correlation property

$$\mathrm{Var}\,|TX|^2 = \sum_{i=1}^n \lambda_i^4 (\mathbb{E}\langle X, \eta_i\rangle^4 - (\mathbb{E}\langle X, \eta_i\rangle^2)^2)$$

$$+ \sum_{i \neq j} \lambda_i^2 \lambda_j^2 (\mathbb{E}\langle X, \eta_i\rangle^2 \langle X, \eta_j\rangle^2 - \mathbb{E}\langle X, \eta_i\rangle^2 \mathbb{E}\langle X, \eta_j\rangle^2)$$

$$\leq \sum_{i=1}^n \lambda_i^4 (\mathbb{E}\langle X, \eta_i\rangle^4 - (\mathbb{E}\langle X, \eta_i\rangle^2)^2)$$

By Borell's lemma (see, for instance, [8], Lemma 3.1 or [22], Appendix III)

$$\mathbb{E}\langle X, \eta_i\rangle^4 \leq C(\mathbb{E}\langle X, \eta_i\rangle^2)^2.$$

Thus

$$\mathrm{Var}\,|TX|^2 \leq C \sum_{i=1}^n \lambda_i^4 (\mathbb{E}\langle X, \eta_i\rangle^2)^2 \leq C\lambda_{TX}^2 \sum_{i=1}^n \lambda_i^2 \mathbb{E}\langle X, \eta_i\rangle^2$$
$$= C\lambda_{TX}^2 \mathbb{E}|TX|^2.$$

Remark 2. Notice that if X satisfies the square negative correlation property with respect to one orthonormal basis $\{\eta_i\}_{i=1}^n$ and U is the orthogonal map such that $U(\eta_i) = e_i$, the same proof gives that $\Lambda U X$ verifies Conjecture 4 for any linear image $\Lambda = [\lambda_1, \ldots, \lambda_n]$ $(\lambda_i > 0)$.

Even though verifying the variance conjecture is not equivalent to satisfy a square negative correlation property, the following lemma shows that it is equivalent to satisfy a "weak averaged square negative correlation" property with respect to one and every orthonormal basis.

Lemma 1. *Let X be a centered log concave random vector in $\mathbb{R}^n$. The following are equivalent*

(i) *X verifies the variance conjecture with constant C_1*

$$\operatorname{Var} |X|^2 \le C_1 \lambda_X^2 \mathbb{E}|X|^2.$$

(ii) *X satisfies the following "weak averaged square negative correlation" property with respect to some orthonormal basis $\{\eta_i\}_{i=1}^n$ with constant C_2*

$$\sum_{i \ne j} (\mathbb{E}\langle X, \eta_i \rangle^2 \langle X, \eta_j \rangle^2 - \mathbb{E}\langle X, \eta_i \rangle^2 \mathbb{E}\langle X, \eta_j \rangle^2) \le C_2 \lambda_X^2 \mathbb{E}|X|^2.$$

(iii) *X satisfies the following "weak averaged square negative correlation" property with respect to every orthonormal basis $\{\eta_i\}_{i=1}^n$ with constant C_3*

$$\sum_{i \ne j} (\mathbb{E}\langle X, \eta_i \rangle^2 \langle X, \eta_j \rangle^2 - \mathbb{E}\langle X, \eta_i \rangle^2 \mathbb{E}\langle X, \eta_j \rangle^2) \le C_3 \lambda_X^2 \mathbb{E}|X|^2,$$

where

$$C_2 \le C_1 \le C_2 + C \qquad C_3 \le C_1 \le C_3 + C$$

with C an absolute constant.

Proof. For any orthonormal basis $\{\eta_i\}_{i=1}^n$ we have

$$\operatorname{Var} |X|^2 = \sum_{i=1}^n (\mathbb{E}\langle X, \eta_i \rangle^4 - (\mathbb{E}\langle X, \eta_i \rangle^2)^2)$$
$$+ \sum_{i \ne j} (\mathbb{E}\langle X, \eta_i \rangle^2 \langle X, \eta_j \rangle^2 - \mathbb{E}\langle X, \eta_i \rangle^2 \mathbb{E}\langle X, \eta_j \rangle^2).$$

Denoting by $A(\eta)$ the second term we have, using Borell's lemma, that

$$A(\eta) \le \operatorname{Var} |X|^2 \le C \lambda_X^2 \mathbb{E}|X|^2 + A(\eta),$$

since $\displaystyle\sum_{i=1}^n \mathbb{E}\langle X, \eta_i \rangle^4 \le C \sup_i \mathbb{E}\langle X, \eta_i \rangle^2 \sum_{i=1}^n \mathbb{E}\langle X, \eta_i \rangle^2 = C \lambda_X^2 \mathbb{E}|X|^2.$ $\qquad\square$

3 Hyperplane Projections of the Cross-Polytope and the Cube

In this section we are going to give some new examples of random vectors verifying the variance conjecture. We will consider the family of random vectors uniformly distributed on a hyperplane projection of some symmetric isotropic convex body K_0. These random vectors will not necessarily be isotropic. However, as we will see in the next proposition, they will be almost isotropic. i.e. the spectral condition number B of their covariance matrix verifies $1 \leq B \leq C$ for some absolute constant C.

Proposition 4. *Let $K_0 \subset \mathbb{R}^n$ be a symmetric isotropic convex body, and let $H = \theta^\perp$ be any hyperplane. Let $K = P_H(K_0)$ and X a random vector uniformly distributed on K. Then, for any $\xi \in S_H = \{x \in H; |x| = 1\}$*

$$\mathbb{E}\langle X, \xi \rangle^2 \sim \frac{1}{\mathrm{Vol}(K)^{1+\frac{2}{n-1}}} \int_K \langle x, \xi \rangle^2 dx \sim L_{K_0}^2.$$

Consequently $\lambda_X \sim L_{K_0}$ and $B(X) \sim 1$.

Proof. The two first expressions are equivalent, since $\mathrm{Vol}(K)^{\frac{1}{n-1}} \sim 1$. Indeed, using Hensley's result [13] and the best general known upper bound for the isotropy constant of an n-dimensional convex body [16], we have

$$\mathrm{Vol}(K)^{\frac{1}{n-1}} \geq \mathrm{Vol}(K_0 \cap H)^{\frac{1}{n-1}} \geq \left(\frac{c}{L_{K_0}} \right)^{\frac{1}{n-1}} \geq \left(\frac{c}{n^{\frac{1}{4}}} \right)^{\frac{1}{n-1}} \geq c.$$

On the other hand, since (see [24] for a proof)

$$\frac{1}{n}\mathrm{Vol}(K)\mathrm{Vol}(K_0 \cap H^\perp) \leq \mathrm{Vol}(K_0) = 1$$

we have

$$\mathrm{Vol}(K)^{\frac{1}{n-1}} \leq \left(\frac{n}{2r(K_0)} \right)^{\frac{1}{n-1}} \leq \left(\frac{n}{2L_{K_0}} \right)^{\frac{1}{n-1}} \leq (cn)^{\frac{1}{n-1}} \leq c,$$

where we have used that $r(K_0) = \sup\{r : rB_2^n \subseteq K_0\} \geq cL_{K_0}$, see [15].

Let us prove the last estimate. Let $S(K_0)$ be the Steiner symmetrization of K_0 with respect to the hyperplane H and let S_1 be its isotropic position. It is known (see [2] or [21]) that for any isotropic n-dimensional convex body L and any linear subspace E of codimension k

$$\mathrm{Vol}(L \cap E)^{\frac{1}{k}} \sim \frac{L_C}{L_L},$$

where C is a convex body in $E^{\perp}$. In particular, we have that

$$\mathrm{Vol}(S_1 \cap H) \sim \frac{1}{L_{S(K_0)}}$$

and

$$\mathrm{Vol}(S_1 \cap H \cap \xi^{\perp}) \sim \frac{1}{L_{S(K_0)}^2}.$$

Since K_0 is symmetric, $S_1 \cap H$ is symmetric and thus centered. Then, by Hensley's result [13]

$$L_{S(K_0)}^2 \sim \frac{\mathrm{Vol}(S_1 \cap H)^2}{\mathrm{Vol}(S_1 \cap H \cap \xi^{\perp})^2} \sim \frac{1}{\mathrm{Vol}(S_1 \cap H)} \int_{S_1 \cap H} \langle x, \xi \rangle^2 dx$$

$$\sim \frac{1}{\mathrm{Vol}(S_1 \cap H)^{1+\frac{2}{n-1}}} \int_{S_1 \cap H} \langle x, \xi \rangle^2 dx,$$

because $\mathrm{Vol}(S_1 \cap H) \sim \frac{1}{L_{S(K_0)}}$ and so $\mathrm{Vol}(S_1 \cap H)^{\frac{1}{n-1}} \sim c$.

But now, $\widetilde{S_1 \cap H} = \mathrm{Vol}(S_1 \cap H)^{-\frac{1}{n-1}}(S_1 \cap H) = \widetilde{S(K_0) \cap H} = \tilde{K}$, because, even though $S(K_0)$ is not isotropic, S_1 is obtained from $S(K_0)$ multiplying it by some λ in H and by $\frac{1}{\lambda^{\frac{1}{n-1}}}$ in $H^{\perp}$. Thus,

$$L_{S(K_0)}^2 \sim \int_{\widetilde{S_1 \cap H}} \langle x, \xi \rangle^2 dx = \int_{\tilde{K}} \langle x, \xi \rangle^2 dx = \frac{1}{\mathrm{Vol}(K)^{1+\frac{2}{n-1}}} \int_K \langle x, \xi \rangle^2 dx$$

and since $L_{S(K_0)} \sim L_{K_0}$ we obtain the result. □

The first examples we consider are the random vectors uniformly distributed on hyperplane projections of the cube. We will see that these random vectors satisfy the negative square correlation property with respect to any orthonormal basis. Consequently, by Proposition 3, any linear image of these random vectors will verify the variance conjecture with an absolute constant.

Theorem 1. *Let $\theta \in S^{n-1}$ and let $K = P_H B_{\infty}^n$ be the projection of B_{∞}^n on the hyperplane $H = \theta^{\perp}$. If X is a random vector uniformly distributed on K then, for any two orthonormal vectors $\eta_1, \eta_2 \in H$, we have*

$$\mathbb{E}\langle X, \eta_1 \rangle^2 \langle X, \eta_2 \rangle^2 \leq \mathbb{E}\langle X, \eta_1 \rangle^2 \mathbb{E}\langle X, \eta_2 \rangle^2.$$

Consequently, X satisfies the negative square correlation property with respect to any orthonormal basis in H.

Proof. Let F_i denote the facet $F_i = \{y \in B_{\infty}^n; y_{|i|} = \mathrm{sgn}\, i\}$, $i \in \{\pm 1, \ldots, \pm n\}$. From Cauchy's formula, it is clear that for any function f

$$\mathbb{E} f(X) = \sum_{i=\pm 1}^{\pm n} \frac{|\theta_{|i|}|}{2\|\theta\|_1} \mathbb{E}(f(P_H Y^i))$$

where Y^i is a random vector uniformly distributed on the facet F_i.

Remark that

$$\mathrm{Vol}(P_H(F_i)) = |\langle \theta, e_{|i|} \rangle| \, \mathrm{Vol}(F_i) = 2^{n-1} |\theta_{|i|}|$$

for $i = \pm 1, \ldots, \pm n$ and $\mathrm{Vol}(K) = 2^{n-1} \|\theta\|_1$.

For any unit vector $\eta \in H$, we have by isotropicity of the facets of B_∞^n,

$$\mathbb{E}\langle X, \eta \rangle^2 = \sum_{i=\pm 1}^{\pm n} \frac{|\theta_{|i|}|}{2\|\theta\|_1} \mathbb{E}\langle Y^i, \eta \rangle^2 = \sum_{i=\pm 1}^{\pm n} \frac{|\theta_{|i|}|}{2\|\theta\|_1} \mathbb{E} \sum_{j=1}^n \eta_j^2 {Y_j^i}^2$$

$$= \frac{1}{2} \sum_{j=1}^n \eta_j^2 \sum_{i=\pm 1}^{\pm n} \frac{|\theta_{|i|}|}{\|\theta\|_1} \mathbb{E}{Y_j^i}^2 = \sum_{j=1}^n \eta_j^2 \left(\frac{|\theta_j|}{\|\theta\|_1} + \frac{1}{3} \sum_{i \neq j} \frac{|\theta_i|}{\|\theta\|_1} \right)$$

$$= \sum_{j=1}^n \eta_j^2 \left(\frac{2|\theta_j|}{3\|\theta\|_1} + \frac{1}{3} \right) = \frac{1}{3} + \frac{2}{3} \sum_{j=1}^n \eta_j^2 \frac{|\theta_j|}{\|\theta\|_1}.$$

Consequently,

$$\mathbb{E}\langle X, \eta_1 \rangle^2 \mathbb{E}\langle X, \eta_2 \rangle^2$$

$$= \frac{1}{9} + \frac{2}{9} \sum_{i=1}^n \frac{|\theta_i|}{\|\theta\|_1} (\eta_1(i)^2 + \eta_2(i)^2) + \frac{4}{9} \sum_{i_1, i_2 = 1}^n \frac{|\theta_{i_1}||\theta_{i_2}|}{\|\theta\|_1^2} \eta_1(i_1)^2 \eta_2(i_2)^2$$

$$\geq \frac{1}{9} + \frac{2}{9} \sum_{i=1}^n \frac{|\theta_i|}{\|\theta\|_1} (\eta_1(i)^2 + \eta_2(i)^2).$$

On the other hand, by symmetry,

$$\mathbb{E}\langle X, \eta_1 \rangle^2 \langle X, \eta_2 \rangle^2$$

$$= \sum_{i=\pm 1}^{\pm n} \frac{|\theta_{|i|}|}{2\|\theta\|_1} \mathbb{E}\langle Y^i, \eta_1 \rangle^2 \langle Y^i, \eta_2 \rangle^2$$

$$= \sum_{i=\pm 1}^{\pm n} \frac{|\theta_{|i|}|}{2\|\theta\|_1} \frac{1}{\mathrm{Vol}(B_\infty^{n-1})} \int_{B_\infty^{n-1}} (\langle y, P_{e_{|i|}} \eta_1 \rangle + \mathrm{sgn}(i) \eta_1(i))^2$$

$$\times (\langle y, P_{e_{|i|}} \eta_2 \rangle + \mathrm{sgn}(i) \eta_2(i))^2 dy$$

$$
= \sum_{i=1}^{n} \frac{|\theta_i|}{\|\theta\|_1} \frac{1}{\mathrm{Vol}(B_\infty^{n-1})} \int_{B_\infty^{n-1}} \Big(\langle y, P_{e_i^\perp} \eta_1 \rangle^2 \langle y, P_{e_i^\perp} \eta_2 \rangle^2 + \eta_2(i)^2 \langle y, P_{e_i^\perp} \eta_1 \rangle^2
$$

$$
+ \eta_1(i)^2 \langle y, P_{e_i^\perp} \eta_2 \rangle^2 + \eta_1(i)^2 \eta_2(i)^2
$$

$$
+ 4\eta_1(i)\eta_2(i)\langle y, P_{e_i^\perp} \eta_1 \rangle \langle y, P_{e_i^\perp} \eta_2 \rangle \Big) \, dy
$$

$$
= \sum_{i=1}^{n} \frac{|\theta_i|}{\|\theta\|_1} \Big(\frac{1}{3}\eta_1(i)^2 |P_{e_i^\perp}\eta_2|^2 + \frac{1}{3}\eta_2(i)^2 |P_{e_i^\perp}\eta_1|^2 + \eta_1(i)^2 \eta_2(i)^2
$$

$$
+ 4\eta_1(i)\eta_2(i) \frac{1}{\mathrm{Vol}(B_\infty^{n-1})} \int_{B_\infty^{n-1}} \langle y, P_{e_i^\perp}\eta_1 \rangle \langle y, P_{e_i^\perp}\eta_2 \rangle dy
$$

$$
+ \frac{1}{\mathrm{Vol}(B_\infty^{n-1})} \int_{B_\infty^{n-1}} \langle y, P_{e_i^\perp}\eta_1 \rangle^2 \langle y, P_{e_i^\perp}\eta_2 \rangle^2 dy \Big)
$$

$$
= \sum_{i=1}^{n} \frac{|\theta_i|}{\|\theta\|_1} \Big(\frac{1}{3}\eta_1(i)^2 + \frac{1}{3}\eta_2(i)^2 + \frac{1}{3}\eta_1(i)^2 \eta_2(i)^2 +
$$

$$
+ 4\eta_1(i)\eta_2(i) \frac{1}{\mathrm{Vol}(B_\infty^{n-1})} \int_{B_\infty^{n-1}} \langle y, P_{e_i^\perp}\eta_1 \rangle \langle y, P_{e_i^\perp}\eta_2 \rangle dy
$$

$$
+ \frac{1}{\mathrm{Vol}(B_\infty^{n-1})} \int_{B_\infty^{n-1}} \langle y, P_{e_i^\perp}\eta_1 \rangle^2 \langle y, P_{e_i^\perp}\eta_2 \rangle^2 dy \Big)
$$

Since

$$
\frac{1}{\mathrm{Vol}(B_\infty^{n-1})} \int_{B_\infty^{n-1}} \langle y, P_{e_i^\perp}\eta_1 \rangle \langle y, P_{e_i^\perp}\eta_2 \rangle dy
$$

$$
= \frac{1}{\mathrm{Vol}(B_\infty^{n-1})} \int_{B_\infty^{n-1}} \Big(\sum_{l_1, l_2 \neq i} y_{l_1} y_{l_2} \eta_1(l_1)\eta_2(l_2) \Big) dy
$$

$$
= \frac{1}{\mathrm{Vol}(B_\infty^{n-1})} \int_{B_\infty^{n-1}} \Big(\sum_{l \neq i} y_l^2 \eta_1(l)\eta_2(l) \Big) dy
$$

$$
= \frac{1}{3} \sum_{l \neq i} \eta_1(l)\eta_2(l)
$$

$$
= \frac{1}{3} (\langle \eta_1, \eta_2 \rangle - \eta_1(i)\eta_2(i))
$$

$$
= -\frac{1}{3}\eta_1(i)\eta_2(i)
$$

the previous sum equals

$$
\sum_{i=1}^{n} \frac{|\theta_i|}{\|\theta\|_1} \Big(\frac{1}{3}\eta_1(i)^2 + \frac{1}{3}\eta_2(i)^2 - \eta_1(i)^2 \eta_2(i)^2
$$

$$
+ \frac{1}{\mathrm{Vol}(B_\infty^{n-1})} \int_{B_\infty^{n-1}} \langle y, P_{e_i^\perp}\eta_1 \rangle^2 \langle y, P_{e_i^\perp}\eta_2 \rangle^2 dy \Big).
$$

Now,

$$\frac{1}{\text{Vol}(B_\infty^{n-1})} \int_{B_\infty^{n-1}} \langle y, P_{e_i^\perp} \eta_1 \rangle^2 \langle y, P_{e_i^\perp} \eta_2 \rangle^2 dy$$

$$= \frac{1}{\text{Vol}(B_\infty^{n-1})} \int_{B_\infty^{n-1}} \left(\sum_{l_1,l_2,l_3,l_4} y_{l_1} y_{l_2} y_{l_3} y_{l_4} \eta_1(l_1)\eta_1(l_2)\eta_2(l_3)\eta_2(l_4) \right) dy$$

$$= \sum_{l \neq i} \eta_1(l)^2 \eta_2(l)^2 \frac{1}{\text{Vol}(B_\infty^{n-1})} \int_{B_\infty^{n-1}} y_l^4 dy$$

$$+ \sum_{l_1 \neq l_2(\neq i)} \frac{1}{\text{Vol}(B_\infty^{n-1})} \int_{B_\infty^{n-1}} y_{l_1}^2 y_{l_2}^2 dy (\eta_1(l_1)^2 \eta_2(l_2)^2$$

$$+ \eta_1(l_1)\eta_1(l_2)\eta_2(l_1)\eta_2(l_2))$$

$$= \frac{1}{5} \sum_{l \neq i} \eta_1(l)^2 \eta_2(l)^2 + \frac{1}{9} \sum_{l_1 \neq l_2(\neq i)} (\eta_1(l_1)^2 \eta_2(l_2)^2$$

$$+ \eta_1(l_1)\eta_1(l_2)\eta_2(l_1)\eta_2(l_2))$$

$$= \frac{1}{5} \sum_{l \neq i} \eta_1(l)^2 \eta_2(l)^2 + \frac{1}{9} \left[\sum_{l \neq i} \left(\eta_1(l)^2 (1 - \eta_2(l)^2 - \eta_2(i)^2) \right) \right.$$

$$\left. + \eta_1(l)\eta_2(l)(\langle \eta_1, \eta_2 \rangle - \eta_1(l)\eta_2(l) - \eta_1(i)\eta_2(i))) \right]$$

$$= \frac{1}{5} \sum_{l \neq i} \eta_1(l)^2 \eta_2(l)^2 + \frac{1}{9} \left(1 - \eta_1(i)^2 - \sum_{l \neq i} \eta_1(l)^2 \eta_2(l)^2 - \eta_2(i)^2 \right.$$

$$\left. + \eta_1(i)^2 \eta_2(i)^2 - \sum_{l \neq i} \eta_1(l)^2 \eta_2(l)^2 + \eta_1(i)^2 \eta_2(i)^2 \right)$$

$$= \frac{1}{9} - \frac{1}{9} \eta_1(i)^2 - \frac{1}{9} \eta_2(i)^2 + \frac{2}{9} \eta_1(i)^2 \eta_2(i)^2 - \frac{1}{45} \sum_{l \neq i} \eta_1(l)^2 \eta_2(l)^2.$$

Consequently

$$\mathbb{E}\langle X, \eta_1 \rangle^2 \langle X, \eta_2 \rangle^2 = \frac{1}{9} + \sum_{i=1}^n \frac{|\theta_i|}{\|\theta\|_1} \left(\frac{2}{9} \eta_1(i)^2 + \frac{2}{9} \eta_2(i)^2 - \frac{7}{9} \eta_1(i)^2 \eta_2(i)^2 \right.$$

$$\left. - \frac{1}{45} \sum_{l \neq i} \eta_1(i)^2 \eta_2(l)^2 \right)$$

$$\leq \frac{1}{9} + \frac{2}{9} \sum_{i=1}^n \frac{|\theta_i|}{\|\theta\|_1} (\eta_1(i)^2 + \eta_2(i)^2)$$

which concludes the proof.   □

By Proposition 3 we obtain the following

Corollary 3. *There exists an absolute constant C such that for every hyperplane H and any linear map T, if X is a random vector uniformly distributed on $P_H B_\infty^n$, then TX verifies the variance conjecture with constant C, i.e.*

$$Var\,|TX|^2 \le C\lambda_{TX}^2 \mathbb{E}|TX|^2$$

The next examples we consider are random vectors uniformly distributed on projections of B_1^n. Even though in this case we are not able to prove that these vectors satisfy a square negative correlation property, we are still able to show that they verify the variance conjecture with some absolute constant.

Theorem 2. *There exists an absolute constant C such that for every hyperplane H, if X is a random vector uniformly distributed on $P_H B_1^n$, X verifies the variance conjecture with constant C, i.e.*

$$Var\,|X|^2 \le C\lambda_X^2 \mathbb{E}|X|^2.$$

Proof. First of all, notice that by Proposition 4 we have that for every $\xi \in S^{n-1} \cap H$

$$\mathbb{E}\langle \mathrm{Vol}(B_1^n)^{-\frac{1}{n}} X, \xi\rangle^2 \sim L_{B_1^n}^2 \sim 1$$

and so

$$\lambda_X^2 \sim \frac{1}{n^2} \qquad \text{and} \qquad \mathbb{E}|X|^2 \sim \frac{1}{n}.$$

Thus, we have to prove that $Var\,|X|^2 \le \dfrac{C}{n^3}$.

By Cauchy formula, denoting by θ the unit vector orthogonal to H, Y a random vector uniformly distributed on $\Delta_{n-1} = \{y \in \mathbb{R}^n \,:\, y_i \ge 0, \sum_{i=1}^n y_i = 1\}$, ε a random vector, independent of Y, in $\{-1, 1\}^n$ distributed according to

$$\mathbb{P}(\varepsilon = \varepsilon_0) = \frac{|\langle \varepsilon_0, \theta\rangle|}{\sum_{\varepsilon \in \{-1,1\}^n} |\langle \varepsilon, \theta\rangle|} = \frac{\mathrm{Vol}_{n-1}(\Delta_{n-1})|\langle \varepsilon_0, \theta\rangle|}{2\sqrt{n}\,\mathrm{Vol}_{n-1}(P_H(B_1^n))}$$

and

$$\varepsilon x = (\varepsilon_1 x_1, \ldots, \varepsilon_n x_n)$$

we have that

$$\begin{aligned}
Var\,|X|^2 &= \mathbb{E}|X|^4 - (\mathbb{E}|X|^2)^2 = \mathbb{E}|P_H(\varepsilon Y)|^4 - (\mathbb{E}|P_H(\varepsilon Y)|^2)^2 \\
&= \mathbb{E}(|\varepsilon Y|^2 - \langle \varepsilon Y, \theta\rangle^2)^2 - (\mathbb{E}(|\varepsilon Y|^2 - \langle \varepsilon Y, \theta\rangle^2))^2 \\
&= \mathbb{E}(|Y|^2 - \langle Y, \varepsilon\theta\rangle^2)^2 - (\mathbb{E}(|Y|^2 - \langle Y, \varepsilon\theta\rangle^2))^2 \\
&\le \mathbb{E}|Y|^4 + \mathbb{E}\langle \varepsilon Y, \theta\rangle^4 - (\mathbb{E}|Y|^2 - \mathbb{E}\langle \varepsilon Y, \theta\rangle^2)^2.
\end{aligned}$$

Since for every $a, b \in \mathbb{N}$ with $a + b = 4$ we have

$$\mathbb{E}Y_1^a Y_2^b = \frac{a!b!}{(n+3)(n+2)(n+1)n}$$

we have

$$\mathbb{E}|Y|^4 = n\mathbb{E}Y_1^4 + n(n-1)\mathbb{E}Y_1^2 Y_2^2$$

$$= \frac{4!}{(n+3)(n+2)(n+1)} + \frac{4(n-1)}{(n+3)(n+2)(n+1)}$$

$$= \frac{4}{n^2} + O\left(\frac{1}{n^3}\right).$$

Denoting by ϵ a random vector uniformly distributed on $\{-1, 1\}^n$ we have, by Khintchine inequality,

$$\mathbb{E}\langle \varepsilon Y, \theta \rangle^4 = \frac{\mathrm{Vol}_{n-1}(\Delta_{n-1})}{2\sqrt{n}\mathrm{Vol}_{n-1}(P_H(B_1^n))}\mathbb{E}_Y \sum_{\varepsilon \in \{-1,1\}^n} |\langle \varepsilon, \theta \rangle| \langle \varepsilon Y, \theta \rangle^4$$

$$= \frac{2^n \mathrm{Vol}_{n-1}(\Delta_{n-1})}{2\sqrt{n}\mathrm{Vol}_{n-1}(P_H(B_1^n))}\mathbb{E}_Y \mathbb{E}_\epsilon |\langle \epsilon, \theta \rangle| \langle \epsilon Y, \theta \rangle^4$$

$$\leq C\mathbb{E}_Y \left(\mathbb{E}_\epsilon \langle \epsilon, \theta \rangle^2\right)^{\frac{1}{2}} \left(\mathbb{E}_\epsilon \langle \epsilon Y, \theta \rangle^8\right)^{\frac{1}{2}}$$

$$\leq C\mathbb{E}_Y \left(\sum_{i=1}^n Y_i^2 \theta_i^2\right)^2$$

$$= C\left(\mathbb{E}Y_1^4 \sum_{i=1}^n \theta_i^4 + \mathbb{E}Y_1^2 Y_2^2 \sum_{i \neq j} \theta_i^2 \theta_j^2\right)$$

$$= \frac{C}{(n+3)(n+2)(n+1)n}\left(24\sum_{i=1}^n \theta_i^4 + 4\sum_{i,j=1}^n \theta_i^2 \theta_j^2\right)$$

$$\leq \frac{C}{n^4}$$

since $\sum_{i=1}^n \theta_i^4 \leq \sum_{i=1}^n \theta_i^2 = 1$.

On the other hand, since

$$\mathbb{E}Y_1^2 = \frac{2}{(n+1)n} \quad \text{and} \quad \mathbb{E}Y_1 Y_2 = \frac{1}{(n+1)n}$$

we have

$$\mathbb{E}|Y|^2 = n\mathbb{E}Y_1^2 = \frac{2}{n+1}$$

and

$$\mathbb{E}\langle \varepsilon Y, \theta \rangle^2 = \mathbb{E}\left(\sum_{i=1}^n Y_i^2 \theta_i^2 + \sum_{i \neq j} \varepsilon_i \varepsilon_j Y_i Y_j \theta_i \theta_j\right)$$

$$= \mathbb{E} Y_1^2 + \sum_{i \neq j} \theta_i \theta_j \mathbb{E}_\varepsilon \varepsilon_i \varepsilon_j \mathbb{E}_Y Y_1 Y_2$$

$$= \frac{2}{(n+1)n} + \frac{1}{(n+1)n}(\mathbb{E}_\varepsilon \langle \varepsilon, \theta \rangle^2 - 1)$$

$$= \frac{1}{(n+1)n}(1 + \mathbb{E}_\varepsilon \langle \varepsilon, \theta \rangle^2) \sim \frac{1}{n^2},$$

since, by Khintchine inequality,

$$\mathbb{E}_\varepsilon \langle \varepsilon, \theta \rangle^2 = \frac{\mathrm{Vol}_{n-1}(\Delta_{n-1})}{2\sqrt{n}\,\mathrm{Vol}_{n-1}(P_H(B_1^n))} \sum_{\varepsilon \in \{-1,1\}^n} |\langle \varepsilon, \theta \rangle|^3$$

$$\leq C\,\mathbb{E}_\epsilon |\langle \varepsilon, \theta \rangle|^3 \sim C$$

Thus

$$(\mathbb{E}|Y|^2 - \mathbb{E}\langle \varepsilon Y, \theta \rangle^2)^2 = \frac{4}{n^2} + O\left(\frac{1}{n^3}\right)$$

and so

$$\mathrm{Var}\,|X|^2 \leq \frac{C}{n^3}.$$

<div style="text-align: right">□</div>

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More Universal Minimal Flows of Groups of Automorphisms of Uncountable Structures

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Abstract In this paper, we compute universal minimal flows of groups of automorphisms of uncountable ω-homogeneous graphs, K_n-free graphs, hypergraphs, partially ordered sets, and their extensions with an ω-homogeneous ordering. We present an easy construction of such structures, expanding the jungle of extremely amenable groups.

1 Introduction

In [12], Pestov first established a connection between extreme amenability of groups of automorphisms of structures and structural Ramsey theory. He showed that the group of automorphisms of an ω-homogeneous linearly ordered set is extremely amenable, which started a list of examples of "natural" extremely amenable groups in contrast to pathological groups of Herer and Christensen [4]. His result was extended to a full theory connecting groups of automorphisms of countable structures, structural Ramsey theory and model theory in [8]. However, Pestov's original example does not distinguish between countable and uncountable cardinality and we show that this is indeed a general phenomenon.

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In [1], we used the ultrafilter flow to compute universal minimal flows of automorphism groups of ω-homogeneous uncountable structures such that the class of finite substructures satisfies the Ramsey property and admits an appropriate extension to a class of linearly ordered structures (e.g. Boolean algebras, vector spaces over a finite field, linearly ordered sets etc.). Here, we apply this result for linearly ordered structures to generalize techniques of Kechris, Pestov and Todorcevic in [8] to compute universal minimal flows of more automorphism groups of uncountable structures (e.g. (linearly ordered) graphs, hypergraphs, posets etc.).

In the first section, we remind the reader of the basic notions from topological dynamics and groups of automorphisms of structures. We recall a theorem characterizing extremely amenable groups of automorphisms and give new examples.

As in [8], we consider the action of G on the compact space $\mathrm{LO}(A)$ of all linear orderings on A. For a linear ordering $<$ in $\mathrm{LO}(A)$, we denote by $\overline{G <}$ the topological closure of the orbit $G < = \{g <: g \in G\}$ of $<$ under the action of G in the space $\mathrm{LO}(A)$. If there exists a linear ordering $<$ in $\mathrm{LO}(A)$ making $(A, <)$ ω-homogeneous, then as in Theorem 7.4 of [8] minimality of the space $\overline{G <}$ corresponds to the ordering property for the class of finite substructure of $(A, <)$ that has been isolated in [8]. In accordance with Theorem 7.5 in [8], universality of $\overline{G <}$ is then implied by the class of finite substructures of $(A, <)$ satisfying the Ramsey property.

In the next two sections, we show that there are many structures in every uncountable cardinality for which we can compute the universal minimal flow of their groups of automorphisms.

In the second section, we recall Jósson's construction of universal homogeneous structures of cardinalities κ whenever $\kappa^{<\kappa} = \kappa$. We show that results of [8] can be extended from Fraïssé structures to Jónsson structures without any obstacles and compute universal minimal flows of groups of automorphisms of universal homogeneous graphs, K_n-free graphs, hypergraphs, $\mathcal{A}$-free hypergraphs and posets of the relevant cardinality.

In the last section, we fill in the gap given by the restriction in Jónsson's construction on cardinality. Given a graph (K_n-free graph, hypergraph, $\mathcal{A}$-free hypergraph, poset) of an arbitrary cardinality, we find a superstructure of the same type and the same cardinality that is ω-homogeneous with an ω-homogeneous linear ordering to which we apply results from the first section to compute the universal minimal flows of their automorphism groups.

2   General Theory

Let G be a topological group with identity element e and X a compact Hausdorff space. We say that X is a G-*flow* via an action π, if $\pi : G \times X \to X$ is a continuous map satisfying $\pi(e, x) = x$ for every $x \in X$ and $\pi(gh, x) = \pi(g, \pi(h, x))$ for every $g, h \in G$ and $x \in X$. Usually, π is understood and we simply say that X is a G-flow and write gx in place of $\pi(g, x)$. A G-flow X is called *minimal* if there is no closed subspace of X invariant under the action of G. Equivalently, X is

minimal if for every $x \in X$ the *orbit* $Gx = \{gx : g \in G\}$ is dense in X. Among all minimal G-flows, there is a maximal one—the *universal minimal flow* $M(G)$. It means that every other minimal G-flow is a factor of $M(G)$. In the study of universal minimal flows, a construction of the greatest ambit turns out to be useful. An *ambit* is a G-flow X with a distinguished point $x_0 \in X$ whose orbit $Gx_0 = \{gx_0 : g \in G\}$ is dense in X. As for minimal flows, there is a maximal ambit— the *greatest ambit* $(S(G), e)$. It means that every other ambit (X, x_0) is a factor of $S(G)$ via a quotient mapping sending e to x_0. The study of $S(G)$ shows that every homomorphisms of $M(G)$ into itself is an isomorphism, which in turn gives that the universal minimal flow is unique up to an isomorphism. For introduction to topological dynamics see [2].

In what follows, we will be interested in automorphism groups of structures in a countable signature and their dense subgroups. If A is a structure, we denote its group of automorphisms by $\mathrm{Aut}(A)$. We consider A as a discrete space and equip $\mathrm{Aut}(A)$ with the topology of pointwise convergence turning it into a topological group. If G is a subgroup of $\mathrm{Aut}(A)$, then the topology on G is generated by pointwise stabilizers of finite substructures of A, where the *pointwise stabilizer* G_F of a substructure F of A is the following clopen subgroup:

$$G_F = \{g \in G : g(a) = a \ \forall \, a \in F\}.$$

For a cardinal κ, denote by S_κ the group of all bijections on κ with the topology of pointwise convergence. If A is a structure of cardinality κ, then $\mathrm{Aut}(A)$ is naturally isomorphic to a closed subgroup of S_κ (see e.g. [5]).

We say that A is ω-*homogeneous*, if every partial isomorphism between two finitely generated substructures of A can be extended to an automorphism of A.

We will identify universal minimal flows of groups of automorphisms of certain structures with spaces of linear orderings. Let us denote by $\mathrm{LO}(\kappa)$ the space of all linear orderings on κ with the topology inherited from the Tychonoff product $2^{\kappa \times \kappa}$. $\mathrm{LO}(\kappa)$ is then a compact space and S_κ naturally acts on $\mathrm{LO}(\kappa)$ as follows:

$$\sigma \in S_\kappa, < \in \mathrm{LO}(\kappa), \alpha, \beta < \kappa \rightarrow (\alpha(\sigma <)\beta \leftrightarrow \sigma^{-1}\alpha < \sigma^{-1}\beta).$$

Whenever we talk about an action of a group $G \leq \mathrm{Aut}(A)$, where A is a structure of cardinality κ, on the space of linear orderings, we always refer to the above action with $\mathrm{Aut}(A)$ identified with a closed subgroup of S_κ.

It follows from [1] that whenever A is a structure and G is a dense subgroup of $\mathrm{Aut}(A)$, then the universal minimal flow of G is the universal minimal flow of $\mathrm{Aut}(A)$ under the restricted action.

We write $B \leq A$ to denote that B is a substructure of A and $\mathrm{Age}(A)$ to denote the class of finitely generated substructures of A up to an isomorphism. We also use $\mathrm{Age}(A)$ to denote the class of all finitely generated substructures of A without ambiguity.

In computations of universal minimal flows of groups of automorphisms, the Ramsey property for finite structures turns out to play a crucial role.

Definition 1 (Ramsey property). A class $\mathcal{K}$ of finite structures satisfies the *Ramsey property* if for every $A \leq B \in \mathcal{K}$ and $k \geq 2$ a natural number there exists $C \in \mathcal{K}$ such that

$$C \to (B)^A_k.$$

It means for every colouring of copies of A in C by k colours, there is a copy B' of B in C, such that all copies of A in B' have the same colour.

Example 1. The following classes of finite structures satisfy the Ramsey property:

- Finite graphs equipped with arbitrary linear orderings,
- Finite K_n-free graphs with arbitrary linear orderings for some $n \in \omega$,
- Finite hypergraphs of a given signature with arbitrary linear orderings,
- Finite $\mathcal{A}$-free hypergraphs of a given signature with arbitrary linear orderings,
- Finite posets with linear orderings extending the partial order.

For graph and hypergraph classes see [9] and [11], for posets see [14].

By a *graph* we mean an unordered graph. A K_n-*free graph* is a graph that does not contain the complete graph on n-vertices as an induced subgraph. A *hypergraph* H is a structure in a finite signature $L = \{R_i : i < k\}$ of relational symbols, where each R_i is closed under permutations, i.e. if R_i has arity l and σ is a permutation on $\{0, 1, \ldots, l-1\}$, then $(h_0, h_1, \ldots, h_{l-1}) \in R_i^H \subset H^l$ implies that all $h_0, h_2, \ldots, h_{l-1}$ are distinct and $(h_{\sigma(0)}, h_{\sigma(1)}, \ldots, h_{\sigma(l-1)}) \in R_i^H$. In other words, we can think of R_i^H as a collection of subsets of H of size l. A hypergraph H is called *irreducible* if it contains at least two elements and whenever $x \neq y$ in H then there exists $i < k$ and $S \in R_i^H$ such that $\{x, y\} \subset S$. Let $\mathcal{A}$ be a class of irreducible hypergraphs in signature L. A hypergraph is $\mathcal{A}$-*free* if no element of $\mathcal{A}$ can be embedded into it. By a *poset*, we mean a partially ordered set.

We are ready to recall a characterization of extremely amenable subgroups of S_κ, that is to say subgroups whose universal minimal flow is trivial.

Definition 2 (Extreme amenability). A topological group G is called *extremely amenable* if the universal minimal flow of G is a single point.

The following theorem was proved in [8] to characterize extremely amenable groups of automorphisms of countable structures. It was verified in [1] that it holds for uncountable structures as well.

Theorem 1. *Let G be a subgroup of S_κ. The following are equivalent:*

(a) G is extremely amenable.
(b) For every finite $F \subset \kappa$ (i) $G_F = G_{(F)}$, where $G_{(F)} = \{g \in G : gF = F\}$ and (ii) for every colouring $c : G/G_F \to \{1, 2, \ldots, k\}$ and for every finite $B \supset F$, there is $g \in G$ and $i \in \{1, 2, \ldots, n\}$ such that $c(hG_F) = i$ whenever $h[F] \subset g[B]$.
(c) (i') G preserves an ordering and (ii) as above.

Remark 1. Let A be an ω-homogeneous structure such that G is dense in its automorphism group. If finitely generated substructures of A are finite, (ii) of (b) simply says that $\text{Age}(A)$ satisfies the Ramsey property. Whence the following corollary.

Corollary 1. *Let A be an ω-homogeneous linearly ordered structure with finitely generated substructures being finite and let G be a dense subgroup of $\text{Aut}(A)$. Then G is extremely amenable if and only if $\text{Age}(A)$ satisfies the Ramsey property.*

Since the classes of finite structures in Example 1 are all relational and satisfy the Ramsey property, we get new examples of extremely amenable groups of uncountable structures (the result for countable structures was proved in [8]).

Corollary 2. *Let A be an uncountable graph (K_n-free graph, hypergraph, $\mathcal{A}$-free hypergraph, poset) and let $<$ be a linear ordering such that $(A, <)$ is ω-homogeneous (and if A is a poset, $<$ extends the partial order). Let G be a dense subgroup of $\text{Aut}(A, <)$. If $\text{Age}(A, <)$ is the class of all finite linearly ordered graphs (K_n-free graphs, hypergraphs, $\mathcal{A}$-free hypergraphs, posets with the linear ordering extending the partial order), then G is extremely amenable.*

We will compute universal minimal flows of groups of automorphisms of ω-homogeneous structures that are not linearly ordered, but admit a suitable linearly ordered ω-homogeneous extension. In [8], it was shown that "suitable" can be expressed in terms of the ordering property.

Definition 3 (Ordering property). Let $L \supset \{>\}$ be a signature and let $\mathcal{K}_<$ be a class of L-structures where $<$ is interpreted as a linear ordering. Let $L_0 = L \setminus \{<\}$ and $\mathcal{K} = \mathcal{K}_< | L_0$. We say that $\mathcal{K}_<$ has the *ordering property* if for every $A \in \mathcal{K}$ there exists $B \in \mathcal{K}$ such that for every linear ordering $\prec$ on A and for every linear ordering $\prec'$ on B, if $(A, \prec) \in \mathcal{K}_<$ and $(B, \prec') \in \mathcal{K}_<$ then $(A, \prec)$ is a substructure of $(B, \prec')$.

All classes of structures in Example 1 satisfying the Ramsey property also satisfy the ordering property. For graph and hypergraph classes see [10] and for posets see [14].

The following three theorems are Theorems 7.1, 7.4 and 7.5 from [8]. Their proofs in [8] are given for countable structures, but with slight modifications they work for uncountable structures as well, since the topology on an automorphism group of arbitrary size is given by finitely generated substructures.

Theorem 2. *Let A be an ω-homogeneous structure, $\prec$ a linear ordering on A and let G be a dense subgroup of $\text{Aut}(A)$. Then $< \in \overline{G \prec}$ if and only if for every $B \in \text{Age}(A)$, $(B, < |B) \in \text{Age}(A, \prec)$.*

Proof. Let $< \in \overline{G \prec}$. It means that for every $B \in \text{Age}(A)$ there exists $g \in G$ such that $g \prec |B =< |B$. Thus $g : g^{-1}(B) \to B$ is an isomorphism between $(g^{-1}(B), \prec |g^{-1}(B))$ and $(B, < |B)$, which shows that $(B, < |B) \in \text{Age}(A, \prec)$.

Conversely, suppose that $< \in \text{LO}(A)$ and $(B, < |B) \in \text{Age}(A, \prec)$ for every $B \in \text{Age}(A)$. This means that for every $B \in \text{Age}(A)$ there is an embedding i :

$(B, < |B) \rightarrow (A, \prec)$. If $C = i(B)$, then i^{-1} is an isomorphism between $(C, \prec |C)$ and $(B, < |B)$ and in particular i^{-1} is an isomorphism between C and B. By ω-homogeneity of A, i^{-1} can be extended to a $g \in G$. Then $< |B = g \prec |B$ and thus $< \in \overline{G \prec}$. $\square$

Theorem 3. *Let A be an ω-homogeneous structure and let $\prec$ be a linear ordering on A such that $(A, \prec)$ is ω-homogeneous as well. Let G be a dense subgroup of* Aut(A). *Then the following are equivalent:*

(a) Age$(A, \prec)$ *satisfies the ordering property,*
(b) $\overline{G \prec}$ *is a minimal G-flow.*

Proof. $(a) \Rightarrow (b)$ Let $< \in \overline{G \prec}$. We would like to show that also $\prec \in \overline{G <}$. By the previous theorem, this is equivalent to showing that Age$(A, \prec) \subset$ Age$(A, <)$.

To that end, let $(B, \prec |B) \in$ Age$(A, \prec)$ and find $D \in$ Age(A) given by the ordering property for B. Let D_0 be an isomorphic copy of D in A and let $i : (B, \prec |B) \rightarrow (D_0, < |D_0)$ be an embedding ensured by the ordering property. Then i is an isomorphism between $(B, \prec |B)$ and $(i(B), < |i(B)) \in$ Age$(A, <)$, showing $(B, \prec |B) \in$ Age$(A, <)$.

$(b) \Rightarrow (a)$ Given $B \in$ Age(A) we need to find $D \in$ Age(A) such that whenever $(B, \prec') \in$ Age$(A, \prec)$ and $(D, <) \in$ Age$(A, \prec)$, there is an embedding of $(B, \prec')$ into $(D, <)$. Fix $(B, \prec') \in$ Age$(A, \prec)$. For every $C \in$ Age(A) consider the set

$$X_C = \{< \in \overline{G \prec} : (B, \prec' |B) \cong (C, < |C)\}.$$

Then $\overline{G \prec} = \bigcup_{C \in \mathrm{Age}(A)} X_C$. Since each X_C is open, there are $C_1, C_2, \ldots, C_n \in$ Age(A) such that $\overline{G \prec} = \bigcup_{i=1}^n X_{C_i}$ by compactness of $\overline{G \prec}$. Let $D_{\prec'}$ be the substructure of A generated by $\bigcup_{i=1}^n C_i$. We show that whenever $(D_{\prec'}, <) \in$ Age$(A, \prec)$, then $(B, \prec') \leq (D_{\prec'}, <)$. It means we need to find $<' \in \overline{G \prec}$ extending $<$. By minimality of $\overline{G \prec}$, we know that there exists an embedding $i : (D_{\prec'}, <) \rightarrow (A, \prec)$. By ω-homogeneity of A, there is $g \in G$ extending i to all of A. Then we get that $g^{-1} \prec |D_{\prec'} = <$, so $g^{-1} \prec \in G \prec$ is the sought for extension of $<$.

Now we repeat the procedure for every linear ordering $\prec'$ on B with $(B, \prec') \in$ Age$(A, \prec)$ and set D to be the substructure of A generated by

$$\bigcup_{(B, \prec') \in \mathrm{Age}(A, \prec)} D_{\prec'}.$$

Then D is a witness of the ordering property for B and we are done. $\square$

Theorem 4. *Let A be an ω-homogeneous structures and let $\prec$ be a linear ordering on A such that $(A, \prec)$ is also ω-homogeneous. Suppose that finitely generated substructures of A are finite. Let $H =$ Aut(A) and $H_\prec =$ Aut($A, \prec$).*

(a) Suppose that Age$(A, \prec)$ *satisfies the Ramsey property. Then $(\overline{H \prec}, \prec)$ is the universal ambit among those H-ambits whose base point is fixed by the action of $H_\prec$.*

(b) Suppose that Age$(A, \prec)$ *satisfies both the Ramsey and the ordering properties.
    Then* $\overline{H \prec}$ *is the universal minimal H-flow.*

Proof. (a) Let (X, x_0) be an H-flow such that $H_{\prec} x_0 = \{x_0\}$. Let Φ be the closure
of the following set in the compact Hausdorff space $\overline{H \prec} \times X$.

$$\{(h \prec, h x_0) : h \in H\} \subset \overline{H \prec} \times X.$$

We will show that Φ is a graph of a function $\phi : \overline{H \prec} \to X$. Having proved this,
it is easy to verify that ϕ works: Since Φ is closed, ϕ is continuous. Also, ϕ is
an H-homomorphism: If $(<, x) \in \Phi$, then there is a net $\{h_i : i \in I\}$ such that
$\{h_i \prec : i \in I\}$ converges to $<$ and $\{h_i x_0 : h_i \in I\}$ converges to x. Let $h \in H$.
Then $(h h_i \prec, h h_i x_0) \in \Phi$ for every $i \in I$, and $\{(h h_i, h h_i x_0) : i \in I\}$ converges to
$(h <, hx) \in \Phi$. It follows that $\phi(h <) = hx = h\phi(\prec)$. Finally, since $H x_0$ is dense
in X, ϕ is surjective.

First, let us show that for every $<$ in $\overline{H \prec}$ there is an $x \in X$ such that $(<, x) \in \Phi$.
Indeed, let $\{h_i : i \in I\}$ be a net such that $\{h_i \prec\}$ converges to $<$. Then $\{h_i x_0\}$ is a
net in X, so by compactness of X there is a subnet $h_i^j x_0$ converging to some $x \in X$.
Then $(h_i^j \prec, h_i^j x_0)$ converges to $(<, x) \in \Phi$.

Second, we prove that if $(<, x_1), (<, x_2) \in \Phi$, then $x_1 = x_2$. Let $\{h_i : i \in I\}$
and $\{g_j : j \in J\}$ be nets such that $(h_i \prec, h_i x_0)$ is a net converging to $(<, x_1)$ and
$(g_j \prec, g_j x_0)$ is a net converging to $(<, x_2)$ and suppose that $x_1 \neq x_2$.

As X is a compact Hausdorff space, it is regular, so there are open neigh-
bourhoods U_1, U_2 of x_1, x_2 respectively and V a neighbourhood of the diagonal
$\Delta = \{(x, x) : x \in X\}$ such that $V \cap (U_1 \times U_2) = \emptyset$. Without loss of generality
we may assume that $h_i x_0 \in U_1$ for every $i \in I$ and $g_j x_0 \in U_2$ for every $j \in J$.
For every $y \in X$, there is a neighbourhood U_y of y such that $U_y \times U_y \subset V$. Again
by regularity, there are open V_y for $y \in X$ such that $\overline{V_y} \subset U_y$. By compactness, we
can find $y_1, y_2, \ldots, y_n \in X$ such that $X = \bigcup_{i=1}^{n} \overline{V_{y_i}}$.

Let Homeo(X) denote the group of homeomorphism of X with the compact open
topology. Since the action of H on X is continuous, the map $\phi : H \to \mathrm{Homeo}(X)$
defined by $\phi(h)(x) = hx$, is continuous. Since $\overline{V_{y_i}} \subset U_{y_i}$ for every i, the set

$$O = \bigcap_{i=1}^{n} \{f \in \mathrm{Homeo}(X) : f(\overline{V_{y_i}}) \subset U_{y_i}\}$$

is an open neighbourhoood of the identity in Homeo(X). Let $O_H = \phi^{-1}(O)$. Then
O_H is an open neighbourhood of the identity element in H and whenever $h \in O_H$,
then $h\overline{V_{y_i}} \subset U_{y_i}$ for $i = 1, 2, \ldots, n$, so $(y, hy) \in V$ for every $y \in X$. Since the
topology on H is determined by finite substructures of A, there exists $B \in \mathrm{Age}(A)$
such that $G_B \subset O_H$. Since $\{(h_i \prec, h_i x_0) : i \in I\}$ converges to $(<, x_1)$ and $\{(g_j \prec,
g_j x_0) : j \in J\}$ converges to $(<, x_2)$, there are $i_0 \in I$ and $j_0 \in J$ such that
$h_{i_0} x_0 \in U_1$, $g_{j_0} x_0 \in U_2$ and

$$h_{i_0} \prec |B = g_{j_0} \prec |B = < |B.$$

That is to say, for every $b_1, b_2 \in B$,

$$h_{i_0}^{-1}(b_1) \prec h_{i_0}^{-1}(b_2) \leftrightarrow g_{j_0}^{-1}(b_1) \prec g_{j_0}^{-1}(b_2).$$

If we denote $h_{i_0}^{-1}B = C$ and $g_{j_0}^{-1}B = D$, then $(C, \prec |C)$ and $(D, \prec |D)$ are isomorphic via $\rho : h_{i_0}^{-1}(b) \mapsto g_{j_0}^{-1}(b)$ for $b \in B$. Since $(A, \prec)$ is ω-homogeneous, there exists $r \in H_\prec$ extending ρ to all $(A, \prec)$. It means that $rh_{i_0}^{-1}(b) = g_{j_0}^{-1}(b)$ for every $b \in B$, in other words $rh_{i_0}^{-1}|B = g_{j_0}^{-1}|B$. By the choice of B, $g_{j_0}rh_{i_0}^{-1} \in O_H$. So $(h_{i_0}x_0, g_{j_0}rh_{i_0}^{-1}(h_{i_0}x_0)) \in V$. But $r \in H_\prec$, so $rx_0 = x_0$ and therefore $(h_{i_0}x_0, g_{j_0}x_0) \in V$. But also $(h_{i_0}x_0, g_{j_0}x_0) \in U_1 \times U_2$, which is a contradiction.

(b) Since $H_\prec$ is extremely amenable, every H-flow has a fixed point under the restricted action by $H_\prec$. In particular, every minimal flow has a fixed point. Every point of a minimal flow has a dense orbit, so $\overline{H \prec}$ is universal among all minimal H-flows by part (a). Since $\text{Age}(A, \prec)$ satisfies the ordering property, $\overline{H \prec}$ is a minimal H-flow by the previous theorem. Altogether we get that $\overline{H \prec}$ is the universal minimal flow of H.   □

Corollary 3. *Let $(A, \prec)$ and H be as above and let G be a dense subgroup of H. Then the universal minimal flow of G is $\overline{G \prec}$.*

3   Jónsson Structures

In [3], Fraïssé showed a correspondence between countable homogeneous structures and countable classes of finitely generated structures satisfying certain properties.

In [6] and [7], Jónsson generalized Fraïssé's construction [3] to uncountable structures of cardinality κ, whenever $\kappa^{<\kappa} = \kappa$. In the first article, [6], Jónsson was looking for conditions on a class $\mathcal{K}$ of relational structures that would give rise to a universal structure for $\mathcal{K}$. In the second article, [6], he used the amalgamation property by Fraïssé to answer in positive a question of R. Baer whether the universal structure would be unique if an additional condition was imposed on the class $\mathcal{K}$. Below, we recall Jónsson's conditions (the third and the fourth are the joint embedding property and the amalgamation property introduced by Fraïssé, the first and the fourth are variations of the original conditions from [6] ensuring homogeneity in [7]. The last condition is a weakening of Fraïssé's condition that the class is hereditary).

I′. For each ordinal ξ there is $A \in \mathcal{K}$ of cardinality greater or equal to $\aleph_\xi$.

II. If $A \in \mathcal{K}$ and A is isomorphic to a structure B then $B \in \mathcal{K}$.

III. For every $A, B \in \mathcal{K}$ there exist $C \in \mathcal{K}$ and embeddings $f : A \to C$ and $g : B \to C$.

IV$'$. For every $A, B, C \in \mathcal{K}$ and embeddings $f : A \to B$ and $g : A \to C$ there exist $D \in \mathcal{K}$ and embeddings $i : B \to D$ and $j : C \to D$ such that $i \circ f = j \circ g$.

V. If λ is a positive ordinal and $\langle A_\xi : \xi \in \lambda \rangle$ is a sequence of structures in $\mathcal{K}$ such that $A_\xi \leq A_\nu$ whenever $\xi < \nu < \lambda$, then $\bigcup_{\xi < \lambda} A_\xi \in \mathcal{K}$.

VI$_\alpha$. If $A \in \mathcal{K}$, $B \leq A$ and the cardinality of B is less than $\aleph_\alpha$, then there exists $C \in \mathcal{K}$ such that $B \leq C \leq A$ and the cardinality of C is less than $\aleph_\alpha$.

We will call a class of relational structures satisfying the conditions I$'$, II, III, VI$'$, V and VI$_\alpha$, for some positive ordinal α a *Jónsson class* (for α).

Let us make precise what we mean by a universal and a homogeneous structure.

Definition 4 (Universality; [6]). Let $\mathcal{K}$ be a class of relational structures and let α be an ordinal. We say that a structure $A \in \mathcal{K}$ is $(\aleph_\alpha, \mathcal{K})$-*universal* if A has cardinality $\aleph_\alpha$ and every $B \in \mathcal{K}$ of cardinality $\leq \aleph_\alpha$ can be embedded into A.

Definition 5 (Homogeneity; [7]). Let $\mathcal{K}$ be a class of relational structures and let α be an ordinal. We say that a structure A is $(\aleph_\alpha, \mathcal{K})$-*homogeneous* if $A \in \mathcal{K}$, the cardinality of A is $\aleph_\alpha$ and the following condition is satisfied. For any substructure $B \in \mathcal{K}$ of A of a smaller cardinality, every embedding of B into A can be extended to an automorphism of A.

The following proposition gives us a tool to check homogeneity of structures. The proof goes by a back-and-forth argument.

Proposition 1 (Extension property). *Let $\mathcal{K}$ be a class of structures and let A be a structure of cardinality $\aleph_\alpha$ for some ordinal α. Then A is $(\aleph_\alpha, \mathcal{K})$-homogeneous if and only if $A \in \mathcal{K}$ and for every $B, C \in \mathcal{K}$ of cardinalities less than $\aleph_\alpha$ and embeddings $i : B \to A$ and $j : B \to C$, there exists and embedding $k : C \to A$ such that $i = k \circ j$.*

The main result of [6] and [7] is the existence of a universal homogeneous structure for a Jónsson class of cardinality $\aleph_\alpha$ whenever $\aleph_\alpha^{<\aleph_\alpha} = \aleph_\alpha$.

Theorem 5 ([7]). *Let α be a positive ordinal with the following two properties:*

(i) If $\lambda < \omega_\alpha$ and if $n_\nu < \aleph_\alpha$ whenever $\nu < \lambda$, then $\sum_{\nu < \lambda} n_\nu < \aleph_\alpha$.
(ii) If $n < \aleph_\alpha$, then $2^n \leq \aleph_\alpha$.

If $\mathcal{K}$ is a Jónsson class for α, then there exists a unique $(\aleph_\alpha, \mathcal{K})$-universal homogeneous structure.

Nowadays, we abbreviate the conditions (i) and (ii) in the previous theorem as

$$\aleph_\alpha^{<\aleph_\alpha} = \aleph_\alpha$$

and say that $\aleph_\alpha$ to the weak power $\aleph_\alpha$ is equal to $\aleph_\alpha$.

An assumption on cardinality is essential, since for instance, it is consistent both that there exists and that there does not exist a universal graph of cardinality $\aleph_1$ (see [13]).

Corollary 4 ([7]). *If the General Continuum Hypothesis holds and $\mathcal{K}$ is a Jónsson class for every positive ordinal α, then there is an $(\aleph_\alpha, \mathcal{K})$-universal homogeneous structure for every positive α.*

In the previous section, we presented a method which shows how to compute universal minimal flows of groups of automorphisms of structures provided that the structures admit a certain linearly ordered extension. To obtain corresponding results for Jónsson classes, we require that the structures of those classes permit an extension to a Jónsson class of linearly ordered structures satisfying a mild condition of *reasonability* as in the case of Fraïssé classes in [8].

Definition 6 (Reasonable class). Let $L \supset \{>\}$ be a language. Let $\mathcal{K}_<$ be a class of L-structures with $<$ interpreted as a linear ordering. Let $L_0 = L \setminus \{<\}$ and $\mathcal{K} = \mathcal{K}_<|L_0$. We say that $K_<$ is *reasonable*, if whenever $A, B \in \mathcal{K}$, A is a substructure of B and $\prec$ is a linear ordering on A with $(A, \prec) \in \mathcal{K}_<$, then there exists a linear ordering $\prec'$ on B such that $(B, \prec') \in \mathcal{K}_<$ and $(A, \prec)$ is a substructures of $(B, \prec')$.

The following proposition is Proposition 5.2 from [8] adjusted to Jónsson classes. It verifies that if we remove the linear order from the universal homogeneous structure for a reasonable Jónsson class, we obtain the universal homogeneous structure for the corresponding class of structures without linear orderings.

Proposition 2. *Suppose that α is a positive ordinal such that $\aleph_\alpha^{<\aleph_\alpha} = \aleph_\alpha$. Let $\mathcal{K}_<$ be a Jónsson class for α in a language $L \supset \{<\}$ with $<$ interpreted as a linear ordering. Suppose that $\mathcal{K}_<$ is closed under substructures. Let $(A, <)$ be the $(\aleph_\alpha, \mathcal{K}_<)$-universal homogeneous structure. Set $L_0 = L \setminus \{<\}$ and $\mathcal{K} = \mathcal{K}_<|L_0$. Then the following are equivalent:*

(a) $\mathcal{K}_<$ is reasonable,
(b) $\mathcal{K}$ is a Jónsson class and $A = (A, <)|L_0$ is an $(\aleph_\alpha, \mathcal{K})$-universal homogeneous structure

Proof. ($\Rightarrow$) Obviously, $\mathcal{K}$ satisfies conditions I', II, V and VI$_\alpha$. To verify the joint embedding property III, let $B, C \in \mathcal{K}$. Find $\prec, \prec'$ linear orderings on B, C respectively such that $(B, \prec), (C, \prec') \in \mathcal{K}_<$. Since $\mathcal{K}_<$ satisfies III, there is $D_< \in \mathcal{K}_<$ such that $(B, \prec), (C, \prec') \leq D_<$. Then $D = D_<|L_0$ is a witness of a joint embedding of B and C in $\mathcal{K}$.

It remains to show the amalgamation property IV': Fix $B, C, D \in \mathcal{K}$ and embeddings $i : B \to C$ and $j : B \to D$. Let $\prec$ be an ordering on B such that $(B, \prec) \in \mathcal{K}_<$. Since $\mathcal{K}_<$ is reasonable, we can find linear orders $\prec', \prec''$ on C, D respectively such that $(C, \prec'), (D, \prec'') \in \mathcal{K}_<$ and $i : (B, \prec) \to (C, \prec'), j : (B, \prec) \to (D, \prec'')$ are still embeddings. Amalgamation property for $\mathcal{K}_<$ provides us with $E_< \in \mathcal{K}_<$ and embeddings $k : (C, \prec') \to E_<, l : (D, \prec'') \to E_<$ such that $k \circ i = l \circ j$. Let $E = E_<|L_0$ and embeddings $k : C \to E, l : D \to E$ are witnesses of amalgamation of C and D over B in $\mathcal{K}$.

Finally, we check that A is $(\aleph_\alpha, \mathcal{K})$-universal and homogeneous. Let $B \in \mathcal{K}$ of cardinality $\leq \aleph_\alpha$ and let $\prec$ be a linear ordering on B such that $(B, \prec) \in \mathcal{K}_<$. Since

$(A, <)$ is $(\aleph_\alpha, K_<)$-universal, there is an embedding $i : (B, \prec) \to (A, <)$ which is also an embedding from B to A. It means that A is $(\aleph_\alpha, K)$-universal.

To show that A is $(\aleph_\alpha, K)$-homogeneous, it is enough to check the extension property in Proposition 1. For that, let $B \leq C$ be structures in K with cardinality $< \aleph_\alpha$ and let $i : B \to A$ and $j : B \to C$ be embeddings. Denote by $\prec = i^{-1}(< |i(B))$. Then $(B, \prec) \in K_<$. Since $K_<$ is reasonable, there exists $\prec'$ on C with $(C, \prec') \in K_<$ such that $j : (B, \prec) \to (C, \prec')$ is also an embedding. Since $(A, <)$ is $(\aleph_\alpha, K_<)$-homogeneous, it satisfies the extension property, i.e. there is an embedding $k : (C, \prec') \to (A, <)$ such that $i = k \circ j$ and we are done.

($\Leftarrow$) Fix $B, C \in K$ and an embedding $i : B \to C$. Let $\prec$ be a linear ordering on B such that $(B, \prec) \in K_<$. Then there is an embedding $j : (B, \prec) \to (A, <)$, which is of course also an embedding from B to A. Since A is homogeneous, it satisfies the extension property in Proposition 1, so there is an embedding $k : C \to A$ extending j. Let $\prec' = j^{-1}(< |j(C))$. Then $(C, \prec') \in K_<$ and $i : (B, \prec) \to (C, \prec')$ is an embedding. □

We are ready to apply Theorem 1 and Theorem 4 to Jónsson structures. We remind the reader that the structures need not be countable.

Theorem 6. *Let $L \supset \{<\}$ be a relational signature and let α be a positive ordinal such that $\aleph_\alpha^{<\aleph_\alpha} = \aleph_\alpha$. Let $K_<$ be a reasonable Jónsson class in the signature L with $<$ interpreted as a linear order and let $K_<$ be closed under substructures. Denote by $(A, <)$ the $(\aleph_\alpha, K_<)$-universal homogeneous structure.*

(a) *If $\mathrm{Age}(A, <)$ satisfies the Ramsey property and $G_<$ is a dense subgroup of $\mathrm{Aut}(A, <)$, then $G_<$ is extremely amenable.*
(b) *If $\mathrm{Age}(A, <)$ satisfies both the Ramsey property and the ordering property, then the universal minimal flow of any dense subgroup G of $\mathrm{Aut}(A)$ is $\overline{G} <$.*

Let us turn to concrete examples of Jónsson classes to which we can apply the above theorem.

Proposition 3. *Let K be one of the following classes:*

- *Graphs or graphs with arbitrary linear orderings,*
- *K_n-free graphs or K_n-free graphs with arbitrary linear orderings,*
- *Hypergraphs or hypergraphs with arbitrary linear orderings,*
- *$\mathcal{A}$-free hypergraphs or $\mathcal{A}$-free hypergraphs with arbitrary linear orderings, where $\mathcal{A}$ is a class of finite irreducible hypergraphs,*
- *Posets or posets with linear orderings extending the partial order.*

Then K is a Jónsson class for every positive ordinal α and it is closed under substructures.

Proof. It is easy to see that in all cases, K satisfies conditions I$'$, II, V and that it is closed under substructures which is a strengthening of VI$_\alpha$ for every α. We show that they also satisfy III and IV$'$:

To satisfy the joint embedding property III for structures $A, B \in K$, we take C to be the disjoint union of A and B and the embeddings to be the identity. If A, B also possess a linear order, then let elements of A precede elements of B in C.

The amalgamation property IV' is proved similarly as in the case of finite structures: Let $A, B, C \in \mathcal{K}$ and let $i : A \to B, j : A \to C$ be embeddings. Let the underlying set of D be a quotient of the disjoint union of B and C via an equivalence relation $\sim$, where $b \sim c$ if and only if $b \in B, c \in C$ and there is an $a \in A$ such that $i(a) = b, j(a) = c$. Let $k : B \to D$ and $l : C \to D$ be the identity injections. Then obviously, $k \circ i = l \circ j$. Now we equip D with a structure of the correct type to make sure that k and l are embeddings:

- If $\mathcal{K}$ is a class of hypergraphs in a signature L and $R_i \in L$ has arity n, then

$$(d_1, d_2, \ldots, d_n) \in R_i^D \subset D^n$$

if and only if either

$$(k^{-1}(d_1), k^{-1}(d_2), \ldots, k^{-1}(d_n)) \in R_i^B \subset B^n$$

or

$$(l^{-1}(d_1), l^{-1}(d_2), \ldots, l^{-1}(d_n)) \in R_i^C \subset C^n.$$

  It means that $R_i^D = R_i^B \cup R_i^C$ when we identify B, C with their corresponding images $k(B), l(C)$. Notice that D will be a graph (K_n-free graph, $\mathcal{A}$-free hypergraph) if A, B, C are.
- If $\mathcal{K}$ is the class of posets with $\prec$ the symbol for the partial order, then $\prec^D$ is the transitive closure of $(k \times k)(\prec^B) \cup (l \times l)(\prec^C)$.

If $\mathcal{K}$ is a class of linearly ordered structures with $<$ the symbol for the linear order, then $<^D$ on D is an arbitrary linear ordering extending the transitive closure of $(k \times k)(<^B) \cup (l \times l)(<^C)$. In the next section, we will need to be more careful when amalgamating linear orders: we will set elements of B to precede elements of C whenever we can. Formally, for $b \in B$ denote by $(-, b) = \{a \in A : k(a) <_B b\}$ and for $c \in C$ denote by $(-, c) = \{a \in A : l(a) <_C c\}$. Let $<_D$ be the extension of both $k \times k(<_B)$ and $l \times l(<_C)$ such that whenever $b \in B, c \in C$ then $k(b) <_D l(c)$ if and only if $(-, b) \subseteq (-, c)$. It is easy to see that $<^D$ extends $\prec^D$ if $\mathcal{K}$ is the class of posets with a linear ordering extending the partial order. So in all cases, $\mathcal{K}$ is a Jónsson class for every α.   □

We know that if $\mathcal{K}$ is one of the ordered classes in the proposition above, then finite structures in $\mathcal{K}$ satisfy the ordering property and the Ramsey property. Also, the ordered classes are obviously reasonable. Therefore, Theorem 6 applies to $\mathcal{K}$.

Corollary 5. *Let κ be a cardinal satisfying $\kappa^{<\kappa} = \kappa$ and let $(A, <)$ be the universal homogeneous ordered graph (K_n-free graph, hypergraph, $\mathcal{A}$-free hypergraph, poset) of cardinality κ. Let $G_<$ be a dense subgroup of $\underline{\mathrm{Aut}}(A, <)$ and let G be a dense subgroup of $\underline{\mathrm{Aut}}(A)$. Then $G_<$ is extremely amenable and the universal minimal flow of G is $\overline{G <}$.*

4 Constructing ω-Homogeneous Structures in Every Cardinality

In this section, we overcome the restriction on the size of structures given by Jónsson's construction for the price of losing universality and only keeping ω-homogeneity. This is however sufficient to compute universal minimal flows of groups of automorphisms of such structures. Hence we obtain many more examples of explicitly computed universal minimal flows.

We present a construction of ω-homogeneous (linearly ordered) graphs, K_n-free graphs, hypergraphs, $\mathcal{A}$-free hypergraphs (where $\mathcal{A}$ is a class of finite irreducible hypergraphs) and posets of arbitrary uncountable cardinality:

Let A be a graph (K_n-free graph, hypergraph, $\mathcal{A}$-free hypergraph, poset) of cardinality κ and let $<$ be an arbitrary linear ordering on A (respectively an ordering extending the partial order if A is a poset). We will construct an ω-homogeneous structure $(A_\kappa, <_\kappa)$ of cardinality κ in which $(A, <)$ is embedded and such that A_κ itself is ω-homogeneous.

By induction, we construct a chain of superstructures $\langle (A_\lambda, <_\lambda) : \lambda < \kappa \rangle$ of $(A, <)$ such that $(A_\lambda, <_\lambda) \leq (A_\mu, <_\mu)$ whenever $\lambda < \mu < \kappa$ and $(A_\kappa, <_\kappa) = \bigcup_{\lambda < \kappa} (A_\lambda, <_\lambda)$. In step λ, we deal with a pair of isomorphic finite substructures $(F_\lambda, <_\lambda)$ and $(G_\lambda, <_\lambda)$ of $(A_\lambda, <_\lambda)$ and an isomorphism $\phi_\lambda : F_\lambda \to G_\lambda$ and we construct $(A_{\lambda+1}, <_{\lambda+1})$ with an automorphism $\psi_{\lambda+1} : A_{\lambda+1} \to A_{\lambda+1}$ extending ϕ_λ that is order-preserving if ϕ_λ is. Moreover, we make sure that if $\xi < \lambda$ and $(F_\xi, <_\xi) = (F_\lambda, <_\lambda)$ and $(G_\xi, <_\xi) = (G_\lambda, <_\lambda)$ and $\phi_\xi = \phi_\lambda$, then $\psi_\xi \subset \psi_\lambda$.

To ensure that A_κ and $(A_\kappa, <_\kappa)$ are both ω-homogeneous, we need to consider every triple $((F_\lambda, <_\lambda), (G_\lambda, <_\lambda), \phi_\lambda : F_\lambda \to G_\lambda)$ cofinality of κ-many times for every $\lambda < \kappa$. In order to do so, we fix a book-keeping function—a bijection $f : \kappa \to \kappa \times \kappa$—which we only need to satisfy that whenever $f(\lambda) = (\mu, \xi)$, then $\mu \leq \lambda$. We will describe f in detail later.

If λ is limit, then $(A_\lambda, <_\lambda) = \bigcup_{\mu < \lambda} (A_\mu, <_\mu)$. If $\lambda = \mu + 1$, then we construct $(A_\lambda, <_\lambda)$ from $(A_\mu, <_\mu)$ as follows.

Let $(F_\mu, <_\mu)$, $(G_\mu, <_\mu)$ be the pair of finite substructures of $(A_\mu, <_\mu)$ and $\phi_\mu : F_\mu \to G_\mu$ the automorphism given by f in step μ (as described later). Denote by $(B, <_B)$ the union of all $(A_\xi, <_\xi)$ such that $\xi \leq \mu$ and $(F_\xi, <_\xi) = (F_\mu, <_\mu)$, $(G_\xi, <_\xi) = (G_\mu, <_\mu)$ and $\phi_\xi = \phi_\mu$ and let ψ_B be the union of the corresponding ψ_ξ's. We need to set $\psi_\lambda | B = \psi_B$. For every $a \in A_\mu \setminus B$ we add two new points a^1 and a^{-1} to be the image and the preimage of a under ψ_λ respectively. In the next step, we need to add for every $a \in A_\mu \setminus B$ another two points a^2, a^{-2} to be the image of a^1 and the preimage of a^{-1} under ψ_λ respectively. We continue in the same manner ω-many times until we have taken care that every point has its image and preimage. Formally, let A_μ^z be a copy of A_μ for $z \in \mathbb{Z}$ and denote by a^z the element of A_μ^z corresponding to an a in A_μ. We identify A_μ^0 with A_μ.

We examine two cases depending on whether the triple $((F_\mu, <_\mu), (G_\mu, <_\mu), \phi_\mu)$ appears for the first time throughout the construction (i.e. $B = \emptyset$) or not. However, in both cases the construction follows a similar pattern.

Case 1: $B = \emptyset$. Let F_μ^z and G_μ^z denote the corresponding copies of F_μ and G_μ in A_μ^z respectively. The underlying set of A_λ will be a quotient of the disjoint union of A_μ^z for $z \in \mathbb{Z}$ by an equivalence relation $\sim$ gluing G_μ^z with F_μ^{z+1} via ϕ_μ :

$$a_1^{z_1} \in F_\mu^{z_1}, a_2^{z_2} \in G_\mu^{z_2} \rightarrow (a_1^{z_1} \sim a_2^{z_2} \leftrightarrow (\phi_\mu(a_1) = a_2 \wedge z_1 = z_2 + 1)).$$

If $a^z \in A_\mu^z$ and $a \notin F_\mu$ then we write a^z to mean its $\sim$-class $[a^z]_\sim = \{a^z\}$. If $a \in F_\mu$, then we write a^z to mean its $\sim$-class $[a^z]_\sim = \{a^z, \phi_\mu(a)^{z-1}\}$.

Let us define $\psi_\lambda : A_\lambda \to A_\lambda$ to be the mapping $a^z \mapsto a^{z+1}$ whenever $a \notin F_\mu$ and $z \in \mathbb{Z}$ and $a^z \mapsto \phi_\mu(a)^z$ whenever $a \in F_\mu$.

If ϕ_μ is order preserving, let us transfer the linear order $<_\mu$ on A_μ to A_μ^z for every $z \in \mathbb{Z}$:

$$a^z <_\mu^z b^z \text{ if and only if } \psi_\lambda^{-z}(a^z) <_\mu \psi_\lambda^{-z}(b^z)$$

We define the structure on A_λ inductively using amalgamation as described in the previous section. If ϕ_μ is order preserving, then we use amalgamation for linearly ordered structures letting elements of $A_\mu^{z_1}$ precede elements of $A_\mu^{z_2}$ for $z_1 < z_2$ whenever we can as described in the previous section. Otherwise, we use amalgamation for unordered structures and in the end obtain $<_\lambda$ as an arbitrary linear extension of $<_\mu$ (and extending the partial order on A_λ if A is a poset).

Let $n \in \omega$ and let A_λ^{n-} denote the subset of A_λ consisting of points in A_μ^z for $z \in \{-n+1, -n+2, \ldots, n\}$, i.e.,

$$A_\lambda^{n-} = \left(\bigcup_{z=-n+1}^{n} A_\mu^z \right) \Big/ \sim .$$

Similarly let A_λ^{n+} denote the quotient

$$A_\lambda^{n+} = \bigcup_{z=-n}^{n} A_\mu^z / \sim$$

of the disjoint union of A_μ^z for $z \in \{-n, -n+1, \ldots, n\}$ by $\sim$. We have that $A_\lambda^0 = A_\mu$. Suppose that the structure on A_λ^{n+} has been defined. Then the structure on $A_\lambda^{(n+1)-}$ is given by amalgamating A_λ^{n+} with A_μ^{n+1} along

$$F_\mu^{n+1} \to A_\lambda^{n+}, a^{n+1} \mapsto (\phi_\mu(a))^n \text{ and } F_\mu^{n+1} \to A_\mu^{n+1}, a^{n+1} \mapsto a^{n+1}.$$

When the structure on A_λ^{n-} has been defined, then the structure on A_λ^{n+} is given by amalgamation of A_λ^{n-} and A_μ^{-n} along

$$F_\mu^{-n+1} \to A_\lambda^{n-}, a^{-n+1} \mapsto a^{-n+1} \text{ and } F_\mu^{-n+1} \to A_\mu^{-n}, a^{-n+1} \mapsto (\phi_\mu(a))^{-n}.$$

If ϕ_μ is order preserving, then A_λ^{n+} (respectively A_μ^{-n}) plays the role of B and A_λ^{n+1} (respectively A_μ^{n-}) the role of C in the amalgamation of linear orders described in the previous section.

We can see that $A_\lambda^{n+} \leq A_\lambda^{n+1-} \leq A_\lambda^{n+1+}$ for every $n \in \omega$, so we can define the structure on A_λ as the union of the chain $\langle A_\lambda^{n+} : n \in \omega \rangle$ (equivalently $\langle A_\lambda^{n-} : n \in \omega \rangle$):

$$A_\lambda = \bigcup_{n \in \omega} A_\lambda^{n+} = \bigcup_{n \in \omega} A_\lambda^{n-}.$$

Case 2: $B \neq \emptyset$. The underlying set of A_λ will then be a quotient of the disjoint union of A_μ^z for $z \in \mathbb{Z}$ by an equivalence relation $\sim$ gluing corresponding copies of B in A_μ^z via ψ_B :

$$a_1^{z_1} \in A_\mu^{z_1}, a_2^{z_2} \in A_\mu^{z_2} \to (a_1^{z_1} \sim a_2^{z_2} \longleftrightarrow a_1, a_2 \in B \wedge \psi_B^{-z_1}(a_1) = \psi_B^{-z_2}(a_2)),$$

We identify ψ_B^0 with ψ_B.

If $a^z \in A_\mu^z$ and $a \notin B$ then we write a^z to mean its $\sim$-class $[a^z]_\sim = \{a^z\}$. If $a \in B$, then we write a to mean its $\sim$-class $[a]_\sim = \{a^z : z \in \mathbb{Z}\}$.

Let us define $\psi_\lambda : A_\lambda \to A_\lambda$ to be the mapping $a^z \mapsto a^{z+1}$ whenever $a \notin B$ and $z \in \mathbb{Z}$ and $a \mapsto \psi_B(a)$ whenever $a \in B$.

As in Case 1, we can obtain the structure on A_λ inductively by amalgamation along B in ω-many steps. However, we can obtain A_λ via amalgamation of $A_\mu^z, z \in \mathbb{Z}$ along $B \to A_\mu^z, b \mapsto \psi_\lambda^z(b)$ in one step:

- If A is a hypergraph in a signature L and $R_i \in L$ has arity n, then

$$(a_1^{z_1}, a_2^{z_2}, \ldots, a_n^{z_n}) \in R_i^{A_\lambda}$$

  if and only if

$$(z_1 = z_2 = \ldots = z_n) \wedge (\psi_\lambda^{-z_1}(a_1^{z_1}), \psi_\lambda^{-z_1}(a_2^{z_1}), \ldots, \psi_\lambda^{-z_1}(a_n^{z_1})) \in R_i^{A_\mu}.$$

  Notice that A_λ will be a graph (K_n-free graph, $\mathcal{A}$-free hypergraph) if A_μ is.
- If A is a poset with a partial order $\prec$, then $\prec_\lambda$ is the transitive closure of the relation obtained as in the case of hypergraphs.

It remains to extend the linear order $<_\mu$ to a linear order $<_\lambda$ on A_λ. If ϕ_μ is not order preserving, let $<_\lambda$ be an arbitrary linear ordering extending $<_\mu$ (and extending the partial order $\prec_\lambda$ on A_λ in case that A is a poset). If ϕ_μ is order preserving, we will extend $<_\mu$ to $<_\lambda$ to make sure that ψ_λ is order preserving.

We can also describe $<_\lambda$ explicitly, without an inductive construction. We again let elements in $A_\mu^{z_1}$ precede elements in $A_\mu^{z_2}$ whenever $z_1 < z_2$ and we are free to do so: Let $a^z \in A_\mu^z$ and denote by

$$(-, a^z) = \{b \in B : b <_\mu \psi_\lambda^{-z}(a)\}.$$

Let $a_1^{z_1} \in A_\mu^z$ and $a_2^{z_2} \in A_\mu^{z_2}$ for some $z_1, z_2 \in \mathbb{Z}$ and $a_1^{z_1} \neq a_2^{z_2}$.

- If $z_1 = z_2$, then $a_1^{z_1} <_\lambda a_2^{z_2}$ if and only if $\psi_\lambda^{-z_1}(a_1^{z_1}) <_\mu \psi_\lambda^{-z_1}(a_2^{z_2})$.
- If $z_1 \neq z_2$, then $a_1^{z_1} <_\lambda a_2^{z_2}$ if and only if $(-, a_1^{z_1}) \subsetneq (-, a_2^{z_2})$ or $(-, a_1^{z_1}) = (-, a_2^{z_2})$ and $z_1 < z_2$.

Obviously, $<_\lambda$ is antireflexive and antisymmetric. Let us check transitivity: Let $a_1^{z_1}, a_2^{z_2}, a_3^{z_3} \in A_\lambda$ and $a_1^{z_1} < a_2^{z_2}$ and $a_2^{z_2} < a_3^{z_3}$. We have five possible cases:

1. $z_1 = z_2 = z_3$. Then $\psi_\lambda^{-z_1}(a_1^{z_1}) <_\mu \psi_\lambda^{-z_1}(a_2^{z_1}) <_\mu \psi_\lambda^{-z_1}(a_3^{z_1})$ so $\psi_\lambda^{-z_2}(a_1^{z_1}) <_\mu \psi_\lambda^{-z_1}(a_3^{z_1})$ by transitivity of $<_\mu$.
2. $z_1 = z_2 \neq z_3$. Then $\psi_\lambda^{-z_1}(a_1^{z_1}) <_\mu \psi_\lambda^{-z_1}(a_2^{z_1})$, so $(-, a_1^{z_1}) \subseteq (-, a_2^{z_1})$, and either $(-, a_2^{z_1}) \subsetneq (-, a_3^{z_3})$ or $(-, a_2^{z_1}) = (-, a_3^{z_3})$ and $z_2 < z_3$.
3. $z_1 \neq z_2 = z_3$. Then $(-, a_1^{z_1}) \subsetneq (-, a_2^{z_2})$ or $(-, a_1^{z_1}) = (-, a_2^{z_2})$ with $z_1 < z_2$ and $\psi_\lambda^{-z_2}(a_2^{z_2}) <_\mu \psi_\lambda^{-z_2}(a_3^{z_2})$, so $(-, a_2^{z_2}) \subseteq (-, a_3^{z_3})$.
4. $z_1 = z_3 \neq z_2$. Then $(-, a_1^{z_1}) \subseteq (-, a_2^{z_2}) \subseteq (-, a_3^{z_1})$ and at least one inclusion is proper, since otherwise $z_1 < z_2 < z_1$. So we get $(-, a_1^{z_1}) \subsetneq (-, a_3^{z_1})$, which implies $\psi_\lambda^{-z_1}(a_1^{z_1}) <_\mu \psi_\lambda^{-z_1}(a_3^{z_1})$.
5. $z_1 \neq z_2 \neq z_3, z_1 \neq z_3$. Then we have the following four cases:

   (i) $(-, a_1^{z_1}) \subsetneq (-, a_2^{z_2}) \subsetneq (-, a_3^{z_3})$.
   (ii) $(-, a_1^{z_1}) \subsetneq (-, a_2^{z_2}) = (-, a_3^{z_3})$ and $z_2 < z_3$.
   (iii) $(-, a_1^{z_1}) = (-, a_2^{z_2}) \subsetneq (-, a_3^{z_3})$ and $z_1 < z_2$
   (iv) $(-, a_1^{z_1}) = (-, a_2^{z_2}) = (-, a_3^{z_3})$ and $z_1 < z_2 < z_3$.

In all cases we get that $a_1^{z_1} <_\lambda a_3^{z_3}$, hence $<_\lambda$ is transitive.

If A is a poset with $\prec$ the symbol for the partial order and $\prec_\lambda$ is the extension of $\prec$ to A_λ described above, let us verify that $<_\lambda$ extends $\prec_\lambda$: If $a_1^z, a_2^z \in A_\mu^z$ for some $z \in \mathbb{Z}$, then $a_1^z <_\lambda a_2^z$ if and only if $a_1^z \prec_\lambda a_2^z$ by definition. If $a_1^{z_1} \in A_\mu^{z_1}, a_2^{z_2} \in A_\mu^{z_2}$ for some $z_1 \neq z_2$ and $a_1^{z_1} \prec_\lambda a_2^{z_2}$, then there is $a \in B$ such that $\psi_\lambda^{-z_1}(a_1^{z_1}) <_\mu a$ and $a <_\mu \psi_\lambda^{-z_2}(a_2^{z_2})$. It means that $a \in (-, a_1^{z_1})$ while $a \notin (-, a_2^{z_2})$, showing that $a_1^{z_1} <_\lambda a_2^{z_2}$.

Let us go back to the definition of the book-keeping function f: When $(A_\mu, <_\mu)$ has been constructed, let $\{((F_\mu^\xi, <_\mu^\xi), (G_\mu^\xi, <_\mu^\xi), \phi_\mu^\xi) : \xi < \kappa\}$ be an enumeration of all triples such that $(F_\mu^\xi, <_\mu^\xi)$ and $(G_\mu^\xi, <_\mu^\xi)$ are finite substructures of $(A_\mu, <_\mu)$ and ϕ_μ^ξ is an isomorphism between F_μ^ξ and G_μ^ξ (this is possible since $\kappa^{<\omega} = \kappa$ for every infinite κ). If $f(\mu) = (\nu, \xi)$, then the triple we consider in step μ is $((F_\nu^\xi, <_\nu^\xi), (G_\nu^\xi, <_\nu^\xi), \phi_\nu^\xi)$. The inequality $\nu \leq \mu$ ensures that F_ν^ξ and G_ν^ξ are finite substructures of already defined $A_\nu \leq A_\mu$. The inclusion $A_\nu \subset A_\mu$ for $\nu < \mu < \kappa$ provides that for every $\mu < \kappa$, every triple $((F, <_\mu), (G_\mu), \phi)$ of two finite substructures of $(A_\mu, <_\mu)$ and an isomorphism between F and G will be considered unboundedly many times throughout our construction.

The outcome of the construction can be formulated as the following theorem.

Theorem 7. *Let A be a graph (K_n-free graph, hypergraph, $\mathcal{A}$-free hypergraph, poset) of an infinite cardinality κ and let $<$ be an arbitrary linear ordering on A (respectively an ordering extending the partial order if A is a poset). Then there*

exists a linearly ordered graph (K_n-free graph, hypergraph, $\mathcal{A}$-free hypergraph, poset) $(A', <')$ of cardinality κ in which $(A, <)$ is embedded and such that both A' and $(A', <')$ are ω-homogeneous (and if A is a poset, $<'$ extends the partial order on A').

If we start with A a graph (K_n-free graph, hypergraph, $\mathcal{A}$-free hypergraph, poset) and a linear order $<$ on A such that $\mathrm{Age}((A, <))$ is the class of all finite linearly ordered graphs (K_n-free graphs, hypergraphs, $\mathcal{A}$-free hypergraphs) or posets with the linear order extending the partial order, the construction provides us with a structure for which we are able to compute the universal minimal flow of its group of automorphisms. This can easily be arranged for instance by requiring that $(A, <)$ contains a copy of the countable ω-homogeneous ordered graph (K_n-free graph, hypergraph, $\mathcal{A}$-free hypergraph, poset).

Theorem 8. *Let A be an ω-homogeneous graph (K_n-free graph, hypergraph, $\mathcal{A}$-free hypergraph, poset) and let $<$ be a linear ordering on A (extending the partial order if A is a poset) such that $(A, <)$ is ω-homogeneous as well.*

Suppose that $\mathrm{Age}(A, <)$ is the class of all finite linearly ordered graphs (K_n-free graphs, hypergraphs, $\mathcal{A}$-free hypergraphs) or posets with linear orderings extending the partial order.

If G is a dense subgroup of $\mathrm{Aut}(A)$, then the universal minimal flow of G is the space of all linear orderings on A (respectively the space of all linear orderings extending the partial order if A is a poset).

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On the Lyapounov Exponents of Schrödinger Operators Associated with the Standard Map

J. Bourgain

Abstract It is shown that Schrödinger operators defined from the standard map have positive (mean) Lyapounov exponents for almost all energies.

Key words Standard map • Lyapounov exponents

Mathematical classification subject (2010): 37D25, 37D50

1 Definitions

Let V be a bounded potential on $\mathbb{Z}_+$. Define

$$\omega(V) = \{W = (W_n)_{n \in \mathbb{Z}}; \text{ there is a sequence } n_j \to \infty \text{ such that}$$
$$d(S^{n_j} V, W) \to 0\} \qquad (1)$$

where

$$d(V, V') = \sum 2^{-n} |V_n - V'_n| \text{ and } (S^k V)(n) = V(n+k).$$

Let $A \subset \mathbb{R}, \text{ mes}(A) > 0$. Denote

$$\mathcal{R}(A) = \{\text{bounded potentials } W = (W_n)_{n \in \mathbb{Z}}$$
$$\text{that are reflectionless on } A\} \qquad (2)$$

and $\mathcal{R}^C(A) = \{W \in \mathbb{R}(A); \sup_n |W_n| \leq C\} W$.

Let $H = W + \Delta$ and let $G(z)$ be the Green's function of H.

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Recall that W is reflectionless on A if

$$ReG(t)(n) = 0 \text{ for a.e. } t \in A \text{ and every } n \in \mathbb{Z}.$$

2 Remling Theorems

We rely on following results from Remling's paper [7]

$$\omega(V) \subset \mathcal{R}\left(\sum_{ac}(V)\right) \quad ([7] \text{ Theorem 1.4}) \tag{3}$$

$$R^C(A) \text{ is compact } ([7] \text{ Proposition 4.1, (d)}) \tag{4}$$

The restriction maps $\mathcal{R}(A) \to \mathcal{R}_\pm(A)$ are injective ($\pm$ refers to $\pi_{\mathbb{Z}_\pm}$), and for any constant C, the map

$$\mathcal{R}^C(A) \to \mathcal{R}_+^C(A) \text{ is uniformly continuous} \tag{5}$$

([R]Proposition 4.1 $(c), (e)$).

$$W \in \mathcal{R}(A) \Rightarrow A \subset \sum_{ac}(E_\pm). \tag{6}$$

To put things in perspective, one should point out the close relation between Remling's results and Kotani's theory [2–5] as well as the paper of Last and Simon [6], where the notion of 'right limits' is introduced.

3 Lyapounov Exponents

Let T be a measure preserving homeomorphism of a compact metric space Ω endowed with a probability measure μ that charges any non-empty open subset (we do not assume T ergodic).

Let $\varphi \in \mathcal{C}(\Omega)$.

Consider the half-line SO

$$H_x = \varphi(T^n x)\delta_n + \Delta \tag{7}$$

and denote

$$M_N(E, x) = \prod_N^1 \begin{pmatrix} E - \varphi(T^n x) & -1 \\ 1 & 0 \end{pmatrix}$$

$$L_N(E) = \frac{1}{N} \int \log \|M_N(E, x)\| \mu(dx).$$

Let $A \subset \mathbb{R}$, mes $(A) > 0$ and assume

$$\lim_N L_N(E) = 0 \text{ on } A. \tag{8}$$

Then

$$A \subset \sum_{a.c} (V_x) \text{ for } \mu \text{ a.e. } x. \tag{9}$$

Proof. Let

$$\mu = \int \beta \, \alpha(d\beta)$$

be the ergodic decomposition of μ.
   By Fubini, for $E \in A$

$$L_N(E) = \int \left\{ \frac{1}{N} \int \log \|M_N(E, x)\| \beta(dx) \right\} \alpha(d\beta)$$

and

$$\int \left\{ \frac{\lim}{N} \left(\frac{1}{N} \int \log \|M_N(E, x)\| \beta(dx) \right) \right\} \alpha(d\beta).$$

Since β is ergodic, it follows that for α - a.e. β

$$\frac{1}{N} \log \|M_N(E, x)\| \to 0 \text{ for } \beta \text{ - a.e. } x \in \Omega \tag{10}$$

Again by Fubini, (10) holds for a.e. $E \in A$ and β - a.e. $x \in \Omega$
   By Kotani theory, this implies that

$$A \subset \sum_{ac} (H_x) \text{ for } \beta - a.e. \ x \in \Omega. \tag{11}$$

   Since (11) is valid for α - a.e. β, (9) follows.

4   Use of Recurrence

Let T be as in §3.
   Let $V_x = \big(\varphi(T^n x) \big)_{n \in \mathbb{Z}_+}$.
   Then

$$V_x \in \pi_{\mathbb{Z}_+} \big(\omega(V_x) \big) \text{ for } \mu \text{ - a.e. } x \in \Omega. \tag{12}$$

Proof. By Poincaré recurrence lemma, for μ - a.e. $x \in \Omega$, there is a sequence $n_j \to \infty$ such that

$$T^{n_j} x \to x. \tag{13}$$

Hence

$$\varphi(T^{n_j+k} x) \to \varphi(T^k x) \equiv W_x(k)$$

for all $k \in \mathbb{Z}$ and

$$d(W_x, V_{T^{n_j} x}) \to 0.$$

It follows that

$$W_x \in \omega(V_x)$$

and obviously $V_x = \pi_{\mathbb{Z}_+}(W_x)$.

By (3) and (12) implies

$$V_x \in \mathcal{R}_+ \left(\sum_{ac} (V_x) \right) \text{ for } \mu \ a.e. \ x. \tag{14}$$

From (14) and (9)

$$V_x \in \mathcal{R}_+(A) \text{ for } \mu \ a.e. \ x \tag{15}$$

thus $V_x \in \mathcal{R}_+^C(A)$ for μ a.e. x and since $\mathcal{R}_+^C(A)$ is compact by (4) and V_x depends continuously on x, we conclude that

$$V_x \in \mathcal{R}_+(A) \text{ for all } x \in \Omega \tag{16}$$

(Ω = closure of any subset of full measure).

Finally, by (6),

$$A \subset \sum_{ac} (V_x) \text{ for all } x \in \Omega. \tag{17}$$

Hence, we established the following general fact.

Proposition 1. *Let T be a measure-preserving automorphism of a compact metric probability space Ω such that every non-empty open subset of Ω has positive measure. Let $\phi \in C(\Omega)$. For $x \in \Omega$, consider the discrete Schrödinger operator $H_x = \Delta + V_x$ on $\ell^2(\mathbb{Z}_+)$, where Δ is discrete Laplacian, and $V_x(n) = \phi(T^n x)$.*

Let $A \subset \mathbb{R}$ be a set of positive Lebesgue measure on which the "lower mean" Lyapounov exponent vanishes. Then A is contained in the absolutely continuous spectrum of every $H_x, x \in \Omega$.

5 Application to Standard Map

Let $T = T_\lambda$ be the standard map on the torus $\mathbb{T}^2 = \Omega$ with sufficiently large λ. Thus

$$T_\lambda(x, y) = (-y + 2x + \lambda \sin 2\pi x, x). \tag{18}$$

Let $\varphi \in \mathcal{C}^1(\mathbb{T}^2)$ be a fixed not-constant function.

If the corresponding SO has vanishing Lyapounov exponents for $E \in A$, (17) implies

$$A \subset \sum_{ac} (\{\varphi(T^n x)\}_{n \in \mathbb{Z}_+} \tag{19}$$

for all $x \in \Omega$.

By Duarte's work (Theorem A in [1]), there is an invariant hyperbolic set $\Lambda = \Lambda_\lambda \subset \Omega$ such that $T|_\Lambda$ is conjugate to a Bernoulli shift. In particular for $x \in \Lambda$, (19) $= \phi$. Furthermore, Duarte's result asserts that each point in $\mathbb{T}^2$ is within a $4\lambda^{-\frac{1}{3}}$-neighborhood of Λ, so that φ will not be constant on Λ for λ large enough. Hence (19) restricted to Λ is non-deterministic and has no a.c. spectrum, by Kotani's theory (cf. [K1, K2, K3, K4]), implying that A is of zero-measure (contradiction).

Hence we proved

Proposition 2. *For $\lambda > \lambda_0$, the SO (7) associated to the standard map T_λ has positive (mean) Lyapounov exponents for a.e. $E \in \mathbb{R}$.*

For comparison, let us recall the major open problem of positivity of the Kolmogorov-Sinai metric entropy of the standard map and which is known to be equivalent to the positivity of the Lyapounov exponent at energy $E = 0$ (for the test function $\phi(x, y) = \cos(2\pi x)$).

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Overgroups of the Automorphism Group of the Rado Graph

Peter Cameron, Claude Laflamme, Maurice Pouzet, Sam Tarzi, and Robert Woodrow

Abstract We are interested in overgroups of the automorphism group of the Rado graph. One class of such overgroups is completely understood; this is the class of reducts. In this article we tie recent work on various other natural overgroups, in particular establishing group connections between them and the reducts.

Key words Rado graph • Automorphism group • Overgroup • Homogeneous structures

Mathematical Subject Classifications (2010): 05C25, 05C80, 05C55, 05C63, 20F28

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1 Introduction

The Rado Graph $\mathcal{R}$ is the countable universal homogeneous graph: it is the unique (up to isomorphism) countable graph with the defining property that for every finite disjoint subsets of vertices A and B there is a vertex adjacent to all vertices in A and not adjacent to any of the vertices in B.

We are interested in overgroups of its automorphism group $\text{Aut}(\mathcal{R})$ in $\text{Sym}(\mathcal{R})$, the symmetric group on the vertex set of $\mathcal{R}$. One class of overgroups of $\text{Aut}(\mathcal{R})$ is completely understood; this is the class of reducts, or automorphism groups of relational structures definable from $\mathcal{R}$ without parameters. Equivalently, this is the class of subgroups of $Sym(\mathcal{R})$ containing $Aut(\mathcal{R})$ which are closed with respect to the product topology. According to a theorem of Thomas [5], there are just five reducts of $\text{Aut}(\mathcal{R})$:

- $\text{Aut}(\mathcal{R})$
- $\mathcal{D}(\mathcal{R})$, the group of dualities (automorphisms and anti-automorphisms) of $\mathcal{R}$
- $\mathcal{S}(\mathcal{R})$, the group of switching automorphisms of $\mathcal{R}$ (see below)
- $\mathcal{B}(\mathcal{R}) = \mathcal{D}(\mathcal{R}).\mathcal{S}(\mathcal{R})$ (the big group)
- $Sym(\mathcal{R})$, the full symmetric group

Given a set X of vertices in a graph G, we denote by $\sigma_X(G)$ the *switching* operation of changing all adjacencies between X and its complement in G, leaving those within or outside X unchanged, thus yielding a new graph. Now a switching automorphism of G is an isomorphism which maps G to $\sigma_X(G)$ for some X, and $\mathcal{S}(G)$ is the group of switching automorphisms. Thus the interesting question is often for which subset X is $\sigma_X(G)$ isomorphic G, and for this reason will sometimes abuse terminology and may call $\sigma_X(G)$ a switching automorphism.

Thomas also showed (see [6], and also the work of Bodirsky and Pinsker [1]) that the group $\mathcal{S}(\mathcal{R})$ can also be understood as the automorphism group of the 3-regular hypergraph whose edges are those 3-element subsets containing an odd number of edges. Similarly, $\mathcal{D}(\mathcal{R})$ is the automorphism group of the 4-regular hypergraph whose edges are those 4-element subsets containing an odd number of edges, and $\mathcal{B}(\mathcal{R})$ is the automorphism group of the 5-regular hypergraph whose edges are those 5-element subsets containing an odd number of edges.

One can see that $\mathcal{G}$ is any subgroup of $\text{Sym}(\mathcal{R})$, then $\mathcal{G}.\text{FSym}(\mathcal{R})$ (the group generated by the union of $\mathcal{G}$ and $\text{FSym}(\mathcal{R})$, the group of all finitary permutations on $\mathcal{R}$) is a subgroup of $\text{Sym}(\mathcal{R})$ containing $\mathcal{G}$ and highly transitive. The reducts $\mathcal{D}(\mathcal{R})$ and $\mathcal{S}(\mathcal{R})$ however are 2-transitive but not 3-transitive, while $\mathcal{B}(\mathcal{R})$ is 3-transitive but not 4-transitive. On the other hand we have the following.

Lemma 1. *Any overgroup of* $\text{Aut}(\mathcal{R})$ *which is not contained in* $\mathcal{B}(\mathcal{R})$ *is highly transitive.*

Proof. Let G with $\text{Aut}(\mathcal{R}) \leq G \nleq \mathcal{B}(\mathcal{R})$, and let $\overline{G}$ be the closure of G in $\text{Sym}(\mathcal{R})$. Since $\overline{G} \nleq \mathcal{B}(\mathcal{R})$, we have $\overline{G} = \text{Sym}(\mathcal{R})$ by Thomas' theorem. Since G and $\overline{G}$ have the same orbits of n-uples, G is highly transitive. □

Now for a bit of notation. With the understanding that $\mathcal{R}$ is the only graph under consideration here, we write $v \sim w$ when v and w are adjacent (in $\mathcal{R}$), $\mathcal{R}(v)$ for the set of vertices adjacent to v (the neighbourhood of v), and will use $\mathcal{R}^c(v)$ for $\mathcal{R} \setminus \mathcal{R}(v)$ (note that $v \in \mathcal{R}^c(v)$). We say that a permutation g changes the adjacency of v and w if $(v \sim w) \Leftrightarrow (v^g \not\sim w^g)$. We say that g changes finitely many adjacencies at v if there are only finitely many points w for which g changes the adjacency of v and w.

Given two groups G_1, G_2 contained in a group H, we write $G_1.G_2$ for the subgroup of H generated by their union.

In Sect. 2, we present various other natural overgroups and tie recent work and in particular establish group connections between them and the reducts.

2 Other Overgroups of Aut($\mathcal{R}$)

Cameron and Tarzi in [2] have studied the following overgroups of $\mathcal{R}$.

(a) $\mathrm{Aut}_1(\mathcal{R})$, the group of permutations which change only a finite number of adjacencies.
(b) $\mathrm{Aut}_2(\mathcal{R})$, the group of permutations which change only a finite number of adjacencies at each vertex.
(c) $\mathrm{Aut}_3(\mathcal{R})$, the group of permutations which change only a finite number of adjacencies at all but finitely many vertices.
(d) $\mathrm{Aut}(\mathcal{F}_\mathcal{R})$, where $\mathcal{F}_\mathcal{R}$ is the neighbourhood filter of $\mathcal{R}$, the filter generated by the neighbourhoods of vertices of $\mathcal{R}$.

One shows that all these sets of permutations really are groups, as claimed. For $\mathrm{Aut}_i(\mathcal{G})$, this is because if $C(g)$ denotes the set of pairs $\{v, w\}$ whose adjacency is changed by g, then one verifies that $C(g^{-1}) = C(g)^{g^{-1}}$ and $C(gh) \subseteq C(g) \cup C(h)^{g^{-1}}$.

The main facts known about these groups are:

Proposition 1 ([2]).

(a) $\mathrm{Aut}(\mathcal{R}) < \mathrm{Aut}_1(\mathcal{R}) < \mathrm{Aut}_2(\mathcal{R}) < \mathrm{Aut}_3(\mathcal{R})$.
(b) $\mathrm{Aut}_2(\mathcal{R}) \leq \mathrm{Aut}(\mathcal{F}_\mathcal{R})$, but $\mathrm{Aut}_3(\mathcal{R})$ and $\mathrm{Aut}(\mathcal{F}_\mathcal{R})$ are incomparable.
(c) $\mathrm{FSym}(\mathcal{R}) < \mathrm{Aut}_3(\mathcal{R}) \cap \mathrm{Aut}(\mathcal{F}_\mathcal{R})$, but $\mathrm{FSym}(\mathcal{R}) \cap \mathrm{Aut}_2(\mathcal{R}) = 1$.
(d) $\mathcal{S}(\mathcal{R}) \not\leq \mathrm{Aut}(\mathcal{F}_\mathcal{R})$, and $\mathrm{Aut}(\mathcal{F}_\mathcal{R}) \cap \mathcal{D}(\mathcal{R}) = \mathrm{Aut}(\mathcal{F}_\mathcal{R}) \cap \mathcal{S}(\mathcal{R}) = \mathrm{Aut}(\mathcal{R})$.

Proof. (a) is clear.
(b) For the first part, let $g \in \mathrm{Aut}_2(\mathcal{R})$. It suffices to show that, for any vertex v, we have $\mathcal{R}(v)^g \in \mathcal{F}(\mathcal{R})$. Now by assumption, $\mathcal{R}(v)^g$ differs only finitely from $\mathcal{R}(v^g)$; let $\mathcal{R}(v)^g \setminus \mathcal{R}(v^g) = \{w_1, \ldots, w_n\}$. If we choose w such that $w_i \notin \mathcal{R}(w)$ for each i, then we have

$$\mathcal{R}(v^g) \cap \mathcal{R}(w) \subseteq \mathcal{R}(v)^g,$$

and we are done.

For the second part, choose a vertex v, and consider the graph $\mathcal{R}'$ obtained by changing all adjacencies at v. Then $\mathcal{R}' \cong \mathcal{R}$. Choose an isomorphism g from $\mathcal{R}$ to $\mathcal{R}'$; since $\mathcal{R}'$ is vertex-transitive, we can assume that g fixes v. So g maps $\mathcal{R}(v)$ to $\mathcal{R}_1(v) = \mathcal{R}^c(v) \setminus \{v\}$. Clearly $g \in \mathrm{Aut}_3(\mathcal{R})$, since it changes only one adjacency at any point different from v. But if $g \in \mathrm{Aut}(\mathcal{F}_{\mathcal{R}})$, then we would have $\mathcal{R}_1(v) \in \mathcal{F}_{\mathcal{R}}$, a contradiction since $\mathcal{R}(v) \cap \mathcal{R}_1(v) = \emptyset$.

In the reverse direction, let $\mathcal{R}''$ be the graph obtained by changing all adjacencies between non-neighbours of v. Again $\mathcal{R}'' \cong \mathcal{R}$, and we can pick an isomorphism g from $\mathcal{R}$ to $\mathcal{R}''$ which fixes v. Now g changes infinitely many adjacencies at all non-neighbours of v (and none at v or its neighbours), so $g \notin \mathrm{Aut}_3(\mathcal{R})$. Also, if w is a non-neighbour of v, then $\mathcal{R}(v) \cap \mathcal{R}(w)^g = \mathcal{R}(v) \cap \mathcal{R}(w^g)$, so $g \in \mathrm{Aut}(\mathcal{F}_{\mathcal{R}})$.

(c) Note that any non-identity finitary permutation belongs to $\mathrm{Aut}_3(\mathcal{R}) \setminus \mathrm{Aut}_2(\mathcal{R})$. For if g moves v, then g changes infinitely many adjacencies at v (namely, all v and w, where w is adjacent to v but not v^g and is not in the support of g). On the other hand, if g fixes v, then g changes the adjacency of v and w only if g moves w, and there are only finitely many such w.

Finally, if $g \in \mathrm{FSym}(\mathcal{R})$, then $\mathcal{R}(v)^g$ differs only finitely from $\mathcal{R}(v)$, for any vertex $v \in V$; so $g \in \mathrm{Aut}(\mathcal{F}_{\mathcal{R}})$.

Thus the left inclusion is proper: $\mathrm{Aut}_2(\mathcal{R})$ is contained in the right-hand side but intersects $\mathrm{FSym}(\mathcal{R})$ in $\{1\}$.

(d) The graph $\mathcal{R}'$ in the proof of (b) is obtained from $\mathcal{R}$ by switching with respect to the set $\{v\}$; so the permutation g belongs to the group $\mathcal{S}(\mathcal{R})$ of switching automorphisms. Thus $S(R) \not\leq \mathrm{Aut}(\mathcal{F}_{\mathcal{R}})$.

Now any anti-automorphism g of $\mathcal{R}$ maps $\mathcal{R}(v)$ to a set disjoint from $\mathcal{R}(v^g)$; so no anti-automorphism can belong to $\mathrm{Aut}(\mathcal{F}_{\mathcal{R}})$. Suppose that $g \in \mathrm{Aut}(\mathcal{F}_{\mathcal{R}})$ is an isomorphism from $\mathcal{R}$ to $\sigma_X(\mathcal{R})$. We may suppose that σ_X is not the identity, that is, $X \neq \emptyset$ and $Y = V \setminus X \neq \emptyset$. Choose x and y so that $x^g \in X$ and $y^g \in Y$. Then $\mathcal{R}(x)^g \triangle Y = \mathcal{R}(x^g)$ and $\mathcal{R}(y)^g \triangle X = \mathcal{R}(y^g)$. Hence $\mathcal{R}(x^g) \cap \mathcal{R}(x)^g \subseteq X$ and $\mathcal{R}(y^g) \cap \mathcal{R}(y)^g \subseteq Y$. Hence

$$\mathcal{R}(x^g) \cap \mathcal{R}(x)^g \cap \mathcal{R}(y^g) \cap \mathcal{R}(y)^g = \emptyset,$$

a contradiction.   □

On the other hand, results of Laflamme, Pouzet and Sauer in [4] concern the hypergraph $\mathcal{H}$ on the vertex set of $\mathcal{R}$ whose edges are those sets of vertices which induce a copy of $\mathcal{R}$. Note that a cofinite subset of an edge is an edge. There are three interesting groups here:

(a) $\mathrm{Aut}(\mathcal{H})$.
(b) $\mathrm{FAut}(\mathcal{H})$, the set of permutations g with the property that there is a finite subset S of $\mathcal{R}$ such that for every edge E, both $(E \setminus S)g$ and $(E \setminus S)g^{-1}$ are edges.
(c) $\mathrm{Aut}^*(\mathcal{H})$, the set of permutations g with the property that, for every edge E, there is a finite subset S of E such that $(E \setminus S)g$ and $(E \setminus S)g^{-1}$ are edges.

Clearly $\mathrm{Aut}(\mathcal{H}) \le \mathrm{FAut}(\mathcal{H}) \le \mathrm{Aut}^*(\mathcal{H})$, and a little thought shows that all three are indeed groups. Moreover one will note that for $\mathrm{Aut}^*(\mathcal{H})$ and $\mathrm{FAut}(\mathcal{H})$ to be groups, both conditions on g and g^{-1} in their definitions are necessary. To see this, choose an infinite clique $C \subset \mathcal{R}$, and also partition $\mathcal{R}$ into two homogeneous edges E_1 and E_2: for every finite disjoint subsets of vertices A and B of $\mathcal{R}$ there is a vertex in E_2 adjacent to all vertices in A and not adjacent to any of the vertices in B. Then it is shown in [4] that there exists $g \in \mathrm{Sym}(\mathcal{R})$ such that $Cg = A$, and Eg is an edge for any edge E. But clearly $(A \setminus S)g^{-1}$ is not an edge for any (finite) S.

As a further remark let $\mathcal{H}^*$ be the hypergraph on the vertex set of $\mathcal{R}$ whose edges are subset of the form $E \cup F$ where E induces a copy of $\mathcal{R}$ and F is a finite subset of $\mathcal{R}$. Equivalently these are the subsets of $\mathcal{R}$ of the form $E \Delta F$ where E induces a copy of $\mathcal{R}$ and F is a finite subset of $\mathcal{R}$ (this follows from the fact that for every copy E and finite set F, $E \setminus F$ is a copy). Then observe that $Aut(\mathcal{H}^*) = Aut^*(\mathcal{H})$.

We now provide some relationships between these LPS groups and the CT groups.

Proposition 2. *(a)* $\mathrm{Aut}(\mathcal{H}) < \mathrm{FAut}(\mathcal{H})$.
(b) $\mathrm{Aut}_2(\mathcal{R}) \le \mathrm{Aut}(\mathcal{H})$ *and* $\mathrm{Aut}_3(\mathcal{R}) \le \mathrm{FAut}(\mathcal{H})$.
(c) $\mathrm{FSym}(\mathcal{R}) \le \mathrm{FAut}(\mathcal{H})$ *but* $\mathrm{FSym}(\mathcal{R}) \cap \mathrm{Aut}(\mathcal{H}) = 1$.

Proof. (a) This follows from part (c).

(b) If we alter a finite number of adjacencies at any point of $\mathcal{R}$, the result is still isomorphic to $\mathcal{R}$. So induced copies of $\mathcal{R}$ are preserved by $\mathrm{Aut}_2(\mathcal{R})$. Similarly, given an element of $\mathrm{Aut}_3(\mathcal{R})$, if we throw away the vertices where infinitely many adjacencies are changed, we are in the situation of $\mathrm{Aut}_2(\mathcal{R})$.

(c) The first part follows from Proposition 1 part (c) and part (b) above. For the second part, choose a vertex v and let E be the set of neighbours of v in $\mathcal{R}$ (this set is an edge of $\mathcal{H}$). Now, for any finitary permutation, there is a conjugate of it whose support contains v and is contained in $\{v\} \cup E$. Then $Eg = E \cup \{v\} \setminus \{w\}$ for some w. But the induced subgraph on this set is not isomorphic to $\mathcal{R}$, since v is joined to all other vertices. □

We shall see later that $\mathrm{FAut}(\mathcal{H}) < \mathrm{Aut}^*(\mathcal{H})$, but we present a bit more information before doing so. In particular we now show that an arbitrary switching is almost a switching isomorphism.

Lemma 2. *Let* $X \subseteq \mathcal{R}$ *arbitrary and* $\sigma = \sigma_X$ *be the operation of switching* $\mathcal{R}$ *with respect to* X. *Then there is a finite set* S *such that* $\sigma(\mathcal{R} \setminus S)$ *is an edge of* $\mathcal{H}$, *namely isomorphic to the Rado graph.*

Proof. For $E \subseteq \mathcal{R}$ and disjoint $U, V \subseteq E$, denote by $W_E(U, V)$ the collection of all witnesses for (U, V) in E. Note that if E is an edge, then $W_E(U, V)$ is an edge for any such sets U and V. Now for $C \subseteq \mathcal{R}$, denote for convenience by C_X the set $C \cap X$, and by C_X^c the set $C \setminus X$.

Thus if $\sigma(\mathcal{R})$ is **not** already an edge of $\mathcal{H}$, then the Rado graph criteria regarding switching yields finite disjoint $U, V \subseteq \mathcal{R}$ such that both:

- $W_{\mathcal{R}}(U_X^c \cup V_X, U_X \cup V_X^c) \subseteq X$
- $W_{\mathcal{R}}(U_X \cup V_X^c, U_X^c \cup V_X) \cap X = \emptyset$

Define $S = U \cup V$ and $E = \mathcal{R} \setminus S$, we show that $\sigma(E)$ is an edge of $\mathcal{H}$.

For this let $\bar{U}, \bar{V} \subseteq E$. But now we have:

$$W_{\mathcal{R}}(\bar{U}_X \cup \bar{V}_X^c \cup U_X^c \cup V_X, \bar{U}_X^c \cup \bar{V}_X \cup U_X \cup V_X^c)$$
$$= W_E(\bar{U}_X \cup \bar{V}_X^c, \bar{U}_X^c \cup \bar{V}_X) \cap W_{\mathcal{R}}(U_X^c \cup V_X, U_X \cup V_X^c)$$
$$\subseteq X$$

In virtue of the Rado graph, the above first set contains infinitely many witnesses, and thus $W_E(\bar{U}_X \cup \bar{V}_X^c, \bar{U}_X^c \cup \bar{V}_X)$ is non empty. Hence $\sigma(E)$ contains a witness for $(\bar{U}, \bar{V})$, and we conclude that $\sigma(E)$ is an edge. $\qquad\square$

The last item above shows that any graph obtained from $\mathcal{R}$ by switching has a cofinite subset inducing a copy of $\mathcal{R}$. This can be formulated as follows: Let G be a graph on the same vertex set as $\mathcal{R}$ and having the same parity of the number of edges in any 3-set as $\mathcal{R}$. Then G has a cofinite subset inducing $\mathcal{R}$.

The next result is about the relation between the LPS-groups and the reducts.

Proposition 3. *(a)* $\mathcal{D}(\mathcal{R}) < \mathrm{Aut}(\mathcal{H})$.
(b) $\mathcal{S}(\mathcal{R}) \nleq \mathrm{Aut}(\mathcal{H})$.
(c) $\mathcal{S}(\mathcal{R}) \leq \mathrm{Aut}^*(\mathcal{H})$.

Proof. (a) Clearly $\mathcal{D}(\mathcal{R}) \leq \mathrm{Aut}(\mathcal{H})$ since $\mathcal{R}$ is self-complementary. We get a strict inequality since $\mathrm{Aut}(\mathcal{H})$ is highly transitive (since $\mathrm{Aut}_2(\mathcal{R}) \leq \mathrm{Aut}(\mathcal{H})$) while $\mathcal{D}(\mathcal{R})$ is not.

(b) We show that $\mathcal{R}$ can be switched into a graph isomorphic to $\mathcal{R}$ in such a way that some induced copy E of $\mathcal{R}$ has an isolated vertex after switching. Then the isomorphism is a switching-automorphism but not an automorphism of $\mathcal{H}$.

Let p, q be two vertices of $\mathcal{R}$. The graph we work with will be $\mathcal{R}_1 = \mathcal{R} \setminus \{p\}$, which is of course isomorphic to $\mathcal{R}$. Let A, B, C, D be the sets of vertices joined to p and q, p but not q, q but not p, and neither p nor q, respectively. Let σ be the operation of switching $\mathcal{R}_1$ with respect to C, and let $E = \{q\} \cup B \cup C$. It is clear that, after the switching σ, the vertex q is isolated in E. So we have to prove two things:

Claim. E induces a copy of $\mathcal{R}$.

Proof. Take U, V to be finite disjoint subsets of E. We may assume without loss of generality that $q \in U \cup V$.

Case 1: $q \in U$. Choose a witness z for $(U, V \cup \{p\})$ in $\mathcal{R}$. Then $z \nsim p$ and $z \sim q$, so $z \in C$; thus z is a witness for (U, V) in E.

Case 2: $q \in V$. Now choose a witness for $(U \cup \{p\}, V)$ in $\mathcal{R}$; the argument is similar. $\qquad\square$

Claim. $\sigma(\mathcal{R}_1)$ is isomorphic to $\mathcal{R}$.

Proof. Choose U, V finite disjoint subsets of $\mathcal{R} \setminus \{p\}$. Again, without loss, $q \in U \cup V$. Set $U_1 = U \cap C$, $U_2 = U \setminus U_1$, and $V_1 = V \cap C$, $V_2 = V \setminus V_1$.

Case 1: $q \in U$, so $q \in U_2$. Take z to be a witness for $(U_2 \cup V_1 \cup \{p\}, U_1 \cup V_2)$ in $\mathcal{R}$. Then $z \sim p, q$, so $z \in A$. The switching σ changes its adjacencies to U_1 and V_1, so in $\sigma(\mathcal{R}_1)$ it is a witness for $(U_1 \cup U_2, V_1 \cup V_2)$.

Case 2: $q \in V$, so $q \in V_2$. Now take z to be a witness for $(U_1 \cup V_2, U_2 \cup V_1 \cup \{p\})$ in $\mathcal{R}$. Then $z \sim q$, $z \not\sim p$, so $z \in C$, and σ changes its adjacencies to U_2 and V_2, making it a witness for $(U_1 \cup U_2, V_1 \cup V_2)$. □

(c) Let $X \subseteq \mathcal{R}$, σ be the operation of switching $\mathcal{R}$ with respect to X, and $g : \mathcal{R} \to \sigma(\mathcal{R})$ an isomorphism. In order to show that $g \in Aut(\mathcal{H}^*)$ we need to show that if E is an edge of $\mathcal{H}$ there is some finite S such that $(E \setminus S)g$ and $(E \setminus S)g^{-1}$ are edges of $\mathcal{H}$.

However the graph Eg (in $\mathcal{R}$) is obtained from switching the graph induced by $\sigma(\mathcal{R})$ on Eg. Since the latter is a copy of $\mathcal{R}$, Lemma 2 yields a finite $S_0 \subset \mathcal{R}$ such that $Eg \setminus S_0$ is an edge of $\mathcal{H}$. If $S_1 = S_0 g^{-1}$, then $(E \setminus S_1)g$ is an edge of $\mathcal{H}$.

Finally notice that g^{-1} is an isomorphism from $\mathcal{R}$ to $\sigma_{Xg^{-1}}(\mathcal{R})$, the above argument shows that there is a finite S_2 such that $(E \setminus S_2)g^{-1}$ is an edge of $\mathcal{H}$. Since cofinite subsets of edges are edges $S := S_1 \cup S_2$ has the required property. □

Corollary 1. $\mathcal{B}(\mathcal{R}) < Aut^*(\mathcal{H})$

Proof. That $\mathcal{B}(\mathcal{R}) \leq Aut^*(\mathcal{H})$ follows from parts (a) and (c) of Proposition 3.

We get a strict inequality since $Aut^*(\mathcal{H})$ is highly transitive (since $Aut_2(\mathcal{R}) \leq Aut^*(\mathcal{H})$) while $\mathcal{D}(\mathcal{R})$ is not. □

Proposition 4. $\mathcal{S}(\mathcal{R}) \not\leq FAut(\mathcal{H})$.

In view of $\mathcal{S}(\mathcal{R}) \leq Aut^*(\mathcal{H})$ (by Proposition 3), this yields the following immediate Corollary.

Corollary 2. $FAut(\mathcal{H}) < Aut^*(\mathcal{H})$.

Clearly we have the following immediate observation:

Note 1.
$$Aut(\mathcal{H}).FSym(\mathcal{R}) \leq FAut(\mathcal{H})$$

Hence, the Corollary yields yet that $Aut(\mathcal{H}).FSym(\mathcal{R}) < Aut^*(\mathcal{H})$.

Proof (of Proposition 4). The argument can be thought of as an infinite version of the one given in part (b) of Proposition 3.

We shall recursively define subsets of $\mathcal{R}$:

- $A = \langle a_n : n \in \mathbb{N} \rangle$
- $B = \langle b_n : n \in \mathbb{N} \rangle$
- $C = \langle c_n : n \in \mathbb{N} \rangle$

and set D so that:

1. $\forall n \forall k \leq n \ a_n \not\sim b_k$ and $c_n \sim b_k$.
2. $\forall n \ E_n := \{a_k : k \geq n\} \cup \{b_n\} \cup \{c_k : k \geq n\}$ is an edge.
3. If σ is the operation of switching $\mathcal{R}$ with respect to C, then $\sigma(\mathcal{R})$ is isomorphic to $\mathcal{R}$.
4. $D = \mathcal{R} \setminus (A \cup B \cup C)$ is infinite.
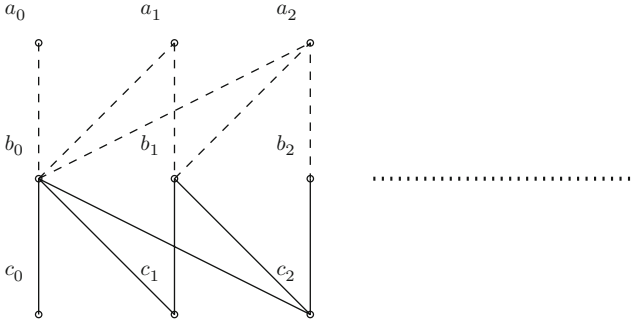
The construction is as follows. First list all pairs (U, V) of disjoint finite subsets of $\mathcal{R}$ so that each one reoccurs infinitely often. Start with $A = B = C = D = \emptyset$ and at stage n, assume we have constructed $A_n = \{a_k : k \leq n\}$, $B_n = \{b_k : k \leq n\}$, and $C_n = \{c_k : k \leq n\}$ satisfying condition (1) above, together with a finite set D disjoint from A_n, B_n and C_n. Then given (U, V), proceed following one of the following cases:

(a) Suppose $U \cup V \subseteq A_n \cup B_n \cup C_n$ and contains at most one b_i (i.e. (U, V) is a type candidate for the eventual E_i). Then, choosing from $\mathcal{R} \setminus (A_n \cup B_n \cup C_n \cup D)$, add a_{n+1} or c_{n+1} as a witness for (U, V) depending as to whether b_i is in V or U (add a_{n+1} if there is no such b_i at all). Then choose two more elements from $\mathcal{R} \setminus D$ to complete the addition of elements a_{n+1}, b_{n+1}, and c_{n+1} as required by condition (1). Also throw a new point in D just to ensure it will become infinite.
(b) Else add the elements of $U \cup V \setminus A_n \cup B_n \cup C_n$ to D, and select an element of $\mathcal{R} \setminus (A_n \cup B_n \cup C_n \cup D)$ as witness to $(U \setminus C_n \cup V \cap C_n, V \setminus C_n \cup U \cap C_n)$.

The construction in part (b) will ensure that $\sigma(\mathcal{R})$ is isomorphic to $\mathcal{R}$. Indeed let U and V be disjoint finite subsets of $\mathcal{R}$, and without loss of generality $U \cap D \neq \emptyset$. Thus when the pair (U, V) is handled at some stage n, part (b) will add a witness d in D to $(U \setminus C_n \cup V \cap C_n, V \setminus C_n \cup U \cap C_n)$. But then d is a witness to (U, V) in $\sigma(\mathcal{R})$.

Let g be the isomorphism from $\sigma(\mathcal{R})$ to $\mathcal{R}$.

Finally the construction in part (a) clearly ensures condition (2). However note that in $\sigma(\mathcal{R})$, b_n is isolated in E_n, and therefore $\sigma(E_n)$ is not an edge.

Finally for any finite set $S \subset \mathcal{R}$, choose n so that $S \cap E_n = \emptyset$. Then $(E_n \setminus S)g$ is not an edge. Thus $g \in \mathcal{S}(\mathcal{R}) \setminus \mathrm{FAut}(\mathcal{H})$. $\qquad\square$

We now go back to $\mathrm{Aut}(\mathcal{F}_{\mathcal{R}})$. One can readily verify that the automorphism g produced in the reverse direction of Proposition 1 is in fact not in $\mathrm{Aut}(\mathcal{H})$, thus $\mathrm{Aut}(\mathcal{F}_{\mathcal{R}}) \not\leq \mathrm{Aut}(\mathcal{H})$. However we have the following.

Proposition 5. $\mathrm{Aut}(\mathcal{F}_{\mathcal{R}}) \not\leq \mathrm{Aut}^*(\mathcal{H})$.

Proof. Fix a vertex $v \in \mathcal{R}$. Now partition $\mathcal{R}^c(v) = E \cup D$, where E is an edge, D is an infinite independent set. This is easily feasible since $\mathcal{R}^c(v)$ is an edge. Now define $g \in \mathrm{Sym}(\mathcal{R})$ such that:

(a) $g \restriction \mathcal{R}(v)$ is the identity.
(b) $g \restriction E$ is a bijection to D.
(c) $g \restriction D$ is a bijection to $\mathcal{R}^c(v)$.

Now for any vertex w, $\mathcal{R}(w)^g \supseteq \mathcal{R}(v) \cap \mathcal{R}(w)$ so $g \in \mathrm{Aut}(\mathcal{F}_{\mathcal{R}})$. However, for any finite set S of E, then $(E \setminus S)g$ is again an independent set, and thus certainly not an edge.

Hence $g \notin \mathrm{Aut}^*(\mathcal{H})$ and the proof is complete. $\qquad\square$

3 Conclusion

The following diagram summarizes the subgroup relationship between the various groups under discussion.
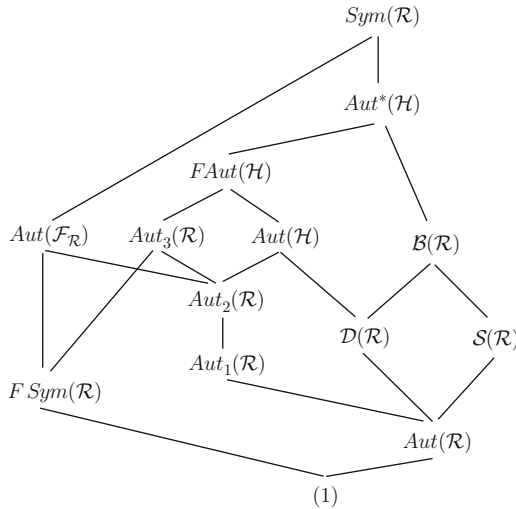
We do not know if the inclusion is strict in Observation 1.

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On a Stability Property of the Generalized Spherical Radon Transform

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Abstract In this note, we study the operator norm of the generalized spherical Radon transform, defined by a smooth measure on the underlying incidence variety. In particular, we prove that for small perturbations of the measure, the spherical Radon transform remains an isomorphism between the corresponding Sobolev spaces.

Key words Radon transform • Sobolev spaces • Pseudodifferential operators • Integral geometry

Mathematical Subject Classifications (2010): 44A12, 53C65

1 Introduction and Background

Throughout the note, we fix a Euclidean space $V = \mathbb{R}^{d+1}$, and consider the Euclidean spheres $X = S^d \subset V$, and $Y = S^d \subset V^*$. For $p \in Y$, $C_p \subset X$ will denote the copy of $S^{d-1} \subset X$ given by $C_p = \{q \in X : \langle q, p \rangle = 0\}$. Let $\sigma_{d-1}(q)$ denote the $SO(d)$-invariant probability measure on C_p. The set $C_q \subset Y$ and the measure $\sigma_{d-1}(p)$ on it are defined similarly. Then the spherical Radon transform is defined as follows:

$$\mathcal{R} : C^\infty(X) \to C^\infty(Y)$$

$$\mathcal{R}f(p) = \int_{C_p} f(q) d\sigma_{d-1}(q).$$

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Let σ be the unique $SO(d + 1)$-invariant probability measure on the incidence variety $Z = \{(q, p) \in X \times Y : \langle q, p \rangle = 0\}$. Assume one is given a smooth, not necessarily positive measure $d\mu$ on Z, given by $\mu(q, p)\sigma$ where $\mu \in C^\infty(Z)$, and which satisfies $\mu(\pm q, \pm p) = \mu(q, p)$ (call such μ symmetric). Introduce $\mathcal{R}_\mu : C^\infty(X) \to C^\infty(Y)$ by

$$(\mathcal{R}_\mu f)(p) = \int_{C_p} f(q)\mu(q, p)d\sigma_{d-1}(q).$$

Introduce also the dual Radon transform $\mathcal{R}_\mu^T : C^\infty(Y) \to C^\infty(X)$ which is formally adjoint to $\mathcal{R}_\mu$ and given by

$$(\mathcal{R}_\mu^T g)(q) = \int_{C_q} g(p)\mu(q, p)d\sigma_{d-1}(p).$$

Let $L_s^2(\mathbb{P}X)$ and $L_s^2(\mathbb{P}Y)$ denote the Sobolev space of even functions on X and Y, respectively. It is well known (see [2]) that the spherical Radon transform extends to an isomorphism of Sobolev spaces:

$$\mathcal{R} : L_s^2(\mathbb{P}X) \to L_{s+\frac{d-1}{2}}^2(\mathbb{P}Y)$$

for every $s \in \mathbb{R}$. For general μ as above, $\mathcal{R}_\mu$ is a Fourier integral operator of order $\frac{d-1}{2}$ (see [3, 5]), and so extends to a bounded map $\mathcal{R}_\mu : L_s^2(\mathbb{P}X) \to L_{s+\frac{d-1}{2}}^2(\mathbb{P}Y)$. We look for conditions on μ so that this is again an isomorphism.

It follows from Guillemin's theorem on general Radon transforms associated to double fibrations [1], that $\mathcal{R}_\mu^T \mathcal{R}_\mu : C^\infty(\mathbb{P}X) \to C^\infty(\mathbb{P}X)$ is an elliptic pseudo-differential operator of order $d - 1$ for all smooth, positive, symmetric measures μ on Z (for completeness, this is verified in the Appendix). The dependence of the principal symbol of $\mathcal{R}_\mu^T \mathcal{R}_\mu$ on μ was investigated in [5]. In this note, we analyze the dependence on μ of the operator norm of $\mathcal{R}_\mu^T \mathcal{R}_\mu : L_s^2(\mathbb{P}X) \to L_{s+(d-1)}^2(\mathbb{P}X)$. We then give a sufficient condition on a perturbation μ of μ_0 so that $\mathcal{R}_\mu : C^\infty(\mathbb{P}X) \to C^\infty(\mathbb{P}Y)$ remains an isomorphism. Namely, we prove the following

Theorem. *The set of C^∞ measures μ on Z for which $\mathcal{R}_\mu : C^\infty(\mathbb{P}X) \to C^\infty(\mathbb{P}Y)$ is an isomorphism, is open in the $C^{2d+1}(Z)$ topology.*

2 Bounding the Norm of $\mathcal{R}_\nu^T \mathcal{R}_\mu$

We start by recalling an equivalent description of the Radon transform. Consider the double fibration

$$Z$$

$$\pi \qquad \rho$$

$$X \qquad\qquad Y$$
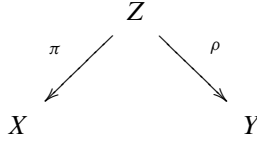
Let $\sigma_X = \pi_*\sigma$ and $\sigma_Y = \rho_*\sigma$ be the rotation-invariant probability measures on X and Y, respectively. Then for $f \in C^\infty(X)$, $(\mathcal{R}f)\sigma_Y = \rho_*(\sigma(\pi^* f))$. For smooth symmetric measures $d\mu$, $d\nu$ on Z, given by $\mu(q, p)\sigma$ and $\nu(q, p)\sigma$ we can define $\mathcal{R}_\mu : C^\infty(X) \to C^\infty(Y)$, $\mathcal{R}_\nu^T : C^\infty(Y) \to C^\infty(X)$ respectively by

$$(\mathcal{R}_\mu f)\sigma_Y = \rho_*(\mu\sigma(\pi^* f))$$

and

$$(\mathcal{R}_\nu^T g)\sigma_X = \pi_*(\nu\sigma(\rho^* g)).$$

It follows from [1] that both $\mathcal{R}_\mu$ and $\mathcal{R}_\nu^T$ are Fourier integral operators of order $\frac{d-1}{2}$. Thus we restrict to even functions and consider $\mathcal{R}_\mu : L_s^2(\mathbb{P}X) \to L_{s+\frac{d-1}{2}}^2(\mathbb{P}Y)$ and $\mathcal{R}_\nu^T : L_{s+\frac{d-1}{2}}^2(\mathbb{P}Y) \to L_{s+(d-1)}^2(\mathbb{P}X)$.

As before, q will denote a point in X and p a point in Y. We will often write $q \in p$ instead of $\langle q, p \rangle = 0 \iff q \in C_p \iff p \in C_q$. In the following, the functions f, g are even. We also assume $d \geq 2$.

Proposition 1. *The Schwartz kernel of $\mathcal{R}_\nu^T \mathcal{R}_\mu : L_s^2(\mathbb{P}X) \to L_{s+(d-1)}^2(\mathbb{P}X)$ is*

$$K(q', q) = \frac{c_d}{\sin \operatorname{dist}(q', q)}\alpha(q, q')$$

that is,

$$\mathcal{R}_\nu^T \mathcal{R}_\mu f(q') = \int_X f(q)K(q', q)d\sigma_X(q).$$

Here c_d is a constant, and $\alpha(q, q')$ is the average over all $p \in Y$ s.t. $q, q' \in p$ of $\mu(q, p)\nu(q', p)$. More precisely,

$$\alpha(q, q') = \int_{SO(d-1)} \mu(q, Mp_0)\nu(q', Mp_0)dM$$

where $SO(d - 1) = \{g \in SO(d + 1) : gq = q, gq' = q'\}$, C_{p_0} is any fixed copy of S^d through q, q', and dM is the Haar probability measure on $SO(d - 1)$.

Proof. Fix some $q' \in X$, and $p_0 \in Y$ s.t. $q' \in p_0$. Let $SO(d) \subset SO(d + 1)$ be the stabilizer of $q' \in X$. For $g \in C^\infty(Y)$ we may write

$$\mathcal{R}_\nu^T g(q') = \int_{p \ni q'} g(p)\nu(q', p)d\sigma_{d-1}(p) = \int_{SO(d)} g(Mp_0)\nu(q', Mp_0)dM$$

where dM is the Haar probability measure on $SO(d)$. Then taking

$$g(p) = \mathcal{R}_\mu f(p) = \int_{\tilde{q} \in p} f(\tilde{q}) \mu(\tilde{q}, p) d\sigma_{d-1}(\tilde{q})$$

we get

$$\mathcal{R}_\nu^T \mathcal{R}_\mu f(q') = \int_{SO(d)} \left(\int_{\tilde{q} \in Mp_0} f(\tilde{q}) \mu(\tilde{q}, Mp_0) d\sigma_{d-1}(\tilde{q}) \right) \nu(q', Mp_0) dM$$

$$= \int_{SO(d)} \left(\int_{\tilde{q} \in p_0} f(M\tilde{q}) \mu(M\tilde{q}, Mp_0) d\sigma_{d-1}(\tilde{q}) \right) \nu(q', Mp_0) dM$$

$$= \int_{\tilde{q} \in p_0} d\sigma_{d-1}(\tilde{q}) \int_{SO(d)} f(M\tilde{q}) \mu(M\tilde{q}, Mp_0) \nu(q', Mp_0) dM.$$

Denote $\theta = \mathrm{dist}(\tilde{q}, q')$, and $S_\theta^{d-1} = \{q : \mathrm{dist}(q', q) = \theta\}$. Let $d\sigma_{d-1}^\theta(q)$ denote the rotationally invariant probability measure on S_θ^{d-1}. The inner integral may be written as

$$\int_{S_\theta^{d-1}} f(q) \alpha(q, q') d\sigma_{d-1}^\theta(q).$$

Here $\alpha(q, q') = \int_{SO(d-1)} \mu(q, Mp_0) \nu(q', Mp_0) dM$ with $SO(d-1) = Stab(q) \cap Stab(q_0)$ is just the average of $\mu(q, p) \nu(q', p)$ over all $(d-1)$-dimensional spheres C_p containing both q and q'. Then

$$\mathcal{R}_\nu^T \mathcal{R}_\mu f(q') = \int_{\tilde{q} \in p_0} d\sigma_{d-1}(\tilde{q}) \int_{S_\theta^{d-1}} f(q) \alpha(q, q') d\sigma_{d-1}^\theta(q)$$

and since the inner integral only depends on $\theta = \mathrm{dist}(\tilde{q}, q')$, this may be rewritten as

$$c_d \int_0^{\pi/2} d\theta \, \sin^{d-2} \theta \int_{S_\theta^{d-1}} f(q) \alpha(q, q') d\sigma_{d-1}^\theta(q).$$

Finally, $d\sigma_d = c_d \sin^{d-1} \theta d\theta d\sigma_{d-1}^\theta$, and so

$$\mathcal{R}_\nu^T \mathcal{R}_\mu f(q') = c_d \int_X \frac{1}{\sin \theta} f(q) \alpha(q, q') d\sigma_d(q).$$

We conclude that the Schwartz kernel is

$$K(q', q) = \frac{c_d}{\sin \mathrm{dist}(q', q)} \alpha(q, q').$$

$\square$

We proceed to estimate the norm of $\mathcal{R}_\nu^T \mathcal{R}_\mu$. Our main tool will be the following proposition proved in Sect. C

Proposition. *Consider a pseudodifferential operator P of order m*

$$P : L^2_{s+m}(\mathbb{R}^n) \to L^2_s(\mathbb{R}^n)$$

between Sobolev spaces with $x-$compactly supported symbol $p(x, \xi)$ in $K \subset \mathbb{R}^n$ s.t.

$$|D^\alpha_x p(x, \xi)| \le C_{\alpha 0}(1 + |\xi|)^\alpha.$$

There exists a constant $C(n, s)$ such that

$$\|P\|_{L^2_{s+m}(\mathbb{R}^n) \to L^2_s(\mathbb{R}^n)} \le C(n, s) \sup_{|\alpha| \le n + \lfloor |s| \rfloor + 1} C_{\alpha 0}|K|.$$

Proposition 2. *The norm of $\mathcal{R}^T_\nu \mathcal{R}_\mu : L^2_{-(d-1)}(\mathbb{P}X) \to L^2_0(\mathbb{P}X)$ is bounded from above by*

$$\|\mathcal{R}^T_\nu \mathcal{R}_\mu\| \le C \sum_{j+k=0}^{2d+1} \|D^j \mu\|_\infty \|D^k \nu\|_\infty$$

for some constant C dependent on the double fibration.

Proof. First introduce coordinate charts. Choose a partition of unity $\chi_i(q')$ corresponding to a covering of X by charts U_i, and a function $\rho : [0, \infty) \to \mathbb{R}_+$ with support in $[0, 1]$ s.t. $\rho(r) = 1$ for $r \le \frac{1}{2}$. Write

$$K(q', q) = \sum_i K_i(q', q) + L_i(q', q)$$

$$K_i(q', q) = \chi_i(q')\rho(\sin \mathrm{dist}(q', q))K(q', q)$$

and

$$L_i(q', q) = \chi_i(q')(1 - \rho(\sin \mathrm{dist}(q', q)))K(q', q).$$

Let $\mathcal{R}^T_\nu \mathcal{R}_\mu = \sum_i T_{K_i} + T_{L_i}$ be the corresponding decomposition for the operators.

First we will bound the norm of the diagonal terms, i.e., the operators defined by K_i. Fix i, and choose some point $q' \in U_i$. Introduce polar coordinates (r, ψ) around q' so that $\psi \in S^{d-1}_1(q')$ and $r = \sin \theta$ for $r \le \frac{1}{2}$, $\theta = \mathrm{dist}(q, q')$. Note that $\alpha(q', (r, \psi)) = \alpha(q', (r, -\psi))$. By Proposition 5, the corresponding symbol is

$$p_1(q', \xi) = \chi_i(q') \int_0^1 \int_{S^{d-1}} \frac{\alpha(q', r\psi)}{r} e^{-i\langle \xi, \psi \rangle} \rho(r) r^{d-1} dr d\psi.$$

For a given q', introduce spherical coordinates $\psi = (\phi, \phi_1, \ldots, \phi_{d-2}) \ 0 \le \phi \le \pi$, on $S^{d-1}_1(q')$ in such a way that $\cos \phi = \psi_1$; Take $\xi_0 = (1, 0, \ldots, 0)$. Then

$$p_1(q', T\xi_0) = C_d \int_0^1 \rho(r) r^{d-2} dr \int_0^\pi m(q', r, \phi)e^{-iTr\cos\phi} \cos^{d-2}\phi d\phi$$

where $m(q', r, \phi) = \chi_i(q') \int_0^\pi \ldots \int_0^\pi \int_0^{2\pi} d\phi_1 \ldots d\phi_{d-2} \alpha(q', r, \phi, \phi_1, \ldots, \phi_{d-2})$.
We then have $m(q', r, \pi/2 + \phi) = m(q', r, \pi/2 - \phi)$. Take $t = \cos\phi$ and
$M(q', r, t) = m(q', r, \arccos t)$. Then $M(q', r, t) = M(q', r, -t)$ and

$$p_1(q', T\xi_0) = C_d \int_0^1 \rho(r) r^{d-2} dr \int_{-1}^1 M(q', r, t) e^{-iTrt} (1 - t^2)^{\frac{d-3}{2}} dt.$$

Since M is even, we may write

$$p_1(q', T\xi_0) = 2 \int_0^1 \rho(r) r^{d-2} dr \int_0^1 M(q', r, t) \cos(Trt)(1 - t^2)^{\frac{d-3}{2}} dt$$

and so for all multi-indices α

$$D_{q'}^\alpha p_1(q', T\xi_0) = 2 \int_0^1 \rho(r) r^{d-2} dr \int_0^1 D_{q'}^\alpha M(q', r, t) \cos(Trt)(1 - t^2)^{\frac{d-3}{2}} dt.$$

Define as in Appendix A

$$I(d, \rho, m) = \int_0^1 \rho(r) r^{d-2} dr \int_0^1 M(q', r, t) \cos(Trt)(1 - t^2)^{\frac{d-3}{2}} dt$$

then we can write

$$D_{q'}^\alpha p_1(q', T\xi_0) = 2I(d, \rho, D_{q'}^\alpha m(q', r, \phi))$$

and conclude by Proposition 3 that

$$|p_1(q', T\xi_0)| \le \frac{C}{T^{d-1}} \sum_{\substack{a + b \le d \\ a \le d/2, b \le d-1}} \sup \left| \frac{\partial^{a+b}}{\partial r^a \partial \phi^b} m \right|$$

$$\le \frac{C}{T^{d-1}} \sum_{j+k \le d} \|D^j \mu\|_\infty \|D^k \nu\|_\infty$$

and similarly for all multi-indices α

$$|D_{q'}^\alpha p_1(q', T\xi_0)| \le \frac{C}{T^{d-1}} \sum_{j+k \le |\alpha|+d} \|D^j \mu\|_\infty \|D^k \nu\|_\infty.$$

It is also immediate that

$$|p_1(q', T\xi_0)| \le C_d \int_0^1 \rho(r) r^{d-2} dr \int_0^\pi |m(q', r, \phi)| d\phi \le C \|\mu\|_\infty \|\nu\|_\infty$$

and similarly

$$|D_{q'}^\alpha p_1(q', T\xi_0)| \le C \sum_{j+k \le |\alpha|} \|D^j \mu\|_\infty \|D^k \nu\|_\infty.$$

So we can write

$$|D_{q'}^\alpha p_1(q', \xi)| \le \frac{C}{(1 + |\xi|)^{d-1}} \sum_{j+k \le |\alpha|+d} \|D^j \mu\|_\infty \|D^k \nu\|_\infty$$

for some universal constant $C = C(d)$. Then choosing $s = 0$ and $|\alpha| = d + 1$ in Proposition 4 we get that

$$\|T_{K_i}\|_{L^2_{-(d-1)}(\mathbb{P}X) \to L^2_0(\mathbb{P}X)} \le C \sum_{j+k=0}^{2d+1} \|D^j \mu\|_\infty \|D^k \nu\|_\infty.$$

Now we bound the norm of the off-diagonal term, namely the sum of operators corresponding to L_i. They constitute a smoothing operator $T_L(\mu, \nu)$; its Schwartz kernel $k(q', q) = (1 - \rho(\text{dist}(q', q)))K(q', q)$ is a smooth function in both arguments. Denoting by $\nabla^j : C^\infty(X) \to (T^*X)^{\otimes j}$ the j-th derivative obtained from the Levi-Civita connection,

$$\left\| \int_X d\sigma_X(q) f(q) k(q', q) \right\|^2_{L^2_{d-1}(\mathbb{P}X)}$$

$$= \int_X \left| \nabla_{q'}^{d-1} \int_X d\sigma_X(q) f(q) k(q', q) \right|^2 d\sigma_X(q')$$

$$= \int_X \left| \int_X d\sigma_X(q) f(q) \nabla_{q'}^{d-1} k(q', q) \right|^2 d\sigma_X(q')$$

$$\le \int_X d\sigma_X(q') \left(\int_X |f(q)^2| d\sigma_X(q) \int_X |\nabla_{q'}^{d-1} k(q', q)|^2 d\sigma_X(q) \right)$$

$$= \|f\|^2_{L^2(X)} \int_X |\nabla_{q'}^{d-1} k(q', q)|^2 d\sigma_X(q)$$

and

$$\sup_{q'} \sqrt{\int_X |\nabla_{q'}^{d-1} k(q', q)|^2 d\sigma_X(q)} \le C \sum_{j+k \le d-1} \|D^j \mu\|_\infty \|D^k \nu\|_\infty.$$

So

$$\|T_L(\mu, \nu)\|_{L^2_0(\mathbb{P}X) \to L^2_{d-1}(\mathbb{P}X)} \le C \sum_{j+k \le d-1} \|D^j \mu\|_\infty \|D^k \nu\|_\infty.$$

It is easy to see that the adjoint operator $T_L(\mu, \nu)^* : L^2_{(d-1)}(\mathbb{P}X)^* \to L^2_0(\mathbb{P}X)^*$ equals $T_L(\nu, \mu)$ after the isomorphic identification $L^2_s(\mathbb{P}X)^* \simeq L^2_{-s}(\mathbb{P}X)$ for $s = 0, d - 1$. Since the bound above is symmetric in μ, ν we conclude

$$\|T_L(\mu, \nu)\|_{L^2_{-(d-1)}(\mathbb{P}X) \to L^2_0(\mathbb{P}X)} \le C \sum_{j+k \le d-1} \|D^j \mu\|_\infty \|D^k \nu\|_\infty.$$

Finally

$$\|\mathcal{R}^T_\nu \mathcal{R}_\mu\| \le \|\sum_i T_{K_i}\| + \|T_L\| \le C \sum_{j+k=0}^{2d+1} \|D^j \mu\|_\infty \|D^k \nu\|_\infty.$$

$\square$

Theorem 1. *Assume $d \ge 2$, and let $\mu_0 \in C^\infty(Z)$ be such that $\mathcal{R}_{\mu_0} : C^\infty(\mathbb{P}X) \to C^\infty(\mathbb{P}Y)$ is an isomorphism. Then there exists $\epsilon_0 > 0$ (depending on the double fibration), such that if $\|\mu - \mu_0\|_{C^{2d+1}(Z)} < \epsilon_0$ then $\mathcal{R}_\mu : C^\infty(\mathbb{P}X) \to C^\infty(\mathbb{P}Y)$ is an isomorphism (for all s).*

Proof. Since $\mathcal{R}^T_{\mu_0} \mathcal{R}_{\mu_0} : L^2_{-(d-1)}(\mathbb{P}X) \to L^2_0(\mathbb{P}X)$ (and likewise for Y) is elliptic, it is an isomorphism. Let us verify that both of the maps $\mathcal{R}^T_\mu \mathcal{R}_\mu : L^2_{-(d-1)}(\mathbb{P}X) \to L^2_0(\mathbb{P}X)$ and $\mathcal{R}_\mu \mathcal{R}^T_\mu : L^2_{-(d-1)}(\mathbb{P}Y) \to L^2_0(\mathbb{P}Y)$ remain an isomorphism for small perturbations μ of μ_0 in the $C^{2d+1}(Z)$ norm:

$$\|\mathcal{R}^T_\mu \mathcal{R}_\mu - \mathcal{R}^T_{\mu_0} \mathcal{R}_{\mu_0}\| = \|(\mathcal{R}^T_{\mu_0} + \mathcal{R}^T_{\mu-\mu_0})(\mathcal{R}_{\mu_0} + \mathcal{R}_{\mu-\mu_0}) - \mathcal{R}^T_{\mu_0} \mathcal{R}_{\mu_0}\|$$
$$\le \|\mathcal{R}^T_{\mu_0} \mathcal{R}_{\mu-\mu_0}\| + \|\mathcal{R}^T_{\mu-\mu_0} \mathcal{R}_{\mu_0}\| + \|\mathcal{R}^T_{\mu-\mu_0} \mathcal{R}_{\mu-\mu_0}\|$$

so by Corollary 2, there is an $\epsilon_0 > 0$ s.t. all norms are indeed small when $\|\mu - \mu_0\|_{C^{2d+1}(Z)} < \epsilon_0$. The operator $\mathcal{R}_\mu \mathcal{R}^T_\mu$ is treated identically, and we take the minimal of the two ϵ_0.

Since both $\mathcal{R}^T_\mu \mathcal{R}_\mu$ and $\mathcal{R}_\mu \mathcal{R}^T_\mu$ are elliptic operators, the dimension of the kernel and cokernel are independent of s. It follows that $\mathcal{R}^T_\mu \mathcal{R}_\mu : L^2_s(\mathbb{P}X) \to L^2_{s+d-1}(\mathbb{P}X)$ and $\mathcal{R}_\mu \mathcal{R}^T_\mu : L^2_s(\mathbb{P}Y) \to L^2_{s+d-1}(\mathbb{P}Y)$ are isomorphisms for all s. In particular, $\mathrm{Ker}(\mathcal{R}_\mu : L^2_s(\mathbb{P}X) \to L^2_{s+\frac{d-1}{2}}(\mathbb{P}Y)) = 0$ and $\mathrm{Coker}(\mathcal{R}_\mu : L^2_s(\mathbb{P}X) \to L^2_{s+\frac{d-1}{2}}(\mathbb{P}Y)) = 0$, implying by the open mapping theorem that $\mathcal{R}_\mu : L^2_s(\mathbb{P}X) \to L^2_{s+\frac{d-1}{2}}(\mathbb{P}Y)$ is an isomorphism for all s. The result follows. $\square$

Remark 1. It is unlikely that the result is sharp. For instance, in the case of $d = 1$ one only needs $\|\mu - 1\|_{C^0}$ to be small to conclude that $\mathcal{R}_\mu$ is an isomorphism, while the statement (although non-applicable for $d = 1$) would suggest bounding the C^3-norm.

Appendix

A   Some Integral Estimates

Fix some real $T > 0$. For an integer $d \geq 2$, a smooth function $\rho : [0, \infty) \to \mathbb{R}$ compactly supported in $[0, 1)$, and smooth functions $m(r, \phi), n(r, \phi) : [0, \infty) \times [0, \pi]\mathbb{R}$ we define the integrals

$$I(d, \rho, m) = \int_0^1 \rho(r)r^{d-2}dr \int_0^1 M(r, t) \cos(Trt)(1 - t^2)^{\frac{d-3}{2}} dt$$

and

$$J(d, \rho, m) = \int_0^1 \rho(r)r^{d-2}dr \int_0^1 N(r, t) \sin(Trt)(1 - t^2)^{\frac{d-3}{2}} dt$$

where $M(r, t) = m(r, \phi)$ and $N(r, t) = n(r, \phi)$ for $t = \cos\phi$. We assume m is even w.r.t. $\frac{\pi}{2}$, namely $m(r, \frac{\pi}{2} + \phi) = m(r, \frac{\pi}{2} - \phi) \iff M(r, t) = M(r, -t)$; while n is odd, i.e. $n(r, \frac{\pi}{2} + \phi) = -n(r, \frac{\pi}{2} - \phi) \iff N(r, t) = -N(r, -t)$.

Proposition 3. *There exists a constant $C = C(d, \rho)$ such that for $d \geq 2$ and all even functions m*

$$|I(d, \rho, m)| \leq \frac{C}{T^{d-1}} \sum_{\substack{a + b \leq d \\ a \leq d/2, b \leq d - 1}} \sup \left| \frac{\partial^{a+b}}{\partial r^a \partial \phi^b} m \right|$$

and for odd functions n

$$|J(d, \rho, n)| \leq \frac{C}{T^{d-1}} \sum_{\substack{a + b \leq d \\ a \leq d/2, b \leq d - 1}} \sup \left| \frac{\partial^{a+b}}{\partial r^a \partial \phi^b} n \right|.$$

Proof. Induction on d. Start by verifying the bounds for $d = 2$. To bound

$$I(2, \rho, m) = \int_0^1 (1 - t^2)^{-\frac{1}{2}} \int_0^1 \rho(r)M(r, t) \cos(Trt)dr$$

we first integrate the inner integral by parts:

$$\int_0^1 M(r, t)\rho(r) \cos(Trt)dr = -\frac{1}{Tt} \int_0^1 \sin(Trt)(M(r, t)\rho'(r) + \frac{\partial M}{\partial r}(r, t)\rho(r))dr$$

Let us bound separately

$$\int_0^1 \frac{dt}{t\sqrt{1-t^2}} \int_0^1 \sin(Trt)M(r,t)\rho'(r)dr$$

and

$$\int_0^1 \frac{dt}{t\sqrt{1-t^2}} \int_0^1 \sin(Trt)\frac{\partial M}{\partial r}(r,t)\rho(r)dr.$$

Now

$$\left| \int_{\frac{1}{2}}^1 \frac{\sin(Trt)M(r,t)dt}{t\sqrt{1-t^2}} \right| \le C \sup|m|$$

and since $\frac{\partial M}{\partial t} = -\frac{\partial m}{\partial \phi}\frac{1}{\sqrt{1-t^2}}$,

$$\int_0^{\frac{1}{2}} \frac{\sin(Trt)M(r,t)dt}{t\sqrt{1-t^2}}$$

$$= Si(Trt)\frac{M(r,t)}{\sqrt{1-t^2}}\Big|_0^{\frac{1}{2}}$$

$$+ \int_0^{\frac{1}{2}} Si(Trt)\left(\frac{\partial m}{\partial \phi}(r,t)\frac{1}{\sqrt{1-t^2}^2} + M(r,t)\frac{t}{(1-t^2)^{3/2}} \right)dt.$$

Now since Si is bounded, it follows that

$$\left| \int_0^{\frac{1}{2}} \frac{\sin(Trt)M(r,t)dt}{t\sqrt{1-t^2}} \right| \le C \left(\sup|m| + \sup\left|\frac{\partial m}{\partial \phi}\right| \right)$$

Thus

$$\left| \int_0^1 \frac{dt}{t\sqrt{1-t^2}} \int_0^1 \sin(Trt)M(r,t)\rho'(r)dr \right|$$

$$\le C\left(\sup|m| + \sup\left|\frac{\partial m}{\partial \phi}\right| \right) \int_0^1 dr|\rho'(r)| \le C\left(\sup|m| + \sup\left|\frac{\partial m}{\partial \phi}\right| \right).$$

Similarly,

$$\left| \int_0^1 \frac{dt}{t\sqrt{1-t^2}} \int_0^1 \sin(Trt)\frac{\partial M}{\partial r}(r,t)\rho(r)dr \right| \le C\left(\sup\left|\frac{\partial m}{\partial r}\right| + \sup\left|\frac{\partial^2 m}{\partial r\partial \phi}\right| \right).$$

Tracing back,

$$|I(2,\rho,m)| \le \frac{C}{T}\left(\sup|m| + \sup\left|\frac{\partial m}{\partial r}\right| + \sup\left|\frac{\partial m}{\partial \phi}\right| + \sup\left|\frac{\partial^2 m}{\partial r \partial \phi}\right|\right).$$

Next we bound $|J(2,\rho,n)|$

$$J(2,\rho,n) = \int_0^1 (1-t^2)^{-1/2}dt \int_0^1 N(r,t)\rho(r)\sin(Trt)dr.$$

Integrate the inner integral by parts:

$$\int_0^1 N(r,t)\rho(r)\sin(Trt)dr = -\frac{1}{Tt}\int_0^1 (1-\cos(Trt))(N(r,t)\rho'(r)$$

$$+\frac{\partial N}{\partial r}(r,t)\rho(r))dr.$$

Let us bound separately

$$\int_0^1 \frac{dt}{t\sqrt{1-t^2}}\int_0^1 (1-\cos(Trt))N(r,t)\rho'(r)dr$$

and

$$\int_0^1 \frac{dt}{t\sqrt{1-t^2}}\int_0^1 (1-\cos(Trt))\frac{\partial N}{\partial r}(r,t)\rho(r)dr.$$

Now

$$\left|\int_{\frac{1}{2}}^1 \frac{(1-\cos(Trt))N(r,t)dt}{t\sqrt{1-t^2}}\right| \le C\sup|n|$$

and since $\frac{\partial N}{\partial t} = -\frac{\partial n}{\partial \phi}\frac{1}{\sqrt{1-t^2}}$ and $N(r,0)=0$ we get that $|\frac{\partial}{\partial t}N(r,t)| \le C\sup|\frac{\partial n}{\partial \phi}|$ for $0 \le t \le \frac{1}{2}$ so $|N(r,t)| \le C\sup|\frac{\partial n}{\partial \phi}|t$ and

$$\left|\int_0^{\frac{1}{2}} \frac{(1-\cos(Trt))N(r,t)dt}{t\sqrt{1-t^2}}\right| \le C\sup\left|\frac{\partial n}{\partial \phi}\right|\int_0^1 \frac{dt}{\sqrt{1-t^2}} = C\sup\left|\frac{\partial n}{\partial \phi}\right|.$$

Thus

$$\left|\int_0^1 \frac{dt}{t\sqrt{1-t^2}}\int_0^1 (1-\cos(Trt))N(r,t)\rho'(r)dr\right|$$

$$\le C\left(\sup|n| + \sup\left|\frac{\partial n}{\partial \phi}\right|\right)\int_0^1 dr|\rho'(r)| \le C\left(\sup|n| + \sup\left|\frac{\partial n}{\partial \phi}\right|\right).$$

Similarly,

$$\left| \int_0^1 \frac{dt}{t\sqrt{1-t^2}} \int_0^1 (1-\cos(Trt)) \frac{\partial N}{\partial r}(r,t)\rho(r)dr \right| \leq C \left(\sup \left| \frac{\partial n}{\partial r} \right| + \sup \left| \frac{\partial^2 n}{\partial r \partial \phi} \right| \right)$$

and putting all together,

$$|J(2,\rho,n)| \leq \frac{C}{T} \left(\sup |n| + \sup \left| \frac{\partial n}{\partial r} \right| + \sup \left| \frac{\partial n}{\partial \phi} \right| + \sup \left| \frac{\partial^2 n}{\partial r \partial \phi} \right| \right)$$

as required.

Next consider the case $d = 3$. We bound I, J simultaneously. Apply integration by parts to the inner integrals:

$$I(3,\rho,m) = \int_0^1 \rho(r)rdr \int_0^1 M(r,t)\cos(Trt)dt$$

$$= \frac{1}{T} \int_0^1 r\rho(r)dr \int_0^1 \frac{\partial m}{\partial \phi}(r,t) \frac{1}{\sqrt{1-t^2}} \frac{\sin(Trt)}{r} dt$$

$$+ \frac{1}{T} \int_0^1 r\rho(r)M(r,1) \frac{\sin(Tr)}{r} dr$$

and similarly

$$J(3,\rho,n) = -\frac{1}{T} \int_0^1 r\rho(r)dr \int_0^1 \frac{\partial n}{\partial \phi}(r,t) \frac{1}{\sqrt{1-t^2}} \frac{\cos(Trt)}{r} dt$$

$$-\frac{1}{T} \int_0^1 r\rho(r) \left(N(r,1) \frac{\cos(Tr)}{r} - \frac{N(r,0)}{r} \right) dr.$$

These first summands are

$$\frac{1}{T} \int_0^1 \rho(r)dr \int_0^1 \frac{\partial m}{\partial \phi}(r,t) \frac{1}{\sqrt{1-t^2}} \sin(Trt)dt = \frac{1}{T} J \left(2,\rho, \frac{\partial m}{\partial \phi} \right)$$

and

$$-\frac{1}{T} \int_0^1 \rho(r)dr \int_0^1 \frac{\partial n}{\partial \phi}(r,t) \frac{1}{\sqrt{1-t^2}} \cos(Trt)dt = -\frac{1}{T} I \left(2,\rho, \frac{\partial n}{\partial \phi} \right)$$

the second summand for I

$$\frac{1}{T} \int_0^1 r\rho(r)M(r,1) \frac{\sin(Tr)}{r} dr$$

$$= \frac{1}{T} \int_0^1 \rho(r)M(r,1)\sin(Tr)dr$$

$$= -\frac{1}{T}\rho(r)M(r,1)\frac{\cos(Tr)}{T}\Big|_0^1 + \frac{1}{T^2}\int_0^1 (\rho'(r)M(r,1)$$

$$+\rho(r)\frac{\partial}{\partial r}M(r,1))\cos(Tr)dr$$

is bounded by

$$\frac{C}{T^2}\left(\sup|m| + \sup\left|\frac{\partial m}{\partial r}\right|\right).$$

Similarly since $N(r,0) = 0$, also the second summand for J

$$\left|\frac{1}{T}\int_0^1 r\rho(r)N(r,1)\frac{\cos(Tr)}{r}dr\right| \leq \frac{C}{T^2}\left(\sup|n| + \sup\left|\frac{\partial n}{\partial r}\right|\right)$$

thus we showed that

$$|I(3,\rho,m)| \leq \frac{C(3,\rho)}{T}J(2,\rho,\frac{\partial m}{\partial\phi}) + \frac{C(3,\rho)}{T^2}\left(\sup|m| + \sup\left|\frac{\partial m}{\partial r}\right|\right)$$

and

$$|J(3,\rho,n)| \leq \frac{C(3,\rho)}{T}I(2,\rho,\frac{\partial n}{\partial\phi}) + \frac{C(3,\rho)}{T^2}\left(\sup|n| + \sup\left|\frac{\partial n}{\partial r}\right|\right)$$

and plugging the already proved estimates for $d = 2$ concludes the case $d = 3$.

Finally, for $d > 3$ we will apply induction. Again consider both integrals simultaneously. Start by integrating by parts the inner integral: the boundary term is zero (for J since n is odd), so

$$\int_0^1 M(r,t)\cos(Trt)(1-t^2)^{\frac{d-3}{2}}dt$$

$$= \frac{1}{Tr}\int_0^1 \sin(Trt)\left(\frac{\partial m}{\partial\phi}(1-t^2)^{\frac{d-4}{2}} + (d-3)M(r,t)t(1-t^2)^{\frac{d-5}{2}}\right)dt$$

$$\int_0^1 N(r,t)\sin(Trt)(1-t^2)^{\frac{d-3}{2}}dt$$

$$= -\frac{1}{Tr}\int_0^1 \cos(Trt)\left(\frac{\partial n}{\partial\phi}(1-t^2)^{\frac{d-4}{2}} + (d-3)N(r,t)t(1-t^2)^{\frac{d-5}{2}}\right)dt.$$

Thus

$$I(d,\rho,m) = \int_0^1 \rho(r)r^{d-2}dr \int_0^1 M(r,t)\cos(Trt)(1-t^2)^{\frac{d-3}{2}}dt$$

$$= \frac{1}{T}\int_0^1 \rho(r)r^{d-3}dr \int_0^1 \sin(Trt)\frac{\partial m}{\partial \phi}(1-t^2)^{\frac{d-4}{2}}dt$$

$$+ \frac{C_d}{T}\int_0^1 \rho(r)r^{d-3}\int_0^1 M(r,t)t(1-t^2)^{\frac{d-5}{2}}\sin(Trt)dt$$

and

$$J(d,\rho,n) = \int_0^1 \rho(r)r^{d-2}dr \int_0^1 N(r,t)\sin(Trt)(1-t^2)^{\frac{d-3}{2}}dt$$

$$= -\frac{1}{T}\int_0^1 \rho(r)r^{d-3}dr \int_0^1 \cos(Trt)\frac{\partial n}{\partial \phi}(1-t^2)^{\frac{d-4}{2}}dt$$

$$- \frac{C_d}{T}\int_0^1 \rho(r)r^{d-3}\int_0^1 N(r,t)t(1-t^2)^{\frac{d-5}{2}}\cos(Trt)dt.$$

The first terms are $\frac{1}{T}J(d-1,\rho,\frac{\partial m}{\partial \phi})$ and $-\frac{1}{T}I(d-1,\rho,\frac{\partial n}{\partial \phi})$, respectively.
In the second term, first change the order of integration:

$$\frac{C_d}{T}\int_0^1 \rho(r)r^{d-3}\int_0^1 M(r,t)t(1-t^2)^{\frac{d-5}{2}}\sin(Trt)dt$$

$$= \frac{C_d}{T}\int_0^1 t(1-t^2)^{\frac{d-5}{2}}dt \int_0^1 M(r,t)\sin(Trt)r^{d-3}\rho(r)dr.$$

Now apply integration by parts to the inner integral. Since $d-3 > 0$ and $\rho(1) = 0$, again there is no boundary term:

$$\int_0^1 M(r,t)\sin(Trt)r^{d-3}\rho(r)dr$$

$$= \int_0^1 dr \frac{\cos(Trt)}{Tt}(r^{d-3}\rho(r)\frac{\partial M}{\partial r}(r,t) + (r^{d-3}\rho'(r)$$

$$+ (d-3)r^{d-4}\rho(r))M(r,t))$$

Thus

$$\frac{C_d}{T}\int_0^1 \rho(r)r^{d-3}\int_0^1 M(r,t)t(1-t^2)^{\frac{d-5}{2}}\sin(Trt)dt$$

$$= \frac{C_d}{T^2}\int_0^1 (1-t^2)^{\frac{d-5}{2}}dt \int_0^1 dr\cos(Trt)r^{d-3}\rho(r)\frac{\partial M}{\partial r}(r,t)$$

$$+\frac{C_d}{T^2}\int_0^1 (1-t^2)^{\frac{d-5}{2}}\,dt\int_0^1 dr\,\cos(Trt)(r^{d-3}\rho'(r)$$

$$+(d-3)r^{d-4}\rho(r))M(r,t)$$

$$=\frac{C_d}{T^2}I\left(d-2,r\rho(r),\frac{\partial m}{\partial r}\right)+\frac{C_d}{T^2}I(d-2,\rho(r)+r\rho'(r),m)$$

and the corresponding term for J:

$$-\frac{C_d}{T}\int_0^1 \rho(r)r^{d-3}\int_0^1 N(r,t)t(1-t^2)^{\frac{d-5}{2}}\cos(Trt)dt$$

$$=-\frac{C_d}{T}\int_0^1 t(1-t^2)^{\frac{d-5}{2}}\,dt\int_0^1 N(r,t)\cos(Trt)r^{d-3}\rho(r)dr$$

$$=-\frac{C_d}{T}\int_0^1 t(1-t^2)^{\frac{d-5}{2}}\,dt\int_0^1 dr\,\frac{\sin(Trt)}{Tt}$$

$$\times\left(r^{d-3}\rho(r)\frac{\partial N}{\partial r}(r,t)+(r^{d-3}\rho'(r)+(d-3)r^{d-4}\rho(r))N(r,t)\right)$$

$$=-\frac{C_d}{T^2}J(d-2,r\rho(r),\frac{\partial n}{\partial r})+\frac{C_d}{T^2}J(d-2,\rho(r)+r\rho'(r),n)$$

and we conclude by induction. $\qquad\qquad\square$

B   Guillemin's Condition

For $q \in X$, we denote by $\bar{q} \in X$ the unique point proportional to q and distinct from it. We will consider the projective space $\mathbb{P}X = \mathbb{R}\mathbb{P}^d$, $\mathbb{P}Y = \mathbb{R}\mathbb{P}^d$ and the projectivized incidence variety $\mathbb{P}Z = \{(q,p)\in\mathbb{P}X\times\mathbb{P}Y : \langle q,p\rangle = 0\}$.
Consider the projectivized double fibration
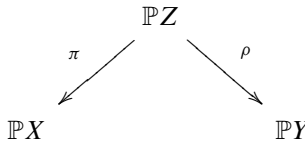
Then any two fibers $F_p(\mathbb{P}X)$ intersect transversally (since before projectivization, the only non-transversal intersection was between fibers over antipodal points). Denote $N_W \subset T^*(\mathbb{P}X\times\mathbb{P}Y)$, $N_E \subset T^*(X\times Y)$ the conormal bundles of W, E respectively. Since $\dim E = 2d-1$ and $\dim(X\times Y) = 2d$, the fibers of N_E, N_W are one-dimensional. Recall that $T_{(q,p)}E = \{(\xi,\eta)\in T_qX\times T_pY :$

$\langle q, \eta \rangle + \langle \xi, p \rangle = 0\}$. Therefore, N_E over $(q, p) \in E$ has its fiber spanned by $(p, q) \in T_q^* X \times T_p^* Y$. One thus has $N_E \backslash 0 \subset (T^* X \backslash 0) \times (T^* Y \backslash 0)$, and $\rho : N_E \backslash 0 \to T^* Y \backslash 0$ given by $((q, p), t(p, q)) \mapsto (p, tq)$ is an immersion, which is two-to-one since $\rho((q, p), t(p, q)) = \rho((\bar{q}, p), (-t)(p, \bar{q}))$. The corresponding map $\rho : N_W \backslash 0 \to T^* \mathbb{P} Y \backslash 0$ is already an injective immersion. Thus Guillemin's condition is satisfied, and we conclude

Corollary 1. *For any smooth positive measure* $\mu \in \mathcal{M}^\infty(\mathbb{P} Z)$, $\mathcal{R}_\mu^T \mathcal{R}_\mu : C^\infty$ $(\mathbb{P} X) \to C^\infty(\mathbb{P} X)$ *is an elliptic pseudodifferential operator.*

C Pseudo-Differential Operators

For a survey of the subject, see for instance [4].
We will study the norm of a pseudodifferential linear operator $P : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ which is given by its symbol $p(x, \xi)$

$$Pf(x) = \int d\xi e^{i \langle x, \xi \rangle} p(x, \xi) \hat{f}(\xi)$$

where $p \in Sym^m(K)$, i.e.,

1. $p \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$
2. p has compact $x-$support $K \subset \mathbb{R}^n$
3. $|D_\xi^\beta D_x^\alpha p(x, \xi)| \le C_{\alpha\beta}(1 + |\xi|)^{m-|\beta|}$

It is well known that for all $s \in \mathbb{R}$, P extends to a bounded operator between Sobolev spaces

$$P : L_{s+m}^2(\mathbb{R}^n) \to L_s^2(\mathbb{R}^n)$$

We will trace the proof of this fact to understand the dependence on p of the operator norm $\|P\|$.

Proposition 4. *There exists a constant* $C(n, s)$ *such that*

$$\|P\|_{L_{s+m}^2 \to L_s^2} \le C(n, s) \sup_{|\alpha| \le n + \lfloor |s| \rfloor + 1} C_{\alpha 0} |K|$$

Proof. All the integrals in the following are over $\mathbb{R}^n$. Start by integrating by parts:

$$\left| \int dx D_x^\alpha p(x, \xi) e^{i \langle x, \zeta \rangle} \right| = \left| \zeta^\alpha \right| \left| \int dx p(x, \xi) e^{i \langle x, \zeta \rangle} \right|.$$

So

$$\left| \int dx p(x, \xi) e^{i\langle x, \zeta \rangle} \right| \leq \min(|\zeta|^{-|\alpha|} C_{\alpha 0}(1 + |\xi|)^m |K|, C_{00}(1 + |\xi|)^m |K|)$$

$$\leq 2^{|\alpha|}(C_{00} + C_{\alpha 0})|K|(1 + |\xi|)^m (1 + |\zeta|)^{-|\alpha|}$$

$$= C_\alpha |K|(1 + |\xi|)^m (1 + |\zeta|)^{-|\alpha|}$$

where $C_\alpha = 2^{|\alpha|}(C_{00} + C_{\alpha 0})$. We want to bound

$$Pu(x) = \int d\xi e^{i\langle x, \xi \rangle} p(x, \xi) \hat{u}(\xi).$$

Take $v \in L^2_{-s}(\mathbb{R}^n)$, then

$$(Pu, v) = \int d\zeta \hat{v}(\zeta) \hat{P}u(\zeta) = \int d\zeta \hat{v}(\zeta) \int dx Pu(x) e^{-i\langle x, \zeta \rangle}$$

$$= \int \int d\zeta dx \hat{v}(\zeta) e^{-i\langle x, \zeta \rangle} \int d\xi \hat{u}(\xi) p(x, \xi) e^{i\langle x, \xi \rangle}$$

$$= \int \int d\zeta d\xi \hat{u}(\xi) \hat{v}(\zeta) \int dx p(x, \xi) e^{i\langle x, \xi - \zeta \rangle}$$

so denoting

$$\Phi(\xi, \zeta) = (1 + |\xi|)^{-m-s}(1 + |\zeta|)^s \left| \int dx p(x, \xi) e^{i\langle x, \xi - \zeta \rangle} \right|$$

we have

$$|(Pu, v)| \leq \int \int d\xi d\zeta \hat{u}(\xi) \hat{v}(\zeta) \Phi(\xi, \zeta)(1 + |\xi|)^{m+s}(1 + |\zeta|)^{-s}$$

$$\leq \left(\int d\xi |\hat{u}(\xi)|^2(1 + |\xi|)^{2(m+s)} \int d\zeta \Phi(\xi, \zeta) \right)^{1/2}$$

$$\times \left(\int \int d\zeta |\hat{v}(\zeta)|^2(1 + |\zeta|)^{-2s} \int d\xi \Phi(\xi, \zeta) \right)^{1/2}.$$

Now

$$\Phi(\xi, \zeta) \leq C_\alpha |K|(1 + |\xi|)^{-m-s}(1 + |\zeta|)^s(1 + |\xi|)^m(1 + |\xi - \zeta|)^{-|\alpha|}$$

$$\leq C_\alpha |K|(1 + |\xi - \zeta|)^{|s|-|\alpha|}.$$

Therefore,

$$\int d\xi \Phi(\xi, \zeta) \le A(n, |s| - |\alpha|) C_\alpha |K|$$

where

$$A(n, l) = \int d\xi (1 + |\xi|)^l$$

similarly

$$\int d\zeta \Phi(\xi, \zeta) \le C(n, |s| - |\alpha|) C_\alpha |K|$$

implying

$$|(Pu, v)| \le A(n, |s| - |\alpha|) C_\alpha |K| \|u\|_{m+s} \|v\|_{-s}$$

$$\Rightarrow \|P\|_{L^2_{s+m} \to L^2_s} \le A(n, |s| - |\alpha|) C_\alpha |K|$$

and this holds for all α s.t. $A(n, |s| - |\alpha|) = \int d\xi (1 + |\xi|)^{|s|-|\alpha|} < \infty$, i.e. $|s| - |\alpha| < -n \iff |\alpha| > n + |s|$. We thus choose α s.t. $|\alpha| = \lfloor |s| \rfloor + n + 1$, and recall that $C_\alpha = 2^{|\alpha|}(C_{00} + C_{\alpha 0})$ to obtain the stated estimate. $\qquad\square$

We will also need the relation between the Schwartz kernel and the symbol.

Proposition 5. *Suppose the Schwartz kernel of P is given by $K(x, y)$, namely*

$$\langle Pf(x), g(x) \rangle = \int dx dy K(x, y) f(y) g(x).$$

Then the symbol $p(x, \xi)$ of P is given by

$$p(x, \xi) = \int e^{-i\langle y, \xi \rangle} K(x, x - y) dy.$$

Proof. Write for smooth compactly supported f, g

$$\langle Pf(x), g(x) \rangle = \int dx d\xi e^{i\langle x, \xi \rangle} p(x, \xi) \hat{f}(\xi) g(x)$$

$$= \int dx dy d\xi e^{i\langle x-y, \xi \rangle} p(x, \xi) f(y) g(x).$$

That is, $K(x, y) = \int d\xi e^{i\langle x-y, \xi \rangle} p(x, \xi)$, and $\langle Pf, g \rangle = \int f(y) g(x) K(x, y) dy dx$. Denoting by $\check{h}(x) = \int d\xi h(\xi) e^{i\langle x, \xi \rangle}$ the inverse Fourier transform, we can also write $K(x, y) = \check{p}(x, \bullet)(x - y) \iff K(x, x - y) = \check{p}(x, \bullet)(y)$, so

$$p(x, \xi) = \int e^{-i\langle y, \xi \rangle} K(x, x - y) dy$$

as claimed. $\qquad\square$

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Banach Representations and Affine Compactifications of Dynamical Systems

Eli Glasner and Michael Megrelishvili

Abstract To every Banach space V we associate a compact right topological affine semigroup $\mathcal{E}(V)$. We show that a separable Banach space V is Asplund if and only if $\mathcal{E}(V)$ is metrizable, and it is Rosenthal (i.e., it does not contain an isomorphic copy of l_1) if and only if $\mathcal{E}(V)$ is a Rosenthal compactum. We study representations of compact right topological semigroups in $\mathcal{E}(V)$. In particular, representations of tame and HNS-semigroups arise naturally as enveloping semigroups of tame and HNS (hereditarily nonsensitive) dynamical systems, respectively. As an application we obtain a generalization of a theorem of R. Ellis. A main theme of our investigation is the relationship between the enveloping semigroup of a dynamical system X and the enveloping semigroup of its various affine compactifications $Q(X)$. When the two coincide we say that the affine compactification $Q(X)$ is E-compatible. This is a refinement of the notion of injectivity. We show that distal non-equicontinuous systems do not admit any E-compatible compactification. We present several new examples of non-injective dynamical systems and examine the relationship between injectivity and E-compatibility.

Key words Affine compactification • Affine flow • Asplund space • Enveloping semigroup • Nonsensitivity • Right topological semigroup • Semigroup compactification • Tame system • Weakly almost periodic

Mathematical Subject Classifications (2010): 37Bxx, 54H20, 54H15, 46xx

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1 Introduction

In this work we pursue our ongoing investigation of representations of dynamical systems on Banach spaces (see [24, 25, 27, 29, 31, 47, 48, 50]).

Recall that a representation of a dynamical system (G, X) on a Banach space V is given by a pair (h, α), where $h : G \to \text{Iso}\,(V)$ is a *co-homomorphism* (i.e., $h(g_1 g_2) = h(g_2) h(g_1)$ for all $g_1, g_2 \in G$) of the group G into the group $\text{Iso}\,(V)$ of linear isometries of V, and $\alpha : X \to V^*$ is a weak* continuous bounded G-map with respect to the dual action of $h(G)$ on V^*. For semigroup actions (S, X) we consider the co-homomorphisms $h : S \to \Theta(V)$, where $\Theta(V)$ is the semigroup of all contractive operators. For every representation (h, α), taking $Q = \overline{co}^{w^*}(\alpha(X))$, we get natural affine S-compactifications $\alpha : X \to Q$. This way of obtaining affine compactifications establishes a direct link to our earlier works which were mainly concerned with representations on reflexive, Asplund, and Rosenthal Banach spaces.

In Sect. 3 we discuss semigroup compactifications which arise from certain linear representations, the so-called *operator compactifications*. These were studied by Witz [74] and Junghenn [39]. In the weakly almost periodic (WAP) case this approach retrieves the classical work of de Leeuw and Glicksberg [15].

To every Banach space V we associate a compact right topological affine semigroup $\mathcal{E}(V)$. This is actually the enveloping semigroup of the natural dynamical system $(\Theta(V)^{\text{op}}, B^*)$, where $B^* \subset V^*$ is the weak* compact unit ball and $\Theta(V)^{\text{op}}$ is the adjoint semigroup of $\Theta(V)$. We show that a separable Banach space V is Asplund if and only if $\mathcal{E}(V)$ is metrizable, and it is Rosenthal (i.e., it does not contain an isomorphic copy of l_1) if and only if $\mathcal{E}(V)$ is a Rosenthal compactum, Theorems 6 and 10, respectively. We note that the first assertion, about Asplund spaces, can in essence be already found in [29].

Among the representations of compact right topological semigroups in $\mathcal{E}(V)$ we are especially interested in tame and HNS-semigroups. These arise naturally in the study of tame and HNS (= hereditarily nonsensitive) dynamical systems.

Tame dynamical metric systems appeared first in the work of Köhler [41] under the name of *regular* systems. In [24] we formulated a dynamical version of the Bourgain-Fremlin-Talagrand (in short: BFT) dichotomy (Fact 14 below). According to this an enveloping semigroup is either *tame*: has cardinality $\leq 2^{\aleph_0}$ and consists of Baire class 1 maps, or it is topologically wild and contains a copy of $\beta\mathbb{N}$, the Čech-Stone compactification of a discrete countable set. This dichotomy combined with a characterization of Rosenthal Banach spaces, Theorem 8, leads to a dichotomy theorem for Banach spaces (Theorem 10).

The enveloping semigroup characterization of (metric) tame systems in [29] led us in [27] to a general, more flexible definition of tame systems. A (not necessarily metrizable) compact dynamical system X is *tame* if every member of its enveloping semigroup is a fragmented (Baire 1, for metrizable X) self-map on X.

In the papers [24, 27, 48] we have shown that a metric system is HNS (tame, WAP) if and only if it admits a faithful representation on an Asplund (respectively, Rosenthal, reflexive) Banach space. The algebra of all Asplund (tame) functions on

a semigroup S is defined as the collection of all functions on S which come from HNS (respectively, tame) S-compactifications $S \rightarrow X$. These algebras are denoted by $\mathrm{Asp}(S)$ and $\mathrm{Tame}(S)$, respectively. Tame and HNS dynamical systems were investigated in several recent publications. See for example the papers by Huang [34] and Kerr-Li [40].

In Sect. 8 we strengthen some of our earlier results regarding representations on Banach spaces. We show in Theorem 15 that the Polish group $G = H_+[0, 1]$, which admits only trivial Asplund representations, is however Rosenthal representable.

One of the main topics treated in this work is a refinement of the notion of "injectivity." The latter was introduced by Köhler [41] (who, in turn, was motivated by a problem of Pym [60]) and examined systematically in [21, 22]. A compact dynamical G-system X is called *injective* if the canonical (restriction) homomorphism $r : E(P(X)) \rightarrow E(X)$—where $E(X)$ denotes the enveloping semigroup of the system (G, X) and $P(X)$ is the compact space of probability measures on X—is an injection, hence an isomorphism. The refinement we investigate in the present work is the following one (Sect. 5). Instead of considering just the space $P(X)$ we consider any embedding $(G, X) \hookrightarrow (G, Q)$ into an affine G-system (G, Q) with $Q = \overline{co}^{w^*}(X)$ and we say that this embedding is *E-compatible* if the homomorphism $r : E(Q) \rightarrow E(X)$ is injective (hence an isomorphism).

Distal affine dynamical systems have quite rigid properties. See for example the work of Namioka [54]. It was shown in [19] that a minimally generated metric distal affine G-flow is equicontinuous. Using a version of this result we show that for a minimal distal dynamical system E-compatibility in any faithful affine compactification implies equicontinuity. Thus such embedding is never E-compatible when the system is distal but not equicontinuous (Proposition 9). In particular this way we obtain in Theorem 3 a concrete example of a semigroup compactification which is not an *operator compactification*. More precisely, for the algebra $D(\mathbb{Z})$ of all distal functions on $\mathbb{Z}$, the corresponding semigroup compactification $\alpha : \mathbb{Z} \rightarrow \mathbb{Z}^{D(\mathbb{Z})}$ is not an operator compactification.

Non-injectivity is not restricted to distal systems. We construct examples of Toeplitz systems which are not injective, Theorem 4. We do not have such examples for a weakly mixing system. We also describe an example of a $\mathbb{Z}^2$-system which admits an E-compatible embedding yet is not injective (Example 1). We do not have such an example for $\mathbb{Z}$-systems.

The notion of a *left introverted* (we say shortly: *introverted*) linear subspace of $C(S)$ was introduced by M.M. Day in 1957. It is an important tool in the study of semigroups of means and affine semigroup compactifications. It also plays a major role in the theory of Banach semigroup algebras and their second duals, see for example [7, 13, 61]. A weaker property of subalgebras of $C(S)$ is being m-introverted. It turns out that a subalgebra of $\mathrm{RUC}(G)$ is m-introverted iff the corresponding dynamical system is point-universal iff it is isomorphic (as a dynamical system) to its own enveloping semigroup. It is well known that the algebras $\mathrm{RUC}(G)$ and $\mathrm{WAP}(G)$ are introverted. In general there is a large room between the algebras $\mathrm{RUC}(G)$ and $\mathrm{WAP}(G)$ for topological groups G. Indeed, by [51] ([1] for monothetic G), $\mathrm{RUC}(G) = \mathrm{WAP}(G)$ iff G is precompact.

We provide new nontrivial examples of introverted spaces. We show that Tame(S) is always introverted. Moreover, all of its m-introverted S-subalgebras (like Asp(S) and WAP(S)) are introverted. As a particular case (Theorem 19) it follows that every m-introverted *separable* S-subalgebra of $C(S)$ is introverted. Note also that, by [26], the algebra Asp(G) (which contains the algebra WAP(G)) is (left) amenable for every topological group G. This is in contrast to the fact that the larger algebra Tame(G) is, in general, non-amenable.

We show that a semigroup compactification $\nu : S \rightarrow P$ is an operator compactification iff the corresponding algebra of this compactification $\mathcal{A}_\nu$ is *intro-generated*. The latter means that $\mathcal{A}_\nu$ contains an introverted subspace $F \subset \mathcal{A}_\nu$ such that the minimal closed subalgebra of $C(S)$ containing F is $\mathcal{A}_\nu$. (This phenomenon reflects the existence of an E-compatible system which is not injective.) The space $D(\mathbb{Z})$ of all distal functions on $\mathbb{Z}$ is not intro-generated (Theorem 3). The $\mathbb{Z}^2$-flow from Example 1 mentioned above provides an intro-generated subspace of $l_\infty(\mathbb{Z}^2)$ which is not introverted.

In Sect. 9 we first show, in Theorem 16, that affine compactifications coming from representations on Rosenthal spaces are E-compatible. The core of the proof is Haydon's characterization of Rosenthal spaces in terms of the w^*-Krein-Milman property. Using results of Sect. 8 about representations of tame systems on Rosenthal spaces we show in Theorem 17 that every tame S-space X is injective. This result was proved by Köhler [41] for metrizable systems. In [21] there is a simple proof of this which uses the fact that for a tame metrizable system X its enveloping semigroup is a Fréchet space.

Next we prove a representation theorem (Theorem 20) according to which the enveloping semigroup of a tame (respectively, HNS) system admits an admissible embedding into $\mathcal{E}(V)$, where V runs over the class of Rosenthal (respectively, Asplund) Banach spaces. These results extend the following well-known theorem: the class of reflexively representable compact right topological semigroups coincides with the class of compact semitopological semigroups (proved in [47, 67]). As an applications of Theorem 20, using Theorem 2, we obtain a *generalized Ellis theorem*: a tame compact right topological group is a topological group (Theorem 21).

Finally, a representation theorem for S-affine compactifications (Theorem 22) shows that for tame (HNS, WAP) compact metrizable S-systems, their S-affine compactifications can be *affinely S-represented* on Rosenthal (Asplund, reflexive) separable Banach spaces.

2 Preliminaries

Topological spaces are always assumed to be Hausdorff and completely regular. The closure of a subset $A \subset X$ is denoted by $\overline{A}$ or cl(A). Banach spaces and locally convex vector spaces are over the field $\mathbb{R}$ of real numbers. For a subset A of a Banach space we denote by $sp(A)$ and $\overline{sp}^{\text{norm}}(A)$ the linear span and the norm-closed linear

span of K, respectively. We denote by $co(A)$ and $\overline{co}(A)$ the convex hull and the closed convex hull of a set A, respectively. If $A \subset V^*$ is a subset of the dual space V^* of V we mostly mean the weak* topology on A and $\overline{co}(A)$ or $\overline{co}^{w^*}(A)$ will denote the w^*-closure of $co(A)$ in V^*. For a topological space X we denote by $C(X)$ the Banach algebra of real valued continuous and bounded functions equipped with the supremum norm. For a subset $A \subset C(X)$ we denote by $\langle A \rangle$ the smallest unital (i.e., containing the constants) closed subalgebra of $C(X)$ containing A.

2.1 Semigroups and Actions

Let S be a semigroup which is also a topological space. By $\lambda_a : S \to S, x \mapsto ax$ and $\rho_a : S \to S, x \mapsto xa$ we denote the left and right a-transitions. The subset $\Lambda(S) := \{a \in S : \lambda_a \text{ is continuous}\}$ is called the *topological center* of S.

Definition 1. A semigroup S as above is said to be:

1. A *right topological semigroup* if every ρ_a is continuous
2. *Semitopological* if the multiplication $S \times S \to S$ is separately continuous
3. [53] *Admissible* if S is right topological and $\Lambda(S)$ is dense in S

Let A be a subsemigroup of a right topological semigroup S. If $A \subset \Lambda(S)$ then the closure $\mathrm{cl}(A)$ is a right topological semigroup. In general, $\mathrm{cl}(A)$ is not necessarily a subsemigroup of S (even if S is compact right topological and A is a left ideal). Also $\Lambda(S)$ may be empty for general compact right topological semigroup S. See [7, p. 29].

Definition 2. Let S be a semitopological semigroup with a neutral element e. Let $\pi : S \times X \to X$ be a *left action* of S on a topological space X. This means that $ex = x$ and $s_1(s_2x) = (s_1s_2)x$ for all $s_1, s_2 \in S$ and $x \in X$, where as usual, we write sx instead of $\pi(s, x) = \lambda_s(x) = \rho_x(s)$. Let $S \times X \to X$ and $S \times Y \to Y$ be two actions. A map $f : X \to Y$ between S-spaces is an *S-map* if $f(sx) = sf(x)$ for every $(s, x) \in S \times X$.

We say that X is a *dynamical S-system* (or an *S-space* or an *S-flow*) if the action π is separately continuous (that is, if all orbit maps $\rho_x : S \to X$ and all translations $\lambda_s : X \to X$ are continuous). We sometimes write it as a pair (S, X).

A *right system* (X, S) can be defined analogously. If S^{op} is the *opposite semigroup* of S with the same topology then (X, S) can be treated as a left system (S^{op}, X) (and vice versa).

Fact 1 ([43]). *Let G be a Čech-complete (e.g., locally compact or completely metrizable) semitopological group. Then every separately continuous action of G on a compact space X is continuous.*

Notation: *All semigroups S are assumed to be monoids, i.e., semigroups with a neutral element which will be denoted by e. Also actions are* monoidal *(meaning*

$ex = x, \forall x \in X$) *and separately continuous. We reserve the symbol G for the case when S is a group. All right topological semigroups below are assumed to be admissible.*

Given $x \in X$, its *orbit* is the set $Sx = \{sx : s \in S\}$ and the closure of this set, $\mathrm{cl}\,(Sx)$, is the *orbit closure* of x. A point x with $\mathrm{cl}\,(Sx) = X$ is called a *transitive point*, and the set of transitive points is denoted by X_{tr}. We say that the system is *point-transitive* when $X_{\mathrm{tr}} \neq \emptyset$. The system is called *minimal* if $X_{\mathrm{tr}} = X$.

2.2   Representations of Dynamical Systems

A *representation* of a semigroup S on a normed space V is a co-homomorphism $h : S \to \Theta(V)$, where $\Theta(V) := \{T \in L(V) : \|T\| \leq 1\}$ and $h(e) = id_V$. Here $L(V)$ is the space of continuous linear operators $V \to V$ and id_V is the identity operator. This is equivalent to the requirement that $h : S \to \Theta(V)^{\mathrm{op}}$ be a monoid homomorphism, where $\Theta(V)^{\mathrm{op}}$ is the opposite semigroup of $\Theta(V)$. If $S = G$, is a group then $h(G) \subset \mathrm{Iso}\,(V)$, where $\mathrm{Iso}\,(V)$ is the group of all linear isometries from V onto V. The adjoint operator $\mathrm{adj} : L(V) \to L(V^*)$ induces an injective co-homomorphism $\mathrm{adj} : \Theta(V) \to \Theta(V^*), \mathrm{adj}(s) = s^*$. We will identify $\mathrm{adj}(L(V))$ and the opposite semigroup $L(V)^{\mathrm{op}}$, as well as $\mathrm{adj}(\Theta(V)) \subset L(V^*)$ and its opposite semigroup $\Theta(V)^{\mathrm{op}}$. Mostly we use the same symbol s instead of s^*. Since $\Theta(V)^{\mathrm{op}}$ acts from the right on V and from the left on V^* we sometimes write vs for $h(s)(v)$ and $s\psi$ for $h(s)^*(\psi)$.

A pair of vectors $(v, \psi) \in V \times V^*$ defines a function (called a *matrix coefficient of h*)

$$m(v, \psi) : S \to \mathbb{R}, \quad s \mapsto \psi(vs) = \langle vs, \psi \rangle = \langle v, s\psi \rangle.$$

The weak operator topology on $\Theta(V)$ (similarly, on $\Theta(V)^{\mathrm{op}}$) is the weak topology generated by all matrix coefficients. So $h : S \to \Theta(V)^{\mathrm{op}}$ is *weakly continuous* iff $m(v, \psi) \in C(S)$ for every $(v, \psi) \in V \times V^*$. The strong operator topology on $\Theta(V)$ (and on $\Theta(V)^{\mathrm{op}}$) is the pointwise topology with respect to its left (respectively, right) action on the Banach space V.

Lemma 1. *Let $h : S \to \Theta(V)$ be a weakly continuous co-homomorphism. Then for every $v \in V$ the following map*

$$T_v : V^* \to C(S), \quad T_v(\psi) = m(v, \psi)$$

is a well-defined linear bounded weak*-pointwise continuous S-map between left S-actions.

Definition 3 (See [24, 48]). Let X be a dynamical S-system.

1. A *representation* of (S, X) on a normed space V is a pair

$$(h, \alpha) : S \times X \rightrightarrows \Theta(V) \times V^*,$$

where $h : S \to \Theta(V)$ is a co-homomorphism of semigroups and $\alpha : X \to V^*$ is a weak* continuous bounded S-mapping with respect to the dual action

$$S \times V^* \to V^*, \ (s\varphi)(v) := \varphi(h(s)(v)).$$

We say that the representation is weakly (strongly) continuous if h is weakly (strongly) continuous. A representation (h, α) is said to be *faithful* if α is a topological embedding.

2. If $\mathcal{K}$ is a subclass of the class of Banach spaces, we say that a dynamical system (S, X) is weakly (respectively, strongly) $\mathcal{K}$-*representable* if there exists a weakly (respectively, strongly) continuous faithful representation of (S, X) on a Banach space $V \in \mathcal{K}$.

3. A subdirect product, i.e., an S-subspace of a direct product, of weakly (strongly) $\mathcal{K}$-representable S-spaces is said to be weakly (strongly) $\mathcal{K}$-*approximable*.

We consider in particular the following classes of Banach spaces: Reflexive, Asplund, and Rosenthal spaces. A reflexively (Asplund) representable compact dynamical system is a dynamical version of the purely topological notion of an *Eberlein* (respectively, a *Radon-Nikodym*) compactum, in the sense of Amir and Lindenstrauss (respectively, in the sense of Namioka).

2.3 Background on Banach Spaces and Fragmentability

Definition 4. Let (X, τ) be a topological space and (Y, μ) a uniform space.

1. [37] X is (τ, μ)-*fragmented* by a (typically, not continuous) function $f : X \to Y$ if for every nonempty subset A of X and every $\varepsilon \in \mu$ there exists an open subset O of X such that $O \cap A$ is nonempty and the set $f(O \cap A)$ is ε-small in Y. We also say in that case that the function f is *fragmented*. Notation: $f \in \mathcal{F}(X, Y)$, whenever the uniformity μ is understood. If $Y = \mathbb{R}$, then we write simply $\mathcal{F}(X)$.

2. [24] We say that a *family of functions* $F = \{f : (X, \tau) \to (Y, \mu)\}$ is *fragmented* if condition (1) holds simultaneously for all $f \in F$. That is, $f(O \cap A)$ is ε-small for every $f \in F$.

3. [28] We say that F is an *eventually fragmented family* if every infinite subfamily $C \subset F$ contains an infinite fragmented subfamily $K \subset C$.

In point 1 of Definition 4 when $Y = X, f = id_X$, and μ is a metric uniform structure, we get the usual definition of fragmentability (more precisely, (τ, μ)-fragmentability) in the sense of Jayne and Rogers [38]. Implicitly it already appears in a paper of Namioka and Phelps [56].

Remark 1 ([24, 27]).

1. It is enough to check the condition of Definition 4 only for closed subsets $A \subset X$ and for $\varepsilon \in \mu$ from a *subbase* γ of μ (that is, the finite intersections of the elements of γ form a base of the uniform structure μ).

2. When X and Y are Polish spaces, $f : X \to Y$ is fragmented iff f is a Baire class 1 function.

3. When X is compact and (Y, ρ) metrizable uniform space then $f : X \to Y$ is fragmented iff f has a *point of continuity property* (i.e., for every closed nonempty $A \subset X$ the restriction $f_{|A} : A \to Y$ has a continuity point).
4. When Y is compact with its unique compatible uniformity μ then $p : X \to Y$ is fragmented if and only if $f \circ p : X \to \mathbb{R}$ has a point of continuity property for every $f \in C(Y)$.

Lemma 2.

1. *Suppose F is a compact space, X is Čech-complete, Y is a uniform space, and we are given a separately continuous map $w : F \times X \to Y$. Then the naturally associated family $\tilde{F} := \{\tilde{f} : X \to Y\}_{f \in F}$ is fragmented, where $\tilde{f}(x) = w(f, x)$.*
2. *Suppose F is a compact metrizable space, X is hereditarily Baire, and M is separable and metrizable. Assume we are given a map $w : F \times X \to M$ such that every $\tilde{x} : F \to M, f \mapsto w(f, x)$ is continuous and $y : X \to M$ is continuous at every $\tilde{y} \in Y$ for some dense subset Y of F. Then the family $\tilde{F}$ is fragmented.*

Proof.

(1) There exists a collection of uniform maps $\{\varphi_i : Y \to M_i\}_{i \in I}$ into metrizable uniform spaces M_i which generates the uniformity on Y. Now for every closed subset $A \subset X$ apply Namioka's joint continuity theorem to the separately continuous map $\varphi_i \circ w : F \times A \to M_i$ and take into account Remark 1 of point 1.
(2) Since every $\tilde{x} : F \to M$ is continuous, the natural map $j : X \to C(F, M), j(x) = \tilde{x}$ is well defined. Every closed nonempty subset $A \subset X$ is Baire. By [29, Proposition 2.4], $j|_A : A \to C(F, M)$ has a point of continuity, where $C(F, M)$ carries the sup-metric. Hence, $\tilde{F}_A = \{\tilde{f} \restriction_A : A \to M\}_{f \in F}$ is equicontinuous at some point $a \in A$. This implies that the family $\tilde{F}$ is fragmented. □

For other properties of fragmented maps and fragmented families refer to [24, 27, 48].

Recall that a Banach space V is an *Asplund* space if the dual of every separable Banach subspace is separable. In the following result the equivalence (1) $\Leftrightarrow$ (2) is a well-known criterion [55], and (3) is a reformulation of (2) in terms of fragmented families. When V is a Banach space we denote by B, or B_V, the closed unit ball of V. $B^* = B_{V*}$ and $B^{**} := B_{V**}$ will denote the weak* compact unit balls in the dual V^* and second dual V^{**} of V, respectively.

Fact 2 ([55, 56]). *Let V be a Banach space. The following conditions are equivalent:*

1. *V is an Asplund space.*
2. *Every bounded subset A of the dual V^* is (weak*,norm)-fragmented.*
3. *B is a fragmented family of real valued maps on the compactum B^*.*

Assertion (3) is a reformulation of (2). Reflexive spaces and spaces of the type $c_0(\Gamma)$ are Asplund. For more details cf. [17, 55].

We say that a Banach space V is *Rosenthal* if it does not contain an isomorphic copy of l_1. Clearly, every Asplund space is Rosenthal.

Definition 5 ([27]). Let X be a topological space. We say that a subset $F \subset C(X)$ is a *Rosenthal family* (for X) if F is norm bounded and the pointwise closure $\mathrm{cl}_p(F)$ of F in $\mathbb{R}^X$ consists of fragmented maps, that is, $\mathrm{cl}_p(F) \subset \mathcal{F}(X)$.

Let $f_n : X \to \mathbb{R}$ be a uniformly bounded sequence of functions on a *set* X. Following Rosenthal we say that this sequence is an l_1-*sequence* on X if there exists a real constant $a > 0$ such that for all $n \in \mathbb{N}$ and all choices of real scalars $c_1, \ldots, c_n$ we have

$$a \cdot \sum_{i=1}^{n} |c_i| \leq || \sum_{i=1}^{n} c_i f_i ||.$$

This is the same as requiring that the closed linear span in $l_\infty(X)$ of the sequence f_n be linearly homeomorphic to the Banach space l_1. In fact, in this case the map

$$l_1 \to l_\infty(X), \ \ (c_n) \to \sum_{n \in \mathbb{N}} c_n f_n$$

is a linear homeomorphic embedding.

The equivalence of (1) and (2) in the following fact is well known. See for example, [68].

Fact 3 ([27, 68]). *Let X be a compact space and $F \subset C(X)$ a bounded subset. The following conditions are equivalent:*

1. *F does not contain a subsequence equivalent to the unit basis of l_1.*
2. *F is a Rosenthal family for X.*
3. *F is an eventually fragmented family.*

We need some known characterizations of Rosenthal spaces.

Fact 4. *Let V be a Banach space. The following conditions are equivalent:*

1. *V is a Rosenthal Banach space.*
2. *(Rosenthal [62]) Every bounded sequence in V has a weak-Cauchy subsequence.*
3. *(Saab and Saab [65]) Each $x^{**} \in V^{**}$ is a fragmented map when restricted to the weak* compact ball B^*. Equivalently, if $B^{**} \subset \mathcal{F}(B^*)$.*
4. *(Haydon [33, Theorem 3.3]) For every weak* compact subset $Y \subset V^*$ the weak* and norm closures of the convex hull $co(Y)$ in V^* coincide: $\overline{co}^{w^*}(Y) = \overline{co}^{\mathrm{norm}}(Y)$.*
5. *B is a Rosenthal family for the weak* compact unit ball B^*.*

Condition (3) is a reformulation (in terms of fragmented maps) of a criterion from [65] which was originally stated in terms of the point of continuity property. (5) can be derived from (3).

Fact 5 (Banach-Grothendieck theorem [4, Corollary 2.6]). *If V is a Banach space then for every continuous linear functional $u : V^* \to \mathbb{R}$ on the dual space V^* the following are equivalent:*

1. *u is w^*-continuous.*
2. *The restriction $u|_{B^*}$ is w^*-continuous.*
3. *u is the evaluation at some point of V. That is, $u \in i(V)$, where $i : V \hookrightarrow V^{**}$ is the canonical embedding.*

Let $\{V_i\}_{i \in I}$ be a family of Banach spaces. The l_2-sum of this family, denoted by $V := (\Sigma_{i \in I} V_i)_{l_2}$, is defined as the space of all functions $(x_i)_{i \in I}$ on I such that $x_i \in V_i$ and

$$||x|| := \left(\sum_{i \in I} ||x_i||^2 \right)^{\frac{1}{2}} < \infty.$$

Lemma 3.

1. *$V^* = (\Sigma_{i \in I} V_i)_{l_2}^* = (\Sigma_{i \in I} V_i^*)_{l_2}$ and the pairing $V \times V^* \to \mathbb{R}$ is defined by $\langle v, f \rangle = \sum_{i \in I} f_i(v_i)$.*
2. *If every V_i is reflexive (Asplund, Rosenthal) then V is reflexive (respectively: Asplund, Rosenthal).*
3. *For every semitopological semigroup S the classes of reflexively (Asplund, Rosenthal) representable compact S-spaces are closed under countable products.*

Proof.

(1) This is well known (see, for example, [56]).
(2) The reflexive case follows easily from (1). For the Asplund case see [56] (or [17] for a simpler proof). Now suppose that each V_i is Rosenthal and $l_1 \subset V = (\Sigma_{i \in I} V_i)_{l_2}$. Since l_1 is separable one may easily reduce the question to the case of countably many Rosenthal spaces V_i. So we can suppose that $V := (\Sigma_{n \in \mathbb{N}} V_n)_{l_2}$. In view of Fact 4 it suffices to show that every element $u = (u_n) \in V^{**}$ is a fragmented map on the weak* compact unit ball B_{V^*}. That is, we need to check that $u \in \mathcal{F}(B_{V^*})$. The set $\mathcal{F}(X) \cap l_\infty(X)$ is a Banach subspace of $l_\infty(X)$ for every topological space X. So the proof can be reduced to the case of coordinate functionals u_{n_0}. Also, $\langle u_{n_0}, (f_n)_{n \in \mathbb{N}} \rangle = f_{n_0}(u_{n_0})$. Now use the fact that u_{n_0} is a fragmented map on $B_{V_{n_0}^*}$ because V_{n_0} is Rosenthal (Fact 4).
(3) Similar to [50, Lemma 3.3] (or [48, Lemma 4.9]) using (2) and the l_2-sum of representations (h_n, α_n) of (S, X_n) on V_n where $||\alpha_n(x)|| \leq 2^{-n}$ for every $x \in X_n$ and $n \in \mathbb{N}$.   □

2.4 S-Compactifications and Functions

A *compactification* of X is a pair (ν, Y) where Y is a compact (Hausdorff, by our assumptions) space and ν is a continuous map with a dense range.

The Gelfand–Kolmogoroff theory [18] establishes an order-preserving bijective correspondence (up to equivalence of compactifications) between Banach *unital* subalgebras $\mathcal{A} \subset C(X)$ and compactifications $\nu : X \to Y$ of X. Every Banach unital S-subalgebra $\mathcal{A}$ induces the *canonical $\mathcal{A}$-compactification* $\alpha_{\mathcal{A}} : X \to X^{\mathcal{A}}$, where $X^{\mathcal{A}}$ is the spectrum (or the Gelfand space—the collection of continuous multiplicative functionals on $\mathcal{A}$). The map $\alpha_{\mathcal{A}} : X \to X^{\mathcal{A}} \subset \mathcal{A}^*$ is defined by the *Gelfand transform*, the evaluation at x functional, $\alpha_{\mathcal{A}}(x)(f) := f(x)$. Conversely, every compactification $\nu : X \to Y$ is equivalent to the *canonical $\mathcal{A}_{\nu}$-compactification* $\alpha_{\mathcal{A}_{\nu}} : X \to X^{\mathcal{A}_{\nu}}$, where the algebra $\mathcal{A}_{\nu}$ is defined as the image $j_{\nu}(C(Y))$ of the embedding $j_{\nu} : C(Y) \hookrightarrow C(X), \ \phi \mapsto \phi \circ \nu$.

Definition 6. Let X be an S-system. An S-compactification of X is a continuous S-map $\alpha : X \to Y$, with a dense range, into a compact S-system Y. An S-compactification is said to be *jointly continuous* (respectively, *separately continuous*) if the action $S \times Y \to Y$ is jointly continuous (respectively, separately continuous).

By S_d we denote the discrete copy of S.

Remark 2. If $\nu_1 : X \to Y_1$ and $\nu_2 : X \to Y_2$ are two compactifications, then ν_2 *dominates* ν_1, that is, $\nu_1 = q \circ \nu_2$ for some (uniquely defined) continuous map $q : Y_2 \to Y_1$ iff $\mathcal{A}_{\nu_1} \subset \mathcal{A}_{\nu_2}$. If in addition, X, Y_1, and Y_2 are S_d-systems (i.e., all the s-translations on X, Y_1, and Y_2 are continuous) and if ν_1 and ν_2 are S-maps, then q is also an S-map. Furthermore, if the action on Y_1 is (separately) continuous then the action on Y_2 is (respectively, separately) continuous. If ν_1 and ν_2 are homomorphisms of semigroups then q is also a homomorphism. See [72, Appendix D].

2.5 From Representations to Compactifications

Representations of dynamical systems (S, X) lead to S-compactifications of X. Let V be a normed space and let

$$(h, \alpha) : (S, X) \rightrightarrows (\Theta(V)^{\mathrm{op}}, V^*)$$

be a representation of (S, X), where α is a weak* continuous map. Consider the induced compactification $\alpha : X \to Y := \overline{\alpha(X)}$, the weak* closure of $\alpha(X)$. Clearly, the induced natural action $S \times Y \to Y$ is well defined and every left translation is continuous. So, Y is an S_d-system.

Remark 3.

1. The induced action $S \times Y \to Y$ is separately continuous iff the matrix coefficient $m(v, y) : S \to \mathbb{R}$ is continuous $\forall\, v \in V,\; y \in Y$.
2. If h is strongly (weakly) continuous then the induced dual action of S on the weak* compact unit ball B^* and on Y is jointly (respectively, separately) continuous.

To every S-space X we associate the *regular representation* on the Banach space $V := C(X)$ defined by the pair (h, α) where $h : S \to \Theta(V), s \mapsto L_s$ (with $L_s f(x) = f(sx)$) is the natural co-homomorphism and $\alpha : X \to V^*, x \mapsto \delta_x$ is the evaluation map $\delta_x(f) = f(x)$. Denote by (WRUC(X)) RUC(X) the set of all (weakly) right uniformly continuous functions. That is functions $f \in C(X)$ such that the orbit map $\check{f} : S \to C(X), s \mapsto fs = L_s(f)$ is (weakly) norm continuous. Then RUC(X) and WRUC(X) are norm-closed S-invariant unital linear subspaces of $C(X)$ and the restriction of the regular representation is continuous on RUC(X) and weakly continuous on WRUC(X). Furthermore, RUC(X) is a Banach subalgebra of $C(X)$. If $S \times X \to X$ is continuous and X is compact then $C(X) = $ RUC(X). In particular, for the left action of S on itself $X := S$ we write simply RUC(S) and WRUC(S). If $X := G$ is a topological group with the left action on itself then RUC(G) is the usual algebra of right uniformly continuous functions on G. Note that WRUC(S) plays a major role in the theory of semigroups being the largest left introverted linear subspace of $C(S)$ (Rao's theorem; see for example, [6]).

We say that a function $f \in C(X)$ on an S-space X *comes from* an S-compactification $\nu : X \to Y$ (recall that we require only that the actions on X, Y are separately continuous) if there exists $\tilde{f} \in C(Y)$ such that $f = \tilde{f} \circ \nu$. Denote by RMC(X) the set (in fact a unital Banach algebra) of all functions on X which come from S-compactifications. The algebra RUC(X) is the set of all functions which come from jointly continuous S-compactifications.

Remark 4. Let X be an S-system.

1. For every S-invariant normed subspace V of WRUC(X) we have the regular weakly continuous V-representation (h, α) of (S, X) on V defined by $\alpha(x)(f) = f(x), \; f \in V$, and the corresponding S-compactification $\alpha : X \to Y := \overline{\alpha(X)}$. The action of S on Y is continuous iff $V \subset $ RUC(X).
2. Let (h, α) be a representation of the S-system X on a Banach space V. The inclusion $\alpha(X) \subset V^*$ induces a *restriction operator*

$$r : V \to C(X), \; r(v)(x) = \langle v, \alpha(x) \rangle.$$

Then r is a linear S-operator (between right actions) with $||r|| \leq 1$. If h is weakly (strongly) continuous then $r(V) \subset $ WRUC(X) (respectively, $r(V) \subset $ RUC(X)).
3. For every topological space X the classical order-preserving Gelfand–Kolmogoroff correspondence between compactifications of X and unital subalgebras has a natural S-space generalization. More precisely, if X is an

S-space, then S-invariant unital Banach subalgebras F of $\mathrm{RUC}(X)$ (respectively, $\mathrm{RMC}(X)$) control the S-compactifications $X \rightarrow Y$ with (respectively, separately) continuous actions $S \times Y \rightarrow Y$.

The correspondence described in point 3 of Remark 4 for Banach subalgebras F of $\mathrm{RUC}(X)$ is well known for topological group actions, [71]. One can easily extend it to the case of topological semigroup actions [4, 49]. Compare this also to the description of jointly continuous *affine S-compactifications* (Sect. 4) in terms of S-invariant closed linear unital subspaces of $\mathrm{RUC}(X)$.

Regarding a description of separately continuous S-compactifications via subalgebras of $\mathrm{RMC}(X)$ and for more details about Remarks 3 and 4 see, for example, [48, 49] and, also, Remark 11 below.

A word of caution about our notation of $\mathrm{WRUC}(S), \mathrm{RUC}(S), \mathrm{RMC}(S)$. Note that in [6] the corresponding notation is $\mathrm{WLUC}(S), \mathrm{LUC}(S), \mathrm{LMC}(S)$ (and sometimes $\mathrm{WLC}(S), \mathrm{LC}(S)$, [7]).

Remark 5. Let $\mathcal{P}$ be a class of compact separately continuous S-dynamical systems. The subclass of S-systems with continuous actions will be denoted by $\mathcal{P}_c$. Assume that $\mathcal{P}$ is closed under products, closed subsystems, and S-isomorphisms. In such cases (following [72, Chap. IV]) we say that $\mathcal{P}$ is *suppable*. Let X be a not necessarily compact S-space and let $\mathcal{P}(X)$ be the collection of functions on X coming from systems having property $\mathcal{P}$. Then, as in the case of jointly continuous actions (see [24, Proposition 2.9]), there exists a universal S-compactification $X \rightarrow X^{\mathcal{P}}$ of X such that $(S, X) \in \mathcal{P}$. Moreover, $j(C(X^{\mathcal{P}})) = \mathcal{P}(X)$. In particular, $\mathcal{P}(X)$ is a uniformly closed, S-invariant subalgebra of $C(X)$. Analogously, one defines $\mathcal{P}_c(X)$. Again it is a uniformly closed, S-invariant subalgebra of $C(X)$, which is in fact a subalgebra of $\mathrm{RUC}(X)$. For the corresponding S-compactification $X \rightarrow X^{\mathcal{P}_c}$ the action of S on $X^{\mathcal{P}_c}$ is continuous.

In particular, for the left action of S on itself we get the definitions of $\mathcal{P}(S)$ and $\mathcal{P}_c(S)$. As in [24, Proposition 2.9] one may show that $\mathcal{P}(S)$ and $\mathcal{P}_c(S)$ are m-introverted Banach subalgebras of $C(S)$ and they define the $\mathcal{P}$-universal and $\mathcal{P}_c$-universal semigroup compactifications $S \rightarrow S^{\mathcal{P}}$ and $S \rightarrow S^{\mathcal{P}_c}$.

In the present paper we are especially interested in the following classes of compact S-systems:

(a) Tame systems (Definition 18).
(b) Hereditarily NonSensitive, HNS in short (Definition 17).
(c) Weakly almost periodic, WAP in short (Sect. 7.2).

See Sects. 7.2–7.4 and also [24, 25, 27].

For the corresponding algebras, defined by Remark 5, we use the following notation: $\mathrm{Tame}(X), \mathrm{Asp}(X), \mathrm{WAP}(X)$. Note that the Tame (respectively, HNS, WAP) systems are exactly the compact systems which admit sufficiently many representations on Rosenthal (respectively, Asplund, reflexive) Banach spaces (Sect. 8).

Lemma 4.

1. *For every S-space X we have $\mathcal{P}_c(X) \subset \mathrm{RUC}(X) \subset \mathrm{WRUC}(X) \subset \mathrm{RMC}(X)$ and $\mathcal{P}_c(X) \subset \mathcal{P}(X) \cap \mathrm{RUC}(X)$. If $\mathcal{P}$ is preserved by factors then $\mathcal{P}_c(X) = \mathcal{P}(X) \cap \mathrm{RUC}(X)$.*
2. *If X is a compact S-system with continuous action then $\mathcal{P}_c(X) = \mathcal{P}(X)$, $\mathrm{RUC}(X) = \mathrm{WRUC}(X) = \mathrm{RMC}(X) = C(X)$.*
3. *If $S = G$ is a Čech-complete semitopological group then for every G-space X we have $\mathcal{P}_c(X) = \mathcal{P}(X)$, $\mathrm{RUC}(X) = \mathrm{WRUC}(X) = \mathrm{RMC}(X)$; in particular, $\mathrm{RUC}(G) = \mathrm{WRUC}(G) = \mathrm{RMC}(G)$.*
4. $\mathrm{WAP}_c(G) = \mathrm{WAP}(G)$ *remains true for every semitopological group G.*
5. *[7, p. 173] If S is a k-space as a topological space then $\mathrm{WRUC}(X) = \mathrm{RMC}(X)$.*

Proof.

(1) Is straightforward. In order to check the less obvious part $\mathcal{P}_c(X) \supset \mathcal{P}(X) \cap \mathrm{RUC}(X)$ we use a fundamental property of cyclic compactifications (see point 1 of Remark 11).
(2) Easily follows from (1). (3) follows from Fact 1, and (4) from Fact 11. (5) is a generalized version of [7, Theorem 5.6] and easily follows from Grothendieck's Lemma [7, Corollary A6]. □

Definition 7. Let X be a compact space with a separately continuous action $\pi : S \times X \to X$. We say that X is WRUC-*compatible* (or that X is WRUC) if $C(X) = \mathrm{WRUC}(X)$. An equivalent condition is that the induced action $\pi_P : S \times P(X) \to P(X)$ be separately continuous (Lemma 13).

Remark 6. We mention three useful sufficient conditions for being WRUC-compatible (compare [48, Definition 7.6] where this concept appears under the name *w-admissible*): (a) the action $S \times X \to X$ is continuous; (b) S, as a topological space, is a k-space (e.g., metrizable); (c) (S, X) is WAP. Below in Proposition 14 we show that $\mathrm{Tame}(X) \subset \mathrm{WRUC}(X)$ for every S-space X. In particular, it follows that every *compact* tame (hence, every WAP) S-system is WRUC-compatible.

2.6   Semigroup Compactifications

Definition 8. Let S be a semitopological semigroup.

1. [7, p. 105] A *right topological semigroup compactification* of S is a pair (γ, T) such that T is a compact right topological semigroup, and γ is a continuous semigroup homomorphism from S into T, where $\gamma(S)$ is dense in T and the left translation $\lambda_s : T \to T$, $x \mapsto \gamma(s)x$ is continuous for every $s \in S$, that is, $\gamma(S) \subset \Lambda(T)$.

    It follows that the *associated action*

$$\pi_\gamma : S \times T \to T, \ (s, x) \mapsto \gamma(s)x = \lambda_s(x)$$

is separately continuous.

2. [64, p. 101] A *dynamical right topological semigroup compactification* of S is a right topological semigroup compactification (γ, T) in the sense of (1) such that, in addition, γ is a jointly continuous S-compactification, i.e., the action $\pi_\gamma : S \times T \to T$ is jointly continuous.

If S is a monoid (as we require in the present paper) with the neutral element e then it is easy to show that necessarily T is a monoid with the neutral element $\gamma(e)$. For a discrete semigroup S, (1) and (2) are equivalent. Directly from Lawson's theorem mentioned above (Fact 1) we have:

Fact 6. *Let G be a Čech-complete (e.g., locally compact or completely metrizable) semitopological group. Then $\gamma : G \to T$ is a right topological semigroup compactification of G if and only if γ is a dynamical right topological semigroup compactification of G.*

For every semitopological semigroup S there exists a maximal right topological (dynamical) semigroup compactification. The corresponding algebra is $\mathrm{RMC}(S)$ (respectively, $\mathrm{RUC}(S)$). If in the definition of a semigroup compactification (γ, T) we remove the condition $\gamma(S) \subset \Lambda(T)$ then maximal compactifications (in this setting) need not exist (see [6, Example V.1.11] which is due to J. Baker).

Let $\mathcal{A}$ be a closed unital subalgebra of $C(X)$ for some topological space X. We let $\nu_{\mathcal{A}} : X \to X^{\mathcal{A}}$ be the associated compactification map (where, as before, $X^{\mathcal{A}}$ is the maximal ideal space of $\mathcal{A}$). For instance, the *greatest ambit* (see, for example, [70, 72]) of a topological group G is the compact G-space $G^{\mathrm{RUC}} := G^{\mathrm{RUC}(G)}$. It defines the *universal dynamical semigroup compactification* of G. For $\mathcal{A} = \mathrm{WAP}(G)$ we get the *universal semitopological compactification $G \to G^{\mathrm{WAP}}$* of G, which is the universal WAP compactification of G (see [15]). Note that by [51] the projection $q : G^{\mathrm{RUC}} \to G^{\mathrm{WAP}}$ is a homeomorphism iff G is precompact.

Remark 7.

1. Recall that $\mathrm{RUC}(G)$ generates the topology of G for every topological group G. It follows that the corresponding canonical representation (*Teleman's representation*)

$$(h, \alpha_{\mathrm{RUC}}) : (G, G) \rightrightarrows (\Theta(V)^{\mathrm{op}}, B^*)$$

on $V := \mathrm{RUC}(G)$ is faithful and h induces a topological group embedding of G into $\mathrm{Iso}\,(V)$. See [58] for details.
2. There exists a nontrivial Polish group G whose universal semitopological compactification G^{WAP} is trivial. This is shown in [46] for the Polish group $G := H_+[0, 1]$ of orientation preserving homeomorphisms of the unit interval. Equivalently: every (weakly) continuous representation $G \to \mathrm{Iso}\,(V)$ of G on a reflexive Banach space V is trivial.
3. A stronger result is shown in [25]: every continuous representation $G \to \mathrm{Iso}\,(V)$ of G on an Asplund space V is trivial and every Asplund function on G is constant (note that $\mathrm{Asp}_c(G) = \mathrm{Asp}(G)$ for Polish G by Lemma 4 of point 3).

Every nontrivial right topological semigroup compactification of the Polish topological group $G := H_+[0, 1]$ is not metrizable [29]. In contrast we show in Theorem 15 that G is Rosenthal representable.

2.7 Enveloping Semigroups

Let X be a compact S-system with a separately continuous action. Consider the natural map $j : S \to C(X, X), s \mapsto \lambda_s$. As usual denote by $E(X) = \mathrm{cl}_p(j(S)) \subset X^X$ the enveloping (Ellis) semigroup of (S, X). The associated homomorphism $j : S \to E(X)$ is a right topological semigroup compactification (say, *Ellis compactification*) of S, $j(e) = id_X$ and the associated action $\pi_j : S \times E(X) \to E(X)$ is separately continuous. Furthermore, if the S-action on X is continuous then π_j is continuous, i.e., $S \to E(X)$ is a dynamical semigroup compactification.

Lemma 5.

1. *Let X be a compact semitopological S-space and L a subset of $C(X)$ such that L separates points of X. Then the Ellis compactification $j : S \to E(X)$ is equivalent to the compactification of S which corresponds to the subalgebra $\mathcal{A}_L := \langle m(L, X) \rangle$, the smallest norm-closed S-invariant unital subalgebra of $C(S)$ which contains the family*

$$\{m(f, x) : S \to \mathbb{R}, \ s \mapsto f(sx)\}_{f \in L, \ x \in X}.$$

2. *Let $q : X_1 \to X_2$ be a continuous onto S-map between compact S-spaces. There exists a (unique) continuous onto semigroup homomorphism $Q : E(X_1) \to E(X_2)$ with $j_{X_1} \circ Q = j_{X_2}$.*
3. *Let Y be a closed S-subspace of a compact S-system X. The map $r_X : E(X) \to E(Y)$, $p \mapsto p|_Y$ is the unique continuous onto semigroup homomorphism such that $r_X \circ j_X = j_Y$.*
4. *Let $\alpha : S \to P$ be a right topological compactification of a semigroup S. Then the enveloping semigroup $E(S, P)$ of the semitopological system (S, P) is naturally isomorphic to P.*
5. *If X is metrizable then $E(X)$ is separable. Moreover, $j(S) \subset E(X)$ is separable.*

Proof.

(1) The proof is straightforward using the Stone-Weierstrass theorem.
(2) By Remark 2 it suffices to show that the compactification $j_{X_1} : S \to E(X_1)$ dominates the compactification $j_{X_2} : S \to E(X_2)$. Equivalently we have to verify the inclusion of the corresponding algebras. Let $q(x) = y$, $f_0 \in C(X_2)$ and $f = f_0 \circ q$. Observe that $m(f_0, y) = m(f, x)$ and use (1).
(3) Is similar to (2).

(4) Since $E(S, P) \to P, a \mapsto a(e)$ is a natural homomorphism, $j_P : S \to E(S, P)$ dominates the compactification $S \to P$. So it is enough to show that, conversely, $\alpha : S \to P$ dominates $j_P : S \to E(S, P)$. By (1) the family of functions

$$\{m(f, x) : S \to \mathbb{R}\}_{f \in C(P), \, x \in P}$$

generates the Ellis compactification $j_P : S \to E(S, P)$. Now observe that each $m(f, x) : S \to \mathbb{R}$ can be extended naturally to the function $P \to \mathbb{R}, \ p \mapsto f(px)$ which is *continuous*.

(5) Since X is a metrizable compactum, $C(X, X)$ is separable and metrizable in the compact open topology. Then $j(S) \subset C(X, X)$ is separable (and metrizable) in the same topology. Hence, the dense subset $j(S) \subset E(X)$ is separable in the pointwise topology. This implies that $E(X)$ is separable. $\qquad \square$

Remark 8. Every enveloping semigroup $E(S, X)$ is an example of a compact right topological admissible semigroup. Conversely, every compact right topological *admissible* semigroup P is an enveloping semigroup (of $(\Lambda(P), P)$, as it follows from point 4 of Lemma 5).

3 Operator Compactifications

Operator compactifications provide an important tool for constructing and studying semigroup compactifications via representations of semigroups on Banach spaces (or, more generally, on locally convex vector spaces). In classical works by Eberlein, de Leeuw, and Glicksberg, it was shown that WAP Banach representations of a semigroup S induce semitopological compactifications of S. In general the situation is more complicated and we have to deal with right topological semigroup compactifications of a semigroup S. We refer to the papers of Witz [74] and Junghenn [39]. We note also that in his book [16] R. Ellis builds his entire theory of abstract topological dynamics using the language of operator representations.

First we reproduce the construction of Witz with some minor changes. Let $h : S \to L(V)$ be a weakly continuous, equicontinuous representation (co-homomorphism) of a semitopological semigroup S into the space $L(V)$ of continuous linear operators on a locally convex vector space V. "Equicontinuous" here means that the subset $h(S) \subset L(V)$ is an equicontinuous family of linear operators. Then the weak* operator closure $\overline{h(S)^{\mathrm{op}}}$ of the adjoint semigroup $h(S)^{\mathrm{op}} \subset L(V)^{\mathrm{op}} = \mathrm{adj}(L(V))$ in $L(V^*)$ is a right topological semigroup compactification of S. We obtain the compactification

$$S \to P := \overline{h(S)^{\mathrm{op}}} \subset L(V^*)$$

which, following Junghenn, we call an *operator compactification* of S (induced by the representation h). The weak* operator topology on $L(V^*)$ is the weakest topology generated by the system

$$\widetilde{m}(v, \psi) : L(V^*) \to \mathbb{R}, \ p \mapsto \langle v, p\psi \rangle = \langle vp, \psi \rangle$$

of maps, where $v \in V, \psi \in V^*, vp = p^*(v) \in V^{**}$, and $p^* : V^{**} \to V^{**}$ is the adjoint of p.

In fact, the semigroup P can be treated also as the weak* operator closure of $h(S)$ in $L(V, V^{**})$. The latter version is found mainly in [39, 74].

The *coefficient algebra A_h* (respectively, *coefficient space M_h*) of the representation $h : S \to \Theta(V)$ is the smallest norm-closed, unital subalgebra (respectively, subspace) of $C(S)$ containing all the matrix coefficients of h

$$m(V, V^*) = \{m(v, \psi) : S \to \mathbb{R}, \ s \mapsto \langle vh(s), \psi \rangle | \ v \in V, \ \psi \in V^*\}.$$

That is, according to our notation

$$A_h = \langle m(V, V^*) \rangle \text{ and } M_h = \overline{sp}^{\text{norm}}(m(V, V^*) \cup \{\mathbf{1}\}).$$

Lemma 6. *Let $S \to P := \overline{h(S)^{\text{op}}} \subset L(V^*)$ be the operator compactification induced by a weakly continuous equicontinuous representation $h : S \to L(V)$ on a locally convex space V. The algebra of this compactification is just the coefficient algebra A_h.*

Proof. For every $(v, \psi) \in V \times V^*$ the function $m(v, \psi) : S \to \mathbb{R}$ is a restriction of the continuous map $\widetilde{m}(v, \psi)|_P : P \to \mathbb{R}, \ p \mapsto \langle v, p\psi \rangle = \psi(vp)$. Such maps separate points of P. Now use the Stone-Weierstrass theorem. $\qquad\qquad\square$

3.1 The Enveloping Semigroup of a Banach Space

Let V be a Banach space and $\Theta(V)$ the semigroup of all non-expanding operators from V to itself. As in Sect. 2.2 consider the natural left action of $\Theta(V)^{\text{op}}$ on the weak* compact unit ball B^*. This action is separately continuous when $\Theta(V)^{\text{op}}$ carries the weak operator topology.

Definition 9. Given a Banach space V we denote by $\mathcal{E}(V)$ the enveloping semigroup of the dynamical system $(\Theta(V)^{\text{op}}, B^*)$. We say that $\mathcal{E}(V)$ is the *enveloping semigroup of V*.

Always, $\mathcal{E}(V)$ is a compact right topological admissible affine semigroup. The corresponding Ellis compactification $j : \Theta(V)^{\text{op}} \to \mathcal{E}(V)$ is a topological embedding. Alternatively, $\mathcal{E}(V)$ can be defined as the weak* operator closure of the adjoint monoid $\Theta(V)^{\text{op}}$ in $L(V^*)$ (point 2 of Lemma 7). So it is the operator compactification of the semigroup $\Theta(V)$.

If V is separable then $\mathcal{E}(V)$ is separable by Lemma 5 of point 5 because B^* is metrizable. $\mathcal{E}(V)$ is metrizable iff V is separable Asplund, Theorem 6.

Every weakly continuous representation $h : S \to \Theta(V)$ of a semitopological semigroup on a Banach space V (by non-expanding operators) gives rise to a right topological semigroup compactification

$$h : S \to \overline{h(S)^{\mathrm{op}}} \subset \mathcal{E}(V),$$

where $\overline{h(S)^{\mathrm{op}}}$ is the closure in $\mathcal{E}(V)$. We sometimes call it the *standard operator compactification* of S (generated by the representation h).

Definition 10. Let $\alpha : P \to K$ be a continuous (not necessarily onto) homomorphism between compact right topological admissible semigroups. Suppose that S is a dense subsemigroup of $\Lambda(P)$. We say that:

1. α is *S-admissible* if $\alpha(S) \subset \Lambda(K)$.
2. α is *admissible* if it is *S*-admissible with respect to some dense subsemigroup $S \subset \Lambda(P)$.
3. P is *representable* on a Banach space V if there exists an admissible embedding α of P into $\mathcal{E}(V)$. If V is Rosenthal (Asplund, reflexive) then we say that P is Rosenthal (Asplund, reflexively) representable.

Every standard operator compactification generated by a representation h of S on V induces an admissible embedding of $\overline{h(S)^{\mathrm{op}}}$ into $\mathcal{E}(V)$ because, $h(S) \subset \Theta(V)^{\mathrm{op}}$ (and $\Theta(V)^{\mathrm{op}} = \Lambda(\mathcal{E}(V))$, Lemma 8 of point 5). In the next lemma, as before, given a subset $A \subset C(S)$, we let $\langle A \rangle$ denote the closed unital subalgebra of $C(S)$ generated by A.

Lemma 7.

1. *Every standard operator compactification $h : S \to \overline{h(S)^{\mathrm{op}}} \subset \mathcal{E}(V)$ is equivalent to the Ellis compactification $j : S \to E = E(S, B^*)$. The algebra of these compactifications is the coefficient algebra $A_h = \langle m(V, V^*) \rangle$.*
2. *$\overline{\Theta(V)^{\mathrm{op}}}$ is isomorphic to $\mathcal{E}(V)$ and the algebra of the compactification $j : \Theta(V)^{\mathrm{op}} \hookrightarrow \mathcal{E}(V)$ is the coefficient algebra A_h for $h : \Theta(V) \hookrightarrow \overline{\Theta(V)^{\mathrm{op}}} = \mathcal{E}(V)$.*
3. *The natural inclusion $\alpha : E(S, B^*) \hookrightarrow \mathcal{E}(V)$ is $j(S)$-admissible.*

Proof. (1) Both of these compactifications have the same algebra A_h. Indeed Lemma 6 implies this for the compactification $h : S \to \overline{h(S)^{\mathrm{op}}} \subset \mathcal{E}(V)$. For $j : S \to E(S, B^*)$ use point 1 of Lemma 5.

Note that (2) is a particular case of (1) for $S = \Theta(V)$.

(3) is trivial because $j(S)$ is dense in $\Lambda(E(S, B^*))$ and $\alpha(j(S)) = h(S) \subset \Theta(V)^{\mathrm{op}}$. $\qquad\square$

Proposition 1. *Every semigroup compactification is a factor of an operator semigroup compactification.*

Proof. Let (γ, P) be a semigroup compactification of S. Take a faithful Banach representation of the S-flow P on V. For example, one can take the regular representation of P on $V := C(P)$. Now the enveloping semigroup $E(S, P)$

is a factor of $E(S, B^*)$ which is an operator semigroup compactification of S (Lemma 7) and $E(S, P)$ is naturally isomorphic to P (point 4 of Lemma 5). □

In Example 3 we show that there exists a right topological semigroup compactifications of the group $\mathbb{Z}$, which is not an operator compactifications. It follows that compact right topological *operator* semigroups are not closed under factors. Indeed the compactification $\mathbb{Z} \to \beta\mathbb{Z} = \mathbb{Z}^{\mathrm{RUC}}$ is an operator compactification (by Remarks 14 of point 1 below) and it is the universal $\mathbb{Z}$-ambit.

Not every admissible compact right topological (semi)group admits a representation on a Banach space (see Theorem 3). On the other hand we will later investigate the question when a "good" semigroup compactification can be realized as a standard operator compactification on "good" Banach spaces (see Sect. 9).

In the sequel whenever V is understood we use the following simple notations $\mathcal{E} := \mathcal{E}(V)$, $\Theta := \Theta(V)$, $\Theta^{\mathrm{op}} := \Theta(V)^{\mathrm{op}}$. By S_V we denote the unit sphere of V.

Lemma 8. *For every Banach space V, every $v \in S_V$ and $\psi \in S_{V^*}$ we have*

1. $\Theta v = B$
2. $v\mathcal{E} = B^{**}$
3. $\mathrm{cl}_{w^*}(\Theta^{\mathrm{op}}\psi) = B^*$
4. $\mathcal{E}\psi = B^*$
5. $\Lambda(\mathcal{E}) = \Theta^{\mathrm{op}}$

Proof.

(1) Take $f \in S_{V^*}$ such that $f(v) = 1$. For every $z \in B$ define the rank 1 operator

$$A(f, z) : V \to V, \quad x \mapsto f(x)z.$$

    Then $A(f, z)(v) = z$ and $A(f, z) \in \Theta$ since $||A(f, z)|| = ||f|| \cdot ||z|| = ||z|| \leq 1$.

(2) By (1), $v\Theta^{\mathrm{op}} = \Theta v = B$ which is pointwise dense in B^{**} by Goldstine theorem. So, $v\mathcal{E} = B^{**}$ because $\mathcal{E} \to (V^{**}, w^*)$, $p \mapsto vp$ is continuous, and $\mathcal{E} = \overline{\Theta^{\mathrm{op}}}$.

(3) We can suppose that V is infinite-dimensional (use (1) for the finite-dimensional case). Then the unit sphere S_{V^*} is norm (hence, weak*) dense in B^*. So it is enough to prove that the weak* closure of $\Theta^{\mathrm{op}}\psi$ contains S_{V^*}. Let $\phi \in S_{V^*}$. We have to show that for every $\varepsilon > 0$ and $v_1, v_2, \ldots, v_n \in V$ there exists $s \in \Theta$ such that $||s^*\psi(v_i) - \phi(v_i)|| < \varepsilon$ for every $i = 1, 2, \ldots, n$, where $s^* \in \Theta^{\mathrm{op}}$ is the adjoint of the operator s. Since $\psi \in V^*$ and $||\psi|| = 1$ one may choose $z \in B_V$ such that

$$|\phi(v_i)(\psi(z) - 1)| < \varepsilon$$

for every $i = 1, 2, \ldots, n$. Define $s := A(\phi, z)$. Then

$$|(s^*\psi)(v_i) - \phi(v_i)| = |\psi(sv_i) - \phi(v_i)| = |\psi(\phi(v_i)z) - \phi(v_i)|$$
$$= |\phi(v_i)(\psi(z) - 1)| < \varepsilon$$

for every i.

(4) Follows from (3) because $\mathcal{E}$ is the weak* operator closure of Θ^{op}.

(5) Trivially, $\Lambda(\mathcal{E}) \supseteq \Theta^{\mathrm{op}}$. Conversely, let $\sigma \in \Lambda(\mathcal{E})$. Then $\sigma \in L(V^*)$ with $||\sigma|| \leq 1$. Consider the adjoint operator $\sigma^* : V^{**} \to V^{**}$. We have to show that $\sigma^*(v) \in V \subset V^{**}$, for every $v \in \Lambda$, where we treat V as a Banach subspace of V^{**}. By Fact 5 it is enough to show that $\sigma^*(v)|_{B^*} : B^* \to \mathbb{R}$ is w*-continuous. By our assumption, $\sigma \in \Lambda(\mathcal{E})$. That is, the left translation $l_\sigma : \mathcal{E} \to \mathcal{E}$ is continuous. Choose a point $z \in S_{V^*}$ and consider the orbit map $\tilde{z} : \mathcal{E} \to B^*$, $p \mapsto pz$. Then, $\tilde{z} \circ l_\sigma = \sigma|_{B^*} \circ \tilde{z}$. By (4) we have $\mathcal{E}z = B^*$; hence, $\tilde{z} : \mathcal{E} \to B^*$ is onto. Since $\mathcal{E}$ is compact, it follows that the map $\sigma|_{B^*} : B^* \to B^*$ is continuous. This implies that $\sigma^*(v)|_{B^*} : B^* \to \mathbb{R}$ is w*-continuous (for any $v \in V$), as desired. $\qquad\square$

4 Affine Compactifications of Dynamical Systems and Introversion

4.1 Affine Compactifications in Terms of State Spaces

Let S be a semitopological semigroup. An S-system Q is an *affine S-system* if Q is a convex subset of a locally convex vector space and each $\lambda_s : Q \to Q$ is affine. If in addition $S = Q$ acts on itself by left translations and if right translations are also affine maps then S is said to be an *affine semigroup*. For every compact affine S-system Q each element of its enveloping semigroup is a (not necessarily, continuous) affine self-map of Q.

Definition 11.

1. [7, p. 123] An *affine semigroup compactification* of a semitopological semigroup S is a pair (ψ, Q), where Q is a compact right topological affine semigroup and $\psi : S \to Q$ is a continuous homomorphism such that $co(\psi(S))$ is dense in Q and $\psi(S) \subset \Lambda(Q)$.

2. By an *affine S-compactification* of an S-space X we mean a pair (α, Q), where $\alpha : X \to Q$ is a continuous S-map and Q is a convex compact affine S-flow such that $\alpha(X)$ *affinely generates Q*, that is, $\overline{co}(\alpha(X)) = Q$ (see [19]).

3. In particular, for a trivial action (or for the trivial semigroup S) we retrieve in (2) the notion of an *affine compactification* of a topological space X.

An affine S-compactification $\alpha : X \to Q$ induces the S-compactification $\alpha : X \to Y := \overline{\alpha(X)} \subset Q$ of X. Of course we have $Y = \overline{\alpha(X)} = \alpha(X)$ when X is compact. Definition 11 of point 2 is a natural extension of Definition 11 of point 1.

Remark 9.

1. For any Banach space V, Θ is an affine semitopological semigroup, $(\Theta^{\mathrm{op}}, B^*)$ is an affine system and the inclusion $\Theta^{\mathrm{op}} \hookrightarrow \mathcal{E}$ is an affine semigroup compactification.

2. Not every semigroup compactification (in contrast to affine semigroup compactifications) comes as an operator compactification. See Theorem 3 (and Proposition 5) below.
3. For every continuous compact S-system X, the weak* compact unit ball $B^* \subset C(X)^*$ and its closed subset $P(X)$ of all probability measures, are continuous affine S-systems (point 2 of Proposition 2).
4. Every Banach representation (h, α) of an S-flow X naturally induces an S-affine compactification $X \to Q := \overline{co}\ \alpha(X)$ (Sect. 5). Conversely, every affine compactification of an S-space X comes from a Banach representation of the S-space X on the Banach space $V \subset C(S)$ which is just the *affine compactification space* (see Lemmas 12 and 13).

As in the case of compactifications of flows one defines notions of preorder, factors, and isomorphisms of affine compactifications. More explicitly, we say that for two affine compactifications, $\alpha_1 : X \to Q_1$ *dominates* $\alpha_2 : X \to Q_2$ if there exists a continuous affine map (a *morphism*) $q : Q_1 \to Q_2$ such that $q \circ \alpha_1 = \alpha_2$. Notation: $\alpha_1 \succeq \alpha_2$. If one may choose q to be a homeomorphism then we say that q is an *isomorphism* of affine compactifications. Notation: $\alpha_2 \cong \alpha_1$. It is easy to see that $\alpha_2 \cong \alpha_1$ iff $\alpha_2 \succeq \alpha_1$ and $\alpha_1 \succeq \alpha_2$.

Lemma 9. *If* $q : Q_1 \to Q_2$ *is a morphism between two* S_d-*affine compactifications* $\alpha_1 : X \to Q_1$ *and* $\alpha_2 : X \to Q_1$ *then* q *is an* S-*map.*

Proof. Since the s-translations in Q_1 and Q_2 are affine it easily follows that the inclusion maps $co(\alpha_1(X)) \hookrightarrow Q_1$, $co(\alpha_2(X)) \hookrightarrow Q_2$ are S_d-compactifications and also the restriction map $q : co(\alpha_1(X)) \to co(\alpha_2(X))$ is an onto S-map. The induced map $co(\alpha_1(X)) \to Q_2$ defines an S_d-compactification. Now Remark 2 yields that $q : Q_1 \to Q_2$ is an S-map. $\square$

Recall that for a normed unital subspace F of $C(X)$ the *state space* of F is the w^*-compact subset

$$M(F) := \{\mu \in F^* : \|\mu\| = \mu(\mathbf{1}) = 1\}$$

of all *means* on F. If in addition $F \subset C(X)$ is a subalgebra, we denote by $MM(F)$ the compact set of all multiplicative means on F. For a compact space X and for $F = C(X)$ the state space $M(C(X))$ is the space of all probability measures on X which we denote as usual by $P(X)$.

Lemma 10 ([6, 7, 59]). *For every topological space X we have:*

1. *State space $M(F)$ is convex and weak* compact in the dual F^* of F.*
2. *The map $\delta : X \to M(F)$, $\delta(x)(f) = f(x)$, is affine and weak*-continuous, and its image $\delta(X)$ affinely generates $M(F)$ (i.e., $\overline{co}^{w^*}(\delta(X)) = M(F)$).*
3. *Every $\mu \in F^*$ is a finite linear combination of members of $M(F)$.*
4. *If $F \subset C(X)$ is a subalgebra then $\delta(X)$ is dense in $MM(F)$.*

Thus $\delta : X \to M(F)$ is an affine compactification of X. We call it the *canonical F-affine compactification* of X. The induced compactification $\delta : X \to \overline{\delta(X)} = Y$

is said to be the *canonical F-compactification* of X. By Stone-Weierstrass theorem it follows that $C(Y)$ is naturally isometrically isomorphic to $\mathcal{A}_\delta := \langle F \rangle$, the closed unital subalgebra of $C(X)$ generated by F.

For every compact convex subset Q of a locally convex vector space V we denote by A(Q) the Banach unital subspace of $C(Q)$ consisting of the affine continuous functions on Q. Of course $f|_Q : Q \to \mathbb{R}$ is affine and continuous for every $f \in V^*$. So by the Hahn-Banach theorem A(Q) always separates points of Q. It is well known (see [3, Corollary 4.8]) that the subspace

$$A_0(Q) := \{ f|_Q + c : \quad f \in V^*, \ c \in \mathbb{R} \}$$

is uniformly dense in A(Q). If V is a Banach space then by Fact 5 every w^*-continuous functional on V^* is the evaluation at some point $v \in V$. This implies the following useful lemma.

Lemma 11. *For every Banach space V and a weak* compact convex set $Q \subset V^*$ the subspace*

$$A_0(Q) := \{ \tilde{v}|_Q + c : \ v \in V, \ c \in \mathbb{R} \} = r_Q(V) + \mathbb{R} \cdot \boldsymbol{1}$$

is uniformly dense in A(Q)*, where $\tilde{v}(\varphi) = \langle v, \varphi \rangle$ and $r_Q : V \to C(Q)$ is the restriction operator.*

Next we classify the affine compactifications of a topological space X in terms of unital closed subspaces of $C(X)$, in the spirit of the Gelfand–Kolmogoroff theorem (compare Remark 4 of point 3). At least for compact spaces X and point-separating subspaces $F \subset C(X)$ versions of Lemma 12 below can be found in several classical sources. See for example [59, Chap. 6], [12, Chap. 6, Sect. 29], [2, Theorem II.2.1], [66, Chap. 6, Sect. 23], and [3, Chap. 1, Sect. 4]. For affine bi-compactifications of transformation semigroups it remains true in a suitable setting, [36, Remark 3.2].

Lemma 12. *Let X be a topological space. The assignment $\Upsilon : F \mapsto \delta_F$, where $\delta_F : X \to M(F))$ is the canonical F-affine compactification, defines an order-preserving bijective correspondence between the collection of unital Banach subspaces F of $C(X)$ and the collection of affine compactifications of X (up to equivalence). In the converse direction, to every affine compactification $\alpha : X \to Q$ corresponds the unital Banach subspace $F := A(Q)|_X \subset C(X)$ (called the affine compactification space). Then the canonical affine compactification $\delta_F : X \to M(F)$ is affinely equivalent to $\alpha : X \to Q$.*

Proof. For a Banach unital subspace F of $C(X)$ define $\Upsilon(F) = (\delta_F, M(F))$ as the canonical F-affine compactification $\delta_F : X \to M(F)$.

Surjectivity of Υ:
Every affine compactification $\alpha : X \to Q$, up to equivalence, is a canonical F-affine compactification. In order to show this consider the set $A(Q)$ of all continuous affine functions on Q, viewed as a (Banach unital) subspace of $C(Q)$. Let $A(Q)|_X$ be the set of all functions on X which are α-extendable to a continuous

affine function on Q. Thus $A(Q)|_X := \alpha_\sharp(A(Q)) \subset C(X)$, where $\alpha_\sharp : C(Q) \to C(X)$, $f \mapsto f \circ \alpha$ is the natural linear operator induced by $\alpha : X \to Q$. Every such operator has norm 1. Moreover, since $\alpha(X)$ affinely generates Q and the functions in $A(Q)$ are affine, it follows that $\alpha_\sharp : A(Q) \to C(X)$ is a linear isometric embedding. Denote by F the Banach unital subspace $\alpha_\sharp(A(Q)) = A(Q)|_X$ of $C(X)$. We are going to show that the affine compactifications $\delta_F : X \to M(F)$ and $\alpha : X \to Q$ are isomorphic. Define the evaluation map

$$e : Q \to M(F) \subset F^*, \;\; e(q)(f) = \tilde{f}(q),$$

where $\tilde{f} := \alpha_\sharp^{-1}(f) \in A(Q)$ is the uniquely defined extension of $f \in F := A(Q)|_X$. Since $\alpha_\sharp^{-1} : F \to A(Q)$ is a linear isometry we easily obtain that $e(q) \in F^*$. Clearly, $\|e(q)\| = e(q)(\mathbf{1}) = 1$ for every $q \in Q$. Hence, indeed $e(q) \in M(F)$ and the map $e : Q \to M(F)$ is well defined. Since $\tilde{f} : Q \to \mathbb{R}$ is an affine map for every $f \in F$, it easily follows that $e : Q \to M(F)$ is an affine map. For every $x \in X$ we have $e(\alpha(x))(q) = \tilde{f}(\alpha(x)) = f(x)$. So, $\delta_F = e \circ \alpha$. It is also clear that e is w^*-continuous. Since $\delta_F(X)$ affinely generates $M(F)$ (Lemma 10), it follows that $e(Q) = M(F)$. Always, $A(Q)$ separates points of Q. This implies that $e : Q \to M(F)$ is injective, hence a homeomorphism.

The *injectivity* and *order-preserving* properties of Υ:

These properties follow from the next claim.

Claim. If $\alpha_1 : X \to Q_1$ and $\alpha_2 : X \to Q_2$ are two affine compactifications then α_2 dominates α_1 if and only if $F_2 \supset F_1$, where F_1 and F_2 are the corresponding affine compactification spaces.

Suppose $F_2 \supset F_1$ and let $j : F_1 \hookrightarrow F_2$ be the inclusion map. Then the restricted adjoint map $j^* : M(F_2) \to M(F_1)$ is a weak* continuous affine map and the following diagram commutes:

$$
\begin{array}{ccc}
X & \xrightarrow{\;\delta_2\;} & M(F_2) \, . \\
 & \searrow{\scriptstyle \delta_1} & \downarrow{\scriptstyle j^*} \\
 & & M(F_1)
\end{array}
$$

Moreover j^* is onto (use for example Lemma 10 of point 2). The second direction is trivial. $\qquad\square$

Corollary 1. *For every compact space X the Banach space $C(X)$ determines the universal (greatest) affine compactification $\alpha_b : X \to P(X) = M(C(X))$. For any other affine compactification $X \to Q$ we have a uniquely determined natural affine continuous onto map, called the* barycenter map, $b : P(X) \to Q$, *such that $\alpha_b = b \circ \delta$.*

Next we deal with affine compactifications of S-systems.

Lemma 13. *Assume that X is endowed with a semigroup action $S \times X \to X$ with continuous translations, i.e., X is an S_d-space. Let $\alpha : X \to Q$ be an affine compactification of the S_d-space X. Denote by F the corresponding affine compactification space.*

1. *$S \times Q \to Q$ is naturally topologically S-isomorphic to the action $S \times M(F) \to M(F)$.*
2. *$S \times Q \to Q$ is separately continuous iff $F \subset \mathrm{WRUC}(X)$.*
3. *$S \times Q \to Q$ is continuous iff $F \subset \mathrm{RUC}(X)$.*

Proof.

(1) All the translations $\lambda_s : X \to X$ are continuous and $F = \mathrm{A}(Q)|_X$ is an S-invariant subset of $C(X)$. So it is clear that the natural dual action $S \times F^* \to F^*$ is well defined and that every translation $\lambda_s : F^* \to F^*$ is weak* continuous. Now observe that $S \times M(F) \to M(F)$ is a restriction of the action $S \times F^* \to F^*$. Since $X \to Q$ and $X \to M(F)$ are S_d-affine compactifications and the evaluation map $e : Q \to M(F)$ from the proof of Lemma 12 is an isomorphism of affine compactifications, we obtain by Lemma 9 that $e : Q \to M(F)$ is an S-map.

(2) Use Lemma 10 and the restriction operator $r : F \to C(X)$ (Remark 4).

(3) Use Remark 4. $\qquad\qquad\square$

Proposition 2.

1. *If X is an S-space then the same assignment Υ, as in Lemma 12, establishes an order-preserving bijection between the collection of S-invariant unital Banach subspaces F of $\mathrm{WRUC}(X)$ and (equivalence classes of) S-affine compactifications of the S-system X. Furthermore, the subspaces of $\mathrm{RUC}(X)$ correspond exactly to the S-affine compactifications $X \to Q$ with continuous actions $S \times Q \to Q$.*
2. *Let X be a compact S-system and $\alpha_b : X \to P(X) = M(C(X))$ be the universal affine compactification of the space X.*

   (a) *For every S_d-affine compactification $\alpha : X \to Q$ the barycenter map $b : P(X) \to Q$ is an S-map.*
   (b) *$S \times P(X) \to P(X)$ is separately continuous iff $C(X) = \mathrm{WRUC}(X)$ (iff X is WRUC).*
   (c) *$S \times X \to X$ is continuous iff $S \times P(X) \to P(X)$ is continuous iff $C(X) = \mathrm{RUC}(X)$.*

Proof.

(1) Use Lemma 13.

(2) (a) Apply Lemma 9 and Corollary 1. For (b) and (c) use Lemma 13. $\qquad\square$

Remark 10.

1. As we already mentioned every S-affine compactification $\alpha : X \to Q$ induces the S-compactification $\alpha_0 : X \to Y := \overline{\alpha(X)}$ of X. Always

the affine compactification space $F := A(Q)|_X \subset C(X)$ generates the induced compactification algebra $\mathcal{A}$ of α_0. That is, $\langle F \rangle = \mathcal{A}$. Indeed, affine continuous functions on Q separate the points. Hence, by Stone-Weierstrass theorem, $A(Q)|_Y$ generates $C(Y)$. It follows that $\alpha_0^*(A(Q)|_Y) = F$ generates $\alpha_0^*(C(Y)) = \mathcal{A}$, where $\alpha_0^* : C(Y) \to C(X)$ is the induced map.

2. For every weakly continuous representation $h : S \to \Theta(V)$ on a Banach space V we have the associated affine semigroup compactification $h : S \to Q := \overline{co}(h(S)^{\mathrm{op}}) \subset \mathcal{E}$. Observe that since $co(h(S)^{\mathrm{op}}) \subset \Theta^{\mathrm{op}} = \Lambda(\mathcal{E})$, the closure $Q := \overline{co}(h(S)^{\mathrm{op}})$ in $\mathcal{E}$ is a semigroup. In this case we say that $S \to Q$ is a *standard affine semigroup compactification*. Every affine semigroup compactification can be obtained in this way (see point 2 of Proposition 3).

3. More generally, every operator compactification (Sect. 3) $h : S \to P := \overline{h(S)^{\mathrm{op}}} \subset L(V^*)$ of S on a locally convex vector space V induces an affine semigroup compactification $\alpha_h : S \to Q := \overline{co}(h(S)^{\mathrm{op}} \subset L(V^*)$. The affine compactification space $\mathcal{A}(Q)|_S$ coincides with the coefficient space M_h. Indeed, $M_h \subset \mathcal{A}(Q)|_S$ because every matrix coefficient $m(v, \psi) : S \to \mathbb{R}$ is a restriction of the map $\widetilde{m}(v, \psi) : Q \to \mathbb{R}$ which is continuous and affine. On the other hand the collection $\{\widetilde{m}(v, \psi)\}_{v \in V, \psi \in V^*}$ separates the points of Q. It follows that the Banach unital subspaces M_h and $\mathcal{A}(Q)|_S$ of $C(S)$ induce the isomorphic affine compactifications of S. By Lemma 12 we obtain that $M_h = \mathcal{A}(Q)|_S$.

4.2   Cyclic Affine S-Compactifications

Let X be a (not necessarily compact) S-system. For every $f \in \mathrm{WRUC}(X)$ denote by $\mathcal{A}_f := \langle fS \cup \{\mathbf{1}\} \rangle \subset C(X)$ the smallest S-invariant unital Banach subalgebra which contains f. The corresponding Gelfand compactification is an S-compactification $\alpha_f : X \to |\mathcal{A}_f| \subset \mathcal{A}_f^*$. We call it the *cyclic compactification* of X (induced by f). Now consider $V_f := \overline{sp}^{\mathrm{norm}}(fS \cup \{\mathbf{1}\})$—the smallest closed linear unital S-subspace of $\mathrm{WRUC}(X)$ generated by f. By Proposition 2 we have the affine S-compactification $\delta_f : X \to M(V_f) \subset V_f^*$ (where, $\delta_f(x)(\varphi) = \varphi(x)$ for every $\varphi \in V_f$) which we call the *cyclic affine S-compactification* of X. A natural idea is to reconstruct α_f from δ_f restricting the codomain of δ_f. In the following technical lemma we also give a useful realization (up to isomorphisms) of these two compactifications in $C(S)$ with the pointwise convergence topology. Note that we have also a left action of S on $C(S)$ defined by $S \times C(S) \to C(S), (sf)(t) = (ts) = R_s f$.

Lemma 14.

1. *The following map*

$$T_f : V_f^* \to C(S), \quad T_f(\psi) = m(f, \psi).$$

   is a well-defined linear bounded weak*-pointwise continuous S-map between left S-actions.

2. *The restriction* $T_f : M(V_f) \to Q_f$ *is an isomorphism of the affine S-compactifications $\delta_f : X \to M(V_f)$ and $\pi_f : X \to Q_f$, where $\pi_f := T_f \circ \delta_f$ and $Q_f := T_f(M(V_f)) \subset C(S)$.*
3. *Consider*

$$X_f := \overline{\pi_f(X)}^p = \mathrm{cl}_p(\{m(f, \delta_f(x)\}_{x \in X}) \subset Q_f.$$

The restriction of the codomain leads to the S-compactification $\pi_f : X \to X_f$ which is isomorphic to the cyclic compactification $\alpha_f : X \to |\mathcal{A}_f|$.

Proof.

(1) $m(f, \psi) \in C(S)$ $\forall \psi \in V_f^*$ because $f \in \mathrm{WRUC}(X)$. Other conditions are also easy.
(2) $T_f : M(V_f) \to Q_f$ is a morphism of the affine S-compactifications $\delta_f : X \to M(V_f)$ and $\pi_f : X \to Q_f$, where $\pi_f := T_f \circ \delta_f$. So, $\delta_f \succeq \pi_f$. In order to establish that $\pi_f \succeq \delta_f$ it is enough to show that our original function $f : X \to \mathbb{R}$ belongs to the affine compactification space of π_f. This follows from the observation that the evaluation at e functional, $\widehat{e} : C(S) \to \mathbb{R}$, restricted to $\overline{co}(X_f) \subset C(S)$, is an affine function such that $f = \widehat{e} \circ \pi_f$.
(3) By Remark 10 the algebra of the cyclic compactification $\pi_f : X \to X_f$ is just $\langle V_f \rangle$, but this is exactly $\mathcal{A}_f$, the algebra of the compactification $\alpha_f : X \to |\mathcal{A}_f|$. $\qquad\square$

Remark 11.

1. Note that $f = F_e \circ \pi_f$, where $F_e := \widehat{e}|_{X_f}$. So f comes from the S-system X_f. Moreover, if f comes from an S-system Y and an S-compactification $\nu : X \to Y$, then there exists a continuous onto S-map $\alpha : Y \to X_f$ such that $\pi_f = \alpha \circ \nu$. The action of S on X_f is continuous iff $f \in \mathrm{RUC}(X)$ (see Remark 4).
2. Moreover, $S\widehat{e} = \{\widehat{s}\}_{s \in S}$ separates points of Q_f (where $\widehat{s}$ is the evaluation at s functional) and $(X_f)_{F_e} = X_f$.
3. The action of S on X_f is separately continuous iff $f \in \mathrm{RMC}(X)$ (use again Remark 4). By definition, $Q_f = \overline{co}^p(X_f)$ in $C(S)$. At the same time the extended action of S on Q_f need not be separately continuous for $f \in \mathrm{RMC}(X)$. This is a reflection of the fact that in general $\mathrm{RMC}(X)$ is strictly larger than $\mathrm{WRUC}(X)$ (see [7, p. 219]). However by Lemma 4 we know that $\mathrm{WRUC}(X) = \mathrm{RMC}(X)$ in many natural cases. Also, $\mathrm{Tame}(X) \subset \mathrm{WRUC}(X)$ for every S-system X by Proposition 14.

Lemma 15. *Let V be a Banach S-invariant unital subspace of $\mathrm{WRUC}(X)$ and $f \in V$. Then*

1. $X_f := \overline{\pi_f(X)}^p = \mathrm{cl}_p(m(f, \delta_f(X)))$.
2. $Q_f = m(f, M(V_f)) = m(f, M(V)) = \overline{co}^p(X_f)$.
3. *In the particular case of $X := S$, with the left action of S on itself, we have $X_f = \overline{Sf}^p$.*

Proof. Straightforward, using Lemma 14.   □

Some other useful properties of cyclic compactifications can be found in [7, 24, 25].

4.3 Introversion and Semigroups of Means

In this section we assume that F is a normed unital subspace of $C(S)$, where S, as before, is a semitopological monoid. Suppose also that F is left translation invariant, that is, the function $(L_s f)(x) = f(sx)$ belongs to F for every $(f, s) \in F \times S$. Then the dual action $S \times M(F) \to M(F)$ is well defined and each s-translation, being the restriction of the adjoint operator L_s^*, is continuous on $M(F)$.

We recall the fundamental definition of introverted subspaces which was introduced by M.M. Day. We follow [6, 7].

Definition 12 (M.M. Day).

1. F is *left introverted* if $m(F, F^*) \subset F$ (equivalently (point 3 of Lemma 10), $m(F, M(F)) \subset F$).
2. When F is an algebra then F is said to be *left m-introverted* if $m(F, MM(F)) \subset F$.

   Causion: *We usually say simply that F is* introverted *(rather than* left introverted*).*

Fact 7 (Evolution product (in the sense of J.S. Pym) [7], [8, Chap. 2.2]).

1. *If F is an introverted closed subspace of $C(S)$ then F^* is a Banach algebra under the dual space norm and multiplication $(\mu, \varphi) \mapsto \mu \odot \varphi$, where*

$$(\mu \odot \varphi)(f) := \mu(m(f, \varphi)) \ (f \in F).$$

   Furthermore, with respect to the weak* topology, F^* is a right topological affine semigroup, $(M(F), \odot)$ is a compact right topological affine subsemigroup, $co(\delta(S)) \subset \Lambda(M(F))$, and $\delta : S \to M(F)$ is an affine semigroup compactification.
2. *If F is an m-introverted closed subalgebra of $C(S)$ then $(MM(F), \odot)$ is a compact right topological subsemigroup of $(M(F), \odot)$. Furthermore, $\delta : S \to MM(F)$ is a right topological semigroup compactification. See also Sect. 4.4 below for another view of m-introverted algebras as the algebras corresponding to enveloping semigroups.*

The following result shows that the m-introverted algebras and introverted subspaces of $C(S)$ correspond to the semigroup compactifications and affine semigroup compactifications of S, respectively (compare Proposition 2).

Fact 8. *Let S be a semitopological semigroup.*

1. *[7, p. 108] If (ψ, K) is a semigroup compactification of S then $\psi^* C(K)$ is an m-introverted closed subalgebra of $C(S)$. Conversely, if F is an m-introverted closed subalgebra of $C(S)$ then there exists a unique semigroup compactification (ψ, K) of S such that $\psi^* C(K) = F$, namely the canonical semigroup compactification $\delta : S \to MM(F)$.*
2. *[7, p. 123] If (ψ, K) is an affine semigroup compactification of S then $\psi^* A(K)$ is an introverted closed subspace of $\mathrm{WRUC}(S)$. Conversely, if F is an introverted closed subspace of $\mathrm{WRUC}(S)$ then there exists a unique affine semigroup compactification (ψ, K) of S such that $\psi^* A(K) = F$. Namely the canonical affine semigroup compactification $\delta : S \to M(F)$.*

Propositions 3 of point 1 and 5 of point 1 cover Fact 8 of point 2.

Remark 12 ([8, p. 123 and p. 172]). $\mathrm{WRUC}(S)$ is the largest introverted subspace of $C(S)$.

The next proposition demonstrates the universality of the standard operator affine semigroup compactifications.

Proposition 3.

1. *For every introverted closed unital subspace F of $C(S)$ there exists a natural $co(\delta(S))$-admissible affine embedding $M(F) \hookrightarrow \mathcal{E}(F)$ of right topological compact affine semigroups.*
2. *Every affine semigroup compactification $\alpha : S \to Q$ is equivalent to a standard operator affine semigroup compactification $\alpha' : S \to Q' \subset \mathcal{E}(V)$ for some Banach space V.*

Proof.

(1) The following natural map

$$i : M(F) \hookrightarrow \mathcal{E}(F), \quad i(m)(\varphi) = m \odot \varphi \ \ \forall \varphi \in F^*$$

is the desired embedding, where $m \odot \varphi$ is the evolution product (point 1 of Fact 7). The continuity is easy to verify, and the injectivity follows from the fact that if e is the neutral element of S then $\delta(e)$ is the neutral element of $M(F)$.

(2) Is a conclusion from (1) and point 2 of Fact 8. $\qquad\qquad\square$

4.4   Point-Universality of Systems

For a point-transitive compact separately continuous S-system (X, x_0) consider the natural S-compactification map $j_{x_0} : S \to X, s \mapsto sx_0$ and the corresponding Banach algebra embedding $j_{x_0}^* : C(X) \hookrightarrow C(S)$. Denote $\mathcal{A}(X, x_0) = j_{x_0}^*(C(X))$. The enveloping semigroup $(E(X), e)$ is always a point-transitive (separately continuous) S-space. Hence, $\mathcal{A}(E(X), e)$ is well defined and $E(X) \to X, p \mapsto px_0$ is the natural surjective S-map. Clearly $\mathcal{A}(X, x_0) \subset \mathcal{A}(E(X), e)$.

Recall that a point-transitive S-flow (X, x_0) is said to be *point-universal* [24, 25] if it has the property that for every $x \in X$ there is a homomorphism $\pi_x : (X, x_0) \to$ (cl $(Sx), x$). This is the case iff $S \to X$, $g \mapsto g x_0$ is a right topological semigroup compactification of S; iff (X, x_0) is isomorphic to its own enveloping semigroup, $(X, x_0) \cong (E(X), e)$; iff the algebra $\mathcal{A}(X, x_0)$ is m-introverted.

5 Operator Enveloping Semigroups

5.1 The Notion of E-Compatibility

In a review article [60, p. 212] J. Pym asks the general question: "how affine flows might be obtained?" and then singles out the canonical construction where, with a given compact S-flow X one associates the induced affine flow on $P(X)$, the compact convex space of probability measures on X, and where X is naturally embedded into $P(X)$ by identifying the points of X with the corresponding dirac measures. Then $P(X)$ is at least S_d-space with respect to the induced affine action $S \times P(X) \to P(X)$. Recall that by Proposition 2, $P(X)$ is an S-space (i.e., the action is separately continuous) iff X is WRUC.

In turn, $P(X)$ can be viewed (via Riesz' representation theorem) as a part of the weak* compact unit ball B^* in the dual space $C(X)^*$. So we have the embeddings of S_d-systems:

$$B^* \supset P(X) \supset X.$$

These embeddings induce the continuous onto homomorphisms of the enveloping semigroups

$$E(B^*) \to E(P(X)) \to E(X).$$

The first homomorphism $E(B^*) \to E(P(X))$ is always an isomorphism (point 3 of Lemma 10). Pym asks when the second homomorphism $\phi : E(P(X)) \to E(X)$ is an isomorphism. The first systematic study of this question is to be found in a paper of Köhler [41]. Since ϕ is an isomorphism iff it is injective, following [21], we say that an S-system X (with continuous action) is *injective* when ϕ is an isomorphism. See Definition 15 for a more general version.

For cascades (dynamical $\mathbb{Z}$-systems) the first non-injective example was constructed by Glasner [21], answering a question of Köhler [41]. Earlier Immervoll [35] gave an example of a non-injective system (S, X) where S is a some special semigroup S.

Now we turn to a more general question. In the construction above instead of the Banach space $C(X)$ and the natural embeddings $X \subset P(X) \subset B^*$ one may consider representations on general Banach spaces V.

Question 1. When is the enveloping semigroup of an affine compactification of a compact system X, arising from a representation on a Banach space V, isomorphic to the enveloping semigroup of the system itself?

More precisely, let

$$(h, \alpha) : (S, X) \rightrightarrows (\Theta(V)^{\mathrm{op}}, V^*)$$

be a weakly continuous representation of a (not necessarily compact) S-system X on a Banach space V. It induces an S-compactification $X \to Y$, where $Y := \overline{\alpha(X)}$ and an S-affine compactification $X \to Q$, where $Q := \overline{co}\,\alpha(X)$. Since h is weakly continuous it follows that the action of S on the weak* compact unit ball B^* (hence also on Y and Q) is separately continuous.

By our definitions, $\alpha(X)$ and hence Y and Q are norm bounded. So for some $r > 0$ we have the embeddings of S-systems:

$$r B^* \supset Q \supset Y = \overline{\alpha(X)}.$$

By Lemma 5, we get the induced continuous surjective homomorphisms of the enveloping semigroups (of course, the S-spaces B^* and $r B^*$ are isomorphic):

$$\psi : E(B^*) \to E(Q), \quad \Phi : E(Q) \to E(Y).$$

Lemma 16.

1. $m(V, \overline{sp}^{\mathrm{norm}}(A)) \subset \overline{sp}^{\mathrm{norm}}(m(V, A))$ for every subset $A \subset V^*$.
2. The algebra of the compactification $S \to E(Y)$ is $\mathcal{A}(E(Y), e) = \langle m(V, Y) \rangle$.
3. The algebra of the compactification $S \to E(Q)$ is $\mathcal{A}(E(Q), e) = \langle m(V, Q) \rangle$.
4. The algebra of the compactification $S \to E(B^*)$ is $\mathcal{A}(E(B^*), e) = \langle m(V, V^*) \rangle = A_h$, where A_h is the coefficient algebra.

Proof. (1) is straightforward. For other assertions use (1) and point 1 of Lemma 5 taking into account the definitions of Sect. 4.4. $\qquad\square$

Definition 13. Let X be an S-flow.

1. We say that an *S-affine compactification* (Definition 11) $\alpha : X \to Q$ is *E-compatible* if the map $\Phi : E(Q) \to E(Y)$ is an isomorphism (equivalently, is injective), where $Y := \overline{\alpha(X)}$. By Lemma 16 it is equivalent to saying that if $\mathcal{A}(E(Q), e) \subset \mathcal{A}(E(Y), e)$ or if $m(V, Q) \subset \langle m(V, Y) \rangle$.
2. We say that a weakly continuous Banach representation $(h, \alpha) : (S, X) \rightrightarrows (\Theta(V)^{\mathrm{op}}, V^*)$ of an S-flow X on a Banach space V is:

    (a) *E-compatible* if the map $\Phi : E(Q) \to E(Y)$ is an isomorphism where $Q := \overline{co}\,\alpha(X)$, that is, if the induced affine compactification of the representation (h, α) is E-compatible.

    (b) *Strongly E-compatible* if the map $\Phi \circ \Psi : E(B^*) \to E(Y)$ is an isomorphism. It is equivalent to saying that $m(V, V^*) \subset \langle m(V, Y) \rangle$ (equivalently, $m(V, V^*) \subset \mathcal{A}(E(Y), e)$).

We say that $K \subset V^*$ is a *w^*-generating* subset of V^* if $sp(\overline{co}^{w^*}(K))$ is norm dense in V^*. A representation (h, α) of a system (S, X) on V is *w^*-generating* (or simply *generating*) if $\alpha(X)$ is a w^*-generating subset of V^*. Later on (in the

proof of Theorem 11) we will have the occasion to use the versatile construction of Davis-Figiel-Johnson-Pelczyński [14]. The second item of the next lemma refers to this construction.

Lemma 17.

1. *For every space X and a closed unital linear subspace $V \subset C(X)$ the regular V-representation $\alpha : X \to V^*$ is generating.*
2. *Let E be a Banach space and let $\| \ \|_n, n \in \mathbb{N}$ be a sequence of norms on E where each of the norms is equivalent to the given norm of E. For $v \in E$, let*

$$N(v) := \left(\sum_{n=1}^{\infty} \|v\|_n^2 \right)^{1/2} \quad and \quad V := \{v \in E \mid N(v) < \infty\}.$$

Denote by $j : V \hookrightarrow E$ the inclusion map. Then (V, N) is a Banach space, $j : V \to E$ is a continuous linear injection such that $j^* : E^* \to V^*$ is norm dense. If $E = C(X)$ and $\alpha = j^* \circ \delta : X \to V^*$ is the induced map then $\alpha(X)$ is a w^*-generating subset of V^*.

Proof.

(1) Indeed by Lemma 10 every $\mu \in V^*$ is a finite linear combination of members of $M(V) = \overline{co}^{w^*}(\alpha(X))$. Hence, $sp(\overline{co}^{w^*}(\alpha(X)) = V^*$.

(2) For the proof that $j^* : E^* \to V^*$ is norm dense see Fabian [17, Lemma 1.2.2]. Since $\delta(X)$ affinely generates $M(C(X)^*)$ and $M(C(X)^*)$ linearly spans $C(X)^*$ (Lemma 10) it follows that $sp(\overline{co}^{w^*}(\alpha(X)))$ is norm dense in V^*, where $\alpha(X) = j^*(\delta(X))$. So, $\alpha(X)$ is w^*-generating in V^*.   □

Lemma 18.

1. *Suppose $\overline{co}^{w^*}(Y) = \overline{co}^{\text{norm}}(Y)$ holds for a weakly continuous representation (h, α) on a Banach space V, where $Y := \overline{\alpha(X)}^{w^*}$. Then the representation is E-compatible.*
2. *For w^*-generating representations, E-compatibility implies strong E-compatibility.*
3. *For every regular representation of an S-space X on a closed unital S-invariant linear subspace $V \subset \text{WRUC}(X)$, E-compatibility implies strong E-compatibility.*
4. *(Monotonicity) Let α_1 and α_2 be two faithful S-affine compactifications of a compact S-space X such that $\alpha_1 \succeq \alpha_2$; then E-compatibility of α_1 implies E-compatibility of α_2.*

Proof.

(1) By Lemma 16 we have $\overline{sp}^{\text{norm}}(m(V, Y)) \subset \mathcal{A}(E(Y), e)$. By our assumption on $Q := \overline{co}^{w^*}(Y) = \overline{co}^{\text{norm}}(Y)$ and using Lemma 16 we get

$$\begin{aligned} m(V, Q) &= m(V, \overline{co}^{w^*}(Y)) \\ &= m(V, \overline{co}^{\text{norm}}(Y)) \subset \overline{sp}^{\text{norm}}(m(V, Y)) \subset \mathcal{A}(E(Y), e). \end{aligned}$$

Since $\langle m(V,Q)\rangle = \mathcal{A}(E(Q),e)$ (point 2 of Lemma 16), we obtain $\mathcal{A}(E(Q),e) \subset \mathcal{A}(E(Y),e)$.

(2) Y is a w^*-generating subset in V^*. Therefore by assertions (1) and (3) of Lemma 16 we get that $\psi : E(B^*) \to E(Q)$ is an isomorphism. So $\Phi \circ \Psi : E(B^*) \to E(Y)$ is an isomorphism iff $\Phi : E(Q) \to E(Y)$ is an isomorphism.

(3) Use (2) and Lemma 17.

(4) Since the affine compactifications are faithful and X is compact, we may identify $E(\alpha_1(X))$ and $E(\alpha_2(X))$ with $E(X)$. Since $\alpha_1 \succeq \alpha_2$ we have the induced homomorphism $E(Q_1) \to E(Q_2)$. The injectivity of the homomorphism $E(Q_1) \to E(X)$ implies the injectivity of $E(Q_2) \to E(X)$.   □

Note that (4) need not remain true if we drop the faithfulness of the affine compactifications.

Lemma 19. *Let X be an S-space and $(h,\alpha) : (S,X) \rightrightarrows (\Theta(V)^{\mathrm{op}}, V^*)$ a weakly continuous representation of (S,X) on a Banach space V. Let $j : S \to E(Y)$ be the Ellis compactification for the S-system $Y := \overline{\alpha(X)}$. The following are equivalent:*

1. *The representation (h,α) is strongly E-compatible.*
2. *There exists a $j(S)$-admissible embedding $h' : E(Y) \hookrightarrow \mathcal{E}(V)$ such that $h' \circ j = h$.*
3. *The semigroup compactifications $j : S \to E(Y)$ and $h : S \to \overline{h(S)}$, where $\overline{h(S)}$ is the closure in $\mathcal{E}(V)$, are naturally isomorphic.*
4. *$m(V,V^*) \subset \langle m(V,Y)\rangle$ (equivalently, $m(V,V^*) \subset \mathcal{A}(E(Y),e)$).*

Proof. By Lemma 7, $E(S,B^*)$ is naturally embedded into $\mathcal{E}(V)$ and the semigroup compactifications $S \to E(S,B^*)$ and $S \to \overline{h(S)}$ are isomorphic. So (1), (2), and (3) are equivalent.

(1) $\Leftrightarrow$ (4): Use Lemma 16.   □

Proposition 4. *The following are equivalent:*

1. *An affine S-compactification $\alpha : X \to Q$ of an S-space X is E-compatible.*
2. *The induced representation of (S,X) on the Banach space $V := A(Q)|_X \subset \mathrm{WRUC}(X)$, the affine compactification space of α, is E-compatible (equivalently, strongly E-compatible).*
3. *$\underline{m(V,V^*)} \subset \langle m(V,Y)\rangle$, (equivalently, $m(V,M(V)) \subset \langle m(V,Y)\rangle$), where $Y := \overline{\alpha(X)}$.*
4. *For every $f \in V$ we have*

$$\overline{co(X_f)}^p \subset \langle m(V,Y)\rangle.$$

Proof.

(1) $\Leftrightarrow$ (2): By Definition 13 and the description of affine compactifications (Lemma 12) taking into account Lemma 18 of point 3.

(2) $\Leftrightarrow$ (3): Use Lemma 19 (taking into account that by Lemma 10 every $\mu \in V^*$ is a finite linear combination of members of $M(V)$).

(3) $\Leftrightarrow$ (4): Clearly, $m(V, M(V)) = \bigcup_{f \in V} m(f, M(V))$. Recall that, by Lemma 15, for each $f \in V$ we have $\overline{co(X_f)}^p = m(f, M(V))$. $\qquad\square$

5.2 Affine Semigroup Compactifications

The second assertion of the next result shows that every affine semigroup compactification is E-compatible.

Proposition 5. *Let* $v : S \to Q$ *be an affine semigroup compactification and let* $P = \overline{v(S)}$. *Then*

1. *[7, p. 123] The space* $V = \mathrm{A}(Q)|_S$ *is introverted.*
2. *The affine compactification* $v : S \to Q$ *is E-compatible, that is, the restriction map* $E(Q) \to E(P) = P$ *is an isomorphism.*

Proof.

(1) We have to show that $m(V, M(V)) \subset V$. As in the proof of Lemma 12 consider the Banach space $V = \mathrm{A}(Q)|_S$ of the affine compactification $S \to Q$. For every $f \in V$ and $\mu \in M(V) = Q$ the corresponding matrix coefficient $m(f, \mu)$ is again in V because $m(f, \mu)$ is a restriction to S of the affine continuous map $Q \to \mathbb{R}$, $q \mapsto \tilde{f}(q \odot \mu)$, where $\tilde{f} \in \mathrm{A}(Q)$ with $f = \tilde{f}|_S$ and $q \odot \mu$ is the evolution product (see Fact 7) in the semigroup $M(V) = Q$.

(2) By (1) we have $m(V, V^*) \subset V$. Observe that $V = \mathrm{A}(Q)|_S \subset \mathcal{A}(P, e)$. Since $\mathcal{A}(P, e) \subset \mathcal{A}(E(P), e)$, we get $m(V, V^*) \subset \mathcal{A}(E(P), e) = \langle m(V, P) \rangle$ (Lemma 16). So, V is E-compatible by Proposition 4. $\qquad\square$

Definition 14. We say that a subalgebra $\mathcal{A} \subset C(S)$ is *intro-generated* if there exists an introverted subspace $V \subset \mathcal{A}$ such that $\langle V \rangle = \mathcal{A}$.

Below, in Theorem 3, we show that the m-introverted algebra of all distal functions $D(\mathbb{Z})$ is not intro-generated. In Example 1 we present an m-introverted intro-generated subalgebra $\mathcal{A}$ of $l_\infty(\mathbb{Z}^2)$ which is not introverted.

Proposition 6. *Let* $v : S \to P$ *be a right topological semigroup compactification and let* $\mathcal{A} = \mathcal{A}(P, e)$ *be the corresponding m-introverted subalgebra of* $C(S)$. *The following are equivalent:*

1. *The compactification* $v : S \to P$ *is equivalent to a standard operator compactification on a Banach space.*
2. *The compactification* $v : S \to P$ *is equivalent to an operator compactification on a locally convex vector space.*
3. *There exists an affine (equivalently,* standard affine*) semigroup compactification* $\psi : S \to Q$ *such that the compactification* $\psi : S \to \overline{\psi(S)}$ *is equivalent to* $v : S \to P$.

4. *The algebra $\mathcal{A}$ of the compactification $\nu : S \to P$ is intro-generated.*
5. *There exists a Banach unital S-subspace $V \subset \mathrm{WRUC}(S)$ such that $\langle V \rangle = \mathcal{A}$ and for every $f \in V$ we have*

$$\overline{co(Sf)}^p \subset \mathcal{A}.$$

Proof.

(1) $\Rightarrow$ (2): Trivial.

(2) $\Rightarrow$ (3): By our assumption $\nu : S \to P$ is equivalent to an operator compactification. Therefore there exists a weakly continuous equicontinuous representation $h : S \to L(V)$ of a semitopological semigroup S on a locally convex vector space V such that ν can be identified with $h : S \to P$ where P is the weak* operator closure $\overline{h(S)^{op}}$ of the adjoint semigroup $h(S)^{op} \subset L(V)^{op}$ in $L(V^*)$. Consider the compact subsemigroup $Q := \overline{co}\ \nu(S) = \overline{co}(P)$ of $L(V^*)$ (in weak* operator topology). Then the map $\psi : S \to Q$, $\psi(s) = \nu(s)$ is an affine compactification of S. Indeed, $\psi(S) \subset \Lambda(Q)$ because $\nu(S) \subset L(V)^{op}$. Observe that ψ induces ν because $\overline{\psi(S)} = P$ and $\psi(s) = \nu(s)$ for every $s \in S$. (By Proposition 3 we can assume that $\psi : S \to Q$ is a standard affine semigroup compactification.)

(3) $\Rightarrow$ (4): By Proposition 5 the space $V := A(Q)|_S$ of the affine compactification ψ is introverted. Always, the affine compactification subspace V generates the induced compactification algebra $\mathcal{A}$ (Remark 10). That is, $\langle V \rangle = \mathcal{A}$. Hence $\mathcal{A}$ is intro-generated.

(4) $\Rightarrow$ (5): If V is an introverted subspace of $\mathcal{A}$ then $m(f, M(V)) \subset V$ for every $f \in V$. Also, $V \subset \mathrm{WRUC}(S)$ by Remark 12. So Lemma 15 implies $m(f, M(V)) = \overline{co(Sf)}^p$. Hence, $\overline{co(Sf)}^p \subset V \subset \langle V \rangle = \mathcal{A}$.

(5) $\Rightarrow$ (1): Consider the regular representation (h, α) of the S-space $X := S$ on the closed unital S-invariant linear subspace $V \subset \mathrm{WRUC}(S)$. By the Stone-Weierstrass theorem it follows that the algebra of the corresponding S-compactification $\alpha : S \to Y := \overline{\alpha(X)}$ is $\langle V \rangle$. By our assumption $\langle V \rangle = \mathcal{A}$. So we obtain that $\alpha : S \to Y$ can be identified with the original S-compactification $\nu : S \to P$. Recall that $\overline{co(X_f)}^p = \overline{co(Sf)}^p$ (Lemma 15) and $\mathcal{A}(E(P), e) = \mathcal{A}(P, e) = \mathcal{A}$ (Lemma 5). Applying the equivalence (4) $\Leftrightarrow$ (1) of Proposition 4 we get that the regular V-representation (h, α) of (S, S) is E-compatible, in fact, strongly E-compatible by Lemma 18. We obtain that $E(S, B^*) \to E(S, P)$ is an isomorphism. Now observe that $E(S, B^*) \subset \mathcal{E}(V)$ (Lemma 7) and $E(S, P) = P$ (Lemma 5). $\qquad\square$

5.3 Injectivity of Compact Dynamical Systems

Every continuous action is WRUC (point 2 of Lemma 4). So the following naturally extends the definition from [21], mentioned in Sect. 5.1.

Definition 15. We say that a compact WRUC S-system X is *injective* if one (hence all) of the following equivalent conditions are satisfied:

1. The greatest affine S-compactification $X \to P(X)$ is E-compatible.
2. The regular representation of (S, X) on the Banach space $C(X)$ is (strongly) E-compatible.
3. $m(C(X), C(X)^*) \subset \langle m(C(X), X) \rangle$ (equivalently, $m(C(X), C(X)^*) \subset \mathcal{A}(E(X), e)$).
4. Every faithful affine S-compactification $X \hookrightarrow Q$ is E-compatible.

Proof. Here we prove that these conditions are equivalent. First of all since X is WRUC the action $S \times P(X) \to P(X)$ is separately continuous by Proposition 2. Regarding (2) note that by Lemma 18 every regular E-compatible representation is strongly E-compatible.

  (1) $\Leftrightarrow$ (2) and (4) $\Rightarrow$ (1): Are trivial.

  (2) $\Leftrightarrow$ (3): Use Lemma 19.

  (2) $\Rightarrow$ (4): Since $X \hookrightarrow P(X)$ is the greatest S-affine compactification of X we can apply the monotonicity of E-compatibility (Lemma 18). $\square$

Proposition 7. *Let X be an injective S-system. Then the enveloping semigroup compactification $S \to E(X)$ is equivalent to a (standard) operator compactification and hence all of the equivalent conditions of Proposition 6 are satisfied.*

Proof. Since X is injective the regular representation of (S, X) on $C(X)$ is weakly continuous and E-compatible. Now apply Lemma 19. $\square$

Theorem 1. *Let $V \subset C(S)$ be an m-introverted Banach S-subalgebra and $\alpha : S \to P$ be the corresponding right topological semigroup compactification. Then the S-system P is injective if and only if V is introverted.*

Proof. By Definition 15 the S-flow P is injective iff $m(C(P), C(P)) \subset \mathcal{A}(E(S, P), e)$. By Lemma 5, $E(S, P)$ is naturally isomorphic to the semigroup P. Hence, $\mathcal{A}(E(P), e) = \mathcal{A}(P.e) = j^*(C(P)) = V$. Observe also that the canonical representations of (S, P) on $C(P)$ and on V are naturally isomorphic. In particular, $m(C(P), C(P)^*) = m(V, V^*)$. Summing up we get: the S-flow P is injective iff $m(V, V^*) \subset V$ iff V is introverted. $\square$

Lemma 20. *Let X be a compact point-transitive S-system. If the enveloping semigroup $E(X)$, as an S-flow, is injective then X is also injective.*

Proof. Let $z \in X$ be a transitive point and let $q : E(X) \to X, q(p) = pz$ be the corresponding onto continuous S-map. It induces the surjective homomorphism $q_E : E(B_{C(E(X))^*}) \to E(B_{C(X)^*})$. By the injectivity of $(S, E(X))$ the S-compactifications $S \to E(E(X))$ and $S \to E(B_{C(E(X))^*})$ are equivalent. On the other hand, $S \to E(E(X))$ is equivalent to $S \to E(X)$. It follows that $S \to E(X)$ dominates $S \to E(B_{C(X)^*})$. Conversely, $S \to E(B_{C(X)^*})$ clearly dominates $S \to E(X)$. Therefore these compactifications are equivalent. $\square$

5.4 Some Examples of Injective Dynamical Systems

Let G be a topological group. As in Remark 5 a property $\mathcal{P}$ of continuous G-dynamical compact systems is said to be *suppable* if it is preserved by the operations of taking products and subsystems. To every suppable property $\mathcal{P}$ corresponds a universal point-transitive G-system $(G, G^{\mathcal{P}})$ (see e.g. [24, Proposition 2.9]).

Remark 13.

1. Suppose $\mathcal{P}$ is a suppable property of dynamical systems such that whenever (G, X) has $\mathcal{P}$ then so does $(G, P(X))$. It then follows immediately that the corresponding $\mathcal{P}$-universal point-transitive system $Y = G^{\mathcal{P}}$ is injective. Indeed, then the G-systems $P(Y)$ and $E(P(Y))$ have $\mathcal{P}$. By the universality of Y it is easy to see that Y and $E(P(Y))$ are naturally isomorphic. On the other hand, Y and $E(Y)$ are naturally isomorphic (point 4 of Lemma 5). Hence, also $E(P(Y))$ and $E(Y)$ are isomorphic. This way we see that, for example, the universal point-transitive (a) equicontinuous, (b) WAP, (c) HNS, and (d) tame dynamical systems, are all injective.

2. Another application of this principle is obtained by regarding the class of $\mathbb{Z}$-flows having zero topological entropy. It is easy to check that this property is suppable and the fact that it is preserved under the functor $X \mapsto P(X)$ follows from a theorem of Glasner and Weiss [30].

3. Let $\Omega = \{0, 1\}^{\mathbb{Z}}$ be the $\{0, 1\}$-Bernoulli system on $\mathbb{Z}$. It is well known that the enveloping semigroup of $(\mathbb{Z}, \Omega)$ is $\beta\mathbb{Z}$, the Čech-Stone compactification of $\mathbb{Z}$ (see [20, Exercise 1.25]). Since $\beta\mathbb{Z}$ is the universal enveloping semigroup, it follows that $E(X) = E(\Omega) = \beta\mathbb{Z}$ for every point-transitive system $(\mathbb{Z}, X)$ which admits $(\mathbb{Z}, \Omega)$ as a factor (e.g., every mixing $\mathbb{Z}$-subshift of finite type will have $\beta\mathbb{Z}$ as its enveloping semigroup since it has a Cartesian power which admits Ω as a factor [11]). Moreover we necessarily have in that case that also $E(P(X)) = \beta\mathbb{Z}$. Thus every such X is injective (see [8, 41]).

4. Every tame S-system is injective (Theorem 17). This result for metrizable systems was obtained by Köhler [41]. In [21] there is a simple proof which uses the fact that the enveloping semigroup $E(X)$ of a tame metric system X is a Fréchet space.

5. Every transitive continuous G-system X is a factor of the injective G-system G^{RUC} (see (1)). Thus injectivity is not preserved by factors. The same assertion holds for subsystems (Remark 15).

Remark 14.

1. Theorem 1 shows that injectivity can serve as a key property in providing introverted subalgebras of $C(S)$. In particular, the algebras in point 1 of Remark 13 are introverted.

   It is well known (see [6, Chap. III, Lemma 8.8]) that every S-invariant unital closed subspace of WAP(S) is introverted. In particular, Hilb(S) and AP(S) are introverted. It is also well known that RUC(S) is an introverted algebra

(see [7, p. 163]). Hence, the corresponding semigroup compactifications, the greatest ambit $S \to S^{\text{RUC}}$, the Bohr compactification $S \to S^{\text{AP}}$, and the universal semitopological semigroup compactification $S \to S^{\text{WAP}}$, respectively, are operator compactifications (Proposition 6) and can be extended to affine semigroup compactifications.

2. In Theorem 19, we show that in fact every m-introverted Banach S-subspace of Tame(S) (hence also of Asp(S) and WAP(S)) is introverted.

3. The Roelcke algebra UC(G) = LUC(G) ∩ RUC(G) is not even m-introverted in general, [25] for Polish topological groups G.

5.5 The Iteration Process

Starting with an arbitrary topological group G and a compact dynamical system (G, X) (with continuous action) we define inductively a sequence of new systems by iterating the operation of passing to the space of probability measures. Explicitly we let $P^{(1)}(X) = P(X)$ and for $n \geq 1$ we let $P^{(n+1)}(X) = P(P^{(n)}(X))$. Each $P^{(n)}(X)$ is an affine dynamical system and thus the barycenter map $b : P^{(n+1)}(X) \to P^{(n)}(X)$ is a well-defined continuous affine homomorphism. Moreover, identifying a measure $\mu \in P^{(n)}(X)$ with the dirac measure $\delta_\mu \in P^{(n+1)}(X)$ we can consider $b : P^{(n+1)}(X) \to P^{(n)}(X)$ as a retract. Next observe that the induced map $b_* : P^{(n+1)}(X) \to P^{(n)}(X)$ coincides with the map $b : P^{(n+1)}(X) \to P^{(n)}(X)$. For convenience we write $Z_n = P^{(n)}(X)$ and we now let $Z = P^{(\infty)}(X) = \varprojlim Z_n$, the inverse limit. We denote by $\pi_n : Z \to Z_n$ the natural projection.

Proposition 8. *There is a natural bijection* $\alpha : P(Z) \to Z$. *In particular* (G, Z) *is injective.*

Proof. Given $\mu \in P(Z)$ we consider, for each n, its image $z_{n+1} = (\pi_n)_*(\mu) = \mu_n \in P(Z_n) = P^{(n+1)}(X)$. Then,

$$z_{n+2} = \mu_{n+1} = (\pi_{n+1})_*(\mu)$$
$$= (b \circ \pi_n)_*(\mu) = b_*((\pi_{n+1})_*(\mu))$$
$$= b_*(z_{n+1}) = b(z_{n+1}).$$

Therefore, the sequence (z_n) defines a unique point $z = \alpha(\mu)$ in Z. Clearly $\alpha : P(Z) \to Z$ is a continuous G-map and it is easy to check that it is one-to-one. Finally for $z \in Z$ we have the sequence of measures $z_{n+1} = \pi_{n+1}(z) \in Z_{n+1} = P^{(n+1)}(X) = P(Z_n)$. Because, as maps from $P^{(n+1)}(X)$ to $P^{(n)}(X)$, the maps b and b_* coincide, this choice of measures on the various Z_n is consistent and defines a measure $\beta(z) \in P(Z)$. One can check now that $\beta \circ \alpha$ is the identity map on $P(Z)$ and our proof is complete. □

Remark 15. We note that in the above construction the compact affine space Z is metrizable when X is metrizable. Moreover, via the maps $z \mapsto \delta_z$ we have a natural embedding of X into the space Z. This iterated construction can serve now as a source of examples. Beginning with an arbitrary compact metric system X with a property, say, $\mathcal{R}$—which is preserved under inverse limits and such that $Y \in \mathcal{R} \Rightarrow P(Y) \in \mathcal{R}$—we obtain in (Z, T), with $Z = P^{(\infty)}(X)$, a metrizable injective system which contains X as a subsystem and has property $\mathcal{R}$. Some properties $\mathcal{R}$ as above are, e.g., "weak-mixing" ([5]), "zero topological entropy" ([30]) and "uniform rigidity" (with respect to a given sequence) ([23]).

6 Examples of Non-Injective Systems

6.1 Minimal Distal Non-Equicontinuos Systems are Not Injective

In this section G denotes a semitopological group. Let Q be an affine compact G-system and let ext Q denote the set of extreme points of Q. The system Q is said to be *minimally generated* [19] if the G-subsystem $\overline{\text{ext } Q}$ is minimal. We recall the following theorem.

Fact 9 ([20, Theorem 1.1]). *Let Q be a minimally generated metric distal affine compact G-system. Then Q is equicontinuous.*

Lemma 21. *Let Q be a compact convex affine G-flow and let X be a compact minimal G-subflow. The following are equivalent:*

1. $X = \overline{\text{ext } Q}$
2. $\overline{co}(X) = Q$

Proof.

- (1) $\Rightarrow$ (2): By Krein-Milman theorem (see, for example, [72, p. 659]), $Q = \overline{co}(\text{ext } Q)$. Hence, we get $Q = \overline{co}(\overline{\text{ext } Q}) = \overline{co}X$.
- (2) $\Rightarrow$ (1): By Krein-Milman theorem ext $Q \neq \emptyset$. Choose $z \in \text{ext } Q$. The orbit Gz is dense in X by minimality of X. By Milman theorem [72, p. 659], ext $Q \subset X$. Since G is a *group* of affine transformations, $gz \in \text{ext } Q$ for every $g \in G$. So ext Q is dense in X. □

Remark 16. Theorem 9 remains true for non-metrizable Q. Indeed, by [57] the general case can be reduced to the metrizable case using Ellis' construction.

At least for every separable G a more direct argument is as follows. We treat Q as an affine compactification of the minimal system $X := \overline{\text{ext } Q}$ [(1) $\Rightarrow$ (2) of Lemma 21] and observe that one may G-approximate the (faithful)

affine compactification $X \hookrightarrow Q$ by metric (not necessarily faithful) affine compactifications $X \to Q_i$. More precisely, let $V \subset C(X)$ be the Banach unital G-invariant space of the affine compactification $X \hookrightarrow Q$. Since G is separable there are sufficiently many *separable* Banach unital G-invariant subspaces V_i to separate points of Q. The corresponding affine distal G-factors $r_i : Q \to Q_i = \overline{co}(X_i)$, $X_i := r_i(X)$ are again minimally generated $((2) \Rightarrow (1)$ of Lemma 21). Since each Q_i is metrizable, it is equicontinuous by Theorem 9, and the same is then true for Q.

Proposition 9. *Let X be a compact distal minimal G-flow. The following are equivalent:*

1. *The G-flow X admits an E-compatible faithful affine compactification $\alpha : X \hookrightarrow Q$.*
2. *The G-flow X admits an E-compatible faithful representation of (S, X) on a Banach space.*
3. *X is an equicontinuous G-flow.*
4. *X is an injective G-flow.*

Proof.

(1) $\Leftrightarrow$ (2): Follows directly from Proposition 4 and Definition 13.
(1) $\Rightarrow$ (3): The distality of the G-flow X means, by Ellis result, that the enveloping semigroup $E(X)$ is a group. Since by (1), $E(Q) \to E(X)$ is an isomorphism we obtain that $E(Q)$ is also a group; hence Q is a distal G-flow. By Theorem 9, taking into account Lemma 21 and Remark 16 we conclude that Q, hence also, X are equicontinuous flows.
(3) $\Rightarrow$ (4): This follows, for example, from Theorem 17 below.
(4) $\Rightarrow$ (1): Apply Definition 15.   □

Theorem 2. *Let P be a compact right topological group and let $v : G \to P$ be a right topological semigroup compactification of a group G. The following are equivalent:*

1. *The compactification $v : G \to P$ is equivalent to an operator compactification.*
2. *There exists a $v(G)$-admissible embedding of P into $\mathcal{E}(V)$ for some Banach space V.*
3. *P is a topological group.*

Proof.

(3) $\Rightarrow$ (1): Every compact topological group is embedded into the unitary group $U(H) \subset \Theta(H)$ for some Hilbert space H and in this case $\Theta(H) = \mathcal{E}(H^*)$.
(1) $\Rightarrow$ (3): By Proposition 6, $G \to P$ can be embedded into an affine compactification $G \to Q$, so that $\overline{co}(P) = Q$. The G-flow P is minimal (being distal and point-transitive) and by Lemma 21 Q is minimally generated. Now by Fact 9 and Remark 16 we get that Q is equicontinuous. This implies that (G, P) is equicontinuous and hence $E(G, P) = P$ is a topological group.
(1) $\Leftrightarrow$ (2): Use Proposition 6 and Lemma 7.   □

Let $D(\mathbb{Z})$ be the algebra of all distal functions on $\mathbb{Z}$ and $\mathbb{Z} \to \mathbb{Z}^{D(\mathbb{Z})}$ the corresponding semigroup compactification (see [7, p. 178]). The right topological group $\mathbb{Z}^{D(\mathbb{Z})}$ is not a topological group so by Proposition 9 and Theorem 2 we get:

Theorem 3.

1. *The semigroup compactification $\mathbb{Z} \to \mathbb{Z}^{D(\mathbb{Z})}$ is not an operator compactification.*
2. *The algebra of all distal functions $D(\mathbb{Z})$ is not intro-generated.*
3. *$(\mathbb{Z}, \mathbb{Z}^{D(\mathbb{Z})})$ is a minimal distal cascade which does not admit faithful E-compatible affine compactifications.*
4. *The compact right topological group $\mathbb{Z}^{D(\mathbb{Z})}$ does not admit faithful admissible representations on Banach spaces.*

The fact that $D(\mathbb{Z})$ is not introverted (weaker than Theorem 3) was shown in [7, p. 179].

6.2 A Toeplitz Non-Injective System

We give here an example of a non-injective metric minimal $\mathbb{Z}$-system which is a transitive almost one-to-one extension of an adding machine. The latter property is actually a characterization of being a *Toepliz dynamical system*. For more details on Toepliz systems we refer to [73].

Theorem 4. *There is a Toeplitz non-injective system.*

Proof. Let $y_0 \in \{0, 1\}^{\mathbb{Z}}$ be the "Heaviside" sequence defined by the rule

$$y_0(n) = \begin{cases} 0 & \text{for } n < 0 \\ 1 & \text{for } n \geq 0, \end{cases}$$

and let $Y = \bar{\mathcal{O}}_S(y_0)$ be its orbit closure in $\{0, 1\}^{\mathbb{Z}}$ under the shift transformation: $S\omega(n) = \omega(n + 1), \omega \in \{0, 1\}^{\mathbb{Z}}, n \in \mathbb{Z}$. Thus $Y = \{S^n y_0 : n \in \mathbb{Z}\} \cup \{\mathbf{0}, \mathbf{1}\}$ is isomorphic to the two-point compactification $\mathbb{Z} \cup \{\pm\infty\}$ of the integers.

Let (X, S) be the (necessarily minimal and non-regular) Toeplitz system corresponding to the subshift Y and a suitable sequence of periods $(p_i)_{i \in \mathbb{N}}$ as described by Williams in Section 4 of [73]. Here, as usual, we write (X, S) for the $\mathbb{Z}$-action $(\mathbb{Z}, X)$ generated by S.

By [73, Theorem 4.5] the system (X, S) admits exactly two S-invariant ergodic probability measures (corresponding to the dirac measures $\delta_{\mathbf{0}}$ and $\delta_{\mathbf{1}}$ on Y), which we will denote by μ_0 and μ_1, respectively.

We will show the existence of a probability measure ν on X for which, in the weak* topology on the compact space $P(X)$ of probability measures on X and the action induced by S, we have

$$\lim_{n \to -\infty} S^n \nu = \mu_0 \quad \text{and} \quad \lim_{n \to +\infty} S^n \nu = \mu_1. \tag{1}$$

The dynamical system (X, S) has a structure of an almost one-to-one extension of an adding machine, say,

$$\pi : X \to G = \varprojlim \mathbb{Z}/p_i \mathbb{Z}.$$

It follows that the proximal relation $\text{Prox}(X)$ on X coincides with the π-relation:

$$R_\pi = \{(x, x') : \pi(x) = \pi(x')\}.$$

In particular this implies that $\text{Prox}(X)$ is an equivalence relation. Now this latter condition is equivalent to the fact that $E(X)$, the enveloping semigroup of (X, S), has a unique minimal ideal. Since $E(E(X)) = E(X)$ we conclude that in any dynamical system (X', T) whose enveloping semigroup is isomorphic to $E(X)$ the proximal relation $\text{Prox}(X')$ is again an equivalence relation.

However Eq. (1) clearly shows that in the dynamical system $(P(X), S)$ the proximal relation is no longer an equivalence relation. It therefore follows that the natural restriction map $r : E(P(X)) \to E(X)$ is not an isomorphism; i.e., (X, S) is not injective.

It thus remains to construct a measure ν as above. For that purpose let us recall the following objects constructed by Williams. Using the notations of [73] we let $C = \{x \in X : 0 \notin Aper(x)\}$, and let $D \subset X$ be the set of $x \in X$ with $Aper(x)$ a two-sided infinite sequence. We have $C = \pi^{-1}(\pi(C))$ so that the subset $Aper(x) \subset \mathbb{Z}$ is well defined on G, with $Aper(x) = Aper(\pi(x))$ for every $x \in X$. By definition the fact that X is a non-regular Toeplitz system means that we have $0 < m(\pi(C)) = d < 1$, with m denoting the Haar measure on G. The (Borel) dynamical system $(G \times Y, T)$ is given by the Borel map $T : G \times Y \to G \times Y$ defined as $T(g, y) = (g + \mathbf{1}, S^{\theta(g)} y)$, where $\theta : G \to \{0, 1\}$ is the function $\mathbf{1}_{\pi(C)}$. Williams shows that $\phi(G \times Y) = X$, that $S \circ \phi = \phi \circ T$, and that the restriction of ϕ to the subset $\pi(D) \times Y$ is a Borel isomorphism from $\pi(D) \times Y$ into X. In our case we have

$$\pi(D) \times Y = (\pi(D) \times \{\mathbf{0}\}) \cup \left(\bigcup_{n \in \mathbb{Z}} \pi(D) \times \{S^n y_0\} \right) \cup (\pi(D) \times \{\mathbf{1}\}).$$

We also have $m(\pi(D)) = 1$ and $\phi(m \times \delta_{\mathbf{0}}) = \mu_0$, $\phi(m \times \delta_{\mathbf{1}}) = \mu_1$.

Now let

$$\nu = \phi(m \times \delta_{y_0}).$$

Iterating the map T we see that for $n \in \mathbb{Z}$

$$T^n(g, y) = (g + n\mathbf{1}, S^{\theta_n(g)} y),$$

where for every $g \in G$, $\theta_0(g) = 0$ and

$$\theta_n(g) = \begin{cases} \theta(g) + \theta(g + \mathbf{1}) + \cdots + \theta(g + (n-1)\mathbf{1}) & \text{for } n \geq 1 \\ -\theta(g + n\mathbf{1}) - \theta(g + (n+1)\mathbf{1})\mathbf{1} - \cdots - \theta(g - \mathbf{1}) & \text{for } n \leq -1. \end{cases}$$

Note that by the ergodic theorem we have

$$\lim_{n \to \pm\infty} \frac{1}{n} \theta_n(g) = m(\pi(C)), \qquad m\text{-a.e..} \tag{2}$$

Next let us consider the integrals $\int f \, dS^n \nu$ for a fixed continuous real valued function f on X. We have

$$\int_X f(x) \, dS^n \nu(x) = \int_X f(S^n x) \, d\nu(x) = \int_{G \times Y} f \circ S^n \circ \phi(g, y) \, d(m \times \delta_{y_0})$$

$$= \int_{G \times Y} f \circ \phi(T^n(g, y)) \, d(m \times \delta_{y_0})$$

$$= \int_G f \circ \phi((g + n\mathbf{1}, S^{\theta_n(g)} y_0)) \, dm(g)$$

$$= \sum_{j \in \mathbb{Z}} \int_{\{g : \theta_n(g) = j\}} f \circ \phi((g + n\mathbf{1}, S^j y_0)) \, dm(g).$$

If we now further assume that the function f depends only on coordinates i with $|i| \le N$ for a fixed N then, taking into account the way the map ϕ is defined on $\pi(D) \times Y$ as well as Eq. (2), we see that indeed

$$\lim_{n \to -\infty} \int f \, dS^n \nu = \int f \, d\mu_0 \quad \text{and} \quad \lim_{n \to +\infty} \int f \, dS^n \nu = \int f \, d\mu_1.$$

Since the collection of functions f depending on finitely many coordinates is uniformly dense in $C(X)$ this proves Eq. (1) and our proof is complete. $\qquad \square$

In view of the last two sections one would like to know how minimal weakly mixing systems behave with respect to injectivity.

Problem 1. Construct examples of minimal weakly mixing $\mathbb{Z}$-flows which are injective (not injective).

6.3 A Non-Injective $\mathbb{Z}^2$-Dynamical System Which Admits an E-Compatible Faithful Affine Compactification

Let $X = \{0, 1\}^{\mathbb{Z}}$ and let σ denote the shift transformation on X. Define two $\mathbb{Z}^2$-actions on X by

$$\Phi_{mn} x = \sigma^m x \quad \text{and} \quad \Psi_{mn} x = \sigma^n x.$$

Since $E(X, \sigma)$ is canonically isomorphic to $\beta\mathbb{Z}$ we clearly have $E(X, \Phi) \cong E(X, \Psi) \cong \beta\mathbb{Z}$. In particular it follows that the two $\mathbb{Z}^2$-systems (X, Φ) and (X, Ψ)

are injective. Next consider the product $\mathbb{Z}^2$-system $(X \times X, \varXi)$, where the action is diagonal; i.e.,

$$\varXi_{mn}(x, y) = (\varPhi_{mn} x, \varPsi_{mn} y) = (\sigma^m x, \sigma^n y).$$

The proof of the next claim is straightforward.

Claim. $E(X \times X, \varXi) \cong E(X, \varPhi) \times E(X, \varPsi) \cong \beta\mathbb{Z} \times \beta\mathbb{Z}$. Moreover, identifying an element p of $E(X \times X, \varXi)$ with a pair $p = (p_\varPhi, p_\varPsi)$, we have $p = (p_\varPhi, p_\varPsi) = (p_\varPhi, \text{id}) \circ (\text{id}, p_\varPsi)$ and if $\varXi_{m_i n_i} \to p$ then, $(\varPhi_{m_i 0}, \text{id}) = (\varPhi_{m_i 0}, \varPsi_{m_i 0}) = \varXi_{m_i 0} \to (p_\varPhi, \text{id})$ and $(\text{id}, \varPsi_{0 n_i}) = (\varPhi_{0 n_i}, \varPsi_{0 n_i}) = \varXi_{0 n_i} \to (\text{id}, p_\varPsi)$.

Proposition 10. *The product dynamical system $(\mathbb{Z}^2, X \times X)$ (of two different $\mathbb{Z}^2$-flows) is not injective.*

Proof. Suppose to the contrary that it is injective, i.e., that we have $E(P(X \times X), \varXi) \cong E(X \times X, \varXi) \cong E(X, \varPhi) \times E(X, \varPsi)$. Let μ be the Bernoulli measure $(1/2(\delta_0 + \delta_1))^\mathbb{Z}$ on X, an element of $P(X)$. We let Δ_μ, an element of $P(X \times X)$, be the corresponding graph measure on $X \times X$ defined as the push-forward of μ via the diagonal map $x \mapsto (x, x)$. Let $A = \{\varXi_{mm} : m \in \mathbb{Z}\}$ and let $p \in \overline{A} \subset E(P(X \times X), \varXi)$ be any element which is not in A so that $p = \lim \varXi_{m_i m_i}$ with $m_i \nearrow \infty$ a net in $\mathbb{Z}$. We let $p_\varPhi$ denote its projection in $E(P(X), \varPhi)$ and $p_\varPsi$ its projection in $E(P(X), \varPsi)$.

Claim.

1. $p\Delta_\mu = \Delta_\mu$
2. $(p_\varPhi, \text{id})\Delta_\mu = \mu \times \mu$
3. $(\text{id}, p_\varPsi)\Delta_\mu = \mu \times \mu$

Proof. The first equality holds trivially as Δ_μ is A-invariant. The second and third equalities follow from the fact that (X, μ, σ) is mixing as a measure dynamical system. $\square$

We complete the proof of the proposition by pointing out the following absurd:

$$\Delta_\mu = p\Delta_\mu = (p_\varPhi, \text{id}) \circ (\text{id}, p_\varPsi)\Delta_\mu = (p_\varPhi, \text{id})\mu \times \mu = \mu \times \mu.$$

$\square$

Example 1.

1. The semigroup $\mathbb{Z}^2$-compactification defined naturally by the embedding

$$\nu : \mathbb{Z}^2 \to Y = \beta\mathbb{Z} \times \beta\mathbb{Z}$$

   is not injective but admits an E-compatible faithful affine compactification.
2. There exists an intro-generated m-introverted Banach subalgebra of $l_\infty(\mathbb{Z}^2)$ which is not introverted.

Proof. Let V be the Banach subspace of $l_\infty(\mathbb{Z}^2)$ consisting of functions of the form $f(x_1, x_2) = f_1(x_1) + f_2(x_2)$ with $f_1, f_2 \in l_\infty(\mathbb{Z})$. It is easy to see that

V is an introverted $\mathbb{Z}^2$-subspace of $C(Y)$. Furthermore, by the Stone-Weierstrass theorem, the closed algebra $\langle V \rangle$ generated by V is the algebra $\mathcal{A} \subset l_\infty(\mathbb{Z}^2)$ which corresponds to the compactification $\nu : \mathbb{Z}^2 \to Y = \beta\mathbb{Z} \times \beta\mathbb{Z}$. In particular, by Propositions 4 and 6, $(\mathbb{Z}^2, Y)$ admits an E-compatible faithful affine compactification. However, the $\mathbb{Z}^2$-flow Y is not injective. Indeed, Y is just the enveloping semigroup $E(\mathbb{Z}^2, X \times X)$ of the $\mathbb{Z}^2$-flow $X \times X$ from Proposition 10. Therefore, by Lemma 20 the injectivity of Y will imply (observe that Y is transitive) that $X \times X$ is injective contradicting Proposition 10. Finally, Theorem 1 shows that $\mathcal{A}$ is not introverted.   □

The non-metrizability of Y in Example 1 and of $\mathbb{Z}^{D(\mathbb{Z})}$ in Theorem 3 is unavoidable by Theorem 18.

Problem 2. Are there examples as above with $\mathbb{Z}$ as the acting group, rather than $\mathbb{Z}^2$?

7   Tame and HNS Systems and Related Classes of Right Topological Semigroups

7.1   Some Classes of Right Topological Semigroups

To the basic classes of right topological semigroups listed in Definition 1 above, we add the following two which have naturally arisen in the study of tame and HNS dynamical systems.

Definition 16 ([24, 27]). A compact admissible right topological semigroup P is said to be:

1. [27] *tame* if the left translation $\lambda_a : P \to P$ is a fragmented map for every $a \in P$.
2. *HNS-semigroup* ($\mathcal{F}$-*semigroup* in [24]) if $\{\lambda_a : P \to P\}_{a \in P}$ is a fragmented family of maps.

These classes are closed under factors. We have the inclusions:

$$\{\text{compact semitopological semigroups}\} \subset \{\text{HNS-semigroups}\}$$

$$\subset \{\text{Tame semigroups}\}.$$

Lemma 22.

1. *Every compact semitopological semigroup P is a HNS-semigroup.*
2. *Every HNS-semigroup is tame.*
3. *If P is a* metrizable *compact right topological admissible semigroup then P is a HNS-semigroup.*

Proof.

(1) Apply Lemma 2 to $P \times P \to P$.
(2) Is trivial.
(3) Apply Lemma 2 to $P \times P \to P$. $\square$

If P is Fréchet, as a topological space, then P is a tame semigroup by Corollary 2 below.

7.2   Compact Semitopological Semigroups and WAP Systems

As usual, a continuous function on a (not necessarily compact) S-space X is said to be WAP if the weakly closure of the orbit fS is weakly compact in $C(X)$. Notation: $f \in \mathrm{WAP}(X)$. It is equivalent that $f = \tilde{f} \circ \alpha$ comes from an S-compactification $\alpha : X \to Y$ such that $\tilde{f} \in \mathrm{WAP}(Y)$. In fact, one may choose the cyclic S-compactification $Y = X_f$. A compact dynamical S-system X is said to be WAP if $C(X) = \mathrm{WAP}(X)$. The latter happens iff every element $p \in E(X)$ is a continuous self-map of X (Ellis and Nerurkar).

Proposition 11. *Let V be a Banach space. The following are equivalent:*

1. *V is reflexive.*
2. *The compact semigroup $\mathcal{E}$ is semitopological.*
3. *$\mathcal{E} = \Theta^{\mathrm{op}}$.*
4. *Θ is compact with respect to the weak operator topology.*
5. *$(\Theta^{\mathrm{op}}, B^*)$ is a WAP system.*

Proof. Use Lemma 8 and the standard characterizations of reflexive Banach spaces. $\square$

Fact 10. ([49, Section 4]). *Let S be a semitopological semigroup.*

1. *A compact (continuous) S-space X is WAP if and only if (S, X) is weakly (respectively, strongly) reflexively approximable.*
2. *A compact (continuous) metric S-space X is WAP if and only if (S, X) is weakly (respectively, strongly) reflexively representable.*

We next recall a version of Lawson's theorem [42] and its soft geometrical proof using representations of dynamical systems on reflexive spaces.

Fact 11 (Ellis-Lawson's Joint Continuity Theorem). *Let G be a subgroup of a compact semitopological monoid S. Suppose that $S \times X \to X$ is a separately continuous action with compact X. Then the action $G \times X \to X$ is jointly continuous and G is a topological group.*

Proof. A sketch of the proof from [48]: It is easy to see by Grothendieck's Lemma that $C(X) = \mathrm{WAP}(X)$. Hence (S, X) is a WAP system. By Theorem 10 the proof can be reduced to the particular case where $(S, X) = (\Theta(V)^{\mathrm{op}}, B_{V^*})$ for

some reflexive Banach space V with $G := \mathrm{Iso}\,(V)$, where $\Theta(V)^{\mathrm{op}}$ is endowed with the weak operator topology. By [47] the weak and strong operator topologies coincide on $\mathrm{Iso}\,(V)$ for reflexive V. In particular, G is a topological group and it acts continuously on B_{V^*}. $\Box$

As a corollary one gets the classical result of Ellis. See also a generalization in Theorem 21.

Fact 12 (Ellis Theorem). *Every compact semitopological group is a topological group.*

Another consequence of Theorem 10 (taking into account Proposition 11) is

Fact 13 ([47, 67]). *Every compact semitopological semigroup S is embedded into $\Theta(V) = \mathcal{E}(V^*)$ for some reflexive V.*

Thus, compact semitopological semigroups S can be characterized as closed subsemigroups of $\mathcal{E}(V)$ for reflexive Banach spaces V. We will show below, in Theorem 20, that analogous statements (for admissible embeddings) hold for HNS and tame semigroups, where the corresponding classes of Banach spaces are Asplund and Rosenthal spaces, respectively.

7.3 HNS-Semigroups and Dynamical Systems

The following definition (for continuous group actions) originated in [24]. One may extend it to separately continuous semigroup actions.

Definition 17. We say that a compact S-system X is *hereditarily non-sensitive* (HNS, in short) if one of the following equivalent conditions are satisfied:

1. For every closed nonempty subset $A \subset X$ and for every entourage ε from the unique compatible uniformity on X there exists an open subset O of X such that $A \cap O$ is nonempty and $s(A \cap O)$ is ε-small for every $s \in S$.
2. The family of translations $\widetilde{S} := \{\tilde{s} : X \to X\}_{s \in S}$ is a fragmented family of maps.
3. $E(S, X)$ is a fragmented family of maps from X into itself.

The equivalence of (1) and (2) is evident from the definitions. Clearly, (3) implies (2). As to the implication (2) $\Rightarrow$ (3), observe that the pointwise closure of a fragmented family is again a fragmented family, [27, Lemma 2.8].

Note that if $S = G$ is a group then in Definition 17 one may consider only closed subsets A which are G-invariant (see the proof of [24, Lemma 9.4]).

Lemma 23.

1. *For every S the class of HNS compact S-systems is closed under subsystems, arbitrary products, and factors.*
2. *For every HNS compact S-system X the corresponding enveloping semigroup $E(X)$ is HNS both as an S-system and as a semigroup.*

3. *Let P be a HNS-semigroup. Assume that $j : S \to P$ be a continuous homomorphism from a semitopological semigroup S into P such that $j(S) \subset \Lambda(P)$. Then the S-system P is HNS.*
4. *{HNS-semigroups}={enveloping semigroups of HNS systems}.*

Proof.

(1) As in [27] using the stability properties of fragmented families.
(2) (S, E) is a HNS system because HNS is preserved by subdirect products. So, by Definition 17, $\{\lambda_a : E \to E\}_{a \in j(S)}$ is a fragmented family of maps. Then its pointwise closure $\{\lambda_a : E \to E\}_{\underline{a \in E}}$ is also a fragmented family.
(3) Since $j(S) \subset \Lambda(P)$ the closure $\overline{j(S)}$ is a subsemigroup of P. We can assume that $\overline{j(S)} = P$. By Lemma 5, the enveloping semigroup $E(S, P) \subset P^P$ can be naturally identified with P so that every $a \in E(S, P)$ is identified with the corresponding left translation $\lambda_a : P \to P$. Since P is a HNS-semigroup the set of all left translations $\{\lambda_a : P \to P\}_{a \in E}$ is a fragmented family. Hence, (S, P) is a HNS system (Definition 17).
(4) Combine (2) and (3) taking into account Lemma 5.   □

Theorem 5. *Let V be a Banach space. The following are equivalent:*

1. *V is an Asplund Banach space.*
2. *$(\Theta^{\mathrm{op}}, B^*)$ is a HNS system.*
3. *$\mathcal{E}$ is a HNS-semigroup.*

Proof.

(1) $\Rightarrow$ (2): Use Definition 17 and the following well-known characterization of Asplund spaces: V is Asplund iff B^* is $(w^*, norm)$-fragmented (Fact 2).
(2) $\Rightarrow$ (1): By Fact 2 we have to show that B is a fragmented family for B^*. Choose a vector $v \in S_V$. Since Θ^{op} is a fragmented family of self-maps on B^* and as $v : B^* \to \mathbb{R}$ is uniformly continuous we get that the system $v\Theta^{\mathrm{op}} = \Theta v$ of maps from B^* to $\mathbb{R}$ is also fragmented. Now recall that $\Theta v = B$ by Lemma 8.
(2) $\Rightarrow$ (3): Follows from Lemma 23 and the fact that $\mathcal{E}$ is the enveloping semigroup $E(\Theta^{\mathrm{op}}, B^*)$.
(3) $\Rightarrow$ (2): $\Lambda(\mathcal{E}) = \Theta^{\mathrm{op}}$ (Lemma 8) and $\mathcal{E}$ is a HNS-semigroup. So, $(S, \mathcal{E})$ is HNS by Lemma 23 with $S = \Theta^{\mathrm{op}}$. Take $\psi \in B^*$ with $||\psi|| = 1$. The map $q : \mathcal{E} \to B^*, \ p \mapsto p\psi$ defines a continuous homomorphism of Θ^{op}-systems. By Lemma 8, we have $\mathcal{E}\psi = B^*$. So q is onto. Now observe that the HNS property is preserved by factors of S-systems (Lemma 23).   □

Our next theorem is based on ideas from [29].

Theorem 6. *Let V be a Banach space. The following are equivalent:*

1. *V is a separable Asplund space.*
2. *$\mathcal{E}$ is homeomorphic to the Hilbert cube $[-1, 1]^{\mathbb{N}}$ (for infinite-dimensional V).*
3. *$\mathcal{E}$ is metrizable.*

Proof.

(1) $\Rightarrow$ (2): Since $\mathcal{E}$ is a compact affine subset in the Fréchet space $\mathbb{R}^{\mathbb{N}}$ we can use Keller's Theorem [9, p. 100].

(2) $\Rightarrow$ (3): Is trivial.

(3) $\Rightarrow$ (1): $\mathcal{E}$ is a HNS-semigroup by Lemma 22. Now Theorem 5 implies that V is Asplund. It is also separable; indeed, by Lemma 8, B^* is a continuous image of $\mathcal{E}$, so that B^* is also w^*-metrizable, which in turn yields the separability of V. □

Now in Theorem 7 we obtain a short proof of one of the main results of [29] (stated there for continuous group actions).

Theorem 7. *Let X be a compact S-system. Consider the following assertions:*

(a) $E(X)$ *is metrizable.*
(b) (S, X) *is HNS.*

Then:

1. (a) $\Rightarrow$ (b).
2. If X, in addition, is metrizable then (a) $\Leftrightarrow$ (b).

Proof.

(1) By Definition 17 we have to show that $E(X)$ is a fragmented family of maps from X into itself. The unique compatible uniformity on the compactum X is the weakest uniformity on X generated by $C(X)$. Using Remark 1 one may reduce the proof to the verification of the following claim: $E^f := \{f \circ p : p \in E(X)\}$ is a fragmented family for every $f \in C(X)$. In order to prove this claim apply Lemma 2 to the induced mapping $E(X) \times X \to \mathbb{R}, (p, x) \mapsto f(px)$ (using our assumption that $E(X)$ is metrizable).

(2) If X is a metrizable HNS S-system then by Theorem 11 below, (S, X) is representable on a separable Asplund space V. We can assume that X is S-embedded into B^*. The enveloping semigroup $E(S, B^*)$ is embedded into $\mathcal{E}$ (Lemma 7). The latter is metrizable by virtue of Theorem 6. Hence $E(S, X)$ is also metrizable, being a continuous image of $E(S, B^*)$. □

Proposition 12. *Let S be a semitopological semigroup and $\alpha : S \to P$ be a right topological semigroup compactification.*

1. If P is metrizable then P is a HNS-semigroup and the system (S, P) is HNS.
2. Let $V \subset C(S)$ be an m-introverted closed subalgebra of $C(S)$. If V is separable then necessarily $V \subset \mathrm{Asp}(S)$.

Proof.

(1) By Lemma 22, P is a HNS-semigroup. By Lemma 23, the system (S, P) is HNS.

(2) By Fact 8, the algebra V induces a semigroup compactification $S \to P$. Since V is separable, P is metrizable. So by (1), (S, P) is HNS. Therefore, $V \subset \mathrm{Asp}(S)$. □

7.4 Tame Semigroups and Tame Systems

Definition 18. A compact separately continuous S-system X is said to be *tame* if the translation $\lambda_a : X \to X$, $x \mapsto ax$ is a fragmented map for every element $a \in E(X)$ of the enveloping semigroup.

This definition is formulated in [27] for continuous group actions.

According to Remark 5 we define, for every S-space X, the S-subalgebras $\mathrm{Tame}(X)$ and $\mathrm{Tame}_c(X)$ of $C(X)$. Recall that in several natural cases we have $\mathcal{P}_c(X) = \mathcal{P}(X)$ (see Lemma 4).

Lemma 24. *Every WAP system is HNS and every HNS is tame. Therefore, for every semitopological semigroup S and every S-space X (in particular, for $X := S$) we have*

$$\mathrm{WAP}(X) \subset \mathrm{Asp}(X) \subset \mathrm{Tame}(X) \quad \mathrm{WAP}_c(X) \subset \mathrm{Asp}_c(X) \subset \mathrm{Tame}_c(X).$$

Proof. We can suppose that X is compact. If (S, X) is WAP then $E(X) \times X \to X$ is separately continuous. By Lemma 2 we obtain that E is a fragmented family of maps from X to X. In particular, its subfamily $\{\tilde{s} : X \to X\}_{s \in S}$ of all translations is fragmented. Hence, (S, X) is HNS.

Directly from the definitions we conclude that every HNS is tame. □

Another proof of Lemma 24 comes also from Banach representations theory for dynamical systems because every reflexive space is Asplund and every Asplund is Rosenthal.

By [28], a compact metrizable S-system X is tame iff S is *eventually fragmented* on X, that is, for every infinite (countable) subset $C \subset G$ there exists an infinite subset $K \subset C$ such that K is a fragmented family of maps $X \to X$.

Lemma 25.

1. *For every S the class of tame S-systems is closed under closed subsystems, arbitrary products, and factors.*
2. *For every tame compact S-system X the corresponding enveloping semigroup $E(X)$ is tame both as an S-system and as a semigroup.*
3. *Let P be a tame right topological compact semigroup and let $v : S \to P$ be a continuous homomorphism from a semitopological semigroup S into P such that $v(S) \subset \Lambda(P)$. Then the S-system P is tame.*
4. *{Tame semigroups}={enveloping semigroups of tame systems}.*

Proof.

(1) As in [27] using the stability properties of fragmented maps.
(2) (S, E) is a tame system because by (1) tameness is preserved by subdirect products. Its enveloping semigroup can be identified with E itself (Lemma 5), so that $\lambda_p : E \to E$ is fragmented for every $p \in E$.
(3) $\nu(S) \subset \Lambda(P)$, so $\overline{\nu(S)}$ is a semigroup. We can assume that $\overline{\nu(S)} = P$. By Lemma 5, the enveloping semigroup $E(S, P) \subset P^P$ can be naturally identified with P in such a way that every $a \in E(S, P)$ is identified with the corresponding left translation $\lambda_a : P \to P$ for some $a \in P$. Since P is a tame semigroup every left translation $\lambda_a : P \to P$ is fragmented. Hence, (S, P) is a tame system.
(4) Combine (2) and (3) taking into account Lemma 5 of point 4.   □

Proposition 13. *Let X be a compact S-space and $f \in C(X)$. The following conditions are equivalent:*

1. *$f \in \text{Tame}(X)$.*
2. *$\text{cl}_p(fS) \subset \mathcal{F}(X)$ (i.e., the orbit fS is a Rosenthal family for X).*

Proof. See [27, Proposition 5.6].   □

Theorem 8. *Let V be a Banach space. The following are equivalent:*

1. *V is a Rosenthal Banach space.*
2. *$(\Theta^{\text{op}}, B^*)$ is a tame system.*
3. *$p : B^* \to B^*$ is a fragmented map for each $p \in \mathcal{E}$.*
4. *$\mathcal{E}$ is a tame semigroup.*

Proof.

(2) $\Leftrightarrow$ (3): Follows from the definition of tame flows because $\mathcal{E} = E(\Theta^{\text{op}}, B^*)$.
(2) $\Rightarrow$ (4): Since $\mathcal{E} = E(\Theta^{\text{op}}, B^*)$, Lemma 25 applies.
(4) $\Rightarrow$ (2): By our assumption, $\mathcal{E}$ is a tame semigroup. Then by Lemma 25 the system $(\Theta^{\text{op}}, \mathcal{E})$ is tame. Its factor (Lemma 8) $(\Theta^{\text{op}}, B^*)$ is tame, too.
(2) $\Rightarrow$ (1): By a characterization of Rosenthal spaces [27, Proposition 4.12] (see also Fact 4) it suffices to show that $B^{**} \subset \mathcal{F}(B^*)$. Since $(\Theta^{\text{op}}, B^*)$ is tame, $p : B^* \to B^*$ is fragmented for every $p \in E(\Theta^{\text{op}}, B^*) = \mathcal{E}$. Pick an arbitrary $v \in B_V$ with $\|v\| = 1$. Then $v\mathcal{E}$ is exactly B^{**} by Lemma 8. So every $\phi \in B^{**}$ is a composition $v \circ p$, where p is a fragmented map. Since $v : B^* \to \mathbb{R}$ is weak* continuous we conclude that $\phi : B^* \to B^*$ is fragmented.
(1) $\Rightarrow$ (3): We have to show that $\mathcal{E} \subset \mathcal{F}(B^*, B^*)$ for every Rosenthal space V. Let $p \in \mathcal{E}$. Then $p \in \Theta(V^*)$. That is, p is a linear map $p : V^* \to V^*$ with norm ≤ 1. Then, for every vector $f \in V$, the composition $f \circ p : V^* \to \mathbb{R}$ is a linear bounded functional on V^*. That is, $f \circ p \in V^{**}$ belongs to the second dual. Again, by the above mentioned characterization of Rosenthal spaces, the corresponding restriction $f \circ p|_{B^*} : B^* \to \mathbb{R}$ is a fragmented function for every $f \in V$. Since V separates points of B^* we can apply [27, Lemma 2.3.3]. It follows that $p : B^* \to B^*$ is fragmented for every $p \in \mathcal{E}$.   □

7.5   A Dynamical BFT Dichotomy

Recall that a topological space K is a *Rosenthal compactum* [32] if it is home-omorphic to a pointwise compact subset of the space $\mathcal{B}_1(X)$ of functions of the first Baire class on a Polish space X. All metric compact spaces are Rosenthal. An example of a separable non-metrizable Rosenthal compactum is the *Helly compact* of all nondecreasing self-maps of $[0, 1]$ in the pointwise topology. Recall that a topological space K is *Fréchet* (or, *Fréchet-Urysohn*) if for every $A \subset K$ and every $x \in \text{cl}(A)$ there exists a sequence of elements of A which converges to x. Every Rosenthal compact space K is Fréchet by a result of Bourgain-Fremlin-Talagrand [10, Theorem 3F], generalizing a result of Rosenthal.

Theorem 9. *If the enveloping semigroup $E(X)$ is a Fréchet (e.g., Rosenthal) space, as a topological space, then (S, X) is a tame system (and $E(X)$ is a tame semigroup).*

Proof. Let $p \in E(X)$. We have to show that $p : X \to X$ is fragmented. By properties of fragmented maps [27, Lemma 2.3.3] it is enough to show that $f \circ p : X \to \mathbb{R}$ is fragmented for every $f \in C(X)$. By the Fréchet property of $E(X)$ we may choose a sequence s_n in S such that the sequence $j(s_n)$ converges to p in E(X). Hence the sequence of continuous functions $f \circ s_n = f \circ j(s_n)$ converges pointwise to $f \circ p$ in $\mathbb{R}^X$. Apply point 2 of Lemma 2 to the evaluation map $F \times X \to \mathbb{R}$, where $F := \{f \circ p\} \cup \{f \circ j(s_n)\}_{n \in \mathbb{N}} \subset \mathbb{R}^X$ carries the pointwise topology. We conclude that F is a fragmented family. In particular, $f \circ p$ is a fragmented map. ($E(X)$ is a tame semigroup by Lemma 25 of point 2.)   □

Corollary 2. *Let P be a compact right topological admissible semigroup. If P is Fréchet (e.g., when it is Rosenthal), as a topological space, then P is a tame semigroup.*

Proof. Applying Theorem 9 to the system (S, P), with $S := \Lambda(P)$ we obtain that $E(S, P) = P$ is a tame semigroup.   □

The following result was proved in [24, Theorem 3.2] using the Bourgain-Fremlin-Talagrand (BFT) dichotomy in the setting of continuous group actions. The same arguments work also for separately continuous semigroup actions. For the sake of completeness we include a simplified proof.

Fact 14 (A dynamical BFT dichotomy). *Let X be a compact metric dynamical S-system and let $E = E(X)$ be its enveloping semigroup. We have the following alternative. Either*

1. *E is a separable Rosenthal compact, hence $card\ E \leq 2^{\aleph_0}$; or*
2. *The compact space E contains a homeomorphic copy of $\beta\mathbb{N}$, hence $card\ E = 2^{2^{\aleph_0}}$.*

The first possibility holds iff X is a tame S-system.

Proof. For every $f \in C(X)$ define $E^f := \{f \circ p : p \in E\}$. Then E^f is a pointwise compact subset of $\mathbb{R}^X$, being a continuous image of E under the map $q_f : E \to E^f$, $p \mapsto f \circ p$. Since X is metrizable by Lemma 5 there exists a sequence $\{s_m\}_{m=1}^{\infty}$ in S such that $\{j(s_m)\}_{m=1}^{\infty}$ is dense in $E(X)$. In particular, the sequence of real valued functions $\{f \circ s_m\}_{m=1}^{\infty}$ is pointwise dense in E^f.

Choose a sequence $\{f_n\}_{n \in \mathbb{N}}$ in $C(X)$ which separates the points of X. For every pair s, t of distinct elements of E there exist a point $x_0 \in X$ and a function f_{n_0} such that $f_{n_0}(sx_0) \neq f_{n_0}(tx_0)$. It follows that the continuous diagonal map

$$\Phi : E \to \prod_{n \in \mathbb{N}} E^{f_n}, \qquad p \mapsto (f_1 \circ p, f_2 \circ p, \dots)$$

separates the points of E and hence is a topological embedding. Now if for each n the space E^{f_n} is a Rosenthal compactum then so is $E \cong \Phi(E) \subset \prod_{n=1}^{\infty} E^{f_n}$, because the class of Rosenthal compacta is closed under countable products and closed subspaces. On the other hand if at least one $E^{f_n} = \mathrm{cl}_p(\{f_n \circ s_m\}_{m=1}^{\infty})$ is not Rosenthal then by a version of the BFT dichotomy (Todorčević [69, Sect. 13]) it contains a homeomorphic copy of $\beta\mathbb{N}$ and it is easy to see that so does its preimage E. In fact if $\beta\mathbb{N} \cong Z \subset E^{f_n}$ then any closed subset Y of E which projects onto Z and is minimal with respect to these properties is also homeomorphic to $\beta\mathbb{N}$.

Now we show the last assertion. If X is tame then every $p \in E(X)$ is a fragmented self-map of X. Hence every $f \circ p \in E^f$ is fragmented. By Remark 1 this is equivalent to saying that every $f \circ p$ is Baire 1. So $E^f \subset \mathcal{B}_1(X)$ is a Rosenthal compactum. Therefore, $E \cong \Phi(E) \subset \prod_{n \in \mathbb{N}} E^{f_n}$ is also Rosenthal. Conversely, if E is a Rosenthal compactum then (S, X) is tame by Theorem 9.   □

Theorem 10 (BFT dichotomy for Banach spaces). *Let V be a separable Banach space and let $\mathcal{E} = \mathcal{E}(V)$ be its (separable) enveloping semigroup. We have the following alternative. Either*

1. *$\mathcal{E}$ is a Rosenthal compactum, hence card $\mathcal{E} \leq 2^{\aleph_0}$; or*
2. *The compact space $\mathcal{E}$ contains a homeomorphic copy of $\beta\mathbb{N}$, hence card $\mathcal{E} = 2^{2^{\aleph_0}}$.*

The first possibility holds iff V is a Rosenthal Banach space.

Proof. Recall that $\mathcal{E} = E(\Theta^{\mathrm{op}}, B^*)$. By Theorem 8, V is Rosenthal iff $(\Theta^{\mathrm{op}}, B^*)$ is tame. Since V is separable, B^* is metrizable. So we can apply Fact 14.   □

7.6 Amenable Affine Compactifications

Let G be a topological group and X a G-space. Let us say that an affine S-compactification $\alpha : X \to Y$ is *amenable* if Y has a G-fixed point. We say that a closed unital linear subspace $\mathcal{A} \subset \mathrm{WRUC}(X)$ is (left) *amenable* if

the corresponding affine G-compactification is amenable. By Ryll-Nardzewski's classical theorem WAP(G) is amenable. Let $f \in \mathrm{RUC}(G)$ and let $\pi_f : X \to Q_f$ be the corresponding cyclic affine G-compactification (Sect. 4.2). In our recent work [26] we show that $\mathrm{Asp}_c(G)$ is amenable and that for every $f \in \mathrm{Asp}_c(G)$ there exists a G-fixed point (a G-average of f) in Q_f. The first result together with Proposition 12 yields the following:

Corollary 3. *Let G be a topological group and $\mathcal{A}$ a (left) m-introverted closed subalgebra of* $\mathrm{RUC}(G)$. *If $\mathcal{A}$ is separable then $\mathcal{A}$ is amenable.*

A topological group G is said to be amenable if $\mathrm{RUC}(G)$ is amenable. By a classical result of von Neumann, the free discrete group $\mathbb{F}_2$ on two symbols is not amenable. So, $\mathrm{RUC}(\mathbb{F}_2) = l_\infty(\mathbb{F}_2)$ is not amenable. By [26], $\mathrm{Tame}(\mathbb{F}_2)$ is not amenable. It would be interesting to study for which non-amenable groups G the algebra $\mathrm{Tame}_c(G)$ is amenable and for which $f \in \mathrm{Tame}_c(G)$ there exists a G-fixed point of Q_f.

Example 2.

1. Results of [28] show that $\varphi_D(n) = sgn \cos(2\pi n \alpha)$ is a tame function on $\mathbb{Z}$ which is not Asplund.
2. As a simple illustration of Proposition 12 note that the two-point semigroup compactifications of $\mathbb{Z}$ and $\mathbb{R}$ are obviously metrizable. So the characteristic function $\xi_{\mathbb{N}} : \mathbb{Z} \to \mathbb{R}$ and $arctg : \mathbb{R} \to \mathbb{R}$ are both Asplund. Grothendieck's double limit criterion shows that these functions are not WAP.

8 Representations of Semigroup Actions on Banach Spaces

As was shown in several of our earlier works some properties of dynamical systems are clearly reflected in analogous properties of their enveloping semigroups on the one hand and in their representations on Banach spaces on the other. Our results from [24, 27, 29] are formulated for group actions. However the main results in these papers remain true for semigroup actions.

For continuous group actions the results (1) and (2) of the following theorem were proved, respectively, in [24, 27] (compare also with Theorem 10). We will show next how the proofs of (1) and (2) can be modified to suit the more general case of semigroup actions, obtaining, in fact, also some new results.

Theorem 11. *Let S be a semitopological semigroup and X a compact S-system with a separately continuous action.*

1. *(S, X) is a tame (continuous) system if and only if (S, X) is weakly (respectively, strongly) Rosenthal-approximable.*
2. *(S, X) is a HNS (continuous) system if and only if (S, X) is weakly (respectively, strongly) Asplund-approximable.*

If X is metrizable then in (1) and (2) "approximable" can be replaced by "representable." Moreover, the corresponding Banach space can be assumed to be separable.

Proof. The proof for continuous actions is the same as in [27]. So below we show only how the proof can be adopted for separately continuous actions and weakly continuous representations.

The "only if" part: For (1) use the fact that $(\Theta^{\mathrm{op}}, B^*)$ is a tame system (Theorem 8) for every Rosenthal V and for (2), the fact that $(\Theta^{\mathrm{op}}, B^*)$ is HNS (Theorem 5) for Asplund V.

The "if" part: (1) For every $f \in C(X) = \mathrm{Tame}(X)$ the orbit fS is a Rosenthal family for X (Proposition 13). Applying Theorem 12 below we conclude that every $f \in C(X) = \mathrm{Tame}(X)$ on a compact G-space X comes from a Rosenthal representation. Since continuous functions separate points of X, this implies that Rosenthal representations of (S, X) separate points of X. So, for (1), it is enough to prove the following result.

Theorem 12. *Let X be a compact S-space and let $F \subset C(X)$ be a Rosenthal family for X such that F is S-invariant; that is, $fS \subset F \ \ \forall f \in F$. Then there exist a Rosenthal Banach space V, an injective mapping $v : F \to B_V$, and a representation*

$$h : S \to \Theta(V), \quad \alpha : X \to V^*$$

of (S, X) on V such that h is weakly continuous, α is a weak* continuous map, and

$$f(x) = \langle v(f), \alpha(x) \rangle \ \ \forall f \in F \ \forall x \in X.$$

Thus the following diagram commutes:

$$
\begin{array}{ccc}
F \times X & \longrightarrow & \mathbb{R} \\
{\scriptstyle v}\downarrow \quad {\scriptstyle \alpha}\downarrow & & \downarrow {\scriptstyle id_{\mathbb{R}}} \\
V \times V^* & \longrightarrow & \mathbb{R}
\end{array}
\tag{3}
$$

If X is metrizable then in addition we can suppose that V is separable.

If the action $S \times X \to X$ is continuous we may assume that h is strongly continuous.

Proof.

Step 1: The construction of V.

For brevity of notation let $\mathcal{A} := C(X)$ denote the Banach space $C(X)$, B will denote its unit ball, and B^* will denote the weak* compact unit ball of the dual space $\mathcal{A}^* = C(X)^*$. Let W be the symmetrized convex hull of F; that is, $W := \mathrm{co}\,(F \cup -F)$. Consider the sequence of sets

$$M_n := 2^n W + 2^{-n} B. \tag{4}$$

Then W is convex and symmetric. We apply the construction of Davis-Figiel-Johnson-Pelczyński [14] as follows. Let $\| \ \|_n$ be the Minkowski functional of the set M_n, that is,

$$\|v\|_n = \inf \{\lambda > 0 \mid v \in \lambda M_n\}.$$

Then $\| \ \|_n$ is a norm on $\mathcal{A}$ equivalent to the given norm of $\mathcal{A}$. For $v \in \mathcal{A}$, set

$$N(v) := \left(\sum_{n=1}^{\infty} \|v\|_n^2 \right)^{1/2} \quad \text{and let} \quad V := \{v \in \mathcal{A} \mid N(v) < \infty\}.$$

Denote by $j : V \hookrightarrow \mathcal{A}$ the inclusion map. Then (V, N) is a Banach space, $j : V \to \mathcal{A}$ is a continuous linear injection, and

$$W \subset j(B_V) = B_V \subset \bigcap_{n \in \mathbb{N}} M_n = \bigcap_{n \in \mathbb{N}} (2^n W + 2^{-n} B). \tag{5}$$

Step 2: The construction of the representation (h, α) of (S, X) on V.

The given action $S \times X \to X$ induces a natural linear norm preserving continuous right action $C(X) \times S \to C(X)$ on the Banach space $\mathcal{A} = C(X)$. It follows by the construction that W and B are S-invariant subsets in $\mathcal{A}$. This implies that V is an S-invariant subset of $\mathcal{A}$ and the restricted natural linear action $V \times S \to V$, $(v, g) \mapsto vg$ satisfies $N(vs) \leq N(v)$. Therefore, the co-homomorphism $h : S \to \Theta(V)$, $h(s)(v) := vs$ is well defined.

Let $j^* : \mathcal{A}^* \to V^*$ be the adjoint map of $j : V \to \mathcal{A}$. Define $\alpha : X \to V^*$ as follows. For every $x \in X \subset C(X)^*$ set $\alpha(x) = j^*(x)$. Then (h, α) is a representation of (S, X) on the Banach space V.

By the construction $F \subset W \subset B_V$. Define $\nu : F \hookrightarrow B_V$ as the natural inclusion. Then

$$f(x) = \langle \nu(f), \alpha(x) \rangle \quad \forall \ f \in F \ \ \forall \ x \in X. \tag{6}$$

Step 3: Weak continuity of $h : S \to \Theta(V)$.

By our construction $j^* : C(X)^* \to V^*$, being the adjoint of the bounded linear operator $j : V \to C(X)$, is a norm and weak* continuous linear operator. By Lemma 17 we obtain that $j^*(C(X)^*)$ is norm dense in V^*. Since V is Rosenthal, Haydon's theorem (Fact 4) gives $Q := \text{cl}_{w^*}(co(Y)) = \text{cl}_{\text{norm}}(co(Y))$, where $Y := j^*(X)$. Now observe that $j^*(P(X)) = Q$. Since $S \times X \to X$ is separately continuous, every orbit map $\tilde{x} : S \to X$ is continuous, and each orbit map $j^*(x) : S \to j^*(X)$ is weak* continuous. Then also $j^*(z) : S \to V^*$ is weak*

continuous for each $z \in \mathrm{cl}_{\mathrm{norm}}(co(j^*(X))) = Q$. Since $sp(Q)$ is norm dense in V^* (and $||h(s)|| \leq 1$ for each $s \in S$) it easily follows that $\tilde{j}^*(z) : S \to V^*$ is weak* continuous for every $z \in V^*$. This is equivalent to the weak continuity of h.

If the action $S \times X \to X$ is continuous we may assume that h is strongly continuous. Indeed, by the definition of the norm N, we can show that the action of S on V is norm continuous (use the fact that, for each $n \in \mathbb{N}$, the norm $\mathrm{norm}||\cdot||_n$ on $\mathcal{A}$ is equivalent to the given norm on $\mathcal{A}$).

Step 4: V is a Rosenthal space.

By results of [27, Sect. 4], W is a Rosenthal family for B^* (and X). Furthermore, a deeper analysis shows (we refer to [27, Theorem 6.3] for details) that B_V is a Rosenthal family for B_{V^*}. Thus V is Rosenthal by Fact 4.

If the compact space X is metrizable then $C(X)$ is separable and it is also easy to see that (V, N) is separable.

This proves Theorem 12 and hence also Theorem 11.   □

Now for point (2) of Theorem 11. (2) For the "Asplund case" one can modify the proof of (1). The main idea is that the corresponding results of [48, Sect. 7] and [24, Sect. 9] can be adopted here, thus obtaining a modification of Theorem 12 which replaces a Rosenthal space by an Asplund space and a "Rosenthal family F" for X by an "Asplund set." The latter means that for every countable subset $A \subset F$ the pseudometric ρ_A on X defined by

$$\rho_A(x, y) := \sup_{f \in A} |f(x) - f(y)|, \ x, y \in X$$

is separable. By [17, Lemma 1.5.3] this is equivalent to saying that $(C(X)^*, \rho_A)$ is separable. Now $co(F \cup -F)$ is an Asplund set for B^* by [17, Lemma 1.4.3]. The rest is similar to the proof of [48, Theorem 7.7]. Checking the weak continuity of h one can apply a similar idea (using again Haydon's theorem as in (1)).

Finally note that if X is metrizable then in (1) and (2) "approximable" can be replaced by "representable" using an l_2-sum of a sequence of separable Banach spaces (see Lemma 3).   □

Remark 17. The fundamental DFJP-factorization construction from [14] has an "isometric modification." According to [44] one may assume in Theorem 12 that the bounded operator $j : V \to \mathcal{A}$ has the property $||j|| \leq 1$. More precisely, we can replace in Eq. 4 the sequence of sets $M_n := 2^n W + 2^{-n} B$ by $K_n := a^{\frac{n}{2}} W + a^{-\frac{n}{2}} B$, where $2 < a < 3$ is the unique solution of the equation $\sum_{n=1}^{\infty} \frac{a^n}{(a^n+1)^2} = 1$. For details see [44]. Taking into account this modification (which is completely compatible with our S-space setting) for a set $F \subset C(X)$ with $\sup\{|f(x)| : x \in X, f \in F\} \leq 1$ we can assume that $\nu(F) \subset B$ and $\alpha(X) \subset B^*$. Hence the following sharper diagram commutes:

$$
\begin{array}{ccc}
F \times X & \longrightarrow & [-1, 1] \\
\Big\downarrow{\scriptstyle v} \quad \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle id} \\
B \times B^* & \longrightarrow & [-1, 1]
\end{array}
\tag{7}
$$

Note also that this modified version from [44] of the DFJP-construction repairs in particular the proof of [48, Theorem 4.5]. The latter was first corrected in the arxiv version of [48, Theorem 4.5] using, however, diagrams like Eq. (3), where $v(F)$ and $\alpha(X)$ are bounded.

Theorem 13.

1. *Let X be a compact S-space. The following conditions are equivalent:*

   (a) *$f \in \mathrm{Tame}(X)$ (respectively, $f \in \mathrm{Tame}_c(X)$).*
   (b) *There exist a weakly (respectively, strongly) continuous representation (h, α) of (S, X) on a Rosenthal Banach space V and a vector $v \in V$ such that $f(x) = \langle v, \alpha(x) \rangle \quad \forall x \in X$.*

2. *Let S be a semitopological semigroup and $f \in C(S)$. The following conditions are equivalent:*

   (a) *$f \in \mathrm{Tame}(S)$ (respectively, $f \in \mathrm{Tame}_c(S)$).*
   (b) *f is a matrix coefficient of a weakly (respectively, strongly) continuous co-representation of S on a Rosenthal space. That is, there exist a Rosenthal space V, a weakly (respectively, strongly) continuous co-homomorphism $h : S \to \Theta(V)$, and vectors $v \in V$ and $\psi \in V^*$ such that $f(s) = \psi(vs)$ for every $s \in S$.*

3. *Similar [to (1) and (2)] results are valid for:*

   (a) *Asplund functions and Asplund Banach spaces;*
   (b) *WAP functions and reflexive Banach spaces.*

Proof.

(1) (b) $\Rightarrow$ (a): $(\Theta(V)^{\mathrm{op}}, B^*)$ is a tame system for every Rosenthal space V by Theorem 8. The action is separately (jointly) continuous for the weak (respectively, strong) operator topology on $\Theta(V)^{\mathrm{op}}$.

   (a) $\Rightarrow$ (b): Let $f \in \mathrm{Tame}(X)$. This means by Proposition 13 that the orbit fS is a Rosenthal family for X. Now we can apply Theorem 12 to the family $F := fS$ (getting X_f as $\alpha(X)$).

(2) (a) $\Rightarrow$ (b): $f \in \mathrm{Tame}(S)$ (respectively, $f \in \mathrm{Tame}_c(S)$) means that there exist a tame S-compactification $\gamma : S \to X$ of the S-space S such that $S \times X \to X$ is separately continuous (respectively, jointly continuous) and a continuous function $f_0 : X \to \mathbb{R}$ such that $f = f_0 \circ v$. Apply Theorem 12 to f_0 getting the desired V and vectors $v := v(f)$ and $\psi := \alpha(\gamma(e))$. Now

$$f(s) = \langle v, \alpha(\gamma(s)) \rangle = m(v, \psi)(s) \quad \forall \, s \in S.$$

(b) $\Rightarrow$ (a): Since $h : S \to \Theta(V)$ is weakly (strongly) continuous the natural action of S on the compact space $X := \mathrm{cl}_{w*}(S\psi)$ is separately (respectively, jointly) continuous. Apply Theorem 8 to establish that (S, X) is tame. Finally observe that $f(s) = \langle v, s\psi \rangle$ comes from the S-compactification $S \to X$, $s \mapsto s\psi$.

(3) (a) is similar to (1) using the Asplund version of Theorem 12. For (b) note that the case of $f \in \mathrm{WAP}(S)$ was proved in [48, Theorem 5.1]. The case of $f \in WAP_c(S)$ is similar using [48, Theorem 4.6]. $\square$

If in Theorem 13, $S := G$ is a semitopological group then for any monoid co-homomorphism $h : G \to \Theta(V)$ we have $h(G) \subset \mathrm{Iso}\,(V)$. Recall also that $\mathrm{WAP}(G) = \mathrm{WAP}_c(G)$ (point 4 of Lemma 4).

Proposition 14. *Let $S \times X \to X$ be a separately continuous action. Then:*

1. *Tame$(X) \subset \mathrm{WRUC}(X)$. In particular, Tame$(S) \subset \mathrm{WRUC}(S)$.*
2. *If X is a compact tame (e.g., HNS or WAP) system then (S, X) is WRUC.*

Proof.

(1) Let $f \in \mathrm{Tame}(X)$. Then there exist a compact tame S-system Y, an S-compactification $\nu : X \to Y$, and $\tilde{f} \in C(Y)$ such that $f = \tilde{f} \circ \nu$. By Theorem 12, $\tilde{f}$ comes from a weakly continuous representation (h, α) of (S, Y) on a Rosenthal space V. That is, $\tilde{f}(y) = \langle \nu(\tilde{f}), \alpha(y) \rangle \,\forall\, y \in Y$. Consider the restriction operator (Remark 4), $r : V \to C(X)$, $r(v)(x) = \langle v, \alpha(x) \rangle$. Then for the vector $r(\nu(\tilde{f})) = f$ the orbit map $S \to C(X), s \mapsto fs$ is weakly continuous.

(2) Since X is tame we have Tame$(X) = C(X)$. On the other hand, by (1), we have Tame$(X) \subset \mathrm{WRUC}(X) \subset C(X)$. Hence, $\mathrm{WRUC}(X) = C(X)$. $\square$

Remark 18. Proposition 14 allows us to strengthen some results of [48]. Namely, in 7.7, 7.11, and 7.12 of [48] one may drop the assumption of WRUC-compatibility of (S, X). Theorem 13 unifies and strengthens some earlier results from [27, 48].

8.1 Representations of Topological Groups

Theorem 14. *Let G be a topological group such that $\mathrm{Tame}_c(G)$ (respectively, $\mathrm{Asp}_c(G)$, $\mathrm{WAP}(G)$) separates points and closed subsets. Then there exists a Rosenthal (respectively, Asplund, reflexive) Banach space V and a topological group embedding $h : G \hookrightarrow \mathrm{Iso}\,(V)$ with respect to the strong topology.*

Proof. We consider only the case of Tame(G). Other cases are similar. The case of $\mathrm{WAP}(G)$ is known [48, 50].

For every topological group G the involution inv : $g \mapsto g^{-1}$ defines a topological isomorphism between G and its opposite group G^{op}. So it is equivalent to show that there exists a topological group embedding $h : G \to \text{Iso}(V)^{op}$. Let $\{f_i\}_{i \in I}$ be a collection of tame functions which come from jointly continuous tame G-compactifications of G and separates points and closed subsets. By Theorem 13 for every $i \in I$ there exist a Rosenthal space V_i, a strongly continuous co-homomorphism $h_i : G \to \text{Iso}(V_i)$, and vectors $v_i \in V_i$ and $\psi_i \in V_i^*$ such that $f_i(g) = \psi_i(v_i g)$ for every $g \in G$. Consider the l_2-type sum $V := (\Sigma_{i \in I} V_i)_{l_2}$ which is Rosenthal by virtue of Lemma 3. We have the natural homomorphism $h : G \to \text{Iso}(V)^{op}$ defined by $h(v) = (h_i(v_i))_{i \in I}$ for every $v = (v_i)_{i \in I} \in V$. It is easy to show that h is a strongly continuous homomorphism. Since $\{f_i\}_{i \in I}$ separates points and closed subsets, the family of matrix coefficients $\{m(v_i, \psi_i)\}_{i \in I}$ generates the topology of G. It follows that $h : G \to \text{Iso}(V)^{op}$ is a topological embedding. $\square$

Recall (see Remark 7) that for the group $G := H_+[0, 1]$ every Asplund (hence also every WAP) function is constant and every continuous representation $G \to \text{Iso}(V)$ on an Asplund (hence also reflexive) space V must be trivial. In contrast one may show that G is Rosenthal representable.

Theorem 15. *The group $G := H_+[0, 1]$ is Rosenthal representable.*

Proof. Consider the natural action of G on the closed interval $X := [0, 1]$ and the corresponding enveloping semigroup $E = E(G, X)$. Every element of G is a (strictly) increasing self-homeomorphism of $[0, 1]$. Hence every element $p \in E$ is a nondecreasing function. It follows that E is naturally homeomorphic to a subspace of the Helly compact space (of all nondecreasing self-maps of $[0, 1]$ in the pointwise topology). Hence E is a Rosenthal compactum. So by the dynamical BFT dichotomy, Fact 14, the G-system X is tame. By Theorem 11 we have a faithful representation (h, α) of (G, X) on a separable Rosenthal space V. Therefore we obtain a G-embedding $\alpha : X \hookrightarrow (V^*, w^*)$. Then the strongly continuous homomorphism $h : G \to \text{Iso}(V)^{op}$ is injective. Since $h(G) \times \alpha(X) \to \alpha(X)$ is continuous (and we may identify X with $\alpha(X)$) it follows, by the minimality properties of the compact open topology, that h is an embedding. Thus $h \circ \text{inv} : G \to \text{Iso(V)}$ is the required topological group embedding. $\square$

Remark 19.

1. Recall that by [45] continuous group representations on Asplund spaces have the *adjoint continuity property*. In contrast this is not true for Rosenthal spaces. Indeed, assuming the contrary we would have, from Theorem 15, that the dual action of the group $H_+[0, 1]$ on V^* is continuous, but this is impossible by the following fact [25, Theorem 10.3] (proved also by Uspenskij (private communication)): every adjoint continuous (co)representation of $H_+[0, 1]$ on a Banach space is trivial.
2. There exists a semigroup compactification $\nu : G = H_+[0, 1] \to P$ into a tame semigroup P such that ν is an embedding. Indeed, the associated enveloping semigroup compactification $j : G \to E$ of the tame system $(G, [0, 1])$ is tame.

Observe that j is a topological embedding because the compact open topology on $j(G) \subset$ Homeo $([0, 1])$ coincides with the pointwise topology.

Question 2. Is it true that every Polish topological group G is Rosenthal representable? Equivalently, is this true for the universal Polish groups $G =$ Homeo $([0, 1]^{\mathbb{N}})$ or $G = Iso(\mathbb{U})$ (the isometry group of the Urysohn space $\mathbb{U}$)? By Theorem 14 a strongly related question is the question whether the algebra Tame(G) separates points and closed subsets.

9 Banach Representations of Right Topological Semigroups and Affine Systems

9.1 Tame Representations

Theorem 16.

1. *Every weakly continuous representation (h, α) of an S-space X on a Rosenthal Banach space is E-compatible.*
2. *If the representation in (1) is w^*-generating then the representation is strongly E-compatible.*

Proof.

(1) Applying Haydon's theorem (Fact 4) we get by Lemma 18 that the representation (h, α) is E-compatible.
(2) Use (1) taking into account Lemma 18.   □

Theorem 17 ([21,41] for metrizable systems). *Every tame compact S-space X is injective. Hence, every affine S-compactification of a tame system is E-compatible.*

Proof. In view of Definition 15 we have to show that $m(f, \phi) \in \mathcal{A}(E(X), e)$ for every $f \in C(X), \phi \in C(X)^*$. By Theorem 12, f comes from a Rosenthal representation. There exist a weakly continuous representation (h, α) of (S, X) on a Rosenthal Banach space V and a vector $v_0 \in V$ such that

$$f(x) = \langle v_0, \alpha(x) \rangle \quad \forall\, x \in X.$$

Consider the restriction linear S-operator (point 2 of Remark 4)

$$r : V \to C(X), \; r(v)(x) = \langle v, \alpha(x) \rangle.$$

Let $r^* : C(X)^* \to V^*$ be the adjoint operator. Since $m(f, \phi) = m(r(v_0), \phi)) = m(v_0, r^*(\phi))$, it is enough to show that $m(v_0, r^*(\phi)) \in \mathcal{A}(E(X), e)$.

Analyzing the proof of Theorem 12 we may assume in addition, in view of Lemma 17 of point 2, that the representation (h, α) is generating. By Theorem 16 the representation (h, α) is strongly E-compatible. So by Lemma 19 we have

$m(v_0, r^*(\phi)) \in \mathcal{A}(E(\alpha(X)), e)$. Since $\alpha : X \to \alpha(X)$ is a surjective S-map we have the natural surjective homomorphism $E(X) \to E(\alpha(X))$. Hence, $\mathcal{A}(E(\alpha(X)), e) \subset \mathcal{A}(E(X), e)$. Thus, $m(v_0, r^*(\phi)) \in \mathcal{A}(E(X), e)$, as required. $\square$

Theorem 18. *Let $v : S \to P$ be a right topological semigroup compactification. If P is a tame semigroup (e.g., HNS-semigroup, semitopological, or metrizable) then the S-system P is injective and the algebra of the compactification v is introverted (in particular, v is an operator compactification).*

Proof. By Lemma 25, P is a tame S-system. Theorem 17 guarantees that it is injective. Hence, by Theorem 1 the algebra of the compactification v is introverted (Proposition 6 implies that v is an operator compactification). $\square$

Theorem 19. *Let $V \subset C(S)$ be an m-introverted Banach subalgebra. If $V \subset$ Tame(S) (e.g., if V is separable) then V is introverted. In particular,* Tame(S), Asp(S), *and* WAP(S) *are introverted (and the same is true for* Tame$_c(S)$, Asp$_c(S)$, *and* WAP$_c(S)$*).*

Proof. Consider the corresponding semigroup compactification $v : S \to P$. Since $V \subset$ Tame(S) the system (S, P) is tame. Then its enveloping semigroup $E(S, P)$ is a tame semigroup (Lemma 25) and $E(S, P)$ can be naturally identified with P (Lemma 5). Now combine Theorems 18 and 1. By Remark 5 the subalgebras above are m-introverted (if V is separable then $V \subset$ Asp(S) by Proposition 12, hence, $V \subset$ Tame(S)). $\square$

9.2   Banach Representation of Enveloping Semigroups

By Theorem 8 the semigroup $\mathcal{E}(V)$ is tame for every Rosenthal space V. We now show that, in the converse direction, every tame (respectively, HNS) semigroup P, or equivalently, every enveloping semigroup of a tame (respectively, HNS) system, admits a faithful representation on a Rosenthal (respectively, Asplund) Banach space V. Fact 13 (for semitopological semigroups and reflexive spaces) is a particular case of the following result.

Theorem 20 (Enveloping semigroup representation theorem).

1. *Let P be a tame semigroup. Then there exist a Rosenthal Banach space V and a $\Lambda(P)$-admissible embedding of P into $\mathcal{E}(V)$.*
2. *If P is a HNS-semigroup then there is a $\Lambda(P)$-admissible embedding of P into $\mathcal{E}(V)$ where V is an Asplund Banach space.*
3. *If P is a semitopological semigroup then there is an embedding of P into $\Theta(V) = \mathcal{E}(V^*)$ where V is a reflexive Banach space.*

Proof. Let $S = \Lambda(P)$ be the topological center of P. Since P is admissible, S is a dense submonoid of P. Denote by $j : S \to P$ the corresponding inclusion.

Now P, as an S-system, is tame (Lemma 25). By Theorem 11 there exists a family of flow representations $\{(h_i, \alpha_i)\}_{i \in I}$

$$h_i : S \to \Theta(V_i)^{\mathrm{op}}, \quad \alpha_i : P \to B_{V_i^*}$$

of (S, P) on Rosenthal Banach spaces V_i, where each h_i is a weakly continuous homomorphism and $\{\alpha_i\}_{i \in I}$ separates points of P. As in the proof of Theorem 17 we may assume (by Lemma 17) that these representations are generating. Then, by Theorem 16, they are strongly E-compatible.

Consider the l_2-type sum $V := (\Sigma_{i \in I} V_i)_{l_2}$. Then we have the natural l_2-sum of representations $h : S \to \Theta(V)^{\mathrm{op}}$ defined by $h(v) = (h_i(v_i))_{i \in I}$ for every $v = (v_i)_{i \in I} \in V$. Since $V^* = (\Sigma_{i \in I} V_i)_{l_2}^* = (\Sigma_{i \in I} V_i^*)_{l_2}$ (point 1 of Lemma 3) and each h_i is weakly continuous it is easy to show that h is a weakly continuous homomorphism. We have the corresponding standard operator compactification $j_K : S \to K = \overline{h(S)} \subset \mathcal{E}(V)$. Since $h(S) \subset \Theta^{\mathrm{op}}(V) = \Lambda(\mathcal{E}(V))$, the embedding $K \subset \mathcal{E}(V)$ is S-admissible (Definition 10). By Lemma 3 we know that V is Rosenthal. So in order to complete the proof for "Rosenthal case" (other cases are similar) we have to check the following claim.

Claim. The semigroup compactifications $j : S \to P$ and $j_K : S \to K$ are equivalent.

Proof of the claim. Let $\mathcal{A}_j$ and $\mathcal{A}_K$ be the corresponding subalgebras of $C(S)$. We will show that each of them equals to

$$\mathcal{A} := \left\langle \bigcup_{i \in I} m(V_i, V_i^*) \right\rangle.$$

Each $Y_i := \alpha_i(P)$ is an S-factor of P. Consider its enveloping semigroup $E(S, Y_i)$ and the compactification $j_i : S \to E(S, Y_i)$. Since the family of S-maps $\{\alpha_i : P \to Y_i\}_{i \in I}$ separates points of P the induced system of homomorphisms $r_{\alpha_i} : E(S, P) \to E(S, Y_i)$ separates points of $P = E(S, P)$. So, $\langle \cup_{i \in I} \mathcal{A}(E(Y_i), e) \rangle = \mathcal{A}_j$. The representations (h_i, α_i) are strongly E-compatible. By Lemma 19 we get $m(V_i, V_i^*) \subset \mathcal{A}(E(Y_i), e)$. By Lemma 16, $\mathcal{A}(E(Y_i), e) = \langle m(V_i, Y_i) \rangle$. So, $\langle m(V_i, V_i^*) \rangle = \mathcal{A}(E(Y_i), e) \quad \forall i \in I$. This implies that $\langle \cup_{i \in I} m(V_i, V_i^*) \rangle = \langle \cup_{i \in I} \mathcal{A}(E(Y_i), e) \rangle$. Therefore, $\mathcal{A} = \mathcal{A}_j$.

Now we show that $\mathcal{A}_K = \mathcal{A}$. First observe that the set $L := \cup_{i \in I} V_i$ separates points of $V^* = (\Sigma_{i \in I} V_i^*)_{l_2}$ (and hence of B_{V^*}). By Lemma 7 the standard operator compactification $j_K : S \to K$ is equivalent to the Ellis compactification $S \to E = E(S, B_{V^*})$. Apply Lemma 5 to the S-system $X = B_{V^*}$ and L. Then $\mathcal{A}_K = \langle m(L, B_{V^*}) \rangle = \langle m(L, V^*) \rangle$. For every $v \in V_i \subset L$, $\phi \in V^*$, $s \in S$ we have $\phi(h(s)(v)) = \phi_i(h_i(s)(v))$. So, $m(v, \phi) = m(v, \phi_i)$. Therefore, $m(\cup_{i \in I} V_i, V^*) = \cup_{i \in I} m(V_i, V^*) = \cup_{i \in I} m(V_i, V_i^*)$. It follows that $\mathcal{A}_K = \langle m(L, V^*) \rangle = \langle \cup_{i \in I} m(V_i, V_i^*) \rangle = \mathcal{A}$, as desired. So the claim is proved.

If P is a HNS-semigroup (or a semitopological semigroup) then one may modify our proof accordingly to ensure that V is an Asplund (or a reflexive) Banach space using Theorem 11 (respectively, 10) and Lemma 3.   □

Theorem 21 (A generalized Ellis' theorem). *Every tame compact right topological group P is a topological group.*

Proof. By Theorem 20 there exists a $\Lambda(P)$-admissible embedding of P into $\mathcal{E}(V)$ for some Rosenthal Banach space V. Since P is a group it is easy to see that its topological center $G := \Lambda(P)$ is a subgroup of P. Now apply Theorem 2 to the compactification $\nu : G \hookrightarrow P$ (defined by the natural inclusion) and conclude that P is a topological group.   □

Since every compact semitopological semigroup is tame, Ellis' classical theorem (Fact 12) now follows as a special case of Theorem 21. (Note that we are not using Ellis' theorem as an intermediate step in the proof of Theorem 21.)

Combining Corollary 2 and Theorem 21 we also have:

Corollary 4. *Let P be a compact admissible right topological group. Assume that P, as a topological space, is Fréchet. Then P is a topological group.*

In particular this holds in each of the following cases:

1. *(Moors and Namioka [52]) P is first countable.*
2. *(Namioka [53], Ruppert [63]) P is metrizable.*

Corollary 5 (Glasner [21] for metrizable X). *A distal minimal (not necessarily, metric) compact G-system is tame if and only if it is equicontinuous.*

Proof. We give the proof for the (nontrivial) "only if" part. When X is distal, E is a group by a well-known theorem of Ellis. Also $E := E(X)$ is a tame semigroup by Lemma 25. By Theorem 21 we get that E is a topological group. Finally, X is equicontinuous because X can be identified with the compact coset E-space E/H, where $H = St(x_0)$ is the stabilizer of some point $x_0 \in X$.   □

Corollary 6. $D(G) \cap \text{Tame}(G) = \text{AP}(G)$ *for every topological group G.*

Proof. Let $f \in D(G) \cap \text{Tame}(G)$. Then the cyclic G-space X_f has the following properties: (a) distal, (b) minimal, and (c) tame. Indeed, for every distal function on a topological group the cyclic system (G, X_f) is minimal (see [7, p.196]). Now Corollary 5 concludes that X_f is equicontinuous. Hence, $f \in AP(G)$. This proves $D(G) \cap \text{Tame}(G) \subset \text{AP}(G)$. The reverse inclusion is trivial.   □

Remark 20 (Non-tame functions).

1. Corollary 6 implies that $(D(G) \setminus \text{AP}(G)) \subset (\text{RUC}(G) \setminus \text{Tame}(G))$. Hence any distal function on G which is not almost periodic is not tame. As a concrete example for $G = \mathbb{Z}$, take $f(n) = \cos(2\pi n^2 \alpha)$ with α any irrational real number.
2. Any function $f \in l_\infty(\mathbb{Z})$ such that the system X_f either has positive entropy, or is minimal and weakly mixing, is non-tame.

9.3 Haydon's Functions

Recall (Sect. 4.2) that for every $f \in \text{WRUC}(X)$ on an S-system X we have the cyclic affine S-compactification $\pi_f : X \to Q_f$, where Q_f is the pointwise closure of $co(X_f)$ in $C(S)$ and $X_f := \text{cl}_p(\{m(f, \delta_f(x)\}_{x \in X})$ is the cyclic S-system generated by f.

Definition 19. We say that $f \in \text{WRUC}(X)$ has the *Haydon's property* (or is a *Haydon function*) if the pointwise and norm closures of $co(X_f)$ in $C(S)$ (equivalently, in $l_\infty(S)$) coincide, that is, if

$$\overline{co}^{\text{norm}}(X_f) = \overline{co}^p(X_f).$$

Proposition 15. *Every tame function $f \in \text{Tame}(X)$ has Haydon's property.*

Proof. By Theorem 13 there exist a weakly continuous representation (h, α) of (S, X) on a Rosenthal Banach space V and a vector $v \in V$ such that $f(x) = \langle v, \alpha(x) \rangle$ $\forall x \in X$.

Consider the linear bounded S-operator (between left S-actions)

$$T : V^* \to C(S), \ \mu \mapsto m(v, \mu).$$

By Lemma 15, $X_f := \text{cl}_p(m(f, \delta_f(X)))$. By the choice of $v \in V$ we have $m(f, \delta_f(X)) = m(v, \alpha(X))$. So, $T(\alpha(X)) = \delta_f(X)$. Then $T(Y) = X_f$, where $Y := \overline{\alpha(X)}^{w^*}$. Since T is weak*-pointwise continuous, the compactness argument implies that $T(\overline{co}^{w^*}(Y)) = \overline{co}^p(X_f)$. By Haydon's theorem (point 4 of Fact 4), we have $\overline{co}^{\text{norm}}(Y) = \overline{co}^{w^*}(Y)$. By the linearity and norm continuity of T we get $T(\overline{co}^{\text{norm}}(Y)) \subset \overline{co}^{\text{norm}}(T(Y))$. Clearly, $\overline{co}^{\text{norm}}(T(Y)) \subset \overline{co}^p(T(Y))$. Summing up (and taking into account that $T(Y) = X_f$) we obtain $\overline{co}^{\text{norm}}(X_f) = \overline{co}^p(X_f)$. □

Example 3. Let $\omega \in \Omega = \{0, 1\}^{\mathbb{Z}}$ be a transitive point under the shift $\sigma : \Omega \to \Omega$. We consider ω as an element of $l_\infty(\mathbb{Z})$. Then by assumption the cyclic flow $X_\omega = \Omega$, and it can be easily checked that $\overline{co}^{\text{norm}}(X_\omega) = \overline{co}^p(X_\omega)$. Thus ω is a Haydon function which is clearly not tame. Thus the converse of Proposition 15 is not true. However we do have the following proposition.

Proposition 16. *For a Haydon function $f : X \to \mathbb{R}$, the cyclic affine compactification*

$$\alpha : X \to Q_f = \overline{co}^{\text{norm}}(X_f) = \overline{co}^p(X_f)$$

is E-compatible.

Proof. By Lemma 14, Q_f is a subset of $C(S)$. Therefore, the evaluation map $w : S \times Q_f \to \mathbb{R}$, where $w(s, \phi) = \phi(s)$, is separately continuous. Since $f \in \text{WRUC}(X)$, $\alpha : X \to Q_f$ is an affine S-compactification. In particular, the action $S \times Q_f \to Q_f$ is separately continuous. So, the function

$$m_w(t, y) : S \to \mathbb{R}, \ s \mapsto \tilde{t}(sy) = y(ts)$$

is continuous for every $y \in Q_f$ and $t \in S$. Clearly, S separates points of Q_f. By Lemma 5 of point 1, $\langle m_w(S, Q_f) \rangle$ and $\langle m_w(S, X_f) \rangle$ are the algebras of the Ellis compactifications $j_Q : S \to E(Q_f)$ and $j_{X_f} : S \to E(X_f)$, respectively. Since all s-translations on Q_f are affine maps we have $m_w(t, \sum_{i=1}^{n} c_i q_i) = \sum_{i=1}^{n} c_i m_w(t, q_i)$ for every $\sum_{i=1}^{n} c_i = 1, c_i > 0$. Also,

$$|m_w(t, y)(s) - m_w(t, y_0)(s)| \leq ||y - y_0||_\infty.$$

Since $Q_f = \overline{co}^{\mathrm{norm}}(X_f)$, it follows that

$$m_w(S, Q_f) \subset \overline{sp}^{\mathrm{norm}}(m_w(S, X_f)) \subset \langle m_w(S, X_f) \rangle.$$

Hence, the Ellis compactifications j_{Q_f} and j_{X_f} are equivalent. □

Example 4. The distal function $f(n) = \cos(2\pi n^2 \alpha)$ in $l_\infty(\mathbb{Z})$ is not a Haydon function. This follows from Propositions 16 and 9.

9.4   Banach Representations of Affine S-Systems

As we have already mentioned in Remark 9, all the affine S-compactifications $\alpha : X \to Q$ of X come, up to equivalence, from representations of dynamical S-systems X on *Banach spaces*. In particular, it follows that Q is *affinely S-isomorphic* to an affine S-subsystem of the weak*-compact unit ball B^* of V^* for some Banach space V. This suggests the following question.

Question 3. Which metric affine S-compactifications $X \to Q$ can be obtained via representations of (S, X) on good Banach spaces V, (say, Rosenthal, Asplund, or reflexive) where Q is a weak* compact *affine S-subset* of V^* (as in Sect. 5.1).

First note that there is no obstruction in the purely topological case (i.e., for trivial actions). Indeed, by Keller's theorem [9, p. 98] any metric compact convex affine set Q in a locally convex linear space is affinely homeomorphic to a compact convex subset K in the Hilbert space l_2.

Theorem 22 (A representation theorem for S-affine compactifications). *Let X be a tame (HNS, WAP) compact metric S-system. Then every S-affine compactification $\gamma : X \to Q$ comes from a weakly continuous representation of (S, X) on a separable Rosenthal (respectively: Asplund, reflexive) Banach space V, where $Q \subset V^*$ is a weak* compact affine subset. If $S \times X \to X$ is continuous we can assume that h is strongly continuous. If $S = G$ is a group then $h(G) \subset \mathrm{Iso}(V)^{\mathrm{op}} \subset \Theta(V)^{\mathrm{op}}$.*

Proof. Let (γ, Q) be an S-affine compactification of a tame system X. As usual let $A(Q)|_X \subset C(X)$ be the corresponding affine compactification space. $A(Q)|_X$ is a closed linear unital subspace of $C(X)$. Moreover, it is separable because X

is compact metrizable. Choose a countable subset $\{f_n\}_{n\in\mathbb{N}} \subset A(Q)|_X$ such that $||f_n|| \leq \frac{1}{2^{n-1}}$ and $sp(\{f_n\}_{n\in\mathbb{N}})$ is norm dense in $A(Q)|_X$. We can suppose that $f_1 = \mathbf{1}$.

Since (S, X) is tame, every $f_n \in \text{Tame}(X) = C(X)$. So $f_n S$ is a Rosenthal family for X (Proposition 13) for any $n \in \mathbb{N}$. Hence, $f_n S$ is an eventually fragmented family of maps $X \to \mathbb{R}$ by Fact 3. Then $F := \cup_{n\in\mathbb{N}}(f_n S)$ is again an eventually fragmented family, as can be shown by diagonal arguments, and the condition $||f_n|| \leq \frac{1}{2^{n-1}}$. Hence, F is a Rosenthal family for X by Fact 3.

Since F is also S-invariant we can apply Theorem 12. We obtain a Rosenthal space V, an injective continuous operator $j : V \to C(X)$, and a weakly continuous representation (h, α) of (S, X) on the Rosenthal Banach space V.

As we have noticed in the proof of Theorem 12, one of the properties of this construction is that $F \subset V$. Hence, $sp(F) \subset V$. Consider the associated S-affine compactification $\gamma_0 : X \to Q_0 \subset V^*$. Here $Q_0 = \overline{co}^{w^*}(\alpha(X))$. We claim that (γ_0, Q_0) is equivalent to (γ, Q). It suffices to show that $A(Q_0)|_X = A(Q)|_X$.

Consider the restriction operators:

$$r_X : V \to A(Q)|_X \subset C(X), \quad r_X(v)(x) := \langle v, \alpha(x) \rangle.$$

$$r_{Q_0} : V \to A(Q_0) \subset C(Q_0), \quad r_Q(v)(y) := \langle v, y \rangle.$$

$$r_0 : C(Q_0) \to C(X), \quad r_0(v)(x) := \langle v, \alpha(x) \rangle = \langle v, \gamma_0(x) \rangle.$$

By the choice of F, clearly, $r_X(sp(F))$ and hence also $r_X(V)$ are norm dense in the Banach space $A(Q)|_X$. Now it suffices to show that $r_X(V)$ is dense also in $A(Q_0)|_X$. First, by Lemma 11, $r_{Q_0}(V) + \mathbb{R} \cdot \mathbf{1}$ is dense in $A(Q_0)$. Since $\mathbf{1} = r_{Q_0}(f_1) \in r_{Q_0}(V_0)$ and $r_{Q_0}(V)$ is a linear subspace we conclude that $r_{Q_0}(V) + \mathbb{R} \cdot \mathbf{1} = r_{Q_0}(V)$. Therefore, $r_{Q_0}(V)$ is norm dense in the Banach space $A(Q_0)$. Then $r_0(r_{Q_0}(V))$ is dense in $r_0(A(Q_0)) = A(Q_0)|_X$. Finally, it is easy to check that $r_0(r_{Q_0}(V)) = r_X(V)$. So we can conclude that indeed $r_X(V)$ is dense also in $A(Q_0)|_X$, as desired. This proves the Tame case.

For the Asplund (respectively, reflexive) case we use the corresponding version of Theorem 12 as explained in the proof of Theorem 11 (respectively, [48, Theorem 4.5]). $\qquad \square$

Theorem 22 can be extended to general (not necessarily metrizable) S-systems X under the assumption that the space $A(Q)|_X$ of the affine compactification $\gamma : X \to Q$ is S-separable. The latter condition means that there exists a countable subset $C \subset A(Q)|_X$ such that $sp(CS)$ is dense in $A(Q)|_X$. In this general case the corresponding Rosenthal space V is not necessarily separable.

Since the space V_f of any cyclic affine S-compactification $\pi_f : S \to Q_f$ is always S-separable we conclude that π_f can be affinely S-represented on a Rosenthal space for every $f \in \text{Tame}(S)$.

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Flag Measures for Convex Bodies

Daniel Hug, Ines Türk, and Wolfgang Weil

Abstract Measures on flag manifolds have been recently used to describe local properties of convex bodies and more general sets in $\mathbb{R}^d$. Here, we provide a systematic account of flag measures for convex bodies, we collect various properties of flag measures and we prove some new results. In particular, we discuss mixed flag measures for several bodies and we present formulas for (mixed) flag measures of generalized zonoids.

Key words Support measure • Mixed area measure • Flag support measure • Grassmannian • Integral geometry • Zonoids

Mathematical Subject Classifications (2010): 52A20, 52A22, 52A39, 53C65

1 Introduction (With Historical Remarks)

The classical Brunn–Minkowski theory of convex geometry is based on the notion of mixed volumes. Special cases of these multilinear expressions lead to the basic functionals of compact convex sets (convex bodies), the intrinsic volumes (quermassintegrals, Minkowski functionals) $V_j(K)$, $j = 0, \ldots, d$, of a convex body K in $\mathbb{R}^d$. Already in the work leading to the celebrated Alexandrov–Fenchel inequalities, local variants of the intrinsic volumes, the area measures $\Psi_j(K, \cdot)$, $j = 0, \ldots, d - 1$, and their mixed versions played an important role. The area measures are finite Borel measures on the unit sphere S^{d-1}. In the full dimensional case, they describe the convex body K uniquely, up to translations. In the case of

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polytopes and for $j = d-1$ this fact goes back even to Minkowski. As a counterpart motivated by differential geometry, curvature measures $\Phi_j(K, \cdot)$, $j = 0, \ldots, d-1$, were introduced and studied by Federer in his seminal paper [6]. The curvature measures also describe K uniquely, but sit on the boundary bd K of K. The two sequences of measures can be unified in the notion of support measures $\Xi_j(K, \cdot)$, which are measures on $\mathbb{R}^d \times S^{d-1}$, concentrated on the *generalized normal bundle*

$$\text{Nor } K = \{(x, u) : x \in \text{bd } K, u \in S^{d-1} \text{ an outer unit normal at } x\}.$$

We refer to Schneider [28] for an excellent survey on the Brunn–Minkowski theory and for background information on most notions and results from convex geometry which we use here and in the sequel.

A fundamental relation between global and local functionals is the integral formula for the special mixed volume

$$V(K[1], M[d-1]) = \frac{2}{d} \int_{S^{d-1}} h(K, u) \, \Psi_{d-1}(M, du) \tag{1}$$

which holds true for arbitrary convex bodies K, M and involves the support function $h(K, \cdot)$ of K. This relation was generalized to mixed volumes

$$V(K[m], M[d-m]) = \frac{2^{d-m} m!}{d!} \int_{G(d,d-m)} V_m(K|E^\perp) \, \rho_{d-m}(M, dE), \tag{2}$$

for $m \in \{0, \ldots, d-1\}$, provided M is centrally symmetric and smooth (differentiable of a sufficiently high order), whereas K may be arbitrary. Here, $G(d, d-m)$ is the Grassmannian of $(d-m)$-dimensional linear subspaces of $\mathbb{R}^d$, $V_m(K|E^\perp)$ is the m-dimensional volume of the orthogonal projection of K on the orthogonal complement $E^\perp \in G(d, m)$ of $E \in G(d, d-m)$, and the measure $\rho_{d-m}(M, \cdot)$ is the $(d-m)$th projection generating measure of M, normalized as in [10, p. 1315]. If both bodies, K and M, are centrally symmetric and smooth (differentiable of a sufficiently high order), then the mixed volume $V(K[m], M[d-m])$ can be expressed in a symmetric way as

$$\begin{aligned} &V(K[m], M[d-m]) \\ &= \frac{2^d}{d!} \int_{G(d,d-m)} \int_{G(d,m)} |\langle E, F^\perp \rangle| \, \rho_m(K, dF) \, \rho_{d-m}(M, dE), \end{aligned} \tag{3}$$

where $|\langle E, F^\perp \rangle| = |\langle E^\perp, F \rangle|$ denotes the absolute value of the determinant of the orthogonal projection of E on $F^\perp$.

It is known that (2) and (3) remain true without smoothness assumptions on M (resp. K and M), if distributions are used. See the introduction of [15], for remarks on the literature. However, the central symmetry of M (or K and M) seems to

be essential, as long as integrals over Grassmannians are considered. In fact, a generalization of (3) to arbitrary bodies K, M was recently obtained in [15], which uses measures on certain flag manifolds. This is strong evidence that flag measures and other flag-type notions can play an important role in the further development of modern Brunn–Minkowski theory.

The first appearance of a flag-type result in convex geometry seems to be in a paper by Firey [8]. He showed that the area measure $\Psi_{d-1-j}(K, A)$ of a convex body K evaluated at a Borel set $A \subset S^{d-1}$ can be interpreted as the natural measure of j-flats touching K in boundary points x with normal directions $u \in A$. Schneider [24] provided a shorter proof of this result and in [25] he gave a corresponding interpretation of the curvature measure $\Phi_{d-1-j}(K, B)$ of a convex body K at a Borel set $B \subset \mathbb{R}^d$. It measures the set of j-flats touching K in boundary points $x \in B$. For these integral-geometric interpretations of area and curvature measures, a measure theoretic foundation was given in [32] by providing, for each $j \in \{0, \ldots, d-1\}$, a finite Borel measure on the space of j-flats touching a convex body K, the measure being obtained by a disintegration of the Haar measure on the affine Grassmannian $A(d, j)$ of affine j-flats in $\mathbb{R}^d$. The results can be reformulated in terms of flats which touch a convex body randomly and are also strongly related to local formulas in integral geometry. The study of collision probabilities (touching probabilities) was continued by Firey and others in a variety of similar situations, for example to a moving convex body M randomly touching a fixed convex body K. Surveys on the subsequent development can be found in [26, 33] and [30, Sect. 8.5]. The measure constructed in [32] can be seen as a measure $\tilde{\Phi}_{d-j-1}^{(j)}(K, \cdot)$ on pairs (x, L), where x is a boundary point of K and $L \in G(d, j)$ such that the affine j-flat $x + L$ touches K. In a diploma thesis, Kropp [17] introduced and studied corresponding measures $\tilde{\Psi}_{d-j-1}^{(j)}(K, \cdot)$ on the flag manifold of pairs $(u, L) \in S^{d-1} \times G(d, j)$ such that $u \perp L$ (that is, u is orthogonal to L). For bodies K, which are strictly convex and have unique support planes, the measures $\tilde{\Phi}_{d-j-1}^{(j)}(K, \cdot)$ and $\tilde{\Psi}_{d-j-1}^{(j)}(K, \cdot)$ are image measures of each other under the Gauss map $x \mapsto u(x)$, where $u(x)$ denotes the (outer) unit normal in $x \in \mathrm{bd}\, K$, respectively its inverse. The curvature and area measures appear as projection images,

$$\Phi_{d-j-1}(K, \cdot) = \tilde{\Phi}_{d-j-1}^{(j)}(K, \cdot \times G(d, j)),$$

$$\Psi_{d-j-1}(K, \cdot) = \tilde{\Psi}_{d-j-1}^{(j)}(K, \cdot \times G(d, j)),$$

for $j = 0, \ldots, d-1$, and the intrinsic volumes equal the total measures,

$$V_{d-j-1}(K) = \tilde{\Phi}_{d-j-1}^{(j)}(K, S^{d-1} \times G(d, j)) = \tilde{\Psi}_{d-j-1}^{(j)}(K, \mathrm{bd}\, K \times G(d, j)).$$

Kropp [17] also introduced flag-type support measures $\tilde{\Xi}_{d-j-1}^{(j)}(K, \cdot)$ which are concentrated on the manifold of triples (x, u, L), where $x \in \mathrm{bd}\, K$, u is a unit normal

of K at x, $L \in G(d, j)$ and $u \perp L$, and thus he unified the measures $\tilde{\Phi}^{(j)}_{d-j-1}(K, \cdot)$ and $\tilde{\Psi}^{(j)}_{d-j-1}(K, \cdot)$. The set of all such triples (x, u, L) is denoted by $\mathrm{Nor}_j\, K$ and can be used to parametrize j-flats touching K.

In the following, it will be convenient to use the different normalization $\Xi^{(j)}_{d-j-1}(K, \cdot)$ of the flag-type support measures and of the corresponding flag area and flag curvature measures, then denoted by $\Psi^{(j)}_{d-1-j}(K, \cdot)$ and $\Phi^{(j)}_{d-1-j}(K, \cdot)$ (see (25) for the reason of this normalization). The two normalizations are related by

$$\tilde{\Xi}^{(j)}_{d-j-1}(K, \cdot) = c(d, j) \cdot \Xi^{(j)}_{d-j-1}(K, \cdot) \tag{4}$$

with

$$c(d, j) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{j+2}{2}\right)\Gamma\left(\frac{d-j+1}{2}\right)} = \binom{d-1}{j}\frac{\Gamma\left(\frac{d-j}{2}\right)\Gamma\left(\frac{j+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{1}{2}\right)},$$

where the second equality follows from Legendre's duplication formula. Note the relations $c(d, 0) = c(d, d-1) = 1$ and $c(d, j) = c(d, d-j-1)$.

The constructions in [17, 32] were based on projection formulas for curvature and area measures, respectively. In his PhD thesis, Hinderer [12] studied the flag measures $\Xi^{(j)}_{d-j-1}(K, \cdot)$ more systematically, starting from a local Steiner-type formula. As a main result, he showed that the projection function

$$L \mapsto V_j(P|L),$$

for $j = 1, \ldots, d-2$, a polytope P and $L \in G(d, j)$, such that P and L are in general relative position, can be expressed as an integral over flag measures (the case $j = d-1$ is a classical consequence of (1)). The results of [12] build the basis of the recent papers [11, 13].

Flag measures were also used by Ambartzumian [1, 2] in his $\sin^2$-representation of the width function of convex bodies in $\mathbb{R}^3$. He introduced a flag measure, different from the one mentioned above, for polytopes and extended it to arbitrary bodies in $\mathbb{R}^3$ by a compactness argument. However, as Hinderer [12] showed, this extension is not continuous and thus lacks an important property which the measure $\Xi^{(j)}_{d-j-1}(K, \cdot)$ has.

From a different point of view, flag-type measures for sets of positive reach were constructed on the natural generalization $\mathrm{Nor}_j\, K$ of the normal bundle $\mathrm{Nor}\, K$, as integrals with respect to the corresponding Hausdorff measures. This development started with Zähle's [35] integral representation of (signed) support measures, then called (Lipschitz–Killing) curvature measures, for sets with positive reach. Seizing a suggestion in [6], Zähle introduced *absolute curvature measures*, corresponding to $\Xi^{(j)}_{d-1-j}(K, \cdot \times G(d, j))$, for sets with positive reach, as non-negative measures

on $\mathbb{R}^d \times S^{d-1}$. These measures are introduced in [36], and further studied in [22], as mean projection measures with the interpretation as the measure of j-flats locally colliding with a set of positive reach in a given subset of $\mathbb{R}^d \times S^{d-1}$. This description admits a comparison of the total absolute curvature measures to corresponding quantities in the setting of smooth $(d-1)$-dimensional submanifolds that have been considered before by Santaló [23] and Baddeley [4]. For convex sets, the absolute curvature measures, as measures on $\mathbb{R}^d \times S^{d-1}$, coincide with the support measures, but they differ from these and from the total variation measures of support measures for sets with positive reach.

In [22, 36], absolute curvature measures are also described as integrals over $\mathrm{Nor}_j K$, for a set K of positive reach, involving generalized (signed) curvature functions on $\mathrm{Nor} K$. At a given point $(x, u, V) \in \mathrm{Nor}_j K$, the definition involves sums of products of generalized curvatures of K at (x, u), and these products are weighted with a quantity that depends on the relative position of the linear subspace V and the directions of curvature. In these papers, some basic properties of absolute curvature measures, corresponding to $\Xi_{d-1-j}^{(j)}(K, \cdot \times G(d, j))$ in the present notation, are explored. In particular, for sets of the form $B \times S^{d-1}$, with a Borel set $B \subset \mathbb{R}^d$, a Crofton formula is established. A more general translative and a kinematic Crofton formula, for certain unions of sets with positive reach, have been obtained by Rataj [20, 21].

In the present context, for each $j \in \{0, \ldots, d-1\}$, we investigate flag support measures $\Xi_r^{(j)}(K, \cdot)$, for all $r = 0, \ldots, d-1-j$, as measures on the product space $\mathbb{R}^d \times S^{d-1} \times G(d, j)$. On a more technical level, a brief measure geometric description of $\Xi_{d-1-j}^{(j)}(K, \cdot)$ for convex bodies K is provided in [15]. An extension of the measure geometric approach to all measures $\Xi_r^{(j)}(K, \cdot)$ and an analogous investigation for convex functions is initiated in [5].

In the following, we study flag measures for convex bodies systematically, we compare the different approaches leading to flag measures and we collect various results which can be obtained in analogy to the well-known theory of curvature and area measures. We include proofs where it is convenient or where results are new, but refer to the literature when recent publications are available. We start with a section which collects the necessary notations and we recall some classical results for support measures. Then we introduce flag measures as coefficients in a local Steiner formula in Sect. 3. In Sect. 4 we describe an approach to flag measures via projection averages and Sect. 5 contains various properties and extensions of flag measures, including mixed flag measures. In Sect. 6, we present formulas for (mixed) flag measures of generalized zonoids, a well-known class of centrally symmetric bodies, and in the final Sect. 7 we mention shortly some recent applications of flag measures to projection functions, mixed volumes and translation invariant valuations of convex bodies.

2 Notations and Classical Results

We work in Euclidean space $\mathbb{R}^d$, $d \geq 2$, with (standard) scalar product $\langle \cdot, \cdot \rangle$ and Euclidean norm $\| \cdot \|$. The closed unit ball in $\mathbb{R}^d$ is denoted by B^d, and the unit sphere by S^{d-1}. We use λ_d for the d-dimensional Lebesgue measure and $\mathcal{H}^j$ for the j-dimensional Hausdorff measure. Let $G(d, k)$ be the Grassmannian of k-dimensional linear subspaces of $\mathbb{R}^d$, supplied with its invariant probability measure ν_k, and let $A(d, k)$ be the corresponding affine Grassmannian with invariant measure μ_k. The latter is normalized as in [30]. Both spaces, $G(d, k)$ and $A(d, k)$, carry the usual Fell–Matheron topology. For a topological space X, we let $\mathcal{B}(X)$ be the σ-algebra of Borel sets in X, and we denote by bd A, relbd A, int A and relint A the boundary, relative boundary, interior and relative interior of a set A. For non-empty, closed sets $A, B \subset \mathbb{R}^d$, the projection point $p(A, B)$ is the point in A closest to B, provided this point exists and is unique. We write $p(A, x)$ if $B = \{x\}$. The standard set class to be used is the class $\mathcal{K}$ of convex bodies (non-empty, compact convex sets in $\mathbb{R}^d$). We endow $\mathcal{K}$ with the Hausdorff metric. For a convex body K and a vector $u \neq o$, the support function of K at u is $h(K, u)$, whereas the support set of K in direction u is denoted by $F(K, u)$ (independent of the length of u). For convex sets we frequently use the volume functional V_d instead of λ_d. The constants $\kappa_d = V_d(B^d)$ and $\omega_d = d\kappa_d$ will often appear in formulas.

Coming now to the more special notions, we start with the *intrinsic volumes* V_j, $j = 0, \ldots, d - 1$, which are conveniently defined through the Steiner formula

$$V_d((K + \rho B^d) \setminus K) = \sum_{j=0}^{d-1} \rho^{d-j} \kappa_{d-j} V_j(K), \tag{5}$$

where $K \in \mathcal{K}$ and $\rho > 0$.

The *support measures* $\Xi_j(K, \cdot)$, $j = 0, \ldots, d - 1$, of a convex body $K \in \mathcal{K}$ can be introduced by a local version of (5). In the following presentation, we follow Schneider [28, Chap. 4]. Let

$$d(A, x) = \min\{\|x - y\| \,:\, y \in A\}$$

be the distance of a point $x \in \mathbb{R}^d$ to a closed set $A \subset \mathbb{R}^d$ and let $p(K, \cdot) : \mathbb{R}^d \to K$, for $K \in \mathcal{K}$ be the *metric projection* onto K, that is, $p(K, x)$ is the (in this case) unique projection point in K closest to x. For $x \notin K$, the direction from $p(K, x)$ to x is denoted by

$$u(K, x) = \frac{x - p(K, x)}{\|x - p(K, x)\|}.$$

The vector $u(K, x)$ is an outer normal of K at $p(K, x)$. Therefore, $(p(K, x), u(K, x)) \in \text{Nor } K$ is called a *support element* of K. For $K \in \mathcal{K}$, a Borel set $\eta \in \mathcal{B}(\mathbb{R}^d \times S^{d-1})$ and $\rho > 0$, we define the *local parallel set* $M_\rho(K, \eta)$ by

$$M_\rho(K, \eta) = \{x \in \mathbb{R}^d \,:\, 0 < d(K, x) \leq \rho, \, (p(K, x), u(K, x)) \in \eta\}.$$

This is a Borel set and $\eta \mapsto \mu_\rho(K, \eta) = \lambda_d(M_\rho(K, \eta))$ is a finite Borel measure which satisfies the following *local Steiner formula*.

Theorem 1 [28, Theorem 4.2.1]. *For $K \in \mathcal{K}$, there are finite (positive) Borel measures $\Xi_0(K, \cdot), \ldots, \Xi_{d-1}(K, \cdot)$ on $\mathbb{R}^d \times S^{d-1}$ such that, for all $\eta \in \mathcal{B}(\mathbb{R}^d \times S^{d-1})$ and $\rho > 0$,*

$$\mu_\rho(K, \eta) = \sum_{j=0}^{d-1} \rho^{d-j} \kappa_{d-j} \, \Xi_j(K, \eta). \tag{6}$$

The measure $\Xi_j(K, \cdot)$, $j \in \{0, \ldots, d-1\}$, is concentrated on Nor K.

For $j = 0, \ldots, d-1$, the mapping $K \mapsto \Xi_j(K, \cdot)$ is weakly continuous and $K \mapsto \Xi_j(K, \eta)$ is measurable, for each $\eta \in \mathcal{B}(\mathbb{R}^d \times S^{d-1})$.

For a polytope $P \in \mathcal{K}$, we have a specific representation of the support measures,

$$\Xi_j(P, \cdot) = \frac{1}{\omega_{d-j}} \sum_{F \in \mathcal{F}_j(P)} \int_F \int_{n(P,F)} \mathbf{1}\{(y, u) \in \cdot\} \, \mathcal{H}^{d-1-j}(du) \, \mathcal{H}^j(dy), \tag{7}$$

for $j = 0, \ldots, d-1$ (see [28, (4.2.2)]). Here, $\mathcal{F}_j(P)$ is the set of j-faces of P and $n(P, F) = n(P, x)$ is the set of unit vectors in the normal cone of P at x, where the point $x \in \operatorname{relint} F$ is arbitrary.

For a general convex body $K \in \mathcal{K}$, the coefficient measures $\Xi_0(K, \cdot), \ldots, \Xi_{d-1}(K, \cdot)$ in (6) are the *support measures* of K. If $\eta \supset$ Nor K is measurable, then (6) turns into (5), hence the total measures equal the intrinsic volumes,

$$\Xi_j(K, \mathbb{R}^d \times S^{d-1}) = V_j(K),$$

for $j = 0, \ldots, d-1$. The classical *area measures* $\Psi_j(K, \cdot)$ appear now as image measures of the support measures under projection onto the second component,

$$\Psi_j(K, \cdot) = \Xi_j(K, \mathbb{R}^d \times \cdot),$$

and the *curvature measures* $\Phi_j(K, \cdot)$ are the image measures under projection onto the first component,

$$\Phi_j(K, \cdot) = \Xi_j(K, \cdot \times S^{d-1}).$$

For polytopes, (7) implies corresponding representations for curvature and area measures.

An important additional property of support measures concerns the behavior under rigid motions $g \in G_d$. If g_0 denotes the rotational part of g and if $g\eta = \{(gx, g_0u) : (x, u) \in \eta\}$, then

$$\Xi_j(gK, g\eta) = \Xi_j(K, \eta).$$

In particular, the curvature measures are translation covariant, whereas the area measures are translation invariant. Another important property of $\Xi_j(K, \cdot)$ is its *homogeneity*, which can be expressed by

$$\Xi_j(\alpha K, \alpha \eta) = \alpha^j \Xi_j(K, \eta),$$

for $\alpha > 0$, where $\alpha \eta = \{(\alpha x, u) : (x, u) \in \eta\}$.

For different normalizations of support measures, area measures and curvature measures, see [28, 30]. We also mention that the support measures are *locally defined*; for a given Borel set $\eta \subset \mathbb{R}^d \times S^{d-1}$ they only depend on an (arbitrarily) small neighborhood of the boundary points x with $(x, u) \in \eta$ (see [28], for more details).

Since support measures arise from a local examination (expansion) of the sum set $K + \rho B^d$, one can expect that support measures can also be expanded, for general combinations $\rho_1 K_1 + \cdots + \rho_k K_k$, where $K_1, \ldots, K_k \in \mathcal{K}$ and $\rho_i \geq 0$. In fact, under suitable additional assumptions, the measure $\Xi_j(\rho_1 K_1 + \cdots + \rho_k K_k, \cdot)$ allows a multilinear expansion. For the area measures this is classical, for the curvature measures it was proved in [16] (see also [9, 14], for special cases). Since the support measure $\Xi_j(K, \cdot)$ corresponds to the mixed curvature measure $\Phi_{j,d-j}(K, B^d, \cdot)$ (cf. [16]), the following multilinear expansion of support measures is a consequence of formula (5.16) in [16]. It holds for convex bodies $K_1, \ldots, K_k$ in *general relative position*. This condition requires that, for each direction $u \in S^{d-1}$, the support set $F(K_1 + \cdots + K_k, u) = F(K_1, u) + \cdots + F(K_k, u)$ satisfies $\dim F(K_1 + \cdots + K_k, u) = \dim F(K_1, u) + \cdots + \dim F(K_k, u)$. It is fulfilled, for example, if $K_1, \ldots, K_k$ are strictly convex. Let $K_1, \ldots, K_k \in \mathcal{K}$ be in general relative position, and let $\beta_1, \ldots, \beta_k \subset \mathbb{R}^d$ be Borel sets with $\beta_i \subset K_i$ for $i = 1, \ldots, k$. In general, $\beta_1 + \cdots + \beta_k \subset \mathbb{R}^d$ need not be a Borel set. However, by [16, Lemma 3.2] it follows that $(\beta_1 + \cdots + \beta_k) \cap \mathrm{bd}(K_1 + \cdots + K_k)$ is a Borel set. Since for a convex body $K \in \mathcal{K}$ we have $\Xi_j(K, \cdot) = \Xi_j(K, \cdot \cap (\mathrm{bd}\, K \times S^{d-1}))$, the left-hand side of (8) below is well defined. For later use we also mention that the orthogonal projection of a Borel set $\beta \subset \mathbb{R}^d$ to a subspace need not be a Borel set, but in case $K \in \mathcal{K}$ and $\beta \subset K$ the intersection $(\beta|L) \cap \mathrm{bd}_L(K|L)$ is a Borel set for ν_k-almost all $L \in G(d, k)$, where bd_L is the boundary with respect to L as the ambient space (see the proof of Theorem 6.2.1 in [30]).

In the following expressions, the notation $K[r]$ means that the corresponding entry K appears r times and $(A)^r$, for a set A, is the r-fold product set.

Theorem 2. *For $k \in \mathbb{N}$, let $\rho_1, \ldots, \rho_k \geq 0$, let $K_1, \ldots, K_k \in \mathcal{K}$ be convex bodies in general relative position, let $\beta_1, \ldots, \beta_k \subset \mathbb{R}^d$ be Borel sets with $\beta_i \subset K_i$ and let $\gamma \subset S^{d-1}$ be a Borel set.*

(a) For $j \in \{0, \ldots, d-1\}$ and $i_1, \ldots, i_k \in \{0, \ldots, j\}$ with $i_1 + \cdots + i_k = j$, there exist measures $\Xi_{i_1, \ldots, i_k}(K_1, \ldots, K_k, \cdot)$ on $(\mathbb{R}^d)^k \times S^{d-1}$ concentrated on $\mathrm{bd}\, K_1 \times \cdots \times \mathrm{bd}\, K_k \times S^{d-1}$ such that

$$
\Xi_j \left(\sum_{i=1}^{k} \rho_i K_i, \left(\sum_{i=1}^{k} \rho_i \beta_i \right) \times \gamma \right)
$$

$$
= \sum_{i_1,\ldots,i_k=0}^{j} \binom{j}{i_1,\ldots,i_k} \rho_1^{i_1} \cdots \rho_k^{i_k} \, \Xi_{i_1,\ldots,i_k}(K_1,\ldots,K_k,\beta_1 \times \cdots \times \beta_k \times \gamma).
$$

(8)

The measure $\Xi_{i_1,\ldots,i_k}(K_1,\ldots,K_k,\cdot)$ depends in a weakly continuous way on $K_1,\ldots,K_k$ in general relative position, and it is symmetric, in the sense that

$$
\Xi_{i_1,\ldots,i_k}(K_1,\ldots,K_k,\beta_1 \times \cdots \times \beta_k \times \gamma)
$$
$$
= \Xi_{i_{\pi(1)},\ldots,i_{\pi(k)}}(K_{\pi(1)},\ldots,K_{\pi(k)},\beta_{\pi(1)} \times \cdots \times \beta_{\pi(k)} \times \gamma),
$$

for all permutations π of $1,\ldots,k$. If $i_1 = 0$, then

$$
\Xi_{i_1,\ldots,i_k}(K_1,\ldots,K_k,\mathrm{bd}\,K_1 \times \cdot) = \Xi_{i_2,\ldots,i_k}(K_2,\ldots,K_k,\cdot).
$$

If $i_1 \neq 0$, then $\Xi_{i_1,\ldots,i_k}(K_1,\ldots,K_k,\cdot)$ is translation covariant and homogeneous of degree i_1 in the first component, hence

$$
\Xi_{i_1,i_2,\ldots,i_k}(\alpha K_1 + x, K_2,\ldots,K_k,(\alpha\beta_1 + x) \times \beta_2 \times \cdots \times \beta_k \times \gamma)
$$
$$
= \alpha^{i_1} \Xi_{i_1,\ldots,i_k}(K_1,\ldots,K_k,\beta_1 \times \cdots \times \beta_k \times \gamma),
$$

for $\alpha > 0$, $x \in \mathbb{R}^d$. Furthermore, $\Xi_{i_1,i_2,\ldots,i_k}(\cdot,K_2,\ldots,K_k,\cdot \times \beta_2 \times \cdots \times \beta_k \times \gamma)$ has a polynomial expansion.

(b) *Let $j \in \{1,\ldots,d-1\}$, $i_1,\ldots,i_k \in \{1,\ldots,j\}$ be such that $i_1 + \cdots + i_k = j$. If $K_1,\ldots,K_k \in \mathcal{K}$ are strictly convex bodies, we have*

$$
\Xi_{i_1,\ldots,i_k}(K_1,\ldots,K_k,\beta_1 \times \cdots \times \beta_k \times \gamma)
$$
$$
= \Xi_{1,\ldots,1}(K_1[i_1],\ldots,K_k[i_k],(\beta_1)^{i_1} \times \cdots \times (\beta_k)^{i_k} \times \gamma),
$$

where the lower index 1 on the right-hand side appears j times.

(c) *For $j \in \{0,\ldots,d-1\}$, $r \in \{0,\ldots,j\}$, a convex body $K \in \mathcal{K}$ and Borel sets $\beta \subset K$, $\gamma \subset S^{d-1}$, we have*

$$
\Xi_r(K,\beta \times \gamma) = \frac{\binom{d}{r}}{\binom{d}{j}} \frac{\kappa_{d-j}}{\kappa_{d-r}} \, \Xi_{r,j-r}(K,B^d,\beta \times S^{d-1} \times \gamma).
$$

In addition, for convex bodies $K_1,\ldots,K_s \in \mathcal{K}$ in general relative position and Borel sets $\gamma \subset S^{d-1}$, $\beta_1,\ldots,\beta_s \subset \mathbb{R}^d$ with $\beta_i \subset K_i$, we have

$$\Xi_{i_1,\ldots,i_s}(K_1,\ldots,K_s,\beta_1 \times \cdots \times \beta_s \times \gamma)$$

$$= \frac{\binom{d}{r}}{\binom{d}{j}}\frac{\kappa_{d-j}}{\kappa_{d-r}}\,\Xi_{i_1,\ldots,i_s,j-r}(K_1,\ldots,K_s,B^d,\beta_1 \times \cdots \times \beta_s \times S^{d-1} \times \gamma),$$

where $i_1,\ldots,i_s \in \{0,\ldots,r\}$ with $i_1 + \cdots + i_s = r$.

The strict convexity is needed, since the i_1 copies of K_1 etc. have to be in general relative position. The condition on the general relative position can be neglected if the projections of the mixed measures onto the last (spherical) component are considered. The resulting expansion then reduces to the classical multilinearity of the mixed area measures.

Proof (of Theorem 2). (a) By [16, (5.16)] and since

$$\Xi_j(L,\cdot) = \frac{\binom{d}{j}}{d\kappa_{d-j}}\Theta_j(L,\cdot)$$

we obtain the polynomial expansion (8) with coefficient measures $\Xi_{i_1,\ldots,i_k}$ which are determined by this expansion. The asserted properties of these mixed measures then follow from [16, displayed formula on p. 328] and from the properties stated in [16, Theorem 5.6 and Corollary 3.6].

(b) Let $j \in \{1,\ldots,d-1\}$ and $i_1,\ldots,i_k \in \{1,\ldots,j\}$ with $j = i_1 + \cdots + i_k$. Let $K_1,\ldots,K_k \in \mathcal{K}$ be strictly convex. We consider $\rho_{11},\ldots,\rho_{1i_1},\ldots,\rho_{k1},\ldots,$ $\rho_{ki_k} \geq 0$, $\beta_i \subset K_i$ for $i = 1,\ldots,k$, $\gamma \subset S^{d-1}$. Then, by (8), in the expansion of

$$\Xi_j\left(\sum_{r=1}^{k}\sum_{s_r=1}^{i_r}\rho_{rs_r}K_r,\left(\sum_{r=1}^{k}\sum_{s_r=1}^{i_r}\rho_{rs_r}\beta_r\right)\times\gamma\right) \qquad (9)$$

the coefficient of $\rho_{11}\cdots\rho_{ki_k}$ is

$$j!\,\Xi_{1,\ldots,1}(K_1[i_1],\ldots,K_k[i_k],(\beta_1)^{i_1} \times \cdots \times (\beta_k)^{i_k} \times \gamma).$$

Here, the index 1 in the measure $\Xi_{1,\ldots,1}$ appears j times. On the other hand, using that K_r is strictly convex and $\beta_r \subset K_r$, we get

$$\left(\sum_{r=1}^{k}\sum_{s_r=1}^{i_r}\rho_{rs_r}\beta_r\right) \cap \mathrm{bd}\left(\sum_{r=1}^{k}\sum_{s_r=1}^{i_r}\rho_{rs_r}K_r\right)$$

$$= \left[\sum_{r=1}^{k}\left(\sum_{s_r=1}^{i_r}\rho_{rs_r}\right)\beta_r\right] \cap \mathrm{bd}\left[\sum_{r=1}^{k}\left(\sum_{s_r=1}^{i_r}\rho_{rs_r}\right)K_r\right].$$

Hence, it follows that (9) is equal to

$$\Xi_j((\rho_{11} + \cdots + \rho_{1i_1})K_1 + \cdots + (\rho_{k1} + \cdots + \rho_{ki_k})K_k,$$
$$[(\rho_{11} + \cdots + \rho_{1i_1})\beta_1 + \cdots + (\rho_{k1} + \cdots + \rho_{ki_k})\beta_k] \times \gamma).$$

Expanding this according to (8), we get

$$\sum_{r_1,\dots,r_k=0}^{j} \binom{j}{r_1,\dots,r_k} (\rho_{11} + \cdots + \rho_{1i_1})^{r_1} \cdots (\rho_{k1} + \cdots + \rho_{ki_k})^{r_k}$$
$$\times \Xi_{r_1,\dots,r_k}(K_1,\dots,K_k,\beta_1 \times \cdots \times \beta_k \times \gamma).$$

Then expanding each of the k expressions $(\dots)^{r_j}$, $j = 1,\dots,k$, we arrive at a polynomial in $\rho_{11},\dots,\rho_{ki_k}$ which is homogeneous of degree j. The monomial $\rho_{11} \cdots \rho_{1i_1} \cdots \rho_{k1} \cdots \rho_{ki_k}$ arises only in the expansion of $r_1 = i_1,\dots,r_k = i_k$ and occurs with multiplicity $\binom{i_1}{1,\dots,1} \cdots \binom{i_k}{1,\dots,1}$. Hence the coefficient of this monomial is

$$\binom{j}{i_1,\dots,i_k}\binom{i_1}{1,\dots,1} \cdots \binom{i_k}{1,\dots,1} \Xi_{i_1,\dots,i_k}(K_1,\dots,K_k,\beta_1 \times \cdots \times \beta_k \times \gamma)$$
$$= j!\, \Xi_{i_1,\dots,i_k}(K_1,\dots,K_k,\beta_1 \times \cdots \times \beta_k \times \gamma).$$

A comparison of coefficients now shows that

$$\Xi_{1,\dots,1}(K_1[i_1],\dots,K_k[i_k],(\beta_1)^{i_1} \times \cdots \times (\beta_k)^{i_k} \times \gamma)$$
$$= \Xi_{i_1,\dots,i_k}(K_1,\dots,K_k,\beta_1 \times \cdots \times \beta_k \times \gamma).$$

(c) If $K \in \mathcal{K}$, we obtain from (a) that

$$\Xi_j(K + \rho B^d, (\beta + \rho B^d) \times \gamma) = \sum_{r=0}^{j} \rho^{j-r} \binom{j}{r} \Xi_{r,j-r}(K, B^d, \beta \times S^{d-1} \times \gamma).$$

Let $T_\rho(x,u) = (x + \rho u, u)$ for $(x,u) \in \mathbb{R}^d \times S^{d-1}$. Since

$$\Xi_j(K + \rho B^d, (\beta + \rho B^d) \times \gamma)$$
$$= \Xi_j(K + \rho B^d, [(\beta + \rho B^d) \times \gamma] \cap \mathrm{Nor}(K + \rho B^d))$$
$$= \Xi_j(K + \rho B^d, T_\rho(\beta \times \gamma))$$

and using the polynomial expansion of the support measures ([28, Theorem 4.2.2] or [30, Theorem 14.2.4])

$$\Xi_j(K + \rho B^d, T_\rho(\beta \times \gamma)) = \sum_{r=0}^{j} \rho^{j-r} \binom{d-r}{j-r} \frac{\kappa_{d-r}}{\kappa_{d-j}} \Xi_r(K, \beta \times \gamma), \quad (10)$$

a comparison of coefficients yields

$$\Xi_r(K, \beta \times \gamma) = \frac{\binom{d}{r}}{\binom{d}{j}} \frac{\kappa_{d-j}}{\kappa_{d-r}} \Xi_{r,j-r}(K, B^d, \beta \times S^{d-1} \times \gamma).$$

Replacing K by a Minkowski combination $\alpha_1 K_1 + \cdots + \alpha_s K_s$, for $K_1, \ldots, K_s \in \mathcal{K}$ in general relative position, expanding both sides and comparing coefficients, we get

$$\Xi_{i_1,\ldots,i_s}(K_1, \ldots, K_s, \beta_1 \times \cdots \times \beta_s \times \gamma)$$

$$= \frac{\binom{d}{r}}{\binom{d}{j}} \frac{\kappa_{d-j}}{\kappa_{d-r}} \Xi_{i_1,\ldots,i_s,j-r}(K_1, \ldots, K_s, B^d, \beta_1 \times \cdots \times \beta_s \times S^{d-1} \times \gamma),$$

where $i_1 + \cdots + i_s = r$. $\qquad\qquad\square$

Remark 1. In the following, we simply write $\Xi(K_1, \ldots, K_j, \cdot)$ for $\Xi_{1,\ldots,1}(K_1, \ldots, K_j, \cdot)$. Here it is clear from the context that the lower index 1 appears j times.

We finish this section with a short description of the proof of Theorem 1, since the structure of this proof also underlies the more general construction of flag measures which we discuss in the next section. Formula (6) is first proved for polytopes K, by discussing the contributions to the measure $\mu_\rho(K, \eta)$ coming from the different faces of K. Here, an important aspect is to see that the summand which is contributed by the faces of dimension j, is homogeneous of degree j. The formula for polytopes then also implies the representation (7) of the support measures. Having proved this polynomial expansion for polytopes K, one considers $\mu_\rho(K, \eta)$, for $\rho = 1, \ldots, d$. This yields a system of linear equations for the coefficients $\Xi_0(K, \eta), \ldots, \Xi_{d-1}(K, \eta)$ which is invertible (the corresponding determinant is a Vandermonde determinant), hence we obtain $\Xi_j(K, \eta)$ as a linear combination of the values $\mu_\rho(K, \eta)$, $\rho = 1, \ldots, d$. Now $K \mapsto \mu_\rho(K, \cdot)$, $K \in \mathcal{K}$, is weakly continuous (this is shown directly by an application of the Portmanteau theorem). Thus, the linear combination of the values $\mu_\rho(K, \eta)$, $\rho = 1, \ldots, d$, which was obtained for polytopes, extends to arbitrary $K \in \mathcal{K}$ by continuity and defines $\Xi_j(K, \eta)$ for $K \in \mathcal{K}$ (in a weakly continuous way). Therefore, the local Steiner formula (6) also extends from polytopes to arbitrary $K \in \mathcal{K}$, by continuity. It is a convenient strategy to establish properties of support measures of general convex bodies first for polytopes in a direct way, and then to deduce the corresponding property for arbitrary bodies by approximation with polytopes.

3 Flag Measures as Coefficients of a Local Steiner Formula

In this section, we introduce flag measures as natural generalizations of support measures by a similar procedure, namely a further variant of a local Steiner formula. The new aspect is that points in the neighborhood of a convex body K in $\mathbb{R}^d$ are replaced by k-flats, $k \in \{0, \ldots, d-1\}$. We mainly follow Hinderer [12, Chap. 4] and mostly skip the proofs since a detailed exposition with proofs is available in the recent paper [13].

Let K be a convex body, $\rho > 0$ and $k \in \{0, \ldots, d-1\}$. For a flat $E \in A(d, k)$, we let $p(K, E)$ and $l(K, E) = p(E, K)$ be the points in K and E closest to each other, provided this pair of closest points is unique. The distance between K and E is then given by $d(K, E) = \|p(K, E) - l(K, E)\|$ and the direction from K to E is

$$u(K, E) = \frac{l(K, E) - p(K, E)}{d(K, E)}$$

provided that $d(K, E) > 0$ (see Fig. 1). Note that this direction is always uniquely defined even if a pair of closest points is not unique. It follows from [28, Corollary 2.3.11] that, for μ_k-almost all flats $E \in A(d, k)$ with $E \cap K = \emptyset$, the distance $d(K, E)$ is realized by a unique pair $(x, y) \in K \times E$, and thus $x = p(K, E)$ and $y = l(K, E)$.

Let $A(d, k, K)$ denote the Borel set of all flats $E \in A(d, k)$ with $E \cap K = \emptyset$ and for which the pair of nearest points is unique. Then, the mappings $d(\cdot, \cdot)$, $p(\cdot, \cdot)$, $l(\cdot, \cdot)$ and $u(\cdot, \cdot)$ are continuous on $\{(K, E) : K \in \mathcal{K}, E \in A(d, k, K)\}$ (see [12, Lemma 23]). We also define a continuous mapping $E \mapsto L(E)$ from $A(d, k)$ to $G(d, k)$, which maps each flat to the parallel linear subspace.

For $K \in \mathcal{K}$, $\rho > 0$ and $\eta \in \mathcal{B}(\mathbb{R}^d \times S^{d-1} \times G(d, k))$, we now consider the *local parallel set*

$$M_\rho^{(k)}(K, \eta)$$
$$= \{E \in A(d, k, K) : 0 < d(K, E) \leq \rho, (p(K, E), u(K, E), L(E)) \in \eta\}.$$
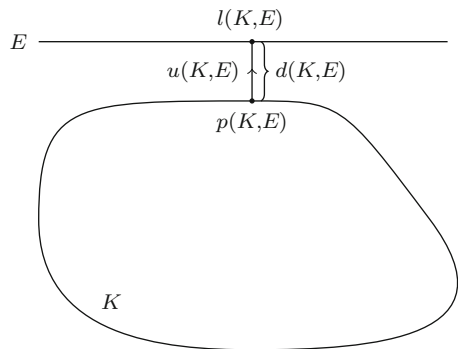
Fig. 1 Points realizing the distance between K and E

Then $M_\rho^{(k)}(K, \eta)$ is a Borel set in $A(d, k)$ and

$$\mu_\rho^{(k)}(K, \cdot) = \mu_k(M_\rho^{(k)}(K, \cdot)) \tag{11}$$

defines a finite Borel measure on $\mathcal{B}(\mathbb{R}^d \times S^{d-1} \times G(d, k))$.

Theorem 3. *For $K \in \mathcal{K}$ and $k \in \{0, \ldots, d-1\}$, there are finite (positive) Borel measures $\Xi_0^{(k)}(K, \cdot), \ldots, \Xi_{d-k-1}^{(k)}(K, \cdot)$ on $\mathbb{R}^d \times S^{d-1} \times G(d, k)$ such that, for all $\rho > 0$,*

$$\mu_\rho^{(k)}(K, \cdot) = \sum_{m=0}^{d-k-1} \rho^{d-k-m} \kappa_{d-k-m} \, \Xi_m^{(k)}(K, \cdot). \tag{12}$$

The measure $\Xi_j^{(k)}(K, \cdot)$, $j \in \{0, \ldots, d-k-1\}$, is concentrated on

$$\text{Nor}_k \, K = \{(x, u, L) : (x, u) \in \text{Nor } K, L \in G(d, k), L \bot u\}.$$

For $j \in \{0, \ldots, d-k-1\}$, the mapping $K \mapsto \Xi_j^{(k)}(K, \cdot)$ is weakly continuous. Moreover, $K \mapsto \Xi_j^{(k)}(K, \eta)$ is measurable, for each $\eta \in \mathcal{B}(\mathbb{R}^d \times S^{d-1} \times G(d, k))$.

The proof of the theorem is based on the following two lemmas.

Lemma 1. *The mapping $K \mapsto \mu_\rho^{(k)}(K, \cdot)$ is weakly continuous.*

This implies, in particular, that $K \mapsto \mu_\rho^{(k)}(K, \eta)$ is measurable, for $\eta \in \mathcal{B}(\mathbb{R}^d \times S^{d-1} \times G(d, k))$.

Lemma 2. *For $m \in \{0, \ldots, d-k-1\}$ and polytopes $K \subset \mathbb{R}^d$, Theorem 3 holds with*

$$\Xi_m^{(k)}(K, \cdot) = \frac{1}{\omega_{d-k-m}} \int_{G(d,k)} \sum_{F \in \mathcal{F}_m(K)} \int_{F|L^\perp} \int_{L^\perp \cap \, n(K,F)}$$

$$\times \mathbf{1}\{(p(F, L+x), u, L) \in \cdot\} \, \mathcal{H}^{d-k-m-1}(du) \, \mathcal{H}^m(dx) \, \nu_k(dL).$$

Note that for a given $F \in \mathcal{F}_m(K)$, with $m \in \{0, \ldots, d-k-1\}$, and ν_k-almost all $L \in G(d, k)$, F and L are in general relative position, that is, $L(F) \cap L = \{o\}$, where $L(F)$ is the linear subspace parallel to F. This fact can be used to show that all expressions involved in the preceding integral are well defined. In particular, for ν_k-almost all $L \in G(d, k)$ we have $\dim(L^\perp \cap \lim n(K, F)) = d-k-m$, where lin denotes the linear hull of a set, and $p(F, L+x)$ is the unique intersection point of F and $L+x$ for a given point $x \in F|L^\perp$.

Using (12) with $\rho = 1, \ldots, d-k$, we obtain a system of linear equations for the values $\Xi_0^{(k)}(K, \eta), \ldots, \Xi_{d-k-1}^{(k)}(K, \eta)$, where K is a polytope and η is a fixed Borel set. This system is invertible (again we have a Vandermonde determinant), hence there is a representation

$$\varXi_m^{(k)}(K,\eta) = \sum_{i=1}^{d-k} a(d,k,m,i)\,\mu_i^{(k)}(K,\eta) \tag{13}$$

with real numbers $a(d,k,m,i)$, which do not depend on η or K.

For an arbitrary body $K \in \mathcal{K}$, we choose a sequence of polytopes $K_r, r \in \mathbb{N}$, converging to K. Then (13) holds, for each polytope K_r. Due to Lemma 1, the right-hand side of (13) converges weakly to $\sum_{i=1}^{d-k} a(d,k,m,i)\mu_i^{(k)}(K,\cdot)$. Therefore, also the left-hand side converges and defines the limit measure $\varXi_m^{(k)}(K,\cdot)$. The measures thus obtained could be signed, but the explicit representation in Lemma 2 shows that they are non-negative (first for polytopes but then for arbitrary bodies by approximation). Lemma 1 also shows that the measures $\varXi_m^{(k)}(K,\cdot)$ depend continuously on $K \in \mathcal{K}$. Since (12) holds for the polytopes $K_r, r \in \mathbb{N}$, a continuity argument shows that it also holds for K, and the proof of Theorem 3 is complete.

The measures $\varXi_m^{(k)}(K,\cdot), m \in \{0,\dots,d-k-1\}$, are called *flag measures* of K. More precisely, $\varXi_m^{(k)}(K,\cdot)$ is called flag support measure of type (k,m).

For each $k \in \{0,\dots,d-1\}$, we thus obtain a sequence $\varXi_0^{(k)}(K,\cdot),\dots,$ $\varXi_{d-k-1}^{(k)}(K,\cdot)$ of flag measures. In particular, for $k = 0$ we get back the classical support measures. More generally, the support measures appear as image measures of the flag measures under a projection map. This fact is expressed by Proposition 1 below.

We also mention that the flag measures $\varXi_m^{(k)}(K,\cdot)$ induce flag area measures $\varPsi_m^{(k)}(K,\cdot)$ and flag curvature measures $\varPhi_m^{(k)}(K,\cdot)$ as image measures under the projection map $(x,u,L) \mapsto (u,L)$, respectively $(x,u,L) \mapsto (x,L)$. The general results on flag measures, which will be discussed in the sequel, always include corresponding assertions on flag area measures and flag curvature measures as special cases, even if we will not point this out explicitly in each case.

4 Flag Measures as Projection Averages

In Lemma 2 we have seen an explicit representation for the flag measure $\varXi_m^{(k)}(K,\cdot)$ of a polytope K as a sum over the m-dimensional faces of K. As an alternative approach, we could take this formula

$$\varXi_m^{(k)}(K,\cdot) = \frac{1}{\omega_{d-k-m}} \int_{G(d,k)} \sum_{F \in \mathcal{F}_m(K)} \int_{F|L^\perp} \int_{L^\perp \cap\, n(K,F)}$$

$$\times \mathbf{1}\{(p(F,L+x),u,L) \in \cdot\}\, \mathcal{H}^{d-k-m-1}(du)\, \mathcal{H}^m(dx)\, \nu_k(dL) \tag{14}$$

as the definition of $\varXi_m^{(k)}(K,\cdot)$ and ask whether this measure has a (weakly) continuous extension to arbitrary bodies $K \in \mathcal{K}$ and whether this extension, if it exists, satisfies a local Steiner formula. In this section, we will provide answers to these questions.

In view of an extension to general bodies $K \in \mathcal{K}$, our first goal is to reformulate the right-hand side of (14) appropriately. For this purpose, we replace $p(F, L + x)$ on the right-hand side of (14) by the value of a more generally defined function which has suitable continuity properties (see Lemma 5). Let $c(M)$ denote the midpoint of the circumsphere of a body $M \in \mathcal{K}$ and define

$$g : \mathbb{R}^d \times G(d,k) \times \mathcal{K} \to \mathbb{R}^d, \qquad (z, L, K) \mapsto c\left((p(K|L^\perp, z) + L) \cap K\right).$$

Then, for $F \in \mathcal{F}_m(K)$, we have $p(F, L + x) = g(x, L, K)$ for ν_k-almost all $L \in G(d,k)$ such that $L^\perp \cap \operatorname{relint} n(K, F) \neq \emptyset$ and all $x \in F|L^\perp$, and therefore

$$\Xi_m^{(k)}(K, \cdot) = \frac{1}{\omega_{d-k-m}} \int_{G(d,k)} \sum_{F \in \mathcal{F}_m(K)} \int_{F|L^\perp} \int_{L^\perp \cap n(K,F)}$$
$$\times \mathbf{1}\{(g(x, L, K), u, L) \in \cdot\} \mathcal{H}^{d-k-m-1}(du) \, \mathcal{H}^m(dx) \, \nu_k(dL). \quad (15)$$

As we shall show now, the inner part of the integral can be expressed in terms of the support measure $\Xi_m^{L^\perp}(K|L^\perp, \cdot)$ of $K|L^\perp$ in $L^\perp$. Namely, from (7) we have

$$\Xi_m^{L^\perp}(K|L^\perp, \cdot) = \frac{1}{\omega_{d-k-m}} \sum_{G \in \mathcal{F}_m(K|L^\perp)} \int_G \int_{n_{L^\perp}(K|L^\perp, G)}$$
$$\times \mathbf{1}\{(x, u) \in \cdot\} \mathcal{H}^{d-k-m-1}(du) \, \mathcal{H}^m(dx), \quad (16)$$

where $n_{L^\perp}(K|L^\perp, G)$ denotes the set of exterior unit normals of $K|L^\perp$ at G with respect to $L^\perp$ as the ambient space. In order to see the connection between (15) and (16), the following lemmas are useful.

Lemma 3. *Let $K \in \mathcal{K}$ be a polytope, $k \in \{0, \dots, d-1\}$, $m \in \{0, \dots, d-k-1\}$, $F \in \mathcal{F}_m(K)$ and $L \in G(d,k)$. If F and L are in general relative position and $u \in L^\perp \cap \operatorname{relint} n(K, F)$, then*

$$F|L^\perp \in \mathcal{F}_m(K|L^\perp)$$

and
$$u \in \operatorname{relint} n_{L^\perp}(K|L^\perp, F|L^\perp).$$

Proof. Let F, L and u be given as in the statement of the lemma. Since $u \in \operatorname{relint} n(K, F)$, [28, (2.4.3)] implies that $F = F(K, u) = H(K, u) \cap K$, where $H(K, u)$ is the supporting hyperplane of K with exterior unit normal u. Since $u \in L^\perp$, $H(K, u) \cap L^\perp$ is a supporting hyperplane of $K|L^\perp$ with outer normal u and $F|L^\perp = (K|L^\perp) \cap H(K, u) \cap L^\perp$. This implies that $F|L^\perp = F_{L^\perp}(K|L^\perp, u)$ is a face of $K|L^\perp$. Since L and F are in general relative position, we have $L \cap L(F) = \{o\}$ and therefore $\dim(F|L^\perp) = m$, which shows that $F|L^\perp \in \mathcal{F}_m(K|L^\perp)$. Using [28, (2.4.3)] again, we see that $u \in \operatorname{relint} n_{L^\perp}(K|L^\perp, F|L^\perp)$.

For a convex polytope $K \in \mathcal{K}$ and a subspace $L \in G(d,k)$ we say that K and L are in general relative position if, for all faces F of K, F and L are in general relative position.

Lemma 4. *Let $K \in \mathcal{K}$ be a polytope, $k \in \{0, \ldots, d-1\}$, and let $L \in G(d,k)$ be such that K and L are in general relative position. If $G \in \mathcal{F}_m(K|L^\perp)$, $m \in \{0, \ldots, d-k-1\}$, and if $u \in \operatorname{relint} n_{L^\perp}(K|L^\perp, G)$, then there is a unique face $F \in \mathcal{F}_m(K)$ with $F|L^\perp = G$ and $u \in \operatorname{relint} n(P, F)$.*

Proof. Let G, L, u be given as in the statement of the lemma. We use [28, (2.4.3)] again to get $F_{L^\perp}(K|L^\perp, u) = G$. Moreover, $F = F(K,u)$ is a face of K with $F|L^\perp = G$. Since K and L are in general relative position, we have $\dim(F) = \dim(G) = m$, hence $F \in \mathcal{F}_m(K)$ with $G = F|L^\perp$ and $u \in \operatorname{relint} n(K, F)$ by another application of [28, (2.4.3)]. The uniqueness assertion is clear.

Now we formulate the main result in this section. It describes the flag support measures of a convex body as mixtures of support measures of projections of the given convex body. Intuitively, it can be interpreted as the measure of k-flats touching a convex body in a given set of support elements and such that the linear subspaces parallel to the k-flats also lie in a prescribed Borel set.

Theorem 4. *Let $K \in \mathcal{K}$ be a polytope and let the measure $\Xi_m^{(k)}(K, \cdot)$ be defined by (15), for $k \in \{0, \ldots, d-1\}$ and $m \in \{0, \ldots, d-k-1\}$. Then*

$$\Xi_m^{(k)}(K, \cdot) = \int_{G(d,k)} \int \mathbf{1}\{(g(x, L, K), u, L) \in \cdot\} \, \Xi_m^{L^\perp}(K|L^\perp, d(x, u)) \, \nu_k(dL). \tag{17}$$

If $\Xi_m^{(k)}(K, \cdot)$ is defined for arbitrary $K \in \mathcal{K}$ by (17), then the extended mapping $K \mapsto \Xi_m^{(k)}(K, \cdot)$ is weakly continuous on $\mathcal{K}$.

Proof. We start with a polytope K and a subspace $L \in G(d,k)$ which are in general relative position. Since $n_{L^\perp}(K|L^\perp, G)$ and $\operatorname{relint} n_{L^\perp}(K|L^\perp, G)$ only differ by a set of $\mathcal{H}^{d-k-m-1}$-measure 0, the representation (16) turns into

$$\Xi_m^{L^\perp}(K|L^\perp, \cdot) = \frac{1}{\omega_{d-k-m}} \sum_{G \in \mathcal{F}_m(K|L^\perp)} \int_G \int_{\operatorname{relint} n_{L^\perp}(K|L^\perp, G)}$$

$$\times \mathbf{1}\{(x, u) \in \cdot\} \, \mathcal{H}^{d-k-m-1}(du) \, \mathcal{H}^m(dx).$$

Using Lemmas 3 and 4, we thus obtain

$$\Xi_m^{L^\perp}(K|L^\perp, \cdot) = \frac{1}{\omega_{d-k-m}} \sum_{F \in \mathcal{F}_m(K)} \int_{F|L^\perp} \int_{L^\perp \cap \operatorname{relint} n(K, F)}$$

$$\times \mathbf{1}\{(x, u) \in \cdot\} \, \mathcal{H}^{d-k-m-1}(du) \, \mathcal{H}^m(dx).$$

The spherical set $n(K, F)$ is the intersection of the unit sphere with a $(d - m)$-dimensional polyhedral cone (the normal cone $N(K, F)$) and the relative boundary of $N(K, F)$ is contained in a finite union of $(d - m - 1)$-dimensional subspaces. Excluding a set in $G(d, k)$ of ν_k-measure 0, we may assume that $L^\perp$ and each of these subspaces bounding $N(K, F)$ are in general relative position. This implies

$$\mathcal{H}^{d-k-m-1}(L^\perp \cap \operatorname{relbd} n(K, F)) = 0,$$

and hence

$$\Xi_m^{L^\perp}(K|L^\perp, \cdot) = \frac{1}{\omega_{d-k-m}} \sum_{F \in \mathcal{F}_m(K)} \int_{F|L^\perp} \int_{L^\perp \cap n(K,F)}$$

$$\times \mathbf{1}\{(x, u) \in \cdot\} \, \mathcal{H}^{d-k-m-1}(du) \, \mathcal{H}^m(dx). \tag{18}$$

Combining (18) with (15) yields (17).

Now let $K \in \mathcal{K}$ be arbitrary and let $\Xi_m^{(k)}(K, \cdot)$ be defined by (17). We show that $K \mapsto \Xi_m^{(k)}(K, \cdot)$ is weakly continuous.

Let $\operatorname{USP}_k(K)$ be the set of all subspaces $L \in G(d, k)$ which have the *unique support property* for K. This means that every k-flat parallel to L, which supports K, meets K only in one point. If the subspace L is fixed, we write $g_L(x, K)$ instead of $g(x, L, K)$, in the following. We first discuss the continuity properties of g_L.

Lemma 5. *Let $K_i \in \mathcal{K}$ and $x_i \in \mathbb{R}^d$, $i \in \mathbb{N}$, be converging sequences, $K_i \to K_0$, $x_i \to x_0$ (as $i \to \infty$). If $L \in \operatorname{USP}_k(K_0)$, then*

$$\lim_{i \to \infty} g_L(x_i, K_i) = g_L(x_0, K_0).$$

In particular, $x \mapsto g_L(x, K_0)$ is continuous and the convergence $g_L(\cdot, K_i) \to g_L(\cdot, K_0)$, as $i \to \infty$, is uniform on compact subsets of $\mathbb{R}^d$.

Proof. The continuity of the metric projection ([28, Lemma 1.8.9]) and [30, Theorem 12.3.5] yield that $p(K_i|L^\perp, x_i) + L \to p(K_0|L^\perp, x_0) + L$, as $i \to \infty$.

Case 1: $p(K_0|L^\perp, x_0) + L$ and K_0 cannot be separated by a hyperplane.

Then the proof of Theorem 1.8.8 in [28] (with Theorem 1.8.7 replaced by Theorem 12.2.2 in [30]) shows that $(p(K_i|L^\perp, x_i) + L) \cap K_i \to (p(K_0|L^\perp, x_0) + L) \cap K_0$, as $i \to \infty$, in $\mathcal{K}$. From the continuity of the map $K \mapsto c(K)$ (see [28, Lemma 4.1.1]) we thus get the assertion.

Case 2: $p(K_0|L^\perp, x_0) + L$ and K_0 can be separated by a hyperplane.

Since $L \in \operatorname{USP}(K_0)$, we get $(p(K_0|L^\perp, x_0) + L) \cap K_0 = \{z_0\}$, for some $z_0 \in \mathbb{R}^d$. We show that

$$(p(K_i|L^\perp, x_i) + L) \cap K_i \to \{z_0\}, \tag{19}$$

which proves the assertion.

For this, we choose points $y_i \in (p(K_i|L^\perp, x_i) + L) \cap K_i$, $i \in \mathbb{N}$, which converge to some $y_0 \in K_0$ (as $i \to \infty$ and possibly after selecting a subsequence). Since $y_i \in p(K_i|L^\perp, x_i) + L \to p(K_0|L^\perp, x_0) + L$, we also have $y_0 \in p(K_0|L^\perp, x_0) + L$ and thus $y_0 \in (p(K_0|L^\perp, x_0) + L) \cap K_0$. This implies $y_0 = z_0$. Hence every sequence $y_i \in (p(K_i|L^\perp, x_i) + L) \cap K_i \neq \emptyset$, $i \in \mathbb{N}$, has a subsequence which converges to z_0. Thus y_i converges to z_0 and so (19) follows from [30, Theorem 12.2.2].

The remaining assertions are clear.

Now we continue the proof of Theorem 4. Let F be a (non-negative) continuous function on $\mathbb{R}^d \times S^{d-1} \times G(d, k)$. We have to show that

$$K \mapsto \int_{G(d,k)} \int F(g(x, L, K), u, L)\, \Xi_m^{L^\perp}(K|L^\perp, d(x,u))\, \nu_k(dL)$$

is continuous on $\mathcal{K}$. For this purpose, we consider $K_i, K_0 \in \mathcal{K}$, $i \in \mathbb{N}$, with $K_i \to K_0$, as $i \to \infty$, and show that

$$\lim_{i \to \infty} \int F(g_L(x, K_i), u, L)\, \Xi_m^{L^\perp}(K_i|L^\perp, d(x,u))$$
$$= \int F(g_L(x, K_0), u, L)\, \Xi_m^{L^\perp}(K_0|L^\perp, d(x,u)) \qquad (20)$$

for ν_k-almost all $L \in G(d, k)$. The assertion then follows by the dominated convergence theorem.

From [28, Corollary 2.3.11] we get $L \in \bigcap_{i \in \mathbb{N}_0} \mathrm{USP}_k(K_i)$ for ν_k-almost all linear subspaces $L \in G(d, k)$. For such an L, we have

$$\left| \int F(g_L(x, K_i), u, L)\, \Xi_m^{L^\perp}(K_i|L^\perp, d(x,u)) \right.$$
$$\left. - \int F(g_L(x, K_0), u, L)\, \Xi_m^{L^\perp}(K_0|L^\perp, d(x,u)) \right|$$
$$= \left| \int F(g_L(x, K_i), u, L) - F(g_L(x, K_0), u, L)\, \Xi_m^{L^\perp}(K_i|L^\perp, d(x,u)) \right.$$
$$+ \int F(g_L(x, K_0), u, L)\, \Xi_m^{L^\perp}(K_i|L^\perp, d(x,u))$$
$$\left. - \int F(g_L(x, K_0), u, L)\, \Xi_m^{L^\perp}(K_0|L^\perp, d(x,u)) \right|$$
$$\leq \int \left| F(g_L(x, K_i), u, L) - F(g_L(x, K_0), u, L) \right| \Xi_m^{L^\perp}(K_i|L^\perp, d(x,u))$$
$$+ \left| \int F(g_L(x, K_0), u, L)\, \Xi_m^{L^\perp}(K_i|L^\perp, d(x,u)) \right.$$
$$\left. - \int F(g_L(x, K_0), u, L)\, \Xi_m^{L^\perp}(K_0|L^\perp, d(x,u)) \right|. \qquad (21)$$

Since $K_i \rightarrow K_0$ and so $K_i|L^\perp \rightarrow K_0|L^\perp$, the support of the measures $\Xi_m^{L^\perp}(K_i|L^\perp, \cdot)$, $i \in \mathbb{N}_0$, is contained in $RB^d \times S^{d-1}$ for some sufficiently large $R > 0$. Therefore, Lemma 5 implies that, for $\epsilon > 0$, we find $i_0 = i_0(\epsilon)$ such that, for $i \geq i_0$,

$$\left| F(g_L(x, K_i), u, L) - F(g_L(x, K_0), u, L) \right| \leq \epsilon$$

for all $(x, u) \in RB^d \times S^{d-1}$. Hence, the first summand in (21) can be made smaller than $c(d, k, R) \cdot \epsilon$.

Since, by Lemma 5, the mapping $(x, u) \mapsto F(g_L(x, K_0), u, L)$ is continuous and bounded on $RB^d \times S^{d-1}$ and since $\Xi_m^{L^\perp}(K_i|L^\perp, \cdot) \rightarrow \Xi_m^{L^\perp}(K_0|L^\perp, \cdot)$, as $i \rightarrow \infty$, in the weak topology, by [28, Theorem 4.2.1], we can also make the second summand in (21) smaller than ϵ, for $i \geq i_1(\epsilon)$.

This proves (20) and finishes the proof of Theorem 4.

We now show that the measures $\Xi_m^{(k)}(K, \cdot)$ defined by (17) satisfy a local Steiner formula. Let $\mu_\rho^{(k)}(K, \eta) = \mu_k(M_\rho^{(k)}(K, \eta))$ be the measure of the outer parallel set which we considered in (11), evaluated at some Borel set $\eta \subset \mathbb{R}^d \times S^{d-1} \times G(d, k)$. By definition of μ_k, we get

$$\mu_\rho^{(k)}(K, \eta) = \int_{G(d,k)} \mathcal{H}^{d-k}(T(L)) \, \nu_k(dL) \tag{22}$$

with

$$T(L) = \{x \in L^\perp : L + x \in M_\rho^{(k)}(K, \eta)\}.$$

For ν_k-almost all $L \in G(d, k)$ and for $x \in L^\perp$, the condition $L + x \in M_\rho^{(k)}(K, \eta)$ is equivalent to

$$0 < d(K|L^\perp, x) \leq \rho, \qquad (g(x, L, K), u(K|L^\perp, x), L) \in \eta.$$

Hence, if $\eta(L)$ denotes the set of pairs $(x|L^\perp, u) \in L^\perp \times (L^\perp \cap S^{d-1})$ with $(x, u, L) \in \eta$, we have

$$\mu_\rho^{(k)}(K, \eta)$$
$$= \int_{G(d,k)} \int_{L^\perp} \mathbf{1}\{0 < d(K|L^\perp, x) \leq \rho, (p(K|L^\perp, x), u(K|L^\perp, x)) \in \eta(L)\}$$
$$\times \mathcal{H}^{d-k}(dx) \, \nu_k(dL)$$
$$= \int_{G(d,k)} \mathcal{H}^{d-k}(M_\rho^{L^\perp}(K|L^\perp, \eta(L))) \, \nu_k(dL), \tag{23}$$

where $M_\rho^{L^\perp}(K|L^\perp, \eta(L))$ denotes an (ordinary) local outer parallel set of $K|L^\perp$ in $L^\perp$. Applying the classical local Steiner formula (6) to $K|L^\perp$ in $L^\perp$, we get

$$\mu_\rho^{(k)}(K, \eta) = \sum_{m=0}^{d-k-1} \rho^{d-k-m} \kappa_{d-k-m} \int_{G(d,k)} \Xi_m^{L^\perp}(K|L^\perp, \eta(L)) \, \nu_k(dL). \tag{24}$$

For v_k-almost all $L \in G(d,k)$, for $x \in L^\perp$ and $u \in L^\perp \cap S^{d-1}$, we have $(g(x,L,K),u,L) \in \eta$ if and only if $(x,u) \in \eta(L)$. Therefore,

$$\int_{G(d,k)} \Xi_m^{L^\perp}(K|L^\perp, \eta(L))\, v_k(dL) = \Xi_m^{(k)}(K,\eta), \tag{25}$$

which shows that (12) holds.

Using [28, Theorem 4.5.10] (see also Note 2 for Sect. 6.2 in [30, p. 223]), we obtain from (25) the following relationship.

Proposition 1. *For $K \in \mathcal{K}$, $k \in \{0, \ldots, d-1\}$ and $m \in \{0, \ldots, d-k-1\}$, the measure $\Xi_m(K, \cdot)$ is the image of $\alpha(d,k,m) \cdot \Xi_m^{(k)}(K, \cdot)$ under the projection $(x,u,L) \mapsto (x,u)$, where*

$$\alpha(d,k,m) = \frac{\Gamma\left(\frac{d+1}{2}\right) \Gamma\left(\frac{d-k-m+1}{2}\right)}{\Gamma\left(\frac{d-k+1}{2}\right) \Gamma\left(\frac{d-m+1}{2}\right)}.$$

Note that $\alpha(d,k,d-k-1) = c(d,k)$ and thus we arrive at the normalization (4) from the introduction.

The arguments just given also lead to a possible introduction of the flag measures $\Xi_m^{(k)}(K, \cdot)$. This is the approach chosen in Kropp [17]. Namely one starts with (22), shows that this implies (23), uses the Steiner formula (6) to obtain formula (24) and then defines the measure $\Xi_m^{(k)}(K, \cdot)$ by (25).

5 Further Properties of Flag Measures

We now collect various additional properties of flag measures and study representations and extensions.

Four major properties are homogeneity, motion covariance, local definedness and additivity. Homogeneity means that $\Xi_m^{(k)}(K, \cdot)$ is *homogeneous of degree m*, in the sense that

$$\Xi_m^{(k)}(\alpha K, \alpha \eta) = \alpha^m \Xi_m^{(k)}(K, \eta),$$

for $K \in \mathcal{K}$, $\eta \in \mathcal{B}(\mathbb{R}^d \times S^{d-1} \times G(d,k))$ and $\alpha > 0$, with

$$\alpha \eta = \{(\alpha x, u, L) : (x,u,L) \in \eta\}.$$

Furthermore, we say that $\Xi_m^{(k)}(K, \cdot)$ is *motion covariant*, if

$$\Xi_m^{(k)}(gK, g\eta) = \Xi_m^{(k)}(K, \eta)$$

holds for $K \in \mathcal{K}$, each Borel set $\eta \in \mathcal{B}(\mathbb{R}^d \times S^{d-1} \times G(d,k))$ and each rigid motion $g \in G_d$, where $g\eta = \{(gx, g_0 u, g_0 L) : (x,u,L) \in \eta\}$ and g_0 is the

rotational part of g. We call $\Xi_m^{(k)}(K, \cdot)$ *locally defined*, if

$$\Xi_m^{(k)}(K, \eta) = \Xi_m^{(k)}(M, \eta)$$

for all bodies $K, M \in \mathcal{K}$ such that there is an open set $A \subset \mathbb{R}^d$ with $K \cap A = M \cap A$ and for all Borel sets η, for which the projection η_1 onto the first component lies in A. Finally, additivity means that the mapping $K \mapsto \Xi_m^{(k)}(K, \cdot)$ is *additive* in the sense that

$$\Xi_m^{(k)}(K \cup M, \cdot) + \Xi_m^{(k)}(K \cap M, \cdot) = \Xi_m^{(k)}(K, \cdot) + \Xi_m^{(k)}(M, \cdot)$$

for all $K, M \in \mathcal{K}$ for which $K \cup M \in \mathcal{K}$.

Theorem 5. *For $k \in \{0, \ldots, d-1\}$ and $m \in \{0, \ldots, d-k-1\}$, the flag measure $\Xi_m^{(k)}(K, \cdot)$ is homogeneous of degree m, it is motion covariant and locally defined and, as a function of K, it is additive.*

Proof. Concerning the homogeneity property, we notice that

$$\Xi_m^{(k)}(\alpha K, \alpha \eta)$$

$$= \int_{G(d,k)} \int \mathbf{1}\{(g(x, L, \alpha K), u, L) \in \alpha \eta\} \, \Xi_m^{L^\perp}(\alpha K | L^\perp, d(x, u)) \, \nu_k(dL)$$

$$= \int_{G(d,k)} \int \mathbf{1}\{(g(x, L, \alpha K), u, L) \in \alpha \eta\} \, \Xi_m^{L^\perp}(\alpha (K | L^\perp), d(x, u)) \, \nu_k(dL)$$

$$= \int_{G(d,k)} \int \mathbf{1}\{(g(\alpha x, L, \alpha K), u, L) \in \alpha \eta\} \, \alpha^m \, \Xi_m^{L^\perp}(K | L^\perp, d(x, u)) \, \nu_k(dL)$$

$$= \alpha^m \int_{G(d,k)} \int \mathbf{1}\{(\alpha g(x, L, K), u, L) \in \alpha \eta\} \, \Xi_m^{L^\perp}(K | L^\perp, d(x, u)) \, \nu_k(dL)$$

$$= \alpha^m \, \Xi_m^{(k)}(K, \eta).$$

Here, we used $\alpha K | L^\perp = \alpha(K | L^\perp)$, the homogeneity of $\Xi_m^{L^\perp}$ and $g(\alpha x, L, \alpha K) = \alpha g(x, L, K)$.

The motion covariance follows from the fact that

$$M_\rho^{(k)}(gK, g\eta)$$

$$= \{E \in A(d, k, gK) : 0 < d(gK, E) \leq \rho, (p(gK, E), u(gK, E), L(E)) \in g\eta\}$$

$$= \{gE : E \in A(d, k, K), 0 < d(gK, gE) \leq \rho, (p(gK, gE), u(gK, gE), L(gE)) \in g\eta\}$$

$$= \{gE : E \in A(d, k, K), 0 < d(K, E) \leq \rho, (gp(K, E), g_0 u(K, E), g_0 L(E)) \in g\eta\}$$

$$= g M_\rho^{(k)}(K, \eta).$$

The motion invariance of μ_k shows that $\mu_\rho^{(k)}(gK, g\eta) = \mu_\rho^{(k)}(K, \eta)$, and so the motion covariance transfers to the coefficient measures in (12), by the procedure described in detail in Sect. 3.

To show that $\Xi_m^{(k)}(K, \cdot)$ is locally defined, let K, M and A be such that $K \cap A = M \cap A$, where $A \subset \mathbb{R}^d$ is open, and let η be a Borel set such that $\{x \in \mathbb{R}^d : (x, u, L) \in \eta\} \subset A$. By definition,

$$
\begin{aligned}
&M_\rho^{(k)}(K, \eta) \\
&\quad = \{E \in A(d, k, K) : 0 < d(K, E) \le \rho, (p(K, E), u(K, E), L(E)) \in \eta\}.
\end{aligned}
$$

Hence, for $E \in M_\rho^{(k)}(K, \eta)$, we have $p(K, E) \in A$. Since A is open, the boundaries of K and M coincide in a neighborhood of $p(K, E)$. Therefore, excluding possibly a set of $E \in A(d, k)$ of μ_k-measure zero, we have $E \in A(d, k, M)$ and $p(K, E) = p(M, E)$. This implies $u(K, E) = u(M, E)$ and thus $E \in M_\rho^{(k)}(K, \eta)$. Interchanging the role of K and M, we conclude that $M_\rho^{(k)}(K, \eta) = M_\rho^{(k)}(M, \eta)$, up to a set of $E \in A(d, k)$ of μ_k-measure zero. With the same argument as above, this implies $\Xi_m^{(k)}(K, \eta) = \Xi_m^{(k)}(M, \eta)$.

Concerning the additivity, assume $K, M, K \cup M \in \mathcal{K}$. It is sufficient, by the same principle, to show that

$$
\begin{aligned}
&\mathbf{1}\{E \in M_\rho^{(k)}(K \cup M, \eta)\} + \mathbf{1}\{E \in M_\rho^{(k)}(K \cap M, \eta)\} \\
&\quad = \mathbf{1}\{E \in M_\rho^{(k)}(K, \eta)\} + \mathbf{1}\{E \in M_\rho^{(k)}(M, \eta)\}
\end{aligned}
\tag{26}
$$

holds for μ_k-almost all E. In proving this, we follow [12, Lemma 28]. We assume $E \in A(d, k, K) \cap A(d, k, M)$ and put $y = p(K, E), z = p(M, E)$.

We first consider the case $d(K \cup M, E) = d(K, E)$, hence $d(K \cup M, E) = d(y, E)$. If $d(M, E) < d(K, E)$, then $p(K \cup M, E) = y$. If $d(M, E) = d(K, E)$, then $[y, z] \subset K \cup M$. Since $E \in A(d, k, K) \cap A(d, k, M)$, it follows that $y = z$. Then, we have again $p(K \cup M, E) = y$. Since $K \cup M$ is convex, $[y, z]$ is a subset of $K \cup M$, and so a point $a \in [y, z] \cap K \cap M$ exists. The mapping

$$
t \mapsto d(tz + (1 - t)y, E)
$$

is convex on $[0, 1]$ and has a minimum at $t = 0$. Therefore, $d(y, E) \le d(a, E) \le d(z, E)$. As $z = p(M, E)$, we have $d(a, E) \ge d(z, E)$ and thus $d(a, E) = d(z, E)$ (and $a, z \in M$). The uniqueness of the nearest point map now implies $z = a \in K \cap M$. We get

$$
d(K \cup M, E) = d(K, E), \quad d(K \cap M, E) = d(M, E)
$$

and

$$
u(K \cup M, E) = u(K, E), \quad u(K \cap M, E) = u(M, E).
$$

Therefore,

$$\mathbf{1}\{E \in M_\rho^{(k)}(K \cup M, \eta)\} = \mathbf{1}\{E \in M_\rho^{(k)}(K, \eta)\},$$

$$\mathbf{1}\{E \in M_\rho^{(k)}(K \cap M, \eta)\} = \mathbf{1}\{E \in M_\rho^{(k)}(M, \eta)\},$$

which implies (26).

In the other case, that is $d(K \cup M, E) = d(M, E)$, we first get $p(K \cup M, E) = z$, and then we conclude in a similar way that

$$\mathbf{1}\{E \in M_\rho^{(k)}(K \cup M, \eta)\} = \mathbf{1}\{E \in M_\rho^{(k)}(M, \eta)\},$$

$$\mathbf{1}\{E \in M_\rho^{(k)}(K \cap M, \eta)\} = \mathbf{1}\{E \in M_\rho^{(k)}(K, \eta)\},$$

which again implies (26).

Since the flag measures are the coefficients in a local Steiner formula, it is a natural question whether they themselves admit a polynomial expansion, if $\Xi_m^{(k)}(K + \rho B^d, \cdot)$ is considered. For the classical support measures, a corresponding result is formula (10), which we used earlier, and the following proposition shows that this carries over to flag measures.

Proposition 2. *Let* $K \in \mathcal{K}$, $\rho > 0$, $k \in \{0, \ldots, d-1\}$ *and* $m \in \{0, \ldots, d-k-1\}$. *Then we have*

$$\Xi_m^{(k)}(K + \rho B^d, t_\rho \eta) = \sum_{j=0}^m \rho^j \binom{d-k+j-m}{j} \frac{\kappa_{d-k+j-m}}{\kappa_{d-k-m}} \Xi_{m-j}^{(k)}(K, \eta),$$

where $t_\rho(x, u, L) = (x + \rho u, u, L)$.

Proof. Using (10) and $(K + \rho B^d)|L^\perp = K|L^\perp + \rho B^d|L^\perp$, we get

$$\Xi_m^{(k)}(K + \rho B^d, t_\rho \eta)$$

$$= \int_{G(d,k)} \int \mathbf{1}\{(g(x, L, K + \rho B^d), u, L) \in t_\rho \eta\}$$

$$\times \Xi_m^{L^\perp}((K + \rho B^d)|L^\perp, d(x, u)) \, \nu_k(dL)$$

$$= \int_{G(d,k)} \int \mathbf{1}\{(g(x - \rho u, L, K) + \rho u, u, L) \in t_\rho \eta\}$$

$$\times \Xi_m^{L^\perp}(K|L^\perp + \rho B^d|L^\perp, d(x, u)) \, \nu_k(dL)$$

$$= \int_{G(d,k)} \int \mathbf{1}\{(g(x - \rho u, L, K), u, L) \in \eta\}$$

$$\times \Xi_m^{L^\perp}(K|L^\perp + \rho B^d|L^\perp, d(x, u)) \, \nu_k(dL)$$

$$= \sum_{j=0}^{m} \rho^j \binom{d-k+j-m}{j} \frac{\kappa_{d-k+j-m}}{\kappa_{d-k-m}} \int_{G(d,k)} \int$$

$$\times \mathbf{1}\{(g(z,L,K),u,L) \in \eta\} \, \Xi_{m-j}^{L^\perp}(K|L^\perp, d(z,u)) \, \nu_k(dL)$$

$$= \sum_{j=0}^{m} \rho^j \binom{d-k+j-m}{j} \frac{\kappa_{d-k+j-m}}{\kappa_{d-k-m}} \, \Xi_{m-j}^{(k)}(K,\eta).$$

Proposition 2 is actually a special case of a multilinear expansion, which is in analogy to (8) and follows from this formula by using the projection average approach. For the latter, we need to know that the condition of general relative position is compatible with projections. Such a result is provided by the following lemma.

Lemma 6. *Let $K_1, K_2 \in \mathcal{K}$ be in general relative position in $\mathbb{R}^d$. Let $L \in G(d,k)$, $k \in \{0, \ldots, d-1\}$, be such that there is no supporting k-flat of $K_1 + K_2$ parallel to L that contains a 1-dimensional convex subset of $K_1 + K_2$. Then $K_1|L^\perp$ and $K_2|L^\perp$ are in general relative position in $L^\perp$.*

Proof. For $k \in \{0, d-1\}$ there is nothing to prove. Hence we assume that L is as in the assumptions of the lemma with $1 \le k \le d-2$.

Assume that $K_1|L^\perp$ and $K_2|L^\perp$ are not in general relative position in $L^\perp$. Then there is a unit vector $v \in L^\perp$ and there are parallel segments $\tilde{S}_i \subset F(K_i|L^\perp, v)$, $i = 1, 2$. Hence there are segments $S_i \subset F(K_i, v)$ with $\tilde{S}_i = S_i|L^\perp, i = 1, 2$. Since K_1, K_2 are in general relative position in $\mathbb{R}^d$, the segments S_1, S_2 are not parallel, and therefore $\dim(S_1 + S_2) = 2$. But then $S_1 + S_2$ contains a segment S parallel to L and $S + L \subset H(K_1 + K_2, v)$ is a supporting k-flat of $K_1 + K_2$ parallel to L which contains a 1-dimensional subset of $K_1 + K_2$. This is a contradiction to the choice of L. $\quad\square$

Theorem 6. *Let $l \in \mathbb{N}$, $K_1, \ldots, K_l \in \mathcal{K}$ be convex bodies in general relative position, $\rho_1, \ldots, \rho_l \ge 0$ and $\beta_1, \ldots, \beta_l$ be Borel sets with $\beta_i \subset K_i$. Let $k \in \{0, \ldots, d-1\}$, and let $\gamma \subset S^{d-1} \times G(d,k)$ be a Borel set.*

(a) For $m \in \{0, \ldots, d-k-1\}$ and $i_1, \ldots, i_l \in \{0, \ldots, m\}$ with $i_1 + \cdots + i_l = m$, there exist measures $\Xi_{i_1,\ldots,i_l}^{(k)}(K_1, \ldots, K_l, \cdot)$ concentrated on $\mathrm{bd}\, K_1 \times \cdots \times \mathrm{bd}\, K_l \times S^{d-1} \times G(d,k)$ such that

$$\Xi_m^{(k)} \left(\sum_{i=1}^{l} \rho_i K_i, \left(\sum_{i=1}^{l} \rho_i \beta_i \right) \times \gamma \right)$$

$$= \sum_{i_1,\ldots,i_l=0}^{m} \binom{m}{i_1, \ldots, i_l} \rho_1^{i_1} \cdots \rho_l^{i_l} \, \Xi_{i_1,\ldots,i_l}^{(k)}(K_1, \ldots, K_l, \beta_1 \times \cdots \times \beta_l \times \gamma).$$

$$(27)$$

The measure $\Xi^{(k)}_{i_1,\ldots,i_l}(K_1,\ldots,K_l,\cdot)$ depends in a weakly continuous way on $K_1,\ldots,K_l$ in general relative position, and it is symmetric, in the sense that

$$\Xi^{(k)}_{i_1,\ldots,i_l}(K_1,\ldots,K_l,\beta_1\times\cdots\times\beta_l\times\gamma)$$

$$= \Xi^{(k)}_{i_{\pi(1)},\ldots,i_{\pi(l)}}(K_{\pi(1)},\ldots,K_{\pi(l)},\beta_{\pi(1)}\times\cdots\times\beta_{\pi(l)}\times\gamma),$$

for all permutations π of $1,\ldots,l$. If $i_1 = 0$, then

$$\Xi^{(k)}_{i_1,\ldots,i_l}(K_1,\ldots,K_l,\operatorname{bd}K_1\times\cdot) = \Xi^{(k)}_{i_2,\ldots,i_l}(K_2,\ldots,K_l,\cdot).$$

If $i_1 \neq 0$, then $\Xi^{(k)}_{i_1,\ldots,i_l}(K_1,\ldots,K_l,\cdot)$ is translation covariant and homogeneous of degree i_1 in the first component, hence

$$\Xi^{(k)}_{i_1,i_2,\ldots,i_l}(\alpha K_1 + x, K_2,\ldots,K_l,(\alpha\beta_1 + x)\times\beta_2\times\cdots\times\beta_l\times\gamma)$$

$$= \alpha^{i_1}\Xi^{(k)}_{i_1,\ldots,i_l}(K_1,\ldots,K_l,\beta_1\times\cdots\times\beta_l\times\gamma),$$

for $\alpha > 0$, $x \in \mathbb{R}^d$. Moreover, $\Xi^{(k)}_{i_1,i_2,\ldots,i_l}(\cdot,K_2,\ldots,K_l,\cdot\times\beta_2\times\cdots\times\beta_l\times\gamma)$ has a polynomial expansion.

(b) Let $m \in \{1,\ldots,d-1-k\}$ and $i_1,\ldots,i_l \in \{1,\ldots,m\}$ be such that $i_1+\cdots+i_l = m$. Then, for strictly convex bodies $K_1,\ldots,K_l$, we have

$$\Xi^{(k)}_{i_1,\ldots,i_l}(K_1,\ldots,K_l,\beta_1\times\cdots\times\beta_l\times\gamma)$$

$$= \Xi^{(k)}_{1,\ldots,1}(K_1[i_1],\ldots,K_l[i_l],(\beta_1)^{i_1}\times\cdots\times(\beta_l)^{i_l}\times\gamma),$$

where the lower index 1 on the right-hand side appears m times.

(c) For $m \in \{0,\ldots,d-1-k\}$, $r \in \{0,\ldots,m\}$, $K \in \mathcal{K}$ and Borel sets $\beta \subset K$, $\gamma \subset S^{d-1}\times G(d,k)$, we have

$$\Xi^{(k)}_r(K,\beta\times\gamma) = \frac{\binom{d-k}{r}}{\binom{d-k}{m}}\frac{\kappa_{d-k-m}}{\kappa_{d-k-r}}\Xi^{(k)}_{r,m-r}(K,B^d,\beta\times S^{d-1}\times\gamma).$$

In addition, for $i_1,\ldots,i_s \in \{0,\ldots,r\}$ with $i_1 + \cdots + i_s = r$, convex bodies $K_1,\ldots,K_s \in \mathcal{K}$ in general relative position and Borel sets $\beta_1,\ldots,\beta_s \subset \mathbb{R}^d$ with $\beta_i \subset K_i$ and $\gamma \subset S^{d-1}\times G(d,k)$, we have

$$\Xi^{(k)}_{i_1,\ldots,i_s}(K_1,\ldots,K_s,\beta_1\times\cdots\times\beta_s\times\gamma)$$

$$= \frac{\binom{d-k}{r}}{\binom{d-k}{m}}\frac{\kappa_{d-k-m}}{\kappa_{d-k-r}}\Xi^{(k)}_{i_1,\ldots,i_s,m-r}(K_1,\ldots,K_s,B^d,\beta_1\times\cdots\times\beta_s\times S^{d-1}\times\gamma).$$

Proof. (a) We may assume $\gamma = \omega \times A$ with Borel sets $\omega \subset S^{d-1}$ and $A \subset G(d,k)$.

In order to use (17) together with Theorem 2, we first need to show that $K_1|L^\perp, \ldots, K_l|L^\perp$ are in general relative position, for ν_k-almost all $L \in G(d,k)$. For this purpose, we combine Lemma 6 with a result due to Zalgaller (see the case $s = 1$ and $r = k$ of the Theorem in Notes for Sect. 2.3 in [28, p. 93]) to obtain that if K_1 and K_2 are in general relative position in $\mathbb{R}^d$, then $K_1|L^\perp$ and $K_2|L^\perp$ are in general relative position in $L^\perp$ for ν_k-almost all $L \in G(d,k)$. A straightforward induction argument yields the extension of this result to finitely many convex bodies in general relative position.

By formulas (17) and (8), we now have

$$
\Xi_m^{(k)}\left(\sum_{i=1}^{l} \rho_i K_i, \left(\sum_{i=1}^{l} \rho_i \beta_i\right) \times \gamma\right)
$$

$$
= \int_{G(d,k)} \int \mathbf{1}\left\{\left(g\left(x, L, \sum_{i=1}^{l} \rho_i K_i\right), u, L\right) \in \left(\sum_{i=1}^{l} \rho_i \beta_i\right) \times \gamma\right\}
$$

$$
\times \Xi_m^{L^\perp}\left(\left(\sum_{i=1}^{l} \rho_i K_i\right)|L^\perp, d(x,u)\right) \nu_k(dL)
$$

$$
= \int_{G(d,k)} \int \mathbf{1}\left\{\left(g\left(x, L, \sum_{i=1}^{l} \rho_i K_i\right), u, L\right) \in \left(\sum_{i=1}^{l} \rho_i \beta_i\right) \times \gamma\right\}
$$

$$
\times \Xi_m^{L^\perp}\left(\sum_{i=1}^{l} \rho_i (K_i|L^\perp), d(x,u)\right) \nu_k(dL)
$$

$$
= \int_{G(d,k)} \mathbf{1}\{L \in A\} \Xi_m^{L^\perp}\left(\sum_{i=1}^{l} \rho_i (K_i|L^\perp), \left(\sum_{i=1}^{l} \rho_i (\beta_i|L^\perp)\right) \times (\omega \cap L^\perp)\right) \nu_k(dL)
$$

$$
= \sum_{i_1,\ldots,i_l=0}^{m} \binom{m}{i_1,\ldots,i_l} \rho_1^{i_1} \cdots \rho_l^{i_l} \int_{G(d,k)} \mathbf{1}\{L \in A\}
$$

$$
\times \Xi_{i_1,\ldots,i_l}^{L^\perp}(K_1|L^\perp,\ldots,K_l|L^\perp, (\beta_1|L^\perp) \times \cdots \times (\beta_l|L^\perp) \times (\omega \cap L^\perp)) \nu_k(dL)
$$

$$
= \sum_{i_1,\ldots,i_l=0}^{m} \binom{m}{i_1,\ldots,i_l} \rho_1^{i_1} \cdots \rho_l^{i_l} \Xi_{i_1,\ldots,i_l}^{(k)}(K_1,\ldots,K_l, \beta_1 \times \cdots \times \beta_l \times \gamma).
$$

Here we denote the mixed support measure of the convex bodies $K_1|L^\perp, \ldots, K_l|L^\perp$ in $L^\perp$ by $\Xi_{i_1,\ldots,i_l}^{L^\perp}(K_1|L^\perp,\ldots,K_l|L^\perp,\cdot)$ and define the mixed flag measure $\Xi_{i_1,\ldots,i_l}^{(k)}(K_1,\ldots,K_l,\cdot)$ by

$$\Xi_{i_1,\ldots,i_l}^{(k)}(K_1,\ldots,K_l,\beta_1\times\cdots\times\beta_l\times\omega\times A)$$

$$=\int_{G(d,k)}\mathbf{1}\{L\in A\}\,\Xi_{i_1,\ldots,i_l}^{L^\perp}(K_1|L^\perp,\ldots,K_l|L^\perp,(\beta_1|L^\perp)\times\cdots$$

$$\cdots\times(\beta_l|L^\perp)\times(\omega\cap L^\perp))\,\nu_k(dL).\tag{28}$$

The remaining assertions of (a) follow from (27), (28) and the corresponding properties of mixed support measures in Theorem 2(a).

(b) and (c) follow from (28) and Theorem 2(b) and (c).

Remark 2. In the following, we simply write $\Xi^{(k)}(K_1,\ldots,K_m,\cdot)$ for $\Xi_{1,\ldots,1}^{(k)}(K_1,\ldots,K_m,\cdot)$, where $m\in\{1,\ldots,d-1-k\}$ and the index 1 in the measure $\Xi_{1,\ldots,1}^{(k)}$ appears m times.

We emphasize two special cases of the preceding theorem.

(1) Let $m=d-1-k$ and $r\in\{0,\ldots,d-1-k\}$ in Theorem 6(c). Then, for $K\in\mathcal{K}$ we have

$$\Xi_r^{(k)}(K,\beta\times\gamma)=\frac{2\binom{d-k}{r}}{(d-k)\kappa_{d-k-r}}\,\Xi_{r,d-1-k-r}^{(k)}(K,B^d,\beta\times S^{d-1}\times\gamma).$$

(2) Let $m=d-1-k\geq1$ and $r\in\{0,\ldots,d-1-k\}$. If $K\in\mathcal{K}$ is strictly convex, Theorem 6(b) and (c) imply that

$$\Xi_r^{(k)}(K,\beta\times\gamma)$$

$$=\frac{2\binom{d-k}{r}}{(d-k)\kappa_{d-k-r}}\,\Xi_{1,\ldots,1}^{(k)}(K[r],B^d[d-1-k-r],(\beta)^r\times(S^{d-1})^{d-1-k-r}\times\gamma).$$

The condition of general relative position is not necessary for the multilinear expansion of the area measures. This carries over to the flag area measures. We list some of the resulting formulas (using the obvious notation for the mixed measures), which are obtained from the preceding results by passing to image measures with respect to the map $(x,u,L)\mapsto(u,L)$ and using the weak continuity of these image measures without the restriction to convex bodies in general relative position. Thus we obtain

$$\Psi_m^{(k)}(\rho_1K_1+\cdots+\rho_lK_l,\cdot)$$

$$=\sum_{i_1,\ldots,i_l=0}^{m}\binom{m}{i_1,\ldots,i_l}\rho_1^{i_1}\cdots\rho_l^{i_l}\,\Psi^{(k)}(K_1[i_1],\ldots,K_l[i_l],\cdot)$$

$$=\sum_{j_1,\ldots,j_m=1}^{l}\rho_{j_1}\cdots\rho_{j_m}\Psi^{(k)}(K_{j_1},\ldots,K_{j_m},\cdot),$$

$$\Psi_m^{(k)}(K,\cdot)=\frac{2\binom{d-k}{m}}{(d-k)\kappa_{d-k-m}}\,\Psi^{(k)}(K[m],B^d[d-1-k-m],\cdot)\tag{29}$$

and

$$\Psi_m^{(k)}(K + \rho B^d, \cdot) = \sum_{r=0}^{m} \rho^r \binom{m}{r} \Psi^{(k)}(K[m-r], B^d[r], \cdot)$$

$$= \sum_{r=0}^{m} \rho^r \binom{m}{r} \Psi_{m-r,r}^{(k)}(K, B^d, \cdot)$$

$$= \sum_{r=0}^{m} \rho^{m-r} \binom{d-k-r}{d-k-m} \frac{\kappa_{d-k-r}}{\kappa_{d-k-m}} \Psi_r^{(k)}(K, \cdot).$$

The latter also follows directly from Proposition 2.

Finally, in this section, we consider relations between the flag measures $\Xi_m^{(k)}(K, \cdot)$, for fixed m but different k. In Theorem 7 we show that these measures are connected by an integral relation. For $k \in \{0, \ldots, d-1\}$, $r \in \{k, \ldots, d-1\}$, $m \in \{0, \ldots, d-r-1\}$ and $L \in G(d, k)$, we denote by $\Xi_m^{L^\perp, (r-k)}(K|L^\perp, \cdot)$ the flag measure of type $(r-k, m)$ of the convex body $K|L^\perp$, computed in $L^\perp$. The same convention applies to $g^{L^\perp}$. Note that the special case $r = k$ of the following theorem boils down to relation (17).

Theorem 7. *For $K \in \mathcal{K}$, $k \in \{0, \ldots, d-1\}$, $r \in \{k, \ldots, d-1\}$, $m \in \{0, \ldots, d-r-1\}$, we have*

$$\Xi_m^{(r)}(K, \cdot) = \int_{G(d,k)} \int \mathbf{1}\{(g(x, L, K), u, V+L) \in \cdot\}$$

$$\times \Xi_m^{L^\perp, (r-k)}(K|L^\perp, d(x, u, V)) \, \nu_k(dL).$$

Proof. In the following, if $V \in G(d, k)$ is a subspace of $\mathbb{R}^d$ and $l \in \{0, \ldots, k\}$, we write $G(V, l)$ for the set of $W \in G(d, l)$ with $W \subset V$. However, if $l \in \{k, \ldots, d\}$, then $G(V, l)$ is the set of all $W \in G(d, l)$ with $V \subset W$. In any case, the Haar probability measure on $G(V, l)$ is denoted by ν_l^V. Let $f : \mathbb{R}^d \times S^{d-1} \times G(d, k) \to [0, \infty)$ be measurable. Then, applying (17) in $L^\perp$, we get

$$\int_{G(d,k)} \int f(g(x, L, K), u, V+L) \, \Xi_m^{L^\perp, (r-k)}(K|L^\perp, d(x, u, V)) \, \nu_k(dL)$$

$$= \int_{G(d,k)} \int_{G(L^\perp, r-k)} \int f\left(g\left(g^{L^\perp}(z, W, K|L^\perp), L, K\right), w, W+L\right)$$

$$\times \Xi_m^{L^\perp \cap W^\perp}\left((K|L^\perp)|(L^\perp \cap W^\perp), d(z, w)\right) \, \nu_{r-k}^{L^\perp}(dW) \, \nu_k(dL)$$

$$= \int_{G(d,k)} \int_{G(L^\perp,r-k)} \int f(g(z, W + L, K), w, W + L)$$

$$\times \, \Xi_m^{(W+L)^\perp} \left(K | (W + L)^\perp, d(z,w) \right) v_{r-k}^{L^\perp}(dW) \, v_k(dL)$$

$$= \int_{G(d,k)} \int_{G(L,r)} \int f(g(z, U, K), w, U) \, \Xi_m^{U^\perp} \left(K | U^\perp, d(z,w) \right) v_r^L(dU) \, v_k(dL),$$

where we put $U = W + L$. Now [30, Theorem 7.1.1] and another application of (17) yield that

$$\int_{G(d,k)} \int f(g(x, L, K), u, V + L) \, \Xi_m^{L^\perp,(r-k)}(K | L^\perp, d(x, u, V)) \, v_k(dL)$$

$$= \int_{G(d,r)} \int f(g(z, U, K), u, U) \, \Xi_m^{U^\perp}(K | U^\perp, d(z, u)) \, v_r(dU)$$

$$= \int f(z, u, U) \, \Xi_m^{(r)}(K, d(z, u, U)).$$

From the preceding result, extensions for mixed measures can be obtained by the procedure described before.

6 Flag Measures for Generalized Zonoids

A *generalized zonoid* is a centrally symmetric convex body $Z \in \mathcal{K}$, the support function of which is of the form

$$h(Z, \cdot) = \int_{S^{d-1}} |\langle \cdot, v \rangle| \, \rho(Z, dv)$$

(up to a linear function). Here, $\rho(Z, \cdot)$ is an even finite signed Borel measure on S^{d-1}, the *generating measure* of Z. For simplicity (and since flag area measures are invariant under translations of the bodies), we assume that the center of Z is at the origin o. Then, the mapping $Z \mapsto \rho(Z, \cdot)$ is injective and the measure $\rho(Z, \cdot)$ is uniquely determined by Z. If $\rho(Z, \cdot) \geq 0$, then Z is a *zonoid*, and if $\rho(Z, \cdot)$ is moreover discrete (a finite combination of Dirac measures), Z is a *zonotope* (a finite sum of line segments).

For generalized zonoids, formulas are known which express mixed area measures (and thus also mixed volumes) in terms of the generating measures (see [31]). For the curvature measures, such representations are not available. In this section, we discuss corresponding results for flag area measures. We first give a general result for the (ordinary) mixed area measure. For vectors $u_1, \ldots, u_k \neq 0$, we denote by $E(u_1, \ldots, u_k)$ their linear hull, if this space is k-dimensional, and we put $E(u_1, \ldots, u_k) = E_0$, for a fixed subspace $E_0 \in G(d, k)$, otherwise. Let

$D_k(u_1, \ldots, u_k)$ denote the k-dimensional volume of the parallelepiped spanned by $u_1, \ldots, u_k$. If $D_k(u_1, \ldots, u_k) > 0$, this is also the absolute value of the determinant of $u_1, \ldots, u_k$ (calculated in $E(u_1, \ldots, u_k)$). For a subspace $M \in G(d, m)$ and $k + m \le d$, we define $D_{k+m}(u_1, \ldots, u_k, M)$ by $D_{k+m}(u_1, \ldots, u_k, v_1, \ldots, v_m)$, where $v_1, \ldots, v_m$ is an orthonormal basis in M. This notion does not depend on the choice of the basis.

In order to simplify the comparison with the literature, we remark that the normalization of the mixed surface area measures $S(K_1, \ldots, K_{d-1}, \cdot)$ and area measures $S_j(K, \cdot)$, as used in [28], is related to the present normalization by

$$2\Psi(K_1, \ldots, K_{d-1}, \cdot) = S(K_1, \ldots, K_{d-1}, \cdot)$$

and

$$\Psi_j(K, \cdot) = \frac{\binom{d}{j}}{d \kappa_{d-j}} S_j(K, \cdot).$$

Since $S_j(K) = S(K[j], B^d[d-1-j], \cdot)$, we get (cf. (29))

$$\Psi_j(K, \cdot) = \frac{2\binom{d}{j}}{d \kappa_{d-j}} \Psi(K[j], B^d[d-1-j], \cdot).$$

Theorem 8. *For $k \in \{1, \ldots, d-2\}$, let $Z_1, \ldots, Z_k$ be generalized zonoids with generating measures $\rho(Z_1, \cdot), \ldots, \rho(Z_k, \cdot)$ and let $K_{k+1}, \ldots, K_{d-1} \in \mathcal{K}$ be arbitrary bodies. Then, for a Borel set $\omega \subset S^{d-1}$, we have*

$$\Psi(Z_1, \ldots, Z_k, K_{k+1}, \ldots, K_{d-1}, \omega)$$

$$= \frac{2^k}{k! \binom{d-1}{k}} \int_{S^{d-1}} \cdots \int_{S^{d-1}} D_k(u_1, \ldots, u_k)$$

$$\times \Psi^{E^\perp}(K_{k+1}|E^\perp, \ldots, K_{d-1}|E^\perp, \omega \cap E^\perp) \, \rho(Z_1, du_1) \ldots \rho(Z_k, du_k)$$

with $E = E(u_1, \ldots, u_k)$.

If in the statement of the theorem, the dimension of the subspace E is smaller than k, then $\Psi^{E^\perp}(\cdots)$ is not properly defined. In this case, however, $D_k(u_1, \ldots, u_k) = 0$ and therefore it is consistent to define the integrand as zero. Similar conventions will be adopted in the following.

Proof. The proof follows the lines in [27, Proposition 3.7].

We define a measure μ by

$$\mu(\omega) = \frac{2^k}{k! \binom{d-1}{k}} \int_{S^{d-1}} \cdots \int_{S^{d-1}} D_k(u_1, \ldots, u_k)$$

$$\times \Psi^{E^\perp}(K_{k+1}|E^\perp, \ldots, K_{d-1}|E^\perp, \omega \cap E^\perp) \, \rho(Z_1, du_1) \ldots \rho(Z_k, du_k),$$

where $\omega \subset S^{d-1}$ is a Borel set.

For $M \in \mathcal{K}$, we use [28, Theorem 5.1.6] and [29, formula (31)] and get

$$
\begin{aligned}
\int_{S^{d-1}} & h(M, v) \, \mu(dv) \\
&= \frac{2^k}{k! \binom{d-1}{k}} \int_{S^{d-1}} \cdots \int_{S^{d-1}} \int_{S^{d-1} \cap E^\perp} D_k(u_1, \ldots, u_k) \, h(M, v) \\
&\quad \times \Psi^{E^\perp}(K_{k+1}|E^\perp, \ldots, K_{d-1}|E^\perp, dv) \, \rho(Z_1, du_1) \ldots \rho(Z_k, du_k) \\
&= \frac{2^k (d-k)}{2k! \binom{d-1}{k}} \int_{S^{d-1}} \cdots \int_{S^{d-1}} D_k(u_1, \ldots, u_k) \\
&\quad \times V^{E^\perp}(K_{k+1}|E^\perp, \ldots, K_{d-1}|E^\perp, M|E^\perp) \, \rho(Z_1, du_1) \ldots \rho(Z_k, du_k) \\
&= \frac{2^k (d-k)}{2k! \binom{d-1}{k}} \frac{k! \binom{d}{k}}{2^k} V(K_{k+1}, \ldots, K_{d-1}, M, Z_1, \ldots, Z_k) \\
&= \frac{d}{2} V(M, K_{k+1}, \ldots, K_{d-1}, Z_1, \ldots, Z_k) \\
&= \int_{S^{d-1}} h(M, v) \, \Psi(Z_1, \ldots, Z_k, K_{k+1}, \ldots, K_{d-1}, dv).
\end{aligned}
$$

Since M was arbitrary and differences of support functions are dense in the Banach space of continuous functions on S^{d-1}, we deduce $\mu = \Psi(Z_1, \ldots, Z_k, K_{k+1}, \ldots, K_{d-1}, \cdot)$.

For $k = d-1$, there would be no bodies $K_{k+1}, \ldots, K_{d-1}$ in the above theorem. If we interpret $\Psi^{E^\perp}(K_{k+1}|E^\perp, \ldots, K_{d-1}|E^\perp, \cdot)$, in this case, as $\frac{1}{2} \mathcal{H}^0 \llcorner (E^\perp \cap S^{d-1})$ ($\llcorner$ denotes the restriction of a measure), then the proof goes through and yields

$$
\begin{aligned}
\Psi(Z_1, \ldots, Z_{d-1}, \omega) = {} & \frac{2^{d-1}}{(d-1)!} \int_{S^{d-1}} \cdots \int_{S^{d-1}} D_{d-1}(u_1, \ldots, u_{d-1}) \\
& \times \frac{1}{2} \mathcal{H}^0(\omega \cap E^\perp) \, \rho(Z_1, du_1) \ldots \rho(Z_{d-1}, du_{d-1}) \quad (30)
\end{aligned}
$$

with $E = E(u_1, \ldots, u_{d-1})$. We can modify this formula slightly by introducing a (measurable) mapping $T : (S^{d-1})^{d-1} \to S^{d-1}$. For linearly independent $u_1, \ldots, u_{d-1}$, we put $T(u_1, \ldots, u_{d-1}) = u$, where $u \in E(u_1, \ldots, u_{d-1})^\perp$ is such that $u_1, \ldots, u_{d-1}, u$ are positively oriented (with respect to a fixed reference basis). For linearly dependent $u_1, \ldots, u_{d-1}$, we put $T(u_1, \ldots, u_{d-1}) = u_0$ for a fixed vector $u_0 \in S^{d-1}$. Let $T\rho$ denote the image measure (on S^{d-1}) of a (signed) measure ρ on $(S^{d-1})^{d-1}$ under T. If $L \in G(d, k)$, we use an upper index T^L to denote the corresponding mapping $T^L : (S^{d-1} \cap L)^{k-1} \to S^{d-1} \cap L$. In this case, we associate with each $L \in G(d, k)$ an orientation in a measurable way. We also use the mappings T and T^L in connection with a subspace $M \in G(d, m)$, with

which we associate an orientation in a measurable way, and in case of T^L, we assume $M \subset L, M \neq L$. Namely, $T(u_1, \ldots, u_{d-m-1}, M)$, for linearly independent unit vectors $u_1, \ldots, u_{d-m-1}$, is defined as $T(u_1, \ldots, u_{d-m-1}, v_1, \ldots, v_m)$ where $v_1, \ldots, v_m$ is a (positively oriented) orthonormal basis of M. In a similar way, $T^L(u_1, \ldots, u_{k-m-1}, M)$ is defined. Since we are integrating with respect to even measures, the particular choice of the orientations is irrelevant.

The following corollary arises as a special case of Theorem 8. The results stated in Corollary 1 were previously obtained in [31].

Corollary 1. *Let $Z, Z_1, \ldots, Z_d$ be generalized zonoids with corresponding generating measures $\rho(Z, \cdot), \rho(Z_1, \cdot), \ldots, \rho(Z_{d-1}, \cdot)$. Then we have*

$$\Psi(Z_1, \ldots, Z_{d-1}, \cdot) = \frac{2^{d-1}}{(d-1)!} T \left[\int_{(\cdot)} D_{d-1} \, d(\rho(Z_1, \cdot) \times \cdots \times \rho(Z_{d-1}, \cdot)) \right]$$

and, for $j \in \{0, \ldots, d-1\}$,

$$\Psi_j(Z, \cdot) = \frac{2^{j+1} \binom{d}{j}}{d! \kappa_{d-1}^{d-j-1} \kappa_{d-j}} T \left[\int_{(\cdot)} D_{d-1} \, d(\rho(Z, \cdot)^j \times (\mathcal{H}^{d-1})^{d-j-1}) \right].$$

The first formula is a reformulation of (30) and the second follows from the first since B^d is a zonoid with generating measure (see [31, Satz 3 and its proof])

$$\rho(B^d, \cdot) = \frac{1}{2\kappa_{d-1}} \mathcal{H}^{d-1} \llcorner S^{d-1}.$$

We remark that the formulas in Theorem 8 and Corollary 1 imply corresponding integral representations for mixed volumes involving generalized zonoids, since

$$V(K_1, \ldots, K_d) = \frac{2}{d} \int_{S^{d-1}} h(K_d, u) \, \Psi(K_1, \ldots, K_{d-1}, du)$$

for $K_1, \ldots, K_d \in \mathcal{K}$.

In order to obtain similar results for mixed flag area measures, we apply a special case of the projection formula (28), namely

$$\Psi^{(k)}(K_1, \ldots, K_{d-k-1}, \cdot)$$
$$= \int_{G(d,k)} \int_{S^{d-1} \cap L^\perp} \mathbf{1}\{(u, L) \in \cdot\} \Psi^{L^\perp}(K_1|L^\perp, \ldots, K_{d-k-1}|L^\perp, du) \, \nu_k(dL),$$
$$\tag{31}$$

where $k \in \{0, \ldots, d-2\}$ and $K_1, \ldots, K_{d-k-1} \in \mathcal{K}$ are arbitrary.

It is therefore important that orthogonal projections $Z|L^\perp$ of generalized zonoids Z to a subspace $L^\perp$, for $L \in G(d, k)$, are generalized zonoids again (in $L^\perp$)

(see [34]). In order to describe the generating measure $\rho(Z|L^\perp, \cdot)$ of $Z|L^\perp$, we introduce the *spherical projection*

$$\Pi_{L^\perp} : S^{d-1} \backslash L \to S^{d-1} \cap L^\perp, \qquad u \mapsto \frac{u|L^\perp}{\|u|L^\perp\|}.$$

Then [34, Theorem 4.1] shows that

$$\rho(Z|L^\perp, \cdot) = \int_{S^{d-1}\backslash L} \mathbf{1}\{\Pi_{L^\perp} u \in \cdot\} \|u|L^\perp\| \, \rho(Z, du). \tag{32}$$

In the integration domain we can replace $S^{d-1} \setminus L$ by S^{d-1}, since $\|u|L^\perp\| = 0$ if $u \in L$, and the integrand is taken as zero in this case. We now extend Theorem 8 to flag measures.

Theorem 9. *For $k \in \{0, \ldots, d-3\}$ and $l \in \{1, \ldots, d-k-2\}$, let $Z_1, \ldots, Z_l$ be generalized zonoids with generating measures $\rho(Z_1, \cdot), \ldots, \rho(Z_l, \cdot)$ and let $K_{l+1}, \ldots, K_{d-k-1} \in \mathcal{K}$ be arbitrary. Then we have*

$$\Psi^{(k)}(Z_1, \ldots, Z_l, K_{l+1}, \ldots, K_{d-k-1}, \cdot)$$
$$= \frac{2^l}{l!\binom{d-k-1}{l}} \int_{S^{d-1}} \cdots \int_{S^{d-1}} \int_{G(d,k)} \int_{S^{d-1}\cap L^\perp \cap E^\perp} D_l(v_1|L^\perp, \ldots, v_l|L^\perp)$$
$$\times \mathbf{1}\{(u, L) \in \cdot\} \Psi^{L^\perp \cap E^\perp}(K_{l+1}|(L^\perp \cap E^\perp), \ldots, K_{d-k-1}|(L^\perp \cap E^\perp), du)$$
$$\times v_k(dL) \, \rho(Z_1, dv_1) \ldots \rho(Z_l, dv_l),$$

with $E = E(v_1|L^\perp, \ldots, v_l|L^\perp)$.

Proof. From (31) and from Theorem 8 applied in $L^\perp$, we get

$$\Psi^{(k)}(Z_1, \ldots, Z_l, K_{l+1}, \ldots, K_{d-k-1}, \cdot)$$
$$= \int_{G(d,k)} \int_{S^{d-1}\cap L^\perp} \mathbf{1}\{(u, L) \in \cdot\}$$
$$\times \Psi^{L^\perp}(Z_1|L^\perp, \ldots, Z_l|L^\perp, K_{l+1}|L^\perp, \ldots, K_{d-k-1}|L^\perp, du) \, v_k(dL)$$
$$= \int_{G(d,k)} \frac{2^l}{l!\binom{d-k-1}{l}} \int_{S^{d-1}\cap L^\perp} \cdots \int_{S^{d-1}\cap L^\perp} \int_{S^{d-1}\cap L^\perp \cap E^\perp} D_l(v_1, \ldots, v_l)$$
$$\times \mathbf{1}\{(u, L) \in \cdot\} \Psi^{L^\perp \cap E^\perp}(K_{l+1}|(L^\perp \cap E^\perp), \ldots, K_{d-k-1}|(L^\perp \cap E^\perp), du)$$
$$\times \rho(Z_1|L^\perp, dv_1) \ldots \rho(Z_l|L^\perp, dv_l) \, v_k(dL),$$

where $E = E(v_1, \ldots, v_l)$. Using (32), we obtain

$$\Psi^{(k)}(Z_1, \ldots, Z_l, K_{l+1}, \ldots, K_{d-k-1}, \cdot)$$

$$= \frac{2^l}{l!\binom{d-k-1}{l}} \int_{G(d,k)} \int_{S^{d-1}\setminus L} \cdots \int_{S^{d-1}\setminus L} \int_{S^{d-1}\cap L^\perp \cap E^\perp} D_l(\Pi_{L^\perp} v_1, \ldots, \Pi_{L^\perp} v_l)$$

$$\times \mathbf{1}\{(u, L) \in \cdot\} \, \|v_1|L^\perp\| \cdots \|v_l|L^\perp\|$$

$$\times \Psi^{L^\perp \cap E^\perp}(K_{l+1}|(L^\perp \cap E^\perp), \ldots, K_{d-k-1}|(L^\perp \cap E^\perp), du)$$

$$\times \rho(Z_1, dv_1) \ldots \rho(Z_l, dv_l) \, \nu_k(dL)$$

$$= \frac{2^l}{l!\binom{d-k-1}{l}} \int_{G(d,k)} \int_{S^{d-1}\setminus L} \cdots \int_{S^{d-1}\setminus L} \int_{S^{d-1}\cap L^\perp \cap E^\perp} D_l(v_1|L^\perp, \ldots, v_l|L^\perp)$$

$$\times \mathbf{1}\{(u, L) \in \cdot\} \Psi^{L^\perp \cap E^\perp}(K_{l+1}|(L^\perp \cap E^\perp), \ldots, K_{d-k-1}|(L^\perp \cap E^\perp), du)$$

$$\times \rho(Z_1, dv_1) \ldots \rho(Z_l, dv_l) \, \nu_k(dL)$$

$$= \frac{2^l}{l!\binom{d-k-1}{l}} \int_{S^{d-1}} \cdots \int_{S^{d-1}} \int_{G(d,k)} \int_{S^{d-1}\cap L^\perp \cap E^\perp} D_l(v_1|L^\perp, \ldots, v_l|L^\perp)$$

$$\times \mathbf{1}\{(u, L) \in \cdot\} \Psi^{L^\perp \cap E^\perp}(K_{l+1}|(L^\perp \cap E^\perp), \ldots, K_{d-k-1}|(L^\perp \cap E^\perp), du)$$

$$\times \nu_k(dL) \, \rho(Z_1, dv_1) \ldots \rho(Z_l, dv_l),$$

where $E = E(v_1|L^\perp, \ldots, v_l|L^\perp)$.

In the last step, we first replaced $S^{d-1}\setminus L$ by S^{d-1}, since the integrand $D_l(v_1|L^\perp, \ldots, v_l|L^\perp) = 0$ if $v_i \in L$ for some $i \in \{1, \ldots, l\}$, and then we applied Fubini's theorem.

Again, the case $l = d - k - 1$, where there are no additional bodies $K_{l+1}, \ldots, K_{d-k-1}$ and also $k = d - 2$ is admitted, can be treated with the same proof, if Corollary 1 instead of Theorem 8 is used. This gives the following result.

Corollary 2. *For $k \in \{0, \ldots, d - 2\}$, let $Z_1, \ldots, Z_{d-k-1}$ be generalized zonoids with generating measures $\rho(Z_1, \cdot), \ldots, \rho(Z_{d-k-1}, \cdot)$. Then we have*

$$\Psi^{(k)}(Z_1, \ldots, Z_{d-k-1}, \cdot)$$

$$= \frac{2^{d-k-1}}{(d - k - 1)!} \int_{S^{d-1}} \cdots \int_{S^{d-1}} \int_{G(d,k)} D_{d-k-1}(u_1|L^\perp, \ldots, u_{d-k-1}|L^\perp)$$

$$\times \mathbf{1}\{(T^{L^\perp}(\Pi_{L^\perp} u_1, \ldots, \Pi_{L^\perp} u_{d-k-1}), L) \in \cdot\}$$

$$\times \nu_k(dL) \, \rho(Z_1, du_1) \ldots \rho(Z_{d-k-1}, du_{d-k-1})$$

$$= \frac{2^{d-k-1}}{(d - k - 1)!} \int_{S^{d-1}} \cdots \int_{S^{d-1}} \int_{G(d,d-k)} D_{d-k-1}(u_1|L, \ldots, u_{d-k-1}|L)$$

$$\times \mathbf{1}\{(T^L(\Pi_L u_1, \ldots, \Pi_L u_{d-k-1}), L^\perp) \in \cdot\}$$

$$\times \nu_{d-k}(dL) \, \rho(Z_1, du_1) \ldots \rho(Z_{d-k-1}, du_{d-k-1}).$$

On the other hand, we can also consider the case of Theorem 9, where the bodies K_i are generalized zonoids as well. Then we get intermediate formulas for $\Psi^{(k)}(Z_1, \ldots, Z_{d-k-1}, \cdot)$.

Remark 3. For generalized zonoids $Z_1, \ldots, Z_{d-k-1}$, $k \in \{0, \ldots, d-3\}$, and $l \in \{1, \ldots, d-k-2\}$, we have

$$
\Psi^{(k)}(Z_1, \ldots, Z_{d-k-1}, \cdot)
$$

$$
= \frac{2^l}{l!\binom{d-k-1}{l}} \int_{S^{d-1}} \cdots \int_{S^{d-1}} \int_{G(d,k)} \int_{S^{d-1} \cap L^\perp \cap E^\perp} D_l(v_1|L^\perp, \ldots, v_l|L^\perp)
$$

$$
\times \mathbf{1}\{(u, L) \in \cdot\} \Psi^{L^\perp \cap E^\perp}(Z_{l+1}|(L^\perp \cap E^\perp), \ldots, Z_{d-k-1}|(L^\perp \cap E^\perp), d u)
$$

$$
\times \nu_k(dL)\, \rho(Z_1, dv_1) \ldots \rho(Z_l, dv_l),
$$

where $E = E(v_1|L^\perp, \ldots, v_l|L^\perp)$.

A case of special interest is given if $Z_{l+1}, \ldots, Z_{d-k-1} = B^d$. Here, we obtain the following result.

Theorem 10. *For $k \in \{1, \ldots, d-2\}$ and $l \in \{1, \ldots, d-k-1\}$, let $Z_1, \ldots, Z_l$ be generalized zonoids with generating measures $\rho(Z_1, \cdot), \ldots, \rho(Z_l, \cdot)$. Then we have*

$$
\Psi^{(k)}(Z_1, \ldots, Z_l, B^d[d-k-l-1], \cdot)
$$

$$
= c(d, k, l) \int_{S^{d-1}} \cdots \int_{S^{d-1}} \int_{G(d,d-k)} \int_{G(L,d-k-l-1)} D_{d-k-1}(u_1|L, \ldots, u_l|L, U)
$$

$$
\times \mathbf{1}\{(T^L(\Pi_L u_1, \ldots, \Pi_L u_l, U), L^\perp) \in \cdot\}
$$

$$
\times \nu_{d-k-l-1}^L(dU)\, \nu_{d-k}(dL)\, \rho(Z_1, du_1) \ldots \rho(Z_l, du_l)
$$

with

$$
c(d, k, l) = \frac{2^{d-k-1}}{(d-k-1)!} \pi^{d-k-l-1} \frac{\omega_{l+2}}{\omega_{d-k+1}}.
$$

Proof. For $l = d - k - 1$ the assertion is implied by Corollary 2. Therefore, we assume that $k \leq d - 3$ and $l < d - k - 1$, in the following. Let $L \in G(d, d-k)$. Since

$$
\rho(B^d|L, \cdot) = \frac{1}{2\kappa_{d-k-1}} \mathcal{H}^{d-k-1} \llcorner (S^{d-1} \cap L),
$$

we obtain for a measurable function $f \geq 0$ on $S^{d-1} \cap L$ that

$$
\int_{S^{d-1} \cap L} f(u)\, \rho(B^d|L, du) = \frac{1}{2\kappa_{d-k-1}} \int_{S^{d-1} \cap L} f(u)\, \mathcal{H}^{d-k-1}(du).
$$

On the other hand,

$$\int_{S^{d-1}\cap L} f(u)\,\rho(B^d|L,du) = \int_{S^{d-1}} f(\Pi_L u)\,\|u|L\|\,\rho(B^d,du)$$

and thus

$$\int_{S^{d-1}} f(\Pi_L u)\,\|u|L\|\,\rho(B^d,du) = \frac{1}{2\kappa_{d-k-1}}\int_{S^{d-1}\cap L} f(u)\,\mathcal{H}^{d-k-1}(du).$$

From Corollary 2, we therefore obtain

$$\Psi^{(k)}(Z_1,\ldots,Z_l,B^d[d-k-l-1],\cdot)$$

$$= \frac{2^{d-k-1}}{(d-k-1)!}\int_{G(d,d-k)}\int_{S^{d-1}}\cdots\int_{S^{d-1}} D_{d-k-1}(u_1|L,\ldots,u_{d-k-1}|L)$$

$$\times \mathbf{1}\{(T^L(\Pi_L u_1,\ldots,\Pi_L u_{d-k-1}),L^\perp)\in\cdot\}$$

$$\times \rho(Z_1,du_1)\ldots\rho(Z_l,du_l)\,\rho(B^d,du_{l+1})\ldots\rho(B^d,du_{d-k-1})\,\nu_{d-k}(dL)$$

$$= a(d,k,l)\int_{G(d,d-k)}\int_{S^{d-1}\cap L}\cdots\int_{S^{d-1}\cap L}\int_{S^{d-1}}\cdots\int_{S^{d-1}}$$

$$\times D_{d-k-1}(u_1|L,\ldots,u_l|L,u_{l+1},\ldots,u_{d-k-1})$$

$$\times \mathbf{1}\{(T^L(\Pi_L u_1,\ldots,\Pi_L u_l,u_{l+1},\ldots,u_{d-k-1}),L^\perp)\in\cdot\}$$

$$\times \rho(Z_1,du_1)\ldots\rho(Z_l,du_l)\,\mathcal{H}^{d-k-1}(du_{l+1})\ldots\mathcal{H}^{d-k-1}(du_{d-k-1})\,\nu_{d-k}(dL)$$

with $a(d,k,l) = \frac{2^{d-k-1}}{(d-k-1)!}\left(\frac{1}{2\kappa_{d-k-1}}\right)^{d-k-l-1}$.

If $u_{l+1},\ldots,u_{d-k-1}$ are linearly independent, they span a $(d-k-l-1)$-dimensional linear subspace $U\subset L$ and

$$D_{d-k-1}(u_1|L,\ldots,u_l|L,u_{l+1},\ldots,u_{d-k-1})$$

$$= D_{d-k-1}(u_1|L,\ldots,u_l|L,U)\,D_{d-k-l-1}(u_{l+1},\ldots,u_{d-k-1}).$$

Applying a special case of [3, Theorem 1] to the $(d-k-l-1)$-fold integral over $S^{d-1}\cap L$, we obtain

$$\Psi^{(k)}(Z_1,\ldots,Z_l,B^d[d-k-l-1],\cdot)$$

$$= a(d,k,l)\mathcal{H}^{(l+1)(d-k-l-1)}(G(L,d-k-l-1))$$

$$\times \int_{G(d,d-k)}\int_{G(L,d-k-l-1)}\int_{S^{d-1}\cap U}\cdots\int_{S^{d-1}\cap U}\int_{S^{d-1}}\cdots\int_{S^{d-1}}$$

$$\times\, D_{d-k-1}(u_1|L,\ldots,u_l|L,U)\, D_{d-k-l-1}(v_1,\ldots,v_{d-k-l-1})^{l+2}$$

$$\times\, \mathbf{1}\{(T^L(\Pi_L u_1,\ldots,\Pi_L u_l,U),L^\perp)\in\cdot\}\, \rho(Z_1,du_1)\ldots\rho(Z_l,du_l)$$

$$\times\, \mathcal{H}^{d-k-l-2}(dv_1)\ldots\mathcal{H}^{d-k-l-2}(dv_{d-k-l-1})\, v^L_{d-k-l-1}(dU)\, v_{d-k}(dL).$$

The integration

$$b(d,k,l) = \int_{S^{d-1}\cap U}\cdots\int_{S^{d-1}\cap U} D_{d-k-l-1}(v_1,\ldots,v_{d-k-l-1})^{l+2}$$

$$\times\, \mathcal{H}^{d-k-l-2}(dv_1)\ldots\mathcal{H}^{d-k-l-2}(dv_{d-k-l-1})$$

$$= \omega^{d-k-l-1}_{d-k+1}\prod_{j=0}^{d-k-l-2}\frac{\omega_{d-k-l-1-j}}{\omega_{d-k+1-j}}$$

follows from [30, Theorem 8.2.2] by introducing polar coordinates. Therefore we obtain

$$\Psi^{(k)}(Z_1,\ldots,Z_l,B^d[d-k-l-1],\cdot)$$

$$= c(d,k,l)\int_{G(d,d-k)}\int_{G(L,d-k-l-1)}\int_{S^{d-1}}\cdots\int_{S^{d-1}} D_{d-k-1}(u_1|L,\ldots,u_l|L,U)$$

$$\times\, \mathbf{1}\{(T^L(\Pi_L u_1,\ldots,\Pi_L u_l,U),L^\perp)\in\cdot\}\, \rho(Z_1,du_1)\ldots\rho(Z_l,du_l)$$

$$\times\, v^L_{d-k-l-1}(dU)\, v_{d-k}(dL),$$

where

$$c(d,k,l) = a(d,k,l)\cdot\mathcal{H}^{(l+1)(d-k-l-1)}(G(L,d-k-l-1))\cdot b(d,k,l)$$

$$= \frac{2^{d-k-1}}{(d-k-1)!}\,\pi^{d-k-l-1}\frac{\omega_{d-k}\cdots\omega_{d-k-l}}{\omega_1\cdots\omega_{l+1}}\prod_{j=0}^{d-k-l-2}\frac{\omega_{d-k-l-1-j}}{\omega_{d-k+1-j}}$$

$$= \frac{2^{d-k-1}}{(d-k-1)!}\,\pi^{d-k-l-1}\frac{\omega_{l+2}}{\omega_{d-k+1}}.$$

The Hausdorff measure of the Grassmannian is determined, for instance, in [7, p. 267].

Choosing $Z_1 = \ldots = Z_l = Z$ and using (29) and Theorem 10, we finally get a formula for the flag area measure $\Psi_l^{(k)}(Z,\cdot)$ of a generalized zonoid Z.

Corollary 3. *For $k\in\{1,\ldots,d-2\}$, $l\in\{1,\ldots,d-k-1\}$, and a generalized zonoid Z, we have*

$$\Psi_l^{(k)}(Z,\cdot) = \frac{2\binom{d-k}{l}}{(d-k)\kappa_{d-k-l}} \cdot c(d,k,l) \int_{S^{d-1}} \cdots \int_{S^{d-1}} \int_{G(d,d-k)} \int_{G(L,d-k-l-1)}$$

$$\times \; D_{d-k-1}(u_1|L,\ldots,u_l|L,U)\mathbf{1}\{(T^L(\Pi_L u_1,\ldots,\Pi_L u_l,U),L^\perp)\in\cdot\}$$

$$\times \; v_{d-k-l-1}^L(dU)\, v_{d-k}(dL)\, \rho(Z,du_1)\ldots\rho(Z,du_l).$$

Remark 4. First choosing $l = d - k - 1$ and then replacing k by $d - k - 1$ in Corollary 3, we obtain

$$\Psi_k^{(d-k-1)}(Z,\cdot) = \frac{2^k}{k!} \int_{S^{d-1}} \cdots \int_{S^{d-1}} \int_{G(d,k+1)} D_k(u_1|L,\ldots,u_k|L)$$

$$\times \; \mathbf{1}\{(T^L(\Pi_L u_1,\ldots,\Pi_L u_k),L^\perp)\in\cdot\}$$

$$\times \; v_{k+1}(dL)\, \rho(Z,du_1)\ldots\rho(Z,du_k).$$

From Proposition 1 (see also (4)) we obtain $\Psi_k(Z,\cdot) = c(d,k) \cdot \Psi_k^{(d-k-1)}(Z,\cdot \times G(d,d-k-1))$. Then a special case of Remark 4 yields that

$$\Psi_k(Z,\cdot) = c(d,k)\frac{2^k}{k!} \int_{S^{d-1}} \cdots \int_{S^{d-1}} \int_{G(d,k+1)} D_k(u_1|L,\ldots,u_k|L)$$

$$\times \; \mathbf{1}\{T^L(\Pi_L u_1,\ldots,\Pi_L u_k)\in\cdot\}$$

$$\times \; v_{k+1}(dL)\, \rho(Z,du_1)\ldots\rho(Z,du_k).$$

This formula does not seem to be available in the literature, but (of course) it could be obtained more directly. Calculating the total measure, we recover a known formula (see [31, Satz 6]) for the quermassintegrals of generalized zonoids, that is

$$W_{d-k}(Z)$$
$$= \frac{2^k(d-k)!\kappa_{d-k}}{d!} \cdot \int_{S^{d-1}} \cdots \int_{S^{d-1}} D_k(u_1,\ldots,u_k)\, \rho(Z,du_1)\ldots\rho(Z,du_k).$$

To derive this from the preceding result, we use that

$$D_k(u_1|L,\ldots,u_k|L) = D_k(u_1,\ldots,u_k) \cdot D_k(\xi_1|L,\ldots,\xi_k|L),$$

where $\xi_1,\ldots,\xi_k$ is an arbitrary orthonormal system with the same span as $u_1,\ldots,u_k$ (assumed to be linearly independent). Then we apply [7, middle of p. 139 and Sect. 3.2.13] to obtain

$$\int_{G(d,k+1)} D_k(\xi_1|L,\ldots,\xi_k|L)\, v_{k+1}(dL) = \frac{\beta_1(d,k)}{\beta_1(k+1,k)} = \frac{1}{c(d,k)},$$

where the constant $\beta_1(d,k)$ is taken from [7, Sect. 3.2.13].

7   Concluding Remarks

As an additive and continuous functional on $\mathcal{K}$, the flag measure $\Xi_m^{(k)}(K, \cdot)$ has an additive extension to the *convex ring* $\mathcal{R}$, the class of finite unions of convex bodies (polyconvex sets). This extension is given by the inclusion-exclusion formula and is a signed measure, in general. Since $\Xi_m^{(k)}(K, \cdot)$ is also locally defined, it can even be extended to countable, locally finite unions of convex bodies, the elements of the *extended convex ring* $\mathcal{S}$. For $K \in \mathcal{S}$, the flag measure $\Xi_m^{(k)}(K, \cdot)$ is then a locally finite, signed Radon measure, that is, it is only defined on bounded Borel sets η. The extended measures are no longer continuous, in general. Also, one has to be cautious with most of the formulas which we discussed so far; they are no longer valid for polyconvex sets. The reason is that the projection approach which we used and the Minkowski addition which underlies the construction of the mixed measures are both not compatible with the additive extension.

As we remarked in the introduction, one major goal in convex geometry is to express global geometric functionals of convex bodies as integrals over locally defined quantities in order to extract local information about the bodies. With respect to flag measures, there is still a variety of open problems, but there are two first results in this direction which deserve to be mentioned.

One result concerns the *projection function* $v_{ij}(K, \cdot)$ of $K \in \mathcal{K}$, where $i \in \{0, \ldots, j\}$ and $j \in \{0, \ldots, d-1\}$. This is a continuous function on $G(d, j)$, defined by

$$v_{ij}(K, L) = V_i(K|L), \quad L \in G(d, j).$$

The case $i = j = 0$ being trivial, let us assume $1 \leq j \leq d-1$. The case $j = d-1$ is well-known, since for $x \neq o$ we have

$$v_{i,d-1}(K, x^{\perp}) = c_i \int_{S^{d-1}} |\langle x, u \rangle| \, \Psi_i(K, du), \quad i = 1, \ldots, d-1,$$

with some constant c_i. Using flag area measures, this formula generalizes as follows (see [11, 12]). In the following, for $k \in \{0, \ldots, d-1\}$ we put $k^* = d-1-k$ to emphasize the symmetry of our statements. Then there is a measurable function g on $G(d, j) \times S^{d-1} \times G(d, j^*)$ such that

$$v_{ij}(K, L) = \int_{S^{d-1} \times G(d, j^*)} g(L, u, U) \Psi_i^{(j^*)}(K, d(u, U)), \quad 1 \leq i \leq j \leq d-1. \tag{33}$$

For the validity of (33), some assumptions on the relative position of K and L are needed. For example, (33) holds, if K is a polytope and K and L are in general relative position. Also, (33) holds for arbitrary $K \in \mathcal{K}$ and v_j-almost all L (see [11, 12], for more details).

A second result concerns the mixed volume $V(K[m], M[d-m]), m = 0, \ldots, d$, of two convex bodies $K, M \in \mathcal{K}$, which we already discussed in the introduction. In [15], it is shown that there is a measurable function $f_{m,d-m}$ on $S^{d-1} \times G(d, m^*) \times S^{d-1} \times G(d, (d-m)^*)$ such that

$$V(K[m], -M[d-m]) = \int_{S^{d-1}\times G(d,m^*)} \int_{S^{d-1}\times G(d,(d-m)^*)} f_{m,d-m}(u, U, v, V)$$

$$\times \Psi_{d-m}^{((d-m)^*)}(M, d(v, V))\Psi_m^{(m^*)}(K, d(u, U)), \qquad (34)$$

for $m = 1, \ldots, d-1$. Equation (34) holds for bodies $K, M \in \mathcal{K}$ in suitable relative position, e.g. if K and M are polytopes in general relative position or if one of the bodies has a support function of class $C^{1,1}$ (see [15], for details). We remark that (34) can be used to provide formulas for general mixed volumes $V(K_1, \ldots, K_d)$ of bodies $K_1, \ldots, K_d \in \mathcal{K}$, based on the concept of mixed flag area measures presented above.

A further field of application for flag measures is the theory of valuations. A *valuation* φ is an additive functional $\varphi : \mathcal{K} \to \mathbb{R}$. If we require φ to be also continuous and motion invariant, then Hadwiger's famous characterization theorem shows that φ is a linear combination of the intrinsic volumes $V_j, j = 0, \ldots, d$. There is an ongoing program to understand the corresponding structure of valuations which are only invariant under translations. McMullen [18, 19] has shown that a valuation φ on the class $\mathcal{P}$ of polytopes is weakly continuous (that is, continuous under parallel displacements of the facets of a polytope) and translation invariant if and only if

$$\varphi(P) = \sum_{j=0}^{d-1} \sum_{F \in \mathcal{F}_j(P)} f_j(n(P, F))V_j(F) + c_d V_d(P), \qquad (35)$$

where c_d is a constant and f_j is a simple additive functional on the class $\wp_{d-j-1}^{d-1}$ of at most $(d-j-1)$-dimensional spherical polytopes, for $j = 0, \ldots, d-1$. Notice that (35) includes a decomposition of φ into homogeneous parts

$$\varphi = \sum_{j=0}^{d-1} \varphi_j,$$

where

$$\varphi_j(P) = \sum_{F \in \mathcal{F}_j(P)} f_j(n(P, F))V_j(F),$$

is homogeneous of degree $j, j = 0, \ldots, d-1$ and $\varphi_d = c_d V_d$. Thus, the structure of (weakly continuous, translation invariant) valuations on polytopes is understood to some extent. For the general picture, still many questions remain open. Which j-homogeneous valuations φ_j on $\mathcal{P}$ admit a continuous extension to $\mathcal{K}$? In [13],

this question is studied for a number of different continuity properties of f_j. In particular, if f_j is *strongly flag-continuous*, in the sense that

$$f_j(p) = \int_p \int_{G([u],d-j)} \langle [u], L \rangle^2 h_j(u, L)\, v_{d-j}^{[u]}(dL)\, \mathcal{H}^{d-j-1}(du), \quad p \in \wp_{d-j-1}^{d-1},$$

for some continuous function h_j on $S^{d-1} \times G(d, d-j)$, where $[u]$ denotes the linear hull of $u \in S^{d-1}$, then

$$\varphi_j(P) = \int_{S^{d-1} \times G(d, d-j)} h_j(u, L)\, \Psi_j^{(d-j-1)}(P, d(u, L)),$$

and a continuous extension to $\mathcal{K}$ is given by

$$\varphi_j(K) = \int_{S^{d-1} \times G(d, d-j)} h_j(u, L)\, \Psi_j^{(d-j-1)}(K, d(u, L)), \quad K \in \mathcal{K}.$$

A special case arises, if $h_j(u, L) = h_j(u)$ does not depend on the subspace L. For such *strongly continuous* valuations φ_j, we get

$$\varphi_j(K) = \int_{S^{d-1}} h_j(u)\, \Psi_j(K, du), \quad K \in \mathcal{K}.$$

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Operator Functional Equations in Analysis

Hermann König and Vitali Milman

Abstract Classical operations in analysis and geometry as derivatives, the Fourier transform, the Legendre transform, multiplicative maps or duality of convex bodies may be characterized, essentially, by very simple properties which may be often expressed as operator equations, like the Leibniz or the chain rule, bijective maps exchanging products with convolutions or bijective order reversing maps on convex functions or convex bodies. We survey and discuss recent results of this type in analysis. The operations we consider act on classical spaces like C^k-spaces or Schwartz spaces $\mathcal{S}(\mathbb{R}^n)$. Naturally, the results strongly depend on the type of the domain and the image space.

Key words Operator equations • Leibniz rule • Chain rule • Multiplicative maps • Fourier transforms

Mathematical Subject Classifications (2010): 39B22, 26A24

1 The Fourier Transform and Multiplicative Maps

Recent results show that various fundamental operations in analysis and geometry are essentially characterized by very elementary and basic properties. In many cases these simple properties may be expressed by operator equations. In this paper, we

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give a survey on these results in analysis for operators acting in classical function spaces. Most of these theorems are very recent and we would like to put them in some order and perspective. We start with multiplicative maps and the Fourier transform. On the real line, we have the standard

Lemma 1. *Assume that* $K : \mathbb{R} \to \mathbb{R}$ *is measurable, not identically zero and multiplicative* in the sense that $K(uv) = K(u)K(v)$ for all $u, v \in \mathbb{R}$. *Then there exists some* $p > 0$ *such that either* $K(u) = |u|^p$ *or* $K(u) = |u|^p \operatorname{sgn}(u)$ *for all* $u \in \mathbb{R}$.

The *additive* version of this lemma, i.e., that measurable additive functions $L : \mathbb{R} \to \mathbb{R}$, $L(u + v) = L(u) + L(v)$, are linear, is due to Banach [5] and Sierpinski [15]; the reduction to this case using logarithms is straightforward, cf. [4]. A classical result of Milgram [13] extends Lemma 1 to bijective transformations on spaces of continuous functions:

Theorem 1. *Let* M *be a real topological manifold where every connected component has dimension* ≥ 1 *and* $C(M)$ *denote the space of real valued continuous functions on* M. *Assume that* $T : C(M) \to C(M)$ *is bijective and* multiplicative, *i.e.*

$$T(f \cdot g) = Tf \cdot Tg \quad ; \quad f, g \in C(M).$$

Then there is a continuous function $p : M \to \mathbb{R}_+$ *and a homeomorphism* $v : M \to M$ *such that*

$$(Tf)(x) = |f(v(x))|^{p(x)} \operatorname{sgn}(f(v(x))) ; \; f \in C(M), \, x \in M.$$

Hence, up to homeomorphic change of variables in M, T is of power type. Further, T is automatically continuous with respect to natural topologics on $C(M)$.

The result is false, however, if T is not assumed to be bijective (or in some other way non-degenerate), as the following example $T : C(\mathbb{R}) \to C(\mathbb{R})$ shows: Let

$$Tf(x) = \begin{cases} f(x) & x \in (-\infty, 0) \\ f(0) & x \in [0, 1] \\ f(x - 1) & x \in (1, \infty) \end{cases} .$$

Next, we move to the spaces of k-times continuously differentiable real-valued functions on C^k-manifolds M, $C^k(M)$, where $k \in \mathbb{N} \cup \{\infty\}$. The analogue of Theorem 1 for C^k-functions when $k \in \mathbb{N}$ is much more recent and due to Mrčun and Šemrl [14]. An alternative proof which works also for $k = \infty$ was given by Artstein-Avidan, Faifman and Milman [3]. In these analogues $p \equiv 1$ so that T turns out to be linear and continuous:

Theorem 2. *Let $k \in \mathbb{N} \cup \{\infty\}$ and M be a C^k-manifold where every connected component has dimension at least one. Assume that $T : C^k(M) \to C^k(M)$ is a bijective and multiplicative operator,*

$$T(f \cdot g) = Tf \cdot Tg \; ; \; f, g \in C^k(M).$$

Then there is a C^k-diffeomorphism $v : M \to M$ such that

$$(Tf)(x) = f(v(x)) \; ; \; f \in C^k(M), \; x \in M.$$

In particular, T is automatically linear and continuous.

Theorems 1 and 2 also hold for maps T on the spaces $C_c^k(M)$ of compactly supported C^k-functions or on the Schwartz spaces $\mathcal{S}(\mathbb{R}^n)$ of complex-valued rapidly decreasing functions on $\mathbb{R}^n$, cf. [3].

An important step in the proof of Theorems 1 and 2 is to show that T is *local*, i.e., pointwise defined. This is shown by considering zero sets $Z(f) = \{x \in M \mid f(x) = 0\}$ of functions $f \in C^k(M)$ and showing that there is a homeomorphism $u : M \to M$ such that the zero sets are transformed by $Z(Tf) = u(Z(f))$ for all $f \in C^k(M)$. This in turn implies that for any open set $V \subset M$, $f = g$ on V implies $Tf = Tg$ on $\overline{u(V)}$: choosing an open ball $B \subset V$ and a "bump" function $h \in C^k(M)$ over B, i.e., with $Z(h) = M \setminus B$, we have $f \cdot h = g \cdot h$ in M, so that $Tf \cdot Th = Tg \cdot Th$ in M. Using $Z(Th) = M \setminus u(B)$ implies $Tf = Tg$ in $u(V)$. This "localization" on open sets V then yields that $(Tf)(u(x))$ is a function of x and function and derivative values $f^{(j)}(x)$, $(Tf)(u(x)) = F(x, f(x), \dots, f^{(k)}(x))$. An analysis of F using the multiplicativity of T shows then that T has the form given in Theorems 1 and 2.

In the case of complex-valued functions, there is an additional solution, namely $(Tf)(x) = \overline{f(v(x))}$.

On the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing functions $f : \mathbb{R}^n \to \mathbb{C}$, there are two natural multiplications: the pointwise multiplication and the convolution. The Fourier transform $\mathcal{F}$ is bijective on $\mathcal{S}(\mathbb{R}^n)$ and exchanges these multiplications with one another. Up to C^∞-diffeomorphisms, it is the only operator with these properties, as shown by Artstein-Avidan, Faifman and Milman [3], cf. also earlier results in [2]:

Theorem 3. *Assume that $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is bijective. Let $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ denote the Fourier transform.*

(a) If T maps products into convolutions,

$$T(f \cdot g) = Tf * Tg \quad ; \; f, g \in \mathcal{S}(\mathbb{R}^n), \tag{1}$$

there is a C^∞-diffeomorphism $v : \mathbb{R}^n \to \mathbb{R}^n$ such that either $Tf = \mathcal{F}(f \circ v)$ or $Tf = \mathcal{F}(\overline{f \circ v})$.

(b) If T maps convolutions into products,

$$T(f * g) = Tf \cdot Tg \quad ; \ f, g \in \mathcal{S}(\mathbb{R}^n), \tag{2}$$

there is a C^∞-diffeomorphism $v : \mathbb{R}^n \to \mathbb{R}^n$ such that either $Tf = \mathcal{F}(f) \circ v$ or $Tf = (\overline{\mathcal{F}f}) \circ v$.

Theorem 3 is a direct consequence of Theorem 2 in the case of $\mathcal{S}(\mathbb{R}^n)$ instead of $C^\infty(\mathbb{R}^n)$: Assuming Eqs. (1) or (2), the map $\mathcal{F} \circ T$ or $T \circ \mathcal{F}$ will be multiplicative with respect to pointwise multiplication so that by Theorem 2 for a suitable C^∞-diffeomorphism $v : \mathbb{R}^n \to \mathbb{R}^n$, $(\mathcal{F} \circ T)(f)(x) = f(v(x))$ or $\overline{f(v(x))}$, the latter being the second possibility for complex-valued functions. This, in turn, implies the formulas in Theorem 3 since $\mathcal{F}^{-1} f = \mathcal{F} \check{f}$, $\check{f}(x) = f(-x)$.

2 The Leibniz Rule

Turning to another basic operation of analysis, the derivative D, its action on products of real-valued C^1-functions obviously is given by the Leibniz rule

$$D(f \cdot g) = Df \cdot g + f \cdot Dg \quad ; \ f, g \in C^1(\mathbb{R}^n).$$

Clearly, on general algebras A, there are many derivations $T : A \to A$ satisfying $T(a \cdot b) = Ta \cdot b + a \cdot Tb : a, b \in A$. However on spaces of continuous functions $C(\mathbb{R}^n)$ or k-times continuously differentiable functions $C^k(\mathbb{R}^n)$, essentially only the entropy function $Tf = f \ln|f|$ gives rise to new derivations as shown by Goldmann–Šemrl [6] for $k = 0$ and König–Milman [8] for $k \in \mathbb{N}$:

Theorem 4. *Let $k \in \mathbb{N} \cup \{0\}$. Assume that $T : C^k(\mathbb{R}) \to C(\mathbb{R})$ is a map satisfying the Leibniz rule*

$$T(f \cdot g) = Tf \cdot g + f \cdot Tg \quad ; \ f, g \in C^k(\mathbb{R}).$$

Then there are continuous functions $a, b \in C(\mathbb{R})$ such that, if $k \in \mathbb{N}$,

$$Tf = b \cdot f' + a \cdot f \ln|f| \quad ; \ f \in C^k(\mathbb{R}).$$

For $k = 0$, the only possibility is $Tf = b \cdot f \ln|f|$.

Note that T is not assumed to be continuous but turns out to be so with respect to natural topologies. It is not assumed to be linear and, in fact, is not linear if $b \neq 0$. For $k \geq 2$, the formula implies that T may be extended to a derivation $T : C^1(\mathbb{R}) \to C(\mathbb{R})$. Suitable initial conditions like $T(c) = 0$ for a suitable constant function with value $c \notin \{0, 1\}$ imply that $b = 0$, and hence that $Tf = a \cdot f'$ is essentially the derivative, up to multiplication by a continuous function.

We give a sketch of the proof of Theorem 4 when $k = 1$, i.e., for operators $T : C^1(\mathbb{R}) \to C(\mathbb{R})$. It is typical for the procedure in other cases which we will discuss below. There are three main steps:

(a) Localization, (b) Analysis of the representing function, (c) Continuity results. Starting with (a) Localization, we want to show that $T : C^1(\mathbb{R}) \to C(\mathbb{R})$ satisfying $T(f \cdot g) = Tf \cdot g + f \cdot Tg$ is a local operator, i.e., such that $(Tf)(x)$ only depends on x, $f(x)$ and $f'(x)$ and not on values of functions at points different from x. Suppose $I \subset \mathbb{R}$ is an open interval and $f_1, f_2 \in C^1(\mathbb{R})$ satisfy $f_1 = f_2$ on I. We claim that $Tf_1 = Tf_2$ on I. Simply choose $g \in C^1(\mathbb{R})$ such that $g \neq 0$ but $\mathrm{supp}\,(g) \subset I$. Then $f_1|_I = f_2|_I$ implies $f_1 \cdot g = f_2 \cdot g$, and by the Leibniz rule equation

$$(Tf_1 - Tf_2) \cdot g = (f_2 - f_1) \cdot Tg,$$

the right side being zero on I, which yields $(Tf_1)|_I = (Tf_2)|_I$. Now let $x_0 \in \mathbb{R}$ be arbitrary and $f \in C^1(\mathbb{R})$. Denote the tangent to f at x_0 by h,

$$h(x) := f(x_0) + f'(x_0)(x - x_0); \; x \in \mathbb{R}.$$

Then $k(x) := \begin{cases} f(x) & x \leq x_0 \\ h(x) & x > x_0 \end{cases}$ defines a C^1-function $k \in C^1(\mathbb{R})$. Let $I_1 = (-\infty, x_0)$, $I_2 = (x_0, \infty)$. Then $f|_{I_1} = k|_{I_1}$, $(Tf)|_{I_1} = (Tk)_{I_1}$ and $k|_{I_2} = h|_{I_2}$, $(Tk)|_{I_2} = (Th)|_{I_2}$. Since Tf, Tk and Th are continuous functions and $x_0 \in \overline{I_1} \cap \overline{I_2}$, we find that

$$(Tf)(x_0) = (Tk)(x_0) = (Th)(x_0).$$

However, h and $(Th)(x_0)$ only depend on x_0, $f(x_0)$ and $f'(x_0)$, so that for a suitable "representing" function $F : \mathbb{R}^3 \to \mathbb{R}$

$$(Tf)(x_0) = F(x_0, f(x_0), f'(x_0)). \tag{3}$$

At this point, F is not yet known to be continuous nor even measurable.

(b) Analysis of F and (c) Continuity of the coefficient functions: Define $Sf := Tf/f$ for positive C^1-functions $f : \mathbb{R} \to \mathbb{R}_+$. Then by the Leibniz rule

$$S(f \cdot g) = \frac{T(f \cdot g)}{f \cdot g} = \frac{Tf}{f} + \frac{Tg}{g} = Sf + Sg \; ; \; f, g : \mathbb{R} \to \mathbb{R}_+.$$

Since $f = \exp(\ln f)$ and $f' = f \, (\ln f)'$, using Eq. (3) we know that there is another function $G : \mathbb{R}^3 \to \mathbb{R}$ such that

$$(Sf)(x) = G(x, (\ln f)(x), (\ln f)'(x)) \; ; \; f : \mathbb{R} \to \mathbb{R}_+.$$

Let $\varphi := \ln f$, $\psi := \ln g$. Then $S(f \cdot g) = Sf + Sg$ translates into

$$G(x, \varphi(x) + \psi(x), \; \varphi'(x) + \psi'(x)) = G(x, \varphi(x), \varphi'(x)) + G(x, \psi(x), \psi'(x)).$$

Since $\varphi(x), \psi(x), \varphi'(x), \psi'(x)$ may be assigned arbitrary values,

$$G(x, \alpha_0 + \beta_0, \alpha_1 + \beta_1) = G(x, \alpha_0, \alpha_1) + G(x, \beta_0, \beta_1)$$

holds for arbitrary $\alpha_0, \alpha_1, \beta_0, \beta_1, x \in \mathbb{R}$. Since $Tf \in C(\mathbb{R})$, we know that $x \mapsto G(x, \varphi(x), \varphi'(x))$ is continuous for any $C^1(\mathbb{R})$-function φ ; note that given such φ, $f = \exp(\varphi)$ is a positive C^1-function. Moreover, $G(x, \cdot, \alpha_1)$ and $G(x, \alpha_0, \cdot)$ are additive functions on $\mathbb{R}$ for any fixed x, α_1 or α_0. A purely analytic lemma due to Faifman, cf. [8], then implies that G has the form

$$G(x, \alpha_0, \alpha_1) = a(x)\alpha_0 + b(x)\alpha_1 \quad ; a, b \in C(\mathbb{R})$$

i.e., it is linear in α_0 and α_1 with continuous coefficients. Hence

$$\begin{aligned}
(Tf)(x) &= f(x)(Sf)(x) \\
&= f(x)G(x, (\ln f)(x), (\ln f)'(x)) \\
&= f(x)(a(x)(\ln f)(x) + b(x)(\ln f)'(x)) \\
&= a(x) \, f(x)(\ln|f|)(x) + b(x) \, f'(x).
\end{aligned}$$

This formula then is extended to all functions $f \in C^1(\mathbb{R})$. Note that $\lim\limits_{x \to 0} x \ln|x| = 0$, so that we put $0 \cdot \ln 0 = 0$.

Remarks.

(a) The proof of the previous result for $k \geq 2$ is very similar, except that for positive C^k-functions one finds solutions of the type $f(\ln|f|)^{(j)}$ for $j = 1, \ldots, k$. However, for $j \geq 2$, $f(\ln|f|)^{(j)}$ cannot be extended to all $C^k(\mathbb{R})$-functions since singularities occur if $f = 0$; one has e.g. $f(\ln|f|)'' = f'' - \frac{f'^2}{f}$.

(b) Suppose $T : C^1(\mathbb{R}) \to C(\mathbb{R})$ satisfies the Leibniz rule. Assume further that for each $x \in \mathbb{R}$ there are $g_1, g_2 \in C^1(\mathbb{R})$ with $g_1(x) = g_2(x)$ and $Tg_1(x) \neq Tg_2(x)$. Then T is surjective. Hence a very weak surjectivity condition implies full surjectivity.

(c) In the case of functions $f \in C^1(\mathbb{R}^n, \mathbb{R})$ on $\mathbb{R}^n$ and operators $T : C^1(\mathbb{R}^n, \mathbb{R}) \to C(\mathbb{R}^n, \mathbb{R}^n)$ satisfying the Leibniz rule

$$T(f \cdot g) = Tf \cdot g + f \cdot Tg \quad ; f, g \in C^1(\mathbb{R}^n, \mathbb{R}),$$

one finds similarly that T has the following form: There are functions $a \in C(\mathbb{R}^n, \mathbb{R}^n)$ and $b \in C(\mathbb{R}^n, L(\mathbb{R}^n, \mathbb{R}^n))$ such that

$$(Tf)(x) = b(x)f'(x) + a(x)f(x)\ln|f|(x) ; \ f \in C^1(\mathbb{R}^n, \mathbb{R}), \ x \in \mathbb{R}^n.$$

Note that $(Tf)(x), f'(x)$ are in $\mathbb{R}^n$ and $b(x) : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map whereas $a(x) \in \mathbb{R}^n$ but $f(x)\ln|f|(x)$ is a scalar.

(d) Remarkably, the Leibniz rule operator equation has two natural domains of definition when the image is in $C(\mathbb{R})$. If the domain $D(T)$ of T lies strictly between $C^1(\mathbb{R})$ and $C(\mathbb{R})$, i.e. $C^1(\mathbb{R}) \subsetneqq D(T) \subsetneqq C(\mathbb{R})$, the Leibniz rule equation has only one type of solutions, namely $Tf = af \ln|f|$ with $a \in C(\mathbb{R})$, and then the domain is extendable to all of $C(\mathbb{R})$ by the same formula. If the domain of T is contained in $C^k(\mathbb{R})$ for some $k \geq 2$, the solutions of the Leibniz rule equation are given by $Tf = bf' + af \ln|f|$, $a, b \in C(\mathbb{R})$, as seen in Theorem 4. Thus in this case, the domain of T can be extended to $C^1(\mathbb{R})$ by the same formula.

The Leibniz rule shows a remarkable degree of stability: perturbations of the Leibniz rule by reasonable small functions involving only function and not derivative values yield solutions which are perturbations of the above solutions $b\ f' + a\ f \ln|f|$, as shown in [12]:

Theorem 5. *Let* $T : C^1(\mathbb{R}) \to C(\mathbb{R})$ *be an operator and* $B : \mathbb{R}^3 \to \mathbb{R}$ *be a function such that for all* $f, g \in C^1(\mathbb{R})$ *and* $x \in \mathbb{R}$

$$T(f \cdot g)(x) = (Tf)(x) \cdot g(x) + f(x) \cdot (Tg)(x) + B(x, f(x), g(x)).$$

Assume there exists a locally bounded function $M : \mathbb{R} \to \mathbb{R}_+$ *such that*

$$|B(x, y, z)| \leq M(x)\,|y|\,|z| \quad, \quad (x, y, z) \in \mathbb{R}^3.$$

Then there are continuous functions $a, b \in C(\mathbb{R})$, *a locally bounded function* $M' : \mathbb{R} \to \mathbb{R}_+$ *and a function* $C : \mathbb{R}^2 \to \mathbb{R}$ *with* $|C(x, y)| \leq M'(x)|y|$ *such that* $x \mapsto C(x, g(x))$ *is continuous for all* $g \in C^1(\mathbb{R})$ *with*

$$(Tf)(x) = b(x)f'(x) + a(x)\ f(x)\ln|f|(x) + C(x, f(x))$$

for all $f \in C^1(\mathbb{R})$, $x \in \mathbb{R}$.

Instead of adding a term, we may also modify the Leibniz rule operator equation by replacing the operator T in each of its three occurrences there by different maps, i.e., by considering the operator equation

$$V(f \cdot g) = T_1 f \cdot g + f \cdot T_2 g \quad;\ f, g \in C^1(\mathbb{R})$$

for operators $V, T_1, T_2 : C^1(\mathbb{R}) \to C(\mathbb{R})$. Note that if $T : C^1(\mathbb{R}) \to C(\mathbb{R})$ solves the Leibniz rule equation and $c_1, c_2 \in C(\mathbb{R})$ are continuous functions, $T_1 f = Tf + c_1 f$, $T_2 f = Tf + c_2 f$, $Vf = Tf + (c_1 + c_2)f$ satisfy the more general equation. The Leibniz rule shows a remarkable degree of *superrigidity* in the sense that this simple modification yields all solutions, as also shown in [12]:

Theorem 6. *Let* $V, T_1, T_2 : C^1(\mathbb{R}) \to C(\mathbb{R})$ *be operators such that*

$$V(f \cdot g) = T_1 f \cdot g + f \cdot T_2 g$$

is satisfied for all $f, g \in C^1(\mathbb{R})$. Then there are continuous functions $a, b, c_1, c_2 \in C(\mathbb{R}$ such that with

$$Tf := b \, f' + a \, f \, \ln|f|$$

we have

$$Vf = Tf + (c_1 + c_2)f$$
$$T_1 f = Tf + c_1 f, \ T_2 f = Tf + c_2 f.$$

If, in addition, $T_1 d = T_2 d = 0$ for two different non-zero constant functions d with values $d_1 \neq d_2$, $d_1 d_2 \neq 1$, then $Vf = T_1 f = T_2 f = b \, f'$.

3 The Laplace Operator

Applying the Leibniz rule twice, we have for functions $f, g \in C^2(\mathbb{R})$ that

$$(f \cdot g)'' = f'' \cdot g + f \cdot g'' + 2 \, f' \cdot g'. \tag{4}$$

Similarly, for functions $f, g \in C^2(\mathbb{R}^n, \mathbb{R})$, the Laplace operator $\Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ satisfies

$$\Delta(f \cdot g) = \Delta f \cdot g + f \cdot \Delta g + 2 \, \langle f', g' \rangle. \tag{5}$$

Generalizing Eq. (5), we consider operators $T : C^2(\mathbb{R}^n, \mathbb{R}) \to C(\mathbb{R}^n, \mathbb{R})$ and $A : C^2(\mathbb{R}^n, \mathbb{R}) \to C(\mathbb{R}^n, \mathbb{R}^n)$ verifying the "second order Leibniz rule"

$$T(f \cdot g)(x) = (Tf)(x) \cdot g(x) + f(x)(Tg)(x) + \langle (Af)(x), (Af)(x) \rangle \tag{6}$$

for all $f, g \in C^2(\mathbb{R}^n, \mathbb{R})$ and $x \in \mathbb{R}^n$. It is easily seen that the "vector equation" (6) is the "sum" of n scalar equations of the type

$$T_i(f \cdot g)(x) = (T_i f)(x) \cdot g(x) + f(x) \cdot (T_i g)(x) + (A_i f)(x) \cdot (A_i g)(x) \tag{7}$$

where $T_i, A_i : C^2(\mathbb{R}^n, \mathbb{R}) \to C(\mathbb{R}^n, \mathbb{R})$ for $i = 1, \ldots, n$ and $(Tf)(x) = \sum_{i=1}^{n} (T_i f)(x)$, $Af(x) = ((A_i f)(x))_{i=1}^{n}$, cf. [10].

Although the operators T and A are only coupled by Eq. (6) but otherwise are a priori arbitrary, it turns out that only few combinations of maps (T, A) are possible, and this may be used to characterize the Laplace operator by a functional equation of the form (6) with some natural additional conditions. Note that for any operators (T, A) satisfying the "inhomogeneous" equation (6), we may add the solutions $bf' + af \ln|f|$ of the "homogeneous" equation

$$S(f \cdot g) = Sf \cdot g + f \cdot Sg$$

studied in the previous section.

To analyze the form of the operators (T, A) satisfying Eq. (6), again a localization step is needed. However, since T and A are rather independent of one another, there are some degenerate situations where localization in the specific point x is impossible. Consider for $n = 1$ the example of $T, A : C^2(\mathbb{R}) \to C(\mathbb{R})$ given by

$$(Tf)(x) = -f(x) + f(x + 1) \quad , (Af)(x) = f(x) - f(x + 1).$$

One quickly verifies that Eq. (6) is satisfied for (T, A), but that $(Tf)(x)$ not only depends on the value of f at x but also at $x + 1$. Note that T and A are proportional but the case we need to exclude has a local nature. To avoid this degenerate situation, we impose the following condition:

Definition 1. Let $n, m \in \mathbb{R}^n$. An operator $A : C^2(\mathbb{R}^n, \mathbb{R}) \to C(\mathbb{R}^n, \mathbb{R}^m)$ is *non-degenerate* if for all open subsets $J \subset \mathbb{R}^n$ and all $x \in J$ there exist $(m+1)$ functions $g_i \in C^2(\mathbb{R}^n, \mathbb{R})$ with support in J such that the $(m+1)$ vectors $(g_i(x), (Ag_i)(x)) \in \mathbb{R}^{m+1}$ $(i = 1, \ldots, m + 1)$ are linearly independent in $\mathbb{R}^{m+1}$.

We will use this for $m = n$ and $m = 1$. For $m = 1$ it means that A should be sufficiently different from the identity on any open interval.

The form of the operators (T, A) satisfying Eqs. (6) and (7) was determined in [9, 10]. The resultfor the "scalar" equation (7) is:

Theorem 7. *Let $n \in \mathbb{N}$. Assume that $T, A : C^2(\mathbb{R}^n, \mathbb{R}) \to C(\mathbb{R}^n, \mathbb{R})$ are operators such that the "second order Leibniz rule equation"*

$$T(f \cdot g)(x) = (Tf)(x) \cdot g(x) + f(x) (Tg)(x) + (Af)(x) \cdot (Ag)(x) \quad (7)$$

is satisfied for all $f, g \in C^2(\mathbb{R}^n, \mathbb{R})$ and $x \in \mathbb{R}^n$. Let A be non-degenerate. Then there are continuous functions $a, d, p \in C(\mathbb{R}^n, \mathbb{R})$ and $b, c \in C(\mathbb{R}^n, \mathbb{R}^n)$ such that T and A have the form

$$(Tf)(x) = (T_1 f)(x) + (Sf)(x),$$

with

$$(Sf)(x) = \langle b(x), f'(x) \rangle + a(x) f(x) \ln |f|(x)$$

solving the "homogeneous equation" and (T_1, A) are of one of the following types

$$(T_1 f)(x) = \frac{1}{2} \langle c(x), f''(x)c(x) \rangle, \ (Af)(x) = \langle c(x), f'(x) \rangle$$

or

$$(T_1 f)(x) = \frac{1}{2} d(x)^2 \ f(x) \ (\ln |f|(x))^2, \ (Af)(x) = d(x) \ln |f|(x)$$

or

$$(T_1 f)(x) = d(x)(Af)(x), (Af)(x) = d(x)f(x)(\{\text{sgn } f(x)\}|f(x)|^{p(x)} - 1).$$

Here $p(x) \geq -1$ and the term $\{\text{sgn } f(x)\}$ may be present or not, yielding two different forms. All these operators satisfy Eq. (7).

Remark. In the last two cases, the operators T and A may be extended to maps $T, A : C^1(\mathbb{R}^n, \mathbb{R}) \to C(\mathbb{R}^n, \mathbb{R})$ by the same formula, or even to $T, A : C(\mathbb{R}^n, \mathbb{R}) \to C(\mathbb{R}^n, \mathbb{R})$ if $b = 0$. The last case is less interesting since T and A are proportional. The second case is a "second order iteration" for entropy-type derivations.

We indicate a few steps of the proof of Theorem 7. For this, we first formulate a result of Faifman, cf. [8, 10].

Proposition 1. *Let $n \in \mathbb{N}$, $d \in \mathbb{N}$ and $H_j : \mathbb{R}^n \times \mathbb{R}^{\binom{n+j-1}{j}} \to \mathbb{R}$ be a family of functions additive in the second variables,*

$$H_j(x, \alpha + \beta) = H_j(x, \alpha) + H_j(x, \beta) \quad ; \alpha, \beta \in \mathbb{R}^{\binom{n+j-1}{j}}, \ x \in \mathbb{R}^n,$$

for $j = 0, \ldots, d - 1$. Denote as usual the Fréchet derivative by D and its j-fold iteration by D^j. Assume that for any function $g \in C^{d-1}(\mathbb{R}^n, \mathbb{R})$

$$H_0(x, g(x)) + H_1(x, Dg(x)) + \cdots + H_{d-1}(x, D^{d-1}g(x))$$

is continuous in $x \in \mathbb{R}^n$. Since $D^j g(x)$ is determined by $\binom{n+j-1}{j}$ j-fold iterated partial derivatives, we use the space of dimension $\binom{n+j-1}{j}$ for these values.

Then for all $j \in \{0, \ldots, d-1\}$, $H_j(x, \alpha) = \langle c_j(x), \alpha \rangle$ is linear in the variable $\alpha \in \mathbb{R}^{\binom{n+j-1}{j}}$ with continuous coefficient $c_j \in C\left(\mathbb{R}^n, \mathbb{R}^{\binom{n+j-1}{j}}\right)$.

Localization: Let $J \subset \mathbb{R}^n$ be open and $f_1, f_2 \in C^2(\mathbb{R}^n, \mathbb{R})$ with $f_1|_J = f_2|_J$. We claim that $(Tf_1)|_J = (Tf_2)|_J$ and $(Af_1)|_J = (Af_2)|_J$. For any $g \in C^2(\mathbb{R}^n, \mathbb{R})$ with support in J we have $f_1 g = f_2 g$ and hence by Eq. (7) for any $x \in J$

$$((Tf_1)(x) - (Tf_2)(x))g(x) + ((Af_i)(x) - (Af_2)(x))(Ag)(x) = 0.$$

By assumption of non-degeneracy of A, we find two such functions $g = g_1$ and $g = g_2$ such that $(g_i(x), (Ag_i)(x)) \in \mathbb{R}^2$ are linearly independent which implies $(Tf_1)(x) = (Tf_2)(x)$, $(Af_1)(x) = (Af_2)(x)$.

This localization on intervals implies pointwise localization using similar ideas as in the previous section. Therefore

$$(Tf)(x) = F(x, f(x), f'(x), f''(x)),$$
$$(Af)(x) = B(x, f(x), f'(x), f''(x))$$

for suitable functions $F, B : \mathbb{R}^4 \to \mathbb{R}$.

Analysis of F, B and continuity of coefficient functions: For positive C^2-functions $f, f_1, f_2 : \mathbb{R}^n \to \mathbb{R}_+$ by Eq. (7)

$$\frac{T(f_1 \cdot f_2)}{f_1 f_2} = \frac{Tf_1}{f_1} + \frac{Tf_2}{f_2} + \frac{Af_1}{f_1} \cdot \frac{Af_2}{f_2}. \tag{8}$$

Since f and its derivatives may be expressed by $\ln f$ and its derivatives we may write

$$\frac{Tf}{f}(x) = G(x, (\ln f)(x), (\ln f)'(x), (\ln f)''(x)),$$

$$\frac{Af}{f}(x) = H(x, (\ln f)(x), (\ln f)'(x), (\ln f)''(x))$$

for suitable functions $G, H : \mathbb{R}^4 \to \mathbb{R}$ originating from F, B by dividing by f and the logarithmic substitution. Equation (8) then yields the real variable functional equation for G and H,

$$G(x, \alpha_0 + \beta_0, \alpha_1 + \beta_1, \alpha_2 + \beta_2)$$
$$= G(x, \alpha_0, \alpha_1, \alpha_2) + G(x, \beta_0, \beta_1, \beta_2) + H(x, \alpha_0, \alpha_1, \alpha_2)H(x, \beta_0, \beta_1, \beta_2),$$

which holds for arbitrary $x, \alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2 \in \mathbb{R}$. Similar functional equations have been considered by Aczél in [1], and this one is solved in [9, 10]: Using the non-degeneracy assumption on A, only two forms of G and H are possible: for any $x \in \mathbb{R}^n$, there are additive functions $b(x), d(x) : \mathbb{R}^3 \to \mathbb{R}$ and there is $e \in C(\mathbb{R}^n, \mathbb{R})$ such that either

$$H(x, \alpha_0, \alpha_1, \alpha_2) = d(x)[\alpha_0, \alpha_1, \alpha_2]$$
$$G(x, \alpha_0, \alpha_1, \alpha_2) = \frac{1}{2}(d(x)[\alpha_0, \alpha_1, \alpha_2])^2 + b(x)[\alpha_0, \alpha_1, \alpha_2]$$

or

$$H(x, \alpha_0, \alpha_1, \alpha_2) = e(x)\exp(d(x)[\alpha_0, \alpha_1, \alpha_2] - 1)$$
$$G(x, \alpha_0, \alpha_1, \alpha_2) = e(x)H(x, \alpha_0, \alpha_1, \alpha_2) + b(x)[\alpha_0, \alpha_1, \alpha_2]$$

The brackets $[\dots]$ indicate that these functions are additive. Moreover, we know that for any $g \in C^2(\mathbb{R}^n, \mathbb{R})$, putting $f = \exp(g)$, i.e., $g = \ln f$ the functions

$$x \in \mathbb{R}^n \mapsto H(x, g(x), g'(x), g''(x))$$
$$x \in \mathbb{R}^n \mapsto G(x, g(x), g'(x), g''(x))$$

are continuous. Faifman's continuity and linearity result (Proposition 1) then implies that the additive functions $b(x), d(x)$ are linear and continuous in $x \in \mathbb{R}^n$, e.g., d is of the form

$$d(x)[\alpha_0, \alpha_1, \alpha_2] = d_0(x)\alpha_0 + d_1(x)\alpha_1 + d_2(x)\alpha_2$$

with $d_0, d_1, d_2 \in C(\mathbb{R}^n)$. This yields the general form of Tf/f and Af/f via G and H for positive C^2-functions $f : \mathbb{R}^n \to \mathbb{R}$. The requirement that T and A are defined on all functions $f \in C^2(\mathbb{R}^n, \mathbb{R})$ then excludes certain cases so that only the two main cases in Theorem 7 and the homogeneous solution S remain.

The "Laplace operator equation" (6) is—as mentioned before—a "sum" of n equations (7) studied in Theorem 7. This enables us to find the "general solution" of Eq. (6). However, to characterize the Laplace operator, it is useful to consider its natural invariance property under the orthogonal group $O(n)$:

Definition 2. A map $T : C^2(\mathbb{R}^n, \mathbb{R}) \to C(\mathbb{R}^n, \mathbb{R})$ is $O(n)$-invariant if for all $f \in C^2(\mathbb{R}^n, \mathbb{R})$ and all orthogonal maps $u \in O(n)$ we have $T(f \circ u) = (Tf) \circ u$.

Further, the Laplace operator is zero on the affine functions, i.e., the linear plus constant functions. Theorem 7 thengives the following characterization of the Laplace operator by the second order Leibniz rule, cf. [10]:

Theorem 8. *Let $n \in \mathbb{N}$. Assume that $T : C^2(\mathbb{R}^n, \mathbb{R}) \to C(\mathbb{R}^n, \mathbb{R})$ and $A : C^2(\mathbb{R}^n, \mathbb{R}) \to C(\mathbb{R}^n, \mathbb{R}^n)$ are operators satisfying*

$$T(f \cdot g)(x) = (Tf)(x) \cdot g(x) + f(x) \cdot (Tg)(x) + \langle (Af)(x), (Ag)(x) \rangle \quad (6)$$

for all $f, g \in C^2(\mathbb{R}^n, \mathbb{R})$ and all $x \in \mathbb{R}^n$. Suppose, in addition, that T is $O(n)$-invariant and vanishes on the affine functions and that A is non-degenerate. Then T is a multiple of the Laplace operator and A a multiple of the derivative: there is a continuous function $d \in C(\mathbb{R}_{\geq 0}, \mathbb{R})$ such that

$$(Tf)(x) = \frac{1}{2} d(||x||)^2 (Af)(x), \ (Af)(x) = d(||x||) f'(x),$$

for all $f \in C^2(\mathbb{R}^n, \mathbb{R})$ and all $x \in \mathbb{R}^n$. Here $|| \cdot ||$ denotes the Euclidean norm on $\mathbb{R}^n$.

Remark. If T is zero on the affine functions but no longer $O(n)$-invariant, there are functions $c_i \in C(\mathbb{R}^n, \mathbb{R}^n)$ for $i = 1, \ldots, n$ such that

$$(Tf)(x) = \frac{1}{2} \sum_{i=1}^{n} \langle c_i(x), f''(x)c_i(x) \rangle, (Af)(x) = (\langle c_i(x), f'(x) \rangle)_{i=1}^{n}.$$

Hence T is a sum of multiples of changing second directional derivatives and A is a vector of the corresponding directional derivatives.

4   The Chain Rule

We now turn to the question to which degree the derivative D is characterized by the chain rule

$$D(f \circ g) = (Df) \circ g \cdot Dg \quad ; f, g \in C^1(\mathbb{R}).$$

This means that we investigate the general form of operators $T : C^1(\mathbb{R}) \to C(\mathbb{R})$ such that the "chain rule operator equation"

$$T(f \circ g) = (Tf) \circ g \cdot Tg \quad ; f, g \in C^1(\mathbb{R}) \tag{9}$$

is satisfied. In addition to the derivative D, also powers $|D|^p$ as well as $|D|^p \operatorname{sgn}(D)$, $\operatorname{sgn}(D)(f) := \operatorname{sgn}(f')$, for $p > 0$ will satisfy Eq. (10). Further given $H \in C(\mathbb{R})$ with $H > 0$, the map given by $Tf = H \circ f/H$ satisfies Eq. (10). Moreover, products of maps satisfying Eq. (10) satisfy Eq. (10).

In addition to these solutions, there are degenerate forms of operators T with Eq. (10): Consider the map $T : C^1(\mathbb{R}) \to C(\mathbb{R})$ given by

$$Tf := \left\{ \begin{array}{ll} f' & f \in C^1(\mathbb{R}) \text{ bijective} \\ 0 & f \in C^1(\mathbb{R}) \text{ not bijective} \end{array} \right\} .$$

This operator satisfies Eq. (10) but is zero on all (half-) bounded functions. To exclude such degenerate cases, we introduce

Definition 3. A map $T : C^1(\mathbb{R}) \to C(\mathbb{R})$ is *non-degenerate* provided T is not identically zero on the space of half-bounded functions, i.e., those bounded from above or from below.

In this case, the previous examples provide all forms of maps T satisfying the chain rule, as shown by Artstein-Avidan, König and Milman [4]:

Theorem 9. *Let $T : C^1(\mathbb{R}) \to C(\mathbb{R})$ be an operator satisfying the chain rule*

$$T(f \circ g) = (Tf) \circ g \cdot Tg \quad ; f, g \in C^1(\mathbb{R}). \tag{10}$$

Assume that T is non-degenerate as in Definition 3. Then there exists some $p > 0$ and a positive continuous function $H \in C(\mathbb{R})$ such that either

$$Tf = \frac{H \circ f}{H} |f'|^p$$

or

$$Tf = \frac{H \circ f}{H} |f'|^p \operatorname{sgn} f'.$$

These operators satisfy Eq. (10). If in addition to Eq. (10), $T(2\,\mathrm{Id}) = c$ is a constant function with $c > 1$, H is constant and $Tf = |f'|^p$ or $Tf = |f'|^p\,\mathrm{sgn}\,f'$ where $p = \log_2 c$.

Remarks.

(1) Hence if $c = 2$ and T also attains negative values, T is the derivative, $Tf = f'$.
(2) For $p > 0$, let G be the anti-derivative of $H^{1/p} > 0$, $G' = H^{1/p}$. Then $G \in C^1(\mathbb{R})$ is strictly monotone and

$$Tf = \left| \frac{d(G \circ f)}{dG} \right|^p \quad \text{or} \quad Tf = \left| \frac{d(G \circ f)}{dG} \right|^p \mathrm{sgn}\left(\frac{d(G \circ f)}{dG} \right).$$

   In this sense, all solutions of the chain rule for non-degenerate maps T are p-th powers of some derivative, up to signs. In the second case, T is surjective.
(3) We emphasize that no linearity or continuity conditions are imposed upon T in Theorem 9; however, some continuity of T is a consequence of the result.
(4) The function H in Theorem 9 is completely determined by $T(2\,\mathrm{Id})$. In fact, let $\varphi(x) := T(2\,\mathrm{Id})(x)/T(2\,\mathrm{Id})(0)$. Then $H(x) = \prod_{n \in \mathbb{N}} \varphi\left(\frac{x}{2^n}\right)$ where the product converges uniformly on compact sets to H, with $H(0) = 1$.
(5) Theorem 9 also holds for operators $T : C^k(\mathbb{R}) \to C(\mathbb{R})$ for $k \in \mathbb{N} \cup \{\infty\}$ with Eq. (10), with the same form of T. The proof of the case $k > 1$ in [4] is more involved than for $k = 1$ since it uses the basic form of the Faà di Bruno formula for $(f \circ g)^{(k)}$. It turns out that $(Tf)(x)$ does not depend on higher derivative values $f^{(k)}(x)$, $k \geq 2$.
(6) For spaces different from $C^k(\mathbb{R})$, e.g. the polynomials $\mathcal{P}$ on $\mathbb{R}$ or real analytic functions, there may be different types of examples: Let $c > 0$. The map $T : \mathcal{P} \to \mathcal{P}$, $Tf := (\deg f)^c$, satisfies the chain rule, if $\deg f$ denotes the degree of the polynomial $f \in \mathcal{P}$.
(7) Assume that T satisfies Eq. (10) with image in $C(\mathbb{R})$. If the domain $D(T)$ of T lies strictly between $C^1(\mathbb{R})$ and $C(\mathbb{R})$, i.e. $C^1(\mathbb{R}) \subsetneq D(T) \subsetneq C(\mathbb{R})$, necessarily $p = 0$ in Theorem 9, and the solution is of the type $Tf = H \circ f / H$ with $H \in C(\mathbb{R})$ and $H > 0$. In particular, no negative values are attained by functions in the image of T. If for some f in the domain of T, $Tf(x) < 0$ at some $x \in \mathbb{R}$, the maximal domain for a non-degenerate operator T satisfying the chain rule is $C^1(\mathbb{R})$. If the domain is between $C^\infty(\mathbb{R})$ and $C^1(\mathbb{R})$, T may be extended by the same formula to $C^1(\mathbb{R})$.

We indicate the main steps in the proof of Theorem 9. Again we start with

Localization. Let $J \subset \mathbb{R}$ be open and $f_1|_J = f_2|_J$. We claim that $(Tf_1)|_J = (Tf_2)|_J$. Then similarly as in Sect. 2 for the Leibniz rule

$$(Tf)(x) = F(x, f(x), f'(x)) \tag{11}$$

for a suitable function $F : \mathbb{R}^3 \to \mathbb{R}$ and all $f \in C^1(\mathbb{R})$, $x \in \mathbb{R}$. Given $f_1|_J = f_2|_J$ and $x \in J$, choose $I \subset J$ open and $g \in C^1(\mathbb{R})$ with $x \in I$, $g|_I = \mathrm{Id}|_I$ and $\mathrm{Im}(g) \subset J$. Then $f_1 \circ g = f_2 \circ g$. As a consequence of the non-degeneracy of T and the chain rule, one shows using scaling and shifts that there is $h \in C^1(\mathbb{R})$ with $\mathrm{Im}(h) \subset I$, $h(x) = x$ and $(Th)(x) \neq 0$. Since $g \circ h = h$, by Eq. (10)

$$0 \neq (Th)(x) = (Tg)(x) \cdot (Th)(x),$$

so that $(Tg)(x) = 1$. This yields with $f_1 \circ g = f_2 \circ g$ that

$$(Tf_1)(x) = (Tf_1)(f(x))(Tg)(x) = T(f_1 \circ g)(x)$$
$$= T(f_2 \circ g)(x) = (Tf_2)(g(x))(Tg)(x) = (Tf_2)(x),$$

i.e. $(Tf_1)|_J = (Tf_2)|_J$.

Analysis of the representing function F. Using Eq. (11), the chain rule Eq. (10) means in terms of $F : \mathbb{R}^3 \to \mathbb{R}$ that

$$F(x, z, \alpha_1 \beta_1) = F(y, z, \alpha_1)\, F(x, y, \beta_1)$$

for all $x, y, z, \alpha_1, \beta_1 \in \mathbb{R}$ by choosing appropriate functions f, g with $y = g(x)$, $z = f(y)$, $\alpha_1 = f'(y)$, $\beta_1 = g'(x)$. Choosing $z = x$, we have

$$F(x, x, \alpha_1 \beta_1) = F(y, x, \alpha_1)\, F(x, y, \beta_1)$$
$$= F(x, y, \beta_1) F(y, x, \alpha_1) = F(y, y, \beta_1 \alpha_1)$$

so that $F(x, x, \alpha) =: K(\alpha)$ is independent of x and, moreover, multiplicative: $K(\alpha\beta) = K(\alpha)\, K(\beta)$. For general $x, y \in \mathbb{R}$, we have

$$F(x, y, \alpha) = G(x, y)K(\alpha), \quad G(x, y) := 1/F(y, x, 1)$$

where $F(y, x, 1) \neq 0$ is a consequence of $F(y, x, 1)F(x, y, 1) = K(1) = 1$. Let $H(y) = G(0, y)$. Then $G(x, y) = H(y)/H(x)$ and

$$F(x, y, \alpha) = H(y)/H(x)\, K(\alpha). \tag{12}$$

Using Eq. (12), apply Eq. (11) to $f = 2\,\mathrm{Id}$. Then

$$(Tf)(x) = F(x, 2x, 2) = H(2x)/H(x)\, K(x)$$

is continuous in x, i.e., $\varphi(x) = H(2x)/H(x)$ defines a function $\varphi \in C(\mathbb{R})$. Then for any $a, b \in \mathbb{R}$

$$\frac{H(b)}{H(a)} = \frac{H\left(\frac{b}{2^k}\right)}{H\left(\frac{a}{2^k}\right)} \prod_{i=1}^{k} \frac{\varphi\left(\frac{b}{2^k}\right)}{\varphi\left(\frac{c}{2^2}\right)}.$$

Choose $a = 1$ and apply Eq. (11) to $g(x) = bx$ to get $(Tg)(x) = \frac{H(bx)}{H(x)} K(b)$. Since Tg is continuous in 0, $\lim_{K \to \infty} H\left(\frac{b}{2^k}\right)/H\left(\frac{1}{2^k}\right) = 1$ exists and hence $H(b) = H(1) \lim_{k \to \infty} \prod_{i=1}^{k} \frac{\varphi\left(\frac{b}{2^k}\right)}{\varphi\left(\frac{1}{2^k}\right)}$ is the pointwise limit of continuous functions and hence measurable. Now Eqs. (11) and (12) imply that K is measurable as well. By Lemma 1, the multiplicative measurable function K on $\mathbb{R}$ has the form $K(\alpha) = |\alpha|^p$ or $K(\alpha) = |\alpha|^p \operatorname{sgn}(\alpha)$ for some $p \geq 0$, so that

$$
\begin{aligned}
Tf(x) &= F(x, f(x), f'(x)) \\
&= H(f(x))/H(x) \, |f'(x)|^p \, \{\operatorname{sgn} f'(x)\}.
\end{aligned}
$$

The proof that H is continuous can be found in [4].

It follows from Theorems 4 and 9 that the derivative is characterized by the Leibniz rule and the chain rule, cf. [8]:

Corollary 1. *Assume $T : C^1(\mathbb{R}) \to C(\mathbb{R})$ is not identically zero and satisfies the Leibniz rule and the chain rule for all $f, g \in C^1(\mathbb{R})$,*

$$T(f \cdot g) = Tf \cdot g + f \cdot Tf,$$
$$T(f \circ g) = (Tf) \circ g \cdot Tg.$$

Then T is the derivative : $Tf = f'$ for all $f \in C^1(\mathbb{R})$.

Proof. By Theorem 4 there are functions $a, b \in C(\mathbb{R})$ such that $Tf = b f' + a f \ln |f|$. Since $T \not\equiv 0$, $(b, a) \not\equiv 0$. This implies that T is non-degenerate in the sense of Definition 3. Hence T also has the form

$$Tf = \frac{H \circ f}{H} |f'|^p \{\operatorname{sgn} f'\}$$

where $H : \mathbb{R} \to \mathbb{R}_{>0}$ is continuous and $p \geq 0$. Both forms coincide for $T \not\equiv 0$ only if $Tf = f'$. □

Remark. Let $C(\mathbb{R}_{>1})_+ := \{f \in C(\mathbb{R}_{>0}) \mid f > 1\}$ and $H(x) = x \ln(x)$, $x \in \mathbb{R}$. Then the operator $T : C(\mathbb{R}_{>0})_+ \to C(\mathbb{R})$ defined by $Tf(x) = H(f(x))/H(x)$ satisfies the Leibniz rule and the chain rule. However, it cannot be defined on the whole space $C(\mathbb{R})$ or $C^1(\mathbb{R})$. Note that these non-linear derivations satisfy $T(x^n) = nx^{n-1}$ and $T(x^x) = x^x$. On this subset of $C(\mathbb{R}_{>1})$, this map satisfying the Leibniz and the chain rule is actually unique. It is a remarkable example different from the derivative satisfying the Leibniz and the chain rule; however, the domain is rather different in both cases.

The chain rule is super-stable in the sense that it does not admit non-trivial perturbations involving function values: for formulating this, we call an operator $T : C^1(\mathbb{R}) \to C(\mathbb{R})$ *locally non-degenerate* provided that for any open interval $J \subset \mathbb{R}$ and any $x \in J$ there exists $g \in C^1(\mathbb{R})$, $y \in \mathbb{R}$ with $g(y) = x$, $\mathrm{Im}(g) \subset J$ and $Tg(y) \neq 0$. We then have the following stability result for the chain rule, cf. [12]:

Theorem 10. *Assume that* $T : C^1(\mathbb{R}) \to C(\mathbb{R})$ *is locally non-degenerate and* $B : \mathbb{R}^3 \to \mathbb{R}$ *is a function such that for all* $f, g \in C^1(\mathbb{R})$

$$T(f \circ g)(x) = (Tf) \circ g(x) \cdot Tg(x) + B(x, f \circ g(x), g(x)).$$

If Tf *depends non-trivially on* f', $B = 0$ *and* T *satisfies the chain rule.*

As in the case of the Leibniz rule, instead of adding a term to the chain rule, we may replace each occurrence of the map T in the chain rule operator equation by different operators, i.e., consider the equation

$$V(f \circ g) = (T_1 f) \circ g \cdot T_2 g \quad ; \quad f, g \in C^1(\mathbb{R})$$

where $V, T_1, T_2 : C^1(\mathbb{R}) \to C(\mathbb{R})$. Multiplying the solution T of the chain rule equation by functions $c_1, c_2 \in C(\mathbb{R})$ in the sense that

$$T_1 f = c_1 \circ f \cdot Tf, \ T_2 = c_2 \cdot Tf, \ Vf = c_1 \circ f \cdot c_2 \cdot Tf$$

obviously yields solutions of the more general equations. The chain rule shows again a high degree of *rigidity* in the sense that for non-degenerate operators the above are all solutions. The following result is shown in [12], under the following assumption of non-degeneracy: We call $V : C^1(\mathbb{R}) \to C(\mathbb{R})$ non-degenerate, provided that

 (i) for any $x \in \mathbb{R}$ there is a half-bounded function $f \in C^1(\mathbb{R})$ with $(Vf)(x) \neq 0$,
(ii) for any $x \in \mathbb{R}$ there is $y \in \mathbb{R}$ with $f(y) = x$ and $(Vf)(y) \neq 0$.

Theorem 11. *Assume that* $V, T_1, T_2 : C^1(\mathbb{R}) \to C(\mathbb{R})$ *are operators such that the equation*

$$V(f \circ g) = (T_1 f) \circ g \cdot T_2 g$$

holds for all $f, g \in C^1(\mathbb{R})$. *Assume that* V *is non-degenerate. Then there is* $p \geq 0$ *and there are continuous functions* $H, c_1, c_2 \in C(\mathbb{R})$ *with* $H > 0$ *such that with*

$$Tf := \frac{H \circ f}{H} |f'|^p \{\mathrm{sgn} \, f'\}$$

we have

$$Vf = c_1 \circ f \cdot c_2 \cdot Tf,$$
$$T_1 f = c_1 \circ f \cdot Tf, \ T_2 f = c_2 \cdot Tf.$$

The term {sgn f'} *may appear in all formulas or not at all. If, in addition, the functions* $V(2\,\mathrm{Id})$, $T_1(2\,\mathrm{Id})$ *and* $T_2(2\mathrm{Id})$ *are constant, the functions* c_1, c_2 *and H are constant.*

We may also use modifications of the chain rule equation studied in Theorem 9 to characterize the integral. If T is the derivative and I the definite integral, $I\,h(x) = \int_c^x h(s)ds$, $h \in C(\mathbb{R})$ with fixed $c \in \mathbb{R}$, we have that I is injective and

$$f \circ g - f \circ g(c) = I(Tf \circ g \cdot Tf).$$

To formulate a characterization of the integral and the derivative by an equation of this type, we will assume that T is *non-degenerate* in the sense that there is $y \in \mathbb{R}$ such that for any $x \in \mathbb{R}$ there is a half-bounded function $f \in C^1(\mathbb{R})$ with $f(x) = y$ and $(Tf)(x) \neq 0$. Then the following result holds, cf. [11].

Theorem 12. *Assume that* $I : C(\mathbb{R}) \to C^1(\mathbb{R})$ *and* $T : C^1(\mathbb{R}) \to C(\mathbb{R})$ *are operators such that for some fixed* $c \in \mathbb{R}$

$$f \circ g - f \circ g(c) = I(Tf \circ g \cdot Tg) \quad ; f, g \in C^1(\mathbb{R})$$

holds. Suppose further that I *is injective and* T *non-degenerate. Then there are constants* $p > 0$, $d \neq 0$ *such that for all* $x \in \mathbb{R}$

$$(Tf)(x) = d\,|f'(x)|^p\,\mathrm{sgn}\,(f'(x)) \quad ; f \in C^1(\mathbb{R})$$

$$(Ih)(x) = d^{-2/p} \int_c^x |h(s)|^{1/p}\,\mathrm{sgn}\,(h(s))\,ds\,; h \in C(\mathbb{R}).$$

If T *satisfies the initial conditions* $T(2\mathrm{Id}) = 2$ *and* $T(3\mathrm{Id}) = 3$, *we have* $p = d = 1$ *and*

$$(Tf)(x) = f'(x), \ (Ih)(x) = \int_c^x h(s)ds.$$

5 The Second Order Chain Rule

Interacting the chain rule, we have for functions $f, g \in C^2(\mathbb{R})$ that

$$(f \circ g)'' = f'' \circ g \cdot g'^2 + f' \circ g \cdot g''.$$

In this section, we consider generalizations of this equation by replacing the second derivatives by an operator T and the first order terms by operators A_1 and A_2.

Thus assume that $T : C^2(\mathbb{R}) \to C(\mathbb{R})$ and $A_1, A_2 : C^1(\mathbb{R}) \to C(\mathbb{R})$ are operators such that for all $f, g \in C^2(\mathbb{R})$ we have

$$T(f \circ g) = (Tf) \circ g \cdot A_1 g + (A_2 f) \circ g \cdot Tg. \tag{13}$$

We study the question to what extent T characterizes the second derivative, and we try to find the general solution for (T, A_1, A_2) of Eq. (13) under a weak non-degeneracy assumption and a weak continuity and isotropicity assumption on the operators A_i. These weak conditions are used to assure that T, A_1 and A_2 are independent in a weak sense.

Definition 4. Let $T : C^2(\mathbb{R}) \to C(\mathbb{R})$ and $A : C^1(\mathbb{R}) \to C(\mathbb{R})$ be operators. We call the pair (T, A) *non-degenerate* provided that

(a) For every open set $J \subset \mathbb{R}$ and $x \in J$, there exist functions $g_1, g_2 \in C^2(\mathbb{R})$ with image in J and points $y_1, y_2 \in \mathbb{R}$ $g_1(y_1) = g_2(y_2) = x$ such that $((Tg_i)(y_i), (Ag_i)(y_i)) \in \mathbb{R}^2$ are linearly independent vectors $(i = 1, 2)$.
(b) For every $x \in \mathbb{R}$ there is $g \in C^2(\mathbb{R})$ with $g(x) = x, (Tg)(x) = 0, (Ag)(x) \neq 1$.
(c) For every $x \in \mathbb{R}$ there are $g_1, g_2 \in C^2(\mathbb{R})$ and $y \in \mathbb{R}$ such that $g_2(y) = x$ and $(Tg_1)(x) \neq 0, (Tg_2)(y) \neq 0$.

Definition 5. An operator $A : C^1(\mathbb{R}) \to C(\mathbb{R})$ is *isotropic* if it commutes with shifts, i.e. $A(f \circ S_c) = (Af) \circ S_c$ for all $f \in C^1(\mathbb{R})$ and all $c \in \mathbb{R}$, where $S_c : \mathbb{R} \to \mathbb{R}$ is the shift $x \mapsto x + c$. We call A pointwise C^1-*continuous* if for any sequence $(f_n)_{n \in \mathbb{N}}$ of C^2-functions on $\mathbb{R}$ and any $f \in C^1(\mathbb{R})$ the uniform convergence of $\lim_{n \to \infty} f_n = f$ and $\lim_{n \to \infty} f'_n = f'$ on $\mathbb{R}$ implies the pointwise convergence of $\lim_{n \to \infty} (Af_n)(x) = (Af)(x)$ for any $x \in \mathbb{R}$.

The main result of [7] states:

Theorem 13. *Consider operators* $T : C^2(\mathbb{R}) \to C(\mathbb{R})$ *and* $A_1, A_2 : C^1(\mathbb{R}) \to C(\mathbb{R})$ *such that the operator equation*

$$T(f \circ g) = (Tf) \circ g \cdot A_1 g + (A_2 f) \circ g \cdot Tg \tag{13}$$

holds for all $f, g \in C^2(\mathbb{R})$. *Assume the pair* (T, A_1) *is non-degenerate and that* A_1 *and* A_2 *are isotropic and pointwise* C^1-*continuous. Suppose also that* $(Tf_1)(0) \neq 0$ *for* $f_1(x) = x + x^2/2$. *Then there is* $p \geq 1$ *such that the operators* A_1 *and* A_2 *have the form*

$$(A_2 f)(x) = |f'(x)|^p \{\text{sgn } f'(x)\}, \quad (A_1 f)(x) = f'(x)(A_2 f)(x).$$

Further, there are $c \in \mathbb{R} \setminus \{0\}$ *and* $H \in C(\mathbb{R})$ *such that*

$$(Tf)(x) = [c \text{ sgn } f'(x)|f'(x)|^{p-1} f''(x)$$
$$+ (H \circ f(x) \cdot f'(x) - H(x))|f'(x)|^p]\{\text{sgn } f'(x)\}$$

for all $f \in C^2(\mathbb{R})$. The brackets $\{\ldots\}$ indicate two solutions, one without and one with the term $\operatorname{sgn} f'(x)$. For $p = 1$, the term $\operatorname{sgn} f'(x)$ on the right side has to be present to avoid discontinuous functions in the range of T. Conversely, any such operators (T, A_1, A_2) satisfy Eq. (13).

Remarks.

(a) The condition that $c = (Tf_1)(0) \neq 0$ holds for $f_1(x) = x + x^2/2$ could be replaced by $(Tf)(0) \neq 0$ for any fixed function $f \in C^2(\mathbb{R})$ with $f(0) = 0$, $f'(0) = 1$, $f''(0) \neq 0$.

(b) In the case that $(Tf_1)(0) = 0$, the operators (T, A_1, A_2) satisfying Eq. (13) do not depend on f'' and are classified in [7] as well. There are three different types of solution operators (T, A_1, A_2).

(c) For $p = 1$, we get the "classical" case

$$(A_1 f)(x) = f'(x)^2, \ (A_2 f)(x) = f'(x),$$
$$(Tf)(x) = c \ f''(x) + ((H \circ f)(x) f'(x) - H(x)) f'(x),$$

with $(Tf)(x) = c \ f''(x)$ if H is zero.

(d) The pointwise C^1-continuity of A_1, A_2 is needed to show that A_1 and A_2 do not depend on f'' when using Eq. (13) for functions $f \in C^2(\mathbb{R})$. The non-degeneracy allows to prove localization of the form

$$(Tf)(x) = F(x, f(x), f'(x), f''(x)),$$
$$(A_i f)(x) = B_i(x, f(x), f'(x)), \ i = 1, 2,$$

for suitable functions $F : \mathbb{R}^4 \to \mathbb{R}$, $B_i : \mathbb{R}^3 \to \mathbb{R}$. The analysis of the functions F and B_i uses similar methods as those used in Sects. 3 and 4, though slightly more involved than those used in Sect. 4 for the "regular" chain rule equation.

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A Remark on the Extremal Non-Central Sections of the Unit Cube

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Abstract In this paper, we investigate extremal volumes of non-central slices of the unit cube. The case of central hyperplane sections is known and was studied by Ball, Hadwiger and Hensley. The case of non-central sections, i.e. when we dictate that the hyperplane must be a certain distance $t > 0$ from the center of the cube, is open in general and the same is true about sections of the unit cube by slabs. In this paper we give a full solution for extremal one-dimensional sections and a partial solution for extremal hyperplane slices for the case $t > \frac{\sqrt{n-1}}{2}$. We also make a remark on minimal volume slices of the cube by slabs of width $2t$, when $t > \frac{\sqrt{n-1}}{2}$.

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1 Introduction

We denote by $Q_n = [-1/2, 1/2]^n$ an n-dimensional cube of volume 1. As usual,
we denote by $x \cdot y$ the inner product of two vectors $x, y \in \mathbb{R}^n$ and by $|x|$ the length
of vector $x \in \mathbb{R}^n$. Let $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be a Euclidean unit sphere,
$t\mathbb{S}^{n-1}$ be a sphere of radius t, $d(A, x) = \inf_{a \in A} |x - a|$ be the distance from a point
$x \in \mathbb{R}^n$ to a set $A \subset \mathbb{R}^n$ and $\mathrm{diam}(A) = \sup\{|x - y| : x, y \in A\}$ be the diameter
of $A \subset \mathbb{R}^n$. We write $|A|$ for the k-dimensional Lebesgue measure (volume) of
a measurable set $A \subset \mathbb{R}^n$, where $k = 1, \dots, n$ is the dimension of the minimal
flat containing A. For $a, b \in \mathbb{R}^n$, we denote by $[a, b]$ the segment joining a to b:
$[a, b] = \{(1-t)a + tb : t \in [0, 1]\}$. A *convex body* is a compact convex subset of $\mathbb{R}^n$
with nonempty interior and Q is *symmetric* if it is centrally symmetric with center
at the origin, i.e. $Q = -Q$. We refer to [12] or [6] for an excellent introduction to
the subject.

In this paper we study the extremal volumes of slices of Q_n. The original question
posed concerns slices by $(n - 1)$-dimension subspaces H of $\mathbb{R}^n$: What are the
minimal and maximal values of $|Q_n \cap H|$? It was shown by Hadwiger [7] that
the section of minimal volume is the canonical section (i.e. the section orthogonal to
one of the canonical unit vectors). The solution to the maximal case was conjectured
by Hensley [8]. Ball [2] (see also [12, 13]) proved Hensley's conjecture by showing
that the maximal volume of hyperplane sections of Q_n is $\sqrt{2}$ and it attains for the
hyperplane H with normal vector $\xi = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \dots, 0)$.

Hensley was motivated by a more general statement, which was conjectured by
Good. The claim stated that, for any k-dimensional subspace P_k of $\mathbb{R}^n$ with $k < n$,
$|P_k \cap Q_n| \geq 1$. Ball's proof addressed the case when $k = n - 1$; Vaaler gave a full
proof of Good's conjecture [15] (see also [16]).

The above results were further extended by Oleszkiewicz and Pełczyński [14],
who proved an analog to Ball's theorem in the complex space $\mathbb{C}^n$. Ball's result was
also generalized to the case of Gaussian and Gaussian-type measures [4, 11, 17].
Much less is known about non-central hyperplane sections (i.e. sections of Q_n by
$\{x \in \mathbb{R}^n : x \cdot \xi = t\}$, for a fixed $t > 0$) and sections of the cube by slabs (i.e.
sections of Q_n by $\{x \in \mathbb{R}^n : |x \cdot \xi| \leqslant t\}$, for a fixed $t > 0$). Barthe and Koldobsky
[3] provided a number of interesting results on sections of Q_n by slabs, including
the complete solutions of the two dimensional case and the case of slabs of small
width. The work was continued in [9] and [10], but the general question of finding
extremal, non-central sections/slabs is still open.

In Sect. 2, below, we describe the extremal lengths of intersection between Q_n and a line ℓ that is a specified distance $t > 0$ away from the origin. Next, in Sect. 3, we provide a partial solution to non-central hyperplane sections problem, by finding extremal hyperplane slices for $t > \frac{\sqrt{n-1}}{2}$. In addition, we also describe the extremal intersection with slabs in this case. Although the methods of solution are rather cumbersome, we hope that the data could be useful to help predict the results of other related problems.

2 One Dimensional Slices

2.1 The Case of $\mathbb{R}^2$

Theorem 1. *For a line ℓ that is distance $t \in [0, 1/\sqrt{2}]$ away from the origin, we have that*

$$\min_{\ell \subset \mathbb{R}^2, d(\ell, 0) = t} |Q_2 \cap \ell| = \begin{cases} 1, & t \in [0, \frac{1}{2}(\sqrt{2}-1)], \\ \sqrt{2} - 2t, & t \in [\frac{1}{2}(\sqrt{2}-1), \frac{1}{2}], \\ 0, & t \in (\frac{1}{2}, \frac{1}{\sqrt{2}}], \end{cases}$$

and

$$\max_{\ell \subset \mathbb{R}^2, d(\ell, 0) = t} |Q_2 \cap \ell| = \begin{cases} \frac{2}{\sqrt{2-(2t)^2}+2t}, & t \in [0, \frac{1}{2}], \\ \frac{1}{2}\left(2t - \sqrt{4t^2 - 1}\right), & t \in (\frac{1}{2}, \frac{3}{4}\frac{1}{\sqrt{2}}], \\ \sqrt{2} - 2t, & t \in (\frac{3}{4}\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}] \end{cases} \cdot$$

Proof. The $\mathbb{R}^2$ case is done by direct calculations, which are very similar to the computation of extremal slab provided in [3]. For the sake of completeness we will sketch it here.

Let ℓ be a line tangent to $t\mathbb{S}^1$. Take the standard polar parametrization and let $(t\cos\theta, t\sin\theta)$ be the tangent point of ℓ and $t\mathbb{S}^1$, where θ is the angle from the positive x-axis. Thus, ℓ is defined by an equation $x\cos\theta + y\sin\theta = t$.

Due to the symmetries of the cube, we need only consider $\theta \in [0, \frac{\pi}{4}]$. We note that for $t > \frac{1}{2}$, we have the trivial result for the minimal case $|Q_2 \cap \ell| = 0$ when we take $\theta = 0$. We will start with $t \le \frac{1}{2}$ and will find the maximal interval for $t > \frac{1}{2}$ in the end.

For $\theta \in [0, \frac{\pi}{4}]$, ℓ must intersect ∂Q_2 by the edge $\{|x| \le \frac{1}{2}, y = \frac{1}{2}\}$. We notice that ℓ will intersect the opposite edge of Q_2 (i.e. $\{|x| \le \frac{1}{2}, y = -\frac{1}{2}\}$) when $\theta \in [0, \frac{\pi}{4} - \arcsin(\sqrt{2}t)]$, yielding $|\ell \cap Q_2| = \sec\theta$, which is an increasing function on the prescribed interval. On the other hand, ℓ will intersect $\{x = \frac{1}{2}, |y| \le \frac{1}{2}\}$ edge of Q_2, when $\theta \in [\frac{\pi}{4} - \arcsin(\sqrt{2}t), \frac{\pi}{4}]$, yielding

$$f_t(\theta) = |Q_2 \cap \ell| = |(\frac{t - \frac{1}{2}\sin\theta}{\cos\theta}, \frac{1}{2}) - (\frac{1}{2}, \frac{t - \frac{1}{2}\cos\theta}{\sin\theta})|$$

$$= \frac{|t - \frac{1}{2}\sin\theta - \frac{1}{2}\cos\theta|}{\sin\theta\cos\theta}$$

$$= \sqrt{2}\frac{\sin\left(\theta + \frac{\pi}{4}\right) - \sqrt{2}t}{\sin 2\theta}.$$

Next we need to compute $f_t'(\theta)$:

$$f_t'(\theta) = \sqrt{2}\frac{\cos\left(\theta + \frac{\pi}{4}\right)\sin 2\theta - 2\cos 2\theta(\sin\left(\theta + \frac{\pi}{4}\right) - \sqrt{2}t)}{\sin^2 2\theta}$$

$$= \sqrt{2}\frac{\cos\left(\theta + \frac{\pi}{4}\right)\sin 2\theta - 2\sin(\frac{\pi}{2} + 2\theta)(\sin\left(\theta + \frac{\pi}{4}\right) - \sqrt{2}t)}{\sin^2 2\theta}$$

$$= \frac{\sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right)}{\sin^2 2\theta}\left[\sin 2\theta - 4\sin(\frac{\pi}{4} + \theta)(\sin\left(\theta + \frac{\pi}{4}\right) - \sqrt{2}t)\right]$$

$$= \frac{\sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right)}{\sin^2 2\theta}\left[-\cos(\frac{\pi}{2} + 2\theta) - 4\sin^2(\frac{\pi}{4} + \theta) + 4\sqrt{2}t\sin\left(\theta + \frac{\pi}{4}\right)\right]$$

$$= \frac{\sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right)}{\sin^2 2\theta}\left[-2\sin^2(\frac{\pi}{4} + \theta) + 4\sqrt{2}t\sin\left(\theta + \frac{\pi}{4}\right) - 1\right].$$

The last expression is nonpositive for $t \in [0, 1/2]$ (indeed, $32t^2 - 8 \leqslant 0$). Thus, $f_t(\theta)$ is a decreasing function on the required interval having the following critical values:

- For $t \in [0, \frac{1}{2}(\sqrt{2} - 1)]$, the minimum value of $|Q_2 \cap \ell| = 1$ occurs when $\theta = 0$.
- For $t \in [\frac{1}{2}(\sqrt{2} - 1), \frac{1}{2}]$, the minimum of $|Q_2 \cap \ell| = \sqrt{2} - 2t$ occurs when $\theta = \frac{\pi}{4}$.
- For $t \in (\frac{1}{2}, \frac{1}{\sqrt{2}}]$, the minimum of $|Q_2 \cap \ell| = 0$ occurs when $\theta = 0$.
- For $t \in [0, \frac{1}{2}]$, the maximum of $|Q_2 \cap \ell| = \frac{2}{\sqrt{2 - (2t)^2} + 2t}$ occurs when $\theta = \frac{\pi}{4} -$ arcsin $t\sqrt{2}$, i.e. when ℓ goes through a vertex of the square to an edge containing the opposite vertex.

Finally, for $t \in (\frac{1}{2}, \frac{1}{\sqrt{2}}]$, we notice that ℓ will intersect Q_2 for $\theta \in [\arcsin(\sqrt{2}t) - \frac{\pi}{4}, \frac{\pi}{4}]$ (i.e. $\sin\left(\theta + \frac{\pi}{4}\right) \in [\sqrt{2}t, 1]$) and must intersect two adjacent edges, yielding

$$f_t(\theta) = |Q_2 \cap \ell| = \sqrt{2}\frac{\sin\left(\theta + \frac{\pi}{4}\right) - \sqrt{2}t}{\sin 2\theta}.$$

From above we get

$$f_t'(\theta) = \frac{\sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right)}{\sin^2 2\theta}\left[-2\sin^2(\frac{\pi}{4} + \theta) + 4\sqrt{2}t\sin\left(\theta + \frac{\pi}{4}\right) - 1\right],$$

which is positive if $\sin\left(\theta + \frac{\pi}{4}\right) \in [\sqrt{2}t - \frac{1}{2}\sqrt{8t^2 - 2}, \sqrt{2}t + \frac{1}{2}\sqrt{8t^2 - 2}]$ and negative otherwise. Thus, $f_t(\theta)$ achieves the maximal value at

$$\sin\left(\theta + \frac{\pi}{4}\right) = \sqrt{2}t + \frac{1}{2}\sqrt{8t^2 - 2} \quad \text{if} \quad \sqrt{2}t + \frac{1}{2}\sqrt{8t^2 - 2} \leqslant 1,$$

and at $\theta = \pi/4$ otherwise.

We notice that $\sqrt{2}t + \frac{1}{2}\sqrt{8t^2 - 2} \leqslant 1$ gives $t \in (\frac{1}{2}, \frac{3}{4}\frac{1}{\sqrt{2}}]$. From $\sin\left(\theta + \frac{\pi}{4}\right) = \sqrt{2}t + \frac{1}{2}\sqrt{8t^2 - 2}$ we get

$$\sin 2\theta = -\cos(\frac{\pi}{2} + 2\theta) = 2\sin^2(\frac{\pi}{4} + \theta) - 1 = 8t^2 + 2\sqrt{2}t\sqrt{8t^2 - 2} - 2$$

$$= \sqrt{8t^2 - 2}\left(\sqrt{8t^2 - 2} + 2\sqrt{2}t\right).$$

Thus, the length of maximal section, for this case, is

$$|Q_2 \cap \ell| = \sqrt{2}\frac{\frac{1}{2}\sqrt{8t^2 - 2}}{\sqrt{8t^2 - 2}\left(\sqrt{8t^2 - 2} + 2\sqrt{2}t\right)}$$

$$= \frac{1}{2\left(\sqrt{4t^2 - 1} + 2t\right)} = \frac{1}{2}\left(2t - \sqrt{4t^2 - 1}\right).$$

Finally, if $t \in [\frac{3}{4}\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$, then the maximal $|Q_2 \cap \ell|$ occurs when $\theta = \frac{\pi}{4}$, and the length, as mentioned before, is $\sqrt{2} - 2t$. $\qquad\square$

2.2 Minimal Case

Theorem 2. *For a line ℓ that is distance t away from the origin, we have that*

$$\min_{\ell \subset \mathbb{R}^n, d(\ell, 0)=t} |Q_n \cap \ell| = \min_{\ell \subset \mathbb{R}^2, d(\ell, 0)=t} |Q_2 \cap \ell| = \begin{cases} 1, & t \in [0, \frac{1}{2}(\sqrt{2} - 1)] \\ \sqrt{2} - 2t, & t \in [\frac{1}{2}(\sqrt{2} - 1), \frac{1}{2}] \\ 0, & t \in (\frac{1}{2}, \infty) \end{cases}.$$

Proof. Our goal is to show that $\mathbb{R}^n$ case is equivalent to the case of the square. We again remark the case $t > 1/2$ is trivial. We also note that by Brunn's theorem (see [6] or [5])

$$\min_{\ell \subset \mathbb{R}^n, d(\ell,0)=t} |\ell \cap Q_n|$$

is a nonincreasing function for $t \geqslant 0$.

Let $[a,b] = Q_n \cap \ell$ be segment of minimal length among lines ℓ which are at distance $t \leqslant 1/2$ from the origin. Notice that from $d([a,b],0) \leqslant 1/2$, we get that $[a,b]$ can not belong to an $(n-2)$-dimensional face of the cube.

Note that $[a,b]$ must pass through two distinct faces of Q_n. By symmetry, we can assume that one face is given by $\{(\frac{1}{2}, x_2, x_3, \ldots, x_n) : |x_i| \leq \frac{1}{2}\}$. Then the other face is either opposite to the first, i.e. $\{(-\frac{1}{2}, x_2, x_3, \ldots, x_n) : |x_i| \leq \frac{1}{2}\}$, or by symmetry again, we can reorient so that it is $\{(x_1, \frac{1}{2}, x_3, x_4, \ldots, x_n) : |x_i| \leq \frac{1}{2}\}$.

Consider the orthogonal projection of Q_n and $[a,b]$ onto the $x_1 x_2$-coordinate plane. The cube will simply project down to Q_2, and $[a,b]$ will project to $[a',b']$, with $|b'-a'| \leqslant |b-a|$. Notice that $[a',b']$ passes either through opposite sides of Q_2 (the first case above), or adjacent sides of Q_2 (the second case). We notice that $[a',b']$ must be a distance $t' \leq t$ away from the origin. Since we've already established that $\min |Q_2 \cap \ell|$ is a non-increasing function of t, we get that there is a line ℓ contained in $x_1 x_2$-coordinate plane and tangent to $t\mathbb{S}^{n-1}$ such that $|b'-a'| \geq |Q_2 \cap \ell|$.

Finally, for $[a,b]$ to be minimal, we must have equalities in above inequalities, i.e.

$$|b-a| = |b'-a'| \text{ and } t = t',$$

which immediately gives that $[a,b]$ must belong to a two dimensional subspace parallel to a two dimensional coordinate plane. □

2.3 Maximal Case

For the maximal case, we will first describe our results in geometric terms, providing numerical answers at the end of this section in Theorem 4.

Theorem 3. *For a line $\ell \subset \mathbb{R}^n$ that is distance $t \in [0, \frac{1}{2}]$ away from the origin, the maximal length $|Q_n \cap \ell|$ occurs when ℓ travels through a vertex of Q_n and passes through an edge containing the opposite vertex. For $t \in [\frac{1}{2}, \frac{\sqrt{n}}{2}]$, the maximal ℓ is confined to a face of Q_n.*

The following elementary lemmas are needed for the proof of the Theorem 3:

Lemma 1. *Consider a convex bounded polytope $P \subset \mathbb{R}^m$, let ℓ be a line in $\mathbb{R}^m$. Then the maximal length $|P \cap \ell|$ occurs only if ℓ is passing through two vertices of P.*

Proof. The lemma is an immediate consequence of the property of convex functions over convex domain. Indeed, assume that $[a,b] = P \cap \ell$ is maximal and that a is not a vertex of P. Then, $f(x) = |x-b|$ is a convex function and thus must achieve its maximal value at an extremal point of P. Moreover, it is also easy to show that a vertex is the only possible point when the maximum can be attained in this case. □

Remark 1. Note that the above lemma is not true if we require the lines to pass through a fixed point in P and would not ask for P to be symmetric, with respect to this fixed point.

Lemma 2. *Let $Q \subset \mathbb{R}^n$ be a convex symmetric polytope and let $\ell \subset \mathbb{R}^n$ be a line that is distance $t < \frac{1}{2}\mathrm{diam}(Q)$ from the origin. The maximal length of $[a, b] = Q \cap \ell$ occurs when a, b belong to the edges of Q.*

Proof. Assume that $[a, b] = Q \cap \ell$, is maximal, but at least one of the endpoints a, b does not belong to an edge of Q. Let $\ell \cap t\mathbb{S}^{n-1} = t\xi$, where $\xi \in \mathbb{S}^{n-1}$. Consider an affine hyperplane $H(\xi, t) = \{x \in \mathbb{R}^n : x \cdot \xi = t\}$ containing ℓ and tangent to $t\mathbb{S}^{n-1}$ at $t\xi$. Let $P = Q \cap H$, then $[a, b]$ is not a maximal segment inside P, otherwise a, b would be vertices of P and thus would each belong to edges of Q, by Lemma 1. Assume $[x, y]$ is a maximal segment inside P. If $t\xi \in [x, y]$ then this contradicts the maximality of $[a, b]$. Otherwise $[x, y]$ is outside of $t\mathbb{S}^{n-1}$. Next we consider a two-dimensional subspace M containing $[x, y]$. We note that $Q \cap M$ is a convex symmetric body and $[x, y]$ is a one-dimensional section of $Q \cap M$ at distance greater than t from the origin. Brunn's theorem tells us that there is a one-dimensional slice of $Q \cap M$ (parallel to $[x, y]$) of length no less than the length of $[x, y]$ and tangent to $t\mathbb{S}^1$, which again contradicts the maximality of $[a, b]$. $\qquad \square$

Proof of Theorem 3: It follows from Lemma 2 that the segment at distance t of maximal length must pass through the edges of Q_n.

Case 0: The two edges share a vertex. Then the maximal segment must belong to a face of Q_n and $t \geq \frac{1}{2}$.

Case I: The two edges are parallel.

We first note that if $t < 1/2$ then the parallel edges must be symmetric to each other with respect to the origin (if this is not the case then the edges will be contained in a face of Q_n and then any segment connecting them will be at the distance greater or equal then $1/2$ from the origin).

We consider the intersection of Q_n and the two-dimensional subspace defined by these edges. The result is that the line ℓ, containing the maximal segment, passes through two opposite of edges of side length 1 in a 1 by $\sqrt{n-1}$ rectangle, and is tangent to a circle of radius t that has its center at the center of the rectangle. We can set this rectangle on the coordinate axes so that the center of the rectangle and the circle is at the origin, and the rectangle has vertices $(\pm\frac{1}{2}, \pm\frac{\sqrt{n-1}}{2})$, ℓ thus intersects the boundary of this rectangle at the points $(x_0, \frac{\sqrt{n-1}}{2})$ and $(y_0, -\frac{\sqrt{n-1}}{2})$.

As with our two-dimensional minimum problem, we can use basic calculus to solve for the optimal length. Again, let θ be the angle between the positive x-axis and the line perpendicular to ℓ. We need only consider $\theta \in \left[0, \arcsin\frac{1}{\sqrt{n}} - \arcsin\frac{2t}{\sqrt{n}}\right]$. Our length of intersection is a function of θ, $|Q_n \cap \ell| = \sqrt{n-1}\sec\theta$. We find that the maximum of $|Q_n \cap \ell|$ occurs at $\theta = \arcsin\frac{1}{\sqrt{n}} - \arcsin\frac{2t}{\sqrt{n}}$ and the maximum value is

$$|Q_n \cap \ell| = \frac{n\sqrt{n-1}}{\sqrt{(n-1)(n-4t^2)} + 2t}. \tag{1}$$

We also note that the maximal ℓ will intersect the boundary of the rectangle at the vertex (which is also a vertex of Q_n), which finishes the proof of this case.

Finally, we note that if $t \geqslant 1/2$, then the segment connecting parallel edges must belong to a face of Q_n.

Case II: The two edges are skew.

We first notice that if edges are contained in the same face then, clearly, $t \geqslant 1/2$. Thus, we only need to consider the case when two edges are not contained in the same face, which gives that the distance between them is $\sqrt{n-2}$. Note that t could be greater than $1/2$ in this case, thus we will need to make sure that the maximal edge will be forced to the face of Q_n for $t \geqslant 1/2$ and endpoints belong to the skew edges.

We remark that in this case the edges must belong to a three-dimensional subspace of $\mathbb{R}^n$. Consider the three-dimensional section of Q_n defined by those edges. The result is a 1 by 1 by $\sqrt{n-2}$ rectangular prism. The section will contain the origin and its intersection with $t\mathbb{S}^{n-1}$ is $t\mathbb{S}^2$. Embedding the prism in to $\mathbb{R}^3$ with its center at the origin, we get a rectangular prism with vertices $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{\sqrt{n-2}}{2})$. The line ℓ intersects the boundary of the prism at $(x_0, \frac{1}{2}, \frac{\sqrt{n-2}}{2})$, and at $(-\frac{1}{2}, y_0, -\frac{\sqrt{n-2}}{2})$ (or at $(\frac{1}{2}, y_0, -\frac{\sqrt{n-2}}{2})$), which, by the symmetry, is an equivalent case).

We immediately notice that if $t > 1/\sqrt{2}$, this case becomes impossible; there is no segment connecting the aforementioned edges of our prism which is also tangent to $t\mathbb{S}^2$. In fact, all such segments will intersect $t\mathbb{S}^2$. Thus, we concentrate on the case when $t \in [0, \frac{1}{\sqrt{2}}]$.

Let $D(x, y)$ be the distance between $(x, \frac{1}{2}, \frac{\sqrt{n-2}}{2})$ and $(\frac{1}{2}, y, -\frac{\sqrt{n-2}}{2})$. We use the method of Lagrange multipliers to optimize the function

$$D^2 = (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 + n - 2$$

with the constraints that $x, y \in [-\frac{1}{2}, \frac{1}{2}]$ and $(x, \frac{1}{2}, \frac{\sqrt{n-2}}{2}), (\frac{1}{2}, y, -\frac{\sqrt{n-2}}{2}) \in \ell$, where ℓ is a line at distance t away from the origin. That is,

$$\frac{|(x, \frac{1}{2}, \frac{\sqrt{n-2}}{2}) \times (\frac{1}{2}, y, -\frac{\sqrt{n-2}}{2})|}{D} = t,$$

or

$$|(x, \frac{1}{2}, \frac{\sqrt{n-2}}{2}) \times (\frac{1}{2}, y, -\frac{\sqrt{n-2}}{2})|^2 - D^2t^2 = 0. \tag{2}$$

To find possible extremal points, we consider the function

$$F_\lambda = D^2 - \lambda \left(|(x, \frac{1}{2}, \frac{\sqrt{n-2}}{2}) \times (\frac{1}{2}, y, -\frac{\sqrt{n-2}}{2})|^2 - D^2 t^2 \right)$$

$$= (1 + \lambda t^2) \left((x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 + n - 2 \right)$$

$$- \lambda \left(\frac{n-2}{4}(y + \frac{1}{2})^2 + \frac{n-2}{4}(x + \frac{1}{2})^2 + (xy - \frac{1}{4})^2 \right).$$

Computing partial derivatives, we get

$$\frac{\partial F_\lambda}{\partial x} = 2(1 + \lambda t^2)(x - \frac{1}{2}) - \lambda \left(\frac{n-2}{2}(x + \frac{1}{2}) + 2y(xy - \frac{1}{4}) \right),$$

$$\frac{\partial F_\lambda}{\partial y} = 2(1 + \lambda t^2)(y - \frac{1}{2}) - \lambda \left(\frac{n-2}{2}(y + \frac{1}{2}) + 2x(xy - \frac{1}{4}) \right).$$

We now set $\frac{\partial F_\lambda}{\partial x} = \frac{\partial F_\lambda}{\partial x} = 0$ and solve $(y - \frac{1}{2})\frac{\partial F_\lambda}{\partial x} - (x - \frac{1}{2})\frac{\partial F_\lambda}{\partial x} = 0$, which, after simplification, becomes

$$(y - x) \left(\frac{n-2}{2} + (xy - \frac{1}{4})(2(y + x) - 1) \right) = 0.$$

We notice that $(xy - \frac{1}{4})(2(y + x) - 1) > -\frac{1}{2}$, for $x, y \in [-\frac{1}{2}, \frac{1}{2}]$. Thus, the only possible solution of the above equation is $x = y$. In this case Eq. (2) becomes

$$\frac{n-2}{2}(x + \frac{1}{2})^2 + (x^2 - \frac{1}{4})^2 = t^2 \left(2(x - \frac{1}{2})^2 + n - 2 \right)$$

or

$$\frac{(x + \frac{1}{2})^2}{2} \left(n - 2 + 2(x - \frac{1}{2})^2 \right) = t^2 \left(2(x - \frac{1}{2})^2 + n - 2 \right)$$

and using $|x| \leq \frac{1}{2}$ and $t \in [0, \frac{1}{\sqrt{2}}]$ we get one possible critical value $x = -\frac{1}{2} + \sqrt{2}t$ and the corresponding segment length

$$D = \sqrt{2(1 - \sqrt{2}t)^2 + n - 2} = \sqrt{4t^2 - 4\sqrt{2}t + n}. \tag{3}$$

Now we will show that Eq. (3) does not represent the global maximum. For $t \in [0, \frac{1}{2}]$ we should compare Eq. (3) with the parallel edges case Eq. (1). We will show that Eq. (1) $\geq$ Eq. (3). To do so we square both quantities and take the difference. Thus, we need to show that

$$n^2(n-1) - (4t^2 - 4\sqrt{2}t + n)(4t^2 + 4t\sqrt{(n-1)(n-4t^2)}) + (n-1)n - 4t^2(n-1)) > 0.$$

We simplify the above inequality and divide by $-4t$ and now need to show that

$$(4t^2 - 4\sqrt{2}t + n)\sqrt{(n-1)(n-4t^2)} + nt - 4\sqrt{2}n(n-1)$$
$$-4(n-2)t^3 + 4\sqrt{2}(n-2)t^2 < 0.$$

Using that $n \geq 3$ and $t \in [0, \frac{1}{2}]$ and writing the maximal value for each of the above summands, it is enough to show that

$$n\sqrt{(n-1)n} + \frac{n}{2} - 4\sqrt{2}n(n-1) + \sqrt{2}(n-2) < 0,$$

which is true for $n \geq 3$.

Finally Eq. (3) must be compared with the boundary values at $x = -\frac{1}{2}$ and $x = \frac{1}{2}$ (or the symmetric cases $y = -\frac{1}{2}$ and $y = \frac{1}{2}$). We notice that we get exactly the same answer as in parallel edges case when $x = -\frac{1}{2}$ (because we get a vertex and an edge containing the opposite vertex), again with the restriction that $t \in [0, \frac{1}{2}]$.

Next we treat the case $x = \frac{1}{2}$. Note that in this case $Q_n \cap \ell$ will belong to a face of our prism (and thus, to a face of Q_n) so we know that $t \geq 1/2$. We claim that this is the maximal case for $t \geq 1/2$. To prove this claim we only need to show that the length of the segment we get here is greater than Eq. (3), with restriction $t \in [1/2, 1/\sqrt{2}]$.

To calculate the length of $Q_n \cap \ell$, we consider the rectangular 1 by $\sqrt{n-2}$ face of our prism that contains $Q_n \cap \ell$. We note that this face intersects $t\mathbb{S}^2$ and creates $r\mathbb{S}^2$, where $r = \sqrt{t^2 - \frac{1}{4}} \leq 1/2$, using a calculation that is similar to the $\mathbb{R}^2$ case and computations from Case I [i.e., we substitute $n-1$ instead of n in Eq. (1)]. This yields

$$\frac{(n-1)\sqrt{n-2}}{\sqrt{(n-2)(n-1-4r^2)} + 2r} = \frac{(n-1)\sqrt{n-2}}{\sqrt{(n-2)(n-4t^2)} + 2\sqrt{t^2 - \frac{1}{4}}}. \tag{4}$$

We claim that above is greater than Eq. (3). To show this we must verify that

$$(n-1)^2(n-2) - (2(1-\sqrt{2}t)^2 + n - 2)(\sqrt{(n-2)(n-4t^2)} + \sqrt{4t^2-1})^2 \geq 0,$$

which, after simplification, becomes

$$(1-\sqrt{2}t)^2\left((n-1)^2 - 2 - 4(n-3)t^2 + 2\sqrt{4t^2-1}\sqrt{(n-2)(n-4t^2)}\right)$$
$$-(n-2)\left(1 + 2(n-3)t^2 - \sqrt{4t^2-1}\sqrt{(n-2)(n-4t^2)}\right) \leq 0,$$

and after division by $(1 - \sqrt{2}t)^2$ becomes

$$\left((n-1)^2 - 2 - 4(n-3)t^2 + 2\sqrt{4t^2 - 1}\sqrt{(n-2)(n-4t^2)}\right)$$

$$-\frac{(n-2)(n-1)^2(\sqrt{2}t + 1)^2}{\left(1 + 2(n-3)t^2 + \sqrt{4t^2 - 1}\sqrt{(n-2)(n-4t^2)}\right)} \leqslant 0.$$

Further simplification shows that it is enough to prove that

$$\left(1 + 2(n-3)t^2 + \sqrt{4t^2 - 1}\sqrt{(n-2)(n-4t^2)}\right)$$

$$-2(2t^2 - 1)^2 - (n-2)(\sqrt{2}t + 1)^2 \leqslant 0.$$

We notice that $(4t^2 - 1)(n - 4t^2) \leqslant (n-2)$ for $t \in [\frac{1}{2}, \frac{1}{\sqrt{2}}]$ and $n \geqslant 3$. Thus, it is enough to show that

$$\left(2(n-3)t^2 + n - 1\right) - 2(2t^2 - 1)^2 - (n-2)(\sqrt{2}t + 1)^2 \leqslant 0,$$

which becomes trivial after simplification. □

We can now apply Theorem 3 [as we did with Eq. (4)] to $(n-m)$-dimensional faces of Q_n (for $m = 0, 1, \ldots, n-3$) and Theorem 1 to the two-dimensional face to give us the general result:

Theorem 4. *Let $m = 0, 1, \ldots, n-2$. Then for a line ℓ that is distance $t \in \left[\frac{\sqrt{m}}{2}, \frac{\sqrt{m+1}}{2}\right]$ away from the origin, we have that*

$$\max_{\ell \subset \mathbb{R}^n, d(\ell,0)=t} |Q_n \cap \ell| = \frac{[n-m]\sqrt{n-m-1}}{\sqrt{(n-m-1)[n-(2t)^2]} + 2\sqrt{t^2 - \frac{m}{4}}}$$

and

$$\max_{\ell \subset \mathbb{R}^n, d(\ell,0)=t} |Q_n \cap \ell| = \begin{cases} \sqrt{t^2 - \frac{n-2}{4}} - \sqrt{t^2 - \frac{n-1}{4}}, & t \in (\frac{\sqrt{n-1}}{2}, \frac{1}{4}\sqrt{\frac{8n-7}{2}}], \\ \sqrt{2} - 2\sqrt{t^2 - \frac{n-2}{4}}, & t \in (\frac{1}{4}\sqrt{\frac{8n-7}{2}}, \frac{\sqrt{n}}{2}] \end{cases}.$$

Proof. In case $0 \leqslant t \leqslant \frac{1}{2}$, Theorem 3 gives

$$\max_{\ell \subset \mathbb{R}^n, d(\ell,0)=t} |Q_n \cap \ell| = \frac{n\sqrt{n-1}}{\sqrt{(n-1)(n-4t^2)} + 2t}.$$

Now we consider the case $t > 1/2$. Let $t \in [\frac{\sqrt{m}}{2}, \frac{\sqrt{m+1}}{2}]$ for $m = 1, \ldots, n-1$. Then the second part of Theorem 3 says that a maximal ℓ should be confined to a

face F of Q_n, and moreover there exist two edges E_1, E_2 of Q_n such that the line ℓ contains two points on each E_1, E_2. Choose a face F in the following way:

1. If E_1, E_2 meet to a vertex, then consider the two-dimensional face F of Q_n containing E_1, E_2
2. Otherwise, then consider the smallest (in dimension) face of Q_n containing E_1, E_2 whose center is the midpoint of two vertices in E_1, E_2.

The dimension of F is equal to $n - m$, and the center of F is $\sqrt{m}/2$ far away from the origin. Thus, the line segment $Q_n \cap \ell$ is contained in F and it is $r = \sqrt{t^2 - m/4}$ away from the center of F. In addition the line ℓ passes through a $(n - m)$-dimensional cube, $Q_n \cap F$, whose center is the same as F.

Next, we use Theorem 3 for the $(n - m)$-dimensional cube and the line which is $r = \sqrt{t^2 - m/4} \in [0, 1/2]$ away from the center of the cube. Then

$$
\max_{\ell \subset \mathbb{R}^n, d(\ell, 0) = t} |Q_n \cap \ell| = \max_{\ell' \subset \mathbb{R}^{n-m}, d(\ell', 0) = r} |Q_{n-m} \cap \ell'|
$$

$$
= \frac{(n - m)\sqrt{(n - m) - 1}}{\sqrt{(n - m - 1)(n - m - 4r^2)} + 2r}
$$

$$
= \frac{(n - m)\sqrt{n - m - 1}}{\sqrt{(n - m - 1)(n - 4t^2)} + 2\sqrt{t^2 - m/4}}.
$$

For the case $m = n - 1$, i.e., $t \in [\frac{\sqrt{n-1}}{2}, \frac{\sqrt{n}}{2}]$, apply Theorem 1 with $r = \sqrt{t^2 - (n - 2)/4}$. $\qquad\square$

3 Hyperplane and Slab Sections at Distance $t > \frac{\sqrt{n-1}}{2}$

In this section we present a partial solution to the problem of finding extremal sections of Q_n by affine hyperplanes tangent to $t\mathbb{S}^{n-1}$ and slabs of width $2t$. Let $H(\xi, t) = \{x \in \mathbb{R}^n : x \cdot \xi = t\}$ be a hyperplane with a normal vector $\xi \in \mathbb{S}^{n-1}$ at the distance $t > 0$ from the origin.

Theorem 5. *Fix $t \in (\frac{\sqrt{n-1}}{2}, \frac{\sqrt{n}}{2}]$. Then, for $n \geqslant 3$,*

$$
|Q_n \cap H(\xi, t)| \leq \frac{n^{\frac{n}{2}}}{(n - 1)!} \left(\frac{\sqrt{n}}{2} - t \right)^{n-1}, \text{ for all } \xi \in \mathbb{S}^{n-1}
$$

with equality occurring when $\xi = \left(\pm\frac{1}{\sqrt{n}}, \ldots, \pm\frac{1}{\sqrt{n}} \right)$.

Remark 2. It is interesting to compare Theorems 1 and 5 and to see the difference between maximal sections of Q_2 and Q_n, $n \geqslant 3$.

Proof. Fix $t \in (\frac{\sqrt{n-1}}{2}, \frac{\sqrt{n}}{2}]$. The main idea of the proof is to notice the geometric implications of this restriction. Indeed, $\frac{\sqrt{n}}{2}$ is the distance from the origin to a vertex of Q_n and $\frac{\sqrt{n-1}}{2}$ is the distance from the origin to an edge of Q_n. Thus, our hyperplane $H(\xi, t)$ cannot separate an entire edge from the origin, nor can it separate two vertices from the origin.

Each vertex is the intersection of precisely n edges, so $H(\xi, t)$ cannot intersect more than n edges. Furthermore, since $H(\xi, t)$ cannot be contained in an $(n-1)$-dimensional face, then it must intersect exactly n edges and separate one vertex from the rest of Q_n (provided that it intersects the interior of Q_n). By symmetry, we may assume that the separated vertex is $\mathbf{v} = (\frac{1}{2}, \ldots, \frac{1}{2})$.

Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \in \mathbb{R}^n$ be the points of intersection of $H(\xi, t)$ with each respective edge extending from $\mathbf{v}$, such that $\mathbf{a}_i$ belongs to the edge parallel to coordinate vector $\mathbf{e}_i$, i.e. all coordinates of $\mathbf{a}_i$ are $\frac{1}{2}$, except for the i-th coordinate, which is $\frac{1}{2} - a_i$, where $a_i = |\mathbf{a}_i - \mathbf{v}| > 0$. Moreover, $a_i < \frac{1}{2}$ because $t\mathbb{S}^{n-1}$ intersects all edges of Q_n.

We notice that $H(\xi, t)$ cuts a simplex $S(\xi, t)$ from Q_n, with an apex $\mathbf{v}$ and a base which is the convex hull of $\{\mathbf{a}_i\}_{i=1}^n$. We also note that $\mathbf{v}$ is an "orthogonal corner" of $S(\xi, t)$, i.e. all facets containing $\mathbf{v}$ are pairwise orthogonal.

Let F be the face of $S(\xi, t)$ opposite from $\mathbf{v}$ and let F_i, for $i = 1, 2, \ldots, n$, be the face that is opposite from the vertex $\mathbf{a}_i$. We can compute $|Q_n \cap H(\xi, t)| = |F|$ as a function of the a_i (instead of ξ and t). First note that

$$|F_i| = \frac{1}{(n-1)!} \prod_{\substack{j=1 \\ j \neq i}}^n a_j.$$

Using a classical n-dimensional analog of the Pythagorean theorem (see, for example, [1]) we claim that $|F|^2 = \sum_{i=1}^n |F_i|^2$ and thus,

$$|H(\xi, t) \cap Q_n|^2 = |F|^2 = \left(\frac{1}{(n-1)!}\right)^2 \sum_{i=1}^n \prod_{\substack{j=1 \\ j \neq i}}^n a_j^2.$$

We want to maximize $|F|$ subject to the constraint that the unique hyperplane containing F is at distance t from the origin. In order to do this, we should express the distance of the (unique) hyperplane containing $\mathbf{a}_1, \ldots, \mathbf{a}_n$ from the origin in terms of $a_1, \ldots, a_n$. This distance can be expressed as $\mathbf{a}_1 \cdot \xi$, so it suffices to find ξ in terms of $\{a_i\}_{i=1}^n$.

The normal vector ξ can be characterized as the unique vector satisfying $|\xi| = 1$ and $\xi \cdot \mathbf{a}_i = t$ for $i = 1, \ldots, n$. Calculation shows:

$$\xi = \frac{(\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n})}{\sqrt{\sum_{i=1}^n \frac{1}{a_i^2}}}.$$

Thus

$$\mathbf{a}_1 \cdot \xi = \frac{\left(\sum_{i=1}^{n} \frac{1}{a_i}\right) - 2}{2\sqrt{\sum_{i=1}^{n} \frac{1}{a_i^2}}}.$$

We will now apply the method of Lagrange multipliers, to optimize the function

$$f(a_1, \ldots, a_n) := \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j \neq i}}^{n} a_j^2 = \prod_{j=1}^{n} a_j^2 \sum_{i=1}^{n} \frac{1}{a_i^2},$$

with the constraint

$$g(a_1, \ldots, a_n) := \sum_{i=1}^{n} \frac{1}{a_i} - 2t\sqrt{\sum_{i=1}^{n} \frac{1}{a_i^2}} = 2. \tag{5}$$

Function f is differentiable everywhere on $\mathbb{R}^n$ and g is differentiable everywhere on $\mathbb{R}^n \setminus \{0\}$, moreover, ∇g is non-zero. From now on, we will consider f and g as functions from $(0, \frac{1}{2})^n$ to $\mathbb{R}$. Under these conditions, every global extremum $\mathbf{c}$ of f with constraint $g = 2$ satisfies $(\nabla f - \lambda \nabla g)(\mathbf{c}) = 0$ for some $\lambda \in \mathbb{R}$.

Let $F_\lambda := f - \lambda g$. Then,

$$\frac{\partial F_\lambda}{\partial a_k} = 2a_k \prod_{\substack{j=1 \\ j \neq k}}^{n} a_j^2 \sum_{i=1}^{n} \frac{1}{a_i^2} - \frac{2}{a_k^3} \prod_{j=1}^{n} a_j^2 - \lambda \left(-\frac{1}{a_k^2} + 2t \frac{\frac{1}{a_k^3}}{\sqrt{\sum_{i=1}^{n} \frac{1}{a_i^2}}} \right).$$

To find possible extremal values we need to solve $\frac{\partial F_\lambda}{\partial a_k} = 0$, for all $k = 1, \ldots, n$, which is:

$$2a_k \prod_{\substack{j=1 \\ j \neq k}}^{n} a_j^2 \sum_{i=1}^{n} \frac{1}{a_i^2} - \frac{2}{a_k^3} \prod_{j=1}^{n} a_j^2 - \lambda \left(-\frac{1}{a_k^2} + 2t \frac{\frac{1}{a_k^3}}{\sqrt{\sum_{i=1}^{n} \frac{1}{a_i^2}}} \right) = 0.$$

Multiplying the above equation by a_k, we get

$$2 \prod_{j=1}^{n} a_j^2 \sum_{i=1}^{n} \frac{1}{a_i^2} - \frac{2}{a_k^2} \prod_{j=1}^{n} a_j^2 - \lambda \left(-\frac{1}{a_k} + 2t \frac{\frac{1}{a_k^2}}{\sqrt{\sum_{i=1}^{n} \frac{1}{a_i^2}}} \right) = 0. \tag{6}$$

To find λ, we will sum up Eqs. (6) for $k = 1, \ldots, n$:

$$2(n-1) \prod_{j=1}^{n} a_j^2 \sum_{i=1}^{n} \frac{1}{a_i^2} - \lambda \left(-\sum_{i=1}^{n} \frac{1}{a_k} + 2t\sqrt{\sum_{i=1}^{n} \frac{1}{a_i^2}} \right) = 0.$$

Thus,

$$\lambda = \frac{2(n-1) \prod_{j=1}^{n} a_j^2 \sum_{i=1}^{n} \frac{1}{a_i^2}}{2t \sqrt{\sum_{i=1}^{n} \frac{1}{a_i^2} - \sum_{i=1}^{n} \frac{1}{a_k}}} = -(n-1) \prod_{j=1}^{n} a_j^2 \sum_{i=1}^{n} \frac{1}{a_i^2}, \tag{7}$$

with the last equality following from Eq. (5). Next, fix two different indices $k, m \in [1, \ldots, n]$. We subtract from Eq. (6) multiplied by $1/a_m^2$ the corresponding equation for m multiplied by $1/a_k^2$ to get

$$\left(2 \prod_{j=1}^{n} a_j^2 \sum_{i=1}^{n} \frac{1}{a_i^2} \right) \left(\frac{1}{a_m^2} - \frac{1}{a_k^2} \right) + \lambda \left(\frac{1}{a_k a_m^2} - \frac{1}{a_m a_k^2} \right) = 0.$$

Thus, we get that the possible extremal point either must satisfy $\frac{1}{a_m} - \frac{1}{a_k} = 0$ or must satisfy

$$\left(2 \prod_{j=1}^{n} a_j^2 \sum_{i=1}^{n} \frac{1}{a_i^2} \right) \left(\frac{1}{a_m} + \frac{1}{a_k} \right) + \frac{\lambda}{a_k a_m} = 0,$$

which is equivalent to

$$\left(2 \prod_{j=1}^{n} a_j^2 \sum_{i=1}^{n} \frac{1}{a_i^2} \right) (a_k + a_m) = -\lambda.$$

Finally, using Eq. (7) we get

$$a_k + a_m = \frac{n-1}{2}.$$

The above equality is only possible for $n = 2$ (indeed, $a_i \in (0, \frac{1}{2})$). Thus, the only critical point is one satisfying $a_1 = a_2 = \cdots = a_n$. This yields that $\xi = (\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}})$.

We must now find whether this critical point is actually a global maximum subject to $g = 2$. To do so we will partition $[0, \frac{1}{2}]^n$ into two pieces: one compact set C containing our critical point, and a set $U = [0, \frac{1}{2}]^n \setminus C$ on which f is small when constrained by $g = 2$. As f is continuous, $f|_C$ attains a (constrained) maximum on C subject to $g = 2$. If f subject to $g = 2$ is uniformly bounded on U by this same (constrained) maximum, then f reaches a global (constrained) maximum when $a_1 = a_2 = \cdots = a_n$.

We consider set $C = [\delta, \frac{1}{2} - \delta]^n$, for some small $\delta > 0$. Fixing $t > (n-1)/2$ and considering only $(a_1 \ldots, a_n)$ satisfying Eq. (5), it is easy to select δ such that

$a_i \leqslant 1/2 - \delta$ for all $i = 1, \ldots, n$. Thus if $(a_1, \cdots, a_n) \in U$, then at least one of the a_i must be less than δ, which gives:

$$f(a_1, \ldots, a_n) \leqslant \delta \frac{n}{4^n}.$$

Thus one can select $\delta > 0$ such that $f(a_1, \ldots, a_n) \leqslant f(\xi)$ for $\xi = (\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}})$, which would guarantee that $f(\xi)$ is the global maximum.

By plugging in $a := a_1 = \cdots = a_n$ into Eq. (5), we find that

$$a = \sqrt{n}(\frac{\sqrt{n}}{2} - t),$$

which yields that for $t \in (\frac{\sqrt{n-1}}{2}, \frac{\sqrt{n}}{2}]$,

$$\max_{\xi \in \mathbb{S}^{n-1}} |Q_n \cap H(\xi, t)| = \frac{1}{(n-1)!} \sqrt{\sum_{i=1}^{n} \prod_{\substack{j=1 \\ j \neq i}}^{n} a_j^2} = \frac{\sqrt{n}}{(n-1)!} a^{(n-1)}$$

$$= \frac{n^{\frac{n}{2}}}{(n-1)!} \left(\frac{\sqrt{n}}{2} - t\right)^{n-1}.$$

$\square$

The ideas used in the proof of Theorem 5 can also be used to give a characterization of minimal volume sections of Q_n by slabs with large width:

Theorem 6. *Fix $t \in (\frac{\sqrt{n-1}}{2}, \frac{\sqrt{n}}{2}]$. Then,*

$$|Q_n \cap \{x \in \mathbb{R}^n : |x \cdot \xi| \leqslant t\}| \geq 1 - \frac{2n^{\frac{n}{2}}}{n!} \left(\frac{\sqrt{n}}{2} - t\right)^n, \text{ for all } \xi \in \mathbb{S}^{n-1}$$

with equality occurring when $\xi = \left(\pm\frac{1}{\sqrt{n}}, \ldots, \pm\frac{1}{\sqrt{n}}\right)$.

Proof. The proof for two-dimensional case was provided in [3]. Thus, we will concentrate on $n \geqslant 3$. Fix $t \in (\frac{\sqrt{n-1}}{2}, \frac{\sqrt{n}}{2}]$. It is clear that the minimal slab must not contain the whole Q_n, and thus, must cut two congruent pyramids from Q_n (see the discussion in the beginning of the proof of Theorem 5). By the symmetry, we may assume that one of the pyramids have $\mathbf{v} = (\frac{1}{2}, \ldots, \frac{1}{2})$ as its apex. Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \in \mathbb{R}^n$ be as define in the proof of Theorem 4 and thus,

$$|Q_n \cap \{x \in \mathbb{R}^n : |x \cdot \xi| \leqslant t\}| = 1 - \frac{2}{n!} \prod_{i=1}^{n} a_i.$$

So, we may again apply the method of Lagrange multipliers, to optimize the function

$$f(a_1, \ldots, a_n) = \prod_{i=1}^{n} a_i, \text{ with the constraint } \sum_{i=1}^{n} \frac{1}{a_i} - 2t \sqrt{\sum_{i=1}^{n} \frac{1}{a_i^2}} = 2. \quad (8)$$

Similarly, as in the proof of Theorem 5, we consider $(a_1, \ldots, a_n) \in (0, \frac{1}{2})^n$ and set

$$F_\lambda(a_1, \ldots, a_n) = \prod_{i=1}^{n} a_i - \lambda \left(\sum_{i=1}^{n} \frac{1}{a_i} - 2 - 2t \sqrt{\sum_{i=1}^{n} \frac{1}{a_i^2}} \right),$$

then

$$\frac{\partial F_\lambda}{\partial a_k} = \prod_{\substack{i=1 \\ i \neq k}}^{n} a_i - \lambda \left(-\frac{1}{a_k^2} + 2t \frac{\frac{1}{a_k^3}}{\sqrt{\sum_{i=1}^{n} \frac{1}{a_i^2}}} \right).$$

To find possible extremal values we need to solve $\frac{\partial F_\lambda}{\partial a_k} = 0$, for all $k = 1, \ldots, n$, which is:

$$\prod_{i=1}^{n} a_i - \lambda \left(-\frac{1}{a_k} + 2t \frac{\frac{1}{a_k^2}}{\sqrt{\sum_{i=1}^{n} \frac{1}{a_i^2}}} \right) = 0. \quad (9)$$

To find λ, we will sum up Eqs. (9) for $k = 1, \ldots, n$ and use Eq. (8) to get:

$$\lambda = -\frac{n}{2} \prod_{j=1}^{n} a_j. \quad (10)$$

Next, we fix two different indices $k, m \in [1, \ldots, n]$. Then, we subtract from Eq. (9) multiplied by $1/a_m^2$ the corresponding equation for m multiplied by $1/a_k^2$ we get

$$\left(\prod_{j=1}^{n} a_j \right) \left(\frac{1}{a_m^2} - \frac{1}{a_k^2} \right) + \lambda \left(\frac{1}{a_k a_m^2} - \frac{1}{a_m a_k^2} \right) = 0.$$

Thus, again, we get that the possible extremal point must satisfy either $\frac{1}{a_m} - \frac{1}{a_k} = 0$ or

$$\left(\prod_{j=1}^{n} a_j \right) \left(\frac{1}{a_m} + \frac{1}{a_k} \right) + \frac{\lambda}{a_k a_m} = 0.$$

The above again gives $a_k + a_m = \frac{n-1}{2}$, which is only possible for $n = 2$. Thus, for $n \geq 3$ the only possible critical point is $a_1 = a_2 = \cdots = a_n$. By arguments similar to those used in Theorem 5 (working with set C and δ), we may show f achieves a (constrained) maximum at $a_1 = a_2 = \cdots = a_n$.

This yields the maximal $\xi = (\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}})$ and $a = \sqrt{n}(\frac{\sqrt{n}}{2} - t)$. This gives, for $t \in (\frac{\sqrt{n-1}}{2}, \frac{\sqrt{n}}{2}]$,

$$|Q_n \cap \{x \in \mathbb{R}^n : |x \cdot \xi| \leq t\}| \geq 1 - \frac{2n^{\frac{n}{2}}}{n!} \left(\frac{\sqrt{n}}{2} - t \right)^n.$$

$\square$

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Universal Flows of Closed Subgroups of S_∞ and Relative Extreme Amenability

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Abstract This paper is devoted to the study of universality for a particular continuous action naturally attached to certain pairs of closed subgroups of S_∞. It shows that three new concepts, respectively called relative extreme amenability, relative Ramsey property for embeddings and relative Ramsey property for structures, are relevant in order to understand this property correctly. It also allows us to provide a partial answer to a question posed in [2] by Kechris, Pestov and Todorcevic (Geom. Funct. Anal. **15**(1), 106–189, 2005).

Key words Extreme amenability • Relative extreme amenability • Relative Ramsey property • Fraïssé theory • Ramsey theory • Universal flow

Mathematical Subject Classifications (2010): 37B05, 03C15, 03E02, 03E15, 05D10, 22F50, 43A07, 54H20

1 Introduction

This note builds on the paper [2] by Kechris, Pestov and Todorcevic, and is devoted to the study of universality for a particular continuous action naturally attached to certain pairs of closed subgroups of S_∞. Recall that if G is a topological group, a *G-flow* is a continuous action of G on a topological space X (in what follows, all topological spaces will be Hausdorff). For those, we will often use the notation $G \curvearrowright X$. The flow $G \curvearrowright X$ is *compact* when the space X is. It is *universal* when

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every compact minimal $G \curvearrowright Y$ is a factor of $G \curvearrowright X$, which means that there exists $\pi : X \longrightarrow Y$ continuous, onto and G-equivariant, i.e., so that

$$\forall g \in G \quad \forall x \in X \quad \pi(g \cdot x) = g \cdot \pi(x).$$

Finally, it is *minimal* when every $x \in X$ has dense orbit in X:

$$\forall x \in X \ \ \overline{G \cdot x} = X.$$

It turns out that when G is Hausdorff, there is, up to isomorphism of G-flows, a unique G-flow that is both minimal and universal. This flow is called *the universal minimal flow* of G and is denoted by $G \curvearrowright M(G)$. When studying universal minimal flows of closed subgroups of S_∞ (here and throughout the paper, S_∞ denotes the symmetric group of the natural numbers $\mathbb{N}$, equipped with the pointwise convergence topology), the authors of [2] showed that certain flows encode remarkable combinatorial properties, called the Ramsey property and the ordering property. This connection also takes place in a slightly broader context and it is in this more general framework that we will present it here, where pure order expansions (i.e., order expansions where the language is enriched with a single binary relation symbol, which is always interpreted as a linear ordering) are replaced by precompact relational expansions and where the ordering property is replaced by the expansion property (see [3]). Given closed subgroups $G^* \leq G$ of S_∞, there are natural classes $\mathcal{K}$ and $\mathcal{K}^*$ of finite objects attached to them, as well as a G-flow $G \curvearrowright X^*$, where X^* is the completion of the quotient G/G^* equipped with the projection of the right-invariant metric. When X^* is compact, the following properties hold (cf [2] for pure order expansions and [3] for the precompact case):

(i)   The flow $G \curvearrowright X^*$ is universal and minimal iff $\mathcal{K}^*$ has the Ramsey property and the expansion property relative to $\mathcal{K}$.
(ii) The flow $G \curvearrowright X^*$ is minimal iff $\mathcal{K}^*$ has the expansion property relative to $\mathcal{K}$.

Question 1 ([2], p.174). Assume that $\mathcal{K}^*$ is a pure order expansion of $\mathcal{K}$. Is universality of $G \curvearrowright X^*$ equivalent to the Ramsey property for $\mathcal{K}^*$?

More generally:

Question 2. Assume that $\mathcal{K}^*$ is a precompact expansion of $\mathcal{K}$. Is universality of $G \curvearrowright X^*$ equivalent to the Ramsey property for $\mathcal{K}^*$?

Apart from the fact that they are very natural in view of the previous results, there is, at least, one other very good reason to study those problems. Indeed, given a class of finite objects, the Ramsey property is very often difficult to prove. Any new tool that would allow to reach it is therefore welcome. However, [2] already provides a spectacular dynamical characterization of the Ramsey property for $\mathcal{K}^*$, as it shows that it is equivalent to $M(G^*)$ being reduced to a single point for some subgroup G^* of G naturally attached to $\mathcal{K}^*$. So what is the real interest of the

questions? We started working on them when we realized that the Ramsey property implies universality of $G \curvearrowright X^*$. Our hope became then that universality would be equivalent to a strictly weaker combinatorial condition for $\mathcal{K}^*$, call it weak Ramsey property for the moment, and that this condition would be, in practice, easier to prove than the Ramsey property. This fact, together with the observation that the expansion property is often much easier to prove than the Ramsey property, would then give an alternate combinatorial approach for further problems involving Ramsey property: in order to prove it for $\mathcal{K}^*$, instead of attacking the problem directly, it would suffice to prove that both the weak Ramsey property and the expansion property are satisfied. Indeed, that would prove that the flow $G \curvearrowright X^*$ is both universal and minimal, and therefore that $\mathcal{K}^*$ has the Ramsey property. So far, this approach is partially successful. It is successful in the sense that universality is indeed equivalent to some combinatorial condition (finally not called weak Ramsey property but, for reasons that should become clear later on, relative Ramsey property) which is, in general, strictly weaker than the Ramsey property. But the success is only partial in the sense that it is unclear whether in practice, this condition is much easier to prove than the Ramsey property.

Let us now turn to a concise description of the results presented in this paper. Let L be a countable first order signature and $L^* = L \cup \{R_i : i \in I^*\}$ a countable relational expansion of L. We refer to Sect. 2 for all other undefined notions. Our first result connects universality of $G \curvearrowright X^*$ to a dynamical statement involving actions of G and G^*.

Definition 1. Let $H \leq G$ be topological groups. Say that the pair (G, H) is *relatively extremely amenable* when every continuous action of G on every compact space admits an H-fixed point.

Theorem 1. *Let $\mathbf{F}$ be a Fraïssé structure in L and $\mathbf{F}^*$ a Fraïssé precompact relational expansion of $\mathbf{F}$ in L^*. The following are equivalent:*

(i)   *The flow $G \curvearrowright X^*$ is universal.*
(ii)   *The pair (G, G^*) is relatively extremely amenable.*

This result allows us to show that Question 2 has a negative answer by exhibiting concrete examples of classes $\mathcal{K}$ and $\mathcal{K}^*$ where the flow $G \curvearrowright X^*$ is universal but the Ramsey property does not hold for $\mathcal{K}^*$ (see Sect. 4). However, quite surprisingly, we are not able to settle the case of Question 1.

Next, we turn to a combinatorial reformulation of relative extreme amenability. Let $\mathcal{K}$ be a class of L-structures and $\mathcal{K}^*$ an expansion of $\mathcal{K}$ in L^*, that is, a class of L^*-structures such that every element of $\mathcal{K}^*$ is an expansion of an element of $\mathcal{K}$. When $\mathbf{A}, \mathbf{B} \in \mathcal{K}$, the set of all embeddings from $\mathbf{A}$ into $\mathbf{B}$ is denoted by

$$\binom{\mathbf{B}}{\mathbf{A}}_{Emb}.$$

Let $\mathbf{B}^*$ be an expansion of $\mathbf{B}$ in $\mathcal{K}^*$ and $a \in \binom{\mathbf{B}}{\mathbf{A}}_{Emb}$. The substructure of $\mathbf{B}^*$ supported by $a(\mathbf{A})$ is an expansion of $\mathbf{A}$ in $\mathcal{K}^*$. Using a, we can then define an expansion of $\mathbf{A}$ in $\mathcal{K}^*$ as follows: for $i \in I$, call $\alpha(i)$ the arity of R_i. Then, set

$$\forall i \in I^* \quad R_i^a(x_1, \ldots, x_{\alpha(i)}) \Leftrightarrow R_i^{\mathbf{B}^*}(a(x_1), \ldots, a(x_{\alpha(i)})).$$

We will refer to $(a(\mathbf{A}), \mathbf{R}^a)$ as the *canonical expansion* induced by a on $\mathbf{A}$. If $a' \in \binom{\mathbf{B}}{\mathbf{A}}_{Emb}$, write $a \cong_{\mathbf{B}^*} a'$ when the canonical expansions on $\mathbf{A}$ induced by a and a' are equal (not only isomorphic).

Definition 2. Let $\mathcal{K}$ be a class of finite L-structures and $\mathcal{K}^*$ an expansion of $\mathcal{K}$ in L^*. Say that the pair $(\mathcal{K}, \mathcal{K}^*)$ has the *relative Ramsey property for embeddings* when for every $k \in \mathbb{N}$, $\mathbf{A} \in \mathcal{K}$, $\mathbf{B}^* \in \mathcal{K}^*$, there exists $\mathbf{C} \in \mathcal{K}$ such that for every coloring $c : \binom{\mathbf{C}}{\mathbf{A}}_{Emb} \longrightarrow [k]$, there exists $b \in \binom{\mathbf{C}}{\mathbf{B}}_{Emb}$ such that:

$$\forall a_0, a_1 \in \binom{\mathbf{B}}{\mathbf{A}}_{Emb} \quad a_0 \cong_{\mathbf{B}^*} a_1 \Rightarrow c(ba_0) = c(ba_1).$$

Theorem 2. *Let F be a Fraïssé structure in L and let F^* be a Fraïssé precompact relational expansion of F in L^*. Then the following are equivalent:*

(i)   *The pair (G, G^*) is relatively extremely amenable.*
(ii)   *The elements of $\mathrm{Age}(F^*)$ are rigid and the pair $(\mathrm{Age}(F), \mathrm{Age}(F^*))$ has the relative Ramsey property for embeddings.*

Then, we investigate the properties of a weakening of the relative Ramsey property for embeddings, which involves only structures, as opposed to embeddings.

Definition 3. Let $\mathcal{K}$ be a class of finite L-structures and $\mathcal{K}^*$ an expansion of $\mathcal{K}$ in L^*. Say that the pair $(\mathcal{K}, \mathcal{K}^*)$ has the *relative Ramsey property for structures* when for every $k \in \mathbb{N}$, $\mathbf{A} \in \mathcal{K}$ and $\mathbf{B}^* \in \mathcal{K}^*$, there exists $\mathbf{C} \in \mathcal{K}$ such that for every coloring $c : \binom{\mathbf{C}}{\mathbf{A}} \longrightarrow [k]$, there exists $b \in \binom{\mathbf{C}}{\mathbf{B}}_{Emb}$ such that:

$$\forall \widetilde{\mathbf{A}}_0, \widetilde{\mathbf{A}}_1 \in \binom{\mathbf{B}}{\mathbf{A}} \quad \left(\widetilde{\mathbf{A}}_0 \cong_{\mathbf{B}^*} \widetilde{\mathbf{A}}_1 \Rightarrow c(b(\widetilde{\mathbf{A}}_0)) = c(b(\widetilde{\mathbf{A}}_1))\right).$$

Above, $\widetilde{\mathbf{A}}_0 \cong_{\mathbf{B}^*} \widetilde{\mathbf{A}}_1$ means that $\mathbf{B}^* \upharpoonright \tilde{A}_0 \cong \mathbf{B}^* \upharpoonright \tilde{A}_1$.

This weakening may be strictly weaker than the embedding version, but at the combinatorial level, it is good enough to play the role of a "weak Ramsey property" as described above:

Theorem 3. *Let F be a Fraïssé structure in L and let F^* be a Fraïssé precompact expansion of F in L^* whose age consists of rigid structures. Assume that the pair $(\mathrm{Age}(F), \mathrm{Age}(F^*))$ has the relative Ramsey property for structures and that*

Age($\boldsymbol{F}^*$) *has the expansion property relative to* Age($\boldsymbol{F}$). *Then* Age($\boldsymbol{F}^*$) *has the Ramsey property.*

Finally, we show that in order to guarantee the existence of an expansion with both the Ramsey and the expansion property, it is enough to prove the existence of an expansion with the relative Ramsey property for structures:

Theorem 4. *Let $\boldsymbol{F}$ be a Fraïssé structure in L^* and let $\boldsymbol{F}^*$ be a Fraïssé precompact relational expansion of $\boldsymbol{F}$ in L^*. Assume that* Age($\boldsymbol{F}^*$) *consists of rigid elements and that the pair* (Age($\boldsymbol{F}$), Age($\boldsymbol{F}^*$)) *has the relative Ramsey property for structures. Then* Age($\boldsymbol{F}^*$) *admits a Fraïssé subclass with the Ramsey property and the expansion property relative to* Age($\boldsymbol{F}$).

The paper is organized as follows: Sect. 2 contains all basic notions concerning Fraïssé theory, structural Ramsey property and precompact relational expansions. Section 3 contains a proof of Theorem 1. Section 4 exhibits concrete classes answering Question 2. Section 5 provides a proof of Theorem 2. In Sect. 6, a more detailed study of the relative Ramsey property for embeddings is carried. Finally, Sect. 7 concentrates on the relative Ramsey property for structures and contains the proofs of Theorem 3 and 4.

2 Ramsey Property, Expansion Property, Precompact Expansions

The purpose of this section is to describe the global framework where our study is taking place. Our main references here are [2] for Fraïssé theory and structural Ramsey theory and [3] for precompact expansions.

2.1 Fraïssé Theory

In what follows, $\mathbb{N}$ denotes the set $\{0, 1, 2, \ldots\}$ of natural numbers and for a natural number m, $[m]$ denotes the set $\{0, \ldots, m - 1\}$. We will assume that the reader is familiar with the concepts of first order logic, first order structures, Fraïssé theory (cf [2], sect. 2), reducts and expansions (cf [2], sect. 5). If L is a first order signature and $\mathbf{A}$ and $\mathbf{B}$ are L-structures, we will write $\mathbf{A} \leq \mathbf{B}$ when $\mathbf{A}$ embeds in $\mathbf{B}$, $\mathbf{A} \subset \mathbf{B}$ when $\mathbf{A}$ is a substructure of $\mathbf{B}$ and $\mathbf{A} \cong \mathbf{B}$ when $\mathbf{A}$ and $\mathbf{B}$ are isomorphic. When L is countable, a *Fraïssé class* in L will be a countable class of finite L-structures of arbitrarily large sizes, satisfying the hereditarity, joint embedding and amalgamation property, and a *Fraïssé structure* (or *Fraïssé limit*) in L will be a countable, locally finite, ultrahomogeneous L-structure.

2.2 Structural Ramsey Theory

In order to define the Ramsey property, let $k, l \in \mathbb{N}$ and $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be L-structures. The set of all *copies* of $\mathbf{A}$ in $\mathbf{B}$ is written

$$\binom{\mathbf{B}}{\mathbf{A}} = \{\widetilde{\mathbf{A}} \subset \mathbf{B} : \widetilde{\mathbf{A}} \cong \mathbf{A}\}.$$

We use the standard arrow partition symbol

$$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{k,l}$$

to mean that for every map $c : \binom{\mathbf{C}}{\mathbf{A}} \longrightarrow [k]$, thought as a k-coloring of the copies of $\mathbf{A}$ in $\mathbf{C}$, there is $\widetilde{\mathbf{B}} \in \binom{\mathbf{C}}{\mathbf{B}}$ such that c takes at most l-many values on $\binom{\widetilde{\mathbf{B}}}{\mathbf{A}}$. When $l = 1$, this is written

$$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{k}.$$

A class $\mathcal{K}$ of finite L-structures is then said to have the *Ramsey property* when

$$\forall k \in \mathbb{N} \quad \forall \mathbf{A}, \mathbf{B} \in \mathcal{K} \quad \exists \mathbf{C} \in \mathcal{K} \quad \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{k}.$$

When $\mathcal{K} = \mathrm{Age}(\mathbf{F})$, where $\mathbf{F}$ is a Fraïssé structure, this is equivalent, via a compactness argument, to:

$$\forall k \in \mathbb{N} \quad \forall \mathbf{A}, \mathbf{B} \in \mathcal{K} \quad \mathbf{F} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{k}.$$

2.3 Precompact Expansions

Assume now that we have a first-order expansion of L, $L^* = L \cup \{R_i : i \in I^*\}$, with I^* countable and every symbol R_i relational and not in L. Let $\mathcal{K}^*$ denote an *expansion* of $\mathcal{K}$ in L^* (that means that all elements of $\mathcal{K}^*$ are of the form $\mathbf{A}^* = (\mathbf{A}, (R_i^{\mathbf{A}^*})_{i \in I^*})$ and that every $\mathbf{A} \in \mathcal{K}$ can be enriched to some element $\mathbf{A}^* = (\mathbf{A}, (R_i^{\mathbf{A}^*})_{i \in I^*})$ of $\mathcal{K}^*$ by adding some relations on A). For $\mathbf{A}^* \in \mathcal{K}^*$, the reduct of $\mathbf{A}^*$ to L is denoted by $\mathbf{A}^* \upharpoonright L = \mathbf{A}$. Then, $\mathcal{K}^*$ satisfies the *expansion property* relative to $\mathcal{K}$ if, for every $\mathbf{A} \in \mathcal{K}$, there exists $\mathbf{B} \in \mathcal{K}$ such that

$$\forall \mathbf{A}^*, \mathbf{B}^* \in \mathcal{K}^* \quad (\mathbf{A}^* \upharpoonright L = \mathbf{A} \ \wedge \ \mathbf{B}^* \upharpoonright L = \mathbf{B}) \Rightarrow \mathbf{A}^* \leq \mathbf{B}^*.$$

Next, consider $\mathbf{F}$, a Fraïssé structure in L. For $i \in I^*$, the arity of the symbol R_i is denoted by $\alpha(i)$. We let $\mathbf{F}^*$ be an expansion of $\mathbf{F}$ in L^*. We assume that $\mathbf{F}^*$ is also Fraïssé and write $\mathbf{F}^* = (\mathbf{F}, (R_i^*)_{i \in I^*})$, or $(\mathbf{F}, \mathbf{R}^*)$. We also assume that $\mathbf{F}$ and $\mathbf{F}^*$ have the set $\mathbb{N}$ of natural numbers as universe. The corresponding automorphism

groups are denoted by G and G^* respectively. The group G^* will be thought as a subgroup of G and both are closed subgroups of S_∞, the permutation group of $\mathbb{N}$ equipped with the topology generated by sets of the form

$$U_{g,F} = \{h \in G : h \restriction F = g \restriction F\},$$

where g runs over G and F runs over all finite subsets of $\mathbb{N}$. Note that the group S_∞ admits two natural metrics: a left invariant one, d_L, defined as

$$d_L(g,h) = \frac{1}{2^m}, \quad m = \min\{n \in \mathbb{N} : g(n) \neq h(n)\},$$

and a right-invariant one, d_R, given by

$$d_R(g,h) = d_L(g^{-1}, h^{-1}).$$

In what follows, we will be interested in the set of all expansions of $\mathbf{F}$ in L^*, which we think as the product

$$P := \prod_{i \in I^*} [2]^{\mathbf{F}^{\alpha(i)}}.$$

In this notation, the factor $[2]^{\mathbf{F}^{\alpha(i)}} = \{0, 1\}^{\mathbf{F}^{\alpha(i)}}$ is thought as the set of all $\alpha(i)$-ary relations on $\mathbf{F}$. Each factor $[2]^{\mathbf{F}^{\alpha(i)}}$ is equipped with an ultrametric d_i, defined by

$$d_i(S_i, T_i) = \frac{1}{2^m}, \quad m = \min\{n \in \mathbb{N} : S_i \restriction [n] \neq T_i \restriction [n]\}$$

where $S_i \restriction [n]$ (resp. $T_i \restriction [n]$) stands for $S_i \cap [n]^{a(i)}$ (resp. $T_i \cap [n]^{a(i)}$).

The group G acts continuously on each factor as follows: if $i \in I^*$, $S_i \in [2]^{\mathbf{F}^{\alpha(i)}}$ and $g \in G$, then $g \cdot S_i$ is defined by

$$\forall y_1 \ldots y_{\alpha(i)} \in \mathbf{F} \quad g \cdot S_i(y_1 \ldots y_{\alpha(i)}) \Leftrightarrow S_i(g^{-1}(y_1) \ldots g^{-1}(y_{\alpha(i)})).$$

This allows us to define a continuous action of G on the product P equipped with the supremum distance d^P of all the distances d_i (where $g \cdot \mathbf{S}$ is simply defined as $(g \cdot S_i)_{i \in I^*}$ whenever $\mathbf{S} = (S_i)_{i \in I^*} \in P$ and $g \in G$). As a set, G/G^* can be thought as $G \cdot \mathbf{R}^*$, the orbit of $\mathbf{R}^*$ in P, by identifying $[g]$, the equivalence class of g, with $g \cdot \mathbf{R}^*$. The metric d_R induces a metric on the quotient G/G^*, which coincides with the restriction of d^P on $G \cdot \mathbf{R}^*$ (see [3], Proposition 1). Therefore, we can really think of the metric space G/G^* as the metric subspace $G \cdot \mathbf{R}^*$ of P and it can be shown that the space $G/G^* \cong G \cdot \mathbf{R}^*$ is precompact (i.e., has a compact completion, or, equivalently, a compact closure in P) iff every element of $\mathrm{Age}(\mathbf{F})$ has finitely many expansions in $\mathrm{Age}(\mathbf{F}^*)$ (see [3], Proposition 2). In that case, we say that $\mathrm{Age}(\mathbf{F}^*)$ *is a precompact expansion of* $\mathrm{Age}(\mathbf{F})$ (or that $\mathbf{F}^*$ is a precompact expansion of $\mathbf{F}$).

3 Relative Extreme Amenability and Universality

In this section, we prove Theorem 1:

Theorem 1. *Let F be a Fraïssé structure and F^* a precompact relational expansion of F. The following are equivalent:*

(i)   The flow $G \curvearrowright X^*$ is universal for minimal compact G-flows.
(ii)   The pair (G, G^*) is relatively extremely amenable.

Proof (Proof of $(i) \Rightarrow (ii)$). Let $G \curvearrowright X$ be a compact G-flow and let $Y \subset X$ be such that $G \curvearrowright Y$ is compact minimal. By universality, find $\pi : X^* \longrightarrow Y$ continuous, G-equivariant and surjective. Denote $\xi = \pi([e])$, where e denotes the neutral element of G (recall that $X^* = \widehat{G/G^*}$). Then, ξ is G^*-fixed, because for $g \in G^*$:

$$g \cdot \xi = g \cdot \pi([e]) = \pi(g \cdot [e]) = \pi([g]) = \pi([e]) = \xi. \qquad \square$$

Proof (Proof of $(ii) \Rightarrow (i)$). Let $G \curvearrowright X$ be a compact minimal G-flow. Find $\xi \in X$, G^*-fixed. Then, let $p : G \longrightarrow X$ be defined by

$$p(g) = g \cdot \xi.$$

This map is G-equivariant, right uniformly continuous and constant on elements of G/G^*. Therefore, it induces $\bar{p} : G/G^* \longrightarrow X$, which is also G-equivariant and right uniformly continuous. Denote by π the continuous extension of $\bar{p}$ to the completion $X^* = \widehat{G/G^*}$. Then π is G-equivariant and surjective because its range is a compact subset of X containing $G \cdot \xi$, which is dense in X. $\qquad \square$

4 Universality vs Ramsey Property

A consequence of Theorem 1 is that for precompact expansions, universality of $G \curvearrowright X^*$ is not equivalent to Ramsey property for Age(F^*). For example, consider the class $\mathcal{U}_S^<$ of finite ordered ultrametric spaces with distances in S, where S is a finite subset of $\mathbb{R}$. The corresponding Fraïssé limit is a countable ordered ultrametric space, denoted by $(\mathbf{U}_S^{ult}, <)$. As a linear ordering, it is isomorphic to $(\mathbb{Q}, <)$. Hence, $(\mathbf{U}_S^{ult}, <)$ can be thought as a precompact relational expansion of $(\mathbb{Q}, <)$, and the group Aut($\mathbf{U}_S^{ult}, <$) can be thought as a closed subgroup of Aut($\mathbb{Q}, <$). Because this latter group is extremely amenable (see [5]), the pair (Aut($\mathbb{Q}, <$), Aut($\mathbf{U}_S^{ult}, <$)) is relatively extremely amenable and the corresponding flow is universal. However, it is known that $\mathcal{U}_S^<$ does not have the Ramsey property, see [4]. A similar situation occurs with finite posets, considering $(\mathbb{Q}, <)$ and the Fraïssé limit $(\mathbb{P}, <)$ of the class of all finite ordered posets. This class does not have the Ramsey property (cf [7, 8]), but the corresponding flow is universal.

In the two previous examples, the pair of groups (G, H) under consideration is proved to be relatively extremely amenable by producing an extremely amenable interpolant, i.e., an extremely amenable closed subgroup K of G containing H. It is a natural question to ask whether every relatively extremely amenable pair of groups admits such an interpolant. The answer in general is negative, as witnessed by the pair $(S_\infty, \mathrm{Aut}(\mathbb{Z}, <))$. This result is due to Gutman. [1] has been withdrawn due to an error in the main argument.

Finally, in view of the original question posed in [2], we do not know whether universality of $G \curvearrowright X^*$ implies Ramsey property of $\mathrm{Age}(\mathbf{F}^*)$ when $\mathbf{F}^*$ is a pure order expansion of $\mathbf{F}$. We believe that the answer should be negative, but were not able to construct any counterexample so far. In fact, results of Sokić (see [7, 9]) provide a positive answer in a number of cases. As a possible strategy for a counterexample, start with a Fraïssé class $\mathrm{Age}(\mathbf{F})$ with the Ramsey property and consisting of rigid elements. Consider then the class $\mathcal{K}^<$ of all finite order expansions of elements of $\mathrm{Age}(\mathbf{F})$ and try to find a Fraïssé subclass $\mathcal{K}^* \subset \mathcal{K}^<$ without the Ramsey property. Then, calling $\mathbf{F}^*$ the Fraïssé limit of $\mathcal{K}^*$ and denoting $G^* = \mathrm{Aut}(\mathbf{F}^*)$, we would have (G, G^*) relatively extremely amenable (because G is extremely amenable), hence $G \curvearrowright X^*$ universal, while $\mathcal{K}^*$ does not have the Ramsey property.

5 Relative Extreme Amenability and Relative Ramsey Property for Embeddings

In view of the two previous sections, it is natural to ask whether relative extreme amenability of a pair (G, G^*) as before can be seen at the level of $\mathrm{Age}(\mathbf{F})$ and $\mathrm{Age}(\mathbf{F}^*)$. The answer is positive, as shown by the following result. Note that the proof has the same pattern as the proof of Kechris-Pestov-Todorcevic theorem as presented in [3], Theorem 1.

Theorem 2. *Let $\mathbf{F}$ be a Fraïssé structure and $\mathbf{F}^*$ a Fraïssé precompact relational expansion of $\mathbf{F}$. Then the following are equivalent:*

(i)   The pair (G, G^*) is relatively extremely amenable.
(ii)   The elements of $\mathrm{Age}(\mathbf{F}^*)$ are rigid and the pair $(\mathrm{Age}(\mathbf{F}), \mathrm{Age}(\mathbf{F}^*))$ has the relative Ramsey property for embeddings.

5.1 Proof of $(i) \Rightarrow (ii)$

Assume that (G, G^*) is relatively extremely amenable. We first prove that all elements of $\mathrm{Age}(\mathbf{F}^*)$ are rigid. To do so, consider the set of all linear orderings

$LO(\mathbf{F})$, seen as a subspace of the space $[2]^{\mathbf{F} \times \mathbf{F}}$. The group G acts continuously on this later space via the logic action. The set $LO(\mathbf{F})$ is then a G-invariant compact subspace. Explicitly, G acts on $LO(\mathbf{F})$ as follows: if $\prec \in LO(\mathbf{F})$ and $g \in G$, then

$$\forall x, y \in \mathbf{F} \quad x (g \cdot \prec) y \Leftrightarrow g^{-1}(x) \prec g^{-1}(y).$$

By relative extreme amenability of (G, G^*), there is a G^*-fixed point in $LO(\mathbf{F})$, call it $<^*$. Consider now a finite substructure $\mathbf{A}^* \subset \mathbf{F}^*$ and let φ be an automorphism of $\mathbf{A}^*$. By ultrahomogeneity of $\mathbf{F}^*$, φ extends to an automorphism ϕ of $\mathbf{F}^*$. Because $<^*$ is G^*-fixed, it is preserved under φ. Thus, on A, $<^*$ is preserved by φ, which means that φ is trivial on A. This proves that $\mathbf{A}^*$ is rigid.

To prove that $(\text{Age}(\mathbf{F}), \text{Age}(\mathbf{F}^*))$ has the relative Ramsey property for embeddings, it suffices to show that $\mathbf{F}$ satisfies the property that is required for $\mathbf{C}$ in Definition 2. A compactness argument allows then to find $\mathbf{C}$. So consider $k \in \mathbb{N}$, $\mathbf{A} \in \text{Age}(\mathbf{F}), \mathbf{B}^* \subset \mathbf{F}^*$ finite and a coloring

$$c : \binom{\mathbf{F}}{\mathbf{A}}_{\text{Emb}} \longrightarrow [k].$$

Consider the compact space $[k]^{\binom{\mathbf{F}}{\mathbf{A}}_{\text{Emb}}}$, acted on continuously by G by shift: if $\chi \in [k]^{\binom{\mathbf{F}}{\mathbf{A}}_{\text{Emb}}}$, $g \in G$ and $a \in \binom{\mathbf{F}}{\mathbf{A}}_{\text{Emb}}$, then

$$g \cdot \chi(a) = \chi(g^{-1}a).$$

The set $\overline{G \cdot c}$ is a G-invariant compact subspace. By relative extreme amenability of G, there is a G^*-fixed point in $\overline{G \cdot c}$, call it c_0. The fact that c_0 is G^*-fixed means that $c_0(a_0) = c_0(a_1)$ whenever $a_0 \cong_{\mathbf{F}^*} a_1$. Consider now the finite set $\binom{\mathbf{B}}{\mathbf{A}}_{\text{Emb}}$, where $\mathbf{B} := \mathbf{B}^* \upharpoonright L$. Because $c_0 \in \overline{G \cdot c}$, there is $g \in G$ so that

$$g \cdot c \upharpoonright \binom{\mathbf{B}}{\mathbf{A}}_{\text{Emb}} = c_0 \upharpoonright \binom{\mathbf{B}}{\mathbf{A}}_{\text{Emb}}.$$

So if $a_0, a_1 \in \binom{\mathbf{B}}{\mathbf{A}}_{\text{Emb}}$ are such that $a_0 \cong_{\mathbf{B}^*} a_1$ then

$$c(g^{-1}a_0) = c(g^{-1}a_1).$$

It follows that g^{-1} witnesses the relative Ramsey property for embeddings. $\qquad \square$

5.2 Proof of (*i i*) ⇒ (*i*)

Assume that Age($\mathbf{F}^*$) consists of rigid elements and the pair (Age($\mathbf{F}$), Age($\mathbf{F}^*$)) has the relative Ramsey property for embeddings. For $A \subset \mathbb{N}$ finite, we denote by Stab(A) the pointwise stabilizer Stab(A) in G and we can make the identification:

$$G/\mathrm{Stab}(A) = \binom{\mathbf{F}}{A}_{\mathrm{Emb}}.$$

Proposition 1. *Let $k \in \mathbb{N}$, $A \subset \mathbb{N}$ finite and supporting a substructure $\mathbf{A}$ of $\mathbf{F}$ and $F \subset G$ finite. Let $\bar{f} : G \longrightarrow [k]$ be constant on elements of $G/\mathrm{Stab}(A)$. Then there exists $g \in G$ such that*

$$\forall h, h' \in F \quad h'h^{-1} \in G^* \Rightarrow \bar{f}(gh) = \bar{f}(gh').$$

Proof. The map $\bar{f}$ induces a map $f : G/\mathrm{Stab}(A) \longrightarrow [k]$, which we may think as a k-coloring of $\binom{\mathbf{F}}{\mathbf{A}}_{\mathrm{Emb}}$. Consider the set $\{[h] : h \in F\}$. It is a finite set of embeddings from $\mathbf{A}$ to $\mathbf{F}$. Therefore, we can find a finite substructure $\mathbf{B} \subset \mathbf{F}$ large enough so that the ranges of all those embeddings are contained in $\mathbf{B}$. Let $\mathbf{B}^*$ denote the substructure of $\mathbf{F}^*$ supported by B. By relative Ramsey property for embeddings applied to $\mathbf{A}$, $\mathbf{B}^*$ and the coloring f, find $g \in G$ such that

$$\forall a_0, a_1 \in \binom{\mathbf{B}}{\mathbf{A}}_{\mathrm{Emb}} \quad (a_0 \cong_{\mathbf{B}^*} a_1 \Rightarrow f(ga_0) = f(ga_1)).$$

Consider now $h, h' \in F$ so that $h'h^{-1} \in G^*$. Then $[h] \cong_{\mathbf{B}^*} [h']$ and so

$$f(g[h]) = f(g[h']).$$

At the level of $\bar{f}$, that means $\bar{f}(gh) = \bar{f}(gh')$. □

Proposition 2. *Let $p \in \mathbb{N}$, $f : G \longrightarrow \mathbb{R}^p$ left uniformly continuous and bounded (where $\mathbb{R}^p$ is equipped with its standard Euclidean structure), $F \subset G$ finite, $\varepsilon > 0$. Then there exists $g \in G$ such that*

$$\forall h, h' \in F \quad h'h^{-1} \in G^* \Rightarrow \|f(gh) - f(gh')\| < \varepsilon.$$

Proof. Let $m \in \mathbb{N}$. Note that as subsets of G, elements of $G/\mathrm{Stab}([m])$ have diameter $1/2^{m+1}$ with respect to the left invariant metric d^L on G. Thus, by left uniform continuity, we can find $m \in \mathbb{N}$ large enough so that f is constant up to ε on each element of $G/\mathrm{Stab}([m])$. By local finiteness of $\mathbf{F}$, let now $A \subset \mathbb{N}$ be finite, supporting a finite substructure $\mathbf{A}$ of $\mathbf{F}$ and such that $[m] \subset A$. Then f is also constant up to ε on each element of $G/\mathrm{Stab}(A)$. Because f is also bounded, we can also find $\bar{f} : G \longrightarrow \mathbb{R}^p$ with finite range, constant on elements of $G/\mathrm{Stab}(A)$

and so that $\|f - \bar{f}\|_\infty < \varepsilon/2$. By Proposition 1, there exists $g \in G$ such that $\bar{f}(gh) = \bar{f}(gh')$ whenever $h'h^{-1} \in G^*$. Then $\|f(gh) - f(gh')\| < \varepsilon$ whenever $h'h^{-1} \in G^*$. $\qquad\square$

We can now show that the pair (G, G^*) is relatively extremely amenable. Let $G \curvearrowright X$ be a continuous action, with X compact. For $p \in \mathbb{N}$, $\phi : G \longrightarrow \mathbb{R}^p$ uniformly continuous and bounded, $F \subset G$ finite, $\varepsilon > 0$, set

$$A_{\phi,\varepsilon,F} = \{x \in X : \forall h \in F \cap G^* \quad \|\phi(h \cdot x) - \phi(x)\| \leq \varepsilon\}.$$

The family $(A_{\phi,\varepsilon,F})_{\phi,\varepsilon,F}$ is a family of closed subsets of X. We claim that it has the finite intersection property. Indeed, if $\phi_1, \ldots, \phi_l, \varepsilon_1, \ldots, \varepsilon_l, F_1, \ldots, F_l$ are given, take

$$\phi = (\phi_1, \ldots, \phi_l), \quad \varepsilon = \min(\varepsilon_1, \ldots, \varepsilon_l), \quad F = F_1^{-1} \cup \ldots \cup F_l^{-1} \cup \{e\}.$$

Fix $x \in X$ and consider the map $f : G \longrightarrow \mathbb{R}^{p_1 + \cdots + p_l}$ defined by

$$\forall g \in G \quad f(g) = (\phi_1(g^{-1} \cdot x), \ldots, \phi_l(g^{-1} \cdot x))).$$

Because the maps ϕ_i's are uniformly continuous and the map $g \mapsto g^{-1} \cdot x$ is left uniformly continuous (cf [6], p40), the map f is left uniformly continuous. By Proposition 2, there exists $g \in G$ so that

$$\forall h, h' \in F \quad h'h^{-1} \in G^* \Rightarrow \|f(gh) - f(gh')\| < \varepsilon.$$

Equivalently,

$$\forall i \leq l \quad \forall h, h' \in F \quad h'h^{-1} \in G^* \Rightarrow \|\phi_i(h^{-1}g^{-1} \cdot x) - \phi_i(h'^{-1}g^{-1} \cdot x)\| \leq \varepsilon_i.$$

Taking $x_0 = g^{-1} \cdot x$ and $h' = e$, we obtain

$$\forall i \leq l \quad \forall h \in F_i \quad \|\phi_i(h \cdot x_0) - \phi_i(x_0)\| \leq \varepsilon_i.$$

This proves the finite intersection property of the family $(A_{\phi,\varepsilon,F})_{\phi,\varepsilon,F}$. By compactness of X, it follows that this family has a non empty intersection. Consider any element x of this intersection. We claim that x is fixed under the action of G^*: if not, we would find $g \in G^*$ so that $g \cdot x \neq x$. Then, there would be a uniformly continuous function $\phi_0 : X \longrightarrow [0, 1]$ so that $\phi_0(x) = 0$ and $\phi_0(g \cdot x) = 1$. That would imply $x \notin A_{\phi_0,1/2,\{g\}}$, a contradiction. $\qquad\square$

6 Versions and Consequences of the Relative Ramsey Property

6.1 Canonical Expansions

Recall that when $\mathbf{A}, \mathbf{B} \in \mathcal{K}$, $\mathbf{B}^*$ is an expansion of $\mathbf{B}$ in $\mathcal{K}^*$ and $a \in \binom{\mathbf{B}}{\mathbf{A}}_{Emb}$, the canonical expansion induced by a on $\mathbf{A}$ is the structure $(a(\mathbf{A}), \mathbf{R}^a)$ defined by

$$\forall i \in I^* \quad R_i^a(x_1, \ldots, x_{\alpha(i)}) \Leftrightarrow R_i^{\mathbf{B}^*}(a(x_1), \ldots, a(x_{\alpha(i)})).$$

Note that a is not completely characterized by the canonical expansion it induces on $\mathbf{A}$ when $a(\mathbf{A})$ possesses a non-trivial automorphism, but that it is when $a(\mathbf{A})$ is rigid. In the case where all expansions of $\mathbf{A}$ are rigid, then the $\cong_{\mathbf{B}^*}$-equivalence classes are those sets of the form $\binom{\mathbf{B}^*}{\mathbf{A}^*}_{Emb}$, where $\mathbf{A}^*$ ranges over the set of all expansions (possibly isomorphic, but based on A) of $\mathbf{A}$ in $\mathcal{K}^*$.

6.2 More on the Relative Ramsey Property for Embeddings

In this section, we present simple facts related to the concept of relative Ramsey property for embeddings.

Proposition 3. *Assume that the pair $(\mathcal{K}, \mathcal{K}^*)$ has the relative Ramsey property for embeddings, then so does the pair $(\mathcal{K}, \mathcal{K}^{**})$ whenever $\mathcal{K}^{**}$ is an expansion of $\mathcal{K}$ in L^* such that $\mathcal{K}^{**} \subset \mathcal{K}^*$.*

Proof. Direct from the definition. □

For anyone familiar with Ramsey theory, this is a rather unexpected feature (most of Ramsey type properties are not preserved when passing to subclasses).

Proposition 4. *Let $\mathbf{F}$ be a Fraïssé structure and $\mathbf{F}^*$ a Fraïssé expansion of $\mathbf{F}$. Then the pair $(\mathrm{Age}(\mathbf{F}), \mathrm{Age}(\mathbf{F}^*))$ has the relative Ramsey property for embeddings when for every $k \in \mathbb{N}$, $\mathbf{A} \in \mathrm{Age}(\mathbf{F})$, $\mathbf{B}^* \subset \mathbf{F}^*$ finite and $c : \binom{\mathbf{F}}{\mathbf{A}}_{Emb} \longrightarrow [k]$, there exists $g \in G$ such that:*

$$\forall a_0, a_1 \in \binom{\mathbf{B}}{\mathbf{A}}_{Emb} \quad a_0 \cong_{\mathbf{B}^*} a_1 \ \Rightarrow \ c(ga_0) = c(ga_1).$$

Proof. A standard compactness argument. □

Note that because $\mathbf{B}^*$ is a substructure of $\mathbf{F}^*$, $a_0 \cong_{\mathbf{B}^*} a_1$ is equivalent to $a_0 \cong_{\mathbf{F}^*} a_1$, which is equivalent to the existence of $g^* \in G^*$ so that $a_1 = g^* a_0$ (use ultrahomogeneity of $\mathbf{F}^*$).

Proposition 5. *Let F be a Fraïssé structure and F^* a Fraïssé expansion of F such that* $\mathrm{Age}(F^*)$ *consists of rigid elements. Then, the following are equivalent:*

(i)   *The pair* $(\mathrm{Age}(F), \mathrm{Age}(F^*))$ *has the relative Ramsey property for embeddings.*

(ii) *For every* $k \in \mathbb{N}$, $A \in \mathrm{Age}(F)$, $B^* \in \mathrm{Age}(F^*)$, *there exists* $C^* \in \mathrm{Age}(F^*)$ *such that for every coloring* $c : \binom{C}{A}_{\mathrm{Emb}} \longrightarrow [k]$, *there exists* $b \in \binom{C}{B}_{\mathrm{Emb}}$ *such that:*

$$\forall a_0, a_1 \in \binom{B}{A}_{\mathrm{Emb}} \qquad a_0 \cong_{B^*} a_1 \;\Rightarrow\; (ba_0 \cong_{C^*} ba_1 \;\wedge\; c(ba_0) = c(ba_1)).$$

Proof. Quite clearly, item ii) implies the relative Ramsey property for embeddings. For the converse, use the standard trick of enriching the coloring c by the $\cong_{C^*}$-isomorphism type. Formally, fix $k \in \mathbb{N}$, $\mathbf{A} \in \mathrm{Age}(\mathbf{F})$, $\mathbf{B}^* \in \mathrm{Age}(\mathbf{F}^*)$. Recall that $E(\mathbf{A})$ denotes the set of all (possibly isomorphic) expansions of $\mathbf{A}$ in $\mathrm{Age}(\mathbf{F}^*)$. Consider $\mathbf{C}$ provided by the relative Ramsey property for embeddings applied to $\mathbf{A}$, $\mathbf{B}^*$ and $k|E(\mathbf{A})|$. Let $\mathbf{C}^*$ denote any expansion of $\mathbf{C}$ in $\mathrm{Age}(\mathbf{F}^*)$. Let $c : \binom{C}{A}_{\mathrm{Emb}} \longrightarrow$ $[k]$ and define $\bar{c}$ by $\bar{c}(a) = (c(a), [a])$, where $[a]$ denotes the $\cong_{C^*}$ equivalence class of a. This is a $k|E(\mathbf{A})|$-coloring of $\binom{C}{A}_{\mathrm{Emb}}$, so we can find $b \in \binom{C}{B}_{\mathrm{Emb}}$ so that

$$\forall a_0, a_1 \in \binom{B}{A}_{\mathrm{Emb}} \qquad a_0 \cong_{B^*} a_1 \;\Rightarrow\; \bar{c}(ba_0) = \bar{c}(ba_1).$$

We are done since

$$\bar{c}(ba_0) = \bar{c}(ba_1) \Leftrightarrow (ba_0 \cong_{C^*} ba_1 \wedge c(ba_0) = c(ba_1)). \qquad \square$$

We now turn to a consequence of the relative Ramsey property for embeddings. Let $\mathbf{A} \in \mathrm{Age}(\mathbf{F})$ and $\mathbf{B}^* \in \mathrm{Age}(\mathbf{F}^*)$. Recall that the set $\binom{B}{A}_{\mathrm{Emb}}$ is partitioned into $\cong_{B^*}$-equivalence classes, which correspond to those sets of the form $\binom{B^*}{A^*}_{\mathrm{Emb}}$, where $\mathbf{A}^*$ ranges over the set of all (possibly isomorphic) expansions of $\mathbf{A}$ in $\mathrm{Age}(\mathbf{F}^*)$ that are based on A. As a direct consequence:

Proposition 6. *Let F be a Fraïssé structure and F^* a Fraïssé expansion of F such that* $\mathrm{Age}(F^*)$ *consists of rigid elements. Assume that the pair* $(\mathrm{Age}(F), \mathrm{Age}(F^*))$ *has the relative Ramsey property for embeddings. Then, for every* $k \in \mathbb{N}$, $A^*, B^* \in \mathrm{Age}(F^*)$, *there exists* $C \in \mathrm{Age}(F)$ *such that for every coloring* $c : \binom{C}{A}_{\mathrm{Emb}} \longrightarrow [k]$, *there exists* $b \in \binom{C}{B}_{\mathrm{Emb}}$ *such that the set* $b \circ \binom{B^*}{A^*}_{\mathrm{Emb}} := \{ba^* : a^* \in \binom{B^*}{A^*}_{\mathrm{Emb}}\}$ *is monochromatic.*

Quite surprisingly, the converse to the previous proposition does not seem to hold. The main obstruction is that we only have a limited control on how b behaves with respect to canonical expansions. As we have seen before, we can make sure that b preserves $\cong_{B^*}$-equivalence. However, we cannot make sure that it

preserves canonical expansions. This detail appears to be problematic when trying to deduce the relative Ramsey property for embeddings from a repeated application of item (ii).

7 The Relative Ramsey Property for Structures

Let $\mathcal{K}$ be a class of L-structures and $\mathcal{K}^*$ an expansion of $\mathcal{K}$ in L^*. Recall that the pair $(\mathcal{K}, \mathcal{K}^*)$ has the *relative Ramsey property for structures* when for every $k \in \mathbb{N}$, $\mathbf{A} \in \mathcal{K}$ and $\mathbf{B}^* \in \mathcal{K}^*$, there exists $\mathbf{C} \in \mathcal{K}$ such that for every coloring $c : \binom{\mathbf{C}}{\mathbf{A}} \longrightarrow [k]$, there exists $b \in \binom{\mathbf{C}}{\mathbf{B}}_{\mathrm{Emb}}$ such that:

$$\forall \widetilde{\mathbf{A}}_0, \widetilde{\mathbf{A}}_1 \in \binom{\mathbf{B}}{\mathbf{A}} \;\; \left(\widetilde{\mathbf{A}}_0 \cong_{\mathbf{B}^*} \widetilde{\mathbf{A}}_1 \;\Rightarrow\; c(b(\widetilde{\mathbf{A}}_0)) = c(b(\widetilde{\mathbf{A}}_1))\right),$$

where $\widetilde{\mathbf{A}}_0 \cong_{\mathbf{B}^*} \widetilde{\mathbf{A}}_1$ means that $\mathbf{B}^* \upharpoonright \tilde{A}_0 \cong \mathbf{B}^* \upharpoonright \tilde{A}_1$.

Again, when $\mathcal{K}$ and $\mathcal{K}^*$ are of the form $\mathrm{Age}(\mathbf{F})$ and $\mathrm{Age}(\mathbf{F}^*)$ respectively, where $\mathbf{F}$ and $\mathbf{F}^*$ are Fraïssé structures and $\mathbf{F}^*$ is an expansion of $\mathbf{F}$, a compactness argument shows that $(\mathrm{Age}(\mathbf{F}), \mathrm{Age}(\mathbf{F}^*))$ has the relative Ramsey property for structures when for every $k \in \mathbb{N}$, $\mathbf{A} \in \mathrm{Age}(\mathbf{F})$, $\mathbf{B}^* \in \mathrm{Age}(\mathbf{F}^*)$ and $c : \binom{\mathbf{F}}{\mathbf{A}} \longrightarrow [k]$, there exists $g \in G$ such that:

$$\forall \widetilde{\mathbf{A}}_0, \widetilde{\mathbf{A}}_1 \in \binom{\mathbf{B}}{\mathbf{A}} \;\; \left(\widetilde{\mathbf{A}}_0 \cong_{\mathbf{B}^*} \widetilde{\mathbf{A}}_1 \;\Rightarrow\; c(g(\widetilde{\mathbf{A}}_0)) = c(g(\widetilde{\mathbf{A}}_1))\right).$$

This property implies (but does not seem to be equivalent to) the following weakening of the Ramsey property for $\mathrm{Age}(\mathbf{F}^*)$: for every $k \in \mathbb{N}$, $\mathbf{A}^*, \mathbf{B}^* \in \mathrm{Age}(\mathbf{F}^*)$ and $c : \binom{\mathbf{F}}{\mathbf{A}} \longrightarrow [k]$, there exists $g \in G$ such that $g \circ \binom{\mathbf{B}^*}{\mathbf{A}^*}$ is monochromatic.

Note that colorings of structures can be seen as particular cases of colorings of embeddings, where elements with isomorphic (and not necessarily equal) canonical expansions receive the same color. For that reason, the relative Ramsey property for embeddings implies the relative Ramsey property for structures. The converse does not seem to hold in general. The only instance for which we could check that the two notions agree is when the elements of $\mathrm{Age}(\mathbf{F})$ are rigid, simply because the sets $\binom{\mathbf{F}}{\mathbf{A}}$ and $\binom{\mathbf{F}}{\mathbf{A}}_{\mathrm{Emb}}$ can be identified. However, from the practical point of view, the following results show that the relative Ramsey property for structures may have some applications in the future.

Proposition 7. *Assume that the pair $(\mathcal{K}, \mathcal{K}^*)$ has the relative Ramsey property for structures, then so does the pair $(\mathcal{K}, \mathcal{K}^{**})$ whenever $\mathcal{K}^{**} \subset \mathcal{K}^*$.*

Proof. Direct from the definition. □

Theorem 3. *Let **F** be a Fraïssé structure and **F*** a precompact expansion of **F** whose age consists of rigid structures. Assume that the pair* (Age(**F**), Age(**F***)) *has the relative Ramsey property for structures and that* Age(**F***) *has the expansion property relative to* Age(**F**). *Then* Age(**F***) *has the Ramsey property.*

Proof. Because of the relative Ramsey property for structures, every **A** ∈ Age(**F**) has a finite Ramsey degree in Age(**F**) whose value is at most equal to the number of non-isomorphic expansions of **A** in Age(**F***). Together with the expansion property relative to Age(**F**), this is known to imply the Ramsey property for Age(**F***) (for a reference, see for example [3], Sect. 5, Proposition 8). □

The preceding result could turn out to be useful in practice, where the Ramsey property is often difficult to prove and the expansion property generally more accessible.

Theorem 4. *Let **F** be a Fraïssé structure and let **F*** be a precompact relational expansion of **F**. Assume that* Age(**F***) *consists of rigid elements and that the pair* (Age(**F**), Age(**F***)) *has the relative Ramsey property for structures. Then* Age(**F***) *admits a Fraïssé subclass with the Ramsey property and the expansion property relative to* Age(**F**).

Proof. Let **S** ∈ $\overline{G \cdot \mathbf{R}^*}$ be such that $G \curvearrowright \overline{G \cdot \mathbf{S}}$ is minimal. Note that Age(**F**, **S**) ⊂ Age(**F***) because **S** ∈ $\overline{G \cdot \mathbf{R}^*}$ (cf [3], section on minimality). We claim that Age(**F**, **S**) is a required. This class clearly has the hereditarity property and the joint embedding property. The expansion property comes from minimality of $G \curvearrowright \overline{G \cdot \mathbf{S}}$ (cf [4], remark following Theorem 4 in Section 4). To prove the Ramsey property, notice first that because Age(**F**, **S**) ⊂ Age(**F***), the pair (Age(**F**), Age(**F**, **S**)) also has the relative Ramsey property for structures (cf Proposition 7), which in turn implies that every **A** ∈ Age(**F**) has a finite Ramsey degree at most equal to the number of non-isomorphic expansions of **A** in Age(**F**, **S**). Because Age(**F**, **S**) has the hereditary property, the joint embedding property and the expansion property relative to Age(**F**), it has the Ramsey property ([3], Sect. 5, Proposition 8). Finally, because of all the previous properties and because Age(**F**, **S**) consists of rigid elements, it is Fraïssé ([2] p.20, or [3] end of Sect. 6). □

Further investigation about the practical status of the relative Ramsey property for structures will decide on its value as a tool to derive the Ramsey property. We have to admit that, so far, we are not aware of any concrete application of any of the preceding results of this section.

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Oscillation of Urysohn Type Spaces

N.W. Sauer

Abstract A metric space M $= (M; \mathrm{d})$ is *homogeneous* if for every isometry α of a finite subspace of M to a subspace of M there exists an isometry of M onto M extending α. The metric space M is *universal* if it isometrically embeds every finite metric space F with dist(F) $\subseteq$ dist(M). (dist(M) being the set of distances between points of M.)

A metric space M is *oscillation stable* if for every $\epsilon > 0$ and every uniformly continuous and bounded function $f : M \to \Re$ there exists an isometric copy M* $= (M^*; \mathrm{d})$ of M in M for which:

$$\sup\{|f(x) - f(y)| \mid x, y \in M^*\} < \epsilon.$$

Theorem. *Every bounded, uncountable, separable, complete, homogeneous, universal metric space* M $= (M; \mathrm{d})$ *is oscillation stable. (Theorem 12.)*

Key words Partitions of metric spaces • Ramsey theory • Metric geometry • Urysohn metric space • Oscillation stability

Mathematical Subject Classifications (2010): 03E02, 22F05, 05C55, 05D10, 22A05, 51F99

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1   Basic Notions and Facts

Find in this section some basic definitions and facts and in the next, introductory section, a general background discussion about the topics of this article.

Definition 1. A metric space M = $(M; d)$ is *homogeneous* if for every isometry α of a finite subspace of M to a subspace of M there exists an isometry of M onto M extending α. A metric space M is *universal* if it embeds every finite metric space F with dist(F) $\subseteq$ dist(M).

Definition 2. A metric space M is an *Urysohn metric space* if it is separable, complete, homogeneous and universal.

It follows easily from the general Fraïssé theory and is stated explicitly as Corollary 2.2 of [1] that

Fact 1. *An Urysohn metric space* **U** *embeds every separable metric space* N *with* dist(N) $\subseteq$ dist(**U**).

See [11] for further background information on homogeneous metric spaces and their partition theory. Two Urysohn metric spaces with the same set of distances are isometric. (Theorem 8). An Urysohn metric space with $\mathcal{R}$ as set of distances will be denoted by $U_{\mathcal{R}}$.

In [8] a very general definition and discussion and applications for the notion that a function $f : G/H \to \Re$ is *oscillation stable* is given. In there, G is a topological group and H a subgroup and f is uniformly continuous with respect to the left uniformity of G. (See [8] for a more complete statement.) Further it is defined in [8] that a pair G, H for H a topological subgroup of a topological group G, is *oscillation stable* if every bounded left uniformly continuous function $f : G/H \to \Re$ is oscillation stable.

In the case of G being the group of isometries of a homogeneous metric space M = $(M; d)$ onto M with H the isotropy subgroup fixing a point v of M and $f : M \to \Re$ (identified with $f : G/H \to \Re$) a uniformly continuous and bounded function, this general definition translates to: The function f is oscillation stable if there exists, for every $\epsilon > 0$, a copy M* = $(M^*; d)$ of M in M for which: $\sup\{|f(x) - f(y)| \mid x, y \in M^*\} < \epsilon$. (Again see [8] where this translation is outlined in the case of the unit sphere of ℓ_2 or see [15] for a short proof.) Hence we may define: A homogeneous metric space M = $(M; d)$ is *oscillation stable* if for every $\epsilon > 0$ and every uniformly continuous and bounded function $f : M \to \Re$ there exists a copy M* = $(M^*; d)$ of M in M for which:

$$\sup\{|f(x) - f(y)| \mid x, y \in M^*\} < \epsilon.$$

Because of the results in Sect. 9 it is natural to extend the use of the notion of oscillation stability to metric spaces in general and to define:

Definition 3. A metric space $M = (M; d)$ is *oscillation stable* if for every $\epsilon > 0$ and every uniformly continuous and bounded function $f : M \to \Re$ there exists a copy $M^* = (M^*; d)$ of M in M for which:

$$\sup\{|f(x) - f(y)| \mid x, y \in M^*\} < \epsilon.$$

Definition 4. Let $M = (M; d)$ be a metric space. Then for $\epsilon > 0$ and $A \subseteq M$ let:

$$\left(A\right)_\epsilon = \{x \in M \mid \exists y \in A \ (d(x, y) < \epsilon)\}. \tag{1}$$

The metric space M is *approximately indivisible* if for every $\epsilon > 0$ and $n \in \omega$ and function $\gamma : M \to n$ there exist $i \in n$ and an isometric copy $M^* = (M^*; d)$ of M in M with

$$M^* \subseteq \left(\gamma^{-1}(i)\right)_\epsilon.$$

The metric space M is *indivisible* if for every $n \in \omega$ and function $\gamma : M \to n$ there exist $i \in n$ and an isometric copy $M^* = (M^*; d)$ of M in M with

$$M^* \subseteq \gamma^{-1}(i).$$

If $\mathcal{R}$ is finite then we obtain from Theorem 9.1 of [2]:

Theorem 1. *Every Urysohn metric space $U_\mathcal{R}$ for which the set of distances $\mathcal{R}$ is finite, is indivisible.*

Theorem 2. *A metric space* $M = (M; d)$ *is* oscillation stable *if and only if it is* approximately indivisible.

Hence we can use the terms oscillation stable and approximately indivisible interchangeably to denote the same phenomenon. When citing theorems we will use original terminology.

Definition 5. Let $\mathcal{R} \subseteq \Re$, then:

$$a \oplus b := \sup\{x \in \mathcal{R} \mid x \leq a + b\}.$$

2   Introduction

Sharpening and reformulating Dvoretzky's Theorem, see [4, 5], Milman, see [10], proved, for $\mathbb{S}^k$ the unit sphere of the $k + 1$-dimensional Euclidean space and for $\mathbb{S}^\infty$, the unit sphere of the Hilbert space ℓ_2:

Theorem 3 (Milman). *Let $n \in \omega$ and $\gamma : \mathbb{S}^\infty \to n$. Then for every $\varepsilon > 0$ and every $k \in \omega$, there is $i \in n$ and an isometric copy $(\mathbb{S}^k)^*$ of $\mathbb{S}^k$ in $\mathbb{S}^\infty$ such that $(\mathbb{S}^k)^* \subseteq \left(\gamma^{-1}(i)\right)_\epsilon.$*

This result then led naturally to the *distortion problem*, asking if the Hilbert sphere $\mathbb{S}^\infty$ is approximately indivisible, that is oscillation stable. Which was settled by:

Theorem 4 (Odell–Schlumprecht [13]). *There is a number $n \in \omega$ and a function $\gamma : \mathbb{S}^\infty \to n$ and $\epsilon > 0$ such for every $i \in n$ the set $\left(\gamma^{-1}(i)\right)_\epsilon$ does not contain a copy of $\mathbb{S}^\infty$. (That is $\mathbb{S}^\infty$ has* distortion.*)*

The notion of oscillation stable topological groups was first introduced and studied by Kechris, Pestov and Todorcevic in [8], see also Pestov [15]. For homogeneous metric spaces, their groups of isometries are oscillation stable if the metric spaces are oscillation stable in the sense of Definition 3. In the special case of metric spaces it is more convenient to use the notion of oscillation stable as given in Definition 3. It is shown in [8], see also [15], that homogeneous metric spaces are oscillation stable if and only if they are approximately indivisible. It follows from Theorem 13 that this equivalence actually holds for all metric spaces.

The Hilbert Sphere $\mathbb{S}^\infty$ and the Urysohn Sphere $\boldsymbol{U}_{[0,1]}$ share many interesting topological properties. For example: The group Iso($\mathbb{S}^\infty$) is extremely amenable, as shown by Gromov and Milman [7]. The group Iso($\boldsymbol{U}_{[0,1]}$ is extremely amenable, as shown by Pestov [14]. Subsequently to the Odell–Schlumprecht result, it was therefore natural to ask whether the Urysohn sphere $\boldsymbol{U}_{[0,1]}$ has distortion as well. The first major step in resolving this question is the main result achieved by Lopez-Abad and Nguyen Van Thé in [9]: The Urysohn sphere is oscillation stable if all Urysohn metric spaces $\boldsymbol{U}_n$ for $n = \{0, 1, \ldots, n-1\}$ are indivisible. This then was proven in [12] by Nguyen Van Thé and Sauer, yielding the result that the Urysohn sphere $\boldsymbol{U}_{[0,1]}$ is oscillation stable.

The main result in this paper, namely that bounded, uncountable Urysohn metric spaces are oscillation stable, required a sequence of results similar to the Abad and Nguyen Van Thé and Sauer sequence of results.

The characterization of the distance sets of Urysohn metric spaces was achieved in [1], see Sect. 3 Theorem 8 of this article. In particular the set of distances $\mathcal{R}$ of an Urysohn metric space is closed and is closed under the operation $\oplus$ which is then associative on $\mathcal{R}$.

Let $\mathcal{R}$ be a closed set of numbers for which $\oplus$ is associative. As a first step we need to approximate $\mathcal{R}$ with a finite subset for which $\oplus$ is associative. This is accomplished in Sect. 7 Theorem 15. It follows from Theorem 9.1 of [2] that Urysohn metric spaces with a finite set of distances are indivisible, extending the result that the Urysohn metric spaces $\boldsymbol{U}_n$ are indivisible.

An essential part of the proof in the Lopez-Abad–Nguyen paper is a clever tree construction. For this construction it was necessary to calculate certain distances. This was possible because the finite approximating sets of numbers for the interval $[0, 1]$ are sets of the form $\{\frac{m}{n} \mid 0 \le m \le n\}$, and hence explicitly available for the calculation. In the general case dealt with in this article, the finite approximating sets are not explicitly available and hence it is impossible to calculate the required distances. It follows that the existence of such distances has to be proven. This necessitated the study of metric $\mathcal{R}$-graphs in Sect. 5.

Finally we have to address the question: What are the closed subsets of the reals having 0 as a limit and for which $\oplus$ is associative? Some initial experiments seemed to indicate that such sets are just finite unions of closed intervals. If this where the case, then the oscillation theorem for Urysohn metric spaces could simply be derived from the Abad–Nguyen Van Thé–Sauer result. But, as will be shown in Sect. 10, there are quite intricate examples, of such sets. (Generalized Cantor type sets for example.) A complete characterization of such sets seems to be beyond our present abilities.

3   The 4-Values Condition

Let $\mathcal{R} \subseteq \mathfrak{R}_{\geq 0}$ be a closed subset of the non negative reals. Then, according to Definition 5, the operation $\oplus$ or $\oplus_{\mathcal{R}}$ if the distinction is needed is given by:

$$a \oplus b := \sup\{x \in \mathcal{R} \mid x \leq a + b\}$$

which, because $\mathcal{R}$ is closed as a subset of $\mathfrak{R}$, is a binary operation on $\mathcal{R}$. A subset $\{a, b, c\} \subseteq \mathcal{R}$ is *metric* if $a \leq b + c$ and $b \leq a + c$ and $c \leq a + b$. A triple (a, b, c) of numbers in $\mathcal{R}$ is *metric* if its set of entries $\{a, b, c\}$ is metric. Note the following immediate consequence:

Observation 1. $\max\{b, c\} \leq b \oplus c \in \mathcal{R}$ and the triple $(b \oplus c, b, c)$ is metric, for all $b, c \in \mathcal{R}$. Also:

If (a, b, c) is a metric triple of numbers in $\mathcal{R}$, then $b \oplus c \geq a$ and hence $b \oplus c$ is the largest number $x \in \mathcal{R}$ for which the triple (x, b, c) is metric. The operation $\oplus$ is commutative and if $a \geq b$ then $a \oplus c \geq b \oplus c$.

Definition 6. The set of quadruples with entries in $\mathcal{R}$ and with $\max\{b, c, d\} \leq a \leq b + c + d$ will be denoted by $\mathcal{Q}(\mathcal{R})$. The arrow $x \rightsquigarrow (a, b, c, d)$ means that the triples (a, b, x) and (c, d, x) are metric. The set $\mathcal{R}$ satisfies the *4-values condition* if:

For all quadruples $(a, b, c, d) \in \mathcal{Q}(\mathcal{R})$ and all $x \in \mathbf{R}$

$x \rightsquigarrow (a, b, c, d)$ implies that there exists $y \in \mathcal{R}$ with $y \rightsquigarrow (a, d, c, b)$.

Theorem 5. *A closed subset $\mathcal{R} \subseteq \mathfrak{R}_{\geq 0}$ of the non negative reals satisfies the 4-values condition if and only if the operation $\oplus$ on $\mathcal{R}$ is associative.*

Proof. Let $\mathcal{R}$ satisfy the 4-values condition and $\{b, c, d\} \subseteq \mathcal{R}$. We will prove that $d \oplus (c \oplus d) = (d \oplus c) \oplus b$.

It follows from Observation 1 that $d \oplus (c \oplus b) = (b \oplus c) \oplus d$, that the triples $(d \oplus c, d, c)$ and $\big((d \oplus c) \oplus b, d \oplus c, b\big)$ are metric and that $d + c + b \geq (d \oplus c) \oplus b \geq \max\{d, c, b\}$. Hence $\big((d \oplus c) \oplus b, b, c, d\big) \in \mathcal{Q}(\mathcal{R})$ and $d \oplus c \rightsquigarrow \big((d \oplus c) \oplus b, b, c, d\big)$.

Because $\mathcal{R}$ satisfies the 4-values condition there exists $y \in \mathcal{R}$ with $y \rightsquigarrow \big((d \oplus c) \oplus b, d, c, b\big)$. Then $y \leq b \oplus c$ because the triple (y, b, c) is metric and $y \oplus d \geq (d \oplus c) \oplus b$ because the triple $\big((d \oplus c) \oplus b, y, d\big)$ is metric. Hence:

$$d \oplus (c \oplus b) = (b \oplus c) \oplus d \geq y \oplus d \geq (d \oplus c) \oplus b,$$

and therefore:

$$(d \oplus c) \oplus b = b \oplus (c \oplus d) \geq (b \oplus c) \oplus d = d \oplus (c \oplus b) \geq (d \oplus c) \oplus b,$$

implying $d \oplus (c \oplus d) = (d \oplus c) \oplus b$.

Let $\oplus$ be associative and $(a, b, c, d) \in \mathcal{Q}(\mathcal{R})$ with $x \in \mathcal{R}$ and $x \rightsquigarrow (a, b, c, d)$. Then $d \oplus c \geq x$ and $x \oplus b \geq a$ and

$$(b \oplus c) \oplus d = d \oplus (c \oplus b) = (d \oplus c) \oplus b \geq x \oplus b \geq a.$$

It follows that $b \oplus c \rightsquigarrow (a, d, c, b)$.

Theorem 5 together with

Theorem 6 ([1] Theorem 2.2). *Any two Urysohn metric spaces* M *and* N *with* $\mathrm{dist}(M) = \mathrm{dist}(N)$ *are isometric.*

and

Theorem 7 ([1] Theorem 4.4). *Let* $0 \in \mathcal{R} \subseteq \Re_{\geq 0}$ *with 0 as a limit. Then there exists an Urysohn metric space* $U_{\mathcal{R}}$ *if and only if* $\mathcal{R}$ *is a closed subset of* $\Re_{\geq 0}$ *which satisfies the 4-values condition.*

Let $0 \in \mathcal{R} \subseteq \Re_{\geq 0}$ *which does not have 0 as a limit. Then there exists an Urysohn metric space* $U_{\mathcal{R}}$ *if and only if* $\mathcal{R}$ *is a countable subset of* $\Re$ *which satisfies the 4-values condition.*

imply:

Theorem 8. *Any two Urysohn metric spaces* M *and* N *with* $\mathrm{dist}(M) = \mathrm{dist}(N)$ *are isometric.*

Let $0 \in \mathcal{R} \subseteq \Re_{\geq 0}$ *with 0 a limit of* $\mathcal{R}$. *Then there exists an Urysohn metric space* $U_{\mathcal{R}}$ *with* $\mathrm{dist}(U_{\mathcal{R}}) = \mathcal{R}$ *if and only if* $\mathcal{R}$ *is a closed subset of* $\Re_{\geq 0}$ *with* $\oplus_{\mathcal{R}}$ *associative.*

An Urysohn metric space with $\mathcal{R}$ as set of distances will be denoted by $U_{\mathcal{R}}$.

4   Ordered Urysohn Metric Spaces

The purpose of this section is to establish Lemma 1, which will be needed later on. For readers familiar with the general Fraïssé theory the Lemma is probably

quite obvious, but unfortunately it does not seem to be stated explicitly in the literature. Using the tools of Fraïssé theory one would use the fact that the class of finite induced substructures of a homogeneous structure has amalgamation and then conclude from Proposition 1.3 of [3] or Theorem 5 of [11] that in the case of metric spaces the amalgamation property implies the disjoint amalgamation property. It is then easily seen that the disjoint amalgamation property implies that every Katětov functions has infinitely many realizations. This then in turn implies Lemma 1 via a standard construction as for the special case proven in [9].

Avoiding Fraïssé theory we use Theorem 2 of [6] which says that for every homogeneous structure for which the class of finite induced substructures has disjoint amalgamation, hence in particular for every Urysohn metric space $U = (U; d)$ and every finite subset $F \subseteq U$, the restriction $U \restriction (U \setminus F)$ is an isometric copy of U. A relational structure having this property is called *strongly inexhaustible*. Hence every Urysohn metric space is strongly inexhaustible.

Definition 7. Let $U = (U; d)$ be a countable Urysohn metric space with $U = \{u_i \mid i \in \omega\}$ an enumeration of U. The *enumeration-order* on U is the linear order given by $u_i \leq u_j$ if and only if $i \leq j$.

Lemma 1. *Let $U = (U; d)$ be a countable Urysohn metric space with $U = \{u_i \mid i \in \omega\}$ an enumeration of U and let $U^* = (U^*; d)$ be an isometric copy of U in U. Then there exists an enumeration-order preserving isometry of U into U^*.*

Proof. For α_0 the empty isometry we will construct an ω-sequence of isometries $\alpha_0 \subseteq \alpha_1 \subseteq \alpha_2 \subseteq \alpha_3 \subseteq$ with $\alpha_n : \{u_i \mid i \in n\} \to U^*$ which is enumeration-order preserving for each $n \in \omega$. Then $\alpha := \bigcup_{n \in \omega} \alpha_n$ will be an order preserving isometry of U into U^*.

For α_n constructed let $l \in \omega$ such that $\alpha(n - 1) = u_l$ and let

$$F = \{u_j \in U^* \mid j < l \text{ and } u_j \notin \{\alpha(u_i) \mid i \in n\}\}$$

and let $\bar{U} = U^* \restriction (U^* \setminus F)$. Let N be the subspace of U induced by the set $\{u_i \mid 0 \leq i \leq n\}$. Because U is strongly inexhaustible, the space $\bar{U}$ is isometric to U and hence universal and therefore there exists an isometry β of the space N into $\bar{U}$. Let γ be the restriction of β to the set $\{u_i \mid i \in n\}$. Then $\gamma^{-1} \circ \alpha_n$ is an isometry and because $\bar{U}$ is homogeneous it has an extension δ to an isometry of $\bar{U}$ onto $\bar{U}$. Let $\alpha_{n+1} = \delta \circ \beta$.

5 Metric $\mathcal{R}$-Graphs

Let $\mathcal{R} \subseteq \Re_{\geq 0}$ be a closed subset of the non negative reals satisfying the 4-values condition. Then the $\oplus$ operation on $\mathcal{R}$ is commutative and associative according to Theorem 5. For $(a_i \in \mathcal{R}; i \in n \in \omega)$ let $\bigoplus_{i \in n} a_i = a_0 \oplus a_1 \oplus a_2 \oplus \cdots \oplus a_{n-1}$.

Lemma 2. *Let $a \in \mathcal{R}$ and $\mathcal{S} \subseteq \mathcal{R}$. Then $a \oplus \inf(\mathcal{S}) = \inf\{a \oplus s \mid s \in \mathcal{S}\}$.*

Proof. If $a + \inf(\mathcal{S}) = a \oplus \inf(\mathcal{S})$ then $a + s \geq a \oplus s \geq a \oplus \inf(\mathcal{S}) = a + \inf(\mathcal{S})$ for all $s \in \mathcal{S}$ and hence $a + \inf(\mathcal{S}) = \inf\{a + s \mid s \in \mathcal{S}\} \geq \inf\{a \oplus s \mid s \in \mathcal{S}\} \geq a \oplus \inf(\mathcal{S}) = a + \inf(\mathcal{S})$ implying $a \oplus \inf(\mathcal{S}) = \inf\{a \oplus s \mid s \in \mathcal{S}\}$.

Otherwise $a + \inf(\mathcal{S}) > a \oplus \inf(\mathcal{S})$. Because $\mathcal{R}$ is closed there exists an $\epsilon > 0$ with $[a \oplus \inf(\mathcal{S}), a + \inf(\mathcal{S}) + \epsilon] \cap \mathcal{R} = \emptyset$. Then $a \oplus s = a \oplus \inf(\mathcal{S})$ for all $s \in \mathcal{S}$ with $s - \inf(\mathcal{S}) < \epsilon$. Hence $\inf\{a \oplus s \mid s \in \mathcal{S}\} = a \oplus \inf(\mathcal{S})$.

A sequence $V = (v_0, v_1, v_2, \ldots, v_{n-1}, v_n)$ of vertices of a graph is a *walk from v_0 to v_n* if $\{x_i, x_{i+1}\}$ is an edge of the graph for all $i \in n$. The walk P is a *trail from v_0 to v_n* if $v_i = v_j$ only for $i = j$. A *path* $\{v_0, v_1, \ldots, v_n\}$ of a graph is a trail in which two vertices x_i and x_j are adjacent if and only if $|i - j| = 1$. Note that every walk from a to b contains a path from a to b. The *length* of a path is the number of its edges. A graph is *connected* if for all vertices a and b there is a path from a to b. A *cycle* is a connected graph in which every vertex is adjacent to exactly two points. (Hence, for our purposes, a single edge is not a cycle of length two.) It follows that the number of edges of a cycle is equal to the number of its vertices, the *length* of the cycle. If $\{a, b\}$ is an edge of a cycle of length n then, after removing the edge $\{a, b\}$ from the cycle, there is a path from a to b of length $n-1$. An isomorphic embedding of a cycle into a graph is a *cycle of the graph.* That is, a cycle of a graph does not have any secants. Note that the vertices of a cycle of a graph can be adjacent to more than two points, just not in the cycle.

Definition 8. An $\mathcal{R}$-*graph* $G = (G; E, d)$ is a triple, for which:

1. $(G; E)$ is a simple graph with G as set of vertices and E as set of edges.
2. $d : E \cup \{(x, x) \mid x \in G\} \to \mathcal{R}$, the *distance function of* H, is a function with $d(x, y) = 0$ if and only if $x = y$.
3. $d(x, y) = d(y, x)$ for all $\{x, y\} \in E$.

The $\mathcal{R}$-graph $G = (G; E, d)$ is *connected* if the graph $(G; E)$ is connected.

Note that if $M = (M; d)$ is a metric space with $\mathrm{dist}(M) \subseteq \mathcal{R}$ then $G = (M; [M]^2, d)$ is an $\mathcal{R}$-graph for which the underlying graph $(M; [M]^2)$ is complete.

For $G = (G; E, d)$ an $\mathcal{R}$-graph and $W = (v_0, v_1, v_2, \ldots, v_{n-1}, v_n)$ a walk let $d(W) = \bigoplus_{i \in n} d(v_i, v_{i+1})$. Note that for every walk W from v_0 to v_n there exists a trail P from v_0 to v_n with $d(P) \leq d(W)$. For $G = (G; E, d)$ and $a, b \in G$ let:

$$d(a, b) := \inf\{d(W) \mid W \text{ is a walk from } a \text{ to } b\}.$$

Note that $d(a, b) = \inf\{d(P) \mid P \text{ is a trail from } a \text{ to } b\}$.

Definition 9. An $\mathcal{R}$-graph $G = (G; E, d)$ is *regular* if $d(a, b) > 0$ for all $(a, b) \in G^2$ with $a \neq b$. (Note that finite $\mathcal{R}$-graphs are regular.)

An $\mathcal{R}$-graph $G = (G; E, d)$ is *metric* if $d(v_0, v_n) \leq d(W)$ for every walk $(v_0, v_1, \ldots, v_{n-1}, v_n)$ from v_0 to v_n and every edge $\{v_0, v_n\} \in E$. That is if $d(a, b) = d(a, b)$ for every edge $\{a, b\} \in E$.

It follows that a cycle C $= (C; E_C, \mathrm{d})$ is metric if and only if for every edge $e \in E_C$:

$$\mathrm{d}(e) \leq \bigoplus_{h \in E_C \setminus \{e\}} \mathrm{d}(h).$$

Lemma 3. *A regular $\mathcal{R}$-graph* G $= (G; E, \mathrm{d})$ *is metric if and only if all of its cycles are metric.*

Proof. If G contains a cycle induced by the set $\{v_i \mid i \in n \in \omega\}$ for which $\mathrm{d}(v_0, v_n) > \bigoplus_{i \in n} \mathrm{d}(v_i, v_{i+1})$ then G is not metric because $v_0, v_1, \ldots, v_n$ is a walk from v_0 to v_n. Assume that all cycles of R are metric and for a contradiction that the $\mathcal{R}$-graph G is not metric.

Let $n \in \omega$ be the smallest number for which there exists an edge $\{v_0, v_1\} \in E$ and a trail T $= (v_0, v_1, \ldots, v_n)$ with $\mathrm{d}(v_0, v_1) > \mathrm{d}(T)$. Then $\{v_i \mid i \in n\}$ does not induce a cycle and hence there exists a secant, that is there are indices i, j with $\{i, j\} \neq \{0, n\}$ and with $i + 1 < j$ so that $\{v_i, v_j\}$ is an edge in E. Then

$$\mathrm{d}(v_i, v_j) \leq \mathrm{d}(v_i, v_{i+1}) \oplus \mathrm{d}(v_{i+1}, v_{i+2}) \oplus \cdots \oplus \mathrm{d}(v_{j-1}, v_j)$$

follows from $j - i < n$ and the minimality condition on n. The trail S $= (v_0, v_1, \ldots, v_i, v_j, v_{j+1}, \ldots, v_n)$ is shorter than the trail T with $\mathrm{d}(S) \leq \mathrm{d}(T) < \mathrm{d}(v_0, v_1)$ in contradiction to the minimality of n.

Corollary 1. *Let* M $= (M; \mathrm{d})$ *be a metric space with* $\mathrm{dist}(M) \subseteq \mathcal{R}$. *Then the $\mathcal{R}$-graph* G $= (M; [M]^2, \mathrm{d})$ *is metric and regular.*

Proof. The only cycles of G are triangles, which are metric. Hence it follows from Lemma 3 that the $\mathcal{R}$-graph G is metric and regular.

For two $\mathcal{R}$-graphs G $= (G; E_G, \mathrm{d}^G)$ and H $= (G; E_H, \mathrm{d}^H)$ let G $\preccurlyeq$ H if $E_G \subseteq E_H$ and d^H restricted to E_G is equal to d^G. If G $= (G; E, \mathrm{d}^G)$ is an R-graph and M $= (G; \mathrm{d}^M)$ a metric space with $\mathrm{dist}(M) \subseteq \mathcal{R}$ and H the $\mathcal{R}$-graph $(G; [G]^2, \mathrm{d}^M)$ then we write G $\preccurlyeq$ M if G $\preccurlyeq$ H. Note that if G $\preccurlyeq$ H and H is metric and regular then G is metric and regular.

Lemma 4. *Let* G $= (G; E, \mathrm{d}^G)$ *be a connected and metric regular $\mathcal{R}$-graph with $a, b \in G$ and $a \neq b$ and $\{a, b\} \notin E$.*

Then H $= (G; E \cup \{a, b\}, \mathrm{d}^H)$ *with G $\preccurlyeq$ H and $\mathrm{d}^H(a, b) = \boldsymbol{d}^G(a, b)$ is a metric regular $\mathcal{R}$-graph with $\boldsymbol{d}^G(x, y) = \boldsymbol{d}^H(x, y)$ for all $\{x, y\} \in [G]^2$.*

Proof. Let $\{x, y\} \in [G]^2$. In order to determine $\boldsymbol{d}^H(x, y)$ let $\mathcal{P}$ be the set of trails in H from x to y. Let $\mathcal{Q}$ be the set of trails from x to y in G and let $\mathcal{Z}$ be the set of trails in G from a to b. Let P $= (x = v_0, v_1, v_2, \ldots, v_{n-1}, v_n = y) \in \mathcal{P}$. If $\{a, b\}$ is not an edge of P then $\mathrm{d}^H(P) = \mathrm{d}^G(P)$ and P $\in \mathcal{Q}$. Otherwise there is exactly one $i \in n$ so that $\{v_i, v_{i+1}\} = \{a, b\}$. Let T be the trail $v_0, v_1, \ldots, v_i$ in G and S the trail $v_{i+1}, v_{i+2}, \ldots, v_n$ in G and let $a = \mathrm{d}^G(T) + \mathrm{d}^G(S) = \mathrm{d}^H(T) + \mathrm{d}^H(S)$.

Then, using Lemma 2

$$\mathrm{d}^{\mathrm{H}}(\mathrm{P}) = a \oplus \boldsymbol{d}^{\,\mathrm{G}}(a,b) = \inf\{a \oplus \mathrm{d}^{\mathrm{G}}(Z) \mid Z \in \mathcal{Z}\}.$$

Hence there exists, for every $\epsilon > 0$, a trail $Z \in \mathcal{Z}$ so that $\mathrm{d}^{\mathrm{G}}(\mathrm{Q}) \leq \mathrm{d}^{\mathrm{H}}(\mathrm{P}) + \epsilon$ for Q the trail in G from $x = v_0$ to $y = v_n$ which goes from v_0 to v_i along T and then to v_{i+1} along Z and then to $v_n = y$ along S. Hence, for every $\epsilon > 0$ and for every trail $\mathrm{P} \in \mathcal{P}$ there exists a trail $\mathrm{Q_P} \in \mathcal{Q}$ with $\mathrm{d}^{\mathrm{G}}(\mathrm{Q_P}) < \mathrm{d}^{\mathrm{H}}(\mathrm{P}) + \epsilon$. This together with $\mathcal{Q} \subseteq \mathcal{P}$ implies that

$$\boldsymbol{d}_{\mathrm{H}}(x,y) = \inf\{\mathrm{d}^{\mathrm{H}}(\mathrm{P}) \mid \mathrm{P} \in \mathcal{P}\} = \inf\{\mathrm{d}^{\mathrm{G}}(\mathrm{Q}) \mid \mathrm{Q} \in \mathcal{Q}\} = \boldsymbol{d}_{\mathrm{G}}(x,y).$$

It follows that H is metric and regular.

Lemma 5. *Let* $\mathrm{G} = (G; E_{\mathrm{G}}, \mathrm{d}^{\mathrm{G}})$ *be a metric regular* $\mathcal{R}$*-graph. There exists a connected, metric regular* $\mathcal{R}$*-graph* $\mathrm{H} = (G; E_{\mathrm{H}}, \mathrm{d}^{\mathrm{H}})$ *with* $\mathrm{G} \preccurlyeq \mathrm{H}$.

Proof. Let $\mathfrak{C}$ be the set of connected components of G. For each $C \in \mathfrak{C}$ let v_C be a vertex in the connected component C. Let $0 < r \in \mathcal{R}$ and $V = \{v_C \mid C \in \mathfrak{C}\}$. Let $\mathrm{H} = (G; E_{\mathrm{H}}, \mathrm{d}^{\mathrm{H}})$ the $\mathcal{R}$-graph with $E_{\mathrm{H}} = E_{\mathrm{G}} \cup [V]^2$ and $\mathrm{d}(v_C, v_D) = r$ for all edges $\{v_C, v_D\} \in [V]^2$ and with $\mathrm{G} \preccurlyeq \mathrm{H}$.

A trail which contains points in different connected components contains an edge of distance $r > 0$. Hence H is regular because each of the connected components is regular. If a cycle of H contains two vertices in V then it is a subset of V and hence a metric triangle. Otherwise the cycle is a subset of one of the connected components and by assumption metric.

Lemma 6. *Let* $\mathrm{G} = (G; E, \mathrm{d})$ *be a connected, regular, metric* $\mathcal{R}$*-graph. Then* $\mathrm{M} = (G, \boldsymbol{d}^{\,\mathrm{G}})$ *is a metric space on* G *with* $\mathrm{d}^{\mathrm{G}}(a,b) = \boldsymbol{d}^{\,\mathrm{G}}(a,b)$ *for all edges* $\{a,b\} \in E(G)$.

Proof. Let $\mathcal{P}$ be the partial $\preccurlyeq$-order of metric and regular $\preccurlyeq$-extensions H of the $\mathcal{R}$-graph $\mathrm{G} = (G; E, \mathrm{d})$ with $\boldsymbol{d}^{\,\mathrm{H}}(a,b) = \boldsymbol{d}^{\,\mathrm{G}}(a,b)$ for all points $a, b \in G$. Let $\mathcal{C} = (\mathrm{H}_i = (G; E_i, \mathrm{d}^i); i \in I)$ be a chain of $\mathcal{P}$ and $\mathrm{K} = (G; E_{\mathrm{K}}, \mathrm{d}^{\mathrm{K}})$ the $\mathcal{R}$-graph with $E_{\mathrm{K}} = \bigcup_{i \in I} E_i$ and $\mathrm{d}^{\mathrm{K}}(x,y) = \mathrm{d}^i(x,y)$ for some $i \in I$ with $\{x,y\} \in E_i$. Let V be a cycle of K. Because V is finite there is an $i \in I$ so that V is a cycle of H_i and hence is metric. It follows now from Lemma 3 that K is metric.

We have to check that K is regular. Assume for a contradiction that there are two vertices $a, b \in G$ with $a \neq b$ and a sequence (T_n) of trails in K from a to b so that the sequence $\mathrm{d}^{\mathrm{K}}(T_n)$ tends to 0. For each of the trails T_n exists an $i \in I$ so that T_n is a trail in H_i and hence $\mathrm{d}^{\mathrm{K}}(T_n) = \mathrm{d}^{\mathrm{H}_i}(T_n) \geq \boldsymbol{d}^{\,\mathrm{H}_i}(a,b) = \boldsymbol{d}^{\,\mathrm{G}}(a,b)$, a contradiction.

A simple application of Zorn's Lemma to the partial order $\mathcal{P}$ together with Lemma 4 shows that the metric $\mathcal{R}$-graph G has a $\preccurlyeq$-extension to a metric space M on G.

Theorem 9. *Let* $\mathcal{R} \subseteq \mathfrak{R}_{\geq 0}$ *be a closed subset of the non negative reals satisfying the 4-values condition and let* $\mathrm{G} = (G; E_{\mathrm{G}}, \mathrm{d}^{\mathrm{G}})$ *be an* $\mathcal{R}$*-graph.*

If there exists a metric space $M = (G; d^M)$ *with* $d^G(a, b) = d^M(a, b)$ *for all edges* $\{a, b\} \in E_G$, *then the* $\mathcal{R}$-*graph* G *is regular and metric.*

If G *is metric and regular, then there exists a connected, metric and regular* $\mathcal{R}$-*graph* $H = (G; E_H, d^G)$ *with* $E_G \subseteq E_H$ *and* $d^G(a, b) = d^H(a, b)$ *for all edges* $\{a, b\} \in E_G$.

If G *is connected, regular and every cycle of* G *is metric then* $M = (G, \boldsymbol{d}^G)$ *is a metric space on* G *with* $d^G(a, b) = \boldsymbol{d}^G(a, b)$ *for all edges* $\{a, b\} \in E_G$.

Proof. If there exists a metric space $M = (G; d^M)$ with $\text{dist}(M) \subseteq \mathcal{R}$ and $G \preccurlyeq M$ then $G \preccurlyeq (G; [G]^2, d^M)$ and the $\mathcal{R}$-graph $(G; [G]^2, d^M)$, being metric and regular according to Corollary 1, implies that G is metric.

The second assertion follows from Lemma 5 and the third from Lemma 6 together with Lemma 3.

6   A Construction

For this section, let: $\mathcal{R} \subseteq \Re_{\geq 0}$ be a closed subset of the non negative reals satisfying the 4-values condition with $0 < r \in \mathcal{R}$. Let $U = (U; d^U)$ and $V = (V; d^V)$ be two disjoint, countable metric spaces with $\text{dist}(U) \cup \text{dist}(V) \subseteq \mathcal{R}$. Let $(u_i; i \in \omega)$ be an enumeration of U. Let $I \subseteq \omega$ and $V = \{v_i \mid i \in I\}$ an indexing of V for which $|d^U(u_i, u_j) - d^W(v_i, v_j)| \leq r$ for all $i, j \in \omega$.

Lemma 7. *There exists a metric space* $W = (W; d^W)$ *with* $\text{dist}(W) \subseteq \mathcal{R}$ *and* $V \subseteq W = \{w_i; i \in \omega\}$ *and* $w_i = v_i$ *for all* $i \in I$ *and* $W \cap U = \emptyset$, *so that* $|d^U(u_i, u_j) - d^W(w_i, w_j)| \leq r$ *for all* $i, j \in \omega$.

Proof. Let $G = (G; E_G, d^G)$ be the $\mathcal{R}$-graph determined by:

1. $G = U \cup V$.
2. $G \upharpoonright U = U$ and $G \upharpoonright V = V$.
3. $E_G = [U]^2 \cup [V]^2 \cup \{\{u_i, v_i\} \mid i \in I\}$.
4. $d^G(u_i, v_i) = r$ for all $i \in I$.

Claim. Every cycle of G is metric. Let $C = \{c_i \mid i \in n \in \omega\}$ induce a cycle of G. If $C \subseteq U$ or $C \subseteq V$ then, because the only cycles in metric spaces are triangles, the set C induces a triangle which is metric. If $C \cap U \neq \emptyset$ and $C \cap V \neq \emptyset$ then $|C \cap U| = |C \cap V| = 2$. (Every subset P with $|P| < n$ of a cycle of length n induces a set of paths and hence a subgraph with fewer than $|P|$ edges. Every finite subset of U or V induces as many edges as it has elements.) Hence if $C \cap U \neq \emptyset$ and $C \cap V \neq \emptyset$ then there are indices $i, j \in I$ with $i \neq j$ and $C = \{u_i, u_j, v_j, v_i\}$. This cycle is metric because $|d^U(u_i, u_j) - d^V(v_i, v_j))| \leq r$.

The $\mathcal{R}$-graph G is regular because if a and b in G are not adjacent then one of them is in V and the other in U and hence every walk from a to b contains an edge of length r. It follows from Theorem 9 that there exists a metric space

$M = (M = U \cup V; d)$ with $G \preccurlyeq M$ and $\text{dist}(M) \subseteq \mathcal{R}$. Let $W = \{w_i \mid i \in \omega\}$ be a set of points with $w_i = v_i$ for all $i \in I$ and $W \cap U = \emptyset$. Let $f : W \to M$ be given by:

$$f(w_i) = \begin{cases} v_i & \text{for all } i \in I, \\ u_i & \text{otherwise.} \end{cases}$$

Let $W = (W; d^W)$ be the metric space on W for which f is an isometry of W into M. Let $i \in I$ and $j \in \omega \setminus I$ then $|d^U(u_i, u_j) - d^W(w_i, w_j)| = |d^M(u_i, u_j) - d^M(v_i, u_j)| \le r$ because (u_i, v_i, u_j) is a metric triangle in the metric space M. Hence $|d^U(u_i, u_j) - d^W(w_i, w_j)| \le r$ for all $i, j \in \omega$.

We fix the metric space W given by Lemma 7.

For $n \in \omega$ let $\boldsymbol{P}_n$ be the set of all order preserving injections $\alpha : \{0, 1, 2, \ldots, n\} \to \omega$ for which the function $\underline{\alpha} : \{u_i \mid 0 \le i \le n\} \to U$ with $\underline{\alpha}(u_i) = u_{\alpha(i)}$ is an isometry. That is $d(u_{\alpha(i)}, u_{\alpha(j)}) = d(u_i, u_j)$ for all $0 \le i, j \le n$. Let $\boldsymbol{P} = \bigcup_{n \in \omega} \boldsymbol{P}_n$. Let $\boldsymbol{P}_\omega$ be the set of all order preserving injections of $\alpha : \omega \to \omega$ for which the function $\underline{\alpha} : U \to U$ with $\underline{\alpha}(u_i) = u_{\alpha(i)}$ is an isometry. Note that $\boldsymbol{P}_\omega$ is the set of order preserving functions α from ω into ω for which the set $\{u_{\alpha(i)} = \underline{\alpha}(u_i) \mid i \in \omega\}$ induces a copy of U in U. This isometric copy of U in U is denoted by $U_\alpha = (U_\alpha; d^U)$.

For $\alpha \in \boldsymbol{P}_n$ and $\beta \in \boldsymbol{P}_m$ with $n \le m \in \omega$ let $\alpha \sqsubseteq \beta$ if $\alpha(i) = \beta(i)$ for all $i \in n$. It follows that the partial order $(\boldsymbol{P}; \sqsubseteq)$ is a disjoint union of trees each one of which having one element of $\boldsymbol{P}_0$ as a root. If $\boldsymbol{B} \subseteq \boldsymbol{P}$ forms a maximal branch of $(\boldsymbol{P}; \sqsubseteq)$ then $\bigcup \boldsymbol{B}$ is a function in $\boldsymbol{P}_\omega$. If $\alpha \in \boldsymbol{P}_\omega$ let $\alpha_n \in \boldsymbol{P}_n$ be the element with $\alpha_n \sqsubseteq \alpha$. Then $\boldsymbol{B} = \{\alpha_n \mid n \in \omega\}$ is a maximal branch.

Let $P = (\boldsymbol{P}; E_P, d^P)$ be the $\mathcal{R}$-graph with:

1. $E_P = \{(\alpha, \beta) \in [\boldsymbol{P}]^2 \mid \alpha \sqsubseteq \beta\}$.
2. $d^P(\alpha, \beta) = d^W(w_n, w_m)$ for all $\alpha \in \boldsymbol{P}_n$ and $\beta \in \boldsymbol{P}_m$ with $\alpha \sqsubseteq \beta$.

Note that if $\boldsymbol{B}$ is a branch of $(\boldsymbol{P}; \sqsubseteq)$ then $\boldsymbol{B}$ induces a metric space in P and if $\boldsymbol{B}$ is an infinite maximal branch then $\boldsymbol{B}$ induces an isometric copy of W in P. (It is not difficult to see that every maximal branch is infinite, but we do not need this fact.)

Lemma 8. *Every induced cycle of the $\mathcal{R}$-graph $P = (\boldsymbol{P}; E_P, d^P)$ is metric.*

Proof. Let C induce a cycle in P and n maximal so that there is a point $\alpha \in C \cap \boldsymbol{P}_n$. Let β and γ be the two points in C adjacent to α. It follows that $n \ge 2$ and that β and γ are on the branch of the tree $(\boldsymbol{P}; \sqsubseteq)$ below α and hence adjacent. Hence $C = \{\alpha, \beta, \gamma\}$ is a triangle in the metric space induced by the branch below α.

Let $H = (H; E_H, d^H)$ be the $\mathcal{R}$-graph with $H = \boldsymbol{P} \cup U$ and $E_H = E_P \cup [U]^2 \cup \bigcup_{n \in \omega} \{\{\alpha, u_{\alpha(n)}\} \mid \alpha \in \boldsymbol{P}_n\}$ so that:

1. $H \upharpoonright \boldsymbol{P} = P$ and $H \upharpoonright U = U$.
2. $d^H(\alpha, u_{\alpha(n)}) = r$ for all $n \in \omega$ and $\alpha \in \boldsymbol{P}_n$.

Note that in H every point $\alpha \in P$ is adjacent to exactly one point in U. Namely, if $\alpha \in P_n$ then it is adjacent to $u_{\alpha(n)}$. Let $u^{\langle \alpha \rangle}$ be the point in U adjacent to $\alpha \in P$. Then $d^H(\alpha, u^{\langle \alpha \rangle}) = r$ and for $\alpha \sqsubseteq \beta$ two points in P:

$$| d^H(\alpha, \beta) - d^H(u^{\langle \alpha \rangle}, u^{\langle \beta \rangle})| \leq r, \tag{2}$$

because for $\alpha \in P_n$ and $\beta \in P_m$:

$| d^H(\alpha, \beta) - d^H(u^{\langle \alpha \rangle}, u^{\langle \beta \rangle})| = | d^W(w_n, w_m) - d^U(u_{\alpha(n)}, u_{\alpha(m)})| =$
$| d^W(w_n, w_m) - d^U(u_n, u_m)| \leq r$. Inequality 2 implies that:

$$d^H(\alpha, \beta) \leq r \oplus d^H(u^{\langle \alpha \rangle}, u^{\langle \beta \rangle}) \text{ and } d^H(u^{\langle \alpha \rangle}, u^{\langle \beta \rangle}) \leq r \oplus d^H(\alpha, \beta). \tag{3}$$

Lemma 9. *Let* $U^* = (U^*; d^U)$ *be an isometric copy of* U *in* U. *Then there exists an isometric copy* $V^* = (V^*; d)$ *of* V *in* H *with:*

$$V^* \subseteq (U^*)_r.$$

Proof. It follows from Lemma 1 that there exists an enumeration order preserving isometry $\underline{\alpha}$ of U into U with $\alpha(U) \subseteq U^*$. Let $\alpha \in P_\omega$ be the order preserving function of ω into ω with $\underline{\alpha}(u_i) = u_{\alpha(i)}$ and let α_n be the restriction of α to n. then $B = \{\alpha_n \mid n \in \omega\}$ is an infinite maximal chain in $(P; \sqsubseteq)$ and hence isometric to W. Because V is a subspace of W there exists an isometric copy $V^* = (V^*; d)$ of V in B. For every point $\beta \in B$ the distance $d^H(\beta, u^{\langle \beta \rangle}) = r$ and hence: $V^* \subseteq B \subseteq (\alpha(U))_r \subseteq (U^*)_r$.

Lemma 10. *The* $\mathcal{R}$-*graph* H *is metric and regular.*

Proof. Let C induce a cycle in H. It follows from Lemma 8 that if $C \subseteq P$ then C is metric. If L induces a metric subspace of H and $C \subseteq L$ then C is a metric triangle. Hence we may assume that C is not a subset of any metric subspace of H. It follows that $|C \cap L| \leq 2$ for every subset L of H which induces a metric subspace of H. In particular $1 \leq |C \cap U| \leq 2$ and $|C \cap B| \leq 2$ for every branch B of P.

Let $\{\alpha_i \mid i \in l \in \omega\}$ be the set of points in $C \cap P$ for which there does not exist a point $\beta \in C \cap P$ with $\alpha_i \sqsubset \beta$, that is those points in $C \cap P$ which are maximal on their branches. Let B_i the set of points $\beta \in P$ with $\beta \sqsubset \alpha_i$. Then $|C \cap B_i| \leq 2$. Hence for each α_i there is at most on other point, say β_i in $C \cap P$, which implies that the other point of C adjacent to α_i is the point $u^{\langle \alpha_i \rangle}$ in U. That is the two points in C adjacent to α_i are $u^{\langle \alpha_i \rangle}$ and β_i.

Let $c \in C$ be the other point of C adjacent to β_0. If $c \in P$ then it is not possible that $c \sqsubset \beta_0$ because otherwise α_0 is adjacent to c. Hence if $c \in P$ then $\beta_0 \sqsubset c$. If there is an α_i with $c \sqsubset \alpha_i$ the branch below or equal to α_i would contain three points of C. Hence $c = \alpha_i$ for some $i \in l$. Say $i = 1$ and then the other point of C adjacent to β_0 is α_1. That is, $\beta_0 = \beta_1$. Similar to before, the other point of C adjacent to α_1 is the point $u^{\langle \alpha_1 \rangle}$ in U. The circle C contains the path $u^{\langle \alpha_1 \rangle}, \alpha_1, \beta_0, \alpha_0, u^{\langle \alpha_0 \rangle}$. If $u^{\langle \alpha_1 \rangle} \neq u^{\langle \alpha_0 \rangle}$ then $C = \{u^{\langle \alpha_1 \rangle}, \alpha_1, \beta_0, \alpha_0, u^{\langle \alpha_0 \rangle}\}$ because

$\{u^{\langle \alpha_1 \rangle}, u^{\langle \alpha_0 \rangle}\}$ is an edge in H. In order to see that C induces a metric cycle we have to check for each edge $\{x, y\}$ induced by C that $d(x, y)$ is less than or equal to the $\bigoplus$-sum over the other edges. Because $d(\alpha_0, u^{\langle \alpha_0 \rangle}) = d(\alpha_1, u^{\langle \alpha_1 \rangle})$ this will be the case for the edges $\{(\alpha_0, u^{\langle \alpha_0 \rangle})\}$ and $\{(\alpha_0, u^{\langle \alpha_0 \rangle})\}$. Using Inequality 3:

$$d^H(u^{\langle \alpha_0 \rangle}, u^{\langle \alpha_1 \rangle}) \leq d^H(u^{\langle \alpha_0 \rangle}, u^{\langle \beta_0 \rangle}) \oplus d^H(u^{\langle \beta_0 \rangle}, u^{\langle \alpha_1 \rangle})$$

$$\leq \left(r \oplus d^H(\alpha_0, \beta_0)\right) \oplus \left(r \oplus d^H(\beta_0, \alpha_1)\right)$$

and

$$d^H(\alpha_0, \beta_0) \leq r \oplus d^H(u^{\langle \alpha_0 \rangle}, u^{\langle \beta_0 \rangle}) \leq r \oplus d^H(u^{\langle \beta_0 \rangle}, u^{\langle \alpha_1 \rangle}) \oplus d^H(u^{\langle \alpha_0 \rangle}, u^{\langle \alpha_1 \rangle})$$

$$\leq r \oplus d^H(\beta_0, \alpha_1) \oplus r \oplus d^H(u^{\langle \alpha_0 \rangle}, u^{\langle \alpha_1 \rangle})$$

and similar for $d^H(\alpha_1, \beta_0)$.

The remaining possibility is that $c \in U$. Then C contains the path $u^{\langle \alpha_0 \rangle}, \alpha_0, \beta_0, u^{\langle \beta_0 \rangle}$ and hence $C = \{u^{\langle \alpha_0 \rangle}, \alpha_0, \beta_0, u^{\langle \beta_0 \rangle}\}$ because $u^{\langle \beta_0 \rangle}$ and $u^{\langle \alpha_0 \rangle}$ are adjacent in H. The cycle induced by C is metric because:

$$d^H(u^{\langle \alpha_0 \rangle}, u^{\langle \beta_0 \rangle}) \leq d^H(\alpha_0, \beta_0) \oplus r \leq d^H(\alpha_0, \beta_0) \oplus r \oplus r \text{ and}$$

$$d^H(\alpha_0, \beta_0) \leq d^H(u^{\langle \alpha_0 \rangle}, u^{\langle \beta_0 \rangle}) \oplus r \leq d^H(u^{\langle \alpha_0 \rangle}, u^{\langle \beta_0 \rangle}) \oplus r \oplus r.$$

Let a and b be two non adjacent points in H. If one is an element of P and the other of U then every walk from a to b contains an edge of length r. If both are in U then every walk from a to b has length at least $d^H(a, b)$. If both points are in P then every walk from a to b which contains a point of U has length at least r. The remaining case then is that a and b are two non adjacent points in the same connected component of P. This connected component is a tree. Let $n \in \omega$ maximal with $\gamma \in P_n$ and $\gamma \sqsubset a$ and $\gamma \sqsubset b$. Then $d^H(a, b) \geq d^H(a, \gamma)$.

Theorem 10. $\mathcal{R} \subseteq \mathfrak{R}_{\geq 0}$ *be a closed subset of the non negative reals satisfying the 4-values condition with* $0 < r \in \mathcal{R}$. *Let* $U = (U; d^U)$ *and* $V = (V; d^V)$ *be two disjoint metric spaces with* $\text{dist}(U) \cup \text{dist}(V) \subseteq \mathcal{R}$ *and let* $(u_i; i \in \omega)$ *be an enumeration of U. Let* $I \subseteq \omega$ *and* $V = \{v_i \mid i \in I\}$ *be an indexing of V. Let*

$$|d^U(u_i, u_j) - d^V(v_i, v_j)| \leq r \text{ for all } i, j \in I.$$

Then there exists a countable metric space $L = (L; d^L)$ *with* $\text{dist}(L) \subseteq \mathcal{R}$, *containing the metric space U as a subspace, so that for every copy* $U^* = (U^*; d^L)$ *of U in U there exists a copy* $V^* = (V^*; d^L)$ *of V in L so that:*

$$V^* \subseteq (U^*)_r.$$

Proof. The $\mathcal{R}$-graph H is metric according to Lemma 10 and hence it follows from Theorem 9 that there is a metric space $L = (L; d)$ with $H \preccurlyeq L$ and $\text{dist}(L) \subseteq \mathcal{R}$ and $L = U \cup \boldsymbol{P}$. Lemma 9 implies that for every copy $U^* = (U^*; d^L)$ of U in U there exists a copy $V^* = (V^*; d^L)$ of V in L so that:

$$V^* \subseteq \left(U^*\right)_r.$$

7   Finite Approximations

Let $\mathcal{R} \subseteq \Re_{\geq 0}$ be a closed and bounded subset of the non negative reals satisfying the 4-values condition and with 0 as a limit. We will write $\oplus$ for $\oplus_{\mathcal{R}}$. Let $\epsilon > 0$ be given.

Definition 10. For A a finite subset of $\mathcal{R}$ with $\max A = \max \mathcal{R}$ and $l \in \mathcal{R}$ let $\bar{l}^{\langle A \rangle}$ or just $\bar{l}$, if A is understood, be the smallest number in A larger than or equal to l.

A finite set $A \subseteq \mathcal{R}$ is a *finite ϵ-approximation* of $\mathcal{R}$ if $\max A = \max \mathcal{R}$ and $\epsilon > \min A > 0$ and $\bar{l} - l < \epsilon$ for all $l \in \mathcal{R}$.

Because $\mathcal{R}$ is compact there exists a finite ϵ-approximation for every $\epsilon > 0$.

Definition 11. For $A \subseteq \mathcal{R}$ let $\mathfrak{C}(A) = \{a \oplus b \mid a, b \in A\} \cup A$ and $\mathfrak{C}(\emptyset) = A$. The set A is *subadditive closed* if $\mathfrak{C}(A) = A$.

Let $\mathfrak{C}_0(A) = \emptyset$ and $\mathfrak{C}_1(A) = A$ and recursively $\mathfrak{C}_{n+1}(A) = \mathfrak{C}(\mathfrak{C}_n(A))$. (Hence $\emptyset$ is not subadditive closed unless A is empty.)

Note that $\min A = \min \mathfrak{C}(A)$ and if $\max A = \max \mathcal{R}$ then $\max A = \max \mathfrak{C}(A)$ and if A is finite then $\mathfrak{C}(A)$ is finite. Hence if A is a finite ϵ-approximation of $\mathcal{R}$ then $\mathfrak{C}_n(A)$ is a finite ϵ-approximation of $\mathcal{R}$. If $\mathfrak{C}_n(A)$ is subadditive closed then $\mathfrak{C}_{n+1}(A) = \mathfrak{C}_n(A)$ is subadditive closed.

For $\mathfrak{C}_n(A)$ not subadditive closed let $w_{n+1} := \min \left(\mathfrak{C}_{n+1}(A) \setminus \mathfrak{C}_n(A)\right)$. (Hence $w_1 = \min A$.)

Lemma 11. *Let A be a finite ϵ-approximation of $\mathcal{R}$. Let $1 \leq n \in \omega$ so that $\mathfrak{C}_n(A)$ is not subadditive closed, then:*

1. $w_{n+1} \geq w_n \oplus w_1$.
2. $w_{n+1} > w_n$
3. *If $w_{n+1} > w_n \oplus w_1$ then $w_{n+1} \geq w_n + w_1$.*

Proof. There exist $t' \in \mathfrak{C}_n(A) \setminus \mathfrak{C}_{n-1}(A)$ and $t'' \in \mathfrak{C}_n(A)$ so that $w_{n+1} = t' \oplus t''$. Hence $w_{n+1} \geq t' \oplus t'' \geq w_n \oplus w_1 \geq w_n$. It is not the case that $w_{n+1} \neq w_n$ because $w_{n+1} \in \mathfrak{C}_{n+1}(A) \setminus \mathfrak{C}_n(A)$ and $w_n \in \mathfrak{C}_n(A)$. Hence $w_{n+1} > w_n$.

There is no element $x \in \mathcal{R}$ with $w_n \oplus w_1 < x \leq w_n + w_1$. Hence if $w_{n+1} > w_n \oplus w_1$ then $w_{n+1} \geq w_n + w_1$.

Lemma 12. *Let A be a finite ϵ-approximation of $\mathcal{R}$. There is a number $n \in \omega$ so that $\mathfrak{C}_n(A)$ is a subadditive closed ϵ-approximation of $\mathcal{R}$.*

Proof. The set $\mathfrak{C}_n(A)$ is an ϵ-approximation of $\mathcal{R}$ for every $1 \leq n \in \omega$. Hence, because $\mathcal{R}$ is bounded above, it suffices to show: If $n \geq 1$ and $\mathfrak{C}_{n+1}(A)$ is not subadditive closed then $w_{n+2} \geq w_n + w_1$.

If $w_n \oplus w_1 < w_{n+1}$ then $w_{n+2} \geq w_{n+1} \geq w_n + w_1$ according to Lemma 11. Because $w_n \oplus w_1 \leq w_{n+1}$ the remaining case is $w_n \oplus w_1 = w_{n+1}$.

Then using Lemma 11: $w_{n+2} > w_{n+1} = w_n \oplus w_1$. There is no element $x \in \mathcal{R}$ with $w_n \oplus w_1 < x < w_n + w_1$ implying $w_{n+2} \geq w_n + w_1$.

Corollary 2. *For every $\epsilon > 0$ and $0 < r < \epsilon$ with $r \in \mathcal{R}$ exists a finite r-approximation $\mathcal{S}$ of $\mathcal{R}$ with $r \in A$.*

Lemma 13. *Let $A \subseteq \mathcal{R}$ be finite and subadditive closed with $\max A = \max \mathcal{R}$ and for $x \in A$ let $\bar{x} := \bar{x}^{\langle A \rangle}$. Then if $\{a, b, c\} \subseteq \mathcal{R}$ is metric the set $\{\bar{a}, \bar{b}, \bar{c}\}$ is metric.*

Proof. Let $\{a, b, c\}$ be metric with $a \leq b \leq c$. Then $\bar{a} + \bar{b} \geq a + b \geq c$ and $c \in \mathcal{R}$ implies $\bar{a} \oplus \bar{b} \geq c$. This in turn implies that $\bar{c} \leq \bar{a} \oplus \bar{b}$ because $\bar{a} \oplus \bar{b} \in \mathcal{R}$. Finally $\bar{c} \leq \bar{a} \oplus \bar{b} \leq \bar{a} + \bar{b}$.

Lemma 14. *Let $A \subseteq \mathcal{R}$ be finite and subadditive closed with $\max A = \max \mathcal{R}$. Then A satisfies the 4-values condition.*

Proof. For $l \in \mathcal{R}$ let $\bar{l} := \bar{l}^{\langle A \rangle}$.

Let $(a, b, c, d) \in \mathcal{Q}(A)$ with $x \in A$ so that $x \rightsquigarrow (a, b, c, d)$. Because $A \subseteq \mathcal{R}$, and A satisfies the 4-values condition, there is a number $y \in \mathcal{R}$ with $y \rightsquigarrow (a, d, c, b)$ implying that the triples (y, c, b) and (y, a, d) are metric. It follows from Lemma 13 that then the triples $(\bar{y}, \bar{c}, \bar{b}) = (\bar{y}, c, b)$ and $(\bar{y}, \bar{a}, \bar{d}) = (\bar{y}, a, d)$ are metric and hence that $\bar{y} \rightsquigarrow (a, d, c, b)$.

8   Oscillation Stability Theorems

Lemma 15. *Let $\mathcal{R} \subseteq \mathfrak{R}_{\geq 0}$ be a closed and bounded subset of the non negative reals satisfying the 4-values condition and with 0 as a limit. Let $U_{\mathcal{R}} = (U_{\mathcal{R}}; \mathrm{d})$ be an Urysohn metric space with $\mathrm{dist}(U_{\mathcal{R}}) = \mathcal{R}$ and let V be a countable dense subset of $U_{\mathcal{R}}$ and $U_{\mathcal{R}} \upharpoonright V$ be equal to the space $\mathrm{V} = (V; \mathrm{d})$.*

Then: There exists, for every $\epsilon > 0$, a finite set $\mathcal{S} \subseteq \mathcal{R}$ satisfying the 4-values condition and a subset $U_{\mathcal{S}}$ of $U_{\mathcal{R}}$ so that $U_{\mathcal{R}} \upharpoonright U_{\mathcal{S}}$ is the Urysohn metric space $U_{\mathcal{S}}$. For every isometric copy $U_{\mathcal{S}}^* = (U_{\mathcal{S}}^*, \mathrm{d})$ of $U_{\mathcal{S}}$ in $U_{\mathcal{S}}$ there exists an isometric copy $V_{U_{\mathcal{S}}^*} = (V_{U_{\mathcal{S}}^*}; \mathrm{d})$ of V in $U_{\mathcal{R}}$ with:

$$V_{U_{\mathcal{S}}^*} \subseteq \left(U_{\mathcal{S}}^*\right)_{\epsilon}.$$

Proof. Let $\epsilon > 0$ be given and let $0 < r \in \mathcal{R}$ with $r < \epsilon$. Using Corollary 2, let $\mathcal{S}$ be a subadditive closed and finite r-approximation of $\mathcal{R}$ with $r \in \mathcal{S}$. It follows from Lemma 14 that $\mathcal{S}$ satisfies the 4-values condition. Hence there exists a countable Urysohn metric space $\mathrm{U} = (U; \mathrm{d}^U)$ with enumeration, say $U = \{u_i \mid i \in \omega\}$.

We may assume that $U \cap U_{\mathcal{R}} = \emptyset$. For $x, y \in U_{\mathcal{R}}$ let $\bar{\mathrm{d}}(x, y) = \bar{a}^{\langle \mathcal{S} \rangle} \in \mathcal{S}$ for $\mathrm{d}(x, y) = a$. It follows from Lemma 13 that $\bar{\mathrm{d}}$ is a metric on $U_{\mathcal{R}}$ with distances in $\mathcal{S}$. Hence, because of Definition 10:

$$| \mathrm{d}(x, y) - \bar{\mathrm{d}}(x, y)| \leq r \quad \text{for all } x, y \in U_{\mathcal{R}}. \tag{4}$$

Because $\bar{V} = (V; \bar{\mathrm{d}})$ is a countable metric space with $\mathrm{dist}(\bar{V}) \subseteq \mathcal{S}$ there exists an isometric embedding f of $\bar{V}$ into U. Let $I = \{i \in \omega \mid \exists v \in V \, (f(v) = u_i)\}$. Let $V = \{v_i \mid i \in I\}$ be the indexing of V with I so that $f(v_i) = u_i$ for all $i \in I$. It follows then from Inequality 4 that:

$$| \mathrm{d}^{\mathrm{U}}(u_i, u_j) - \mathrm{d}^{\mathrm{V}}(v_i, v_j)| \leq r \quad \text{for all } i, j \in I.$$

We are now in the position to apply Theorem 10 which yields a countable metric space $\mathrm{L} = (L; \mathrm{d}^{\mathrm{L}})$ with $\mathrm{dist}(\mathrm{L}) \subseteq \mathcal{R}$, containing the metric space U as a subspace, so that for every copy $\mathrm{U}^* = (U^*; \mathrm{d}^{\mathrm{L}})$ of U in U there exists a copy $\mathrm{V}^* = (V^*; \mathrm{d}^{\mathrm{L}})$ of V in L so that:

$$V^* \subseteq \left(U^*\right)_r.$$

There exists an isometric embedding g of L into $U_{\mathcal{R}}$, according to Fact 1. Let $U_{\mathcal{S}}$ be the image of U under g. Then $U_{\mathcal{R}}$ restricted to $U_{\mathcal{S}}$ is an Urysohn space $U_{\mathcal{S}}$. If $U_{\mathcal{S}}^* = (U_{\mathcal{S}}^*, \mathrm{d})$ is an isometric copy of $U_{\mathcal{S}}$ in $U_{\mathcal{S}}$, the inverse isometry of g yields an isometric copy of U in U and hence a copy of V which projected via g into $U_{\mathcal{R}}$ gives a copy $V_{U_{\mathcal{S}}^*} = (V_{U_{\mathcal{S}}^*}; \mathrm{d})$ of V in $U_{\mathcal{R}}$ with:

$$V_{U_{\mathcal{S}}^*} \subseteq \left(U_{\mathcal{S}}^*\right)_r \subseteq \left(U_{\mathcal{S}}^*\right)_\epsilon.$$

Theorem 11. *Every bounded, separable, complete, homogeneous, universal metric space $U_{\mathcal{R}}$ with 0 a limit of $\mathcal{R}$ is oscillation stable.*

Proof. On account of Theorem 13 it suffices to prove that $U_{\mathcal{R}} = (U_{\mathcal{R}}; \mathrm{d})$ is approximately indivisible. Let $\epsilon > 0$ and the function $\gamma : U_{\mathcal{R}} \to n \in \omega$ be given and let V be a dense subset of $U_{\mathcal{R}}$. Lemma 15 yields a finite set $\mathcal{S} \subseteq \mathcal{R}$ satisfying the 4-values condition for which the Urysohn space $U_{\mathcal{S}}$ is a subspace of $U_{\mathcal{R}}$ with: For every isometric copy $U_{\mathcal{S}}^* = (U_{\mathcal{S}}^*, \mathrm{d})$ of $U_{\mathcal{S}}$ in $U_{\mathcal{S}}$ there exists an isometric copy $V_{U_{\mathcal{S}}^*} = (V_{U_{\mathcal{S}}^*}; \mathrm{d})$ of V in $U_{\mathcal{R}}$ with:

$$V_{U_{\mathcal{S}}^*} \subseteq \left(U_{\mathcal{S}}^*\right)_{\frac{\epsilon}{2}}.$$

The restriction of γ to $U_{\mathcal{S}}$ maps $U_{\mathcal{S}}$ to n. Because $U_{\mathcal{S}}$ is indivisible according to Theorem 1 there exists $i \in n$ and an isometric copy $U_{\mathcal{S}}^* = (U_{\mathcal{S}}^*; \mathrm{d})$ of $U_{\mathcal{S}}$ in $U_{\mathcal{S}}$ with $U_{\mathcal{S}}^* \subseteq \gamma^{-1}(i)$ and hence a copy $V_{U_{\mathcal{S}}^*} = (V_{U_{\mathcal{S}}^*}; \mathrm{d})$ of V in $U_{\mathcal{R}}$ so that

$$V_{U_{\mathcal{S}}^*} \subseteq \left(U_{\mathcal{S}}^*\right)_{\frac{\epsilon}{2}} \subseteq \left(\gamma^{-1}(i)\right)_{\frac{\epsilon}{2}}.$$

The completion C of $V_{U_{\mathbb{S}}^*}$ in $U_{\mathcal{R}}$ induces in $U_{\mathcal{R}}$ a subspace $\mathrm{C} = (C; \mathrm{d})$ isometric to $U_{\mathcal{R}}$ with:

$$C \subseteq \left(V_{U_{\mathbb{S}}^*}\right)_{\frac{\epsilon}{2}} \subseteq \left(\left(\gamma^{-1}(i)\right)_{\frac{\epsilon}{2}}\right)_{\frac{\epsilon}{2}} \subseteq \left(\gamma^{-1}(i)\right)_{\epsilon}.$$

Let $U_{\mathcal{R}}$ be a bounded, uncountable, separable, complete, homogeneous, universal metric space. Because $U_{\mathcal{R}}$ is separable, 0 is a limit of the set of distances $\mathcal{R}$. Hence we obtain:

Theorem 12. *Every bounded, uncountable, separable, complete, homogeneous, universal metric space is oscillation stable.*

9   Oscillation Stable and Approximately Indivisible

It follows from a general discussion of "oscillation stable" topological groups in [8,15] that a homogeneous metric space $\mathrm{M} = (M; \mathrm{d})$ is oscillation stable if and only if it is approximately indivisible. We will prove in this section the more general fact that a metric space $\mathrm{M} = (M; \mathrm{d})$ is oscillation stable if and only if it is approximately indivisible.

Lemma 16. *If $\mathrm{M} = (M; \mathrm{d})$ is an approximately indivisible metric space then M is oscillation stable.*

Proof. Let $f : M \to \Re$ be bounded and uniformly continuous and let $\epsilon > 0$ be given.

Let $\delta > 0$ be such that $\mathrm{d}(p, q) < \delta$ implies $|f(p) - f(q)| < \frac{1}{3}\epsilon$ for all $p, q \in M$. Let $a, b \in \Re$ be such that $f(M) \subseteq [a, b]$ and let $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ be a partition of $[a, b]$ with $x_{i+1} - x_i < \frac{1}{3}\epsilon$ for all $i \in n$. Let $\gamma : M \to n$ be the function with $\gamma(p) = i$ if $f(p) \in [x_i, x_{i+1})$.

Because M is approximately indivisible there exists an $i \in n$ and a copy $\mathrm{M}^* = (M^*, \mathrm{d})$ of M in M with $M^* \subseteq \left(\gamma^{-1}(i)\right)_{\delta}$. Let $p, q \in M^*$. There exist $u, v \in \gamma^{-1}(i)$ with $\mathrm{d}(p, u) < \delta$ and $\mathrm{d}(q, v) < \delta$. Then:

$$|f(p) - f(q)| \le |f(p) - f(u)| + |f(u) - f(v)| + |f(v) - f(q)| = \epsilon.$$

Lemma 17. *Oscillation stable metric spaces are bounded.*

Proof. Let $\mathrm{M} = (M; \mathrm{d})$ be an oscillation stable metric space and assume for a contradiction that M is unbounded. Fix a sequence $(p_i; i \in \omega)$ of points in M with:

$$\forall (i \in \omega) \left(\mathrm{d}(p_0, p_i) < \mathrm{d}(p_0, p_{i+1})\right) \text{ and } \lim_{i \to \infty} |\mathrm{d}(p_0, p_i) - \mathrm{d}(p_0, p_{i+1})| = \infty.$$

Let $l(i) = d(p_0, p_{i+1} - d(p_0, p_i)$ and $f : M \to \Re$ be the function so that if $d(p_0, x) \in [d(p_0, p_i), d(p_0, p_{i+1})]$, then:

$$f(x) = \frac{1}{l(i)}(d(p_0, x) - d(p_0, p_i)) \text{ if } i \text{ is even,}$$

$$f(x) = \frac{1}{l(i)}(d(p_0, p_{i+1}) - d(p_0, p_x)) \text{ if } i \text{ is odd.}$$

The function f is the composition of a uniformly continuous piecewise linear map from $\Re \to \Re$ with the uniformly continuous distance function $d(p_0, x)$ and hence is uniformly continuous. The range of f is a subset of the interval $[0, 1]$ with $f(p_i) = 0$ if i is even and $f(p_i) = 1$ if i is odd.

Note. If $0 \leq r \in \Re$ and $\epsilon > 0$ then there exists an $i \in \omega$, even, so that $0 \leq f(x) < \epsilon$ for all x with $| d(p_0, x) - d(p_0, p_i)| \leq r$. If $0 \leq r \in \Re$ and $\epsilon > 0$ then there exists an $i \in \omega$, odd, so that $0 \leq 1 - f(x) < \epsilon$ for all x with $| d(p_0, x) - d(p_0, p_i)| \leq r$.

Given $\epsilon > 0$. Because M is oscillation stable there exists an isometric embedding α of M to a copy $M^* = (M^*; d)$ of M in M so that $|f(x) - f(y)| < \frac{1}{3}\epsilon$ for all $x, y \in M^*$. Let $r = d(p_0, \alpha(p_0))$ and i even be such that $0 \leq f(x) < \frac{1}{3}\epsilon$ for all x with $| d(p_0, x) - d(p_0, p_i)| \leq r$. Then $0 \leq f(\alpha(p_i)) < \frac{1}{3}\epsilon$ because $| d(p_0, \alpha(p_i)) - d(p_0, p_i)| = | d(p_0, \alpha(p_i)) - d(\alpha(p_0), \alpha(p_i))| \leq r$. Let j be odd such that $0 \leq 1 - f(x) < \frac{1}{3}\epsilon$ for all x with $| d(p_0, x) - d(p_0, p_j)| \leq r$. Then $0 \leq 1 - f(\alpha(p_j)) < \frac{1}{3}\epsilon$ because $| d(p_0, \alpha(p_j)) - d(p_0, p_j)| = | d(p_0, \alpha(p_j)) - d(\alpha(p_0), \alpha(p_j))| \leq r$.

But then we arrived at the contradiction $|f(\alpha(p_i)) - f(\alpha(p_j))| > \frac{1}{3}\epsilon$.

Lemma 18. *If for every partition of M into two parts (X, Y) and for every $\epsilon > 0$ there exists a copy $M^* = (M^*, d)$ of M in M with $M^* \subseteq (X)_\epsilon$ or $M^* \subseteq (Y)_\epsilon$ then M is indivisible.*

Proof. By induction on the number of parts in the partition. Let $(B_0, B_1, B_2, \ldots, B_{n-1}, B_n)$ be a partition of M and $\epsilon > 0$ be given. Let $X = \bigcup_{i \in n} B_i$ and $Y = B_n$. If there exists a copy $M^* = (M^*, d)$ of M with $M \subseteq (B_n)_{\frac{\epsilon}{2}}$ we are done. Otherwise there exists a copy $M^* = (M^*; d)$ of M with $M^* \subseteq (X)_{\frac{\epsilon}{2}}$.

Lemma 19. *Let $M = (M; d)$ be a metric space and (X, Y) a partition of M and $f : M \to \Re$ with*

$$f(x) = \inf\{d(x, y) \mid y \in Y\}$$

and for all $y \in Y$

$$f(y) = \inf\{d(x, y) \mid x \in X\}.$$

then f is uniformly continuous.

Proof. Let $\epsilon > 0$ and $\delta = \frac{1}{2}\epsilon$ and let $p, q \in M$ with $d(p, q) \leq \frac{1}{2}\epsilon$. If $p \in X$ and $q \in Y$ then $f(p) \leq \frac{1}{2}\epsilon$ and $f(q) \leq \frac{1}{2}\epsilon$ and hence $|f(p) - f(q)| \leq \epsilon$. If $p, q \in X$ there exists a point $u \in Y$ with $d(p, u) < f(p) + \frac{1}{2}\epsilon$ and hence $f(q) \leq d(q, u) < f(p) + \epsilon$ implying $|f(q) - f(p)| < \epsilon$.

Lemma 20. *Let* $M = (M; d)$ *be a metric space and* $k > 0$ *and* (X, Y) *a partition of* M *so that* $d(x, y) \geq \delta$ *for all* $x \in X$ *and* $y \in Y$. *Then the function* $f : M \to \Re$ *with* $f(x) = 0$ *for all* $x \in X$ *and* $f(y) = 1$ *for all* $y \in Y$ *is uniformly continuous.*

Proof. $|f(x) - f(y)| = 0$ for all $x, y \in M$ with $d(x, y) < \delta$.

Theorem 13. *A metric space* $M = (M; d)$ *is oscillation stable if and only if it is approximately indivisible.*

Proof. On account of Lemma 16 it remains to prove that if M is oscillation stable then it is approximately indivisible.

Let $\epsilon > 0$ be given and (X, Y) a partition of M into two parts.

Let $f : M \to \Re$ be the function so that for all $x \in X$

$$f(x) = \inf\{d(x, y) \mid y \in Y\}$$

and for all $y \in Y$

$$f(y) = \inf\{d(x, y) \mid x \in X\}.$$

The function f is uniformly continuous because of Lemma 19.

Hence there exists a copy $M^* = (M^*; d)$ of M in M so that $|f(p) - f(q)| < \frac{\epsilon}{2}$ for all $p, q \in M^*$. If $f(p) < \epsilon$ for all $p \in X \cap M^*$ then $M^* \subseteq (Y)_\epsilon$.

Let $p \in X \cap M^*$ with $f(p) \geq \epsilon$. If there is no $q \in Y \cap M^*$ then $M^* \subseteq X \subseteq (X)_\epsilon$. Let $M^* \cap Y \neq \emptyset$. Let $x \in X \cap M^*$ and $y \in Y \cap M^*$ with $d(x, y) < \frac{\epsilon}{2}$. Then $f(y) < \frac{\epsilon}{2}$ and hence $|f(p) - f(y)| > \frac{\epsilon}{2}$ a contradiction. It follows that $d(x, y) \geq \frac{\epsilon}{2}$ for all $x \in X \cap M^*$ and all $y \in Y \cap M^*$.

The function $g : M^* \to \Re$ with $g(x) = 0$ for all $x \in X \cap M^*$ and $g(y) = 1$ for all $y \in Y \cap M^*$ is uniformly continuous according to Lemma 20. It follows that there exists a copy $M^{**} = (M^{**}; d)$ of M^* in M^* so that $|g(a) - g(b)| < \frac{1}{2}$ for all $a, b \in M^{**}$. This in turn implies that $g(a) = g(b)$ for all $a, b \in M^{**}$ and hence that $M^{**} \subseteq X$ or $M^{**} \subseteq Y$.

10   Cantor Sets and the 4-Values Condition

Let $\mathcal{R} \subseteq \Re_{\geq 0}$. Theorem 5 states that if $\mathcal{R}$ is closed, then it satisfies the 4-values condition if and only if the operation $\oplus$ is associative on $\mathcal{R}$. If $\mathcal{R}$ is not a closed subset of $\Re_{\geq 0}$ then $\mathcal{R}$ can satisfy the 4-values condition but might not be closed under the operation $\oplus$. Let for example $\mathcal{R} = [0, 1) \cup [2, 3]$ then $\mathcal{R}$ satisfies the 4-values condition. We aim to show that there are quite interesting closed subsets $\mathcal{R}$ of the reals which satisfy the 4-values condition and hence quite intricate homogeneous

metric spaces which are oscillation stable. Lemma 21 together with Lemma 22 are the basic tools to generate subsets of the reals which satisfy the 4-values condition. The sets $\mathcal{R} \subseteq \mathfrak{R}_{\geq 0}$ constructed below have 0 as a limit are closed and satisfy the 4-values condition. Hence, according to Theorem 8, for each such set $\mathcal{R}$ there exists a unique Urysohn metric space $\boldsymbol{U}_{\mathcal{R}}$.

The ordinary Cantor set obtained by successively removing the middle open third of the closed intervals does not satisfy the 4-values condition because $(\frac{1}{3} \oplus_{\mathcal{R}} \frac{2}{9}) \oplus_{\mathcal{R}} \frac{1}{9} = \frac{1}{3}$ but $\frac{1}{3} \oplus_{\mathcal{R}} (\frac{2}{9} \oplus_{\mathcal{R}} \frac{1}{9}) = \frac{2}{3}$. But if $(l_1, l_2, l_3, l_4, \dots)$ is a finite or infinite sequence of numbers with $\frac{1}{3} < l_i < 1$ for all i then the set obtained by successively removing the middle open intervals of length l_i times the length of the interval, satisfies the 4-values condition. See below for a more precise account.

Lemma 21. *Let $\mathcal{R}$ be closed and satisfy the 4-values condition and let $c > 0$, then: The set $c\mathcal{R} = \{cr \mid r \in \mathcal{R}\}$ satisfies the 4-values condition. The set $\{x \in \mathcal{R} \mid x \leq c\}$ satisfies the 4-values condition. The interval $[0, 1]$ satisfies the 4-values condition.*

Proof. Let $c > 0$ and $r, s \in \mathcal{R}$. Then $cr \oplus_{c\mathcal{R}} cs = c(r \oplus_{\mathcal{R}} s)$, implying the first assertion. Let $r, s, t \in \{x \in \mathcal{R} \mid x \leq c\} := \mathcal{S}$ and let $m = \max(\mathcal{S})$. If $r + s \geq m$, then $(r \oplus_{\mathcal{S}} s) \oplus_{\mathcal{S}} t = m \oplus_{\mathcal{S}} t = m$ and $s \oplus_{\mathcal{S}} t \geq s$ and hence $r \oplus_{\mathcal{S}} (s \oplus_{\mathcal{S}} t) = m$. Similarly we obtain $(r \oplus_{\mathcal{S}} s) \oplus_{\mathcal{S}} t = r \oplus_{\mathcal{S}} (s \oplus_{\mathcal{S}} t)$ if $t + s \geq m$ or $r + t \geq m$.

Let $r + s \leq m$ and $r + t \leq m$ and $s + t \leq m$. Then $r \oplus_{\mathcal{S}} s = r \oplus_{\mathcal{R}} s$ and $r \oplus_{\mathcal{S}} t = r \oplus_{\mathcal{R}} t$ and $t \oplus_{\mathcal{S}} s = t \oplus_{\mathcal{R}} s$. Hence if $(r \oplus_{\mathcal{S}} s) + t \geq m$ and $r + (s \oplus_{\mathcal{S}} t) < m$ then $(r \oplus_{\mathcal{R}} s) \oplus_{\mathcal{R}} t \geq m$ but $r \oplus_{\mathcal{R}} (s \oplus_{\mathcal{R}} t) < m$. Hence if $(r \oplus_{\mathcal{S}} s) + t \geq m$ then $r + (s \oplus_{\mathcal{S}} t) \geq m$ implying $(r \oplus_{\mathcal{S}} s) \oplus_{\mathcal{S}} t = m = r \oplus_{\mathcal{S}} (s \oplus_{\mathcal{S}} t) \geq m$. Similarly $(r \oplus_{\mathcal{S}} s) \oplus_{\mathcal{S}} t = m = r \oplus_{\mathcal{S}} (s \oplus_{\mathcal{S}} t) \geq m$ if $r + (s \oplus_{\mathcal{S}} t) \geq m$. If $(r \oplus_{\mathcal{S}} s) + t < m$ and $r + (s \oplus_{\mathcal{S}} t) < m$ then $(r \oplus_{\mathcal{S}} s) \oplus_{\mathcal{S}} t = (r \oplus_{\mathcal{R}} s) \oplus_{\mathcal{R}} t = r \oplus_{\mathcal{R}} (s \oplus_{\mathcal{R}} t) = r \oplus_{\mathcal{S}} (s \oplus_{\mathcal{S}} t)$.

The set $\mathfrak{R}_{\geq 0}$ satisfies the 4-values condition, because $r \oplus_{\mathfrak{R}_{\geq 0}} s = r + s$, implying that the set $[0, 1]$ satisfies the 4-values condition.

Lemma 22. *If $\mathcal{R} \subseteq \mathfrak{R}_{\geq 0}$ is closed and satisfies the 4-values condition and $l > 2 \cdot \max(\mathcal{R})$ then the set $\{r + nl \mid n \in \omega \text{ and } r \in \mathcal{R}\}$ is closed and satisfies the 4-values condition.*

Proof. Let $n, m \in \omega$ and $r, s \in \mathcal{R}$. Then $(nl + r) + (ml + s) = (n + m)l + r + s < (n + m)l + l$. Hence $(nl + r) \oplus (ml + s) = (n + m)l + (r \oplus s)$ and we have, for $n, m, k \in \omega$ and $r, s, t \in \mathcal{R}$:

$$\big((nl + r) \oplus (ml + s)\big) \oplus (kl + t) = \big((n + m)l + (r \oplus s)\big) \oplus (kl + t)$$
$$= (n + m + k)l + \big((r \oplus s) \oplus t\big) = (n + m + k)l + \big(r \oplus (s \oplus t)\big)$$
$$= (nl + r) \oplus \big((m + k)l + (s \oplus t)\big) = (nl + r) \oplus \big((ml + s) \oplus (kl + t)\big).$$

Given an interval $[a, b]$ of the reals and $0 \leq w \leq 1$ let $[a(w)b]$ denote the subset of $[a, b]$ obtained by removing the middle open interval of length $w(b - a)$ from $[a, b]$. Note that $[a(0)b] = [a, b]$ and $[a(1)b] = \emptyset$ and that:

$$[a(w)b] = [a, \gamma(a, w, b)] \cup [\delta(a, w, b), b], \text{ for:} \tag{5}$$

$$\gamma(a, w, b) = \frac{1}{2}\big((1 + w)a + (1 - w)b\big), \tag{6}$$

$$\delta(a, w, b) = \frac{1}{2}\big((1 - w)a + (1 + w)b\big). \tag{7}$$

For finite sequences $\mathbf{w} = (w_i; i \in n \in \omega)$ we define recursively the set of disjoint intervals $[a(\mathbf{w})b]$. If $\mathbf{w}$ is the empty sequence then $[a(\mathbf{w})b] = [a, b]$. If $\mathbf{w} = (w)$, the sequence consisting of a single entry, then $[a(\mathbf{w})b] = [a(w)b]$. In general, for $\mathbf{w}_* = (w_i; 1 \leq i \in n)$:

$$[a(\mathbf{w})b] = [a(\mathbf{w}_*)\gamma(a, l_0, b)] \cup [\delta(a, l_0, b), b](\mathbf{w}_*)b].$$

Note that the set $[0(\mathbf{w})b]$ is a scaled version of the set $[0(\mathbf{w})c]$, that is $[0(\mathbf{w})b] = \frac{b}{c} \cdot [0(\mathbf{w})c]$. Hence:

Lemma 23. *Let* $\mathbf{w} = (w_i; i \in n \in \omega)$ *be a sequence of numbers and* b, c *be two positive numbers. Then* $[0(\mathbf{w})b]$ *satisfies the 4-values condition if and only if* $[0(\mathbf{w})c]$ *satisfies the 4-values condition.*

Note that the set $[a(\mathbf{w})b+a]$ is a translation of the set $[0(\mathbf{w})b]$, that is $[a(\mathbf{w})b+a] = \{x + a \mid x \in [0(\mathbf{w})b]\}$. Hence we obtain by induction on n:

Lemma 24. *For all* $n \in \omega$ *and sequences* $\mathbf{w} = (w_i; i \in n \in \omega)$ *with* $\frac{1}{3} < w_i < 1$ *for all* $i \in n$, *the set* $[0(\mathbf{w})1]$ *satisfies the 4-values condition.*

Proof. The interval $[0, 1]$ satisfies the 4-values condition. Let $\frac{1}{3} < w < 1$ and $\mathbf{w} = (w_i; i \in n \in \omega)$. If $[0(\mathbf{w})1]$ satisfies the 4-values condition, the set $[0(\mathbf{w})\frac{1}{2}(1 - w)]$ satisfies the 4-values condition and $[\frac{1}{2}(1 + w)(\mathbf{w})1]$ is an $\frac{1}{2}(1 + w)$ translation of $[0(\mathbf{w})\frac{1}{2}(1 - w)]$. Hence $[0(\mathbf{w})\frac{1}{2}(1 - w)] \cup [\frac{1}{2}(1 + w)(\mathbf{w})1]$ satisfies the 4-values condition, because $\frac{1}{2}(1 + w) > 2 \cdot \frac{1}{2}(1 - w)$ and therefore Lemma 22 applies.

Let $\mathbf{v}$ be the sequence $(w, w_0, w_1, w_2, \ldots, w_{n-1})$. Then $[0, (\mathbf{v})1] = [0(\mathbf{w})\frac{1}{2}(1 - w)] \cup [\frac{1}{2}(1 + w)(\mathbf{w})1]$ and hence $[0(\mathbf{v})1]$ satisfies the 4-values condition.

Definition 12. Let $\mathbf{w} = (w_i; i \in \omega)$ be a sequence with $0 < w_i < 1$ for all $i \in \omega$. Then

$$[0(\mathbf{w})1] = \bigcap_{n \in \omega} \mathbf{W}_n \quad \text{for } \mathbf{W}_n = (w_i \ i \in n).$$

Theorem 14. *Let* $\mathbf{w}$ *be a finite or infinite sequence with* $\frac{1}{3} < w_i < 1$ *for all indices* i. *Then the Cantor type set* $[0(\mathbf{w})1]$ *satisfies the 4-values condition.*

Proof. If **w** is finite, the Theorem follows from Lemma 24. Let $\mathbf{w} = (w_i; i \in \omega)$ and $\mathbf{W}_n = (w_i \ i \in n)$. Let $\mathcal{R}_n = [0(\mathbf{W}_n)1]$ and $\oplus_{\mathcal{R}_n} := \oplus_n$ and let $\mathcal{R}_\infty = [0(\mathbf{w})1]$ and $\oplus = \oplus_{\mathcal{R}_\infty}$.

Note that if x is a boundary point of $\mathcal{R}_n$ for some n, then $x \in \mathcal{R}_m$ and it is a boundary point of $\mathcal{R}_m$ for all $m > n$ and hence an element of $\mathcal{R}_\infty$ and it is a boundary point of $\mathcal{R}_\infty$. This implies that for all $a, b \in \mathcal{R}_\infty$ exists an index n so that $a \oplus_n b = a \oplus_m b$ for all $m \geq n$ and hence $a \oplus b = a \oplus_n b$. Therefore, for all triples $\{a, b, c\}$ of numbers in $\mathcal{R}_\infty$, there exists an index n so that $\oplus_n$ agrees with $\oplus$ on the set $\{a, b, c\}$. Because $\oplus_n$ is associative:

$$(a \oplus b) \oplus c = (a \oplus_n b) \oplus_n c = a \oplus_n (b \oplus_n c) = a \oplus (b \oplus c).$$

We did not use the full strength of Lemma 22 when splitting closed intervals into two parts separated by an open interval. We could have instead split the intervals into finitely many parts separated by open intervals of the same length. Hence for infinite sequences of the type $\mathbf{w} = ((l_i, m_i); i \in \omega)$, with m_i giving the number of parts, defined Cantor type sets of the form $[0(\mathbf{w})1]$. Of course the construction can then also be extended to any countable sequence of this type, yielding quite intricate Urysohn type spaces.

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Euclidean Sections of Convex Bodies

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Abstract This is a somewhat expanded form of a 4 h course given, with small variations, first at the educational workshop Probabilistic methods in geometry, Bedlewo, Poland, July 6–12, 2008 and a few weeks later at the Summer school on Fourier analytic and probabilistic methods in geometric functional analysis and convexity, Kent, Ohio, August 13–20, 2008.

The main part of these notes gives yet another exposition of Dvoretzky's theorem on Euclidean sections of convex bodies with a proof based on Milman's. This material is by now quite standard. Towards the end of these notes we discuss issues related to fine estimates in Dvoretzky's theorem and there are some results that didn't appear in print before. In particular there is an exposition of an unpublished result of Figiel (Claim 1) which gives an upper bound on the possible dependence on ϵ in Milman's theorem. We would like to thank Tadek Figiel for allowing us to include it here. There is also a better version of the proof of one of the results from Schechtman (Adv. Math. **200**(1), 125–135, 2006) giving a lower bound on the dependence on ϵ in Dvoretzky's theorem. The improvement is in the statement and proof of Proposition 2 here which is a stronger version of the corresponding Corollary 1 in Schechtman (Adv. Math. **200**(1), 125–135, 2006).

Key words Dvoretzky's theorem • Euclidean sections • Fine estimates

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1 Lecture 1

By a convex, symmetric body $K \subset \mathbb{R}^n$ we shall refer to a compact set with non-empty interior which is convex and symmetric about the origin (i.e., $x \in K$ implies that $-x \in K$).

This series of lectures will revolve around the following theorem of Dvoretzky.

Theorem 1 (A. Dvoretzky, 1960). *There is a function* $k : (0,1) \times \mathbb{N} \to \mathbb{N}$ *satisfying, for all* $0 < \varepsilon < 1$, $k(\varepsilon, n) \to \infty$ *as* $n \to \infty$, *such that for every* $0 < \varepsilon < 1$, *every* $n \in \mathbb{N}$ *and every convex symmetric body in* $K \subset \mathbb{R}^n$ *there exists a subspace* $V \subseteq \mathbb{R}^n$ *satisfying:*

1. $\dim V = k(\varepsilon, n)$.
2. $V \cap K$ *is "ε-euclidean," which means that there exists* $r > 0$ *such that:*

$$r \cdot V \cap B_2^n \subset V \cap K \subset (1 + \varepsilon) r \cdot V \cap B_2^n.$$

The theorem was proved by Aryeh Dvoretzky [3], answering a question of Grothendieck. The question of Grothendieck was asked in [8] in relation with a paper of Dvoretzky and Rogers [4]. Grothendieck [8] gives another proof of the main application (the existence, in any infinite-dimensional Banach space, of an unconditionally convergent series which is not absolutely convergent) of the result of Dvoretzky and Rogers [4] a version of which is used bellow (Lemma 2).

The original proof of Dvoretzky is very involved. Several simplified proofs were given in the beginning of the 1970s, one by Figiel [5], one by Szankowski [17] and the earliest one, a version of which we will present here, by Milman [10]. This proof which turns out to be very influential is based on the notion of *concentration of measure*. Milman was also the first to get the right estimate ($\log n$) of the dimension $k = k(\varepsilon, n)$ of the almost Euclidean section as the function of the dimension n. The dependence of k on ε is still wide open and we will discuss it in detail later in this survey. Milman's version of Dvoretzky's theorem is the following.

Theorem 2. *For every* $\varepsilon > 0$ *there exists a constant* $c = c(\varepsilon) > 0$ *such that for every* $n \in \mathbb{N}$ *and every convex symmetric body in* $K \subset \mathbb{R}^n$ *there exists a subspace* $V \subseteq \mathbb{R}^n$ *satisfying:*

1. $\dim V = k$, *where* $k \geq c \cdot \log n$.
2. $V \cap K$ *is* ε-euclidean:

$$r \cdot V \cap B_2^n \subset V \cap K \subset (1 + \varepsilon) r \cdot V \cap B_2^n.$$

For example, the unit ball of ℓ_∞^n—the n-dimensional cube—is far from the Euclidean ball. It is easy to see that the ratio of radii of the bounding and the bounded ball is $\sqrt{n}$:

$$B_2^n \subset B_\infty^n \subset \sqrt{n} B_2^n$$

and $\sqrt{n}$ is the best constant. Yet, according to Theorem 2, we can find a subspace of $\mathbb{R}^n$ of dimension proportional to $\log n$ in which the ratio of bounding and bounded balls will be $1 + \varepsilon$.

There is a simple correspondence between symmetric convex sets in $\mathbb{R}^n$ and norms on $\mathbb{R}^n$ given by $\|x\|_K = \inf\{\lambda > 0 \ : \ \frac{x}{\lambda} \in K\}$. The following is an equivalent formulation of Theorem 2 in terms of norms.

Theorem 3. *For every $\varepsilon > 0$ there exists a constant $c = c(\varepsilon) > 0$ such that for every $n \in \mathbb{N}$ and every norm $\|\cdot\|$ in $\mathbb{R}^n$ ℓ_2^k $(1 + \varepsilon)$-embeds in $(\mathbb{R}^n, \|\cdot\|)$ for some $k \geq c \cdot \log n$.*

By "X C-embed in Y" I mean that there exists a one-to-one bounded operator $T : X \to Y$ with $\|T\|\|(T_{|TX})^{-1}\| \leq C$.

Clearly, Theorem 2 implies Theorem 3. Also, Theorem 3 clearly implies a weaker version of Theorem 2, with B_2^n replaced by some ellipsoid (which by definition is an invertible linear image of B_2^n). But, since any k-dimensional ellipsoid easily seen to have a $k/2$-dimensional section which is a multiple of the Euclidean ball, we see that also Theorem 3 implies Theorem 2. This argument also shows that proving Theorem 2 for K is equivalent to proving it for some invertible linear image of K. Before starting the actual proof of Theorem 3 here is *a very vague sketch of the proof:* Consider the unit sphere of ℓ_2^n and the surface of B_2^n, which we will denote by $S^{n-1} = \{x \in \mathbb{R}^n \ : \ \|x\|_2 = 1\}$. Let $\|x\|$ be some arbitrary norm in $\mathbb{R}^n$. The first task will be to show that there exists a "large" set $S_{\text{good}} \subset S^{n-1}$ satisfying $\forall x \in S_{\text{good}}. \|\|x\| - M\| < \varepsilon M$ where M is the average of $\|x\|$ on S^{n-1}. Moreover, we shall see that, depending on the Lipschitz constant of $\|\cdot\|$, the set S_{good} is "almost all" the sphere in the measure sense. This phenomenon is called *concentration of measure.*

The next stage will be to pass from the "large" set to a large dimensional subspace of $\mathbb{R}^n$ contained in it. Denote $O(n)$—the group of orthogonal transformations from $\mathbb{R}^n$ into itself. Choose some subspace V_0 of appropriate dimension k and fix an ε-net N on $V_0 \cap S^{n-1}$. For some $x_0 \in N$, "almost all" transformations $U \in O(n)$ will send it into some point in S_{good}. Moreover, if the "almost all" notion is good enough, we will be able to find a transformation that sends all the points of the ε-net into S_{good}. Now there is a standard approximation procedure that will let us pass from the ε-net to all points in the subspace.

In preparation for the actual proof denote by μ the normalized Haar measure on S^{n-1}—the unique, probability measure which is invariant under the group of orthogonal transformations. The main tool will be the following concentration of measure theorem of Paul Levy (for a proof see e.g. [14]).

Theorem 4 (P. Levy). *Let $f : S^{n-1} \longrightarrow \mathbb{R}$ be a Lipschitz function with a constant L; i.e.,*

$$\forall x, y \in S^{n-1} \ |f(x) - f(y)| \leq L\|x - y\|_2.$$

Then,

$$\mu\{x \in S^{n-1} \ : \ |f(x) - Ef| > \varepsilon\} \leq 2\mathrm{e}^{-\frac{\varepsilon^2 n}{2L^2}}.$$

Remark. The theorem also holds with the expectation of f replaced by its median.

Our next goal is to prove the following theorem of Milman which, gives some lower bound on the dimension of almost Euclidean section in each convex body. It will be the main tool in the proof of Theorem 3.

Theorem 5 (V. Milman). *For every $\varepsilon > 0$ there exists a constant $c = c(\varepsilon) > 0$ such that for every $n \in \mathbb{N}$ and every norm $\|\cdot\|$ in $\mathbb{R}^n$ there exists a subspace $V \subseteq \mathbb{R}^n$ satisfying:*

1. $\dim V = k$, where $k \geq c \cdot \left(\frac{E}{b}\right)^2 n$.
2. For every $x \in V$

$$(1 - \varepsilon)E \cdot \|x\|_2 \leq \|x\| \leq (1 + \varepsilon)E \cdot \|x\|_2.$$

Here $E = \int_{S^{n-1}} \|x\| d\mu$ and b is the smallest constant satisfying $\|x\| \leq b\|x\|_2$.

The definition of b implies that the function $\|\cdot\|$ is Lipschitz with constant b on S^{n-1}. Applying Theorem 4 we get a subset of S^{n-1} of probability very close to one ($\geq 1 - 2e^{-\varepsilon^2 E^2 n/2}$), assuming E is not too small, on which

$$(1 - \varepsilon)E \leq \|x\| \leq (1 + \varepsilon)E. \tag{1}$$

We need to replace this set of large measure with a set which is large in the algebraic sense: a set of the form $V \cap S^{n-1}$ for a subspace V of relatively high dimension. The way to overcome this difficulty is to fix an ε-net in $V_0 \cap S^{n-1}$ (i.e., a finite set such that any other point in $V_0 \cap S^{n-1}$ is of distance at most ε from one of the points in this set) for some *fixed* subspace V_0 (of dimension k to be decided upon later) and show that we can find an orthogonal transformation U such that $\|Ux\|$ satisfies Eq. (1) for each x in the ε-net. A successive approximation argument (the details of which can be found, e.g., in [11], as all other details which are not explained here) then gives a similar inequality (maybe with 2ε replacing ε) for all $x \in V_0 \cap S^{n-1}$, showing that $V = UV_0$ can serve as the needed subspace.

To find the required $U \in O(n)$ we need two simple facts. The first is to notice that if we denote by ν the normalized Haar measure on the orthogonal group $O(n)$, then, using the uniqueness of the Haar measure on S^{n-1}, we get that, for each fixed $x \in S^{n-1}$, the distribution of Ux, where U is distributed according to ν, is μ. It follows that, for each fixed $x \in S^{n-1}$, with ν-probability at least $1 - 2e^{-\varepsilon^2 E^2 n/2}$,

$$(1 - \varepsilon)E \leq \|Ux\| \leq (1 + \varepsilon)E.$$

Using a simple union bound we get that for any finite set $N \subset S^{n-1}$, with ν-probability $\geq 1 - 2|N|e^{-\varepsilon^2 E^2 n/2}$, U satisfies

$$(1 - \varepsilon)E \leq \|Ux\| \leq (1 + \varepsilon)E$$

for all $x \in N$ ($|N|$ denotes the cardinality of N).

Lemma 1. *For every* $0 < \varepsilon < 1$ *there exists an* ε-*net* N *on* S^{k-1} *of cardinality* $\leq \left(\frac{3}{\varepsilon} \right)^k$.

So as long as $2 \left(\frac{3}{\varepsilon} \right)^k e^{-\varepsilon^2 E^2 n/2} < 1$ we can find the required U. This translates into $k \geq c \frac{\varepsilon^2}{\log \frac{3}{\varepsilon}} E^2 n$ for some absolute $c > 0$ as is needed in the conclusion of Theorem 5.

Remark. This proof gives that the $c(\varepsilon)$ in Theorem 5 can be taken to be $c \frac{\varepsilon^2}{\log \frac{3}{\varepsilon}}$ for some absolute $c > 0$. This can be improved to $c(\varepsilon) \geq c\varepsilon^2$ as was done first by Gordon in [7]. (See also [13] for a proof that is more along the lines here.) This later estimate can't be improved as we shall see below in Claim 1.

To prove the lemma, let $N = \{x_i\}_{i=1}^m$ be a maximal set in S^{k-1} such that for all $x, y \in N$ $\|x - y\|_2 \geq \varepsilon$. The maximality of N implies that it is an ε-net for S^{k-1}. Consider $\{B(x_i, \frac{\varepsilon}{2})\}_{i=1}^m$—the collection of balls of radius $\frac{\varepsilon}{2}$ around the x_i-s. They are mutually disjoint and completely contained in $B(0, 1 + \frac{\varepsilon}{2})$. Hence:

$$m \mathrm{Vol}\left(B\left(x_1, \frac{\varepsilon}{2} \right) \right) = \sum \mathrm{Vol}\left(B\left(x_i, \frac{\varepsilon}{2} \right) \right) = \mathrm{Vol}\left(\bigcup B\left(x_i, \frac{\varepsilon}{2} \right) \right)$$
$$\leq \mathrm{Vol}\left(B\left(0, 1 + \frac{\varepsilon}{2} \right) \right).$$

The k homogeneity of the Lebesgue measure in $\mathbb{R}^k$ implies now that $m \leq \left(\frac{1 + \varepsilon/2}{\varepsilon/2} \right)^k = \left(1 + \frac{2}{\varepsilon} \right)^k$.

This completes the sketch of the proof of Theorem 5.

2 Lecture 2

In order to prove Theorem 3 we need to estimate E and b for a general symmetric convex body. Since the problem is invariant under invertible linear transformation we may assume that S^{n-1} is included in K, i.e., $b = 1$. It remains to estimate E from below. As we will see this can be done quite effectively for many interesting examples (we will show the computation for the ℓ_p^n balls). However in general it may happen that E is very small even if we assume as we may that S^{n-1} touches the boundary of K. This is easy to see.

The way to overcome this difficulty is to assume in addition that S^{n-1} is the ellipsoid of maximal volume inscribed in K. An ellipsoid is just an invertible linear image of the canonical Euclidean ball. Given a convex body one can find by compactness an ellipsoid of maximal volume inscribed in it. It is known that this maximum is attained for a unique inscribed ellipsoid but this fact will not be

used in the reasoning below. The invariance of the problem lets us assume that the canonical Euclidean ball is such an ellipsoid. The advantage of this special situation comes from the following lemma.

Lemma 2 (Dvoretzky–Rogers). *Let* $\|\cdot\|$ *be some norm on* $\mathbb{R}^n$ *and denote its unit ball by* $K = B_{\|\cdot\|}$. *Assume the Euclidean ball* $B_2^n = B_{\|\cdot\|_2}$ *is (the) ellipsoid of maximal volume inscribed in* K. *Then there exists an orthonormal basis* $x_1, \ldots, x_n$ *such that*

$$\mathrm{e}^{-1}\left(1 - \frac{i-1}{n}\right) \leq \|x_i\| \leq 1, \quad \textit{for all } 1 \leq i \leq n.$$

Remark 1. This is a weaker version of the original Dvoretzky–Rogers lemma. It shows in particular that half of the x_i-s have norm bounded from below: for all $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ $\|x_i\| \geq (2e)^{-1}$. This is what will be used in the proof of the main theorem.

Proof. First of all choose an arbitrary $x_1 \in S^{n-1}$ of maximal norm. Of course, $\|x_1\| = 1$. Suppose we have chosen $\{x_1, \ldots, x_{i-1}\}$ that are orthonormal. Choose x_i as the one having the maximal norm among all $x \in S^{n-1}$ that are orthogonal to $\{x_1, \ldots, x_{i-1}\}$. Define a new ellipsoid which is smaller in some directions and bigger in others:

$$\mathcal{E} = \left\{ \sum_{i=1}^n a_i x_i \; : \; \sum_{i=1}^{j-1} \frac{a_i^2}{a^2} + \sum_{i=j}^n \frac{a_i^2}{b^2} \leq 1 \right\}.$$

Suppose $\sum_{i=1}^n b_i x_i \in \mathcal{E}$. Then $\sum_{i=1}^{j-1} b_i x_i \in a B_2^n$; hence $\|\sum_{i=1}^{j-1} b_i x_i\| \leq a$. Moreover, for each $x \in span\{x_j, \ldots, x_n\} \cap B_2^n$ we have $\|x\| \leq \|x_j\|$ and since $\sum_{i=j}^n b_i x_i \in b B_2^n$, $\|\sum_{i=j}^n b_i x_i\| \leq \|x_j\| b$. Thus,

$$\left\| \sum_{i=1}^n b_i x_i \right\| \leq \left\| \sum_{i=1}^{j-1} b_i x_i \right\| + \left\| \sum_{i=j}^n b_i x_i \right\| \leq a + \|x_j\| \cdot b.$$

The relation between the volumes of $\mathcal{E}$ and B_2^n is $\mathrm{Vol}(\mathcal{E}) = a^{j-1} b^{n-j+1} \mathrm{Vol}(B_2^n)$. If $a + \|x_j\| \cdot b \leq 1$, then $\mathcal{E} \subseteq K$. Using the fact that B_2^n is the ellipsoid of the maximal volume inscribed in K we conclude that

$$\forall a, b, j \text{ s.t. } a + \|x_j\| \cdot b = 1, \quad a^{j-1} b^{n-j+1} \leq 1.$$

Substituting $b = \frac{1-a}{\|x_j\|}$ and $a = \frac{j-1}{n}$ it follows that for every $j \geq 2$

$$\|x_j\| \geq a^{\frac{j-1}{n-j+1}}(1-a) = \left(\frac{j-1}{n}\right)^{\frac{j-1}{n-j+1}}\left(1 - \frac{j-1}{n}\right) \geq \mathrm{e}^{-1}\left(1 - \frac{j-1}{n}\right).$$

$\square$

We are now ready to prove Theorem 3 and consequently also Theorem 2.

As we have indicated, using Theorem 5, and assuming as we may that B_2^n is the ellipsoid of maximal volume inscribed in $K = B_{\|\cdot\|}$, it is enough to prove that

$$E = \int_{S^{n-1}} \|x\|_d x \geq c \sqrt{\frac{\log n}{n}}, \tag{2}$$

for some absolute constant $c > 0$.

This will prove Theorems 2 and 3 with the bound $k \geq c \frac{\varepsilon^2}{\log \frac{1}{\varepsilon}} \log n$.

We now turn to prove Inequality (2). According to Dvoretzky–Rogers Lemma 2 there are orthonormal vectors $x_1, \ldots, x_n$ such that for all $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ $\|x_i\| \geq 1/2e$

$$\int_{S^{n-1}} \|x\| \, d\mu(x) = \int_{S^{n-1}} \left\| \sum_{i=1}^{n} a_i x_i \right\| \, d\mu(a)$$

$$= \int_{S^{n-1}} \frac{1}{2} \left(\left\| \sum_{i=1}^{n-1} a_i x_i + a_n x_n \right\| + \left\| \sum_{i=1}^{n-1} a_i x_i - a_n x_n \right\| \right) d\mu(a)$$

$$\geq \int_{S^{n-1}} \max \left\{ \left\| \sum_{i=1}^{n-1} a_i x_i \right\|, \|a_n x_n\| \right\} d\mu(a)$$

$$\geq \int_{S^{n-1}} \max \left\{ \left\| \sum_{i=1}^{n-2} a_i x_i \right\|, \|a_{n-1} x_{n-1}\|, \|a_n x_n\| \right\} d\mu(a) \geq \ldots$$

$$\geq \int_{S^{n-1}} \max_{1 \leq i \leq n} \|a_i x_i\| \, d\mu(a) \geq \frac{1}{2e} \int_{S^{n-1}} \max_{1 \leq i \leq \lfloor \frac{n}{2} \rfloor} |a_i| \, d\mu(a).$$

To evaluate the last integral we notice that because of the invariance of the canonical Gaussian distribution in $\mathbb{R}^n$ under orthogonal transformation and (again!) the uniqueness of the Haar measure on S^{n-1}, the vector $(\sum g_i^2)^{-1/2}(g_1, g_2, \ldots, g_n)$ is distributed μ. Here $g_1, g_2, \ldots, g_n$ are i.i.d. $N(0, 1)$ variables. Thus

$$\int_{S^{n-1}} \max_{1 \leq i \leq \lfloor \frac{n}{2} \rfloor} |a_i| \, d\mu(a) = \mathbb{E} \frac{\max_{1 \leq i \leq \lfloor \frac{n}{2} \rfloor} |g_i|}{(\sum_{i=1}^{n} g_i^2)^{1/2}} = \frac{\mathbb{E} \max_{1 \leq i \leq \lfloor \frac{n}{2} \rfloor} |g_i|}{\mathbb{E}(\sum_{i=1}^{n} g_i^2)^{1/2}}. \tag{3}$$

(The last equation follows from the fact that the random vector $(\sum g_i^2)^{-1/2}$ $(g_1, g_2, \ldots, g_n)$ and the random variable $(\sum g_i^2)^{1/2}$ are independent.)

To evaluate the denominator from above note that by Jensen's inequality:

$$\mathbb{E} \left(\sum_{i=1}^{n} g_i^2 \right)^{1/2} \leq \left(\mathbb{E} \sum_{i=1}^{n} g_i^2 \right)^{1/2} = \sqrt{n}.$$

The numerator is known to be of order $\sqrt{\log n}$ (estimate the tail behaviour of $\max_{1 \leq i \leq \lfloor \frac{n}{2} \rfloor} |g_i|$).

This gives the required estimate and concludes the proof of Theorems 2 and 3.

As another application of Theorem 5 we will estimate the almost Euclidean sections of the ℓ_p^n balls $B_p^n = \{x \in \mathbb{R}^n; \|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p} \leq 1\}$.

Using the connection between the Gaussian distribution and μ we can write

$$E_p = \int_{S^{n-1}} \|x\|_p \mathrm{d}\mu = \mathbb{E}\frac{(\sum |g_i|^p)^{1/p}}{(\sum g_i^2)^{1/2}} = \frac{\mathbb{E}(\sum |g_i|^p)^{1/p}}{\mathbb{E}(\sum g_i^2)^{1/2}}.$$

To bound the last quantity from below we will use the following inequality:

$$\sqrt{2/\pi} \cdot n^{1/r} = \left(\sum (\mathbb{E}|g_i|)^r\right)^{1/r} \leq \mathbb{E}\left(\sum |g_i|^r\right)^{1/r} \leq \left(\mathbb{E}\sum |g_i|^r\right)^{1/r} = c_r \cdot n^{1/r}$$

Hence:

$$E_p \geq c_p \cdot n^{\frac{1}{p} - \frac{1}{2}}.$$

For $p > 2$ we have $\|x\|_p \leq \|x\|_2$. For $1 \leq p < 2$ we have $\|x\|_p \leq n^{\frac{1}{p} - \frac{1}{2}} \cdot \|x\|_2$. It now follows from Theorem 5 that the dimension of the largest ε Euclidean section of the ℓ_p^n ball is

$$k \geq \begin{cases} c_p(\varepsilon)n^{\frac{2}{p}}, & 2 < p < \infty \\ c(\varepsilon)n, & 1 \leq p < 2. \end{cases}$$

3 Lecture 3

In this section we will mostly be concerned with the question of how good the estimates we got are. We begin with the last result of the last section concerning the dimension of almost Euclidean sections of the ℓ_p^n balls.

Clearly, for $1 \leq p < 2$ the dependence of k on n is best possible. The following proposition of Bennett, Dor, Goodman, Johnson and Newman [2] shows that this is the case also for $2 < p < \infty$.

Proposition 1. *Let $2 < p < \infty$ and suppose that ℓ_2^k C-embeds into ℓ_p^n, meaning that there exists a linear operator $T : \mathbb{R}^k \to \mathbb{R}^n$ such that*

$$\|x\|_2 \leq \|Tx\|_p \leq C\|x\|_2,$$

then $k \leq c(p, C)n^{2/p}$.

Proof. Let $T : \mathbb{R}^k \to \mathbb{R}^n$, $T = (a_{ij})_{i=1}^n{}_{j=1}^k$ be the linear operator from the statement of the claim. Then for every $x \in \mathbb{R}^k$

$$\left(\sum_{j=1}^k x_j^2 \right)^{1/2} \leq \left(\sum_{i=1}^n \left| \sum_{j=1}^k a_{ij} x_j \right|^p \right)^{1/p} \leq C \left(\sum_{j=1}^k x_j^2 \right)^{1/2}. \qquad (4)$$

In particular, for every $1 \leq l \leq n$, substituting instead of x the lth row of T we get

$$\left(\sum_{j=1}^k a_{lj}^2 \right)^p \leq \sum_{i=1}^n \left| \sum_{j=1}^k a_{ij} a_{lj} \right|^p \leq C^p \left(\sum_{j=1}^k a_{lj}^2 \right)^{p/2}.$$

Hence, for every $1 \leq l \leq n$

$$\left(\sum_{j=1}^k a_{lj}^2 \right)^{p/2} \leq C^p.$$

Let $g_1, \ldots, g_k$ be independent standard normal random variables. Then using the fact that $\sum_{j=1}^k g_j a_j$ has the same distribution as $(\sum_{j=1}^k a_j^2)^{1/2} g_1$ and the left-hand side of the Inequality (4) we have

$$\mathbb{E} \left(\sum_{j=1}^k g_j^2 \right)^{p/2} \leq \mathbb{E} \left(\sum_{i=1}^n \left| \sum_{j=1}^k g_j a_{ij} \right|^p \right)$$

$$= \sum_{i=1}^n \mathbb{E} \left(|g_1|^p \left(\sum_{j=1}^k a_{ij}^2 \right)^{p/2} \right) \leq C^p \mathbb{E} |g_1|^p n.$$

On the other hand we can evaluate $\mathbb{E}(\sum_{j=1}^k g_j^2)^{p/2}$ from below using the convexity of the exponent function for $p/2 > 1$:

$$\mathbb{E} \left(\sum_{j=1}^k g_j^2 \right)^{p/2} \geq \left(\mathbb{E} \sum_{j=1}^k g_j^2 \right)^{p/2} = k^{p/2}.$$

Combining the last two inequalities we get an upper bound for k:

$$k \leq C^2 (\mathbb{E} |g_1|^p)^{2/p} n^{2/p}. \qquad \square$$

Remarks.

1. There exist absolute constants $0 < \alpha \le A < \infty$ such that $\alpha \sqrt{p} \le (\mathbb{E}|g_1|^p)^{1/p} \le A\sqrt{p}$. Hence the estimate we get for $c(p, C)$ is $c(p, C) \le ApC^2$. In particular, for $p = \log n$, we have

$$k \le AC^2 \log n$$

   for an absolute A. $\ell_{\log n}^n$ is e-isomorphic to ℓ_∞^n. Hence, if we C-embed ℓ_2^k into ℓ_∞^n, then $k \le Ac^2 \log n$, which means that the $\log n$ bound in Theorem 2 is sharp.

2. The exact dependence on ε in Theorem 2 is an open question. From the proof we got an estimation $k \ge \frac{c\varepsilon^2}{\log(1/\varepsilon)} \log n$. We will deal more with this issue below.

Although the last result doesn't directly give good results concerning the dependence on ε in Dvoretzky's theorem it can be used to show that one can't expect any better behaviour on ε than ε^2 in Milman's Theorem 5. This was observed by Tadek Figiel and didn't appear in print before. We thank Figiel for permitting us to include it here.

Claim 1 (Figiel). *For any $0 < \epsilon < 1$ and n large enough ($n > \epsilon^{-4}$ will do), there is a 1-symmetric norm, $\|\cdot\|$, on $\mathbb{R}^n$ which is 2-equivalent to the ℓ_2 norm and such that if V is a subspace of $\mathbb{R}^n$ on which the $\|\cdot\|$ and $\|\cdot\|_2$ are $(1+\epsilon)$-equivalent then $\dim V \le C\epsilon^2 n$ (C is an absolute constant).*

Proof. Given ϵ and $n > \epsilon^{-4}$ (say) let $2 < p < 4$ be such that $n^{\frac{1}{p}-\frac{1}{2}} = 2\epsilon$. Put

$$\|x\| = \|x\|_2 + \|x\|_p$$

on $\mathbb{R}^n$. Assume that for some A and all $x \in V$,

$$A\|x\|_2 \le \|x\| \le (1+\epsilon)A\|x\|_2.$$

Clearly, $1 + \frac{\epsilon}{2} \le \frac{1+n^{\frac{1}{p}-\frac{1}{2}}}{1+\epsilon} \le A \le 2$ and be get that for all $x \in V$,

$$(A-1)\|x\|_2 \le \|x\|_p \le ((1+\epsilon)A - 1)\|x\|_2 = (A - 1 + \epsilon A)\|x\|_2.$$

Since $\epsilon A \le n^{\frac{1}{p}-\frac{1}{2}} \le 4(A-1)$, we get that, for $B = A - 1$,

$$B\|x\|_2 \le \|x\|_p \le 5B\|x\|_2.$$

It follows from [BDGJN] that for some absolute C,

$$\dim V \le Cn^{2/p} = C(n^{\frac{1}{p}-\frac{1}{2}})^2 n = 4C\epsilon^2 n. \qquad \square$$

Next we will see another relatively simple way of obtaining an upper bound on k in Dvoretzky's theorem, which, unlike the estimate in Remark 1, tends to 0 as $\varepsilon \to 0$. It still leaves a big gap with the lower bound above.

Claim 2. *If ℓ_2^k $(1 + \varepsilon)$-embeds into ℓ_∞^n, then*

$$k \le \frac{C \log n}{\log(1/c\varepsilon)},$$

for some absolute constants $0 < c, C < \infty$.

Proof. Assume we have $(1-\varepsilon)^{-1}$-embedding of ℓ_2^k into ℓ_∞^n, i.e., we have an operator $T = (a_{ij})_{i=1}^n{}_{j=1}^k$ satisfying, for every $x \in \mathbb{R}^k$,

$$(1 - \varepsilon) \left(\sum_{j=1}^k x_j^2 \right)^{1/2} \le \max_{1 \le i \le n} \left| \sum_{j=1}^k a_{ij} x_j \right| \le \left(\sum_{j=1}^k x_j^2 \right)^{1/2}. \tag{5}$$

This means that there exist vectors $v_1, \ldots, v_n \in \mathbb{R}^k$ such that for every $x \in \mathbb{R}^k$:

$$(1 - \varepsilon)\|x\|_2 \le \max_{1 \le i \le n} < v_i, x > \le \|x\|_2. \tag{6}$$

In particular, $\|v_i\|_2 \le 1$ for every $1 \le i \le n$.

Suppose $x \in S^{k-1}$, then the left-hand side of Eq. (6) states that there exists an $1 \le i \le n$ such that $< v_i, x > \ge (1 - \varepsilon)$; hence

$$\|x - v_i\|_2^2 = \|x\|_2^2 + \|v_i\|_2^2 - 2 < v_i, x > \le 2 - 2(1 - \varepsilon) = 2\varepsilon.$$

Thus, the vectors $v_1, \ldots, v_n$ form a $\sqrt{2\varepsilon}$-net on the S^{k-1}, which means that n is much larger (exponentially) than k.
Indeed, we have

$$\bigcup_{i=1}^n B(v_i, 2\sqrt{2\varepsilon}) \supseteq B_2^k \setminus (1 - \sqrt{2\varepsilon}) B_2^k$$

$$\Rightarrow \quad n \, Vol \, B(0, 2\sqrt{2\varepsilon}) \ge Vol \, B(0, 1) - Vol \, B(0, 1 - \sqrt{2\varepsilon})$$

$$\Rightarrow \quad n(2\sqrt{2\varepsilon})^k \ge 1 - (1 - \sqrt{2\varepsilon})^k \ge \sqrt{2\varepsilon} k (1 - \sqrt{2\varepsilon})^{k-1}.$$

This gives for $\varepsilon < \frac{1}{32}$ and $k \ge 12$

$$n \ge \frac{k}{2} \left(\frac{1}{4\sqrt{2\varepsilon}} \right)^{k-1} \ge \left(\frac{1}{4\sqrt{2\varepsilon}} \right)^{k/2}$$

or

$$k \leq \frac{4 \log n}{\log \frac{1}{32\varepsilon}}. \qquad \qquad \square$$

This shows that the $c(\epsilon)$ in the statement of Theorem 2 can't be larger than $\frac{C}{\log(1/c\varepsilon)}$.

Our last objective in this survey is to improve somewhat the lower estimate on $c(\epsilon)$ in the version of Dvoretzky's theorem we proved. For that we will need the inverse to Claim 2.

Claim 3. ℓ_2^k $(1 + \varepsilon)$-*embeds into* ℓ_∞^n *for*

$$k = \frac{c \log n}{\log(1/c\varepsilon)}$$

for some absolute constants $0 < c, C < \infty$.

The proof is very simple and we only state the embedding. Use Lemma 1 to find an ϵ-net $\{x_i\}_{i=1}^n$ on s^{k-1} where k and n are related as in the statement of the claim. The embedding of ℓ_2^k into ℓ_∞^n is given by $x \to \{\langle x, x_i \rangle\}_{i=1}^n$.

4   Lecture 4

In this last section we will prove a somewhat improved version of Dvoretzky's theorem, replacing the ϵ^2 dependence by ϵ (except for a log factor).

Theorem 6. *There is a constant* $c > 0$ *such that for all* $n \in \mathbb{N}$ *and all* $\epsilon > 0$, *every n-dimensional normed space* ℓ_2^k $(1 + \varepsilon)$-*embeds in* $(\mathbb{R}^n, \|\cdot\|)$ *for some* $k \geq \frac{c\epsilon}{(\log \frac{1}{\epsilon})^2} \log n$.

The idea of the proof is the following: We start as in the proof of Milman's Theorem 5, assuming S^{n-1} is the ellipsoid of maximal volume inscribed in the unit ball of $B_{\|\cdot\|}$. If E is large enough (so that $\epsilon^2 E^2 n \geq \frac{\epsilon}{(\log \frac{1}{\epsilon})^2} \log n$) we get the result from Milman's theorem. If not, we will show that the space actually contains a relatively high dimensional ℓ_∞^m and then use Claim 1 to get an estimate on the dimension of the embedded ℓ_2^k.

The main proposition is the following one which improves the main proposition of [15]:

Proposition 2. *Let* $(X, \| \cdot \|)$ *be a normed space and let* $x_1, \ldots, x_n$ *be a sequence in X satisfying* $\|x_i\| \geq 1/10$ *for all i and*

$$\mathbb{E}\Big(\| \sum_{i=1}^n g_i x_i \| \Big) \leq L \sqrt{\log n}. \qquad (7)$$

Then, there is a subspace of X of dimension $k \geq \frac{n^{1/4}}{CL}$ which is CL-isomorphic to ℓ_∞^k. C is a universal constant.

Let us assume the proposition and continue with the

Proof (Proof of Theorem 6). We start as in the proof of Theorem 2, assuming B_2^n is the ellipsoid of maximal volume inscribed in the unit ball of $(\mathbb{R}^n, \| \cdot \|)$. As we already said we may assume $\epsilon^2 E^2 n \leq \frac{\epsilon}{(\log \frac{1}{\epsilon})^2} \log n$ or $E\sqrt{n} \leq \frac{\sqrt{\log n}}{\sqrt{\epsilon} \log \frac{1}{\epsilon}}$. Let $x_1, \ldots, x_n$ be the orthonormal basis given by the Dvoretzky–Rogers lemma, so that in particular $\|x_i\| \geq 1/10$ for $i = 1, \ldots, n/2$. It follows from the triangle inequality for the first inequality and from the relation between the distribution of a canonical Gaussian vector and the Haar measure on the sphere that

$$
\mathbb{E}\left(\| \sum_{i=1}^{n/2} g_i x_i \| \right) \leq \mathbb{E}\left(\| \sum_{i=1}^{n} g_i x_i \| \right) \leq CE\sqrt{n}
$$

So

$$
\mathbb{E}\left(\| \sum_{i=1}^{n/2} g_i x_i \| \right) \leq \frac{\sqrt{\log n}}{\sqrt{\epsilon} \log \frac{1}{\epsilon}}
$$

and by Proposition 2 there is a subspace of $(\mathbb{R}^n, \| \cdot \|)$ of dimension $k \geq \frac{n^{1/4}}{CL}$ which is CL-isomorphic to ℓ_∞^k where $L = \frac{1}{\sqrt{\epsilon} \log \frac{1}{\epsilon}}$. It now follows from an iteration result of James (see Lemma 3 below and Corollary 1 following it) that for any $0 < \epsilon < 1$ there is a subspace of $(\mathbb{R}^n, \| \cdot \|)$ of dimension $k \geq cn^{\frac{c\epsilon}{\log L}}$ which is $1 + \epsilon$ - isomorphic to ℓ_∞^k. $c > 0$ is a universal constant. We now use Claim 1 to conclude that ℓ_2^k embeds in our space for some $k \geq \frac{c \log(cn^{\frac{c\epsilon}{\log L}})}{\log(1/c\epsilon)} = \frac{c'\epsilon \log n}{(\log(1/c\epsilon))^2}$. $\qquad\square$

The following simple lemma is due to R.C. James:

Lemma 3. *Let $x_1, \ldots, x_m$ be vectors in some normed space X such that $\|x_i\| \geq 1$ for all i and*

$$
\left\| \sum_{i=1}^{m} a_i x_i \right\| \leq L \max_{1 \leq i \leq m} |a_i|
$$

for all sequences of coefficients $a_1, \ldots, a_m \in \mathbb{R}$. Then X contains a sequence $y_1, \ldots, y_{\lfloor \sqrt{m} \rfloor}$ satisfying $\|y_i\| \geq 1$ for all i and

$$
\left\| \sum_{i=1}^{\lfloor \sqrt{m} \rfloor} a_i y_i \right\| \leq \sqrt{L} \max_{1 \leq i \leq \lfloor \sqrt{m} \rfloor} |a_i|
$$

for all sequences of coefficients $a_1, \ldots, a_{\lfloor \sqrt{m} \rfloor} \in \mathbb{R}$.

Proof. Let σ_j, $j = 1, \ldots, \lfloor \sqrt{m} \rfloor$ be disjoint subsets of $\{1, \ldots, m\}$ each of cardinality $\lfloor \sqrt{m} \rfloor$. If for some j

$$\left\| \sum_{i \in \sigma_j} a_i x_i \right\| \leq \sqrt{L} \max_{i \in \sigma_j} |a_i|$$

for all sequences of coefficients, we are done. Otherwise, for each j we can find a vector $y_j = \sum_{i \in \sigma_j} a_i x_i$ such that $\|y_j\| = 1$ and $\sqrt{L} \max_{i \in \sigma_j} |a_i| < 1$. But then,

$$\left\| \sum_{j=1}^{\lfloor \sqrt{m} \rfloor} b_j y_j \right\| \leq L \max_{j, i \in \sigma_j} |b_j a_i| \leq L \max_j |b_j| \sqrt{L^{-1}} = \sqrt{L} \max_j |b_j|. \qquad \square$$

Corollary 1. *If ℓ_∞^m L-embeds into a normed space X, then for all $0 < \epsilon < 1$, ℓ_∞^k $\frac{1+\epsilon}{1-\epsilon}$-embeds into X for $k \sim m^{\epsilon/\log L}$.*

Proof. By iterating the lemma (pretending for the sake of simplicity of notation that $m^{2^{-s}}$ is an integer for all the relevant s-s), for all positive integer t there is a sequence of length $k = m^{2^{-t}}$ of norm one vectors $x_1, \ldots, x_k$ in X satisfying

$$\left\| \sum_{i=1}^{k} a_i x_i \right\| \leq L^{2^{-t}} \max |a_i|$$

for all coefficients. Pick a t such that $L^{2^{-t}} = 1 + \epsilon$ (approximately); i.e., $2^{-t} = \frac{\log 1+\epsilon}{\log L} \sim \frac{\epsilon}{\log L}$. Thus $k \sim m^{\epsilon/\log L}$ and

$$\left\| \sum_{i=1}^{k} a_i x_i \right\| \leq (1 + \varepsilon) \max |a_i|.$$

To get a similar lower bound on $\| \sum_{i=1}^{k} a_i x_i \|$, assume without loss of generality that $\max |a_i| = a_1$. Then

$$\begin{aligned}
\| \textstyle\sum_{i=1}^{k} a_i x_i \| &= \| 2a_1 x_1 - (a_1 x_1 - \textstyle\sum_{i=2}^{k} a_i x_i) \| \\
&\geq 2a_1 - \| a_1 x_1 - \textstyle\sum_{i=2}^{k} a_i x_i \| \\
&\geq 2a_1 - (1 + \epsilon)a_1 = (1 - \epsilon) \max |a_i|. \qquad \square
\end{aligned}$$

We are left with the task of proving Proposition 2. We begin with

Claim 4. *Let $x_1, \ldots, x_n$ be normalized vectors in a normed space. Then for all real $a_1, \ldots, a_n$,*

$$\text{Prob}_{\epsilon_i = \pm 1}\left(\left\|\sum_{i=1}^{n} \epsilon_i a_i x_i\right\| < \max_{1 \le i \le n} |a_i|\right) \le 1/2.$$

Proof. Assume as we may $a_1 = \max_{1 \le i \le n} |a_i|$. If $\|a_1 x_1 + \sum_{i=2}^{n} \epsilon_i a_i x_i\| < a_1$ then

$$\left\|a_1 x_1 - \sum_{i=2}^{n} \epsilon_i a_i x_i\right\| \ge 2a_1 - \left\|a_1 x_1 + \sum_{i=2}^{n} \epsilon_i a_i x_i\right\| > a_1$$

and thus

$$P\left(\left\|\sum_{i=1}^{n} \epsilon_i a_i x_i\right\| > a_1\right) \ge P\left(\left\|\sum_{i=1}^{n} \epsilon_i a_i x_i\right\| < a_1\right).$$

So

$$
\begin{aligned}
1 &\ge P(\|\textstyle\sum_{i=1}^{n} \epsilon_i a_i x_i\| \ne \max |a_i|) \\
&= P(\|\textstyle\sum_{i=1}^{n} \epsilon_i a_i x_i\| < a_1) + P(\|\textstyle\sum_{i=1}^{n} \epsilon_i a_i x_i\| > a_1) \\
&\ge 2P(\|\textstyle\sum_{i=1}^{n} \epsilon_i a_i x_i\| < a_1).
\end{aligned}
$$
□

Remark 2. If $x_1 = x_2$, $a_1 = a_2 = 1$ and $a_3 = \cdots = a_n = 0$ then the $1/2$ in the statement of Claim 4 cannot be replaced by any smaller constant.

Proposition 3. *Let $x_1, \ldots, x_n$ be vectors in a normed space with $\|x_i\| \ge 1/10$ for all i and let $g_1, \ldots, g_n$ be a sequence of independent standard Gaussian variables. Then, for n large enough,*

$$P\left(\left\|\sum_{i=1}^{n} g_i x_i\right\| < \frac{\sqrt{\log n}}{100}\right) \le 2/3.$$

Proof. Note first that it follows from Claim 4 that

$$P\left(\left\|\sum_{i=1}^{n} g_i x_i\right\| < \max_{1 \le i \le n} |g_i| \|x_i\|\right) \le \frac{1}{2}. \tag{8}$$

This is easily seen by noticing that $(g_1 \ldots, g_n)$ is distributed identically to $(\varepsilon_1 |g_1| \ldots, \varepsilon_n |g_n|)$ where $\varepsilon_1 \ldots, \varepsilon_n$ are independent random signs independent of the g_i-s. Now compute

$$P\left(\left\|\sum_{i=1}^{n} \varepsilon_i |g_i| x_i\right\| < \max_{1 \le i \le n} |g_i| \|x_i\|\right)$$

by first conditioning on the g_i-s. We use (8) in the following sequence of inequalities:

$$P\left(\left\|\sum_{i=1}^{n} g_i x_i\right\| < \frac{\sqrt{\log n}}{100}\right)$$

$$\leq P\left(\left\|\sum_{i=1}^{n} g_i x_i\right\| < \frac{\sqrt{\log n}}{100} \ \& \ \frac{\sqrt{\log n}}{100} < \max_{1\leq i\leq n} |g_i| \|x_i\|\right)$$

$$+ P\left(\max_{1\leq i\leq n} |g_i| \|x_i\| \leq \frac{\sqrt{\log n}}{100}\right)$$

$$\leq P\left(\left\|\sum_{i=1}^{n} g_i x_i\right\| < \max_{1\leq i\leq n} |g_i| \|x_i\|\right)$$

$$+ P\left(\max_{1\leq i\leq n} |g_i| \leq \frac{\sqrt{\log n}}{10}\right)$$

$$\leq \frac{1}{2} + (1 - e^{-c\log n})^n \text{ for } n \text{ large enough}$$

$$\leq \frac{1}{2} + e^{-n^{1-c}} \leq \frac{2}{3}. \qquad\qquad \square$$

In the proof of Proposition 2 we shall use a theorem of Alon and Milman [1] (see [18] for a simpler proof) which has a very similar statement: Gaussians are replaced by random signs and $\sqrt{\log n}$ by a constant.

Theorem 7 (Alon and Milman). *Let $(X, \|\cdot\|)$ be a normed space and let $x_1, \ldots, x_n$ be a sequence in X satisfying $\|x_i\| \geq 1$ for all i and*

$$\mathbb{E}_{\epsilon_i = \pm 1}\left(\left\|\sum_{i=1}^{n} \epsilon_i x_i\right\|\right) \leq L. \tag{9}$$

Then, there is a subspace of X of dimension $k \geq \frac{n^{1/2}}{CL}$ which is CL-isomorphic to ℓ_∞^k. C is a universal constant.

Proof of Proposition 2. Let $\sigma_1, \ldots, \sigma_{\lfloor\sqrt{n}\rfloor} \subset \{1, \ldots, n\}$ be disjoint with $|\sigma_j| = \lfloor\sqrt{n}\rfloor$ for all j. We will show that there is a subset $J \subset \{1, \ldots, \lfloor\sqrt{n}\rfloor\}$ of cardinality at least $\frac{\sqrt{n}}{4}$ and there are $\{y_j\}_{j\in J}$ with y_j supported on σ_j such that $\|y_j\| = 1$ for all $j \in J$ and

$$\mathbb{E}_{\epsilon_i = \pm 1}\left(\left\|\sum_{j\in J} \epsilon_j y_j\right\|\right) \leq 80L.$$

We then apply the theorem above.

To show this notice that the events $\|\sum_{i\in\sigma_j} g_i x_i\| < \frac{\sqrt{\log n}}{200}$, $j = 1,\ldots,\lfloor\sqrt{n}\rfloor$, are independent and by Proposition 3 have probability at most $2/3$ each. So with probability at least $1/2$ there is a subset $J \subset \{1,\ldots,\lfloor\sqrt{n}\rfloor\}$ with $|J| \geq \frac{\lfloor\sqrt{n}\rfloor}{4}$ such that $\|\sum_{i\in\sigma_j} g_i x_i\| > \frac{1}{200}\sqrt{\log n}$ for all $j \in J$. Denote the event that such a J exists by A. Let $\{r_j\}_{j=1}^{\lfloor\sqrt{n}\rfloor}$ be a sequence of independent signs independent of the original Gaussian sequence. We get that

$$
\begin{aligned}
L\sqrt{\log n} &\geq \mathbb{E}_g\left(\|\sum_{j=1}^{\lfloor\sqrt{n}\rfloor}\sum_{i\in\sigma_j} g_i x_i\|\right) = \mathbb{E}_r\mathbb{E}_g\left(\|\sum_{j=1}^{\lfloor\sqrt{n}\rfloor} r_j \sum_{i\in\sigma_j} g_i x_i\|\right)\\
&\geq \mathbb{E}_r\mathbb{E}_g\left(\|\sum_{j=1}^{\lfloor\sqrt{n}\rfloor} r_j \sum_{i\in\sigma_j} g_i x_i\|\mathbf{1}_A\right)\\
&\geq \tfrac{1}{2}\mathbb{E}_g\left(\left(\mathbb{E}_r\|\sum_{j=1}^{\lfloor\sqrt{n}\rfloor} r_j \sum_{i\in\sigma_j} g_i e_i\|\right)\Big/A\right).
\end{aligned}
$$

It follows that for some $\omega \in A$, there exists a $J \subset \{1,\ldots,\lfloor\sqrt{n}\rfloor\}$ with $|J| \geq \frac{\lfloor\sqrt{n}\rfloor}{4}$ such that putting $\bar{y}_j = \sum_{i\in\sigma_j} g_i(\omega)x_i$, one has $\|\bar{y}_j\| > \frac{1}{200}\sqrt{\log n}$ for all $j \in J$ and

$$
\mathbb{E}_r\left(\left\|\sum_{j\in J} r_j \bar{y}_j\right\|\right) \leq 2L\sqrt{\log n}.
$$

Take $y_j = \bar{y}_j/\|\bar{y}_j\|$.

In the list of references below we included also some books and expository papers not directly referred to in the text above.

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Duality on Convex Sets in Generalized Regions

Alexander Segal and Boaz A. Slomka

Abstract Recently, the duality relation on several families of convex sets was shown to be completely characterized by the simple property of reversing order. The families discussed in aforementioned results were convex sets in $\mathbb{R}^n$. Our goal in this note is to generalize this type of results to regions in $\mathbb{R}^n$ bounded between two convex sets.

Key words Duality of convex bodies • Fractional linear transformations • Order isomorphism

Mathematical Subject Classifications (2010): 26B25, 06D50, 52A20

1 Introduction

Let A and B be two convex sets in $\mathbb{R}^n$ such that $A \subset B$. Denote by $\mathcal{K}^n(A, B)$ the class of all closed convex sets containing A and contained in B:

$$\mathcal{K}^n(A, B) = \{K \in \mathcal{K}^n : A \subseteq K \subseteq B\}.$$

Note that we may consider simply all sets contained in a given set B, or alternatively all sets containing A, if we allow for $A = \phi$ and $B = \mathbb{R}^n$, so that for example $\mathcal{K}^n = \mathcal{K}^n(\emptyset, \mathbb{R}^n)$. Our goal is to determine all order-preserving isomorphisms and as a result order-reversing isomorphisms on this class. Note that characterizations for the special cases $\mathcal{K}^n(0, \mathbb{R}^n)$ and $\mathcal{K}^n(\emptyset, \mathbb{R}^n)$ were done by Artstein–Milman in [1, 2]. In addition, the family of convex bodies with zero in the interior was considered by Böröczky and Schneider in [4]. A special case of the aforementioned

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family where the bodies are origin symmetric was considered by Gruber in [5]. A somewhat different family of closed convex cones was also treated by Schneider in [9]. We remark that in the aforementioned papers [4, 5, 9] the authors actually consider a more general setting, namely the classification of the endomorphisms of corresponding lattices. In this paper we generalize these theorems to the class $\mathcal{K}^n(A, B)$ for any convex A, B such that A is compact. To state our first result, we introduce the class of n-dimensional fractional linear maps, which will play a central role in this paper. In the sequel we assume we have some fixed Euclidean structure $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$.

Definition 1. A map $F : \mathbb{R}^n \to \mathbb{R}^n$ will be called fractional linear if

$$F(x) = \frac{Ax + b}{\langle c, x \rangle + d},$$

where A is a $n \times n$ matrix, $b, c \in \mathbb{R}^n$ and $d \in \mathbb{R}$, such that the matrix

$$\hat{A} = \begin{pmatrix} A & b \\ c & d \end{pmatrix}$$

is invertible.

We denote the family of fractional linear maps by $F.L(n)$. Note that a fractional linear map is actually the restriction of a projective map. For further information of such maps, we refer the reader to [3, 10]. We will use a result by Shiffman, roughly stating that any injective map, that preserves, in some sense, most of the intervals, on some open set must be fractional linear. More precisely, given an open set U, denote by $\mathcal{L}(U)$ the set of lines in $\mathbb{R}^n$ intersecting U. Then:

Theorem 1 (Shiffman). *Let $n \geq 2$. Let $U \subset \mathbb{R}^n$ be an open connected set and let $\mathcal{L}_0$ be an open subset of $\mathcal{L}(U)$ that covers U. Assume that $F : U \to \mathbb{R}^n$ is an injective continuous map and that $F(l \cap U)$ is contained in a line for every $l \in \mathcal{L}_0$. Then F is a fractional linear map.*

Let us state our main result.

Theorem 2. *Let $n \geq 2$ and $A_1, B_1, A_2, B_2 \in \mathcal{K}^n$ such that A_1, A_2 are compact and $A_i \subset B_i$ for $i = 1, 2$. Also, assume that $int(B_1) \neq \emptyset$. Let $T : \mathcal{K}^n(A_1, B_1) \to \mathcal{K}^n(A_2, B_2)$ be a bijective mapping satisfying for all $K, L \in \mathcal{K}^n(A_1, B_1)$:*

$$K \subset L \Leftrightarrow T(K) \subset T(K).$$

Then there exists $F \in F.L(n)$ such that $T(K) = F(K)$ for all $K \in \mathbb{K}$. Moreover, $A_2 = F(A_1)$ and $B_2 = F(B_1)$.

As a corollary of Theorem 2 we get a characterization of duality on the class $\mathcal{K}^n(A, B)$ in case where $0 \in A$. The result is presented in Sect. 3.6.

Remark 1. A theorem in the spirit of Theorem 2 was proved for the case of convex functions by Artstein–Florentin–Milman in [3]. For the sake of completeness, we state the result here; given two compact sets $K \subset T$ define the class

$$Cvx_T(K) = \{f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} : f \text{ is convex }, 1_T^\infty \leq f \leq 1_K^\infty\},$$

where $1_K^\infty = -\log 1_K$ is the convex indicator function.

Theorem 3 (Artstein–Florentin–Milman). *Let $n \geq 2$. Let $A_1 \subset B_1$ and $A_2 \subset B_2$ be compact convex sets. Let $T : Cvx_{A_1}(B_1) \to Cvx_{A_2}(B_2)$ be an order-preserving bijection. Then, there exists a fractional linear map $F : B_1 \times \mathbb{R}^+ \to B_2 \times \mathbb{R}^+$ such that for every $f \in Cvx_{A_1}(B_1)$ we have*

$$epi(Tf) = F(epif),$$

where $epi(f)$ is the epigraph of f.

We remark that in the heart of all previous results (where $B = \mathbb{R}^n$) lies the use of extremal sets of the classes in hand that satisfy some useful properties. Using these extremal sets leads to a construction of a point map, inducing the orderisomorphism. Then the use of the fundamental theorem of affine geometry (see [8]) essentially completes the proof. In our setting, the use of extremal families is also central and, in fact, an extremal property which holds in general for all of our cases is described and proved in Sect. 2. Using Theorem 1 that generalizes the classical fundamental theorem of projective geometry, we will conclude that the inducing map is a fractional linear map.

2 Extremal Families

We will denote by $A \vee B$ the closed convex hull of A and B, i.e. $\overline{\text{conv}\{A, B\}}$.

Definition 2. Let K be a closed convex set. A closed convex subset $L \subset K$ is said to be *extremal* in K if for every $x \in L, a, b \in K$,

$$x \in (a, b) \qquad \Longrightarrow \qquad [a, b] \subset L.$$

Definition 3. Let F be a family of closed convex sets. A set $K \in F$ is said to be *extremal* in F if for every $A, B \in F$,

$$K = A \vee B \qquad \Longrightarrow \qquad A = K \quad \text{or} \quad B = K.$$

Characterization of extremal sets of $\mathcal{K}^n(A, B)$ is given in Sect. 3.2.

Recall Klee's theorem [6, 7], which states that if a closed convex set in $\mathbb{R}^n$ does not contain a full line, it is the convex hull of its extremal points and extremal rays (extremal subsets, which are a point/ray respectively).

Definition 4. Consider the set $\mathcal{K}^n(A, B)$. Given two points $x, y \in B \setminus A$, we will say that they are comparable if $x \in A \vee \{y\}$ or $y \in A \vee \{x\}$.

3 Proof of Theorem 2

The plan of the proof is as follows. First, we will describe all the possibilities for extremal elements in $\mathcal{K}^n(A, B)$. Then, we will show that they are preserved under T. This will provide us with a point map F that induces the map T. After that, we will show that the point map preserves, in some sense, most of the intervals and apply a stronger version of the fundamental theorem of affine geometry to conclude that F is a fractional linear map.

3.1 Largest and Smallest Elements

Since $A_1 \subset K$ for all $K \in \mathcal{K}^n(A_1, B_1)$ it must hold that $T(A_1) \subset T(K)$. Since T is onto we get that $T(A_1)$ is a subset of every element of $\mathcal{K}^n(A_2, B_2)$ and thus $T(A_1) = A_2$. In the same way, $T(B_1) = B_2$.

3.2 Extremal Elements

Remark 2. Several proofs in this section are based on separation of convex sets by hyperplanes. Unless stated otherwise, separation is assumed to be strict, i.e. if a hyperplane H separates sets X, Y then it is assumed that H, X and Y are disjoint.

Lemma 1. *Let $A \in \mathcal{K}^n$ be a compact convex set and $B \in \mathcal{K}^n$. Then, if $K \in \mathcal{K}^n(A, B)$ is extremal, then either $K = A \vee \{x\}$ for some point x or $K = A \vee R$ for some ray R.*

Proof. Case 1: Assume that K contains some ray R. If $K \neq A \vee R$, then there exists some point $x \in K \setminus (A \vee R)$. Due to compactness of A there exists a hyperplane H that separates x from $A \vee R$ and contains no translate of R. Denote by H^+ the closed half-space that contains x and by H^- the half-space that contains $A \vee R$. Denote by $K^+ = A \vee (K \cap H^+)$ and $K^- = A \vee (K \cap H^-)$. Then, by the choice of H we know that K^+ does not contain R, K^- does not contain x and $K = K^+ \vee K^-$, which is a contradiction to extremality of K.

Case 2: Assume that K contains no rays. In this case it is not hard to check that K is a compact set (see e.g. [6]). Thus, K can be written as $A \vee E$ where E is the set of extremal exposed points in K which are not in A. Take a point $x_1 \in E$. If $E \setminus \{x_1\}$ is empty, we are done. Otherwise, there exists a point $x_2 \in E \setminus \{x_1\}$. Choose a hyperplane H that separates x_2 from $A \vee \{x_1\}$. Define H^+ the closed half-space containing x_2 and H^- the closed half-space containing $A \vee \{x_1\}$. Consider $K^+ = A \vee (K \cap H^+)$ and $K^- = A \vee (K \cap H^-)$. Obviously, $K = K^+ \vee K^-$. It is also clear that K^+ does not contain x_1. Indeed, since A and $K \cap H^+$ are both compact, their convex hull is the union of all convex combinations of $a \in A$ and $k \in K \cap H^+$. Since x_1 is an extremal point of K it cannot be written as convex combination of points in K. Thus $x_1 \notin K^+$. Obviously $x_2 \notin K^-$; hence we get a contradiction to extremality of K. $\qquad\square$

3.3 Segments Are Mapped to Segments

Let $T : \mathcal{K}^n(A_1, B_1) \rightarrow \mathcal{K}^n(A_2, B_2)$ be an order-preserving isomorphism. As we already noticed, $T(A_1) = A_2$ and $T(B_1) = B_2$. Since T is order isomorphism, it is easy to see that the following holds:

1. $T(K_1 \vee K_2) = T(K_1) \vee T(K_2)$.
2. $T(K_1 \cap K_2) = T(K_1) \cap T(K_2)$.

Thus, we conclude that extremal elements of $\mathcal{K}^n(A_1, B_1)$ are mapped to extremal elements of $\mathcal{K}^n(A_2, B_2)$.

Denote by E_i the set $B_i \setminus A_i$, for $i = 1, 2$. Before we define our point map, let us check that an extremal element of the form $A_1 \vee \{x\}$ cannot be mapped to $A_2 \vee R$ for some ray:

Lemma 2. *For every point $x_1 \in E_1$ there exists $x_2 \in E_2$ such that $T(A_1 \vee \{x_1\}) = A_2 \vee \{x_2\}$.*

Proof. Assume the contrary: There exists a point $x \in E_1$ and a ray R such that $T(A_1 \vee \{x\}) = A_2 \vee \{R\}$. Since E_1 is open, there exists $y \in E_1$ such that $A_1 \vee \{x\} \subsetneq A_1 \vee \{y\}$. The image of $A_1 \vee \{y\}$ must be an extremal element that contains $A_2 \vee R$, but this is impossible, unless $T(A_1 \vee \{y\}) = A_2 \vee R$ and thus $y = x$, which is a contradiction. $\qquad\square$

Now, since T is bijective, we get a well-defined point map $F : B_1 \setminus A_1 \rightarrow B_2 \setminus A_2$ as follows:

$$T(A_1 \vee \{x\}) = A_2 \vee \{F(x)\}.$$

Clearly F is also bijective. Our main goal now is to show that F is a fractional linear map.

Lemma 3. *Let $a, b \in E_1$ two points such that the line passing through them does not intersect A_1. Then, the line passing through $F(a), F(b)$ does not intersect A_2.*

Proof. First notice that $F(a)$ and $F(b)$ are not comparable since T^{-1} preserves order. Assume the claim does not hold. Then, there exist two points $a, b \in E_1$, such that the line passing through a, b does not intersect A_1, but the line through $F(a), F(b)$ does intersect A_2. Since $F(a), F(b)$ are not comparable it must hold that the segment $[F(a), F(b)]$ intersects A_2. Let us check that $A_2 \vee F(a) \vee F(b) = (A_2 \vee F(a)) \cup (A_2 \vee F(b))$. Obviously, $[F(a), F(b)] \subset (A_2 \vee F(a)) \cup (A_2 \vee F(b))$. Indeed, denote by z_0 the point where $[F(a), F(b)]$ intersects the boundary of A_2. Then, $[F(a), z_0] \subset A_2 \vee F(a)$ and $[z_0, F(b) \subset A_2 \vee F(b)$. Consider a point z_1 on the boundary of $A_2 \vee F(b)$ and denote by z_2 the point where the line through $F(b)$ and z_1 intersects the boundary of A_2. Clearly, $[F(b), z_2] \subset A_2 \vee F(b)$ and $[z_2, F(a)] \subset A_2 \vee F(a)$. Thus, the triangle created by $F(a), F(b), z_2$ is contained in $(A_2 \vee F(a)) \cup (A_2 \vee F(b))$, which implies that $[F(a), z_1] \subset (A_2 \vee F(a)) \cup (A_2 \vee F(b))$. Since A_2 is compact we get that the convex hull $A_2 \vee F(a) \vee F(b) = (A_2 \vee F(a)) \cup (A_2 \vee F(b))$, which is a contradiction since $A_1 \vee a \vee b \neq (A_1 \vee a) \cup (A_1 \vee b)$. □

Lemma 4. *Let $a, b \in E_1$ be two incomparable points. Then $F([a, b]) = [F(a), F(b)]$.*

Proof. By the previous argument $[F(a), F(b)]$ does not intersect A_2. Define the set $K = A_1 \vee \{a\} \vee \{b\}$ and consider some point $z \in (a, b)$. Since T is an order isomorphism, we have that $T(K) = T(A_1 \vee \{a\}) \vee T(A_1 \vee \{b\})$ and since $K = K \vee (A \vee \{z\})$ we have that $T(K) = T(K) \vee (A_2 \vee F(z))$. Thus $F(z) \in T(K)$. If $F(z) \notin [F(a), F(b)]$, we could find some point $z' \in [F(a), F(b)]$ such that $A_2 \vee F(z) \subset A_2 \vee z'$. Since T^{-1} preserves order, we know that the pre-image of $A_2 \vee z'$ contains $A_1 \vee z$. On the other hand, $A_2 \vee z' \subset T(K)$ and thus its pre-image is contained in K. Since $T^{-1}(A_2 \vee z')$ is an extremal element, we conclude that $T^{-1}(A_2 \vee z') = A_1 \vee z$, which is a contradiction to injectivity. □

At this stage we have a point map that induces T and sends, in some sense, a large set of intervals to intervals. In order to show that F is a fractional linear map we must show that it is continuous. Since F is defined on some subset of $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ with the standard Euclidean structure, we get a naturally induced metric. The continuity we discuss is with respect to this metric.

3.4 Continuity of F

Lemma 5. *Let $K_n \in \mathcal{K}^n(A_1, B_1)$ be a decreasing sequence such that $K_n \searrow K$. Then, $T(K_n) \searrow T(K)$.*

Proof. Obviously, since T is order preserving it holds that $T(K_n) \searrow M$ for some $M \in \mathcal{K}^n(A_2, B_2)$. Since $K \subset K_n$ we have that $T(K) \subset T(K_n)$ which implies that

$T(K) \subset M$. On the other hand, since $T(K_n) \supset M$ and T^{-1} is order-preserving isomorphism, we know that $K_n \supset T^{-1}(M)$. Thus $T^{-1}(M) \subset \cap_1^\infty K_n$ and $M \subset T(\cap_1^\infty K_n)$, which implies that $M = T(\cap_1^\infty K_n) = T(K)$. □

Lemma 6. *Let $\{K_n\} \subset \mathcal{K}^n(A_1, B_1)$ be an increasing sequence such that $K_n \nearrow K$. Then, $T(K_n) \nearrow T(K)$.*

Proof. The proof is similar to the proof of Lemma 5. □

Lemma 7. *Let $\{x_n\} \subset E_1$ be a sequence that converges to some point x. For a given n, consider the set*

$$K_n = A_1 \vee \{x_n, x_{n+1}, \ldots\}.$$

Then,

$$A_1 \vee x = \bigcap_{n=1}^\infty K_n.$$

Proof. Denote by $K_{n,m} = A_1 \vee x_n \ldots \vee x_m$. Then, $K_n = \bigcup_{m \geq n+1} K_{n,m}$. Assume we have a point $z \notin A_1 \vee x$ and $z \in \bigcap_{n=1}^\infty \bigcup_{m=n+1}^\infty K_{n,m}$. This implies that for each n there exists p such that $z \in K_{n,p}$. Since $z \notin A_1 \vee x$, there exists $\epsilon_0 > 0$ such that $d(z, A_1 \vee x) > \epsilon_0$. On the other hand, since $\{x_n\}$ converges to x, there exists n_0 such that for all $n > n_0$ $|x - x_n| < \epsilon$ and thus $d_H(K_{n,p}, A_1 \vee x) < \epsilon_0$ for all $p > n$ (where d_H stands for the Hausdorff distance). Since $z \in K_{n,p}$ for some p, we get a contradiction.

Now assume that $x \notin \bigcap_{n=1}^\infty \bigcup_{m=n+1}^\infty K_{n,m}$. Thus, there exists n_0 such that $x \notin \bigcup_{m=n_0+1}^\infty K_{n_0,m}$. This implies that $d(x, K_{n_0,m}) > \epsilon_0$ for some $\epsilon_0 > 0$ and for all $m > n_0$. In particular, for all $m > n_0$ we have that $d(x, x_m) > \epsilon_0$, but this cannot hold since the sequence $\{x_n\}$ converges to x. Thus, since $A_1 \subset \bigcap_{n=1}^\infty \bigcup_{m=n+1}^\infty K_{n,m}$ and $x \in \bigcap_{n=1}^\infty \bigcup_{m=n+1}^\infty K_{n,m}$ we have that $A_1 \vee x \subset \bigcap_{n=1}^\infty \bigcup_{m=n+1}^\infty K_{n,m}$. This completes the proof. □

Lemma 8. *The map F is continuous on E_1.*

Proof. We would like to show that for any sequence $\{x_n\} \subset E_1$ such that $x_n \to x$ we have that $F(x_n) \to F(x)$. For a given n, we may consider an increasing sequence (with respect to m) of convex sets defined in Lemma 7:

$$K_{n,m} := A_1 \vee x_n \vee x_{n+1} \ldots \vee x_m$$

and a decreasing sequence

$$K_n = \lim_{m \to \infty} K_{n,m}.$$

By Lemma 7 we have that $K_n \searrow K$, where $K = A_1 \vee x$. By Lemma 5 $T(K_n) \searrow T(K) = A_2 \vee F(x)$. Assume that $F(x_n)$ converges to some point y. Since T is an order isomorphism we have that $T(K_{n,m}) = A_2 \vee F(x_n) \ldots \vee F(x_m)$, $T(K_{n,m})$ is

a sequence increasing to $T(K_n)$ while $T(K_n)$ is a sequence decreasing to $T(K) = A_2 \vee F(x)$. On the other hand, using Lemma 7 we get that $T(K_n)$ converges to $A_2 \vee y$. This implies that $A_2 \vee F(x) = T(K) = A_2 \vee y$ which means that $y = F(x)$.

$\square$

3.5 Completing the Proof

Now we know that the map $F : E_1 \rightarrow E_2$ is continuous and maps intervals connecting non-comparable points to intervals. Since we do not know that any interval is mapped to interval under F we cannot apply the classical fundamental theorem of projective geometry. Thus, we would like to apply Theorem 1. Since we are considering open sets in a family of lines in $\mathbb{R}^n$, we must discuss the relevant topology. A line $l \in \mathcal{L}(\mathbb{R}^n)$ is defined uniquely by its direction (up to a sign) and the distance from the origin. Hence a line l can be determined by a non-negative number d_l and a vector (u_l) on the sphere S^{n-1}. The metric on $\mathcal{L}(\mathbb{R}^d)$ is inherited from the Grassmannian. A neighbourhood of the line defined by (d, u) is given by a small perturbation of both d and u.

Consider the set $\tilde{E}_1 = \mathrm{int}(E_1)$ and $\mathcal{L}_1 := \mathcal{L}(\tilde{E}_1) \setminus \mathcal{L}(A_1)$. Notice that $\mathcal{L}_1$ is the set of lines that intersect the interior of E_1 but do not intersect A_1. Now, we will show that the interior of $\mathcal{L}_1$ is an open set that covers the interior of E_1.

Remark 3. The open set $\mathcal{L}'_1 = \mathrm{int}\mathcal{L}_1$ satisfies:

$$\tilde{E}_1 \subset \bigcup_{l_0 \in \mathcal{L}'_1} l.$$

Indeed, take some point $x \in \tilde{E}_1$. Since $\tilde{E}_1$ is open, there exists a point y such that the line l_0 passing through x and y does not intersect A_1. Since both $x, y \in \tilde{E}_1$ it is clear that $l_0 \in \mathcal{L}'_1$.

Clearly, two points $x, y \in \tilde{E}_1$ are comparable if and only if the line $l_{x,y}$ passing through them is not in $\mathcal{L}'_1$. Thus, applying Lemma 4 we get that for any $l \in \mathcal{L}'_1$, $F(l \cap \tilde{E}_1)$ is contained in a line. Therefore, we have shown that on the interior of E_1, all the conditions of Theorem 1 are satisfied. Hence, we conclude that $F|_{\tilde{E}_1}$ is a fractional linear map. If the defining hyperplane of $F|_{\tilde{E}_1}$ has no common points with E_1, then it is obvious that F is fractional linear on E_1 (two continuous functions that coincide on a dense subset are equal). If the defining hyperplane of $F|_{\tilde{E}_1}$ has some common point x with E_1, then for any sequence $\{x_n\} \subset \tilde{E}_1$ that converges to x it must hold that $F(x_n)$ converges to ∞ which is a contradiction to continuity. To see that $T(K) = F(K)$ for $K \in \mathcal{K}^n(A_1, B_1)$ notice that every $K \in \mathcal{K}^n(A_1, B_1)$ can be written as the convex hull of all the extremal elements contained in K.

3.6 Duality

After we have a characterization of all known order-preserving isomorphism on the classes mentioned above, we may easily write characterizations of duality on such classes. Recall that given a convex set K in $\mathbb{R}^n$, the polar set is defined as follows:

$$K^\circ = \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1, \quad \forall y \in K\}.$$

Theorem 4. *Let* $n \geq 2$, A_1, A_2, B_1, B_2 *convex sets in* $\mathbb{R}^n$ *such that* $0 \in A_1$. *Let* $T : \mathcal{K}^n(A_1, B_1) \to \mathcal{K}^n(A_2, B_2)$ *be a bijection satisfying for all* $K, L \in \mathcal{K}^n(A_1, B_1)$

$$K \subset L \Leftrightarrow T(K) \supset T(L).$$

Then, there exists a fractional linear map $F : A_1^\circ \setminus B_1^\circ \to B_2 \setminus A_2$ *such that for every* $K \in \mathcal{K}^n(A_1, B_1)$ *we have* $T(K) = F(K^\circ)$.

Proof. Consider the map $T_1(K) = T(K^\circ)$. Obviously, the domain of T_1 is $\mathcal{K}^n(B_1^\circ, A_1^\circ)$ and T is an order-preserving isomorphism. Thus, by Theorem 2, we know that there exists a fractional linear map $F : A_1^\circ \setminus B_1^\circ \to B_2 \setminus A_2$ such that $T_1(K) = F(K)$ for all $K \in \mathcal{K}^n(B_1^\circ, A_1^\circ)$. Hence $T(K) = F(K^\circ)$. This means that $T(A_1) = B_2$ and $T(B_1) = A_2$. □

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On Polygons and Injective Mappings
of the Plane

Boaz A. Slomka

Abstract We give an affirmative answer to a question asked by Gardner and Mauldin (Geom. Dedicata **26**, 323–332, 1988) about bijections of the plane taking each polygon with *n* sides onto a polygon with *n* sides. We also state and prove more general results in this spirit. For example, we show that an injective mapping taking each convex *n*-gon onto a non-degenerate *n*-gon (not necessarily convex or even simple) must be affine.

Key words Polygons • Injective maps • Fundamental theorem of affine geometry

Mathematical Subject Classifications (2010): 52A10, 52A25

1 Introduction

1.1 *Background and Definitions*

In a paper by Gardner and Mauldin [2] the authors present a few problems in the end of their note. Problem 4 asks the following.

"Let f be a bijection of $\mathbb{R}^2$ onto itself taking polygons with n sides onto polygons with n sides. Must f be affine?"

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Fig. 1 Example of polygons: (**a**) convex polygons, (**b**) simple polygons, (**c**) non-degenerate polygons, (**d**) general polygons
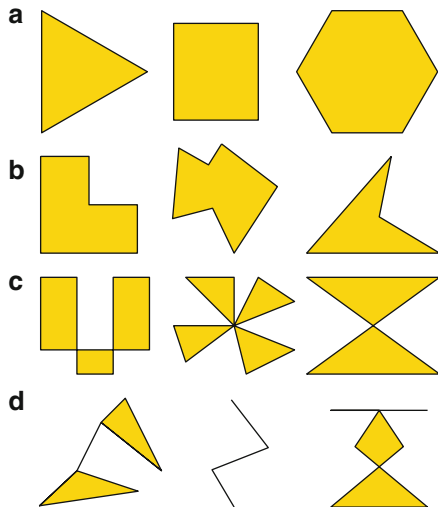
However, the term "polygon" could be interpreted in several ways, where each definition yields a different problem, not comparable with the others. The classes of polygons that we consider are: convex polygons, simple polygons, non-degenerate polygons and general polygons (see Fig. 1), defined as follows.

Definition 1. A set $P \subset \mathbb{R}^2$ is said to be a *general polygon* if there exists a continuous piecewise linear function $l : [0, 1] \rightarrow \mathbb{R}^2$, with $l(0) = l(1)$, such that $P = l([0, 1])$. If l is injective everywhere but a finite set then P is said to be a *non-degenerate polygon*. If l is injective on $[0, 1)$ then P is said to be a *simple polygon* and if P is also the boundary of a convex set then P is said to be a *convex polygon*. The sides of P are the images of the segments in which l is linear and its vertices are their endpoints.

The number of sides of a polygon may depend on its parametrization, unless the polygon is simple (this may imply that the original problem refers to simple polygons), and so we will always mean for the minimal possible number of sides. As customary, a polygon with n sides will be called an n-gon.

We say that two sides of a polygon are adjacent if they share a vertex of the polygon. Similarly, any two vertices of a polygon that share a side are said to be adjacent vertices.

Denote the family of all convex polygons by $\mathrm{Poly}_{\mathrm{conv}}$, the family of all simple polygons by $\mathrm{Poly}_{\mathrm{simp}}$, the family of all non-degenerate polygons by $\mathrm{Poly}_{\mathrm{nond}}$, and the family of all general polygons by $\mathrm{Poly}_{\mathrm{gene}}$. Obviously, we have that

$$\mathrm{Poly}_{\mathrm{conv}} \subset \mathrm{Poly}_{\mathrm{simp}} \subset \mathrm{Poly}_{\mathrm{nond}} \subset \mathrm{Poly}_{\mathrm{gene}} .$$

Each polygon P divides the plane into finitely many open connected sets. In other words, the complement of P, denoted by $P^c = \mathbb{R}^2 \setminus P$, is the union of a finite number of disjoint open (polygonally) connected components.

1.2 Main Results

For all of the above definitions of polygons, we give an affirmative answer for Problem 4 in [2]. In fact, for three of the cases, we prove a much stronger result. Let us state the first result about general n-gons.

Theorem 1. *Let $n > 2$ be fixed. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a bijective mapping taking each general n-gon onto a general n-gon. Then f is an affine transformation.*

The remaining classes of polygons are dealt with in a single result which is much stronger. We show that it is actually sufficient to assume only that each convex n-gon is mapped onto a non-degenerate n-gon. Moreover, we also remove the assumption that the function is onto. We prove the following.

Theorem 2. *Let $n > 2$ be fixed. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an injective mapping taking each convex n-gon onto a non-degenerate n-gon. Then f is an affine transformation.*

The proof of Theorem 2 is reduced to the case where convex n-gons are mapped to simple n-gons:

Theorem 3. *Let $n > 2$ be fixed. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an injective mapping taking each convex n-gon onto a simple n-gon. Then f is an affine transformation.*

The proof of Theorem 3 is reduced to the case where convex n-gons are mapped to convex n-gons:

Theorem 4. *Let $n > 2$ be fixed. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an injective mapping taking each convex n-gon onto a convex n-gon. Then f is an affine transformation.*

In fact, the assumption in Theorem 4 that the image of a convex n-gon is a convex n-gon may be relaxed and replaced by the assumption that its image is a convex polygon which is not necessarily an n-gon. We prove the following.

Theorem 5. *Let $n > 2$ be fixed. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an injective mapping taking each convex n-gon onto a convex polygon (not necessarily an n-gon). Then f is affine.*

Adding the assumption that f is onto, we provide a simpler proof to Theorem 5 (conveniently restated as follows) that requires no further preparation besides recalling the classical fundamental theorem of affine geometry.

Theorem 6. *Let $n > 2$ be fixed. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a bijection taking each convex n-gon onto a convex polygon (not necessarily an n-gon). Then f is an affine transformation.*

We remark here that often a simple polygon is defined as the area bounded by a closed simple polygonal path in the plane as opposed to Definition 1 where the polygon is identified as the bounding curve itself. This alternative definition does not affect our results whatsoever, as explained in Remark 1.

This note is organised as follows. In Sect. 2 we gather some preliminary needed results; in Sect. 2.1 we recall the well-known fundamental theorem of affine geometry which, as in [2], plays a central role in our proofs. In Sect. 2.2 we prove two geometric observations about simple polygons and in Sect. 2.3 we prove two statements about polygon-preserving maps that we will later on use multiple times. In Sect. 3 we prove the main results, Theorems 1–6, in reversed order, and conclude the section with a few remarks.

2 Preliminary Results

In this section we recall some known results and prove some new ones.

2.1 The Fundamental Theorem of Affine Geometry

A mapping from $\mathbb{R}^n$ into itself is called a collineation if it maps any three collinear points (that is, points on one straight line) to collinear points.

As mentioned in [2], the classical fundamental theorem of affine geometry states that a bijective collineation of $\mathbb{R}^n$ onto itself is an affine transformation. This classical result has various strengthened variations, as well as versions for the projective setting (in this case the theorem is referred to as the fundamental theorem of projective geometry). For a list of such results (as well as further related results) we refer the reader to [1]. In the sequel, we shall refer to the following version as "the fundamental theorem of affine geometry".

Theorem [The fundamental theorem of affine geometry]. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an injective collineation. Assume that the image of f is not contained in a straight line. Then f is an affine transformation.*

2.2 Some Facts About Simple Polygons

We will need two geometric observations about simple polygons. Let us first recall some basic facts about simple polygons and introduce more notations.
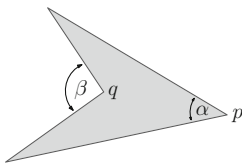
Fig. 2 A simple 4-gon. Here, p is a convex corner with internal angle α and q is a reflex corner with external angle β. The grey area is the polygon's interior and the white area is its exterior

Due to Jordan's curve theorem, a simple polygon P divides the plane into two connected components; a bounded component and an unbounded component. Denote the union of P and its bounded component by $\overline{P}$. Then its bounded component is the interior of $\overline{P}$, denoted by $\text{int } \overline{P}$ and its unbounded component is the exterior of $\overline{P}$, denoted by $\text{ext } \overline{P}$. A simple polygon P will be called a convex polygon if $\text{int } \overline{P}$ is a convex set.

There are two angles between two adjacent sides of a simple polygon P, one internal angle facing $\text{int } \overline{P}$ and one external angle facing $\text{ext } \overline{P}$. If an internal angle has measure less than π radians, then the common vertex (of the angle forming sides) is called a convex corner, and otherwise it is called a reflex corner (Fig. 2).

The following proposition is a self-evident fact (and thus its proof is left for the reader as an easy exercise).

Proposition 1. *Let P be a convex n-gon and let $a, b \in \text{int}\overline{P}$ (or $a, b \in \text{ext } \overline{P}$). Then there exists an n-gon Q so that $a, b \in Q$ and $P \cap Q = \emptyset$.*

Although not needed in this note, it is worth mentioning that Proposition 1 may be proven even if the polygon P is only assumed to be a simple n-gon. Moreover, one may also ask how many points can be added to a and b (such that the polygon Q will pass through them as well). For example, it is not hard to check that for a 4-gon, we may add one more point.

The following geometric observation is the key proposition for the proof of Theorem 3.

Proposition 2. *Let P and Q be two simple n-gons that satisfy*

(i) either $Q \subset \text{int } \overline{P} \cup P$ or $Q \subset \text{ext } \overline{P} \cup P$, and
(ii) Q and P intersect at exactly n points, none of which are vertices of P.

Then $Q \subset \text{int } \overline{P} \cup P$ and P and Q are convex n-gons.

Proof. Let us first give a brief sketch of the proof. We will see that Q must be the path connecting the intersection points of P and Q that belong to adjacent sides of P, each point from a different side of P (see Fig. 3). Consequently, as Fig. 3 illustrates, Q would intersect both connected components of P^c, unless P is convex.

Fig. 3 A sketch of the proof: Q is the path orderly connecting the intersection points of P and Q. Thus, only if P is convex, Condition (i) may hold
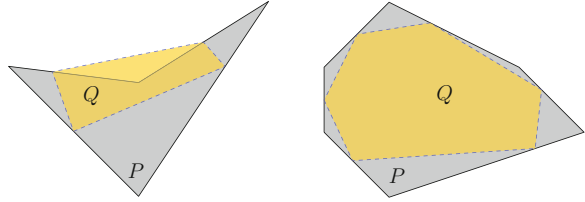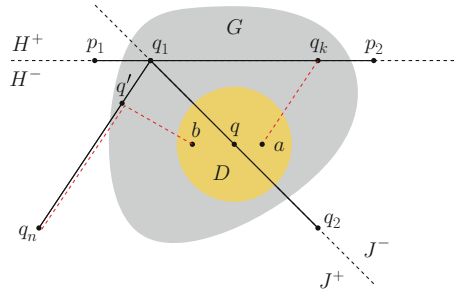

Fig. 4 The dotted segment $[a, q_k]$ lies in V and the dotted path connecting b, q' and q_n lies in W

Let U denote the connected component of P^c for which $Q \subset U \cup P$. Let U' denote the other connected component of P^c (later on in the proof we will show that $U = \text{int } \overline{P}$ and $U' = \text{ext } \overline{P}$). Notice that since $Q \subset U \cup P$, any polygonal path in U' has no intersection with Q. Let $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$ be the (consecutive) vertices of P and Q, respectively.

First, it is easy to verify that all n intersecting points of P and Q are vertices of Q. Indeed, otherwise Q would either intersect P at infinitely many points or would not be contained in $U \cup P$. Since Q is an n-gon, there are no other vertices of Q.

Next, we show that there must be exactly one vertex of Q on each side of P. To this end, assume that there exist two vertices, say q_1 and q_k, of Q that lie both on one side of P, say (without loss of generality) $[p_1, p_2]$. By assumption, $q_1, q_k \notin \{p_1, p_2\}$. Moreover, notice that $2 < k < n$ as P and Q intersect at finitely many points. Without loss of generality we may also assume that $q_1 \in [p_1, q_k]$ and that no vertex of Q lies on $[q_1, q_k]$. The infinite straight line containing $[p_1, p_2]$ divides the plane into two open half planes, denoted by H^- and H^+ (see Fig. 4). Since $[p_1, p_2]$ is a side of P, there exists an open convex set G containing $[q_1, q_k]$ so that $H^- \cap G \subset U$ and $H^+ \cap G \subset U'$. Hence, as $Q \subset U \cup P$, we have that $q_2, q_n, q_{k-1}, q_{k+1} \in H^-$. The line passing through $[q_1, q_2]$ divides the plane into two open half planes, denoted by J^+ and J^-. Without loss of generality, assume that $q_k \in J^-$ and $q_n \in J^+$. The segment $[q_1, q_2]$ divides $U \cup P$ into two connected components. We claim that q_k and q_n belong to different components. Let $q \in U \cap [q_1, q_2]$ and let $D \subset G \cap H^-$ be an open ball around q. Take $a \in J^- \cap D$ and $b \in J^+ \cap D$, so that a belongs to one of the aforementioned components, denoted by V, and b belongs to the other component, denoted by W.

By our construction, the segment $[a, q_k]$ is in $G \cap H^-$ and so is in $U \cup P$. Hence $q_k \in V$. Let q' be a point on $[q_1, q_n] \cap (G \cap H^-) \subset U$. Then both the segment $[b, q']$ and the segment $[q', q_n]$ belong to $U \cup P$ and so $q_n \in W$. By definition, there

are two polygonal paths from q_k to q_n that lie entirely in Q. One contains $[q_1, q_2]$ and the other does not, and since $Q \subset U \cup P$, this is a contradiction to the fact that $q_k \in V$ whereas $q_n \in W$. Thus on each side of P lies only one vertex of Q.

Next, we show that any two vertices of Q lying on adjacent sides of P must share a side of Q. To this end, assume that there exist two vertices of Q, say (without loss of generality) q_1 and q_2, connecting non-adjacent sides of P, denoted by A and B. Hence, the line segment $[q_1, q_2] \subset U \cup P$ divides $U \cup P$ into two connected components, each containing a side of P different from both A and B, and so each component contains a vertex of Q different from both q_1 and q_2. Let q and q' be two vertices of Q, one from each component. Since Q is a simple polygon, it can be decomposed into two polygonal paths from q to q', intersecting only at q and q'. As one of these paths must contain $[q_1, q_2]$, the second cannot intersect it, a contradiction to the fact that q and q' belong to different connected components into which $[q_1, q_2]$ divides $U \cup P$.

Next, we show that $U = \operatorname{int} \overline{P}$. Assume that $U = \operatorname{ext} \overline{P}$. Let p' be a convex corner of P. Such a corner exists and in fact every simple polygon has at least 3 convex corners; see, for example, [3].[1] Let A' and B' be sides of P with p' as a common vertex and assume that $q_1' \in A'$ and $q_2' \in B'$ are the corresponding adjacent vertices of Q that lie on A' and B'. Since q_1' and q_2' are not vertices of P, it follows that the line segment $[q_1', q_2'] \subset Q$ has non-empty intersection with $\operatorname{int} \overline{P}$, a contradiction to the fact that $Q \subset \operatorname{ext} \overline{P} \cup P$. Thus, $Q \subset \operatorname{int} \overline{P} \cup P$.

Finally, we show that P is convex. Assume that P is not a convex n-gon. Then it has reflex vertex $\tilde{p}$. Let $\tilde{A}$ and $\tilde{B}$ be sides of P with $\tilde{p}$ as a common vertex and let $\tilde{q}_1 \in \tilde{A}$ and $\tilde{q}_2 \in \tilde{B}$ be the adjacent vertices of Q that lie on $\tilde{A}$ and $\tilde{B}$. Since $\tilde{q}_1$ and $\tilde{q}_2$ are not vertices of P, it follows that the line segment $[\tilde{q}_1, \tilde{q}_2] \subset Q$ has non-empty intersection with $\operatorname{ext} \overline{P}$, a contradiction to the fact that $Q \subset \overline{P}$. Thus, P and Q are convex n-gons. $\square$

2.3 Polygon-Preserving Injective Maps

In this section we prove two statements about polygon-preserving injective maps, which we will repeatedly use in the proofs of the main results.

Lemma 1. *Let $n > 2$. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an injective mapping, taking each convex n-gon onto a general polygon. Let P be a convex n-gon, mapped onto a polygon $f(P)$. Then $f(\operatorname{int} \overline{P})$ and $f(\operatorname{ext} \overline{P})$ are contained each in a single connected component of $f(P)^c$.*

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[1]Alternatively, this can be easily seen by considering its convex hull whose boundary is a convex polygon with at least 3 corners and observing that all its corners are convex corners of the original polygon.

Fig. 5 $f(T)$ (from the proof of Proposition 3) is contained in the cone K
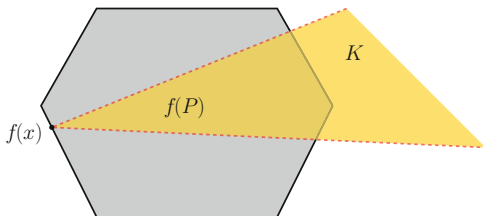
Proof. Let $a, b \in \text{int}\,\overline{P}$. By Proposition 1, there exists a convex n-gon Q such that $a, b \in Q$ and $Q \cap P = \emptyset$. Moreover, since f is injective, $f(Q) \cap f(P) = \emptyset$. Since $f(Q)$ is a polygon such that $f(a), f(b) \in f(Q)$ it follows that there exists a polygonal path γ connecting $f(a)$ and $f(b)$ and satisfying that $\gamma \cap f(P) = \emptyset$ and so $f(a)$ and $f(b)$ both belong to the same connected component of $f(P)^c$. Thus, $f(\text{int}\,\overline{P})$ is contained in the same connected component. Repeating the exact same proof for $\text{ext}\,\overline{P}$ yields that $f(\text{ext}\,\overline{P})$ is also contained in one connected component of $f(P)^c$. □

Proposition 3. *Let $n > 2$. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an injective mapping, taking each convex n-gon onto a convex polygon (not necessarily an n-gon). Then for every convex n-gon P, $f(\text{int}\,\overline{P}) \subset \text{int}\,f(P)$ and $f(\text{ext}\,\overline{P}) \subset \text{ext}\,f(P)$. Moreover, given a convex n-gon P, if $x, y \in P$ are points such that $f(x)$ is not a vertex of $f(P)$ then $f(x)$ and $f(y)$ do not belong to a common side of $f(P)$.*

Proof. Let P be a convex n-gon. Clearly, there exist many points in P that their images are not vertices of $f(P)$. Fix such a point $x \in P$ and fix a point y which does not share a side of P with x. By Lemma 1, we have that $f(\text{int}\,\overline{P})$ is contained in one of the two connected components of $f(P)^c$, which we denote by U.

Let $T \subset \overline{P}$ be a convex n-gon such that $T \cap P = \{x, y\}$. Then $f(T)$ is a convex polygon that intersects $f(P)$ only at $f(x)$ and $f(y)$. Since $f(\text{int}\,\overline{P}) \subset U$, we have that $f(T) \subset U \cup f(P)$. We claim that $f(x)$ is a vertex of $f(T)$. Indeed, assume that $f(x)$ is not a vertex of $f(T)$ and let A be the side of $f(T)$ containing $f(x)$. Since $f(x)$ is not a vertex of $f(P)$, it follows that either A intersects both connected components of $f(P)^c$ or A is contained in a side of P. The first possibility contradicts the fact that $f(T) \subset U \cup f(P)$ and the second possibility contradicts the fact that $f(T) \cap f(P) = \{f(x), f(y)\}$. Thus, $f(x)$ is a vertex of $f(T)$.

Next, we show that $U = \text{int}\,\overline{f(P)}$. To this end, denote the two vertices of $f(T)$ that are adjacent to $f(x)$ by w and z. Since $f(T)$ is a convex polygon, it follows that $f(T)$ is contained in the cone

$$K = \{f(x) + \alpha(w - f(x)) + \beta(z - f(x)) : \alpha, \beta \geq 0\}$$

as illustrated in Fig. 5. Since $f(y) \in f(T) \cap f(P) \subset K \cap f(P)$, it follows that $f(T) \cap \text{int}\,\overline{f(P)} \neq \emptyset$ which implies that $U = \text{int}\,\overline{f(P)}$. Moreover, the fact that

$$f(T) \cap f(P) = \{f(x), f(y)\}$$

implies that $f(x)$ and $f(y)$ cannot belong to a common side of $f(P)$.

To show that $f(\mathrm{ext}\overline{P}) \subset \mathrm{ext}\overline{f(P)}$, take another convex n-gon $Q \neq P$ such that $\overline{P} \subset \overline{Q}$. By the above arguments, we have that $f(\mathrm{int}\,\overline{Q}) \subset \mathrm{int}\,\overline{f(Q)}$, and so $f(P) \subset \overline{f(Q)}$. Since $f(P) \neq f(Q)$, it follows that $f(Q) \cap \mathrm{ext}\overline{f(P)} \neq \emptyset$. Thus, there exists a point $z \in Q \cap \mathrm{ext}\overline{P}$ such that $f(z) \in \mathrm{ext}\overline{f(P)}$. Thus, by Lemma 1, $f(\mathrm{ext}\overline{P}) \subset \mathrm{ext}\overline{f(P)}$. □

3 Proofs of the Main Results

Proof of Theorem 6. We show that f^{-1} is a bijective collineation. Let $a', b', c' \in \mathbb{R}^2$ be collinear points, and assume that $f^{-1}(a')$, $f^{-1}(b')$ and $f^{-1}(c')$, respectively denoted by a, b, c, are not collinear. Construct a convex n-gon P as follows (see Fig. 6); take a triangle containing a, b, c, each on a different side, such that a, b, c are not vertices of the triangle. Then trim the corners to get a 4-gon, a 5-gon, or a 6-gon. Repeat the trimming process until you get a convex n-gon P containing a, b, c, where none of them are its vertices.

In addition to a, b, c, pick $n - 3$ points $v_1, \ldots, v_{n-3}$ from the remaining $n - 3$ sides of P which do not contain a, b, c, one point from each side (none of which is a vertex of P). Let Q denote the convex n-gon with vertices $a, b, c, v_1, \ldots, v_{n-3}$. By assumption, $f(P)$ and $f(Q)$ are both convex polygons which intersect at exactly n points, including a', b', c'. Since a', b', c' are collinear, it follows that the whole (smallest) segment containing them is contained in both $f(P)$ and $f(Q)$, a contradiction. Thus, f^{-1} is a bijective collineation. By the classical fundamental theorem of affine geometry, f^{-1}, and so f, is an affine transformation. □

Proof of Theorem 5. We show that f is a collineation. To this end, let $a, b, c \in \mathbb{R}^2$ be collinear points. Let P be a convex n-gon containing a, b, c in one of its sides, where none of the three are vertices of P. Let $I \subset P$ be a segment containing a, b, c and let Q be a convex n-gon such that $Q \subset \mathrm{ext}\,\overline{P} \cup P$ and $P \cap Q = I$, as in Fig. 7. By Proposition 3, it follows that $f(Q) \subset \mathrm{ext}\overline{P} \cup P$ and $f(P) \subset \mathrm{ext}\overline{f(Q)} \cup f(Q)$. Since both $f(P)$ and $f(Q)$ are convex, it follows that their intersection must be
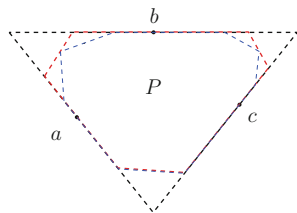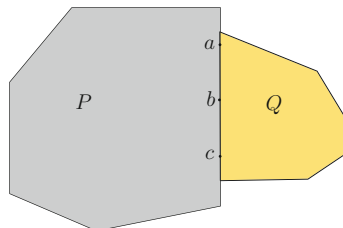

Fig. 6 The construction of P in the proof of Theorem 6

Fig. 7 The construction of Q
in the proof of Theorem 5

contained in a segment in $f(P)$ (just as P and Q intersect in Fig. 7). In particular, $f(a)$, $f(b)$, $f(c)$ are collinear.

Concluding the above, collinear points are mapped to collinear points and so, by the fundamental theorem of affine geometry, f is affine. $\qquad\square$

Proof of Theorem 4. We show that f is a collineation. To this end, let $a, b, c \in \mathbb{R}^2$ be collinear points. Let P be a convex n-gon containing a, b, c in one of its sides, where none of the three are vertices of P. Denote the sides of P by $A_1, \ldots, A_n$. Without loss of generality, $a, b, c \in A_1$. Let $x_1, x_2, \ldots, x_n \in P$ be additional n points such that for all $i \in \{1, \ldots, n\}$, $x_i \in A_i$ and $f(x_i)$ is not a vertex of $f(P)$. By Proposition 3, it follows that for $i \neq j$, $f(x_i)$ and $f(x_j)$ do not belong to a common side of $f(P)$ and so $f(x_1), \ldots, f(x_n)$ belong to n different sides of $f(P)$, respectively denoted by $B_1, \ldots, B_n$. Since $f(P)$ is an n-gon, replacing x_1 by any member of $\{a, b, c\}$ and repeating the previous argument yields that $f(a), f(b), f(c) \in B_1$. In particular, the three are collinear. Concluding the above, collinear points are mapped to collinear points and so, by the fundamental theorem of affine geometry, f is affine. $\qquad\square$

Proof of Theorem 3. Let P be a convex n-gon, mapped to $f(P)$. Our goal is to show that $f(P)$ is a convex n-gon. Pick n points $v_1, \ldots, v_n \in P$, each from a different side of P, such that $v_1, \ldots, v_n$ are not vertices of P and their images $f(v_1), \ldots f(v_n)$ are not vertices of $f(P)$. Let Q be the convex n-gon with vertices $v_1, \ldots, v_n$. Then $Q \subset \overline{P}$ and $P \cap Q = \{v_1, \ldots, v_n\}$. By Lemma 1, we have that either $f(Q) \subset \overline{f(P)}$ or $f(Q) \subset \text{ext } \overline{f(P)} \cup f(P)$, and so $f(P)$ and $f(Q)$ satisfy the assumptions of Proposition 2. This implies that $f(P)$ is a convex n-gon. By Theorem 4, f is affine. $\qquad\square$

Proof of Theorem 2. Let P be convex n-gon, mapped to $f(P)$. Our goal is to show that $f(P)$ is a simple n-gon. By Lemma 1, $f(\text{int } \overline{P})$ is contained in one of the connected components of P, which we denote by U.

Next, we show that there exists another connected component $V \neq U$ of $f(P)^c$ for which $f(\text{ext } \overline{P}) \subset V$. To this end, choose $n - 1$ points $x_1, \ldots, x_{n-1}$ from $n - 1$ consecutive sides of P and another two points x_n, x_{n+1} from the remaining nth side of P, such that none of $x_1, \ldots, x_{n+1}$ are vertices of P and none of their images are vertices of $f(P)$. Construct a convex n-gon Q as in Fig. 8, so that $P \cap Q = \{x_1, \ldots, x_{n+1}\}$, $Q \cap \text{int } \overline{P} \neq \emptyset$ and $Q \cap \text{ext } \overline{P} \neq \emptyset$. As $f(Q)$ is an n-gon, there must

Fig. 8 The construction of Q in the proof of Theorem 2
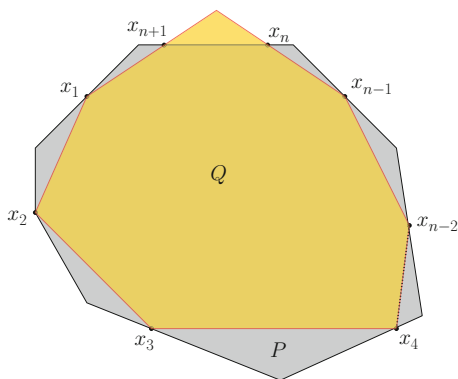
Fig. 9 The construction of T in the proof of Theorem 2
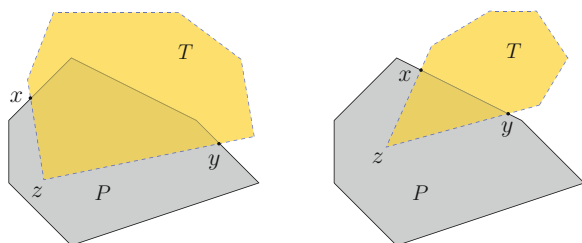
be a point $f(a) \in \{f(x_1), \ldots, f(x_{n+1})\}$ which is not a vertex of $f(Q)$. Therefore, $f(a)$ is neither a vertex of $f(P)$ nor a vertex of $f(Q)$, and so $f(Q)$ intersects at least two different connected components of $f(P)^c$. Thus, there exist a point $q \in Q \setminus \{P\}$ and a connected component $V \neq U$ of $f(P)^c$ so that $f(q) \in V$. Since $f(\mathrm{int}\overline{P}) \subset U$, $q \in \mathrm{ext}\,\overline{P}$ and so, by Lemma 1, $f(\mathrm{ext}\,\overline{P}) \subset V$.

To show that $f(P)$ is a simple n-gon, assume towards a contradiction that it is a self-intersecting n-gon. Since $f(P)$ is a non-degenerate polygon, there exists a third connected component $W \neq U, V$ of $f(P)^c$. Denote the closure of W by $\mathrm{cl}(W)$.

By the definition of a non-degenerate polygon, only points that belong to multiple sides of $f(P)$ may belong to the closure of more than two connected components of $f(P)^c$. As there are only finitely many such points, we may choose two points $x', y' \in f(P) \cap \mathrm{cl}(W)$ that belong to the closure of exactly two connected components of $f(P)^c$, one of which is W. Let $x, y \in P$ be the corresponding points for which $f(x) = x'$ and $f(y) = y'$. Let T be a convex n-gon such that $T \cap P = \{x, y\}$, $T \cap \mathrm{int}\,\overline{P} \neq \emptyset$ and $T \cap \mathrm{ext}\,\overline{P} \neq \emptyset$. Indeed, it is easy to construct such a convex n-gon T as in Fig. 9; pick any point $z \in \mathrm{int}\,\overline{P}$ that does not belong to the segment $[x, y]$ and take two long enough segments emanating from z and passing through x and y as sides of T where the remaining $n - 2$ sides are constructed in $\mathrm{ext}\,\overline{P}$. Since $f(\mathrm{int}\overline{P}) \subset U$ and $f(\mathrm{ext}\,\overline{P}) \subset V$, we have that $f(T) \subset U \cup V \cup f(P)$.

Let $s \in \mathrm{int}\,\overline{P} \cap T$ and $t \in \mathrm{ext}\,\overline{P} \cap T$. By the above, we have that $f(s) \in U$, $f(t) \in V$ and $f(T) \subset U \cup V \cup f(P)$. Let $\gamma : [0, 1] \to f(T)$ be a polygonal path

connecting $f(s)$ and $f(t)$. That is, $\gamma(0) = f(s)$, $\gamma(1) = f(t)$ and $\gamma(\theta) \in f(T)$ for all $0 \leq \theta \leq 1$. Define

$$\theta_0 = \sup\{\theta < 1 : \gamma(\theta) \notin V\}.$$

Then $\gamma((\theta_0, 1]) \subset V$ and, for some $0 < \delta < 1$, $\gamma([\theta_0 - \delta, \theta_0))$ is contained in a connected component of $f(P)^c$ which is different from V. Obviously, $\gamma(\theta_0) \in f(P)$, and as $\gamma \subset f(T)$, it follows that $\gamma(\theta_0) \in f(P) \cap f(T) = \{f(x), f(y)\}$. Since both $f(x)$ and $f(y)$ are in the closure of exactly two connected components of $f(P)^c$, one of which is W, it follows that

$$\gamma([\theta_0 - \delta, \theta_0)) \subset W,$$

a contradiction to the fact that $f(T) \subset U \cup V \cup f(P)$. Thus, $f(P)$ is a simple n-gon. By Theorem 3, f is affine. □

We remark that under the further assumption that f is onto, the above proof of Theorem 2, which reduces to Theorem 3, becomes almost trivial, as one can verify.

Proof of Theorem 1. The plan of the proof if to show that each convex n-gon is mapped to a simple n-gon and apply Theorem 3 to complete the proof.

Let P be a convex n-gon. First, we show that $f(P)$ divides the plane into at least two connected components. To this end, assume that $f(P)^c$ is connected. Then for any two points $x, y \in f(P)$ there exists a single path $\Gamma \subset f(P)$ connecting x and y. This means that every closed path in $f(P)$ must pass through each point an even number of times, and so $f(P)$ has an even number of sides, that is $n = 2m$. Let $P_1 \subset P$ be the union of m consecutive sides of P, and let $P_2 \subset P$ be the union of the remaining m sides of P. Then P_1 and P_2 are (degenerate) n-gons that intersect at exactly two points, say a, b, and so $f(P_1)$ and $f(P_2)$ are two n-gons that intersect exactly at $f(a)$ and $f(b)$. Thus, there exists a path

$$\gamma_1 \subset f(P_1) \subset f(P)$$

connecting $f(a)$ and $f(b)$, and another path

$$\gamma_2 \subset f(P_2) \subset f(P)$$

connecting $f(a)$ and $f(b)$, where $\gamma_1 \cap \gamma_2 = \{f(a), f(b)\}$, a contradiction to the fact that $f(P)^c$ is connected. Therefore, $f(P)^c$ has at least two connected components.

Next, we show that $f(P)$ is simple. Since f is onto and since $f(P)^c$ has at least two connected components, it follows by Lemma 1 that $U := f(\text{int } \overline{P})$ and $V := f(\text{ext } \overline{P})$ are the only two connected components of $f(P)^c$. To show that $f(P)$ is simple, we need to prove that each point in $f(P)$ is contained in the closure of both U and V. This fact will follow by, essentially, the same argument used in the last part of the proof of Theorem 2; assume that there exists a segment $I \subset f(P)$

which is contained in the closure of exactly one connected component of $f(P)^c$, denoted by W, which is either U or V.

Choose two points $x', y' \in I$. Let $x, y \in P$ be the corresponding points for which $f(x) = x'$ and $f(y) = y'$. Let T be a convex n-gon such that $T \cap P = \{x, y\}$, $T \cap \text{int } \overline{P} \neq \emptyset$ and $T \cap \text{ext } \overline{P} \neq \emptyset$. The construction of T is the same as in the proof of Theorem 2 (see Fig. 9).

Let $s \in \text{int } \overline{P} \cap T$ and $t \in \text{ext } \overline{P} \cap T$. Then $f(s) \in U$ and $f(t) \in V$. Let $\gamma : [0, 1] \to f(T)$ be a polygonal path connecting $f(s)$ and $f(t)$. That is, $\gamma(0) = f(s)$, $\gamma(1) = f(t)$ and $\gamma(\theta) \in f(T)$ for all $0 \le \theta \le 1$. Define

$$\theta_0 = \sup\{\theta < 1 \, : \, \gamma(\theta) \notin V\}.$$

Then $\gamma((\theta_0, 1]) \subset V$ and, for some $0 < \delta < 1$, $\gamma([\theta_0 - \delta, \theta_0))$ is contained in U. Obviously, $\gamma(\theta_0) \in f(P)$, and as $\gamma \subset f(T)$, it follows that

$$\gamma(\theta_0) \in f(P) \cap f(T) = \{f(x), f(y)\}.$$

Since both $f(x)$ and $f(y)$ are in the closure of only one connected component of $f(P)^c$, we get a contradiction. Thus, $f(P)$ is a simple n-gon. By Theorem 3, f is affine. $\qquad\qquad\square$

Note that for a fixed odd number of sides $2n + 1$, one can verify that the above proof of Theorem 1 actually proves a stronger statement, namely that if a bijection f maps each convex $(2n + 1)$-gon onto a general $(2n + 1)$-gon then f must be affine. This statement probably holds also for even number of sides, however proving it seems quite technical.

Remark 1. As mentioned in Sect. 1.2, sometimes a simple polygon is defined as the area bounded by a closed simple polygonal path in the plane as opposed to Definition 1. This alternative definition for polygons does not affect the validity of Theorems 3, 4, 5, 6 since one may easily show that a bijection $f : \mathbb{R}^2 \to \mathbb{R}^2$ that maps each simple polygon (in the alternative sense) onto a polygon must map its boundary onto the boundary of its image. Indeed, for any point x on the boundary of a simple polygon $\overline{P}$, there exists a simple polygon $\overline{Q}$ intersecting $\overline{P}$ only at x. Thus $f(\overline{P})$ and $f(\overline{Q})$ intersect only at $f(x)$ and so $f(x)$ must be on the boundary of $f(\overline{P})$.

One may consider yet another class of polygons, namely general polygons with a locally injective parametrization. Such polygons will be called locally injective. It can be proven that a bijection of the plane taking each convex n-gon onto a locally injective n-gon must be affine. In fact, it can be checked that every locally injective polygon divides the plane into at least two connected components, and then one can repeat verbatim the second part of the proof of Theorem 1 to complete the proof.

We remark that it is worth thinking about other problems posed in [2]. In particular, Problems 1, 3 and 6 which are closely related to the problem settled in this note, ask the following (respectively.)

1. Are there bijections of $\mathbb{R}^n$ which map each circle onto a finite union of line segments? Or even polygonal arcs?
2. Let f be a bijection of $\mathbb{R}^2$ onto itself which maps polygons onto polygons. Is f piecewise affine on every bounded set?
3. Is there a bijection of $\mathbb{R}^2$ onto itself taking each circle into a triangle (or into a square)?

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Abstract Approach to Ramsey Theory and Ramsey Theorems for Finite Trees

Sławomir Solecki

Abstract I will give a presentation of an abstract approach to finite Ramsey theory found in an earlier paper of mine. I will prove from it a common generalization of Deuber's Ramsey theorem for regular trees and a recent Ramsey theorem of Jasiński for boron tree structures. This generalization appears to be new. I will also show, in exercises, how to deduce from it the Milliken Ramsey theorem for strong subtrees.

Key words Ramsey theory • Ramsey theorems for trees

Mathematical Subject Classifications (2010): 05D10, 05C55

1 Introduction

The first result of pure finite Ramsey theory and a prototype of the many later results of this area (see [5]) is the theorem proved by Ramsey in 1930. We recall it now to remind the reader of the flavor of pure finite Ramsey theory. We will also refer to this statement later on. For a natural number n, let $[n] = \{1, \ldots, n\}$; in particular, $[0] = \emptyset$. The classical Ramsey theorem says that given natural numbers d, l, and m, there exists a natural number n such that for each d-coloring, that is, a coloring with d colors, of all l element subsets of $[n]$, there exists an m element subset z of $[n]$ such that all l element subsets of z have the same color.

In Sect. 2, we give an exposition of the abstract approach to pure finite Ramsey theory developed in [7]; the main theorem, saying that a general pigeonhole

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principle implies a general Ramsey property, is stated as Theorem 9 (see also Appendix 1). Most pure finite Ramsey theoretic results can be viewed as instances of the machinery presented here. In the exposition, we make an effort to motivate the main abstract notions and we also illustrate them with examples.

In Sects. 3, 4, 5, using arguments consisting mostly of formulating appropriate definitions, we show that certain Ramsey-theoretic results for finite trees, one of which is new, are particular instances of the general Theorem 9. These applications of Theorem 9 to concrete situations are similar to each other, with the main differences lying in the derivations used (more on it in the next paragraph). Therefore, in the first two applications (the illustrations in Sect. 2 and Illustration 10), we explicitly check all the details and provide pictures; in the third application (a generalization of Deuber's and Jasiński's theorems, Sect. 4), we give all the definitions, but carefully check the pigeonhole principle only; in the last application (Milliken's theorem, Sect. 5), we state all the definitions, but leave checking the details to the reader in exercises. Recall that in [7], it is shown how, for example, the classical Ramsey theorem, the Graham–Rothschild theorem, and a new self-dual Ramsey theorem can be obtained as instances of Theorem 9.

In each of the many concrete Ramsey theorems (considered here and in [7]), the same underlying algebraic structure turns out to be present, the structure of a *normed background* given by Definition 1 (see also Appendix 1). A crucial element of such structures is a *truncation operator*, which forms a basis for inductive arguments. In the concrete situations involving trees and considered in the present paper, there is a close connection between truncation operators and *derivations on trees*. Roughly speaking there are two natural derivations on trees: cutting off the rightmost branch and cutting off the highest leaves. These two derivations give rise to two types of truncation operators, which lead to two types of normed backgrounds, which in turn lead to two Ramsey theorems. Namely, the branch cutting derivation gives a generalization of Deuber's and Jasiński's theorems, while the leaf cutting derivation gives Milliken's theorem.

For convenience, we adopt the following modification to the notation for the operation of subtracting 1 *among natural numbers: we set* $0 - 1$ *to be equal to* 0*; for* $k > 0$*,* $k - 1$ *retains its usual meaning.*

2 Abstract Approach with Illustrations

2.1 Abstract Ramsey Theory

A typical Ramsey-type theorem has the following form. We start with two families $\mathcal{F}$ and $\mathcal{P}$. (Elements of $\mathcal{F}$ and $\mathcal{P}$ are usually finite sets of functions, most frequently some type of morphisms.) A set P from $\mathcal{P}$ and a number of colors d are given. The conclusion of the theorem then asserts that there is a set F from $\mathcal{F}$ with a given mapping (usually a type of composition) defined on $F \times P$,

$$F \times P \ni (f, x) \rightarrow f \cdot x,$$

such that for each d-coloring of the image $\{f \cdot x : f \in F, \ x \in P\}$ of $F \times P$ under the mapping there exists $f_0 \in F$ with $\{f_0 \cdot x : x \in P\}$ monochromatic.

Below in the paper, we formalize this vague idea and we also give several concrete examples that should convince the reader that Ramsey-type theorems do indeed have this form. Here, as an illustration, we only phrase the classical Ramsey theorem in a way that is compatible with the general framework above. It may be useful for the reader to recall here the Ramsey theorem from the first paragraph of the introduction. In the restatement of the Ramsey theorem to which we now proceed, for natural numbers p and q, we identify p element subsets of $[q]$ with increasing injections from $[p]$ to $[q]$ so that a subset z is identified with the unique increasing injection whose range is equal to z. One can take $\mathcal{F} = \mathcal{P}$ to be the family of all sets produced as follows: fix natural numbers p and q and form the set of all increasing injections from $[p]$ to $[q]$. Fix natural numbers d, l, and m, and let $P \in \mathcal{P}$ be the set of all increasing injections from $[l]$ to $[m]$. Then the classical Ramsey theorem says that there is an n with the following property. For the set $F \in \mathcal{F}$ of all increasing injections from $[m]$ to $[n]$, if we d-color the set

$$\{f \circ x : f \in F, \ x \in P\} = \text{ all increasing injections from } [l] \text{ to } [n],$$

then there exists $f_0 \in F$ such that $\{f_0 \circ x : x \in P\}$ is monochromatic.

Now we start the description of the abstract approach. Let A and X be sets. Assume we are given a *partial* function from $A \times X$ to X:

$$(a, x) \rightarrow a \cdot x.$$

Such a function $\cdot$ will be called an **action (of A on X)**. No properties of the function $\cdot$ are assumed to hold at this point. For $F \subseteq A$ and $P \subseteq X$, we say that $F \cdot P$ *is defined* if $a \cdot x$ is defined for all $a \in F$ and $x \in P$, and we let

$$F \cdot P = \{a \cdot x : a \in F, \ x \in P\}.$$

We also write $a \cdot P$ for $\{a\} \cdot P$.

We will give a sequence of illustrations that contain the most rudimentary examples of the general notions being introduced. The illustrations depend on each other and lead to the classical Ramsey theorem.

Illustration 1 Let $A = X$ be the set of all *(strictly) increasing* functions from $[k] = \{1, \ldots, k\}$ to $\mathbb{N} \setminus \{0\}$, where k ranges over $\mathbb{N}$. Given $a, x \in A = X$ with $a : [l] \rightarrow \mathbb{N} \setminus \{0\}$ and $x : [k] \rightarrow \mathbb{N} \setminus \{0\}$, let $a \cdot x$ be defined precisely when $[l]$ contains the image of x and put

$$a \cdot x = a \circ x.$$

Going back to the general situation, let $\mathcal{F}$ and $\mathcal{P}$ be families of non-empty subsets of A and X, respectively. Assume we have a partial function from $\mathcal{F} \times \mathcal{P}$ to $\mathcal{P}$:

$$(F, P) \to F \bullet P$$

such that if $F \bullet P$ is defined, then it is given by the point-wise action of F on P, that is, $F \cdot P$ is defined and

$$F \bullet P = F \cdot P.$$

In such a situation, we say that $(\mathcal{F}, \mathcal{P}, \bullet)$ is a **pair of families over** $(A, X, \cdot)$. Introducing a restriction $\bullet$ of the point operation of sets in $\mathcal{F}$ on sets in $\mathcal{P}$ makes the Ramsey condition (R) below more flexible, while a careful calibration of the resteriction makes it possible to satisfy condition $(*)$ of the next subsection. In concrete situations, definitions of restrictions $\bullet$ are very natural.

Illustration 2 For $k, l \in \mathbb{N}$ with $0 < k \le l$, let $\binom{l}{k}$ stand for the set of all (strictly) increasing functions from $[k]$ to $[l]$. Let also $\binom{0}{0}$ consist of one element—the empty function. Since an increasing function from $[k]$ to $[l]$ is determined by its range, $\binom{l}{k}$ can be identified with the set of all k element subsets of $[l]$. Let $\mathcal{F} = \mathcal{P}$ be the set of all $\binom{l}{k}$ with $0 < k \le l$ or $k = l = 0$. Declare $\binom{n}{m} \bullet \binom{l}{k}$ to be defined precisely when $m = l$, and let

$$\binom{n}{l} \bullet \binom{l}{k} = \binom{n}{k}.$$

It is clear that $\binom{n}{l} \bullet \binom{l}{k} = \binom{n}{l} \cdot \binom{l}{k}$. Note, however, that $\binom{n}{m} \cdot \binom{l}{k}$ is defined if we assume only $m \ge l$.

The following condition is our Ramsey statement, which is just a formalization of the statement from the beginning of this subsection:

(R) given $d > 0$, for each $P \in \mathcal{P}$, there is an $F \in \mathcal{F}$ such that $F \bullet P$ is defined, and for every d-coloring of $F \bullet P$ there is an $f \in F$ such that $f \cdot P$ is monochromatic.

Illustration 3 In the special case of Illustrations 1 and 2, condition (R) says in particular that given $d > 0$ and $0 < k \le l$ there exists $m \ge l$ such that for each d-coloring of $\binom{m}{l} \bullet \binom{l}{k} = \binom{m}{k}$ there exists $a \in \binom{m}{l}$ such that the set

$$\left\{ a \circ x : x \in \binom{l}{k} \right\}$$

is monochromatic. This is the classical Ramsey theorem.

2.2   Abstract Pigeonhole Principle

We introduce here our pigeonhole principle. The name is purely conventional as the principle is not a simple abstraction of the well known pigeonhole principle of Dirichlet. Rather it is a condition that is easy to check in concrete situations and that implies, through inductive arguments encoded in Theorem 9, the Ramsey condition (R).

   We will need an important additional piece of structure. Let A, X, and an action . be as above. Let $\partial\colon X \to X$ be a function such that for $a \in A$ and $x \in X$, if $a \, . \, x$ is defined, then $a \, . \, \partial x$ is defined and

$$\partial(a \, . \, x) = a \, . \, \partial x. \tag{1}$$

Such a function ∂ is called a **truncation**. For $P \subseteq X$, we write

$$\partial P = \{\partial x \colon x \in P\}. \tag{2}$$

   Introduction of the operator ∂ equips X with an additional structure and Eq. (1) states that the action of A on X is implemented by homomorphism of this structure. In applications to concrete Ramsey theorems, ∂ is always a form of derivation leading from an object in X to another, less complex object in X. In this fashion, in proofs, ∂ provides a foothold for inductive arguments.

Illustration 4   We continue the pervious illustrations, in particular, our notation is as in Illustration 1. For $x \in X$ with $x\colon [k] \to \mathbb{N} \setminus \{0\}$, define

$$\partial x = x \upharpoonright [k-1].$$

(Recall here the convention for the notation $k - 1$ adopted in the introduction.) It is easy to check that condition (1) is satisfied. Note also that, by Eq. (2), $\partial\binom{l}{k} = \binom{l-1}{k-1}$, if $k > 1$, and $\partial\binom{l}{k} = \binom{0}{0}$, if $k \leq 1$.

   Let $(\mathcal{F}, \mathcal{P}, \bullet)$ be a pair of families over $(A, X, .)$ equipped with a truncation ∂. We are ready to formulate our pigeonhole principle. For $P \subseteq X$ and $y \in X$, put

$$P_y = \{x \in P \colon \partial x = y\}. \tag{3}$$

So P_y is the set consisting of those elements of P that truncate to the same simpler object y. Given $a, b \in A$, we say that b **extends** a if for each x with $a \, . \, x$ defined, we have that $b \, . \, x$ is defined and that it is equal to $a \, . \, x$. For $F \in \mathcal{F}$ and $a \in A$, let

$$F_a = \{f \in F \colon f \text{ extends } a\}. \tag{4}$$

   The Ramsey statement (R) above requires, upon coloring of $F \bullet P$, stabilizing the coloring on a copy $f \, . \, P$ of P obtained by acting on P by some element f

Fig. 1 Condition (P) in
Illustration 5
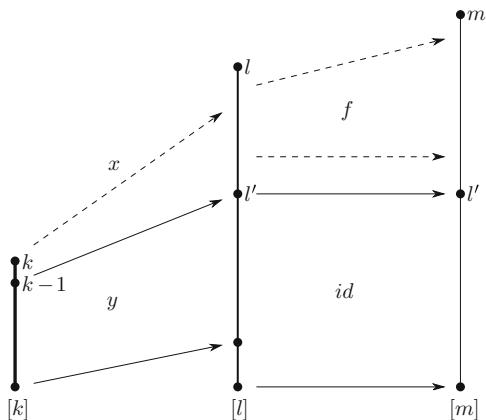
of F. Pigeonhole principle (P) below asks us to perform the following much easier task. We fix an object $y \in X$, which can be assumed to be simpler than objects in P. We consider the elements of P that truncate to this fixed y, that is, we consider P_y, and require stabilizing the coloring *only* on a copy $f \cdot P_y$ of P_y obtained by acting on P_y by an element f from F. The price to pay is that f has to act on y in a way prescribed by an element $a \in A$ chosen in advance, that is, f is actually taken from F_a for some a for which $a \cdot y$ is defined.

It is surprising that various concrete pigeonhole principles occurring in the finite pure Ramsey theory have this form. We illustrate it below by the classical pigeonhole principle used to prove the classical Ramsey theorem. In the following sections, we will give more complex examples involving trees. Paper [7] contains a number of further examples.

The following criterion on $(\mathcal{F}, \mathcal{P}, \bullet)$ is our pigeonhole principle:

(P)   given $d > 0$, for all $P \in \mathcal{P}$ and $y \in \partial P$, there are $F \in \mathcal{F}$ and $a \in A$ such that $F \bullet P$ is defined, $a \cdot y$ is defined, and for every d-coloring of $F_a \cdot P_y$ there is an $f \in F_a$ such that $f \cdot P_y$ is monochromatic.

Note that in the condition above $F_a \cdot P_y$ is defined since $F \bullet P$ is assumed to be defined and $F_a \subseteq F$ and $P_y \subseteq P$. Also, of course, the condition would not have changed if we required the coloring to be defined on $F \cdot P_y$ or even on $F \cdot P$. It is, however, crucial that f be found in F_a.

Illustration 5 In our special case from the earlier illustrations, a moment of thought and a picture convince one that condition (P) boils down to the classical pigeonhole principle. For the sake of practice, however, let us look at it carefully in detail. We will be helped by Fig. 1.

For notational simplicity, in the argument below, we assume that $k > 1$ and leave checking that the same argument works for $k \leq 1$ to the reader. We state condition (P) in our special case:

let $d > 0$, $1 < k \le l$ and $y \in \binom{l-1}{k-1}$ be given; let l' be the maximum of the range of y; there exists $m \ge l$ and an increasing function $a \colon [l'] \to \mathbb{N} \setminus \{0\}$ such that for each d-coloring of

$$\left\{ f \circ x \colon f \in \binom{m}{l}, \ f \upharpoonright [l'] = a, \ x \in \binom{l}{k}, \ x \upharpoonright [k-1] = y \right\}$$

there exists $f \in \binom{m}{l}$ with $f \upharpoonright [l'] = a$ and such that

$$\left\{ f \circ x \colon x \in \binom{l}{k}, \ x \upharpoonright [k-1] = y \right\}$$

is monochromatic.

We claim that the condition above holds with a being the identity function from $[l']$ to itself. Indeed, with this choice of a, the conclusion of the condition reads: there exists $m \ge l$ such that for each d-coloring of

$$\left\{ f \circ x \colon f \in \binom{m}{l}, \ f(i) = i \text{ for } i \in [l'], \ x \in \binom{l}{k}, \ x \upharpoonright [k-1] = y \right\}$$

there exists $f \in \binom{m}{l}$ with $f(i) = i$, for $i \in [l']$, and with

$$\{ f \circ x \colon x \in \binom{l}{k}, \ x \upharpoonright [k-1] = y \}$$

monochromatic.
The elements of the set

$$\left\{ f \circ x \colon f \in \binom{m}{l}, \ f(i) = i \text{ for } i \in [l'], \ x \in \binom{l}{k}, \ x \upharpoonright [k-1] = y \right\}$$

differ only in the single value $f(x(k))$ and this value comes from the set $[m] \setminus [l']$. Also $x(k)$ is an arbitrary element of $[l] \setminus [l']$. So, in essence, we are d-coloring $[m] \setminus [l']$ and are looking for an increasing function from $[l] \setminus [l']$ to $[m] \setminus [l']$ whose values take the same color. This is just the classical pigeonhole condition, and we can take m to be any number strictly bigger than $l' + d \cdot (l - l' - 1)$.

Our goal is to state a theorem that condition (P) implies condition (R). Achieving this goal, in Theorem 9, will require introducing more structure on $(A, X, \cdot, \partial)$ and imposing additional conditions on $(\mathcal{F}, \mathcal{P}, \bullet)$.

2.3   Additional Structure and Additional Conditions

Let A, X, an action $.$, and a truncation ∂ be as above.

Let again $(\mathcal{F}, \mathcal{P}, \bullet)$ be a pair of families over $(A, X, .)$. Recall the notion of extension for elements of A defined in the discussion preceding (4). We first state two conditions on $(\mathcal{F}, \mathcal{P}, \bullet)$ that do not require introducing any additional structure:

(A)   if $P \in \mathcal{P}$, then $\partial P \in \mathcal{P}$;
(B)   if $F \in \mathcal{F}$, $P \in \mathcal{P}$, and $F \bullet \partial P$ is defined, then there is $G \in \mathcal{F}$ such that $G \bullet P$ is defined and for each $f \in F$ there is $g \in G$ extending f.

Strictly speaking conditions (A) and (B) are not needed to prove Theorem 9; one can dispense with them at the expense of strengthening condition (P) slightly. (We elaborate on it in Appendix 1.) However, in some situations, for example, in all the situations in this note, conditions (A) and (B) hold, and whenever they hold they do so in an obvious way (and they make strengthening of (P) unnecessary). Condition (A) simply requires closure of $\mathcal{P}$ under truncation. As for condition (B), note that if $F . P$ is defined, then $F . \partial P$ is defined. The reverse implication is false in general. Condition (B) gives a substitute for this reverse implication: assuming something stronger, namely that $F \bullet \partial P$ is defined, we can infer that $G \bullet P$ is defined for a G that can simulate the action of every element of F.

Illustration 6   Recall that we have

$$\mathcal{F} = \mathcal{P} = \left\{ \binom{n}{m} : 0 < m \leq n \text{ or } m = n = 0 \right\}.$$

We check conditions (A) and (B). It follows from the remark in Illustration 4 that $\mathcal{P}$ is closed under ∂, so (A) holds. To check (B), let $F = \binom{n}{m}$ and $P = \binom{l}{k}$. We assume $k > 1$ and leave the trivial case $k \leq 1$ to the reader. We have

$$F \bullet \partial P = \binom{n}{m} \bullet \binom{l-1}{k-1}$$

is defined precisely when $m = l - 1$, and we can take $G = \binom{n+1}{l}$ to witness the conclusion of (B) since $\binom{n+1}{l} \bullet \binom{l}{k}$ is defined and each element of $\binom{n}{l-1}$ is extended by an element of $\binom{n+1}{l}$. We elaborate on this last point. Note that for each $f \in \binom{n}{l-1}$, that is, for each increasing $f : [l-1] \to [n]$, there is increasing $g : [l] \to [n+1]$ with $g \upharpoonright [l-1] = f$. In this situation, for each $x \in X$ (recall that X is the set of all increasing functions from some $[k]$ to $\mathbb{N}$), if $f . x$ is defined, then the image of x is included in $[l-1]$, and so $g . x$ is defined and obviously

$$g . x = g \circ x = f \circ x = f . x.$$

So, in our example, g extending f as an increasing function is equivalent to g extending f as an element of A. A similar coincidence will be present also in the subsequent illustrations.

To make the partial function . from $A \times X$ to X into an honest action, we assume that we also have a *partial* function from $A \times A$ to A:

$$(a, b) \rightarrow a \cdot b,$$

such that for $a, b \in A$ and $x \in X$ if $a . (b . x)$ and $(a \cdot b) . x$ are both defined, then

$$a . (b . x) = (a \cdot b) . x. \tag{5}$$

The operation $\cdot$ as above will be called **multiplication**. Equation (5) is the usual equation defining, say, a group action on a set. As before, for F, $G \subseteq A$, we say that $F \cdot G$ *is defined* if $a \cdot b$ is defined for all $a \in F$ and $b \in G$ and we let

$$F \cdot G = \{a \cdot b : a \in F, b \in G\}.$$

Now, again as before, in addition to the partial function $\bullet$ from $\mathcal{F} \times \mathcal{P}$ to $\mathcal{P}$, assume that we have a partial function $\bullet$ from $\mathcal{F} \times \mathcal{F}$ to $\mathcal{F}$ with the property that if $G \bullet F$ is defined, then it is given point-wise, that is, $G \cdot F$ is defined and

$$G \bullet F = G \cdot F.$$

We now call $(\mathcal{F}, \mathcal{P}, \bullet, \bullet)$ a **pair of families over** $(A, X, ., \cdot)$.

We can now state our final condition on $\mathcal{F}, \mathcal{P}, \bullet$ and $\bullet$:

$(\ast)$ if $F, G \in \mathcal{F}$, $P \in \mathcal{P}$, and $F \bullet (G \bullet P)$ is defined, then so is $(F \bullet G) \bullet P$.

This condition is crucial. It says that $F \bullet (G \bullet P)$ is never defined "by chance;" if it is defined, then the product $F \bullet G$ is defined, as is its action on P. In concrete situations, this condition is guaranteed by a natural calibration of the domains of the operations $\bullet$ and $\bullet$. Note that under the assumptions of $(\ast)$, from Eq. (5), we have

$$F \bullet (G \bullet P) = (F \bullet G) \bullet P.$$

In [7], a pair of families $(\mathcal{F}, \mathcal{P}, \bullet, \bullet)$ over $(A, X, ., \cdot)$ fufilling condition $(\ast)$ is called an *actoid of sets*.

Illustration 7 Recall again that

$$\mathcal{F} = \mathcal{P} = \left\{ \binom{n}{m} : 0 < m \leq n \text{ or } m = n = 0 \right\} .$$

Declare $\binom{n}{m} \bullet \binom{l}{k}$ on $\mathcal{F}$ to be defined precisely when $m = l$ and let

$$\binom{n}{l} \bullet \binom{l}{k} = \binom{n}{k}.$$

So $\bullet$ is equal to $\boldsymbol{\cdot}$ defined earlier in Illustration 2. It follows that $\bullet$ is given pointwise.

To check $(*)$, note that if

$$\binom{q}{p} \bullet \left(\binom{n}{m} \bullet \binom{l}{k} \right)$$

is defined, then $m = l$ and $p = n$, but in this situation

$$\left(\binom{q}{p} \bullet \binom{n}{m} \right) \bullet \binom{l}{k}$$

is defined.

We require one more piece of structure that, roughly speaking, measures complexity of objects in X. A function $|\cdot|: X \to D$, where $(D, \leq)$ is a linear order, is called a **norm** if for $x, y \in X$, $|x| \leq |y|$ implies that for all $a \in A$

$$a \cdot y \text{ defined} \Rightarrow (a \cdot x \text{ defined and } |a \cdot x| \leq |a \cdot y|). \qquad (6)$$

Illustration 8 In our special case, X is the set of all increasing injections $x: [k] \to \mathbb{N} \setminus \{0\}$ for $k \in \mathbb{N}$. Define $|\cdot|: X \to \mathbb{N}$, where $\mathbb{N}$ is taken with its natural linear order, to be

$$|x| = \begin{cases} \max \text{image}(x) = x(k), & \text{if } k > 0; \\ 0, & \text{if } k = 0. \end{cases}$$

We check that this definition gives a norm. Let $a \in A$, $a: [l] \to \mathbb{N} \setminus \{0\}$. Note that, for $x \in X$, $a \cdot x$ is defined precisely when $|x| \leq l$ and $|a \cdot x| = a(|x|)$, if $|x| > 0$, and $|a \cdot x| = 0$, if $|x| = 0$. So given $x_1, x_2 \in X$ with $|x_1| \leq |x_2|$, it is clear that if $a \cdot x_2$ is defined, then so is $a \cdot x_1$ and

$$|a \cdot x_1| = a(|x_1|) \leq a(|x_2|) = |a \cdot x_2|, \quad \text{if } |x_1| > 0,$$

or

$$|a \cdot x_1| = 0 \leq |a \cdot x_2|, \quad \text{if } |x_1| = 0.$$

The additional conditions required to prove our theorem were stated as (A), (B), and $(*)$. The additional structure introduced above is consolidated in the following notion.

Definition 1. A **normed background** is a pair of sets A, X equipped with a multiplication $\cdot$ and an action . fulfilling (5), with a truncation ∂ fulfilling (1), and with a norm $|\cdot|$ fulfilling (6).

With some abuse of notation, a normed background as above will be denoted by (A, X).

2.4 The Theorem

Now we can phrase our theorem. To see how it follows from the somewhat more general results of [7], the reader should consult Appendix 1. We write $\partial^t P$, $t \in \mathbb{N}$, for the result of applying truncation ∂ to P t times.

Theorem 9. *Let $(\mathcal{F}, \mathcal{P}, \bullet, \bullet)$ be a pair of families over a normed background fulfilling conditions (A), (B), and ($*$). Assume that each $P \in \mathcal{P}$ is finite and for each $P \in \mathcal{P}$ there is $t \in \mathbb{N}$ such that $\partial^t P$ consist of one element. If $(\mathcal{F}, \mathcal{P})$ fulfills (P), then it fulfills (R).*

Note that the theorem above gives the classical Ramsey theorem on the basis of Illustrations 1–8. In them, we checked all the assumptions of Theorem 9 except: for $P \in \mathcal{P}$, P is finite and $\partial^t P$ has one element for some $t \in \mathbb{N}$. Finiteness of P is clear. Note that $\partial^k \binom{l}{k} = \binom{0}{0}$, so this last assumption is also fulfilled.

3 Trees and Another Illustration

Trees and Embeddings We state here basic definitions concerning trees. By a **tree** we understand a *finite*, possibly empty, partial order such that each two elements have a common predecessor and the set of predecessors of each element is linearly ordered. So trees for us are *finite trees*. If a tree is non-empty, it has a smallest element, which we call the **root**. Maximal elements of a tree are called **leaves**. By convention, we regard every node of a tree as one of its own predecessors and as one of its own successors.

Each tree T carries a binary function $\wedge_T$ that assigns to each $v, w \in T$ the largest element $v \wedge_T w$ of T that is a predecessor of both v and w. After Deuber [2], we say that a function $f \colon S \to T$, for trees S and T, is a **morphism** if for all $v, w \in S$,

$$f(v \wedge_S w) = f(v) \wedge_T f(w).$$

So strictly speaking f is a morphism from the functional structure $(S, \wedge_S)$ to the functional structure $(T, \wedge_T)$.

For a tree T and $v \in T$, let $\mathrm{im}_T(v)$ be the set of all **immediate successors** of v, and we do not regard v as one of them. Let $T(v)$ be the tree whose elements are all

the successors of v (with v among them, of course). Let $\mathrm{ht}_T(v)$ be the cardinality of the set of all predecessors of v (including v), and let

$$\mathrm{ht}(T) = \max\{\mathrm{ht}_T(v) : v \in T\}.$$

For a non-empty tree T, let $\mathrm{br}(T)$ be the maximum of cardinalities of $\mathrm{im}_T(v)$ for $v \in T$.

We will occasionally suppress the subscripts from various pieces of notation introduced above if we deem them clear from the context.

A tree T is called **ordered** if for each $v \in T$ there is a fixed linear order of $\mathrm{im}(v)$. Such an assignment allows us to define the lexicographic linear order $\leq_T$ on all the nodes of T by stipulating that $v \leq_T w$ if v is a predecessor of w and, in case v is not a predecessor of w and w is not a predecessor of v, that $v \leq_T w$ if the predecessor of v in $\mathrm{im}(v \wedge w)$ is less than or equal to the predecessor of w in $\mathrm{im}(v \wedge w)$ in the given order on $\mathrm{im}(v \wedge w)$.

The simplest ordered trees are $[n]$ for $n \in \mathbb{N}$ with their natural successor relation and the unique ordering of the immediate successors of each vertex.

An **embedding** f **from an ordered tree** S **to an ordered tree** T is an injective tree morphism such that

(i)   It is order preserving between $\leq_S$ and $\leq_T$;
(ii)   For each $v \in S$, the set $\{w \in \mathrm{im}_T(f(v)) : w$ is a predecessor of $f(v')$ for some $v' \in \mathrm{im}_S(v)\}$ forms an initial segment with respect to $\leq_T$ of $\mathrm{im}_T(f(v))$.

Note that preservation of order by f is equivalent to saying that for every $v \in S$ and all $w_1, w_2 \in \mathrm{im}_S(v)$ with $w_1 \leq_S w_2$ if w_1', w_2' in $\mathrm{im}_T(f(v))$ are predecessors of $f(w_1)$ and $f(w_2)$, respectively, then $w_1' \leq_T w_2'$. An embedding is **leaf preserving** if each leaf of the domain is mapped to a leaf of the range. An embedding $f : S \to T$ is called **strong** if for $v, w \in S$ with $\mathrm{ht}(v) = \mathrm{ht}(w)$ we have that $\mathrm{ht}(f(v)) = \mathrm{ht}(f(w))$. Note that each embedding from $[n]$, $n \in \mathbb{N}$, to an ordered tree is a strong embedding.

Derivations on Trees There are two natural ways to trim an ordered tree. Let an ordered tree T be given. Put

$$T^* = \{v \in T : \mathrm{ht}(v) < \mathrm{ht}(T)\}, \tag{7}$$

that is, T^* is obtained from T by removing all of its highest leaves. Note that T^* with $\leq_T$ restricted to it is an ordered tree, and that the inclusion from T^* to T is a strong embedding.

Let x be the rightmost with respect to $\leq_T$ leaf of T, that is, x is the $\leq_T$-largest element of T, and let

$$T' = \{v \in T : T(v) \text{ has a leaf different from } x\}, \tag{8}$$

that is, T' is obtained from T by removing from it a final segment of its rightmost branch. The tree T' with $\leq_T$ restricted to it forms an ordered tree and the inclusion $T' \subseteq T$ is a leaf preserving embedding. If $T \neq \emptyset$, then the set $T \setminus T'$ with the inherited tree structure can be identified with $[p]$ for some $p \in \mathbb{N}$, $p > 0$, with its natural tree order. If $T' \neq \emptyset$, there is a unique node $v_0 \in T'$ that has an immediate successor in $T \setminus T'$. We call v_0 the **splitting node of** T.

Examples of Trees and Embeddings We fix some notation concerning trees. After Deuber [2], a non-empty tree T is called **regular** if for each $v \in T$ that is not a leaf, $|\mathrm{im}(v)| = \mathrm{br}(T)$ and for each leaf $x \in T$, $\mathrm{ht}(x) = \mathrm{ht}(T)$. Of course, each such tree is fully determined by the value of two parameters: $\mathrm{br}(T)$ and $\mathrm{ht}(T)$. For $k, n \in \mathbb{N}$, $k > 0$, $n > 1$, let $T^{k,n}$ be the regular tree of height n and with branching number k. By convention, for $k \in \mathbb{N}$, let $T^{k,1}$ have exactly one node and $T^{k,0}$ be equal to the empty tree, and for $n \in \mathbb{N}$, $n > 1$, let $T^{0,n}$ have exactly one node. We consider $T^{k,n}$ to be an ordered tree with some linear order $\leq_{T^{k,n}}$. (All possible orders making $T^{k,n}$ into an ordered tree lead to isomorphic ordered trees.) The tree $T^{1,n}$ can be identified with $[n]$ as an ordered tree.

We fix two natural ways of embedding $T^{k,n}$ into $T^{k,n+1}$. First, there is a unique embedding ι^* of $T^{k,n}$ into $T^{k,n+1}$ with

$$\iota^*(\mathrm{im}_{T^{k,n}}(v)) \subseteq \mathrm{im}_{T^{k,n+1}}(\iota^*(v)),$$

for $v \in T^{k,n}$, and with ι^* mapping the root of $T^{k,n}$ to the root of $T^{k,n+1}$, if $T^{k,n}$ is non-empty. Note that $\mathrm{ht}_{T^{k,n}}(v) = \mathrm{ht}_{T^{k,n+1}}(\iota^*(v))$ for $v \in T^{k,n}$. We write

$$T^{k,n} \subseteq^* T^{k,n+1} \tag{9}$$

to indicate that we consider $T^{k,n}$ identified with its image under ι^*. There is also a unique embedding ι' of $T^{k,n}$ into $T^{k,n+1}$ with

$$\iota'(\mathrm{im}_{T^{k,n}}(v)) \subseteq \mathrm{im}_{T^{k,n+1}}(\iota'(v)),$$

for $v \in T^{k,n}$, and with ι' mapping the $\leq_{T^{k,n}}$-smallest leaf of $T^{k,n}$ to the $\leq_{T^{k,n+1}}$-smallest leaf of $T^{k,n+1}$, if $T^{k,n}$ is non-empty. This embedding comes from the isomorphism between $T^{k,n}$ and $T^{k,n+1}(v_0)$, where v_0 is the $\leq_{T^{k,n+1}}$-smallest immediate successor of the root of $T^{k,n+1}$. Note that the image of the set of all leaves of $T^{k,n}$ under ι' is an initial segment with respect to $\leq_{T^{k,n+1}}$ of the set of leaves of $T^{k,n+1}$. We write

$$T^{k,n} \subseteq' T^{k,n+1} \tag{10}$$

to indicate that we consider $T^{k,n}$ identified with its image under ι'.

We give one more illustration. Its conclusion will be used in the sequel.

Illustration 10 We prove the following, possibly folklore, generalization of the classical Ramsey theorem.

Given $d > 0$, $s \in \mathbb{N}$, and a non-empty ordered tree S, there is a non-empty ordered tree T with $\mathrm{br}(T) = \mathrm{br}(S)$ such that for each d-coloring of all leaf preserving embeddings of $[s]$ to T there exists a leaf preserving embedding $g_0 \colon S \to T$ such that

$$\{g_0 \circ f \colon f \colon [s] \to S \text{ a leaf preserving embedding}\}$$

is monochromatic.

The proof below consists essentially of stating definitions. All the checking that needs to be done is routine and would be probably best left to the reader. However, since this is the first example involving trees, we will perform all the verifications carefully and explicitly.

Let $k = \mathrm{br}(S)$. Since there is a leaf preserving embedding from S to $T^{k,m}$ with $m = \mathrm{ht}(S)$, we can assume that $S = T^{k,m}$ for some m. For $n \in \mathbb{N}$, set

$$T^n = T^{k,n}.$$

Normed Background Let X be the set of all (not necessarily leaf preserving) embeddings from some $[m]$ to some T^n. Let A consist of all strong embeddings from some T^m to some T^n. (Strong embeddings were defined earlier in this section. We will need this more restrictive notion of embedding for the normed background we are defining to work.) For $f \in X$ and $g, g_1, g_2 \in A$ declare that $g \cdot f$ is defined if the image of f is included with respect to $\subseteq^*$ (as defined in Eq. (9)) in the domain of g, and similarly declare that $g_2 \cdot g_1$ is defined if the image of g_1 is included with respect to $\subseteq^*$ in the domain of g_2, and let

$$g \cdot f = g \circ f \quad \text{and} \quad g_2 \cdot g_1 = g_2 \circ g_1.$$

Define ∂^* on X be letting for $f \colon [m] \to T^n$,

$$\partial^* f = f \upharpoonright [m-1].$$

Note, after recalling the derivation (7), that $\partial^* f = f \upharpoonright [m]^*$. Let

$$|f| = \begin{cases} \mathrm{ht}(f(m)), & \text{if } m > 0; \\ 0, & \text{if } m = 0. \end{cases} \tag{11}$$

It is checked without any difficulty that (A, X) with the operations defined above is a normed background. The requirement that embeddings in A be strong is used in checking that $|\cdot|$ is a norm.

A Pair of Families Over (A, X) Let S, T be ordered trees. Let

$$\left(\frac{T}{S}\right)^s \quad \text{and} \quad \left[\frac{T}{S}\right]^s \tag{12}$$

stand for the set of all strong embeddings and strong, leaf preserving embeddings, respectively, from S to T. Since all embeddings from $[m]$, $m \in \mathbb{N}$, to an ordered tree are strong, we simplify our notation by setting

$$\left(\frac{T}{m}\right) = \left(\frac{T}{[m]}\right)^s \quad \text{and} \quad \left[\frac{T}{m}\right] = \left[\frac{T}{[m]}\right]^s, \tag{13}$$

that is, $\left(\frac{T}{m}\right)$ and $\left[\frac{T}{m}\right]$ stand for the set of all embeddings from $[m]$ to T and for the set of all leaf preserving embeddings from $[m]$ to T, respectively.

Let $\mathcal{F}$ consist of sets of the form $\left(\frac{T^n}{T^m}\right)^s$ and $\left[\frac{T^n}{T^m}\right]^s$ with $0 < m \le n$ or $m = n = 0$. Declare $\bullet$ to be defined precisely in the following situations: $\left(\frac{T^n}{T^m}\right)^s \bullet \left(\frac{T^m}{T^l}\right)^s$ and $\left[\frac{T^n}{T^m}\right]^s \bullet \left[\frac{T^m}{T^l}\right]^s$, and define them to be

$$\left(\frac{T^n}{T^m}\right)^s \bullet \left(\frac{T^m}{T^l}\right)^s = \left(\frac{T^n}{T^l}\right)^s \quad \text{and} \quad \left[\frac{T^n}{T^m}\right]^s \bullet \left[\frac{T^m}{T^l}\right]^s = \left[\frac{T^n}{T^l}\right]^s.$$

Let $\mathcal{P}$ consist of sets of the form $\left(\frac{T^n}{m}\right)$ and $\left[\frac{T^n}{m}\right]$ with $0 < m \le n$ or $m = n = 0$. Declare $\bullet$ to be defined precisely in the following situations: $\left(\frac{T^n}{T^m}\right)^s \bullet \left(\frac{T^m}{l}\right)$ and $\left[\frac{T^n}{T^m}\right]^s \bullet \left[\frac{T^m}{l}\right]$, and let

$$\left(\frac{T^n}{T^m}\right)^s \bullet \left(\frac{T^m}{l}\right) = \left(\frac{T^n}{l}\right) \quad \text{and} \quad \left[\frac{T^n}{T^m}\right]^s \bullet \left[\frac{T^m}{l}\right] = \left[\frac{T^n}{l}\right].$$

It is easy to see that these $\bullet$ and $\bullet$ when defined are given point-wise. This checking boils down to showing that each strong embedding from T^l to T^n factors through T^m, and the same for strong, leaf preserving embeddings. Such factorizations are easy to produce. Arguing by induction, we see that it is suffices to show their existence for $l < m = l + 1 \le n$. Since $l < n$, given a strong (leaf preserving, respectively) embedding $g: T^l \to T^n$, there is $1 \le j \le n$ such that $\text{ht}(g(v)) \ne j$, for each $v \in T^l$. Fix the largest $1 \le i \le l$ such that $\text{ht}(g(v)) < j$ for all $v \in T^l$ with $\text{ht}(v) = i$, or let $i = 0$ if no such i exists. Let $g_1: T^l \to T^{l+1}$ be an arbitrary strong (leaf preserving, respectively) embedding such that for $v \in T^l$

$$\text{ht}(g_1(v)) = \begin{cases} \text{ht}(v), & \text{if } \text{ht}(v) \le i; \\ \text{ht}(v) + 1, & \text{if } \text{ht}(v) \ge i + 1. \end{cases}$$

So there is no element of T^l that gets mapped to a $w \in T^{l+1}$ with $\mathrm{ht}(w) = i + 1$. Now it is easy to find a strong (leaf preserving, respectively) embedding $g_2 \colon T^{l+1} \to T^n$ such that $g = g_2 \circ g_1$. (We do it so that $\mathrm{ht}(g_2(w)) = j$ for all $w \in T^{l+1}$ with $\mathrm{ht}(w) = i + 1$.)

Note that

$$
\partial^* \binom{T^n}{m} = \partial^* \left[\begin{matrix} T^n \\ m \end{matrix} \right] = \begin{cases} \binom{T^{n-1}}{m-1}, & \text{if } m > 1; \\ \binom{T^0}{0}, & \text{if } m \leq 1. \end{cases} \tag{14}
$$

Using Eq. (14), we verify that $\mathcal{F}$ and $\mathcal{P}$ is a pair of families over (A, X) fulfilling conditions (A) and (B). Condition (A) is clear from Eq. (14). We verify condition (B) for $k > 1$ in the calculation below, and leave the trivial case $k \leq 1$ to the reader. Note that by Eq. (14), if

$$
\binom{T^n}{T^m}^s \bullet \partial^* \binom{T^l}{k} = \binom{T^n}{T^m}^s \bullet \binom{T^{l-1}}{k-1}
$$

is defined, then $l = m + 1$, so $\binom{T^{n+1}}{T^{m+1}}^s \bullet \binom{T^l}{k}$ is defined and each $g \in \binom{T^n}{T^m}^s$ is extended by some $h \in \binom{T^{n+1}}{T^{m+1}}^s$ (that is, for each $f \in X$ if $g \, . \, f$ is defined, then so is $h \, . \, f$ and $h \, . \, f = g \, . \, f$); simply view T^m as included in T^{m+1} via $\subseteq^*$ and take $h \colon T^{m+1} \to T^{n+1}$ to be any strong embedding with $h \upharpoonright T^m = g$, that is, h extends $g \colon T^m \to T^n$ as an embedding. We handle the situation when

$$
\binom{T^n}{T^m}^s \bullet \partial^* \left[\begin{matrix} T^l \\ k \end{matrix} \right] = \binom{T^n}{T^m}^s \bullet \binom{T^{l-1}}{k-1}
$$

is defined in the same way, except that in this case $\left[\begin{matrix} T^{n+1} \\ T^{m+1} \end{matrix} \right]^s$ witnesses that (B) holds.

To see condition $(*)$, note that if

$$
\binom{T^q}{T^p}^s \bullet \left(\binom{T^n}{T^m}^s \bullet \binom{T^l}{k} \right)
$$

is defined, then $m = l$ and $p = n$, so

$$
\left(\binom{T^q}{T^p}^s \bullet \binom{T^n}{T^m}^s \right) \bullet \binom{T^l}{k}
$$

is defined, as required. We handle the situation when

$$
\left[\begin{matrix} T^q \\ T^p \end{matrix} \right]^s \bullet \left(\left[\begin{matrix} T^n \\ T^m \end{matrix} \right]^s \bullet \left[\begin{matrix} T^l \\ k \end{matrix} \right] \right)
$$

is defined in the same way.

Condition (R) for the above defined pair of families clearly gives the statement from the beginning of this illustration. Since for $\left[\begin{smallmatrix} T^n \\ m \end{smallmatrix}\right] \in \mathcal{P}$ we have $(\partial^*)^n \left[\begin{smallmatrix} T^n \\ m \end{smallmatrix}\right] = \left(\begin{smallmatrix} T^0 \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} T^0 \\ 0 \end{smallmatrix}\right)$ has exactly one element, by Theorem 9, it suffices to see condition (P).

Condition (P) We carefully check condition (P). Fix an element of $\mathcal{P}$, which must be of the form $\left(\begin{smallmatrix} T^q \\ p \end{smallmatrix}\right)$ or $\left[\begin{smallmatrix} T^q \\ p \end{smallmatrix}\right]$. We consider the first case first. We assume $p > 1$ and leave $p \leq 1$ to the reader. To check (P), recall the pieces of notation set up in Eqs. (3) and (4). Fix $f_0 \in \partial^* \left(\begin{smallmatrix} T^q \\ p \end{smallmatrix}\right) = \left(\begin{smallmatrix} T^{q-1} \\ p-1 \end{smallmatrix}\right)$. We need to find an element $\left(\begin{smallmatrix} T^r \\ T^q \end{smallmatrix}\right)^s$ of $\mathcal{F}$ (it suffices, of course, to find r) and $g_0 \in A$ so that for each d-coloring of $\left(\begin{smallmatrix} T^r \\ T^q \end{smallmatrix}\right)^s_{g_0} \cdot \left(\begin{smallmatrix} T^q \\ p \end{smallmatrix}\right)_{f_0}$ there is $g \in \left(\begin{smallmatrix} T^r \\ T^q \end{smallmatrix}\right)^s_{g_0}$ such that $g \cdot \left(\begin{smallmatrix} T^q \\ p \end{smallmatrix}\right)_{f_0}$ is monochromatic. Note that $\left(\begin{smallmatrix} T^r \\ T^q \end{smallmatrix}\right)^s \bullet \left(\begin{smallmatrix} T^q \\ p \end{smallmatrix}\right)$ is automatically defined.

We claim that $g_0 \in A$ equal the identity function $T^{|f_0|+1} \to T^{|f_0|+1}$ does the job, where $|f_0|$ is defined by Eq. (11). Checking (P) boils down to stating precisely what elements the sets $\left(\begin{smallmatrix} T^q \\ p \end{smallmatrix}\right)_{f_0}$, $\left(\begin{smallmatrix} T^r \\ T^q \end{smallmatrix}\right)^s_{g_0}$, and $g \cdot \left(\begin{smallmatrix} T^q \\ p \end{smallmatrix}\right)_{f_0}$, for $g \in \left(\begin{smallmatrix} T^r \\ T^q \end{smallmatrix}\right)^s_{g_0}$, consist of. Let v_0 be the smallest with respect to $\leq_{T^q}$ element of the set $\mathrm{im}_{T^q}(f_0(p-1))$, and keep in mind that we are looking for r.

The set $\left(\begin{smallmatrix} T^q \\ p \end{smallmatrix}\right)_{f_0}$ consists of all $f \in \left(\begin{smallmatrix} T^q \\ p \end{smallmatrix}\right)$ with $\partial^* f = f_0$. This last condition is equivalent to saying that $f \restriction [p-1] = f_0$ and

$$f(p) \in T^q(v_0), \tag{15}$$

where Eq. (15) is a consequence of point (ii) in the definition of embedding between ordered trees. Each such embedding f is completely determined by the value of $f(p)$. Fix $r \geq q$, arbitrary for the moment. Let g be a strong embedding in $\left(\begin{smallmatrix} T^r \\ T^q \end{smallmatrix}\right)^s_{g_0}$. It is equal to the identity on $T^{|f_0|+1}$ and it is determined by strong embeddings g_v from $T^q(v)$ to $T^r(v)$, where v varies over the nodes of T^q with $\mathrm{ht}(v) = |f_0| + 1$. Now, elements of $g \cdot \left(\begin{smallmatrix} T^q \\ p \end{smallmatrix}\right)_{f_0}$ are embeddings $g \circ f : [p] \to T^r$ with f for which Eq. (15) holds. Each such embedding is completely determined by the value

$$(g \circ f)(p) = g_{v_0}(f(p)) \in T^r(v_0).$$

Therefore, solving the problem of fixing the color on $g \cdot \left(\begin{smallmatrix} T^q \\ p \end{smallmatrix}\right)_{f_0}$ amounts to the following: d-color $T^r(v_0)$ (this is where the values of $(g \circ f)(p)$ are coming from), then find a strong embedding (this is g_{v_0}) of $T^q(v_0)$ (this is where the values $f(p)$ are located) to $T^r(v_0)$ so that the image of $T^q(v_0)$ is monochromatic. This can be arranged using a form of the Halpern–Läuchli theorem ((HL2) with $t = 1$ from Appendix 2) by taking r large enough since $T^q(v_0)$ and $T^r(v_0)$ are isomorphic to T^m and T^n, where $m = q - (|f_0| + 1)$ and $n = r - (|f_0| + 1)$, respectively. For $v \neq v_0$, after identifying $T^q(v)$ with $T^q(v_0)$ and $T^r(v)$ with $T^r(v_0)$ via the unique isomorphisms, we let g_v be equal to g_{v_0}. Note that so defined g is strong (Fig. 2).
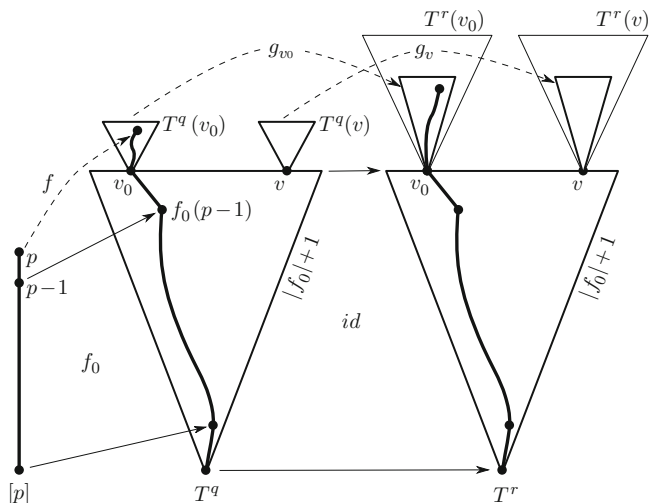
Fig. 2 Condition (P) in Illustration 10

The case $P = \begin{bmatrix} T^q \\ p \end{bmatrix}$ is handled analogously with the exception that for F one takes $\begin{bmatrix} T^r \\ T^q \end{bmatrix}^s$ for large enough r and one uses another form of the Halpern–Läuchli theorem ((HL1) from Appendix 2). We leave it to the reader to re-check the details.

4   A Ramsey Theorem for Finite Trees

We prove the following theorem that extends the results of Deuber [2] and of Jasiński [3]. Our proof differs from the arguments of these two papers.

Proposition 1. *For non-empty ordered trees S, T and $d > 0$, there exists a non-empty ordered tree V with* $\mathrm{br}(V) = \mathrm{br}(T)$ *such that for each d-coloring of all leaf preserving embeddings from S to V there is a leaf preserving embedding $g_0 \colon T \to V$ such that*

$$\{g_0 \circ f \colon f \colon S \to T \text{ a leaf preserving embedding}\}$$

is monochromatic.

As a direct consequence of the above result, one gets its version for embeddings that are not necessarily leaf preserving by the following argument. Given ordered trees S, T, let S_+, T_+ be the trees obtained from S and T by adding one node on top of each leaf of S and T, respectively. Apply now the above statement to S_+, T_+ obtaining V. Let V_- be gotten from V by deleting from it all of its leaves. It is easy to check that V_- works by using the obvious observation that embeddings from S

to T, from S to V_-, and from T to V_- are precisely restrictions of leaf preserving embeddings from S_+ to T_+, from S_+ to V, and from T_+ to V, respectively.

Deuber's theorem [2] is the above result for embeddings that are not necessarily leaf preserving and under the additional assumptions that $\mathrm{br}(S) = \mathrm{br}(T)$ and that S is regular as defined in Sect. 3. Jasiński's theorem is originally [3] stated for boron structures as defined in [1], but can easily be rephrased in terms of trees, and then it becomes equivalent to the above result with the additional assumptions that $\mathrm{br}(S) = \mathrm{br}(T)$ and that for each $v \in S$ that is not a leaf $|\mathrm{im}(v)| = 2$.

We show now how to derive Proposition 1 from Theorem 9.

Let $k = \mathrm{br}(T)$ and set $T^n = T^{k,n}$. Note that it is enough to prove the theorem for T equal to some T^n since every ordered tree T with $\mathrm{br}(T) = k$ embeds leaf-preservingly into some T^n.

We define an analogue of the set of natural numbers for the present Ramsey situation. We view T^n as an ordered subtree T^{n+1} via the inclusion $\subseteq'$ defined by Eq. (10). This convention gives an increasing sequence $(T^n)_{n \in \mathbb{N}}$ of ordered trees. Let the direct limit (that is, the union, if T^n is identified with its image in T^{n+1}) of this sequence be denoted by T^∞. Observe that T^∞ carries a linear order induced from the linear orders $\leq_{T^n}$ on the T^n-s. We denote this linear order by $\leq^\infty$. Each element v of T^∞ belongs to some T^n. We call v a **leaf** if v is a leaf in some, or equivalently all, T^n to which it belongs. For an ordered tree S, each function $f \colon S \to T^\infty$ has its range included in some T^n. We call f a **leaf preserving embedding** if f is a leaf preserving embedding to some, or equivalently all, T^n in which the image of f is included. Further, $g \colon D \to T^\infty$ for a subset D of T^∞ is called a **leaf preserving embedding** if the restriction of g to each $D \cap T^n$, $n \in \mathbb{N}$, is a leaf preserving embedding, where $D \cap T^n$ is taken with the tree order inherited from T^n. For a leaf $x \in T^\infty$, let

$$T_x = \{v \in T^\infty : v \leq^\infty x\}.$$

Note that T_x is an infinite set.

Normed Background Let Y consist of all leaf preserving embeddings $f \colon S \to T^\infty$, where S is an ordered tree. Let B consist of the empty function and of all leaf preserving embeddings $g \colon T_x \to T^\infty$, where x is a leaf of T^∞. It is easy to see that for such a $g \colon T_x \to T^\infty$, we have $g(T_x) \subseteq T_{g(x)}$. As always, for $f \in Y$ and $g \in B$, let $g \, . \, f$ to be defined precisely when the image of f is included in the domain of g and let

$$g \, . \, f = g \circ f.$$

Similarly for $g_1, g_2 \in B$, define $g_2 \cdot g_1$ to be defined precisely when the image of g_1 is contained in the domain of g_2 and let

$$g_2 \cdot g_1 = g_2 \circ g_1.$$

We define a truncation using the branch cutting derivation on trees given by Eq. (8). For $f \in Y$ with $f \colon S \to T^\infty$ define

$$\partial' f = f \upharpoonright S',$$

where S' is given by Eq. (8). We define a norm $|\cdot|: Y \to T^\infty \cup \{-\infty\}$, where T^∞ is considered as a linear order with $\leq^\infty$ and $-\infty$ is an element that is less than all the elements of T^∞, by letting for $f \in Y$ with $f: S \to T^\infty$

$$|f| = \begin{cases} \max \text{image}(f), & \text{if } S \neq \emptyset; \\ -\infty, & \text{if } S = \emptyset. \end{cases}$$

Observe that if $S \neq \emptyset$, then $|f|$ is the $\leq^\infty$-minimal leaf $x \in T^\infty$ such that image$(f) \subseteq T_x$. It is easy to check that with so defined operations, (B, Y) becomes a normed background.

A Pair of Families Over (B, Y) For $n \in \mathbb{N}$, let $x_n \in T^\infty$ be the rightmost leaf of T^n and let $v_n \in T^\infty$ be the root of T^n. Note that

$$T_{x_n} = T^n \cup \{v_{n+k}: k \in \mathbb{N}, \, k > 0\}.$$

Define for $0 < m \leq n$

$$\begin{bmatrix} T^n \\ T^m \end{bmatrix}^\infty = \{g \in B : g : T_{x_m} \to T_{x_n}, \, g(T^m) \subseteq T^n, \text{ and}$$

$$g(v_{m+k}) = v_{n+k} \text{ for all } k \in \mathbb{N}, k > 0\}.$$

Additionally, let $\begin{bmatrix} T^0 \\ T^0 \end{bmatrix}^\infty$ consist of the empty function. Observe that the function

$$\begin{bmatrix} T^n \\ T^m \end{bmatrix}^\infty \ni g \to g \restriction T^m$$

is a bijection from $\begin{bmatrix} T^n \\ T^m \end{bmatrix}^\infty$ to all leaf preserving embeddings from T^m to T^n.

Let $\mathcal{G}$ consist be the family of all subsets of B of the form $\begin{bmatrix} T^n \\ T^m \end{bmatrix}^\infty$, where $n, m \in \mathbb{N}$ and $0 < m \leq n$ or $m = n = 0$. Let $\mathcal{Q}$ be the family of all non-empty finite sets $Q \subseteq Y$ of the following form: there is an ordered tree S such that Q consists of some leaf preserving embeddings from S to T^∞. In such a situation, we say that Q **is based on** S. As usual, declare $\begin{bmatrix} T^n \\ T^m \end{bmatrix}^\infty \bullet \begin{bmatrix} T^l \\ T^k \end{bmatrix}^\infty$ to be defined precisely when $m = l$ and let

$$\begin{bmatrix} T^n \\ T^l \end{bmatrix}^\infty \bullet \begin{bmatrix} T^l \\ T^k \end{bmatrix}^\infty = \begin{bmatrix} T^n \\ T^k \end{bmatrix}^\infty.$$

Declare $\begin{bmatrix} T^n \\ T^m \end{bmatrix}^\infty \bullet Q$ to be defined precisely when m is the smallest natural number with the property that the images of all elements of Q are included in T^m, and let

$$\begin{bmatrix} T^n \\ T^m \end{bmatrix}^{\infty} \bullet Q = \begin{bmatrix} T^n \\ T^m \end{bmatrix}^{\infty} \cdot Q.$$

We leave to the reader the easy check that $(\mathcal{F}, \mathcal{Q}, \bullet, \bullet)$ is a pair of families over (B, Y), that is, the operations $\bullet$ and $\bullet$ are given pointwise. The pair of families fulfills conditions (A), (B), and ($*$). Condition (A) is clear. To see condition (B), assume that $\begin{bmatrix} T^n \\ T^m \end{bmatrix}^{\infty} \bullet \partial Q$ is defined, that is, m is smallest such that the image of all elements of ∂Q is included in T^m. Let $l \in \mathbb{N}$ be smallest such that the image of each element of Q is included in T^{m+l}. Then $\begin{bmatrix} T^{n+l} \\ T^{m+l} \end{bmatrix}^{\infty}$ witnesses that (B) holds since $\begin{bmatrix} T^{n+l} \\ T^{m+l} \end{bmatrix}^{\infty} \bullet Q$ is defined and, as is easy to check, each leaf preserving embedding from $\begin{bmatrix} T^n \\ T^m \end{bmatrix}^{\infty}$ extends (as a function) to a leaf preserving embedding from $\begin{bmatrix} T^{n+l} \\ T^{m+l} \end{bmatrix}^{\infty}$. Condition ($*$) follows immediately from an easy observation that if m is the smallest natural number such that the image of each function in Q is included in T^m, then n is the smallest natural number with each function in $\begin{bmatrix} T^n \\ T^m \end{bmatrix}^{\infty} \cdot Q$ having its image included in T^n.

Note that condition (R) in this case is the theorem we are proving. Observe also that if $Q \in \mathcal{Q}$ is based on S, then ∂Q is based on S', and S' has one leaf fewer than S if $S \neq \emptyset$. Thus, $\partial^t Q$ has exactly one element (the empty function) for t equal to the number of leaves in S. It follows that to get (R) it remains to check condition (P).

Condition (P) Let $Q \in \mathcal{Q}$ be based on S and let $q \in \mathbb{N}$ be smallest such that all elements of Q have ranges included in T^q. The set ∂Q is based on S'. We assume S' is not the empty tree. (The case $S' = \emptyset$ is easier, and we ask the reader to handle it after reading the current argument.) Let $u_0 \in S'$ be the splitting node of S, and identify $S \setminus S'$ with $[p]$ for some non-zero $p \in \mathbb{N}$. (Recall here the discussion following (8).) Fix $f_0 \in \partial' Q$. Then $f_0 : S' \to T^q$, $f_0 \in Y$. Let

$$v_0 = f_0(u_0) \in T^q. \tag{16}$$

To check (P), we need to find $r \in \mathbb{N}$ and $g_0 \in B$ such that for each d-coloring of $\begin{bmatrix} T^r \\ T^q \end{bmatrix}^{\infty}_{g_0} \cdot Q_{f_0}$ there is $g \in \begin{bmatrix} T^r \\ T^q \end{bmatrix}^{\infty}_{g_0}$ with $g \cdot Q_{f_0}$ monochromatic. We will show that large enough r works. Fix $r \geq q$. Now, we define g_0. Find the $\leq^{\infty}$-smallest leaf x in T^q such that the image of f_0 is included in T_x. Note that v_0 is a predecessor of x.

First we define $g_0 : T_x \to T^{\infty}$. Note that

$$T_x = (T_x \cap T^q) \cup \{v_{q+k} : k \in \mathbb{N}, k > 0\}.$$

For the moment, we view T^q as a subset of T^r in the sense $T^q \subseteq^* T^r$, as defined by Eq. (9), and we let g_0 be the identity on the elements of $T_x \cap T^q$ that are not leaves. Let g_0 map leaves of $T_x \cap T^q$ to leaves of T^r in such a way that g_0 on $T_x \cap T^q$ is a leaf preserving embedding to T^r. Let $g_0(v_{q+k}) = v_{r+k}$ for $k \in \mathbb{N}$, $k > 0$. It is clear that $g_0 \in B$.

Consider the set E of all $w \in T^q$ such that w is an immediate successor of a predecessor of x and $x <^\infty w$. The set $T^q \setminus T^q_x$ is partitioned into trees $T^q(w)$ with $w \in E$. Therefore, each $g \in \left[{T^r \atop T^q} \right]^\infty_{g_0}$ is equal to g_0 in T_x and is completely determined by leaf preserving embeddings

$$g_w = g \upharpoonright T^q(w) \colon T^q(w) \to T^r(w), \ w \in E.$$

Note that v_0 given by Eq. (16) has an immediate successor in E. Let w_0 be the $\leq^\infty$-smallest among them. For each $f \in Q_{f_0}$, $f \upharpoonright S'$ is equal to f_0 whose image is included in T_x, while the image of $f \upharpoonright (S \setminus S')$ is included in $T^q(w_0)$. So each element of $g \cdot Q_{f_0}$ being of the form $g \circ f \colon S \to T^r$ is completely determined by

$$g_{w_0} \circ (f \upharpoonright (S \setminus S')) \colon S \setminus S' \to T^r(w_0).$$

Note that the identification of $S \setminus S'$ with $[p]$ makes $f \upharpoonright (S \setminus S')$ into a leaf preserving embedding from $[p]$ to $T^q(w_0)$. Thus, fixing the color on $g \cdot Q_{f_0}$ amounts to the following (with notation as in Eq. (13)): d-color $\left[{T^r(w_0) \atop p} \right]$, find a leaf preserving embedding $g_{w_0} \colon T^q(w_0) \to T^r(w_0)$ so that $g_{w_0} \cdot \left[{T^q(w_0) \atop p} \right]$ is monochromatic. This can be achieved from Illustration 10 by taking r large enough as $T^r(w_0)$ and $T^q(w_0)$ are isomorphic to T^n and T^m, where $n = r - \mathrm{ht}(w_0)$ and $m = q - \mathrm{ht}(w_0)$. We can let $g_w \colon T^q(w) \to T^r(w)$ be arbitrary leaf preserving embeddings for $w \in E$, $w \neq w_0$.

5   Milliken's Theorem in Exercises

We prove in this section the following result due to Milliken [4]. The reader may consult [6] for another purely finitary proof of Milliken's theorem.

Proposition 2. *Let S and T be ordered trees. Assume that all leaves in T have the same height. For $d > 0$, there exists an ordered tree V with $\mathrm{br}(V) = \mathrm{br}(T)$ such that for each d-coloring of all strong, leaf preserving embeddings from S to V there is a strong, leaf preserving embedding $g_0 \colon T \to V$ such that*

$$\{g_0 \circ f \colon f \colon S \to T \ a \ strong, \ leaf \ preserving \ embedding\}$$

is monochromatic.

The proof of Proposition 2 that we will give yields also the statement obtained from Proposition 2 by replacing strong, leaf preserving embeddings by strong embeddings in all places. This statement can also be obtained from Proposition 2 by a proof that is identical to the argument following Proposition 1. It suffices to notice that, with the notation as in that argument, strong embeddings from S to T, from S to V_-, and from T to V_- are precisely restrictions of strong, leaf preserving embeddings from S_+ to T_+, from S_+ to V, and from T_+ to V, respectively.

The proof of Proposition 2 is a somewhat more sophisticated version of the argument in Illustration 10. Let $k = \mathrm{br}(T)$. As before set $T^n = T^{k,n}$. View T^n as a subtree of T^{n+1} via the inclusion $\subseteq^*$ defined in Eq. (9). This inclusion is a strong embedding. This way we obtain an increasing sequence $(T^n)_{n \in \mathbb{N}}$ of ordered trees. Let T_∞ be the union (direct limit) of this sequence. The range of each function $f \colon S \to T_\infty$ on an ordered tree S is included in some T^n. We call f a **strong embedding** if f is a strong embedding as a function from S to T^n for some, or equivalently all, T^n in which the image of f is included. For $v \in T_\infty$, let $\mathrm{ht}(v)$ be equal to $\mathrm{ht}_{T^n}(v)$ for some, or equivalently, all T^n with $v \in T^n$.

Normed Background Let Z consist of all strong embeddings $f \colon S \to T_\infty$, where S is an ordered tree. Let C consist of all strong embeddings $g \colon T^m \to T^n$, for some $m \le n$. For $f \in Z$ and $g \in C$, let $g \cdot f$ be defined precisely when the image of f is included in the domain of g and let

$$g \cdot f = g \circ f.$$

Similarly for $g_1, g_2 \in C$, let $g_2 \cdot g_1$ be defined precisely when the image of g_1 is contained in the domain of g_2, and let

$$g_2 \cdot g_1 = g_2 \circ g_1.$$

For $f \in Z$ with $f \colon S \to T^n$ define

$$\partial^* f = f \restriction S^*.$$

Define a norm $| \cdot | \colon Z \to \mathbb{N}$, by letting for $f \in Y$ with $f \colon S \to T_\infty$

$$|f| = \max_{v \in S} \mathrm{ht}(f(v)).$$

Exercise. Check that (C, Z) is a normed background.

A Pair of Families Over (C, Z) The pair of families described below extends the one described in Illustration 10. Recall the sets $\binom{T}{S}^s$ and $\left[\begin{smallmatrix} T \\ S \end{smallmatrix}\right]^s$ defined in Eq. (12). Let $\mathcal{H}$ consist of all $\binom{T^n}{T^m}^s$ and $\left[\begin{smallmatrix} T^n \\ T^m \end{smallmatrix}\right]^s$ where $m, n \in \mathbb{N}$ and $0 < m \le n$ or $m = n = 0$. Let $\mathcal{R}$ consist of all non-empty sets of the form $\binom{T^n}{S}^s$ and $\left[\begin{smallmatrix} T^n \\ S \end{smallmatrix}\right]^s$, where S is an ordered tree. Declare $\binom{T^n}{T^m}^s \bullet \binom{T^l}{T^k}^s$ and $\left[\begin{smallmatrix} T^n \\ T^m \end{smallmatrix}\right]^s \bullet \left[\begin{smallmatrix} T^l \\ T^k \end{smallmatrix}\right]^s$ to be defined precisely when $m = l$, and let

$$\binom{T^n}{T^l}^s \bullet \binom{T^l}{T^k}^s = \binom{T^n}{T^k}^s \quad \text{and} \quad \left[\begin{smallmatrix} T^n \\ T^l \end{smallmatrix}\right]^s \bullet \left[\begin{smallmatrix} T^l \\ T^k \end{smallmatrix}\right]^s = \left[\begin{smallmatrix} T^n \\ T^k \end{smallmatrix}\right]^s.$$

Similarly, declare $\binom{T^n}{T^m}^s \bullet \binom{T^l}{S}^s$ and $\left[\begin{smallmatrix} T^n \\ T^m \end{smallmatrix}\right]^s \bullet \left[\begin{smallmatrix} T^l \\ S \end{smallmatrix}\right]^s$ to be defined precisely when $m = l$, and let

$$\binom{T^n}{T^l}^s \bullet \binom{T^l}{S}^s = \binom{T^n}{S}^s \text{ and } \left[\begin{matrix} T^n \\ T^l \end{matrix}\right]^s \bullet \left[\begin{matrix} T^l \\ S \end{matrix}\right]^s = \left[\begin{matrix} T^n \\ S \end{matrix}\right]^s.$$

The operations $\bullet$ and $_\bullet$ are undefined in situations not specified above.

Exercise. Check that $(\mathcal{H}, \mathcal{R}, _\bullet, \bullet)$ is a pair of families over (C, Z) fulfilling conditions (A), (B), and ($*$). (Hint. This is almost identical to the argument in Illustration 10.)

Exercise. Note that it suffices to prove Proposition 2 for T of the form T^n (this is where the assumption that all leaves in T have the same height is used) and check that condition (R) for $(\mathcal{H}, \mathcal{R}, _\bullet, \bullet)$ implies Proposition 2 (as well as the statement obtained from Proposition 2 by replacing strong, leaf preserving embeddings by strong embeddings).

Exercise. Check condition (P) for $(\mathcal{H}, \mathcal{R}, \bullet, _\bullet)$. (Hint. This follows from the Halpern–Läuchli theorem for strong subtrees (HL1) and (HL2) from Appendix 2 and is similar to the argument for (P) in Illustration 10.)

6 Appendix 1: Conditions (A) and (B) Removed and the Final Word on Normed Backgrounds

1. The following criterion (P+) is the strengthening of condition (P) allowing us to get rid of conditions (A) and (B). It is obtained from (P) by replacing all occurrences of P, except the one in $F _\bullet P$, by $\partial^t P$ for a fixed but arbitrary $t \in \mathbb{N}$.

(P+)   given $d > 0$ and t, for all $P \in \mathcal{P}$ and $x \in \partial^{t+1} P$, there are $F \in \mathcal{F}$ and $a \in A$ such that $F _\bullet P$ is defined, $a \,.\, x$ is defined, and for every d-coloring of $F_a \,.\, (\partial^t P)_x$ there is $f \in F_a$ such that $f \,.\, (\partial^t P)_x$ is monochromatic.

The following result is [7, Corollary 4.4].

Theorem 11. *Let $(\mathcal{F}, \mathcal{S}, _\bullet, \bullet)$ be a pair of families with ($*$) over a normed background. Assume that each $P \in \mathcal{P}$ is finite and for each $P \in \mathcal{P}$ there is $t \in \mathbb{N}$ such that $\partial^t P$ consist of one element. If $(\mathcal{F}, \mathcal{P})$ fulfills (P+), then it fulfills (R).*

To see that Theorem 9 is a consequence Theorem 11, we note that (P) in the presence of (A) and (B) implies (P+). To see this implication, we proceed by induction on t. Condition (P+) for $t = 0$ is just (P). Assuming that (P+) holds for t, we prove it for $t + 1$. Let $P \in \mathcal{P}$ and $x \in \partial^{t+2} P$. By condition (A), $\partial P \in \mathcal{P}$. So condition (P+) for t applied to ∂P and x gives $F \in \mathcal{F}$ and $a \in A$ such that $F _\bullet \partial P$ is defined, $a \,.\, x$ is defined, and for every d-coloring of $F_a \,.\, (\partial^{t+1} P)_x$ there is $f \in F_a$ such that $f \,.\, (\partial^{t+1} P)_x$ is monochromatic. Now condition (B) gives $G \in \mathcal{F}$ such that $G _\bullet P$ is defined and such that each element of F is extended by an element of G. It follows that each element of F_a is extended by an element of

G_a. Now it is clear that G and a witness that (P+) holds for $t + 1$.

2. The main algebraic structures in the paper are normed backgrounds. We list below conditions that are more symmetric than those defining normed backgrounds. As indicated by Lemma 1, they give a notion that is in essence equivalent to normed background. All the normed backgrounds in the present paper and in [7] fulfill the conditions below.

Let $(A, X, \cdot, \,.\,, \partial, |\cdot|)$ be such that $\cdot$ is a partial function from $A \times A$ to A, $.$ is a partial function from $A \times X$ to X, ∂ is a function from X to X and $|\cdot|$ is a function from X to a set with a linear order $\leq$. Assume the following axioms hold for all $a, b \in A$ and $x, y \in X$:

(i)   if $a \,.\, (b \,.\, x)$ and $(a \cdot b) \,.\, x$ are defined, then $a \,.\, (b \,.\, x) = (a \cdot b) \,.\, x$;
(ii)   if $a \,.\, x$ and $a \,.\, \partial x$ are defined, then $\partial(a \,.\, x) = a \,.\, \partial x$;
(iii) $|\partial x| \leq |x|$;
(iv)   if $|x| \leq |y|$ and $a \,.\, x$ and $a \,.\, y$ are defined, then $|a \,.\, x| \leq |a \,.\, y|$;
(v)   if $|x| \leq |y|$ and $a \,.\, y$ is defined, then so is $a \,.\, x$.

The following result is [7, Lemma 4.5].

Lemma 1. *(a)   Assume $(A, X, \cdot, \,.\,, \partial, |\cdot|)$ fulfills conditions (i)–(v) above, then (A, X) with $\cdot, \,.\,, \partial$ and $|\cdot|$ is a normed background*
(b)   If (A, X) with $\cdot, \,.\,, \partial$ and $|\cdot|$ is a normed background, then there is a function $|\cdot|_1$ on X such that $(A, X, \cdot, \,.\,, \partial, |\cdot|_1)$ fulfills conditions (i)–(v) above.

7   Appendix 2: The Halpern–Läuchli Theorem for Strong Subtrees as a Restatement of the Hales–Jewett Theorem

We point out here that the Halpern–Läuchli theorem for strong subtrees (there are other, more difficult, versions) and the Hales–Jewett theorem are identical statements phrased in different languages. The importance of this translation for the presentation here comes from the fact that the Hales–Jewett theorem is shown in [7] to be one of the results that follow from the abstract approach to Ramsey theory. So when using the Halpern–Läuchli theorem in the present paper we stay within this approach. Justin Moore remarks that equivalence of these two statements (that is, of the Hales–Jewett and the Halpern–Läuchli theorems) has been known for some time.

We set up a dictionary for translating the Hales–Jewett theorem to the Halpern–Läuchli theorem. Let S and T be ordered trees. Let $f : \text{leaves}(S) \rightarrow \text{leaves}(T)$ be strictly increasing with respect the orders $\leq_S$ and $\leq_T$ (restricted to the leaves), and be such that for each $v \in S$ there is $w \in T$ such that for any two leaves x, y of S with $v = x \wedge y$ we have $w = f(x) \wedge f(y)$. Then there is a unique leaf preserving embedding from S to T whose restriction to leaves(S) is equal to f. We, therefore, refer to such an f itself as a **leaf preserving embedding**. If in the above definition

ht(w) depends only on ht(v), then the induced embedding is strong and again we call f **strong**. A sequence $f_1, \ldots, f_r : \text{leaves}(S) \to \text{leaves}(T)$ of strong embeddings is called a **strong sequence** if for $x, y \in \text{leaves}(S)$ and $1 \le i, j \le r$

$$\text{ht}(f_i(x) \wedge f_i(y)) = \text{ht}(f_j(x) \wedge f_j(y)).$$

Fix a linearly ordered finite set A that is disjoint from $\mathbb{N}$. For $n \in \mathbb{N}$, we consider ordered trees

$$A^{\le n} = \{v : [l] \to A : l \le n\},$$

where the tree relation is equal to the extension relation and the order relation is the one coming from the linear order on A. So $A^{\le n}$ is a version of the trees $T^{k,n}$ defined in Sect. 3, where $k = |A|$. Note that the set of leaves of this tree is equal to the set A^n of all functions from $[n]$ to A.

For any function $v : [l] \to A$, let $v' : A \cup [l] \to A$ be equal to the identity function on A and to v on $[l]$. Assume we have a function $w : [n] \to A \cup [m]$ such that

(i) $[m]$ is included in the image of w;
(ii) $w([l]) \cap [m]$ is an initial segment of $[m]$, for each $l \le n$.

Such w gives rise to a strong embedding $g_w : A^m \to A^n$ (recall that A^m and A^n are the sets of leaves of $A^{\le m}$ and $A^{\le n}$, respectively) defined by

$$g_w(x) = x' \circ w,$$

for $x \in A^m$. It is easy to check, using property (ii) of w, that g_w preserves the lexicographic order. Property (i) ensures that g_w is injective. Further note that for $x, y \in A^m$, if $x \wedge y = v_0$ with ht(v_0) $= i_0$, then $g_w(x) \wedge g_w(y) = v_1$ with ht(v_1) $= i_1$, where

$$i_1 = \max\{i : w([i]) \cap [m] \subseteq [i_0]\} \quad \text{and} \quad v_1(i) = v_0'(w(i)), \text{ for } i \in [i_1]. \tag{17}$$

Note that i_1 depends only on i_0. Thus, g_w is indeed a strong embedding.

Now assume that for $r \in \mathbb{N}$, we have $w : [n] \to A^r \cup [m]$ with properties (i) and (ii) above. Such a w gives rise to r functions $w_i = \pi_i \circ w$, where $\pi_i : A^r \cup [m] \to A \cup [m]$ is the i-th projection on A^r and the identity on $[m]$, also fulfilling conditions (i) and (ii). We therefore get a sequence of strong embeddings $g_w^1, \ldots, g_w^r : A^m \to A^n$ defined by

$$g_w^i(x) = x' \circ w_i,$$

where $x \in A^m$. Formulas (17) imply that this is a strong sequence.

The following result is a version of the Halpern–Läuchli theorem (for strong subtrees). Recall the definition of the trees $T^{k,n}$ from Sect. 3. Fix k and let $T^n = T^{k,n}$. Note that we can take $T^n = A^{\le n}$ for A with $|A| = k$.

(HL1) *Given $d > 0$, t and m there exists n such that for each d-coloring of* leaves(T^n) $\times \cdots \times$ leaves(T^n) *(t factors) there exists a strong sequence of leaf*

preserving embeddings $g_i: T^m \to T^n$, *for* $i = 1, \ldots, t$, *such that the set*

$$g_1(\text{leaves}(T^m)) \times \cdots \times g_t(\text{leaves}(T^m))$$

is monochromatic.

(HL2) *Given* $d > 0$, t *and* m *there exists* n *such that for each* d*-coloring of*

$$\{(w_1, \ldots, w_t): w_1, \ldots, w_r \in T^n, \text{ht}(w_i) = \text{ht}(w_j), \text{ for } 1 \le i, j \le t\}$$

there exists a strong sequence of embeddings $g_i: T^m \to T^n$ *for* $i = 1, \ldots, t$ *such that the set*

$$\{(g_1(v_1), \ldots g_t(v_t)): v_1, \ldots, v_t \in T^m, \text{ht}(v_i) = \text{ht}(v_j), \text{ for } 1 \le i, j \le t\}$$

is monochromatic.

We show that the above statements are re-phrasings of the Hales–Jewett theorem. The Hales–Jewett theorem can be stated as below in points (a) and (b). (It is stated this way in [7, Sect. 7], and it is proved there using the abstract approach to Ramsey theory.)

(a) Let B be a finite set not including any natural numbers. Given $d > 0$ and m there is n such that for each d-coloring of functions from $[n]$ to B there is a function $w_0: [n] \to B \cup [m]$ with properties (i) and (ii) such that the set

$$\{v \circ w_0: v: B \cup [m] \to B, v \upharpoonright B = \text{id}_B\}$$

is monochromatic.

(b) Let B be a finite set not including any natural numbers. Given $d > 0$ and m there is n such that for each d-coloring of functions from $[q]$ to B for all $q \le n$ there is $n_0 \le n$ and a function $w_0: [n_0] \to B \cup [m]$ with properties (i) and (ii) such that the set

$$\{v \circ w_0: v: B \cup [p] \to B, \ p \le m, \ v \upharpoonright B = \text{id}_B\}$$

is monochromatic.

By the discussion at the beginning of this appendix, it is clear that (HL1) and (HL2) follow from (a) and (b), respectively, by taking $B = A^t$.

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Some Affine Invariants Revisited

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Abstract We present several sharp inequalities for the $SL(n)$ invariant $\Omega_{2,n}(K)$ introduced in our earlier work on centro-affine invariants for smooth convex bodies containing the origin. A connection arose with the Paouris-Werner invariant Ω_K defined for convex bodies K whose centroid is at the origin. We offer two alternative definitions for Ω_K when $K \in C^2_+$. The technique employed prompts us to conjecture that any $SL(n)$ invariant of convex bodies with continuous and positive centro-affine curvature function can be obtained as a limit of normalized p-affine surface areas of the convex body.

Key words Affine surface area • Brunn-Minkowski-Firey theory • Centro-affine curvature • Centro-affine surface area • p-affine surface area

1 Introduction

Besides the intrinsic interest in affine invariants originating in Felix Klein's Erlangen Program, the extension to the Brunn-Minkowski-Firey theory [20, 21], and very recent connections between affine invariants and fields like stochastic geometry [3, 7] and information theory [17, 27, 30], led to an intense activity in this area of geometric analysis. The renewed interest in affine invariants has benefited also from a systematic approach classifying them, as for example in [8, 15, 16, 18], and

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from their use in affine and affine Sobolev inequalities [10, 11, 19, 23–26, 28, 39, 40] and problems arising in differential geometry [4–6, 9, 22, 35–38] which rely on isoperimetric-type functional inequalities. The study of such inequalities constitutes one of our primary goals of an on-going project.

The present paper spun as a follow-up of [34] in which we introduced new $SL(n)$-invariants for smooth convex bodies. We started by searching for sharp affine inequalities satisfied by one such invariant derived, in a certain sense, from the centro-affine surface area. The resulting inequalities are the subject of the next section. In the process, we encountered a connection to another $SL(n)$ invariant of convex bodies defined by Paouris and Werner who also related it to information theory [30]. In Sect. 3, we present two alternative definitions of this invariant. We noted that an additional $SL(n)$ invariant of convex bodies of class C_+^2 is defined with analogous techniques. This prompted us to conjecture that $SL(n)$ invariants for convex bodies with continuous and positive centro-affine curvature function can be obtained as limits of normalized p-affine surface areas of the convex body.

The setting for this paper is the Euclidean space $\mathbb{R}^n$, $n \geq 2$, in which we consider convex bodies containing the origin in their interior. Most of the time, we will also require that the convex bodies have smooth boundary, i.e. C^∞, with positive Gauss curvature. We will denote the set of such convex bodies by $\mathcal{K}_{\text{reg}}$. However, on several occasions, we will relax the regularity of the boundary to class C^2 with positive Gauss curvature and we will use the notation C_+^2 to indicate this latter class of convex bodies. The preferred parametrization of a convex body K will be with respect to the unit normal vector, $u \mapsto X_K(u)$, making many functions on the boundary ∂K to be considered as functions on the unit sphere $\mathbb{S}^{n-1}$.

We will denote the Gauss curvature of a convex body by $\mathcal{K}$ and its centro-affine curvature by $\mathcal{K}_0$. Geometrically, $\mathcal{K}_0^{-1/2}$ at a given point of ∂K is, up to a dimension dependent constant, the volume of the centered osculating ellipsoid at that point. Note that the centro-affine curvature is constant if and only if K is a centered ellipsoid. This can also be seen from a lemma due to Petty [31] since, analytically, as a function on the unit sphere, the centro-affine curvature is the ratio $\mathcal{K}_0(u) = \dfrac{\mathcal{K}(u)}{h^{n+1}(u)}$, $u \in \mathbb{S}^{n-1}$, where h is the support function of K: $h(u) = \max\{x \cdot u \mid \mathbf{x} \in K\}$ with $x \cdot u$ denoting the usual inner product in $\mathbb{R}^n$. Two additional notations are deemed necessary. First, $\mathcal{N}_0(u) := \mathcal{K}_0^{-\frac{1}{n+1}}(u)\mathcal{N}(u)$ is the centro-affine normal which is, pointwise, proportional to the (classical) affine normal $\mathcal{N}(u)$, [14]. Finally, we will use $d\mu_K$ to denote the cone measure of ∂K which, given that the Gauss curvature of K is positive, can be expressed by $d\mu_K(x) = h(\nu(x))\dfrac{1}{\mathcal{K}}(\nu(x))\,d\mu_{\mathbb{S}^{n-1}}(\nu(x))$, where $\nu : \partial K \to \mathbb{S}^{n-1}$ is the Gauss map of the boundary of K, hence the inverse of the parametrization X.

2 Inequalities for a Second Order Centro-Affine Invariant

In [34], we introduced a class of $SL(n)$ invariants for smooth convex bodies in $\mathbb{R}^n$. For a fixed convex body K, these invariants were the first, second, and, for an arbitrary integer k, the k-th variation of the volume of K while the boundary of the body was subject to a pointwise deformation in the direction of the centro-affine normal by a speed equal to a power of the centro-affine curvature at each specific point. The p-affine surface areas introduced by Lutwak [21] for p greater than one (later extended to the range $0 \leq p < 1$ by Hug [13], to $-n \leq p < 0$ by Meyer-Werner [29], and to $-\infty \leq p < n$ by Schütt-Werner [33]) are, via this method, part of this class of invariants. To exemplify, and also bring the reader's attention to a particular such invariant which is one of the main objects of this paper, let us consider the following deformation of a convex body K with smooth boundary:

$$\begin{cases} \dfrac{\partial X(u,t)}{\partial t} = \mathcal{K}_0^{\frac{1}{2}}(u,t)\, \mathcal{N}_0(u,t) \\ X(u,0) = X_K(u). \end{cases} \quad (1)$$

Then, the first variation of $Vol(K)$ is the centro-affine surface area of K:

$$\frac{d}{dt}\left(Vol(K)\right)_{t=0} = -\int_{\partial K} \mathcal{K}_0^{\frac{1}{2}}(\nu(x))\, d\mu_K(x) = -\Omega_n(K) =: \Omega_{1,n}(K), \quad (2)$$

see [34]. Recall that the centro-affine surface area of a convex body is the only one among the p-affine surface areas, $\Omega_p(K) = \displaystyle\int_{\partial K} \mathcal{K}_0^{\frac{p}{n+p}}\, d\mu_K$, invariant under $GL(n)$ transformations of the Euclidean space. Moreover, pursuing an additional variation, we obtain:

$$\begin{aligned} \Omega_{2,n}(K) \\ := -\left(\frac{d^2\, Vol(K(t))}{dt^2}\right)_{|t=0} \\ = \frac{n(n-1)}{2}\, Vol(K^\circ) - \frac{n-1}{2} \int_{\mathbb{S}^{n-1}} h\sqrt{\mathcal{K}_0}\, s(h\sqrt{\mathcal{K}_0}, h, \dots, h)\, d\mu_{\mathbb{S}^{n-1}}, \end{aligned} \quad (3)$$

where $s(f_1, f_2, \cdots, f_{n-1})$ is an extension of the mixed curvature function usually defined on C^2, here smooth, support functions to arbitrary smooth functions on the unit sphere $\mathbb{S}^{n-1}$, see [32] page 115 and also [34]. For the reader familiar with mixed determinants, the following can be taken as definition for the function $s(f_1, f_2, \cdots, f_{n-1})(u) := D(((f_1)_{ij} + \delta_{ij} f_1)(u), ((f_2)_{ij} + \delta_{ij} f_2)(u), \dots, ((f_{n-1})_{ij} + \delta_{ij} f_{n-1})(u)), u \in \mathbb{S}^{n-1}$, where D is a mixed determinant and $(.)_i$ represents the covariant differentiation with respect to the i-th vector of a positively oriented orthonormal frame on the unit sphere $\mathbb{S}^{n-1}$.

We will show in Proposition 1 that, in a certain sense, $\Omega_{2,n}(K)$ measures how far K is from being a centered ellipsoid. In preparation, we call the Aleksandrov body, A_f, associated with a continuous positive function f on the unit sphere the convex body whose support function h_f is the maximal element of

$$\{h \leq f \mid h : \mathbb{S}^{n-1} \to \mathbb{R} \text{ support function of a convex body}\}.$$

If f is itself a support function of a convex body L, then A_f is precisely the body L. Moreover, in general, $f = h_f$ almost everywhere with respect to the surface area measure of A_f. We could not find where this notion first surfaced in the literature, yet the work [9] gives an excellent background on this notion. We are now ready to state the following comparison result which we will use in analyzing $\Omega_{2,n}$:

Lemma 1 (Monotonicity Lemma). *Suppose that f is a strictly positive smooth function on the unit sphere $\mathbb{S}^{n-1}$ and that h is the support function of a convex body $K \subset \mathbb{R}^n$ which belongs to $\mathcal{K}_{reg}$. Then, denoting by $m := \min\limits_{\mathbb{S}^{n-1}} \dfrac{f}{h}$, respectively, $M := \max\limits_{\mathbb{S}^{n-1}} \dfrac{f}{h}$, we have*

$$m \cdot n \, Vol(K) \leq \int_{\mathbb{S}^{n-1}} fs(h, h, \ldots, h) \, d\mu_{\mathbb{S}^{n-1}} \leq M \cdot n \, Vol(K) \qquad (4)$$

and, if the Aleksandrov body associated with f has continuous positive curvature function, then

$$m^2 \cdot n \, Vol(K) \leq \int_{\mathbb{S}^{n-1}} fs(f, h, \ldots, h) \, d\mu_{\mathbb{S}^{n-1}} \leq M^2 \cdot n \, Vol(K). \qquad (5)$$

Proof. Since K belongs to $\mathcal{K}_{\text{reg}}$, $s(h, h, \ldots, h) > 0$ on $\mathbb{S}^{n-1}$, thus $mh \leq f \leq Mh$ implies directly (4). In fact, we will show that we also have

$$m \cdot V(h, g, h, \ldots, h) \leq V(f, g, h, \ldots, h) \leq M \cdot V(h, g, h, \ldots, h), \qquad (6)$$

for any g support function of a convex body, denoted for later use by K_2. Indeed, if f itself would be a support function of a convex body, this claim is simply due to the monotonicity of mixed volumes. If f is not a support function, then there exists a large enough constant c so that $f + ch$ is a support function of a convex body, say L, with the Gauss parametrization. Moreover, $L \subseteq K_1$, where the latter is the dilation of K by the factor $M + c$. Then, from the monotonicity of mixed volumes, we have that $V(L, K_2, K, \ldots, K) \leq V(K_1, K_2, K, \ldots, K)$. Choosing to represent these mixed volumes through the notation emphasizing the support functions of the two convex bodies, we have $V(f + ch, g, h, \ldots, h) \leq V((M + c)h, g, h, \ldots, h)$. Finally, using the linearity of mixed volumes, we obtain $V(f, g, h, \ldots, h) + cV(h, g, h, \ldots, h) \leq (M + c)V(h, g, h, \ldots, h)$ which is, after a trivial simplification, the right-hand side inequality of Eq. (6). Similarly,

by considering the dilation K of factor $(m + c)$, we obtain a convex body K_3 such that $K_3 \subseteq L$ and an argument analogous with the one above will imply $mV(h, g, h, \ldots, h) \leq V(f, g, h, \ldots, h)$.

We will now proceed to prove Eq. (5). Note again that if f would be a support function of a convex body, the claim follows from the monotonicity of mixed volumes. If f is not a support function, consider the Aleksandrov body associated to f, A_f, whose support function we denote by h_f. Thus $Mh \geq f \geq h_f \geq mh$ and, S_{A_f}-a.e., $f \circ \nu_{A_f}(x) = h_f(x)$, where ν_{A_f} is the Gauss map of ∂A_f. As, by hypothesis, A_f has a continuous positive curvature function, and by using Eq. (6), we have

$$\int_{\mathbb{S}^{n-1}} f s(f, h, \ldots, h) \, d\mu_{\mathbb{S}^{n-1}} = \int_{\partial A_f} f(\nu_{A_f}^{-1}(x)) s(f, h, \ldots, h)(\nu_{A_f}^{-1}(x)) \, dS_{A_f}(x)$$

$$= \int_{\partial A_f} h_f(\nu_{A_f}^{-1}(x)) s(f, h, \ldots, h)(\nu_{A_f}^{-1}(x)) \, dS_{A_f}(x)$$

$$= \int_{\mathbb{S}^{n-1}} h_f s(f, h, \ldots, h) \, d\mu_{\mathbb{S}^{n-1}}$$

$$= nV(f, h_f, h, \ldots, h)$$

$$\geq m \cdot n V(h, h_f, h, \ldots, h)$$

$$= m \int_{\mathbb{S}^{n-1}} h_f s(h, h, \ldots, h) \, d\mu_{\mathbb{S}^{n-1}}$$

$$\geq m \int_{\mathbb{S}^{n-1}} m \, h s(h, h, \ldots, h) \, d\mu_{\mathbb{S}^{n-1}}$$

$$= m^2 \cdot n \, Vol(K). \tag{7}$$

The second inequality can be proved similarly. $\square$

Consequently, we obtain the following inequalities for $\Omega_{2,n}(K)$.

Proposition 1. *Let $K \in \mathcal{K}_{reg}$ with the usual notations of h and $\mathcal{K}_0$ for the support function, respectively, the centro-affine curvature of K as functions on the sphere $\mathbb{S}^{n-1}$. Then*

1. *$\Omega_{n,2}(K) \geq 0$ with equality if and only if K is a centered ellipsoid.*
2. *If, in addition, the Aleksandrov body associated with $f := h\sqrt{\mathcal{K}_0}$ has continuous positive curvature function, then $\Omega_{n,2}(K) \leq \dfrac{(n-1)n}{2}(M - m)Vol(K)$, where M, m are the maximum and minimum of the centro-affine curvature of K. The equality occurs if and only if K is a centered ellipsoid.*

Proof. 1. The first claim follows immediately from the Minkowski-type inequality we detailed in Lemma 4.3 of [34]

$$\left(\int_{\mathbb{S}^{n-1}} f s(f, h, \ldots, h) \, d\mu_{\mathbb{S}^{n-1}} \right) \left(\int_{\mathbb{S}^{n-1}} h s(h, h, \ldots, h) d\mu_{\mathbb{S}^{n-1}} \right)$$

$$\leq \left(\int_{\mathbb{S}^{n-1}} f s(h, h, \ldots, h) \, d\mu_{\mathbb{S}^{n-1}} \right)^2 ,$$

where f is an arbitrary smooth function on the sphere, while h is a smooth support function of a convex body. It suffices to apply this inequality to the second term of $\Omega_{n,2}(K)$ with $f := h\sqrt{\mathcal{K}_0}$ to obtain

$$\Omega_{2,n}(K) \geq \frac{n(n-1)}{2} Vol(K^\circ) - \frac{n-1}{2n} \frac{\Omega_n^2(K)}{Vol(K)}$$

from which the result follows by Hölder's inequality

$$Vol(K^\circ) \cdot Vol(K)$$

$$= \frac{1}{n^2} \left(\int_{\partial K} \mathcal{K}_0 \, d\mu_K \right) \cdot \left(\int_{\partial K} d\mu_K \right) \geq \frac{1}{n^2} \left(\int_{\partial K} \sqrt{\mathcal{K}_0} \, d\mu_K \right)^2 . \quad (8)$$

Note that the equality is attained if and only if $\mathcal{K}_0$ is constant on $\mathbb{S}^{n-1}$, hence if and only if K is a centered ellipsoid.

2. By taking $f = h\sqrt{\mathcal{K}_0}$ with $m \leq \mathcal{K}_0 \leq M$, we can apply Eq. (5),

$$\Omega_{2,n}(K) = \frac{n(n-1)}{2} Vol(K^\circ)$$

$$- \frac{n-1}{2} \int_{\mathbb{S}^{n-1}} h\sqrt{\mathcal{K}_0} \, s(h\sqrt{\mathcal{K}_0}, h, \ldots, h) \, d\mu_{\mathbb{S}^{n-1}},$$

$$\leq \frac{n(n-1)}{2} \frac{1}{n} \int_{\partial K} \mathcal{K}_0 \, d\mu_K - \frac{n(n-1)}{2} m \, Vol(K)$$

$$\leq \frac{n(n-1)}{2} (M - m) \, Vol(K). \quad (9)$$

Equality is attained if and only if $M = m$ which implies, as before, that K is a centered ellipsoid. Note that we have only used the left-hand side inequality of Eq. (5). It so happens that the right-hand side inequality of Eq. (5) follows for this choice of function f from the positivity of $\Omega_{2,n}(K)$ for any $K \in \mathcal{K}_{\text{reg}}$. $\quad \square$

Further, the previous result implies additional isoperimetric-type inequalities.

Theorem 1. *If $K \in \mathcal{K}_{reg}$, the following $Gl(n)$-invariant inequality holds*

$$\frac{1}{n^2}\,\Omega_n^2(K) \le Vol(K) \cdot Vol(K^\circ)$$

$$\le \frac{2}{n(n-1)}\,\min\{Vol(K) \cdot \Omega_{n,2}(K), Vol(K^\circ) \cdot \Omega_{n,2}(K^\circ)\} + \frac{1}{n^2}\,\Omega_n^2(K),$$

and equality occurs if and only if K is a centered ellipsoid.

If, in addition, K is such that the Aleksandrov body associated with $f := h\sqrt{\mathcal{K}_0}$ has continuous positive curvature function and $\dfrac{M}{m} \le \dfrac{1 + \sqrt{5}}{2}$, the golden ratio, then the following $Gl(n)$-invariant inequality holds:

$$\frac{1}{n^2}\,\Omega_n^2(K) \le Vol(K) \cdot Vol(K^\circ) \le \frac{1}{n^2}\,\Omega_n^2(K)\left[1 - \frac{M-m}{\sqrt{Mm}}\right]^{-1}$$

with equality if and only if K is a centered ellipsoid.

Proof. The left-hand inequality follows immediately from Hölder's inequality. In fact, this easy remark motivated a search for an upper bound of the volume product $Vol(K) \cdot Vol(K^\circ)$ in terms of the centro-affine surface area or, in other words, a reverse isoperimetric-type inequality.

Toward this goal, note that the sign of $\Omega_{2,n}(K)$ translates into the following $Gl(n)$-invariant inequality:

$$\frac{1}{n^2}\,\Omega_n^2(K) \le Vol(K) \cdot Vol(K^\circ) \le \frac{2}{n(n-1)}Vol(K) \cdot \Omega_{n,2}(K) + \frac{1}{n^2}\,\Omega_n^2(K),$$

with equality if and only if K is a centered ellipsoid. Apply the same inequality with the roles of K and K° reversed and use the fact that $\Omega_n(K) = \Omega_n(K^\circ)$, [12,18,39]. Therefore,

$$\frac{1}{n^2}\,\Omega_n^2(K) \le Vol(K) \cdot Vol(K^\circ)$$

$$\le \frac{2}{n(n-1)}\,\min\{Vol(K) \cdot \Omega_{n,2}(K), Vol(K^\circ) \cdot \Omega_{n,2}(K^\circ)\} + \frac{1}{n^2}\,\Omega_n^2(K),$$

with equality if and only if K is a centered ellipsoid.

From Proposition 1,

$$\frac{2}{n(n-1)}\,Vol(K) \cdot \Omega_{n,2}(K) \le (M-m)\,Vol^2(K)$$

and

$$\frac{2}{n(n-1)}\,Vol(K^\circ) \cdot \Omega_{n,2}(K^\circ) \le (M^\circ - m^\circ)\,Vol^2(K^\circ),$$

thus

$$\frac{2}{n(n-1)} \min\{Vol(K) \cdot \Omega_{n,2}(K), Vol(K^{\circ}) \cdot \Omega_{n,2}(K^{\circ})\}$$

$$\leq \sqrt{(M-m)(M^{\circ}-m^{\circ})}\, Vol(K) \cdot Vol(K^{\circ}).$$

Here m° and M° are the minimum, respectively, the maximum of the centro-affine curvature of ∂K°.

For any point of ∂K, x, there exists a point y on ∂K° such that $\mathcal{K}_0(x) \cdot \mathcal{K}_0^{\circ}(y) = 1$, see [12], thus $M \cdot m^{\circ} = 1$ and $m \cdot M^{\circ} = 1$ otherwise a contradiction with one of the definitions of m°, M° occurs. Hence

$$\sqrt{(M-m)(M^{\circ}-m^{\circ})} = \sqrt{(M-m)\left(\frac{1}{m}-\frac{1}{M}\right)} = \frac{M-m}{\sqrt{Mm}},$$

which is less or equal to 1 if and only if M/m is less or equal to the golden ratio above.

Thus

$$Vol(K) \cdot Vol(K^{\circ}) \leq \frac{M-m}{\sqrt{Mm}} \cdot Vol(K) \cdot Vol(K^{\circ}) + \frac{1}{n^2}\, \Omega_n^2(K)$$

which implies the right-hand side inequality. The equalities follow as before from $M = m$ equivalent to constant centro-affine curvature along the boundary ∂K.   □

Note that in the next proposition we drop the smoothness assumption on the boundary of K to class C^2.

Proposition 2. *For any $p > 1$, and any $K \in C_+^2$ with the origin in its interior, we have*

$$\frac{\Omega_p^{n+p}(K)}{Vol^{n-p}(K)} \leq n^{p-1}\left(Vol(K) \cdot Vol(K^{\circ})\right)^{p-1} \cdot \frac{\Omega^{n+1}(K)}{Vol^{n-1}(K)}. \tag{10}$$

The equality holds if and only if $p = 1$ or K is a centered ellipsoid.
The opposite inequality holds for $p < 1$, $p \neq -n$.

Proof. Note that, for any $p \neq -n$,

$$\Omega_p(K) = \int_{\partial K} \mathcal{K}_0^{\frac{p}{n+p}}\, d\mu_K = \int_{\partial K} \left(\mathcal{K}_0^{\frac{n}{n+1}}\right)^{\frac{p-1}{n+p}}\, d\sigma_K, \tag{11}$$

where $d\sigma_K$ is the affine surface area measure, in other words the Blaschke metric, of K. As the function $x \mapsto x^{\frac{p-1}{n+p}}$, $x > 0$, is concave for $p \geq 1$ and convex for $p \leq 1$, we apply the appropriate Jensen's inequality for each range and the normalized measure $\frac{1}{\Omega(K)}\, d\sigma_K$. If $p \geq 1$, we obtain

$$\left(\frac{nVol(K^\circ)}{\Omega(K)}\right)^{\frac{p-1}{n+p}} \geq \frac{\Omega_p(K)}{\Omega(K)} \quad \Leftrightarrow \quad \Omega_p(K) \leq \left(n\,Vol(K^\circ)\right)^{\frac{p-1}{n+p}} \cdot \Omega^{\frac{n+1}{n+p}}(K) \quad (12)$$

with equality if and only if $p = 1$ or K is a centered ellipsoid. A re-arrangement of terms, gives Eq. (10). The proof of the reverse inequality in the case $p \leq 1$ is perfectly similar. □

Corollary 1. *For any convex body* $K \in \mathcal{K}_{reg}$,

$$n^n \left[\frac{2}{n-1} Vol(K) \cdot \Omega_{n,2}(K) + \frac{1}{n} \Omega_n^2(K)\right]$$

$$\geq \frac{\Omega^{n+1}(K)}{Vol^{n-1}(K)} \geq \frac{\Omega_n^{2n}(K)}{[(2/(n-1))\,\Omega_{n,2}(K)Vol(K) + \Omega_n^2(K)/n]^{n-1}},$$

with equality iff K *is a centered ellipsoid.*

Proof. Apply the previous result for $p = 0$ and, respectively, $p = n$, and use the bounds on $Vol(K) \cdot Vol(K^\circ)$ from Theorem 1. □

Corollary 2 (Isoperimetric-like Inequality). *For any* $K \in C_+^2$ *with the centroid at the origin, and any* $T \in Sl(n)$,

$$\frac{S^n(TK)}{Vol^{n-1}(K)} \geq \frac{n}{\omega_n^{2n-3}} \max\left\{\frac{\Omega_n^{2n}(K)}{\Omega^{n+1}(K)/Vol^{n-1}(K)}, \left(\frac{\Omega^{n+1}(K)}{Vol^{n-1}(K)}\right)^{n-1}\right\}, \quad (13)$$

where $S(TK)$ *stands for the surface area of* TK *and* ω_n *is the volume of the unit ball* $x_1^2 + \ldots + x_n^2 = 1$ *in* $\mathbb{R}^n$. *Equality occurs if and only if* K *is a centered ellipsoid and* T *is the linear transformation of determinant one such that* TK *is a ball.*
Hence

Proof. Consider $p = n$ in the inequality of Proposition 2 to obtain

$$\Omega_n^{2n}(K) \leq n^{n-1}[Vol(K) \cdot Vol(K^\circ)]^{n-1} \cdot \frac{\Omega^{n+1}(K)}{Vol^{n-1}(K)}. \quad (14)$$

From the classical isoperimetric inequality,

$$Vol^{n-1}(K) \leq \left(Vol^{n-1}(B)/S^n(B)\right) S^n(K),$$

where B is the unit ball as above. On the other hand, by Blaschke-Santaló inequality, $Vol(K) \cdot Vol(K^\circ) \leq (Vol(B))^2$.
Therefore

$$\Omega_n^{2n}(K) \leq n^{n-1} \frac{Vol^{3(n-1)}(B)}{S^n(B)} \frac{S^n(K)}{Vol^{n-1}(K)} \cdot \frac{\Omega^{n+1}(K)}{Vol^{n-1}(K)}, \quad (15)$$

where all quantities, except $S(K)$, are invariant under linear transformations of determinant one. Hence, the conclusion follows as $n^{n-1} \dfrac{Vol^{3(n-1)}(B)}{S^n(B)} = \dfrac{\omega_n^{2n-3}}{n}$. To analyze the equality case one needs to take T to be the linear transformation of determinant one minimizing the surface area of K and note that all other equalities hold if and only if K is a centered ellipsoid.

We will now use $p = 0$ in Proposition 2, to obtain

$$
\begin{aligned}
\frac{\Omega^{n+1}(K)}{Vol^{n-1}(K)} &\le n\,Vol(K) \cdot Vol(K^\circ) \le n\,\frac{V(B)}{S(B)^{n/(n-1)}} \cdot S(K)^{n/(n-1)} \cdot Vol(K^\circ) \\
&\le n\,\frac{V(B)^3}{S(B)^{n/(n-1)}} \cdot \frac{S(K)^{n/(n-1)}}{Vol(K)} = n^{1-\frac{n}{n-1}}\,\omega_n^{3-\frac{n}{n-1}}\,\frac{S(K)^{n/(n-1)}}{Vol(K)},
\end{aligned}
\tag{16}
$$

relying again on Blaschke-Santaló inequality.

From here,

$$
\frac{S^n(TK)}{Vol^{n-1}(K)} \ge \frac{n}{\omega_n^{2n-3}} \left(\frac{\Omega^{n+1}(K)}{Vol^{n-1}(K)} \right)^{n-1},
\tag{17}
$$

with the same condition for the equality case as above.   □

One can use K. Ball's reverse isoperimetric ratio which gives an upper bound on $\dfrac{S^n(TK)}{Vol^{n-1}(K)}$ by the corresponding ratio for the regular solid simplex in $\mathbb{R}^n$ (or the solid cube in the centrally-symmetric case), [1,2], in the above corollary to get lower bounds on the affine isoperimetric ratio of bodies in C_+^2. However, these bounds will not be sharp.

As in Corollary 1, one can drop the requirement that the centroid of K is at the origin, consider $K \in \mathcal{K}_{\text{reg}}$, and use the upper bound on the volume product from Theorem 1 instead of Blaschke-Santaló inequality, to obtain $SL(n)$ invariant lower bounds on the isoperimetric ratio $S(TK)^n / Vol(K)^{n-1}$.

Finally, we include the next corollary, due to [30], which follows immediately from Proposition 2.

Corollary 3. *For any convex body K of class C_+^2 containing the origin in its interior,*

$$
\Omega_K \le \frac{\Omega^{n+1}(K)}{(nVol(K^\circ))^{n+1}},
\tag{18}
$$

where $\Omega_K := \lim\limits_{p \to \infty} \left(\dfrac{\Omega_p(K)}{nVol(K^\circ)} \right)^{n+p}$ is the affine invariant introduced by Paouris and Werner in [30]. The equality occurs if and only if K is a centered ellipsoid.

Note that in [30], for certain considerations, the invariant Ω_K has been defined for convex bodies whose centroid is at the origin, yet the above definition makes sense for any convex body K of class C_+^2 containing the origin in its interior for which one can show as in [30] that the limit exists.

3   More on the Paouris-Werner Invariant

Motivated by the earlier occurrence of Ω_K, we would like to give here a couple of other definitions of this invariant when K belongs to C_+^2. To do so, let us also recall that Paouris and Werner showed in [30] that Ω_K is related to the Kullback-Leibler divergence D_{KL} of two specific probability measures P, Q on ∂K via the relation $D_{KL}(P\|Q) = \ln\left(\dfrac{Vol(K)}{Vol(K^\circ)}\Omega_K^{-1/n}\right)$, where, in slightly different terms than in [30],

$$D_{KL}(P\|Q) := \frac{1}{nVol(K^\circ)}\int_{\partial K}\mathcal{K}_0\ln\left(\mathcal{K}_0\frac{Vol(K)}{Vol(K^\circ)}\right)d\mu_K.$$

Hence, it is useful to note the identity

$$\ln(\Omega_K) = -\frac{1}{Vol(K^\circ)}\int_{\partial K}\mathcal{K}_0\ln\mathcal{K}_0\,d\mu_K, \tag{19}$$

and note that, in this paper, we assume only that the origin is contained in the interior of the convex body K.

Proposition 3. *For any K of class C_+^2 containing the origin in its interior, and any integer $p > 1$, the following $Gl(n)$-invariant inequalities hold*

$$\Omega_n^2(K) \geq \frac{\left(\Omega_{n/3}(K)\right)^4}{(nVol(K))^2} \geq \frac{\left(\Omega_{n/7}(K)\right)^8}{(nVol(K))^6} \geq \ldots \geq \frac{\left(\Omega_{n/(2^p-1)}(K)\right)^{2^p}}{(nVol(K))^{2^p-2}} \geq \ldots, \tag{20}$$

or, alternately,

$$\Omega_n^2(K) \geq \frac{\left(\Omega_{3n}(K^\circ)\right)^4}{(nVol(K))^2} \geq \frac{\left(\Omega_{7n}(K^\circ)\right)^8}{(nVol(K))^6} \geq \ldots \geq \frac{\left(\Omega_{n(2^p-1)}(K^\circ)\right)^{2^p}}{(nVol(K))^{2^p-2}} \geq \ldots, \tag{21}$$

$$\Omega_n^2(K) \geq \frac{\left(\Omega_{3n}(K)\right)^4}{(nVol(K^\circ))^2} \geq \frac{\left(\Omega_{7n}(K)\right)^8}{(nVol(K^\circ))^6} \geq \ldots \geq \frac{\left(\Omega_{n(2^p-1)}(K)\right)^{2^p}}{(nVol(K^\circ))^{2^p-2}} \geq \ldots. \tag{22}$$

In all sequences, all equalities hold if and only if K is a centered ellipsoid (which is the only reason why we did not include $p = 1$ in the statement).

Proof. Note that Eqs. (20) and (21) are equivalent through the equality $\Omega_q(K) = \Omega_{n^2/q}(K^\circ)$, [12, 18, 39]. The same goes for Eq. (22) due to $\Omega_n(K) = \Omega_n(K^\circ)$ and interchanging the roles of K and K° in the previous sequence of inequalities. Thus, it suffices to prove (20).

We will use the concavity of the function $x \mapsto \sqrt{x}$ on $(0, \infty)$ and Jensen's inequality as follows:

$$\left(\frac{\Omega_n(K)}{n \, Vol(K)} \right)^{1/2} = \left(\int_{\partial K} \sqrt{\mathcal{K}_0} \, \frac{d\mu_K}{n \, Vol(K)} \right)^{1/2} \geq \int_{\partial K} \sqrt[4]{\mathcal{K}_0} \, \frac{1}{n \, Vol(K)} \, d\mu_K, \quad (23)$$

thus

$$\left(\frac{\Omega_n(K)}{n \, Vol(K)} \right)^{1/2} \geq \frac{\Omega_{n/3}(K)}{n \, Vol(K)},$$

which is, after raising both sides to power four, the first inequality of Eq. (20). Re-iterate now the same argument for $\Omega_{n/3}(K)$:

$$\left(\frac{\Omega_{n/3}(K)}{n \, Vol(K)} \right)^{1/2} = \left(\int_{\partial K} \sqrt[4]{\mathcal{K}_0} \, \frac{d\mu_K}{n \, Vol(K)} \right)^{1/2} \geq \int_{\partial K} \sqrt[8]{\mathcal{K}_0} \, \frac{1}{n \, Vol(K)} \, d\mu_K, \quad (24)$$

which translates into

$$\left(\frac{\Omega_{n/3}(K)}{n \, Vol(K)} \right)^{1/2} \geq \frac{\Omega_{n/7}(K)}{n \, Vol(K)}.$$

Hence

$$\Omega_n(K) \geq \frac{\Omega_{n/3}^2(K)}{n \, Vol(K)} \geq \frac{\Omega_{n/7}^4(K)}{(n \, Vol(K))^3}$$

and so on, the sequence is obtained by iterating the argument. □

Theorem 2 (Alternative Definition of Ω_K). *For any K of class C_+^2 containing the origin in its interior, the scaling invariant sequence*

$$\left\{ \frac{\left(\Omega_{n(2^p-1)}(K) \right)^{2^p}}{(n \, Vol(K^\circ))^{2^p}} \right\}_{p \in \mathbb{N}, \, p \geq 1}$$

converges and

$$\lim_{p \to \infty} \left(\frac{\Omega_{n(2^p-1)}(K)}{n \, Vol(K^\circ)} \right)^{2^p} = \Omega_K. \quad (25)$$

Proof. By Eq. (22), the positive sequence $\dfrac{\left(\Omega_{n(2^p-1)}(K) \right)^{2^p}}{(n \, Vol(K^\circ))^{2^p-2}}$ is decreasing, thus converges. Therefore, so does the sequence above whose general term differs from general term of the former sequence by a factor of $(n \, Vol(K^\circ))^{-2}$.

Let $q := n(2^p - 1)$, and, similarly with Proposition 3.6 in [30], consider

$$\ln\left[\lim_{p\to\infty} \frac{\left(\Omega_{n(2^p-1)}(K)\right)^{2^p}}{(nVol(K^\circ))^{2^p}}\right]$$

$$= \lim_{p\to\infty} 2^p \ln\left(\frac{\Omega_{n(2^p-1)}(K)}{nVol(K^\circ)}\right)$$

$$= -\frac{2^p}{\ln 2} \frac{\frac{d}{dp}\left(\Omega_{n(2^p-1)}(K)\right)}{\Omega_{n(2^p-1)}(K)}$$

$$= -\lim_{p\to\infty} \frac{2^p}{\ln 2} \frac{\frac{d}{dq}\left(\int_{\partial K} \exp\left(\ln \mathcal{K}_0^{\frac{q}{n+q}}\right) d\mu_K\right) \frac{dq}{dp}}{\Omega_{n(2^p-1)}(K)}$$

$$= -\lim_{p\to\infty} 2^{2p} \frac{\frac{d}{dq}\left(\int_{\partial K} \exp\left(\ln \mathcal{K}_0^{\frac{q}{n+q}}\right) d\mu_K\right)}{\Omega_{n(2^p-1)}(K)}$$

$$= -\lim_{p\to\infty} 2^{2p} \frac{\left(\int_{\partial K} \exp\left(\ln \mathcal{K}_0^{\frac{q}{n+q}}\right) \ln(\mathcal{K}_0) \frac{n}{(n+q)^2} d\mu_K\right)}{\Omega_{n(2^p-1)}(K)}$$

$$= -n \lim_{p\to\infty} \frac{\int_{\partial K} \mathcal{K}_0^{\frac{2^p-1}{2^p}} \ln(\mathcal{K}_0) d\mu_K}{\Omega_{n(2^p-1)}(K)}$$

$$= -n \frac{\int_{\partial K} \mathcal{K}_0 \ln(\mathcal{K}_0) d\mu_K}{n \, Vol(K^\circ)} = \ln(\Omega_K).$$

The last equality, due to Eq. (19), completes the proof.   □

Following from the monotonicity of the sequence (22), we have

Corollary 4. *For any K of class C_+^2 containing the origin in its interior, and any integer $p \geq 1$,*

$$\Omega_K \cdot (nVol(K^\circ))^2 \leq \frac{(\Omega_{n(2^p-1)}(K))^{2^p}}{(nVol(K^\circ))^{2^p-2}}, \tag{26}$$

in particular $\Omega_K \cdot (nVol(K^\circ))^2 \leq \Omega_n^2(K)$, with equalities everywhere if and only if K is a centered ellipsoid.

Corollary 5. *For any K of class C_+^2 containing the origin in its interior, and any integer $p \geq 1$,*

$$\Omega_K \cdot \Omega_{K^\circ} \leq \frac{(\Omega_{n(2^p-1)}(K) \cdot \Omega_{n(2^p-1)}(K^\circ))^{2^p}}{(n^2 Vol(K) \cdot Vol(K^\circ))^{2^p}}, \tag{27}$$

in particular $\Omega_K \cdot \Omega_{K^\circ} \le \dfrac{\Omega_n^2(K) \cdot \Omega_n^2(K^\circ)}{(n^2 Vol(K) \cdot Vol(K^\circ))^2}$, *with equalities everywhere if and only if K is a centered ellipsoid in which case the right-hand sides of the two inequalities are equal to* 1.

The definition of Ω_K can be extended to affine surface areas of negative exponent using a similar result with Proposition 3:

Theorem 3 (Second Alternative Definition of Ω_K). *For any K of class C_+^2 containing the origin in its interior, the sequence*

$$\left\{ \left(\frac{\Omega_{-(n+2^p)}(K^\circ)}{n Vol(K)} \right)^{2^p} \right\}_{p \in \mathbb{N}, \ p \ge 1}$$

converges and

$$\lim_{p \to \infty} \left(\frac{\Omega_{-(n+2^p)}(K^\circ)}{n Vol(K)} \right)^{2^p} = \Omega_K^{-1}. \tag{28}$$

Proof. By applying again Jensen's inequality for the concave function $x \mapsto \sqrt{x}$, $x > 0$, we have, for any integer $p \ge 1$,

$$\int_{\partial K} \mathcal{K}_0^{-\frac{n}{2}} d\mu_K \ge \frac{\left(\int_{\partial K} \mathcal{K}_0^{-\frac{n}{4}} d\mu_K \right)^2}{n\, Vol(K)} \ge \frac{\left(\int_{\partial K} \mathcal{K}_0^{-\frac{n}{8}} d\mu_K \right)^4}{(n\, Vol(K))^3}$$

$$\ge \cdots \ge \frac{\left(\int_{\partial K} \mathcal{K}_0^{-\frac{n}{2^p}} d\mu_K \right)^{2^p}}{(n\, Vol(K))^{2^p - 1}} \ge \cdots \tag{29}$$

therefore the sequence of general term

$$\left(\frac{\Omega_{-(n+2^p)}(K^\circ)}{n Vol(K)} \right)^{2^p} = \left(\frac{\Omega_{-n^2/(n+2^p)}(K)}{n Vol(K)} \right)^{2^p}$$

$$= \left(\frac{1}{n\, Vol(K)} \cdot \frac{(\Omega_{-n^2/(n+2^p)}(K))^{2^p}}{(n Vol(K))^{2^p - 1}} \right)$$

is monotone. Interchanging K with K°, we conclude that the sequence $\left\{ \left(\dfrac{\Omega_{-(n+2^p)}(K)}{n Vol(K^\circ)} \right)^{2^p} \right\}_{p \in \mathbb{N}, \ p \ge 1}$ is monotone.

We now proceed as in the previous theorem with

$$\ln \left[\lim_{p \to \infty} \frac{(\Omega_{-(n+2^p)}(K))^{2^p}}{(n Vol(K^\circ))^{2^p}} \right] = \lim_{p \to \infty} 2^p \ln \left(\frac{\Omega_{-(n+2^p)}(K)}{n Vol(K^\circ)} \right)$$

$$= -\frac{2^p}{\ln 2} \frac{\frac{d}{dp}\left(\Omega_{-(n+2^p)}(K)\right)}{\Omega_{-(n+2^p)}(K)}$$

$$= -\lim_{p\to\infty} \frac{2^p}{\ln 2} \frac{\frac{d}{dp}\left(\int_{\partial K} \exp\left(\ln \mathcal{K}_0^{\frac{n}{2^p}+1}\right) d\mu_K\right)}{\Omega_{-(n+2^p)}(K)}$$

$$= n \lim_{p\to\infty} \frac{\left(\int_{\partial K} \exp\left(\ln \mathcal{K}_0^{\frac{n+2^p}{2^p}}\right) \ln(\mathcal{K}_0)\, d\mu_K\right)}{\Omega_{-(n+2^p)}(K)}$$

$$= n \frac{\int_{\partial K} \mathcal{K}_0 \ln(\mathcal{K}_0)\, d\mu_K}{n\, Vol(K^\circ)} = -\ln(\Omega_K),$$

and, using Eq. (19), we complete the proof of the theorem. $\qquad\square$

While it is known that integrals of the form $\displaystyle\int_{\partial K} \phi(\mathcal{K}_0)\, d\mu_K$ are $SL(n)$-invariant, see also [16, 18], considering the results in [30], and others, including for example the next theorem, we conjecture that the set of p-affine surface areas, with algebraic operations, can generate, by taking the *closure*, all integrals of the above form.

Theorem 4. *For any K of class C_+^2 containing the origin in its interior, the $SL(n)$-invariant $\Lambda(K) := \exp\left[\frac{1}{nVol(K)} \int_{\partial K} \ln(\mathcal{K}_0)\, d\mu_K\right]$ is the limit, as $p \to +\infty$, of the sequence $\left\{\left(\frac{\Omega_{-\frac{n}{2^p}}(K)}{n\, Vol(K)}\right)^{2^p}\right\}_{p\in\mathbb{N},\ p>1}$.*

Proof. The claim follows directly from

$$\ln\left[\lim_{p\to\infty} \frac{\left(\Omega_{-n/2^p}(K)\right)^{2^p}}{(nVol(K^\circ))^{2^p}}\right]$$

$$= \lim_{p\to\infty} 2^p \ln\left(\frac{\Omega_{-n/2^p}(K)}{nVol(K^\circ)}\right)$$

$$= -\frac{2^p}{\ln 2} \frac{\frac{d}{dp}\left(\Omega_{-n/2^p}(K)\right)}{\Omega_{-n/2^p}(K)}$$

$$= -\lim_{p\to\infty} \frac{2^p}{\ln 2} \frac{\frac{d}{dp}\left(\int_{\partial K} \exp\left(\ln \mathcal{K}_0^{-\frac{1}{2^p-1}}\right) d\mu_K\right)}{\Omega_{-n/2^p}(K)}$$

$$= \lim_{p\to\infty} \frac{2^{2p}}{(2^p-1)^2} \frac{\left(\int_{\partial K} \exp\left(\ln \mathcal{K}_0^{-\frac{1}{2^p-1}}\right) \ln(\mathcal{K}_0)\, d\mu_K\right)}{\Omega_{-n/2^p}(K)}$$

$$= \lim_{p \to \infty} \frac{2^{2p}}{(2^p - 1)^2} \frac{\left(\int_{\partial K} \mathcal{K}_0^{-\frac{1}{2^p - 1}} \ln(\mathcal{K}_0) \, d\mu_K \right)}{\Omega_{-n/2^p}(K)}$$

$$= \frac{\int_{\partial K} \ln(\mathcal{K}_0) \, d\mu_K}{n \, Vol(K)} = \ln(\Lambda_K). \qquad \square$$

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On the Geometry of Log-Concave Probability Measures with Bounded Log-Sobolev Constant

P. Stavrakakis and P. Valettas

Abstract Let $\mathcal{LS}_{lc}(\kappa)$ denote the class of log-concave probability measures μ on $\mathbb{R}^n$ which satisfy the logarithmic Sobolev inequality with a given constant $\kappa > 0$. We discuss $\mathcal{LS}_{lc}(\kappa)$ from a geometric point of view and we focus on related open questions.

Key words Log-Sobolev inequality • (τ)-property • ψ_2-measures

Mathematical Subject Classifications (2010): 52A20, 52A40, 60E15

1 Introduction

The general setting of this article is the class of log-concave probability measures on $\mathbb{R}^n$; these are the Borel probability measures μ on $\mathbb{R}^n$ with the property that

$$\mu((1-\lambda)A + \lambda B) \geq (\mu(A))^{1-\lambda}(\mu(B))^{\lambda}. \tag{1}$$

for any pair of Borel subsets A, B of $\mathbb{R}^n$ and any $\lambda \in (0, 1)$. The study of geometric properties of log-concave probability measures is a central topic in asymptotic geometric analysis and several questions asking for universal bounds for important geometric parameters of these measures remain open. Let us briefly introduce two of them, the *hyperplane conjecture* and the *Kannan-Lovász-Simonovits conjecture*.

A log-concave probability measure μ on $\mathbb{R}^n$ is called *isotropic* if the barycentre of μ is at the origin and its covariance matrix $\mathrm{Cov}(\mu)$ with entries

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$$\mathrm{Cov}(\mu)_{ij} := \int_{\mathbb{R}^n} x_i\, x_j\, f_\mu(x)\, dx \tag{2}$$

is the identity matrix. Then the isotropic constant of μ is defined by $f(0)^{1/n}$ where f is the density of μ with respect to Lebesgue measure. The hyperplane conjecture asks if there exists an absolute constant $C > 0$ such that $L_\mu \leq C$ for all $n \geq 1$ and all isotropic log-concave probability measures μ. Bourgain in [9] proved that one always has $L_\mu \leq C \sqrt[4]{n} \log n$, and Klartag [14] improved this bound to $L_\mu \leq C \sqrt[4]{n}$; a second proof of this estimate appears in [15]. On the other hand, one of the equivalent versions of the Kannan–Lovász–Simonovits conjecture asks if there exists an absolute constant $C > 0$ such that the Poincaré inequality

$$\int \varphi^2 d\mu \leq C \int \|\nabla\varphi\|_2^2 d\mu \tag{3}$$

holds true (with constant C) for all isotropic log-concave probability measures and all smooth enough functions φ satisfying $\int \varphi d\mu = 0$.

Both questions are known to have an affirmative answer if we restrict our attention to special classes of log-concave probability measures. One way to introduce such a class is to impose some assumption of uniform boundedness on one geometric parameter for this subclass and to study other main geometric parameters of the measures in this subclass, trying to obtain uniform estimates for them which should depend on the bound for the chosen parameter only.

The purpose of this article is to provide a survey on the basic geometric properties of the class $\mathcal{LS}(\kappa)$ of probability measures μ on $\mathbb{R}^n$ which satisfy the logarithmic Sobolev inequality with a given constant $\kappa > 0$. We obtain bounds in terms of κ, but independent of the dimension, for several of these parameters and we emphasize some questions which remain open even if we impose this additional assumption.

A Borel probability measure μ on $\mathbb{R}^n$ is said to satisfy the logarithmic Sobolev inequality with constant $\kappa > 0$ if for any (locally) Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ one has,

$$\mathrm{Ent}_\mu(f^2) \leq 2\kappa \int \|\nabla f\|_2^2 \, d\mu, \tag{4}$$

where $\mathrm{Ent}_\mu(g) = \mathbb{E}_\mu(g \log g) - \mathbb{E}_\mu g \log(\mathbb{E}_\mu g)$ is the *entropy* of g with respect to μ. It is well-known (see e.g. [18, Chap. 5]) that the log-Sobolev inequality implies normal concentration. For every measurable function f on $\mathbb{R}^n$ consider the *logarithmic Laplace transform*

$$L_f(u) = \log\left(\int e^{uf} d\mu\right), \quad u \in \mathbb{R}. \tag{5}$$

Then, the Herbst argument shows that if f is 1-Lipschitz and $\mathbb{E}_\mu(f) = 0$, one has $L_f(u) \leq \kappa u^2/2$ for all $u \in \mathbb{R}$, and hence, from Markov's inequality,

$$\mu(x : |f(x)| \geq t) \leq 2e^{-t^2/2\kappa}, \qquad t > 0. \tag{6}$$

It is also known that the log-Sobolev inequality implies Poincaré inequality, namely: if μ belongs to the class $\mathcal{LS}(\kappa)$, then for any (locally) Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ we have

$$\text{Var}_\mu(f) \leq \kappa \int \|\nabla f\|_2^2 \, d\mu, \tag{7}$$

where $\text{Var}_\mu(g) = \mathbb{E}_\mu(g^2) - (\mathbb{E}_\mu(g))^2$ denotes the *variance* of g with respect to μ. We denote the class of probability measures satisfying Poincaré inequality with a given constant $\kappa > 0$ by $\mathcal{P}(\kappa)$.

We are mainly interested in the subclasses $\mathcal{LS}_{lc}(\kappa)$ and $\mathcal{P}_{lc}(\kappa)$ of isotropic log-concave probability measures that belong to $\mathcal{LS}(\kappa)$ and $\mathcal{P}(\kappa)$ respectively. In particular, we study the dependence on κ of various parameters that play a crucial role in recent works about isotropic log-concave measures—see the next section for definitions and background information. It turns out that, from this point of view, $\mathcal{LS}_{lc}(\kappa)$ is a rather restricted class with very nice properties:

Theorem 1. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$ which satisfies the logarithmic Sobolev inequality with constant $\kappa > 0$. Then,*

(i)   *All directions are sub-Gaussian: μ is a ψ_2-measure with constant $c_1 \sqrt{\kappa}$.*
(ii) *The isotropic constant of μ is bounded: $L_\mu \leq c_2 \sqrt{\kappa}$.*
(iii) *Let $I_q(\mu) = \left(\int \|x\|_2^q d\mu \right)^{1/q}$, $-n < q < \infty$, $q \neq 0$. Then, $I_q(\mu) \leq I_2(\mu) + \sqrt{\kappa} \sqrt{q}$ for all $2 \leq q < \infty$. In particular, $I_q(\mu) \leq c_3 \sqrt{n}$ for all $q \leq c_4 n/\kappa$. Also, $I_{-q}(\mu) \geq c_5 \sqrt{n}$ for all $q \leq c_6 n/\kappa$.*
(iv) *Most directions are "regular" and super-Gaussian: there exists a subset A of S^{n-1} with measure $\sigma(A) > 1 - e^{-c_7 n/\kappa}$ such that for any $\theta \in A$ we have*

$$\left(\int |\langle x, \theta \rangle|^q \, d\mu(x) \right)^{1/q} \leq c_8 \sqrt{\kappa} \sqrt{q/p} \left(\int |\langle x, \theta \rangle|^p \, d\mu(x) \right)^{1/p} \tag{8}$$

for any $1 \leq p \leq c_9 n/\kappa$ and any $q \geq p$, and also,

$$\mu(x : |\langle x, \theta \rangle| \geq t) \geq e^{-c_{10} t^2/\kappa}, \tag{9}$$

for all $1 \leq t \leq c_{11} \sqrt{n}/\kappa$.

The proofs of the previous statements are given in Sect. 3. Our basic tools are the classical Herbst argument and the theory of L_q-centroid bodies as it is developed in [10, 12, 27–29]. All these assertions show that measures belonging to $\mathcal{LS}_{lc}(\kappa)$ (with $\kappa \simeq 1$) share many of the properties of the standard n-dimensional Gaussian measure γ_n (recall that γ_n satisfies the log-Sobolev inequality with $\kappa = 1$). We close Sect. 3 with a strengthened version of a recent result of Latała (see [16]) about the

tails of order statistics of log-concave isotropic probability measures μ in $\mathbb{R}^n$: Latała showed that

$$\mu(x : x_m^* \geq t) \leq \exp(-\sqrt{m}t/c) \tag{10}$$

for all $1 \leq m \leq n$ and $t \geq \log(en/m)$, where $(x_1^*, \ldots, x_n^*)$ is the decreasing rearrangement of $(|x_1|, \ldots, |x_n|)$. We show that if $\mu \in \mathcal{LS}(\kappa)$ is centered then, for every $1 \leq m \leq n$ and for any $t \geq C\sqrt{\kappa}\log(en/m)$, we have

$$\mu(x : x_m^* \geq t) \leq e^{-cmt^2/\kappa}. \tag{11}$$

In fact, using a recent result from [1], one can obtain a similar estimate in the setting of log-concave isotropic probability measures with bounded ψ_2-constant, but for a slightly different range of t's.

   According to Theorem 1 (i), if $\mu \in \mathcal{LS}_{lc}(\kappa)$ then μ is a ψ_2-measure. It is natural to ask what is the exact relation of ψ_2-measures with this class: more precisely, what is the best upper bound $m(b, n)$—with respect to b and the dimension n—that one can have for the log-Sobolev constant of an isotropic measure on $\mathbb{R}^n$ with ψ_2 constant less than or equal to b. In Sect. 4 we show that a transportation of measure argument from [17] allows one to show that the log-Sobolev constants of the ℓ_q^n balls for $2 \leq q \leq \infty$ are uniformly bounded. It is well known that these bodies are ψ_2 (actually, the list of known ψ_2 measures is also rather poor).

Theorem 2. *There exists an absolute constant $C > 0$ such that for every n and every $2 \leq q \leq \infty$ one has $\mu_{q,n} \in \mathcal{LS}_{lc}(C)$, where $\mu_{q,n}$ is the Lebesgue measure on the normalized ℓ_q^n ball $\overline{B}_q^n$.*

   In Sect. 5 we discuss the infimum convolution conjecture of Latała and Wojtaszczyk for the class $\mathcal{LS}_{lc}(\kappa)$. We first recall property (τ) which was introduced by Maurey in [21]. If μ is a probability measure on $\mathbb{R}^n$ and $\varphi : \mathbb{R}^n \to [0, \infty]$ is a measurable function, then the pair (μ, φ) is said to have property (τ) if

$$\int e^{f \square \varphi} \, d\mu \int e^{-f} \, d\mu \leq 1 \tag{12}$$

for any bounded measurable function $f : \mathbb{R}^n \to \mathbb{R}$, where

$$(f \square g)(x) := \inf\{f(x - y) + g(y) : y \in \mathbb{R}^n\} \tag{13}$$

is the infimum convolution of two functions $f, g : \mathbb{R}^n \to \mathbb{R}$. Since (12) is clearly satisfied with $\varphi \equiv 0$, the question is to find the largest cost function φ for which it is still true. In [17] it is proved that if μ is symmetric and (μ, φ) has property (τ) for some convex cost function φ, then

$$\varphi(y) \leq 2\Lambda_\mu^*(y/2) \leq \Lambda_\mu^*(y), \tag{14}$$

where

$$\Lambda_\mu^*(y) = \mathcal{L}\Lambda_\mu(y) = \sup_{x \in \mathbb{R}^n} \left\{ \langle x, y \rangle - \log \int e^{\langle x, z \rangle} d\mu(z) \right\} \tag{15}$$

is the Legendre transform of the logarithmic Laplace transform Λ_μ of μ. Thus, Λ_μ^* is the best cost function that might satisfy property (τ) with a given measure μ. Latała and Wojtaszczyk conjecture that there exists an absolute constant $b > 0$ such that $(\mu, \Lambda_\mu^*(\frac{\cdot}{b}))$ has property (τ) for every symmetric log-concave probability measure μ on $\mathbb{R}^n$. This is a very strong conjecture. If true in full generality, this optimal infimum convolution inequality would imply a positive answer to the Kannan–Lovasz–Simonovits conjecture and the hyperplane conjecture.

We study the conjecture of Latała and Wojtaszczyk for the class of log-concave probability measures with log-Sobolev constant κ. It is not hard to check that $\Lambda_\mu^*(y) \geq \frac{\|y\|_2^2}{2\kappa}$. Therefore, a weaker answer would be to show that, for any bounded measurable function f we have (12) for a function φ which is proportional to $\|y\|_2^2$. At this point we are able to give a proof of this fact using the equivalence of the logarithmic Sobolev inequality and the Gaussian isoperimetric inequality in the context of log-concave measures, first established by Bakry and Ledoux (see [3]).

Theorem 3. *Let μ be a log-concave probability measure which satisfies the log-Sobolev inequality with constant $\kappa > 0$. Then, (μ, φ) has property (τ), where $\varphi(y) = \frac{c}{\kappa}\|y\|_2^2$ and $c > 0$ is an absolute constant.*

Theorem 3 is close in spirit to a result due to Maurey proved in [21] stating that if (μ, φ) has property (τ) with $\varphi(y) = \frac{\|y\|_2^2}{2\kappa}$, then μ satisfies Poincaré inequality with constant κ (see Sect. 5 for the exact statement).

2 Notation and Background Material

We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle \cdot, \cdot \rangle$. We denote by $\| \cdot \|_2$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball, and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. We write ω_n for the volume of B_2^n and σ for the rotationally invariant probability measure on S^{n-1}. The Grassmann manifold $G_{n,k}$ of k-dimensional subspaces of $\mathbb{R}^n$ is equipped with the Haar probability measure $\nu_{n,k}$. Let $k \leq n$ and $F \in G_{n,k}$. We will denote by P_F the orthogonal projection from $\mathbb{R}^n$ onto F. We also write $\overline{A}$ for the homothetic image of volume 1 of a compact set $A \subseteq \mathbb{R}^n$ of positive volume, i.e. $\overline{A} := \frac{A}{|A|^{1/n}}$. The letters c, c', c_1, c_2 etc. denote absolute positive constants which may change from line to line. Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$.

A convex body in $\mathbb{R}^n$ is a compact convex subset C of $\mathbb{R}^n$ with non-empty interior. We say that C is symmetric if $x \in C$ implies that $-x \in C$. We say that C is centered if it has barycentre at the origin, i.e. $\int_C \langle x, \theta \rangle \, dx = 0$ for every $\theta \in S^{n-1}$. The support function $h_C : \mathbb{R}^n \to \mathbb{R}$ of C is defined by $h_C(x) = \max\{\langle x, y \rangle :$

$y \in C\}$. We define the mean width of C by $w(C) = \int_{S^{n-1}} h_C(\theta)\sigma(d\theta)$, and more generally, for each $-\infty < q < \infty$, $q \neq 0$, we define the q-mean width of C by

$$w_q(C) = \left(\int_{S^{n-1}} h_C^q(\theta)\sigma(d\theta) \right)^{1/q}. \tag{16}$$

The radius of C is the quantity $R(C) = \max\{\|x\|_2 : x \in C\}$ and, if the origin is an interior point of C, the polar body C° of C is

$$C^\circ := \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in C\}. \tag{17}$$

Let C be a symmetric convex body in $\mathbb{R}^n$. Define $k_*(C)$ as the largest positive integer $k \leq n$ for which

$$\frac{1}{2}w(C)(B_2^n \cap F) \subseteq P_F(C) \subseteq 2w(C)(B_2^n \cap F) \tag{18}$$

with probability greater than $\frac{n}{n+k}$ with respect to the Haar measure $\nu_{n,k}$ on $G_{n,k}$. It is known (see [23, 26]) that the parameter $k_*(C)$ is completely determined by $w(C)$ and $R(C)$: There exist $c_1, c_2 > 0$ such that

$$c_1 n \frac{w(C)^2}{R(C)^2} \leq k_*(C) \leq c_2 n \frac{w(C)^2}{R(C)^2} \tag{19}$$

for every symmetric convex body C in $\mathbb{R}^n$. The same parameter is crucial for the behavior of the q-mean width of C: it is proved in [19] that for any symmetric convex body C in $\mathbb{R}^n$ one has (1) $w_q(C) \simeq w(C)$ if $1 \leq q \leq k_*(C)$, (2) $w_q(C) \simeq \sqrt{q/n}\,w(C)$ if $k_*(C) \leq q \leq n$ and (3) $w_q(C) \simeq R(C)$ if $q \geq n$.

Recall that a Borel probability measure μ on $\mathbb{R}^n$ is called log-concave if $\mu((1-\lambda)A + \lambda B) \geq (\mu(A))^{1-\lambda}(\mu(B))^\lambda$ for any pair of Borel subsets A, B of $\mathbb{R}^n$ and any $\lambda \in (0, 1)$. It is known that if μ is log-concave and if $\mu(H) < 1$ for every hyperplane H, then μ is absolutely continuous with respect to Lebesgue measure and its density f_μ is a log-concave function, i.e. $\log f$ is concave (see [8]).

A well-known consequence of Borell's lemma (see [25, Appendix III]) states that if $f : \mathbb{R}^n \to \mathbb{R}$ is a seminorm and μ is a log-concave probability measure on $\mathbb{R}^n$, then, for any $1 \leq p < q$,

$$\|f\|_{L_q(\mu)} \leq \frac{cq}{p}\|f\|_{L_p(\mu)}, \tag{20}$$

where $c > 0$ is an absolute constant. In particular, (20) holds true for any linear functional $f(x) = \langle x, \theta \rangle$ and any norm $f(x) = \|x\|$.

Let $\alpha \in [1, 2]$. The ψ_α-norm of f is defined by

$$\|f\|_{\psi_\alpha} = \inf\left\{ t > 0 : \int \exp(|f|/t)^\alpha \, d\mu \leq 2 \right\}. \tag{21}$$

One can check that $\|f\|_{\psi_\alpha} \simeq \sup_{q \geq \alpha} \frac{\|f\|_{L_q(\mu)}}{q^{1/\alpha}}$. We say that a log-concave probability measure μ on $\mathbb{R}^n$ satisfies a ψ_α-estimate with constant b_α in the direction of θ if

$$\|\langle \cdot, \theta \rangle\|_{\psi_\alpha} \leq b_\alpha \|\langle \cdot, \theta \rangle\|_2. \tag{22}$$

The measure μ is called ψ_α with constant $b = b_\alpha$ if it satisfies a ψ_α estimate with constant b in every direction $\theta \in S^{n-1}$. The following are equivalent:

1. μ satisfies a ψ_α-estimate with constant b in the direction of θ.
2. For all $t > 0$ we have $\mu(x : |\langle x, \theta \rangle| \geq t \|\langle \cdot, \theta \rangle\|_2) \leq 2e^{-t^a/b^\alpha}$.

From (20) we see that every log-concave probability measure has ψ_1 constant $b \leq C$, where $C > 0$ is an absolute constant.

Let μ be a probability measure on $\mathbb{R}^n$. For every $q \geq 1$ and $\theta \in S^{n-1}$ we define

$$h_{Z_q(\mu)}(\theta) := \left(\int_{\mathbb{R}^n} |\langle x, \theta \rangle|^q d\mu(x) \right)^{1/q}. \tag{23}$$

Note that if μ is log-concave then $h_{Z_q(\mu)}(\theta) < \infty$. We define the L_q-centroid body $Z_q(\mu)$ of μ to be the centrally symmetric convex set with support function $h_{Z_q(\mu)}$. L_q-centroid bodies were introduced in [20]. Here we follow the normalization (and notation) that appeared in [27]. The original definition concerned the class of measures $\mathbf{1}_K$ where K is a convex body of volume 1. In this case, we also write $Z_q(K)$ instead of $Z_q(\mathbf{1}_K)$. Additional information on L_q-centroid bodies can be found in [28, 29].

An absolutely continuous (with respect to Lebesgue measure) probability measure μ on $\mathbb{R}^n$ with density f_μ is called isotropic if it is centered and $Z_2(\mu) = B_2^n$. Equivalently, if $\int \langle x, \theta \rangle^2 d\mu(x) = 1$ for all $\theta \in S^{n-1}$. In the log-concave case we define the isotropic constant of μ by $L_\mu := f_\mu(0)^{\frac{1}{n}}$. We refer to [11, 24, 29] for additional information on isotropic convex bodies and measures.

For every $-n < q \leq \infty$, $q \neq 0$, we define

$$I_q(\mu) := \left(\int_{\mathbb{R}^n} \|x\|_2^q d\mu(x) \right)^{1/q}. \tag{24}$$

Observe that if μ is isotropic then $I_2(\mu) = \sqrt{n}$. Next, we consider the parameter

$$q_*(\mu) = \max\{k \leq n : k_*(Z_k(\mu)) \geq k\}. \tag{25}$$

The main result of [28] asserts that the moments of the Euclidean norm on log-concave isotropic measures satisfy a strong reverse Hölder inequality up to the value q_*: for every $q \leq q_*(\mu)$,

$$I_q(\mu) \leq C I_{-q}(\mu), \tag{26}$$

where $C > 0$ is an absolute constant. In other words, $I_q(\mu) \simeq \sqrt{n}$ if $1 \leq |q| \leq q_*(\mu)$. Moreover, one has a non-trivial estimate for the parameter q_*: if μ is a ψ_α-measure with constant b_α, then

$$q_*(\mu) \geq c n^{\alpha/2} / b_\alpha^\alpha, \tag{27}$$

where $c > 0$ is an absolute constant. In particular, $q_*(\mu) \geq c \sqrt{n}$ for every isotropic log-concave probability measure μ in $\mathbb{R}^n$.

3 Isotropic Log-Concave Measures with Bounded Logarithmic Sobolev Constant

3.1 Geometric Properties

In this section we assume that μ is an isotropic log-concave measure on $\mathbb{R}^n$ with logarithmic Sobolev constant κ and provide short proofs of the statements in Theorem 1. In some cases the results hold true under weaker assumptions on μ.

Let us first recall the classical Herbst argument (for a proof see [18]):

Lemma 1 (Herbst). *Let μ be a Borel probability measure on $\mathbb{R}^n$ such that $\mu \in \mathcal{LS}(\kappa)$. Then, for any 1-Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ with $\mathbb{E}_\mu(f) = 0$, we have*

$$L_f(t) \leq \frac{\kappa}{2} t^2, \tag{28}$$

for any $t \in \mathbb{R}$. $\square$

Proposition 1. *Let μ be an isotropic measure in $\mathcal{LS}(\kappa)$. Then, for any $\theta \in S^{n-1}$ we have:*

$$\|\langle \cdot, \theta \rangle\|_{\psi_2} \leq c \sqrt{\kappa}, \tag{29}$$

where $c > 0$ is an absolute constant.

Proof. From Herbst's Lemma and Markov's inequality we conclude that $\mu(x : |f(x)| \geq t) \leq 2e^{-t^2/2\kappa}$ for every 1-Lipschitz function f with $\mathbb{E}_\mu(f) = 0$. Since μ is assumed isotropic, μ is centered and this result applies to the function $x \mapsto \langle x, \theta \rangle$, where $\theta \in S^{n-1}$. Thus, we get

$$\mu(x : |\langle x, \theta \rangle| \geq t) \leq 2e^{-t^2/2\kappa} \tag{30}$$

for every $t > 0$, and this implies that

$$\|\langle \cdot, \theta \rangle\|_{\psi_2} \leq c \sqrt{\kappa}. \tag{31}$$

In other words, μ is a ψ_2-measure with constant $O(\sqrt{\kappa})$. Note that the log-concavity of μ is not necessary for this claim.   □

Next we prove that the isotropic constant of μ is bounded in terms of κ.

Proposition 2. *Let $\mu \in \mathcal{LS}_{ls}(\kappa)$. Then, one has*

$$L_\mu \le c\sqrt{\kappa}, \tag{32}$$

where $c > 0$ is an absolute constant.

Proof. It is known that ψ_2-isotropic log-concave measures have bounded isotropic constant. Actually, it was recently proved in [15] that the dependence on the ψ_2-constant is linear. This follows from the following main result of [15]: if $q \le q_*(\mu)$ then

$$|Z_q(\mu)|^{1/n} \ge c\sqrt{\frac{q}{n}}. \tag{33}$$

Since μ is a ψ_2-measure with constant $\sqrt{\kappa}$ we have $q_*(\mu) \ge cn/\kappa$. Thus, using also the fact that $|Z_n(\mu)|^{1/n}[f_\mu(0)]^{1/n} \simeq 1$ (see [28]) we get

$$L_\mu = [f_\mu(0)]^{1/n} \simeq \frac{1}{|Z_n(\mu)|^{1/n}} \le \frac{1}{|Z_q(\mu)|^{1/n}} \le C\sqrt{\frac{n}{q}}, \tag{34}$$

for all $q \le cn/\kappa$ and the result follows.   □

Remark. Let μ be a measure which satisfies Poincaré inequality with constant κ. Note that

$$\langle \mathrm{Cov}(\mu)(u), u \rangle = \mathrm{Var}_\mu(f) \le \kappa \int \|\nabla f\|_2^2 \, d\mu = \kappa \|u\|_2^2, \tag{35}$$

where $f(x) = \langle x, u \rangle$. Thus, for any probability measure $\mu \in \mathcal{P}(\kappa)$ we have

$$L_\mu := \|\mu\|_\infty^{\frac{1}{n}} [\det \mathrm{Cov}(\mu)]^{\frac{1}{2n}} \le \|\mu\|_\infty^{1/n} \sqrt{k}, \tag{36}$$

where $\|\mu\|_\infty = \sup_x f_\mu(x)$ and f_μ is the density of μ.

Proposition 3. *Let $\mu \in \mathcal{LS}(\kappa)$. Then, μ satisfies the following moment estimate: For any $q \ge 2$ one has*

$$I_q(\mu) \le I_2(\mu) + \sqrt{\kappa}\sqrt{q}. \tag{37}$$

In particular, if μ is isotropic then we have:

$$I_q(\mu) \le (1 + \delta)I_2(\mu), \tag{38}$$

for $2 \le q \le \delta^2 n/\kappa$ and $\delta > (2\kappa/n)^{1/2}$.

Proof. We prove a more general result following [2]: if μ satisfies the logarithmic Sobolev inequality with constant $\kappa > 0$ then, for any Lipschitz function f on $\mathbb{R}^n$ and for any $2 \leq p \leq q$, we have

$$\|f\|_q^2 - \|f\|_p^2 \leq \kappa \|f\|_{\text{Lip}}^2 (q - p). \tag{39}$$

For the proof we may assume that $\|f\|_{\text{Lip}} = 1$. Let $g(p) = \|f\|_p$. Differentiating g we see

$$g'(p) = \|f\|_p \left[\frac{1}{p} \frac{\int |f|^p \log |f| \, d\mu}{\int |f|^p \, d\mu} - \frac{1}{p^2} \log \int |f|^p \, d\mu \right]. \tag{40}$$

On the other hand using the logarithmic Sobolev inequality for $|f|^{p/2}$, $p > 2$, after some calculations we arrive at:

$$\frac{1}{p} \frac{\int f^p \log f \, d\mu}{\int f^p \, d\mu} - \frac{1}{p^2} \log \int |f|^p \, d\mu \leq \frac{\kappa}{2} \frac{\int |f|^{p-2} \, d\mu}{\int |f|^p \, d\mu}. \tag{41}$$

That is

$$\frac{g'(p)}{g(p)} \leq \frac{\kappa}{2} \frac{g(p-2)^{p-2}}{g^p(p)}. \tag{42}$$

Then, using Hölder's inequality, we get

$$2g'(p)g(p) \leq \kappa \tag{43}$$

for all $p > 2$. Thus, for any $2 \leq p \leq q$ we get $g(q)^2 - g(p)^2 \leq \kappa(q - p)$. Choosing $f(x) = \|x\|_2$ and using the elementary inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, we see that

$$I_q(\mu) \leq I_2(\mu) + \sqrt{\kappa} \sqrt{q} \tag{44}$$

for all $2 \leq q < \infty$. Note that the log-concavity assumption is not needed for the proof of this claim. □

Proposition 4. *Let μ be an isotropic measure in $\mathcal{LS}_{lc}(\kappa)$. Then,*

$$I_{-q}(\mu) \geq c_1 I_2(\mu) \tag{45}$$

for all $q \leq c_2 n/\kappa$, where $c_1, c_2 > 0$ are absolute constants.

Proof. For the negative values of q we use the fact that $q_*(\mu) \geq c_6 n/\kappa$. This is a consequence of (27) because μ is a ψ_2-measure with constant $O(\sqrt{\kappa})$ from Proposition 1. Then, from (26) we conclude that $I_{-q}(\mu) \geq C^{-1} I_q(\mu) \geq c_5 \sqrt{n}$ for all $2 \leq q \leq c_6 n/\kappa$. □

Proposition 5. *Let μ be an isotropic measure in $\mathcal{LS}_{lc}(\kappa)$. Then, most directions are "regular" and super-Gaussian: There exists a subset A of S^{n-1} with measure $\sigma(A) > 1 - e^{-c_7 n/\kappa}$ such that for any $\theta \in A$ we have*

$$\left(\int |\langle x,\theta\rangle|^q \, d\mu(x)\right)^{1/q} \le c_8 \sqrt{\kappa}\, \sqrt{q/p} \left(\int |\langle x,\theta\rangle|^p \, d\mu(x)\right)^{1/p} \quad (46)$$

for any $1 \le p \le c_9 n/\kappa$ *and any* $q \ge p$, *and also,*

$$\mu(x : |\langle x,\theta\rangle| \ge t) \ge e^{-c_{10}t^2/\kappa}, \quad (47)$$

for all $1 \le t \le c_{11}\sqrt{n}/\kappa$.

Proof. Under the weaker assumption that μ is an isotropic log-concave ψ_2-measure with constant b in $\mathbb{R}^n$, we show that there exists a subset A of S^{n-1} of measure $\sigma(A) > 1 - e^{-c_1 n/b^2}$ such that for any $\theta \in A$ and for any $1 \le p \le c_2 n/b^2$ we have:

$$\left(\int |\langle x,\theta\rangle|^p \, d\mu(x)\right)^{1/p} \simeq \sqrt{p}. \quad (48)$$

The argument has more or less appeared in [12] (see also [29]). Since μ has ψ_2 constant b, from (27) we have $q_*(\mu) \ge cn/b^2$. Let $k \le cn/b^2$. Then, if we fix $p \le k$, applying Dvoretzky's theorem for $Z_p(\mu)$ we have

$$\frac{1}{2}w(Z_p(\mu))(B_2^n \cap F) \subseteq P_F(Z_p(\mu)) \subseteq 2w(Z_p(\mu))(B_2^n \cap F) \quad (49)$$

for all F in a subset $B_{k,p}$ of $G_{n,k}$ of measure

$$\nu_{n,k}(B_{k,p}) \ge 1 - e^{-c_3 k_*(Z_p(\mu))} \ge 1 - e^{-c_4 n/b^2}. \quad (50)$$

Applying this argument for $p = 2^i$, $i = 1,\ldots,\lfloor \log_2 k\rfloor$, and taking into account the fact that, by (20), $Z_q(\mu) \subseteq cZ_p(\mu)$ if $p < q \le 2p$, we conclude that there exists $B_k \subset G_{n,k}$ with $\nu_{n,k}(B_k) \ge 1 - e^{-c_5 n/b^2}$ such that (48) holds true for every $F \in B_k$ and every $1 \le p \le k$. On the other hand, since $I_p(\mu) \simeq I_2(\mu) = \sqrt{n}$ for all $2 \le p \le q_*(\mu)$, we see that

$$w(Z_p(\mu)) \simeq \sqrt{p} \quad (51)$$

for all $p \le cn/b^2$. Therefore, (48) can be written in the form

$$h_{Z_p(\mu)}(\theta) \simeq \sqrt{p} \quad (52)$$

for all $F \in B_k$, $\theta \in S_F$ and $1 \le p \le k$. To conclude the proof, let $k = \lfloor cn/b^2\rfloor$. Then, if we set $A = \{\theta \in S^{n-1} : h_{Z_p}(\theta) \simeq \sqrt{p},$ for all $1 \le p \le k\}$, Fubini's theorem gives:

$$\sigma(A) = \int_{G_{n,k}} \sigma_F(A \cap F)\, d\nu_{n,k}(F) \ge \int_{B_k} \sigma_F(A \cap F)\, d\nu_{n,k}(F)$$

$$\ge 1 - e^{-cn/b^2}. \quad (53)$$

Now, let $\theta \in A$ and let $p \leq cn/b^2$ and $q \geq p$. From (48) we have $\|\langle \cdot, \theta \rangle\|_p \simeq \sqrt{p}$. Since μ is a ψ_2 measure we have $\|\langle \cdot, \theta \rangle\|_q \leq cb\sqrt{q}$ for all $\theta \in S^{n-1}$ and all $q \geq 1$. This shows that

$$\|\langle \cdot, \theta \rangle\|_q \leq cb\sqrt{q/p}\,\|\langle \cdot, \theta \rangle\|_p. \tag{54}$$

In the case $\mu \in \mathcal{LS}_{lc}(\kappa)$ we know that $b = O(\sqrt{\kappa})$, and this proves Proposition 5.

For the second part we use an argument which has essentially appeared in [12]. Using the fact that for all $\theta \in A$ and for all $1 \leq q \leq cn/b^2$ we have $h_{Z_q(\mu)}(\theta) \simeq \sqrt{q}$, we write

$$\mu\left(x : |\langle x, \theta \rangle| \geq \frac{1}{2}\|\langle \cdot, \theta \rangle\|_q\right) \geq (1 - 2^{-q})^2 \frac{\|\langle \cdot, \theta \rangle\|_q^{2q}}{\|\langle \cdot, \theta \rangle\|_{2q}^{2q}} \geq e^{-cq}, \tag{55}$$

where we have used Paley–Zygmund inequality and (20). Therefore, for all $\theta \in A$ and all $q \leq cn/b^2$, we get

$$\mu(x : |\langle x, \theta \rangle| \geq c_1\sqrt{q}) \geq e^{-c_2 q}. \tag{56}$$

Writing $c_1\sqrt{q} = t$ we have that for all $1 \leq t \leq c_3\sqrt{n}/b$ one has $\mu(x : |\langle x, \theta \rangle| \geq t)$ $\geq e^{-ct^2}$ for all $\theta \in A$, and $\sigma(A) \geq 1 - e^{-cn/b^2}$. $\square$

Remark. For a general measure $\mu \in \mathcal{LS}_{lc}(\kappa)$ one cannot expect that every direction θ will be super-Gaussian (with a constant depending on κ). To see this, consider the uniform measure $\mu_{\infty,n}$ on the unit cube $C_n = \left[-\frac{1}{2}, \frac{1}{2}\right]^n$. This is a product log-concave probability measure, and hence, it satisfies the logarithmic Sobolev inequality with an absolute constant κ (see [18, Corollary 5.7]). On the other hand, it is clearly not super-Gaussian in the directions of e_i, because $h_{C_n}(e_i) \simeq 1$. The same is true for all $\theta \in S^{n-1}$ for which $h_{C_n}(\theta)/\sqrt{n} = o_n(1)$.

3.2   Tail Estimates for Order Statistics

The starting point for the next property is a result of Latała from [16]: if μ is a log-concave isotropic probability measure on $\mathbb{R}^n$ then

$$\mu(x : x_m^* \geq t) \leq \exp(-\sqrt{m}t/c) \tag{57}$$

for all $1 \leq m \leq n$ and $t \geq \log(en/m)$, where $(x_1^*, \ldots, x_n^*)$ is the decreasing rearrangement of $(|x_1|, \ldots, |x_n|)$. We will show that if $\mu \in \mathcal{LS}(\kappa)$ and is centered, then a much better estimate holds true. The idea of the proof comes from [16, Proposition 2].

Proposition 6. *Let μ be a centered probability measure on $\mathbb{R}^n$ which belongs to the class $\mathcal{LS}(\kappa)$. For every $1 \leq m \leq n$ and for any $t \geq C\sqrt{\kappa \log(en/m)}$, we have*

$$\mu(x : x_m^* \geq t) \leq e^{-cmt^2/\kappa}. \tag{58}$$

Proof. Since μ satisfies the log-Sobolev inequality with constant κ, the following isoperimetric inequality holds (for a proof see [18]): if $\mu(A) \geq 1/2$, then for any $t > 0$ one has

$$1 - \mu(A + tB_2^n) \leq e^{-t^2/8\kappa}. \tag{59}$$

Applying Herbst's argument to the function $x \mapsto x_i$ (that is (30) for $\theta = e_i$) we have

$$\mu(x : |x_i| \geq t) \leq 2e^{-t^2/2\kappa} \tag{60}$$

for $t > 0$. Given $1 \leq m \leq n$, for any $t > 0$ we define the set

$$A(t) := \{x : \text{card}(i : |x_i| \geq t) < m/2\}. \tag{61}$$

Claim. For every $t \geq \sqrt{6\kappa \log(en/m)}$ we have $\mu(A(t)) \geq 1/2$.

Indeed, using Markov's inequality and (60) we obtain:

$$1 - \mu(A(t)) = \mu(x : \text{card}(i : |x_i| \geq t) \geq m/2)$$

$$= \mu\left(x : \sum_{i=1}^n \mathbf{1}_{\{|x_i| \geq t\}}(x) \geq \frac{m}{2}\right)$$

$$\leq \frac{2}{m} \sum_{i=1}^n \mu(x : |x_i| \geq t)$$

$$\leq \frac{4n}{m} e^{-\frac{t^2}{2\kappa}} \leq \frac{4n}{m}\left(\frac{en}{m}\right)^{-3} < \frac{1}{2}. \tag{62}$$

Now, let $t_0 := \sqrt{6\kappa \log(en/m)}$. For any $s > 0$, if we write $z = x + y \in A(t_0) + s\sqrt{m}B_2^n$ then less than $m/2$ of the $|x_i|$'s are greater than t_0 and less than $m/2$ of the $|y_i|$'s are greater than $s\sqrt{2}$. Using the isoperimetric inequality once again, we get:

$$\mu(x : x_m^* \geq t_0 + \sqrt{2}s) \leq 1 - \mu(A(t_0) + s\sqrt{m}B_2^n) \leq e^{-ms^2/8\kappa}. \tag{63}$$

Choosing $s \geq 2t_0$ we get the result with $C = 2\sqrt{6}$ and $c = 1/64$. $\quad\square$

Note that for previous argument neither isotropicity nor log-concavity is needed. Nevertheless, one can actually obtain the strong estimate of Proposition 6 in the setting of log-concave isotropic probability measures with bounded ψ_2-constant, using a more general result from [1, Theorem 3.3]: For every log-concave isotropic probability measure μ in $\mathbb{R}^n$, for every $1 \leq m \leq n$ and every $t \geq c \log(en/m)$, one has

$$\mu(x : x_m^* \geq t) \leq \exp\left(\sigma_\mu^{-1}(\sqrt{m}t/c)\right), \tag{64}$$

where

$$\sigma_\mu(p) = \max_{\theta \in S^{n-1}} \|\langle \cdot, \theta \rangle\|_p = R(Z_p(\mu)), \quad p \geq 1. \tag{65}$$

Assuming that μ is a ψ_2-measure with constant $b > 0$, we have $\sigma_\mu(p) \leq c_1 b \sqrt{p}$, and hence $\sigma_\mu^{-1}(\sqrt{m}t/c) \geq c_2 m t^2/b^2$. Then, we get the following.

Proposition 7. *Let μ be a* log-*concave isotropic probability measure on $\mathbb{R}^n$ with ψ_2-constant $b > 0$. For every $1 \leq m \leq n$ and for any $t \geq C \log(en/m)$, we have*

$$\mu(x : x_m^* \geq t) \leq e^{-cmt^2/b^2}. \tag{66}$$

$\square$

Similarly, we can state [1, Theorem 3.4] in the setting of ψ_2-measures with constant b (in particular, for all $\mu \in \mathcal{LS}_{lc}(\kappa)$):

Proposition 8. *Let μ be a* log-*concave isotropic probability measure on $\mathbb{R}^n$ with ψ_2-constant $b > 0$. For every $1 \leq m \leq n$ and for any $t \geq 1$, we have*

$$\mu \left(x : \max_{|\sigma|=m} \|P_\sigma(x)\|_2 \geq ct \sqrt{m} \log(en/m) \right)$$
$$\leq \exp \left(-cmt^2 \log^2(en/m)/b^2 \log b \right), \tag{67}$$

where P_σ denotes the orthogonal projection onto $\mathbb{R}^\sigma$ and the maximum is over all $\sigma \subseteq \{1, \ldots, n\}$ with $|\sigma| = m$. $\square$

4 Log-Sobolev Constant of ψ_2-Measures

The question whether log-concave probability measures with bounded ψ_2-constant exhibit a good behavior with respect to the Poincaré or log-Sobolev constant seems to be open. In fact, the Kannan–Lovasz–Simonovits conjecture, which asks if the Poincaré constants of all log-concave probability measures are uniformly bounded, has been verified only in some special cases: these include the Euclidean ball, the unit cube and log-concave product measures (see [22] for a complete picture of what is known). The "KLS-conjecture" was also confirmed for the normalized ℓ_p^n balls by S. Sodin [30] in the case $1 \leq p \leq 2$ and by Latała and Wojtaszczyk [17] in the case $p \geq 2$.

It is well-known (see [4]) that the ψ_2-constants of the ℓ_q^n-balls, $2 \leq q \leq \infty$, are uniformly bounded. Below we show that the argument of [17] allows one to show that their log-Sobolev constants are also uniformly bounded.

Proof of Theorem 2. Recall that if μ, ν are Borel probability measures on $\mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ is a Borel measurable function, we say that T *transports* μ *to* ν if, for every Borel subset A of $\mathbb{R}^n$,

$$\mu(T^{-1}(A)) = \nu(A). \tag{68}$$

Equivalently, if for every Borel measurable function $f : \mathbb{R}^n \to \mathbb{R}$,

$$\int f(Tx)\, d\mu(x) = \int f(x)\, d\nu(x). \tag{69}$$

Let $1 \le q < \infty$. We consider the probability distribution ν_q on $\mathbb{R}$ with density $(2\delta_q)^{-1} \exp(-|x|^q)$, where $\delta_q = \Gamma(1 + 1/q)$, and write ν_q^n for the product measure $\nu_q^{\otimes n}$ on $\mathbb{R}^n$, with density $(2\delta_q)^{-n} \exp(-\|x\|_q^q)$. We define a function $w_q : \mathbb{R} \to \mathbb{R}$ by the equation

$$\frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2}\, dt = \frac{1}{2\delta_q} \int_{w_q(x)}^\infty e^{-|t|^q}\, dt. \tag{70}$$

We also define $W_{q,n} : \mathbb{R}^n \to \mathbb{R}^n$ by $W_{q,n}(x_1, \ldots, x_n) = (w_q(x_1), \ldots, w_q(x_n))$. It is proved in [17] that $W_{q,n}$ transports γ_n to ν_q^n: for every Borel subset A of $\mathbb{R}^n$ we have $\gamma_n(W_{q,n}^{-1}(A)) = \nu_q^n(A)$. Moroever, $W_{q,n}$ is Lipschitz: for any $r \ge 1$ and for all $x, y \in \mathbb{R}^n$ we have

$$\|W_{q,n}(x) - W_{q,n}(y)\|_r \le \frac{2\delta_q}{\sqrt{2\pi}} \|x - y\|_r. \tag{71}$$

Next, we consider the radial transformation $T_{q,n}$, which transports ν_q^n to $\mu_{q,n}$—the uniform probability measure on $\overline{B}_q^n$, the normalized ball of ℓ_q^n. For every $1 \le q < \infty$ and $n \in \mathbb{N}$ we define $f_{q,n} : [0, \infty) \to [0, \infty)$ by the equation

$$\frac{1}{(2\delta_q)^n} \int_0^s r^{n-1} e^{-r^q}\, dr = \int_0^{f_{q,n}(s)} r^{n-1}\, dr \tag{72}$$

and $T_{q,n} : \mathbb{R}^n \to \mathbb{R}^n$ by $T_{q,n}(x) = f_{q,n}(\|x\|_q)\frac{x}{\|x\|_q}$. One can check that $T_{q,n}$ transports the probability measure ν_q^n to the measure $\mu_{q,n}$.

In the case $2 \le q < \infty$, the composition $S_{q,n} = T_{q,n} \circ W_{q,n}$ transports the Gaussian measure γ_n to $\mu_{p,n}$ and is a Lipschitz map with respect to the standard Euclidean norm, with a Lipschitz norm which is bounded by an absolute constant: for every Borel subset A of $\mathbb{R}^n$ we have $\gamma_n(S_{q,n}^{-1}(A)) = \mu_q^n(A)$, and for all $x, y \in \mathbb{R}^n$ we have:

$$\|S_{q,n}(x) - S_{q,n}(y)\|_2 \le C\|x - y\|_2, \tag{73}$$

where $C > 0$ is an absolute constant.

Now, we use the following simple Lemma.

Lemma 2. *Let μ, ν be two Borel probability measures on $\mathbb{R}^n$. Assume that μ satisfies the log-Sobolev inequality with constant κ and that there exists a Lipschitz map $T : (\mathbb{R}^n, \mu) \to (\mathbb{R}^n, \nu)$, with respect to the Euclidean metric, that transports μ to ν. Then, ν satisfies the log-Sobolev inequality with constant $\kappa \|T\|_{\mathrm{Lip}}^2$.*

Proof. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz map. Then, $f \circ T$ is Lipschitz. Since, μ satisfies the log-Sobolev inequality with constant κ, we get:

$$\mathrm{Ent}_\mu((f \circ T)^2) \leq 2\kappa \int \|\nabla(f \circ T)\|_2^2 \, d\mu. \tag{74}$$

From (69) we obtain:

$$\mathrm{Ent}_\mu((f \circ T)^2) = \mathrm{Ent}_\nu(f^2), \tag{75}$$

while for the right-hand side we have:

$$\int \|\nabla(f \circ T)\|_2^2 \, d\mu \leq \|T\|_{\mathrm{Lip}}^2 \int \|(\nabla f) \circ T\|_2^2 \, d\mu = \|T\|_{\mathrm{Lip}}^2 \int \|\nabla f\|_2^2 \, d\nu. \tag{76}$$

Combining the above, we conclude the proof. $\qquad\square$

We can now complete the proof of Theorem 2. The Gaussian measure γ_n satisfies the log-Sobolev inequality with constant 1: for any Lipschitz function f in $\mathbb{R}^n$ we have

$$\mathrm{Ent}_{\gamma_n}(f^2) \leq 2 \int \|\nabla f\|_2^2 \, d\gamma_n. \tag{77}$$

Then, the result follows from (73) and Lemma 2. $\qquad\square$

Problem 1. Determine the smallest constant $m(b, n)$ such that every isotropic log-concave probability measure μ on $\mathbb{R}^n$, which is ψ_2 with constant less than or equal to b, satisfies the log-Sobolev (resp. Poincaré) inequality with constant $m(b, n)$.

Remark 1. At this point we should mention that Bobkov [5] has proved that if μ is a log-concave, centered probability measure on $\mathbb{R}^n$, then μ satisfies the log-Sobolev inequality with constant $O(d^2)$, where

$$d = \inf \left\{ t > 0 : \int \exp(\|x\|_2/t)^2 \, d\mu(x) \leq 2 \right\}, \tag{78}$$

that is, the ψ_2 norm of the Euclidean norm $x \mapsto \|x\|_2$ with respect to the measure μ. In [5] he also proves that any log-concave, centered probability measure satisfies Poincaré inequality with constant $I_2^2(\mu)$, where

$$I_2(\mu) = \left(\int \|x\|_2^2 \, d\mu(x) \right)^{1/2}. \tag{79}$$

Thus, if K is an isotropic convex body in $\mathbb{R}^n$ and $\mu = \mu_K$ is the uniform measure on K, then by Alesker's theorem (see for example [11, Theorem 2.2.4]) we have that $d \simeq \sqrt{n} L_K = I_2(K)$, thus we obtain that μ satisfies the log-Sobolev inequality and Poincaré inequality with constant $O(n L_K^2)$. Actually a better dependence for the Poincaré constant is known, due to recent developments on the central limit theorem for convex bodies (see [6]).

5 Infimum Convolution

In this paragraph we discuss the relation between the logarithmic Sobolev inequality and the infimum convolution conjecture, as formulated by Latała and Wojtaszczyk in [17]. By the classical Herbst's argument we can easily verify that if μ is centered and satisfies the log-Sobolev inequality with constant κ, then

$$e^{\Lambda_\mu(\xi)} = \int e^{\langle x, \xi \rangle} \, d\mu(x) \leq e^{\kappa \|\xi\|_2^2 / 2}, \tag{80}$$

for all $\xi \in \mathbb{R}^n$. (Actually, this can be easily verified for all log-concave, isotropic, ψ_2 probability measures with ψ_2 constant $\sqrt{\kappa}$ without the assumption on the log-Sobolev constant). This in turn gives that

$$\Lambda_\mu^*(\xi) \geq \frac{\|\xi\|_2^2}{4\kappa}, \tag{81}$$

for all $\xi \in \mathbb{R}^n$. The main question is the following:

Problem 2. We say that μ has the infimum convolution property with constant α (which we denote by IC(α)) if the pair $(\mu, \Lambda_\mu^*(\frac{\cdot}{\alpha}))$ has property (τ). Given $\kappa > 0$, determine if there is a positive constant $c(\kappa)$ such that every isotropic, log-concave probability measure μ on $\mathbb{R}^n$ which belongs to $\mathcal{LS}_{lc}(\kappa)$ satisfies IC($c(\kappa)$).

Since the infimum convolution property is of "maximal" nature, one could ask, in view of (81), if a probability measure μ which satisfies the log-Sobolev inequality with constant κ also has property (τ) with a cost function w of the form $w(y) = \frac{c}{\kappa}\|y\|_2^2$ for some absolute constant $c > 0$. Below we give a proof of this fact under the assumption that μ is a log-concave probability measure. We first recall some well known facts.

Let μ be a Borel probability measure on $\mathbb{R}^n$. For every Borel subset A of $\mathbb{R}^n$ we define its surface area as follows:

$$\mu^+(A) = \liminf_{t \to 0^+} \frac{\mu(A_t) - \mu(A)}{t}, \tag{82}$$

where $A_t = A + tB_2^n$ is the t-extension of A with respect to $\| \cdot \|_2$. In other words,

$$A_t \equiv A + tB_2^n = \left\{ x \in \mathbb{R}^n : \inf_{a \in A} \|x - a\|_2 < t \right\}. \tag{83}$$

We say that μ satisfies a Gaussian isoperimetric inequality with constant $c > 0$ if

$$\mu^+(A) \geq c I(\mu(A)) \tag{84}$$

for every Borel subset A of $\mathbb{R}^n$, where I is the Gaussian isoperimetric function

$$I(x) = \phi \circ \Phi^{-1}(x). \tag{85}$$

Here, Φ is the standard normal distribution function

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} \, dt \tag{86}$$

and $\phi = \Phi'$ is its density. Assuming that μ satisfies (84) with constant $c = c(\kappa)$ we will show that (μ, φ) has property (τ), where $\varphi(x) = \frac{c^2}{4}\|x\|_2^2$. Note that this condition is in general more restrictive than the condition $\mu \in \mathcal{LS}_{lc}(\kappa)$: It is known that if μ satisfies (84) with constant $c > 0$ then $\mu \in \mathcal{LS}(1/c^2)$ (see [5]). Nevertheless, in the context of log-concave probability measures on $\mathbb{R}^n$, (84) and the log-Sobolev inequality are equivalent. This was first established by Bakry and Ledoux in [3]. Below, we first sketch an argument for the sake of completeness.

Assume that μ is a log-concave probability measure on $\mathbb{R}^n$. Then, the density of μ with respect to the Lebesgue measure is of the form e^{-U}, where $U : \mathbb{R}^n \to [-\infty, \infty)$ is a convex function. If we consider the differential operator

$$Lu = \Delta u - \langle \nabla U, \nabla u \rangle$$

for $u \in C^2$ and $u \in L_2(\mu)$, then using integration by parts we easily check that the log-Sobolev inequality can be written in the form

$$\mathrm{Ent}_\mu(f^2) \le 2\kappa \int f(-Lf) \, d\mu. \tag{87}$$

Using a hypercontractivity result of Gross [13] and semigroup arguments we can arrive at the following parametrized variant of the log-Sobolev inequality:

Theorem 4 (Bakry-Ledoux, 1996[3]). *Let μ be a probability measure with density e^{-U} with respect to Lebesgue measure, where $U : \mathbb{R}^n \to [-\infty, \infty)$ is a convex function. If μ satisfies the log-Sobolev inequality with constant $\kappa > 0$ then, for any $t \ge 0$ and any smooth function f, we have*

$$\|f\|_2^2 - \|f\|_{p(t)}^2 \le \sqrt{2t}\|f\|_\infty \int \|\nabla f\|_2 \, d\mu, \tag{88}$$

where $p(t) = 1 + e^{-t/\kappa}$.

Using this Theorem we can derive the Gaussian isoperimetric inequality with constant $c = O(\kappa^{-1/2})$.

Proposition 9. *Let μ be a probability measure with density e^{-U} with respect to the Lebesgue measure, where $U : \mathbb{R}^n \to [-\infty, \infty)$ is a convex function. If μ satisfies the log-Sobolev inequality with constant $\kappa > 0$ then, for any Borel set A in $\mathbb{R}^n$ we have*

$$\mu^+(A) \ge c(\kappa)I(\mu(A)). \tag{89}$$

Furthermore, we can have $c(\kappa) = c/\sqrt{\kappa}$, where $c > 0$ is an absolute constant.

Proof. Let A be a Borel set in $\mathbb{R}^n$. It is enough to consider the case $0 < \mu(A) \le 1/2$. For any $t \ge 0$, approximating χ_A with smooth functions $f_\varepsilon : \mathbb{R}^n \to [0, 1]$ and passing to the limit, from Theorem 4 we get

$$\mu(A) \left(1 - \mu(A)^{\frac{2}{p(t)}-1}\right) \le \sqrt{2t}\mu^+(A). \tag{90}$$

Note that

$$\frac{2}{p(t)} - 1 = \tanh\left(\frac{t}{2\kappa}\right) \ge \frac{t}{2\kappa}\tanh(1), \tag{91}$$

for all $0 \le t \le 2\kappa$. Therefore, we have

$$\mu(A) \left(1 - e^{-\frac{c_1 t}{2\kappa}\log(1/\mu(A))}\right) \le \sqrt{2t}\mu^+(A). \tag{92}$$

Computing at time $t_0 = \frac{\kappa}{\log(1/\mu(A))} \in (0, 2\kappa)$ we see that

$$\mu^+(A) \ge \frac{1 - e^{-c_1/2}}{\sqrt{2}} \frac{1}{\sqrt{\kappa}}\mu(A)\sqrt{\log\frac{1}{\mu(A)}}. \tag{93}$$

Using the fact that $I(x) \le c_2 x\sqrt{\log(1/x)}$ for all $x \in (0, 1/2)$ and some absolute constant $c_2 > 0$, we get the result with constant $c = \frac{1-e^{-c_1/2}}{2c_2}\kappa^{-1/2}$. $\qquad\square$

Proof of Theorem 3. Let γ denote the standard 1-dimensional Gaussian measure. It is known that (γ, w) has property (τ), where $w(x) = x^2/4$—see [21]. Let f be a bounded measurable function on $\mathbb{R}^n$. We consider a function $g : \mathbb{R} \to \mathbb{R}$ which is increasing, continuous from the right, and such that, for any $t \in \mathbb{R}$,

$$\mu(f < t) = \gamma(g < t). \tag{94}$$

Then, for the proof of (12) we just need to verify that

$$\int e^{f\square\varphi}\,d\mu \le \int e^{g\square w}\,d\gamma. \tag{95}$$

To this end, it suffices to prove that for any $u > 0$,

$$\mu(f\square\varphi < u) \ge \gamma(g\square w < u). \tag{96}$$

Since g is increasing we get that $g\square w$ is also increasing, thus the set $D_u = \{x : (g\square w)(x) < u\}$ is a half-line. For every $x \in D_u$ there exist $x_1, x_2 \in \mathbb{R}$ such that $x_1 + x_2 = x$ and $g(x_1) + w(x_2) < u$. By a limiting argument, for the proof of (96) it is

enough to prove that for any $x \in D_u$ and for any $x_1, x_2 \in \mathbb{R}$ with $g(x_1) + w(x_2) < u$ we have

$$\mu(f \square \varphi < u) \geq \gamma(-\infty, x_1 + x_2] = \Phi(x_1 + x_2). \tag{97}$$

For any $g(x_1) < s_1 < u - w(x_2)$ the definition of g implies that: $\mu(f < s_1) = \gamma(g < s_1) \geq \gamma(-\infty, x_1] = \Phi(x_1)$. Moreover, the inclusion

$$\{f < s_1\} + \beta \frac{|x_2|}{2} B_2^n \subseteq \{f \square \varphi < u\} \tag{98}$$

is valid with $\varphi(x) = \|x\|_2^2/\beta^2$ for any $\beta > 0$.

In order to get the result, we ought to verify an inequality of the following form: if $\mu(A) \geq \Phi(x_1)$ one has $\mu(A + \frac{\beta}{2}|x_2|B_2^n) \geq \Phi(x_1 + x_2)$. Equivalently, for any $t > 0$ and any Borel subset A in $\mathbb{R}^n$ we would like to have $\mu(A + tB_2^n) \geq \Phi(\Phi^{-1}(\mu(A)) + \frac{2}{\beta}t)$.

To finish the proof we just observe that our assumption is equivalent to $\mu^+(A) \geq c(\kappa)I(\mu(A))$ for any Borel subset A of $\mathbb{R}^n$ and this in turn to the fact that for any Borel subset A of $\mathbb{R}^n$ and any $t > 0$ we have

$$\mu(A + tB_2^n) \geq \Phi(\Phi^{-1}(\mu(A)) + tc(\kappa)). \tag{99}$$

A proof of this last assertion can be found in [18]. Thus, we have proved Theorem 3 with $\varphi(y) = \frac{c(\kappa)^2}{4}\|y\|_2^2 = \frac{c}{\kappa}\|y\|_2^2$, where $c > 0$ is an absolute constant. $\qquad\square$

Remark. We should mention here that Maurey has proved in [21] that if (μ, w) has property (τ) and $w : \mathbb{R}^n \to \mathbb{R}^+$ is a convex function such that $w(x) \geq \frac{1}{2\kappa}\|x\|_2^2$ in some neighborhood of 0, then μ satisfies Poincaré inequality with constant κ. Thus, the previous Theorem shows that in the context of log-concave measures, the class $\mathcal{LS}_{lc}(\kappa)$ is contained in the class of measures μ satisfying (τ) with a convex cost function which satisfies this hyperquadratic condition near zero, and in turn, this class is contained in $\mathcal{P}_{lc}(c\kappa)$, where $c > 0$ is an absolute constant.

A weaker version of Problem 2 is the following:

Problem 3. We say that μ has comparable weak and strong moments with constant α (which we denote by $\mathrm{CWSM}(\alpha)$) if for any norm $\|\cdot\|$ in $\mathbb{R}^n$ and any $q \geq p \geq 2$ one has:

$$\left[\int \left|\|x\| - \left(\int \|x\|^p \, d\mu(x)\right)^{1/p}\right|^q\right]^{1/q} \leq \alpha \sup_{\|z\|_* \leq 1} \left(\int |\langle z, x\rangle|^p \, d\mu(x)\right)^{1/p}. \tag{100}$$

Determine if every measure $\mu \in \mathcal{LS}_{lc}(\kappa)$ satisfies $\mathrm{CWSM}(c)$ for some constant $c = c(\kappa)$.

It was communicated to us by R. Latała that a positive answer to Problems 2 and 3 is not known even if we restrict our attention to the following case:

Problem 4. Given a probability measure μ of the form $d\mu(x) = e^{-W(x)}\,dx$ with $W : \mathbb{R}^n \to \mathbb{R}$ a convex function such that $\mathrm{Hess}\,W \geq \alpha^{-1}I$ for some given constant $\alpha > 0$, determine if Problems 2 and 3 have a positive answer up to constants $c(\alpha)$.

This class of measures has been studied systematically, and it is well known that it is a subclass to $\mathcal{LS}_{lc}(\alpha)$ (see for example [7]).

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f-Divergence for Convex Bodies

Elisabeth M. Werner

Abstract We introduce f-divergence, a concept from information theory and statistics, for convex bodies in $\mathbb{R}^n$. We prove that f-divergences are $SL(n)$ invariant valuations and we establish an affine isoperimetric inequality for these quantities. We show that generalized affine surface area and in particular the L_p affine surface area from the L_p Brunn Minkowski theory are special cases of f-divergences.

Key words f-divergence • Relative entropy • Affine surface area

Mathematical Subject Classifications (2010): 52A20, 53A15

1 Introduction

In information theory, probability theory and statistics, an f-divergence is a function $D_f(P, Q)$ that measures the difference between two probability distributions P and Q. The divergence is intuitively an average, weighted by the function f, of the odds ratio given by P and Q. These divergences were introduced independently by Csiszár [2], Morimoto [36] and Ali & Silvey [1]. Special cases of f-divergences are the Kullback Leibler divergence or relative entropy and the Rényi divergences (see Sect. 1).

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Due to a number of highly influential works (see, e.g., [4–11, 14, 15, 18, 19, 21–27, 29, 31, 34, 35, 37, 41–53, 55–57]), the L_p-Brunn–Minkowski theory is now a central part of modern convex geometry. A fundamental notion within this theory is L_p affine surface area, introduced by Lutwak in the ground breaking paper [26].

It was shown in [51] that L_p affine surface areas are entropy powers of Rényi divergences of the cone measures of a convex body and its polar, thus establishing further connections between information theory and convex geometric analysis. Further examples of such connections are e.g. several papers by Lutwak, Yang, and Zhang [28, 30, 32, 33] and the recent article [38] where it is shown how relative entropy appears in convex geometry.

In this paper we introduce f-divergences to the theory of convex bodies and thus strengthen the already existing ties between information theory and convex geometric analysis. We show that generalizations of the L_p affine surface areas, the L_ϕ and L_ψ affine surface areas introduced in [20, 22], are in fact f-divergences for special functions f. We show that f-divergences are $SL(n)$ invariant valuations and establish an affine isoperimetric inequality for these quantities. Finally, we give geometric characterizations of f-divergences.

Usually, in the literature, f-divergences are considered for convex functions f. A similar theory with the obvious modifications can be developed for concave functions. Here, we restrict ourselves to consider the convex setting.

1.1 Further Notation

We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle \cdot, \cdot \rangle$. We write B_2^n for the Euclidean unit ball centered at 0 and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$ or, if we want to emphasize the dimension, by $\mathrm{vol}_d(A)$ for a d-dimensional set A.

Let $\mathcal{K}_0$ be the space of convex bodies K in $\mathbb{R}^n$ that contain the origin in their interiors. Throughout the paper, we will only consider such K. For $K \in \mathcal{K}_0$, $K^\circ = \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in K\}$ is the polar body of K. For a point $x \in \partial K$, the boundary of K, $N_K(x)$ is the outer unit normal in x to K and $\kappa_K(x)$, or, in short κ, is the (generalized) Gauss curvature in x. We write $K \in C_+^2$, if K has C^2 boundary ∂K with everywhere strictly positive Gaussian curvature κ_K. By μ or μ_K we denote the usual surface area measure on ∂K and by σ the usual surface area measure on S^{n-1}.

Let K be a convex body in $\mathbb{R}^n$ and let $u \in S^{n-1}$. Then $h_K(u)$ is the support function of K in direction $u \in S^{n-1}$, and $f_K(u)$ is the curvature function, i.e. the reciprocal of the Gaussian curvature $\kappa_K(x)$ at the point $x \in \partial K$ that has u as outer normal.

2 *f*-Divergences

Let (X, μ) be a measure space and let $dP = pd\mu$ and $dQ = qd\mu$ be probability measures on X that are absolutely continuous with respect to the measure μ. Let $f : (0, \infty) \to \mathbb{R}$ be a convex function. The $*$-adjoint function $f^* : (0, \infty) \to \mathbb{R}$ of f is defined by (e.g. [16])

$$f^*(t) = tf(1/t), \ t \in (0, \infty). \tag{1}$$

It is obvious that $(f^*)^* = f$ and that f^* is again convex if f is convex. Csiszár [2], and independently Morimoto [36] and Ali & Silvery [1] introduced the f-divergence $D_f(P, Q)$ of the measures P and Q which, for a convex function $f : (0, \infty) \to \mathbb{R}$ can be defined as (see [16])

$$D_f(P, Q) = \int_{\{pq>0\}} f\left(\frac{p}{q}\right) qd\mu + f(0) Q (\{x \in X : p(x) = 0\})$$
$$+ f^*(0) P (\{x \in X : q(x) = 0\}), \tag{2}$$

where

$$f(0) = \lim_{t\downarrow 0} f(t) \quad \text{and} \quad f^*(0) = \lim_{t\downarrow 0} f^*(t). \tag{3}$$

We make the convention that $0 \cdot \infty = 0$.

Please note that

$$D_f(P, Q) = D_{f^*}(Q, P). \tag{4}$$

With (3) and as

$$f^*(0) P (\{x \in X : q(x) = 0\}) = \int_{\{q=0\}} f^*\left(\frac{q}{p}\right) pd\mu = \int_{\{q=0\}} f\left(\frac{p}{q}\right) qd\mu,$$

we can write in short

$$D_f(P, Q) = \int_X f\left(\frac{p}{q}\right) qd\mu. \tag{5}$$

For particular choices of f we get many common divergences. E.g. for $f(t) = t \ln t$ with $*$-adjoint function $f^*(t) = -\ln t$, the f-divergence is the classical *information divergence*, also called *Kullback–Leibler divergence* or *relative entropy* from P to Q (see [3])

$$D_{KL}(P \| Q) = \int_X p \ln \frac{p}{q} d\mu. \tag{6}$$

For the convex or concave functions $f(t) = t^\alpha$ we obtain the *Hellinger integrals* (e.g. [16])

$$H_\alpha(P, Q) = \int_X p^\alpha q^{1-\alpha} d\mu. \tag{7}$$

Those are related to the Rényi divergence of order α, $\alpha \neq 1$, introduced by Rényi
[40] (for $\alpha > 0$) as

$$D_{\alpha}(P \| Q) = \frac{1}{\alpha - 1} \ln \left(\int_X p^{\alpha} q^{1-\alpha} d\mu \right) = \frac{1}{\alpha - 1} \ln \left(H_{\alpha}(P, Q) \right). \qquad (8)$$

The case $\alpha = 1$ is the relative entropy $D_{KL}(P \| Q)$.

3 f-Divergences for Convex Bodies

We will now consider f-divergences for convex bodies $K \in \mathcal{K}_0$. Let

$$p_K(x) = \frac{\kappa_K(x)}{\langle x, N_K(x) \rangle^n \, n |K^\circ|}, \quad q_K(x) = \frac{\langle x, N_K(x) \rangle}{n \, |K|}. \qquad (9)$$

Usually, in the literature, the measures under consideration are probability measures.
Therefore we have normalized the densities. Thus

$$P_K = p_K \, \mu_K \quad \text{and} \quad Q_K = q_K \, \mu_K \qquad (10)$$

are measures on ∂K that are absolutely continuous with respect to μ_K. Q_K is a
probability measure and P_K is one if K is in C_+^2.

Recall that the normalized cone measure cm_K on ∂K is defined as follows: For
every measurable set $A \subseteq \partial K$

$$cm_K(A) = \frac{1}{|K|} \left| \{ ta : a \in A, t \in [0, 1] \} \right|. \qquad (11)$$

The next proposition is well known. See e.g. [38] for a proof. It shows that the
measures P_K and Q_K defined in (10) are the cone measures of K and K°. $N_K :
\partial K \to S^{n-1}$, $x \to N_K(x)$ is the Gauss map.

Proposition 1. *Let K be a convex body in $\mathbb{R}^n$. Let P_K and Q_K be the probability
measures on ∂K defined by (10). Then*

$$Q_K = cm_K,$$

or, equivalently, for every measurable subset A in ∂K $Q_K(A) = cm_K(A)$.

If K is in addition in C_+^2, then

$$P_K = N_K^{-1} N_{K^\circ} cm_{K^\circ}$$

or, equivalently, for every measurable subset A in ∂K

$$P_K(A) = cm_{K^\circ} \left(N_{K^\circ}^{-1} \left(N_K(A) \right) \right). \qquad (12)$$

It is in the sense (12) that we understand P_K to be the "cone measure" of K° and we write $P_K = cm_{K^\circ}$.

We now define the f-divergences of $K \in \mathcal{K}_0$. Note that $\langle x, N_K(x) \rangle > 0$ for all $x \in \partial K$ and therefore $\{x \in \partial K : q_K(x) = 0\} = \emptyset$. Hence, possibly also using our convention $0 \cdot \infty = 0$,

$$f^*(0) \, P_K (\{x \in \partial K : q_K(x) = 0\}) = 0.$$

Definition 1. Let K be a convex body in $\mathcal{K}_0$ and let Let $f : (0, \infty) \to \mathbb{R}$ be a convex function. The f-divergence of K with respect to the cone measures P_K and Q_K is

$$D_f(P_K, Q_K) = \int_{\partial K} f\left(\frac{p_K}{q_K}\right) q_K d\mu_K$$

$$= \int_{\partial K} f\left(\frac{|K|\kappa_K(x)}{|K^\circ|\langle x, N_K(x)\rangle^{n+1}}\right) \frac{\langle x, N_K(x)\rangle}{n|K|} d\mu_K. \quad (13)$$

Remarks. By (4) and (13)

$$D_f(Q_K, P_K) = \int_{\partial K} f\left(\frac{q_K}{p_K}\right) p_K d\mu_K = D_{f^*}(P_K, Q_K)$$

$$= \int_{\partial K} f^*\left(\frac{p_K}{q_K}\right) q_K d\mu_K$$

$$= \int_{\partial K} f\left(\frac{|K^\circ|\langle x, N_K(x)\rangle^{n+1}}{|K|\kappa_K(x)}\right) \frac{\kappa_K(x) \, d\mu_K}{n|K^\circ|\langle x, N_K(x)\rangle^n}. \quad (14)$$

f-divergences can also be expressed as integrals over S^{n-1},

$$D_f(P_K, Q_K) = \int_{S^{n-1}} f\left(\frac{|K|}{|K^\circ| f_K(u) h_K(u)^{n+1}}\right) \frac{h_K(u) f_K(u)}{n|K|} d\sigma \quad (15)$$

and

$$D_f(Q_K, P_K) = \int_{S^{n-1}} f\left(\frac{|K^\circ| f_K(u) h_K(u)^{n+1}}{|K|}\right) \frac{d\sigma_K}{n|K^\circ| h_K(u)^n}. \quad (16)$$

Examples. If K is a polytope, the Gauss curvature κ_K of K is 0 a.e. on ∂K. Hence

$$D_f(P_K, Q_K) = f(0) \quad \text{and} \quad D_f(Q_K, P_K) = f^*(0). \quad (17)$$

For every ellipsoid $\mathcal{E}$,

$$D_f(P_\mathcal{E}, Q_\mathcal{E}) = D_f(Q_\mathcal{E}, P_\mathcal{E}) = f(1) = f^*(1). \quad (18)$$

Denote by $Conv(0, \infty)$ the set of functions $\psi : (0, \infty) \to (0, \infty)$ such that ψ is convex, $\lim_{t \to 0} \psi(t) = \infty$, and $\lim_{t \to \infty} \psi(t) = 0$. For $\psi \in Conv(0, \infty)$, Ludwig [20] introduces the L_ψ affine surface area for a convex body K in $\mathbb{R}^n$

$$\Omega_\Psi(K) = \int_{\partial K} \psi \left(\frac{\kappa_K(x)}{\langle x, N_K(x) \rangle^{n+1}} \right) \langle x, N_K(x) \rangle d\mu_K. \tag{19}$$

Note that in (19) the densities corresponding to p_k and q_K of (9) are not normalized. However, we could equally have used non-normalized densities in Definition 1. Thus, in this sense, L_ψ affine surface areas are special cases of (non-normalized) f-divergences for $f = \psi$.

For $\psi \in Conv(0, \infty)$, the $*$-adjoint function ψ^* is convex, $\lim_{t \to 0} \psi(t) = 0$, and $\lim_{t \to \infty} \psi(t) = \infty$. Thus ψ^* is an *Orlicz function* (see [17]), and gives rise to the corresponding *Orlicz-divergences* $D_{\psi^*}(P_K, Q_K)$ and $D_{\psi^*}(Q_K, P_K)$.

Let $p \leq 0$. Then the function $f : (0, \infty) \to (0, \infty)$, $f(t) = t^{\frac{p}{n+p}}$, is convex. The corresponding (non-normalized) f-divergence (which is also an Orlicz-divergence) is the L_p *affine surface area*, introduced by Lutwak [26] for $p > 1$ and by Schütt and Werner [46] for $p < 1$, $p \neq -n$. See also [12].

It was shown in [51] that all L_p affine surface areas are entropy powers of Rényi divergences.

For $p \geq 0$, the function $f : (0, \infty) \to (0, \infty)$, $f(t) = t^{\frac{p}{n+p}}$ is concave. The corresponding L_p affine surface areas $\int_{\partial K} \frac{\kappa_K^{\frac{p}{n+p}} d\mu_K}{\langle x, N_K(x) \rangle^{\frac{n(p-1)}{n+p}}}$ are examples of L_ϕ affine surface areas which were considered in [22] and [20]. Those, in turn are special cases of (non-normalized) f-divergences for *concave* functions f.

Let $f(t) = t \ln t$. Then the $*$-adjoint function is $f^*(t) = -\ln t$. The corresponding f-divergence is the *Kullback Leibler divergence* or *relative entropy* $D_{KL}(P_K \| Q_K)$ from P_K to Q_K

$$D_{KL}(P_K \| Q_K) = \int_{\partial K} \frac{\kappa_K(x)}{n|K^\circ|\langle x, N_K(x) \rangle^n} \ln \left(\frac{|K|\kappa_K(x)}{|K^\circ|\langle x, N_K(x) \rangle^{n+1}} \right) d\mu_K. \tag{20}$$

The relative entropy $D_{KL}(Q_K \| P_K)$ from Q_K to P_K is

$$D_{KL}(Q_K \| P_K) = D_{f^*}(P_K, Q_K)$$
$$= \int_{\partial K} \frac{\langle x, N_K(x) \rangle}{n|K|} \log \left(\frac{|K^\circ|\langle x, N_K(x) \rangle^{n+1}}{|K|\kappa_K(x)} \right) d\mu_K. \tag{21}$$

Those were studied in detail in [38].

Similar to (15) and (16), one can define mixed f-divergences for n convex bodies in $\mathbb{R}^n$. We will not treat those here but will concentrate on f-divergence for one convex body. We also refer to [54], where they have been investigated for functions in $Conv(0, \infty)$.

The observation (17) about polytopes holds more generally.

Proposition 2. *Let K be a convex body in $\mathcal{K}_0$ and let $f : (0, \infty) \to \mathbb{R}$ be a convex function. If K is such that $\mu_K(\{p_K > 0\}) = 0$, then*

$$D_f(P_K, Q_K) = f(0) \ \text{ and } \ D_f(Q_K, P_K) = f^*(0).$$

Proof. $\mu_K(\{p_K > 0\}) = 0$ iff $Q_K(\{p_K > 0\}) = 0$. Hence the assumption implies that $Q_K(\{p_K = 0\}) = 1$. Therefore,

$$
\begin{aligned}
D_f(P_K, Q_K) &= \int_{\partial K} f\left(\frac{p_K}{q_K}\right) q_K d\mu_K \\
&= \int_{\{p_K > 0\}} f\left(\frac{p_K}{q_K}\right) q_K d\mu_K + \int_{\{p_K = 0\}} f\left(\frac{p_K}{q_K}\right) q_K d\mu_K \\
&= f(0).
\end{aligned}
$$

By (4), $D_f(Q_K, P_K) = D_{f^*}(P_K, Q_K) = f^*(0)$.

The next proposition complements the previous one. In view of (18) and (24), it corresponds to the affine isoperimetric inequality for f-divergences. It was proved in [16] in a different setting and in the special case of $f \in Conv(0, \infty)$ by Ludwig [20]. We include a proof for completeness.

Proposition 3. *Let K be a convex body in $\mathcal{K}_0$ and let $f : (0, \infty) \to \mathbb{R}$ be a convex function. If K is such that $\mu_K(\{p_K > 0\}) > 0$, then*

$$D_f(P_K, Q_K) \geq f\left(\frac{P_K(\{p_K > 0\})}{Q_K(\{p_K > 0\})}\right) Q_K(\{p_K > 0\}) + f(0) Q_K(\{p_K = 0\}) \tag{22}$$

and

$$D_f(Q_K, P_K) \geq f^*\left(\frac{P_K(\{p_K > 0\})}{Q_K(\{p_K > 0\})}\right) Q_K(\{p_K > 0\}) + f^*(0) Q_K(\{p_K = 0\}). \tag{23}$$

If K is in C_+^2, or if f is decreasing, then

$$D_f(P_K, Q_K) \geq f(1) \ \text{ and } \ D_f(Q_K, P_K) \geq f^*(1) = f(1). \tag{24}$$

Equality holds in (22) and (23) iff f is linear or K is an ellipsoid. If K is in C_+^2, equality holds in both inequalities (24) iff f is linear or K is an ellipsoid. If f is decreasing, equality holds in both inequalities (24) iff K is an ellipsoid.

Remarks.

(i) It is possible for f to be decreasing and linear without having equality in (24). To see that, let $f(t) = at + b, a < 0, b > 0$. Then, for polytopes K (for which $\mu_K(\{p_K > 0\}) = 0$), $D_f(P_K, Q_K) = f(0) = b > f(1) = a + b$. But, also in the case when $0 < \mu_K(\{p_K > 0\}) < 1$, strict inequality may hold.

Indeed, let $\varepsilon > 0$ be sufficiently small and let $K = B_\infty^n(\varepsilon)$ be a "rounded" cube, where we have "rounded" the corners of the cube B_∞^n with sidelength 2 centered at 0 by replacing each corner with εB_2^n Euclidean balls. Then $D_f(P_K, Q_K) = b + a \, P_K(\{p_K > 0\}) > b + a = f(1)$.

(ii) Proposition 3 corresponds, in a sense, to the isoperimetric inequality. This can be seen, e.g., from Eq. (24), which, by (18), for any ellipsoid $\mathcal{E}$ can be written as

$$D_f(P_K, Q_K) \geq D_f(P_\mathcal{E}, Q_\mathcal{E}) \text{ and } D_f(Q_K, P_K) \geq D_f(Q_\mathcal{E}, P_\mathcal{E}).$$

Proof of Proposition 3. Let K be such that $\mu_K(\{p_K > 0\}) > 0$, which is equivalent to $Q_K(\{p_K > 0\}) > 0$. Then, by Jensen's inequality,

$$D_f(P_K, Q_K) = Q_K(\{p_K > 0\}) \int_{\{p_K > 0\}} f\left(\frac{p_K}{q_K}\right) \frac{q_K d\mu_K}{Q_K(\{p_K > 0\})}$$

$$+ f(0) \, Q_K(\{p_K = 0\})$$

$$\geq Q_K(\{p_K > 0\}) f\left(\frac{P_K(\{p_K > 0\})}{Q_K(\{p_K > 0\})}\right) + f(0) Q_K(\{p_K = 0\}).$$

Inequality (23) follows by (4), as $D_f(Q_K, P_K) = D_{f^*}(P_K, Q_K)$.

If K is in C_+^2, $Q_K(\{p_K > 0\}) = 1$, $Q_K(\{p_K = 0\}) = 0$, $P_K(\{p_K > 0\}) = 1$ and $P_K(\{p_K = 0\}) = 0$. Thus we get that $D_f(P_K, Q_K) \geq f(1)$ and $D_f(Q_K, P_K) \geq f^*(1) = f(1)$.

If f is decreasing, then, by Jensen's inequality

$$D_f(P_K, Q_K) = \int_{\partial K} f\left(\frac{p_K}{q_K}\right) q_K d\mu_K \geq f\left(\int_{\partial K} p_K d\mu_K\right) \geq f(1).$$

The last inequality holds as $\int_{\partial K} p_K d\mu_K \leq 1$ and as f is decreasing.

Equality holds in Jensen's inequality iff either f is linear or $\frac{p_K}{q_K}$ is constant. Indeed, if $f(t) = at + b$, then

$$D_f(P_K, Q_K) = \int_{\{p_K > 0\}} \left(a\frac{p_K}{q_K} + b\right) q_K d\mu_K + f(0) \, Q_K(\{p_K = 0\})$$

$$= a P_K(\{p_K > 0\}) + f(0).$$

If f is not linear, equality holds iff $\frac{p_K}{q_K} = c$, c a constant. By a theorem of Petty [39], this holds iff K is an ellipsoid. Note that the constant c is different from 0, as we assume that $\mu_K (\{p_K > 0\}) > 0$.

The next proposition can be found in [16] in a different setting. Again, we include a proof for completeness.

Proposition 4. *Let K be a convex body in $\mathcal{K}_0$ and let $f : (0, \infty) \to \mathbb{R}$ be a convex function. Then*

$$D_f(P_K, Q_K)$$

$$\leq f(0) + f^*(0) + f(1) \left[Q_K(\{0 < p_K \leq q_K\}) + P_K(\{0 < q_K \leq p_K\}) \right]$$

and

$$D_f(Q_K, P_K)$$

$$\leq f(0) + f^*(0) + f(1) \left[Q_K(\{0 < p_K \leq q_K\}) + P_K(\{0 < q_K \leq p_K\}) \right].$$

If f is decreasing, the inequalities reduce to $D_f(P_K, Q_K) \leq f(0)$ respectively, $D_f(Q_K, P_K) \leq f^*(0)$.

Proof. It is enough to prove the first inequality. The second one follows immediately from the first by (4).

$$D_f(P_K, Q_K) = \int_{\partial K} f\left(\frac{p_K}{q_K}\right) q_K d\mu$$

$$= \int_{\{p_K > 0\}} f\left(\frac{p_K}{q_K}\right) q_K d\mu + f(0)\, Q_K(\{p_K = 0\})$$

$$= f(0)\, Q_K(\{p_K = 0\}) + \int_{\{0 < p_K\} \cap \{f' \geq 0\}} f\left(\frac{p_K}{q_K}\right) q_K d\mu$$

$$+ \int_{\{0 < p_K\} \cap \{f' < 0\}} f\left(\frac{p_K}{q_K}\right) q_K d\mu$$

$$\leq f(0) \left[Q_K(\{p_K = 0\}) + Q_K\left(\{p_K > 0\} \cap \{f' < 0\}\right) \right]$$

$$+ \int_{\{0 < p_K \leq q_K\} \cap \{f' \geq 0\}} f\left(\frac{p_K}{q_K}\right) q_K d\mu$$

$$+ \int_{\{0 < q_K \leq p_K\} \cap \{f' \geq 0\}} f\left(\frac{p_K}{q_K}\right) q_K d\mu$$

$$\leq f(0) + f(1)\, Q_K\left(\{0 < p_K \leq q_K\} \cap \{f' \geq 0\}\right)$$

$$+ \int_{\{0 < q_K \le p_K\} \cap \{f' \ge 0\}} f^* \left(\frac{q_K}{p_K} \right) p_K d\mu$$

$$= f(0) + f(1) \, Q_K \left(\{0 < p_K \le q_K\} \cap \{f' \ge 0\} \right)$$

$$+ \int_{\{0 < q_K \le p_K\} \cap \{f' \ge 0\} \cap \{(f^*)' \ge 0\}} f^* \left(\frac{q_K}{p_K} \right) p_K d\mu$$

$$+ \int_{\{0 < q_K \le p_K\} \cap \{f' \ge 0\} \cap \{(f^*)' < 0\}} f^* \left(\frac{q_K}{p_K} \right) p_K d\mu$$

$$\le f(0) + f(1) \, Q_K \left(\{0 < p_K \le q_K\} \cap \{f' \ge 0\} \right)$$

$$+ f^*(1) \, P_K \left(\{0 < q_K \le p_K\} \cap \{f' \ge 0\} \cap \{(f^*)' \ge 0\} \right)$$

$$+ f^*(0) \, P_K \left(\{0 < q_K \le p_K\} \cap \{f' \ge 0\} \cap \{(f^*)' < 0\} \right)$$

$$\le f(0) + f^*(0) \, P_K \left(\{0 < q_K \le p_K\} \cap \{f' \ge 0\} \right)$$

$$+ f(1) \Bigg[Q_K(\{0 < p_K \le q_K\} \cap \{f' \ge 0\})$$

$$+ P_K(\{0 < q_K \le p_K\} \cap \{f' \ge 0\}) \Bigg].$$

It follows from the last expression that, if f is decreasing, the inequality reduces to $D_f(P_K, Q_K) \le f(0)$.

The next proposition shows that f-divergences are $GL(n)$ invariant and that non-normalized f-divergences are $SL(n)$ invariant valuations. For functions in $Conv(0, \infty)$, this was proved by Ludwig [20].

For functions in $Conv(0, \infty)$ the expressions are also lower semicontinuous, as it was shown in [20]. However, this need not be the case anymore if we assume just convexity of f. Indeed, let $f(t) = t^2$ and let $K = B_2^n$ be the Euclidean unit ball. Let $(K_j)_{j \in \mathbb{N}}$ be a sequence of polytopes that converges to B_2^n. As observed above, $D_f(P_{K_j}, Q_{K_j}) = f(0) = 0$ for all j. But $D_f(P_{B_2^n}, Q_{B_2^n}) = f(1) = 1$.

Let $\tilde{P}_K = \frac{\kappa_K \mu_K}{\langle x, N_K(x) \rangle^n}$ and $\tilde{Q}_K = \langle x, N_K(x) \rangle \mu_K$. Then we will denote by $D_f(\tilde{P}_K, \tilde{Q}_K)$ and $D_f(\tilde{Q}_K, \tilde{P}_K)$ the non-normalized f-divergences. We will also use the following lemma from [46] for the proof of Proposition 5.

Lemma 1. *Let K be a convex body in $\mathcal{K}_0$. Let $h : \partial K \to \mathbb{R}$ be an integrable function, and $T : \mathbb{R}^n \to \mathbb{R}^n$ an invertible, linear map. Then*

$$\int_{\partial K} h(x) d\mu_K = |\det(T)|^{-1} \int_{\partial T(K)} \frac{f(T^{-1}(y))}{\|T^{-1t}(N_K(T^{-1}(y)))\|} d\mu_{T(K)}.$$

Proposition 5. *Let K be a convex body in $\mathcal{K}_0$ and let $f : (0, \infty) \to \mathbb{R}$ be a convex function. Then $D_f(P_K, Q_K)$ and $D_f(Q_K, P_K)$ are $GL(n)$ invariant and $D_f(\tilde{P}_K, \tilde{Q}_K)$ and $D_f(\tilde{Q}_K, \tilde{P}_K)$ are $SL(n)$ invariant valuations.*

Proof. We use (e.g. [46]) that

$$\langle T(x), N_{T(K)}(T(x)) \rangle = \frac{\langle x, N_K(x) \rangle}{\|T^{-1t}(N_K(x))\|},$$

and

$$\kappa_K(x) = \|T^{-1t}(N_K(x))\|^{n+1} \det(T)^2 \kappa_{T(K)}(T(x))$$

and Lemma 1 to get that

$$D_f(P_K, Q_K) = \int_{\partial K} f\left(\frac{p_K(x)}{q_K(x)}\right) q_K(x) d\mu(x)$$

$$= \frac{1}{|\det(T)|} \int_{\partial T(K)} \frac{f\left(\frac{p_K(T^{-1}(y))}{q_K(T^{-1}(y))}\right) q_K(T^{-1}(y)) d\mu_{T(K)}}{\|T^{-1t}(N_K(T^{-1}(y)))\|}$$

$$= D_f(P_{T(K)}, Q_{T(K)}).$$

The formula for $D_f(Q_K, P_K)$ follows immediately from this one and (4). The $SL(n)$ invariance for the non-normalized f-divergences is shown in the same way.

Now we show that $D_f(\tilde{P}_K, \tilde{Q}_K)$ and $D_f(\tilde{Q}_K, \tilde{P}_K)$ are valuations, i.e. for convex bodies K and L in $\mathcal{K}_0$ such that $K \cup L \in \mathcal{K}_0$,

$$D_f(\tilde{P}_{K\cup L}, \tilde{Q}_{K\cup L}) + D_f(\tilde{P}_{K\cap L}, \tilde{Q}_{K\cap L}) = D_f(\tilde{P}_K, \tilde{Q}_K) + D_f(\tilde{P}_L, \tilde{Q}_L). \quad (25)$$

Again, it is enough to prove this formula and the one for $D_f(\tilde{Q}_K, \tilde{P}_K)$ follows with (4). To prove (25), we proceed as in Schütt [43]. For completeness, we include the argument. We decompose

$$\partial(K \cup L) = (\partial K \cap \partial L) \cup (\partial K \cap L^c) \cup (K^c \cap \partial L),$$

$$\partial(K \cap L) = (\partial K \cap \partial L) \cup (\partial K \cap \text{int} L) \cup (\text{int} K \cap \partial L),$$

$$\partial K = (\partial K \cap \partial L) \cup (\partial K \cap L^c) \cup (\partial K \cap \text{int} L),$$

$$\partial L = (\partial K \cap \partial L) \cup (\partial K^c \cap \partial L) \cup (\text{int} K \cap \partial L),$$

where all unions on the right hand side are disjoint. Note that for x such that the curvatures $\kappa_K(x)$, $\kappa_L(x)$, $\kappa_{K\cup L}(x)$ and $\kappa_{K\cap L}(x)$ exist,

$$\langle x, N_K(x) \rangle = \langle x, N_L(x) \rangle = \langle x, N_{K\cap L}(x) \rangle = \langle x, N_{K\cup L}(x) \rangle \quad (26)$$

and

$$\kappa_{K\cup L}(x) = \min\{\kappa_K(x), \kappa_L(x)\}, \quad \kappa_{K\cap L}(x) = \max\{\kappa_K(x), \kappa_L(x)\}. \quad (27)$$

To prove (25), we split the involved integral using the above decompositions and (26) and (27).

4   Geometric Characterization of f-Divergences

In [51], geometric characterizations were proved for Rényi divergences. Now, we want to establish such geometric characterizations for f-divergences as well. We use the *surface body* [46] but the *illumination surface body* [53] or the *mean width body* [13] can also be used.

Let K be a convex body in $\mathbb{R}^n$. Let $g : \partial K \to \mathbb{R}$ be a nonnegative, integrable, function. Let $s \geq 0$.

The *surface body* $K_{g,s}$, introduced in [46], is the intersection of all closed half-spaces H^+ whose defining hyperplanes H cut off a set of $f\mu_K$-measure less than or equal to s from ∂K. More precisely,

$$K_{g,s} = \bigcap_{\int_{\partial K \cap H^-} g d\mu_K \leq s} H^+.$$

For $x \in \partial K$ and $s > 0$

$$x_s = [0, x] \cap \partial K_{g,s}.$$

The *minimal function* $M_g : \partial K \to \mathbb{R}$

$$M_g(x) = \inf_{0 < s} \frac{\int_{\partial K \cap H^-(x_s, N_{K_{g,s}}(x_s))} g \, d\mu_K}{\mathrm{vol}_{n-1}\left(\partial K \cap H^-(x_s, N_{K_{g,s}}(x_s))\right)} \tag{28}$$

was introduced in [46]. $H(x, \xi)$ is the hyperplane through x and orthogonal to ξ. $H^-(x, \xi)$ is the closed halfspace containing the point $x + \xi$, $H^+(x, \xi)$ the other halfspace.

For $x \in \partial K$, we define $r(x)$ as the maximum of all real numbers ρ so that $B_2^n(x - \rho N_K(x), \rho) \subseteq K$. Then we formulate an integrability condition for the minimal function

$$\int_{\partial K} \frac{d\mu_K(x)}{\left(M_g(x)\right)^{\frac{2}{n-1}} r(x)} < \infty. \tag{29}$$

If g is bounded, then M_g is bounded as well. Therefore condition (29) is satisfied, if, e.g., g is bounded and K is in C_+^2.

The following theorem was proved in [46].

Theorem 1. *Let K be a convex body in $\mathbb{R}^n$. Suppose that $f : \partial K \to \mathbb{R}$ is an integrable, almost everywhere strictly positive function that satisfies the integrability condition (29). Then*

$$c_n \lim_{s \to 0} \frac{|K| - |K_{g,s}|}{s^{\frac{2}{n-1}}} = \int_{\partial K} \frac{\kappa_K^{\frac{1}{n-1}}}{g^{\frac{2}{n-1}}} d\mu_K,$$

where $c_n = 2|B_2^{n-1}|^{\frac{2}{n-1}}$.

Theorem 1 was used in [46] to give geometric interpretations of L_p affine surface area and in [51] to give geometric interpretations of Rényi divergences. Now we use this theorem to give geometric interpretations of *f*-divergence for cone measures of convex bodies.

For a convex function $f : (0, \infty) \to \mathbb{R}$, let $g_f, h_f : \partial K \to \mathbb{R}$ be defined as

$$
g_f(x) = \left[n|K^\circ|^n |K|^n \frac{p_K q_K}{\left(f \left(\frac{p_K}{q_K} \right) \right)^{n-1}} \right]^{\frac{1}{2}}
\tag{30}
$$

and

$$
h_f(x) = g_{f*}(x) = \left[n|K^\circ|^n |K|^n \frac{q_K^n / p_K^{n-2}}{\left(f \left(\frac{p_K}{q_K} \right) \right)^{n-1}} \right]^{\frac{1}{2}}.
\tag{31}
$$

Corollary 1. *Let K be a convex body in $\mathcal{K}_0$ and let $f : (0, \infty) \to \mathbb{R}$ be convex. Let $g_f, h_f : \partial K \to \mathbb{R}$ be defined as in (30) and (31). If g_f and h_f are integrable, almost everywhere strictly positive functions that satisfy the integrability condition (29), then*

$$
c_n \lim_{s \to 0} \frac{|K| - |K_{g_f, s}|}{s^{\frac{2}{n-1}}} = D_f(P_K, Q_K)
$$

and

$$
c_n \lim_{s \to 0} \frac{|K| - |K_{h_f, s}|}{s^{\frac{2}{n-1}}} = D_f(Q_K, P_K).
$$

Note that if, e.g., $K \in C_+^2$ and f is bounded, then g_f and h_f are bounded and condition (29) is satisfied.

Proof of Corollary 1. The proof of the corollary follows immediately from Theorem 1.

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