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Advances in Applied Mathematics and Approximation Theory

Contributions from AMAT 2012

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George A. Anastassiou • Oktay Duman
Editors

Advances in Applied Mathematics and Approximation Theory

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AMAT 2012 Conference, TOBB University of Economics and Technology,
Ankara, Turkey, May 17–20, 2012

George A. Anastassiou and Oktay Duman
Ankara, Turkey, May 18, 2012

Preface

This volume was prepared in connection with the proceedings of AMAT 2012—International Conference on Applied Mathematics and Approximation Theory—which was held during May 17–20, 2012 in Ankara, Turkey, at TOBB University of Economics and Technology.

AMAT 2012 conference brought together researchers from all areas of applied mathematics and approximation theory. Previous conferences which had a similar approach were held at the University of Memphis (1991, 1997, 2008), UC Santa Barbara (1993) and the University of Central Florida at Orlando (2002).

Around 200 scientists coming from 30 different countries (Algeria, Azerbaijan, China, Cyprus, Egypt, Georgia, Germany, Greece, Hungary, India, Indonesia, Iran, Iraq, Jordan, Kazakhstan, Korea, Kuwait, Libya, Lithuania, Malaysia, Morocco, Nigeria, Poland, Russia, Saudi Arabia, Taiwan, Thailand, Turkey, UAE, USA) participated in the conference. They presented 110 papers in three parallel sessions.

We are particularly indebted to the organizing committee, the scientific committee and our plenary speakers: George A. Anastassiou (University of Memphis, USA), Dumitru Baleanu (Çankaya University, Turkey), Martin Bohner (Missouri University of Science and Technology, USA), Jerry L. Bona (University of Illinois at Chicago, USA), Weimin Han (University of Iowa, USA), Margareta Heilmann (University of Wuppertal, Germany), and Cihan Orhan (Ankara University, Turkey).

We would also like thank the anonymous reviewers who helped us select the best articles for inclusion in this proceedings volume and the authors for their valuable contributions.

Finally, we are grateful to "TOBB University of Economics and Technology", which was hosting this conference and provided all of its facilities and also to "Central Bank of Turkey" and "The Scientific and Technological Research Council of Turkey" for financial support.

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Chapter 1
Approximation by Neural Networks Iterates

George A. Anastassiou

Abstract Here we study the multivariate quantitative approximation of real valued continuous multivariate functions on a box or $\mathbb{R}^N$, $N \in \mathbb{N}$, by the multivariate quasi-interpolation sigmoidal and hyperbolic tangent iterated neural network operators. This approximation is derived by establishing multidimensional Jackson type inequalities involving the multivariate modulus of continuity of the engaged function or its high order partial derivatives. Our multivariate iterated operators are defined by using the multidimensional density functions induced by the logarithmic sigmoidal and the hyperbolic tangent functions. The approximations are pointwise and uniform. The related feed-forward neural networks are with one hidden layer.

1.1 Introduction

The author in [1–3], see Chaps. 2–5, was the first to establish neural network approximations to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet–Euvrard and "Squashing" Types, by employing the modulus of continuity of the engaged function or its high-order derivative and producing very tight Jackson-type inequalities. He treats both the univariate and multivariate cases. Defining these operators "bell-shaped" and "squashing" functions are assumed to be of compact support. Also in [3] he gives the Nth-order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions; see Chaps. 4–5 there.

This article is a continuation of [4–8].

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The author here performs multivariate sigmoidal and hyperbolic tangent iterated neural network approximations to continuous functions over boxes or over the whole $\mathbb{R}^N$, $N \in \mathbb{N}$.

All convergences are with rates expressed via the multivariate modulus of continuity of the involved function or its high-order partial derivatives and given by very tight multidimensional Jackson-type inequalities.

Many times for accuracy computer processes repeat themselves. We prove that the speed of the convergence of the iterated approximation remains the same, as the original, even if we increase the number of neurons per cycle.

Feed-forward neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma\left(\left\langle a_j \cdot x\right\rangle + b_j\right), \quad x \in \mathbb{R}^s, \quad s \in \mathbb{N},$$

where for $0 \le j \le n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in \mathbb{R}$ are the coefficients, $\left\langle a_j \cdot x\right\rangle$ is the inner product of a_j and x, and σ is the activation function of the network. In many fundamental network models, the activation functions are the hyperbolic tangent and the sigmoidal. About neural networks see [9–12].

1.2 Basics

(I) Here all come from [7, 8].

We consider the sigmoidal function of logarithmic type

$$s_i(x_i) = \frac{1}{1+e^{-x_i}}, \quad x_i \in \mathbb{R}, i = 1, \ldots, N; \; x := (x_1, \ldots, x_N) \in \mathbb{R}^N,$$

each has the properties $\lim\limits_{x_i \to +\infty} s_i(x_i) = 1$ and $\lim\limits_{x_i \to -\infty} s_i(x_i) = 0$, $i = 1, \ldots, N$.

These functions play the role of activation functions in the hidden layer of neural networks.

As in [9], we consider

$$\Phi_i(x_i) := \frac{1}{2}\left(s_i(x_i+1) - s_i(x_i-1)\right), \quad x_i \in \mathbb{R}, i = 1, \ldots, N.$$

We notice the following properties:

(i) $\Phi_i(x_i) > 0, \; \forall \, x_i \in \mathbb{R}$
(ii) $\sum_{k_i=-\infty}^{\infty} \Phi_i(x_i - k_i) = 1, \; \forall \, x_i \in \mathbb{R}$
(iii) $\sum_{k_i=-\infty}^{\infty} \Phi_i(nx_i - k_i) = 1, \; \forall \, x_i \in \mathbb{R}; n \in \mathbb{N}$
(iv) $\int_{-\infty}^{\infty} \Phi_i(x_i)\, dx_i = 1$
(v) Φ_i is a density function
(vi) Φ_i is even: $\Phi_i(-x_i) = \Phi_i(x_i), \; x_i \ge 0$, for $i = 1, \ldots, N$

We see that [9]

$$\Phi_i(x_i) = \left(\frac{e^2-1}{2e^2}\right)\frac{1}{(1+e^{x_i-1})(1+e^{-x_i-1})}, \quad i=1,\ldots,N.$$

(vii) Φ_i is decreasing on $\mathbb{R}_+$ and increasing on $\mathbb{R}_-$, $i=1,\ldots,N$

Notice that $\max \Phi_i(x_i) = \Phi_i(0) = 0.231$.
Let $0<\beta<1$, $n\in\mathbb{N}$. Then as in [8] we get

(viii)

$$\sum_{\begin{cases} k_i=-\infty \\ : |nx_i-k_i| > n^{1-\beta}\end{cases}}^{\infty} \Phi_i(nx_i-k_i) \le 3.1992 e^{-n^{(1-\beta)}}, \quad i=1,\ldots,N$$

Denote by $\lceil\cdot\rceil$ the ceiling of a number and by $\lfloor\cdot\rfloor$ the integral part of a number. Consider here $x\in\left(\prod_{i=1}^N [a_i,b_i]\right)\subset\mathbb{R}^N$, $N\in\mathbb{N}$ such that $\lceil na_i\rceil\le\lfloor nb_i\rfloor$, $i=1,\ldots,N$; $a:=(a_1,\ldots,a_N)$, $b:=(b_1,\ldots,b_N)$.

As in [8] we obtain

(ix)

$$0<\frac{1}{\sum_{k_i=\lceil na_i\rceil}^{\lfloor nb_i\rfloor}\Phi_i(nx_i-k_i)}<\frac{1}{\Phi_i(1)}=5.250312578,$$

$\forall\, x_i\in[a_i,b_i]$, $i=1,\ldots,N$

(x) As in [8], we see that

$$\lim_{n\to\infty}\sum_{k_i=\lceil na_i\rceil}^{\lfloor nb_i\rfloor}\Phi_i(nx_i-k_i)\neq 1,$$

for at least some $x_i\in[a_i,b_i]$, $i=1,\ldots,N$

We will use here

$$\Phi(x_1,\ldots,x_N) := \Phi(x) := \prod_{i=1}^N \Phi_i(x_i), \quad x\in\mathbb{R}^N \tag{1.1}$$

It has the properties:

(i)′ $\Phi(x)>0$, $\forall\, x\in\mathbb{R}^N$

(ii)′

$$\sum_{k=-\infty}^{\infty}\Phi(x-k) := \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\ldots\sum_{k_N=-\infty}^{\infty}\Phi(x_1-k_1,\ldots,x_N-k_N)=1 \tag{1.2}$$

$k:=(k_1,\ldots,k_N)$, $\forall\, x\in\mathbb{R}^N$

(iii)′

$$\sum_{k=-\infty}^{\infty}\Phi(nx-k):=$$

$$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\cdots\sum_{k_N=-\infty}^{\infty}\Phi(nx_1-k_1,\ldots,nx_N-k_N)=1, \quad (1.3)$$

$\forall\, x\in\mathbb{R}^N;\ n\in\mathbb{N}$

(iv)′

$$\int_{\mathbb{R}^N}\Phi(x)\,dx=1,$$

that is, Φ is a multivariate density function

Here $\|x\|_\infty:=\max\{|x_1|,\ldots,|x_N|\}$, $x\in\mathbb{R}^N$, also set $\infty:=(\infty,\ldots,\infty)$, $-\infty:=(-\infty,\ldots,-\infty)$ upon the multivariate context, and

$$\lceil na\rceil:=(\lceil na_1\rceil,\ldots,\lceil na_N\rceil)$$
$$\lfloor nb\rfloor:=(\lfloor nb_1\rfloor,\ldots,\lfloor nb_N\rfloor)$$

In general $\|\cdot\|_\infty$ stands for the supremum norm.
We also have

(v)′

$$\sum_{\substack{k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_\infty>\frac{1}{n^\beta}}}^{\lfloor nb\rfloor}\Phi(nx-k)\le 3.1992e^{-n^{(1-\beta)}}$$

$0<\beta<1$, $n\in\mathbb{N}$, $x\in\left(\prod_{i=1}^N[a_i,b_i]\right)$

(vi)′

$$0<\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)}<(5.250312578)^N$$

$\forall\, x\in\left(\prod_{i=1}^N[a_i,b_i]\right),\ n\in\mathbb{N}$

(vii)′

$$\sum_{\substack{k=-\infty \\ \left\|\frac{k}{n}-x\right\|_\infty>\frac{1}{n^\beta}}}^{\infty}\Phi(nx-k)\le 3.1992e^{-n^{(1-\beta)}}$$

$0<\beta<1$, $n\in\mathbb{N}$, $x\in\mathbb{R}^N$

(viii)′

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\neq 1$$

for at least some $x\in\left(\prod_{i=1}^N[a_i,b_i]\right)$

Let $f\in C\left(\prod_{i=1}^N[a_i,b_i]\right)$ and $n\in\mathbb{N}$ such that $\lceil na_i\rceil\le\lfloor nb_i\rfloor$, $i=1,\ldots,N$.

We introduce and define the multivariate positive linear neural network operator $(x := (x_1, \ldots, x_N) \in (\prod_{i=1}^{N} [a_i, b_i]))$

$$G_n(f, x_1, \ldots, x_N) := G_n(f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \Phi(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx - k)} \tag{1.4}$$

$$:= \frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} \cdots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, \ldots, \frac{k_N}{n}\right) \left(\prod_{i=1}^{N} \Phi_i(nx_i - k_i)\right)}{\prod_{i=1}^{N} \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \Phi_i(nx_i - k_i)\right)}.$$

For large enough n we always obtain $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, \ldots, N$. Also $a_i \leq \frac{k_i}{n} \leq b_i$, iff $\lceil na_i \rceil \leq k_i \leq \lfloor nb_i \rfloor$, $i = 1, \ldots, N$.

We need, for $f \in C\left(\prod_{i=1}^{N} [a_i, b_i]\right)$, the first multivariate modulus of continuity

$$\omega_1(f, h) := \sup_{\substack{x, y \in \left(\prod_{i=1}^{N} [a_i, b_i]\right) \\ \|x - y\|_\infty \leq h}} |f(x) - f(y)|, \ \ h > 0. \tag{1.5}$$

Similarly it is defined for $f \in C_B(\mathbb{R}^N)$ (continuous and bounded functions on $\mathbb{R}^N$). We have that $\lim_{h \to 0} \omega_1(f, h) = 0$, when f is uniformly continuous.

When $f \in C_B(\mathbb{R}^N)$ we define

$$\overline{G}_n(f, x) := \overline{G}_n(f, x_1, \ldots, x_N) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) \Phi(nx - k) \tag{1.6}$$

$$:= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} f\left(\frac{k_1}{n}, \frac{k_2}{n}, \ldots, \frac{k_N}{n}\right) \left(\prod_{i=1}^{N} \Phi_i(nx_i - k_i)\right),$$

$n \in \mathbb{N}$, $\forall\, x \in \mathbb{R}^N$, $N \geq 1$, the multivariate quasi-interpolation neural network operator.

We mention from [7]:

Theorem 1.1. *Let* $f \in C\left(\prod_{i=1}^{N} [a_i, b_i]\right)$, $0 < \beta < 1$, $x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$, n, $N \in \mathbb{N}$. *Then*

i)

$$|G_n(f, x) - f(x)| \leq (5.250312578)^N$$

$$\left\{ \omega_1\left(f, \frac{1}{n^\beta}\right) + (6.3984) \|f\|_\infty e^{-n^{(1-\beta)}} \right\} =: \lambda_1 \tag{1.7}$$

ii)

$$\|G_n(f) - f\|_\infty \leq \lambda_1 \tag{1.8}$$

Theorem 1.2. *Let $f \in C_B\left(\mathbb{R}^N\right)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, n, $N \in \mathbb{N}$. Then*

i)

$$\left|\overline{G}_n(f,x) - f(x)\right| \leq \omega_1\left(f, \frac{1}{n^\beta}\right) + (6.3984)\|f\|_\infty e^{-n^{(1-\beta)}} =: \lambda_2 \tag{1.9}$$

ii)

$$\left\|\overline{G}_n(f) - f\right\|_\infty \leq \lambda_2 \tag{1.10}$$

(II) Here we follow [5, 6].

We also consider the hyperbolic tangent function $\tanh x$, $x \in \mathbb{R}$:

$$\tanh x := \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

It has the properties $\tanh 0 = 0$, $-1 < \tanh x < 1$, $\forall\, x \in \mathbb{R}$, and $\tanh(-x) = -\tanh x$. Furthermore $\tanh x \to 1$ as $x \to \infty$, and $\tanh x \to -1$, as $x \to -\infty$, and it is strictly increasing on $\mathbb{R}$.

This function plays the role of an activation function in the hidden layer of neural networks.

We further consider

$$\Psi(x) := \frac{1}{4}(\tanh(x+1) - \tanh(x-1)) > 0, \quad \forall\, x \in \mathbb{R}.$$

We easily see that $\Psi(-x) = \Psi(x)$, that is, Ψ is even on $\mathbb{R}$. Obviously Ψ is differentiable, thus continuous.

Proposition 1.3. *([5]) $\Psi(x)$ for $x \geq 0$ is strictly decreasing.*

Obviously $\Psi(x)$ is strictly increasing for $x \leq 0$. Also it holds $\lim_{x \to -\infty} \Psi(x) = 0 = \lim_{x \to \infty} \Psi(x)$.

Infact Ψ has the bell shape with horizontal asymptote the x-axis. So the maximum of Ψ is zero, $\Psi(0) = 0.3809297$.

Theorem 1.4. *([5]) We have that $\sum_{i=-\infty}^{\infty} \Psi(x-i) = 1$, $\forall\, x \in \mathbb{R}$.*

Thus

$$\sum_{i=-\infty}^{\infty} \Psi(nx-i) = 1, \quad \forall\, n \in \mathbb{N}, \forall\, x \in \mathbb{R}.$$

Also it holds

$$\sum_{i=-\infty}^{\infty} \Psi(x+i) = 1, \quad \forall x \in \mathbb{R}.$$

Theorem 1.5. *([5]) It holds $\int_{-\infty}^{\infty} \Psi(x)\, dx = 1$.*

So $\Psi(x)$ is a density function on $\mathbb{R}$.

Theorem 1.6. *([5]) Let $0 < \alpha < 1$ and $n \in \mathbb{N}$. It holds*

$$\sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}}^{\infty} \Psi(nx-k) \leq e^4 \cdot e^{-2n^{(1-\alpha)}}.$$

Theorem 1.7. *([5]) Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Psi(nx-k)} < \frac{1}{\Psi(1)} = 4.1488766.$$

Also by [5] we get that

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Psi(nx-k) \neq 1,$$

for at least some $x \in [a,b]$.

In this article we will use

$$\Theta(x_1, \ldots, x_N) := \Theta(x) := \prod_{i=1}^{N} \Psi(x_i), \quad x = (x_1, \ldots, x_N) \in \mathbb{R}^N, \; N \in \mathbb{N}. \tag{1.11}$$

It has the properties:

(i) $\Theta(x) > 0, \; \forall\, x \in \mathbb{R}^N$

(ii)

$$\sum_{k=-\infty}^{\infty} \Theta(x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \ldots \sum_{k_N=-\infty}^{\infty} \Theta(x_1-k_1, \ldots, x_N-k_N) = 1 \tag{1.12}$$

where $k := (k_1, \ldots, k_N)$, $\forall\, x \in \mathbb{R}^N$

(iii)

$$\sum_{k=-\infty}^{\infty} \Theta(nx-k) :=$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \ldots \sum_{k_N=-\infty}^{\infty} \Theta(nx_1-k_1, \ldots, nx_N-k_N) = 1 \tag{1.13}$$

$\forall\, x \in \mathbb{R}^N$; $n \in \mathbb{N}$.

(iv)

$$\int_{\mathbb{R}^N} \Theta(x)\, dx = 1$$

that is, Θ is a multivariate density function.

(v)

$$\sum_{\begin{cases} k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\lfloor nb\rfloor} \Theta(nx-k) \le e^4 \cdot e^{-2n^{(1-\beta)}}$$

$0<\beta<1$, $n\in\mathbb{N}$, $x\in\left(\prod_{i=1}^N [a_i,b_i]\right)$.

(vi)

$$0 < \frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Theta(nx-k)} < \frac{1}{(\Psi(1))^N} = (4.1488766)^N$$

$\forall\, x\in\left(\prod_{i=1}^N [a_i,b_i]\right)$, $n\in\mathbb{N}$

(vii)

$$\sum_{\begin{cases} k=-\infty \\ \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\infty} \Theta(nx-k) \le e^4 \cdot e^{-2n^{(1-\beta)}}$$

$0<\beta<1$, $n\in\mathbb{N}$, $x\in\mathbb{R}^N$

Also we get that

$$\lim_{n\to\infty} \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Theta(nx-k) \neq 1,$$

for at least some $x\in\left(\prod_{i=1}^N [a_i,b_i]\right)$.

Let $f\in C\left(\prod_{i=1}^N [a_i,b_i]\right)$ and $n\in\mathbb{N}$ such that $\lceil na_i\rceil \le \lfloor nb_i\rfloor$, $i=1,\dots,N$.

We introduce and define the multivariate positive linear neural network operator $\left(x:=(x_1,\dots,x_N)\in\left(\prod_{i=1}^N [a_i,b_i]\right)\right)$

$$F_n(f,x_1,\dots,x_N) := F_n(f,x) := \frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right)\Theta(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Theta(nx-k)} \tag{1.14}$$

$$:= \frac{\sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1\rfloor} \sum_{k_2=\lceil na_2\rceil}^{\lfloor nb_2\rfloor} \cdots \sum_{k_N=\lceil na_N\rceil}^{\lfloor nb_N\rfloor} f\left(\frac{k_1}{n},\dots,\frac{k_N}{n}\right)\left(\prod_{i=1}^N \Psi(nx_i-k_i)\right)}{\prod_{i=1}^N \left(\sum_{k_i=\lceil na_i\rceil}^{\lfloor nb_i\rfloor} \Psi(nx_i-k_i)\right)}.$$

When $f\in C_B\left(\mathbb{R}^N\right)$ we define

$$\overline{F}_n(f,x) := \overline{F}_n(f,x_1,\dots,x_N) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right)\Theta(nx-k) := \tag{1.15}$$

$$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\cdots\sum_{k_N=-\infty}^{\infty} f\left(\frac{k_1}{n},\frac{k_2}{n},\dots,\frac{k_N}{n}\right)\left(\prod_{i=1}^N \Psi(nx_i-k_i)\right),$$

$n\in\mathbb{N}$, $\forall\, x\in\mathbb{R}^N$, $N\ge 1$, the multivariate quasi-interpolation neural network operator.

We mention from [6]:

Theorem 1.8. *Let* $f \in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $0<\beta<1$, $x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$, *n*, $N \in \mathbb{N}$. *Then*

i)

$$|F_n(f,x) - f(x)| \leq (4.1488766)^N$$
$$\left\{\omega_1\left(f, \frac{1}{n^\beta}\right) + 2e^4 \|f\|_\infty e^{-2n^{(1-\beta)}}\right\} =: \lambda_1 \tag{1.16}$$

ii)

$$\|F_n(f) - f\|_\infty \leq \lambda_1 \tag{1.17}$$

Theorem 1.9. *Let* $f \in C_B\left(\mathbb{R}^N\right)$, $0<\beta<1$, $x \in \mathbb{R}^N$, *n*, $N \in \mathbb{N}$. *Then*

i)

$$\left|\overline{F}_n(f,x) - f(x)\right| \leq \omega_1\left(f, \frac{1}{n^\beta}\right) + 2e^4 \|f\|_\infty e^{-2n^{(1-\beta)}} =: \lambda_2 \tag{1.18}$$

ii)

$$\left\|\overline{F}_n(f) - f\right\|_\infty \leq \lambda_2 \tag{1.19}$$

Let $r \in \mathbb{N}$, in this article, we study the uniform convergence with rates to the unit operator I of the iterates G_n^r, $\overline{G}_n^r$, F_n^r, and $\overline{F}_n^r$.

1.3 Preparatory Results

We need

Theorem 1.10. *Let* $f \in C_B\left(\mathbb{R}^N\right)$, $N \geq 1$. *Then* $\overline{G}_n(f) \in C_B\left(\mathbb{R}^N\right)$.

Proof. We have that

$$\left|\overline{G}_n(f,x)\right| \leq \sum_{k=-\infty}^{\infty}\left|f\left(\frac{k}{n}\right)\right| \Phi(nx-k)$$

$$\leq \|f\|_\infty \left(\sum_{k=-\infty}^{\infty} \Phi(nx-k)\right) \overset{(1.3)}{=} \|f\|_\infty, \ \forall\, x \in \mathbb{R}^N.$$

So that $\overline{G}_n(f)$ is bounded.

Next we prove the continuity of $\overline{G}_n(f)$. We will use the Weierstrass M-test: If a sequence of positive constants $M_1, M_2, M_3, \ldots$ can be found such that in some interval

(a) $|u_n(x)| \leq M_n$, $n = 1,2,3,\ldots$
(b) $\sum M_n$ converges,

then $\sum u_n(x)$ is uniformly and absolutely convergent in the interval.

Also we will use:

If $\{u_n(x)\}$, $n = 1, 2, 3, \ldots$ are continuous in $[a,b]$, and if $\sum u_n(x)$ converges uniformly to the sum $S(x)$ in $[a,b]$, then $S(x)$ is continuous in $[a,b]$, that is, a uniformly convergent series of continuous functions is a continuous function. First we prove claim for $N = 1$.

We will prove that $\sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) \Phi(nx-k)$ is continuous in $x \in \mathbb{R}$.

There always exists $\lambda \in \mathbb{N}$ such that $nx \in [-\lambda, \lambda]$.

Since $nx \leq \lambda$, then $-nx \geq -\lambda$ and $k - nx \geq k - \lambda \geq 0$, when $k \geq \lambda$. Therefore

$$\sum_{k=\lambda}^{\infty} \Phi(nx-k) = \sum_{k=\lambda}^{\infty} \Phi(k-nx) \leq \sum_{k=\lambda}^{\infty} \Phi(k-\lambda) = \sum_{k'=0}^{\infty} \Phi\left(k'\right) \leq 1.$$

So for $k \geq \lambda$ we get

$$\left| f\left(\frac{k}{n}\right) \right| \Phi(nx-k) \leq \|f\|_{\infty} \Phi(k-\lambda)$$

and

$$\|f\|_{\infty} \sum_{k=\lambda}^{\infty} \Phi(k-\lambda) \leq \|f\|_{\infty}.$$

Hence by Weierstrass M-test we obtain that $\sum_{k=\lambda}^{\infty} f\left(\frac{k}{n}\right) \Phi(nx-k)$ is uniformly and absolutely convergent on $\left[-\frac{\lambda}{n}, \frac{\lambda}{n}\right]$.

Since $f\left(\frac{k}{n}\right) \Phi(nx-k)$ is continuous in x, then $\sum_{k=\lambda}^{\infty} f\left(\frac{k}{n}\right) \Phi(nx-k)$ is continuous on $\left[-\frac{\lambda}{n}, \frac{\lambda}{n}\right]$.

Because $nx \geq -\lambda$, then $-nx \leq \lambda$, and $k - nx \leq k + \lambda \leq 0$, when $k \leq -\lambda$. Therefore

$$\sum_{k=-\infty}^{-\lambda} \Phi(nx-k) = \sum_{k=-\infty}^{-\lambda} \Phi(k-nx) \leq \sum_{k=-\infty}^{-\lambda} \Phi(k+\lambda) = \sum_{k'=-\infty}^{0} \Phi\left(k'\right) \leq 1.$$

So for $k \leq -\lambda$ we get

$$\left| f\left(\frac{k}{n}\right) \right| \Phi(nx-k) \leq \|f\|_{\infty} \Phi(k+\lambda)$$

and

$$\|f\|_{\infty} \sum_{k=-\infty}^{-\lambda} \Phi(k+\lambda) \leq \|f\|_{\infty}.$$

Hence by Weierstrass M-test we obtain that $\sum_{k=-\infty}^{-\lambda} f\left(\frac{k}{n}\right) \Phi(nx-k)$ is uniformly and absolutely convergent on $\left[-\frac{\lambda}{n}, \frac{\lambda}{n}\right]$.

Since $f\left(\frac{k}{n}\right) \Phi(nx-k)$ is continuous in x, then $\sum_{k=-\infty}^{-\lambda} f\left(\frac{k}{n}\right) \Phi(nx-k)$ is continuous on $\left[-\frac{\lambda}{n}, \frac{\lambda}{n}\right]$.

So we proved that $\sum_{k=\lambda}^{\infty} f\left(\frac{k}{n}\right) \Phi(nx-k)$ and $\sum_{k=-\infty}^{-\lambda} f\left(\frac{k}{n}\right) \Phi(nx-k)$ are continuous on $\mathbb{R}$. Since $\sum_{k=-\lambda+1}^{\lambda-1} f\left(\frac{k}{n}\right) \Phi(nx-k)$ is a finite sum of continuous functions on $\mathbb{R}$, it is also a continuous function on $\mathbb{R}$.

Writing

$$\sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) \Phi(nx-k) = \sum_{k=-\infty}^{-\lambda} f\left(\frac{k}{n}\right) \Phi(nx-k) +$$

$$\sum_{k=-\lambda+1}^{\lambda-1} f\left(\frac{k}{n}\right) \Phi(nx-k) + \sum_{k=\lambda}^{\infty} f\left(\frac{k}{n}\right) \Phi(nx-k)$$

we have it as a continuous function on $\mathbb{R}$. Therefore $\overline{G}_n(f)$, when $N=1$, is a continuous function on $\mathbb{R}$.

When $N=2$ we have

$$\overline{G}_n(f,x_1,x_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} f\left(\frac{k_1}{n}, \frac{k_2}{n}\right) \Phi_1(nx_1-k_1) \Phi(nx_2-k_2) =$$

$$\sum_{k_1=-\infty}^{\infty} \Phi_1(nx_1-k_1) \left(\sum_{k_2=-\infty}^{\infty} f\left(\frac{k_1}{n}, \frac{k_2}{n}\right) \Phi_2(nx_2-k_2) \right)$$

(there always exist $\lambda_1, \lambda_2 \in \mathbb{N}$ such that $nx_1 \in [-\lambda_1, \lambda_1]$ and $nx_2 \in [-\lambda_2, \lambda_2]$)

$$= \sum_{k_1=-\infty}^{\infty} \Phi_1(nx_1-k_1) \left[\sum_{k_2=-\infty}^{-\lambda_2} f\left(\frac{k_1}{n}, \frac{k_2}{n}\right) \Phi_2(nx_2-k_2) + \right.$$

$$\left. \sum_{k_2=-\lambda_2+1}^{\lambda_2-1} f\left(\frac{k_1}{n}, \frac{k_2}{n}\right) \Phi_2(nx_2-k_2) + \sum_{k_2=\lambda_2}^{\infty} f\left(\frac{k_1}{n}, \frac{k_2}{n}\right) \Phi_2(nx_2-k_2) \right] =$$

$$= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{-\lambda_2} f\left(\frac{k_1}{n}, \frac{k_2}{n}\right) \Phi_1(nx_1-k_1) \Phi_2(nx_2-k_2) +$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\lambda_2+1}^{\lambda_2-1} f\left(\frac{k_1}{n}, \frac{k_2}{n}\right) \Phi_1(nx_1-k_1) \Phi_2(nx_2-k_2) +$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=\lambda_2}^{\infty} f\left(\frac{k_1}{n}, \frac{k_2}{n}\right) \Phi_1(nx_1-k_1) \Phi_2(nx_2-k_2).$$

(for convenience call

$$F(k_1,k_2,x_1,x_2) := f\left(\frac{k_1}{n}, \frac{k_2}{n}\right) \Phi_1(nx_1-k_1) \Phi_2(nx_2-k_2) \;)$$

$$= \sum_{k_1=-\infty}^{-\lambda_1} \sum_{k_2=-\infty}^{-\lambda_2} F(k_1,k_2,x_1,x_2) + \sum_{k_1=-\lambda_1+1}^{\lambda_1-1} \sum_{k_2=-\infty}^{-\lambda_2} F(k_1,k_2,x_1,x_2) +$$

$$\sum_{k_1=\lambda_1}^{\infty} \sum_{k_2=-\infty}^{-\lambda_2} F(k_1,k_2,x_1,x_2) + \sum_{k_1=-\infty}^{-\lambda_1} \sum_{k_2=-\lambda_2+1}^{\lambda_2-1} F(k_1,k_2,x_1,x_2) +$$

$$\sum_{k_1=-\lambda_1+1}^{\lambda_1-1} \sum_{k_2=-\lambda_2+1}^{\lambda_2-1} F(k_1,k_2,x_1,x_2) + \sum_{k_1=\lambda_1}^{\infty} \sum_{k_2=-\lambda_2+1}^{\lambda_2-1} F(k_1,k_2,x_1,x_2) +$$

$$\sum_{k_1=-\infty}^{-\lambda_1} \sum_{k_2=\lambda_2}^{\infty} F(k_1,k_2,x_1,x_2) + \sum_{k_1=-\lambda_1+1}^{\lambda_1-1} \sum_{k_2=\lambda_2}^{\infty} F(k_1,k_2,x_1,x_2) +$$

$$\sum_{k_1=\lambda_1}^{\infty} \sum_{k_2=\lambda_2}^{\infty} F(k_1,k_2,x_1,x_2).$$

Notice that the finite sum of continuous functions $F(k_1,k_2,x_1,x_2)$, $\sum_{k_1=-\lambda_1+1}^{\lambda_1-1} \sum_{k_2=-\lambda_2+1}^{\lambda_2-1} F(k_1,k_2,x_1,x_2)$ is a continuous function.

The rest of the summands of $\overline{G}_n(f,x_1,x_2)$ are treated all the same way and similarly to the case of $N=1$. The method is demonstrated as follows.

We will prove that $\sum_{k_1=\lambda_1}^{\infty} \sum_{k_2=-\infty}^{-\lambda_2} f\left(\frac{k_1}{n},\frac{k_2}{n}\right) \Phi_1(nx_1-k_1)\Phi_2(nx_2-k_2)$ is continuous in $(x_1,x_2) \in \mathbb{R}^2$.

The continuous function

$$\left| f\left(\frac{k_1}{n},\frac{k_2}{n}\right)\right| \Phi_1(nx_1-k_1)\Phi_2(nx_2-k_2) \le \|f\|_\infty \Phi_1(k_1-\lambda_1)\Phi_2(k_2+\lambda_2),$$

and

$$\|f\|_\infty \sum_{k_1=\lambda_1}^{\infty} \sum_{k_2=-\infty}^{-\lambda_2} \Phi_1(k_1-\lambda_1)\Phi_2(k_2+\lambda_2) =$$

$$\|f\|_\infty \left(\sum_{k_1=\lambda_1}^{\infty} \Phi_1(k_1-\lambda_1)\right)\left(\sum_{k_2=-\infty}^{-\lambda_2} \Phi_2(k_2+\lambda_2)\right) \le$$

$$\|f\|_\infty \left(\sum_{k_1'=0}^{\infty} \Phi_1\left(k_1'\right)\right)\left(\sum_{k_2'=-\infty}^{0} \Phi_2\left(k_2'\right)\right) \le \|f\|_\infty.$$

So by the Weierstrass M-test we get that $\sum_{k_1=\lambda_1}^{\infty} \sum_{k_2=-\infty}^{-\lambda_2} f\left(\frac{k_1}{n},\frac{k_2}{n}\right) \Phi_1(nx_1-k_1)\Phi_2(nx_2-k_2)$ is uniformly and absolutely convergent. Therefore it is continuous on $\mathbb{R}^2$.

Next we prove continuity on $\mathbb{R}^2$ of $\sum_{k_1=-\lambda_1+1}^{\lambda_1-1} \sum_{k_2=-\infty}^{-\lambda_2} f\left(\frac{k_1}{n},\frac{k_2}{n}\right) \Phi_1(nx_1-k_1)\Phi_2(nx_2-k_2)$.

Notice here that

$$\left|f\left(\frac{k_1}{n},\frac{k_2}{n}\right)\right| \Phi_1(nx_1-k_1)\,\Phi_2(nx_2-k_2) \le \|f\|_\infty \Phi_1(nx_1-k_1)\,\Phi_2(k_2+\lambda_2)$$

$$\le \|f\|_\infty \Phi_1(0)\,\Phi_2(k_2+\lambda_2) = (0.231)\,\|f\|_\infty \Phi_2(k_2+\lambda_2),$$

and

$$(0.231)\,\|f\|_\infty \left(\sum_{k_1=-\lambda_1+1}^{\lambda_1-1} 1\right)\left(\sum_{k_2=-\infty}^{-\lambda_2} \Phi_2(k_2+\lambda_2)\right) =$$

$$(0.231)\,\|f\|_\infty (2\lambda_1-1)\left(\sum_{k_2'=-\infty}^{0} \Phi_2\left(k_2'\right)\right) \le (0.231)\,(2\lambda_1-1)\,\|f\|_\infty .$$

So the double series under consideration is uniformly convergent and continuous. Clearly $\overline{G}_n(f,x_1,x_2)$ is proved to be continuous on $\mathbb{R}^2$.

Similarly reasoning one can prove easily now, but with more tedious work, that $\overline{G}_n(f,x_1,\ldots,x_N)$ is continuous on $\mathbb{R}^N$, for any $N \ge 1$. We choose to omit this similar extra work. □

Theorem 1.11. *Let* $f \in C_B\left(\mathbb{R}^N\right)$, $N \ge 1$. *Then* $\overline{F}_n(f) \in C_B\left(\mathbb{R}^N\right)$.

Proof. We notice that

$$\left|\overline{F}_n(f,x)\right| \le \sum_{k=-\infty}^{\infty} \left|f\left(\frac{k}{n}\right)\right| \theta(nx-k) \le \|f\|_\infty \left(\sum_{k=-\infty}^{\infty} \theta(nx-k)\right) \overset{(1.13)}{=} \|f\|_\infty ,$$

$\forall\, x \in \mathbb{R}^N$, so that $\overline{F}_n(f)$ is bounded. The continuity is proved as in Theorem 1.10. □

Theorem 1.12. *Let* $f \in C\left(\prod_{i=1}^N [a_i,b_i]\right)$, *then* $\|G_n(f)\|_\infty \le \|f\|_\infty$ *and* $\|F_n(f)\|_\infty \le \|f\|_\infty$, *also* $G_n(f)$, $F_n(f) \in C\left(\prod_{i=1}^N [a_i,b_i]\right)$.

Proof. By (1.4) we get

$$|G_n(f,x)| = \frac{\left|\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right)\Phi(nx-k)\right|}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Phi(nx-k)} \le$$

$$\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \left|f\left(\frac{k}{n}\right)\right| \Phi(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Phi(nx-k)} \le \|f\|_\infty,\ \forall\, x \in \mathbb{R}^N,$$

so that $G_n(f)$ is bounded.

Similarly we act for the boundedness of F_n; see (1.14). Continuity of both is obvious. □

We make

Remark 1.13. Notice that

$$\left\|G_n^2(f)\right\|_\infty = \|G_n(G_n(f))\|_\infty \leq \|G_n(f)\|_\infty \leq \|f\|_\infty, \text{ etc.}$$

Therefore we get

$$\left\|G_n^k(f)\right\|_\infty \leq \|f\|_\infty, \ \forall\, k \in \mathbb{N}, \tag{1.20}$$

the contraction property.

Similarly we obtain

$$\left\|F_n^k(f)\right\|_\infty \leq \|f\|_\infty, \ \forall\, k \in \mathbb{N}, \tag{1.21}$$

Similarly by Theorems 1.10 and 1.11 we obtain

$$\left\|\overline{G}_n^k(f)\right\|_\infty \leq \|f\|_\infty, \tag{1.22}$$

and

$$\left\|\overline{F}_n^k(f)\right\|_\infty \leq \|f\|_\infty, \ \forall\, k \in \mathbb{N}, \tag{1.23}$$

Infact here we have

$$\left\|G_n^k(f)\right\|_\infty \leq \left\|G_n^{k-1}(f)\right\|_\infty \leq \ldots \leq \|G_n(f)\|_\infty \leq \|f\|_\infty, \tag{1.24}$$

$$\left\|F_n^k(f)\right\|_\infty \leq \left\|F_n^{k-1}(f)\right\|_\infty \leq \ldots \leq \|F_n(f)\|_\infty \leq \|f\|_\infty, \tag{1.25}$$

$$\left\|\overline{G}_n^k(f)\right\|_\infty \leq \left\|\overline{G}_n^{k-1}(f)\right\|_\infty \leq \ldots \leq \left\|\overline{G}_n(f)\right\|_\infty \leq \|f\|_\infty, \tag{1.26}$$

and

$$\left\|\overline{F}_n^k(f)\right\|_\infty \leq \left\|\overline{F}_n^{k-1}(f)\right\|_\infty \leq \ldots \leq \left\|\overline{F}_n(f)\right\|_\infty \leq \|f\|_\infty. \tag{1.27}$$

We need

Notation 1.14. Call $L_n = G_n, \overline{G}_n, F_n, \overline{F}_n$. Denote by

$$c_N = \begin{cases} (5.250312578)^N, \ \text{if } L_n = G_n, \\ (4.1488766)^N, \ \text{if } L_n = F_n, \\ 1, \ \text{if } L_n = \overline{G}_n, \overline{F}_n, \end{cases} \tag{1.28}$$

$$\mu = \begin{cases} 6.3984, \ \text{if } L_n = G_n, \overline{G}_n, \\ 2e^4, \ \text{if } L_n = F_n, \overline{F}_n, \end{cases} \tag{1.29}$$

and

$$\gamma = \begin{cases} 1, & \text{when } L_n = G_n, \overline{G}_n, \\ 2 & \text{when } L_n = F_n, \overline{F}_n. \end{cases} \tag{1.30}$$

Based on the above notations Theorems 1.1, 1.2, 1.8, and 1.9 can be put in a unified way as follows:

Theorem 1.15. *Let* $f \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$ *or* $f \in C_B\left(\mathbb{R}^N\right)$*;* $n, N \in \mathbb{N}$, $0 < \beta < 1$, $x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$ *or* $x \in \mathbb{R}^N$*. Then*

(i)

$$|L_n(f, x) - f(x)| \leq c_N \left\{ \omega_1\left(f, \frac{1}{n^\beta}\right) + \mu \|f\|_\infty e^{-\gamma n^{(1-\beta)}} \right\} =: \rho_n \tag{1.31}$$

(ii)

$$\|L_n(f) - f\|_\infty \leq \rho_n \tag{1.32}$$

Remark 1.16. We have

$$\left\| L_n^k f \right\|_\infty \leq \|f\|_\infty, \ \forall\, k \in \mathbb{N}, \tag{1.33}$$

the contraction property.

Also it holds

$$L_n 1 = 1, \tag{1.34}$$

and

$$L_n^k 1 = 1, \ \forall\, k \in \mathbb{N}. \tag{1.35}$$

Here L_n^k are positive linear operators.

1.4 Main Results

We present

Theorem 1.17. *Let* $f \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$ *or* $f \in C_B\left(\mathbb{R}^N\right)$*;* $r, n, N \in \mathbb{N}$, $0 < \beta < 1$, $x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$ *or* $x \in \mathbb{R}^N$*. Then*

$$|L_n^r(f, x) - f(x)| \leq \|L_n^r f - f\|_\infty \leq r \|L_n f - f\|_\infty$$

$$\leq r c_N \left\{ \omega_1\left(f, \frac{1}{n^\beta}\right) + \mu \|f\|_\infty e^{-\gamma n^{(1-\beta)}} \right\}. \tag{1.36}$$

Proof. We observe that

$$L_n^r f - f = \left(L_n^r f - L_n^{r-1} f\right) + \left(L_n^{r-1} f - L_n^{r-2} f\right) +$$

$$\left(L_n^{r-2} f - L_n^{r-3} f\right) + \ldots + \left(L_n^2 f - L_n f\right) + \left(L_n f - f\right).$$

Then

$$\left\|L_n^r f - f\right\|_\infty \leq \left\|L_n^r f - L_n^{r-1} f\right\|_\infty + \left\|L_n^{r-1} f - L_n^{r-2} f\right\|_\infty +$$

$$\left\|L_n^{r-2} f - L_n^{r-3} f\right\|_\infty + \ldots + \left\|L_n^2 f - L_n f\right\|_\infty + \left\|L_n f - f\right\|_\infty =$$

$$\left\|L_n^{r-1}\left(L_n f - f\right)\right\|_\infty + \left\|L_n^{r-2}\left(L_n f - f\right)\right\|_\infty + \left\|L_n^{r-3}\left(L_n f - f\right)\right\|_\infty$$

$$+ \ldots + \left\|L_n\left(L_n f - f\right)\right\|_\infty + \left\|L_n f - f\right\|_\infty \overset{(1.33)}{\leq}$$

$$r\left\|L_n f - f\right\|_\infty \overset{(1.32)}{\leq} r\rho_n,$$

proving the claim. □

More generally we have

Theorem 1.18. *Let* $f \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$ *or* $f \in C_B\left(\mathbb{R}^N\right)$*;* $n, N, m_1, \ldots, m_r \in \mathbb{N} : m_1 \leq m_2 \leq \ldots \leq m_r$*,* $0 < \beta < 1$*,* $x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$ *or* $x \in \mathbb{R}^N$*. Then*

$$\left|L_{m_r}\left(L_{m_{r-1}}\left(\ldots L_{m_2}\left(L_{m_1} f\right)\right)\right)(x) - f(x)\right| \leq \tag{1.37}$$

$$\left\|L_{m_r}\left(L_{m_{r-1}}\left(\ldots L_{m_2}\left(L_{m_1} f\right)\right)\right) - f\right\|_\infty \leq \sum_{i=1}^r \left\|L_{m_i} f - f\right\|_\infty \leq$$

$$c_N \sum_{i=1}^r \left\{\omega_1\left(f, \frac{1}{m_i^\beta}\right) + \mu\left\|f\right\|_\infty e^{-\gamma m_i^{(1-\beta)}}\right\} \leq$$

$$r c_N \left\{\omega_1\left(f, \frac{1}{m_1^\beta}\right) + \mu\left\|f\right\|_\infty e^{-\gamma m_1^{(1-\beta)}}\right\}.$$

Clearly, we notice that the speed of convergence of the multiply iterated operator equals to the speed of L_{m_1}*.*

Proof. We write

$$L_{m_r}\left(L_{m_{r-1}}\left(\ldots L_{m_2}\left(L_{m_1} f\right)\right)\right) - f =$$

$$L_{m_r}\left(L_{m_{r-1}}\left(\ldots L_{m_2}\left(L_{m_1} f\right)\right)\right) - L_{m_r}\left(L_{m_{r-1}}\left(\ldots L_{m_2} f\right)\right) +$$

$$L_{m_r}\left(L_{m_{r-1}}\left(\ldots L_{m_2} f\right)\right) - L_{m_r}\left(L_{m_{r-1}}\left(\ldots L_{m_3} f\right)\right) +$$

$$L_{m_r}\left(L_{m_{r-1}}\left(\ldots L_{m_3} f\right)\right) - L_{m_r}\left(L_{m_{r-1}}\left(\ldots L_{m_4} f\right)\right) + \ldots +$$

$$L_{m_r}\left(L_{m_{r-1}} f\right) - L_{m_r} f + L_{m_r} f - f =$$

$$L_{m_r}\left(L_{m_{r-1}}\left(\ldots L_{m_2}\right)\right)\left(L_{m_1} f - f\right) + L_{m_r}\left(L_{m_{r-1}}\left(\ldots L_{m_3}\right)\right)\left(L_{m_2} f - f\right) +$$

$$L_{m_r}\left(L_{m_{r-1}}\left(\ldots L_{m_4}\right)\right)\left(L_{m_3} f - f\right) + \ldots + L_{m_r}\left(L_{m_{r-1}} f - f\right) + L_{m_r} f - f.$$

Hence by the triangle inequality property of $\|\cdot\|_\infty$ we get

$$\left\|L_{m_r}\left(L_{m_{r-1}}\left(\ldots L_{m_2}\left(L_{m_1} f\right)\right)\right) - f\right\|_\infty \leq$$

$$\left\|L_{m_r}\left(L_{m_{r-1}}\left(\ldots L_{m_2}\right)\right)\left(L_{m_1}f-f\right)\right\|_\infty+\left\|L_{m_r}\left(L_{m_{r-1}}\left(\ldots L_{m_3}\right)\right)\left(L_{m_2}f-f\right)\right\|_\infty+$$
$$\left\|L_{m_r}\left(L_{m_{r-1}}\left(\ldots L_{m_4}\right)\right)\left(L_{m_3}f-f\right)\right\|_\infty+\ldots+$$
$$\left\|L_{m_r}\left(L_{m_{r-1}}f-f\right)\right\|_\infty+\left\|L_{m_r}f-f\right\|_\infty$$

(repeatedly applying (1.33))

$$\leq\left\|L_{m_1}f-f\right\|_\infty+\left\|L_{m_2}f-f\right\|_\infty+\left\|L_{m_3}f-f\right\|_\infty+\ldots+$$

$$\left\|L_{m_{r-1}}f-f\right\|_\infty+\left\|L_{m_r}f-f\right\|_\infty=\sum_{i=1}^{r}\left\|L_{m_i}f-f\right\|_\infty\overset{(1.32)}{\leq}$$

$$c_N\sum_{i=1}^{r}\left\{\omega_1\left(f,\frac{1}{m_i^\beta}\right)+\mu\left\|f\right\|_\infty e^{-\gamma m_i^{(1-\beta)}}\right\}=:(*).$$

We have

$$\frac{1}{m_r}\leq\frac{1}{m_{r-1}}\leq\ldots\leq\frac{1}{m_2}\leq\frac{1}{m_1},$$

and

$$\frac{1}{m_r^\beta}\leq\frac{1}{m_{r-1}^\beta}\leq\ldots\leq\frac{1}{m_2^\beta}\leq\frac{1}{m_1^\beta}.$$

Therefore

$$\omega_1\left(f,\frac{1}{m_r^\beta}\right)\leq\omega_1\left(f,\frac{1}{m_{r-1}^\beta}\right)\leq\ldots\leq\omega_1\left(f,\frac{1}{m_2^\beta}\right)\leq\omega_1\left(f,\frac{1}{m_1^\beta}\right).$$

Also it holds

$$\gamma m_1^{(1-\beta)}\leq\gamma m_2^{(1-\beta)}\leq\ldots\leq\gamma m_r^{(1-\beta)}$$

and

$$e^{\gamma m_1^{(1-\beta)}}\leq e^{\gamma m_2^{(1-\beta)}}\leq\ldots\leq e^{\gamma m_r^{(1-\beta)}},$$

so that

$$e^{-\gamma m_r^{(1-\beta)}}\leq e^{-\gamma m_{r-1}^{(1-\beta)}}\leq\ldots\leq e^{-\gamma m_2^{(1-\beta)}}\leq e^{-\gamma m_1(1-\beta)}.$$

Therefore

$$(*)\leq rc_N\left\{\omega_1\left(f,\frac{1}{m_1^\beta}\right)+\mu\left\|f\right\|_\infty e^{-\gamma m_1^{(1-\beta)}}\right\},$$

proving the claim. □

Next we give a partial global smoothness preservation result of operators L_n^r.

Theorem 1.19. *Same assumptions as in Theorem 1.17,* $\delta>0$*. Then*

$$\omega_1\left(L_n^r f,\delta\right)\leq 2rc_N\left\{\omega_1\left(f,\frac{1}{n^\beta}\right)+\mu\left\|f\right\|_\infty e^{-\gamma n^{(1-\beta)}}\right\}+\omega_1\left(f,\delta\right). \tag{1.38}$$

In particular for $\delta = \frac{1}{n^{\beta}}$, *we obtain*

$$\omega_1 \left(L_n^r f, \frac{1}{n^{\beta}} \right) \leq (2rc_N + 1) \omega_1 \left(f, \frac{1}{n^{\beta}} \right) + 2rc_N \mu \|f\|_{\infty} e^{-\gamma n^{(1-\beta)}}. \tag{1.39}$$

Proof. We write

$$L_n^r f(x) - L_n^r f(y) = L_n^r f(x) - L_n^r f(y) + f(x) - f(x) + f(y) - f(y) =$$

$$(L_n^r f(x) - f(x)) + (f(y) - L_n^r f(y)) + (f(x) - f(y)).$$

Hence

$$|L_n^r f(x) - L_n^r f(y)| \leq |L_n^r f(x) - f(x)| + |L_n^r f(y) - f(y)| + |f(x) - f(y)|$$

$$\leq 2 \|L_n^r f - f\|_{\infty} + |f(x) - f(y)|.$$

Let $x, y \in \left(\prod_{i=1}^{N} [a_i, b_i] \right)$ or $x, y \in \mathbb{R}^N : |x - y| \leq \delta$, $\delta > 0$. Then

$$\omega_1 (L_n^r f, \delta) \leq 2 \|L_n^r f - f\|_{\infty} + \omega_1 (f, \delta).$$

That is

$$\omega_1 (L_n^r f, \delta) \overset{(1.36)}{\leq} 2r \|L_n f - f\|_{\infty} + \omega_1 (f, \delta),$$

proving the claim. □

Notation 1.20. Let $f \in C^m \left(\prod_{i=1}^{N} [a_i, b_i] \right)$, $m, N \in \mathbb{N}$. Here f_{α} denotes a partial derivative of f, $\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, N$, and $|\alpha| := \sum_{i=1}^{N} \alpha_i = l$, where $l = 0, 1, \ldots, m$. We write also $f_{\alpha} := \frac{\partial^{\alpha} f}{\partial x^{\alpha}}$ and we say it is of order l. We denote

$$\omega_{1,m}^{\max} (f_{\alpha}, h) := \max_{\alpha : |\alpha| = m} \omega_1 (f_{\alpha}, h). \tag{1.40}$$

Call also

$$\|f_{\alpha}\|_{\infty,m}^{\max} := \max_{|\alpha| = m} \{\|f_{\alpha}\|_{\infty}\}. \tag{1.41}$$

We discuss higher-order approximation next.
We mention from [7] the following result.

Theorem 1.21. *Let* $f \in C^m \left(\prod_{i=1}^{N} [a_i, b_i] \right)$, $0 < \beta < 1$, $n, m, N \in \mathbb{N}$. *Then*

$$\|G_n(f) - f\|_{\infty} \leq (5.250312578)^N \cdot \tag{1.42}$$

$$\left\{ \sum_{j=1}^{N} \left(\sum_{|\alpha| = j} \left(\frac{\|f_{\alpha}\|_{\infty}}{\prod_{i=1}^{N} \alpha_i!} \right) \left[\frac{1}{n^{\beta j}} + \left(\prod_{i=1}^{N} (b_i - a_i)^{\alpha_i} \right) (3.1992) e^{-n^{(1-\beta)}} \right] \right) + \right.$$

$$\left. \frac{N^m}{m! n^{m\beta}} \omega_{1,m}^{\max} \left(f_{\alpha}, \frac{1}{n^{\beta}} \right) + \left(\frac{(6.3984) \|b - a\|_{\infty}^m \|f_{\alpha}\|_{\infty,m}^{\max} N^m}{m!} \right) e^{-n^{(1-\beta)}} \right\} =: M_n.$$

Using Theorem 1.17 we derive

Theorem 1.22. *Let* $f \in C^m\left(\prod_{i=1}^N [a_i,b_i]\right)$, $0<\beta<1$, $r,n,m,N\in\mathbb{N}$, $x\in\left(\prod_{i=1}^N [a_i,b_i]\right)$. *Then*

$$\left|G_n^r(f,x)-f(x)\right| \le \left\|G_n^r(f)-f\right\|_\infty \le r\left\|G_n(f)-f\right\|_\infty \le rM_n. \tag{1.43}$$

One can have a similar result for the operator F_n but we omit it.

Next we specialize on Lipschitz classes of functions. We apply Theorem 1.18 to obtain

Theorem 1.23. *Let* $f\in C\left(\prod_{i=1}^N [a_i,b_i]\right)$ *or* $f\in C_B\left(\mathbb{R}^N\right)$; n, N, $m_1,\ldots,m_r\in\mathbb{N}: m_1\le m_2\le\ldots\le m_r$, $0<\beta<1$. *We further assume that* $|f(x)-f(y)|\le M\|x-y\|_\infty^\alpha$, $\forall$ $x,y\in\left(\prod_{i=1}^N [a_i,b_i]\right)$ *or* $x,y\in\mathbb{R}^N$ *(respectively),* $0<\alpha\le 1$, $M>0$. *Then*

$$\left\|L_{m_r}\left(L_{m_{r-1}}\left(\ldots L_{m_2}\left(L_{m_1}f\right)\right)\right)-f\right\|_\infty \le \tag{1.44}$$

$$\sum_{i=1}^r \left\|L_{m_i}f-f\right\|_\infty \le$$

$$c_N\sum_{i=1}^r\left\{\frac{M}{m_i^{\alpha\beta}}+\mu\|f\|_\infty e^{-\gamma m_i^{(1-\beta)}}\right\}.$$

Example 1.24. Let $f(x_1,\ldots,x_N)=\sum_{i=1}^N\cos x_i$, $(x_1,\ldots,x_N)\in\mathbb{R}^N$, $N\in\mathbb{N}$. Denote $\overline{x}=(x_1,\ldots,x_N)$, $\overline{y}=(y_1,\ldots,y_N)$ and observe that

$$\left|\sum_{i=1}^N\cos x_i-\sum_{i=1}^N\cos y_i\right|\le\sum_{i=1}^N\left|\cos x_i-\cos y_i\right|$$

$$\le\sum_{i=1}^N|x_i-y_i|\le N\|\overline{x}-\overline{y}\|_\infty.$$

That is

$$|f(\overline{x})-f(\overline{y})|\le N\|\overline{x}-\overline{y}\|_\infty.$$

Consequently by (1.5) we get that

$$\omega_1(f,h)\le Nh,\ \ h>0.$$

Therefore by (1.9) we derive

$$\left\|\overline{G}_n\left(\sum_{i=1}^N\cos x_i\right)-\left(\sum_{i=1}^N\cos x_i\right)\right\|_\infty\le N\left(\frac{1}{n^\beta}+(6.3984)\,e^{-n^{(1-\beta)}}\right), \tag{1.45}$$

where $0<\beta<1$ and $n\in\mathbb{N}$.

Let now $m_1, \ldots, m_r \in \mathbb{N} : m_1 \leq m_2 \leq \ldots \leq m_r$. Then by (1.44) we get

$$\left\| \overline{G}_{m_r}\left(\overline{G}_{m_{r-1}}\left(\ldots\left(\overline{G}_{m_2}\left(\overline{G}_{m_1}\left(\sum_{i=1}^{N}\cos x_i\right)\right)\right)\right)\right) - \left(\sum_{i=1}^{N}\cos x_i\right)\right\|_\infty \leq \quad (1.46)$$

$$\sum_{i=1}^{r}\left\|\overline{G}_{m_i}\left(\sum_{i=1}^{N}\cos x_i\right) - \left(\sum_{i=1}^{N}\cos x_i\right)\right\|_\infty \overset{\text{(by (1.45))}}{\leq}$$

$$N\sum_{i=1}^{r}\left(\frac{1}{m_i^{\beta}} + (6.3984)\, e^{-m_i^{(1-\beta)}}\right) \leq rN\left(\frac{1}{m_1^{\beta}} + (6.3984)\, e^{-m_1^{(1-\beta)}}\right). \quad (1.47)$$

One can give easily many other interesting applications.

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Chapter 2
Univariate Hardy-Type Fractional Inequalities

George A. Anastassiou

Abstract Here we present integral inequalities for convex and increasing functions applied to products of functions. As applications we derive a wide range of fractional inequalities of Hardy type. They involve the left and right Riemann-Liouville fractional integrals and their generalizations, in particular the Hadamard fractional integrals. Also inequalities for left and right Riemann-Liouville, Caputo, Canavati and their generalizations fractional derivatives. These application inequalities are of L_p type, $p \geq 1$, and exponential type, as well as their mixture.

2.1 Introduction

We start with some facts about fractional derivatives needed in the sequel; for more details, see, for instance, [1, 9].

Let $a < b$, $a,b \in \mathbb{R}$. By $C^N([a,b])$, we denote the space of all functions on $[a,b]$ which have continuous derivatives up to order N, and $AC([a,b])$ is the space of all absolutely continuous functions on $[a,b]$. By $AC^N([a,b])$, we denote the space of all functions g with $g^{(N-1)} \in AC([a,b])$. For any $\alpha \in \mathbb{R}$, we denote by $[\alpha]$ the integral part of α (the integer k satisfying $k \leq \alpha < k+1$), and $\lceil \alpha \rceil$ is the ceiling of α ($\min\{n \in \mathbb{N}, n \geq \alpha\}$). By $L_1(a,b)$, we denote the space of all functions integrable on the interval (a,b), and by $L_\infty(a,b)$ the set of all functions measurable and essentially bounded on (a,b). Clearly, $L_\infty(a,b) \subset L_1(a,b)$.

We start with the definition of the Riemann–Liouville fractional integrals; see [12]. Let $[a,b]$, $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$. The Riemann–Liouville fractional integrals $I_{a+}^{\alpha} f$ and $I_{b-}^{\alpha} f$ of order $\alpha > 0$ are defined by

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$$\left(I_{a+}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}f(t)(x-t)^{\alpha-1}dt,\quad (x>a), \tag{2.1}$$

$$\left(I_{b-}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}f(t)(t-x)^{\alpha-1}dt,\quad (x<b), \tag{2.2}$$

respectively. Here $\Gamma(\alpha)$ is the Gamma function. These integrals are called the left-sided and the right-sided fractional integrals. We mention some properties of the operators $I_{a+}^{\alpha}f$ and $I_{b-}^{\alpha}f$ of order $\alpha>0$; see also [13]. The first result yields that the fractional integral operators $I_{a+}^{\alpha}f$ and $I_{b-}^{\alpha}f$ are bounded in $L_p(a,b)$, $1\le p\le\infty$, that is,

$$\left\|I_{a+}^{\alpha}f\right\|_{p}\le K\|f\|_{p},\quad \left\|I_{b-}^{\alpha}f\right\|_{p}\le K\|f\|_{p}, \tag{2.3}$$

where

$$K=\frac{(b-a)^{\alpha}}{\alpha\Gamma(\alpha)}. \tag{2.4}$$

Inequality (2.3), which is the result involving the left-sided fractional integral, was proved by H.G. Hardy in one of his first papers; see [10]. He did not write down the constant, but the calculation of the constant was hidden inside his proof.

Next we follow [11].

Let $(\Omega_1,\Sigma_1,\mu_1)$ and $(\Omega_2,\Sigma_2,\mu_2)$ be measure spaces with positive σ-finite measures, and let $k:\Omega_1\times\Omega_2\to\mathbb{R}$ be a nonnegative measurable function, $k(x,\cdot)$ measurable on Ω_2 and

$$K(x)=\int_{\Omega_2}k(x,y)\,d\mu_2(y),\quad x\in\Omega_1. \tag{2.5}$$

We suppose that $K(x)>0$ a.e. on Ω_1, and by a weight function (shortly: a weight), we mean a nonnegative measurable function on the actual set. Let the measurable functions $g:\Omega_1\to\mathbb{R}$ with the representation

$$g(x)=\int_{\Omega_2}k(x,y)f(y)\,d\mu_2(y), \tag{2.6}$$

where $f:\Omega_2\to\mathbb{R}$ is a measurable function.

Theorem 2.1 ([11]). *Let u be a weight function on Ω_1, k a nonnegative measurable function on $\Omega_1\times\Omega_2$, and K be defined on Ω_1 by (2.5). Assume that the function $x\mapsto u(x)\frac{k(x,y)}{K(x)}$ is integrable on Ω_1 for each fixed $y\in\Omega_2$. Define ν on Ω_2 by*

$$\nu(y):=\int_{\Omega_1}u(x)\frac{k(x,y)}{K(x)}d\mu_1(x)<\infty. \tag{2.7}$$

If $\Phi:[0,\infty)\to\mathbb{R}$ is convex and increasing function, then the inequality

$$\int_{\Omega_1}u(x)\,\Phi\left(\left|\frac{g(x)}{K(x)}\right|\right)d\mu_1(x)\le\int_{\Omega_2}\nu(y)\,\Phi(|f(y)|)\,d\mu_2(y) \tag{2.8}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$ *such that:*

(i) $f, \Phi(|f|)$ *are both* $k(x,y)\,d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$.
(ii) $\nu(y)\,\Phi(|f|)$ *is* μ_2*-integrable, and for all corresponding functions g given by (2.6).*

Important assumptions (i) and (ii) are missing from Theorem 2.1 of [11].

In this article we generalize Theorem 2.1 for products of several functions and we give wide applications to fractional calculus.

2.2 Main Results

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive σ-finite measures, and let $k_i : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be nonnegative measurable functions, $k_i(x,\cdot)$ measurable on Ω_2, and

$$K_i(x) = \int_{\Omega_2} k_i(x,y)\,d\mu_2(y), \quad \text{for any } x \in \Omega_1, \tag{2.9}$$

$i = 1,\dots,m$. We assume that $K_i(x) > 0$ a.e. on Ω_1 and the weight functions are nonnegative measurable functions on the related set.

We consider measurable functions $g_i : \Omega_1 \to \mathbb{R}$ with the representation

$$g_i(x) = \int_{\Omega_2} k_i(x,y)\,f_i(y)\,d\mu_2(y), \tag{2.10}$$

where $f_i : \Omega_2 \to \mathbb{R}$ are measurable functions, $i = 1,\dots,m$.

Here u stands for a weight function on Ω_1.

The first introductory result is proved for $m = 2$.

Theorem 2.2. *Assume that the function* $x \mapsto \left(\frac{u(x)k_1(x,y)k_2(x,y)}{K_1(x)K_2(x)}\right)$ *is integrable on* Ω_1*, for each* $y \in \Omega_2$*. Define* λ_2 *on* Ω_2 *by*

$$\lambda_2(y) := \int_{\Omega_1} \frac{u(x)\,k_1(x,y)\,k_2(x,y)}{K_1(x)\,K_2(x)}\,d\mu_1(x) < \infty. \tag{2.11}$$

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$*,* $i = 1,2$*, are convex and increasing functions.*

Then

$$\int_{\Omega_1} u(x)\,\Phi_1\left(\left|\frac{g_1(x)}{K_1(x)}\right|\right)\Phi_2\left(\left|\frac{g_2(x)}{K_2(x)}\right|\right)d\mu_1(x) \le$$
$$\left(\int_{\Omega_2} \Phi_2(|f_2(y)|)\,d\mu_2(y)\right)\left(\int_{\Omega_2} \Phi_1(|f_1(y)|)\,\lambda_2(y)\,d\mu_2(y)\right), \tag{2.12}$$

true for all measurable functions, $i = 1,2$*,* $f_i : \Omega_2 \to \mathbb{R}$ *such that:*

(i) f_i*,* $\Phi_i(|f_i|)$*, are both* $k_i(x,y)\,d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$.
(ii) $\lambda_2\Phi_1(|f_1|)$*,* $\Phi_2(|f_2|)$*, are both* μ_2 *-integrable,*

and for all corresponding functions g_i *given by (2.10).*

Proof. Notice here that Φ_1, Φ_2 are continuous functions. Here we use Jensen's inequality and Fubini's theorem and that Φ_i are increasing. We have

$$\int_{\Omega_1} u(x)\,\Phi_1\left(\left|\frac{g_1(x)}{K_1(x)}\right|\right)\Phi_2\left(\left|\frac{g_2(x)}{K_2(x)}\right|\right)d\mu_1(x) =$$

$$\int_{\Omega_1} u(x)\,\Phi_1\left(\left|\frac{1}{K_1(x)}\int_{\Omega_2} k_1(x,y)\,f_1(y)\,d\mu_2(y)\right|\right)\cdot \tag{2.13}$$

$$\Phi_2\left(\left|\frac{1}{K_2(x)}\int_{\Omega_2} k_2(x,y)\,f_2(y)\,d\mu_2(y)\right|\right)d\mu_1(x) \leq$$

$$\int_{\Omega_1} u(x)\,\Phi_1\left(\frac{1}{K_1(x)}\int_{\Omega_2} k_1(x,y)\,|f_1(y)|\,d\mu_2(y)\right)\cdot$$

$$\Phi_2\left(\frac{1}{K_2(x)}\int_{\Omega_2} k_2(x,y)\,|f_2(y)|\,d\mu_2(y)\right)d\mu_1(x) \leq$$

$$\int_{\Omega_1} u(x)\,\frac{1}{K_1(x)}\left(\int_{\Omega_2} k_1(x,y)\,\Phi_1(|f_1(y)|)\,d\mu_2(y)\right)\cdot$$

$$\frac{1}{K_2(x)}\left(\int_{\Omega_2} k_2(x,y)\,\Phi_2(|f_2(y)|)\,d\mu_2(y)\right)d\mu_1(x) =$$

(calling $\gamma_1(x) := \int_{\Omega_2} k_1(x,y)\,\Phi_1(|f_1(y)|)\,d\mu_2(y)$)

$$\int_{\Omega_1}\int_{\Omega_2} \frac{u(x)\,\gamma_1(x)}{K_1(x)\,K_2(x)} k_2(x,y)\,\Phi_2(|f_2(y)|)\,d\mu_2(y)\,d\mu_1(x) =$$

$$\int_{\Omega_2}\int_{\Omega_1} \frac{u(x)\,\gamma_1(x)}{K_1(x)\,K_2(x)} k_2(x,y)\,\Phi_2(|f_2(y)|)\,d\mu_1(x)\,d\mu_2(y) = \tag{2.14}$$

$$\int_{\Omega_2} \Phi_2(|f_2(y)|)\left(\int_{\Omega_1} \frac{u(x)\,\gamma_1(x)}{K_1(x)\,K_2(x)} k_2(x,y)\,d\mu_1(x)\right)d\mu_2(y) =$$

$$\int_{\Omega_2} \Phi_2(|f_2(y)|)\cdot$$

$$\left(\int_{\Omega_1} \frac{u(x)\,k_2(x,y)}{K_1(x)\,K_2(x)}\left(\int_{\Omega_2} k_1(x,y)\,\Phi_1(|f_1(y)|)\,d\mu_2(y)\right)d\mu_1(x)\right)d\mu_2(y) =$$

$$\int_{\Omega_2} \Phi_2(|f_2(y)|)\cdot$$

$$\left[\int_{\Omega_1}\left(\int_{\Omega_2} \frac{u(x)\,k_1(x,y)\,k_2(x,y)}{K_1(x)\,K_2(x)}\Phi_1(|f_1(y)|)\,d\mu_2(y)\right)d\mu_1(x)\right]d\mu_2(y) = \tag{2.15}$$

$$\left(\int_{\Omega_2} \Phi_2(|f_2(y)|)\,d\mu_2(y)\right)\cdot$$

$$\left[\int_{\Omega_1}\left(\int_{\Omega_2}\frac{u(x)\,k_1(x,y)\,k_2(x,y)}{K_1(x)\,K_2(x)}\Phi_1(|f_1(y)|)\,d\mu_2(y)\right)d\mu_1(x)\right]=$$

$$\left(\int_{\Omega_2}\Phi_2(|f_2(y)|)\,d\mu_2(y)\right)\cdot$$

$$\left[\int_{\Omega_2}\left(\int_{\Omega_1}\frac{u(x)\,k_1(x,y)\,k_2(x,y)}{K_1(x)\,K_2(x)}\Phi_1(|f_1(y)|)\,d\mu_1(x)\right)d\mu_2(y)\right]=$$

$$\left(\int_{\Omega_2}\Phi_2(|f_2(y)|)\,d\mu_2(y)\right)\cdot$$

$$\left[\int_{\Omega_2}\Phi_1(|f_1(y)|)\left(\int_{\Omega_1}\frac{u(x)\,k_1(x,y)\,k_2(x,y)}{K_1(x)\,K_2(x)}d\mu_1(x)\right)d\mu_2(y)\right]= \tag{2.16}$$

$$\left(\int_{\Omega_2}\Phi_2(|f_2(y)|)\,d\mu_2(y)\right)\left[\int_{\Omega_2}\Phi_1(|f_1(y)|)\,\lambda_2(y)\,d\mu_2(y)\right],$$

proving the claim. □

When $m=3$, the corresponding result follows.

Theorem 2.3. *Assume that the function* $x\mapsto\left(\frac{u(x)k_1(x,y)k_2(x,y)k_3(x,y)}{K_1(x)K_2(x)K_3(x)}\right)$ *is integrable on* Ω_1*, for each* $y\in\Omega_2$*. Define* λ_3 *on* Ω_2 *by*

$$\lambda_3(y):=\int_{\Omega_1}\frac{u(x)\,k_1(x,y)\,k_2(x,y)\,k_3(x,y)}{K_1(x)\,K_2(x)\,K_3(x)}d\mu_1(x)<\infty. \tag{2.17}$$

Here $\Phi_i:\mathbb{R}_+\to\mathbb{R}_+$*,* $i=1,2,3$*, are convex and increasing functions.*

Then

$$\int_{\Omega_1}u(x)\prod_{i=1}^{3}\Phi_i\left(\left|\frac{g_i(x)}{K_i(x)}\right|\right)d\mu_1(x)\leq \tag{2.18}$$

$$\left(\prod_{i=2}^{3}\int_{\Omega_2}\Phi_i(|f_i(y)|)\,d\mu_2(y)\right)\left(\int_{\Omega_2}\Phi_1(|f_1(y)|)\,\lambda_3(y)\,d\mu_2(y)\right),$$

true for all measurable functions, $i=1,2,3$*,* $f_i:\Omega_2\to\mathbb{R}$ *such that:*

(i) f_i*,* $\Phi_i(|f_i|)$*, are both* $k_i(x,y)\,d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x\in\Omega_1$*.*
(ii) $\lambda_3\Phi_1(|f_1|),\Phi_2(|f_2|),\Phi_3(|f_3|)$*, are all* μ_2 *-integrable,*

and for all corresponding functions g_i *given by (2.10).*

Proof. Here we use Jensen's inequality, Fubini's theorem, and that Φ_i are increasing. We have

$$\int_{\Omega_1}u(x)\prod_{i=1}^{3}\Phi_i\left(\left|\frac{g_i(x)}{K_i(x)}\right|\right)d\mu_1(x)=$$

$$\int_{\Omega_1}u(x)\prod_{i=1}^{3}\Phi_i\left(\left|\frac{1}{K_i(x)}\int_{\Omega_2}k_i(x,y)\,f_i(y)\,d\mu_2(y)\right|\right)d\mu_1(x)\leq \tag{2.19}$$

$$\int_{\Omega_1} u(x) \prod_{i=1}^{3} \Phi_i \left(\frac{1}{K_i(x)} \int_{\Omega_2} k_i(x,y) |f_i(y)| d\mu_2(y) \right) d\mu_1(x) \leq$$

$$\int_{\Omega_1} u(x) \prod_{i=1}^{3} \left(\frac{1}{K_i(x)} \int_{\Omega_2} k_i(x,y) \Phi_i(|f_i(y)|) d\mu_2(y) \right) d\mu_1(x) =$$

$$\int_{\Omega_1} \left(\frac{u(x)}{\prod_{i=1}^{3} K_i(x)} \right) \left(\prod_{i=1}^{3} \int_{\Omega_2} k_i(x,y) \Phi_i(|f_i(y)|) d\mu_2(y) \right) d\mu_1(x) =$$

(calling $\theta(x) := \frac{u(x)}{\prod_{i=1}^{3} K_i(x)}$)

$$\int_{\Omega_1} \theta(x) \left(\prod_{i=1}^{3} \int_{\Omega_2} k_i(x,y) \Phi_i(|f_i(y)|) d\mu_2(y) \right) d\mu_1(x) = \tag{2.20}$$

$$\int_{\Omega_1} \theta(x) \left[\int_{\Omega_2} \left(\prod_{i=1}^{2} \int_{\Omega_2} k_i(x,y) \Phi_i(|f_i(y)|) d\mu_2(y) \right) \right.$$

$$\left. k_3(x,y) \Phi_3(|f_3(y)|) d\mu_2(y) \right] d\mu_1(x) =$$

$$\int_{\Omega_1} \left(\int_{\Omega_2} \theta(x) \left(\prod_{i=1}^{2} \int_{\Omega_2} k_i(x,y) \Phi_i(|f_i(y)|) d\mu_2(y) \right) \right.$$

$$\left. k_3(x,y) \Phi_3(|f_3(y)|) d\mu_2(y) \right) d\mu_1(x) =$$

$$\int_{\Omega_2} \left(\int_{\Omega_1} \theta(x) \left(\prod_{i=1}^{2} \int_{\Omega_2} k_i(x,y) \Phi_i(|f_i(y)|) d\mu_2(y) \right) \right.$$

$$\left. k_3(x,y) \Phi_3(|f_3(y)|) d\mu_1(x) \right) d\mu_2(y) =$$

$$\int_{\Omega_2} \Phi_3(|f_3(y)|) \left(\int_{\Omega_1} \theta(x) k_3(x,y) \left(\prod_{i=1}^{2} \int_{\Omega_2} k_i(x,y) \Phi_i(|f_i(y)|) d\mu_2(y) \right) \right. \tag{2.21}$$

$$\left. d\mu_1(x) \right) d\mu_2(y) =$$

$$\int_{\Omega_2} \Phi_3(|f_3(y)|) \left[\int_{\Omega_1} \theta(x) k_3(x,y) \left(\int_{\Omega_2} \left\{ \int_{\Omega_2} k_1(x,y) \Phi_1(|f_1(y)|) d\mu_2(y) \right\} \cdot \right. \right.$$

$$k_2(x,y)\,\boldsymbol{\Phi}_2(|f_2(y)|)\,d\mu_2(y)\Bigg)\,d\mu_1(x)\Bigg]\,d\mu_2(y) =$$

$$\int_{\Omega_2}\boldsymbol{\Phi}_3(|f_3(y)|)\left[\int_{\Omega_1}\left(\int_{\Omega_2}\theta(x)\,k_2(x,y)\,k_3(x,y)\,\boldsymbol{\Phi}_2(|f_2(y)|)\cdot\right.\right. \tag{2.22}$$

$$\left.\left.\left\{\int_{\Omega_2}k_1(x,y)\,\boldsymbol{\Phi}_1(|f_1(y)|)\,d\mu_2(y)\right\}d\mu_2(y)\right)d\mu_1(x)\right]d\mu_2(y) =$$

$$\left(\int_{\Omega_2}\boldsymbol{\Phi}_3(|f_3(y)|)\,d\mu_2(y)\right)\left[\int_{\Omega_1}\left(\int_{\Omega_2}\theta(x)\,k_2(x,y)\,k_3(x,y)\,\boldsymbol{\Phi}_2(|f_2(y)|)\cdot\right.\right.$$

$$\left.\left.\left\{\int_{\Omega_2}k_1(x,y)\,\boldsymbol{\Phi}_1(|f_1(y)|)\,d\mu_2(y)\right\}d\mu_2(y)\right)d\mu_1(x)\right] =$$

$$\left(\int_{\Omega_2}\boldsymbol{\Phi}_3(|f_3(y)|)\,d\mu_2(y)\right)\left[\int_{\Omega_2}\left(\int_{\Omega_1}\theta(x)\,k_2(x,y)\,k_3(x,y)\,\boldsymbol{\Phi}_2(|f_2(y)|)\cdot\right.\right.$$

$$\left.\left.\left\{\int_{\Omega_2}k_1(x,y)\,\boldsymbol{\Phi}_1(|f_1(y)|)\,d\mu_2(y)\right\}d\mu_1(x)\right)d\mu_2(y)\right] = \tag{2.23}$$

$$\left(\int_{\Omega_2}\boldsymbol{\Phi}_3(|f_3(y)|)\,d\mu_2(y)\right)\left[\int_{\Omega_2}\boldsymbol{\Phi}_2(|f_2(y)|)\left(\int_{\Omega_1}\theta(x)\,k_2(x,y)\,k_3(x,y)\cdot\right.\right.$$

$$\left.\left.\left(\int_{\Omega_2}k_1(x,y)\,\boldsymbol{\Phi}_1(|f_1(y)|)\,d\mu_2(y)\right)d\mu_1(x)\right)d\mu_2(y)\right] =$$

$$\left(\int_{\Omega_2}\boldsymbol{\Phi}_3(|f_3(y)|)\,d\mu_2(y)\right)\left[\int_{\Omega_2}\boldsymbol{\Phi}_2(|f_2(y)|)\left\{\int_{\Omega_1}\left(\int_{\Omega_2}\theta(x)\prod_{i=1}^{3}k_i(x,y)\cdot\right.\right.\right.$$

$$\left.\left.\left.\boldsymbol{\Phi}_1(|f_1(y)|)\,d\mu_2(y)\right)d\mu_1(x)\right\}d\mu_2(y)\right] =$$

$$\left(\int_{\Omega_2}\boldsymbol{\Phi}_3(|f_3(y)|)\,d\mu_2(y)\right)\left(\int_{\Omega_2}\boldsymbol{\Phi}_2(|f_2(y)|)\,d\mu_2(y)\right)\cdot$$

$$\left(\int_{\Omega_1}\left(\int_{\Omega_2}\theta(x)\prod_{i=1}^{3}k_i(x,y)\,\boldsymbol{\Phi}_1(|f_1(y)|)\,d\mu_2(y)\right)d\mu_1(x)\right) = \tag{2.24}$$

$$\left(\prod_{i=2}^{3}\int_{\Omega_2}\boldsymbol{\Phi}_i(|f_i(y)|)\,d\mu_2(y)\right)\cdot$$

$$\left(\int_{\Omega_2}\left(\int_{\Omega_1}\theta(x)\prod_{i=1}^{3}k_i(x,y)\,\boldsymbol{\Phi}_1(|f_1(y)|)\,d\mu_1(x)\right)d\mu_2(y)\right) =$$

$$\left(\prod_{i=2}^{3}\int_{\Omega_2}\boldsymbol{\Phi}_i(|f_i(y)|)\,d\mu_2(y)\right)\cdot$$

$$\left(\int_{\Omega_2} \Phi_1\left(\left|f_1(y)\right|\right)\left(\int_{\Omega_1} \theta(x) \prod_{i=1}^{3} k_i(x,y)\, d\mu_1(x)\right) d\mu_2(y)\right) =$$

$$\left(\prod_{i=2}^{3} \int_{\Omega_2} \Phi_i\left(\left|f_i(y)\right|\right) d\mu_2(y)\right)\left(\int_{\Omega_2} \Phi_1\left(\left|f_1(y)\right|\right) \lambda_3(y)\, d\mu_2(y)\right), \tag{2.25}$$

proving the claim. □

For general $m \in \mathbb{N}$, the following result is valid.

Theorem 2.4. *Assume that the function* $x \mapsto \left(\frac{u(x) \prod_{i=1}^{m} k_i(x,y)}{\prod_{i=1}^{m} K_i(x)}\right)$ *is integrable on* Ω_1, *for each* $y \in \Omega_2$. *Define* λ_m *on* Ω_2 *by*

$$\lambda_m(y) := \int_{\Omega_1} \left(\frac{u(x) \prod_{i=1}^{m} k_i(x,y)}{\prod_{i=1}^{m} K_i(x)}\right) d\mu_1(x) < \infty. \tag{2.26}$$

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, \ldots, m$, *are convex and increasing functions.*
Then

$$\int_{\Omega_1} u(x) \prod_{i=1}^{m} \Phi_i\left(\left|\frac{g_i(x)}{K_i(x)}\right|\right) d\mu_1(x) \leq \tag{2.27}$$

$$\left(\prod_{i=2}^{m} \int_{\Omega_2} \Phi_i\left(\left|f_i(y)\right|\right) d\mu_2(y)\right)\left(\int_{\Omega_2} \Phi_1\left(\left|f_1(y)\right|\right) \lambda_m(y)\, d\mu_2(y)\right),$$

true for all measurable functions, $i = 1, \ldots, m$, $f_i : \Omega_2 \to \mathbb{R}$ *such that:*

(i) f_i, $\Phi_i(|f_i|)$, *are both* $k_i(x,y)\, d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$.
(ii) $\lambda_m \Phi_1(|f_1|), \Phi_2(|f_2|), \Phi_3(|f_3|), \ldots, \Phi_m(|f_m|)$, *are all* μ_2 *-integrable,*

and for all corresponding functions g_i *given by (2.10).*

When $k(x,y) = k_1(x,y) = k_2(x,y) = \ldots = k_m(x,y)$, then $K(x) := K_1(x) = K_2(x) = \ldots = K_m(x)$. Then from Theorem 2.4 we get:

Corollary 2.5. *Assume that the function* $x \mapsto \left(\frac{u(x) k^m(x,y)}{K^m(x)}\right)$ *is integrable on* Ω_1, *for each* $y \in \Omega_2$. *Define* U_m *on* Ω_2 *by*

$$U_m(y) := \int_{\Omega_1} \left(\frac{u(x) k^m(x,y)}{K^m(x)}\right) d\mu_1(x) < \infty. \tag{2.28}$$

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, \ldots, m$, *are convex and increasing functions.*

Then

$$\int_{\Omega_1} u(x) \prod_{i=1}^{m} \Phi_i \left(\left| \frac{g_i(x)}{K(x)} \right| \right) d\mu_1(x) \leq \tag{2.29}$$

$$\left(\prod_{i=2}^{m} \int_{\Omega_2} \Phi_i (|f_i(y)|) \, d\mu_2(y) \right) \left(\int_{\Omega_2} \Phi_1 (|f_1(y)|) \, U_m(y) \, d\mu_2(y) \right),$$

true for all measurable functions, $i = 1, \ldots, m$, $f_i : \Omega_2 \to \mathbb{R}$ *such that:*

(i) f_i, $\Phi_i(|f_i|)$, *are both* $k(x,y)\,d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$.
(ii) $U_m \Phi_1(|f_1|), \Phi_2(|f_2|), \Phi_3(|f_3|), \ldots, \Phi_m(|f_m|)$, *are all* μ_2 *-integrable,*

and for all corresponding functions g_i *given by (2.10).*

When $m = 2$ from Corollary 2.5, we obtain

Corollary 2.6. *Assume that the function* $x \mapsto \left(\frac{u(x) k^2(x,y)}{K^2(x)} \right)$ *is integrable on* Ω_1, *for each* $y \in \Omega_2$. *Define* U_2 *on* Ω_2 *by*

$$U_2(y) := \int_{\Omega_1} \left(\frac{u(x) k^2(x,y)}{K^2(x)} \right) d\mu_1(x) < \infty. \tag{2.30}$$

Here $\Phi_1, \Phi_2 : \mathbb{R}_+ \to \mathbb{R}_+$, *are convex and increasing functions.*
Then

$$\int_{\Omega_1} u(x) \, \Phi_1 \left(\left| \frac{g_1(x)}{K(x)} \right| \right) \Phi_2 \left(\left| \frac{g_2(x)}{K(x)} \right| \right) d\mu_1(x) \leq \tag{2.31}$$

$$\left(\int_{\Omega_2} \Phi_2 (|f_2(y)|) \, d\mu_2(y) \right) \left(\int_{\Omega_2} \Phi_1 (|f_1(y)|) \, U_2(y) \, d\mu_2(y) \right),$$

true for all measurable functions, $f_1, f_2 : \Omega_2 \to \mathbb{R}$ *such that:*

(i) f_1, f_2, $\Phi_1(|f_1|)$, $\Phi_2(|f_2|)$ *are all* $k(x,y)\,d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$.
(ii) $U_2 \Phi_1(|f_1|), \Phi_2(|f_2|)$, *are both* μ_2 *-integrable,*

and for all corresponding functions g_1, g_2 *given by (2.10).*

For $m \in \mathbb{N}$, the following more general result is also valid.

Theorem 2.7. *Let* $j \in \{1, \ldots, m\}$ *be fixed. Assume that the function* $x \mapsto \left(\frac{u(x) \prod_{i=1}^{m} k_i(x,y)}{\prod_{i=1}^{m} K_i(x)} \right)$ *is integrable on* Ω_1, *for each* $y \in \Omega_2$. *Define* λ_m *on* Ω_2 *by*

$$\lambda_m(y) := \int_{\Omega_1} \left(\frac{u(x) \prod_{i=1}^{m} k_i(x,y)}{\prod_{i=1}^{m} K_i(x)} \right) d\mu_1(x) < \infty. \tag{2.32}$$

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, \ldots, m$, *are convex and increasing functions.*

Then

$$I := \int_{\Omega_1} u(x) \prod_{i=1}^{m} \Phi_i \left(\left| \frac{g_i(x)}{K_i(x)} \right| \right) d\mu_1(x) \leq \tag{2.33}$$

$$\left(\prod_{\substack{i=1 \\ i \neq j}}^{m} \int_{\Omega_2} \Phi_i(|f_i(y)|) \, d\mu_2(y) \right) \left(\int_{\Omega_2} \Phi_j \left(\left| f_j(y) \right| \right) \lambda_m(y) \, d\mu_2(y) \right) := I_j,$$

true for all measurable functions, $i = 1, \ldots, m$, $f_i : \Omega_2 \to \mathbb{R}$ *such that:*

(i) f_i, $\Phi_i(|f_i|)$, *are both* $k_i(x,y) \, d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$.

(ii) $\lambda_m \Phi_j \left(\left| f_j \right| \right); \Phi_1(|f_1|), \Phi_2(|f_2|), \Phi_3(|f_3|), \ldots, \widehat{\Phi_j \left(\left| f_j \right| \right)}, \ldots, \Phi_m(|f_m|)$, *are all* μ_2 *-integrable,*

and for all corresponding functions g_i *given by (2.10). Above* $\widehat{\Phi_j \left(\left| f_j \right| \right)}$ *means missing item.*

We make

Remark 2.8. In the notations and assumptions of Theorem 2.7, replace assumption (ii) by the assumption,

(iii) $\Phi_1(|f_1|), \ldots, \Phi_m(|f_m|); \lambda_m \Phi_1(|f_1|), \ldots, \lambda_m \Phi_m(|f_m|)$, are all μ_2 -integrable functions.

Then, clearly it holds,

$$I \leq \frac{\sum_{j=1}^{m} I_j}{m}. \tag{2.34}$$

An application of Theorem 2.7 follows.

Theorem 2.9. *Let* $j \in \{1, \ldots, m\}$ *be fixed. Assume that the function* $x \mapsto \left(\frac{u(x) \prod_{i=1}^{m} k_i(x,y)}{\prod_{i=1}^{m} K_i(x)} \right)$ *is integrable on* Ω_1, *for each* $y \in \Omega_2$. *Define* λ_m *on* Ω_2 *by*

$$\lambda_m(y) := \int_{\Omega_1} \left(\frac{u(x) \prod_{i=1}^{m} k_i(x,y)}{\prod_{i=1}^{m} K_i(x)} \right) d\mu_1(x) < \infty. \tag{2.35}$$

Then

$$\int_{\Omega_1} u(x) \, e^{\sum_{i=1}^{m} \left| \frac{g_i(x)}{K_i(x)} \right|} d\mu_1(x) \leq \tag{2.36}$$

$$\left(\prod_{\substack{i=1 \\ i \neq j}}^{m} \int_{\Omega_2} e^{|f_i(y)|} d\mu_2(y) \right) \left(\int_{\Omega_2} e^{\left| f_j(y) \right|} \lambda_m(y) \, d\mu_2(y) \right),$$

true for all measurable functions, $i = 1, \ldots, m$, $f_i : \Omega_2 \to \mathbb{R}$ *such that:*

(i) f_i, $e^{|f_i|}$, *are both* $k_i(x,y)\,d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$.
(ii) $\lambda_m e^{|f_j|}; e^{|f_1|}, e^{|f_2|}, e^{|f_3|}, \ldots, \widehat{e^{|f_j|}}, \ldots, e^{|f_m|}$, *are all* μ_2 *-integrable,*

and for all corresponding functions g_i *given by (2.10). Above* $\widehat{e^{|f_j|}}$ *means absent item.*

Another application of Theorem 2.7 follows.

Theorem 2.10. *Let* $j \in \{1,\ldots,m\}$ *be fixed,* $\alpha \geq 1$. *Assume that the function* $x \mapsto \left(\frac{u(x)\prod_{i=1}^{m} k_i(x,y)}{\prod_{i=1}^{m} K_i(x)}\right)$ *is integrable on* Ω_1, *for each* $y \in \Omega_2$. *Define* λ_m *on* Ω_2 *by*

$$\lambda_m(y) := \int_{\Omega_1} \left(\frac{u(x)\prod_{i=1}^{m} k_i(x,y)}{\prod_{i=1}^{m} K_i(x)} \right) d\mu_1(x) < \infty. \tag{2.37}$$

Then

$$\int_{\Omega_1} u(x) \left(\prod_{i=1}^{m} \left| \frac{g_i(x)}{K_i(x)} \right|^{\alpha} \right) d\mu_1(x) \leq \tag{2.38}$$

$$\left(\prod_{\substack{i=1 \\ i \neq j}}^{m} \int_{\Omega_2} |f_i(y)|^{\alpha} d\mu_2(y) \right) \left(\int_{\Omega_2} |f_j(y)|^{\alpha} \lambda_m(y)\, d\mu_2(y) \right),$$

true for all measurable functions, $i = 1,\ldots,m$, $f_i : \Omega_2 \to \mathbb{R}$ *such that:*

(i) $|f_i|^{\alpha}$ *is* $k_i(x,y)\,d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$.
(ii) $\lambda_m |f_j|^{\alpha}; |f_1|^{\alpha}, |f_2|^{\alpha}, |f_3|^{\alpha}, \ldots, \widehat{|f_j|^{\alpha}}, \ldots, |f_m|^{\alpha}$, *are all* μ_2 *-integrable,*

and for all corresponding functions g_i *given by (2.10). Above* $\widehat{|f_j|^{\alpha}}$ *means absent item.*

We make

Remark 2.11. Let f_i be Lebesgue measurable functions from (a,b) into $\mathbb{R}$, such that $\left(I_{a+}^{\alpha_i}(|f_i|)\right)(x) \in \mathbb{R}$, $\forall\, x \in (a,b)$, $\alpha_i > 0$, $i = 1,\ldots,m$, e.g., when $f_i \in L_\infty(a,b)$.
Consider

$$g_i(x) = \left(I_{a+}^{\alpha_i} f_i\right)(x), \quad x \in (a,b),\, i = 1,\ldots,m, \tag{2.39}$$

we remind

$$\left(I_{a+}^{\alpha_i} f_i\right)(x) = \frac{1}{\Gamma(\alpha_i)} \int_a^x (x-t)^{\alpha_i - 1} f_i(t)\, dt.$$

Notice that $g_i(x) \in \mathbb{R}$ and it is Lebesgue measurable.

We pick $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$, the Lebesgue measure.

We see that

$$\left(I_{a+}^{\alpha_i} f\right)(x) = \int_a^b \frac{\chi_{(a,x]}(t)(x-t)^{\alpha_i - 1}}{\Gamma(\alpha_i)} f_i(t)\,dt, \tag{2.40}$$

where χ stands for the characteristic function.

So, we pick here

$$k_i(x,t) := \frac{\chi_{(a,x]}(t)(x-t)^{\alpha_i - 1}}{\Gamma(\alpha_i)}, \quad i = 1,\dots,m. \tag{2.41}$$

In fact

$$k_i(x,y) = \begin{cases} \frac{(x-y)^{\alpha_i - 1}}{\Gamma(\alpha_i)}, & a < y \le x, \\ 0, & x < y < b. \end{cases} \tag{2.42}$$

Clearly it holds

$$K_i(x) = \int_{(a,b)} \frac{\chi_{(a,x]}(y)(x-y)^{\alpha_i - 1}}{\Gamma(\alpha_i)} dy = \frac{(x-a)^{\alpha_i}}{\Gamma(\alpha_i + 1)}, \tag{2.43}$$

$a < x < b$, $i = 1,\dots,m$.

Notice that

$$\prod_{i=1}^m \frac{k_i(x,y)}{K_i(x)} = \prod_{i=1}^m \left(\frac{\chi_{(a,x]}(y)(x-y)^{\alpha_i - 1}}{\Gamma(\alpha_i)} \cdot \frac{\Gamma(\alpha_i + 1)}{(x-a)^{\alpha_i}} \right) =$$

$$\prod_{i=1}^m \left(\frac{\chi_{(a,x]}(y)(x-y)^{\alpha_i - 1}\alpha_i}{(x-a)^{\alpha_i}} \right) = \frac{\chi_{(a,x]}(y)(x-y)^{\left(\sum_{i=1}^m \alpha_i - m\right)} \left(\prod_{i=1}^m \alpha_i\right)}{(x-a)^{\left(\sum_{i=1}^m \alpha_i\right)}}. \tag{2.44}$$

Calling

$$\alpha := \sum_{i=1}^m \alpha_i > 0, \;\; \gamma := \prod_{i=1}^m \alpha_i > 0, \tag{2.45}$$

we have that

$$\prod_{i=1}^m \frac{k_i(x,y)}{K_i(x)} = \frac{\chi_{(a,x]}(y)(x-y)^{\alpha - m}\gamma}{(x-a)^{\alpha}}. \tag{2.46}$$

Therefore, for (2.32), we get for appropriate weight u that

$$\lambda_m(y) = \gamma \int_y^b u(x) \frac{(x-y)^{\alpha - m}}{(x-a)^{\alpha}} dx < \infty, \tag{2.47}$$

for all $a < y < b$.

Let $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, \ldots, m$, be convex and increasing functions. Then by (2.33) we obtain

$$\int_a^b u(x) \prod_{i=1}^m \Phi_i \left(\left| \frac{\left(I_{a+}^{\alpha_i} f_i\right)(x)}{(x-a)^{\alpha_i}} \right| \Gamma(\alpha_i + 1) \right) dx \leq$$

$$\left(\prod_{\substack{i=1 \\ i \neq j}}^m \int_a^b \Phi_i \left(|f_i(x)| \right) dx \right) \left(\int_a^b \Phi_j \left(\left| f_j(x) \right| \right) \lambda_m(x)\, dx \right), \tag{2.48}$$

with $j \in \{1, \ldots, m\}$, true for measurable f_i with $I_{a+}^{\alpha_i}(|f_i|)$ finite ($i = 1, \ldots, m$) and with the properties:

(i) $\Phi_i(|f_i|)$ is $k_i(x,y)\,dy$ -integrable, a.e. in $x \in (a,b)$.

(ii) $\lambda_m \Phi_j\left(\left|f_j\right|\right); \Phi_1(|f_1|), \Phi_2(|f_2|), \ldots, \widehat{\Phi_j\left(\left|f_j\right|\right)}, \ldots, \Phi_m(|f_m|)$ are all Lebesgue integrable functions,

where $\widehat{\Phi_j\left(\left|f_j\right|\right)}$ means absent item.

Let now

$$u(x) = (x-a)^{\alpha}, \quad x \in (a,b). \tag{2.49}$$

Then

$$\lambda_m(y) = \gamma \int_y^b (x-y)^{\alpha - m}\, dx = \frac{\gamma (b-y)^{\alpha - m + 1}}{\alpha - m + 1}, \tag{2.50}$$

$y \in (a,b)$, where $\alpha > m - 1$.

Hence (2.48) becomes

$$\begin{aligned} &\int_a^b (x-a)^{\alpha} \prod_{i=1}^m \Phi_i \left(\left| \frac{\left(I_{a+}^{\alpha_i} f_i\right)(x)}{(x-a)^{\alpha_i}} \right| \Gamma(\alpha_i + 1) \right) dx \leq \\ &\left(\frac{\gamma}{\alpha - m + 1} \right) \left(\prod_{\substack{i=1 \\ i \neq j}}^m \int_a^b \Phi_i(|f_i(x)|)\, dx \right) \left(\int_a^b (b-x)^{\alpha - m + 1} \Phi_j\left(\left|f_j(x)\right|\right) dx \right) \leq \\ &\left(\frac{\gamma (b-a)^{\alpha - m + 1}}{\alpha - m + 1} \right) \left(\prod_{i=1}^m \int_a^b \Phi_i(|f_i(x)|)\, dx \right), \end{aligned} \tag{2.51}$$

where $\alpha > m - 1$, f_i with $I_{a+}^{\alpha_i}(|f_i|)$ finite, $i = 1, \ldots, m$, under the assumptions (i), (ii) following (2.48).

If $\Phi_i = id$, then (2.51) turns to

$$\int_a^b \prod_{i=1}^m \left| \left(I_{a+}^{\alpha_i} f_i\right)(x) \right| dx \leq$$

$$\left(\frac{\gamma}{\left(\prod_{i=1}^{m}\Gamma\left(\alpha_i+1\right)\right)\left(\alpha-m+1\right)}\right)\left(\prod_{\substack{i=1\\i\neq j}}^{m}\int_a^b\left|f_i\left(x\right)\right|dx\right)\cdot$$

$$\left(\int_a^b\left(b-x\right)^{\alpha-m+1}\left|f_j\left(x\right)\right|dx\right)\leq$$

$$\left(\frac{\gamma\left(b-a\right)^{\alpha-m+1}}{\left(\prod_{i=1}^{m}\Gamma\left(\alpha_i+1\right)\right)\left(\alpha-m+1\right)}\right)\left(\prod_{i=1}^{m}\int_a^b\left|f_i\left(x\right)\right|dx\right), \tag{2.52}$$

where $\alpha>m-1$, f_i with $I_{a+}^{\alpha_i}\left(|f_i|\right)$ finite and f_i Lebesgue integrable, $i=1,\ldots,m$.

Next let $p_i>1$, and $\Phi_i\left(x\right)=x^{p_i}$, $x\in\mathbb{R}_+$. These Φ_i are convex, increasing, and continuous on $\mathbb{R}_+$.

Then, by (2.48), we get

$$I_1:=\int_a^b\left(x-a\right)^{\alpha}\prod_{i=1}^{m}\left|\frac{\left(I_{a+}^{\alpha_i}f_i\right)\left(x\right)}{\left(x-a\right)^{\alpha_i}}\right|^{p_i}dx\leq$$

$$\left(\frac{\gamma}{\left(\prod_{i=1}^{m}\left(\Gamma\left(\alpha_i+1\right)\right)^{p_i}\right)\left(\alpha-m+1\right)}\right)\left(\prod_{\substack{i=1\\i\neq j}}^{m}\int_a^b\left|f_i\left(x\right)\right|^{p_i}dx\right)\cdot$$

$$\left(\int_a^b\left(b-x\right)^{\alpha-m+1}\left|f_j\left(x\right)\right|^{p_j}dx\right)\leq$$

$$\left(\frac{\gamma\left(b-a\right)^{\alpha-m+1}}{\left(\prod_{i=1}^{m}\left(\Gamma\left(\alpha_i+1\right)\right)^{p_i}\right)\left(\alpha-m+1\right)}\right)\left(\prod_{i=1}^{m}\int_a^b\left|f_i\left(x\right)\right|^{p_i}dx\right). \tag{2.53}$$

Notice that $\sum_{i=1}^{m}\alpha_i p_i>\alpha$; thus, $\beta:=\alpha-\sum_{i=1}^{m}\alpha_i p_i<0$. Since $0<x-a<b-a$ ($x\in(a,b)$), then $(x-a)^{\beta}>(b-a)^{\beta}$.

Therefore

$$I_1:=\int_a^b\left(x-a\right)^{\beta}\prod_{i=1}^{m}\left|\left(I_{a+}^{\alpha_i}f_i\right)\left(x\right)\right|^{p_i}dx\geq$$

$$\left(b-a\right)^{\beta}\int_a^b\prod_{i=1}^{m}\left|\left(I_{a+}^{\alpha_i}f_i\right)\left(x\right)\right|^{p_i}dx. \tag{2.54}$$

Consequently, by (2.53) and (2.54), it holds

$$\int_a^b \prod_{i=1}^m \left| \left(I_{a+}^{\alpha_i} f_i \right)(x) \right|^{p_i} dx \leq \tag{2.55}$$

$$\left(\frac{\gamma (b-a)^{\left(\left(\sum_{i=1}^m \alpha_i p_i \right) - m + 1 \right)}}{\left(\prod_{i=1}^m \left(\Gamma(\alpha_i + 1) \right)^{p_i} \right) (\alpha - m + 1)} \right) \left(\prod_{i=1}^m \int_a^b |f_i(x)|^{p_i} dx \right),$$

where $p_i > 1$, $i = 1,..,m$, $\alpha > m-1$, true for measurable f_i with $I_{a+}^{\alpha_i}(|f_i|)$ finite, with the properties ($i = 1, \ldots, m$):

(i) $|f_i|^{p_i}$ is $k_i(x,y)\,dy$ -integrable, a.e. in $x \in (a,b)$.
(ii) $|f_i|^{p_i}$ is Lebesgue integrable on (a,b).

If $p = p_1 = p_2 = \ldots = p_m > 1$, then by (2.55), we get

$$\left\| \prod_{i=1}^m \left(I_{a+}^{\alpha_i} f_i \right) \right\|_{p,(a,b)} \leq \tag{2.56}$$

$$\left(\frac{\gamma^{\frac{1}{p}} (b-a)^{\left(\alpha - \frac{m}{p} + \frac{1}{p} \right)}}{\left(\prod_{i=1}^m \left(\Gamma(\alpha_i + 1) \right) \right) (\alpha - m + 1)^{\frac{1}{p}}} \right) \left(\prod_{i=1}^m \|f_i\|_{p,(a,b)} \right),$$

$\alpha > m-1$, true for measurable f_i with $I_{a+}^{\alpha_i}(|f_i|)$ finite, and such that ($i = 1, \ldots, m$):

(i) $|f_i|^p$ is $k_i(x,y)\,dy$ -integrable, a.e. in $x \in (a,b)$.
(ii) $|f_i|^p$ is Lebesgue integrable on (a,b).

Using (ii) and if $\alpha_i > \frac{1}{p}$, by Hölder's inequality we derive that $I_{a+}^{\alpha_i}(|f_i|)$ is finite on (a,b). If we set $p = 1$ to (2.56) we get (2.52).

If $\Phi_i(x) = e^x$, $x \in \mathbb{R}_+$, then from (2.51) we get

$$\int_a^b (x-a)^{\alpha} e^{\sum_{i=1}^m \left(\left| \frac{\left(I_{a+}^{\alpha_i} f_i \right)(x)}{(x-a)^{\alpha_i}} \right| \Gamma(\alpha_i + 1) \right)} dx \leq$$

$$\left(\frac{\gamma (b-a)^{\alpha - m + 1}}{\alpha - m + 1} \right) \left(\prod_{i=1}^m \left(\int_a^b e^{|f_i(x)|} dx \right) \right), \tag{2.57}$$

where $\alpha > m-1$, f_i with $I_{a+}^{\alpha_i}(|f_i|)$ finite, $i = 1, \ldots, m$, under the assumptions:

(i) $e^{|f_i|}$ is $k_i(x,y)\,dy$ -integrable, a.e. in $x \in (a,b)$.
(ii) $e^{|f_i|}$ is Lebesgue integrable on (a,b).

We continue with

Remark 2.12. Let f_i be Lebesgue measurable functions : $(a,b) \to \mathbb{R}$, such that $I_{b-}^{\alpha_i}(|f_i|)(x) < \infty$, $\forall\, x \in (a,b)$, $\alpha_i > 0$, $i = 1, \ldots, m$, e.g., when $f_i \in L_\infty(a,b)$.

Consider

$$g_i(x) = \left(I_{b-}^{\alpha_i} f_i\right)(x), \quad x \in (a,b), i = 1, \ldots, m, \tag{2.58}$$

we remind

$$\left(I_{b-}^{\alpha_i} f_i\right)(x) = \frac{1}{\Gamma(\alpha_i)} \int_x^b f_i(t)(t-x)^{\alpha_i - 1} dt, \tag{2.59}$$

$(x < b)$.

Notice that $g_i(x) \in \mathbb{R}$ and it is Lebesgue measurable.

We pick $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$, the Lebesgue measure.

We see that

$$\left(I_{b-}^{\alpha_i} f_i\right)(x) = \int_a^b \chi_{[x,b)}(t) \frac{(t-x)^{\alpha_i - 1}}{\Gamma(\alpha_i)} f_i(t) dt. \tag{2.60}$$

So, we pick here

$$k_i(x,t) := \chi_{[x,b)}(t) \frac{(t-x)^{\alpha_i - 1}}{\Gamma(\alpha_i)}, \quad i = 1, \ldots, m. \tag{2.61}$$

In fact

$$k_i(x,y) = \begin{cases} \frac{(y-x)^{\alpha_i - 1}}{\Gamma(\alpha_i)}, & x \le y < b, \\ 0, & a < y < x. \end{cases} \tag{2.62}$$

Clearly it holds

$$K_i(x) = \int_{(a,b)} \chi_{[x,b)}(y) \frac{(y-x)^{\alpha_i - 1}}{\Gamma(\alpha_i)} dy = \frac{(b-x)^{\alpha_i}}{\Gamma(\alpha_i + 1)}, \tag{2.63}$$

$a < x < b$, $i = 1, \ldots, m$.

Notice that

$$\prod_{i=1}^m \frac{k_i(x,y)}{K_i(x)} = \prod_{i=1}^m \left(\chi_{[x,b)}(y) \frac{(y-x)^{\alpha_i - 1}}{\Gamma(\alpha_i)} \cdot \frac{\Gamma(\alpha_i + 1)}{(b-x)^{\alpha_i}} \right) =$$

$$\prod_{i=1}^m \left(\chi_{[x,b)}(y) \frac{(y-x)^{\alpha_i - 1} \alpha_i}{(b-x)^{\alpha_i}} \right) = \chi_{[x,b)}(y) \frac{(y-x)^{\left(\sum_{i=1}^m \alpha_i - m\right)} \left(\prod_{i=1}^m \alpha_i\right)}{(b-x)^{\left(\sum_{i=1}^m \alpha_i\right)}}. \tag{2.64}$$

Calling

$$\alpha := \sum_{i=1}^m \alpha_i > 0, \ \gamma := \prod_{i=1}^m \alpha_i > 0, \tag{2.65}$$

we have that

$$\prod_{i=1}^m \frac{k_i(x,y)}{K_i(x)} = \frac{\chi_{[x,b)}(y)(y-x)^{\alpha - m} \gamma}{(b-x)^{\alpha}}. \tag{2.66}$$

Therefore, for (2.32), we get for appropriate weight u that

$$\lambda_m(y) = \gamma \int_a^y u(x) \frac{(y-x)^{\alpha-m}}{(b-x)^{\alpha}} dx < \infty, \tag{2.67}$$

for all $a < y < b$.

Let $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, \ldots, m$, be convex and increasing functions. Then by (2.33) we obtain

$$\int_a^b u(x) \prod_{i=1}^m \Phi_i \left(\left| \frac{\left(I_{b-}^{\alpha_i} f_i\right)(x)}{(b-x)^{\alpha_i}} \right| \Gamma(\alpha_i+1) \right) dx \leq$$

$$\left(\prod_{\substack{i=1 \\ i \neq j}}^m \int_a^b \Phi_i(|f_i(x)|)\, dx \right) \left(\int_a^b \Phi_j(|f_j(x)|)\, \lambda_m(x)\, dx \right), \tag{2.68}$$

with $j \in \{1, \ldots, m\}$,

true for measurable f_i with $I_{b-}^{\alpha_i}(|f_i|)$ finite ($i = 1, \ldots, m$) and with the properties:

(i) $\Phi_i(|f_i|)$ is $k_i(x,y)\, dy$ -integrable, a.e. in $x \in (a,b)$.

(ii) $\lambda_m \Phi_j(|f_j|)$; $\Phi_1(|f_1|), \ldots, \widehat{\Phi_j(|f_j|)}, \ldots, \Phi_m(|f_m|)$ are all Lebesgue integrable functions,

where $\widehat{\Phi_j(|f_j|)}$ means absent item.

Let now

$$u(x) = (b-x)^{\alpha}, \quad x \in (a,b). \tag{2.69}$$

Then

$$\lambda_m(y) = \gamma \int_a^y (y-x)^{\alpha-m} dx = \frac{\gamma (y-a)^{\alpha-m+1}}{\alpha-m+1}, \tag{2.70}$$

$y \in (a,b)$, where $\alpha > m-1$.

Hence (2.68) becomes

$$\int_a^b (b-x)^{\alpha} \prod_{i=1}^m \Phi_i \left(\frac{\left| \left(I_{b-}^{\alpha_i} f_i\right)(x) \right|}{(b-x)^{\alpha_i}} \Gamma(\alpha_i+1) \right) dx \leq$$

$$\left(\frac{\gamma}{\alpha-m+1} \right) \left(\prod_{\substack{i=1 \\ i \neq j}}^m \int_a^b \Phi_i(|f_i(x)|)\, dx \right) \left(\int_a^b (x-a)^{\alpha-m+1} \Phi_j(|f_j(x)|)\, dx \right) \leq$$

$$\left(\frac{\gamma (b-a)^{\alpha-m+1}}{\alpha-m+1} \right) \left(\prod_{i=1}^m \int_a^b \Phi_i(|f_i(x)|)\, dx \right), \tag{2.71}$$

where $\alpha > m-1$, f_i with $I_{b-}^{\alpha_i}(|f_i|)$ finite, $i = 1, \ldots, m$, under the assumptions (i), (ii) following (2.68).

If $\Phi_i = id$, then (2.71) turns to

$$\int_a^b \prod_{i=1}^m \left| \left(I_{b-}^{\alpha_i} f_i \right)(x) \right| dx \leq$$

$$\left(\frac{\gamma}{\left(\prod_{i=1}^m \Gamma(\alpha_i + 1) \right)(\alpha - m + 1)} \right) \left(\prod_{\substack{i=1 \\ i \neq j}}^m \int_a^b |f_i(x)| \, dx \right) \cdot$$

$$\left(\int_a^b (x-a)^{\alpha - m + 1} \left| f_j(x) \right| dx \right) \leq$$

$$\left(\frac{\gamma (b-a)^{\alpha - m + 1}}{\left(\prod_{i=1}^m \Gamma(\alpha_i + 1) \right)(\alpha - m + 1)} \right) \left(\prod_{i=1}^m \int_a^b |f_i(x)| \, dx \right), \tag{2.72}$$

where $\alpha > m - 1$, f_i with $I_{b-}^{\alpha_i}(|f_i|)$ finite and f_i Lebesgue integrable, $i = 1, \ldots, m$.

Next let $p_i > 1$, and $\Phi_i(x) = x^{p_i}$, $x \in \mathbb{R}_+$.

Then, by (2.68), we get

$$I_2 := \int_a^b (b-x)^{\alpha} \frac{\left(\prod_{i=1}^m \left| \left(I_{b-}^{\alpha_i} f_i \right)(x) \right|^{p_i} \right)}{(b-x)^{\sum_{i=1}^m \alpha_i p_i}} dx \leq$$

$$\left(\frac{\gamma}{\left(\prod_{i=1}^m (\Gamma(\alpha_i + 1))^{p_i} \right)(\alpha - m + 1)} \right) \left(\prod_{\substack{i=1 \\ i \neq j}}^m \int_a^b |f_i(x)|^{p_i} \, dx \right) \cdot$$

$$\left(\int_a^b (x-a)^{\alpha - m + 1} \left| f_j(x) \right|^{p_j} dx \right) \leq$$

$$\left(\frac{\gamma (b-a)^{\alpha - m + 1}}{\left(\prod_{i=1}^m (\Gamma(\alpha_i + 1))^{p_i} \right)(\alpha - m + 1)} \right) \left(\prod_{i=1}^m \int_a^b |f_i(x)|^{p_i} \, dx \right). \tag{2.73}$$

Notice here that $\beta := \alpha - \sum_{i=1}^m \alpha_i p_i < 0$. Since $0 < b - x < b - a$ ($x \in (a,b)$), then $(b-x)^{\beta} > (b-a)^{\beta}$.

Therefore

$$I_2 := \int_a^b (b-x)^{\beta} \left(\prod_{i=1}^m \left| \left(I_{b-}^{\alpha_i} f_i \right)(x) \right|^{p_i} \right) dx \geq$$

$$(b-a)^{\beta} \int_a^b \left(\prod_{i=1}^m \left| \left(I_{b-}^{\alpha_i} f_i \right)(x) \right|^{p_i} \right) dx. \tag{2.74}$$

Consequently, by (2.73) and (2.74), it holds

$$\int_a^b \prod_{i=1}^m \left| \left(I_{b-}^{\alpha_i} f_i \right)(x) \right|^{p_i} dx \leq \tag{2.75}$$

$$\left(\frac{\gamma (b-a)^{\left(\left(\sum_{i=1}^m \alpha_i p_i \right) - m + 1 \right)}}{\left(\prod_{i=1}^m \left(\Gamma(\alpha_i + 1) \right)^{p_i} \right) (\alpha - m + 1)} \right) \left(\prod_{i=1}^m \int_a^b |f_i(x)|^{p_i} dx \right),$$

where $p_i > 1$, $i = 1, .., m$, $\alpha > m - 1$,

true for measurable f_i with $I_{b-}^{\alpha_i}(|f_i|)$ finite, with the properties ($i = 1, \ldots, m$):

(i) $|f_i|^{p_i}$ is $k_i(x,y)\,dy$ -integrable, a.e. in $x \in (a,b)$.
(ii) $|f_i|^{p_i}$ is Lebesgue integrable on (a,b).

If $p := p_1 = p_2 = \ldots = p_m > 1$, then by (2.75), we get

$$\left\| \prod_{i=1}^m \left(I_{b-}^{\alpha_i} f_i \right) \right\|_{p,(a,b)} \leq \tag{2.76}$$

$$\left(\frac{\gamma^{\frac{1}{p}} (b-a)^{\left(\alpha - \frac{m}{p} + \frac{1}{p} \right)}}{\left(\prod_{i=1}^m \left(\Gamma(\alpha_i + 1) \right) \right) (\alpha - m + 1)^{\frac{1}{p}}} \right) \left(\prod_{i=1}^m \|f_i\|_{p,(a,b)} \right),$$

$\alpha > m - 1$, true for measurable f_i with $I_{b-}^{\alpha_i}(|f_i|)$ finite, and such that ($i = 1, \ldots, m$):

(i) $|f_i|^p$ is $k_i(x,y)\,dy$ -integrable, a.e. in $x \in (a,b)$.
(ii) $|f_i|^p$ is Lebesgue integrable on (a,b).

Using (ii) and if $\alpha_i > \frac{1}{p}$, by Hölder's inequality, we derive that $I_{b-}^{\alpha_i}(|f_i|)$ is finite on (a,b).

If we set $p = 1$ to (2.76) we obtain (2.72).

If $\Phi_i(x) = e^x$, $x \in \mathbb{R}_+$, then from (2.71), we obtain

$$\int_a^b (b-x)^{\alpha} e^{\sum_{i=1}^m \left(\left| \frac{\left(I_{b-}^{\alpha_i} f_i \right)(x)}{(b-x)^{\alpha_i}} \right| \Gamma(\alpha_i + 1) \right)} dx \leq$$

$$\left(\frac{\gamma (b-a)^{\alpha - m + 1}}{\alpha - m + 1} \right) \left(\prod_{i=1}^m \left(\int_a^b e^{|f_i(x)|} dx \right) \right), \tag{2.77}$$

where $\alpha > m - 1$, f_i with $I_{b-}^{\alpha_i}(|f_i|)$ finite, $i = 1, \ldots, m$, under the assumptions:

(i) $e^{|f_i|}$ is $k_i(x,y)\,dy$ -integrable, a.e. in $x \in (a,b)$.
(ii) $e^{|f_i|}$ is Lebesgue integrable on (a,b).

We mention

Definition 2.13 ([1], p. 448). The left generalized Riemann–Liouville fractional derivative of f of order $\beta > 0$ is given by

$$D_a^{\beta} f(x) = \frac{1}{\Gamma(n-\beta)} \left(\frac{d}{dx}\right)^n \int_a^x (x-y)^{n-\beta-1} f(y)\, dy, \tag{2.78}$$

where $n = [\beta] + 1$, $x \in [a,b]$.

For $a, b \in \mathbb{R}$, we say that $f \in L_1(a,b)$ has an L_∞ fractional derivative $D_a^{\beta} f$ $(\beta > 0)$ in $[a,b]$, if and only if:

(1) $D_a^{\beta-k} f \in C([a,b])$, $k = 2, \ldots, n = [\beta] + 1$
(2) $D_a^{\beta-1} f \in AC([a,b])$
(3) $D_a^{\beta} f \in L_\infty(a,b)$

Above we define $D_a^0 f := f$ and $D_a^{-\delta} f := I_{a+}^{\delta} f$, if $0 < \delta \leq 1$.

From [1, p. 449] and [9] we mention and use

Lemma 2.14. *Let $\beta > \alpha \geq 0$ and let $f \in L_1(a,b)$ have an L_∞ fractional derivative $D_a^{\beta} f$ in $[a,b]$ and let $D_a^{\beta-k} f(a) = 0$, $k = 1, \ldots, [\beta] + 1$, then*

$$D_a^{\alpha} f(x) = \frac{1}{\Gamma(\beta-\alpha)} \int_a^x (x-y)^{\beta-\alpha-1} D_a^{\beta} f(y)\, dy, \tag{2.79}$$

for all $a \leq x \leq b$.

Here $D_a^{\alpha} f \in AC([a,b])$ for $\beta - \alpha \geq 1$, and $D_a^{\alpha} f \in C([a,b])$ for $\beta - \alpha \in (0,1)$.

Notice here that

$$D_a^{\alpha} f(x) = \left(I_{a+}^{\beta-\alpha}\left(D_a^{\beta} f\right)\right)(x), \quad a \leq x \leq b. \tag{2.80}$$

We give

Theorem 2.15. *Let $f_i \in L_1(a,b)$, $\alpha_i, \beta_i : \beta_i > \alpha_i \geq 0$, $i = 1, \ldots, m$. Here (f_i, α_i, β_i) fulfill terminology and assumptions of Definition 2.13 and Lemma 2.14. Let $\overline{\alpha} := \sum_{i=1}^{m} (\beta_i - \alpha_i)$, $\overline{\gamma} := \prod_{i=1}^{m} (\beta_i - \alpha_i)$, assume $\overline{\alpha} > m - 1$, and $p \geq 1$. Then*

$$\left\| \prod_{i=1}^{m} \left(D_a^{\alpha_i} f_i\right) \right\|_{p,(a,b)} \leq \tag{2.81}$$

$$\left(\frac{\overline{\gamma}^{\frac{1}{p}} (b-a)^{\left(\overline{\alpha} - \frac{m}{p} + \frac{1}{p}\right)}}{\left(\prod_{i=1}^{m} \left(\Gamma(\beta_i - \alpha_i + 1)\right)\right) (\overline{\alpha} - m + 1)^{\frac{1}{p}}} \right) \left(\prod_{i=1}^{m} \left\| D_a^{\beta_i} f_i \right\|_{p,(a,b)} \right).$$

Proof. By (2.52) and (2.56). □

We continue with

Theorem 2.16. *All here as in Theorem 2.15. Then*

$$\int_a^b (x-a)^{\overline{\alpha}} e^{\sum_{i=1}^m \left(\left| \frac{\left(D_a^{\alpha_i} f_i\right)(x)}{(x-a)^{\left(\beta_i - \alpha_i\right)}} \right| \Gamma(\beta_i - \alpha_i + 1) \right)} dx \leq$$

$$\left(\frac{\overline{\gamma}(b-a)^{\overline{\alpha}-m+1}}{\overline{\alpha}-m+1} \right) \left(\prod_{i=1}^m \left(\int_a^b e^{\left| \left(D_a^{\beta_i} f_i\right)(x) \right|} dx \right) \right). \tag{2.82}$$

Proof. By (2.57), assumptions there (i) and (ii) are easily fulfilled. □

We need

Definition 2.17 ([6], p. 50, [1], p. 449). Let $\nu \geq 0$, $n := \lceil \nu \rceil$, $f \in AC^n([a,b])$. Then the left Caputo fractional derivative is given by

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t) \, dt$$

$$= \left(I_{a+}^{n-\nu} f^{(n)} \right)(x), \tag{2.83}$$

and it exists almost everywhere for $x \in [a,b]$, in fact $D_{*a}^{\nu} f \in L_1(a,b)$, ([1], p. 394).

We have $D_{*a}^{n} f = f^{(n)}$, $n \in \mathbb{Z}_+$.

We also need

Theorem 2.18 ([4]). *Let $\nu \geq \rho + 1$, $\rho > 0$, $\nu, \rho \notin \mathbb{N}$. Call $n := \lceil \nu \rceil$, $m^* := \lceil \rho \rceil$. Assume $f \in AC^n([a,b])$, such that $f^{(k)}(a) = 0$, $k = m^*, m^*+1, \ldots, n-1$, and $D_{*a}^{\nu} f \in L_{\infty}(a,b)$. Then $D_{*a}^{\rho} f \in AC([a,b])$ (where $D_{*a}^{\rho} f = \left(I_{a+}^{m^*-\rho} f^{(m^*)} \right)(x)$), and*

$$D_{*a}^{\rho} f(x) = \frac{1}{\Gamma(\nu-\rho)} \int_a^x (x-t)^{\nu-\rho-1} D_{*a}^{\nu} f(t) \, dt$$

$$= \left(I_{a+}^{\nu-\rho} \left(D_{*a}^{\nu} f \right) \right)(x), \tag{2.84}$$

$\forall\, x \in [a,b]$.

We give

Theorem 2.19. *Let (f_i, ν_i, ρ_i), $i = 1, \ldots, m$, $m \geq 2$, as in the assumptions of Theorem 2.18. Set $\overline{\alpha} := \sum_{i=1}^m (\nu_i - \rho_i)$, $\overline{\gamma} := \prod_{i=1}^m (\nu_i - \rho_i)$, and let $p \geq 1$. Here $a, b \in \mathbb{R}$, $a < b$. Then*

$$\left\| \prod_{i=1}^m \left(D_{*a}^{\rho_i} f_i \right) \right\|_{p,(a,b)} \leq \tag{2.85}$$

$$\left(\frac{\overline{\gamma}^{\frac{1}{p}}(b-a)^{\left(\overline{\alpha}-\frac{m}{p}+\frac{1}{p}\right)}}{\left(\prod_{i=1}^{m}\left(\Gamma\left(\nu_i-\rho_i+1\right)\right)\right)\left(\overline{\alpha}-m+1\right)^{\frac{1}{p}}}\right)\left(\prod_{i=1}^{m}\left\|D_{*a}^{\nu_i}f_i\right\|_{p,(a,b)}\right).$$

Proof. By (2.52) and (2.56), see here $\overline{\alpha}\geq m>m-1$. □

We also give

Theorem 2.20. *Here all as in Theorem 2.19, let $p_i\geq 1$, $i=1,\ldots,l$; $l<m$. Then*

$$\int_a^b (x-a)^{\left(\overline{\alpha}-\sum_{i=1}^{l}p_i(\nu_i-\rho_i)\right)}\left(\prod_{i=1}^{l}\left|D_{*a}^{\rho_i}f_i(x)\right|^{p_i}\right)\cdot$$

$$e^{\left(\sum_{i=l+1}^{m}\left|D_{*a}^{\rho_i}f_i(x)\right|\left(\frac{\Gamma(\nu_i-\rho_i+1)}{(x-a)^{(\nu_i-\rho_i)}}\right)\right)}dx\leq$$

$$\left(\frac{\overline{\gamma}(b-a)^{\overline{\alpha}-m+1}}{\left(\prod_{i=1}^{l}\left(\Gamma\left(\nu_i-\rho_i+1\right)\right)^{p_i}\right)\left(\overline{\alpha}-m+1\right)}\right)\left(\prod_{i=1}^{l}\int_a^b\left|D_{*a}^{\nu_i}f_i(x)\right|^{p_i}dx\right)\cdot \tag{2.86}$$

$$\left(\prod_{i=l+1}^{m}\int_a^b e^{\left|D_{*a}^{\nu_i}f_i(x)\right|}dx\right).$$

Proof. By (2.51). □

We need

Definition 2.21 ([2, 7, 8]). Let $\alpha\geq 0$, $n:=\lceil\alpha\rceil$, $f\in AC^n([a,b])$. We define the right Caputo fractional derivative of order $\alpha\geq 0$, by

$$\overline{D}_{b-}^{\alpha}f(x):=(-1)^n I_{b-}^{n-\alpha}f^{(n)}(x), \tag{2.87}$$

we set $\overline{D}_{-}^{0}f:=f$, i.e.,

$$\overline{D}_{b-}^{\alpha}f(x)=\frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b (J-x)^{n-\alpha-1}f^{(n)}(J)\,dJ. \tag{2.88}$$

Notice that $\overline{D}_{b-}^{n}f=(-1)^n f^{(n)}$, $n\in\mathbb{N}$.

In [3] we introduced a balanced fractional derivative combining both right and left fractional Caputo derivatives.

We need

Theorem 2.22 ([4]). *Let $f\in AC^n([a,b])$, $\alpha>0$, $n\in\mathbb{N}$, $n:=\lceil\alpha\rceil$, $\alpha\geq\rho+1$, $\rho>0$, $r=\lceil\rho\rceil$, $\alpha,\rho\notin\mathbb{N}$. Assume $f^{(k)}(b)=0$, $k=r,r+1,\ldots,n-1$, and $\overline{D}_{b-}^{\alpha}f\in L_\infty([a,b])$. Then*

$$\overline{D}_{b-}^{\rho} f(x) = \left(I_{b-}^{\alpha-\rho}\left(\overline{D}_{b-}^{\alpha} f\right)\right)(x) \in AC([a,b]), \tag{2.89}$$

that is,

$$\overline{D}_{b-}^{\rho} f(x) = \frac{1}{\Gamma(\alpha-\rho)} \int_x^b (t-x)^{\alpha-\rho-1} \left(\overline{D}_{b-}^{\alpha} f\right)(t)\, dt, \tag{2.90}$$

$\forall\, x \in [a,b]$.

We give

Theorem 2.23. *Let (f_i, α_i, ρ_i), $i = 1, \ldots, m$, $m \geq 2$, as in the assumptions of Theorem 2.22. Set $\overline{\alpha} := \sum_{i=1}^{m} (\alpha_i - \rho_i)$, $\overline{\gamma} := \prod_{i=1}^{m} (\alpha_i - \rho_i)$, and let $p \geq 1$. Here $a, b \in \mathbb{R}$, $a < b$. Then*

$$\left\| \prod_{i=1}^{m} \left(\overline{D}_{b-}^{\rho_i} f_i\right) \right\|_{p,(a,b)} \leq \tag{2.91}$$

$$\left(\frac{\overline{\gamma}^{\frac{1}{p}} (b-a)^{\left(\overline{\alpha} - \frac{m}{p} + \frac{1}{p}\right)}}{\left(\prod_{i=1}^{m} \left(\Gamma(\alpha_i - \rho_i + 1)\right)\right) (\overline{\alpha} - m + 1)^{\frac{1}{p}}} \right) \left(\prod_{i=1}^{m} \left\| \overline{D}_{b-}^{\nu_i} f_i \right\|_{p,(a,b)} \right).$$

Proof. By (2.72) and (2.76), see here $\overline{\alpha} \geq m > m-1$. □

We make

Remark 2.24. Let $r_1, r_2 \in \mathbb{N}$; $A_j > 0$, $j = 1, \ldots, r_1$; $B_j > 0$, $j = 1, \ldots, r_2$; $x \geq 0$, $p \geq 1$. Clearly $e^{A_j x^p}, e^{B_j x^p} \geq 1$, and $\sum_{j=1}^{r_1} e^{A_j x^p} \geq r_1$, $\sum_{j=1}^{r_2} e^{B_j x^p} \geq r_2$. Hence, $\varphi_1(x) := \ln\left(\sum_{j=1}^{r_1} e^{A_j x^p}\right)$, $\varphi_2(x) := \ln\left(\sum_{j=1}^{r_2} e^{B_j x^p}\right) \geq 0$. Clearly here $\varphi_1, \varphi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are increasing, convex, and continuous.

We give

Theorem 2.25. *Let (f_i, α_i, ρ_i), $i = 1, 2$, as in the assumptions of Theorem 2.22. Set $\overline{\alpha} := \sum_{i=1}^{2} (\alpha_i - \rho_i)$, $\overline{\gamma} := \prod_{i=1}^{2} (\alpha_i - \rho_i)$. Here $a, b \in \mathbb{R}$, $a < b$, and φ_1, φ_2 as in Remark 2.24. Then*

$$\int_a^b (b-x)^{\overline{\alpha}} \prod_{i=1}^{2} \varphi_i \left(\frac{\left|\overline{D}_{b-}^{\rho_i} f_i(x)\right|}{(b-x)^{(\alpha_i-\rho_i)}} \Gamma(\alpha_i - \rho_i + 1) \right) dx \leq \tag{2.92}$$

$$\left(\frac{\overline{\gamma}(b-a)^{\overline{\alpha}-1}}{\overline{\alpha}-1} \right) \left(\prod_{i=1}^{2} \int_a^b \varphi_i \left(\left| \overline{D}_{b-}^{\alpha_i} f_i(x) \right| \right) dx \right),$$

under the assumptions ($i = 1, 2$):

(i) $\varphi_i\left(\left|\overline{D}_{b-}^{\alpha_i} f_i(t)\right|\right)$ *is* $\left(\chi_{[x,b)}(t) \frac{(t-x)^{\alpha_i-\rho_i-1}}{\Gamma(\alpha_i-\rho_i)} dt\right)$ *-integrable, a.e. in* $x \in (a,b)$.

(ii) $\varphi_i\left(\left|\overline{D}_{b-}^{\alpha_i} f_i\right|\right)$ *is Lebesgue integrable on* (a,b).

We make

Remark 2.26. (i) Let now $f \in C^n([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$. Clearly $C^n([a,b]) \subset AC^n([a,b])$. Assume $f^{(k)}(a) = 0$, $k = 0,1,\ldots,n-1$. Given that $D_{*a}^{\nu} f$ exists, then there exists the left generalized Riemann–Liouville fractional derivative $D_a^{\nu} f$ (see (2.78)) and $D_{*a}^{\nu} f = D_a^{\nu} f$ (see also [6], p. 53). In fact here $D_{*a}^{\nu} f \in C([a,b])$, see [6], p. 56.

So Theorems 2.19 and 2.20 can be true for left generalized Riemann–Liouville fractional derivatives.

(ii) Let also $\alpha > 0$, $n := \lceil \alpha \rceil$, and $f \in C^n([a,b]) \subset AC^n([a,b])$. From [2] we derive that $\overline{D}_{b-}^{\alpha} f \in C([a,b])$. By [2], we obtain that the right Riemann–Liouville fractional derivative $D_{b-}^{\alpha} f$ exists on $[a,b]$. Furthermore if $f^{(k)}(b) = 0$, $k = 0,1,\ldots,n-1$, we get that $\overline{D}_{b-}^{\alpha} f(x) = D_{b-}^{\alpha} f(x)$, $\forall\ x \in [a,b]$; hence $D_{b-}^{\alpha} f \in C([a,b])$.

So Theorems 2.23 and 2.25 can be valid for right Riemann–Liouville fractional derivatives. To keep this article short we avoid details.

We give

Definition 2.27. Let $\nu > 0$, $n := [\nu]$, $\alpha := \nu - n$ $(0 \le \alpha < 1)$. Let $a,b \in \mathbb{R}$, $a \le x \le b$, $f \in C([a,b])$. We consider $C_a^{\nu}([a,b]) := \{f \in C^n([a,b]) : I_{a+}^{1-\alpha} f^{(n)} \in C^1([a,b])\}$. For $f \in C_a^{\nu}([a,b])$, we define the left generalized ν-fractional derivative of f over $[a,b]$ as

$$\Delta_a^{\nu} f := \left(I_{a+}^{1-\alpha} f^{(n)}\right)'; \tag{2.93}$$

see [1], p. 24, and Canavati derivative in [5].

Notice here $\Delta_a^{\nu} f \in C([a,b])$.

So that

$$\left(\Delta_a^{\nu} f\right)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{-\alpha} f^{(n)}(t)\, dt, \tag{2.94}$$

$\forall\, x \in [a,b]$.

Notice here that

$$\Delta_a^{n} f = f^{(n)}, \quad n \in \mathbb{Z}_+. \tag{2.95}$$

We need

Theorem 2.28 ([4]). *Let* $f \in C_a^{\nu}([a,b])$, $n = [\nu]$, *such that* $f^{(i)}(a) = 0$, $i = r, r+1, \ldots, n-1$, *where* $r := [\rho]$, *with* $0 < \rho < \nu$. *Then*

$$\left(\Delta_a^{\rho} f\right)(x) = \frac{1}{\Gamma(\nu-\rho)} \int_a^x (x-t)^{\nu-\rho-1} \left(\Delta_a^{\nu} f\right)(t)\, dt, \tag{2.96}$$

i.e.,

$$\left(\Delta_a^{\rho} f\right) = I_{a+}^{\nu-\rho}\left(\Delta_a^{\nu} f\right) \in C\left([a,b]\right). \tag{2.97}$$

Thus $f \in C_a^{\rho}\left([a,b]\right)$.

We present

Theorem 2.29. *Let* (f_i, ν_i, ρ_i), $i = 1, \ldots, m$, *as in Theorem 2.28 and fractional derivatives as in Definition 2.27. Let* $\alpha := \sum_{i=1}^{m} (\nu_i - \rho_i)$, $\gamma := \prod_{i=1}^{m} (\nu_i - \rho_i)$, $p_i \geq 1$, $i = 1, \ldots, m$, *assume* $\alpha > m - 1$. *Then*

$$\int_a^b \prod_{i=1}^{m} \left|\Delta_a^{\rho_i} f_i(x)\right|^{p_i} dx \leq \tag{2.98}$$

$$\left(\frac{\gamma (b-a)^{\left(\left(\sum_{i=1}^{m} (\nu_i - \rho_i) p_i\right) - m + 1\right)}}{\left(\prod_{i=1}^{m} \left(\Gamma(\nu_i - \rho_i + 1)\right)^{p_i}\right)(\alpha - m + 1)} \right) \left(\prod_{i=1}^{m} \int_a^b \left|\Delta_a^{\nu_i} f_i(x)\right|^{p_i} dx \right).$$

Proof. By (2.52) and (2.55). □

We continue with

Theorem 2.30. *Let all here as in Theorem 2.29. Consider* λ_i, $i = 1, \ldots, m$, *distinct prime numbers. Then*

$$\int_a^b (x-a)^{\alpha} \prod_{i=1}^{m} \lambda_i^{\left(\left|\Delta_a^{\rho_i} f_i(x)\right| \frac{\Gamma(\nu_i - \rho_i + 1)}{(x-a)^{(\nu_i - \rho_i)}} \right)} dx \leq$$

$$\left(\frac{\gamma (b-a)^{\alpha - m + 1}}{\alpha - m + 1} \right) \left(\prod_{i=1}^{m} \int_a^b \lambda_i^{\left|\Delta_a^{\nu_i} f_i(x)\right|} dx \right). \tag{2.99}$$

Proof. By (2.51). □

We need

Definition 2.31 ([2]). Let $\nu > 0$, $n := [\nu]$, $\alpha = \nu - n$, $0 < \alpha < 1$, $f \in C\left([a,b]\right)$. Consider

$$C_{b-}^{\nu}\left([a,b]\right) := \{ f \in C^n\left([a,b]\right) : I_{b-}^{1-\alpha} f^{(n)} \in C^1\left([a,b]\right) \}. \tag{2.100}$$

Define the right generalized ν-fractional derivative of f over $[a,b]$, by

$$\Delta_{b-}^{\nu} f := (-1)^{n-1} \left(I_{b-}^{1-\alpha} f^{(n)} \right)'. \tag{2.101}$$

We set $\Delta_{b-}^{0} f = f$. Notice that

$$\left(\Delta_{b-}^{\nu} f\right)(x)=\frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{x}^{b}(J-x)^{-\alpha} f^{(n)}(J)\, dJ, \tag{2.102}$$

and $\Delta_{b-}^{\nu} f \in C([a,b])$.

We also need

Theorem 2.32 ([4]). *Let $f \in C_{b-}^{\nu}([a,b])$, $0<\rho<\nu$. Assume $f^{(i)}(b)=0$, $i=r,r+1,\ldots,n-1$, where $r:=[\rho]$, $n:=[\nu]$. Then*

$$\Delta_{b-}^{\rho} f(x)=\frac{1}{\Gamma(\nu-\rho)} \int_{x}^{b}(J-x)^{\nu-\rho-1}\left(\Delta_{b-}^{\nu} f\right)(J)\, dJ, \tag{2.103}$$

$\forall\, x \in [a,b]$, i.e.,

$$\Delta_{b-}^{\rho} f=I_{b-}^{\nu-\rho}\left(\Delta_{b-}^{\nu} f\right) \in C([a,b]), \tag{2.104}$$

and $f \in C_{b-}^{\rho}([a,b])$.

We give

Theorem 2.33. *Let (f_i,ν_i,ρ_i), $i=1,\ldots,m$, and fractional derivatives as in Theorem 2.32 and Definition 2.31. Let $\alpha:=\sum_{i=1}^{m}(\nu_i-\rho_i)$, $\gamma:=\prod_{i=1}^{m}(\nu_i-\rho_i)$, $p_i \geq 1$, $i=1,\ldots,m$, and assume $\alpha>m-1$. Then*

$$\int_{a}^{b} \prod_{i=1}^{m}\left|\Delta_{b-}^{\rho_i} f_i(x)\right|^{p_i} dx \leq \tag{2.105}$$

$$\left(\frac{\gamma(b-a)^{\left(\left(\sum_{i=1}^{m}(\nu_i-\rho_i)p_i\right)-m+1\right)}}{\left(\prod_{i=1}^{m}\left(\Gamma(\nu_i-\rho_i+1)\right)^{p_i}\right)(\alpha-m+1)}\right)\left(\prod_{i=1}^{m}\int_{a}^{b}\left|\Delta_{b-}^{\nu_i} f_i(x)\right|^{p_i} dx\right).$$

Proof. By (2.72) and (2.75). □

We continue with

Theorem 2.34. *Let all here as in Theorem 2.33. Consider λ_i, $i=1,\ldots,m$, distinct prime numbers. Then*

$$\int_{a}^{b}(b-x)^{\alpha} \prod_{i=1}^{m} \lambda_i^{\left(\left|\Delta_{b-}^{\rho_i} f_i(x)\right| \frac{\Gamma(\nu_i-\rho_i+1)}{(b-x)^{(\nu_i-\rho_i)}}\right)} dx \leq$$

$$\left(\frac{\gamma(b-a)^{\alpha-m+1}}{\alpha-m+1}\right)\left(\prod_{i=1}^{m}\int_{a}^{b} \lambda_i^{\left|\Delta_{b-}^{\nu_i} f_i(x)\right|} dx\right). \tag{2.106}$$

Proof. By (2.71). □

We make

Definition 2.35. [12, p. 99] The fractional integrals of a function f with respect to given function g are defined as follows:

Let $a,b \in \mathbb{R}$, $a < b$, $\alpha > 0$. Here g is an increasing function on $[a,b]$ and $g \in C^1([a,b])$. The left- and right-sided fractional integrals of a function f with respect to another function g in $[a,b]$ are given by

$$\left(I_{a+;g}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t) f(t) dt}{(g(x) - g(t))^{1-\alpha}}, \ x > a, \tag{2.107}$$

$$\left(I_{b-;g}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t) f(t) dt}{(g(t) - g(x))^{1-\alpha}}, \ x < b, \tag{2.108}$$

respectively.

We make

Remark 2.36. Let f_i be Lebesgue measurable functions from (a,b) into $\mathbb{R}$, such that $\left(I_{a+;g}^{\alpha_i}(|f_i|)\right)(x) \in \mathbb{R}$, $\forall\, x \in (a,b)$, $\alpha_i > 0$, $i = 1, \ldots, m$.

Consider

$$g_i(x) := \left(I_{a+;g}^{\alpha_i} f_i\right)(x), \ x \in (a,b), i = 1, \ldots, m, \tag{2.109}$$

where

$$\left(I_{a+;g}^{\alpha_i} f_i\right)(x) = \frac{1}{\Gamma(\alpha_i)} \int_a^x \frac{g'(t) f_i(t) dt}{(g(x) - g(t))^{1-\alpha_i}}, \ \ x > a. \tag{2.110}$$

Notice that $g_i(x) \in \mathbb{R}$ and it is Lebesgue measurable.

We pick $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$, the Lebesgue measure.

We see that

$$\left(I_{a+;g}^{\alpha_i} f_i\right)(x) = \int_a^b \frac{\chi_{(a,x]}(t) g'(t) f_i(t)}{\Gamma(\alpha_i)(g(x) - g(t))^{1-\alpha_i}} dt, \tag{2.111}$$

where χ is the characteristic function.

So, we pick here

$$k_i(x,t) := \frac{\chi_{(a,x]}(t) g'(t)}{\Gamma(\alpha_i)(g(x) - g(t))^{1-\alpha_i}}, \quad i = 1, \ldots, m. \tag{2.112}$$

In fact

$$k_i(x,y) = \begin{cases} \frac{g'(y)}{\Gamma(\alpha_i)(g(x)-g(y))^{1-\alpha_i}}, & a < y \leq x, \\ 0, & x < y < b. \end{cases} \tag{2.113}$$

Clearly it holds

$$K_i(x) = \int_a^b \frac{\chi_{(a,x]}(y) g'(y)}{\Gamma(\alpha_i)(g(x) - g(y))^{1-\alpha_i}} dy =$$

$$\int_a^x \frac{g'(y)}{\Gamma(\alpha_i)(g(x)-g(y))^{1-\alpha_i}}dy = \frac{1}{\Gamma(\alpha_i)}\int_a^x (g(x)-g(y))^{\alpha_i-1}dg(y) = \tag{2.114}$$
$$\frac{1}{\Gamma(\alpha_i)}\int_{g(a)}^{g(x)}(g(x)-z)^{\alpha_i-1}dz = \frac{(g(x)-g(a))^{\alpha_i}}{\Gamma(\alpha_i+1)}.$$

So for $a<x<b$, $i=1,\ldots,m$, we get

$$K_i(x) = \frac{(g(x)-g(a))^{\alpha_i}}{\Gamma(\alpha_i+1)}. \tag{2.115}$$

Notice that

$$\prod_{i=1}^m \frac{k_i(x,y)}{K_i(x)} = \prod_{i=1}^m \left(\frac{\chi_{(a,x]}(y)\,g'(y)}{\Gamma(\alpha_i)(g(x)-g(y))^{1-\alpha_i}}\cdot\frac{\Gamma(\alpha_i+1)}{(g(x)-g(a))^{\alpha_i}}\right) =$$
$$\frac{\chi_{(a,x]}(y)\,(g(x)-g(y))^{\left(\sum_{i=1}^m \alpha_i - m\right)}(g'(y))^m\left(\prod_{i=1}^m \alpha_i\right)}{(g(x)-g(a))^{\left(\sum_{i=1}^m \alpha_i\right)}}. \tag{2.116}$$

Calling

$$\alpha := \sum_{i=1}^m \alpha_i > 0, \;\; \gamma := \prod_{i=1}^m \alpha_i > 0, \tag{2.117}$$

we have that

$$\prod_{i=1}^m \frac{k_i(x,y)}{K_i(x)} = \frac{\chi_{(a,x]}(y)\,(g(x)-g(y))^{\alpha-m}(g'(y))^m\gamma}{(g(x)-g(a))^{\alpha}}. \tag{2.118}$$

Therefore, for (2.32), we get for appropriate weight u that (denote λ_m by λ_m^g)

$$\lambda_m^g(y) = \gamma\big(g'(y)\big)^m \int_y^b u(x)\frac{(g(x)-g(y))^{\alpha-m}}{(g(x)-g(a))^{\alpha}}dx < \infty, \tag{2.119}$$

for all $a<y<b$.

Let $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i=1,\ldots,m$, be convex and increasing functions. Then by (2.33) we obtain

$$\int_a^b u(x)\prod_{i=1}^m \Phi_i\left(\left|\frac{\left(I_{a+;g}^{\alpha_i} f_i\right)(x)}{(g(x)-g(a))^{\alpha_i}}\right|\Gamma(\alpha_i+1)\right)dx \leq$$
$$\left(\prod_{\substack{i=1\\ i\neq j}}^m \int_a^b \Phi_i(|f_i(x)|)\,dx\right)\left(\int_a^b \Phi_j\left(|f_j(x)|\right)\lambda_m^g(x)\,dx\right), \tag{2.120}$$

with $j \in \{1,\ldots,m\}$,

true for measurable f_i with $I^{\alpha_i}_{a+;g}(|f_i|)$ finite, $i = 1, \ldots, m$, and with the properties:

(i) $\Phi_i(|f_i|)$ is $k_i(x,y)\,dy$ -integrable, a.e. in $x \in (a,b)$.

(ii) $\lambda^g_m \Phi_j(|f_j|); \Phi_1(|f_1|), \Phi_2(|f_2|), \ldots, \widehat{\Phi_j(|f_j|)}, \ldots, \Phi_m(|f_m|)$ are all Lebesgue integrable functions, where $\widehat{\Phi_j(|f_j|)}$ means absent item.

Let now

$$u(x) = (g(x) - g(a))^{\alpha} g'(x), \quad x \in (a,b). \tag{2.121}$$

Then

$$\lambda^g_m(y) = \gamma(g'(y))^m \int_y^b (g(x) - g(y))^{\alpha - m} g'(x)\,dx =$$

$$\gamma(g'(y))^m \int_{g(y)}^{g(b)} (z - g(y))^{\alpha - m} dz = \tag{2.122}$$

$$\gamma(g'(y))^m \frac{(g(b) - g(y))^{\alpha - m + 1}}{\alpha - m + 1},$$

with $\alpha > m - 1$. That is,

$$\lambda^g_m(y) = \gamma(g'(y))^m \frac{(g(b) - g(y))^{\alpha - m + 1}}{\alpha - m + 1}, \tag{2.123}$$

$\alpha > m - 1$, $y \in (a,b)$.

Hence (2.120) becomes

$$\int_a^b g'(x)(g(x) - g(a))^{\alpha} \prod_{i=1}^m \Phi_i\left(\left|\frac{\left(I^{\alpha_i}_{a+;g} f_i\right)(x)}{(g(x) - g(a))^{\alpha_i}}\right| \Gamma(\alpha_i + 1)\right) dx \leq$$

$$\left(\frac{\gamma}{\alpha - m + 1}\right) \left(\prod_{\substack{i=1 \\ i \neq j}}^m \int_a^b \Phi_i(|f_i(x)|)\,dx\right) \cdot$$

$$\left(\int_a^b (g'(x))^m (g(b) - g(x))^{\alpha - m + 1} \Phi_j(|f_j(x)|)\,dx\right) \leq \tag{2.124}$$

$$\left(\frac{\gamma(g(b) - g(a))^{\alpha - m + 1} \|g'\|^m_{\infty}}{\alpha - m + 1}\right) \left(\prod_{i=1}^m \int_a^b \Phi_i(|f_i(x)|)\,dx\right),$$

where $\alpha > m - 1$, f_i with $I^{\alpha_i}_{a+;g}(|f_i|)$ finite, $i = 1, \ldots, m$, under the assumptions:

(i) $\Phi_i(|f_i|)$ is $k_i(x,y)\,dy$ -integrable, a.e. in $x \in (a,b)$.

(ii) $\Phi_i(|f_i|)$ is Lebesgue integrable on (a,b).

If $\Phi_i(x) = x^{p_i}$, $p_i \geq 1$, $x \in \mathbb{R}_+$, then by (2.124), we have

$$\int_a^b g'(x)(g(x) - g(a))^{\left(\alpha - \sum_{i=1}^m p_i \alpha_i\right)} \prod_{i=1}^m \left|\left(I^{\alpha_i}_{a+;g} f_i\right)(x)\right|^{p_i} dx \leq \tag{2.125}$$

$$\left(\frac{\gamma\left(g\left(b\right)-g\left(a\right)\right)^{\alpha-m+1}\left\|g'\right\|_{\infty}^{m}}{\left(\prod_{i=1}^{m}\left(\Gamma\left(\alpha_i+1\right)\right)^{p_i}\right)\left(\alpha-m+1\right)}\right)\left(\prod_{i=1}^{m}\int_a^b\left|f_i\left(x\right)\right|^{p_i}dx\right),$$

but we see that

$$\int_a^b g'\left(x\right)\left(g\left(x\right)-g\left(a\right)\right)^{\left(\alpha-\sum_{i=1}^{m}p_i\alpha_i\right)}\prod_{i=1}^{m}\left|\left(I_{a+;g}^{\alpha_i}f_i\right)\left(x\right)\right|^{p_i}dx\geq$$

$$\left(g\left(b\right)-g\left(a\right)\right)^{\left(\alpha-\sum_{i=1}^{m}p_i\alpha_i\right)}\int_a^b g'\left(x\right)\prod_{i=1}^{m}\left|\left(I_{a+;g}^{\alpha_i}f_i\right)\left(x\right)\right|^{p_i}dx. \tag{2.126}$$

By (2.125) and (2.126) we get

$$\int_a^b g'\left(x\right)\prod_{i=1}^{m}\left|\left(I_{a+;g}^{\alpha_i}f_i\right)\left(x\right)\right|^{p_i}dx\leq \tag{2.127}$$

$$\left(\frac{\gamma\left(g\left(b\right)-g\left(a\right)\right)^{\left(\sum_{i=1}^{m}p_i\alpha_i-m+1\right)}\left\|g'\right\|_{\infty}^{m}}{\left(\prod_{i=1}^{m}\left(\Gamma\left(\alpha_i+1\right)\right)^{p_i}\right)\left(\alpha-m+1\right)}\right)\left(\prod_{i=1}^{m}\int_a^b\left|f_i\left(x\right)\right|^{p_i}dx\right),$$

$\alpha>m-1$, f_i with $I_{a+;g}^{\alpha_i}\left(\left|f_i\right|\right)$ finite, $i=1,\ldots,m$, under the assumptions:

(i) $\left|f_i\right|^{p_i}$ is $k_i\left(x,y\right)dy$ -integrable, a.e. in $x\in\left(a,b\right)$.
(ii) $\left|f_i\right|^{p_i}$ is Lebesgue integrable on $\left(a,b\right)$.

We need

Definition 2.37 ([11]). Let $0<a<b<\infty$, $\alpha>0$. The left- and right-sided Hadamard fractional integrals of order α are given by

$$\left(J_{a+}^{\alpha}f\right)\left(x\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_a^x\left(\ln\frac{x}{y}\right)^{\alpha-1}\frac{f\left(y\right)}{y}dy,\quad x>a, \tag{2.128}$$

and

$$\left(J_{b-}^{\alpha}f\right)\left(x\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_x^b\left(\ln\frac{y}{x}\right)^{\alpha-1}\frac{f\left(y\right)}{y}dy,\quad x<b, \tag{2.129}$$

respectively.

Notice that the Hadamard fractional integrals of order α are special cases of left- and right-sided fractional integrals of a function f with respect to another function, here $g\left(x\right)=\ln x$ on $[a,b]$, $0<a<b<\infty$.

Above f is a Lebesgue measurable function from $\left(a,b\right)$ into $\mathbb{R}$, such that $\left(J_{a+}^{\alpha}\left(\left|f\right|\right)\right)\left(x\right)$ and/or $\left(J_{b-}^{\alpha}\left(\left|f\right|\right)\right)\left(x\right)\in\mathbb{R}$, $\forall\, x\in\left(a,b\right)$.

We give

Theorem 2.38. *Let (f_i, α_i), $i = 1, \ldots, m$; $J_{a+}^{\alpha_i} f_i$ as in Definition 2.37. Set $\alpha := \sum_{i=1}^{m} \alpha_i$, $\gamma := \prod_{i=1}^{m} \alpha_i$; $p_i \geq 1$, $i = 1, \ldots, m$, assume $\alpha > m - 1$. Then*

$$\int_a^b \prod_{i=1}^{m} \left| \left(J_{a+}^{\alpha_i} f_i \right)(x) \right|^{p_i} dx \leq \tag{2.130}$$

$$\left(\frac{b\gamma \left(\ln\left(\frac{b}{a}\right)\right)^{\left(\sum_{i=1}^{m} p_i\alpha_i - m + 1\right)}}{a^m (\alpha - m + 1) \left(\prod_{i=1}^{m} (\Gamma(\alpha_i + 1))^{p_i} \right)} \right) \left(\prod_{i=1}^{m} \int_a^b |f_i(x)|^{p_i} dx \right),$$

where $J_{a+}^{\alpha_i}(|f_i|)$ is finite, $i = 1, \ldots, m$, under the assumptions:

(i) $|f_i(y)|^{p_i}$ is $\left(\frac{\chi_{(a,x]}(y)dy}{\Gamma(\alpha_i) y \left(\ln\left(\frac{x}{y}\right)\right)^{1-\alpha_i}} \right)$-integrable, a.e. in $x \in (a,b)$.

(ii) $|f_i|^{p_i}$ is Lebesgue integrable on (a,b).

We also present

Theorem 2.39. *Let all as in Theorem 2.38. Consider $p := p_1 = p_2 = \ldots = p_m \geq 1$. Then*

$$\left\| \prod_{i=1}^{m} \left(J_{a+}^{\alpha_i} f_i \right) \right\|_{p,(a,b)} \leq \tag{2.131}$$

$$\left(\frac{(b\gamma)^{\frac{1}{p}} \left(\ln\left(\frac{b}{a}\right)\right)^{\left(\alpha - \frac{m}{p} + \frac{1}{p}\right)}}{a^{\frac{m}{p}} (\alpha - m + 1)^{\frac{1}{p}} \left(\prod_{i=1}^{m} (\Gamma(\alpha_i + 1)) \right)} \right) \left(\prod_{i=1}^{m} \|f_i\|_{p,(a,b)} \right),$$

where $J_{a+}^{\alpha_i}(|f_i|)$ is finite, $i = 1, \ldots, m$, under the assumptions:

(i) $|f_i(y)|^{p}$ is $\left(\frac{\chi_{(a,x]}(y)dy}{\Gamma(\alpha_i) y \left(\ln\left(\frac{x}{y}\right)\right)^{1-\alpha_i}} \right)$-integrable, a.e. in $x \in (a,b)$.

(ii) $|f_i|^{p}$ is Lebesgue integrable on (a,b).

We make

Remark 2.40. Let f_i be Lebesgue measurable functions from (a,b) into $\mathbb{R}$, such that $\left(I_{b-;g}^{\alpha_i}(|f_i|) \right)(x) \in \mathbb{R}$, $\forall\, x \in (a,b)$, $\alpha_i > 0$, $i = 1, \ldots, m$.

Consider

$$g_i(x) := \left(I_{b-;g}^{\alpha_i} f_i \right)(x), \;\; x \in (a,b), \, i = 1, \ldots, m, \tag{2.132}$$

where

$$\left(I_{b-;g}^{\alpha_i} f_i \right)(x) = \frac{1}{\Gamma(\alpha_i)} \int_x^b \frac{g'(t) f(t) dt}{(g(t) - g(x))^{1-\alpha_i}}, \;\; x < b. \tag{2.133}$$

Notice that $g_i(x) \in \mathbb{R}$ and it is Lebesgue measurable.

We pick $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$, the Lebesgue measure.
We see that

$$\left(I_{b-;g}^{\alpha_i} f_i\right)(x) = \int_a^b \frac{\chi_{[x,b)}(t)\, g'(t)\, f(t)\, dt}{\Gamma(\alpha_i)\,(g(t) - g(x))^{1-\alpha_i}}, \tag{2.134}$$

where χ is the characteristic function.

So, we pick here

$$k_i(x,y) := \frac{\chi_{[x,b)}(y)\, g'(y)}{\Gamma(\alpha_i)\,(g(y) - g(x))^{1-\alpha_i}}, \quad i = 1, \ldots, m. \tag{2.135}$$

In fact

$$k_i(x,y) = \begin{cases} \frac{g'(y)}{\Gamma(\alpha_i)(g(y)-g(x))^{1-\alpha_i}}, & x \le y < b, \\ 0, \quad a < y < x. \end{cases} \tag{2.136}$$

Clearly it holds

$$K_i(x) = \int_a^b \frac{\chi_{[x,b)}(y)\, g'(y)\, dy}{\Gamma(\alpha_i)\,(g(y) - g(x))^{1-\alpha_i}} =$$

$$\frac{1}{\Gamma(\alpha_i)} \int_x^b g'(y)\,(g(y) - g(x))^{\alpha_i - 1}\, dy = \tag{2.137}$$

$$\frac{1}{\Gamma(\alpha_i)} \int_{g(x)}^{g(b)} (z - g(x))^{\alpha_i - 1}\, dg(y) = \frac{(g(b) - g(x))^{\alpha_i}}{\Gamma(\alpha_i + 1)}.$$

So for $a < x < b$, $i = 1, \ldots, m$, we get

$$K_i(x) = \frac{(g(b) - g(x))^{\alpha_i}}{\Gamma(\alpha_i + 1)}. \tag{2.138}$$

Notice that

$$\prod_{i=1}^m \frac{k_i(x,y)}{K_i(x)} = \prod_{i=1}^m \left(\frac{\chi_{[x,b)}(y)\, g'(y)}{\Gamma(\alpha_i)\,(g(y) - g(x))^{1-\alpha_i}} \cdot \frac{\Gamma(\alpha_i + 1)}{(g(b) - g(x))^{\alpha_i}} \right) =$$

$$\frac{\chi_{[x,b)}(y)\,(g'(y))^m\,(g(y) - g(x))^{\left(\sum_{i=1}^m \alpha_i - m\right)} \prod_{i=1}^m \alpha_i}{(g(b) - g(x))^{\sum_{i=1}^m \alpha_i}}. \tag{2.139}$$

Calling

$$\alpha := \sum_{i=1}^m \alpha_i > 0, \;\; \gamma := \prod_{i=1}^m \alpha_i > 0, \tag{2.140}$$

we have that

$$\prod_{i=1}^{m}\frac{k_i(x,y)}{K_i(x)}=\frac{\chi_{[x,b)}(y)\,(g'(y))^m\,(g(y)-g(x))^{\alpha-m}\,\gamma}{(g(b)-g(x))^{\alpha}}. \tag{2.141}$$

Therefore, for (2.32), we get for appropriate weight u that (denote λ_m by λ_m^g)

$$\lambda_m^g(y)=\gamma\left(g'(y)\right)^m\int_a^y u(x)\,\frac{(g(y)-g(x))^{\alpha-m}}{(g(b)-g(x))^{\alpha}}\,dx<\infty, \tag{2.142}$$

for all $a<y<b$.

Let $\Phi_i:\mathbb{R}_+\to\mathbb{R}_+$, $i=1,\dots,m$, be convex and increasing functions. Then by (2.33) we obtain

$$\int_a^b u(x)\prod_{i=1}^{m}\Phi_i\left(\left|\frac{\left(I_{b-;g}^{\alpha_i}f_i\right)(x)}{(g(b)-g(x))^{\alpha_i}}\right|\Gamma(\alpha_i+1)\right)dx\leq$$

$$\left(\prod_{\substack{i=1\\ i\neq j}}^{m}\int_a^b\Phi_i(|f_i(x)|)\,dx\right)\left(\int_a^b\Phi_j\left(\left|f_j(x)\right|\right)\lambda_m^g(x)\,dx\right), \tag{2.143}$$

with $j\in\{1,\dots,m\}$,

true for measurable f_i with $I_{b-;g}^{\alpha_i}(|f_i|)$ finite, $i=1,\dots,m$, and with the properties:

(i) $\Phi_i(|f_i|)$ is $k_i(x,y)\,dy$ -integrable, a.e. in $x\in(a,b)$.

(ii) $\lambda_m^g\Phi_j\left(\left|f_j\right|\right);\Phi_1(|f_1|),\dots,\widehat{\Phi_j\left(\left|f_j\right|\right)},\dots,\Phi_m(|f_m|)$ are all Lebesgue integrable functions, where $\widehat{\Phi_j\left(\left|f_j\right|\right)}$ means absent item.

Let now

$$u(x)=(g(b)-g(x))^{\alpha}g'(x),\quad x\in(a,b). \tag{2.144}$$

Then

$$\lambda_m^g(y)=\gamma\left(g'(y)\right)^m\int_a^y g'(x)\,(g(y)-g(x))^{\alpha-m}\,dx=$$

$$\gamma\left(g'(y)\right)^m\int_a^y(g(y)-g(x))^{\alpha-m}\,dg(x)=\gamma\left(g'(y)\right)^m\int_{g(a)}^{g(y)}(g(y)-z)^{\alpha-m}\,dz= \tag{2.145}$$

$$\gamma\left(g'(y)\right)^m\frac{(g(y)-g(a))^{\alpha-m+1}}{\alpha-m+1},$$

with $\alpha>m-1$. That is,

$$\lambda_m^g(y)=\gamma\left(g'(y)\right)^m\frac{(g(y)-g(a))^{\alpha-m+1}}{\alpha-m+1}, \tag{2.146}$$

$\alpha>m-1$, $y\in(a,b)$.

Hence (2.143) becomes

$$\int_a^b g'(x)(g(b)-g(x))^{\alpha}\prod_{i=1}^m \Phi_i\left(\left|\frac{\left(I_{b-;g}^{\alpha_i}f_i\right)(x)}{(g(b)-g(x))^{\alpha_i}}\right|\Gamma(\alpha_i+1)\right)dx \leq$$

$$\left(\frac{\gamma}{\alpha-m+1}\right)\left(\prod_{\substack{i=1\\ i\neq j}}^m \int_a^b \Phi_i(|f_i(x)|)\,dx\right)\cdot$$

$$\left(\int_a^b \Phi_j\left(\left|f_j(x)\right|\right)\left(g'(x)\right)^m (g(x)-g(a))^{\alpha-m+1}dx\right) \leq \tag{2.147}$$

$$\left(\frac{\gamma(g(b)-g(a))^{\alpha-m+1}\|g'\|_{\infty}^m}{\alpha-m+1}\right)\left(\prod_{i=1}^m \int_a^b \Phi_i(|f_i(x)|)\,dx\right),$$

where $\alpha > m-1$, f_i with $I_{b-;g}^{\alpha_i}(|f_i|)$ finite, $i = 1,\ldots,m$, under the assumptions:

(i) $\Phi_i(|f_i|)$ is $k_i(x,y)\,dy$ -integrable, a.e. in $x \in (a,b)$.
(ii) $\Phi_i(|f_i|)$ is Lebesgue integrable on (a,b).

If $\Phi_i(x) = x^{p_i}$, $p_i \geq 1$, $x \in \mathbb{R}_+$, then by (2.147), we have

$$\int_a^b g'(x)(g(b)-g(x))^{\left(\alpha-\sum_{i=1}^m \alpha_i p_i\right)}\prod_{i=1}^m \left|\left(I_{b-;g}^{\alpha_i}f_i\right)(x)\right|^{p_i}dx \leq \tag{2.148}$$

$$\left(\frac{\gamma(g(b)-g(a))^{\alpha-m+1}\left(\|g'\|_{\infty}\right)^m}{(\alpha-m+1)\prod_{i=1}^m(\Gamma(\alpha_i+1))^{p_i}}\right)\left(\prod_{i=1}^m \int_a^b |f_i(x)|^{p_i}dx\right),$$

but we see that

$$\int_a^b g'(x)(g(b)-g(x))^{\left(\alpha-\sum_{i=1}^m \alpha_i p_i\right)}\prod_{i=1}^m \left|\left(I_{b-;g}^{\alpha_i}f_i\right)(x)\right|^{p_i}dx \geq$$

$$(g(b)-g(a))^{\left(\alpha-\sum_{i=1}^m \alpha_i p_i\right)}\int_a^b g'(x)\prod_{i=1}^m \left|\left(I_{b-;g}^{\alpha_i}f_i\right)(x)\right|^{p_i}dx. \tag{2.149}$$

Hence by (2.148) and (2.149) we derive

$$\int_a^b g'(x)\prod_{i=1}^m \left|\left(I_{b-;g}^{\alpha_i}f_i\right)(x)\right|^{p_i}dx \leq \tag{2.150}$$

$$\left(\frac{\gamma(g(b)-g(a))^{\left(\sum_{i=1}^m p_i\alpha_i-m+1\right)}\|g'\|_{\infty}^m}{\left(\prod_{i=1}^m(\Gamma(\alpha_i+1))^{p_i}\right)(\alpha-m+1)}\right)\left(\prod_{i=1}^m \int_a^b |f_i(x)|^{p_i}dx\right),$$

$\alpha > m-1$, f_i with $I_{b-;g}^{\alpha_i}(|f_i|)$ finite, $i = 1,\ldots,m$, under the assumptions:

(i) $|f_i|^{p_i}$ is $k_i(x,y)\,dy$ -integrable, a.e. in $x \in (a,b)$.
(ii) $|f_i|^{p_i}$ is Lebesgue integrable on (a,b).

We give

Theorem 2.41. *Let (f_i,α_i), $i=1,\ldots,m$; $J_{b-}^{\alpha_i} f_i$ as in Definition 2.37. Set $\alpha := \sum_{i=1}^{m} \alpha_i$, $\gamma := \prod_{i=1}^{m} \alpha_i$; $p_i \geq 1$, $i=1,\ldots,m$, assume $\alpha > m-1$. Then*

$$\int_a^b \prod_{i=1}^{m} \left| \left(J_{b-}^{\alpha_i} f_i \right)(x) \right|^{p_i} dx \leq \tag{2.151}$$

$$\left(\frac{b\gamma \left(\ln\left(\frac{b}{a}\right)\right)^{\left(\sum_{i=1}^{m} p_i\alpha_i - m+1\right)}}{a^m (\alpha - m+1) \left(\prod_{i=1}^{m} \left(\Gamma(\alpha_i+1)\right)^{p_i} \right)} \right) \left(\prod_{i=1}^{m} \int_a^b |f_i(x)|^{p_i}\, dx \right),$$

where $J_{b-}^{\alpha_i}(|f_i|)$ is finite, $i=1,\ldots,m$, under the assumptions:

(i) $|f_i(y)|^{p_i}$ is $\left(\frac{\chi_{[x,b)}(y)\,dy}{\Gamma(\alpha_i)\, y \left(\ln\left(\frac{y}{x}\right)\right)^{1-\alpha_i}} \right)$ -integrable, a.e. in $x \in (a,b)$.
(ii) $|f_i|^{p_i}$ is Lebesgue integrable on (a,b).

We finish with

Theorem 2.42. *Let all as in Theorem 2.41. Take $p := p_1 = p_2 = \ldots = p_m \geq 1$. Then*

$$\left\| \prod_{i=1}^{m} \left(J_{b-}^{\alpha_i} f_i \right) \right\|_{p,(a,b)} \leq \tag{2.152}$$

$$\left(\frac{(b\gamma)^{\frac{1}{p}} \left(\ln\left(\frac{b}{a}\right)\right)^{\left(\alpha - \frac{m}{p} + \frac{1}{p}\right)}}{a^{\frac{m}{p}} (\alpha - m+1)^{\frac{1}{p}} \left(\prod_{i=1}^{m} \left(\Gamma(\alpha_i+1)\right) \right)} \right) \left(\prod_{i=1}^{m} \|f_i\|_{p,(a,b)} \right),$$

where $J_{b-}^{\alpha_i}(|f_i|)$ is finite, $i=1,\ldots,m$, under the properties:

(i) $|f_i(y)|^p$ is $\left(\frac{\chi_{[x,b)}(y)\,dy}{\Gamma(\alpha_i)\, y \left(\ln\left(\frac{y}{x}\right)\right)^{1-\alpha_i}} \right)$ -integrable, a.e. in $x \in (a,b)$.
(ii) $|f_i|^p$ is Lebesgue integrable on (a,b).

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Chapter 3
Statistical Convergence on Timescales and Its Characterizations

Ceylan Turan and Oktay Duman

Abstract In this paper, we introduce the concept of statistical convergence of delta measurable real-valued functions defined on time scales. The classical cases of our definition include many well-known convergence methods and also suggest many new ones. We obtain various characterizations on statistical convergence.

3.1 Introduction

The main features of the timescales calculus which was first introduced by Hilger [14] are to unify the discrete and continuous cases and to extend them in order to obtain some new methods. This method of calculus is also effective in modeling some real-life problems. For example, in modeling insect populations, one may need both discrete and continuous time variables. There are also many applications of timescales on dynamic equations (see, for instance, [6]). However, so far, there is no any usage of timescale in the summability theory. The aim of this paper is to fill this gap in the literature and to generate a new research area. More precisely, in this paper, we study the concept of statistical convergence of functions defined on appropriate timescales. Recall that the statistical convergence of number sequences (i.e., the case of a discrete timescale) introduced by Fast [10] is the well-known topic in the summability theory and also its continuous version was studied by Móricz [15].

It is well known from the classical analysis that if a number sequence is convergent, then almost all terms of the sequence have to belong to arbitrarily small neighborhood of the limit. The main idea of statistical convergence (of a number sequence) is to weaken this condition and to demand validity of the convergence condition only for a majority of elements. Note that the classical limit implies the statistical convergence, but the converse does not hold true. This method of

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convergence has been investigated in many areas of mathematics, such as measure theory, approximation theory, fuzzy logic theory, and summability theory. These studies demonstrate that the concept of statistical convergence provides an important contribution to improvement of the classical analysis.

Firstly we recall some basic concepts and notations from the theory of timescales. A timescale $\mathbb{T}$ is any nonempty closed subset of $\mathbb{R}$, the set of real numbers. The forward and backward jump operators from $\mathbb{T}$ into itself are defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$. A closed interval in a timescale $\mathbb{T}$ is given by $[a,b]_{\mathbb{T}} := \{t \in \mathbb{T} : a \leq t \leq b\}$. Open intervals or half-open intervals are defined accordingly.

Now let $\mathscr{T}_1$ denote the family of all left closed and right open intervals of $\mathbb{T}$ of the form $[a,b)_{\mathbb{T}}$. Let $m_1 : \mathscr{T}_1 \to [0,\infty]$ be a set function on $\mathscr{T}_1$ such that $m_1([a,b)_{\mathbb{T}}) = b-a$. Then, it is known that m_1 is a countably additive measure on $\mathscr{T}_1$. Now, the Carathéodory extension of the set function m_1 associated with family $\mathscr{T}_1$ is said to be the Lebesgue Δ-measure on $\mathbb{T}$ and is denoted by μ_Δ (see [3, 13] for details). In this case, we know from [13] that:

- If $a \in \mathbb{T}\backslash\{\max\mathbb{T}\}$, then the single-point set $\{a\}$ is Δ-measurable and $\mu_\Delta(a) = \sigma(a) - a$.
- If $a,b \in \mathbb{T}$ and $a \leq b$, then $\mu_\Delta([a,b)_{\mathbb{T}}) = b-a$ and $\mu_\Delta((a,b)_{\mathbb{T}}) = b - \sigma(a)$.
- If $a,b \in \mathbb{T}\backslash\{\max\mathbb{T}\}$ and $a \leq b$, $\mu_\Delta((a,b]_{\mathbb{T}}) = \sigma(b) - \sigma(a)$ and $\mu_\Delta([a,b]_{\mathbb{T}}) = \sigma(b) - a$.

3.2 Density and Statistical Convergence on Timescales

In this section, we focus on constructing a concept of statistical convergence on timescales. To see that we first need a definition of density function on timescales. So, we mainly use the Lebesgue Δ-measure μ_Δ introduced by Guseinov [13].

We should note that throughout the paper, we assume that $\mathbb{T}$ is a timescale satisfying $\inf\mathbb{T} = t_0 > 0$ and $\sup\mathbb{T} = \infty$.

Definition 3.1. Let Ω be a Δ-measurable subset of $\mathbb{T}$. Then, for $t \in \mathbb{T}$, we define the set $\Omega(t)$ by

$$\Omega(t) := \{s \in [t_0,t]_{\mathbb{T}} : s \in \Omega\}.$$

In this case, we define the density of Ω on $\mathbb{T}$, denoted by $\delta_{\mathbb{T}}(\Omega)$, as follows:

$$\delta_{\mathbb{T}}(\Omega) := \lim_{t\to\infty} \frac{\mu_\Delta(\Omega(t))}{\mu_\Delta([t_0,t]_{\mathbb{T}})}$$

provided that the above limit exists.

We should note that the discrete case of Definition 3.1, i.e., the case of $\mathbb{T} = \mathbb{N}$, reduces to the concept of asymptotic density (see, for instance, [16]); also, the continuous case of this definition, i.e., the case of $\mathbb{T} = [0,\infty)$, turns out to be the concept of approximate density which was first considered by Denjoy [9] (see also [15]). So, by choosing suitable timescales, our definition fulfills the gap between the discrete and continuous cases.

It follows from Definition 3.1 that:

- $\delta_{\mathbb{T}}(\mathbb{T}) = 1$
- $0 \leq \delta_{\mathbb{T}}(\Omega) \leq 1$ for any Δ-measurable subset Ω of $\mathbb{T}$

Assume now that A and B are Δ-measurable subsets of $\mathbb{T}$ and that $\delta_{\mathbb{T}}(A)$, $\delta_{\mathbb{T}}(B)$ exist. Then, it is easy to check the following properties of the density:

- $\delta_{\mathbb{T}}(A \cup B) \leqslant \delta_{\mathbb{T}}(A) + \delta_{\mathbb{T}}(B)$
- If $A \cap B = \varnothing$, then $\delta_{\mathbb{T}}(A \cup B) = \delta_{\mathbb{T}}(A) + \delta_{\mathbb{T}}(B)$
- $\delta_{\mathbb{T}}(\mathbb{T} \backslash A) = 1 - \delta_{\mathbb{T}}(A)$
- If $A \subset B$, then $\delta_{\mathbb{T}}(A) \leqslant \delta_{\mathbb{T}}(B)$
- If A is bounded, then $\delta_{\mathbb{T}}(A) = 0$

Furthermore, we get the next lemma.

Lemma 3.2. *Assume that A and B are Δ-measurable subsets of $\mathbb{T}$ for which $\delta_{\mathbb{T}}(A) = \delta_{\mathbb{T}}(B) = 1$ hold. Then, we have*

$$\delta_{\mathbb{T}}(A \cup B) = \delta_{\mathbb{T}}(A \cap B) = 1.$$

Proof. Since $A \subset A \cup B$, it follows from the above properties that $\delta_{\mathbb{T}}(A) \leqslant \delta_{\mathbb{T}}(A \cup B)$, which implies $\delta_{\mathbb{T}}(A \cup B) = 1$. On the other hand, since $A \cup B = (A \backslash B) \cup (B \backslash A) \cup (A \cap B)$, we see that $\delta_{\mathbb{T}}(A \cup B) = \delta_{\mathbb{T}}(A \backslash B) + \delta_{\mathbb{T}}(B \backslash A) + \delta_{\mathbb{T}}(A \cap B)$. Also, using the fact that $A \backslash B \subset \mathbb{T} \backslash B$, we obtain $\delta_{\mathbb{T}}(A \backslash B) \leq \delta_{\mathbb{T}}(\mathbb{T} \backslash B) = 0$, which gives $\delta_{\mathbb{T}}(A \backslash B) = 0$. Similarly, one can show that $\delta_{\mathbb{T}}(B \backslash A) = 0$. Then, combining them, we see that $\delta_{\mathbb{T}}(A \cap B) = 1$, which completes the proof. □

Now we are ready to give the definition of statistical convergence of real-valued function f defined on a timescale $\mathbb{T}$ satisfying $\inf \mathbb{T} = t_0$ and $\sup \mathbb{T} = \infty$.

Definition 3.3. Let $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function. We say that f is statistically convergent on $\mathbb{T}$ to a number L if, for every $\varepsilon > 0$,

$$\delta_{\mathbb{T}}(\{t \in \mathbb{T} : |f(t) - L| \geqslant \varepsilon\}) = 0 \tag{3.1}$$

holds. Then, we denote this statistical limit as follows:

$$st_{\mathbb{T}} - \lim_{t \to \infty} f(t) = L.$$

It is not hard to see that (3.1) can be written as follows:

$$\lim_{t \to \infty} \frac{\mu_{\Delta}(\{s \in [t_0, t]_{\mathbb{T}} : |f(s) - L| \geqslant \varepsilon\})}{\mu_{\Delta}([t_0, t]_{\mathbb{T}})} = 0.$$

Definition 3.4. Let $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function. We say that the function f is statistical Cauchy on $\mathbb{T}$ if, for every $\varepsilon > 0$, there exists a number $t_1 > t_0$ such that

$$\lim_{t \to \infty} \frac{\mu_{\Delta}(\{s \in [t_0, t]_{\mathbb{T}} : |f(s) - f(t_1)| \geqslant \varepsilon\})}{\mu_{\Delta}([t_0, t]_{\mathbb{T}})} = 0.$$

A few obvious properties of Definition 3.3 are as follows:
Let $f,g\colon \mathbb{T}\to\mathbb{R}$ be Δ-measurable functions and $\alpha\in\mathbb{R}$. Then, we have:

- If $st_{\mathbb{T}}-\lim_{t\to\infty} f(t)=L_1$ and $st_{\mathbb{T}}-\lim_{t\to\infty} f(t)=L_2$, then $L_1=L_2$
- If $st_{\mathbb{T}}-\lim_{t\to\infty} f(t)=L$, then $st_{\mathbb{T}}-\lim_{t\to\infty} (\alpha f(t))=\alpha L$
- If $st_{\mathbb{T}}-\lim_{t\to\infty} f(t)=L$ and $st_{\mathbb{T}}-\lim_{t\to\infty} g(t)=M$, then $st_{\mathbb{T}}-\lim_{t\to\infty} f(t).g(t)=LM$

We should note that after searching the website, arxiv.org, we discovered that Definitions 3.1–3.4 were also obtained in a non-published article by Seyyidoglu and Tan [19]. They only proved the next result (see Theorem 3.5). However, in this paper, we obtain many new characterizations and applications of statistical convergence on timescales.

Theorem 3.5. *(see also [19]) Let $f:\mathbb{T}\to\mathbb{R}$ be a Δ-measurable function. Then the following statements are equivalent:*

(i) f is statistical convergent on $\mathbb{T}$.
(ii) f is statistical Cauchy on $\mathbb{T}$.
(iii) f can be represented as the sum of two Δ-measurable functions g and h such that $\lim_{t\to\infty} g(t)=st_{\mathbb{T}}-\lim_{t\to\infty} f(t)$ and $\delta_{\mathbb{T}}(\{t\in\mathbb{T}:h(t)\neq 0\})=0$. Moreover, if f is bounded, then both g and h are also bounded.

It is not hard to see that the discrete version of Theorem 3.5 reduces to Theorem 1 introduced by Fridy [12] and also the continuous one turns out to be Theorem 1 proved by Móricz [15]. The above results can be easily obtained by using the same proof techniques in [12, 15].

Now we display some applications of Definition 3.3. We will see that many well-known convergence methods can be obtained from Definition 3.3. Some of them are as follows:

Example 3.6. Let $\mathbb{T}=\mathbb{N}$ in Definition 3.3. In this case, replacing t with n and using the fact that $t_0=1$, we get

$$\mu_\Delta([1,n]_{\mathbb{N}})=\mu_\Delta(\{1,2,3,\ldots,n\})=\sigma(n)-1=(n+1)-1=n.$$

Also, we see that

$$\begin{aligned}\mu_\Delta(\{k\in[1,n]_{\mathbb{N}}:|f(k)-L|\geqslant\varepsilon\}) &= \mu_\Delta(\{1\leqslant k\leqslant n:|f(k)-L|\geqslant\varepsilon\}) \\ &= \#\{1\leqslant k\leqslant n:|f(k)-L|\geqslant\varepsilon\},\end{aligned}$$

where $\#B$ denotes the cardinality of the set B. Then, we can write, for $\mathbb{T}=\mathbb{N}$, that

$$st_{\mathbb{N}}-\lim_{n\to\infty} f(n)=L$$

is equivalent to

$$\lim_{n\to\infty}\frac{\#\{1\leqslant k\leqslant n:|f(k)-L|\geqslant\varepsilon\}}{n}=0, \tag{3.2}$$

which is the classical statistical convergence of the sequence $(x_k) := (f(k))$ to L (see [10]). Note that the statistical convergence in (3.2) is denoted by

$$st - \lim_{k\to\infty} x_k = L$$

in the literature.

Example 3.7. If we choose $\mathbb{T} = [a,\infty)$ $(a > 0)$ in Definition 3.3, then we immediately obtain the convergence method introduced by Móricz [15]. Indeed, since $t_0 = a$, observe that

$$\mu_\Delta\left([a,t]_{[a,\infty)}\right) = \mu_\Delta([a,t]) = \sigma(t) - a = t - a,$$

and also since $\mathbb{T} = [a,\infty)$,

$$\begin{aligned}\mu_\Delta\left(s \in [a,t]_{[a,\infty)} : |f(s) - L| \geqslant \varepsilon\right) &= \mu_\Delta(\{a \leqslant s \leqslant t : |f(s) - L| \geqslant \varepsilon\}) \\ &= m(\{a \leqslant s \leqslant t : |f(s) - L| \geqslant \varepsilon\}),\end{aligned}$$

where $m(B)$ denotes the classical Lebesgue measure of the set B. Hence, we obtain that

$$st_{[a,\infty)} - \lim_{t\to\infty} f(t) = L$$

is equivalent to

$$\lim_{t\to\infty} \frac{m(\{a \leqslant s \leqslant t : |f(s) - L| \geqslant \varepsilon\})}{t - a} = 0,$$

which was first introduced by Móricz [15].

Example 3.8. Now let $\mathbb{T} = q^{\mathbb{N}}$ $(q > 1)$ in Definition 3.3. Then, using $t_0 = q$ and replacing t with q^n, we observe that

$$\mu_\Delta\left([q,q^n]_{q^{\mathbb{N}}}\right) = \mu_\Delta\left(\{q,q^2,\ldots,q^n\}\right) = \sigma(q^n) - q = q(q^n - 1),$$

and letting $K(\varepsilon) := \left\{q^k \in [q,q^n]_{q^{\mathbb{N}}} : \left|f(q^k) - L\right| \geqslant \varepsilon\right\}$ we get

$$\begin{aligned}\mu_\Delta(K(\varepsilon)) &= \sum_{k=1}^{n}\left(\sigma(q^k) - q^k\right)\chi_{K(\varepsilon)}(q^k) \\ &= (q-1)\sum_{k=1}^{n} q^k \chi_{K(\varepsilon)}(q^k).\end{aligned}$$

Hence, we deduce that

$$st_{q^{\mathbb{N}}} - \lim_{k\to\infty} f(q^k) = L$$

is equivalent to

$$\lim_{n\to\infty} \frac{(q-1)\sum_{k=1}^{n} q^k \chi_{K(\varepsilon)}(q^k)}{q(q^n - 1)} = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} q^{k-1} \chi_{K(\varepsilon)}(q^k)}{[n]_q} = 0, \tag{3.3}$$

where $[n]_q$ denotes the q-integer given by

$$[n]_q := 1+q+q^2+\ldots+q^{n-1} = \frac{q^n-1}{q-1}. \tag{3.4}$$

The limit in (3.3) can be represented via matrix summability method as follows:

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} q^{k-1}\chi_{K(\varepsilon)}(q^k)}{[n]_q} = \lim_{n\to\infty} C_1(q)\chi_{K(\varepsilon)}(q^n),$$

where $C_1(q) := [c_{n,k}(q)]$, $k,n \in \mathbb{N}$ denotes the q-Cesáro matrix of order one defined by

$$c_{n,k}(q) = \begin{cases} \dfrac{q^{k-1}}{[n]_q}, & \text{if } 1 \le k \le n \\ 0, & \text{otherwise.} \end{cases} \tag{3.5}$$

Recall that the q-Cesáro matrix in (3.5) was first introduced by Aktuğlu and Bekar [2]. So, it follows from (3.3) to (3.5) that

$$st_{q^{\mathbb{N}}} - \lim_{k\to\infty} f(q^k) = L \Leftrightarrow \lim_{n\to\infty} C_1(q)\chi_{K(\varepsilon)}(q^n) = 0. \tag{3.6}$$

In [2], the last convergence method was called as q-statistical convergence of the function f to L.

Before closing this section, we should note that it is also possible to derive many new convergence methods from our Definitions 3.1 and 3.3 by choosing appropriate timescales.

3.3 Some Characterizations of Statistical Convergence

In this section we obtain many characterizations of the statistical convergence in Definition 3.3.

In the next result, we generalize Šalát's theorem in [17].

Theorem 3.9. *Let f be a Δ-measurable function. Then, $st_{\mathbb{T}} - \lim_{t\to\infty} f(t) = L$ if and only if there exists a Δ-measurable set $\Omega \subset \mathbb{T}$ such that $\delta_{\mathbb{T}}(\Omega) = 1$ and $\lim_{t\to\infty\ (t\in\Omega)} f(t) = L$.*

Proof. Necessity. Setting

$$\Omega_j = \left\{ t \in \mathbb{T} : |f(t) - L| < \frac{1}{j} \right\}, \quad j = 1,2,\ldots,$$

we may write from hypothesis that $\delta_{\mathbb{T}}(\Omega_j) = 1$ for every $j \in \mathbb{N}$. Also, we see that (Ω_j) is decreasing. Now, for $j = 1$, choose $t_1 \in \Omega_1$. Since $\delta_{\mathbb{T}}(\Omega_1) = 1$, there exists a number $t_2 \in \Omega_2$ with $t_2 > t_1$ such that $\dfrac{\mu_\Delta(\Omega_2(t))}{\mu_\Delta\left([t_0,t]_{\mathbb{T}}\right)} > \dfrac{1}{2}$ holds for each $t \geq t_2$ with $t \in \mathbb{T}$. Also, since again $\delta_{\mathbb{T}}(\Omega_2) = 1$, there exists a number $t_3 \in \Omega_3$ with $t_3 > t_2$ such that $\dfrac{\mu_\Delta(\Omega_3(t))}{\mu_\Delta\left([t_0,t]_{\mathbb{T}}\right)} > \dfrac{2}{3}$ holds for each $t \geq t_3$ with $t \in \mathbb{T}$. By repeating the same process, one can construct an increasing sequence (t_j) such that, for each $t \geq t_j$ with $t \in \mathbb{T}$, $\dfrac{\mu_\Delta(\Omega_j(t))}{\mu_\Delta\left([t_0,t]_{\mathbb{T}}\right)} > \dfrac{j-1}{j}$, where $\Omega_j(t) := \{s \in [t_0,t]_{\mathbb{T}} : s \in \Omega_j\}$, $j \in \mathbb{N}$. With the help of the sets Ω_j, we can construct a set Ω as in the following way:

- If $t \in [t_0,t_1]_{\mathbb{T}}$, then $t \in \Omega$.
- If $t \in \Omega_j \cap \left[t_j,t_{j+1}\right]_{\mathbb{T}}$ for $j = 1,2,\ldots$, then $t \in \Omega$, i.e.,

Hence, we get

$$\Omega := \left\{t \in \mathbb{T} : t \in [t_0,t_1]_{\mathbb{T}} \text{ or } t \in \Omega_j \cap \left[t_j,t_{j+1}\right]_{\mathbb{T}},\ j = 1,2,\ldots\right\}$$

Then, we may write that

$$\frac{\mu_\Delta(\Omega(t))}{\mu_\Delta\left([t_0,t]_{\mathbb{T}}\right)} \geq \frac{\mu_\Delta(\Omega_j(t))}{\mu_\Delta\left([t_0,t]_{\mathbb{T}}\right)} > \frac{j-1}{j}$$

holds for each $t \in [t_j,t_{j+1})_{\mathbb{T}}$ $(j = 1,2,\ldots)$. The last inequality implies that $\delta_{\mathbb{T}}(\Omega) = 1$. Now we show that $\lim_{t\to\infty\ (t\in\Omega)} f(t) = L$. To see this, for a given $\varepsilon > 0$, choose a number j such that $\dfrac{1}{j} < \varepsilon$. Also, let $t \geqslant t_j$ with $t \in \Omega$. Then there exists a number $n \geq j$ such that $t \in [t_n,t_{n+1})_{\mathbb{T}}$. It follows from the definition of Ω that $t \in \Omega_n$, and hence

$$|f(t) - L| < \frac{1}{n} \leqslant \frac{1}{j} < \varepsilon.$$

Therefore, we see that $|f(t) - L| < \varepsilon$ for each $t \in \Omega$ with $t \geq t_j$, which gives the result $\lim_{t\to\infty\ (t\in\Omega)} f(t) = L$.

Sufficiency. By the hypothesis, for a given $\varepsilon > 0$, there exists a number $t_* \in \mathbb{T}$ such that for every $t \geq t_*$ with $t \in \Omega$, one can obtain that $|f(t) - L| < \varepsilon$. Hence, if we put $A(\varepsilon) := \{t \in \mathbb{T} : |f(t) - L| \geq \varepsilon\}$ and $B := \Omega \cap [t_*,\infty)_{\mathbb{T}}$, then it is easy to see that $A(\varepsilon) \subset \mathbb{T} \setminus B$. Furthermore, using the facts that

$$\Omega = (\Omega \cap [t_0,t_*)_{\mathbb{T}}) \cup B \quad \text{and} \quad \delta_{\mathbb{T}}(\Omega) = 1,$$

and also observing $\delta_{\mathbb{T}}(\Omega \cap [t_0,t_*)_{\mathbb{T}}) = 0$ due to boundedness, Lemma 3.2 immediately yields that $\delta_{\mathbb{T}}(B) = 1$, and therefore we get $\delta_{\mathbb{T}}(A(\varepsilon)) = 0$, which completes the proof. □

Note that the discrete version of Theorem 3.9 was proved by Šalát [17].

In order to get a new characterization for statistical convergence on timescales, we first need the following two lemmas:

Lemma 3.10. *Let $f:\mathbb{T}\to\mathbb{R}$ be a Δ-measurable function. If $st_{\mathbb{T}}-\lim_{t\to\infty} f(t)=L$ and f is bounded above by M, then we have*

$$\lim_{t\to\infty}\frac{1}{\mu_\Delta([t_0,t]_{\mathbb{T}})}\int_{[t_0,t]_{\mathbb{T}}} f(s)\,\Delta s = L,$$

where we use the Lebesgue Δ-integral on timescales introduced by Cabada and Vivero [7].

Proof. Without loss of generality, we may assume that $L=0$. Now let $\varepsilon>0$ and $\Omega(t):=\{s\in[t_0,t]_{\mathbb{T}}:|f(s)|\geq\varepsilon\}$. Since $st_{\mathbb{T}}-\lim_{t\to\infty} f(t)=L$, we get

$$\lim_{t\to\infty}\frac{\mu_\Delta(\Omega(t))}{\mu_\Delta([t_0,t]_{\mathbb{T}})}=0,$$

which means that $\dfrac{\mu_\Delta(\Omega(t))}{\mu_\Delta([t_0,t]_{\mathbb{T}})}<\dfrac{\varepsilon}{M}$ for sufficiently large t. Now, we may write that

$$\begin{aligned}
&\left|\frac{1}{\mu_\Delta([t_0,t]_{\mathbb{T}})}\int_{[t_0,t]_{\mathbb{T}}} f(s)\,\Delta s\right| \\
&\leq \frac{1}{\mu_\Delta([t_0,t]_{\mathbb{T}})}\left\{\int_{\Omega(t)}|f(s)|\,\Delta s+\int_{[t_0,t]_{\mathbb{T}}\setminus\,\Omega(t)}|f(s)|\,\Delta s\right\} \\
&\leqslant \frac{1}{\mu_\Delta([t_0,t]_{\mathbb{T}})}\left\{M\int_{\Omega(t)}\Delta s+\varepsilon\int_{[t_0,t]_{\mathbb{T}}}\Delta s\right\}.
\end{aligned}$$

We know from [7] that $\int_A \Delta s=\mu_\Delta(A)$ for any measurable subset $A\subset\mathbb{T}$. Hence, the last inequality implies that

$$\left|\frac{1}{\mu_\Delta([t_0,t]_{\mathbb{T}})}\int_{[t_0,t]_{\mathbb{T}}} f(s)\,\Delta s\right|\leq\frac{M\mu_\Delta(\Omega(t))+\varepsilon\mu_\Delta([t_0,t]_{\mathbb{T}})}{\mu_\Delta([t_0,t]_{\mathbb{T}})}\leq 2\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, the proof is completed. □

Lemma 3.11. *Let $f:\mathbb{T}\to\mathbb{R}$ be a Δ-measurable function and $st_{\mathbb{T}}-\lim_{t\to\infty} f(t)=L$. If $g:\mathbb{R}\to\mathbb{R}$ is a continuous function at L, then we have*

$$st_{\mathbb{T}} - \lim_{t\to\infty} g\left(f\left(t\right)\right) = g\left(L\right)$$

Proof. By the continuity of g at L, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|g(y) - g(L)| < \varepsilon$ whenever $|y - L| < \delta$. But then $|g(y) - g(L)| \geqslant \varepsilon$ implies $|y - L| \geq \delta$, and hence

$$|g(f(t)) - g(L)| \geq \varepsilon \text{ implies } |f(t) - L| \geqslant \delta.$$

So, we get

$$\{t \in \mathbb{T} : |g(f(t)) - g(L)| \geqslant \varepsilon\} \subset \{t \in \mathbb{T} : |f(t) - L| \geq \delta\},$$

which yields that

$$\delta_{\mathbb{T}}(\{t \in \mathbb{T} : |g(f(t)) - g(L)| \geqslant \varepsilon\}) \leqslant \delta_{\mathbb{T}}(\{t \in \mathbb{T} : |f(t) - L| \geq \delta\}) = 0,$$

whence the result. □

Now we are ready to give our new characterization.

Theorem 3.12. *Let $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function. Then,*

$$st_{\mathbb{T}} - \lim_{t\to\infty} f\left(t\right) = L$$

if and only if, for every $\alpha \in \mathbb{R}$,

$$\lim_{t\to\infty} \frac{1}{\mu_\Delta\left([t_0,t]_{\mathbb{T}}\right)} \int\limits_{[t_0,t]_{\mathbb{T}}} e^{i\alpha f(s)} \Delta s = e^{i\alpha L}. \tag{3.7}$$

Proof. Necessity. Assume that $st_{\mathbb{T}} - \lim_{t\to\infty} f(t) = L$ holds. It is easy to see that $e^{i\alpha t}$ is a continuous function for any fixed $\alpha \in \mathbb{R}$. Thus, by Lemma 3.11, we can write that

$$st_{\mathbb{T}} - \lim_{t\to\infty} e^{i\alpha f(t)} = e^{i\alpha L}$$

Also, since $e^{i\alpha f(t)}$ is a bounded function, it follows from Lemma 3.10 that

$$\lim_{t\to\infty} \frac{1}{\mu_\Delta\left([t_0,t]_{\mathbb{T}}\right)} \int\limits_{[t_0,t]_{\mathbb{T}}} e^{i\alpha f(s)} \Delta s = e^{i\alpha L}.$$

Sufficiency. Assume now that (3.7) holds for any $\alpha \in \mathbb{R}$. As in [18], define the following continuous function:

$$M(x) = \begin{cases} 0, & \text{if } x < 1 \\ 1 + x, & \text{if } -1 \leqslant x < 0 \\ 1 - x, & \text{if } 0 \leqslant x < 1 \\ 0, & \text{if } x \geqslant 1. \end{cases}$$

Then, we know from [18] (see also [11]) that $M(x)$ has the following integral representation:

$$M(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin(\alpha/2)}{\alpha/2} \right)^2 e^{ix\alpha} d\alpha \text{ for } x \in \mathbb{R}.$$

Without loss of generality, we can assume that $L = 0$ in (3.7). So, we get

$$\lim_{t\to\infty} \frac{1}{\mu_\Delta([t_0,t]_\mathbb{T})} \int_{[t_0,t]_\mathbb{T}} e^{i\alpha f(s)} \Delta s = 1 \text{ for every } \alpha \in \mathbb{R}. \tag{3.8}$$

Now let $\Omega := \{t \in \mathbb{T} : |f(t)| \geqslant \varepsilon\}$ for a given $\varepsilon > 0$. Then, to complete the proof, we need to show $\delta_\mathbb{T}(\Omega) = 0$. To see this, firstly, we write that

$$M\left(\frac{f(s)}{\varepsilon}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin(\alpha/2)}{\alpha/2} \right)^2 e^{i\alpha f(s)/\varepsilon} d\alpha$$

After making an appropriate change of variables, we obtain that

$$M\left(\frac{f(s)}{\varepsilon}\right) = \frac{\varepsilon}{2\pi} \int_{\mathbb{R}} \left(\frac{\sin(\alpha\varepsilon/2)}{\alpha\varepsilon/2} \right)^2 e^{if(s)\alpha} d\alpha, \tag{3.9}$$

and hence

$$\begin{aligned} &\frac{1}{\mu_\Delta([t_0,t]_\mathbb{T})} \int_{[t_0,t]_\mathbb{T}} M\left(\frac{f(s)}{\varepsilon}\right) \Delta s \\ &= \frac{\varepsilon}{2\pi} \frac{1}{\mu_\Delta([t_0,t]_\mathbb{T})} \int_{[t_0,t]_\mathbb{T}} \left\{ \int_{\mathbb{R}} \left(\frac{\sin(\alpha\varepsilon/2)}{\alpha\varepsilon/2} \right)^2 e^{if(s)\alpha} d\alpha \right\} \Delta s. \end{aligned}$$

Observe that the integral in (3.9) is an absolutely convergent. Now, by the Fubini theorem on timescales (see [1, 4, 5]), we have

$$\begin{aligned} &\frac{1}{\mu_\Delta([t_0,t]_\mathbb{T})} \int_{[t_0,t]_\mathbb{T}} M\left(\frac{f(s)}{\varepsilon}\right) \Delta s \\ &= \frac{\varepsilon}{2\pi} \int_{\mathbb{R}} \left(\frac{\sin(\alpha\varepsilon/2)}{\alpha\varepsilon/2} \right)^2 \left\{ \frac{1}{\mu_\Delta([t_0,t]_\mathbb{T})} \int_{[t_0,t]_\mathbb{T}} e^{if(s)\alpha} \Delta s \right\} d\alpha. \end{aligned}$$

Moreover, for all $\alpha \in \mathbb{R}$ and $t \in \mathbb{T}$,

$$\left| \frac{1}{\mu_\Delta([t_0,t]_\mathbb{T})} \int_{[t_0,t]_\mathbb{T}} e^{if(s)\alpha} \Delta s \right| \leqslant 1.$$

Hence, if we consider (3.8) and also use the Lebesgue dominated convergence theorem we obtain that

$$\lim_{t\to\infty}\frac{1}{\mu_\Delta([t_0,t]_{\mathbb{T}})}\int_{[t_0,t]_{\mathbb{T}}} M\left(\frac{f(s)}{\varepsilon}\right)\Delta s$$

$$=\frac{\varepsilon}{2\pi}\int_{\mathbb{R}}\left(\frac{\sin(\alpha\varepsilon/2)}{\alpha\varepsilon/2}\right)^2\left\{\lim_{t\to\infty}\frac{1}{\mu_\Delta([t_0,t]_{\mathbb{T}})}\int_{[t_0,t]_{\mathbb{T}}} e^{if(s)\alpha}\Delta s\right\}d\alpha$$

$$=\frac{\varepsilon}{2\pi}\int_{\mathbb{R}}\left(\frac{\sin(\alpha\varepsilon/2)}{\alpha\varepsilon/2}\right)^2 d\alpha.$$

Now, the definition of the function M implies that

$$\lim_{t\to\infty}\frac{1}{\mu_\Delta([t_0,t]_{\mathbb{T}})}\int_{[t_0,t]_{\mathbb{T}}} M\left(\frac{f(s)}{\varepsilon}\right)\Delta s = M(0)=1. \tag{3.10}$$

Observe now that for any $s\in\Omega(t)$, $\dfrac{f(s)}{\varepsilon}\geqslant 1$, where $\Omega(t):=\{s\in[t_0,t]_{\mathbb{T}}: s\in\Omega\}$ as stated before. Then, we get

$$\int_{\Omega(t)} M\left(\frac{f(s)}{\varepsilon}\right)\Delta s = 0.$$

Furthermore, since

$$\begin{aligned}\int_{[t_0,t]_{\mathbb{T}}} M\left(\frac{f(s)}{\varepsilon}\right)\Delta s &= \int_{[t_0,t]_{\mathbb{T}}\setminus\,\Omega(t)} M\left(\frac{f(s)}{\varepsilon}\right)\Delta s + \int_{\Omega(t)} M\left(\frac{f(s)}{\varepsilon}\right)\Delta s\\ &\leqslant \int_{[t_0,t]_{\mathbb{T}}\setminus\,\Omega(t)} \Delta s\\ &= \mu_\Delta([t_0,t]_{\mathbb{T}})-\mu_\Delta(\Omega(t)),\end{aligned}$$

we have

$$\frac{\mu_\Delta(\Omega(t))}{\mu_\Delta([t_0,t]_{\mathbb{T}})}\leqslant 1-\frac{1}{\mu_\Delta([t_0,t]_{\mathbb{T}})}\int_{[t_0,t]_{\mathbb{T}}} M\left(\frac{f(s)}{\varepsilon}\right)\Delta s.$$

Now taking limit as $t\to\infty$ on both sides of the last equality and also using (3.10), we see that

$$\delta_{\mathbb{T}}(\Omega)=0,$$

which completes the proof. □

Note that if take $\mathbb{T}=\mathbb{N}$ in Theorem 3.12, then we immediately get Schoenberg's result in [18]; also if $\mathbb{T}=[a,\infty)$, $a>0$, then Theorem 3.12 reduces to the univariate version of Theorem 1 in [11]. The next result indicates the special case $\mathbb{T}=q^{\mathbb{N}}$ $(q>1)$ of Theorem 3.12.

Corollary 3.13. *Let $f: q^{\mathbb{N}}\to\mathbb{R}$ $(q>1)$ be a Δ-measurable function. Then,*

$$st_{q^{\mathbb{N}}}-\lim_{t\to\infty} f(t)=L$$

if and only if, for every $\alpha\in\mathbb{R}$,

$$\lim_{n\to\infty}\frac{1}{[n]_q}\sum_{k=1}^{n} e^{i\alpha f(q^k)}q^{k-1}=e^{i\alpha L},$$

where $[n]_q$ is the same as in (3.4).

Now to obtain a new characterization we consider the next definition.

Definition 3.14. Let f: $\mathbb{T}\to\mathbb{R}$ be a Δ-measurable function and $0<p<\infty$. We say that f is strongly p-Cesáro summable on the timescale $\mathbb{T}$ if there exists some $L\in\mathbb{R}$ such that

$$\lim_{t\to\infty}\frac{1}{\mu_\Delta\left([t_0,t]_{\mathbb{T}}\right)}\int\limits_{[t_0,t]_{\mathbb{T}}}|f(s)-L|^p\,\Delta s=0.$$

Observe that our Definition 3.14 covers the well-known concepts on strongly p-Cesáro summability for discrete and continuous cases. Furthermore, for example, one can deduce from Definition 3.14 that f is strongly p-Cesáro summable on the timescale $q^{\mathbb{N}}$ $(q>1)$ if there exists a real number L such that

$$\lim_{n\to\infty}\frac{1}{[n]_q}\sum_{k=1}^{n} q^{k-1}\left|f\left(q^k\right)-L\right|^p=0,$$

which is a new concept on summability theory.

We first need the next lemma which gives Markov's inequality on timescales.

Lemma 3.15. *Let f: $\mathbb{T}\to\mathbb{R}$ be a Δ-measurable function and let $\Omega(t):=\{s\in[t_0,t]_{\mathbb{T}}:|f(s)-L|\geq\varepsilon\}$ for $\varepsilon>0$. In this case, we have*

$$\mu_\Delta(\Omega(t))\leqslant\frac{1}{\varepsilon}\int\limits_{\Omega(t)}|f(s)-L|\,\Delta s\leq\frac{1}{\varepsilon}\int\limits_{[t_0,t]_{\mathbb{T}}}|f(s)-L|\,\Delta s.$$

Proof. For all $s\in[t_0,t]_{\mathbb{T}}$ and $\varepsilon>0$, we can write that

$$0\leqslant\varepsilon\chi_{\Omega(t)}(s)\leqslant|f(s)-L|\,\chi_{\Omega(t)}(s)\leqslant|f(s)-L|,$$

which implies that

$$\varepsilon \int_{\Omega(t)} \Delta s \leq \int_{\Omega(t)} |f(s) - L| \, \Delta s \leq \int_{[t_0,t]_{\mathbb{T}}} |f(s) - L| \, \Delta s.$$

Therefore, we obtain that

$$\varepsilon \mu_{\Delta}(\Omega(t)) \leq \int_{\Omega(t)} |f(s) - L| \, \Delta s \leq \int_{[t_0,t]_{\mathbb{T}}} |f(s) - L| \, \Delta s,$$

which proves the lemma. □

Then, we get the following result.

Theorem 3.16. *Let $f \colon \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function, $L \in \mathbb{R}$ and $0 < p < \infty$. Then, we get:*

(i) *If f is strongly p-Cesáro summable to L, then $st_{\mathbb{T}} - \lim\limits_{t\to\infty} f(t) = L$.*
(ii) *If $st_{\mathbb{T}} - \lim\limits_{t\to\infty} f(t) = L$ and f is a bounded function, then f is strongly p-Cesáro summable to L.*

Proof. (i) Let f be strongly p-Cesáro summable to L. For a given $\varepsilon > 0$, on timescale, let $\Omega(t) := \{s \in [t_0,t]_{\mathbb{T}} : |f(s) - L| \geqslant \varepsilon\}$. Then, it follows from Lemma 3.15 that

$$\varepsilon^p \mu_{\Delta}(\Omega(t)) \leqslant \int_{[t_0,t]_{\mathbb{T}}} |f(s) - L|^p \, \Delta s.$$

Dividing both sides of the last equality by $\mu_{\Delta}([t_0,t]_{\mathbb{T}})$ and taking limit as $t \to \infty$, we obtain that

$$\lim_{t\to\infty} \frac{\mu_{\Delta}(\Omega(t))}{\mu_{\Delta}([t_0,t]_{\mathbb{T}})} \leqslant \frac{1}{\varepsilon^p} \lim_{t\to\infty} \frac{1}{\mu_{\Delta}([t_0,t]_{\mathbb{T}})} \int_{[t_0,t]_{\mathbb{T}}} |f(s) - L|^p \, \Delta s = 0,$$

which yields that $st_{\mathbb{T}} - \lim\limits_{t\to\infty} f(t) = L$.

(ii) Let f be bounded and statistically convergent to L on $\mathbb{T}$. Then, there exists a positive number M such that $|f(s)| \leqslant M$ for all $s \in \mathbb{T}$, and also

$$\lim_{t\to\infty} \frac{\mu_{\Delta}(\Omega(t))}{\mu_{\Delta}([t_0,t]_{\mathbb{T}})} = 0, \tag{3.11}$$

where $\Omega(t) := \{s \in [t_0, t]_{\mathbb{T}} : |f(s) - L| \geqslant \varepsilon\}$ as stated before. Since

$$\begin{aligned}\int_{[t_0,t]_{\mathbb{T}}} |f(s)-L|^p \Delta s &= \int_{\Omega(t)} |f(s)-L|^p \Delta s + \int_{[t_0,t]_{\mathbb{T}}\setminus\Omega(t)} |f(s)-L|^p \Delta s \\ &\leqslant (M+|L|)^p \int_{\Omega(t)} \Delta s + \varepsilon^p \int_{[t_0,t]_{\mathbb{T}}} \Delta s \\ &= (M+|L|)^p \mu_\Delta(\Omega(t)) + \varepsilon^p \mu_\Delta([t_0,t]_{\mathbb{T}}),\end{aligned}$$

we obtain that

$$\lim_{t\to\infty} \frac{1}{\mu_\Delta([t_0,t]_{\mathbb{T}})} \int_{[t_0,t]_{\mathbb{T}}} |f(s)-L|^p \Delta s \leqslant (M+|L|)^p \lim_{t\to\infty} \frac{\mu_\Delta(A)}{\mu_\Delta([t_0,t]_{\mathbb{T}})} + \varepsilon^p. \quad (3.12)$$

Since ε is arbitrary, the proof follows from (3.11) and (3.12). □

Observe that the discrete and continuous cases of Theorem 3.16 were presented in [8] and [15], respectively. Furthermore, it is not hard to see that, for $\mathbb{T} = q^{\mathbb{N}}$ $(q > 1)$, Theorem 3.16 implies the following result.

Corollary 3.17. *Let f: $q^{\mathbb{N}} \to \mathbb{R}$ $(q > 1)$ be a Δ-measurable and bounded function on $\mathbb{T}$. Then, we get*

$$st_{q^{\mathbb{N}}} - \lim_{n\to\infty} f(q^n) = L \Leftrightarrow \lim_{n\to\infty} \frac{1}{[n]_q} \sum_{k=1}^{n} q^{k-1} \left| f\left(q^k\right) - L \right|^p = 0.$$

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Chapter 4
On the *g*-Jacobi Matrix Functions

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Abstract In this paper, we introduce a matrix version of the generalized Jacobi (g-Jacobi) function, which is a solution of fractional Jacobi differential equation, and study its fundamental properties. We also present the fractional hypergeometric matrix function as a solution of the matrix generalization of the fractional Gauss differential equation. Some special cases are discussed.

4.1 Introduction

The theory of fractional calculus has recently been applied in many areas of pure and applied mathematics and engineerings, such as biology, physics, electrochemistry, economics, probability theory, and statistics [7, 9]. In the present paper, we mainly use the fractional calculus in the theory of special functions. More precisely, we study on a matrix version of the Jacobi function which gives via the Riemann–Liouville (fractional) operator. Furthermore we define the matrix version of the fractional hypergeometric function which is a solution of the fractional analogue of the Gauss matrix differential equation.

Throughout the paper, we consider the Riemann–Liouville fractional derivative of a function f with order μ, which is defined by

$$D^{\mu} f(t) := D^{m}[J^{m-\mu} f(t)],$$

where $m \in \mathbb{N}$, $m-1 \leq \mu < m$ and

$$J^{m-\mu} f(t) := \frac{1}{\Gamma(m-\mu)} \int_0^t (t-\tau)^{m-\mu-1} f(\tau)\, d\tau$$

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is the Riemann–Liouville fractional integral of f with order $m-\mu$. Here Γ denotes the classical gamma function. It is easy to see that the fractional derivative of the power function $f(t)=t^{\alpha}$ is given by

$$D^{\mu}t^{\alpha}=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\mu+1)}t^{\alpha-\mu}$$

where $\alpha\geq -1,\ \mu\geq 0,\ t>0$. We know from [10] that if f is a continuous function in $[0,t]$ and φ has $n+1$ continuous derivatives in $[0,t]$, then the fractional derivative of the product φf, that is, the Leibniz rule, is given as follows:

$$D^{\mu}[\varphi(t)f(t)]=\sum_{k=0}^{\infty}\binom{\mu}{k}\varphi^{(k)}(t)D^{\mu-k}f(t). \tag{4.1}$$

It is well known that the classical Gauss differential equation is given as follows:

$$x(1-x)y''+[c-(a+b+1)x]y'-aby=0. \tag{4.2}$$

As usual, (4.2) has a solution of the hypergeometric function defined by

$$F(a,b;c;x)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k k!}x^k, \tag{4.3}$$

where $(\lambda)_k$ is the Pochhammer symbol

$$(\lambda)_k=\lambda(\lambda+1)\dots(\lambda+k-1),\quad (\lambda)_0=1.$$

Jacobi polynomials $P_n^{(\alpha,\beta)}$ are defined by the Rodrigues formula

$$P_n^{(\alpha,\beta)}(x)=\frac{(1-x)^{-\alpha}(1+x)^{-\beta}}{(-2)^n n!}D_x^n\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right], \tag{4.4}$$

where $\alpha,\beta>-1$ [3]. In [8], Mirevski et al. gave the fractional generalizations of (4.2)–(4.4).

On the other hand, it is well known that special matrix functions appear in lots of studies [1, 2, 4]. The aim of this paper is to study the matrix versions of the results in [8]. And also some properties of Jacobi matrix functions and some special cases are obtained. To see that we consider the following terminology on the matrix theory of special functions.

If A is a matrix in $\mathbb{C}^{r\times r}$, then by $\sigma(A)$ we denote the set of all the eigenvalues of A. It follows from [5] that if $f(z),g(z)$ are holomorphic functions in an open set Ω of the complex plane and if $\sigma(A)\subset\mathbb{C}$, we denote by $f(A)$, $g(A)$, respectively, the image by the Riesz–Dunford functional calculus of the functions $f(z),g(z)$, respectively, acting on the matrix A, and

$$f(A)g(A)=g(A)f(A).$$

Let $\|A\|$ denote the two norms of A defined by

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2},$$

where $\|y\|_2 = \left(y^T y\right)^{1/2}$ for a vector $y \in \mathbb{C}^r$ is the Euclidean norm of y. It is easy to check that

$$\begin{aligned} \|A+B\| &\leq \|A\| + \|B\| \\ \|AB\| &\leq \|A\| \,.\, \|B\| \end{aligned} \tag{4.5}$$

for all $A, B \in \mathbb{C}^{r\times r}$. The reciprocal scalar Gamma function, $\Gamma^{-1}(z) = 1/\Gamma(z)$, is an entire function of the complex variable z. Thus, for any $C \in \mathbb{C}^{r\times r}$, the Riesz–Dunford functional calculus [5] shows that $\Gamma^{-1}(C)$ is well defined and is, indeed, the inverse of $\Gamma(C)$. Hence, if $C \in \mathbb{C}^{r\times r}$ is such that $C + nI$ is invertible for every integer $n \geq 0$, then

$$\Gamma^{-1}(C) = C(C+I)(C+2I)\dots(C+kI)\Gamma^{-1}(C+(k+1)I).$$

The hypergeometric matrix function $F(A,B;C;z)$ is given in [6] as follows:

$$F(A,B;C;z) = \sum_{n=0}^{\infty} \frac{(A)_n (B)_n}{n!} \left[(C)_n\right]^{-1} z^n, \tag{4.6}$$

where $|z| < 1$ and $A, B, C \in \mathbb{C}^{r\times r}$ such that $C + nI$ is invertible for all integer $n \geq 0$ and also $(A)_n$ denotes the Pochhammer symbol:

$$(A)_n = A(A+I)\dots(A+(n-1)I), \;\; n \geq 1\;, \; (A)_0 = I\;. \tag{4.7}$$

4.2 Fractional Hypergeometric Matrix Function

In this section, we give the matrix version of (4.3) by solving the matrix version of the linear homogeneous hypergeometric differential equation (4.2).

Definition 4.1. We define fractional hypergeometric matrix differential equation as follows:

$$t^{\mu}(1-t^{\mu})D^{2\mu}Y(t) - t^{\mu}AD^{\mu}\left[Y(t)\right] + D^{\mu}\left[Y(t)\right]\left(C - t^{\mu}(B+I)\right) - AY(t)B = \mathbf{0}, \tag{4.8}$$

where $0 < \mu \leq 1$ and $C + kI$ is invertible for every integer $k \geq 0$.

Definition 4.2. The fractional hypergeometric matrix function is defined as

$${}_2^{\mu}F_1(A,B;C;t) = Y_0 t^{\theta} + \sum_{k=1}^{\infty} \left[\prod_{j=0}^{k-1} G_j(\theta)\right] Y_0 \left[\prod_{j=0}^{k-1} F_{j+1}^{-1}(\theta)\right] t^{\theta + k\mu I}, \tag{4.9}$$

where $0 < \mu \leq 1$ and

$$\begin{aligned} F_k(\theta) &= \Gamma((k\mu+1)I+\theta)\Gamma^{-1}([(k-2)\mu+1]I+\theta) \\ &\quad + C\Gamma((k\mu+1)I+\theta)\Gamma^{-1}([(k-1)\mu+1]I+\theta) \end{aligned} \tag{4.10}$$

$$\begin{aligned} G_k(\theta) &= \Gamma((k\mu+1)I+\theta)\Gamma^{-1}([(k-2)\mu+1]I+\theta) + AB \\ &\quad + (A+B+I)\Gamma((k\mu+1)I+\theta)\Gamma^{-1}([(k-1)\mu+1]I+\theta) \end{aligned} \tag{4.11}$$

and also $Re(\rho) > -1$ for $\forall \rho \in \sigma(\theta)$ $(\theta \in \mathbb{C}^{r\times r})$ yields the following properties:

$$\begin{aligned} F_0(\theta) &= \Gamma(I+\theta)\Gamma^{-1}((1-2\mu)I+\theta) \\ &\quad + C\Gamma(I+\theta)\Gamma^{-1}((1-\mu)I+\theta) = \mathbf{0} \end{aligned} \tag{4.12}$$

where $\theta A = A\theta,\ \theta B = B\theta, AB = BA,\ \theta Y_0 = \theta Y_0, BY_0 = Y_0 B$ and $(1-2\mu)I+\theta$ and $(1-\mu)I+\theta$ are invertible for $0 < \mu \leq 1$.

If we take $\mu = 1$ and $A = a, B = b, C = c$ in (4.9) for $r = 1$, we obtain the classical hypergeometric function.

Theorem 4.3. *The fractional hypergeometric matrix function is a solution of (4.8).*

Proof. We find a solution of (4.8) in the form

$$Y(t) = \sum_{k=0}^{\infty} Y_k t^{\theta + k\mu I},$$

where $\theta, Y_k \in \mathbb{C}^{r\times r}$and also $Re(\rho) > -1$ for all $\rho \in \sigma(\theta)$. If we make fractional derivatives of $Y(t)$ with orders μ and 2μ, then left-hand side of (4.8) gives that

$$\begin{aligned} & LHS\ of\ (4.8) \\ = & \sum_{k=0}^{\infty} Y_k \Gamma((k\mu+1)I+\theta)\Gamma^{-1}([(k-2)\mu+1]I+\theta) t^{\theta+(k-1)\mu I} \\ & - \sum_{k=0}^{\infty} Y_k \Gamma((k\mu+1)I+\theta)\Gamma^{-1}([(k-2)\mu+1]I+\theta) t^{\theta+k\mu I} \\ & - A\sum_{k=0}^{\infty} Y_k \Gamma((k\mu+1)I+\theta)\Gamma^{-1}([(k-1)\mu+1]I+\theta) t^{\theta+k\mu I} \\ & + \sum_{k=0}^{\infty} Y_k \Gamma((k\mu+1)I+\theta)\Gamma^{-1}([(k-1)\mu+1]I+\theta) t^{\theta+(k-1)\mu I} C \\ & - \sum_{k=0}^{\infty} Y_k \Gamma((k\mu+1)I+\theta)\Gamma^{-1}([(k-1)\mu+1]I+\theta) t^{\theta+k\mu I}(B+I) \\ & - A\sum_{k=0}^{\infty} Y_k t^{\theta+k\mu I} B \\ = & \mathbf{0}, \end{aligned}$$

where $\theta C = C\theta,\ \theta A = A\theta,\ \theta B = B\theta, AB = BA,\ \theta Y_k = Y_k\theta$ and $BY_k = Y_kB,\ (k = 0,1,\ldots)$. Thus we obtain that

$$\begin{aligned}
&LHS\ of\ (4.8)\\
&= \sum_{k=0}^{\infty} Y_k \left\{\Gamma((k\mu+1)I+\theta)\Gamma^{-1}([(k-2)\mu+1]I+\theta)\right.\\
&\quad \left. + C\Gamma((k\mu+1)I+\theta)\Gamma^{-1}([(k-1)\mu+1]I+\theta)\right\} t^{\theta+(k-1)\mu I}\\
&\quad - \sum_{k=0}^{\infty} \left\{\Gamma((k\mu+1)I+\theta)\Gamma^{-1}([(k-2)\mu+1]I+\theta)\right.\\
&\quad \left. + (A+B+I)\Gamma((k\mu+1)I+\theta)\Gamma^{-1}([(k-1)\mu+1]I+\theta)+AB\right\} Y_k t^{\theta+k\mu I}\\
&= Y_0F_0(\theta)t^{\theta-\mu I} + \sum_{k=0}^{\infty}[Y_{k+1}F_{k+1}(\theta)-G_k(\theta)Y_k]\,t^{\theta+k\mu I}\\
&= \mathbf{0}.
\end{aligned}$$

Assuming $Y_0 \neq \mathbf{0}$, we have to choose $F_0(\theta) = \mathbf{0}$. θ has to be chosen such that (4.12) holds. Thus, from

$$Y_{k+1}F_{k+1}(\theta) - G_k(\theta)Y_k = \mathbf{0},$$

then we have

$$Y_k = \left[\prod_{j=0}^{k-1} G_j(\theta)\right] Y_0 \left[\prod_{j=0}^{k-1} F_{j+1}^{-1}(\theta)\right].$$

We understand from (4.12) that it doesn't need to hold the equality $\theta C = C\theta$. Furthermore, from $\theta Y_k = Y_k\theta$ and $BY_k = Y_kB,\ (k = 0,1,\ldots)$, it is sufficient that $\theta Y_0 = Y_0\theta$ and $BY_0 = Y_0B$. So, the proof is completed. □

It is clear that the case $\theta = \mathbf{0},\ \mu = 1,\ Y_0 = I$ in (4.9) is reduced ${}_2^1F_1(A,B;C;t) = {}_2F_1(A,B;C;t)$.

4.3 g-Jacobi Matrix Functions

In this section, we define the g-Jacobi matrix functions and obtain their some significant properties.

Definition 4.4. Assume that all eigenvalues z of the matrices A and B satisfy the conditions

$$\begin{aligned}
Re(z) &> -1 \text{ for } \forall z \in \sigma(A)\\
Re(z) &> -1 \text{ for } \forall z \in \sigma(B)\\
AB &= BA.
\end{aligned} \tag{4.13}$$

The g-Jacobi matrix functions are defined to be as the following Rodrigues formula:

$$P_{\upsilon}^{(A,B)}(x) = (-2)^{-\upsilon}\Gamma^{-1}(\upsilon+1)(1-x)^{-A}(1+x)^{-B} D_x^{\upsilon}\left[(1-x)^{A+\upsilon I}(1+x)^{B+\upsilon I}\right], \tag{4.14}$$

where $\upsilon > 0$.

Theorem 4.5. *The explicit form of the g-Jacobi matrix functions is given by*

$$P_{\upsilon}^{(A,B)}(x) = 2^{-\upsilon}\Gamma(A+(\upsilon+1)I)\Gamma(B+(\upsilon+1)I) \times \sum_{k=0}^{\infty}\frac{\Gamma^{-1}(A+(k+1)I)\Gamma^{-1}(B+(\upsilon-k+1)I)}{\Gamma(\upsilon-k+1)k!}(x-1)^k(x+1)^{\upsilon-k} \tag{4.15}$$

where $z_1 \notin \mathbb{N}$ *for* $\forall z_1 \in \sigma(A+\upsilon I)$ *and* $z_2 \notin \mathbb{N}$ *for* $\forall z_2 \in \sigma(B+\upsilon I)$.

Proof. If we use the Leibniz rule (4.1) in (4.14), then we have

$$P_{\upsilon}^{(A,B)}(x) = (-2)^{-\upsilon}\Gamma^{-1}(\upsilon+1)(1-x)^{-A}(1+x)^{-B} \times \sum_{k=0}^{\infty}\binom{\upsilon}{k}\left\{D_x^k\left[(1+x)^{B+\upsilon I}\right]\right\}\left\{D_x^{\upsilon-k}\left[(1-x)^{A+\upsilon I}\right]\right\}. \tag{4.16}$$

It follows from the definition of fractional derivative that

$$\begin{aligned} & D_x^{\upsilon-k}\left[(1-x)^{A+\upsilon I}\right] \\ &= \sum_{r=0}^{\infty}\frac{\Gamma(A+(\upsilon+1)I)\Gamma^{-1}(A+(\upsilon-r+1)I)}{r!}(-1)^{A+(\upsilon-r)I}D_x^{\upsilon-k}\left[x^{A+(\upsilon-r)I}\right] \\ &= \sum_{r=0}^{\infty}\left\{\frac{\Gamma(A+(\upsilon+1)I)\Gamma^{-1}(A+(\upsilon-r+1)I)}{r!}\right. \\ & \left.\times(-1)^{A+(\upsilon-r)I}\Gamma(A+(\upsilon-r+1)I)\Gamma^{-1}(A+(k-r+1)I)x^{A+(k-r)I}\right\}. \end{aligned} \tag{4.17}$$

From (4.17) and (4.16), we get that

$$\begin{aligned} P_{\upsilon}^{(A,B)}(x) = {} & (-2)^{-\upsilon}\Gamma^{-1}(\upsilon+1)(1-x)^{-A}(1+x)^{-B} \times \\ & \sum_{k=0}^{\infty}\binom{\upsilon}{k}\Gamma(B+(\upsilon+1)I)\Gamma^{-1}(B+(\upsilon-k+1)I)(x+1)^{B+(\upsilon-k)I} \times \\ & \sum_{r=0}^{\infty}\left\{\frac{\Gamma(A+(\upsilon+1)I)\Gamma^{-1}(A+(\upsilon-r+1)I)}{r!}\right. \\ & \left.\times(-1)^{A+(\upsilon-r)I}\Gamma(A+(\upsilon-r+1)I)\Gamma^{-1}(A+(k-r+1)I)x^{A+(k-r)I}\right\}. \end{aligned}$$

By the following property

$$\sum_{r=0}^{\infty}\frac{\Gamma(A+(k+1)I)\Gamma^{-1}(A+(k-r+1)I)}{r!}(-1)^r x^{A+(k-r)I}=(x-1)^{A+kI}$$

the proof is completed. □

Theorem 4.6. *The g-Jacobi matrix functions have the following representation:*

$$P_{\upsilon}^{(A,B)}(x)=\frac{\Gamma(A+(\upsilon+1)I)\Gamma^{-1}(A+I)}{\Gamma(\upsilon+1)}\,F\left(-\upsilon I,A+B+(\upsilon+1)I;A+I;\frac{1-x}{2}\right),\tag{4.18}$$

where F is a hypergeometric matrix function defined in (4.6).

Proof. Writing $(x-1)+2$ instead of $(x+1)$ in (4.15) and using binomial expansion, we obtain

$$\begin{aligned}
P_{\upsilon}^{(A,B)}(x)
&=2^{-\upsilon}\sum_{k=0}^{\infty}\frac{\Gamma(A+(\upsilon+1)I)\Gamma^{-1}(A+(k+1)I)\Gamma(B+(\upsilon+1)I)}{\Gamma(\upsilon-k+1)k!}\\
&\quad\times\Gamma^{-1}(B+(\upsilon-k+1)I)(x-1)^k((x-1)+2)^{\upsilon-k}\\
&=2^{-\upsilon}\sum_{k=0}^{\infty}\frac{\Gamma(A+(\upsilon+1)I)\Gamma^{-1}(A+(k+1)I)\Gamma(B+(\upsilon+1)I)}{\Gamma(\upsilon-k+1)k!}\\
&\quad\times\Gamma^{-1}(B+(\upsilon-k+1)I)(x-1)^k\sum_{r=0}^{\infty}\binom{\upsilon-k}{r}(x-1)^r 2^{\upsilon-k-r}\\
&=2^{-\upsilon}\sum_{r=0}^{\infty}\sum_{k=0}^{r}\frac{\Gamma(A+(\upsilon+1)I)\Gamma^{-1}(A+(k+1)I)\Gamma(B+(\upsilon+1)I)}{\Gamma(\upsilon-k+1)k!}\\
&\quad\times\Gamma^{-1}(B+(\upsilon-k+1)I)(x-1)^k\binom{\upsilon-k}{r-k}(x-1)^{r-k}2^{\upsilon-r}\\
&=\sum_{r=0}^{\infty}\left(\frac{x-1}{2}\right)^r\sum_{k=0}^{r}\frac{\Gamma(A+(\upsilon+1)I)\Gamma^{-1}(A+(k+1)I)\Gamma(B+(\upsilon+1)I)}{\Gamma(\upsilon-r+1)\Gamma(r-k+1)k!}\\
&\quad\times\Gamma^{-1}(B+(\upsilon-k+1)I)\Gamma(A+(r+1)I)\Gamma^{-1}(A+(r+1)I).
\end{aligned}\tag{4.19}$$

For $AB=BA$, using the following identity

$$(1-x)^{B+\upsilon I}(1-x)^{A+rI}=(1-x)^{B+A+(\upsilon+r)I},$$

we have

$$\begin{aligned}
&\frac{\Gamma(B+A+(r+\upsilon+1)I)\Gamma^{-1}(B+A+(\upsilon+1)I)}{r!}\\
&=\sum_{k=0}^{r}\frac{\Gamma(B+(\upsilon+1)I)\Gamma^{-1}(B+(\upsilon-k+1)I)}{(r-k)!}\\
&\times\frac{\Gamma(A+(r+1)I)\Gamma^{-1}(A+(k+1)I)}{k!}.
\end{aligned}\tag{4.20}$$

Substituting (4.20) in (4.19), we get

$$
\begin{aligned}
&P_{\upsilon}^{(A,B)}(x) \\
&= \frac{1}{\Gamma(\upsilon+1)} \sum_{r=0}^{\infty} \frac{\Gamma(\upsilon+1)}{r!\Gamma(\upsilon-r+1)} \Gamma(B+A+(r+\upsilon+1)I)\Gamma^{-1}(B+A+(\upsilon+1)I) \\
&\quad \times \Gamma(A+(\upsilon+1)I)\Gamma^{-1}(A+I)\Gamma(A+I)\Gamma^{-1}(A+(r+1)I)\left(\frac{1-x}{2}\right)^{r}(-1)^{r} \\
&= \frac{\Gamma(A+(\upsilon+1)I)\Gamma^{-1}(A+I)}{\Gamma(\upsilon+1)} F\left(-\upsilon I, A+B+(\upsilon+1)I; A+I; \frac{1-x}{2}\right)
\end{aligned}
$$

which is the desired result. □

Corollary 4.7. *The g-Jacobi matrix functions $P_{\upsilon}^{(A,B)}(x)$ can be presented as*

$$
P_{\upsilon}^{(A,B)}(x) = (-1)^{\upsilon} \frac{\Gamma(B+(\upsilon+1)I)\Gamma^{-1}(B+I)}{\Gamma(\upsilon+1)} F\left(-\upsilon I, A+B+(\upsilon+1)I; B+I; \frac{1+x}{2}\right).
$$

Theorem 4.8. *The g-Jacobi matrix functions $P_{\upsilon}^{(A,B)}(x)$ satisfy the matrix differential equation of second order*

$$
\begin{aligned}
\left(1-x^2\right) Y''(x) - 2Y'(x)A + (A+B-x(A+B+2I))\,Y'(x)& \\
+\upsilon(A+B+(\upsilon+1)I)\,Y(x) = \mathbf{0}& \qquad (4.21)
\end{aligned}
$$

or

$$
\begin{aligned}
&\frac{d}{dx}\left[(1-x)(1+x)^{A+B+I}\,Y'(x)\left(\frac{1-x}{1+x}\right)^{A}\right] \\
&\quad +\upsilon(A+B+(\upsilon+1)I)(1+x)^{A+B}\,Y(x)\left(\frac{1-x}{1+x}\right)^{A} = \mathbf{0}. \qquad (4.22)
\end{aligned}
$$

Proof. *Note that hypergeometric matrix function $Y = F(A,B;C;t)$ satisfies hypergeometric matrix differential equation*

$$
t(1-t)F'' - tAF' + F'(C-t(B+I)) - AFB = \mathbf{0}\;,\quad 0 \leq |\,t\,| < 1
$$

Also hypergeometric matrix function $F(\upsilon I + A + B + I, -\upsilon I; A+I; t)$ satisfies

$$
t(1-t)F'' - t(\upsilon I + A + B + I)F' + F'(A+I-t(-\upsilon I+I)) + \upsilon(A+B+(\upsilon+1)I)F = \mathbf{0}
$$

where $0 \leq |\,t\,| < 1$. Writing $\frac{1-x}{2}$ instead of t in this equation, we get

$$
\begin{aligned}
\left(1-x^2\right) F''(x) - 2F'(x)A + (A+B-x(A+B+2I))\,F'(x)& \\
+\upsilon(A+B+(\upsilon+1)I)\,F = \mathbf{0}.&
\end{aligned}
$$

$P_{\upsilon}^{(A,B)}(x)$ *having hypergeometric matrix function (4.18) satisfies the above matrix differential equation. Premultiplying (4.21) by* $(1+x)^{A+B}$ *and postmultiplying it by* $\left(\frac{1-x}{1+x}\right)^{A}$ *and rearranging, we have the second matrix differential equation.* □

Theorem 4.9. *The g-Jacobi matrix functions satisfy the following properties:*

$$(i) \qquad \lim_{\upsilon\to n} P_{\upsilon}^{(A,B)}(x) = P_{n}^{(A,B)}(x)$$

$$(ii) \qquad P_{\upsilon}^{(A,B)}(-x) = (-1)^{\upsilon} P_{\upsilon}^{(B,A)}(x)$$

$$(iii) \qquad P_{\upsilon}^{(A,B)}(1) = \frac{\Gamma(A+(\upsilon+1)I)\Gamma^{-1}(A+I)}{\Gamma(\upsilon+1)}$$

$$(iv) \qquad P_{\upsilon}^{(A,B)}(-1) = \frac{\Gamma(B+(\upsilon+1)I)\Gamma^{-1}(B+I)}{\Gamma(\upsilon+1)}$$

$$(v) \quad \frac{d}{dx} P_{\upsilon}^{(A,B)}(x) = \frac{1}{2}(A+B+(\upsilon+1)I)P_{\upsilon-1}^{(A+I,B+I)}(x).$$

Proof. (i) From (4.18), we have

$$\begin{aligned}
\lim_{\upsilon\to n} P_{\upsilon}^{(A,B)}(x) &= \lim_{\upsilon\to n} \frac{\Gamma(A+(\upsilon+1)I)\Gamma^{-1}(A+I)}{\Gamma(\upsilon+1)}\, F\left(-\upsilon, A+B+(\upsilon+1)I; A+I; \frac{1-x}{2}\right) \\
&= \frac{\Gamma(A+(n+1)I)\Gamma^{-1}(A+I)}{\Gamma(n+1)}\, F\left(-n, A+B+(n+1)I; A+I; \frac{1-x}{2}\right) \\
&= P_{n}^{(A,B)}(x).
\end{aligned}$$

(ii) From (4.15), we have $P_{\upsilon}^{(A,B)}(-x)$

$$\begin{aligned}
&= 2^{-\upsilon} \sum_{k=0}^{\infty} \frac{\Gamma(A+(\upsilon+1)I)\Gamma^{-1}(A+(k+1)I)\Gamma(B+(\upsilon+1)I)\Gamma^{-1}(B+(\upsilon-k+1)I)}{\Gamma(\upsilon-k+1)k!} \\
&\quad \times (-x-1)^{k}(-x+1)^{\upsilon-k} \\
&= (-1)^{\upsilon} P_{\upsilon}^{(B,A)}(x).
\end{aligned}$$

(iii) The proof is enough for $x=1$ in (4.15).
(iv) Using (ii) and (iii), we obtain the desired result.
(v) Using (4.18) and differentiating with respect to x, the result follows. □

4.4 Generalized g-Jacobi Matrix Function

In this section, we define fractional g-Jacobi matrix differential equation and its solution which is generalized g-Jacobi matrix function.

Definition 4.10. Fractional g-Jacobi matrix differential equation is defined as

$$t^{\mu}(1-t^{\mu})D^{2\mu}Y(t)-t^{\mu}(A+B+(\upsilon+1)I)D^{\mu}[Y(t)]+D^{\mu}[Y(t)](A+I+(\upsilon-1)It^{\mu}) + \upsilon(A+B+(\upsilon+1)I)Y(t)=\mathbf{0} \tag{4.23}$$

where $0<\mu\leq 1$.

Definition 4.11. Generalized g-Jacobi matrix functions are defined as

$$ {}_2^{\mu}F_1(A+B+(\upsilon+1)I,-\upsilon I;A+I;t)=Y_0t^{\theta}+\sum_{k=1}^{\infty}\left[\prod_{j=0}^{k-1}G_j(\theta)\right]Y_0\left[\prod_{j=0}^{k-1}F_{j+1}^{-1}(\theta)\right]t^{\theta+k\mu I}$$

where $0<\mu\leq 1$, $\theta Y_0=Y_0\theta$, $\theta B=B\theta$, and

$$\begin{aligned} F_k(\theta) &= \Gamma((k\mu+1)I+\theta)\Gamma^{-1}([(k-2)\mu+1]I+\theta) \\ &\quad +(A+I)\Gamma((k\mu+1)I+\theta)\Gamma^{-1}([(k-1)\mu+1]I+\theta) \end{aligned}$$

$$\begin{aligned} G_k(\theta) &= \Gamma((k\mu+1)I+\theta)\Gamma^{-1}([(k-2)\mu+1]I+\theta)-\upsilon(A+B+(\upsilon+1)I) \\ &\quad +(A+B+2I)\Gamma((k\mu+1)I+\theta)\Gamma^{-1}([(k-1)\mu+1]I+\theta) \end{aligned}$$

and $Re(\rho)>-1$ for all $\rho\in\sigma(\theta)$ satisfies the equation

$$\begin{aligned} F_0(\theta) &= \Gamma(I+\theta)\Gamma^{-1}((1-2\mu)I+\theta) \\ &\quad +(A+I)\Gamma(I+\theta)\Gamma^{-1}((1-\mu)I+\theta)=\mathbf{0} \end{aligned}$$

where $(1-2\mu)I+\theta$ and $(1-\mu)I+\theta$ are invertible for $0<\mu\leq 1$.

Theorem 4.12. *Generalized g-Jacobi matrix function is a solution of (4.23).*

4.5 Special Cases

Case 1. If we take matrix $C-I$ instead of A and $-C$ instead of B in $P_{\upsilon}^{(A,B)}(x)$, we define Chebyshev matrix functions $T_{\upsilon}(x,C)$ as follows:

$$\begin{aligned} & P_{\upsilon}^{(C-I,-C)}(x) \\ &= \frac{(-2)^{-\upsilon}}{\Gamma(\upsilon+1)}(1-x)^{I-C}(1+x)^{C}\,D_x^{\upsilon}\left[(1-x)^{C+(\upsilon-1)I}(1+x)^{-C+\upsilon I}\right] \\ &= \frac{\Gamma^{-1}(C)\Gamma(C+\upsilon I)}{\Gamma(\upsilon+1)}T_{\upsilon}(x,C) \end{aligned}$$

where C is a matrix in $\mathbb{C}^{r\times r}$ satisfying the condition $0 < Re(z) < 1$ for $\forall z \in \sigma(C)$. Chebyshev matrix functions have the following properties:

(a) *Rodrigues formula:*

$$T_\upsilon(x,C) = (-2)^{-\upsilon}(1-x)^{I-C}(1+x)^{C}\Gamma(C)\Gamma^{-1}(C+\upsilon I)D_x^{\upsilon} \times \left[(1-x)^{C+(\upsilon-1)I}(1+x)^{-C+\upsilon I}\right].$$

(b) *Hypergeometric matrix representations:*

$$T_\upsilon(x,C) = F\left(-\upsilon I, \upsilon I\, ;\, C\, ;\, \frac{1-x}{2}\right),$$

$$T_\upsilon(x,C) = \left(\frac{x+1}{2}\right)^{\upsilon} F\left(-\upsilon I, C-\upsilon I\, ;\, C\, ;\, \frac{x-1}{x+1}\right).$$

(c) *Matrix differential equation:*

$$(1-x^2)Y'' + Y'(-2C + (1-x)I) + \upsilon^2 Y = \mathbf{0}.$$

(d) *Limit relation:*

$$\lim_{\upsilon\to n} T_\upsilon(x,C) = T_n(x,C),$$

where $T_n(x,C)$ is the Chebyshev matrix polynomial.

Case 2. If we take matrix $A-\frac{I}{2}$ instead of A and $A-\frac{I}{2}$ instead of B in $P_\upsilon^{(A,B)}(x)$, we define Gegenbauer matrix functions $C_\upsilon^A(x)$ as follows:

$$P_\upsilon^{\left(A-\frac{I}{2},\, A-\frac{I}{2}\right)}(x) = (-2)^{-2\upsilon}\Gamma(2A+2\upsilon I)\Gamma^{-1}(2A+\upsilon I)\Gamma^{-1}(A+\upsilon I)\Gamma(A)C_\upsilon^A(x),$$

where A is a matrix in $\mathbb{C}^{r\times r}$ satisfying the condition $Re(z) > 0$ for $\forall z \in \sigma(A)$. Gegenbauer matrix functions have the following properties:

(a) *Rodrigues formula:*

$$C_\upsilon^A(x) = \frac{(-2)^{\upsilon}}{\Gamma(\upsilon+1)}\Gamma^{-1}(2A+2\upsilon I)\Gamma(2A+\upsilon I) \times\Gamma(A+\upsilon I)\Gamma^{-1}(A)\left(1-x^2\right)^{\frac{I}{2}-A} D_x^{\upsilon}\left[\left(1-x^2\right)^{A+\left(\upsilon-\frac{1}{2}\right)I}\right].$$

(b) *Hypergeometric matrix representations:*

$$C_\upsilon^A(x) = \frac{\Gamma(2A+\upsilon I)\Gamma^{-1}(2A)}{\Gamma(\upsilon+1)}F\left(-\upsilon I, 2A+\upsilon I\, ;\, A+\frac{I}{2}\, ;\, \frac{1-x}{2}\right),$$

$$C_\upsilon^A(x) = \frac{\Gamma(2A+\upsilon I)\Gamma^{-1}(2A)}{\Gamma(\upsilon+1)}\left(\frac{x+1}{2}\right)^{\upsilon} F\left(-\upsilon I, -A+\left(-\upsilon+\frac{1}{2}\right)I\, ;\, A+\frac{I}{2}\, ;\, \frac{x-1}{x+1}\right).$$

(c) *Matrix differential equation:*

$$(1-x^2)Y'' - xY'(2A+I) + \upsilon(2A+\upsilon I)Y = \mathbf{0}.$$

(d) *Limit relation:*

$$\lim_{\upsilon\to n} C_\upsilon^A(x) = C_n^A(x),$$

where $C_n^A(x)$ is the Gegenbauer matrix polynomial.

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Chapter 5
Linear Combinations of Genuine Szász–Mirakjan–Durrmeyer Operators

Margareta Heilmann and Gancho Tachev

Abstract We study approximation properties of linear combinations of the genuine Szász–Mirakjan–Durrmeyer operators which are also known as Phillips operators. We prove a full quantitative Voronovskaja-type theorem generalizing and improving earlier results by Agrawal, Gupta, and May. A Voronovskaja-type result for simultaneous approximation is also established. Furthermore global direct theorems for the approximation and weighted simultaneous approximation in terms of the Ditzian–Totik modulus of smoothness are proved.

5.1 Introduction

We consider linear combinations of a variant of Szász–Mirakjan operators which are known as Phillips operators or genuine Szász–Mirakjan–Durrmeyer operators, which for $n \in \mathbb{R}$, $n > 0$, are given by

$$\widetilde{S}_n(f,x) = n \sum_{k=1}^{\infty} s_{n,k}(x) \int_0^{\infty} s_{n,k-1}(t) f(t) dt + e^{-nx} f(0), \tag{5.1}$$

where

$$s_{n,k}(x) = \frac{(nx)^k}{k!} e^{-nx}, \; k \in \mathbb{N}_0, \; x \in [0,\infty),$$

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for every function f, for which the right-hand side of (5.1) makes sense. For $n > \alpha$ this is the case for real-valued continuous functions on $[0,\infty)$ satisfying an exponential growth condition, i.e.,

$$f \in C_\alpha[0,\infty) = \{f \in C[0,\infty) : |f(t)| \le Me^{\alpha t}, t \in [0,\infty)\}$$

for a constant $M > 0$ and an $\alpha > 0$ and for $\alpha = 0$ for bounded continuous functions, i.e.,

$$f \in C_B[0,\infty) = \{f \in C[0,\infty) : |f(t)| \le M, t \in [0,\infty)\}.$$

We also will consider L_p-integrable functions f possessing a finite limit at 0^+, i.e.,

$$f \in L_{p,0}[0,\infty) = \{f \in L_p[0,\infty) : \lim_{x\to 0^+} f(x) = f_0 \in \mathbb{R}\},$$

$1 \le p \le \infty$ and define $f(0) := f_0$.

The operators $\widetilde{S}_n$ were first considered in a paper by Phillips [20] in the context of semi-groups of linear operators and therefore often are called Phillips operators.

A strong converse result of type B in the terminology of Ditzian and Ivanov [6] can be found in a paper by Finta and Gupta [8] and also in a more general setting in another paper by Finta [9]. Recently the authors proved a strong converse result of type A improving the former results by Finta and Gupta. Up to our current knowledge linear combinations of these operators were first considered by May [17]. There are two other papers by Agrawal and Gupta [2, 3] dealing with a generalization of May's linear combinations and iterative combinations.

The operators $\widetilde{S}_n$ are closely related to the Szász–Mirakjan operators (see [22]) defined by

$$S_n(f,x) = \sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right),$$

to its Kantorovich variants

$$\widehat{S}_n(f,x) = n \sum_{k=0}^{\infty} s_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt$$

and the Durrmeyer version

$$\overline{S}_n(f,x) = n \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} s_{n,k}(t) f(t)dt$$

first considered by Mazhar and Totik in [18].

All these operators are positive linear operators. Comparing the different variants of Szász–Mirakjan operators, we see that all preserve constants, but the classical Szász–Mirakjan operators S_n and the genuine Szász–Mirakjan–Durrmeyer operators $\widetilde{S}_n$ also preserve all linear functions and interpolate at the point 0. In [16, (19)] the authors proved that $\widetilde{S}_n$ and $\overline{S}_n$ are connected in the same way as S_n and $\widehat{S}_n$, i.e.,

$$(\widetilde{S}_n f)' = \overline{S}_n f', \tag{5.2}$$

for sufficiently smooth f. In [16, Sect. 3] it is also proved that the operators $\widetilde{S}_n$ commute and that they commute with the differential operators $\widetilde{D}^{2l} := D^{l-1}\varphi^{2l}D^{l+1}$, $l \in \mathbb{N}$, where $\varphi(x) := \sqrt{x}$ and D denotes the ordinary differentiation of a function with respect to its variable. So the operators $\widetilde{S}_n$ combine nice properties of the classical Szász–Mirakjan operators and their Durrmeyer variant.

The term "genuine" is by now often used in the context of Bernstein–Durrmeyer operators and the corresponding variants, which also preserve linear functions and interpolate at the endpoints of the interval. They commute and also commute with certain differential operators and they can be considered as the limit case for Jacobi-weighted Bernstein–Durrmeyer operators. As analogous properties are fulfilled by appropriate variants of Baskakov and Szász–Mirakjan operators, we call them also "genuine".

We would like to mention that the iterates of the operators $\overline{S}_n$ and $\widetilde{S}_n$ can be expressed by the operators itself, i.e.,

$$\overline{S}_n^l = \overline{S}_{\frac{n}{l}}, \qquad \widetilde{S}_n^l = \widetilde{S}_{\frac{n}{l}}. \tag{5.3}$$

These representations are special for Durrmeyer-type modifications of the Szász–Mirakjan operators. For $\overline{S}_n^l$ the result was proved by Abel and Ivan in [1], for $\widetilde{S}_n^l$ by the authors in [16, Theorem 3,1, Corollary 3.1].

In this paper, we consider linear combinations $\widetilde{S}_{n,r}$ of order r of the operators $\widetilde{S}_{n_i}$, i.e.,

$$\widetilde{S}_{n,r} = \sum_{i=0}^{r} \alpha_i(n)\widetilde{S}_{n_i}, \tag{5.4}$$

where n_i, $i = 0, \ldots, r$, denote different positive numbers. In general the coefficients $\alpha_i(n)$ may depend on n.

In [17] May considered the case

$$n_i = 2^i n, \quad \alpha_i = \prod_{k=0,k\neq i}^{r} \frac{2^i}{2^i - 2^k}$$

which was generalized in [2] to

$$n_i = d_i n, \quad \alpha_i = \prod_{k=0,k\neq i}^{r} \frac{d_i}{d_i - d_k}$$

with different positive numbers d_i, $i = 0, \ldots, r$, independent of n. In [3, 4] also the iterative combinations

$$I - (I - \widetilde{S}_n)^{r+1}$$

are considered.

We will show that all these above-mentioned combinations suit into the following general approach. We determine the coefficients $\alpha_i(n)$ in (5.4) such that all polynomials of degree at most $r+1$ are reproduced, i.e.,

$$\widetilde{S}_{n,r}p = p \text{ for all } p \in \mathscr{P}_{r+1}.$$

This seems to be natural as the operators $\widetilde{S}_n$ preserve the linear functions. For the monomials $e_\nu(t) = t^\nu$, $\nu \in \mathbb{N}_0$, we have proved in [16, Lemma 2.1] that

$$\widetilde{S}_n(e_0,x) = 1, \quad \widetilde{S}_n(e_\nu,x) = \sum_{j=1}^{\nu} \binom{\nu-1}{j-1} \frac{\nu!}{j!} n^{j-\nu} x^j, \ \nu \in \mathbb{N}.$$

Thus the requirement that each polynomial of degree at most $r+1$ should be reproduced leads to a system of linear equations, i.e.,

$$\sum_{i=0}^{r} \alpha_i(n) = 1, \quad \sum_{i=0}^{r} n_i^{-l} \alpha_i(n) = 0, \ 1 \le l \le r,$$

which has the unique solution

$$\alpha_i(n) = \prod_{k=0,k\neq i}^{r} \frac{n_i}{n_i - n_k}. \tag{5.5}$$

Note that $\widetilde{S}_{n,0} = \widetilde{S}_n$.

Obviously the choice $n_i = d_i n$ is a special case of the general construction. Now we look at a special case of this special choice. For $n_i = d_i n$ with $d_i = \frac{1}{i+1}$ we get

$$\alpha_i(n) = \prod_{k=0,k\neq i}^{r} \frac{\frac{1}{i+1}}{\frac{1}{i+1} - \frac{1}{k+1}} = \prod_{k=0,k\neq i}^{r} \frac{k+1}{k-i} = (-1)^i \binom{r+1}{i+1}.$$

Thus for the corresponding linear combinations we get by using the representation (5.3) for the iterates

$$\begin{aligned}\widetilde{S}_{n,r} &= \sum_{i=0}^{r} (-1)^i \binom{r+1}{i+1} \widetilde{S}_{\frac{n}{i+1}} \\ &= \sum_{i=0}^{r} (-1)^i \binom{r+1}{i+1} \widetilde{S}_n^{i+1} = I - (I - \widetilde{S}_n)^{r+1}.\end{aligned}$$

So it turns out that the iterative combinations of the operators $\widetilde{S}_n$ are a special case of linear combinations. Note that the same arguments hold true for the linear combinations of the Szász–Mirakjan–Durrmeyer operators considered, e.g., in [13].

Now we state some useful properties for the coefficients of the linear combinations.

Lemma 5.1. *For $l \in \mathbb{N}$ the coefficients in (5.5) have the properties*

$$\sum_{i=0}^{r} n_i^{-(r+l)} \alpha_i(n) = (-1)^r \tau_{l-1}\left(\frac{1}{n_0}, \dots, \frac{1}{n_r}\right) \prod_{k=0}^{r} \frac{1}{n_k}, \tag{5.6}$$

$$\sum_{i=0}^{r} n_i^{l} \alpha_i(n) = \tau_l(n_0, \dots, n_r). \tag{5.7}$$

where $\tau_j(x_0, \dots, x_m)$ denotes the complete symmetric function which is the sum of all products of $x_0, \dots, x_m$ of total degree j for $j \in \mathbb{N}$ and $\tau_0(x_0, \dots, x_m) := 1$.

Proof.
(5.6): Let $t_i = \frac{1}{n_i}$, $0 \le i \le r$. Then the left-hand side of (5.6) is equal to

$$\sum_{i=0}^{r} t_i^l \prod_{k=0,k\neq i}^{r} \frac{t_i t_k}{t_k - t_i} = (-1)^r \prod_{k=0}^{r} t_k \sum_{i=0}^{r} t_i^{l+r-1} \prod_{k=0,k\neq i}^{r} \frac{1}{t_i - t_k}$$
$$= (-1)^r \prod_{k=0}^{r} t_k \sum_{i=0}^{r} \frac{f(t_i)}{\omega'(t_i)},$$

where $f(t) = t^{l+r-1}$ and $\omega(t) = \prod_{k=0}^{r}(t - t_k)$. We apply the well-known identity for divided differences

$$\sum_{i=0}^{r} \frac{f(t_i)}{\omega'(t_i)} = f[t_0, t_1, \ldots, t_r].$$

For $f(t) = t^{l+r-1}$ it is valid that

$$f[t_0, t_1, \ldots, t_r] = \tau_{l-1}(t_0, \ldots, t_r)$$

(see [19, Theorem 1.2.1]). Thus we have proved (5.6).
(5.7):The left-hand side of (5.7) is equal to

$$\sum_{i=0}^{r} n_i^l n_i^r \prod_{k=0,k\neq i}^{r} \frac{1}{n_i - n_k} = \sum_{i=0}^{r} \frac{f(n_i)}{\omega'(n_i)} = f[n_0, n_1, \ldots, n_r] = \tau_l(n_0, \ldots, n_r)$$

with $f(t) = t^{l+r}$ and application of the same identity for the divided differences as above. □

For the proofs of our theorems we need two additional assumptions for the coefficients. The first condition is

$$an \le n_0 < n_1 < \cdots < n_r \le An, \tag{5.8}$$

where a, A denote positive constants independent of n. With (5.6) it is clear that this guarantees that

$$\sum_{i=0}^{r} n_i^{-l} \alpha_i(n) = \mathcal{O}\left(n^{-l}\right),\ l \ge r+1.$$

Secondly we assume that

$$\sum_{i=0}^{r} |\alpha_i(n)| \le C \tag{5.9}$$

with a constant C independent of n. This condition is due to the fact that the linear combinations are no longer positive operators. Especially for the considerations of remainder terms of Taylor expansions in our proofs this assumption is important. These assumptions are fulfilled for all the special cases mentioned above.

The paper is organized as follows. In Sect. 5.2 we define an auxiliary operator useful in the context of simultaneous approximation and list some basic results, such as the moments, estimates for the moments, and some identities which will be used throughout the paper. Section 5.3 is devoted to the Voronovskaja-type results and Sect. 5.4 to the global direct theorems for the approximation and weighted simultaneous approximation. For the latter we will need some technical definitions and results which are given in Sect. 5.5. Note that throughout this paper C always denotes a positive constant not necessarily the same at each occurrence.

5.2 Auxiliary Results

For the proofs of our results concerning simultaneous approximation we will make use of the auxiliary operators

$$_m\widetilde{S}_n = n\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^{\infty} s_{n,k+m-1}(t)f(t)dt,\ m\in\mathbb{N}.$$

For $m=1$ we have $_m\widetilde{S}_n=\overline{S}_n$. Due to the relation (5.2) between $\widetilde{S}_n$ and $\overline{S}_n$ the operators $_m\widetilde{S}_n$ coincide with the auxiliary operators $_{m-1}\overline{S}_n$ which were used in [11, 13]. Thus, for sufficiently smooth f, we have

$$(\widetilde{S}_n f)^{(m)} = {}_m\widetilde{S}_n f^{(m)} = (\overline{S}_n f')^{(m-1)} = {}_{m-1}\overline{S}_n f^{(m)}. \tag{5.10}$$

The corresponding linear combinations of order r are given by

$$_m\widetilde{S}_{n,r} = \sum_{i=0}^{r}\alpha_i(n)\,{}_m\widetilde{S}_{n_i} = \sum_{i=0}^{r}\alpha_i(n)\,{}_{m-1}\overline{S}_{n_i} = {}_{m-1}\overline{S}_{n,r}$$

with the same coefficients $\alpha_i(n)$ given in (5.5) and the additional assumptions (5.8) and (5.9).

From the moments of $\widetilde{S}_n$ in [16, Lemma 2.1], Lemma 5.1 and the moments for the auxiliary operators in [11, Lemma 4.7] we derive the following result.

Lemma 5.2. *For $\mu\in\mathbb{N}_0$ let $f_{\mu,x}=(t-x)^{\mu}$. Then*

$$(\widetilde{S}_{n,r}f_{0,x})(x) = 1,\quad (\widetilde{S}_{n,r}f_{\mu,x})(x) = 0, 1\le\mu\le r+1,$$

$$(\widetilde{S}_{n,r}f_{\mu,x})(x) = (-1)^r\prod_{k=0}^{r}\frac{1}{n_k}$$
$$\times\begin{cases}\displaystyle\sum_{j=1}^{\mu-(r+1)}\binom{\mu-j-1}{j-1}\frac{\mu!}{j!}x^j\tau_{\mu-j-r-1}\left(\frac{1}{n_0},\dots,\frac{1}{n_r}\right), & r+2\le\mu\le 2r+2,\\ \displaystyle\sum_{j=1}^{\left[\frac{\mu}{2}\right]}\binom{\mu-j-1}{j-1}\frac{\mu!}{j!}x^j\tau_{\mu-j-r-1}\left(\frac{1}{n_0},\dots,\frac{1}{n_r}\right), & \mu\ge 2r+2,\end{cases}$$

$$({}_m\widetilde{S}_{n,r}f_{0,x})(x) = 1,\quad ({}_m\widetilde{S}_{n,r}f_{\mu,x})(x) = 0, 1\le\mu\le r,$$

$$({}_m\widetilde{S}_{n,r}f_{\mu,x})(x) = (-1)^r \prod_{k=0}^{r} \frac{1}{n_k}$$

$$\times \begin{cases} \sum_{j=0}^{\mu-(r+1)} \binom{\mu-j+m-1}{j+m-1} \frac{\mu!}{j!} x^j \tau_{\mu-j-r-1}\left(\frac{1}{n_0},\dots,\frac{1}{n_r}\right), \; r+1 \le \mu \le 2r+2, \\ \sum_{j=0}^{\left[\frac{\mu}{2}\right]} \binom{\mu-j+m-1}{j+m-1} \frac{\mu!}{j!} x^j \tau_{\mu-j-r-1}\left(\frac{1}{n_0},\dots,\frac{1}{n_r}\right), \; \mu \ge 2r+2. \end{cases}$$

From these representations of the moments we obtain some needed estimates.

Corollary 5.3. *For $\mu \ge r+2$ we have*

$$|\widetilde{S}_{n,r}(f_{\mu,x},x)| \le C \begin{cases} n^{-\mu}, & x \in \left[0,\frac{1}{n}\right], \\ n^{-(r+1)} x^{\mu-r-1}, & x \in \left[\frac{1}{n},\infty\right), \; r+2 \le \mu \le 2r+2, \\ n^{-\left[\frac{\mu+1}{2}\right]} x^{\left[\frac{\mu}{2}\right]}, & x \in \left[\frac{1}{n},\infty\right), \; 2r+2 \le \mu, \end{cases}$$

and for $\mu \ge r+1$

$$|{}_m\widetilde{S}_{n,r}(f_{\mu,x},x)| \le C \begin{cases} n^{-\mu}, & x \in \left[0,\frac{1}{n}\right], \\ n^{-(r+1)} x^{\mu-r-1}, & x \in \left[\frac{1}{n},\infty\right), \; r+1 \le \mu \le 2r+2, \\ n^{-\left[\frac{\mu+1}{2}\right]} x^{\left[\frac{\mu}{2}\right]}, & x \in \left[\frac{1}{n},\infty\right), \; 2r+2 \le \mu. \end{cases}$$

Now we list some basic identities for the basis functions $s_{n,k}$ which follow directly from their definition. For simplicity we set $s_{n,k} = 0$ for $k < 0$.

$$\sum_{k=0}^{\infty} s_{n,k} = 1, \tag{5.11}$$

$$n \int_0^{\infty} t^{\nu} s_{n,k}(t)dt = \frac{1}{n^{\nu}} \cdot \frac{(k+\nu)!}{k!}, \; \nu \in \mathbb{N}_0, \tag{5.12}$$

$$s'_{n,k}(x) = n(s_{n,k-1}(x) - s_{n,k}(x)), \tag{5.13}$$

$$\varphi(x)^{2m} s_{n,k}(x) s_{n,k+2m-1}(t) = \beta(k,m) s_{n,k+m}(x) \varphi(t)^{2m} s_{n,k+m-1}(t), \; m \in \mathbb{N}, \tag{5.14}$$

with $\frac{(m-1)!}{(2m-1)!} \le \beta(k,m) := \frac{(k+m)!(k+m-1)!}{k!(k+2m-1)!} \le 1$. Proofs can be found, for example, in [14, 18, 22].

5.3 Voronovskaja-Type Theorems

In this section we present a Voronovskaja-type theorem for the linear combinations of the genuine Szász–Mirakjan–Durrmeyer operators. Similar results were stated earlier in [17, Lemma 2.5], [2, Theorem 1], and [3, Theorem 1]. Our Theorem 5.4

now improves and generalizes these results. Furthermore, in Theorem 5.5 we prove a Voronovskaja-type result for simultaneous approximation by linear combinations. In both theorems explicit formulas for the limits are given.

Theorem 5.4. *Let $f \in C_B[0,\infty)$ be $(2r+2)$-times differentiable at a fixed point x. Then with $\widetilde{D}^{2(r+1)} = D^r \varphi^{2(r+1)} D^{r+2}$ we have*

$$\lim_{n\to\infty} \left\{ \prod_{k=0}^{r} n_k \right\} \left(\widetilde{S}_{n,r} f - f \right)(x) = \frac{(-1)^r}{(r+1)!} \left(\widetilde{D}^{2(r+1)} f \right)(x).$$

Proof. For the function f we use the Taylor expansion

$$\begin{aligned} f(t) &= \sum_{\mu=0}^{2(r+1)} \frac{(t-x)^\mu}{\mu!} f^{(\mu)}(x) + (t-x)^{2(r+1)} R(t,x) \\ &:= \widetilde{f}(t) + (t-x)^{2(r+1)} R(t,x), \end{aligned}$$

where

$$|R(t,x)| \le C \text{ for every } t \in [0,\infty) \text{ and } \lim_{t\to x} R(t,x) = 0.$$

From Lemma 5.2 we get

$$\begin{aligned} \widetilde{S}_{n,r}(\widetilde{f},x) - f(x) &= \sum_{\mu=r+2}^{2(r+1)} \frac{f^{(\mu)}(x)}{\mu!} \widetilde{S}_n(f_{\mu,x},x) \\ &= (-1)^r \prod_{k=0}^{r} \frac{1}{n_k} \sum_{\mu=r+2}^{2(r+1)} f^{(\mu)}(x) \\ &\quad \times \sum_{j=1}^{\mu-(r+1)} \binom{\mu-j-1}{j-1} \frac{1}{j!} x^j \tau_{\mu-j-(r+1)} \left(\frac{1}{n_0}, \dots, \frac{1}{n_r} \right) \\ &= (-1)^r \prod_{k=0}^{r} \frac{1}{n_k} \sum_{j=r+1}^{2r+1} \tau_{j-(r+1)} \left(\frac{1}{n_0}, \dots, \frac{1}{n_r} \right) \\ &\quad \times \sum_{\mu=j+1}^{2(r+1)} f^{(\mu)}(x) \binom{j-1}{\mu-j-1} \frac{1}{(\mu-j)!} x^{\mu-j}. \end{aligned}$$

From (5.6) and the additional assumption (5.8) for the numbers n_i it is clear that we only have to consider the summand with $j = r+1$ for the following limit. Thus we get

$$\begin{aligned} &\lim_{n\to\infty} \left\{ \prod_{k=0}^{r} n_k \right\} \left(\widetilde{S}_{n,r} \widetilde{f} - f \right)(x) \\ &= (-1)^r \sum_{\mu=r+2}^{2(r+1)} f^{(\mu)}(x) \binom{r}{\mu-r-2} \frac{1}{(\mu-r-1)!} x^{\mu-r-1} \end{aligned}$$

$$= \frac{(-1)^r}{(r+1)!} D^r\{x^{r+1} D^{r+2} f(x)\}$$
$$= \frac{(-1)^r}{(r+1)!} \widetilde{D}^{2(r+1)} f(x),$$

where we used Leibniz's rule.
For the remainder term we have to show that

$$\lim_{n\to\infty} \left\{\prod_{k=0}^{r} n_k\right\} \left[\widetilde{S}_{n,r}((t-x)^{2(r+1)} R(t,x),x)\right] = 0. \tag{5.15}$$

For $\varepsilon > 0$ let $\delta > 0$ be a positive number, such that

$$|R(t,x)| < \varepsilon \ \text{ for } |t-x| < \delta.$$

Thus for every $t \in [0,\infty)$ we have

$$|R(t,x)| < \varepsilon + C\frac{(t-x)^2}{\delta^2}.$$

Therefore, due to the assumptions (5.8) and (5.9), we have

$$\begin{aligned}
&\left|\widetilde{S}_{n,r}((t-x)^{2(r+1)} R(t,x),x)\right| \\
&\le C\widetilde{S}_n((t-x)^{2(r+1)}|R(t,x)|,x) \\
&\le C\left(\varepsilon \widetilde{S}_n((t-x)^{2(r+1)},x) + \frac{M}{\delta^2}\widetilde{S}_n((t-x)^{2(r+2)},x)\right).
\end{aligned}$$

From the estimates for the moments in Corollary 5.3 we get (5.15). □

Next we show a Voronovskaja-type result for simultaneous approximation.

Theorem 5.5. *Let $f \in C_B[0,\infty)$ be $(m+2r+2)$-times differentiable at a fixed point x. Then with $\widetilde{D}^{2(r+1)} = D^r \varphi^{2(r+1)} D^{r+2}$ we have*

$$\lim_{n\to\infty} \left\{\prod_{k=0}^{r} n_k\right\} \left(\widetilde{S}_{n,r} f - f\right)^{(m)}(x) = \frac{(-1)^r}{(r+1)!}\left(D^m \widetilde{D}^{2(r+1)} f\right)(x).$$

Proof. We use the Taylor expansion of $f^{(m)}$

$$\begin{aligned}
f^{(m)}(t) &= \sum_{\mu=0}^{2(r+1)} \frac{(t-x)^\mu}{\mu!} f^{(\mu+m)}(x) + (t-x)^{2(r+1)} R(t,x) \\
&:= \widetilde{f}^{(m)}(t) + (t-x)^{2(r+1)} R(t,x),
\end{aligned}$$

with the same properties for $|R(x,t)|$ as in the proof of Theorem 5.4. With the relation (5.10) we get

$$\begin{aligned}
(\widetilde{S}_{n,r}(\widetilde{f},x)-f(x))^{(m)} &= {}_m\widetilde{S}_{n,r}(\widetilde{f}^{(m)},x)-f^{(m)}(x)\\
&= \sum_{\mu=r+1}^{2(r+1)} \frac{f^{(\mu+m)}(x)}{\mu!}\,{}_m\widetilde{S}_n(f_{\mu,x},x)\\
&= (-1)^r \prod_{k=0}^{r}\frac{1}{n_k}\sum_{\mu=r+1}^{2(r+1)} f^{(\mu+m)}(x)\\
&\quad \times \sum_{j=0}^{\mu-(r+1)} \binom{\mu-j+m-1}{j+m-1}\frac{1}{j!}x^j\tau_{\mu-j-(r+1)}\left(\frac{1}{n_0},\dots,\frac{1}{n_r}\right)\\
&= (-1)^r \prod_{k=0}^{r}\frac{1}{n_k}\sum_{j=r+1}^{2r+1}\tau_{j-(r+1)}\left(\frac{1}{n_0},\dots,\frac{1}{n_r}\right)\\
&\quad \times \sum_{\mu=j}^{2(r+1)} f^{(\mu+m)}(x)\binom{j+m-1}{\mu-j+m-1}\frac{1}{(\mu-j)!}x^{\mu-j}
\end{aligned}$$

The rest follows analogously to the proof of Theorem 5.4 □

5.4 Global Direct Results

In this section we prove some global direct results for the approximation and weighted simultaneous approximation by the linear combinations $\widetilde{S}_n$. The estimates are formulated in terms of weighted and nonweighted Ditzian–Totik moduli of smoothness (see [7]). We choose the step-weight $\varphi(x)=\sqrt{x}$ and assume $t>0$ sufficiently small to define

$$\begin{aligned}
\omega_\varphi^r(f,t)_p &= \sup_{0<h\le t}\|\Delta_{h\varphi}^r f\|_p,\\
\omega_\varphi^r(f,t)_{\varphi^m,p} &= \sup_{0<h\le t}\|\,\varphi^m\Delta_{h\varphi}^r f\|_p^{[t^*,\infty)} + \sup_{0<h\le t^*}\|\,\varphi^m\overrightarrow{\Delta}_h^r f\|_p^{[0,12t^*]},
\end{aligned}$$

where $t^*=r^2t^2$. The symmetric and forward differences are given by

$$\begin{aligned}
\Delta_{h\varphi(x)}^r f(x) &= \sum_{k=0}^{r}(-1)^k\binom{r}{k} f\left(x+\left(\frac{r}{2}-k\right)h\varphi(x)\right),\\
\overrightarrow{\Delta}_h^r f(x) &= \sum_{k=0}^{r}(-1)^k\binom{r}{k} f\left(x+(r-k)h\right),
\end{aligned}$$

whenever the arguments of the function f are contained in the corresponding interval. Otherwise, they are defined to be zero. In [7, Chaps. 2, 3, 6.1] Ditzian and Totik proved that these moduli are equivalent to the K-functionals

$$K_\varphi^r(f,t^r)_p = \inf\{\|f-g\|_p + t^r\|\varphi^r g^{(r)}\|_p; g, \varphi^r g^{(r)} \in L_p[0,\infty)\},$$
$$\overline{K}_\varphi^r(f,t^r)_p = \inf\Big\{\|f-g\|_p + t^r\|\varphi^r g^{(r)}\|_p + t^{2r}\|g^{(r)}\|_p; \quad g, g^{(r)}, \varphi^r g^{(r)} \in L_p[0,\infty)\Big\},$$
$$K_\varphi^r(f,t^r)_{\varphi^m,p} = \inf\{\|\varphi^m(f-g)\|_p + t^r\|\varphi^{m+r} g^{(r)}\|_p; \varphi^m g, \varphi^{m+r} g^{(r)} \in L_p[0,\infty)\}$$

for the nonweighted and weighted case, respectively.

For the proof of Theorem 5.8 we will use the equivalence of the weighted modulus to the modified weighted K-functional (see [15])

$$\overline{K}_\varphi^r(f,t^r)_{\varphi^m,p} = \inf\Big\{\|\varphi^m(f-g)\|_p + t^r\|\varphi^{m+r} g^{(r)}\|_p + t^{2r}\|\varphi^m g^{(r)}\|_p; \quad \varphi^m g, \varphi^m g^{(r)}, \varphi^{m+r} g^{(r)} \in L_p[0,\infty)\Big\}.$$

For the proofs of the main theorems we need the Hardy inequality (see [21, Chap. V, Lemma 3.14])

$$\left\{\int_0^\infty \left(\int_x^\infty h(y)dy\right)^p x^{s-1}dx\right\}^{1/p} \le \frac{p}{s}\left\{\int_0^\infty (yh(y))^p y^{s-1}dy\right\}^{1/p} \tag{5.16}$$

where $h \ge 0$, $p \ge 1$ and $s > 0$.

Theorem 5.6. *Let $\varphi(x) = \sqrt{x}$, $f \in L_{p,0}[0,\infty)$, $1 \le p < \infty$. Then*

$$\|\widetilde{S}_{n,r}f - f\|_p \le C\omega_\varphi^{2(r+1)}\left(f, \frac{1}{\sqrt{n}}\right)_p,$$

where C denotes a constant independent of n.

Proof. For every $g \in L_p[0,\infty)$ with $g(0) := f(0)$, $g^{(2(r+1))}, \varphi^{2(r+1)}g^{(2(r+1))} \in L_p[0,\infty)$ we get

$$\|\widetilde{S}_{n,r}f - f\|_p \le C\|f-g\|_p + \|\widetilde{S}_{n,r}g - g\|_p. \tag{5.17}$$

We look at the second term on the right-hand side of (5.17) and prove that

$$\|\widetilde{S}_{n,r}g - g\|_p \le C\left(n^{-(r+1)}\|\varphi^{2(r+1)}g^{(2(r+1))}\|_p + n^{-2(r+1)}\|g^{(2(r+1))}\|_p\right). \tag{5.18}$$

To do so, we consider the Taylor expansion of g and define

$$\begin{aligned} g(t) &= \sum_{\mu=0}^{r+1}\frac{(t-x)^\mu}{\mu!}g^{(\mu)}(x) + \sum_{\mu=r+2}^{2(r+1)}\frac{(t-x)^\mu}{\mu!}g^{(\mu)}(x) + R(t,x) \\ &:= g_1(t) + g_2(t) + R(t,x), \end{aligned}$$

with the remainder

$$R(t,x) = \frac{1}{(2r+1)!} \int_x^t (t-u)^{2r+1} g^{(2(r+1))}(u)du.$$

As all polynomials of degree at most $r+1$ are reproduced, it is enough to show the estimates

$$\|\widetilde{S}_{n,r} g_2\|_p \le C \left(n^{-(r+1)} \|\varphi^{2(r+1)} g^{(2(r+1))}\|_p + n^{-2(r+1)} \|g^{(2(r+1))}\|_p \right) \tag{5.19}$$

and

$$\|\widetilde{S}_{n,r} R(t,\cdot)\|_p \le C \left(n^{-(r+1)} \|\varphi^{2(r+1)} g^{(2(r+1))}\|_p + n^{-2(r+1)} \|g^{(2(r+1))}\|_p \right). \tag{5.20}$$

We first prove (5.19) separately for the intervals $\left[0, \frac{1}{n}\right]$ and $\left[\frac{1}{n}, \infty\right)$.
For $x \in \left[0, \frac{1}{n}\right]$ we get with Corollary 5.3

$$\|\widetilde{S}_{n,r} g_2\|_p^{[0,1/n]} \le C \sum_{\mu=r+2}^{2(r+1)} n^{-\mu} \|g^{(\mu)}\|_p. \tag{5.21}$$

Similar as in the proof of [13, Theorem 6] we apply Hardy's inequality (5.16) $2(r+1)-\mu$ times with $s=(l-1)p+1, h=|g^{(\mu+l)}|$ in the l-th step, $l=1,\dots,2(r+1)-\mu$. This leads to

$$\|g^{(\mu)}\|_p \le C \|\varphi^{2(2r+2-\mu)} g^{(2r+2)}\|_p.$$

So, together with (5.21), $x^{2(r+1)-\mu} \le n^{-2(r+1)+\mu}$ for $x \in [0, 1/n]$ and $x^{r+1-\mu} \le n^{-(r+1)+\mu}$ for $x \in \left[\frac{1}{n}, \infty\right)$, it follows

$$\begin{aligned} &\|\widetilde{S}_{n,r} g_2\|_p^{[0,1/n]} \qquad (5.22)\\ &\le C \sum_{\mu=r+2}^{2(r+1)} n^{-\mu} \left\{ \|\varphi^{2(2r+2-\mu)} g^{(2(r+1))}\|_p^{[0,1/n]} + \|\varphi^{2(2r+2-\mu)} g^{(2(r+1))}\|_p^{[1/n,\infty)} \right\} \\ &\le C \left\{ n^{-(r+1)} \|\varphi^{2(r+1)} g^{(2(r+1))}\|_p + n^{-2(r+1)} \|g^{(2(r+1))}\|_p \right\}. \end{aligned}$$

For $x \in \left[\frac{1}{n}, \infty\right)$ we derive from Corollary 5.3

$$\|\widetilde{S}_{n,r} g_2\|_p^{[1/n,\infty)} \le C n^{-(r+1)} \sum_{\mu=r+2}^{2(r+1)} \|\varphi^{2(\mu-r-1)} g^{(\mu)}\|_p. \tag{5.23}$$

Again applying Hardy's inequality (5.16) $2(r+1)-\mu$ times now with $s = (\mu - r - 2 + l)p + 1, h = |g^{(\mu+l)}|$ in the l-th step, $l = 1, \dots, 2(r+1) - \mu$, leads to

$$\|\varphi^{2(\mu-r-1)} g^{(\mu)}\|_p \le C \|\varphi^{2(r+1)} g^{(2(r+1))}\|_p.$$

Together with (5.23), this implies

$$\|\widetilde{S}_{n,r}g_2\|_p^{[1/n,\infty)} \leq Cn^{-(r+1)}\|\varphi^{2(r+1)}g^{(2(r+1))}\|_p. \tag{5.24}$$

With (5.22) and (5.24) we have proved (5.19).

Next we prove the estimate (5.20) explicitly for $p=1$ and $p=\infty$. The cases $1<p<\infty$ then follow by the Riesz–Thorin interpolation theorem [5, Theorem 1.1.1]. Due to the assumptions (5.8) and (5.9) for the coefficients of the linear combinations it is enough to prove

$$\|\widetilde{S}_nR(t,\cdot)\|_p \leq C\left(n^{-(r+1)}\|\varphi^{2(r+1)}g^{(2(r+1))}\|_p + n^{-2(r+1)}\|g^{(2(r+1))}\|_p\right). \tag{5.25}$$

$\underline{p=\infty}$: Note that

$$|R(t,x)| \leq \frac{1}{(2r+1)!}\|g^{(2r+2)}\|_\infty (t-x)^{2r+2},$$
$$|R(t,x)| \leq \frac{1}{(2r+1)!}\|\varphi^{2r+2}g^{(2r+2)}\|_\infty \frac{(t-x)^{2r+2}}{x^{r+1}},$$

as $\frac{|t-u|^{2r+1}}{u^{r+1}} \leq \frac{|t-x|^{2r+1}}{x^{r+1}}$. Thus, with Corollary 5.3 we derive

$$\widetilde{S}_n(|R(t,x)|,x) \leq Cn^{-2(r+1)}\|g^{(2r+2)}\|_\infty \text{ for } x\in\left[0,\frac{1}{n}\right],$$
$$\widetilde{S}_n(|R(t,x)|,x) \leq Cn^{-(r+1)}\|\varphi^{2r+2}g^{(2r+2)}\|_\infty \text{ for } x\in\left[\frac{1}{n},\infty\right),$$

i.e., we have proved (5.25) for $p=\infty$.

$\underline{p=1}$: By applying Fubini's theorem twice we first obtain

$$\begin{aligned}
&\|\widetilde{S}_n(R(t,\cdot)\|_1 \\
&\leq Cn\Bigg\{\int_0^\infty \sum_{k=1}^\infty s_{n,k}(x)\int_0^x s_{n,k-1}(t)\int_t^x (u-t)^{2r+1}\left|g^{(2r+2)}(u)\right| du\,dt\,dx \\
&\qquad + \int_0^\infty \sum_{k=1}^\infty s_{n,k}(x)\int_x^\infty s_{n,k-1}(t)\int_x^t (t-u)^{2r+1}\left|g^{(2r+2)}(u)\right| du\,dt\,dx \\
&\qquad + \int_0^\infty s_{n,0}(x)\int_0^x u^{2r+1}\left|g^{(2r+2)}(u)\right| du\,dx\Bigg\}
\end{aligned}$$

$$= C\int_0^\infty \left|g^{(2r+2)}(u)\right|\left[\frac{1}{n}u^{2r+1}e^{-nu}\right.$$

$$+\left\{\int_u^\infty\int_0^u - \int_0^u\int_u^\infty\right\}(u-t)^{2r+1} n\sum_{k=1}^\infty s_{n,k}(x)s_{n,k-1}(t)dtdx\Bigg] du$$

$$= C\int_0^\infty \left|g^{(2r+2)}(u)\right| H_{n,2r+2}(u)du.$$

From this estimate (5.25) now follows for the case $p=1$ by using Corollary 5.11. □

For the proof of the next theorem we need the following result:

Lemma 5.7. *Let $h\in L_p[0,\infty)$ such that $\varphi^{2m}h\in L_p[0,\infty)$, $m\in\mathbb{N}$, $1\le p\le\infty$. Then*

$$\|\varphi^{2m}\ {}_{2m}\widetilde{S}_n h\|_p \le \|\varphi^{2m}h\|_p.$$

Proof. By using (5.14) we first get that

$$\varphi(x)^{2m}\ {}_{2m}\widetilde{S}_n(h,x) = n\sum_{k=0}^\infty \beta(k,m)s_{n,k+m}(x)\int_0^\infty s_{n,k+m-1}(t)\varphi(t)^{2m}h(t)dt.$$

Thus

$$\|\varphi^{2m}\ {}_{2m}\widetilde{S}_n h\|_p \le \|\widetilde{S}_n(\varphi^{2m}h)\|_p \le \|\varphi^{2m}h\|_p.$$

□

In our next theorem we prove a global direct theorem for simultaneous approximation.

Theorem 5.8. *Let $f\in L_p[0,\infty)$, $1\le p<\infty$, $m\in\mathbb{N}$ such that $\varphi^{2m}f^{(2m)}\in L_p[0,\infty)$. Then*

$$\|\varphi^{2m}(\widetilde{S}_{n,r}f-f)^{(2m)}\|_p \le C\omega_\varphi^{2(r+1)}\left(f^{(2m)},\frac{1}{\sqrt{n}}\right)_{\varphi^{2m},p}.$$

Proof. For every function g with $\varphi^{2m}g$, $\varphi^{2m}g^{(2(r+1))}$, $\varphi^{2m+2(r+1)}g^{(2(r+1))}\in L_p[0,\infty)$ we derive by using (5.10) and Lemma 5.7

$$\begin{aligned}&\|\varphi^{2m}(\widetilde{S}_{n,r}f-f)^{(2m)}\|_p\\ &=\|\varphi^{2m}({}_{2m}\widetilde{S}_{n,r}f^{(2m)}-f^{(2m)})\|_p\\ &\le C\|\varphi^{2m}(f^{(2m)}-g)\|_p+\|\varphi^{2m}({}_{2m}\widetilde{S}_{n,r}g-g)\|_p.\end{aligned}\tag{5.26}$$

Similar to the proof of Theorem 5.6 we use the Taylor expansion

$$\begin{aligned}g(t) &= \sum_{\mu=0}^r \frac{(t-x)^\mu}{\mu!}g^{(\mu)}(x)+\sum_{\mu=r+1}^{2(r+1)}\frac{(t-x)^\mu}{\mu!}g^{(\mu)}(x)+R(t,x)\\ &:= g_1(t)+g_2(t)+R(t,x),\end{aligned}$$

${}_{2m}\widetilde{S}_{n,r}(g_1,x)=0$ and

$$\|\varphi^{2m}{}_{2m}\widetilde{S}_{n,r}g_2\|_p \\ \leq C\left(n^{-(r+1)}\|\varphi^{2(m+r+1)}g^{(2(r+1))}\|_p+n^{-2(r+1)}\|\varphi^{2m}g^{(2(r+1))}\|_p\right).$$

Thus in view of (5.8) and (5.9) it remains to prove

$$\|\varphi^{2m}{}_{2m}\widetilde{S}_nR(t,\cdot)\|_p \\ \leq C\left(n^{-(r+1)}\|\varphi^{2(m+2r+1)}g^{(2(r+1))}\|_p+n^{-2(r+1)}\|\varphi^{2m}g^{(2(r+1))}\|_p\right). \tag{5.27}$$

Again we look at the cases $p=1$ and $p=\infty$ separately and use the Riesz–Thorin interpolation theorem for $1<p<\infty$.
$p=\infty$: First we observe that by using (5.14) we have

$$|\varphi(x)^{2m}{}_{2m}\widetilde{S}_n(R(t,x),x)| \\ \leq \frac{n}{(2r+1)!}\left\{\sum_{k=0}^{\infty}s_{n,k+s}(x)\int_0^x\varphi(t)^{2m}s_{n,k+m-1}(t)\int_t^x(u-t)^{2r+1}|g^{(2(r+1))}(u)|dudt\right. \\ \left.+\varphi(x)^{2m}\sum_{k=0}^{\infty}s_{n,k}(x)\int_x^{\infty}s_{n,k+2m-1}(t)\int_x^t(t-u)^{2r+1}|g^{(2(r+1))}(u)|dudt\right\}.$$

Thus with $\varphi(t)^{2m}\leq\varphi(u)^{2m}$ in the first and $\varphi(x)^{2m}\leq\varphi(u)^{2m}$ in the second term on the right-hand side we get as $|u-t|\leq|x-t|$ and $\frac{|t-u|^{2r+1}}{u^{r+1}}\leq\frac{|t-x|^{2r+1}}{x^{r+1}}$
for $x\in\left[0,\frac{1}{n}\right]$

$$|\varphi(x)^{2m}{}_{2m}\widetilde{S}_n(R(t,x),x)| \\ \leq C\|\varphi^{2m}g^{(2(r+1))}\|_\infty\left\{\widetilde{S}_n(f_{2(r+1),x},x)+{}_{2m}\widetilde{S}_n(f_{2(r+1),x},x)\right\} \\ \leq Cn^{-2(r+1)}\|\varphi^{2m}g^{(2(r+1))}\|_\infty$$

and for $x\in\left[\frac{1}{n},\infty\right)$

$$|\varphi(x)^{2m}{}_{2m}\widetilde{S}_n(R(t,x),x)| \\ \leq C\|\varphi^{2(m+r+1)}g^{(2(r+1))}\|_\infty x^{-r-1}\left\{\widetilde{S}_n(f_{2(r+1),x},x)+{}_{2m}\widetilde{S}_n(f_{2(r+1),x},x)\right\} \\ \leq Cn^{-r-1}\|\varphi^{2(m+r+1)}g^{(2(r+1))}\|_\infty,$$

where we again used the estimates in Corollary 5.3.

$p=1$: Similar as in the proof of Theorem 5.6 we apply first Fubini's theorem twice, then split the second term into a sum of two integrals for the variable x over the interval $\left[0,\frac{1}{n}\right]$ and $\left[\frac{1}{n},\infty\right)$ and afterwards use (5.14) in the first and last integral to derive

$$\|\varphi^{2m}{}_{2m}\widetilde{S}_n(R(t,\cdot)\|_1$$
$$\le C\left\{\int_0^\infty \left|g^{(2(r+1))}(u)\right| \int_u^\infty \int_0^u \varphi(t)^{2m}(u-t)^{2r+1} n\sum_{k=0}^\infty s_{n,k+m}(x)s_{n,k+m-1}(t)dtdxdu\right.$$
$$+\int_0^{\frac{1}{n}} \left|g^{(2(r+1))}(u)\right| \int_0^u \int_u^\infty (t-u)^{2r+1} n\sum_{k=0}^\infty \varphi(x)^{2m} s_{n,k}(x)s_{n,k+2m-1}(t)dtdxdu$$
$$\left.+\int_{\frac{1}{n}}^\infty \left|g^{(2(r+1))}(u)\right| \int_0^u \int_u^\infty \varphi(t)^{2m}(t-u)^{2r+1} n\sum_{k=0}^\infty s_{n,k+m}(x)s_{n,k+m-1}(t)dtdxdu\right\}.$$

Now we apply $\varphi(t)^{2m} \le \varphi(u)^{2m}$ in the first and $\varphi(x)^{2m} \le \varphi(u)^{2m}$ in the second integral on the right-hand side to get

$$\|\varphi^{2m}{}_{2m}\widetilde{S}_n(R(t,\cdot)\|_1$$
$$\le C\left\{\int_0^\infty \left|\varphi(u)^{2m}g^{(2(r+1))}(u)\right| \int_u^\infty \int_0^u (u-t)^{2r+1} n\sum_{k=1}^\infty s_{n,k}(x)s_{n,k-1}(t)dtdxdu\right.$$
$$+\int_0^{\frac{1}{n}} \left|\varphi(u)^{2m}g^{(2(r+1))}(u)\right| \int_0^u \int_u^\infty (t-u)^{2r+1} n\sum_{k=0}^\infty s_{n,k}(x)s_{n,k+2m-1}(t)dtdxdu$$
$$\left.+\int_{\frac{1}{n}}^\infty \left|g^{(2(r+1))}(u)\right| \int_0^u \int_u^\infty \varphi(t)^{2m}(t-u)^{2r+1} n\sum_{k=1}^\infty s_{n,k}(x)s_{n,k-1}(t)dtdxdu\right\}$$
$$= C\left\{\int_0^\infty \left|\varphi(u)^{2m}g^{(2(r+1))}(u)\right| H_{n,2(r+1)}(u)du\right.$$
$$\left.+\int_0^{\frac{1}{n}} \left|\varphi(u)^{2m}g^{(2(r+1))}(u)\right| \widetilde{H}_{n,2(r+1),2m}(u)du + \int_{\frac{1}{n}}^\infty \left|g^{(2(r+1)}(u)\right| H_{n,2(r+1),m}(u)du\right\}.$$

From this estimate we derive (5.27) for $p=1$ by using Corollarys 5.11, 5.15 and 5.13. □

In [13, Theorem 6] the first author proved for the linear combinations $\overline{S}_{n,r} = \sum_{i=0}^r \alpha_i(n)\overline{S}_{n_i}$ with the $\alpha_i(n)$ given in (5.5) the direct result for simultaneous approximation for derivatives of even order

$$\|\varphi^{2m}(\overline{S}_{n,r}h-h)^{(2m)}\|_p \le C\omega_\varphi^{2(r+1)}\left(h^{(2m)}, \frac{1}{\sqrt{n}}\right)_{\varphi^{2m},p} \tag{5.28}$$

for $h \in L_p[0,\infty)$, $1 \le p < \infty$, $m \in \mathbb{N}$ such that $\varphi^{2m} h^{(2m)} \in L_p[0,\infty)$. With this result we can now prove a theorem for weighted simultaneous approximation by the linear combinations $\widetilde{S}_{n,r}$ also for odd derivatives.

Theorem 5.9. *Let $f \in L_p[0,\infty)$, $1 \le p < \infty$, $m \in \mathbb{N}$ such that $\varphi^{2m} f^{(2m+1)} \in L_p[0,\infty)$. Then*

$$\|\varphi^{2m}(\widetilde{S}_{n,r} f - f)^{(2m+1)}\|_p \le C\omega_{\varphi}^{2(r+1)}\left(f^{(2m+1)}, \frac{1}{\sqrt{n}}\right)_{\varphi^{2m},p}.$$

Proof. With (5.2) and (5.28) we get immediately

$$\begin{aligned}\|\varphi^{2m}(\widetilde{S}_{n,r} f - f)^{(2m+1)}\|_p &= \|\varphi^{2m}(\overline{S}_{n,r} f' - f')^{(2m)}\|_p \\ &\le C\omega_{\varphi}^{2(r+1)}\left(f^{(2m+1)}, \frac{1}{\sqrt{n}}\right)_{\varphi^{2m},p}.\end{aligned}$$

□

5.5 Technical Lemmas

In this section we show some technical lemmas and corresponding corollaries used in the estimates of the remainder terms in the proofs of the global direct results in Sect. 5.4. For $l, m \in \mathbb{N}$ we define

$$\begin{aligned}H_{n,l}(u) &= \frac{l}{n} u^{l-1} e^{-nu} + l\left\{\int_u^\infty \int_0^u - \int_0^u \int_u^\infty\right\} (u-t)^{l-1} n \sum_{k=1}^\infty s_{n,k}(x) s_{n,k-1}(t)\,dt\,dx,\\ H_{n,l,m}(u) &= l\left\{\int_u^\infty \int_0^u - \int_0^u \int_u^\infty\right\} \varphi(t)^{2m} (u-t)^{l-1} n \sum_{k=1}^\infty s_{n,k}(x) s_{n,k-1}(t)\,dt\,dx,\\ \widetilde{H}_{n,l,m}(u) &= l\left\{\int_u^\infty \int_0^u - \int_0^u \int_u^\infty\right\} (u-t)^{l-1} n \sum_{k=0}^\infty s_{n,k}(x) s_{n,k+m-1}(t)\,dt\,dx.\end{aligned}$$

Lemma 5.10.

$$\begin{aligned}H_{n,2}(u) &= \frac{2}{n} u,\ H_{n,3}(u) = 0,\\ H_{n,l}(u) &= (-1)^l \sum_{j=2}^{[\frac{l}{2}]} \binom{l-j-2}{j-2} \frac{l!}{j!} n^{j-l} u^j,\ l \ge 4.\end{aligned}$$

Proof.

$$H_{n,l}(u) = \frac{l}{n}u^{l-1}e^{-nu} + l\left\{\int_0^\infty\int_0^u - \int_0^u\int_0^\infty\right\}(u-t)^{l-1}n\sum_{k=1}^{\infty}s_{n,k}(x)s_{n,k-1}(t)\,dt\,dx$$
$$=: \quad \frac{l}{n}u^{l-1}e^{-nu} + I_1 - I_2. \tag{5.29}$$

With (5.11) and (5.12)

$$I_1 = l\int_0^u\left\{(u-t)^{l-1}\sum_{k=1}^{\infty}s_{n,k-1}(t)\left[n\int_0^\infty s_{n,k}(x)dx\right]\right\}dt$$
$$= l\int_0^u(u-t)^{l-1}dt = u^l. \tag{5.30}$$

By partial integration and then using (5.13) twice

$$I_2$$
$$= u^l n\int_0^u s_{n,1}(x)dx + n\sum_{k=1}^{\infty}\left\{\int_0^u s_{n,k}(x)dx\right\}\left\{\int_0^\infty(u-t)^l s'_{n,k-1}(t)dt\right\}$$
$$= u^l(1-e^{-nu}-nue^{-nu}) - n\sum_{k=1}^{\infty}\left\{\int_0^u s'_{n,k+1}(x)dx\right\}\left\{\int_0^\infty(u-t)^l s_{n,k-1}(t)dt\right\}$$
$$= u^l(1-e^{-nu}-nue^{-nu}) - \sum_{\nu=0}^{l}\binom{l}{\nu}u^{l-\nu}(-1)^\nu\sum_{k=1}^{\infty}s_{n,k+1}(u)n\int_0^\infty t^\nu s_{n,k-1}(t)dt$$
$$= u^l(1-e^{-nu}-nue^{-nu}) - \sum_{\nu=0}^{l}\binom{l}{\nu}u^{l-\nu}(-1)^\nu n^{-\nu}\sum_{k=2}^{\infty}s_{n,k}(u)\frac{(k+\nu-2)!}{(k-2)!}, \tag{5.31}$$

where we used (5.12) for the last equation.
Inner sum of the last term on the right-hand side $S_\nu := \sum_{k=2}^{\infty}s_{n,k}(u)\frac{(k+\nu-2)!}{(k-2)!}$.

Direct calculation gives

$$S_0 = 1-e^{-nu}-nue^{-nu},$$
$$S_1 = -1+e^{-nu}+nu,$$
$$S_2 = n^2u^2.$$

For $\nu \geq 3$ we use that

$$\prod_{l=1}^{\nu-2}(k+2+l) = \sum_{j=0}^{\nu-2}\binom{\nu-2}{j}\frac{\nu!}{(j+2)!}\prod_{l=0}^{j-1}(k-l),$$

where empty products are defined to be 1. This formula can be derived by evaluating the Newton form of the interpolation polynomial of $\prod_{l=1}^{\nu-2}(x+2+l)$ to the knots $x_i = i$, $i = 0,\dots,\nu-2$, at $x = k$. Thus with (5.11)

$$S_\nu = (nu)^2 \sum_{j=0}^{\nu-2} \binom{\nu-2}{j} \frac{\nu!}{(j+2)!}(nu)^j \sum_{k=j}^{\infty} s_{n,k-j}(u)$$
$$= \sum_{j=2}^{\nu} \binom{\nu-2}{j-2} \frac{\nu!}{j!}(nu)^j.$$

Putting the terms for S_ν into (5.31), we get

$$\begin{aligned} I_2 = {} & u^l(1-e^{-nu}-nue^{-nu}) - u^l(1-e^{-nu}-nue^{-nu}) \\ & + lu^{l-1}n^{-1}(-1+e^{-nu}+nu) \\ & - \sum_{\nu=2}^{l} \binom{l}{\nu} u^{l-\nu}(-1)^\nu n^{-\nu} \sum_{j=2}^{\nu} \binom{\nu-2}{j-2} \frac{\nu!}{j!}(nu)^j. \end{aligned} \tag{5.32}$$

Next we calculate $T_l := \sum_{\nu=2}^{l} \binom{l}{\nu} u^{l-\nu}(-1)^\nu n^{-\nu} \sum_{j=2}^{\nu} \binom{\nu-2}{j-2} \frac{\nu!}{j!}(nu)^j$. For $l = 2$ and $l = 3$ we have

$$T_1 = u^2,\; T_3 = 2u^3 - \frac{3}{n}u^2.$$

For $l \geq 4$ we get

$$\begin{aligned} T_l = {} & \sum_{\nu=2}^{l} \binom{l}{\nu} u^{l-\nu}(-1)^\nu n^{-\nu} \sum_{j=2}^{\nu} \binom{\nu-2}{j-2} \frac{\nu!}{j!}(nu)^j \\ = {} & \sum_{\nu=2}^{l} \binom{l}{\nu} u^{l-\nu}(-1)^\nu \sum_{j=l-\nu+2}^{l} \binom{\nu-2}{l-j} \frac{\nu!}{(j-l+\nu)!} n^{j-l}u^j \\ = {} & u^l \sum_{\nu=2}^{l} (-1)^\nu \binom{l}{\nu} + n^{-1}u^{l-1} \sum_{\nu=3}^{l} (-1)^\nu \binom{l}{\nu}(\nu-2)\nu \\ & + \sum_{j=2}^{l-2} n^{j-l}u^j \sum_{\nu=0}^{j-2} (-1)^{\nu-j+l} \binom{l}{j-2-\nu} \binom{\nu-j+l}{\nu} \frac{(\nu-j+l+2)!}{(\nu+2)!} \\ = {} & (l-1)u^l - ln^{-1}u^{l-1} \\ & + \sum_{j=2}^{l-2} n^{j-l}u^j(-1)^{l-j} \frac{l!}{(l-j)(l-j-1)(j-2)!} \sum_{\nu=0}^{j-2} (-1)^\nu \binom{j-2}{\nu} \binom{\nu-j+l}{\nu+2} \\ = {} & (l-1)u^l - ln^{-1}u^{l-1} + (-1)^l \sum_{j=2}^{\left[\frac{l}{2}\right]} \binom{l-j-2}{j-2} \frac{l!}{j!} n^{j-l}u^j, \end{aligned}$$

where the last equation follows from [10, (3.48)]. Together with the definition of $H_{n,l}(u)$, (5.30) and inserting T_l into (5.32) now prove our proposition. □

Corollary 5.11. *For $l \in \mathbb{N}$ we have*

$$H_{n,2l}(u) \le \begin{cases} n^{-2l}, & u \in \left[0, \frac{1}{n}\right], \\ n^{-l}u^l, & u \in \left[\frac{1}{n}, \infty\right). \end{cases}$$

Lemma 5.12.

$$H_{n,l,m}(u) = \sum_{\nu=0}^{m} \binom{m}{\nu} (-1)^\nu u^{m-\nu} \frac{l}{l+\nu} \left\{ H_{n,l+\nu}(u) - \frac{l+\nu}{n} u^{l+\nu-1} e^{-nu} \right\}.$$

Proof. The result follows immediately by rewriting $\varphi(t)^{2m}$ into

$$\varphi(t)^{2m} = \sum_{\nu=0}^{m} \binom{m}{\nu} (-1)^\nu u^{m-\nu} (u-t)^\nu.$$

□

Corollary 5.13. *For $u \in \left[\frac{1}{n}, \infty\right)$ we have*

$$H_{n,2l,m}(u) \le C u^{m+l} n^{-l}.$$

Proof. As

$$\frac{1}{n} \sum_{\nu=0}^{m} u^{2l+\nu-1} e^{-nu} \le C u^{l+m} u^l e^{-nu} \le C u^{l+m} n^{-l}$$

for $u \in \left[\frac{1}{n}, \infty\right)$ we derive the proposition by using the same arguments as in [11, Korollar 6.9]. □

For $m = 1$ we have that $\widetilde{H}_{n,l,1}$ coincide with the functions in [12, Lemma 4.10] for the case $c = 0$ and beside a factor $(-1)^l$ are the moments of the genuine Szász–Mirakjan–Durrmeyer operators (see also [16, below (10)]. For $m \ge 2$ $\widetilde{H}_{n,l,m}$ coincides with the functions considered in [11, (6.3)] for $c = 0$ with $m = 2s+1$. Thus, rewriting [11, Lemma 6.10, Korollar 6.11], we get the following results.

Lemma 5.14. *For $l \in \mathbb{N}$ we have*

$$\widetilde{H}_{n,l,1}(u) = (-1)^l \widetilde{S}_n(f_{l,u}, u),$$

and with $m \ge 2$

$$\begin{aligned} &\widetilde{H}_{n,l,m}(u) \\ &= -l \sum_{k=0}^{m-2} \int_0^u (u-t)^{l-1} s_{n,k}(t)\,dt \\ &\quad + l \sum_{\nu=1}^{l-1} u^{l-\nu} n^{-\nu} \sum_{j=\nu}^{l-1} \binom{l-1}{j} \binom{j}{\nu} \frac{(j+m-1)!}{(j-\nu+m-1)!} \cdot \frac{1}{j-\nu+1} (-1)^{j+1}. \end{aligned}$$

Corollary 5.15. *For* $u \in \left[0, \frac{1}{n}\right]$ *we have*

$$\widetilde{H}_{n,2l,2m}(u) \leq Cn^{-2l}.$$

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Chapter 6
Extensions of Schur's Inequality for the Leading Coefficient of Bounded Polynomials with Two Prescribed Zeros

Heinz-Joachim Rack

Abstract We extend Schur's Chebyshev-type inequality([18], p. 285) for the leading coefficient of polynomials that are uniformly bounded on the interval $[-1,1]$ and vanish at its endpoints. Our extension is threefold: We obtain sharp V.A. Markov-type estimates for all single coefficients as well as sharp Szegö-type estimates for consecutive pairs of coefficients of such polynomials, and both these estimates imply Schur's inequality for the leading coefficient. Thirdly, we consider a larger class of admissible polynomials by replacing uniform with pointwise boundedness on $[-1,1]$.

6.1 The Inequalities of Chebyshev and Schur for the Leading Coefficient of Bounded Polynomials

Issai Schur (1875–1941) was an eminent mathematician who has made fundamental contributions to many areas of mathematics, see [19, 21]. His investigations in algebraic properties of Chebyshev polynomials have influenced the second edition of Rivlin's book [15]. We focus here on Schur's classical paper [18] which has been a source of inspiration for many authors, for example, [5, 8, 13]. In particular, we turn to Theorem IV* of [18]. To make the coefficient estimate stated there comparable to related coefficient estimates by other authors, we normalize the occurring quantities as follows: $M = 1, z_0 = -1, z_1 = 1$ and consider the linear space $\boldsymbol{\Phi_n}$ of real algebraic (univariate) polynomials P_n of degree $\leq n$ given by $P_n(x) = \sum_{k=0}^{n} a_k x^k$ (note that, contrary to [18], we are indexing both the coefficients and the monomials

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in ascending order). Let $\mathbf{B_n}$ denote the unit ball in $\boldsymbol{\Phi_n}$ with respect to the interval $\mathbf{I} = [-1,1] \subset \mathbb{R}$ and with respect to the uniform norm $||P_n||_{\mathbf{I},\infty} = \sup_{x\in\mathbf{I}} |P_n(x)|$, i.e.,

$$\mathbf{B_n} = \{P_n \in \boldsymbol{\Phi_n} : ||P_n||_{\mathbf{I},\infty} \leq 1\}. \tag{6.1}$$

The n-th Chebyshev polynomial of the first kind with respect to $\mathbf{I}$, T_n with $T_n(x) = \sum_{k=0}^{n} t_{n,k}x^k$, can be defined on $\mathbf{I}$ as ([15], p. 2)

$$T_n(x) = \cos(n \arccos(x)). \tag{6.2}$$

T_n hence belongs to $\mathbf{B_n}$ and is an even resp. odd polynomial, depending on the parity of n, so that $t_{n,k} = 0$, if $n-k$ is odd, whereas, if $n-k$ is even, the coefficients $t_{n,k}$ are nonzero integers given by

$$t_{n,k} = t_{n,n-2q} = \frac{(-1)^q}{n-q} n 2^{n-2q-1} \binom{n-q}{q}, 0 \leq q \leq \lfloor n/2 \rfloor. \tag{6.3}$$

Let $\mathbf{B_{n,\pm 1}}$ denote the subset of $\mathbf{B_n}$ consisting of polynomials which vanish at both endpoints of $\mathbf{I}$, i.e.,

$$\mathbf{B_{n,\pm 1}} = \{P_n \in \mathbf{B_n} : P_n(\pm 1) = 0\}. \tag{6.4}$$

It is well known that T_n is an extremizer for various linear functionals defined on $\mathbf{B_n}$. In particular, Chebyshev's celebrated inequality of 1854 [2] for the leading coefficient of uniformly bounded polynomials (on $\mathbf{I}$) holds (see also [9], p. 385 or [14], p. 672 or [15], p. 68):

Theorem 6.1. *For all $P_n \in \mathbf{B_n}$ with $P_n(x) = \sum_{k=0}^{n} a_k x^k$ there holds the coefficient estimate*

$$|a_n| \leq t_{n,n} = 2^{n-1}, \text{ with equality if } P_n = \pm T_n \in \mathbf{B_n}. \tag{6.5}$$

In 1919 Schur added ([18], Theorem IV*) that within the restricted class $\mathbf{B_{n,\pm 1}}$ of polynomials from $\mathbf{B_n}$ with two prescribed zeros, at -1 and at 1, the polynomial S_n defined by

$$S_n(x) = T_n\left(\cos\frac{\pi}{2n}x\right) = \sum_{k=0}^{n} t_{n,k}\left(\cos\frac{\pi}{2n}\right)^k x^k \tag{6.6}$$

is extremal for the leading coefficient:

Theorem 6.2. *For all $P_n \in \mathbf{B_{n,\pm 1}}$ with $P_n(x) = \sum_{k=0}^{n} a_k x^k$ there holds the coefficient estimate*

$$|a_n| \leq 2^{n-1}\left(\cos\frac{\pi}{2n}\right)^n, \text{ with equality if } P_n = \pm S_n \in \mathbf{B_{n,\pm 1}}. \tag{6.7}$$

Note that $||T_n||_{\mathbf{I},\infty} = 1 = |T_n(\pm 1)|$, whereas $||S_n||_{\mathbf{I},\infty} = 1 \neq |S_n(\pm 1)| = 0, n \geq 2$.

Example 6.3. The first few polynomials T_n resp. S_n read as follows:

$$\begin{aligned}
&T_2(x) = -1 + 2x^2 && \text{resp. } S_2(x) = -1 + x^2 \\
&T_3(x) = -3x + 4x^3 && \text{resp. } S_3(x) = \frac{3}{2}\sqrt{3}(-x + x^3) \\
&T_4(x) = 1 - 8x^2 + 8x^4 && \text{resp. } S_4(x) = 1 - (4 + 2\sqrt{2})x^2 + (3 + 2\sqrt{2})x^4 \\
&T_5(x) = 5x - 20x^3 + 16x^5 && \text{resp.} \\
& && S_5(x) = \frac{5}{4}\sqrt{(10 + 2\sqrt{5})}(x - \frac{5 + \sqrt{5}}{2}x^3 + \frac{3 + \sqrt{5}}{2}x^5).
\end{aligned} \tag{6.8}$$

In this paper we are going to extend Schur's theorem 6.2 which covers, analogously to Chebyshev's coefficient inequality (Theorem 6.1), only the leading coefficient. Our goal is, guided by classical coefficient inequalities of V.A. Markov and Szegö valid for $P_n \in \mathbf{B_n}$, to provide sharp estimates for each coefficient $|a_k|$ $(0 \le k \le n)$ and for each pair of consecutive coefficients $|a_{k-1}| + |a_k|$ (if $n - k$ even) of $P_n \in \mathbf{B_{n,\pm 1}}$ and, even more, of $P_n \in \mathbf{D_{n,\pm 1}}$, where the encompassing set $\mathbf{D_{n,\pm 1}}$ of pointwise bounded polynomials (on $\mathbf{I}$) is defined below. In particular, we reveal new extremal properties of the polynomial S_n deployed by Schur and hence of T_n.

6.2 A Schur-Type Analog to V.A. Markov's Estimates for Arbitrary Coefficients, Part 1

Once the sharp upper bound (6.5) for the leading coefficient of $P_n \in \mathbf{B_n}$ was established, it was natural to ask for the sharp upper bounds for all $n + 1$ coefficients of $P_n \in \mathbf{B_n}$. This question was explicitly raised in 1887 (for the case $n = 2$) by the famous chemist Mendeleev, see [11] for details. The definitive answer was provided by Markov [7] (see also [1], p. 248 or [9], p. 423 or [17], p. 167):

Theorem 6.4. *For all $P_n \in \mathbf{B_n}$ with $P_n(x) = \sum_{k=0}^{n} a_k x^k$ there hold the coefficient estimates*

$$|a_k| \le |t_{n,k}| = \frac{n 2^{k-1}(\frac{n+k}{2} - 1)!}{k!(\frac{n-k}{2})!}, \text{ if } n - k \text{ is even } (\text{equality if } P_n = \pm T_n \in \mathbf{B_n}), \tag{6.9}$$

$$|a_k| \le |t_{n-1,k}|, \text{ if } n - k \text{ is odd } (\text{equality if } P_n = \pm T_{n-1} \in \mathbf{B_n}). \tag{6.10}$$

It is well known that the sharp upper bounds in (6.9) and (6.10) are reciprocal to the best approximations to x^k by means of linear combinations of the remaining monomials $1, x, x^2, \ldots, x^{k-1}, x^{k+1}, \ldots, x^n$, see [7] or [17], Satz 1.2. We analogously ask for the sharp upper bounds for all $n + 1$ coefficients of $P_n \in \mathbf{B_{n,\pm 1}}$. The answer is contained in the next theorem, the proof of which we postpone to Sect. 6.6 below:

Theorem 6.5. *For all* $P_n \in \mathbf{B_{n,\pm 1}}$ *with* $P_n(x) = \sum_{k=0}^{n} a_k x^k$ *there hold the coefficient estimates*

$$|a_k| \le |t_{n,k}| \left(\cos\frac{\pi}{2n}\right)^k, \text{ if } n-k \text{ is even } (\text{equality if } P_n = \pm S_n \in \mathbf{B_{n,\pm 1}}). \quad (6.11)$$

$$|a_k| \le |t_{n-1,k}| \left(\cos\frac{\pi}{2n-2}\right)^k, \text{ if } n-k \text{ is odd } (\text{equality if } P_n = \pm S_{n-1} \in \mathbf{B_{n,\pm 1}}). \quad (6.12)$$

Thus Theorem 6.5 is a complete analog within $\mathbf{B_{n,\pm 1}}$ to V.A. Markov's theorem 6.4 which is valid for $P_n \in \mathbf{B_n}$. The special case $k = n$ in (6.11) takes us back to Schur's inequality (6.7).

6.3 A Schur-Type Analog to Szegö's Estimates for Pairs of Coefficients, Part 1

Although Theorem 6.4 gives the sharp upper bounds for each coefficient of $P_n \in \mathbf{B_n}$, the first part of Theorem 6.4 still leaves room for refinement. The following striking extension of (6.9) to pairs of consecutive coefficients was communicated orally by Szegö to Erdös who published it (without proof) in 1947 [6]. A concise proof is to be found in [14], Theorem 16.3.3, see also [11]:

Theorem 6.6. *For all* $P_n \in \mathbf{B_n}$ *with* $P_n(x) = \sum_{k=0}^{n} a_k x^k$ *there hold the coefficient estimates*

$$|a_{k-1}| + |a_k| \le |t_{n,k}|, \text{ if } n-k \text{ is even } (\text{equality if } P_n = \pm T_n \in \mathbf{B_n};\ \text{set } a_{-1} = 0). \quad (6.13)$$

Obviously, (6.13) implies (6.9). We analogously ask for sharp upper bounds for the corresponding pairs of coefficients of $P_n \in \mathbf{B_{n,\pm 1}}$. The answer is contained in the next theorem, the proof of which we postpone to Sect. 6.6:

Theorem 6.7. *For all* $P_n \in \mathbf{B_{n,\pm 1}}$ *with* $P_n(x) = \sum_{k=0}^{n} a_k x^k$ *there hold the coefficient estimates*

$$|a_{k-1}| + |a_k| \le |t_{n,k}| \left(\cos\frac{\pi}{2n}\right)^k, \text{ if } n-k \text{ is even} \quad (6.14)$$

$(\text{equality if } P_n = \pm S_n \in \mathbf{B_{n,\pm 1}};\ \text{set } a_{-1} = 0).$

This theorem is thus a complete analog within $\mathbf{B_{n,\pm 1}}$ to Szegö's Theorem 6.6 which is valid for $P_n \in \mathbf{B_n}$. The special case $k = n$ in (6.14) yields $|a_{n-1}| + |a_n| \le 2^{n-1}\left(\cos\frac{\pi}{2n}\right)^n$, and this refined inequality implies Schur's inequality (6.7).

6.4 A Schur-Type Analog to V.A. Markov's Estimates for Arbitrary Coefficients, Part 2

The goal of this section is to improve on Theorem 6.5 by enlarging the set of admissible polynomials. Consider the first part of V.A. Markov's two-staged inequality in Theorem 6.4. An alternative refinement of (6.9) is due to Shohat [20]. In 1929 he observed that (6.9) will hold true even if P_n satisfies the relaxed condition $|P_n(x^*_{n,i})| \le 1$ (pointwise boundedness), where the $x^*_{n,i} = \cos\dfrac{(n-i)\pi}{n}, 0 \le i \le n$, are the extremal points of T_n on $\mathbf{I}$ with $T_n(x^*_{n,i}) = (-1)^{n-i}$. We note in passing that Duffin and Schaeffer [4] succeeded to refine V.A. Markov's celebrated inequality for the k-th derivatives of $P_n \in \mathbf{B_n}$ under this relaxed condition, see also [15], p. 136.

As simple examples show, the second part of V.A. Markov's inequality, (6.10), does not hold true (i.e., T_{n-1} is not extremal) if the condition $P_n \in \mathbf{B_n}$ is relaxed to $|P_n(x^*_{n,i})| \le 1$. To the best of our knowledge, it was Rogosinski [16] in 1955 who first constructed the extremal polynomials which satisfy $|P_n(x^*_{n,i})| \le 1$ and maximize $|a_k|$, if $n-k$ is odd. Building on the ideas of Shohat and Rogosinski we are now going to relax the condition $P_n \in \mathbf{B_{n,\pm1}}$ (uniform boundedness) to pointwise boundedness on $\mathbf{I}$ in order to get Schur-type analogs of the two-staged coefficient inequality of V.A. Markov. It follows from the definition of S_n that the points $x_{n,i} = \dfrac{x^*_{n,i}}{\cos\frac{\pi}{2n}}, 1 \le i \le n-1$, are the extremal points of S_n on $\mathbf{I}$ with $S_n(x_{n,i}) = (-1)^{n-i}$. Consider now, in place of $\mathbf{B_{n,\pm1}}$, the set $\mathbf{D_{n,\pm1}}$ given by

$$\mathbf{D_{n,\pm1}} = \{P_n \in \boldsymbol{\Phi_n} : |P_n(x_{n,i})| \le 1 \text{ for } 1 \le i \le n-1 \text{ and } P_n(\pm1) = 0\}. \tag{6.15}$$

Note that $S_n \in \mathbf{D_{n,\pm1}}$ and $\mathbf{B_{n,\pm1}} \subset \mathbf{D_{n,\pm1}}, n \ge 3$.

Example 6.8. The polynomial P_4 given by

$$P_4(x) = 1 - \frac{1}{2}(2+\sqrt{2})^{\frac{3}{2}}x - (2+\frac{1}{\sqrt{2}})x^2 + \frac{1}{2}(2+\sqrt{2})^{\frac{3}{2}}x^3 + (1+\frac{1}{\sqrt{2}})x^4 \tag{6.16}$$

belongs to $\mathbf{D_{4,\pm1}}$ since we have $P_4(-1) = S_4(-1) = 0$, $P_4(x_{4,1}) = -S_4(x_{4,1}) = 1$, $P_4(x_{4,2}) = S_4(x_{4,2}) = 1$, $P_4(x_{4,3}) = S_4(x_{4,3}) = -1$, $P_4(1) = S_4(1) = 0$, but P_4 does not belong to $\mathbf{B_{4,\pm1}}$ because $P_4(-\frac{1}{2}) > 1$.

We can now state a second, more general analog to (6.9), compare with (6.11):

Theorem 6.9. *For all $P_n \in \mathbf{D_{n,\pm1}}$ with $P_n(x) = \sum\limits_{k=0}^{n} a_k x^k$ there hold the coefficient estimates*

$$|a_k| \le |t_{n,k}|(\cos\frac{\pi}{2n})^k, \textit{ if } n-k \textit{ is even } (\textit{equality if } P_n = \pm S_n \in \mathbf{D_{n,\pm1}}). \tag{6.17}$$

The proof of this theorem is postponed to Sect. 6.6.

We next proceed to find out how a second, more general analog to (6.10) may look like. To this end, we define, following [16], a polynomial Π_{n-1} by interpolatory constraints which will turn out as extremal within $\mathbf{D_{n,\pm 1}}$ for $|a_k|$, if $n-k$ is odd: Set

$$\Pi_{n-1}(\pm 1) = 0 \tag{6.18}$$

$$\Pi_{n-1}(x_{n,i}) = (-1)^i, \text{ if } 1 \leq i \leq \frac{n-1}{2}, \text{ when } n \text{ is odd} \tag{6.19}$$

$$\Pi_{n-1}(x_{n,i}) = (-1)^{i+1}, \text{ if } \frac{n+1}{2} \leq i \leq n-1, \text{ when } n \text{ is odd} \tag{6.20}$$

$$\Pi_{n-1}(x_{n,i}) = (-1)^{i+1}, \text{ if } 1 \leq i \leq \frac{n}{2} - 1, \text{ when } n \text{ is even} \tag{6.21}$$

$$\Pi_{n-1}(x_{n,i}) = 0, \text{ if } i = \frac{n}{2}, \text{ when } n \text{ is even} \tag{6.22}$$

$$\Pi_{n-1}(x_{n,i}) = (-1)^i, \text{ if } \frac{n}{2} + 1 \leq i \leq n-1, \text{ when } n \text{ is even.} \tag{6.23}$$

Example 6.10. For $n=4$ the polynomial $\Pi_3 \in \mathbf{D_{4,\pm 1}}$ is given by

$$\Pi_3(x) = \sqrt{1 + \frac{1}{\sqrt{2}}}(1+\sqrt{2})(-x+x^3), \tag{6.24}$$

and for $n=3$ the polynomial $\Pi_2 \in \mathbf{D_{3,\pm 1}}$ is given by

$$\Pi_2(x) = \frac{3}{2}(-1+x^2). \tag{6.25}$$

We are now in a position to state the second analog to (6.10), compare with (6.12):

Theorem 6.11. *Let $\Pi_{n-1} \in \mathbf{D_{n,\pm 1}}$ with $\Pi_{n-1}(x) = \sum_{k=0}^{n-1} A_{n-1,k}x^k$ and $n \geq 3$ be defined as in (6.18)–(6.23). For all $P_n \in \mathbf{D_{n,\pm 1}}$ with $P_n(x) = \sum_{k=0}^{n} a_k x^k$ there hold the coefficient estimates*

$$|a_k| \leq |A_{n-1,k}|, \text{ if } n-k \text{ is odd (equality if } P_n = \pm\Pi_{n-1}). \tag{6.26}$$

The proof of this theorem is postponed to Sect. 6.6.

Example 6.12. For $n=4$ and $P_4(x) = \sum_{k=0}^{4} a_k x^k$ we deduce from the above theorems and examples the following sharp coefficient estimates:

If $P_4 \in \mathbf{B_4}$, then	If $P_4 \in \mathbf{B_{4,\pm1}}$, then	If $P_4 \in \mathbf{D_{4,\pm1}}$, then
$\lvert a_0\rvert \le 1.000$	$\lvert a_0\rvert \le 1.000$	$\lvert a_0\rvert \le 1.000$
$\lvert a_1\rvert \le 3.000$	$\lvert a_1\rvert \le \frac{3}{2}\sqrt{3} = 2.598\ldots$	$\lvert a_1\rvert \le \sqrt{1+\frac{1}{\sqrt{2}}}(1+\sqrt{2}) = 3.154\ldots$
$\lvert a_2\rvert \le 8.000$	$\lvert a_2\rvert \le 4+2\sqrt{2} = 6.828\ldots$	$\lvert a_2\rvert \le 4+2\sqrt{2} = 6.828\ldots$
$\lvert a_3\rvert \le 4.000$	$\lvert a_3\rvert \le \frac{3}{2}\sqrt{3} = 2.598\ldots$	$\lvert a_3\rvert \le \sqrt{1+\frac{1}{\sqrt{2}}}(1+\sqrt{2}) = 3.154\ldots$
$\lvert a_4\rvert \le 8.000$	$\lvert a_4\rvert \le 3+2\sqrt{2} = 5.828\ldots$	$\lvert a_4\rvert \le 3+2\sqrt{2} = 5.828\ldots$

6.5 A Schur-Type Analog to Szegö's Estimates for Pairs of Coefficients, Part 2

The goal of this section is to improve on Theorem 6.7 by enlarging the set of admissible polynomials. As in the previous section we replace $\mathbf{B_{n,\pm1}}$ by its superset $\mathbf{D_{n,\pm1}}$:

Theorem 6.13. *For all $P_n \in \mathbf{D_{n,\pm1}}$ with $P_n(x) = \sum_{k=0}^{n} a_k x^k$ there hold the coefficient estimates*

$$|a_{k-1}| + |a_k| \le |t_{n,k}| \left(\cos\frac{\pi}{2n}\right)^k, \text{ if } n-k \text{ is even} \tag{6.27}$$

(*equality if $P_n = \pm S_n \in \mathbf{D_{n,\pm1}}$; set $a_{-1} = 0$*).

This theorem is thus a complete analog within $\mathbf{D_{n,\pm1}}$ to Szegö's Theorem 6.6 and to our first extension of it, Theorem 6.7. The proof of Theorem 6.13 builds on a result from [3] and is postponed to the next section. The special case $k = n$ in (6.27) yields $|a_{n-1}| + |a_n| \le 2^{n-1}\left(\cos\frac{\pi}{2n}\right)^n$ for $P_n \in \mathbf{D_{n,\pm1}}$ and hence for $P_n \in \mathbf{B_{n,\pm1}}$, and this inequality for the pair of leading coefficients of P_n in particular implies Schur's inequality (6.7).

Example 6.14. For $n = 4$ and $P_4(x) = \sum_{k=0}^{4} a_k x^k$ we deduce from the above theorems and examples the following sharp estimates for consecutive pairs of coefficients:

If $P_4 \in \mathbf{B_4}$, then	If $P_4 \in \mathbf{B_{4,\pm1}}$ or $P_4 \in \mathbf{D_{4,\pm1}}$, then
$\lvert a_0\rvert \le 1.000$	$\lvert a_0\rvert \le 1.000$
$\lvert a_1\rvert + \lvert a_2\rvert \le 8.000$	$\lvert a_1\rvert + \lvert a_2\rvert \le 4+2\sqrt{2} = 6.828\ldots$
$\lvert a_3\rvert + \lvert a_4\rvert \le 8.000$	$\lvert a_3\rvert + \lvert a_4\rvert \le 3+2\sqrt{2} = 5.828\ldots$

6.6 Proofs

We now provide proofs to our Theorems 6.5, 6.7, 6.9, 6.11, and 6.13. The Theorems 6.1, 6.2, 6.4, and 6.6 reflect historical results.

Proof. *(of Theorem 6.5)* Since $S_n \in \mathbf{B}_{\mathbf{n},\pm\mathbf{1}} \subset \mathbf{D}_{\mathbf{n},\pm\mathbf{1}}$, the estimates (6.11) follow from (6.17). To prove (6.12), consider the polynomial P_{n-1} defined by $P_{n-1}(x) = (P_n(x) + (-1)^{n-1}P_n(-x))/2 = \sum_{k=0}^{n-1} d_k x^k$, where $P_n \in \mathbf{B}_{\mathbf{n},\pm\mathbf{1}}$. We deduce that $|P_{n-1}(x)| \le |P_n(x)|/2 + |P_n(-x)|/2 \le \frac{1}{2} + \frac{1}{2} = 1$ for $x \in \mathbf{I}$, i.e., $P_{n-1} \in \mathbf{B_n}$, and obviously $P_{n-1}(\pm 1) = 0$ holds, so that in fact $P_{n-1} \in \mathbf{B}_{\mathbf{n},\pm\mathbf{1}}$. The coefficients d_k of P_{n-1} with $(n-1)-k$ even coincide with the coefficients a_k of P_n with $n-k$ odd. Applying the estimates (6.11) to P_{n-1} with $n-1$ in place of n eventually gives (6.12). □

Proof. *(of Theorem 6.7)* Since $S_n \in \mathbf{B}_{\mathbf{n},\pm\mathbf{1}} \subset \mathbf{D}_{\mathbf{n},\pm\mathbf{1}}$, the estimates (6.14) follow from (6.27). □

Proof. *(of Theorem 6.9)* We will make use of the following general assumption, denoted by $(\mathscr{A})$: *Let there be given non-negative real numbers, M_i, $0 \le i \le n$, satisfying $\sum_{i=0}^{n} M_i > 0$ and $M_i = M_{n-i}$. Let there be given a zero-symmetric partition of* $\mathbf{I}$, $-1 = z_{n,0} < z_{n,1} < \cdots < z_{n,n-1} < z_{n,n} = 1$, *satisfying $z_{n,i} + z_{n,n-i} = 0$. Let Q_n with $Q_n(x) = \sum_{k=0}^{n} b_k x^k$ be a polynomial satisfying $|Q_n(z_{n,i})| \le M_i$, $0 \le i \le n$.*

Furthermore, let R_n with $R_n(x) = \sum_{k=0}^{n} B_k x^k$ denote the polynomial which satisfies the oscillating interpolatory condition $R_n(z_{n,i}) = (-1)^{n-i} M_i$ for $0 \le i \le n$. A result of Rogosinski [16], Theorem III, states that under these assumptions the V.A. Markov-type coefficient estimate $|b_k| \le |B_k|$ holds true (equality if $Q_n = \pm R_n$), provided $n-k$ is even. To deduce (6.17) from this result we set equal $M_0 = M_n = 0$ and $M_i = 1$, $1 \le i \le n-1$; $z_{n,0} = -z_{n,n} = -1$ and $z_{n,i} = x_{n,i}$, $1 \le i \le n-1$; $Q_n = P_n$ and $R_n = S_n$. □

Proof. *(of Theorem 6.11)* Under the assumption $(\mathscr{A})$, let W_{n-1} with $W_{n-1}(x) = \sum_{k=0}^{n-1} C_k x^k$ denote the polynomial which satisfies the interpolatory conditions (depending on the parity of n)

(i) $W_{n-1}(z_{n,i}) = (-1)^i M_i$ for $0 \le i \le \frac{n-1}{2}$, when n is odd
(ii) $W_{n-1}(z_{n,i}) = (-1)^{i+1} M_i$ for $\frac{n+1}{2} \le i \le n$, when n is odd
(iii) $W_{n-1}(z_{n,i}) = (-1)^{i+1} M_i$ for $0 \le i \le \frac{n}{2} - 1$, when n is even
(iv) $W_{n-1}(z_{n,i}) = 0$ for $i = \frac{n}{2}$, when n is even
(v) $W_{n-1}(z_{n,i}) = (-1)^i M_i$ for $\frac{n}{2} + 1 \le i \le n$, when n is even.

A result of Rogosinski [16], Theorem IV, states that then the V.A. Markov-type coefficient estimate $|b_k| \le |C_k|$ holds true (equality if $Q_n = \pm W_{n-1}$), provided $n-k$ is odd. To deduce (6.26) from this result we set equal $M_0 = M_n = 0$ and $M_i = 1$, $1 \le i \le n-1$; $z_{n,0} = -z_{n,n} = -1$ and $z_{n,i} = x_{n,i}$, $1 \le i \le n-1$; $Q_n = P_n$ and $W_{n-1} = \Pi_{n-1}$. □

Proof. *(of Theorem 6.13)* Under the assumption $(\mathscr{A})$, let R_n with $R_n(x) = \sum_{k=0}^{n} B_k x^k$ denote the polynomial which satisfies the oscillating interpolatory condition

$R_n(z_{n,i}) = (-1)^{n-i}M_i$ for $0 \le i \le n$. A result of Dryanov et al. [3], Theorem 3 (restated in [10], Theorem F), for which we have provided an alternative proof and a bivariate extension ([12], Theorems 2.4.1 and 3.2), states that then the Szegö-type coefficient estimate $|b_{k-1}| + |b_k| \le |B_k|$ holds true (equality if $Q_n = \pm R_n$), provided $n-k$ is even. To deduce (6.27) from this result we set equal $M_0 = M_n = 0$ and $M_i = 1, 1 \le i \le n-1$; $z_{n,0} = -z_{n,n} = -1$ and $z_{n,i} = x_{n,i}$, $1 \le i \le n-1$; $Q_n = P_n$ and $R_n = S_n$. □

Remark 6.15. Schur [18], Theorem III*, also obtained an inequality for the leading coefficient of a polynomial from $\mathbf{B_n}$ which additionally has one prescribed zero on **I**, either at -1 or at 1 (asymmetric case). Our deployed method of proof relies on the symmetries stated in the assumption $(\mathscr{A})$ and hence cannot be applied to obtain extensions of this asymmetric case. It requires a different approach that we intend to expose in a separate manuscript.

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Chapter 7
An Example of Optimal Nodes for Interpolation Revisited

Heinz-Joachim Rack

Abstract A famous unsolved problem in the theory of polynomial interpolation is that of explicitly determining a set of nodes which is optimal in the sense that it leads to minimal Lebesgue constants. In [11] a solution to this problem was presented for the first non-trivial case of cubic interpolation. We add here that the quantities that characterize optimal cubic interpolation (in particular: the minimal Lebesgue constant) can be compactly expressed as real roots of certain cubic polynomials with integral coefficients. This facilitates the presentation and impartation of the subject matter and may guide extensions to optimal higher-degree interpolation.

7.1 Introduction

The Bernstein conjecture of 1931 and Kilgore's theorem of 1977 [6] characterize, by means of the equioscillation property of the Lebesgue function, the optimal nodes which minimize the Lebesgue constant for n-th degree Lagrange polynomial interpolation. The Bernstein conjecture has been settled to the affirmative in 1978 [2, 7].

However, as put in [3]: *In spite of this nice characterization, the optimal nodes as well as the optimal Lebesgue constants are not known explicitly*.
Although the knowledge of these quantities may be of little practical importance, since they can be computed numerically for the first few values of n (see [1, 3, 9, 15]), and near-optimal nodes are explicitly known (see [3]), *...the problem of analytical description of the optimal matrix of nodes is considered by pure mathematicians as a great challenge* [3]. In [8] (p. xlvii) it is put more dramatically: *The nature of the optimal set X^* remains a mystery*.

But at least the first non-trivial case of cubic interpolation has been demystified so that for $n = 3$ the desired analytical solution to the problem of explicitly

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determining the optimal nodes and the minimal Lebesgue constant is known [11]. To facilitate the presentation and impartation of this instructive example we add here alternative expositions of the minimal cubic Lebesgue constant and of the (positive) extremum point at which the local maximum of the optimal cubic Lebesgue function occurs: we identify them as roots of certain intrinsic cubic polynomials with integral coefficients. The third determining quantity, the (positive) optimal node for cubic interpolation, has already been described in this concise way [11].
Such a description is in the spirit of the open question raised in [4] (p. 21): *Is there a set of relatively simple functions f_n such that the roots of f_n are the optimal nodes for Lagrange interpolation*?

We will provide as simple functions f_3 three cubic polynomials with integral coefficients whose roots yield the solution to the optimal cubic interpolation problem.

7.2 Three Cubic Polynomials with Integral Coefficients Whose Roots Yield the Solution to the Optimal Cubic Interpolation Problem

The situation is as follows ($n = 3$): It suffices to consider (algebraic) Lagrange interpolation on the zero-symmetric partition

$$-1 = x_0 < x_1 = -x_2 < x_2 < x_3 = 1 \tag{7.1}$$

of the canonical interval $[-1,1]$, so that only the placement of the positive node x_2 remains critical. The sampled values $y_i = f(x_i), 0 \le i \le 3$, of some (continuous) function f which is to be interpolated on (7.1) by a cubic polynomial, do not enter into the discussion. We know from [11] that the following holds:
The square of the optimal node $x_2 = x_2^*$ is given as the unique real root of a cubic polynomial with integral coefficients:

$$P_3(z) = -1 + 2z + 17z^2 + 25z^3. \tag{7.2}$$

Proposition 7.1. *We add here that the analytic expression for x_2^* as given in* $($*[11]*$, (22))$ *can alternatively be restated as*

$$x_2^* = \frac{1}{5\sqrt{3}} \sqrt{-17 + \left(\frac{14699 + 1725\sqrt{69}}{2}\right)^{\frac{1}{3}} + \left(\frac{14699 - 1725\sqrt{69}}{2}\right)^{\frac{1}{3}}} \tag{7.3}$$

$$= 0.4177913013\ldots$$

Proof. The verification that the expression (7.3) equals the expression (22) given in [11] is straightforward and is left to the reader. □

Proposition 7.2. L_3^*, *the sought-for minimal value of the cubic Lebesgue constant*

$$L_3(x_2) = \max_{|x|\le 1} F_3(x,x_2) \text{ with } F_3(x,x_2) = \sum_{i=0}^{3} |l_{i,3}(x)| \text{ and } l_{i,3}(x) = \prod_{j=0, j\neq i}^{3} \frac{x-x_j}{x_i-x_j}, \tag{7.4}$$

can likewise be identified with the unique real root of a cubic polynomial with integral coefficients:

$$Q_3(z) = -11 + 53z - 93z^2 + 43z^3. \tag{7.5}$$

The analytic expression for L_3^* *as deduced in* ([11](23)) *can alternatively be restated as*

$$L_3^* = \frac{1}{129}\left(93 + \left(125172 + 11868\sqrt{69}\right)^{\frac{1}{3}} + \left(125172 - 11868\sqrt{69}\right)^{\frac{1}{3}}\right)$$
$$= 1.4229195732\ldots \tag{7.6}$$

Proof. The verification that L_3^* in its identical forms ([11], (23)) or (7.6) coincides with the real root of Q_3 is by straightforward insertion and is left to the reader. □

Proposition 7.3. *The square of the maximum point* $x = \bar{x} \in [x_2^*, 1]$, *at which the first derivative of the optimal cubic Lebesgue function* $F_3(x, x_2^*)$ *vanishes, can likewise be identified with the unique real root of a cubic polynomial with integral coefficients:*

$$R_3(z) = -1 + 7z - 23z^2 + 25z^3. \tag{7.7}$$

The analytic expression for $\bar{x}$ *as given in* ([11], (14)), *after insertion of* $x_2 = x_2^*$, *reads as*

$$\bar{x} = \frac{1}{5\sqrt{3}}\sqrt{23 + 2\left(\frac{623+75\sqrt{69}}{2}\right)^{\frac{1}{3}} + 2\left(\frac{623-75\sqrt{69}}{2}\right)^{\frac{1}{3}}} \tag{7.8}$$
$$= 0.7331726239\ldots$$

Proof. The verification that the square of $\bar{x}$, where $\bar{x}$ is given by (7.8), coincides with the real root of R_3 is again by straightforward insertion. □

By symmetry, the first derivative of $F_3(x, x_2^*)$ also vanishes at $-\bar{x} \in [-1, -x_2^*]$ and at $x = 0 \in [-x_2^*, x_2^*]$ which gives the three equal local maxima $F_3(-\bar{x}, x_2^*) = F_3(0, x_2^*) = F_3(\bar{x}, x_2^*)$ of the optimal cubic Lebesgue function (equioscillation property). These maxima are identical with the value $\min\limits_{0<x_2<1} L_3(x_2) = L_3(x_2^*) = L_3^*$.

The three polynomials P_3, Q_3, and R_3, respectively their unique real roots, thus completely describe the solution to the problem of optimal cubic interpolation on $[-1, 1]$.

7.3 Concluding Remarks

We point out that already in 1968 the polynomial P_3 (in the variable $z = t^2$) has appeared as part of a posed problem in the not easily accessible source [14] (p. 89, Problem 6.43).

However, no analytic expressions for x_2^* or L_3^* or $\bar{x}$ are given there. At the time of writing [11] the source [14], which we had learned from [7], was not available to us.

We believe that the polynomials Q_3 and R_3 appear here for the first time in connection with optimal cubic polynomial interpolation and we hope that they may guide, together with P_3, the finding of extensions to optimal n-th degree polynomial interpolation, $n \geq 4$.

Additional recommended reading is [5, 9, 10] (especially Example 2.5.3), [12, 13].

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Chapter 8
Theory of Differential Approximations of Radiative Transfer Equation

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Abstract The radiative transfer equation (RTE) arises in a variety of applications. The equation is challenging to solve numerically for a couple of reasons: high dimensionality, integro-differential form, highly forward-peaked scattering in application. In the literature, various approximations of RTE have been proposed in the literature. In an earlier publication, we explored a family of differential approximations to RTE, to be called RT/DA equations. In this paper, we study the RT/DA equations and investigate numerically the closeness of solutions of the RT/DA equations to that of the RTE.

8.1 Introduction

The radiative transfer equation (RTE) arises in a variety of applications, such as neutron transport, heat transfer, stellar atmospheres, optical molecular imaging, infrared and visible light in space and the atmosphere, and so on. We refer the reader to [1, 14, 15, 19, 20]. Recently, there is much interest in analysis and numerical simulation of the RTE and its related inverse problems, motivated by applications in biomedical optics [4, 7, 8].

We proceed to give a brief description of RTE as follows. Let X be a domain in $\mathbb{R}^3$ with a Lipschitz boundary ∂X. The unit outward normal $n(x)$ exists a.e. on ∂X. Denote by Ω the unit sphere in $\mathbb{R}^3$. For each fixed direction $\omega \in \Omega$, introduce a new Cartesian coordinate system (z_1, z_2, s) by the relations

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$$x = z + s\omega, \quad z = z_1\omega_1 + z_2\omega_2,$$

where $(\omega_1, \omega_2, \omega)$ is an orthonormal basis of $\mathbb{R}^3$, $z_1, z_2, s \in \mathbb{R}$. With respect to this new coordinate system, we denote by X_ω the projection of X on the plane $s = 0$ in $\mathbb{R}^3$ and by $X_{\omega,z}$ $(z \in X_\omega)$ the intersection of the straight line $\{z + s\omega \mid s \in \mathbb{R}\}$ with X. We assume that the domain X is such that for any (ω, z) with $z \in X_\omega$, $X_{\omega,z}$ is the union of a finite number of line segments:

$$X_{\omega,z} = \cup_{i=1}^{N(\omega,z)} \{z + s\omega \mid s \in (s_{i,-}, s_{i,+})\}.$$

Here $s_{i,\pm} = s_{i,\pm}(\omega, z)$ depend on ω and z, and $x_{i,\pm} := z + s_{i,\pm}\omega$ are the intersection points of the line $\{z + s\omega \mid s \in \mathbb{R}\}$ with ∂X. We further assume $\sup_{\omega,z} N(\omega, z) < \infty$, known as a generalized convexity condition. As an example, for a convex domain X, $\sup_{\omega,z} N(\omega, z) = 1$. We then introduce the following subsets of ∂X:

$$\partial X_{\omega,-} = \{z + s_{i,-}\omega \mid 1 \le i \le N(\omega, z),\, z \in X_\omega\},$$

$$\partial X_{\omega,+} = \{z + s_{i,+}\omega \mid 1 \le i \le N(\omega, z),\, z \in X_\omega\}.$$

It can be shown that for a.e. $z \in X_\omega$, $n(z + s_{i,-}\omega)\cdot\omega \le 0$; if $x \in \partial X$ and $n(x)\cdot\omega < 0$, then $x \in \partial X_{\omega,-}$. Likewise, for a.e. $z \in X_\omega$, $n(z + s_{i,+}\omega)\cdot\omega \ge 0$; if $x \in \partial X$ and $n(x)\cdot\omega > 0$, then $x \in \partial X_{\omega,+}$. Then the incoming boundary Γ_- and outgoing boundary Γ_+ are

$$\Gamma_- = \{(x, \omega) \mid x \in \partial X_{\omega,-},\, \omega \in \Omega\}, \qquad \Gamma_+ = \{(x, \omega) \mid x \in \partial X_{\omega,+},\, \omega \in \Omega\}.$$

Denote by $d\sigma(\omega)$ the infinitesimal area element on the unit sphere Ω. For the spherical coordinate system

$$\omega = (\sin\theta\cos\psi, \sin\theta\sin\psi, \cos\theta)^T, \quad 0 \le \theta \le \pi,\, 0 \le \psi \le 2\pi, \tag{8.1}$$

$d\sigma(\omega) = \sin\theta\, d\theta\, d\psi$. We will need an integral operator S defined by

$$(Su)(x, \omega) = \int_\Omega k(\omega\cdot\hat{\omega})\, u(x, \hat{\omega})\, d\sigma(\hat{\omega}) \tag{8.2}$$

with k a nonnegative normalized phase function:

$$\int_\Omega k(\omega\cdot\hat{\omega})\, d\sigma(\hat{\omega}) = 1 \quad \forall\, \omega \in \Omega. \tag{8.3}$$

One well-known example is the Henyey–Greenstein phase function (cf. [10])

$$k(t) = \frac{1 - g^2}{4\pi(1 + g^2 - 2gt)^{3/2}}, \quad t \in [-1, 1], \tag{8.4}$$

where the parameter $g \in (-1, 1)$ is the anisotropy factor of the scattering medium. Note that $g = 0$ for isotropic scattering, $g > 0$ for forward scattering, and $g < 0$ for backward scattering.

With the above notation, a boundary value problem of the RTE reads (cf. [1, 13])

$$\omega\cdot\nabla u(x,\omega)+\sigma_t(x)\,u(x,\omega)=\sigma_s(x)\,(Su)(x,\omega)+f(x,\omega), \quad (x,\omega)\in X\times\Omega, \tag{8.5}$$
$$u(x,\omega)=u_{\rm in}(x,\omega), \quad (x,\omega)\in\Gamma_-. \tag{8.6}$$

Here $\sigma_t=\sigma_a+\sigma_s$, σ_a is the macroscopic absorption cross section, σ_s is the macroscopic scattering cross section, and f and $u_{\rm in}$ are source functions in X and on Γ_-, respectively. We assume that these given functions satisfy

$$\sigma_t,\sigma_s\in L^\infty(X), \quad \sigma_s\ge 0 \text{ and } \sigma_t-\sigma_s\ge c_0 \text{ in } X \text{ for some constant } c_0>0, \tag{8.7}$$
$$f\in L^2(X\times\Omega), \quad u_{\rm in}\in L^2(\Gamma_-). \tag{8.8}$$

These assumptions are naturally valid in applications; the last part of (8.7) means that the absorption effect is not negligible. For a vacuum setting around X, the incoming flux boundary condition $u_{\rm in}(x,\omega)=0$ on Γ_-.

It can be shown [1] that the problem (8.5)–(8.6) has a unique solution $u\in H_2^1(X\times\Omega)$, where

$$H_2^1(X\times\Omega):=\{v\in L^2(X\times\Omega)\mid \omega\cdot\nabla v\in L^2(X\times\Omega)\}$$

with $\omega\cdot\nabla v$ denoting the generalized directional derivative of v in the direction ω.

It is challenging to solve the RTE problem numerically for a couple of reasons. First, it is a high-dimensional problem. The spatial domain is three dimensional and the region for the angular variable is two dimensional. Second, when the RTE is discretized by the popular discrete-ordinate method, the integral term $Su(x,\omega)$ on the right side of the equation is approximated by a summation that involves all the numerical integration points on the unit sphere. Consequently, for the resulting discrete system, the desired locality property is not valid, and many of the solution techniques for solving sparse systems from discretization of partial differential equations cannot be applied efficiently to solve the discrete systems of RTE. Moreover, in applications involving highly forward-peaked media, which are typical in biomedical imaging, the phase function tends to be numerically singular. Take the Henyey–Greenstein phase function (8.4) as an example: $k(1)=(1+g)/[4\pi(1-g)^2]$ blows up as $g\to 1-$. In such applications, it is even more difficult to solve RTE since accurate numerical solutions require a high resolution of the direction variable, leading to prohibitively large amount of computations. For these reasons, various approximations of RTE have been proposed in the literature, e.g., the delta-Eddington approximation [11], the Fokker–Planck approximation [16, 17], the Boltzmann–Fokker–Planck approximation [5, 18], the generalized Fokker–Planck approximation [12], the Fokker–Planck–Eddington approximation, and the generalized Fokker–Planck–Eddington approximation [6]. In [9], we provided a preliminary study of a family of differential approximations of the RTE. For convenience, we will call these approximation equations as RT/DA (radiative transfer/differential approximation) equations. An RT/DA equation with j terms for the approximation of the integral operator will be called an RT/DA$_j$ equation.

This paper is devoted to a mathematical study of the RT/DA equations, as well as numerical experiments on how accurate are the RT/DA equations as approximations of the RTE. We prove the well posedness of the RT/DA equations and provide numerical examples to show the increased improvement in solution accuracy when the number of terms, j, increases in RT/DA$_j$ equations.

8.2 Differential Approximation of the Integral Operator

The idea of the derivation of the RT/DA equations is based on the approximation of the integral operator S by a sequence of linear combinations of the inverse of linear elliptic differential operators on the unit sphere [9]. The point of departure of the approach is the knowledge of eigenvalues and eigenfunctions of the operator S. Specifically, for a spherical harmonic of order n, $Y_n(\omega)$ (cf. [3] for an introduction and spherical harmonics),

$$(SY_n)(\omega) = k_n Y_n(\omega), \tag{8.9}$$

$$k_n = 2\pi \int_{-1}^{1} k(s)\, P_n(s)\, ds, \quad P_n: \text{ Legendre polynomial of deg. } n. \tag{8.10}$$

In other words, k_n is an eigenvalue of S with spherical harmonics of order n as corresponding eigenfunctions. The eigenvalues have the property that

$$\{k_n\} \text{ is bounded and } k_n \to 0 \text{ as } n \to \infty. \tag{8.11}$$

Denote by Δ^* the Laplace–Beltrami operator on the unit sphere Ω. Then,

$$-(\Delta^* Y_n)(\omega) = n\,(n+1)\, Y_n(\omega).$$

Let $\{Y_{n,m} \mid -n \le m \le n,\, n \ge 0\}$ be an orthonormalized basis in $L^2(\Omega)$. We have the expansion

$$u(\omega) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} u_{n,m} Y_{n,m}(\omega) \text{ in } L^2(\Omega), \quad u_{n,m} = \int_{\Omega} u(\omega)\, Y_{n,m}(\omega)\, d\sigma(\omega).$$

With such an expansion of $u \in L^2(\Omega)$, we have an expansion for Su:

$$Su(\omega) = \sum_{n=0}^{\infty} k_n \sum_{m=-n}^{n} u_{n,m} Y_{n,m}(\omega) \text{ in } L^2(\Omega).$$

Suppose there are real numbers $\{\lambda_i, \alpha_i\}_{i \ge 1}$ such that

$$\sum_{i=1}^{\infty} \frac{\lambda_i}{1 + n\,(n+1)\, \alpha_i} = k_n, \quad n = 0, 1, \cdots. \tag{8.12}$$

Then formally,

$$S = \sum_{i=1}^{\infty} \lambda_i (I - \alpha_i \Delta^*)^{-1}. \tag{8.13}$$

The formal equality (8.13) motivates us to consider approximating S by

$$S_j = \sum_{i=1}^{j} \lambda_{j,i} (I - \alpha_{j,i} \Delta^*)^{-1}, \quad j = 1, 2, \cdots. \tag{8.14}$$

The eigenvalues of S_j are $\sum_{i=1}^{j} \lambda_{j,i} (1 + n(n+1)\alpha_{j,i})^{-1}$ with associated eigenfunctions the spherical harmonics of order n:

$$(S_j Y_n)(\omega) = \left[\sum_{i=1}^{j} \frac{\lambda_{j,i}}{1 + n(n+1)\alpha_{j,i}} \right] Y_n(\omega).$$

Note that for a fixed j,

$$\sum_{i=1}^{j} \frac{\lambda_{j,i}}{1 + n(n+1)\alpha_{j,i}} \to 0 \quad \text{as } n \to \infty.$$

Thus, the eigenvalue sequence of S_j has a unique accumulation point 0, a property for the operator S [cf. (8.11)]. Hence, we choose the parameters $\{\lambda_{j,i}, \alpha_{j,i}\}_{i=1}^{j}$ so that for some integer n_j depending on j,

$$\sum_{i=1}^{j} \frac{\lambda_{j,i}}{1 + n(n+1)\alpha_{j,i}} = k_n, \quad n = 0, 1, \cdots, n_j - 1. \tag{8.15}$$

We require $n_j \to \infty$ as $j \to \infty$.

The following results are shown in [9]:

Theorem 8.1. *Under the assumption* (8.15) *and*

$$\sup_{n \geq n_j} \left| \sum_{i=1}^{j} \lambda_{j,i} (1 + n(n+1)\alpha_{j,i})^{-1} \right| \to 0 \text{ as } j \to \infty, \tag{8.16}$$

we have the convergence $\|S_j - S\|_{\mathscr{L}(L^2(\Omega), L^2(\Omega))} \to 0$ *as* $j \to \infty$.

A sufficient condition for (8.16) is that all $\lambda_{j,i}$ and $\alpha_{j,i}$ are positive.

Theorem 8.2. *Assume* (8.15) *and* $\lambda_{j,i} > 0$ *and* $\alpha_{j,i} > 0$ *for* $i = 1, \cdots, j$. *Then* (8.16) *holds.*

Notice that $\alpha_{j,i} > 0$ is needed to ensure ellipticity of the differential operator $(I - \alpha_{j,i}\Delta^*)$. When we discretize the operator S_j, the positivity of $\{\lambda_{j,i}\}_{i=1}^{j}$ is desirable for numerical stability in computing approximations of S_j.

Consider an operator S_j of the form (8.14) to approximate S. From now on, we drop the letter j in the subscripts for $\lambda_{j,i}$ and $\alpha_{j,i}$. As noted after Theorem 8.2, to maintain ellipticity of the differential operator $(I-\alpha_i\Delta^*)$ and for stable numerical approximation of the operator S_j, we require

$$\alpha_i > 0,\ \lambda_i > 0, \quad 1 \le i \le j. \tag{8.17}$$

Recall the property (8.11); for the numbers $\{k_n\}$ defined in (8.10), we assume $k_0 \ge k_1 \ge \cdots$. This assumption is quite reasonable and is valid for phase functions in practical use.

To get some idea about the operators S_j, we consider the special cases $j=1$ and 2 next. For $j=1$, we have

$$S_1 Y_n(\omega) = k_{1,n} Y_n(\omega), \qquad k_{1,n} = \frac{\lambda_1}{1+\alpha_1 n\,(n+1)}. \tag{8.18}$$

Equating the first two eigenvalues of S and S_1, we can find

$$\lambda_1 = k_0, \qquad \alpha_1 = \frac{1}{2}\left(\frac{k_0}{k_1} - 1\right). \tag{8.19}$$

Observe that (8.17) is satisfied.

For $j=2$, $S_2 = \lambda_1 (I-\alpha_1\Delta^*)^{-1} + \lambda_2 (I-\alpha_2\Delta^*)^{-1}$ with the parameters satisfying $\lambda_1 > 0$, $\lambda_2 > 0$, $\alpha_1 > 0$, $\alpha_2 > 0$, and $\alpha_1 \ne \alpha_2$. We have

$$S_2 Y_n(\omega) = k_{2,n} Y_n(\omega), \qquad k_{2,n} = \frac{\lambda_1}{1+\alpha_1 n\,(n+1)} + \frac{\lambda_2}{1+\alpha_2 n\,(n+1)}. \tag{8.20}$$

Require the parameters to match the first three eigenvalues $k_{2,i} = k_i$, $i = 0,1,2$, i.e.,

$$\lambda_1 + \lambda_2 = k_0, \tag{8.21}$$

$$\frac{\lambda_1}{1+2\,\alpha_1} + \frac{\lambda_2}{1+2\,\alpha_2} = k_1, \tag{8.22}$$

$$\frac{\lambda_1}{1+6\,\alpha_1} + \frac{\lambda_2}{1+6\,\alpha_2} = k_2. \tag{8.23}$$

Consider the system (8.21)–(8.23) for a general form solution. Use α_1 as the parameter for the solution. It is shown in [9] that

$$\alpha_2 = \frac{1}{6} \cdot \frac{(3\,k_1 - 2\,k_0 - k_2) + 6\,(k_1 - k_2)\,\alpha_1}{(k_2 - k_1) + 2\,(3\,k_2 - k_1)\,\alpha_1}, \tag{8.24}$$

$$\lambda_2 = \frac{2\,[(k_1 - k_0) + 2\,k_1\alpha_1]\,[(k_2 - k_0) + 6\,k_2\alpha_1]}{(2\,k_0 + k_2 - 3\,k_1) + 12\,(k_2 - k_1)\,\alpha_1 + 12\,(3\,k_2 - k_1)\,\alpha_1^2}, \tag{8.25}$$

$$\lambda_1 = 1 - \lambda_2. \tag{8.26}$$

The issue of positivity of the solution $(\alpha_1, \alpha_2, \lambda_1, \lambda_2)$ is also discussed in [9].

Next, we take the Henyey–Greenstein phase function as an example; in this case,

$$k_n = g^n, \quad n = 0, 1, \cdots.$$

For the one-term approximation $S_1 = \lambda_1(I - \alpha_1 \Delta^*)^{-1}$, from (8.19), we have

$$\lambda_1 = 1, \qquad \alpha_1 = \frac{1-g}{2g}. \tag{8.27}$$

For the two-term approximation $S_2 = \lambda_1(I - \alpha_1 \Delta^*)^{-1} + \lambda_2(I - \alpha_2 \Delta^*)^{-1}$, we have

$$\alpha_2 = \frac{1-g}{6g} \cdot \frac{g-2+6g\,\alpha_1}{g-1+2\,(3g-1)\,\alpha_1}, \tag{8.28}$$

$$\lambda_2 = \frac{2\,(g-1+2g\alpha_1)\left(g^2-1+6g^2\alpha_1\right)}{(1-g)\,(2-g)+12g\,(g-1)\,\alpha_1+12g\,(3g-1)\,\alpha_1^2}, \tag{8.29}$$

$$\lambda_1 = \frac{g\,(1-g)\,(2g-1)\left(1+8\alpha_1+12\alpha_1^2\right)}{(1-g)\,(2-g)+12g\,(g-1)\,\alpha_1+12g\,(3g-1)\,\alpha_1^2}. \tag{8.30}$$

On the issue of positivity of the one parameter solution $(\alpha_1, \alpha_2, \lambda_1, \lambda_2)$ given by the formulas (8.28)–(8.30), with $\alpha_1 > 0$, it is shown in [9] that under the assumption $g > 1/2$, valid in applications with highly forward-peaked scattering, the condition for a positive solution $(\alpha_1, \alpha_2, \lambda_1, \lambda_2)$ is

$$\alpha_1 > \frac{2-g}{6g}. \tag{8.31}$$

Since $\alpha_1 = 1/2$ satisfies (8.31), one solution is

$$\alpha_1 = \frac{1}{2}, \qquad \alpha_2 = \frac{1-g}{6g}, \qquad \lambda_1 = \frac{4g\,(1-g)}{4g-1}, \qquad \lambda_2 = \frac{4g^2-1}{4g-1}. \tag{8.32}$$

Now consider the case $j = 3$:

$$S_3 = \lambda_1(I - \alpha_1 \Delta^*)^{-1} + \lambda_2(I - \alpha_2 \Delta^*)^{-1} + \lambda_3(I - \alpha_3 \Delta^*)^{-1} \tag{8.33}$$

with the parameters α_1, α_2, and α_3 pairwise distinct. We want to match the first four eigenvalues

$$k_{3,0} = k_0, \quad k_{3,1} = k_1, \quad k_{3,2} = k_2, \quad k_{3,3} = k_3,$$

i.e., for the special case of the Henyey–Greenstein phase function,

$$\lambda_1 + \lambda_2 + \lambda_3 = 1, \tag{8.34}$$

$$\frac{\lambda_1}{1+2\,\alpha_1} + \frac{\lambda_2}{1+2\,\alpha_2} + \frac{\lambda_3}{1+2\,\alpha_3} = g, \tag{8.35}$$

$$\frac{\lambda_1}{1+6\,\alpha_1} + \frac{\lambda_2}{1+6\,\alpha_2} + \frac{\lambda_3}{1+6\,\alpha_3} = g^2, \tag{8.36}$$

$$\frac{\lambda_1}{1+12\,\alpha_1} + \frac{\lambda_2}{1+12\,\alpha_2} + \frac{\lambda_3}{1+12\,\alpha_3} = g^3. \tag{8.37}$$

We choose α_1 and α_2, positive and distinct, as the parameters and express the other quantities in terms of them. There are many positive solution sets to the system (8.34)–(8.37) with positive parameters α_1 and α_2. For the numerical examples in Sect. 8.6, we use parameter sets so that overall the eigenvalues of S_3 are close to those of S. In particular, for $g = 0.9$, we choose

$$\alpha_1 = 0.00957621, \quad \alpha_2 = 0.08, \quad \alpha_3 = 0.712,$$
$$\lambda_1 = 0.660947, \quad \lambda_2 = 0.248262, \quad \lambda_3 = 0.0907913;$$

for $g = 0.95$, we choose

$$\alpha_1 = 0.00325598, \quad \alpha_2 = 0.06, \quad \alpha_3 = 0.701,$$
$$\lambda_1 = 0.78042, \quad \lambda_2 = 0.174622, \quad \lambda_3 = 0.0449584;$$

and for $g = 0.99$, we choose

$$\alpha_1 = 0.000306188, \quad \alpha_2 = 0.05, \quad \alpha_3 = 0.95,$$
$$\lambda_1 = 0.940247, \quad \lambda_2 = 0.0526772, \quad \lambda_3 = 0.00707558.$$

For $g = 0.9$, we compare the eigenvalues of S_j for $j = 1,2,3$ with those of S in Figs. 8.1, 8.2, and 8.3, respectively. From these figures, we can tell that the approximation of S_3 should be more accurate than that of S_2, which should be in turn more accurate than S_1. This observation is valid for other values of g below.

For $g = 0.95$, the eigenvalues of S_1, S_2, and S_3 are shown in Figs. 8.4–8.6.

For $g = 0.99$, the eigenvalues of S, S_1, S_2, and S_3 are shown in Fig. 8.7. Evidently, because of the strong singular nature of the phase function for $g = 0.99$, a higher value j will be needed for S_j to be a good approximation of S.

8.3 Analysis of the RT/DA Problems

We use S_j of (8.14) for the approximation of the integral operator S. In the following, we drop the subscript j in the parameters $\lambda_{j,i}$ and $\alpha_{j,i}$ for S_j and write

$$S_j u(x,\omega) = \sum_{i=1}^{j} \lambda_i (I - \alpha_i \Delta^*)^{-1} u(x,\omega).$$

Then the RT/DA$_j$ problem is

$$\omega\cdot\nabla u(x,\omega) + \sigma_t(x)\,u(x,\omega) = \sigma_s(x)\,S_j u(x,\omega) + f(x,\omega), \quad (x,\omega) \in X \times \Omega, \tag{8.38}$$
$$u(x,\omega) = u_{\text{in}}(x,\omega), \quad (x,\omega) \in \Gamma_-. \tag{8.39}$$

Let us consider the well posedness of (8.38)–(8.39). Introduce

$$w_i(x,\omega) = (I - \alpha_i \Delta^*)^{-1} u(x,\omega), \quad 1 \le i \le j, \tag{8.40}$$

$$w(x,\omega) = \sum_{i=1}^{j} \lambda_i w_i(x,\omega). \tag{8.41}$$

Then (8.38) can be rewritten as

$$\omega\cdot\nabla u(x,\omega) + \sigma_t(x)\, u(x,\omega) = \sigma_s(x)\, w(x,\omega) + f(x,\omega), \quad (x,\omega) \in X \times \Omega. \tag{8.42}$$

For simplicity we limit the analysis to the case where X is a convex domain in $\mathbb{R}^3$. The argument can be extended to a domain X satisfying the generalized convexity condition without problem [1]. Then for each $\omega \in \Omega$ and each $z \in X_\omega$, $X_{\omega,z}$ is the line segment

$$X_{\omega,z} = \{z + s\omega \mid s \in (s_-, s_+)\},$$

where $s_\pm = s_\pm(\omega,z)$ depend on ω and z and $x_\pm := z + s_\pm\omega$ are the intersection points of the line $\{z + s\omega \mid s \in \mathbb{R}\}$ with ∂X.

In the following, we write $s_\pm$ instead of $s_\pm(\omega,z)$ wherever there is no danger for confusion. We write (8.42) as

$$\frac{\partial}{\partial s} u(z+s\omega,\omega) + \sigma_t(z+s\omega)\, u(z+s\omega,\omega) = \sigma_s(z+s\omega)\, w(z+s\omega,\omega) + f(z+s\omega,\omega)$$

and multiply it by $\exp(\int_{s_-}^{s} \sigma_t(z+s\omega)\, ds)$ to obtain

$$\frac{\partial}{\partial s}\left(e^{\int_{s_-}^{s} \sigma_t(z+s\omega)\, ds} u(z+s\omega,\omega)\right)$$
$$= e^{\int_{s_-}^{s} \sigma_t(z+s\omega)\, ds} \left(\sigma_s(z+s\omega)\, w(z+s\omega,\omega) + f(z+s\omega,\omega)\right).$$

Integrate this equation from s_- to s:

$$e^{\int_{s_-}^{s} \sigma_t(z+s\omega)\, ds} u(z+s\omega,\omega) - u_{\text{in}}(z+s_-\omega,\omega)$$
$$= \int_{s_-}^{s} e^{\int_{s_-}^{t} \sigma_t(z+s\omega)\, ds} \left(\sigma_s(z+t\omega)\, w(z+t\omega,\omega) + f(z+t\omega,\omega)\right) dt.$$

Thus, (8.38) and (8.39) is converted to a fixed-point problem

$$u = Au + F, \tag{8.43}$$

where

$$Au(z+s\omega,\omega) = \int_{s_-}^{s} e^{-\int_t^s \sigma_t(z+s\omega)\, ds} \sigma_s(z+t\omega)\, w(z+t\omega,\omega) dt,$$

$$F(z+s\omega,\omega) = e^{-\int_{s_-}^{s} \sigma_t(z+s\omega)\, ds} u_{\text{in}}(z+s_-\omega,\omega)$$
$$+ \int_{s_-}^{s} e^{-\int_t^s \sigma_t(z+s\omega)\, ds} f(z+t\omega,\omega)\, dt.$$

We will show that A is a contractive mapping in a weighted $L^2(X\times\Omega)$ space. Denote $\kappa=\sup\{\sigma_s(x)/\sigma_t(x)\mid x\in X\}$. By (8.7), we know that $\kappa<1$. Consider

$$\begin{aligned}
&\int_{s_-}^{s_+}\sigma_t(z+s\omega)\,|Au(z+s\omega,\omega)|^2\,ds\\
&\quad=\int_{s_-}^{s_+}\sigma_t(z+s\omega)\left|\int_{s_-}^{s}e^{-\int_t^s\sigma_t(z+s\omega)\,ds}\sigma_s(z+t\omega)\,w(z+t\omega,\omega)dt\right|^2ds\\
&\quad\le\int_{s_-}^{s_+}\sigma_t(z+s\omega)\left(\int_{s_-}^{s}e^{-\int_t^s\sigma_t(z+s\omega)\,ds}\sigma_s(z+t\omega)\,dt\right)\\
&\qquad\cdot\left(\int_{s_-}^{s}e^{-\int_t^s\sigma_t(z+s\omega)\,ds}\sigma_s(z+t\omega)\,|w(z+t\omega,\omega)|^2\,dt\right)ds.
\end{aligned}$$

Since

$$\begin{aligned}
\int_{s_-}^{s}e^{-\int_t^s\sigma_t(z+s\omega)\,ds}\sigma_s(z+t\omega)\,dt&\le\kappa\int_{s_-}^{s}e^{-\int_t^s\sigma_t(z+s\omega)\,ds}\sigma_t(z+t\omega)\,dt\\
&=\kappa\left(1-e^{-\int_{s_-}^{s}\sigma_t(z+s\omega)\,ds}\right)<\kappa,
\end{aligned}$$

we have

$$\begin{aligned}
&\int_{s_-}^{s_+}\sigma_t(z+s\omega)\,|Au(z+s\omega,\omega)|^2\,ds\\
&\quad\le\kappa\int_{s_-}^{s_+}\sigma_t(z+s\omega)\int_{s_-}^{s}e^{-\int_t^s\sigma_t(z+s\omega)\,ds}\sigma_s(z+t\omega)\,|w(z+t\omega,\omega)|^2\,dt\,ds\\
&\quad=\kappa\int_{s_-}^{s_+}\sigma_s(z+t\omega)\,|w(z+t\omega,\omega)|^2\left(\int_{t}^{s_+}e^{-\int_t^s\sigma_t(z+s\omega)\,ds}\sigma_t(z+s\omega)\,ds\right)dt.
\end{aligned}$$

Now

$$\int_{t}^{s_+}e^{-\int_t^s\sigma_t(z+s\omega)\,ds}\sigma_t(z+s\omega)\,ds=1-e^{-\int_t^{s_+}\sigma_t(z+s\omega)\,ds}<1,$$

we obtain

$$\begin{aligned}
\int_{s_-}^{s_+}\sigma_t(z+s\omega)\,|Au(z+s\omega,\omega)|^2\,ds&\le\kappa\int_{s_-}^{s_+}\sigma_s(z+t\omega)\,|w(z+t\omega,\omega)|^2\,dt\\
&\le\kappa^2\int_{s_-}^{s_+}\sigma_t(z+t\omega)\,|w(z+t\omega,\omega)|^2\,dt.
\end{aligned}$$

Thus, we have proved the inequality

$$\|\sigma_t^{1/2}Au\|_{L^2(X\times\Omega)}\le\kappa\,\|\sigma_t^{1/2}w\|_{L^2(X\times\Omega)}. \tag{8.44}$$

Returning to the definition (8.40), we have, equivalently,

$$(I-\alpha_i\Delta^*)\,w_i=u\quad\text{in }X\times\Omega.$$

For a.e. $x \in X$, $w_i(x,\cdot) \in H^1(\Omega)$ and

$$\int_\Omega (w_i v + \alpha_i \nabla^* w_i \cdot \nabla^* v)\, d\sigma(\omega) = \int_\Omega u v\, d\sigma(\omega) \quad \forall v \in H^1(\Omega). \tag{8.45}$$

Since $\alpha_i > 0$, this problem has a unique solution by the Lax–Milgram Lemma. Take $v(\omega) = w_i(x,\omega)$ in (8.45):

$$\int_\Omega \left(|w_i|^2 + \alpha_i |\nabla^* w_i|^2\right) d\sigma(\omega) = \int_\Omega u w_i\, d\sigma(\omega).$$

Thus,

$$\int_\Omega \left(|w_i|^2 + 2\alpha_i |\nabla^* w_i|^2\right) d\sigma(\omega) \le \int_\Omega |u|^2 d\sigma(\omega). \tag{8.46}$$

In particular,

$$\int_\Omega |w_i|^2 d\sigma(\omega) \le \int_\Omega |u|^2 d\sigma(\omega).$$

Therefore,

$$\|\sigma_t^{1/2} w_i\|_{L^2(X\times\Omega)} \le \|\sigma_t^{1/2} u\|_{L^2(X\times\Omega)}. \tag{8.47}$$

Since $\lambda_i > 0$ and $\sum_{i=1}^j \lambda_i = 1$, from the definitions (8.41) and (8.47), we get

$$\|\sigma_t^{1/2} w\|_{L^2(X\times\Omega)} \le \|\sigma_t^{1/2} u\|_{L^2(X\times\Omega)}. \tag{8.48}$$

Combining (8.44) and (8.48), we see that the operator $A : L^2(X\times\Omega) \to L^2(X\times\Omega)$ is contractive with respect to the weighted norm $\|\sigma_t^{1/2} v\|_{L^2(X\times\Omega)}$:

$$\|\sigma_t^{1/2} A u\|_{L^2(X\times\Omega)} \le \kappa \|\sigma_t^{1/2} u\|_{L^2(X\times\Omega)}. \tag{8.49}$$

By an application of the Banach fixed-point theorem, we conclude that (8.43) has a unique solution $u \in L^2(X\times\Omega)$. By (8.42), we also have $\omega \cdot \nabla u(x,\omega) \in L^2(X\times\Omega)$. Therefore, the solution $u \in H_2^1(X\times\Omega)$.

In summary, we have shown the following existence and uniqueness result:

Theorem 8.3. *Under the assumptions* (8.7)*,* (8.8)*,* (8.15)*, and* (8.17)*, the problem* (8.38) *and* (8.39) *has a unique solution* $u \in H_2^1(X\times\Omega)$*.*

Next we show a positivity property required for the model (8.38) and (8.39) to be physically meaningful.

Theorem 8.4. *Under the assumptions of Theorem 8.3,*

$$f \ge 0 \text{ in } X\times\Omega,\ u_{\rm in} \ge 0 \text{ on } \Gamma_- \quad \Longrightarrow \quad u \ge 0 \text{ in } X\times\Omega. \tag{8.50}$$

Proof. From (8.43),

$$u = (I-A)^{-1} F = \sum_{j=0}^\infty A^j F.$$

By the given condition, $F \geq 0$. So the proof is done if we can show that $u \geq 0$ implies $Au \geq 0$. This property follows from the implication $u \geq 0 \Longrightarrow w_i \geq 0$ for the solution w_i of the problem (8.45). In (8.45), take $v = w_i^- = \min(w_i, 0)$ to obtain

$$\int_\Omega (|w_i^-|^2 + \alpha_i |\nabla^* w_i^-|^2)\, d\sigma(\omega) = \int_\Omega u\, w_i^-\, d\sigma(\omega) \leq 0.$$

Hence, $w_i^- = 0$, i.e., $w_i \geq 0$. □

We now derive an error estimate for the approximation (8.38)–(8.39) of the RTE problem (8.5)–(8.6). Denote the solution of the problem (8.38)–(8.39) by u_j and consider the error $e := u - u_j$. From (8.38)–(8.39) and (8.5)–(8.6), we obtain the following problem for the error:

$$\omega\cdot\nabla e + \sigma_t e = \sigma_s e_0 + \sigma_s \sum_{i=1}^{j} \lambda_i (I - \alpha_i \Delta^*)^{-1} e \quad \text{in } X \times \Omega, \tag{8.51}$$

$$e = 0 \quad \text{in } \Gamma_-, \tag{8.52}$$

where

$$e_0 = Su - \sum_{i=1}^{j} \lambda_i (I - \alpha_i \Delta^*)^{-1} u. \tag{8.53}$$

Since $\lambda_i > 0$ and $\sum_{i=1}^{j} \lambda_i = 1$, we obtain from (8.51) to (8.52) that, as in (8.43),

$$e = A\, e + E$$

with

$$E(z + s\omega, \omega) = \int_{s_-}^{s} e^{-\int_t^s \sigma_t(z+s\omega)\, ds} (\sigma_s e_0)(z + t\omega, \omega)\, dt.$$

Thus,

$$\begin{aligned} \|\sigma_t^{1/2} e\|_{L^2(X\times\Omega)} &\leq \|\sigma_t^{1/2} A e\|_{L^2(X\times\Omega)} + \|\sigma_t^{1/2} E\|_{L^2(X\times\Omega)} \\ &\leq \kappa \|\sigma_t^{1/2} e\|_{L^2(X\times\Omega)} + \|\sigma_t^{1/2} E\|_{L^2(X\times\Omega)}. \end{aligned}$$

Therefore,

$$\|\sigma_t^{1/2} e\|_{L^2(X\times\Omega)} \leq \frac{1}{1-\kappa} \|\sigma_t^{1/2} E\|_{L^2(X\times\Omega)} \leq c\, \|e_0\|_{L^2(X\times\Omega)}. \tag{8.54}$$

By expanding functions in terms of the spherical harmonics, we have

$$\|e_0\|_{L^2(X\times\Omega)} \leq c_j \|u\|_{L^2(X\times\Omega)}, \quad c_j = \max_n \left| k_n - \sum_{i=1}^{j} \frac{\lambda_i}{1 + \alpha_i n\,(n+1)} \right|. \tag{8.55}$$

Hence, from (8.54), we get the error bound

$$\|\sigma_t^{1/2}(u-u_j)\|_{L^2(X\times\Omega)} \le c\,c_j\|u\|_{L^2(X\times\Omega)}. \tag{8.56}$$

Theorem 8.5. *Under the assumptions of Theorem 8.3, we have the error bound* (8.56) *with* c_j *given in* (8.55).

8.4 An Iteration Method

We now consider the convergence of an iteration method for solving the problem defined by (8.42) and (8.39)–(8.41). Let $w^{(0)}$ be an initial guess, e.g., we may take $w^{(0)} = 0$. Then, for $n = 1, 2, \cdots$, define $u^{(n)}$ and $w^{(n)}$ as follows:

$$\omega\cdot\nabla u^{(n)} + \sigma_t u^{(n)} = \sigma_s w^{(n-1)} + f \quad \text{in } X\times\Omega, \tag{8.57}$$

$$u^{(n)} = u_{\text{in}} \quad \text{on } \Gamma_-, \tag{8.58}$$

$$w_i^{(n)} = (I-\alpha_i\Delta^*)^{-1}u^{(n)}, \quad 1\le i\le j, \tag{8.59}$$

$$w^{(n)} = \sum_{i=1}^{j}\lambda_i w_i^{(n)}. \tag{8.60}$$

Denote the iteration errors $e_u^{(n)} := u - u^{(n)}$, $e_w^{(n)} = w - w^{(n)}$. Then we have the error relations

$$\begin{aligned}
\omega\cdot\nabla e_u^{(n)} + \sigma_t e_u^{(n)} &= \sigma_s e_w^{(n-1)} \quad \text{in } X\times\Omega,\\
e_u^{(n)} &= 0 \quad \text{on } \Gamma_-,\\
e_{w_i}^{(n)} &= (I-\alpha_i\Delta^*)^{-1}e_u^{(n)}, \quad 1\le i\le j,\\
e_w^{(n)} &= \sum_{i=1}^{j}\lambda_i e_{w_i}^{(n)}.
\end{aligned}$$

Similar to (8.44) and (8.48), we have

$$\begin{aligned}
\|\sigma_t^{1/2}e_u^{(n)}\|_{L^2(X\times\Omega)} &\le \kappa\|\sigma_t^{1/2}e_w^{(n-1)}\|_{L^2(X\times\Omega)},\\
\|\sigma_t^{1/2}e_w^{(n-1)}\|_{L^2(X\times\Omega)} &\le \|\sigma_t^{1/2}e_u^{(n-1)}\|_{L^2(X\times\Omega)}.
\end{aligned}$$

Thus,

$$\|\sigma_t^{1/2}e_u^{(n)}\|_{L^2(X\times\Omega)} \le \kappa\|\sigma_t^{1/2}e_u^{(n-1)}\|_{L^2(X\times\Omega)},$$

and so we have

$$\|\sigma_t^{1/2}e_u^{(n)}\|_{L^2(X\times\Omega)} \le \kappa^n\|\sigma_t^{1/2}e_u^{(0)}\|_{L^2(X\times\Omega)} \to 0 \quad \text{as } n\to\infty.$$

Moreover, we also have the convergence of the sequence $\{w^{(n)}\}$:

$$\|\sigma_t^{1/2}e_w^{(n)}\|_{L^2(X\times\Omega)} \le \|\sigma_t^{1/2}e_u^{(n)}\|_{L^2(X\times\Omega)} \to 0 \quad \text{as } n\to\infty.$$

8.5 Error Analysis of a Hybrid Analytic/Finite Element Method

To focus on the main idea, in this section, we perform the analysis for the case of solving an RT/DA$_1$ equation with $u_{\text{in}} = 0$. The same argument can be extended straightforward to an RT/DA$_j$ equation for an arbitrary $j \geq 1$. Thus, consider the problem

$$\omega\cdot\nabla u(x,\omega) + \sigma_t(x)\,u(x,\omega) = \sigma_s(x)\,w(x,\omega) + f(x,\omega), \quad (x,\omega) \in X\times\Omega, \tag{8.61}$$

$$u(x,\omega) = 0, \quad (x,\omega) \in \Gamma_-, \tag{8.62}$$

$$(I - \alpha\Delta^*)\,w(x,\omega) = u(x,\omega), \quad (x,\omega) \in X\times\Omega. \tag{8.63}$$

A weak formulation of (8.63) is $w(x,\cdot) \in H^1(\Omega)$ and

$$\int_\Omega (wv + \alpha\,\nabla^* w\cdot\nabla^* v)\,d\sigma(\omega) = \int_\Omega u\,v\,d\sigma(\omega) \quad \forall\, v \in H^1(\Omega) \tag{8.64}$$

for a.e. $x \in X$, where ∇^* is the first-order Beltrami operator. Let V_ω^h be a finite element subspace of $H^1(\Omega)$. Then a finite element approximation of (8.64) is to find $w_h(x,\cdot) \in V_\omega^h$ such that

$$\int_\Omega (w_h v_h + \alpha\nabla^* w_h\cdot\nabla^* v_h)\,d\sigma(\omega) = \int_\Omega u_h v_h d\sigma(\omega) \quad \forall\, v_h \in V_\omega^h, \tag{8.65}$$

where the numerical solution u_h is defined by (8.61) with w replaced with w_h and (8.62). We have, similar to (8.43),

$$\begin{aligned} u_h(z+s\omega,\omega) = &\int_{s_-}^{s} e^{-\int_t^s \sigma_t(z+s\omega)\,ds}\sigma_s(z+t\omega)\,w_h(z+t\omega,\omega)dt \\ &+ \int_{s_-}^{s} e^{-\int_t^s \sigma_t(z+s\omega)\,ds} f(z+t\omega,\omega)\,dt. \end{aligned} \tag{8.66}$$

Denote the error functions

$$e_{u,h}(x,\omega) = u(x,\omega) - u_h(x,\omega), \qquad e_{w,h}(x,\omega) = w(x,\omega) - w_h(x,\omega). \tag{8.67}$$

Subtract (8.66) from (8.43):

$$e_{u,h}(z+s\omega,\omega) = \int_{s_-}^{s} e^{-\int_t^s \sigma_t(z+s\omega)\,ds}\sigma_s(z+t\omega)\,e_{w,h}(z+t\omega,\omega)\,dt. \tag{8.68}$$

Similar to derivation of (8.44), we then deduce from (8.68) that

$$\int_{s_-}^{s_+} \sigma_t(z+s\omega)\left|e_{u,h}(z+s\omega,\omega)\right|^2 ds \leq \kappa^2 \int_{s_-}^{s_+} \sigma_t(z+s\omega)\left|e_{w,h}(z+s\omega,\omega)\right|^2 ds. \tag{8.69}$$

To bound the error $e_{w,h}$, we subtract (8.65) from (8.64) with $v = v_h$:

$$\int_\Omega (e_{w,h}v_h + \alpha\nabla^* e_{w,h}\cdot\nabla^* v_h)\,d\sigma(\omega) = \int_\Omega e_{u,h}v_h d\sigma(\omega) \quad \forall\, v_h \in V_\omega^h. \tag{8.70}$$

Thus,

$$\int_{\Omega} \left(|e_{w,h}|^2 + \alpha|\nabla^* e_{w,h}|^2\right) d\sigma(\omega) = \int_{\Omega} \left[e_{w,h}(w - v_h) + \alpha\nabla^* e_{w,h} \cdot \nabla^*(w - v_h)\right] d\sigma(\omega) + \int_{\Omega} e_{u,h}(v_h - w + e_{w,h})\, d\sigma(\omega).$$

For any $\varepsilon > 0$, we have positive constants $C_1(\varepsilon)$ and $C_2(\varepsilon)$ such that

$$\int_{\Omega} e_{w,h}(w - v_h) d\sigma(\omega) \le \varepsilon \int_{\Omega} |e_{w,h}|^2 d\sigma(\omega) + C_1(\varepsilon) \int_{\Omega} |w - v_h|^2 d\sigma(\omega),$$

$$\int_{\Omega} e_{u,h}(v_h - w + e_{w,h})\, d\sigma(\omega) \le \frac{1}{2}\int_{\Omega} |e_{u,h}|^2 d\sigma(\omega) + \frac{1+\varepsilon}{2}\int_{\Omega} |e_{w,h}|^2 d\sigma(\omega) + C_2(\varepsilon)\int_{\Omega} |w - v_h|^2 d\sigma(\omega).$$

Moreover,

$$\int_{\Omega} \nabla^* e_{w,h} \cdot \nabla^*(w - v_h)\, d\sigma(\omega) \le \frac{1}{2}\int_{\Omega} |\nabla^* e_{w,h}|^2 d\sigma(\omega) + \frac{1}{2}\int_{\Omega} |\nabla^*(w - v_h)|^2 d\sigma(\omega).$$

Then,

$$\begin{aligned}\int_{\Omega} \left(|e_{w,h}|^2 + \alpha|\nabla^* e_{w,h}|^2\right) d\sigma(\omega) &\le (1+\varepsilon)\int_{\Omega} |e_{u,h}|^2 d\sigma(\omega) + C_3(\varepsilon)\int_{\Omega} |w - v_h|^2 d\sigma(\omega) \\ &\quad + \alpha\int_{\Omega} |\nabla^*(w - v_h)|^2 d\sigma(\omega) \\ &\le (1+\varepsilon)\int_{\Omega} |e_{u,h}|^2 d\sigma(\omega) + C(\varepsilon)\|w - v_h\|^2_{H^1(\Omega)}.\end{aligned}$$

Since $v_h \in V^h_\omega$ is arbitrary, we have

$$\int_{\Omega} \left(|e_{w,h}|^2 + \alpha|\nabla^* e_{w,h}|^2\right) d\sigma(\omega) \le (1+\varepsilon)\int_{\Omega} |e_{u,h}|^2 d\sigma(\omega) + C(\varepsilon) \inf_{v_h \in V^h_\omega} \|w - v_h\|^2_{H^1(\Omega)}. \tag{8.71}$$

We now integrate (8.69) and apply (8.71):

$$\begin{aligned}\|\sigma_t^{1/2} e_{u,h}\|^2_{L^2(X\times\Omega)} &\le \kappa^2 \|\sigma_t^{1/2} e_{w,h}\|^2_{L^2(X\times\Omega)} \\ &= \kappa^2 \int_X \sigma_t(x)\, dx \int_{\Omega} |e_{w,h}(x,\omega)|^2 d\sigma(\omega) \\ &\le (1+\varepsilon)\kappa^2 \|\sigma_t^{1/2} e_{u,h}\|^2_{L^2(X\times\Omega)} + C(\varepsilon)\int_X \left[\inf_{v_h \in V^h_\omega} \|w - v_h\|^2_{H^1(\Omega)}\right] dx.\end{aligned}$$

Choose $\varepsilon > 0$ small enough to obtain

$$\|\sigma_t^{1/2} e_{u,h}\|^2_{L^2(X\times\Omega)} \le C \int_X \left[\inf_{v_h \in V^h_\omega} \|w - v_h\|^2_{H^1(\Omega)}\right] dx. \tag{8.72}$$

In a typical error estimate, if $w \in L^2(X, H^{k+1}(\Omega))$ and piecewise polynomials of degree less than or equal to k are used for the finite element space V_ω^h, then

$$\int_X \left[\inf_{v_h \in V_\omega^h} \|w - v_h\|^2_{H^1(\Omega)} \right] dx \le c\, h^{2k} \|w\|^2_{L^2(X, H^{k+1}(\Omega))}. \tag{8.73}$$

From (8.72), we then have the error bound

$$\|e_{u,h}\|_{L^2(X\times\Omega)} \le c\, h^k \|w\|_{L^2(X, H^{k+1}(\Omega))}. \tag{8.74}$$

8.6 Numerical Experiments

Here we report some numerical results on the differences between numerical solutions of RTE and those of RT/DA equations. For definiteness, we use the Henyey–Greenstein phase function and consider the approximations S_j, $1 \le j \le 3$, specified in Sect. 8.2.

For the discretization of the unit sphere Ω for the direction variable ω, we use the finite element method described in [2]. The angular discretizations used all have $n_\phi = 8$ and have various values of n_θ. For reference, the total number of angular nodes in each discretization is listed in Table 8.1.

Table 8.1: Number of angular nodes

n_θ	Nodes
4	26
8	98
16	386
32	1538
64	6146
128	24578

Experiment 8.6.1. We first make sure that the numerical methods behave as expected. Let us comment on the discretization of S used in approximating the RTE. For ease, we compare the numerical solution of the RTE with the numerical solutions to the RT/DA$_1$ equation calculated on the same mesh. This leaves us with a choice of weights when solving the RTE. Initially, the choice was made that $w_i = \frac{4\pi}{N}$ where N is the number of angular nodes. However, this is not a good quadrature rule, as the nodes are not quite evenly spaced on the sphere. This point is illustrated in Table 8.2. In this table, we numerically integrate

$$\int_\Omega Y_1(\omega') k_{.5}(\omega_0 \cdot \omega') d\sigma(\omega')$$

using both uniform weights and the weights introduced below. Here $k_{.5}$ is the HG phase function with anisotropy factor $g = 0.5$, ω_0 is rather arbitrarily chosen to be $\frac{1}{\sqrt{3}}(1,1,1)^T$, and $Y_1(\omega)$ is the order 1 spherical harmonic:

$$Y_1(\omega) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos(\theta).$$

The true value of this integral is $.5Y_1(\omega_0) \approx 0.14105$.

When solving the approximation to the RT/DA$_1$ equation, a matrix A is formed with the property that

$$f^T Ag = \int_\Omega f(\omega)g(\omega)d\sigma(\omega)$$

if f is a vector containing function values of f at the nodes ω_i and f, g are elements of the finite element space associated with the angular mesh. We choose the weight vector w to be $w = Ae$, where e is the vector with all components 1. This quadrature rule will be exact for all functions in the finite element space associated with the angular mesh. Since this rule will correctly integrate any piecewise linear function in the finite element space, it may be thought of as an analogue of the trapezoidal rule for the sphere. Quick investigation shows that for the example integral above, this method is order 2 in terms of n_θ, which makes it order 1 in terms of the number of nodes. It seems likely that this is true in general.

Table 8.2: Comparison of trapezoidal weights vs. uniform weights in evaluating $\int_\Omega Y_1(\omega')k_{.5}(\omega_0 \cdot \omega')d\sigma(\omega')$ for specific choice of ω_0

n_θ	Trapezoidal rule	Trapezoidal error	Uniform rule	Uniform error
4	1.51038e-01	9.99030e-03	1.29518e-01	1.15298e-02
8	1.42538e-01	1.49073e-03	1.28156e-01	1.28914e-02
16	1.41424e-01	3.76434e-04	1.28242e-01	1.28050e-02
32	1.41141e-01	9.39943e-05	1.28244e-01	1.28030e-02
64	1.41071e-01	2.34917e-05	1.28244e-01	1.28035e-02

We take $\mu_t(x) = 2$, $\mu_s(x) = 1$, $g = 0.9$, and $f = (\mu_t - g\mu_s)Y_1(\omega)$. Under these choices and with appropriate choice of boundary conditions, the solution to both the RTE and RT/DA$_1$ equation is Y_1. We report the errors

$$e_S := \left\{ \sum_i w_i \int_X (u_S(x,\omega_i) - Y_1(\omega_i))^2 dx \right\}^{1/2} \tag{8.75}$$

$$e_{S_1} := \left\{ \sum_i w_i \int_X (u_{S_1}(x,\omega_i) - Y_1(\omega_i))^2 dx \right\}^{1/2} \tag{8.76}$$

in Tables 8.3 and 8.4. We see that both methods converge in the above norm. We report the maximum difference between u_S and u_{S_1} in Table 8.5. Unless specified otherwise, all meshes have 96 space elements. A "–" in the tables reflects the fact that the iteration algorithm used to solve the discrete systems does not converge within a fixed (large) number of iterations.

Table 8.3: Experiment 8.6.1: error between u_S, u_{S_1}, and Y_1

n_θ	e_S	e_{S_1}
4	–	0.301621
8	–	0.236104
16	0.227244	0.172195
32	0.129529	0.123071
64	0.088994	0.087421

Table 8.4: Experiment 8.6.1: different errors

n_θ	$\max\|u_S-Y_1\|$	$\text{Mean}\|u_S-Y_1\|$	$\max\|u_{S_1}-Y_1\|$	$\text{Mean}\|u_{S_1}-Y_1\|$
4	–	–	6.070e-03	8.433e-04
8	–	–	3.363e-03	1.668e-04
16	1.077e-01	1.275e-02	1.306e-03	3.851e-05
32	1.558e-02	3.707e-04	4.410e-04	9.002e-06
64	4.315e-03	4.332e-05	1.381e-04	2.212e-06

Table 8.5: Experiment 8.6.1: maximum error at the nodes of the mesh between u_S and u_{S_1}

n_θ	$\max\|u_S-u_{S_1}\|$
16	0.127953
32	0.019912
64	0.007467

To investigate the relative error, we introduce new notation. Define the set of all nodes as

$$\mathscr{N} = \{(x,\omega) \mid x \text{ is a node of the spatial mesh},\ \omega \text{ is a node of the angular mesh}\}.$$

For a given relative error level e, define

$$\mathscr{N}_e = \{(x,\omega) \in \mathscr{N} \mid |u_S(x,\omega) - u_{S_1}(x,\omega)|/|u_S(x,\omega)| < e\}.$$

Finally, define $f(e) = |\mathscr{N}_e|/|\mathscr{N}|$ for the fraction of nodes at which the solution to the RT/DA$_1$ equation agrees with the RTE within relative error e. Here we use the convention that $|\cdot|$ applied to a set denotes cardinality.

We plot $f(e)$ in Fig. 8.8. Note that there is no logical upper bound on the domain e. However, we will only plot $0 < e < 1$, as it makes the graphs more readable. □

Experiment 8.6.2. The spatial domain is $X = [0,1]^3$. We choose $\mu_t = 2$, $\mu_s = 1$, and the Henyey–Greenstein phase function with several different choices of scattering parameter g. The source function f is taken to be

$$f(x,\omega) = \begin{cases} 1 \text{ if } x \in R \\ 0 \text{ otherwise} \end{cases}$$

where R is approximately a sphere of radius $1/4$ centered at $(0.5, 0.5, 0.5)$. To do the numerical simulations the domain X is partitioned into 324 tetrahedrons and we use various angular discretizations to investigate the effect of angular discretization.

Again let $\mathscr{N}$ be the set of all nodes of the mesh. Let u_S^h be the numerical solution to the RTE and let $u_{S_j}^h$ be the numerical solution to the RT/DA$_j$ equation. For a given relative error level, e, define the set of all nodes on which the numerical solution to the RT/DA$_j$ equation agrees with the RTE within relative error e. That is,

$$\mathscr{N}_{e,j} = \{(x,\omega) \in \mathscr{N} \mid |u_S^h(x,\omega) - u_{S_j}^h(x,\omega)| < e|u_S^h(x,\omega)|\}.$$

Define $f_j(e) = |\mathscr{N}_{e,j}|/|\mathscr{N}|$, giving the fraction of nodes at which the solution to the RT/DA$_j$ equation agrees with the RTE within relative error e. Obviously, we would like $f(e) \approx 1$ for as small e as possible.

Plots of $f_j(e)$ are shown with scattering parameter $\eta = 0.9, 0.95$, and 0.99 for the RT/DA$_j$ $(j = 1,2,3)$ equations in Figs. 8.9–8.16. We observe that (1) as j increases, the RT/DA$_j$ equation with properly chosen parameter values provides increasingly accurate solution to the RTE, and (2) as g gets close to $1-$, higher value of j will be needed for the RT/DA$_j$ equation to be a good approximation of the RTE. □

Acknowledgement

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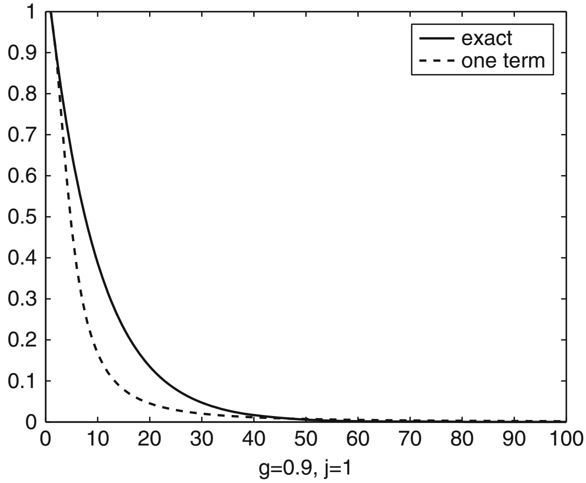

Fig. 8.1: Eigenvalues of S (*solid line*) and S_1 (*broken line*)

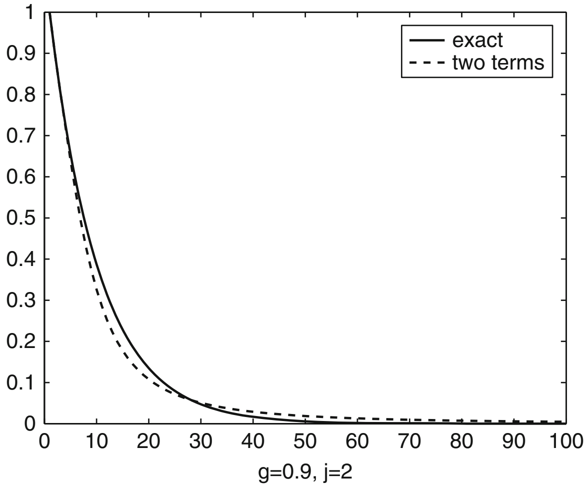

Fig. 8.2: Eigenvalues of S (*solid line*) and S_2 with the choice (8.32) (*broken line*)

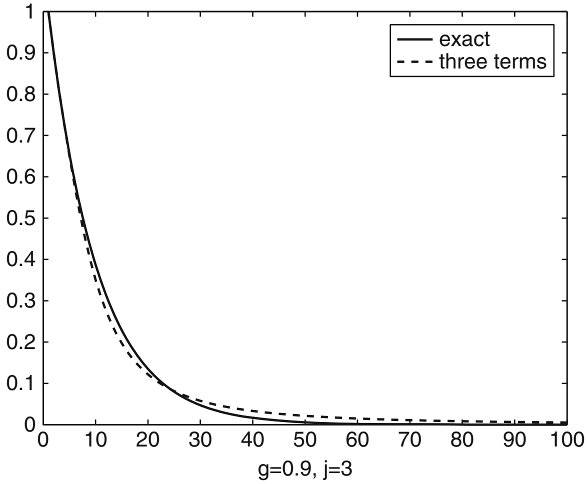

Fig. 8.3: Eigenvalues of S (*solid line*) and S_3 (*broken line*)

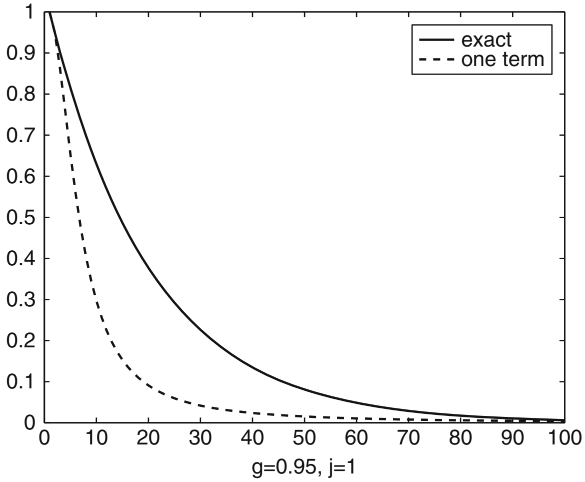

Fig. 8.4: Eigenvalues of S (*solid line*) and S_1 (*broken line*)

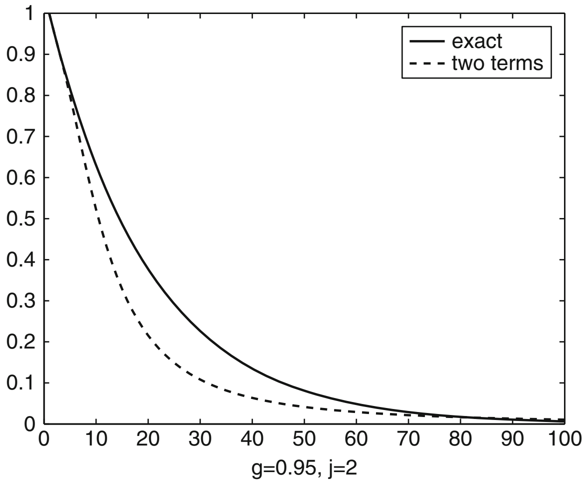

Fig. 8.5: Eigenvalues of S (*solid line*) and S_2 (*broken line*)

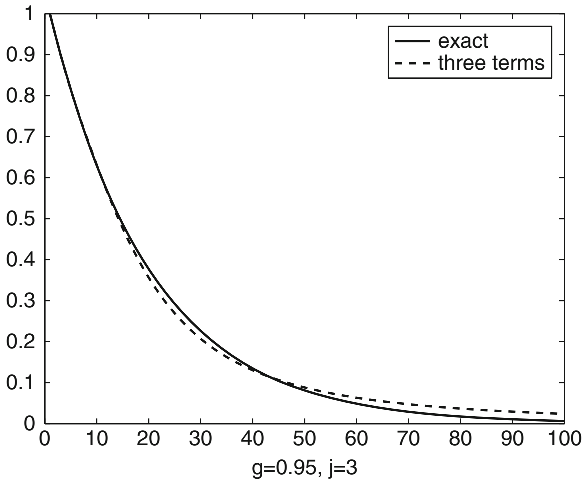

Fig. 8.6: Eigenvalues of S (*solid line*) and S_3 (*broken line*)

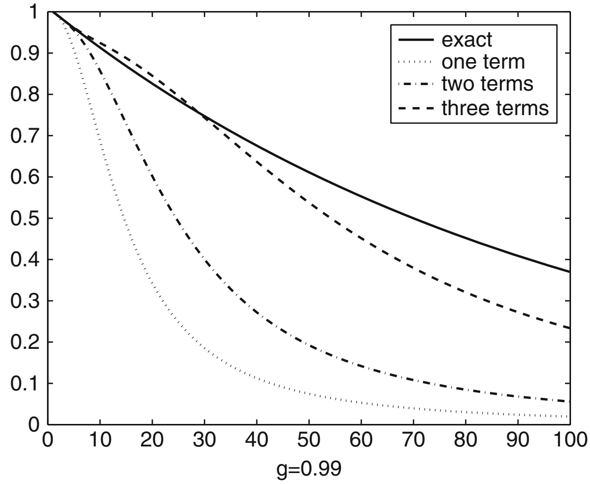

Fig. 8.7: Eigenvalues of S, S_1, S_2, and S_3

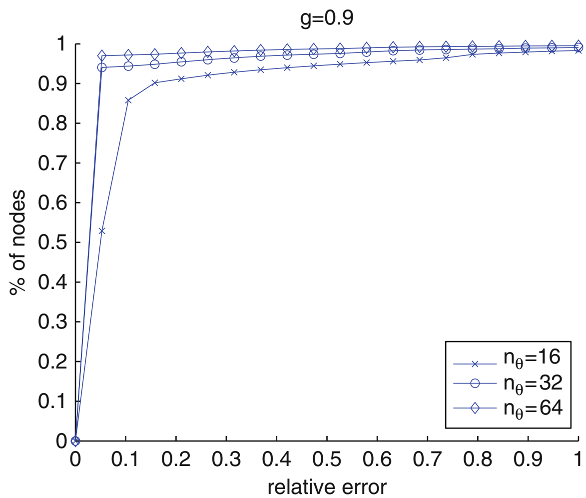

Fig. 8.8: f vs e for Experiment 8.6.1

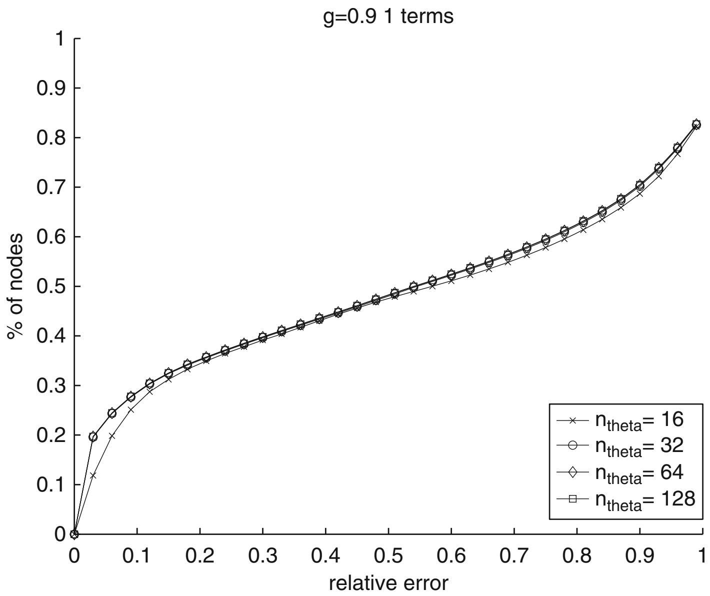

Fig. 8.9: Experiment 8.6.2: f vs. e for $g = 0.9$ using one-term approximation S_1

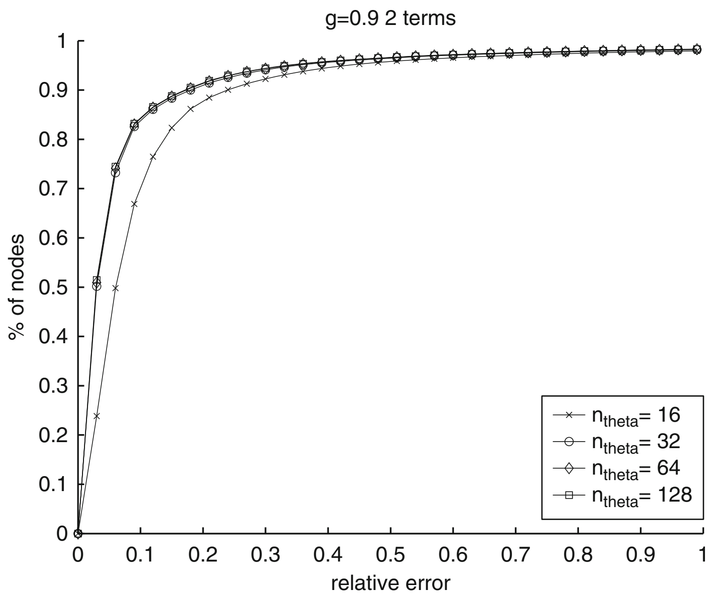

Fig. 8.10: Experiment 8.6.2: f vs. e for $g = 0.9$ using two-term approximation S_2

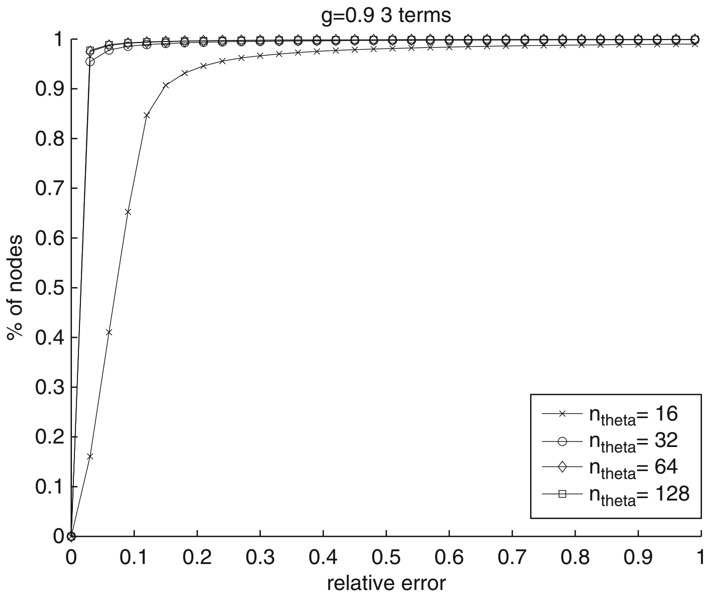

Fig. 8.11: Experiment 8.6.2: f vs. e for $g = 0.9$ using three term approximation S_3

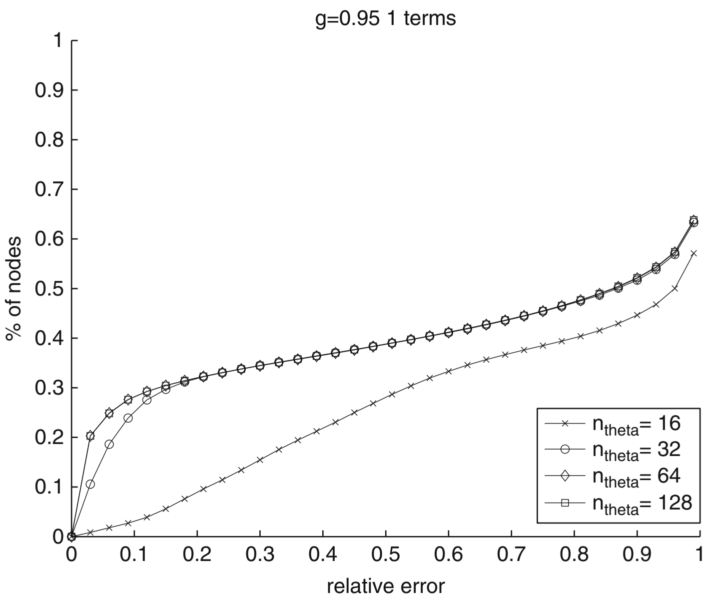

Fig. 8.12: Experiment 8.6.2: f vs. e for $g = 0.95$ using one-term approximation S_1

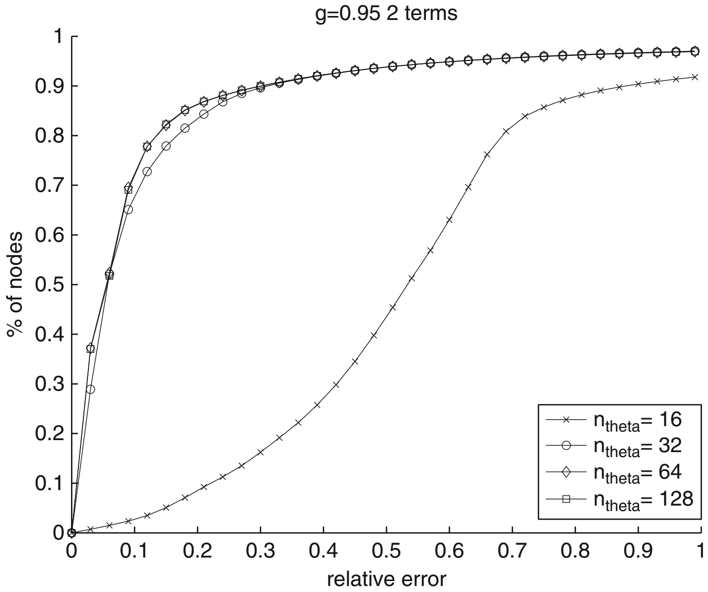

Fig. 8.13: Experiment 8.6.2: f vs. e for $g = 0.95$ using two-term approximation S_2

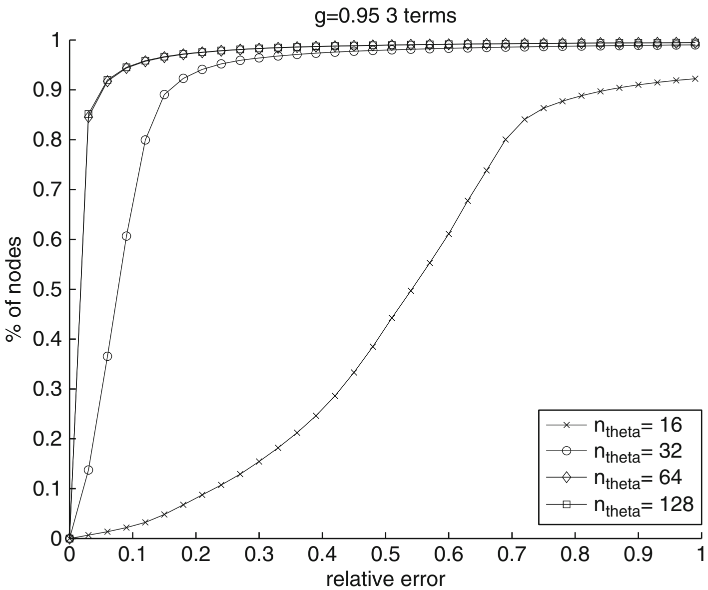

Fig. 8.14: Experiment 8.6.2: f vs. e for $g = 0.95$ using three term approximation S_3

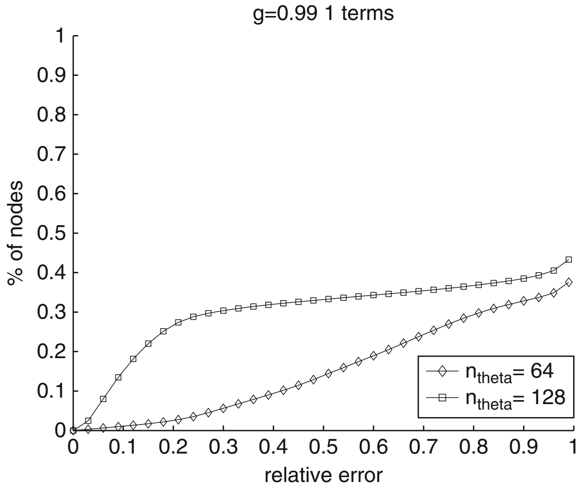

Fig. 8.15: Experiment 8.6.2: f vs. e for $g = 0.99$ using one-term approximation S_1

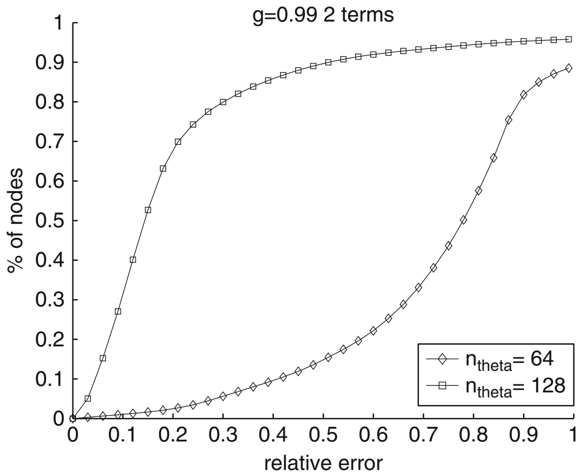

Fig. 8.16: Experiment 8.6.2: f vs. e for $g = 0.99$ using two-term approximation S_2

Chapter 9
Inverse Spectral Problems for Complex Jacobi Matrices

Gusein Sh. Guseinov

Abstract The paper deals with two versions of the inverse spectral problem for finite complex Jacobi matrices. The first is to reconstruct the matrix using the eigenvalues and normalizing numbers (spectral data) of the matrix. The second is to reconstruct the matrix using two sets of eigenvalues (two spectra), one for the original Jacobi matrix and one for the matrix obtained by deleting the last row and last column of the Jacobi matrix. Uuniqueness and existence results for solution of the inverse problems are established and an explicit procedure of reconstruction of the matrix from the spectral data is given. It is shown how the results can be used to solve finite Toda lattices subject to the complex-valued initial conditions.

9.1 Introduction

An $N \times N$ complex Jacobi matrix is a matrix of the form

$$J = \begin{bmatrix} b_0 & a_0 & 0 & \cdots & 0 & 0 & 0 \\ a_0 & b_1 & a_1 & \cdots & 0 & 0 & 0 \\ 0 & a_1 & b_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b_{N-3} & a_{N-3} & 0 \\ 0 & 0 & 0 & \cdots & a_{N-3} & b_{N-2} & a_{N-2} \\ 0 & 0 & 0 & \cdots & 0 & a_{N-2} & b_{N-1} \end{bmatrix}, \tag{9.1}$$

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where for each n, a_n and b_n are arbitrary complex numbers such that a_n is different from zero:

$$a_n, b_n \in \mathbb{C}, \quad a_n \neq 0. \tag{9.2}$$

The general inverse spectral problem is to reconstruct the matrix J given some of its spectral characteristics (spectral data). Many versions of the inverse spectral problem for finite and infinite Jacobi matrices have been investigated in the literature and many procedures and algorithms for their solution have been proposed (see [1–4, 6–15]). Some of them form analogs of problems of inverse Sturm–Liouville theory [5, 17] in which a coefficient function or "potential" in a second-order differential equation is to be recovered, either given the spectral function or alternatively given two sets of eigenvalues corresponding to two given boundary conditions at one end, the boundary condition at the other end being fixed.

A distinguishing feature of the Jacobi matrix (9.1) from other matrices is that the eigenvalue problem $Jy = \lambda y$ for a column vector $y = \{y_n\}_{n=0}^{N-1}$ is equivalent to the second-order linear difference equation

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \quad n \in \{0, 1, \ldots, N-1\}, \quad a_{-1} = a_{N-1} = 1, \tag{9.3}$$

for $\{y_n\}_{n=-1}^{N}$, with the boundary conditions

$$y_{-1} = y_N = 0. \tag{9.4}$$

This allows using techniques from the theory of three-term linear difference equations [1], to develop a thorough analysis of the eigenvalue problem $Jy = \lambda y$.

Problem (9.3), (9.4) arises, for example, in the discretization of the (continuous) Sturm–Liouville eigenvalue problem

$$\frac{d}{dx}\left[p(x)\frac{dy(x)}{dx}\right] + q(x)y(x) = \lambda y(x), \quad x \in [a,b],$$

$$y(a) = y(b) = 0,$$

where $[a,b]$ is a finite interval.

In the case of real entries the finite Jacobi matrix is self-adjoint and its eigenvalues are real and distinct. In the complex case the Jacobi matrix is, in general, no longer self-adjoint and its eigenvalues may be complex and multiple. In [9] the author introduced the concept of spectral data for finite complex Jacobi matrices and investigated the inverse spectral problem in which it is required to recover the matrix from its spectral data. The spectral data consist of the complex-valued eigenvalues and associated normalizing numbers derived by decomposing the resolvent function (Weyl–Titchmarsh function) into partial fractions using the eigenvalues. Let $R(\lambda) = (J - \lambda I)^{-1}$ be the resolvent of the matrix J (by I we denote the identity matrix of needed dimension) and e_0 be the N-dimensional column vector with the components $1, 0, \ldots, 0$. The rational function

$$w(\lambda) = -\langle R(\lambda)e_0, e_0\rangle = \left\langle (\lambda I - J)^{-1} e_0, e_0 \right\rangle, \tag{9.5}$$

introduced earlier in [15], we call the *resolvent function* of the matrix J, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{C}^N$. This function is known also as the Weyl–Titchmarsh function of J.

Denote by $\lambda_1, \ldots, \lambda_p$ all the distinct eigenvalues of the matrix J and by $m_1, \ldots, m_p$ their multiplicities, respectively, as the zeros of the characteristic polynomial $\det(J - \lambda I)$, so that $1 \le p \le N$, $m_1 + \ldots + m_p = N$, and

$$\det(\lambda I - J) = (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_p)^{m_p}. \tag{9.6}$$

We can decompose the rational function $w(\lambda)$ into partial fractions to get

$$w(\lambda) = \sum_{k=1}^{p} \sum_{j=1}^{m_k} \frac{\beta_{kj}}{(\lambda - \lambda_k)^j},$$

where β_{kj} are some complex numbers uniquely determined by the matrix J. For each $k \in \{1, \ldots, p\}$, the (finite) sequence $\{\beta_{k1}, \ldots, \beta_{km_k}\}$, we call the *normalizing chain* (of the matrix J) associated with the eigenvalue λ_k.

The collection of the eigenvalues and normalizing numbers

$$\{\lambda_k,\ \beta_{kj}\ (j = 1, \ldots, m_k;\ k = 1, \ldots, p)\},$$

of the matrix J of the form (9.1), (9.2) is called the *spectral data* of this matrix.

The first inverse problem is to reconstruct the matrix using the eigenvalues and normalizing numbers (spectral data) of the matrix.

Let J_1 be the $(N-1) \times (N-1)$ matrix obtained from J by deleting its last row and last column:

$$J_1 = \begin{bmatrix} b_0 & a_0 & 0 & \cdots & 0 & 0 \\ a_0 & b_1 & a_1 & \cdots & 0 & 0 \\ 0 & a_1 & b_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & b_{N-3} & a_{N-3} \\ 0 & 0 & 0 & \cdots & a_{N-3} & b_{N-2} \end{bmatrix}. \tag{9.7}$$

Denote by $\mu_1, \ldots, \mu_q$ all the distinct eigenvalues of the matrix J_1 and by $n_1, \ldots, n_q$ their multiplicities, respectively, as the roots of the characteristic polynomial $\det(J_1 - \lambda I)$ so that $1 \le q \le N-1$ and $n_1 + \ldots + n_q = N-1$.

The collections

$$\{\lambda_k,\ m_k\ (k = 1, \ldots, p)\} \quad \text{and} \quad \{\mu_k,\ n_k\ (k = 1, \ldots, q)\}$$

form the spectra (together with their multiplicities) of the matrices J and J_1, respectively. We call these collections the two spectra of the matrix J.

The second inverse problem (inverse problem about two spectra) consists in the reconstruction of the matrix J by its two spectra.

This paper consists, besides this introductory section, of three sections. Section 9.2 presents solution of the inverse problem for eigenvalues and normalizing numbers (spectral data) of the matrix and Sect. 9.3 presents a uniqueness result for solution of the inverse problem for two spectra. Finally, in Sect. 9.4, we show how to solve finite Toda lattices subject to the complex-valued initial conditions by the method of inverse spectral problem.

9.2 Inverse Problem for Eigenvalues and Normalizing Numbers

Given a Jacobi matrix J of the form (9.1) with the entries (9.2), consider the eigenvalue problem $Jy = \lambda y$ for a column vector $y = \{y_n\}_{n=0}^{N-1}$, which is equivalent to the problem (9.3), (9.4). Denote by $\{P_n(\lambda)\}_{n=-1}^{N}$ and $\{Q_n(\lambda)\}_{n=-1}^{N}$ the solutions of (9.3) satisfying the initial conditions

$$P_{-1}(\lambda) = 0, \quad P_0(\lambda) = 1; \tag{9.8}$$

$$Q_{-1}(\lambda) = -1, \quad Q_0(\lambda) = 0. \tag{9.9}$$

For each $n \geq 0$, $P_n(\lambda)$ is a polynomial of degree n and is called a polynomial of first kind and $Q_n(\lambda)$ is a polynomial of degree $n-1$ and is known as a polynomial of second kind. These polynomials can be found recurrently from (9.3) using initial conditions (9.8) and (9.9). The leading terms of the polynomials $P_n(\lambda)$ and $Q_n(\lambda)$ have the forms

$$P_n(\lambda) = \frac{\lambda^n}{a_0 a_1 \cdots a_{n-1}} + \ldots, \; n \geq 0; \quad Q_n(\lambda) = \frac{\lambda^{n-1}}{a_0 a_1 \cdots a_{n-1}} + \ldots, \; n \geq 1. \tag{9.10}$$

The equality

$$\det(J - \lambda I) = (-1)^N a_0 a_1 \cdots a_{N-2} P_N(\lambda) \tag{9.11}$$

holds (see [9, 11]) so that the eigenvalues of the matrix J coincide with the zeros of the polynomial $P_N(\lambda)$.

The Wronskian of the solutions $P_n(\lambda)$ and $Q_n(\lambda)$:

$$a_n[P_n(\lambda)Q_{n+1}(\lambda) - P_{n+1}(\lambda)Q_n(\lambda)],$$

does not depend on $n \in \{-1, 0, 1, \ldots, N-1\}$. On the other hand, the value of this expression at $n = -1$ is equal to 1 by (9.8), (9.9), and $a_{-1} = 1$. Therefore

$$a_n[P_n(\lambda)Q_{n+1}(\lambda) - P_{n+1}(\lambda)Q_n(\lambda)] = 1 \quad \text{for all} \quad n \in \{-1, 0, 1, \ldots, N-1\}.$$

Putting, in particular, $n = N-1$, we arrive at

$$P_{N-1}(\lambda)Q_N(\lambda) - P_N(\lambda)Q_{N-1}(\lambda) = 1. \tag{9.12}$$

The entries $R_{nm}(\lambda)$ of the matrix $R(\lambda) = (J - \lambda I)^{-1}$ (resolvent of J) are of the form

$$R_{nm}(\lambda) = \begin{cases} P_n(\lambda)[Q_m(\lambda) + M(\lambda)P_m(\lambda)], & 0 \le n \le m \le N-1, \\ P_m(\lambda)[Q_n(\lambda) + M(\lambda)P_n(\lambda)], & 0 \le m \le n \le N-1, \end{cases} \tag{9.13}$$

(see [9, 11]) where

$$M(\lambda) = -\frac{Q_N(\lambda)}{P_N(\lambda)}. \tag{9.14}$$

According to (9.5), (9.13), (9.14) and using initial conditions (9.8), (9.9), we get

$$w(\lambda) = -R_{00}(\lambda) = -M(\lambda) = \frac{Q_N(\lambda)}{P_N(\lambda)}. \tag{9.15}$$

By (9.11) and (9.6) we have

$$P_N(\lambda) = c(\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_p)^{m_p},$$

where c is a nonzero constant. Therefore we can decompose the rational function $w(\lambda)$ into partial fractions to get

$$w(\lambda) = \sum_{k=1}^{p} \sum_{j=1}^{m_k} \frac{\beta_{kj}}{(\lambda - \lambda_k)^j}, \tag{9.16}$$

where

$$\beta_{kj} = \frac{1}{(m_k - j)!} \lim_{\lambda \to \lambda_k} \frac{d^{m_k - j}}{d\lambda^{m_k - j}} \left[(\lambda - \lambda_k)^{m_k} \frac{Q_N(\lambda)}{P_N(\lambda)} \right] \tag{9.17}$$

are called the normalizing numbers of the matrix J.

The collection of the eigenvalues and normalizing numbers

$$\{\lambda_k, \ \beta_{kj} \ (j = 1, \dots, m_k; \ k = 1, \dots, p)\}, \tag{9.18}$$

of the matrix J of the form (9.1), (9.2) is called the *spectral data* of this matrix.

Determination of the spectral data of a given Jacobi matrix is called the *direct spectral problem* for this matrix.

Thus, the spectral data consist of the eigenvalues and associated normalizing numbers derived by decomposing the resolvent function (Weyl–Titchmarsh function) $w(\lambda)$ into partial fractions using the eigenvalues.

It follows from (9.15) by (9.10) that $\lambda w(\lambda)$ tends to 1 as $\lambda \to \infty$. Therefore multiplying (9.16) by λ and passing then to the limit as $\lambda \to \infty$, we find that

$$\sum_{k=1}^{p} \beta_{k1} = 1. \tag{9.19}$$

The *inverse spectral problem* for spectral data is stated as follows:

(a) Is the matrix J determined uniquely by its spectral data?
(b) To indicate an algorithm for the construction of the matrix J from its spectral data.
(c) To find necessary and sufficient conditions for a given collection (9.18) to be the spectral data for some matrix J of the form (9.1) with entries from class (9.2).

This problem was solved by the author in [9] and we will present here the final result.

Let us set

$$s_l = \sum_{k=1}^{p} \sum_{j=1}^{m_k} \binom{l}{j-1} \beta_{kj} \lambda_k^{l-j+1}, \quad l = 0, 1, 2, \dots , \tag{9.20}$$

where $\binom{l}{j-1}$ is a binomial coefficient and we put $\binom{l}{j-1} = 0$ if $j-1 > l$. Next, using these numbers s_l, we introduce the determinants

$$D_n = \begin{vmatrix} s_0 & s_1 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_n & s_{n+1} & \cdots & s_{2n} \end{vmatrix}, \quad n = 0, 1, 2, \dots . \tag{9.21}$$

Let us bring two important properties of the determinants D_n in the form of two lemmas.

Lemma 9.1. *For any collection (9.18), for the determinants D_n defined by (9.21), (9.20), we have $D_n = 0$ for $n \geq N$, where $N = m_1 + \ldots + m_p$.*

Proof. Given a collection (9.18), define a linear functional Ω on the linear space of all polynomials in λ with complex coefficients as follows: if $G(\lambda)$ is a polynomial then the value $\langle \Omega, G(\lambda) \rangle$ of the functional Ω on the element (polynomial) G is

$$\langle \Omega, G(\lambda) \rangle = \sum_{k=1}^{p} \sum_{j=1}^{m_k} \beta_{kj} \frac{G^{(j-1)}(\lambda_k)}{(j-1)!}, \tag{9.22}$$

where $G^{(n)}(\lambda)$ denotes the n-th order derivative of $G(\lambda)$ with respect to λ. Let $m \geq 0$ be a fixed integer and set

$$T(\lambda) = \lambda^m (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_p)^{m_p}$$

$$= t_m \lambda^m + t_{m+1} \lambda^{m+1} + \ldots + t_{m+N-1} \lambda^{m+N-1} + \lambda^{m+N}. \tag{9.23}$$

Then, according to (9.22),

$$\left\langle \Omega, \lambda^l T(\lambda) \right\rangle = 0, \quad l = 0, 1, 2, \dots . \tag{9.24}$$

Consider (9.24) for $l = 0, 1, 2, \ldots, N+m$, and substitute (9.23) in it for $T(\lambda)$. Taking into account that

$$\left\langle \Omega, \lambda^l \right\rangle = s_l, \quad l = 0, 1, 2, \ldots\ , \tag{9.25}$$

where s_l is defined by (9.20), we get

$$t_m s_{l+m} + t_{m+1} s_{l+m+1} + \ldots + t_{m+N-1} s_{l+m+N-1} + s_{l+m+N} = 0,$$

$$l = 0, 1, 2, \ldots, N+m.$$

Therefore $(0, \ldots, 0, t_m, t_{m+1}, \ldots, t_{m+N-1}, 1)$ is a nontrivial solution of the homogeneous system of linear algebraic equations

$$x_0 s_l + x_1 s_{l+1} + \ldots + x_m s_{l+m} + x_{m+1} s_{l+m+1} + \ldots + x_{m+N-1} s_{l+m+N-1}$$

$$+ x_{m+N} s_{l+m+N} = 0, \quad l = 0, 1, 2, \ldots, N+m,$$

with the unknowns $x_0, x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+N-1}, x_{m+N}$. Therefore, the determinant of this system, which coincides with D_{N+m}, must be zero. □

Lemma 9.2. *If collection (9.18) is the spectral data of the matrix J of the form (9.1) with entries belonging to the class (9.2), then for the determinants D_n defined by (9.21), (9.20), we have $D_n \neq 0$ for $n \in \{0, 1, \ldots, N-1\}$.*

Proof. We have

$$D_0 = s_0 = \sum_{k=1}^{p} \beta_{k1} = 1 \neq 0$$

by (9.19). Consider now D_n for $n \in \{1, \ldots, N-1\}$. For any $n \in \{1, \ldots, N-1\}$ let us consider the homogeneous system of linear algebraic equations

$$\sum_{k=0}^{n} g_k s_{k+m} = 0, \quad m = 0, 1, \ldots, n, \tag{9.26}$$

with unknowns $g_0, g_1, \ldots, g_n$. The determinant of system (9.26) coincides with the D_n. Therefore, to prove $D_n \neq 0$, it is sufficient to show that system (9.26) has only a trivial solution. Assume the contrary: let (9.26) have a nontrivial solution $\{g_0, g_1, \ldots, g_n\}$. For each $m \in \{0, 1, \ldots, n\}$ take an arbitrary complex number h_m. Multiply both sides of (9.26) by h_m and sum the resulting equation over $m \in \{0, 1, \ldots, n\}$ to get

$$\sum_{m=0}^{n} \sum_{k=0}^{n} h_m g_k s_{k+m} = 0.$$

Substituting expression (9.25) for s_{k+m} in this equation and denoting

$$G(\lambda) = \sum_{k=0}^{n} g_k \lambda^k, \quad H(\lambda) = \sum_{m=0}^{n} h_m \lambda^m,$$

we obtain

$$\langle \Omega, G(\lambda)H(\lambda)\rangle = 0. \tag{9.27}$$

Since $\deg G(\lambda) \le n$, $\deg H(\lambda) \le n$ and the polynomials $P_0(\lambda), P_1(\lambda), \ldots, P_n(\lambda)$ form a basis (their degrees are different) of the linear space of polynomials of degree $\le n$, we have expansions

$$G(\lambda) = \sum_{k=0}^{n} c_k P_k(\lambda), \quad H(\lambda) = \sum_{k=0}^{n} d_k P_k(\lambda).$$

Substituting these in (9.27) and using the orthogonality relations (see [9])

$$\langle \Omega, P_m(\lambda)P_n(\lambda)\rangle = \delta_{mn}, \quad m,n \in \{0,1,\ldots,N-1\},$$

where δ_{mn} is the Kronecker delta [at this place we use the condition that collection (9.18) is the spectral data for a matrix J of the form (9.1), (9.2)], we get

$$\sum_{k=0}^{n} c_k d_k = 0.$$

Since the polynomial $H(\lambda)$ is arbitrary, we can take $d_k = \overline{c_k}$ in the last equality and get that $c_0 = c_1 = \ldots = c_n = 0$, that is, $G(\lambda) \equiv 0$. But this is a contradiction and the proof is complete. □

The solution of the above inverse problem is given by the following theorem (see [9]):

Theorem 9.3. *Let an arbitrary collection (9.18) of numbers be given, where $1 \le p \le N$, $m_1, \ldots, m_p$ are positive integers with $m_1 + \ldots + m_p = N$, $\lambda_1, \ldots, \lambda_p$ are distinct complex numbers. In order for this collection to be the spectral data for a Jacobi matrix J of the form (9.1) with entries belonging to the class (9.2), it is necessary and sufficient that the following two conditions be satisfied:*

(i) $\sum_{k=1}^{p} \beta_{k1} = 1$;
(ii) $D_n \neq 0$, *for* $n \in \{1,2,\ldots,N-1\}$, *where D_n is the determinant defined by (9.21), (9.20).*

Under the conditions (i) and (ii) the entries a_n and b_n of the matrix J for which the collection (9.18) is spectral data are recovered by the formulae

$$a_n = \frac{\pm\sqrt{D_{n-1}D_{n+1}}}{D_n}, \quad n \in \{0,1,\ldots,N-2\},\ D_{-1} = 1, \tag{9.28}$$

$$b_n = \frac{\Delta_n}{D_n} - \frac{\Delta_{n-1}}{D_{n-1}}, \quad n \in \{0,1,\ldots,N-1\},\ \Delta_{-1} = 0,\ \Delta_0 = s_1, \tag{9.29}$$

where D_n is defined by (9.21), (9.20) and Δ_n is the determinant obtained from the determinant D_n by replacing in D_n the last column by the column with the components $s_{n+1}, s_{n+2}, \ldots, s_{2n+1}$.

It follows from the above solution of the inverse problem that the matrix (9.1) is not uniquely restored from the spectral data. This is linked with the fact that the a_n are determined from (9.28) uniquely up to a sign. To ensure that the inverse problem is uniquely solvable, we have to specify additionally a sequence of signs $+$ and $-$. Namely, let $\{\sigma_0, \sigma_1, \ldots, \sigma_{N-2}\}$ be a given finite sequence, where for each $n \in \{0, 1, \ldots, N-2\}$, the σ_n is $+$ or $-$. We have 2^{N-1} such different sequences. Now to determine a_n uniquely from (9.28) for $n \in \{0, 1, \ldots, N-2\}$ we can choose the sign σ_n when extracting the square root. In this way we get precisely 2^{N-1} distinct Jacobi matrices possessing the same spectral data. The inverse problem is solved uniquely from the data consisting of the spectral data and a sequence $\{\sigma_0, \sigma_1, \ldots, \sigma_{N-2}\}$ of signs $+$ and $-$. Thus, we can say that the inverse problem with respect to the spectral data is solved uniquely up to signs of the off-diagonal elements of the recovered Jacobi matrix.

9.3 Inverse Problem for Two Spectra

Let J be an $N \times N$ Jacobi matrix of the form (9.1) with entries satisfying (9.2). Denote by $\lambda_1, \ldots, \lambda_p$ all the distinct eigenvalues of the matrix J and by $m_1, \ldots, m_p$ their multiplicities, respectively, as the roots of the characteristic polynomial $\det(J - \lambda I)$ so that $1 \leq p \leq N$, $m_1 + \ldots + m_p = N$, and

$$\det(\lambda I - J) = (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_p)^{m_p}. \tag{9.30}$$

Further, let J_1 be the $(N-1) \times (N-1)$ truncated matrix obtained from J by (9.7). Denote by $\mu_1, \ldots, \mu_q$ all the distinct eigenvalues of the matrix J_1 and by $n_1, \ldots, n_q$ their multiplicities, respectively, as the roots of the characteristic polynomial $\det(J_1 - \lambda I)$ so that $1 \leq q \leq N-1$, $n_1 + \ldots + n_q = N-1$ and

$$\det(\lambda I - J_1) = (\lambda - \mu_1)^{n_1} \cdots (\lambda - \mu_q)^{n_q}. \tag{9.31}$$

The collections

$$\{\lambda_k,\ m_k\ (k = 1, \ldots, p)\} \quad \text{and} \quad \{\mu_k,\ n_k\ (k = 1, \ldots, q)\} \tag{9.32}$$

form the spectra (together with their multiplicities) of the matrices J and J_1, respectively. We call these collections the two spectra of the matrix J.

The inverse problem about two spectra consists in the reconstruction of the matrix J by its two spectra.

In this section, we reduce the inverse problem for two spectra to the inverse problem for the spectral data consisting of the eigenvalues and normalizing numbers solved above in Sect. 9.2 and show in this way that the complex Jacobi matrix is determined from the two its spectra uniquely up to signs of the off-diagonal elements of the matrix.

First let us study some necessary properties of the two spectra of the Jacobi matrix J.

Let $P_n(\lambda)$ and $Q_n(\lambda)$ be the polynomials of the first and second kind for the matrix J. By (9.11) we have

$$\det(J-\lambda I)=(-1)^N a_0 a_1\cdots a_{N-2}P_N(\lambda), \tag{9.33}$$

$$\det(J_1-\lambda I)=(-1)^{N-1} a_0 a_1\cdots a_{N-2}P_{N-1}(\lambda). \tag{9.34}$$

Note that we have used the fact that $a_{N-1}=1$. Therefore the eigenvalues $\lambda_1,\ldots,\lambda_p$ and $\mu_1,\ldots,\mu_q$ of the matrices J and J_1 and their multiplicities coincide with the zeros and their multiplicities of the polynomials $P_N(\lambda)$ and $P_{N-1}(\lambda)$, respectively.

Dividing both sides of (9.12) by $P_{N-1}(\lambda)P_N(\lambda)$ gives

$$\frac{Q_N(\lambda)}{P_N(\lambda)}-\frac{Q_{N-1}(\lambda)}{P_{N-1}(\lambda)}=\frac{1}{P_{N-1}(\lambda)P_N(\lambda)}.$$

Therefore, by formula (9.15) for the resolvent function $w(\lambda)$, we obtain

$$w(\lambda)=\frac{Q_{N-1}(\lambda)}{P_{N-1}(\lambda)}+\frac{1}{P_{N-1}(\lambda)P_N(\lambda)}. \tag{9.35}$$

Lemma 9.4. *The matrices J and J_1 have no common eigenvalues, that is, $\lambda_k\neq\mu_j$ for all values of k and j.*

Proof. Suppose that λ is a common eigenvalue of the matrices J and J_1. Then by (9.33) and (9.34), we have $P_N(\lambda)=P_{N-1}(\lambda)=0$. But this is impossible by (9.12). □

The following lemma gives a formula for calculating the normalizing numbers β_{kj} $(j=1,\ldots,m_k;\ k=1,\ldots,p)$ in terms of the two spectra.

Lemma 9.5. *For each $k\in\{1,\ldots,p\}$ and $j\in\{1,\ldots,m_k\}$ the formula*

$$\beta_{kj}=\frac{a}{(m_k-j)!}\lim_{\lambda\to\lambda_k}\frac{d^{m_k-j}}{d\lambda^{m_k-j}}\frac{1}{\prod\limits_{l=1,l\neq k}^{p}(\lambda-\lambda_l)^{m_l}\prod\limits_{i=1}^{q}(\lambda-\mu_i)^{n_i}} \tag{9.36}$$

holds, where

$$\frac{1}{a}=\sum_{k=1}^{p}\frac{1}{(m_k-1)!}\lim_{\lambda\to\lambda_k}\frac{d^{m_k-1}}{d\lambda^{m_k-1}}\frac{1}{\prod\limits_{l=1,l\neq k}^{p}(\lambda-\lambda_l)^{m_l}\prod\limits_{i=1}^{q}(\lambda-\mu_i)^{n_i}}. \tag{9.37}$$

Proof. Substituting (9.16) in the left-hand side of (9.35) we can write

$$\sum_{k=1}^{p}\sum_{j=1}^{m_k}\frac{\beta_{kj}}{(\lambda-\lambda_k)^j}=\frac{Q_{N-1}(\lambda)}{P_{N-1}(\lambda)}+\frac{1}{P_{N-1}(\lambda)P_N(\lambda)}.$$

Hence, taking into account that $P_{N-1}(\lambda_k) \neq 0$, we get

$$\beta_{kj} = \frac{1}{(m_k-j)!} \lim_{\lambda\to\lambda_k} \frac{d^{m_k-j}}{d\lambda^{m_k-j}} \left[(\lambda-\lambda_k)^{m_k} \left(\frac{Q_{N-1}(\lambda)}{P_{N-1}(\lambda)} + \frac{1}{P_{N-1}(\lambda)P_N(\lambda)} \right) \right]$$
$$= \frac{1}{(m_k-j)!} \lim_{\lambda\to\lambda_k} \frac{d^{m_k-j}}{d\lambda^{m_k-j}} \left[(\lambda-\lambda_k)^{m_k} \frac{1}{P_{N-1}(\lambda)P_N(\lambda)} \right]. \tag{9.38}$$

Next, by (9.30), (9.31), (9.33), and (9.34), we have

$$(-1)^N a_0 a_1 \cdots a_{N-2} P_N(\lambda) = \prod_{l=1}^{p} (\lambda_l - \lambda)^{m_l},$$

$$(-1)^{N-1} a_0 a_1 \cdots a_{N-2} P_{N-1}(\lambda) = \prod_{i=1}^{q} (\mu_i - \lambda)^{n_i}.$$

Substituting these in the right-hand side of (9.38), we obtain

$$\beta_{kj} = \frac{a}{(m_k-j)!} \lim_{\lambda\to\lambda_k} \frac{d^{m_k-j}}{d\lambda^{m_k-j}} \frac{1}{\prod\limits_{l=1, l\neq k}^{p} (\lambda-\lambda_l)^{m_l} \prod\limits_{i=1}^{q} (\lambda-\mu_i)^{n_i}}, \tag{9.39}$$

where

$$a = (a_0 a_1 \cdots a_{N-2})^2.$$

Thus (9.36) is proved. Next, putting $j = 1$ in (9.39) and then summing this equation over $k = 1, \ldots, p$ and taking into account (9.19), we get (9.37). The lemma is proved. □

Theorem 9.6. *(Uniqueness Result). The two spectra in (9.32) determine the matrix J uniquely up to signs of the off-diagonal elements of J.*

Proof. Given the two spectra in (9.32) we uniquely determine the normalizing numbers β_{kj} of the matrix J by (9.36), (9.37). Since the inverse problem for the spectral data (9.18) is solved uniquely up to signs of the off-diagonal elements of the recovered matrix (see Theorem 9.3), the proof is complete. □

The procedure of reconstruction of the matrix J from the two spectra consists in the following: If we are given the two spectra in (9.32), we find the quantities β_{kj} from (9.36), (9.37) and then solve the inverse problem with respect to the spectral data

$$\{\lambda_k,\ \beta_{kj}\ (j = 1, \ldots, m_k;\ k = 1, \ldots, p)\}$$

to recover the matrix J by using formulae (9.28) and (9.29).

9.4 Solving of the Toda Lattice

The (open) *finite Toda lattice* is a nonlinear Hamiltonian system which describes the motion of N particles moving in a straight line, with "exponential interactions". Adjacent particles are connected by strings. Let the positions of the particles at time t be $q_0(t)$, $q_1(t)$, $\dots$, $q_{N-1}(t)$, where $q_n = q_n(t)$ is the displacement at the moment t of the n-th particle from its equilibrium position. We assume that each particle has mass 1. The momentum of the n-th particle at time t is therefore $p_n = \dot{q}_n$. The Hamiltonian function is defined to be

$$H = \frac{1}{2}\sum_{n=0}^{N-1} p_n^2 + \sum_{n=0}^{N-2} e^{q_n - q_{n+1}}.$$

The Hamiltonian system

$$\dot{q}_n = \frac{\partial H}{\partial p_n}, \quad \dot{p}_n = -\frac{\partial H}{\partial q_n}$$

becomes

$$\dot{q}_n = p_n, \quad n = 0, 1, \dots, N-1,$$
$$\dot{p}_0 = -e^{q_0 - q_1},$$
$$\dot{p}_n = e^{q_{n-1} - q_n} - e^{q_n - q_{n+1}}, \quad n = 1, 2, \dots, N-2,$$
$$\dot{p}_{N-1} = e^{q_{N-2} - q_{N-1}},$$

where the dot denotes differentiation with respect to t. Let us set

$$a_n = \frac{1}{2} e^{(q_n - q_{n+1})/2}, \quad n = 0, 1, \dots, N-2,$$

$$b_n = -\frac{1}{2} p_n, \quad n = 0, 1, \dots, N-1.$$

Then the above system can be written in the form

$$\dot{a}_n = a_n(b_{n+1} - b_n),\ \dot{b}_n = 2(a_n^2 - a_{n-1}^2), \quad n = 0, 1, \dots, N-1, \tag{9.40}$$

with the boundary conditions

$$a_{-1} = a_{N-1} = 0. \tag{9.41}$$

If we define the $N \times N$ matrices J and A by

$$J=\begin{bmatrix} b_0 & a_0 & 0 & \cdots & 0 & 0 & 0 \\ a_0 & b_1 & a_1 & \cdots & 0 & 0 & 0 \\ 0 & a_1 & b_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b_{N-3} & a_{N-3} & 0 \\ 0 & 0 & 0 & \cdots & a_{N-3} & b_{N-2} & a_{N-2} \\ 0 & 0 & 0 & \cdots & 0 & a_{N-2} & b_{N-1} \end{bmatrix}, \tag{9.42}$$

$$A=\begin{bmatrix} 0 & -a_0 & 0 & \cdots & 0 & 0 & 0 \\ a_0 & 0 & -a_1 & \cdots & 0 & 0 & 0 \\ 0 & a_1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -a_{N-3} & 0 \\ 0 & 0 & 0 & \cdots & a_{N-3} & 0 & -a_{N-2} \\ 0 & 0 & 0 & \cdots & 0 & a_{N-2} & 0 \end{bmatrix}, \tag{9.43}$$

then the system (9.40) with the boundary conditions (9.41) is equivalent to the Lax equation

$$\frac{d}{dt}J=[J,A]=JA-AJ. \tag{9.44}$$

The system (9.40), (9.41) is considered subject to the initial conditions

$$a_n(0)=a_n^0,\ b_n(0)=b_n^0, \quad n=0,1,\dots,N-1, \tag{9.45}$$

where a_n^0, b_n^0 are given complex numbers such that $a_n^0\neq 0$ $(n=0,1,\dots,N-2)$, $a_{N-1}^0=0$.

In this section we present a procedure for solving the problem (9.40), (9.41), (9.45) by the method of inverse spectral problem.

Let $\{a_n(t),b_n(t)\}$ be a solution of (9.40), (9.41) and $J=J(t)$ be the Jacobi matrix defined by this solution according to (9.42). In [16] it is shown that then the eigenvalues of the matrix $J(t)$, as well as their multiplicities, do not depend on t; however, the normalizing numbers β_{kj} of the matrix $J(t)$ depend on t and for the normalizing numbers $\beta_{kj}(t)$ $(j=1,\dots,m_k;\ k=1,\dots,p)$ of the matrix $J(t)$ the following time evolution holds:

$$\beta_{kj}(t)=\frac{e^{2\lambda_k t}}{S(t)}\sum_{s=j}^{m_k}\beta_{ks}(0)\frac{(2t)^{s-j}}{(s-j)!}, \tag{9.46}$$

where

$$S(t)=\sum_{k=1}^{p}e^{2\lambda_k t}\sum_{j=1}^{m_k}\beta_{kj}(0)\frac{(2t)^{j-1}}{(j-1)!}. \tag{9.47}$$

Therefore we get the following procedure for solving the problem (9.40), (9.41), (9.45). We construct from the initial data $\{a_n(0),\ b_n(0)\}$ the Jacobi matrix

$$J(0)=\begin{bmatrix} b_0(0) & a_0(0) & 0 & \cdots & 0 & 0 & 0 \\ a_0(0) & b_1(0) & a_1(0) & \cdots & 0 & 0 & 0 \\ 0 & a_1(0) & b_2(0) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b_{N-3}(0) & a_{N-3}(0) & 0 \\ 0 & 0 & 0 & \cdots & a_{N-3}(0) & b_{N-2}(0) & a_{N-2}(0) \\ 0 & 0 & 0 & \cdots & 0 & a_{N-2}(0) & b_{N-1}(0) \end{bmatrix}$$

and determine its spectral data

$$\{\lambda_k,\ \beta_{kj}(0)\ (j=1,\dots,m_k, k=1,\dots,p)\}.$$

Then we calculate for each $t \geq 0$ the numbers $\beta_{kj}(t)$ dependent on t by (9.46), (9.47). Finally, solving the inverse spectral problem with respect to

$$\{\lambda_k,\ \beta_{kj}(t)\ (j=1,\dots,m_k, k=1,\dots,p)\},$$

we construct a Jacobi matrix $J(t)$. The entries $\{a_n(t),\ b_n(t)\}$ of the matrix $J(t)$ give a solution of problem (9.40), (9.41), (9.45). We can write the explicit expressions for $a_n(t)$, $b_n(t)$ through the moments

$$s_l(t)=\sum_{k=1}^{p}\sum_{j=1}^{m_k}\binom{l}{j-1}\beta_{kj}(t)\lambda_k^{l-j+1}, \quad l=0,1,2,\dots,$$

using (9.21), (9.28), (9.29) (s_l in them should be replaced by $s_l(t)$).

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Chapter 10
To Approximate Solution of Ordinary Differential Equations, I

Tamaz S. Vashakmadze

Abstract This article is dedicated to approximate solution of two-point boundary value problems for linear and nonlinear normal systems of ordinary differential equations. We study problems connected with solvability, construction of high order finite difference and finite sums schemes, error estimation and investigate the order of arithmetic operations for finding approximate solutions. Corresponding results refined and generalized well-known classical achievements in this field.

10.1 Introduction: Nonlinear ODE of 2nd Order with Dirichlet Conditions

We consider the problem of approximate solution (AOS) of two-point boundary value problems (BVPs) for ordinary differential equations (ODEs) by using multipoint finite-difference method. Let us divide BVP into two classes. We include in the first class the problems satisfying the *Banach–Picard–Schauder* conditions and in the second class those satisfying the maximum principle. We remark that the basic apparatus are special spline-functions (named as$(P),(Q)$ formulae which are the high-order finite elements too) and *Cesáro–Stieltjes*-type method of finite sums (see [4]). These results for first class of BVP refined and generalized the corresponding results of *Shröder* [2], *Collatz, Berezin, and Jidkov, and Quarteroni et al., Buthcher and Stetter*, having first order of convergence and arithmetic operations for finding of AOS of $O(n^2 \log n)$. First order of convergence with respect to n, where n is the number of subintervals, has the multiple shooting method (*Keller, Osborne, Bulirsch*), but the order of AOS is not less than $O(n^2)$. For the

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second class, the corresponding results which are cited in the classical textbooks of *Collats, Henrici, Keller, Richtmaier, Engel-Miugler and Router, Berezin and Jidkov, Marchuk, Kantorovich and Krilov, Strang and Fix, de Boor* and recent manuals (e.g., of *Quarteroni and Buthcher and Stetter, Bulirsch and Stoer, Ascher et al.*) may be formulated in the following form: by finite-difference or FEM methods, the AOSs converge to exact solutions with no more than fourth order with respect to mesh width and order of AOS is $O(1/\ h)$. Further the high-order-accuracy three-point schemes were obtained by *Tikhonov and Samarski, and Volkov*. The constructions of these models contain unstable processes and the orders of AOSs are no less than two because an employment of multipoint formulae of numerical differentiation is necessary for them.

For the first class of BVPs we proved the following statement:

Theorem 10.1. *The order of arithmetic operations for calculation of AOS and its derivative of BVP for nonlinear second-order DE or for system with two equations of normal form with Sturm–Liouville boundary conditions is O(n log n) Horner unit. The convergence of the AOS and its derivative has (p - 1) order with respect to mesh width h = 1/n if exact solution y(x) has (p + 1) order continuously differentiable derivatives. If the order is less than p, the remainder member of corresponding scheme has the best constant in Sard's sense.*

Proof. For the proof of this theorem, let us consider second-order nonlinear ODE

$$y''(x) = f(x, y(x), y'(x))(0 < x < 1), \tag{10.1}$$

for simplicity with the following boundary conditions:

$$y(0) = \alpha, y(1) = \beta. \tag{10.2}$$

Let us consider a uniform step $h = (2ks)^{-1}$ $(p = 2s+1 = 3,5,7)$ or Gauss' $(p = 2s+3)$ partion of interval [0,1]. In the last case subinterval $(x_{(t-1)(s+1)+1}, x_{t(s+1)+1})$ $(t = 1,2,\ldots,2k-1)$ are divided into s parts such that knots are s degree Legendre polynomial zeroes distributed here.

Now we use the approach of [5] (Chap. 3, point 13.1), which represents the summary analogous of Green formula in the netpoints expressions presenting linear form with respect to $y''(x_i)$, reminder terms and $y(0)$ and $y(1)$. Introducing artificial parameter z, which is equal to s for a uniform mesh and $s+1$ if the mesh is Gaussian, we have the following general representation:

$$y_{(t-1)z+i} = \alpha_{(t-1)z+i}y(0) + \beta_{(t-1)z+i}y(1) + \sigma_{(t-1)z+i}, \tag{10.3}$$

when $t = 1,2,\ldots,2k-2$ then $i = 2,3,\ldots,z+1$; when $t = 2k-1$ then $i = 2,3,\ldots,2z$. Here

$$\alpha_{(t-1)(s+1)+i} = \frac{2k - 2kx_i - t + 1}{2k}, \beta_{(t-1)(s+1)+i} = \frac{2kx_i + t - 1}{2k},$$

$$\Phi(x) \equiv y''(x) - R''_{2z}(x; y),$$

$$\sigma_{(t-1)z+i} = (1-kx_i)\sigma_{(t-1)z+1} + kx_i\sigma_{(t+1)z+1} + \sum_{j=2}^{2z} b_{ij}\left(y''_{(t-1)z+j} - R''_{2z}\left(x_{(t-1)z+j};y\right)\right),$$

$$\sigma_{tz+1} = \frac{t}{k}\sigma_{kz+1} + \frac{t}{k-1}\Sigma^{[k-1]} + \ldots.. + \frac{t}{t+1}\Sigma^{[t+1]} + \Sigma^{[t]}, t<k,$$

$$\sigma_{tz+1} = \frac{2k-t}{k}\sigma_{kz+1} + \frac{2k-t}{k-1}\Sigma^{[k+1]} + \ldots.. + \frac{2k-t}{2k-t+1}\Sigma^{[t-1]} + \Sigma^{[t]}, t>k,$$

$$\sigma_{kz+1} = \sum_{t=1}^{k-1} t\sum_{j=2}^{2z} b_{z+1,j}\Phi_{(t-1)z+j} + k\sum_{j=2}^{2s} b_{z+1,j}\Phi_{(t-1)z+j} + \sum_{t=1}^{k-1} t\sum_{j=2}^{2s} b_{z+1,j}\Phi_{(2k-1-t)z+j},$$

$$\Sigma^{[t]} = \frac{2}{t+1}\sum_{i=1}^{t} i\sum_{j=2}^{2z} b_{z+1,j}\Phi_{(i-1)z+j}, \Sigma^{[2k-t]} = \frac{2}{t+1}\sum_{i=1}^{t} i\sum_{j=2}^{2z} b_{z+1,j}\Phi_{(2k-i-1)z+j}.$$

These formulae are equivalence of the following recurrence relations:

$$y_{kz+1} = \frac{1}{2}y(0) + \frac{1}{2}y(1) + \sigma_{kz+1}; \tag{10.4}$$

$$y_{tz+1} = \frac{t}{t+1}y_{(t+1)z+1} + \frac{1}{t+1}y(0) + \Sigma^{[t]},\ t<k; \tag{10.5}$$

$$y_{tz+1} = \frac{2k-t}{2k-t+1}y_{(t-1)z+1} + \frac{1}{2k-t+1}y(1) + \Sigma^{[t]},\ t>k; \tag{10.6}$$

$$y_{(t-1)z+i} = (1-kx_i)y_{(t-1)z+1} + kx_iy_{(t+1)z+1} + \sum_{j=2}^{2z} b_{ij}\Phi_{(t-1)z+j}. \tag{10.7}$$

If we neglect the remainder members in these (10.3) or (10.4)–(10.7) expressions it is possible to use immediately a simple iteration method. But here arisen two problems:

1. Let us define high-accuracy scheme for slopes too; if we use the high-order schemes of numerical differentiations, these processes are unstable in ordinary sense.
2. Construction of the following approximation by scheme (10.3) having an AOS of $O(n\ln n)$, $(n = 2kz)$.

Let us investigate the first problem. For this we used in the interval $(x_{(t-1)z+1}, x_{(t+1)z+1})$ by (Q)-special splines ([4], p. 160) and formula (10.4)–(10.7) which give

$$y'_{(k-1)z+i} = k[y_{(k+1)z+1} - y_{(k-1)z+1}] - k\sum_{j=2}^{2z} c_{ij}y''_{(k-1)z+j} - \rho_{2z-2,(k-1)z+i}, \tag{10.8}$$

where c_{ij} are known coefficients, $\rho_{2z-2,(k-1)z+i}$ are the remainder members and $i = 1,2,\ldots,2z+1$. (10.8) contains the first-order differences, for which from (10.4)–(10.7) immediately follows

$$y_{(k+1)z+1} - y_{(k-1)z+1} =$$
$$\frac{1}{k}\left[y(1) - y(0)\right] + \frac{2}{k}\sum_{r=1}^{k-1} r \sum_{j=2}^{2z} b_{z+1,j}\left[\Phi_{(2k-r-1)z+j} - \Phi_{(r-1)z+j}\right]. \tag{10.9}$$

The construction of (10.8)-type formula corresponding to $x_{tz=i} \in [0,1]/(x_{(t-1)z+1}$,
$x_{(t+1)z+1})$ netpoints is easy, but they define unstable processes. In this connection let us consider two Cauchy problems:

$$y_1'(x) = f(x,\lambda(x),y_1(x)), l_1 \le x \le 1, \tag{10.10}$$
$$y_1(l_1) = \gamma;$$
$$y_1'(x) = f(x,\mu(x),y_1(x)), l_2 \ge x \ge 0, \tag{10.11}$$
$$y_1(l_2) = \delta,$$

where $x_{(k-1)z+1} = l_1, x_{k-1)z+1} = l_2, y'(x) = y_1(x)$.

Let us consider (10.3) or (10.4)–(10.7) and (10.8) and to initial value problems (10.10) and (10.11). Such expressions approximate the (10.1) and (10.2) BVP. For remainder vector we have explicit expressions. Now we choose the AOS and slopes (ApS and Sl), so let us have two sequences $(y_1^{[0]}, y_2^{[0]}, \ldots, y_{2kz}^{[0]}, y_{2kz+1}^{[0]})^T$ and $(y_1'^{[0]}, y_2'^{[0]}, \ldots, y_{2kz}'^{[0]}, y_{2kz+1}'^{[0]})^T$, by which using (10.4)–(10.7) and (10.8), we find the first approximations of ApS in each netpoints of (0,1) and slopes on the netpoints of interval $(x_{(t-1)z+1}, x_{(t+1)z+1})$. To define slopes in the other netpoints, we use *Hermite–Gauss* numerical processes [4] for initial value problems of (10.10) and (10.11); the functions $\lambda(x)$, $\mu(x)$ in discrete points will be same with ApS defined by the first approximation; for the initial table for slopes, we use the expressions (10.8) without the reminder terms. Continuing this process we shall find $y_i^{[m]}, y_i'^{[m]}$, $m = 2,3,\ldots$.

Now we consider the problem (ii). It is evident that from schemes (10.3) the calculation of each y_{tz+1} value request AOS of order $O(n)$ multiplications, as well as for all y_{tz+1}, is equal to $O\left(n^2\right)$. We will have different results if we use (10.4)–(10.7) schemes. They represent the recurrence-type relations and for approximate calculation of any ordinate corresponding to central points x_{tz+1}request no more than five operations, because we know that sums $\forall \sum^{[t]}$, $t \ne k$ are subsumes of σ_{kz+1}. For other ordinates if we apply (10.7), they request no more than $2z+1$ AOS. Since the process of solution is realized for finite z and variable k, AOS of finding AOS for each step of iteration is $O(n) = O(k)$. The same results are true while finding the slopes corresponding to netpoints $x_{(k-1)z+i}$. This fact is stipulated by (10.8). For Cauchy problems (10.10) (10.11) when high-order finite-difference method is used for them, it is evident that the order of AOS is $O(n)$.

The remainder member of corresponding scheme has the best constant in Sard's sense. For simplicity and clearness let $p=5$, and then when $i=3$, main part of the (P)-formulae, we have

$$y(2h)=\frac{1}{2}(y(0)+y(4h))-\frac{1}{2}\int_0^{2h} t\left(y''(t)+y''(4h-t)\right)dt,$$

$$\int_0^{2h} t\left(y''(t)+y''(4h-t)\right)dt=\int_0^{2h}\Phi(t)dt=$$

$$\frac{4h^2}{3}\left[y''(h)+y''(2h)+y''(3h)\right]+\int_0^{2h}F_r(t)\Phi^{(r)}(t)dt,$$

where $F_r(t)$ is a well-known piecewise polynomial of degree r and corresponds to Simpson's rule in the interval $(0,2h)$ (according to Sard's technology). Then, for $i=3$, from (P)-type formulae follows Sard-type best constant estimation of arbitrary $r\le p+1$. It is evident that this scheme is typical and the same results are true for all (P) formulae corresponding to central netpoints for uniform $p\le 7$ or all $z\ge 3$ for Gaussian grid. □

The process of defining by representations (10.3)–(10.9) for AOS of (10.1)–(10.2) is realized by two independent parallel procedure, as well as for definition of ordinates for noncentral points $x_{tz+i}, i\neq 1$ may be used by computers with k multiprocessors. The slope finding processes are automatically parallel procedure as Cauchy problems (10.10) and (10.11) must be solved in different intervals $(x_{(k+1)z+1},0)$[1] and $(x_{(k-1)z+1},1)$. The initial table for slopes is defined by (10.8) on $2z+1$ knots.

10.2 Linear 2nd Order ODE of Self-adjoint Type

According to [4], for the numerical solution of BVP,

$$-(Au+qu)=\frac{d}{dt}\left(k(t)u'(t)\right)-q(t)\cdot u(t)=f(t),\ k>0, q\ge 0, 0<t<1,\quad (10.12)$$

$$u(0)-k_1u'(0)=\alpha, u(1)+k_2u'(1)=\beta,(k_i\ge 0).\qquad (10.13)$$

The method of any order of accuracy, depending on the order of the smoothness of the unknown solution $u(t)$, will be given below. These numerical schemes are contained as particular case of the corresponding results presented in [1, Chap. 2, point 2.2].

Preliminarily we shall put the auxiliary formula. They are generalized (P) and (Q) formulae of [4, Sect. 13.1]. Thus we suppose that $u(t)\in C^{(p+1)}(0,1)$, $p=2s+1$.

[1] Here, please note that the notation $(x_{(k+1)z+1},0)$ underlines that the corresponding Cauchy problem solved from the initial point $x_{(k+1)z+1}$ to the point zero.

(P) formulae have a form (the notation here and below are borrowed from [4, Sect. 13]):

$$u(t_i) = \alpha_i^{p,1}(k)\,u(t_1) + \beta_i^{p,1}(k)\,u(t_{i+s}) - \sum_{j=2}^{p-1} b_{ij}^{p,1}(k)\left[Au(t_j) - R_{p-1}(t_i)\right], \quad (10.14)$$

where

$$b_{i,j}^{p,1}(\tau) = \frac{1}{\tau_p - \tau_1}\left[(\tau_p - \tau_1)\int_{\tau_1}^{\tau_i} dt \int_{\tau_1}^{t} l_j(t)\,dt - (\tau_i - \tau_1)\int_{\tau_1}^{\tau_p} dt \int_{\tau_1}^{t} l_j(t)\,dt\right],$$

$$i,j = 2,3,\dots,p-1,\ \tau_i = \int_{t_i}^{t_p} k^{-1}(t)dt,$$

$$\alpha_i^{p,1}(k) = \left(\int_{t_1}^{t_p} k^{-1}(t)\,dt\right)^{-1} \cdot \int_{t_i}^{t_p} k^{-1}(t)\,dt, \beta_i^{p,1}(k) = 1 - \alpha_i^{p,1}(k),$$

$$l_j(t) = \prod_{\substack{i=2\\ j\neq i}}^{p-1} \frac{t - t_i}{t_j - t_i}.$$

(Q) formulae are presented as follows:

$$\begin{aligned} u'(t_i) =& \gamma_i^{p,1}(k)\left[u(t_p) - u(t_1)\right] + \\ & \sum_{j=2}^{p-1} c_{i,j}^{p,1}(k)\,Au(t_j) + \int_{\tau_1}^{\tau_p} dt \int_{\tau_i}^{t} k(t)\,Ar_{p-3}(t)\,dt,\ i = 1,2,\dots,p, \end{aligned} \quad (10.15)$$

$$c_{i,j}^{p,1}(\tau) = \int_{\tau_1}^{\tau_p} dt \int_{\tau_i}^{t} l_j(t)\,dt, (i = 1,2,\dots,p, j = 2,3,\dots,p-1),$$

$$\gamma_i^{p,1}(k) = 1/\left(\tau_1 k(t_i)\right).$$

Now let ω_h designate the net area as: $\omega_h = \{0 = t_1, t_2, \dots, t_n, t_{n+1} = 1;$ $h_i = t_i - t_{i-1}\}$. As bounding points of the net ω_h we shall name those t_i knots, for which $i \le s+1$ or $i \ge n-s+1$. For relation (10.14) for bounding points it follows that

$$\begin{aligned} u(t_i) =& \alpha_i^{i+s,1}(k)\,u(0) + \beta_i^{i+s,1}(k)\,u(t_{i+s}) - \\ & \sum_{j=2}^{2s} b_{i,j}^{i+s,1}(k)\,k(t_j)\,Au(t_j) + O\left(h^{2s+1}\right), i \le s+1; \end{aligned} \quad (10.16)$$

$$u(t_i) = \alpha_i^{n+1,i-s}(k)\,u(t_{i-s}) + \beta_i^{n+1,i-s}(k)\,u(1) - \sum_{j=2}^{2s} b_{i,j}^{n+1,i-s}(k)\,k(t_j)\,Au(t_j) + O\left(h^{2s+1}\right). \tag{10.17}$$

The above relations permit to receive expressions of the following form:

$$u(t_i) = \frac{\alpha_i}{1+k_1\gamma_1}\left[u(0) - k_1 u'(0)\right] + \frac{\beta_i + k_1\gamma_1}{1+k_1\gamma_1} u(t_{i+s}) - \sum_{j=2}^{2s}\left(b_{i,j}^{i+s,1}(k) - \frac{k_1\alpha_i}{1+k_1\gamma_1} c_{i,j}\right) k(t_j)\,Au(t_j) + O\left(h^{2s+1}\right), i \le s+1, \tag{10.18}$$

$$u(t_i) = \frac{\beta_i}{1+k_2\gamma_{n+1}}\left[u(1) + k_2 u'(1)\right] + \frac{\alpha_i + k_2\gamma_{n+1}}{1+k_2\gamma_{n+1}} u(t_{i-s}) -$$

$$\sum_{j=n-2s+1}^{n}\left(b_{i,j}^{i+s,1}(k) - \frac{k_2\beta_i\gamma_{n+1}}{1+k_2\gamma_{n+1}} c_{n+1,j}\right) k(t_j)\,Au(t_j) + O\left(h^{2s+1}\right), i \ge n-s+1.$$

Here and below, in the coefficients, the top indexes and the dependence of factors on the function $k(t)$ are omitted. In addition the designation $h = \max_i (t_{i+1} - t_i)$ is introduced. A feature of the formulae (10.18) is that the right parts contain the same expression (from conditions (10.14)), as data of initial problem. Obviously, the approach of construction of the formulae of a type (10.18) allows generalization for other conditions.

Let $t_i - t_{i-j} = t_{i+j} - t_i$, $(s+2 \le i \le n-s)$. Then the residual member of the formula (10.14) allows
the valuation:

$$\left|\sum_{j=i-s+1}^{i+s-1} b_{i,j}(k)\,AR_{p-1}(t_j)\right| < c_1 M_{p+1} h^{p+1}, M_{p+1} = \max_{(0,1)}\left|u^{(p+1)}(t)\right|.$$

For interior knots $t_i \in \omega_h$ from expression (10.14) follows:

$$u(t_i) = \alpha_i u(t_{i-s}) + \beta_i u(t_{i+s}) - \sum_{j=i-s+1}^{i+s-1} b_{i,j}(k)\,Au + O\left(h^{2s+2}\right), i \ge n-s+1. \tag{10.19}$$

If now in the formulae (10.19) we replace the expression Au by $qu+f$ and then omit the remainder term, we shall obtain algebraic system of linear equations, the solution of which shall be designated through u_i, $(i = 2,3,\ldots,n)$. The matrix appropriate to this system is a multi-diagonal matrix depending on s. For the solution of such systems it is easy to apply the classical factorization method, which is done below.

For convenience we shall rewrite the system of equations concerning the values u_i, received from (10.18), as:

$$u_i = \frac{\beta_i + k_1\gamma_1}{1+k_1\gamma_1} u_{i+s} + \sum_{j=2}^{2s} d_{ij} u_j + F_i,\ i = 2,3,\dots,s+1, \tag{10.20}$$

$$u_i = \alpha_i u_{i-s} + \beta_i u_{i+s} + \sum_{j=i-s+1}^{i+s-1} d_{ij} u_j + F_i,\ i = s+2,\dots,n-s, \tag{10.21}$$

$$u_i = \frac{\alpha_i + k_2\gamma_{n+1}}{1+k_2\gamma_{n+1}} u_{i-s} + \sum_{j=n-2s+1}^{n} d_{ij} u_j + F_i,\ i = n-s+1,\dots,n, \tag{10.22}$$

where, for example,

$$F_i = \frac{\alpha_i}{1+k_1\gamma_1}\alpha + \sum_{j=2}^{2s} d_{ij} f(t_j), i \le s+1.$$

The first s of the formulae give the following recurrence expression:

$$u_i = A_i u_{i+s} + \sum_{\substack{j=i+1 \\ j\neq i+s}}^{2s} A_{ij} u_j + B_i,\ i = 2,3,\dots,s+1, \tag{10.23}$$

where

$$A_{ij} = \frac{e_{ij}}{1-e_{ij}},\ j = i+1,\dots,2s,\ j \neq i+s,$$

$$A_i = A_{i,i+s} = \frac{\beta_i + k_1\gamma_1}{(1-e_{ii})(1+k_1\gamma_1)},$$

$$e_{ii} = d_{ij} + \sum_{k=2}^{i-1} d_{ik} \sum_{l=k}^{i-1} A_{ij} \prod_{m=k}^{l-1} A_{m,m+1}, \prod_{m=k}^{l-1} \cdot = 1,\ k > l-1, \tag{10.24}$$

$$B_i = \frac{F_i + \sum_{k=2}^{i-1} d_{ik} \sum_{l=k}^{i-1} B_l \prod_{m=k}^{l-1} A_{m,m+1}}{1-e_{ii}}, i = 2,3,\dots,s+1.$$

Let i be the number of any internal point of the net areaω_h. Then from expressions (10.20)–(10.24) follows:

$$u_i = A_i u_{i+s} + \sum_{j=i+1}^{i+s-1} A_{ij} u_j + B_j, i = s+2,\dots,n-s, \tag{10.25}$$

where

$$A_{ij} = \frac{e_{ij}}{1-e_{ij}}, j = i+1,\dots,2s, j \neq i+s,$$

$$A_i = A_{i,i+s} = \frac{\beta_i}{(1-e_{ii})}$$

$$e_{ii} = d_{ij} + \sum_{k=i-s+1}^{i-1} d_{ik} \sum_{l=k}^{i-1} A_{lj} \prod_{m=k}^{l-1} A_{m,m+1} + d_i A_{i-s,j}, \tag{10.26}$$

$$B_i = \frac{F_i + \sum_{k=i-s+1}^{i-1} d_{ik} \sum_{l=k}^{i-1} B_l \prod_{m=k}^{l-1} A_{m,m+1} + d_i B_{i-s}}{1-e_{ii}}, i = 2,3,\ldots,s+1.$$

The values $u_i, i = n-s+1,\ldots,n$ satisfy the following equalities:

$$u_i = \sum_{j=i+1}^{i+s-1} A_{ij} u_j + B_i, i = n-s+1,\ldots,n-1, \tag{10.27}$$

where

$$A_{ij} = \frac{e_{ij}}{1-e_{ij}}, j \neq i+s, A_i = A_{i,i+s} = \frac{\beta_i}{(1-e_{ii})} \tag{10.28}$$

$$e_{ii} = d_{ij} + \sum_{k=i-s+1}^{i-1} d_{ik} \sum_{l=k}^{i-1} A_{lj} \prod_{m=k}^{l-1} A_{m,m+1} + \frac{\alpha_i + k_2 \gamma_{n+1}}{1+k_2 \gamma_{n+1}} A_{i-s,j},$$

$$B_i = \frac{F_i + \sum_{k=i-s+1}^{i-1} d_{ik} \sum_{l=k}^{i-1} B_l \prod_{m=k}^{l-1} A_{m,m+1} + \frac{\alpha_i + k_2 \gamma_{n+1}}{1+k_2 \gamma_{n+1}} B_{i-s}}{1-e_{ii}}.$$

At last, the value u_n is defined explicitly:

$$u_n = B_n, \tag{10.29}$$

$$B_n = \frac{F_n + \frac{\alpha_n + k_2 \gamma_{n+1}}{1+k_2 \gamma_{n+1}} B_{n-s} + \sum_{k=n-s+1}^{n-1} d_{nk} \sum_{l=k}^{n-1} B_l \prod_{m=k}^{l-1} A_{m,m+1}}{1-e_{nn}},$$

$$e_{nn} = d_{nn} + \sum_{k=n-s+1}^{n-1} d_{nk} \sum_{l=k}^{n-1} A_{\ln} \prod_{m=k}^{l-1} A_{m,m+1} + \frac{\alpha_n + k_2 \gamma_{n+1}}{1+k_2 \gamma_{n+1}} A_{n-s,n}. \tag{10.30}$$

Let α_i and β_i satisfy the following bilateral inequalities:

$$\frac{1}{s} < \beta_i, \alpha_{n+1-i} < \frac{1}{2},\ i = 2,3,\ldots,s+1,$$

$$1 - c_3 h^2 < \alpha_i \beta_i^{-1} < 1 + c_4 h^2,\ c_3, c_4 > 0,\ i = s+2,\ldots,n-s,$$

where $c_3\, and\, c_4$ are constants. Obviously, these inequalities are true with the appropriate choice of h [see expressions for α and β in (10.14)]. Then from (10.24), (10.28), and (10.30) follows:

$$A_{ij} < (1 - c_5 h) \max_{i \le s+1} \{A_i, A_{s+2}\},\ B_i < c_6 \alpha + c_7 \beta + c_8 \max_j \left| f(t_j) \right|, \tag{10.31}$$

where nonnegative constants c_5, c_6, c_7, and c_8 do not depend on h.

The conditions (10.31) are the definition of stability of computing process by formulae (10.25) and (10.27) concerning to initial data and the right part accordingly. The stability of process (10.23), (10.25) and (10.27) for calculation of values u_i is also obvious, as the appropriate operator corresponding to these expressions is an operator of compression. From the above-stated formulae it follows that the method of generalized factorization is optimum, as the number of arithmetic operations necessary for calculation of AOS u_i is directly proportional to the number of points of the net area ω_h.

Finally we remark that when $k(t) \equiv 1$, $p = 3$ we have the classical cases. For $p = 4, 5$ and $k(t) \equiv 1$ these schemes are different from well-known Strang and Fix unstable FEMs [3].

10.3 Nonlinear ODE of 2nd Order with Newton's Conditions

Now we consider more general case when we have nonlinear differential equation with Newton's boundary conditions

$$u''(x) = f(x, u(x), u'(x)), 0 < x < 1, \; -M < u, \; u' < M, \tag{10.32}$$

$$k_1 u(0) - u'(0) = \alpha, k_2 u(1) + u'(1) = \beta, k_1^2 + k_2^2 > 0, \, (k_i \geq 0)\,. \tag{10.33}$$

Here we use some expressions from the first two parts and construct one parametrical class of schemes which are equivalence of BVP (10.32)–(10.33).
Let us give the partition of [0,1] as a uniform if $z \leq 8$ and arbitrary z if a grid is Gaussian. Then for the central knots we have

$$u_{tz+1} = \frac{1}{2} u_{(t-1)z+1} + \frac{1}{2} u_{(t+1)z+1} + A_t, \;\; t = 2, 3, \ldots, k-1, \tag{10.34}$$

where $A_t = \sum_{j=2}^{2z} b_{z+1,j} u''_{(t-1)z+j} + O\left(h_{z-s}^{p+1}\right)$.
By using formulae of type (10.18) we have

$$u_{z+1} = \frac{1}{2} \frac{1}{k+k_2} (k_1 u(0) - u'(0)) + \frac{1}{2} \frac{2k+k_1}{k+k_1} u_{2z+1} + A_1, \tag{10.35}$$

$$u_{(2k-1)z+1} = \frac{1}{2} \frac{1}{k+k_2} (k_2 u(1) + u'(1)) + \frac{1}{2} \frac{2k+k_2}{k+k_2} u_{(2k-2)z+1} + A_{2k-1}, \tag{10.36}$$

where

$$A_1 = \sum_{j=2}^{2z} (b_{z+1,j} - k^2 \frac{x_{z+1}}{k+k_1} c_{1,j}) u''_j + O\left(h_{z-s}^{p+1}\right),$$

$$A_{2k-1} = \sum_{i=2(k-1)z+2}^{2kz} \left(b_{z+1,j} - k^2 \frac{x_{z+1}}{k+k_2} c_{2z+1} \right) u''_j + O\left(h_{z-s}^{p+1}\right).$$

Now if we multiply the expressions (10.34) by unknown numbers $\alpha_i (i = 1, \dots, 2k-1)$ and select these numbers so that the following ratios were executed:

$$u_{kz+1} = \frac{2+k_2}{2(k_1+k_2+k_1k_2)}\alpha + \frac{2+k}{2(k_1+k_2+k_1k_2)}\beta + \sigma_{kz+1}, \tag{10.37}$$

$$\sigma_{kz+1} = \frac{1}{k_1+k_2+k_1k_2}\Bigg[(2+k_2)\Big((k+k_1)A_1 + \sum_{i=2}^{k-1}(2k+ik_1)A_i + \frac{1}{2}(2+k_1)A_k\Big) + (2+k_1)\Big(\frac{1}{2}(2+k_2)A_k + \sum_{i=2}^{k-1}(2k+ik_2)A_{2k-i} + (k+k_2)A_{2k-1}\Big)\Bigg]$$

$$u_{tz+1} = (2k+(t+1)k_1)^{-1}\alpha + (2k+(t+1)k_1)^{-1}(2k+tk_1)u_{(t+1)z+1} + \Sigma^{[t]},$$

where

$$\Sigma^{[t]} = 2(2k+(t+1)k_1)^{-1}\left[(k+k_1)A_1 + \sum_{i=2}^{t}(2k+ik_1)A_i\right],\ t = \overline{1,k-1}$$

$$u_{(2k-t)z+1} = \frac{\beta}{2k+(t+1)k_2} + \frac{2k+k_1}{2k+(t+1)k_1}u_{(2k-t+1)z+1} + \Sigma^{[2k-t]}, t = \overline{1,k-1}$$

$$\Sigma^{[2k-t]} = \frac{2}{2k+(t+1)k_2}\left[(k_1+k_2)A_{2k-1} + \sum_{i=2}^{t}(2k+ik_2)A_{i2k-i}\right], t = \overline{1,2k-1}.$$

From expressions (10.37), after some calculations, follows

$$u_{tz+1} = \frac{2k+(2k-t)k_2}{2k(k_1+k_2+k_1k_2)}\alpha + \frac{2k+tk_1}{2k(k_1+k_2+k_1k_2)}\beta + \sigma_{tz+1}, t = \overline{1,2k-1}, \tag{10.38}$$

where

$$\sigma_{tz+1} = \frac{2k+tk_1}{2k+kk_1}\sigma_{kz+1} + \sum_{j=t}^{k-1}\frac{2k-tk_1}{2k+jk_1}\Sigma^{[j]},$$

$$\sigma_{(2k-1)z+1} = \frac{2k+tk_2}{2k+kk_2}\sigma_{kz+1} + \sum_{j=t}^{k-1}\frac{2k+tk_2}{2k+jk_2}\Sigma^{[2k-j]}.$$

Now from (10.14) and (10.38) for $u_{(t-1)z+i} (i \neq z+1)$ follows

$$u_{(t-1)z+i} = \frac{2k+(2k-2kx_i-t+1)k_2}{2k(k_1+k_2+k_1k_2)}\alpha + \frac{2k+(2kx_i+t-1)k_1}{2k(k_1+k_2+k_1k_2)}\beta + \sigma_{(t-1)z+i}, \tag{10.39}$$

where

$$\sigma_{(t-1)z+i} = (1-kx_i)\sigma_{(t-1)z+1} + kx_i\sigma_{(t+1)z+1} + \sum_{j=2}^{2s} b_{ij}\Phi_{(t-1)z+n},$$

$$t = \overline{2,2k-1},\ i = \overline{2,z+1}.$$

If we use the formulae of type

$$u_i = k\frac{x_{2z+1}-x_i}{k+k_1}\alpha + k\frac{1+x_ik_1}{k+k_1}y_{2z+1} + \sum_{j=2}^{2s}\left(b_{ij} - k^2\frac{x_{2z+1}-x_i}{k+k_1}c_{ij}\right)u_j'' + O(h_{z-s}^p)$$

$$u_{2kz+1-i} = k\frac{x_{2z+1}}{k+k_2}\beta + k\frac{1+x_ik_2}{k+k_2}y_{(2k-1)z+1} +$$

$$\sum_{j=2(k-1)z+2}^{2s}\left(b_{2z+2,j} + k^2\frac{x_{2z+1}-x_i}{k+k_2}c_{2z+1j}\right)u_j'' + O(h_{z-s}^p)$$

for boarding points $x_i, 1-x_i$, $(i=\overline{2,z})$ analogously to the last formulae we will have

$$u_i = \frac{1+(1-x_i)k_2}{k_1+k_2+k_1k_2}\alpha + \frac{1+x_ik_1}{k_1+k_2+k_1k_2}\beta + \sigma_i, \tag{10.40}$$

$$u_{2kz+1-i} = \frac{1+x_ik_2}{2(k_1+k_2+k_1k_2)}\alpha + \frac{1+(1-x_i)k_1}{2(k_1+k_2+k_1k_2)}\beta + \sigma_{2kz+1-i}, \tag{10.41}$$

where

$$\sigma_i = \frac{k+x_ikk_1}{k+k_1}\sigma_{2z+1} + \sum_{j=2}^{2z}\left(b_{ij} - k_2\frac{x_{2z+1}-x_i}{k+k_1}c_{ij}\right)u_j'' + O(h_{z-s}^p),$$

$$\sigma_{2kz+1-i} = \frac{k+x_ikk_2}{k+k_2}\sigma_{2(k-1)z+1} +$$

$$\sum_{j=2(k-1)z+2}^{2kz}\left(b_{2z+2-i,j} + k^2\frac{x_{2z+1}-x_i}{k+k_2}c_{2z+1,j}\right)u_j'' + O(h_{z-s}^p).$$

The expressions (10.38)–(10.41) are difference analogue of Green's function for any arbitrary (fixed) degree of exactness with respect to mesh wide. To these expressions, the finite-difference-type formulae with respect to slopes should be added. As in the first part we use the same scheme for slopes and in this case we have

$$u'_{(k-1)z+i} = \frac{k_1\beta - k_2\alpha}{k_1+k_2+k_1k_2} + \sigma'_{(k-1)z+i}[f], \tag{10.42}$$

$$\sigma'_{(k-1)z+i} = \frac{2}{(k_1+k_2+k_1k_2)}\Bigg\{k_1\Bigg[(2+k_{21})A_k + \sum_{i=2}^{k-1}(2k+ik_2)A_{2k-i}+$$

$$(k+k_2)A_{2k-1}\Bigg] - k_2\left[\frac{1}{2}k(2+k_1)A_k + \sum_{i=2}^{k-1}(2k+ik_1)A_i + (k+k_1)A_1\right]\Bigg\} -$$

$$k\sum_{j=2}^{2z} c_{ij}u''_{(k-1)z+j} + O(h^{p-1}),$$

and as above the Cauchy problems:

$$u'_1(x) = f(x,\ \lambda(x),\ u_1(x)), l_1 \le x \le 1, u_1(l_1) = \gamma, l_1 = x_{(k-1)z+1},$$

$$u'_1(x) = f(x,\ \mu(x),\ u_1(x)), l_2 \ge x \ge 0, u_1(l_2) = \delta, l_2 = x_{(k+1)z+1}.$$

Now we return to study the problem (10.32)–(10.33) and introduce the following values:

$$\omega_1 = \frac{1}{8} + \frac{1}{4\,(k_1+k_2+k_1k_2)}\left(4+k_1+k_2+\frac{(k_2-k_1)^2}{k_1+k_2+k_1k_2}\right),$$

$$\omega_2 = \frac{1}{2\,(k_1+k_2+k_1k_2)}\,(k_1k_2 + 2\max\{k_1,k_2\}), \tag{10.43}$$

$$\omega'_2 = \frac{1}{2} - \frac{k_1k_2}{4\,(k_1+k_2+k_1k_2)}, \omega = \max\left\{\omega_2, \omega'_2\right\}.$$

The above expressions of this part and the methodology of first part give possibility to prove the truthiness of following theorems:

Theorem 10.2. *Let the function $f(x, u(x), u'(x))$ be continuous with respect to x and satisfy a Lipschitz's condition relative to u and u' with constants L and L', respectively; in addition, let one of two conditions be executed:*

$$\omega\left(L+L'\right) < 1, \omega_1 L + \omega_2 L' < 1. \tag{10.44}$$

Then the initial problem has the unique solution which can be constructed by an iterative method.

Proof. The proof of this theorem coincides with the scheme of the proofs of Theorems 13.2 and 13.3 in [4]. □

Now in the formulae of the type (10.38)–(10.42), we omit the remainder terms. We get the expressions for construction of the initial table. We shall replace the Cauchy problem by the multistage methods. We shall name the resulting system as the difference scheme.

Theorem 10.3. *For the problem (10.32)–(10.33), let one of the conditions (10.44) be true. Then:*

1) The difference scheme has a unique solution and the iteration method converges.
2) As in the case of the uniform grid $(p = 3, 5, 7)$ and in the case of Gaussian grid $(p > 3)$, convergence of the solution of algebraic analogue to the solution of a problem (10.1)–(10.2) and its derivative have $(p-1)$-degree with respect to h.

Proof. The proof of this theorem coincides with the scheme of the proof of Theorem 13.4 in [4]. □

Theorem 10.4. *The number of arithmetic operations which is necessary for the calculation of AOS $\overline{u}(x)$ and its derivative $\overline{u'}(x)$ is $O(k \ln k)$.*

Proof. A proof of this theorem as in the first part is based on the specific character of sums σ_{tz+1}. If we calculate σ_{kz+1}, then σ_{tz+1} $\forall t \neq k$ will be calculated, as it is contained in σ_{kz+1} as subsums. □

10.4 The BVPs of Normal Type System of ODEs

Now we consider the BVP for system of DEs of normal form:

$$y'(x) = f(x, y(x)), y = (y_1, y_2, \ldots, y_{2m})^T, 0 < x < 1, \tag{10.45}$$

with boundary conditions:

$$y_i(0) = l_i[y(0)] + \alpha_i, i = 1, 2, \ldots n, y_i(1) = l_{n+i}[y(1)] + \beta_i,\ i = 1, 2, \ldots, 2m-n, \tag{10.46}$$

where $I - l_i$and $I - l_{i+n}$ are the matrix operators with ranks of n and $2m - n$, respectively. Below we consider the case when the BVP (10.45)–(10.46) belongs to the first class, satisfying the *Banach–Picard–Schauder*-type conditions.

Let us consider the following three problems:

$$\begin{gathered} y'(x) = f(x, y(x)), 0 < x < l, y_i(0) = l_i[y(0)] + \alpha_i, \\ y_{n+j}(0) = \bar{\alpha}_j (j = 1, 2, \ldots, 2m-n; \end{gathered} \tag{10.47}$$

$$y'(x) = f(x, y(x)),\ l < x < 1-l,\ y_i(l) = a_i,\ y_i(1-l) = b_i,\ \ i = 1, 2, \ldots, 2m; \tag{10.48}$$

$$\begin{gathered} y'(x) = f(x, y(x)), 1-l < x < 1, y_i(1) = l_{n+i}[y(1)] + \beta_i, \\ y_j(1) = \bar{\beta}_j, j = 2m-n+1, \ldots, 2m. \end{gathered} \tag{10.49}$$

Here we count that $0 < l < 1/2$; numbers $\bar{\alpha},\ \bar{\beta},\ a,\ b$ will be defined below when we consider the processes of investigation of above problems.

The scheme of AOS of these problems by iteration is such: we solve at first (i_-), (i_+) as Cauchy problems (s is the number of iterations):

$$y'^{[s]}(x) = f\left(x, y^{[s]}(x)\right),\ 0 < x < l,$$

$$y_i^{[s]}(0) = l_i[y(0)] + \alpha_i,\ y_{n+j}^{[s]}(0) = \bar{\alpha}_j,\ \ j = 1, 2, \ldots, 2m - n;$$

$$y'^{[s]}(x) = f\left(x, y^{[s]}(x)\right),\ 1 - l < x < 1,$$

$$y_i^{[s]}(1) = l_{n+i}[y^{[s]}(1)] + \beta_i,\ y_j^{[s]}(1) = \bar{\beta}_j,\ j = 2m - n + 1, \ldots, 2m.$$

Then we solve the following BVP:

$$y'^{[s]}(x) = f\left(x, y^{[s]}(x)\right),\ l < x < 1 - x,$$

$$y_i^{[s]}(l) = a_i,\ y_i^{[s]}(1 - l) = b_i,\ \ i = 1, 2, \ldots, 2m,$$

by which we define new initial values in the points $l,\ 1 - l$ and solve two Cauchy problems into intervals $(l, 0)$, $(1 - l, 1)$; we denote these solutions as $y^{[s+1]}(x),\ l > x \geq 0$ and $y^{[s+1]}(x),\ 1 - l < x \leq 1$. By these values we define the following iteration relative to conditions (10.46):

$$y_i^{[s+2]}(0) = l_i[y^{[s+1]}(0)] + \alpha_i, y_{n+i}^{[s+2]}(0) = y_{n+i}^{[s+1]}(0);$$

$$y_i^{[s+2]}(1) = l_{n+i}[y^{[s+1]}(1)] + \beta_i,\ y_j^{[s+2]}(1) = y_j^{[s+1]}(1),$$

solve again two Cauchy problems in intervals $(0, l)$, $(1, 1 - l)$ and so on.

For solution of the BVP type (ii) we construct the scheme which will be same with the method which we considered in parts 1 and 3.

Let us separate $(l,\ 1 - l)$ interval into $2k$ subintervals, each of them we divide into z parts. Thus, as above we have : $x_{tz+i+1} = x_{tz+i} + h_i,\ t = 0, 1, 2, \ldots, 2k - 1,\ i = 1, 2, \ldots, z,\ h_i > 0$ are mesh widths. In such case from (ii) follows

$$\begin{aligned} y\left(x_{tz+1}\right) = \frac{1}{2}\left(y\left(x_{(t-1)z+1}\right) + y\left(x_{(t+1)z+1}\right)\right) + \\ + \frac{1}{2}\sum_{j=2}^{z} b_{z+1,j}\left(f(x_{(t-1)z+j}, y\left(x_{(t-1)z+j}\right) - f(x_{tz+j}, y\left(x_{tz+j}\right)\right) + O\left(h^{2z-2}\right) \end{aligned} \tag{10.50}$$

if $z \geq 5, h = \max h_i$,$b_{z+1,J}$are weights of quadrature formulae *Gauss* or *Clenshaw–Curtis* type. When $z \leq 4$ more preferable ones are a uniform grid and trapezoid, *Simpson's* and of 3/8 rules (*Newton–Cotes* formula with four nodes) as so we have

$$y\left(x_{t+1}\right) = \frac{1}{2}\left(y\left(x_t\right) + y\left(x_{t+2}\right)\right) + \frac{h}{2}\left(f(x_t, y\left(x_t\right) - f(x_{t+2}, y\left(x_{t+2}\right)\right) + O(h^4),$$

$$y(x_{2t+1}) = \tfrac{1}{2}\left(y(x_{2t-1}) + y(x_{2t+3})\right) +$$
$$\tfrac{h}{3}\left(f_{2t-1} - f_{2t+3} + 4(f_{2t} - f_{2t+2})\right) + O\left(h^6\right), \tag{10.51}$$

$$y(x_{3t+1}) = \tfrac{1}{2}\left(y(x_{3t-2}) + y(x_{3t+4})\right) +$$
$$\tfrac{3h}{8}\left(f_{3t-2} - f_{3t+4} + 3(f_{3t-1} + f_{3t} - f_{3t+2} - f_{3t+3})\right) + O\left(h^6\right).$$

As we see the expressions (10.50) and (10.51) have a form by which it is easy to construct direct relations of type (10.3) or recurrence expressions of kind (10.4)–(10.6). Instead of the formula (10.7) for *Gaussian* grid we have

$$y(x_{tz+i}) = \frac{1}{2}\left(y\left(x_{(t-1)z+1}\right) + y\left(x_{(t+1)z+1}\right)\right) +$$
$$\frac{1}{2}\sum_{j=1}^{2z+1} b_{ij}\left(f(x_{(t-1)z+j}, y\left(x_{(t-1)z+j}\right)\right) + O\left(h^{2z+2}\right),$$

where $b_{ij}, i = 1, 2, \ldots, z$ are weights of quadrature formulae following from expressions of type (10.50) with respect to knots x_{tz+i}. Similarly, as early as we saw in parts 1 or 3, the AOS of finding ApS for each step of iteration would be $O(k)$.

When the grid is uniform for all points, it is possible to construct relations of type (10.4)–(10.6) by enlarging the network in left and right sides no more than at two knots.

With respect to problems of solvability, error estimation, and convergence we must study the members of σ_{tz+i} type [see (10.4)–(10.7)]. For decision of this question typical and most important is the case when $t = k$. If we denote here the corresponding member as early as we have

$$\sigma_{kz+1} = \frac{1}{2}\sum_{t=1}^{k-1} t(A_{t+1} + A_{2k-t+1}) + kB_{k+1},$$
$$B_t = \frac{1}{2}\sum_{j=2}^{z} b_{z+1,j}\left(f\left(x_{(t-1)z+j}, y\left(x_{(t-1)z+j}\right)\right) - f\left(x_{tz+j}, y(x_{tz+j})\right)\right). \tag{10.52}$$

If we consider (10.52) for differences $\delta f = f(x, y(x)) - f(t, y(t)$ in the parallelepiped $D = (l, i-l) \times \prod_{i=1}^{2m}(-Y < y_i < Y)$, we have:

$$|\delta f_i| \le \frac{1}{2k}M, M = \max_i \sup_D \left[\left|\frac{\partial f_i}{\partial x}\right| + \sum_{j=1}^{2m}\left|f_j\frac{\partial f_i}{\partial y_j}\right|\right] \text{ and } |\sigma_{kz+1}| \le \frac{1-2l}{8}M.$$

We underline that when $l = 0.5$, the scheme of AOS of the BVP is almost the same with well-known simple shooting method.

10.5 Remark

In [6] with respect to numerical solution of Cauchy problem basing on applications of *Gauss* and *Clenshaw–Curtis type* quadratures and *Hermite* interpolation we formulated that the new (*Adam's* type) multistep finite-difference schemes converge as $O(h^{2n})$ for any finite integer n and they are absolutely stable if the matrices of nodes are normal types in *Fejer's* sense. The creation of corresponding schemes and proof of these results are described in the article "To Approximate Solution of Ordinary Differential Equations, II" (in appear).

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Chapter 11
A Hybrid Method for Inverse Scattering Problem for a Dielectric

Ahmet Altundag

Abstract The inverse problem under consideration is to reconstruct the shape of a homogeneous dielectric infinite cylinder from the far field pattern for scattering of a time-harmonic E-polarized electromagnetic plane wave. We propose an inverse algorithm that extends the approach suggested by Kress [14] and further investigated by Kress and Serranho [17, 18] and Serranho [22] for the case of the inverse problem for a perfectly conducting scatterer. It is based on a system of nonlinear boundary integral equations associated with a single-layer potential approach to solve the forward scattering problem. We present the mathematical foundations of the method and exhibit its feasibility by numerical examples.

11.1 Introduction

In inverse obstacle scattering problems for time-harmonic waves, the scattering object is a homogeneous obstacle and the inverse problem is to obtain an image of the scattering object, i.e., an image of the shape of the obstacle from a knowledge of the scattered wave at large distances, i.e., from the far-field pattern. In this paper we deal with dielectric scatterers and confine ourselves to the case of infinitely long cylinders.

Assume that the simply connected bounded domain $D \subset \mathbb{R}^2$ with C^2 boundary ∂D represents the cross section of a dielectric infinite cylinder having constant wave number k_d with $\operatorname{Re} k_d > 0$ and $\operatorname{Im} k_d \geq 0$ embedded in a homogeneous background with positive wave number k_0. Denote by ν the outward unit normal to ∂D. Then, given an incident plane wave $u^i(x) = e^{ik_0 x \cdot d}$ with incident direction given by the unit vector d, the direct scattering problem for E-polarized electromagnetic waves

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is modeled by the following transmission problem for the Helmholtz equation: Find solutions $u \in H^1_{\text{loc}}(\mathbb{R}^2 \setminus \bar{D})$ and $v \in H^1(D)$ to the Helmholtz equations

$$\triangle u + k_0^2 u = 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{D}, \quad \triangle v + k_d^2 v = 0 \quad \text{in } D \tag{11.1}$$

satisfying the transmission conditions

$$u = v, \quad \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad \text{on } \partial D \tag{11.2}$$

in the trace sense such that $u = u^i + u^s$ with the scattered wave u^s fulfilling the Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{1/2} \left(\frac{\partial u^s}{\partial r} - ik_0 u^s \right) = 0, \quad r = |x|, \tag{11.3}$$

uniformly with respect to all directions. The latter is equivalent to an asymptotic behavior of the form

$$u^s(x) = \frac{e^{ik_0|x|}}{\sqrt{|x|}} \left\{ u_\infty\left(\frac{x}{|x|}\right) + O\left(\frac{1}{|x|}\right) \right\}, \quad |x| \to \infty, \tag{11.4}$$

uniformly in all directions, with the far-field pattern u_∞ defined on the unit circle S^1 in $\mathbb{R}^2$ (see [4]). In the above, u and v represent the electric field that is parallel to the cylinder axis, (11.1) corresponds to the time-harmonic Maxwell equations and the transmission conditions (11.2) model the continuity of the tangential components of the electric and magnetic field across the interface ∂D.

The inverse obstacle problem we are interested in is given the far-field pattern u_∞ for one incident plane wave with incident direction $d \in S^1$ to determine the boundary ∂D of the scattering dielectric D. More generally, we also consider the reconstruction of ∂D from the far-field patterns for a small finite number of incident plane waves with different incident directions. This inverse problem is nonlinear and ill-posed, since the solution of the scattering problem (11.1)–(11.3) is nonlinear with respect to the boundary and since the mapping from the boundary into the far-field pattern is extremely smoothing.

At this point we note that uniqueness results for this inverse transmission problem are only available for the case of infinitely many incident waves (see [11]). A general uniqueness result based on the far-field pattern for one or finitely many incident waves is still lacking. More recently, a uniqueness result for recovering a dielectric disk from the far-field pattern for scattering of one incident plane wave was established by Altundag and Kress [2].

For a stable solution of the inverse transmission problem we propose an algorithm that extends the approach suggested by Kress [14] and further investigated by Kress and Serranho [17, 18] and Serranho [22] for the case of the inverse problem for a perfectly conducting scatterer. Representing the solution v and u^s to the forward scattering problem in terms of single-layer potentials in D and in $\mathbb{R}^2 \setminus \bar{D}$ with densities φ_d and φ_0, respectively, the transmission condition (11.2) provides a system of two boundary integral equations on ∂D for the corresponding densities that in the sequel we will denote as field equations. For the inverse problem, the required coincidence of the far field of the single-layer potential representing u^s and the given far field u_∞ provides a further equation that we denote as data equation. The system of the field and data equations can be viewed as three equations for three unknowns, i.e., the two densities and the boundary curve. They are linear with respect to the densities and nonlinear with respect to the boundary curve.

In the spirit of [14, 17, 18], given a current approximation ∂D_{approx} for the unknown boundary ∂D. In a first step, the ill-posed data equation can be regularized via Tikhonov regularization and one of the density can be solved on ∂D_{approx}. Then in a second step, keeping the density fixed we can solve the other density from one of the field equation. In a third step, keeping the densities fixed we linearize the remaining field equation with respect to boundary ∂D. In a fourth step, the solution of the ill-posed linearized equation can be utilized to update the boundary approximation. Because of the ill-posedness the solution of this update equation requires stabilization, for example, by Tikhonov regularization. These four steps can be iterated until some suitable stopping criterion is satisfied.

We also consider the inverse problem for the physical parameter such as reconstructing the interior wave number k_d. The direct and inverse problem proceed the same line as the shape reconstruction with difference that the unknown boundary ∂D is replaced by interior wave number k_d.

In principle, one can also think of linearizing both the field and the data equations simultaneously with respect to the densities and the boundary curve. Such a full linearization of a corresponding system for the perfect conductor boundary condition has been considered by Ivanyshyn and Kress [8]. For a recent survey on the connections of the different approaches Ivanyshyn and Johansson [7] and Ivanyshyn, Kress and Serranho [9]. For related work for the Laplace equation we refer to Kress and Rundell [16] for the Dirichlet boundary condition and Eckel and Kress [5], Hohage and Schormann [6], Altundag and Kress [2] for the transmission condition. Finally, for a recent survey on the hybrid method see Kress [14], Kress and Serranho [17, 18] and Serranho [22].

The plan of the paper is as follows: In Sect. 11.2, as ingredient of our inverse algorithm we describe the solution of the forward scattering problem via a single-layer approach followed by a corresponding numerical solution method in Sect. 11.3. The details of the inverse algorithm are presented in Sect. 11.4, and in Sect. 11.5 we demonstrate the feasibility of the method by some numerical examples.

11.2 The Direct Problem

The forward scattering problem (11.1)–(11.3) has at most one solution (see [3, 15] for the three-dimensional case). Existence can be proven via boundary integral equations by a combined single- and double-layer approach (see [3, 15] for the three-dimensional case). Here, we base the solution of the forward problem on a single-layer approach as investigated in [2]. For this we denote by

$$\Phi_k(x,y) := \frac{i}{4}\, H_0^{(1)}(k|x-y|), \quad x \neq y,$$

the fundamental solution to the Helmholtz equation with wave number k in $\mathbb{R}^2$ in terms of the Hankel function $H_0^{(1)}$ of order zero and of the first kind. Adopting the notation of [4], in a Sobolev space setting, for $k = k_d$ and $k = k_0$, we introduce the single-layer potential operators

$$S_k : H^{-1/2}(\partial D) \to H^{1/2}(\partial D)$$

by

$$(S_k\varphi)(x) := 2\int_{\partial D} \Phi_k(x,y)\varphi(y)\, ds(y), \quad x \in \partial D \tag{11.5}$$

and the normal derivative operators

$$K_k' : H^{-1/2}(\partial D) \to H^{-1/2}(\partial D)$$

by

$$(K_k'\varphi)(x) := 2\int_{\partial D} \frac{\partial \Phi_k(x,y)}{\partial \nu(x)}\, \varphi(y)\, ds(y), \quad x \in \partial D. \tag{11.6}$$

For the Sobolev spaces and the mapping properties of these operators we refer to [13, 20]. Then, from the jump relations it can be seen that the single-layer potentials

$$\begin{aligned} v(x) &= \int_{\partial D} \Phi_{k_d}(x,y)\varphi_d(y)\, ds(y), \quad x \in D, \\ u^s(x) &= \int_{\partial D} \Phi_{k_0}(x,y)\varphi_0(y)\, ds(y), \quad x \in \mathbb{R}^2 \setminus \bar{D}, \end{aligned} \tag{11.7}$$

solve the scattering problem (11.1)–(11.3) provided the densities φ_d and φ_0 satisfy the system of integral equations

$$\begin{aligned} S_{k_d}\varphi_d - S_{k_0}\varphi_0 &= 2u^i|_{\partial D}, \\ \varphi_d + K_{k_d}'\varphi_d + \varphi_0 - K_{k_0}'\varphi_0 &= 2\left.\frac{\partial u^i}{\partial \nu}\right|_{\partial D}, \end{aligned} \tag{11.8}$$

which in the sequel we will call the field equations. Provided k_0 is not a Dirichlet eigenvalue of the negative Laplacian for the domain D, with the aid of the Riesz–Fredholm theory, in [2] it has been shown that the system (11.8) has a unique solution in $H^{-1/2}(\partial D) \times H^{-1/2}(\partial D)$. Thus, throughout this paper we shall assume that k_0 is not a Dirichlet eigenvalue of the negative Laplacian for the domain D.

After introducing the far-field operator $S_\infty : H^{-1/2}(\partial D) \to L^2(S^1)$ by

$$(S_\infty \varphi)(\hat{x}) := \gamma \int_{\partial D} e^{-ik_0 \hat{x}\cdot y} \varphi(y)\, ds(y), \quad \hat{x} \in S^1, \tag{11.9}$$

from (11.7) and asymptotics of the Hankel function we observe that the far-field pattern for the solution to the scattering problem (11.1)–(11.3) is given by

$$u_\infty = S_\infty \varphi_0 \tag{11.10}$$

in terms of the solution to (11.8).

11.3 Numerical Solution

For the numerical solution of (11.8) and the presentation of our inverse algorithm we assume that the boundary curve ∂D is given by a regular 2π-periodic parameterization

$$\partial D = \{z(t) : 0 \le t \le 2\pi\}. \tag{11.11}$$

Then, via $\psi = \varphi \circ z$ emphasizing the dependence of the operators on the boundary curve, we introduce the parameterized single-layer operator

$$\widetilde{S}_k : H^{-1/2}[0,2\pi] \times C^2[0,2\pi] \to H^{1/2}[0,2\pi]$$

by

$$\widetilde{S}_k(\psi,z)(t) := \frac{i}{2} \int_0^{2\pi} H_0^{(1)}(k|z(t)-z(\tau)|)\, |z'(\tau)|\, \psi(\tau)\, d\tau$$

and the parameterized normal derivative operators

$$\widetilde{K}'_k : H^{-1/2}[0,2\pi] \times C^2[0,2\pi] \to H^{-1/2}[0,2\pi]$$

by

$$\widetilde{K}'_k(\psi,z)(t) := \frac{ik}{2} \int_0^{2\pi} \frac{[z'(t)]^\perp \cdot [z(\tau)-z(t)]}{|z'(t)|\,|z(t)-z(\tau)|}\, H_1^{(1)}(k|z(t)-z(\tau)|)\, |z'(\tau)|\, \psi(\tau)\, d\tau$$

for $t \in [0,2\pi]$. Here we made use of $H_0^{(1)\prime} = -H_1^{(1)}$ with the Hankel function $H_1^{(1)}$ of order zero and of the first kind. Furthermore, we write $a^\perp = (a_2, -a_1)$ for any vector $a = (a_1, a_2)$, that is, $a^\perp$ is obtained by rotating a clockwise by 90 degrees. Then the parameterized form of (11.8) is given by

$$\begin{aligned} \widetilde{S}_{k_d}(\psi_d, z) - \widetilde{S}_{k_0}(\psi_0, z) &= 2\, u^i \circ z, \\ \psi_d + \widetilde{K}'_{k_d}(\psi_d, z) + \psi_0 - \widetilde{K}'_{k_0}(\psi_0, z) &= \frac{2}{|z'|}\, [z']^\perp \cdot \operatorname{grad} u^i \circ z. \end{aligned} \tag{11.12}$$

The kernels

$$M(t,\tau) := \frac{i}{2}\, H_0^{(1)}(k|z(t)-z(\tau)|)\,|z'(\tau)|$$

and

$$L(t,\tau) := \frac{ik}{2}\, \frac{[z'(t)]^{\perp}\cdot[z(\tau)-z(t)]}{|z'(t)|\,|z(t)-z(\tau)|}\, H_1^{(1)}(k|z(t)-z(\tau)|)\,|z'(\tau)|$$

of the operators $\widetilde{S}_k$ and $\widetilde{K}'_k$ can be written in the form

$$\begin{aligned} M(t,\tau) &= M_1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right) + M_2(t,\tau),\\ L(t,\tau) &= L_1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right) + L_2(t,\tau), \end{aligned} \tag{11.13}$$

where

$$\begin{aligned} M_1(t,\tau) &:= -\frac{1}{2\pi}\, J_0(k|z(t)-z(\tau)|)|z'(\tau)|,\\ M_2(t,\tau) &:= M(t,\tau) - M_1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right),\\ L_1(t,\tau) &:= -\frac{k}{2\pi}\, \frac{[z'(t)]^{\perp}\cdot[z(\tau)-z(t)]}{|z'(t)|\,|z(t)-z(\tau)|}\, J_1(k|z(t)-z(\tau)|)\,|z'(\tau)|,\\ L_2(t,\tau) &:= L(t,\tau) - L_1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right). \end{aligned}$$

The functions M_1, M_2, L_1, and L_2 turn out to be smooth with diagonal terms

$$M_2(t,t) = \left[\frac{i}{2} - \frac{C}{\pi} - \frac{1}{\pi}\ln\left(\frac{k}{2}|z'(t)|\right)\right]|z'(t)|$$

in terms of Euler's constant C and

$$L_2(t,t) = -\frac{1}{2\pi}\, \frac{[z'(t)]^{\perp}\cdot z''(t)}{|z_1'(t)|^2}\,.$$

For integral equations with kernels of the form (11.13) a combined collocation and quadrature method based on trigonometric interpolation as described in Sect. 3.5 of [4] or in [19] is at our disposal. We refrain from repeating the details. For a related error analysis we refer to [13] and note that we have exponential convergence for smooth, i.e., analytic boundary curves ∂D.

For a numerical example, we consider the scattering of a plane wave by a dielectric cylinder with a non-convex kite-shaped cross section with boundary ∂D described by the parametric representation

$$z(t) = (\cos t + 0.65\cos 2t - 0.65, 1.5\sin t), \quad 0 \le t \le 2\pi. \tag{11.14}$$

From the asymptotics for the Hankel functions, it can be deduced that the far-field pattern of the single-layer potential u^s with density φ_0 is given by

$$u_\infty(\hat{x}) = \gamma \int_{\partial D} e^{-ik_0 \hat{x}\cdot y} \varphi_0(y)\, ds(y), \quad \hat{x} \in S^1, \tag{11.15}$$

where

$$\gamma = \frac{e^{i\frac{\pi}{4}}}{\sqrt{8\pi k_0}}.$$

The latter expression can be evaluated by the composite trapezoidal rule after solving the system of integral equations (11.8) for φ_0, i.e., after solving (11.12) for ψ_0. Table 11.1 gives some approximate values for the far-field pattern $u_\infty(d)$ and $u_\infty(-d)$ in the forward direction d and the backward direction $-d$. The direction d of the incident wave is $d = (1,0)$ and the wave numbers are $k_0 = 1$ and $k_d = 2+3i$. Note that the exponential convergence is clearly exhibited.

Table 11.1: Numerical results for direct scattering problem

n	$\mathrm{Re}\, u_\infty(d)$	$\mathrm{Im}\, u_\infty(d)$	$\mathrm{Re}\, u_\infty(-d)$	$\mathrm{Im}\, u_\infty(-d)$
8	−0.6017247940	−0.0053550779	−0.2460323014	0.3184957768
16	−0.6018967551	−0.0056192337	−0.2461831740	0.3186052686
32	−0.6019018135	−0.0056277492	−0.2461946976	0.3186049949
64	−0.6019018076	−0.0056277397	−0.2461946846	0.3186049951

11.4 The Inverse Problem

We now proceed describing an iterative algorithm for approximately solving the inverse scattering problem by extending the method proposed by Kress [14] and further investigated by Kress and Serranho [17, 18] and Serranho [22] for the case of the inverse problem for a perfectly conducting scatterer.

After introducing the far-field operator $S_\infty : H^{-1/2}(\partial D) \to L^2(S^1)$ by

$$(S_\infty \varphi)(\hat{x}) := \gamma \int_{\partial D} e^{-ik_0 \hat{x}\cdot y} \varphi(y)\, ds(y), \quad \hat{x} \in S^1, \tag{11.16}$$

from (11.7) and (11.15) we observe that the far-field pattern for the solution to the scattering problem (11.1)–(11.3) is given by

$$u_\infty = S_\infty \varphi_0 \tag{11.17}$$

in terms of the solution to (11.8).

11.4.1 The Inverse Problem for Shape Reconstruction

We can state the following theorem as theoretical basis of our inverse algorithm. For this we note that all our integral operators depend on the boundary curve ∂D.

Theorem 11.1. *For a given incident field u^i and a given far-field pattern u_∞, assume that ∂D and the densities φ_d and φ_0 satisfy the system of three integral equations*

$$\begin{aligned} S_{k_d}\varphi_d - S_{k_0}\varphi_0 &= 2u^i, \\ \varphi_d + K'_{k_d}\varphi_d + \varphi_0 - K'_{k_0}\varphi_0 &= 2\,\frac{\partial u^i}{\partial \nu}\,, \\ S_\infty\varphi_0 &= u_\infty. \end{aligned} \tag{11.18}$$

Then ∂D solves the inverse problem.

The ill-posedness of the inverse problem is reflected through the ill-posedness of the third integral equation, the far-field equation that we denote as *data equation*. Note that (11.18) is linear with respect to the densities and nonlinear with respect to the boundary ∂D. This opens up a variety of approaches to solve (11.18) by linearization and iteration. In [2] we investigated an extension of the approach suggested by Johansson and Sleeman [10] for a perfectly conducting scatterer. Given a current approximation ∂D_{approx} for the unknown boundary ∂D we first solved the first two equations, or field equations, of system (11.18) for the unknown densities φ_d and φ_0. Then, keeping φ_0 fixed we linearized the third equation, or data equation, of system (11.18) with respect to the boundary ∂D to update the approximation. Here, following [14, 17, 18] we are going to proceed differently. Given a current approximation ∂D_{approx} the unknown boundary ∂D. In a first step, the data equation regularized via Tikhonov regularization, the density φ_0 can be found on ∂D_{approx}. Then in a second step, keeping the density φ_0 fixed we find the density φ_d from the second equation of (11.18). In a third step, keeping the densities φ_0 and φ_d fixed we linearize the first equation of (11.18) with respect to boundary ∂D. In a fourth step, the solution of ill-posed linearized equation can be utilized to update the boundary approximation.

To describe this in more detail, we also require the parameterized version

$$\widetilde{S}_\infty : H^{-1/2}[0,2\pi] \times C^2[0,2\pi] \to L^2(S^1)$$

of the far-field operator as given by

$$\widetilde{S}_\infty(\psi,z)(\hat{x}) := \gamma\int_0^{2\pi} e^{-ik_0\,\hat{x}\cdot z(\tau)}\,\psi(\tau)\,d\tau, \quad \hat{x}\in S^1. \tag{11.19}$$

Then the parameterized form of (11.18) is given by

$$\begin{aligned}
\widetilde{S}_{k_d}(\psi_d,z) - \widetilde{S}_{k_0}(\psi_0,z) &= 2\,u^i\circ z,\\
\psi_d + \widetilde{K}'_{k_d}(\psi_d,z) + \psi_0 - \widetilde{K}'_{k_0}(\psi_0,z) &= \frac{2}{|z'|}\,[z']^\perp\cdot \operatorname{grad} u^i\circ z,\\
\widetilde{S}_\infty(\psi_0,z) &= u_\infty.
\end{aligned} \tag{11.20}$$

For a fixed ψ the Fréchet derivative of the operator $\widetilde{S}_k$ with respect to the boundary z in the direction h is given by (see [21])

$$\begin{aligned}
\partial\widetilde{S}_k(\psi,z;h)(t) = {} & \frac{-ik}{2}\int_0^{2\pi}\frac{(z(t)-z(\tau))\cdot(h(t)-h(\tau))}{|z(t)-z(\tau)|}|z'(\tau)|\,H_1^{(1)}(k|z(t)-z(\tau)|)\,\psi(\tau)d\tau\\
& + \frac{i}{2}\int_0^{2\pi}\frac{z'(\tau)\cdot h'(\tau)}{|z'(\tau)|}H_0^{(1)}(k|z(t)-z(\tau)|)\,\psi(\tau)d\tau.
\end{aligned} \tag{11.21}$$

Then the linearization of the first equation in (11.20) with respect to z in the direction h reads

$$\partial\widetilde{S}_{k_d}(\psi_d,z;h) - \partial\widetilde{S}_{k_0}(\psi_0,z;h) - 2\operatorname{grad} u^i\circ z\cdot h = 2u^i\circ z - \widetilde{S}_{k_d}(\psi_d,z) + \widetilde{S}_{k_0}(\psi_0,z)$$

and is a linear equation for the update h.

Now, given an approximation for the boundary curve ∂D with parameterization z, each iteration step of the proposed inverse algorithm consists of four parts:

1. We find the density ψ_0 from the regularized data equation via Tikhonov regularization
$$(\alpha I + \widetilde{S}_\infty^*\widetilde{S}_\infty)\psi_0 = \widetilde{S}_\infty^*\,u_\infty, \tag{11.22}$$
where $\widetilde{S}_\infty^*$ is the adjoint operator of $\widetilde{S}_\infty$.
2. We keep ψ_0 fixed and find the density ψ_d from
$$(I+\widetilde{K}'_{k_d})(\psi_d,z) = \frac{2}{|z'|}[z'(t)]^\perp\cdot\operatorname{grad} u^i\circ z - \psi_0 + \widetilde{K}'_{k_0}(\psi_0,z).$$
3. We keep the densities ψ_d and ψ_0 fixed and find the perturbed boundary h form the linearized equation
$$\partial\widetilde{S}_{k_d}(\psi_d,z;h) - \partial\widetilde{S}_{k_0}(\psi_0,z;h) - 2\operatorname{grad} u^i\circ z\cdot h = 2u^i\circ z - \widetilde{S}_{k_d}(\psi_d,z) + \widetilde{S}_{k_0}(\psi_0,z). \tag{11.23}$$

4. Updating the boundary $z := z + h$ then we go to first step. We continue this procedure until some stopping criteria is achieved. The stopping criterion for the iterative scheme is given by the relative error

$$\frac{\| u_{\infty;N} - u_\infty \|}{\| u_\infty \|} \leq \varepsilon_1, \tag{11.24}$$

where $u_{\infty;N}$ is the computed far-field pattern after N iteration steps.

In principle, the parameterization of the update is not unique. To cope with this ambiguity, one possibility that we will pursue in our numerical examples of the subsequent section is to allow only parameterizations of the form

$$z(t) = r(t) \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}, \quad 0 \leq t \leq 2\pi, \tag{11.25}$$

with a non-negative function r representing the radial distance of ∂D from the origin. Consequently, the perturbations are of the form

$$h(t) = q(t) \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}, \quad 0 \leq t \leq 2\pi, \tag{11.26}$$

with a real function q. In the approximations we assume r and its update q to have the form of a trigonometric polynomial of degree J, in particular,

$$q(t) = \sum_{j=0}^{J} a_j \cos jt + \sum_{j=1}^{J} b_j \sin jt. \tag{11.27}$$

Then the update equation (11.23) is solved in the least squares sense, penalized via Tikhonov regularization, for the unknown coefficients $a_0, \ldots, a_J$ and $b_1, \ldots, b_J$ of the trigonometric polynomial representing the update q. As experienced in the application of the above approach for related problems, it is advantageous to use an H^p Sobolev penalty term rather than an L^2 penalty in the Tikhonov regularization, i.e., to interpret $\partial \widetilde{S}_k$ as an ill-posed linear operator

$$\partial \widetilde{S}_k : H^p[0, 2\pi] \to L^2[0, 2\pi] \tag{11.28}$$

for some small $p \in \mathbb{N}$.

As a theoretical basis for the application of Tikhonov regularization from [4] we cite that, after the restriction to starlike boundaries, the operator $\partial \widetilde{S}_k$ is injective provided k_0^2 is not a Neumann eigenvalue for the negative Laplacian in D.

The above algorithm has a straightforward extension for the case of more than one incident wave. Assume that $u_1^i, \ldots, u_M^i$ are M incident waves with different incident directions and $u_{\infty,1}, \ldots, u_{\infty,M}$ the corresponding far-field patterns for scattering from ∂D. Then the inverse problem to determine the unknown boundary ∂D from these given far-field patterns and incident fields is equivalent to solve the following iterative scheme: Given a current approximation to the boundary ∂D, parametrized by z:

1. We find the densities $\psi_{0,1},\dots,\psi_{0,M}$ from the regularized data equations via Tikhonov regularization

$$(\alpha I + \widetilde{S}_\infty^* \widetilde{S}_\infty)\psi_{0,m} = \widetilde{S}_\infty^* u_{\infty,m}, \quad \text{for} \quad m = 1,\dots,M.$$

2. We keep the $\psi_{0,1},\dots,\psi_{0,M}$ fixed and find densities $\psi_{d,1},\dots,\psi_{d,M}$ from

$$(I + \widetilde{K}'_{k_d})(\psi_{d,m}, z) = \frac{2}{|z'|}[z'(t)]^\perp \cdot \mathrm{grad} u_m^i \circ z - \psi_{0,m} + \widetilde{K}'_{k_0}(\psi_{0,m}, z).$$

   for $m = 1,\dots,M$.
3. We keep the densities $\psi_{0,1},\dots,\psi_{0,M}$ and $\psi_{d,1},\dots,\psi_{d,M}$ fixed and find the perturbed boundary h form the linearized equation

$$\partial\widetilde{S}_{k_d}(\psi_{d,m}, z; h) - \partial\widetilde{S}_{k_0}(\psi_{0,m}, z; h) - 2\mathrm{grad} u_m^i \circ z \cdot h = 2u_m^i \circ z - \widetilde{S}_{k_d}(\psi_{d,m}, z) + \widetilde{S}_{k_0}(\psi_{0,m}, z),$$

   for $m = 1,\dots,M$. For the update h by interpreting them as one ill-posed equation with an operator from $H^p[0,2\pi] \mapsto (L^2[0,2\pi])^M$ and applying Tikhonov regularization.
4. We update the boundary $z := z + h$ then go to first step. We continue this procedure until some stopping criteria is achieved. The stopping criterion for the iterative scheme is given by the relative error (11.24).

For the numerical implementation we need to discretize the boundary operator ∂S_k in (11.21). The kernels of the operator ∂S_k can be written in the form

$$A(t,\tau) := -\frac{ik}{2} \frac{(z(t) - z(\tau)) \cdot (h(t) - h(\tau))}{|z(t) - z(\tau)|} |z'(\tau)| H_1^{(1)}(k|z(t) - z(\tau)|),$$
$$B(t,\tau) := \frac{i}{2} \frac{z'(\tau) \cdot h'(\tau)}{|z'(\tau)|} H_0^{(1)}(k|z(t) - z(\tau)|).$$

The kernels A and B can be expressed of the form

$$A(t,\tau) = A_1(t,\tau) \ln(4\sin^2 \frac{t-\tau}{2}) + A_2(t,\tau),$$
$$B(t,\tau) = B_1(t,\tau) \ln(4\sin^2 \frac{t-\tau}{2}) + B_2(t,\tau),$$

where

$$A_1(t,\tau) := \frac{k}{2\pi} \frac{(z(t) - z(\tau)) \cdot (h(t) - h(\tau))}{|z(t) - z(\tau)|} |z'(\tau)| J_1(k|z(t) - z(\tau)|),$$
$$A_2(t,\tau) := A(t,\tau) - A_1(t,\tau) \ln(4\sin^2 \frac{t-\tau}{2}),$$
$$B_1(t,\tau) := -\frac{1}{2\pi} \frac{z'(\tau) \cdot h'(\tau)}{|z'(\tau)|} J_0(k|z(t) - z(\tau)|),$$
$$B_2(t,\tau) := B(t,\tau) - B_1(t,\tau) \ln(4\sin^2 \frac{t-\tau}{2})$$

in terms of Bessel functions J_0 and J_1. The functions A_1, A_2, B_1 and B_2 turn out to be smooth with diagonal terms. Their diagonal terms are in the form

$$A_1(t,t) = 0, \quad \& \; A_2(t,t) = -\frac{1}{\pi}\frac{z'(t)\cdot h'(t)}{|z'(t)|}.$$

$$B_1(t,t) = -\frac{1}{2\pi}\frac{z'(t)\cdot h'(t)}{|z'(t)|}, \quad \& \; B_2(t,t) = \left\{\frac{i}{2} - \frac{1}{\pi} ln\left(\frac{k}{2}|z'(t)|\right) - \frac{C}{\pi}\right\} z'(t)\cdot h'(t).$$

in terms of Euler's constant C.

11.4.2 The Inverse Problem for the Interior Wave Number k_d Reconstruction

The inverse problem we are interested is that given an incident plane wave u^i, far-field pattern u_∞ and the shape of the scatterer, we would like to determine the interior wave number of the field that occurs inside the obstacle.

We can state the following theorem as theoretical basis of our inverse algorithm:

Theorem 11.2. *For a given incident field u^i and a given far-field pattern u_∞ and the shape of the scatterer, assume that k_d and the densities φ_d and φ_0 satisfy the system of three integral equations*

$$\begin{aligned} S_{k_d}\varphi_d - S_{k_0}\varphi_0 &= 2u^i, \\ \varphi_d + K'_{k_d}\varphi_d + \varphi_0 - K'_{k_0}\varphi_0 &= 2\,\frac{\partial u^i}{\partial \nu}\,, \\ S_\infty \varphi_0 &= u_\infty. \end{aligned} \tag{11.29}$$

Then k_d solves the inverse problem.

The ill-posedness of the inverse problem is reflected through the ill-posedness of the third integral equation, the far-field equation that we denote as *data equation*. Note that (11.29) is linear with respect to the densities and nonlinear with respect to the interior wave number k_d. In the spirit of [14, 17, 18] we are going to describe the iterative scheme for the inverse problem.

Given a current approximation $k_{d_{approx}}$ to unknown interior wave number k_d. In a first step, after the data equation regularized via Tikhonov regularization, the density φ_0 can be found for $k_{d_{approx}}$. Then in a second step, keeping the density φ_0 fixed we find the density φ_d from the second equation of (11.29). In a third step, keeping the densities φ_0 and φ_d fixed we linearize the first equation of (11.29) with respect to boundary k_d to update the approximation.

We now proceed describing an iterative algorithm for approximately solving this inverse problem for the interior wave number. Now we consider an operator

$$\widetilde{S}_{k_d} : L^2[0,2\pi] \times \mathbb{C} \to L^2[0,2\pi]. \tag{11.30}$$

Then the parameterized form of (11.29) is given by (11.20). For a fixed ψ_d the Fréchet derivative of the operator $\widetilde{S}_{k_d}$ with respect to the interior wave number k_d in the direction σ is given by

$$\partial \widetilde{S}_{k_d}(\psi_d,k_d;\sigma) = -\frac{i\sigma}{2}\int_0^{2\pi} H_1^{(1)}(k_d|z(t)-z(\tau)|)|z(t)-z(\tau)||z'(\tau)|\psi_d(\tau)d\tau, \tag{11.31}$$

for $t \in [0,2\pi]$.

Now, the first iteration step of the proposed inverse algorithm consists of four parts and the rest of iteration steps consist of three parts:

1. In a first part, we find the density ψ_0 from the stabilized data equation, i.e., from

$$(\alpha I + \widetilde{S}_\infty^* \widetilde{S}_\infty)\psi_0 = \widetilde{S}_\infty^* u_\infty.$$

2. Give a current approximation for the interior wave number k_d. In a second part, we find ψ_d from the following equation

$$(I+\widetilde{K}'_{k_d})(\psi_d,k_d) = \frac{2}{|z'|}[z'(t)]^\perp \cdot \mathrm{grad} u^i \circ z - \psi_0 + \widetilde{K}'_{k_0}(\psi_0,z).$$

3. In a third part, we keep the density ψ_d fixed and linearize the first field equation with respect to interior wave number k_d in the direction of σ, and then we find perturbed interior wave number σ from the following linearized equation:

$$\partial \widetilde{S}_{k_d}(\psi_d,k_d;\sigma) = 2u^i \circ z - \widetilde{S}_{k_d}(\psi_d,z) + \widetilde{S}_{k_0}(\psi_0,z). \tag{11.32}$$

4. In a fourth part, we update the interior wave number as $k_d := k_d + \sigma$ and then we return to the second part and repeat this procedure until some stopping criteria is achieved. The stopping criterion for the iterative scheme is given by the relative error

$$\frac{|k_{d;N} - k_d|}{|k_d|} \le \varepsilon_2, \tag{11.33}$$

   where $k_{d;N}$ is the computed interior wave number after N iteration steps.

The kernel,

$$P(t,\tau;k_d) := -\frac{i}{2}|z(t)-z(\tau)||z'(\tau)|H_1^{(1)}(k_d|z(t)-z(\tau)|),$$

of the operator $\partial \widetilde{S}_{k_d}(\psi_d,k_d;\sigma)$ in (11.31) can be written in the form

$$P(t,\tau;k_d) = P_1(t,\tau;k_d)\ln(4\sin^2\frac{t-\tau}{2}) + P_2(t,\tau;k_d),$$

where

$$P_1(t,\tau;k_d) = \frac{1}{2\pi}|z(t)-z(\tau)||z'(\tau)|J_1(k_d|z(t)-z(\tau)|),$$
$$P_2(t,\tau;k_d) = P(t,\tau;k_d) - P_1(t,\tau;k_d)\ln(4\sin^2\frac{t-\tau}{2}).$$

The functions P_1 and P_2 turn out to be smooth with diagonal terms. Their diagonal terms are in the form

$$P_1(t,t;k_d) = 0 \qquad \& \qquad P_2(t,t;k_d) = -\frac{1}{\pi k_d}|z'(t)|.$$

11.5 Numerical Examples

To avoid an inverse crime, in our numerical examples the synthetic far-field data were obtained by a numerical solution of the boundary integral equations based on a combined single- and double-layer approach (see [3, 15]) using the numerical schemes as described in [4, 12, 13]. In each iteration step of the inverse algorithm for the solution of the field equations we used the numerical method described in Sect. 11.3 using 64 quadrature points. The data equation was solved via Tikhonov regularization with an L^2 penalty term with α regularization parameter. The linearized first field equation (11.23) with respect to boundary was solved by Tikhonov regularization with an H^2 penalty term, i.e., $p = 2$ in (11.28) and with a λ regularization parameter. The linearized first field equation (11.32) with respect to interior wave number was solved by Tikhonov regularization with an L^2 penalty term in (11.30) and with a μ regularization parameter. The regularized data equation is solved by Nyström's method with the composite trapezoidal rule again using 64 quadrature points.

11.5.1 Numerical Examples of Shape Reconstruction

In all our five examples we used $M = 8$ as a number of incident waves with the directions $d = (\cos(2\pi m/M), \sin(2\pi m/M))$, $m = 1,\dots,M$ and $J = 5$ as degree for the approximating trigonometric polynomials in (11.27) and N as the number of recursion and the wave numbers $k_0 = 1$ and $k_d = 10+10i$. The initial guess is given by the green curve, the exact boundary curves are given by the dashed (blue) lines, and the reconstructions by the full (red) lines. The iteration numbers and the regularization parameters α and λ for the Tikhonov regularization of (11.22) and (11.23), respectively, were chosen by trial and error and their values are indicated in the following description of the individual examples.

In order to obtain noisy data, random errors are added point-wise to u_∞,

$$\widetilde{u}_\infty = u_\infty + \delta\xi\frac{||u_\infty||}{|\xi|} \tag{11.34}$$

with the random variable $\xi \in \mathbb{C}$ and $\{\mathrm{Re}\,\xi, \mathrm{Im}\,\xi\} \in (0,1)$.

Table 11.2: Parametric representation of boundary curves

Counter type		Parametric representation
Apple-shaped	:	$z(t) = \{\frac{0.5+0.4\cos t+0.1\sin 2t}{1+0.7\cos t}(\cos t, \sin t) : t \in [0, 2\pi]\}$
Dropped-shaped	:	$z(t) = \{(-0.5+0.75\sin\frac{t}{2}, -0.75\sin t) : t \in [0, 2\pi]\}$
Kite-shaped	:	$z(t) = \{(\cos t + 1.3\cos^2 t - 1.3, 1.5\sin t) : t \in [0, 2\pi]\}$
Peanut-shaped	:	$z(t) = \{\sqrt{\cos^2 t + 0.25\sin t}\,(\cos t, \sin t) : t \in [0, 2\pi]\}$
Rounded triangle	:	$z(t) = \{(2+0.3\cos 3t)(\cos t, \sin t) : t \in [0, 2\pi]\}$

In the first example Fig. 11.1 shows reconstructions after $N = 12$ iterations with the regularization parameters $\alpha = 10^{-7}$ and $\lambda = 0.8^j$ decreasing with the iteration steps j. For the stopping criteria (11.24), $\varepsilon_1 = 10^{-3}$ is chosen.

In the second example Fig. 11.2 shows reconstructions after $N = 10$ iterations with the regularization parameter chosen as in the first example. For the stopping criteria (11.24), $\varepsilon_1 = 10^{-3}$ is chosen.

In the third example the reconstructions in Fig. 11.3 were obtained after $N = 15$ iterations with the regularization parameter chosen as in the first example. For the stopping criteria (11.24), $\varepsilon = 10^{-2}$ is chosen.

In the fourth example the reconstructions in Fig. 11.4 were obtained after $N = 10$ iterations with the regularization parameters chosen as $\alpha = 10^{-6}$ and $\lambda = 0.7^j$. For the stopping criteria (11.24), $\varepsilon_1 = 10^{-3}$ is chosen. In the final example Fig. 11.5 shows reconstructions after $N = 8$ iterations with the regularization parameters chosen as $\alpha = 10^{-6}$ and $\lambda = 0.8^j$. For the stopping criteria (11.24), $\varepsilon_1 = 10^{-3}$ is chosen.

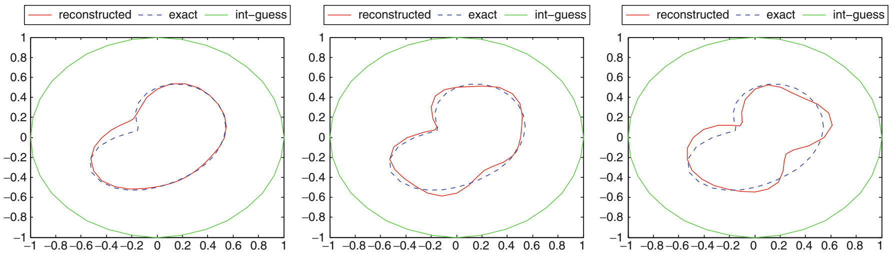

Fig. 11.1: Reconstruction of the apple-shaped contour (Table 11.2) for exact data (*left*), 1% noise (*middle*) and 2% noise (*right*)

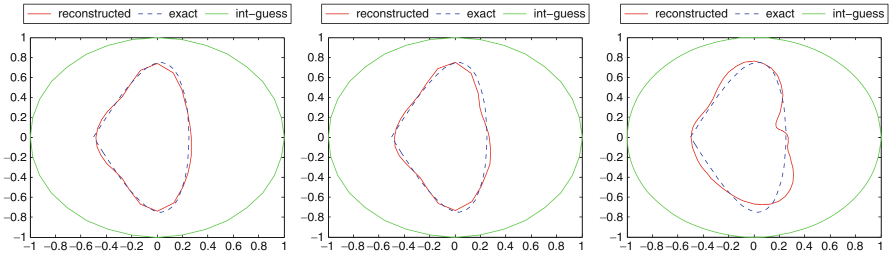

Fig. 11.2: Reconstruction of dropped-shaped contour (Table 11.2) for exact data (*left*), 1% noise (*middle*) and 2% noise (*right*)

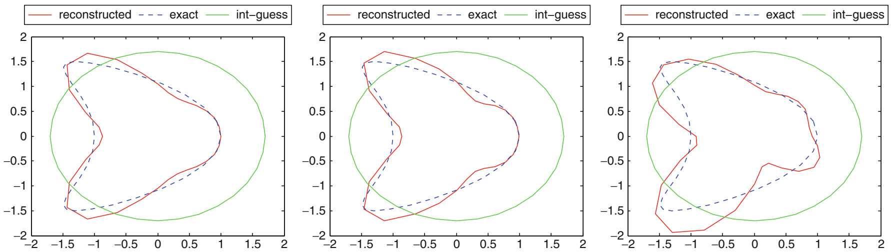

Fig. 11.3: Reconstruction of kite-shaped contour (Table 11.2) for exact data (*left*), 1% noise (*middle*) and 2% noise (*right*)

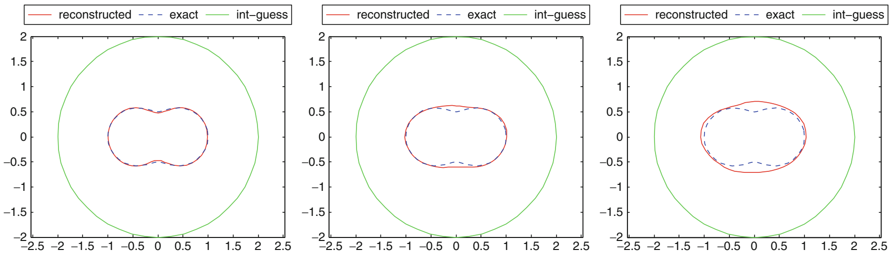

Fig. 11.4: Reconstruction of peanut-shaped contour (Table 11.2) for exact data (*left*), 1% noise (*middle*) and 2% noise (*right*)

Our examples clearly indicate the feasibility of the proposed algorithm. From our further numerical experiments it is observed that using more than one incident wave improved on the accuracy of the reconstruction and the stability. Furthermore, an appropriate initial guess was important to ensure numerical convergence of the iterations. Our examples also indicate that the proposed algorithm with the numerical reconstructions is superior to those obtained by Johansson and Sleeman [10] in [2]. This behavior is confirmed by a number of further numerical examples in [1]. However, the proposed algorithm is sensitive to noise level. It only tolerates 1% noise level.

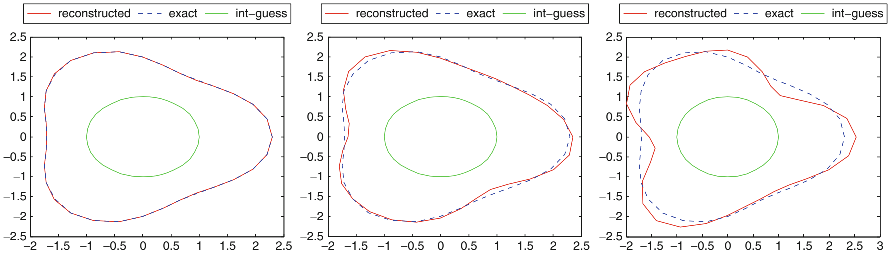

Fig. 11.5: Reconstruction of rounded-triangle-shaped contour (Table 11.2) for exact data (*left*), 1% noise (*middle*) and 2% noise (*right*)

11.5.2 Numerical Example of Interior Wave Number k_d Reconstruction

The table shows the reconstruction of the interior wave number k_d after $N = 12$ iteration steps. The regularization parameter μ is chosen by trial and error. For the numerical example, $\mu = 10^{-8}$ is chosen. For the stopping criteria (11.33), $\varepsilon_2 = 10^{-4}$ is chosen. $k_d = 5 + 3.5i$ is the initial guess for the interior wave number. $k_d = 6 + 3i$ is the exact value of the interior wave number.

j	$\text{Re}\, k_d$	$\text{Im}\, k_d$
1	7.1183586985	0.7713567402
2	7.0978164742	1.1142157466
3	6.7877484915	1.7601039965
4	6.4182938086	2.4436691343
5	6.0756484419	2.8851109484
6	6.0007133176	2.9782074624
7	5.9963101217	2.9978738350
8	5.9987548077	3.0003488868
9	5.9997523215	3.0002332761
10	5.9999618896	3.0000776416
11	5.9999839302	3.0000316851
12	5.9999813584	3.0000238253

Further research will be directed towards applying the algorithm to real data, to extend the numerics to the three-dimensional case and to a simultaneous linearization of the field and data equations with respect to the boundary and the densities in the spirit of [8, 16].

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Chapter 12
Solving Second-Order Discrete Sturm–Liouville BVP Using Matrix Pencils

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Abstract This paper deals with discrete second order Sturm-Liouville Boundary Value Problems (DSLBVP) where the parameter λ, as part of the difference equation, appears nonlinearly in the boundary conditions. We focus on the case where the boundary condition is given by a cubic equation in λ. We first describe the problem by a matrix equation with nonlinear variables such that solving the DSLBVP is equivalent to solving the matrix equation. We develop methods to finding roots of the characteristic polynomial (in the variable λ) of the involved matrix. We further reduce the problem to finding eigenvalues of a matrix pencil in the form of $A - \lambda B$. Under certain conditions, such a matrix pencil eigenvalue problem can be reduced to a stabdard eigenvalue problem, so that existing computational tools can be used to solve the problem. The main results of the paper provide the reduction procedure and rules to identify the cubic DSLBVPs which can be reduced to standard eigenvalue problems. We also investigate the structure of the matrix form of a DSLBVP and its effect on the reality of the eigenvalues of the problem. We give a class of DSLBVPs which have only real eigenvalues.

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12.1 Introduction

12.1.1 History of Sturm–Liouville Problems

Named after Jacques Charles Francois Sturm (1803–1855) and Joseph Liouville (1809–1882), a second-order Sturm–Liouville equation is a real second-order differential equation of the form:

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + (\lambda w(x) - q(x))\,y = 0,$$

where λ is a constant and $w(x)$, $p(x)$, and $q(x)$ are known real functions. A solution pair (λ, y) to the equation (with appropriate boundary conditions) is called an eigenpair, where λ is called an eigenvalue and y the corresponding eigenfunction or eigenvector. The solutions of this equation satisfy important mathematical properties under appropriate boundary conditions [5].

The classical Sturm–Liouville Boundary Value Problem over a finite closed interval $[a,b]$ can be described in the following form:

$$\begin{cases} Ly = (1/r)(-py')' + qy = \lambda wy \\ A_1 y(a) + A_2 p(a) y'(a) = 0 \\ B_1 y(b) + B_2 p(b) y'(b) = 0. \end{cases}$$

Here L is an operator, r, p, q, w are real functions on $[a,b]$, and r, p are positive valued functions.

The continuous version of Sturm–Liouville Boundary Value Problems with the parameter appearing linearly in the boundary conditions has been dealt with by Walters [21], Hinton [13], Fulton [9], Schneider [19], Belinskiy and Graef [3], and many others. This type of boundary condition arises from various applied problems such as the study of stability of rotating axles [1], heat conduction [15], and diffusion through porous membranes [17]. The Sturm–Liouville Boundary Value Problems with the parameter appearing nonlinearly in the boundary conditions also have many applications in science and engineering. For example, they were discussed in the study of waves of ice-covered oceans in [2, 4]. In particular, when considering an acoustic wave guide covered by an ice cover, an SLBVP arises in which the parameter occurs quadratically at one end [4]. The continuous version of this problem with quadratic boundary conditions was dealt with by Paul A. Binding [5], Patrick J. Browne and Bruce A. Watson [6, 7], Yoko Shioji [20], and Leon Greenberg and I. Babuska [10].

The discrete version (DSLBVP) of the problem in which the parameter appears linearly in the boundary conditions was dealt with by Harmsen and Li [11]. They further studied DSLBVP with the parameter appearing quadratically in the boundary condition [12]. They proved that the eigenvalues of the DSLBVP are simple, distinct, and real under certain conditions.

This paper focuses on DSLBVPs in which the parameter appears nonlinearly in the boundary condition given by a cubic polynomial equation.

12.1.2 Statement of the Problem

In the continuous case, if we choose the interval $[0,1]$ and $w \equiv 0$, the problem is simplified as

$$\begin{cases} Ly = (1/r)(-py')' + qy = \lambda y \\ y(0) = 0 \\ C(\lambda)y(1) = D(\lambda)y'(1), \end{cases}$$

where $C(\lambda)$ and $D(\lambda)$ are fixed real functions. We focus on the discrete version of the above problem. Consider the equalized partition of the time interval $[0,1]$. For an integer $N > 1$, let $t_0 = 0 < t_1 < \cdots < t_{N-1} < 1 = t_N$ and $T = [t_0, t_1, \ldots, t_{N-1}, t_1]$. We use a constant step size $h = t_{n+1} - t_n$ for $n = 0, 1, \ldots, N-1$. Let y be a complex valued function on T. We will use the shorthand notation y_n for $y(t_n)$. Corresponding to the derivative notation, the delta difference is commonly used:

Definition 12.1. The delta difference is defined as

$$\Delta y_n = \frac{y_{n+1} - y_n}{t_{n+1} - t_n} = \frac{y_{n+1} - y_n}{h}.$$

For simplicity, we assume that r, p, q are all constant real-valued functions on T and r and p are positive. By applying the "product rule" for the delta difference, that is, $\Delta(y_n z_n) = y_n \Delta z_n + z_{n-1} \Delta y_n$, the operator L is discretized as

$$Ly_n = \frac{1}{r}\left(\nabla(-p\Delta y_n) + qy_n\right) = -ay_{n+1} + \sigma y_n - ay_{n-1} = \lambda y_n,$$

where

$$a = -\frac{p}{rh^2}, \quad \sigma = \frac{2p}{rh^2} + q.$$

Now we add the cubic boundary condition to the problem. Let $C(\lambda) = c_0 + c_1\lambda + c_2\lambda^2 + c_3\lambda^3$ and $D(\lambda) = d_0 + d_1\lambda + d_2\lambda^2 + d_3\lambda^3$ be cubic real polynomials. The discrete version of the second-order Sturm–Liouville problem with cubic boundary condition has the following form:

$$\begin{cases} Ly_n = \lambda y_n \text{ for } n \text{ from 1 to } N-1 \\ y_0 = 0 \\ C(\lambda)y_N = -pD(\lambda)\Delta y_{N-1}. \end{cases} \tag{12.1}$$

To simplify the notations, we define

$$\begin{aligned} \alpha(\lambda) &= pD(\lambda)/h = \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0 \\ \beta(\lambda) &= C(\lambda) + pD(\lambda)/h = \beta_3\lambda^3 + \beta_2\lambda^2 + \beta_1\lambda + \beta_0. \end{aligned} \tag{12.2}$$

The boundary condition in (12.1) then can be rewritten as $\alpha(\lambda)y_{N-1} = \beta(\lambda)y_N$. The corresponding discrete boundary problem (12.1) can be formulated as

$$\text{(DSLBVP)} \begin{cases} Ly_n = \lambda y_n \text{ for } n \text{ from } 1 \text{ to } N-1 \\ y_0 = 0 \\ \alpha(\lambda)y_{N-1} + \beta(\lambda)y_N = 0. \end{cases} \tag{12.3}$$

This is the discrete version of the Sturm–Liouville problem we concentrate in this paper.

12.2 The Matrix Form of DSLBVP

It is straightforward to check that the top operation in DSLBVP (12.3) is equivalent to the matrix equation $\Gamma_\lambda \mathbf{y} = \mathbf{0}$, where $\mathbf{y} = (y_1, \cdots, y_N)^T$ and Γ_λ is an $N \times N$ matrix given by

$$\Gamma_\lambda = \begin{bmatrix} \sigma-\lambda & -a & 0 & \cdots & 0 & 0 & 0 \\ -a & \sigma-\lambda & -a & \cdots & 0 & 0 & 0 & \ddots & \ddots & \ddots & 0\ 0\ 0 \cdots & -a \\ & \ddots & \ddots\ \ddots & & 0 & 0 & 0 & \cdots & -a\ \sigma-\lambda & -a & 0\ 0\ 0 \cdots & 0 \\ 0 & 0 & 0 & \cdots & -a\ \sigma-\lambda & -a \\ 0 & 0 & 0 & \cdots & 0 & \alpha(\lambda) & \beta(\lambda) \end{bmatrix}, \tag{12.4}$$

where $\alpha(\lambda) = \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0$ and $\beta(\lambda) = \beta_3\lambda^3 + \beta_2\lambda^2 + \beta_1\lambda + \beta_0$.

Every eigenvalue of DSLBVP (12.3) is a root of the polynomial $|\Gamma_\lambda|$ in λ. Thus, the matrix Γ_λ will help us to find solutions to (12.3) and analyze the eigenvalues. The following example shows how to find the solutions to a DSLBVP (12.3) by converting it to a matrix equation and then solve the equation.

Example 12.2. Let $N = 4,\ h = \frac{1}{4},\ r = 16,\ p = 1,\ q = 0$.
Then the operator L satisfies $Ly_n = -y_{n+1} + 2y_n - y_{n-1} = \lambda y_n$ and the boundary condition is $(-1 + 3\lambda + 2\lambda^3)y_4 = (-1 + 2\lambda^3)(-\Delta y_3)$. Thus we have

$$\begin{aligned} Ly_1 &= -y_2 + 2y_1 = \lambda y_1 \\ Ly_2 &= -y_3 + 2y_2 - y_1 = \lambda y_2 \\ Ly_3 &= -y_4 + 2y_3 - y_2 = \lambda y_3 \\ (-1 + 3\lambda + 2\lambda^3)y_4 &= -2(-1 + 2\lambda^3)(y_4 - y_3). \end{aligned}$$

From the equations above, we obtain the matrix form of the DSLBVP as

$$\Gamma_\lambda \mathbf{y} = \begin{bmatrix} 2-\lambda & -1 & 0 & 0 \\ -1 & 2-\lambda & -1 & 0 \\ 0 & -1 & 2-\lambda & -1 \\ 0 & 0 & 4-8\lambda^3 & -5+3\lambda+10\lambda^3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \mathbf{0}.$$

The determinant $|\Gamma_\lambda| = -8+46\lambda-56\lambda^2+39\lambda^3-71\lambda^4+52\lambda^5-10\lambda^6$ has five distinct roots:

$$1.6646, 3.2731, 0.73058, -0.3507+0.85963i, -0.3507-0.85963i.$$

For $\lambda_1 = 3.2731$ we solve the matrix equation:

$$\Gamma_\lambda \mathbf{y} = \begin{bmatrix} -1.2731 & -1 & 0 & 0 \\ -1 & -1.2731 & -2 & 0 \\ 0 & -1 & -1.2731 & -1 \\ 0 & 0 & -276.52 & 355.47 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \mathbf{0}.$$

An eigenvector is given by $\mathbf{y}^T = [2.0712, -2.6366, 1.2855, 1]^T$. One can easily verify that

$$\begin{aligned} Ly_n &= -y_{n+1}+2y_n-y_{n-1} = 3.2731y_n \quad \text{for } n = 1, 2, 3 \\ \textit{and} \quad (-1+3\lambda_1+2\lambda_1^3)y_4 &= -2(-1+2\lambda_1^3)(y_4-y_3). \end{aligned}$$

We now focus on the matrix problem derived from (12.3) and apply linear algebraic techniques to find solutions and analyze them. We state an iterative formula for finding determinant of a tridiagonal matrix which will be used later.

Lemma 12.3. *(Mikkawy and Karawia [16]) Consider the tridiagonal matrix $T_n = [t_{ij}]$ in which $t_{ij} = 0$ for $|i-j| \geq 2$:*

$$T_n = \begin{bmatrix} \sigma_1 & a_1 & 0 & \cdots & \cdots & 0 \\ b_2 & \sigma_2 & a_2 & \cdots & \cdots & 0 \\ 0 & b_3 & \sigma_3 & a_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & b_{n-1} & \sigma_{n-1} & a_{n-1} \\ 0 & 0 & \cdots & 0 & b_n & \sigma_n \end{bmatrix},$$

and assume $a_1a_2\cdots a_{n-1} \neq 0$ and $b_2b_3\cdots b_n \neq 0$ ($n > 2$). Then

$$|T_i| = \begin{cases} \sigma_1 & \text{if } i = 1 \\ \sigma_1\sigma_2 - a_1b_2 & \text{if } i = 2 \\ \sigma_i|T_{i-1}| - b_ia_{i-1}|T_{i-2}| & \text{if } i = 3, 4, \cdots, n. \end{cases}$$

12.3 Matrix Pencils from DSLBVP

From the last section, the matrix form of DSLBVP (12.3) is $\Gamma_\lambda \mathbf{y}^T = \mathbf{0}$, where Γ_λ is given in (12.4). Note that the last row of Γ_λ involves cubic polynomials $\alpha(\lambda) = \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0$ and $\beta(\lambda) = \beta_3\lambda^3 + \beta_2\lambda^2 + \beta_1\lambda + \beta_0$. Thus finding the roots of Γ_λ is not a standard eigenvalue problem. It needs special treatment so that the existing algorithms for finding standard eigenvalues can be applied. Let I_n denote the $n \times n$ identity matrix. The following matrices play important roles when we examine behavior of eigenvalues.

Definition 12.4. Refer to the matrix Γ_λ. We define

$$A_3 = \begin{bmatrix} \mathbf{0}_{N-2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & 0 \\ \mathbf{0} & \alpha_3 & \beta_3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} \mathbf{0}_{N-2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & 0 \\ \mathbf{0} & \alpha_2 & \beta_2 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -I_{N-2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -1 & 0 \\ \mathbf{0} & \alpha_1 & \beta_1 \end{bmatrix}, A_0 = \begin{bmatrix} \sigma & a & 0 & \cdots & \cdots & 0 \\ a & \sigma & a & \cdots & \cdots & 0 \\ 0 & a & \sigma & a & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & a & \sigma & a \\ 0 & 0 & \cdots & 0 & \alpha_0 & \beta_0 \end{bmatrix},$$

where A_0 is an $N \times N$ matrix.

Now we represent $\Gamma_\lambda \mathbf{y}^T = \mathbf{0}$ as an equation involving the above block matrices, which gives the same set of eigenvalues.

Lemma 12.5. *The equation $\Gamma_\lambda \mathbf{y}^T = \mathbf{0}$ is equivalent to the following matrix equation:*

$$\left(\begin{bmatrix} A_2 & A_1 & A_0 \\ \mathbf{0} & I_N & \mathbf{0} \\ I_N & \mathbf{0} & \mathbf{0} \end{bmatrix} - \lambda \begin{bmatrix} -A_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_N \\ \mathbf{0} & I_N & \mathbf{0} \end{bmatrix} \right) \begin{bmatrix} \lambda^2 \mathbf{y} \\ \lambda \mathbf{y} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

It is straightforward to check the equivalence. We skip the proof.

In [8], a matrix in the form of $A - \lambda B$ is defined as a matrix pencil, where λ is an indeterminate. From Lemma 12.5 we can immediately claim that

Lemma 12.6. *The set of eigenvalues of DSLBVP (12.3) and the set of eigenvalues of the matrix pencil $A - \lambda B$ are identical, where*

$$A = \begin{bmatrix} A_2 & A_1 & A_0 \\ \mathbf{0} & I_N & \mathbf{0} \\ I_N & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} -A_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_N \\ \mathbf{0} & I_N & \mathbf{0} \end{bmatrix}.$$

Now we focus on investigating eigenvalues of matrix pencils.

12.4 Solving the DSLBVP as a Standard Eigenvalue Problem

When the matrix A is nonsingular, all roots of the polynomial $|A-\lambda B|$ are nonzero. Thus it is obvious that

$$\begin{aligned} |A-\lambda B| = 0 &\iff \left|\frac{1}{\lambda}A^{-1}(A-\lambda B)\right| = 0 \\ &\iff \left|\frac{1}{\lambda}I - A^{-1}B\right| = 0 \\ &\iff \left|A^{-1}B - \mu I\right| = 0, \text{ where } \mu = \frac{1}{\lambda}. \end{aligned}$$

Now finding eigenvalues of the matrix pencil $A-\lambda B$ is converted to finding the standard (nonzero) eigenvalues of the matrix $A^{-1}B$. In this case, using the matrix pencil, we can reduce the DSLBVP into a standard eigenvalue problem. The requirements of A being nonsingular is a key here. Our next task then is to investigate conditions for A to be nonsingular.

From the configuration of A in Lemma 12.6, we note that $|A| = \pm|A_0|$; thus, A is nonsingular if and only if A_0 is nonsingular. Refer to Definition 12.5. A_0 is a tridiagonal matrix with the diagonal element $\sigma > 0$ and the other nonzero number $a < 0$ which appears on the second diagonal above or below the main diagonal, except for the last row. The numbers in the last row are α_0 and β_0 which are the constant terms of $\alpha(\lambda)$ and $\beta(\lambda)$, respectively. For $0 < i < N$, let U_i be the determinant of the $i \times i$ main diagonal submatrix of A_0, that is,

$$U_i = \begin{vmatrix} \sigma & a & 0 & \cdots & \cdots & 0 \\ a & \sigma & a & \cdots & \cdots & 0 \\ 0 & a & \sigma & a & \cdots & 0 \\ & & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & a & \sigma & a \\ 0 & 0 & \cdots & 0 & a & \sigma \end{vmatrix}, \ 0 < i < N.$$

We implement a result on tridiagonal matrices obtained from [16] (see Lemma 12.3 to give an iterative formula for U_i and a formula for $|A_0|$.

Lemma 12.7. *Consider the $(N\times N)$ $(N>2)$ tridiagonal matrix A_0 defined before and the determinant U_i as above. Then:*

1. *$U_1 = \sigma$, $U_2 = \sigma^2 - a^2$, and $U_i = \sigma U_{i-1} - a^2 U_{i-2}$ for $3 \le i \le N-1$.*
2. *The determinant $U_N = |A_0| = \beta_0 U_{N-1} - a\alpha_0 U_{N-2}$.*

The proof is immediately from Lemma 12.3 and the determinant rules. Lemma 12.7 helps us to develop explicit formulas for U_i and $|A_0|$.

Lemma 12.8. *Consider the tridiagonal matrix* A_0 *and the values* U_i *as above. Let* $s_1 = \frac{\sigma+\sqrt{\sigma^2-4a^2}}{2}$ *and* $s_2 = \frac{\sigma-\sqrt{\sigma^2-4a^2}}{2}$ *be the solutions to the equation* $x^2-\sigma x+a^2=0$. *Then for* $1 \le i < N$,

$$U_i = \begin{cases} \frac{s_1^{i+1}-s_2^{i+1}}{s_1-s_2} & \textit{if} \quad \sigma^2 \neq 4a^2; \\ (1+i)s_1^i & \textit{if} \quad \sigma^2 = 4a^2. \end{cases}$$

Proof. The characteristic equation for the recursive relation $U_i = \sigma U_{i-1} - a^2 U_{i-2}$ is $x^2-\sigma x+a^2=0$. Note that $s_1 = s_2 \Longleftrightarrow \sigma^2 = 4a^2$. Thus, for each $i = 1,2,\cdots,N-1$, U_i has the following form:

$$U_i = \begin{cases} u_1 s_1^i + u_2 s_2^i & \text{if} \quad \sigma^2 \neq 4a^2; \\ u_1 s_1^i + u_2 i s_1^i & \text{if} \quad \sigma^2 = 4a^2, \end{cases}$$

where u_1, u_2 are constant complex numbers. By applying the initial conditions $U_1 = \sigma$ and $U_2 = \sigma^2 - a^2$, we obtain $u_1 = s_1/(s_1-s_2)$ and $u_2 = -s_2/(s_1-s_2)$ when $\sigma^2 \neq 4a^2$ and $u_1 = 1 = u_2$ when $\sigma^2 = 4a^2$. The result follows immediately. □

We next focus on the cases when $\sigma^2 - 4a^2 \geq 0$, that is, when s_1, s_2 are real numbers. We give an explicit formula for $|A_0|$:

Theorem 12.9. *Let* s_1, s_2 *be as above, which are the solutions to* $x^2-\sigma x+a^2=0$. *If* $\sigma^2 > 4a^2$, *then*

$$|A_0| = \frac{1}{s_1-s_2}\left[\left(\beta_0 - \frac{\alpha_0}{a}s_2\right)s_1^N - \left(\beta_0 - \frac{\alpha_0}{a}s_1\right)s_2^N\right].$$

If $\sigma^2 = 4a^2$, *then*

$$|A_0| = s_1^{N-2}\left(\beta_0 N s_1 - \alpha_0 a(N-1)\right).$$

Proof. Recall that s_1, s_2 are the real solutions of the equation $x^2-\sigma x+a^2$ with $s_1 \geq s_2$. In addition, $|A_0| = U_N = \beta_0 U_{N-1} - a\alpha_0 U_{N-2}$.
Case 1. $\sigma^2 > 4a^2$. In this case, $s_1 > s_2 > 0$ since $s_1 s_2 = a^2$ and $a < 0$. By Lemmas 12.7 and 12.8,

$$\begin{aligned} |A_0| &= \frac{\beta_0\left(s_1^N - s_2^N\right)}{s_1-s_2} - \frac{a\alpha_0\left(s_1^{N-1}-s_2^{N-1}\right)}{s_1-s_2} \\ &= \frac{1}{s_1-s_2}\left[\left(\beta_0 - \frac{\alpha_0}{a}s_2\right)s_1^N - \left(\beta_0 - \frac{\alpha_0}{a}s_1\right)s_2^N\right]. \end{aligned}$$

Case 2. $\sigma^2 = 4a^2 \Longrightarrow s_1 = s_2$, $\sigma = 2s_1$, and $a^2 = s_1^2$. Thus

$$\begin{aligned} |A_0| &= \beta_0 N s_1^{N-1} - \alpha_0 a\left(N-1\right)s_1^{N-2} \\ &= s_1^{N-2}\left[\beta_0 N s_1 - \alpha_0 a\left(N-1\right)\right]. \end{aligned}$$

□

After establishing the above results about the determinant of A_0, we now discuss conditions for A_0 to be nonsingular. We require A_0 to be nonsingular because we can then reduce the DSLBVP to a regular eigenvalue problem. We summarize these conditions in the following remark and theorem.

Theorem 12.10. *Consider the $(N \times N)$ tridiagonal matrix A_0 as above with $N > 2$ and $(\alpha_0, \beta_0) \neq (0,0)$. Then A_0 is nonsingular in any of the following cases:*

1. $\alpha_0\beta_0 = 0$, *or*
2. $\alpha_0\beta_0 \neq 0$, $\sigma^2 = 4a^2$, *and* $\alpha_0(N-1) \neq N\beta_0$, *or*
3. $\alpha_0\beta_0 \neq 0$, $\sigma^2 > 4a^2$, *and* $\frac{\alpha_0}{\beta_0} \neq \frac{s_1^N - s_2^N}{a(s_1^{N-1} - s_2^{N-1})}$.

Proof. 1. Assume $\alpha_0 = 0$. Then $\beta_0 \neq 0$. By Theorem 12.9, $|A_0| = \frac{\beta_0(s_1^N - s_2^N)}{s_1 - s_2}$ when $\sigma^2 > 4a^2$ or $|A_0| = \beta_0 N s_1^{N-1}$ when $\sigma^2 = 4a^2$. In the first case, $s_1 > s_2 > 0$ and in the second case, $s_1^2 = a^2 > 0$. It is obvious that in either case, $|A_0| \neq 0$. Similarly, $\beta_0 = 0 \Longrightarrow |A_0| \neq 0$.

2. Let $\sigma^2 = 4a^2$ and assume $|A_0| = 0$. It is equivalent to $\beta_0 N s_1 - \alpha_0 a(N-1) = 0$. As before $a < 0$ and $s_1^2 = a^2 \Longrightarrow s_1 = -a > 0$. Thus, if $\alpha_0\beta_0 \neq 0$,

$$|A_0| = 0 \Longleftrightarrow \beta_0 N + \alpha_0(N-1) = 0 \Longleftrightarrow \frac{\alpha_0}{\beta_0} = \frac{N}{1-N}.$$

By the condition $\beta_0 N + \alpha_0(N-1) \neq 0$, we claim that $|A_0| \neq 0$.

3. In case $\alpha_0\beta_0 \neq 0$ and $\sigma^2 > 4a^2$, we have $s_1 > s_2 > 0$ and

$$|A_0| = 0 \iff (\beta_0 a - \alpha_0 s_2)s_1^{N-1} = (\beta_0 a - \alpha_0 s_1)s_2^{N-1}.$$

If $|A_0| = 0$, then $\beta_0 a - \alpha_0 s_2 \neq 0$ and $\beta_0 a - \alpha_0 s_1 \neq 0$ because $s_1 \neq s_2$ and $\alpha_0 \neq 0$. With $a = s_1 s_2$, it implies

$$(\beta_0 a - \alpha_0 s_2)s_1^{N-1} = (\beta_0 a - \alpha_0 s_1)s_2^{N-1} \Longrightarrow$$
$$\frac{\alpha_0}{\beta_0} = \frac{s_1^N - s_2^N}{a(s_1^{N-1} - s_2^{N-1})}, \quad \text{a contradiction.}$$

So $|A_0| \neq 0$.

□

A quick test on the singularity of A_0 s given below:

Corollary 12.11. *Let N be an integer greater than 2 and $\sigma^2 \geq 4a^2$. Then A_0 is nonsingular if $\alpha_0\beta_0 > 0$.*

Proof. Let $\alpha_0\beta_0 > 0$. By Theorem 12.9, when $\sigma^2 = 4a^2$, $|A_0| = 0 \Longrightarrow \alpha_0/\beta_0 = N/(1-N)$, which is negative because N is a positive integer > 2. In case $\sigma^2 > 4a^2$, $|A_0| = 0 \Longrightarrow$

$$\frac{\alpha_0}{\beta_0} = \frac{s_1^N - s_2^N}{a(s_1^{N-1} - s_2^{N-1})},$$

which is also negative. In either case, a contradiction to $\alpha_0\beta_0 > 0$ occurs. Thus $|A_0| \neq 0$. □

With the above results on the nonsingularity of A_0 established, we can proceed to discuss $\left|A^{-1}B - \mu I\right|$, where

$$A = \begin{bmatrix} A_2 & A_1 & A_0 \\ \mathbf{0} & I_N & \mathbf{0} \\ I_N & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -A_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_N \\ \mathbf{0} & I_N & \mathbf{0} \end{bmatrix}.$$

By applying matrix operations,

$$A^{-1} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & I_N \\ \mathbf{0} & I_N & \mathbf{0} \\ A_0^{-1} & -A_0^{-1}A_1 & -A_0^{-1}A_2 \end{bmatrix} \quad \text{and furthermore}$$

$$A^{-1}B = \begin{bmatrix} \mathbf{0} & I_N & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_N \\ -A_0^{-1}A_3 & -A_0^{-1}A_2 & -A_0^{-1}A_1 \end{bmatrix}.$$

Therefore we solve $\left|A^{-1}B - \mu I\right| = 0$, which is a standard eigenvalue problem, to determine the solutions of the DSLBVP (12.3).

12.5 Reality of Eigenvalues

We now discuss conditions for all the eigenvalues to be real. We begin with a Lemma which shows the conditions when A_0 is similar to a symmetric matrix.

Lemma 12.12. *If $a\alpha_0 > 0$, then A_0 is similar to a symmetric matrix.*

Proof. Write A_0 in the form

$$A_0 = \begin{bmatrix} E_1 & E_2 \\ E_3 & \beta_0 \end{bmatrix},$$

where E_1 is the $(N-1)\times(N-1)$ major diagonal submatrix of A_0, $E_2 = [0 \ \cdots \ 0 \ a]^T$, and $E_3 = [0 \ \cdots \ 0 \ \alpha_0]$.

Define $Q = \begin{bmatrix} I_{N-1} & \mathbf{0} \\ \mathbf{0} & \sqrt{\frac{\alpha_0}{a}} \end{bmatrix}$ and $A_0'^{-1}A_0Q$; then

$$A_0'^{-1}A_0Q = \begin{bmatrix} E_1 & \sqrt{\frac{\alpha_0}{a}}E_2 \\ \sqrt{\frac{a}{\alpha_0}}E_3 & \beta_0 \end{bmatrix}.$$

Obviously E_1 is symmetric and $\left(\sqrt{\frac{\alpha_0}{a}}E_2\right)^T = [0 \cdots 0 \sqrt{\alpha_0 a}] = \sqrt{\frac{a}{\alpha_0}}E_3$. Therefore A_0' is symmetric and is similar to A_0. □

Example 12.13. We show an example here to demonstrate Lemma 12.5. Consider the matrix G_2 as below:

$$G_2 = \begin{bmatrix} 8 & -4 & 0 & 0 \\ -4 & 8 & -4 & 0 \\ 0 & -4 & 8 & -4 \\ 0 & 0 & -2 & 3 \end{bmatrix}, \text{which implies} \quad E_1 = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix},$$

$$E_2 = [0 \quad 0 \ -4]^T, \quad \text{and} \quad E_3 = [0 \quad 0 \ -2].$$

One can check that

$$Q^{-1}G_2Q = \begin{bmatrix} 8 & -4 & 0 & 0 \\ -4 & 8 & -4 & 0 \\ 0 & -4 & 8 & -2\sqrt{2} \\ 0 & 0 & -2\sqrt{2} & 3 \end{bmatrix}, \quad \text{which is symmetric,}$$

where $Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$.

We end the paper by giving a type of DSLBVP where all the eigenvalues are real.

Theorem 12.14. *If $D(\lambda) \equiv d_0 < 0$ and A_0 is nonsingular, then all the eigenvalues of the DSLBVP (12.3) are real.*

Proof. As stated earlier, the eigenvalues of the DSLBVP with cubic boundary condition are the eigenvalues of the matrix pencil $A - \lambda B$, where

$$A = \begin{bmatrix} A_2 & A_1 & A_0 \\ \mathbf{0} & I_N & \mathbf{0} \\ I_N & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -A_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_N \\ \mathbf{0} & I_N & \mathbf{0} \end{bmatrix}.$$

We know from linear algebra that if A and B are symmetric matrices with real entries, then the eigenvalues of the pencil are all real [8].

Given $D(\lambda) \equiv d_0 < 0$, then $\alpha(\lambda) \equiv pd_0/h = \alpha_0 < 0$, since h and p are both positive. Since $a < 0$, we have $a\alpha_0 > 0$. We can thus apply Lemma 12.12 to construct a matrix

Q_0 so that $Q_0^{-1}A_0Q_0 = A_0'$ is a nonsingular symmetric matrix. Also, $D(\lambda) = d_0$ implies that $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Thus A_1, A_2, A_3 are all diagonal and so symmetric matrices.

We use $N \times N$ matrix Q_0 as above to define a new $(3N) \times (3N)$ matrix:

$$Q = \begin{bmatrix} Q_0^{-1} & & \\ & Q_0^{-1} & \\ & & Q_0^{-1} \end{bmatrix}.$$

Then A is similar to the matrix A' by Q as follows:

$$Q^{-1} \begin{bmatrix} A_2 & A_1 & A_0 \\ \mathbf{0} & I_N & \mathbf{0} \\ I_N & \mathbf{0} & \mathbf{0} \end{bmatrix} Q = \begin{bmatrix} A_2 & A_1 & A_0' \\ \mathbf{0} & I_N & \mathbf{0} \\ I_N & \mathbf{0} & \mathbf{0} \end{bmatrix} = A'.$$

Similarly, B is similar to a symmetric matrix B' by Q:

$$Q \begin{bmatrix} -A_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_N \\ \mathbf{0} & I_N & \mathbf{0} \end{bmatrix} Q = \begin{bmatrix} -A_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_N \\ \mathbf{0} & I_N & \mathbf{0} \end{bmatrix} = B'.$$

Next we define $P = \begin{bmatrix} I_N & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_0' & A_1 \\ \mathbf{0} & \mathbf{0} & A_0' \end{bmatrix}$ and compute:

$$P(A' - \lambda B') = \begin{bmatrix} A_2 & A_1 & A_0' \\ A_1 & A_0' & \mathbf{0} \\ A_0' & \mathbf{0} & \mathbf{0} \end{bmatrix} - \lambda \begin{bmatrix} -A_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_1 & A_0' \\ \mathbf{0} & A_0' & \mathbf{0} \end{bmatrix}.$$

Denote

$$A'' = \begin{bmatrix} A_2 & A_1 & A_0' \\ A_1 & A_0' & \mathbf{0} \\ A_0' & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad \text{and} \quad B'' = \begin{bmatrix} -A_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_1 & A_0' \\ \mathbf{0} & A_0' & \mathbf{0} \end{bmatrix}.$$

We have $P(A' - \lambda B') = A'' - \lambda B''$. Since the matrix P is invertible,

$$(A' - \lambda B')\mathbf{y} = \mathbf{0} \iff P(A' - \lambda B')\mathbf{y} = \mathbf{0} \iff (A'' - \lambda B'')\mathbf{y} = \mathbf{0}.$$

Therefore, since A_3, A_2, and A_1 are symmetric and A_0 is similar to a symmetric matrix with real entries, A'' and B'' are both similar to symmetric matrices with real entries, which implies that the DSLBVP has all distinct real eigenvalues. $\square$

12.6 Conclusion and Future Directions

In this paper we discuss discrete second-order Sturm–Liouville Boundary Value Problems (DSLBVP) where the parameter λ appears nonlinearly in the boundary conditions. We focus on analyzing a DSLBVP with cubic nonlinearity in the boundary condition. We first describe the problem with a matrix equation ($\Gamma_\lambda \mathbf{y} = \mathbf{0}$) which involves the parameter λ in a cubic polynomial. We then construct a new matrix equation $(A - B\lambda)\,\mathbf{y} = \mathbf{0}$ which has the same solution space. Thus finding the eigenvalues of $\Gamma_\lambda \mathbf{y} = \mathbf{0}$ is equivalent to finding the eigenvalues of the matrix pencil $A - B\lambda$.

Since several key matrices involved are tridiagonal, we apply linear algebraic results on tridiagonal matrices and combinatorial results of general forms of iterative numbers to obtain explicit formulas for the determinants of the involved matrices. With these formulas we are able to give conditions under which the matrix A is nonsingular. When A is nonsingular, we can formulate a process of reducing the DSLBVP into a regular eigenvalue problem so that many powerful existing tools of solving eigenvalue problems can be implemented. In the last part of the paper we discuss the reality of the eigenvalues of the DSLBVP. We give conditions on the boundary constraints under which all the eigenvalues are real.

Many questions remain open. For example, what can we say when the matrix A is singular? Can we give less restricted conditions to guarantee the reality of the eigenvalues of the problem? Under what conditions will all the eigenvalues be distinct? We can further explore similar problems where the boundary conditions have higher degrees.

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Chapter 13
Approximation Formulas for the Ergodic Moments of Gaussian Random Walk with a Reflecting Barrier

Tahir Khaniyev, Basak Gever and Zulfiyya Mammadova

Abstract In this study, Gaussian random walk process with a generalized reflecting barrier is constructed mathematically. Under some weak conditions, the ergodicity of the process is discussed and exact form of the first four moments of the ergodic distribution is obtained. After, the asymptotic expansions for these moments are established. Moreover, the coefficients of the asymptotic expansions are expressed by means of numerical characteristics of a residual waiting time.

13.1 Introduction

The random walk processes with one or two barriers are being used to solve a number of very interesting problems in the fields of inventory, mathematical biology, queueing and reliability theories, etc. These barriers can be reflecting, delaying, absorbing, elastic, etc., depending on concrete problems at hand. Numerous studies have been done about random walks with one or two barriers because of their practical and theoretical importance ([1–5, 7–10, 12, 13, 15–17, 19, 21, 25], etc.). Moreover, some special real-world problems can be expressed by random walks with reflecting barriers. For example, motion of the particle with high energy in a diluted environment can be expressed by means of random walk with reflecting barriers. There are some studies in this subject in literature, as well (e.g., [3, 6, 7, 11, 14, 22–24, 26], etc.). However, these studies are generally in a theoretical character and they don't have useful results for application because of their complex structure. To remove these difficulties, it is tried to obtain simple but approximate formulas lately.

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Thus, in this study, a generalized Gaussian random walk with a reflecting barrier is investigated, and approximation formulas for the ergodic moments of this process are obtained.

The Model. In this study, we assume that the capital amount of a company is $\lambda z > 0$ at the start time. At the random times $T_n = \sum_{i=1}^{n} \xi_i, n \geq 1$ the capital of the company is increasing by coming premiums or is decreasing because of accidents. Amount of decrease or increase is represented by $\{-\eta_n\}, n \geq 1$. According to definition, the random variables $\{\eta_n\}, n \geq 1$ can take both positive and negative values. The capital level of system increases or decreases until it drops to a negative value. However, when the capital level is negative, company makes decision to take credit or debt. This amount of credit or debt is λ times of the amount of the negative part $(-\zeta_1)$ of capital level. After that, the company starts working with a new initial level of capital $(\lambda \zeta_1)$ and the changes continue until the capital level becomes negative, again. When the capital level decreases to the position $(-\zeta_2)$, the company determines new initial level of the capital $(\lambda \zeta_2)$. Next, the system continues in similar way. At a company which serves like that, the variation of the capital amount is expressed by means of a stochastic process which is called “Random walk with a generalized reflecting barrier.” Our aim is to define this process mathematically and to obtain the asymptotic results for the ergodic moments of this process.

13.2 Mathematical Construction of the Process $X(t)$

Let $\{(\xi_n, \eta_n)\}, n = 1,2,3,\ldots$, be a sequence of independent and identically distributed random variables defined on any probability space $(\Omega, \Im, I\!P)$ where the ξ_ns take only positive values and η_ns have normal distribution with the parameters $(m,1), (m>0)$. Suppose that ξ_n and η_n are mutually independent random variables. Define the renewal sequence $\{T_n\}$ and random walk $\{S_n\}$ as follows:

$$T_0 \equiv S_0 \equiv 0; \quad T_n = \sum_{i=1}^{n} \xi_i; \quad S_n = \sum_{i=1}^{n} \eta_i, \quad n = 1,2,\ldots$$

Additionally, define the following random variables:

$$N_0 = 0; \quad \zeta_0 = z \geq 0; \quad N_1 \equiv N_1(\lambda z) = \inf\{k \geq 1 : \lambda z - S_k < 0\};$$

$$\zeta_1 \equiv \zeta_1(\lambda z) = |\lambda \zeta_0 - S_{N_1}|;$$

$$N_2 \equiv N_2(\lambda \zeta_1) = \inf\{k \geq N_1 + 1 : \lambda \zeta_1 - (S_k - S_{N_1}) < 0\};$$

$$\zeta_2 = \zeta_2(\lambda \zeta_1) = |\lambda \zeta_1 - (S_{N_2} - S_{N_1})|;$$

$$\ldots$$

$$N_n \equiv N_n(\lambda \zeta_{n-1}) = \inf\{k \geq N_{n-1} + 1 : \lambda \zeta_{n-1} - (S_k - S_{N_{n-1}}) < 0\};$$

$$\zeta_n \equiv \zeta_n(\lambda \zeta_{n-1}) = |\lambda \zeta_{n-1} - (S_{N_n} - S_{N_{n-1}})|, \quad n = 1,2,\ldots$$

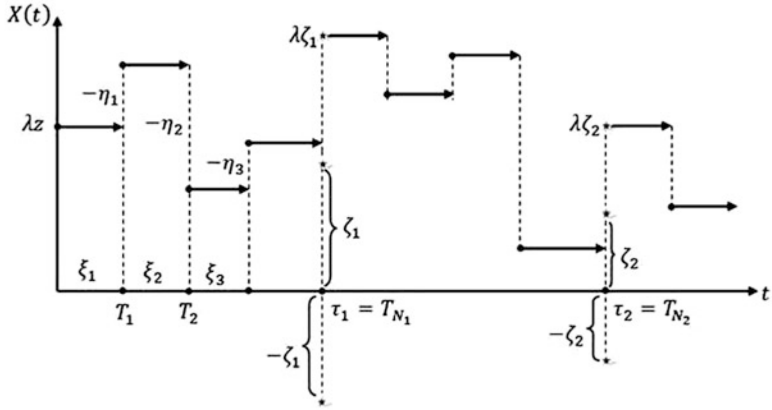

Fig. 13.1: A sample trajectory of the process $X(t)$

Using the sequence of integer-valued random variables $\{N_n\}, \quad n = 1,2,\ldots$, the following sequence $\{\tau_n\}, n = 1,2,\ldots$ is constructed:

$$\tau_0 \equiv 0, \quad \tau_1 = \tau_1(\lambda z) = \sum_{i=1}^{N_1} \xi_i, \quad \tau_2 = \sum_{i=1}^{N_2} \xi_i, \ldots, \quad \tau_n = \sum_{i=1}^{N_n} \xi_i, \ldots$$

Moreover, we put $\nu(t) = max\{n \geq 0 : T_n \leq t\}, \quad t > 0.$

Now, we can construct the desired stochastic process which is as follows:

$$X(t) \equiv \lambda \zeta_n - (S_{\nu(t)} - S_{N_n}),$$

for each $t \in [\tau_n; \tau_{n+1}), \quad n = 0,1,2,\ldots$

The following alternative representation can be given for the process $X(t)$:

$$X(t) \equiv \sum_{n=0}^{\infty} \Big(\lambda \zeta_n - (S_{\nu(t)} - S_{N_n})\Big) I_{[\tau_n;\tau_{n+1})}(t),$$

where $I_A(t)$ is indicator function of set A.

The process $X(t)$ is called as "Gaussian Random Walk with a Generalized Reflecting Barrier." The process $X(t)$ is known as "Gaussian Random Walk with a Reflecting Barrier," when $\lambda = 1$, in literature.

A sample trajectory of the considered process can be seen as in the following Fig. 13.1.

13.3 The Ergodicity of the Process $X(t)$

Before investigating the stationary characteristics of the process, we need to show that this process is ergodic. For this reason, let's state the following theorem on the ergodicity of the process $X(t)$:

Theorem 13.1. *Let the initial random variables $\{\xi_n\}$ and $\{\eta_n\}$ be satisfied the following supplementary conditions:*

1. $0 < E(\xi_1) < \infty$.
2. Random variable η_1 has a Gaussian distribution with parameters $(m,1)$, i.e.,

$$F(x) = P\{\eta_1 \leq x\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} exp\Big\{ - \frac{(u-m)^2}{2} \Big\} du.$$

3. $E(\eta_1) \equiv m > 0$.

Then, the process $X(t)$ is ergodic.

Proof. Considered process belongs to a wide class which is called "semi-Markov processes with a discrete interference of chance" in literature. The general ergodic theorem is proved in the monography of Gihman and Skorohod [8] for this class.

In order to apply this theorem, the following two assumptions should be satisfied:

Assumption 1. It is required to choose a sequence of ascending random times $(\tau_0 \equiv 0 < \tau_1 < \tau_2 < \ldots < \tau_n < \ldots)$, such that the values of the process $X(t)$ at these times $\Big(X(\tau_n)\Big)$ form an ergodic Markov chain.

For this aim, it is enough to choose the sequence $\{\tau_n\}$ defined as in the Sect. 13.2. Then, $X(\tau_n) = \lambda \zeta_n$. According to the definition of the process $X(t)$, the sequence $\{\zeta_n\}$ is an ergodic Markov chain. Hence, the first assumption is satisfied.

Assumption 2. For each $z \in (0,\infty)$, $\quad E(\tau_1) = E\Big(\tau_1(\lambda z)\Big) < \infty$ and $E(\tau_n - \tau_{n-1}) = \int_0^\infty E\Big(\tau_1(\lambda z)\Big) d\pi_\lambda(z) < \infty$, $\quad n = 2,3,\ldots$ should be hold. Here $\pi_\lambda(z)$ is the ergodic distribution of the Markov chain $\{\zeta_n\}$, $\quad n = 0,1,2,\ldots$, i.e.,

$$\pi_\lambda(z) \equiv \lim_{n\to\infty} P\{\zeta_n \leq z\}.$$

Under the conditions of Theorem 13.1, it is not difficult to see that Assumption 2 is also satisfied. Therefore, the second assumption of general ergodic theorem is hold. This means that the process $X(t)$ is ergodic and Theorem 13.1 is proved. □

Theorem 13.2. *Suppose that the conditions of Theorem 13.1 are satisfied. Then, for each measurable function $f(x)(f : [0,\infty) \to R)$, the following relation is hold, with probability 1:*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f\Big(X(s)\Big) ds = \frac{1}{E(\tau_1)} \int_{z=0}^{\infty} \int_{x=0}^{\infty} \int_{t=0}^{\infty} f(x) P_{\lambda z}\{\tau_1 \geq t; X(t) \in dx\} dt d\pi_\lambda(z), \tag{13.1}$$

where $\pi_\lambda(z)$ is the ergodic distribution of the Markov chain $\{\zeta_n\}$, $\quad n = 0,1,2.$

By the help with the basic identity for the random walk process [7] and the Theorem 13.2, we can give the following form for the characteristic function of the ergodic distribution of the process $X(t)$:

Theorem 13.3. *Assume that the conditions of Theorem 13.1 are satisfied. Then, the characteristic function $\Big(\varphi_X(\alpha)\Big)$ of the ergodic distribution of the process $X(t)$ can* *be expressed by means of the characteristics of the boundary functionals* $N_1(\lambda z)$ *and* $S_{N_1(\lambda z)}$*, as follows:*

$$\varphi_X(\alpha) = \frac{1}{E\Big(N_1(\lambda\zeta)\Big)} \int_0^\infty e^{i\alpha\lambda z} \frac{\varphi_{S_{N_1(\lambda z)}}(-\alpha) - 1}{\varphi_\eta(-\alpha) - 1} d\pi_\lambda(z), \tag{13.2}$$

where ζ *is a random variable having distribution* $\pi_\lambda(z)$*, i.e.,* $P\{\zeta \leq z\} \equiv \pi_\lambda(z) \equiv \lim_{n\to\infty} P\{\zeta_n \leq z\}$*. Moreover,* $\varphi_\eta(-\alpha) \equiv E\Big(exp(-i\alpha\eta_1)\Big)$*,* $\varphi_{S_{N_1(\lambda z)}}(-\alpha) \equiv E\Big(exp(-i\alpha S_{N_1(\lambda z)})\Big)$*,* $\quad E(N_1(\lambda\zeta)) \equiv \int_0^\infty E\Big(N_1(\lambda z)\Big) d\pi_\lambda(z)$*.*

We can acquire many valuable results by means of the relation (13.2). In this study, from the relation (13.2), the exact expressions for the first four ergodic moments of the process $X(t)$ are derived.

13.4 The Exact Expressions for the Ergodic Moments of the Process $X(t)$

The aim of this section is to obtain the exact expressions for the moments of ergodic distribution of the process $X(t)$. Therefore, let's give the following notations: $m_n = E(\eta_1)$; $\quad m_{n1} = \frac{m_n}{nm_1}$; $\quad M_n(z) \equiv E\Big(S^n_{N_1(z)}\Big)$; $\quad E(X^n) \equiv \lim_{t\to\infty} E(X^n(t))$;

$$E\Big(\zeta^r M_n(\zeta)\Big) \equiv \int_0^\infty z^r M_n(z) d\pi_\lambda(z), \quad r = 0,1,2,\ldots, \quad n = 1,2,\ldots$$

Theorem 13.4. *In addition to the conditions of Theorem 13.1, assume that* $E\Big(|\eta_1|^{n+1}\Big) < \infty$ *is satisfied. Then, the* n^{th} *moment of the ergodic distribution exists, and it can be shown as follows, for each* $n = 1,2,3,4$*:*

$$E(X) = \frac{1}{E\Big(M_1(\lambda\zeta)\Big)} \Big\{E\Big(\lambda\zeta M_1(\lambda\zeta)\Big) - \frac{1}{2}E\Big(M_2(\lambda\zeta)\Big) + A_1 E\Big(M_1(\lambda\zeta)\Big)\Big\};$$

$$E(X^2) = \frac{1}{E\Big(M_1(\lambda\zeta)\Big)} \Big\{E\Big(\lambda^2\zeta^2 M_1(\lambda\zeta)\Big) - E\Big(\lambda\zeta M_2(\lambda\zeta)\Big) + \frac{1}{3}E\Big(M_3(\lambda\zeta)\Big)$$
$$+A_1[2E\Big(\lambda\zeta M_1(\lambda\zeta)\Big) - E\Big(M_2(\lambda\zeta)\Big)] + A_2 E\Big(M_1(\lambda\zeta)\Big)\Big\};$$

$$E(X^3) = \frac{1}{E\Big(M_1(\lambda\zeta)\Big)} \Big\{E\Big(\lambda^3\zeta^3 M_1(\lambda\zeta)\Big) - \frac{3}{2}E\Big(\lambda^2\zeta^2 M_1(\lambda\zeta)\Big) + E\Big(\lambda\zeta M_3(\lambda\zeta)\Big)$$

$$
-\frac{1}{4}E\Big(M_4(\lambda\zeta)\Big)+A_1\Big[3E\Big(\lambda^2\zeta^2M_1(\lambda\zeta)\Big)-3E\Big(\lambda\zeta M_2(\lambda\zeta)\Big)+E\Big(M_3(\lambda\zeta)\Big)\Big]
$$
$$
+3A_2\Big[E\Big(\lambda\zeta M_1(\lambda\zeta)\Big)-\frac{1}{2}E\Big(2M_2(\lambda\zeta_1)\Big)\Big]+3A_3E\Big(M_1(\lambda\zeta)\Big)\Big\}
$$

$$
E(X^4)=\frac{1}{E\Big(M_1(\lambda\zeta)\Big)}\Big\{E\Big(\lambda^4\zeta^4M_1(\lambda\zeta)\Big)-2E\Big(\lambda^3\zeta^3M_2(\lambda\zeta)\Big)-E\Big(\lambda^2\zeta^2M_3(\lambda\zeta)\Big)
$$
$$
-E\Big(\lambda\zeta M_4(\lambda\zeta)\Big)+\frac{1}{5}E\Big(M_5(\lambda\zeta)\Big)+A_1\Big[4E\Big(\lambda^3\zeta^3M_1(\lambda\zeta)\Big)
$$
$$
-6E\Big(\lambda^2\zeta^2M_2(\lambda\zeta)\Big)+4E\Big(\lambda\zeta M_3(\lambda\zeta)\Big)-E\Big(M_4(\lambda\zeta)\Big)\Big]
$$
$$
+2A_2\Big[3E\Big(\lambda^2\zeta^2M_1(\lambda\zeta)\Big)-3E\Big(\lambda\zeta M_2(\lambda\zeta)\Big)+E\Big(M_3(\lambda\zeta)\Big)\Big]
$$
$$
+6A_3\Big[2E\Big(\lambda\zeta M_1(\lambda\zeta)\Big)-E\Big(M_2(\lambda\zeta)\Big)\Big]+3A_4E\Big(M_1(\lambda\zeta)\Big)\Big\}
$$

where $A_1=m_{21}$; $A_2=2m_{21}^2-m_{31}$; $A_3=(1/3)m_{41}-2m_{21}m_{31}+2m_{21}^3$; $A_4=4m_{21}^4-6m_{21}^2m_{31}+m_{31}^2-(1/6)m_{51}$; $m_{k1}=m_k/km_1$; $m_k=E(\eta_1^k), k=\overline{1,5}$.

Remark 13.5. As seen in Theorem 13.4, the exact expressions for the first four moments are written. It is difficult to apply these exact expressions in practice. Consequently, instead of the exact expressions for the ergodic moments, it is advisable to have asymptotic expansions or approximation expressions. In order to get asymptotic expansions, first of all, the moments of the boundary functional $S_{N_1(z)}$ should be investigated.

13.5 Asymptotic Expansions for the Moments of Boundary Functional $S_{N_1(z)}$

In the previous section, the first four moments of the process $X(t)$ have been expressed by means of the first five moments of the boundary functional $S_{N_1(z)}$. However, it is difficult to compute these expressions. Therefore, in this study, using asymptotic methods, we aim to obtain asymptotic expansions for the moments of the process $X(t)$. For this reason, we will first get the asymptotic expansions for the moments of the boundary functional $S_{N_1(z)}$, when $z\to\infty$. We will use the ladder heights and ladder epochs of the random walk $\{S_n\}, n\geq 0$. Hence, let's give the definition of the first ladder epoch (ν_1^+) and the first ladder height (χ_1^+) as follows, respectively:

$$
\nu_1^+=\inf\{n\geq 1: S_n>0\};\quad \chi_1^+=S_{\nu_1^+}=\sum_{i=0}^{\nu_1^+}\eta_i.
$$

These special random variables ν_1^+ and χ_1^+ have an important role in the investigation of random walks (see, [7], p. 391). Suppose that $\{\nu_n^+\}$ and $\{\chi_n^+\}, n\geq 1$ are sequences of independent and positive-valued random variables having the same distribution with the random variables ν_1^+ and χ_1^+, respectively. Let $H(z)$ represent the renewal process which is generated by the random variables $\chi_n^+, n \geq 1$, i.e.,

$$H(z) = \inf\Big\{n \geq 1 : \sum_{i=1}^{n} \chi_i^+ > z\Big\}, z \geq 0.$$

According to the principle of E. Dynkin, the boundary functionals $N_1(z)$ and $S_{N_1(z)}$ can be expressed by means of the random variables $\{\nu_n^+\}$ and $\{\chi_n^+\}, n \geq 1$, as follows:

$$N_1(z) = \sum_{i=1}^{H(z)} \nu_i^+; \quad S_{N_1(z)} = \sum_{i=1}^{H(z)} \chi_i^+$$

The renewal function generated by ladder heights $\chi_n^+, n \geq 1$, is denoted by the notation $U_+(z)$, i.e.,

$$U_+(z) \equiv E(H(z)) = 1 + \sum_{n=0}^{\infty} F_+^{*(n)}(z), \quad z \geq 0,$$

where $F_+^{*(n)}(z)$ represents the n-fold convolution multiplication of the distribution function $F_+(z) \equiv P\{\chi_1^+ \leq z\}$. In this part, the aim is to obtain the asymptotic results for the following integrals:

$$E\Big(\lambda^n \zeta^n M_k(\lambda\zeta)\Big) = \int_{z=0}^{\infty} (\lambda z)^n M_k(\lambda z) d\pi_\lambda(z), \quad n = 0, 1, 2, \ldots, \quad k = 1, 2, \ldots, \tag{13.3}$$

where $M_k(z) \equiv E\Big(S_{N_1(z)}^k\Big), \quad k = \overline{1,5}$ and $\pi_\lambda(z)$ is the ergodic distribution of the Markov chain $\{\zeta_n\}$ and at the same time it is the distribution of the random variable ζ. The main aim of this section is to get the asymptotic expansion with two terms for the moments $M_k(z)$, when $z \to \infty$. Using the study of Rogozin [20], the following lemmas can be given:

Lemma 13.6. *Suppose that* $E\Big(|\eta_1^3|\Big) < \infty$. *Then, the following asymptotic expansions with two terms for the first five moments* $\Big(M_k(z)\Big)$ *of the boundary functional* $S_{N_1(z)}$ *can be written:*

$$M_n(z) \equiv E\Big(S_{N_1}^n(z)\Big) = z^n + n\mu_{21} z^{n-1} + \frac{1}{2} n(n-1)\mu_{31} z^{n-2} + o(z^{n-2}), \quad n = \overline{1,5},$$

where $\mu_k \equiv E(\chi_1^{+k}); \mu_{k1} = \mu_k / k\mu_1, k = 2,3$ *and the random variable* χ_1^+ *is the first ladder height of the random walk* $\{S_n\}$. □

Our aim is to obtain the asymptotic expansions with two terms for the integrals in Eq. (13.3), when $\lambda \to \infty$. Hence, we should first give the following lemma:

Lemma 13.7. *Assume that $E(\chi_1^{+2}) < \infty$. Then, for each $n = 0,1,2,3\ldots$, the following asymptotic expressions with three terms can be written, when $\lambda \to \infty$:*

$$E\Big(\zeta^n M_k(\lambda\zeta)\Big) = \lambda^k \beta_{n+k} + \lambda^{k-1}\mu_{21}\beta_{n+k-1} + o(\lambda^{k-1}), \quad k = \overline{1,5}.$$

Here, $\beta_r \equiv E(\zeta^r), r = 1,2,3;$ $\quad \mu_{21} = \mu_2/2\mu_1$.

Proof. • According to Lemma 13.6, the following equality is true:

$$M_1(\lambda z) = \lambda z + \mu_{21} + g_1(\lambda z).$$

Here $\lim_{\lambda\to\infty} g_1(\lambda z) = 0$. Taking this expansion into account,

$$E\Big(M_1(\lambda\zeta)\Big) \equiv \lambda\beta_1 + \mu_{21} + o(1)$$

can be obtained. Here $\beta_n \equiv E(\zeta^n), \quad n = 1,2,\ldots; \quad \beta \equiv \beta_1 \equiv E(\zeta)$; $\mu_{21} = \mu_2/2\mu_1$. On the other hand, for each $n = 1,2,3,\ldots$, the following expansion can be obtained:

$$\begin{aligned} E\Big(\lambda^n\zeta^n M_1(\lambda\zeta)\Big) &= \int_0^\infty (\lambda z)^n M_1(\lambda z) d\pi_\lambda(z) \\ &= \int_0^\infty (\lambda z)^n \Big(\lambda z + \mu_{21} + g_1(\lambda z)\Big) d\pi_\lambda(z) \\ &= \int_0^\infty \lambda^{n+1} z^{n+1} d\pi_\lambda(z) + \mu_{21}\int_0^\infty \lambda^n z^n d\pi_\lambda(z) \\ &\quad + \int_0^\infty \lambda^n z^n g_1(\lambda z) d\pi_\lambda(z) \\ &= \lambda^{n+1}E(\zeta^{n+1}) + \lambda^n\mu_{21}E(\zeta^n) + o(\lambda^n) \\ &= \lambda^{n+1}\beta_{n+1} + \lambda^n\mu_{21}\beta_n + o(\lambda^n), \end{aligned}$$

where $\beta_n \equiv E(\zeta^n) = \int_0^\infty z^n d\pi_\lambda(z)$.

• In Lemma 13.6, it can be shown that

$$M_2(\lambda z) = (\lambda z)^2 + 2\mu_{21}\lambda z + \lambda z g_2(\lambda z).$$

Here $\lim_{\lambda\to\infty} g_2(\lambda z) = 0$. In this case, for each $n = 0,1,2,\ldots$, the following expansion can be found:

$$\begin{aligned} E\Big(\lambda^n\zeta^n M_2(\lambda\zeta)\Big) &= \int_0^\infty (\lambda z)^n M_2(\lambda z) d\pi_\lambda(z) \\ &= \int_0^\infty (\lambda z)^n \Big((\lambda z)^2 + 2\mu_{21}\lambda z + \lambda z g_2(\lambda z)\Big) d\pi_\lambda(z) \\ &= \int_0^\infty (\lambda z)^{n+2} d\pi_\lambda(z) + 2\mu_{21}\int_0^\infty (\lambda z)^{n+1} d\pi_\lambda(z) \\ &\quad + \int_0^\infty (\lambda z)^{n+1} g_2(\lambda z) d\pi_\lambda(z) \end{aligned}$$

$$= \lambda^{n+2}E(\zeta^{n+2}) + \lambda^{n+1}2\mu_{21}E(\zeta^{n+1}) + o(\lambda^{n+1})$$
$$\lambda^{n+2}\beta_{n+2} + \lambda^{n+1}2\mu_{21}\beta_{n+1} + o(\lambda^{n+1}).$$

Thus, the second part of the proof is completed.

- In Lemma 13.6, the following expansion is indicated:

$$M_3(z) = (\lambda z)^3 + 3\mu_{21}(\lambda z)^2 + (\lambda z)^2 g_3\Big((\lambda z)\Big)$$

Here, $\lim_{z\to\infty} g_3(\lambda z) = 0$. In this case, for each $n = 0, 1, 2, \ldots$, the following expansion can be written:

$$\begin{aligned}
E\Big(\lambda^n \zeta^n M_3(\lambda\zeta)\Big) &= \int_0^\infty (\lambda z)^n M_3(\lambda z) d\pi_\lambda(z) \\
&= \int_0^\infty (\lambda z)^n \Big((\lambda z)^3 + 3\mu_{21}(\lambda z)^2 + (\lambda z)^2 g_3\Big((\lambda z)\Big)\Big) d\pi_\lambda(z) \\
&= \int_0^\infty (\lambda z)^{n+3} d\pi_\lambda(z) + 3\mu_{21}\int_0^\infty (\lambda z)^{n+2} d\pi_\lambda(z) \\
&\quad + \int_0^\infty (\lambda z)^{n+2} g_3\Big((\lambda z)\Big) d\pi_\lambda(z) \\
&= \lambda^{n+3}E(\zeta^{n+3}) + \lambda^{n+2}3\mu_{31}E(\zeta^{n+2}) + o(\lambda^{n+2}) \\
&= \lambda^{n+3}\beta_{n+3} + \lambda^{n+2}3\mu_{21}\beta_{n+2} + o(\lambda^{n+2}).
\end{aligned}$$

Hence, the third part of the proof is completed.

- In Lemma 13.6, the following expansion is shown:

$$M_4(z) = (\lambda z)^4 + 4\mu_{21}(\lambda z)^3 + (\lambda z)^3 g_4\Big((\lambda z)\Big),$$

where $\lim_{\lambda\to\infty} g_4(\lambda z) = 0$. In this case, for each $n = 0, 1, 2, \ldots$, the following expansion can be obtained:

$$\begin{aligned}
E\Big(\lambda^n \zeta^n M_4(\lambda\zeta)\Big) &= \int_0^\infty (\lambda z)^n M_4(\lambda z) d\pi_\lambda(z) \\
&= \int_0^\infty (\lambda z)^n \Big((\lambda z)^4 + 4\mu_{21}(\lambda z)^3 + (\lambda z)^3 g_4\Big((\lambda z)\Big)\Big) d\pi_\lambda(z) \\
&= \int_0^\infty (\lambda z)^{n+4} d\pi_\lambda(z) + 4\mu_{21}\int_0^\infty (\lambda z)^{n+3} d\pi_\lambda(z) \\
&\quad + \int_0^\infty (\lambda z)^{n+3} g_4\Big((\lambda z)\Big) d\pi_\lambda(z) \\
&= \lambda^{n+4}E(\zeta^{n+4}) + \lambda^{n+3}4\mu_{21}E(\zeta^{n+3}) + o(\lambda^{n+3}) \\
&= \lambda^{n+4}\beta_{n+4} + \lambda^{n+3}4\mu_{21}\beta_{n+3} + o(\lambda^{n+3}).
\end{aligned}$$

This completes the fourth part of the proof.

- In Lemma 13.6, $M_5(z)$ can be shown as follows:

$$M_5(z) = (\lambda z)^5 + 5\mu_{21}(\lambda z)^4 + (\lambda z)^4 g_5\Big((\lambda z)\Big)$$

Here, $\lim_{\lambda\to\infty} g_5(\lambda z) = 0$ In this case, for each $n = 0, 1, 2, \ldots$, the following expansion can be obtained:

$$\begin{aligned}
E\Big(\lambda^n \zeta^n M_5(\lambda\zeta)\Big) &= \int_0^\infty (\lambda z)^n M_5(\lambda z) d\pi_\lambda(z) \\
&= \int_0^\infty (\lambda z)^n \Big((\lambda z)^5 + 5\mu_{21}(\lambda z)^4 + (\lambda z)^4 g_5\Big((\lambda z)\Big)\Big) d\pi_\lambda(z) \\
&= \int_0^\infty (\lambda z)^{n+5} d\pi_\lambda(z) + 5\mu_{21} \int_0^\infty (\lambda z)^{n+4} d\pi_\lambda(z) \\
&\quad + \int_0^\infty (\lambda z)^{n+4} g_5\Big((\lambda z)\Big) d\pi_\lambda(z) \\
&= \lambda^{n+5} E(\zeta^{n+5}) + \lambda^{n+4} 5\mu_{21} E(\zeta^{n+4}) + o(\lambda^{n+4}) \\
&= \lambda^{n+5}\beta_{n+5} + \lambda^{n+4} 5\mu_{21}\beta_{n+4} + o(\lambda^{n+4})
\end{aligned}$$

is hold and this completes the fifth part of the proof.

Since the five part is proved, the proof of Lemma 13.7 is completed. Thus, the behavior of the integrals related with the boundary functional of $S_{N_1(\lambda z)}$ is investigated. □

Using the asymptotic expansions for the moments of boundary functional $S_{N_1(z)}$ above, it is possible to obtain the asymptotic expansions for the ergodic moments of the process $X(t)$.

13.6 The Asymptotic Expansions for the Moments of the Process $X(t)$

The aim of this section is to obtain the asymptotic expansions with two terms for the first four ergodic moments $\Big(E(X^k), k = \overline{1,4}\Big)$ of the process $X(t)$, when $\lambda \to \infty$. The main result of this section can be given with the following theorem:

Theorem 13.8. *Under the conditions $E(\eta_1) > 0$ and $E(|\eta_1^3|) < \infty$, the following asymptotic expansions can be written, when $\lambda \to \infty$:*

$$E(X^n) = \lambda^n \beta_{(n+1),1} + \lambda^{n-1} D_n + o(\lambda^{n-1}), n = \overline{1,4},$$

where $D_n = n m_{21}\beta_{n1} - \mu_{21} c_{(n+1),1}$, $n = \overline{1,4}$; $\beta_n = E(\zeta^n)$; $\beta_{n1} = \beta_n / n\beta_1$; $c_{n1} = \beta_n / n\beta_1^2$, $n = \overline{2,5}$; $m_{n1} = m_n / n m_1$; $\mu_{n1} = \mu_n / n\mu_1$; $m_n = E(\eta_1^n)$; $\mu_n = E(\chi_1^{+n})$.

Proposition 13.9. *Assume that $E(\chi_1^{+(n+1)}) < \infty$. Then, the following asymptotic relation can be written, when $\lambda \to \infty$*

$$\beta_n \equiv E(\zeta^n) \to \frac{\mu_{n+1}}{(n+1)\mu_1}, \quad n = 1, 2, \dots,$$

where $\mu_n = E(\chi_1^{+n})$.

Proof. The random variable ζ has the distribution $\pi_\lambda(z)$ and $\pi_\lambda(z)$ is an ergodic distribution of the Markov chain $\{\zeta_n\}$. On the other hand, $\pi_\lambda(z)$ converges $\pi_0(z)$, when $\lambda \to \infty$, i.e., (Feller [7])

$$\pi_\lambda(z) \to \pi_0(z) \equiv \frac{1}{\mu_1} \int_0^z \Big(1 - F_+(x)\Big) dx.$$

In other words, the random variable ζ expresses the residual waiting time of the renewal process generated by ladder heights $\{\chi_n^+\}$. According to Rogozin [20], in this case, the following relation is true, when $\lambda \to \infty$:

$$\beta_n \equiv E(\zeta^n) \to \frac{\mu_{n+1}}{(n+1)\mu_1}.$$

Therefore, $\beta_n = \frac{\mu_{n+1}}{(n+1)\mu_1} + o(1)$, as $\lambda \to \infty$. □

Corollary 13.10. *Under the conditions of Theorem 13.8, the following asymptotic relation can be written:*

$$\beta_{n1} = \frac{\beta_n}{n\beta_1} = \frac{2\mu_{n+1}}{n(n+1)\mu_2} + o(1) \tag{13.4}$$

where $\mu_n = E(\chi_1^{+n})$.

Theorem 13.11. *Suppose that $E(\chi_1^{+(n+2)}) < \infty$. Then, the following expansions can be written, when $\lambda \to \infty$:*

$$E(X^n) = \frac{2}{(n+1)(n+2)\mu_2} \{\mu_{n+2}\lambda^n + [(n+2)m_{21}\mu_{n+1} - \mu_{n+2}]\lambda^{n-1}\} + o(\lambda^{n-1}).$$

Proof. Taking the Corollary 13.10 into consideration, we can get the proof of Theorem 13.11. □

To compute $E(X^n)$, it is necessary to know $\mu_n = E(\chi_1^{+n})$. The random variable χ_1^+ is a ladder height of the random walk $S_n = \sum_{i=1}^n \eta_i$. In this study, $m_1 = E(\eta_1) \equiv m > 0$. As known, computing the moments of the ladder heights is a very complicated problem. However, for some cases (e.g., Gaussian random walk), when $m = 0$, the exact expressions for the first five moments of the first ladder height (χ_1^+) have been obtained. To be able to use these results, let's express the moments

$\mu_n \equiv \mu_n(m), m > 0$ by means of $\mu_n(0) \equiv \mu_n(m)|_{m=0}$. In the study of Siegmund [21], the relation between these two various types of moments is established as follows:

$$\mu_n(m) = \mu_n(0) + \frac{n}{n+1}\mu_{n+1}(0)m + o(m), n = 1,2,\ldots$$

On the other hand, in the studies of Spitzer [22], Chang and Peres [5], Lotov [17], and Nagaev [18], the following exact expressions have been obtained for $\mu_n(0)$, $n = \overline{1,5}$:

$$\mu_1(0) = \frac{\sqrt{2}}{2} \quad (Spitzer); \quad \mu_2(0) = A \quad \text{(Lotov)};$$

$$\mu_3(0) = \frac{3\sqrt{2}}{8}(1+2A^2) \quad \text{(Chang and Peres)};$$

$$\mu_4(0) = \frac{3}{2}A + A^3 + B \quad \text{(Chang and Peres)}$$

$$\mu_5(0) = \frac{5\sqrt{2}}{32}\{5 + 12A^2 + 4A^4 + 16AB\} \quad \text{(Nagaev)}.$$

Here $A = \frac{-\zeta(1/2)}{\sqrt{\pi}}; \quad B = \frac{\zeta(3/2)}{\pi^{3/2}}$. Moreover, $\zeta(x)$ is Riemann zeta function in here.

Using these exact expressions, for the moments $\mu_n(0), n = \overline{1,5}$, the following values can be obtained:

$$\mu_1(0) = 0.707106781\ldots; \quad \mu_2(0) = 0.823893771\ldots; \quad \mu_3(0) = 1.250307211\ldots;$$

$$\mu_4(0) = 2.264330947\ldots; \quad \mu_5(0) = 4.678835252\ldots$$

Using these knowledge, we can state the following theorem:

Theorem 13.12. *Suppose that $E(\chi_1^{+(n+3)}) < \infty$. Then, the following expansions with approximation coefficients are hold, when $m \to 0$ and $\lambda \to \infty$, for each $n = 1,2,3,4$:*

$$E(X^n) = \frac{2}{(n+1)(n+2)\mu_2(0)}\left\{[A_n + mB_n]\lambda^n - [C_n + mD_n]\lambda^{n-1} + o(\lambda^{n-1})\right\},$$

where $A_n = \mu_{n+2}(0); \quad B_n = [(n+2)/(n+3)]\mu_{n+3}(0) - [(2\mu_3(0))/(3\mu_2(0)]\mu_{n+2}(0);$
$C_n = \mu_{n+2}(0) - (n+2)m_{21}\mu_{n+1}(0); \quad D_n = [(n+2)/(n+3)]\mu_{n+3}(0)$
$-\Big((n+1)m_{21} + [(2\mu_3(0))/(3\mu_2(0))]\Big)\mu_{n+2}(0) + [(2\mu_3(0))/(3\mu_2(0))](n+2)m_{21}$
$\mu_{n+1}(0); m_{21} = m_2/(2m_1); \quad m_k = E(\eta_1^k); \quad \mu_k(0) = E(\chi_1^{+k})|_{m=0}, \quad k = 1,2,\ldots$

13.7 Conclusion

It is known that many interesting problems in stock control, queuing, reliability theory, etc. can be expressed by means of Gaussian random walk and its modifications. In addition, the probability and numerical characteristics of these processes are generally expressed by the Wiener–Hopf factorization components. To compute the factorization components is quite difficult. In order to remove this difficulty, in this study, a random walk with a reflecting barrier is investigated by using a new asymptotic approach. Additionally, the exact and asymptotic expansions for the first four moments of the ergodic distribution are found. Moreover, using the formula of Siegmund, asymptotic expansions with approximated coefficient for the first four moments of the process are also obtained. The main result which is quite useful for application is stated by Theorem 13.12. Especially, the leading terms of the asymptotic expansions coincided with the moments of a residual waiting time. This information gives us a clue that the ergodic distribution of the process can converge to the limit distribution of a residual waiting time. Finally, the approximation methods used in this study can be applied to random walk with another type of barrier, e.g., delaying, elastic, and absorbing.

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Chapter 14
A Generalization of Some Orthogonal Polynomials

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Abstract In this paper we show how the action of operators $L^k_{e_1e_2}$ to the sequences $\sum_{j=0}^{\infty} a j\ e_1^j z^j$ allows us to obtain an alternative approach of Fibonacci numbers and some results of Foata and other results on Tchebychev polynomials of first and second kind.

14.1 Introduction

By studying the Fibonacci sequence, we note its close connection with the equation $x^2 = x+1$, where whose roots are the golden numbers Φ_1 and Φ_2, and with the matrix

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

where the eigenvalues are also two golden numbers [4], we then have (Vieta formulas) $1 = \sigma_1 = \lambda_1 + \lambda_2$ and $1 = \sigma_2 = -\lambda_1\lambda_2$ (where λ_1 and λ_2 are the two roots of the equation (real)). So the eigenvectors of M are proportional to $v_1 = \binom{\lambda_1}{1}$ and $v_2 = \binom{\lambda_2}{1}$.

If we assume that $|\lambda_1| > |\lambda_2|$, we have (see [1])

$$M^n = \begin{pmatrix} S_n(\lambda_1+\lambda_2) & -\lambda_1\lambda_2 S_{n-1}(\lambda_1+\lambda_2) \\ S_{n-1}(\lambda_1+\lambda_2) & -\lambda_1\lambda_2 S_{n-2}(\lambda_1+\lambda_2) \end{pmatrix} \text{ avec : } S_n(\lambda_1+\lambda_2) = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2}.$$

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In this chapter, given an alphabet $E = \{e_1, e_2\}$ and define in Sect. 14.3 the operator $L^k_{e_1 e_2}$ to the sequences $\sum_{j=0}^{\infty} a_j e_1^j z^j$. After that, we will give an important result (Theorem 14.2) which allows us doing some customization on the above alphabet to obtain the results of Foata [3] and other results obtained.

14.2 Preliminaries

14.2.1 Definition of Symmetric Functions in Several Variables

A function of several variables is said to be symmetric if its value does not change when we permute the variables.

Consider an equation of degree n :

$$(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)\ldots(x-\lambda_n)=0,$$

with n real or complex roots λ_1, λ_2,..., λ_n. If we develop the left side, we obtain [5]

$$x^n - \sigma_1 x^{n-1} + \sigma_2 x^{n-2} - \sigma_3 x^{n-3} + \ldots + (-1)^n \sigma_n = 0, \tag{14.1}$$

where $\sigma_1, \sigma_2, \ldots, \sigma_n$ are homogeneous and symmetrical polynomials in $\lambda_1, \lambda_2, \ldots, \lambda_n$. To be more accurate, these polynomial can be noted $\sigma_i(\lambda_1, \lambda_2, \ldots, \lambda_n)$, or $\sigma_i^{(n)}$ (if we want to specify only the number of roots):

$$(X-\lambda_1)(X-\lambda_2)(X-\lambda_3)\ldots(X-\lambda_n).$$

These polynomials σ are called elementary symmetric functions of roots [5]. For an equation of degree 2 ($n=2$), we have two roots: λ_1 and λ_2, where

$$\begin{cases} \sigma_0 = 1 \\ \sigma_1 = \lambda_1 + \lambda_2 \\ \sigma_2 = \lambda_1 \lambda_2. \end{cases} \tag{14.2}$$

The general formula is $\sigma_m(\lambda_1, \lambda_2, \ldots, \lambda_n) = \sum\limits_{i_1+i_2+\ldots+i_n=m} \lambda_1^{i_1} \lambda_2^{i_2} \ldots \lambda_n^{i_n}$ with $i_1, i_2, \ldots, i_n = 0$ or 1. $\sigma_m^{(n)}$ is the sum of all distinct products that can be formed by monomials polynomial C_n^m of degree n. ($C_n^m = \frac{n!}{m!(n-m)!}$) $\sigma_m^{(n)}$ vanishes for $m > n$.

14.2.2 Symmetric Functions

Let A, B two finished sets of indeterminates (called alphabets), we denote by $S_j(A-B)$ the coefficients of the rational sequence of poles A and zeros B [1, 2]:

$$\sum_{j=0}^{\infty} S_j(A-B)z^j = \frac{\prod_{b\in B}(1-bz)}{\prod_{a\in A}(1-az)}. \tag{14.3}$$

The polynomial whose roots are B is written as

$$S_j(x-B), \text{with } cardB = j$$

or in the case where A has cardinality 1 (that is to say $A = \{x\}$). It is clear that

$$\frac{\prod_{b\in B}(1-bz)}{(1-xz)} = 1+\ldots..+z^{j-1}S_{j-1}(x-B)+z^j\frac{S_j(x-B)}{(1-xz)}, \tag{14.4}$$

where $S_{j+k}(x-B) = x^k S_j(x-B)$ and this equality for all $k \geq 0$. The separation of the numerator and denominator of equality (14.3), obtained by successively placing $A = \phi$ and $B = \phi$, gives

$$S_n(A-B) = \sum_{j=0}^{n} S_{n-j}(-B)S_j(A). \tag{14.5}$$

The summation is actually limited to a finite number of terms, since for all $k > 0$, $S_{-k}(.) = 0$ [1, 2]. In particular,

$$\prod_{b\in B}(x-b) = S_j(x-B) = x^j S_0(-B)+x^{j-1}S_1(-B)+x^{j-2}S_2(-B)+\ldots, \tag{14.6}$$

where the $S_k(-B)$ are the coefficients of the polynomials $S_j(x-B)$, $0 \leq k \leq j$; these coefficients are zero for $k > j$, for example, if all $b \in B$ are equal (that is to write $B = nb$). We have $S_j(x-nb) = (x-b)^n$, and by specializing $b = 1$, i.e., $B = \{1,1,\ldots,1\}$, we obtain

$$S_k(-j) = (-1)^k\binom{j}{k} \quad \text{et } S_k(j) = \binom{j+k-1}{k}. \tag{14.7}$$

There is another manner of writing the previous polynomials by showing the binomial coefficients. If n is a positive integer, E is an alphabet and x an indeterminate. According to (14.4) and (14.7) we have [1, 2]

$$S_j(E-nx) = S_j(E) - \binom{j}{1}xS_{j-1}(E)+\ldots\ldots\pm\binom{j}{j}x^j, \tag{14.8}$$

where $S_k(-jx) = (-x)^k\binom{j}{k}$.

14.2.3 Divided Difference

We define operators on the polynomial ring that extend to these rings many properties of symmetric functions. So for any pair (x_i, x_{i+1}) we can associate the divided difference $\partial_{x_i x_{i+1}}$, defined by

$$\partial_{x_i x_{i+1}}(f) = \frac{f\ (x_1, x_2, ..x_i, x_{i+1} \ldots) - f\ (x_1, x_2, ..x_{i+1}, x_i \ldots)}{x_i - x_{i+1}}. \tag{14.9}$$

14.3 The Major Formulas

We deduce that the inverse of the sequence $\sum_{j=0}^{\infty} a_j\, z^j$ is the sequence $\frac{1}{\sum_{j=0}^{\infty} b_j z^j}$ that is

$$\sum_{j=0}^{\infty} a_j z^j = \frac{1}{\sum_{j=0}^{\infty} b_j z^j}. \tag{14.10}$$

We define the symmetric operator L_{xy}^k by

$$L_{xy}^k f\ (x) = \frac{x^k\ f\ (x) - y^k\ f\ (y)}{x - y}. \tag{14.11}$$

If $f\ (x) = x$, the operator (14.11) gives us

$$L_{xy}^k x = S_k\ (x + y). \tag{14.12}$$

Proposition 14.1. *Let* $E = \{e_1, e_2\}$ *,we define for any integer natural* k *the operator* $L_{e_1 e_2}^k$:

$$L_{e_1 e_2}^k f\ (e_1) = S_{k-1}(e_1 + e_2) f\ (e_1) + e_2^k \partial_{e_1 e_2} f\ (e_1). \tag{14.13}$$

Our result is as follows:

Theorem 14.2. *Given an alphabet* $E = \{e_1, e_2\}$ *and two sequences* $\sum_{j=0}^{\infty} a_j z^j$ *and* $\sum_{j=0}^{\infty} b_j z^j$ *as* $\left(\sum_{j=0}^{\infty} a_j z^j\right)\left(\sum_{j=0}^{\infty} b_j z^j\right) = 1$*, then*

$$\sum_{j=0}^{\infty} a_j S_{k+j-1}(e_1 + e_2) z^j = \frac{\sum_{j=0}^{k-1} b_j e_1^j e_2^j S_{k-j-1}(e_1 + e_2) z^j - e_1^k e_2^k z^{k+1} \sum_{j=0}^{\infty} b_{j+k+1} S_j (e_1 + e_2) z^j}{\left(\sum_{j=0}^{\infty} b_j e_1^j z^j\right)\left(\sum_{j=0}^{\infty} b_j e_2^j z^j\right)}. \tag{14.14}$$

Proof (Proof of the Main Theorem). Let $\sum_{j=0}^{\infty} a_j z^j$ and $\sum_{j=0}^{\infty} b_j z^j$ be two sequences as $\sum_{j=0}^{\infty} a_j z^j = \frac{1}{\sum_{j=0}^{\infty} b_j z^j}$; then 1st member of formula (14.13) is written:

$$
\begin{aligned}
Ł_{e_1e_2}^k f(e_1) &= Ł_{e_1e_2}^k \left(\sum_{j=0}^{\infty} a_j e_1^j z^j \right) \\
&= \sum_{j=0}^{\infty} a_j S_{k+j-1}(e_1+e_2) z^j
\end{aligned}
$$

and the second member of the formula (14.13) can be written:

$$
\begin{aligned}
& S_{k-1}(e_1+e_2) f(e_1) + e_2^k \partial_{e_1 e_2} f(e_1) \\
&= \frac{S_{k-1}(e_1+e_2)}{\sum_{j=0}^{\infty} b_j e_1^j z^j} + e_2^k \partial_{e_1 e_2} \frac{1}{\sum_{j=0}^{\infty} b_j e_1^j z^j} \\
&= \frac{S_{k-1}(e_1+e_2)}{\sum_{j=0}^{\infty} b_j e_1^j z^j} - \frac{\sum_{j=0}^{\infty} b_j S_{j-1}(e_1+e_2) z^j}{\left(\sum_{j=0}^{\infty} b_j e_1^j z^j \right) \left(\sum_{j=0}^{\infty} b_j e_2^j z^j \right)} \\
&= \frac{\sum_{j=0}^{\infty} b_j \left[e_2^j S_{k-1}(e_1+e_2) - e_2^k S_{j-1}(e_1+e_2) \right] z^j}{\left(\sum_{j=0}^{\infty} b_j e_1^j z^j \right) \left(\sum_{j=0}^{\infty} b_j e_2^j z^j \right)} \\
&= \frac{\sum_{j=0}^{k-1} b_j \left[e_2^j S_{k-1}(e_1+e_2) - e_2^k S_{j-1}(e_1+e_2) \right] z^j}{\left(\sum_{j=0}^{\infty} b_j e_1^j z^j \right) \left(\sum_{j=0}^{\infty} b_j e_2^j z^j \right)} \\
&\quad + \frac{\sum_{j=k+1}^{\infty} b_j \left[e_2^j S_{k-1}(e_1+e_2) - e_2^k S_{j-1}(e_1+e_2) \right] z^j}{\left(\sum_{j=0}^{\infty} b_j e_1^j z^j \right) \left(\sum_{j=0}^{\infty} b_j e_2^j z^j \right)} \\
&= \frac{\sum_{j=0}^{k-1} b_j e_1^j e_2^j S_{k-j-1}(e_1+e_2) z^j - e_1^k e_2^k z^{k+1} \sum_{j=0}^{\infty} b_{j+k+1} S_j(e_1+e_2) z^j}{\left(\sum_{j=0}^{\infty} b_j e_1^j z^j \right) \left(\sum_{j=0}^{\infty} b_j e_2^j z^j \right)}.
\end{aligned}
$$

□

14.4 Applications

In the case

$$\frac{1}{1-z}=\sum_{j=0}^{\infty} z^j,$$

the coefficients of the two series in question (in the theorems below) are $\{1,-1,0,0,\ldots\}$ and $\{1,1,1,\ldots\}$:

Corollary 14.3. *Given an alphabet $E=\{e_1,e_2\}$ and an integer k, then we have*

$$\sum_{j=0}^{\infty} S_{k+j-1}(e_1+e_2)z^j=\frac{S_{k-1}(e_1+e_2)-e_1e_2S_{k-2}(e_1+e_2)z}{(1-ze_1)(1-ze_2)}. \tag{14.15}$$

Note:

Taking $e_1=1$ and $e_2=x$ then (14.15) are written:

$$\sum_{j=0}^{\infty}(1+x+\ldots\ldots\ldots x^{k+j-1})z^j=\frac{(1+x+\ldots\ldots\ldots x^{k-1})-x(1+x+\ldots\ldots\ldots x^{k-2})z}{(1-z)(1-zx)}.$$

In the case $k=1$ Corollary 14.3 can be written as follows:

$$\sum_{j=0}^{\infty} S_j(e_1+e_2)z^j=\frac{1}{(1-ze_1)(1-ze_2)}. \tag{14.16}$$

By replacing e_2 by $(-e_2)$, Corollary 14.3 becomes

$$\sum_{j=0}^{\infty} S_j(e_1+[-e_2])z^j=\frac{1}{(1-ze_1)(1+ze_2)}. \tag{14.17}$$

If $\begin{cases} e_1e_2=1 \\ e_1-e_2=1 \end{cases}$ *formula (14.17) becomes*

$$\sum_{j=0}^{\infty} S_j(e_1+[-e_2])z^j=\frac{1}{1-z-z^2}. \tag{14.18}$$

Formula (14.18) is given by Foata [3]. We note that in the Fibonacci numbers can be written as

$$F_j=S_j(e_1+[-e_2]). \tag{14.19}$$

By the formula (14.17), we can deduce it by replacing e_1 on $2e_1$and e_2 on $2e_2$ and under the condition $4e_1e_2=-1$ Formula (14.18) becomes

$$\sum_{j=0}^{\infty} S_j(2e_1+[-2e_2])z^j=\frac{1}{1-2(e_1-e_2)z+z^2}. \tag{14.20}$$

The formula (14.20) *is similar to the one proved by Foata [3]. Consequently, the Chebyshev polynomials of the second kind are written:*

$$U_j = S_j(2e_1 + [-2e_2]). \tag{14.21}$$

By the formula (14.20) *we can deduce*

$$\sum_{j=0}^{\infty} \left[S_j(2e_1 + [-2e_2]) - (e_1 - e_2)S_{j-1}(2e_1 + [-2e_2])\right] z^j = \frac{1-(e_1-e_2)z}{1-2(e_1-e_2)z+z^2}. \tag{14.22}$$

We find the formula (14.22) *in Foata [3]. Consequently, the Chebyshev polynomials of first kind are written:*

$$T_j(e_1 - e_2) = \left[S_j(2e_1 + [-2e_2]) - (e_1 - e_2)S_{j-1}(2e_1 + [-2e_2])\right]. \tag{14.23}$$

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Chapter 15
Numerical Study of the High-Contrast Stokes Equation and Its Robust Preconditioning

Burak Aksoylu and Zuhal Unlu

Abstract We numerically study the Stokes equation with high-contrast viscosity coefficients. The high-contrast viscosity values create complications in the convergence of the underlying solver methods. To address this complication, we construct a preconditioner that is robust with respect to contrast size and mesh size simultaneously based on the preconditioner proposed by Aksoylu et al. (Comput. Vis. Sci. 11:319–331, 2008). We examine the performance of our preconditioner against multigrid and provide a comparative study reflecting the effect of the underlying discretization and the aspect ratio of the mesh by utilizing the preconditioned inexact Uzawa and Minres solvers. Our preconditioner turns out to be most effective when used as a preconditioner to the inexact p-Uzawa solver and we observe contrast size and mesh size robustness simultaneously. As the contrast size grows asymptotically, we numerically demonstrate that the inexact p-Uzawa solver converges to the exact one. We also observe that our preconditioner is contrast size and mesh size robust under p-Minres when the Schur complement solve is accurate enough. In this case, the multigrid preconditioner loses both contrast size and mesh size robustness.

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15.1 Introduction

The Stokes equation plays a fundamental role in the modeling of several problems in emerging geodynamics applications. Numerical solutions to the Stokes flow problems especially with high-contrast variations in viscosity are critically needed in the computational geodynamics community; see recent studies [27, 28, 33, 43]. The high-contrast viscosity corresponds to a small Reynolds number regime because the Reynolds number is inversely proportional to the viscosity value. One of the main applications of the high-contrast Stokes equation is the study of earth's mantle dynamics. The processes such as the long timescale dynamics of the earth's convecting mantle and the formation and subsequent evolution of plate tectonics can be satisfactorily modeled by the Stokes equation; see [28, 33, 34] for further details. Realistic simulation of mantle convection critically relies on the treatment of the two essential components of simulation: *the contrast size in viscosity* and *the mesh resolution*. Hence, our aim is to achieve robustness of the underlying preconditioner with respect to the contrast size and the mesh size simultaneously, which we call as m- and h-robustness, respectively.

Roughness of PDE coefficients causes loss of robustness of preconditioners. In [3, 4] Aksoylu and Beyer have studied the diffusion equation with such coefficients in the operator theory framework and have showed that the roughness of coefficients creates serious complications. For instance, in [4], they have shown that the standard elliptic regularity in the smooth coefficient case fails to hold. Moreover, the domain of the diffusion operator heavily depends on the regularity of the coefficients. Similar complications also arise in the Stokes case. This article came about from a need to address solver complications through the help of robust preconditioning. For that, we construct a robust preconditioner based on the one proposed in [2], which we call as the Aksoylu–Graham–Klie–Scheichl (AGKS) preconditioner. The AGKS preconditioner originates from the family of robust preconditioners constructed in [5]. It was proven and numerically verified to be m- and h-robust simultaneously.

The AGKS preconditioner was originally designed for the high-contrast diffusion equation under finite element discretization. In [6] we extended the AGKS preconditioner from finite element discretization to cell-centered finite volume discretization. Hence, we have shown that the same preconditioner could be used for different discretizations with minimal modification. Furthermore, in [7], we applied the same family of preconditioners to high-contrast biharmonic plate equation. Therefore, we have accomplished a desirable preconditioning design goal by using the same family of preconditioners to solve the elliptic family of PDEs with varying discretizations. In this article, we aim to bring the same preconditioning technology to *vector -valued* problems such as the Stokes equation. We extend the usage of AGKS preconditioner to the solution of the stationary Stokes equation in a domain $\Omega \subset \mathbb{R}^2$:

$$\begin{aligned} -\nabla \cdot (\nu \nabla u) + \nabla p = f \quad &\text{in} \quad \Omega, \\ \nabla \cdot u = 0 \quad &\text{in} \quad \Omega, \end{aligned} \tag{15.1}$$

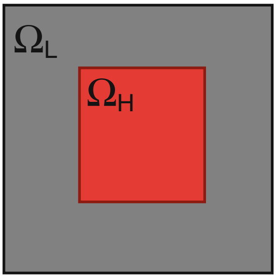

Fig. 15.1: $\Omega = \overline{\Omega}_H \cup \Omega_L$ where Ω_H and Ω_L are highly and lowly viscous regions, respectively

with piecewise constant high-contrast viscosity used in the slab subduction referred as the *Sinker* model by [33] :

$$\nu(x) = \begin{cases} m \gg 1, & x \in \Omega_H, \\ 1, & x \in \Omega_L. \end{cases} \tag{15.2}$$

see Fig. 15.1.

Here, u, p, and f stand for the velocity, pressure, and body force, respectively.

The discretization of (15.1) gives rise to the following saddle-point matrix:

$$\mathscr{A} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} K(m) & B^t \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}. \tag{15.3}$$

The velocity vector can be treated componentwise which allows the usage of a single finite element space for each component. The extension of AGKS preconditioner from diffusion to Stokes equation is accomplished by the following crucial block partitioning of (15.3); see [21, p. 226]:

$$\begin{bmatrix} K^x(m) & 0 & (B^x)^t \\ 0 & K^y(m) & (B^y)^t \\ B^x & B^y & 0 \end{bmatrix} \begin{bmatrix} u^x \\ u^y \\ p \end{bmatrix} = \begin{bmatrix} f^x \\ f^y \\ 0 \end{bmatrix}, \tag{15.4}$$

where $K^* = K^x = K^y$ are the scalar diffusion matrices and B^x and B^y represent the weak derivatives in x and y directions, respectively. We apply the AGKS preconditioning idea to the K^x and K^y blocks by further decomposing each of them as the following 2×2 block system; see [7, Eqn. 11], [6, Eqn. 4], [2, Eqn. 3]:

$$K^*(m) = \begin{bmatrix} K^*_{HH}(m) & K^*_{HL} \\ K^*_{LH} & K^*_{LL} \end{bmatrix}, \tag{15.5}$$

where the degrees of freedom (DOF) are identified as *high* and *low* based on the viscosity value in (15.2) and $K^*_{HH}, K^*_{HL}, K^*_{LH}$, and K^*_{LL} denote couplings between the high–high, high–low, low–high, and low–low DOF, respectively. The exact inverse of K^* can be written as

$$K^{*^{-1}} = \begin{bmatrix} I_{HH} & -K_{HH}^{*^{-1}} K_{HL}^{*} \\ 0 & I_{LL} \end{bmatrix} \begin{bmatrix} K_{HH}^{*^{-1}} & 0 \\ 0 & S^{*^{-1}} \end{bmatrix} \begin{bmatrix} I_{HH} & 0 \\ -K_{LH}^{*} K_{HH}^{*^{-1}} & I_{LL} \end{bmatrix},$$

where I_{HH} and I_{LL} denote the identity matrices of the appropriate dimension and the Schur complement S^* is explicitly given by

$$S^*(m) = K_{LL}^* - K_{LH}^* K_{HH}^{*^{-1}}(m) K_{HL}^*. \tag{15.6}$$

The AGKS preconditioner is defined as follows:

$$\hat{K}^{*^{-1}}(m) := \begin{bmatrix} I_{HH} & -K_{HH}^{\infty^{\dagger}} K_{HL}^{*} \\ 0 & I_{LL} \end{bmatrix} \begin{bmatrix} K_{HH}(m)^{*^{-1}} & 0 \\ 0 & S^{\infty^{-1}} \end{bmatrix} \begin{bmatrix} I_{HH} & 0 \\ -K_{LH}^{*} K_{HH}^{\infty^{\dagger}} & I_{LL} \end{bmatrix}, \tag{15.7}$$

where $K_{HH}^{\infty^{\dagger}}$ and S^{∞} are the asymptotic values of $K_{HH}^{*^{-1}}$ and S^*, respectively; see [2, Lemma 1].

15.1.1 Literature Review

There are many solution methods proposed for the system of equations in (15.3); see the excellent survey article [15]. Based on where the emphasis is put in the design of a solution method, solving a saddle-point matrix system can be classified into two approaches: *preconditioning and solver*. The *preconditioning approach* aims to construct novel preconditioners for standard solver methods such as Uzawa and Minres. A vast majority of the articles on the *preconditioning* approach focuses on the preconditioning of Schur complement matrix; see [18, 31–33, 36, 38, 43]. It is well known that the Schur complement matrix S is spectrally equivalent to the pressure mass matrix (PMM) for the steady Stokes equation; see [17]. For rigorous convergence analysis of Krylov solvers with PMM preconditioner, see [40, 44]. Elman and Silvester [24] established that scaled PMM lead to h-robustness for the Stokes equation with large constant viscosity. Using a new inner product, Olshanskii and Reusken [36] introduced a robust preconditioner for the Schur complement matrix $S = BK^{-1}B^t$ for discontinuous viscosity $0 < \nu \leq 1$ and showed that the preconditioned Uzawa (p-Uzawa) and Minres (p-Minres) became h-robust with this new PMM preconditioner. Further properties of this preconditioner such as clustering in the spectrum of preconditioned S-system were shown in [30]. It was pointed out in [31] that Elman [19] designed LSQR commutator (BFBt) preconditioner in order to overcome the m-robustness issues by using $\hat{S} = (BB^t)^{-1}BKB^t(BB^t)^{-1}$ preconditioner for S. This preconditioner is further studied in [18, 20]. Additionally, the usage of variants of the BFBt preconditioner for the high-contrast Stokes equation is popularized with $\nu|_{\Omega_H} = m \gg 1$ in geodynamics applications in [27, 28, 33, 43]. May and Moresi [33] established that this preconditioner was m-robust when used along with a preconditioned Schur Complement Reduction solver and h-robustness of this preconditioner when used with the Schur method and generalized conjugate residual method with block triangular preconditioners was obtained by a further study in [43].

There have been studies focusing on different ways of preconditioning K for the Stokes equation restricted to constant viscosity case; see [16, 23, 42]. It was observed that a single multigrid (MG) cycle with an appropriate smoother was usually a good preconditioner for K because MG is sufficiently effective as a preconditioner for the constant viscosity case; see [21]. For discontinuous coefficient case, however, there has not been much study to analyze the performance of preconditioners for K in a Stokes solver framework. Since MG loses h-robustness, there is an imminent need for the robustness study of preconditioners for the case of discontinuous coefficients and we present the AGKS preconditioner to address this need.

The *solver method approach* aims to construct a solver by sticking with standard preconditioners such as MG for the K matrix and PMM or BFBt for the S matrix. The performance of the solver depends heavily on the choice of the inner preconditioner; see [10, 11, 23, 26]. The Uzawa solver is one of the most popular iterative methods for the saddle-point problems in fluid dynamics; see [8, 26, 29]. Since this method requires the solution of K-system in each step, this leads to the utilization of an inexact Uzawa method involving an approximate evaluation of K^{-1}; see [13, 45]. This method involves an inner and outer iteration (in our context, S- and outer-solve, respectively), and the convergence of this method is studied extensively in [13, 16, 23, 38].

Another commonly used iterative method is Minres; see [37]. The usage of block diagonal preconditioner for the p-Minres solver was suggested in [25] and further results were presented for this type of preconditioning in [39]. For constant viscosity case, there have been many studies for different choices of the preconditioners for K and S blocks; see [14, 15, 38, 40, 44]. For the discontinuous viscosity case, on the other hand, Olshanskii and Reusken [35, 36] studied the performance of p-Minres with a new PMM preconditioner.

The remainder of this paper is structured as follows. In Sect. 15.2, we describe p-Uzawa and p-Minres solvers. In Sect. 15.3, we comparatively study the performance of the AGKS preconditioner against MG used under the above solvers. We highlight important aspects of robust preconditioning and draw some conclusions in Sect. 15.4.

15.2 Solver Methods

The LBB stability of Stokes discretizations has been extensively studied due to utilization of weak formulations to solve (15.1). We are interested in the LBB stability in the case of high-contrast coefficients. In [35], the LBB stability was proved only for the case $0 < \nu \leq 1$. Later, in [36], this restriction was eliminated, and the results were extended to cover general viscosity, thereby, immediately establishing the LBB stability of the discretization under consideration as the following:

$$\sup_{u_h \in V_h} \frac{(\text{div } u_h, p_h)}{\|u_h\|_V} \geq c_{LBB} \|p_h\|_Q, \quad p_h \in Q_h, \tag{15.8}$$

The associated spaces and weighted norms are defined as follows:

$$\begin{aligned} V &:= [H_0^1(\Omega)]^d, \\ Q &:= \left\{p \in L^2(\Omega) : (\nu^{-1} p, 1) = 0\right\}, \\ \|u\|_V &:= (\nu \nabla u, \nabla v)^{\frac{1}{2}}, \quad u \in V, \\ \|p\|_Q &:= (\nu^{-1} p, p)^{\frac{1}{2}}, \quad p \in Q. \end{aligned}$$

Here $V_h \subset V$ and $Q_h \subset Q$ are finite element spaces that are LBB stable. To be precise, we utilize the $Q2$-$Q1$ (the so-called Taylor-Hood finite element) discretization for numerical experiments in Sect. 15.3.

There are many solution methods for the indefinite saddle-point problem (15.3). We concentrate on two different solver methods: the p-Uzawa and p-Minres. We test the performance of the AGKS preconditioner with these solver methods. First, we establish two spectral equivalences: between the velocity stiffness matrix K and the AGKS preconditioner and between the Schur complement matrix S and the scaled PMM. Note that the constant c_{LBB} in (15.8) is directly used for the spectral equivalence of S in the following.

Lemma 15.1. *Let $\hat{K}$ and $\hat{S}$ denote the AGKS preconditioner and the scaled PMM. Then, for sufficiently large m, the following spectral equivalences hold:*

(a)

$$(1 - cm^{-1/2})(\hat{K}u, u) \leq (Ku, u) \leq (1 + cm^{-1/2})(\hat{K}u, u), \tag{15.9}$$

for some constant c independent of m.

(b)

$$c_{LBB}^2(\hat{S}p, p)_Q \leq (Sp, p) \leq d(\hat{S}p, p)_Q, \tag{15.10}$$

where c_{LBB} is the constant in (15.8) which is independent of m and h.

Proof. One can extract a symmetric positive semidefinite matrix $\mathcal{N}_{HH}^*$ with a rank one kernel from K_{HH}^* in (15.5). $\mathcal{N}_{HH}^*$ is the so-called Neumann matrix and the extraction leads to the following decomposition:

$$K_{HH}^*(m) = m\mathcal{N}_{HH}^* + \Delta.$$

Δ corresponds to the coupling between the DOF in Ω_L and on the boundary of Ω_H. Since $\ker(\mathcal{N}_{HH}^*)$ has rank one, $\mathcal{N}_{HH}^*$ has a simple zero eigenvalue and the below spectral decomposition holds with $\lambda_i > 0, \quad i = 1, \ldots, n_H - 1$ where n_H denotes the order of $\mathcal{N}_{HH}^*$:

$$Z^t \mathcal{N}_{HH}^* Z = \operatorname{diag}(\lambda_1, \ldots, \lambda_{n_H - 1}, 0).$$

Although the eigenvectors in the columns of Z and the eigenvalues λ_i can change according to the underlying discretization, there is always one simple zero eigenvalue and its corresponding constant eigenvector independent of the discretization. This is a direct consequence of the diffusion operator corresponding to a Neumann problem. Therefore, the spectral equivalence established for the $P1$ finite element in [2, Thm. 1] extends to $Q2$ and $Q1$ discretizations, thereby, completing the proof of part (a) for K^*. The spectral equivalence of K easily follows from that of K^* because of the decomposition in (15.4).

The proof of (b) follows from [36, Thm. 6]. □

15.2.1 The Preconditioned Uzawa Solver

The Uzawa algorithm is a classical solution method which involves block factorization with forward and backward substitution. Here, we use the preconditioned inexact Uzawa method described in [12, 38]. The system (15.3) can be block factorized as follows:

$$\begin{bmatrix} K(m) & 0 \\ B & -I \end{bmatrix} \begin{bmatrix} I & K(m)^{-1}B^t \\ 0 & S(m) \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}. \tag{15.11}$$

Let (u^k, p^k) be a given approximation of the solution (u, p). Using the block factorization (15.11) combined with a preconditioned Richardson iteration, one obtains

$$\begin{bmatrix} u^{k+1} \\ p^{k+1} \end{bmatrix} = \begin{bmatrix} u^k \\ p^k \end{bmatrix} + \begin{bmatrix} I & -K^{-1}B^tS^{-1} \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} K^{-1} & 0 \\ BK^{-1} & -I \end{bmatrix} \left(\begin{bmatrix} f \\ 0 \end{bmatrix} - \mathscr{A} \begin{bmatrix} u^k \\ p^k \end{bmatrix} \right). \tag{15.12}$$

This leads to the following iterative method:

$$u^{k+1} = u^k + w^k - \hat{K}^{-1}B^t z^k, \tag{15.13a}$$

$$p^{k+1} = p^k + z^k, \tag{15.13b}$$

where $w^k := \hat{K}^{-1}r_1^k$, $r_1^k := f - Ku^k - B^t p^k$, and $z^k := \hat{S}B(w^k + u^k)$. Here, $\hat{K}$ and $\hat{S}$ are the AGKS and PMM preconditioners for K and S, respectively. Computing z^k involves ℓ iterations of pCG. In this computation, since the assembly of S is prohibitively expensive, first we replace it by $\tilde{S}$. Then, we utilize the preconditioner $\hat{K}$ for K and $\hat{S}$ for $\tilde{S}$ where the explicit formula is given by

$$\tilde{S} := B\hat{K}^{-1}B^t. \tag{15.14}$$

Thus, the total number of applications of $\hat{K}^{-1}$ in (15.13a) and (15.13b) becomes $\ell + 2$. We refer the outer-solve (one Uzawa iteration) as steps (15.13a) and (15.13b) combined. In particular, we call the computation of z^k as an S-solve; see Table 15.1. The stopping criterion of the S-solve plays an important role for the efficiency of the Uzawa method and it is affected by the accuracy of $\hat{K}$; see the analysis in [38, Sec. 4]. When the AGKS preconditioner is used for velocity stiffness matrix, the stopping criterion of the S-solve is determined as follows:

Let r_p^i be the residual of the S-solve at iteration i. Then, we abort the iteration when $\frac{\|r_p^i\|}{\|r_p^0\|} \leq \delta_{tol}$ where:

- $\delta_{tol} = 0.5$ or
- maximum iteration reaches 4.

15.2.2 The Preconditioned Minres Solver

The p-Minres is a popular iterative method applied to the system (15.3). Let $v := \begin{bmatrix} u \\ p \end{bmatrix}$. With the given initial guess $v^0 := \begin{bmatrix} u^0 \\ p^0 \end{bmatrix}$ where $p^0 \in e^{\perp_Q}$ and with the corresponding error $r^0 := v - v^0$, the p-Minres solver computes:

$$v^k = \underset{v \in v^0 + \mathscr{K}^k(\mathscr{B}^{-1}\mathscr{A}, \tilde{r}^0)}{\text{argmin}} \|\mathscr{B}^{-1} \left(\begin{bmatrix} f \\ 0 \end{bmatrix} - \mathscr{A}\, v \right) \|.$$

Here, $\tilde{r}^0 = \mathscr{B}^{-1} r^0$ and $\mathscr{K}^k = span\{\tilde{r}^0, \mathscr{B}^{-1}\mathscr{A}\tilde{r}^0, \ldots (\mathscr{B}^{-1}\mathscr{A})^k \tilde{r}^0\}$, and the preconditioner has the following block diagonal structure:

$$\mathscr{B} = \begin{bmatrix} \hat{K} & 0 \\ 0 & \hat{S} \end{bmatrix}, \qquad (15.15)$$

where $\hat{K}$ and $\hat{S}$ are the preconditioners for K and S, respectively. In each step of the p-Minres solver the above preconditioner is applied in the following fashion: for the K-block one application of $\hat{K}$ and for the S-block several applications of pCG to the $\tilde{S}$-system with $\hat{S}$ as the preconditioner. Here, $\tilde{S} = B\hat{K}^{-1}B^t$ stands for the approximation of S. Since S is replaced by $\tilde{S}$, this turns the p-Minres algorithm to an inexact one; see the inexactness discussion in Sect. 15.3.2. The p-Minres iterations are called outer-solve whereas the pCG solve for the $\tilde{S}$-system is called inner-solve.

The convergence rate of the p-Minres method depends on the condition number of the preconditioned matrix, $\mathscr{B}^{-1}\mathscr{A}$. Combining the spectral equivalences given in (15.9) and (15.10) with the well-known condition number estimate [9], we obtain

$$\kappa_{\mathscr{B}}(\mathscr{B}^{-1}\mathscr{A}) \leq \frac{\max\{(1 + cm^{-1/2}), d\}}{\min\{(1 - cm^{-1/2}), c_{LBB}^2\}}$$

It immediately follows that the convergence rate of the p-Minres method is independent of m asymptotically.

15.3 Numerical Experiments

The goal of the numerical experiments is to compare the performance of the AGKS and MG preconditioners by using two different solvers: p-Uzawa and p-Minres. We use a four-level hierarchy in which the numbers of DOF, N_1, N_2, N_3, and N_4, are $659, 2467, 9539$, and 37507 from coarsest to finest level. We consider cavity flow with enclosed boundary conditions with right-hand side functions $f = 1$ and $g = 0$ on a 2D domain $[-1, 1] \times [-1, 1]$.

For discretization, we use the Q_2-Q_1 (the Taylor-Hood) stable finite elements and stabilized Q_1-Q_1 finite elements for the velocity-pressure pair. We consider

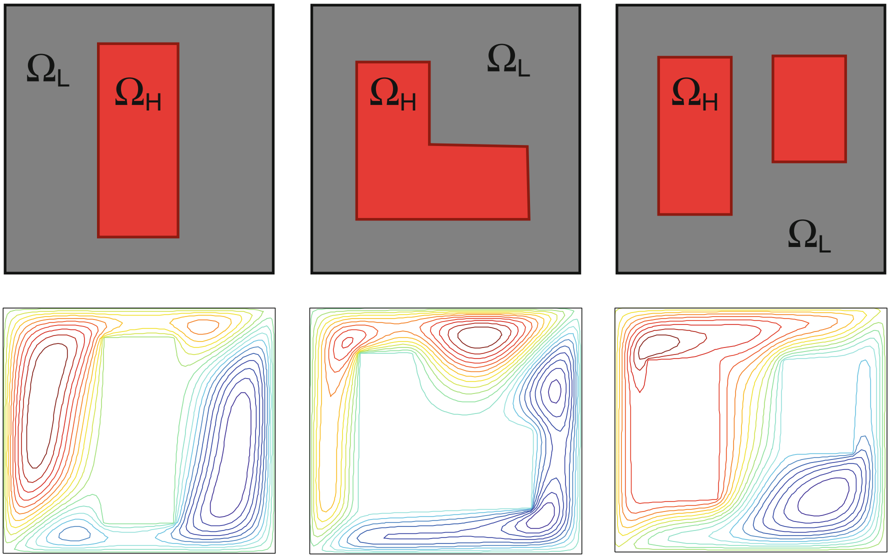

Fig. 15.2: The streamline plot of the high-contrast Stokes equation for three different high-viscosity island configurations; (*left*) rectangular, (*middle*) L-shaped, and (*right*) two disconnected islands

the case of a single island (viscous inclusion) located at the region $[-1/4,1/4] \times [-3/4,3/4]$. For an extension, we also consider the cases of L-shaped island and two disconnected islands; see Fig. 15.2. The observation about these cases is given in Sect. 15.4. The implementation of discretization is based on ifiss3.1 software provided in [41]; also see [22]. The AGKS preconditioner implementation is based on our implementation in [2, 6, 7]. The implementation of the MG preconditioner is derived from the one in [1]. We employ a V(1,1)-cycle, with point Gauss–Seidel (GS) smoother. A direct solver is used for the coarsest level. For each level of refinement, we present the number of iteration and average reduction factor corresponding to each solve (outer-solve and S-solve; outer-solve and inner-solve for p-Uzawa and p-Minres iterations, respectively). In the tables, N stands for the number of DOF in $\mathscr{A}$ for the outer-solves and the number of DOF in S for the S- and inner-solves. We enforce an iteration bound of 200. If the method seems to converge slightly beyond this bound, we denote it by $*$. A zero initial guess is used. The numerical experiments were performed on a dual core Macbook Pro, running at 2.4 GHz with 4GB RAM.

In analyzing m-robustness, we observe a special feature. The iteration count remains fixed when m becomes larger than a certain threshold value. We define the notion *asymptotic regime* to indicate m values bigger than this threshold. Identifying an asymptotic regime is desirable because it immediately indicates m-robustness.

15.3.1 The Preconditioned Uzawa Solver

We use pCG solver with scaled PMM as a preconditioner, 0.5 as tolerance and 4 as maximum number of iterations, for the S-system in each iteration of p-Uzawa. The tolerance for the outer-solve is set to be 5×10^{-6}. We report the performance of the p-Uzawa solver applied to a rectangular and skewed mesh with Q_2-Q_1 discretization. We observe that the p-Uzawa method is m-robust as long as the optimal stopping criterion is used for the S-solve; see Tables 15.1–15.6. The performances of the AGKS and MG preconditioners are observed as follows. When the MG preconditioner is used, the p-Uzawa solver loses m- and h-robustness and the iteration count increases dramatically when the mesh aspect ratio or the island configuration changes; see Tables 15.1,15.3, and15.5. Especially for viscosity values larger than 10^5, we further observe that the iteration number of pCG method for the S-solve, denoted by ℓ, reaches the maximum iteration count 4. Since the MG preconditioner is applied $\ell + 2$ times at each iteration of the outer-solve, we illustrate how this results in an unreasonable number of applications of the MG preconditioner; see Fig. 15.3. For instance, in Table 15.1, for the case of $m = 10^8$, we have $\ell = 4$. Therefore, in each outer iteration, we apply the MG preconditioner $\ell + 2 = 6$ times. At level $= 4$, since the total number of MG application is the product of the outer-solve count with $\ell + 2$, it becomes $48 \times 6 = 288$. The iteration increases even more rapidly as we refine the mesh. Therefore, the loss of h-robustness sets a major drawback as larger size problems are considered.

On the other hand, the AGKS preconditioner maintains m- and h-robustness simultaneously.of the discretization type or Asymptotically, only one iteration of pCG is sufficient to obtain an accurate S-solve for a rectangular mesh; see Table 15.2. When we do the above calculation, we find that for a rectangular mesh, the total number of AGKS applications is $15 \times (1 + 2) = 45$. Since this application count remains fixed as the mesh is refined, we infer the h-robustness of the AGKS preconditioner; see Fig. 15.3. When the mesh aspect ratio or the island configuration changes, the number of pCG iterations required to have an accurate S-solve becomes 2 or 3. However, this is reasonable since the outer-solve maintains h- and m-robustness; see Tables 15.4 and 15.6. Hence, the AGKS preconditioner will acceleratedly outperform the MG preconditioner as more mesh refinements are introduced regardless of the island or mesh configuration.

15.3.2 The Preconditioned Minres Solver

We notice that the p-Minres has not been the solver of choice for high-contrast problems due to its unfavorable performance with PMM for the S-system; see [35]. We have taken a novel approach for the S-system. First, we replace S by $\tilde{S} = B\hat{K}^{-1}B^t$ where $\hat{K}^{-1}$ step is one application of the AGKS preconditioner. This makes the solver method *inexact*. Then, we solve $\tilde{S}$-system by using a pCG solver with scaled PMM preconditioner with tolerance 0.05 with a maximum of 20 iterations. The

Table 15.1: Number of iterations and average reduction factors for p-Uzawa, $Q2$-$Q1$, rectangular mesh, and MG

$N\backslash m$	10^0	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9
					Outer-solve					
659	*13*, 0.546	*15*, 0.594	*15*, 0.517	*17*, 0.517	*19*, 0.506	*19*, 0.596	*19*, 0.589	*19*, 0.588	*22*, 0.588	*22*, 0.589
2467	*13*, 0.516	*17*, 0.531	*17*, 0.552	*18*, 0.456	*20*, 0.456	*21*, 0.455	*21*, 0.455	*21*, 0.455	*21*, 0.456	*21*, 0.460
9539	*18*, 0.345	*20*, 0.460	*20*, 0.487	*23*, 0.491	*25*, 0.492	*26*, 0.491	*27*, 0.683	*28*, 0.677	*31*, 0.678	*32*, 0.698
37507	*13*, 0.371	*23*, 0.476	*23*, 0.509	*26*, 0.508	*27*, 0.503	*38*, 0.502	*35*, 0.500	*40*, 0.499	*48*, 0.800	*50*,0.825
					S-solve					
81	*2*, 0.797	*3*, 0.703	*2*, 0.715	*2*, 0.726	*3*, 0.729	*3*, 0.729	*3*, 0.729	*3*, 0.729	*2*, 0.729	*2*, 0.729
289	*4*, 0.899	*4*, 0.903	*4*, 0.912	*4*, 0.915	*4*, 0.915	*4*, 0.915	*4*, 0.915	*4*, 0.915	*4*, 0.915	*4*, 0.915
1089	*1*, 0.997	*2*, 0.802	*3*, 0.914	*4*, 0.919	*4*, 0.920	*4*, 0.920	*4*, 0.920	*3*, 0.920	*3*, 0.920	*3*, 0.920
4225	*1*, 0.995	*1*, 0.800	*1*, 0.913	*1*, 0.920	*3*, 0.920	*2*, 0.921	*4*, 0.981	*3*, 0.921	*4*, 0.941	*3*, 0.921

Table 15.2: Number of iterations and average reduction factors for p-Uzawa, $Q2$-$Q1$, rectangular mesh, and AGKS

$N \backslash m$	10^0	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9
					Outer-solve					
659	*24*, 0.546	*15*, 0.394	*14*, 0.417	*14*, 0.417	*14*, 0.406	*14*, 0.396	*14*, 0.389	*14*, 0.388	*14*, 0.388	*14*, 0.389
2467	*38*, 0.316	*21*, 0.431	*18*, 0.452	*19*, 0.456	*18*, 0.456	*18*, 0.455	*18*, 0.455	*18*, 0.455	*18*, 0.456	*18*, 0.460
9539	*47*, 0.745	*31*, 0.660	*16*, 0.487	*16*, 0.491	*15*, 0.492	*15*, 0.491	*15*, 0.483	*15*, 0.477	*15*, 0.478	*15*, 0.480
37507	*70*, 0.871	*50*, 0.476	*17*, 0.509	*16*, 0.508	*15*, 0.503	*15*, 0.502	*15*, 0.500	*15*, 0.499	*15*, 0.500	*15*, 0.501
					S-solve					
81	*3*, 0.420	*2*, 0.495	*3*, 0.420	*3*, 0.427	*3*, 0.408	*3*, 0.403	*3*, 0.401	*3*, 0.403	*3*, 0.403	*3*, 0.403
289	*3*, 0.420	*3*, 0.495	*3*, 0.420	*3*, 0.427	*3*, 0.408	*3*, 0.403	*3*, 0.401	*3*, 0.403	*3*, 0.403	*3*, 0.403
1089	*1*, 0.620	*1*, 0.695	*3*, 0.620	*1*, 0.627	*1*, 0.608	*1*, 0.603	*1*, 0.601	*1*, 0.603	*1*, 0.603	*1*, 0.603
4225	*1*, 0.620	*1*, 0.695	*3*, 0.620	*1*, 0.627	*1*, 0.608	*1*, 0.603	*1*, 0.601	*1*, 0.603	*1*, 0.603	*1*, 0.603

Table 15.3: Number of iterations and average reduction factors for p-Uzawa, $Q2$-$Q1$, skewed mesh, and MG

$N\backslash m$	10^0	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9
					Outer-solve					
659	*16*, 0.477	*18*, 0.488	*70*, 0.817	∗, 0.977	∗, 0.996	∗, 0.996	∗, 0.989	∗, 0.988	∗,0.988	∗, 0.989
2467	*18*, 0.616	*21*, 0.631	*24*, 0.652	*34*, 0.706	*80*, 0.856	∗, 0.955	∗, 0.955	∗, 0.965	∗, 0.956	∗, 0.990
9539	*19*, 0.515	*23*, 0.570	*32*, 0.687	∗, 0.991	∗, 0.952	∗, 0.991	∗, 0.983	∗, 0.687	∗, 0.978	∗, 0.998
37507	*17*, 0.471	*27*, 0.576	*27*, 0.569	*72*, 0.808	*97*, 0.883	∗, 0.962	∗, 0.990	∗, 0.999	∗, 0.980	∗,0.985
					S-solve					
81	*2*, 0.797	*3*, 0.703	*2*, 0.715	*2*, 0.726	*3*, 0.729	*3*, 0.729	*3*, 0.729	*3*, 0.729	*3*, 0.729	*3*, 0.729
289	*4*, 0.899	*4*, 0.903	*4*, 0.912	*4*, 0.915	*4*, 0.915	*3*, 0.915	*4*, 0.915	*4*, 0.915	*4*, 0.915	*4*, 0.915
1089	*1*, 0.997	*2*, 0.802	*3*, 0.914	*4*, 0.919	*4*, 0.920	*3*, 0.920	*4*, 0.920	*3*, 0.920	*3*, 0.920	*3*, 0.920
4225	*2*, 0.995	*1*, 0.800	*1*, 0.913	*2*, 0.920	*3*, 0.920	*3*, 0.921	*4*, 0.981	*3*, 0.921	*3*, 0.941	*3*, 0.921

Table 15.4: Number of iterations and average reduction factors for p-Uzawa, $Q2$-$Q1$, skewed mesh, and AGKS

$N\backslash m$	10^0	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9
Outer-solve										
659	*33*, 0.695	*23*, 0.580	*21*, 0.556	*21*, 0.553	*22*, 0.564	*22*, 0.564	*22*, 0.564	*21*, 0.545	*19*, 0.529	*25*, 0.603
2467	*57*, 0.805	*29*, 0.649	*23*, 0.592	*27*, 0.632	*29*, 0.653	*33*, 0.684	*35*, 0.700	*24*, 0.639	*32*, 0.720	*33*, 0.693
9539	*72*, 0.875	*39*, 0.733	*31*, 0.666	*38*, 0.728	*32*, 0.682	*40*, 0.737	*33*, 0.691	*31*, 0.667	*31*, 0.701	*31*, 0.708
37507	*91*, 0.921	*59*, 0.811	*31*, 0.679	*36*, 0.708	*28*, 0.633	*32*, 0.762	*30*, 0.700	*29*, 0.669	*31*, 0.698	*31*,0.701
S-solve										
81	*2*, 0.750	*2*, 0.722	*2*, 0.769	*2*, 0.806	*2*, 0.915	*2*, 0.918	*2*, 0.918	*2*, 0.922	*5*, 0.897	*5*, 0.916
289	*3*, 0.814	*3*, 0.791	*2*, 0.800	*3*, 0.727	*3*, 0.708	*3*, 0.703	*4*, 0.711	*3*, 0.723	*2*, 0.703	*3*, 0.703
1089	*2*, 0.700	*3*, 0.695	*3*, 0.720	*2*, 0.727	*2*, 0.708	*2*, 0.688	*2*, 0.701	*2*, 0.713	*2*, 0.693	*2*, 0.703
4225	*3*, 0.720	*2*, 0.695	*3*, 0.720	*2*, 0.697	*2*, 0.688	*2*, 0.603	*2*, 0.701	*2*, 0.703	*2*, 0.723	*2*, 0.713

Table 15.5: Number of iterations and average reduction factors for p-Uzawa, $Q2$-$Q1$, rectangular mesh, L-shaped island, and MG

$N \backslash m$	10^0	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9
					Outer-solve					
659	*12*, 0.377	*16*, 0.478	*27*, 0.617	*67*, 0.717	*97*, 0.817	$*$, 0.976	$*$, 0.969	$*$, 0.980	$*$,0.988	$*$, 0.989
2467	*13*, 0.316	*17*, 0.431	*17*, 0.452	*20*, 0.546	*80*, 0.836	$*$, 0.951	$*$, 0.965	$*$, 0.962	$*$, 0.976	$*$, 0.992
9539	*18*, 0.525	*24*, 0.575	*25*, 0.607	*29*, 0.657	*87*, 0.887	$*$, 0.981	$*$, 0.978	$*$, 0.987	$*$, 0.988	$*$, 0.998
37507	*18*, 0.491	*27*, 0.667	*27*, 0.649	*49*, 0.778	*50*, 0.793	$*$, 0.962	$*$, 0.995	$*$, 0.993	$*$, 0.998	$*$,0.985
					S-solve					
81	*2*, 0.797	*3*, 0.703	*2*, 0.715	*2*, 0.726	*3*, 0.729	*3*, 0.729	*3*, 0.729	*3*, 0.729	*3*, 0.729	*3*, 0.729
289	*4*, 0.899	*4*, 0.903	*4*, 0.912	*4*, 0.915	*4*, 0.915	*3*, 0.915	*4*, 0.915	*4*, 0.915	*4*, 0.915	*4*, 0.915
1089	*1*, 0.997	*2*, 0.802	*3*, 0.914	*4*, 0.919	*4*, 0.920	*3*, 0.920	*4*, 0.920	*3*, 0.920	*3*, 0.920	*3*, 0.920
4225	*2*, 0.995	*1*, 0.800	*1*, 0.913	*2*, 0.920	*3*, 0.920	*3*, 0.921	*4*, 0.981	*3*, 0.921	*3*, 0.941	*3*, 0.921

Table 15.6: Number of iterations and average reduction factors for p-Uzawa, $Q2$-$Q1$, rectangular mesh, L-shaped island, and AGKS

$N\backslash m$	10^0	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9
					Outer-solve					
659	*30*, 0.660	*23*, 0.583	*22*, 0.568	*23*, 0.580	*24*, 0.596	*26*, 0.616	*25*, 0.611	*27*, 0.628	*25*, 0.613	*25*, 0.609
2467	*52*, 0.791	*27*, 0.631	*18*, 0.503	*16*, 0.454	*19*, 0.525	*18*, 0.509	*19*, 0.527	*21*, 0.581	*21*, 0.556	*21*, 0.560
9539	*68*, 0.875	*50*, 0.780	*24*, 0.591	*22*, 0.591	*23*, 0.582	*28*, 0.611	*28*, 0.653	*29*, 0.687	*31*, 0.687	*32*, 0.718
37507	*73*, 0.871	*53*, 0.796	*25*, 0.599	*24*, 0.510	*26*, 0.524	*34*, 0.512	*35*, 0.502	*33*, 0.490	*42*, 0.804	*42*,0.815
					S-solve					
81	*2*, 0.787	*3*, 0.703	*2*, 0.717	*2*, 0.726	*3*, 0.729	*3*, 0.725	*3*, 0.729	*3*, 0.729	*2*, 0.739	*2*, 0.729
289	*4*, 0.895	*4*, 0.904	*4*, 0.922	*4*, 0.911	*4*, 0.935	*4*, 0.921	*4*, 0.915	*4*, 0.915	*4*, 0.912	*4*, 0.915
1089	*2*, 0.997	*3*, 0.802	*3*, 0.914	*3*, 0.921	*4*, 0.920	*4*, 0.920	*4*, 0.919	*3*, 0.920	*3*, 0.920	*3*, 0.920
4225	*2*, 0.995	*2*, 0.810	*2*, 0.913	*2*, 0.920	*3*, 0.919	*2*, 0.921	*3*, 0.981	*3*, 0.921	*4*, 0.941	*3*, 0.921

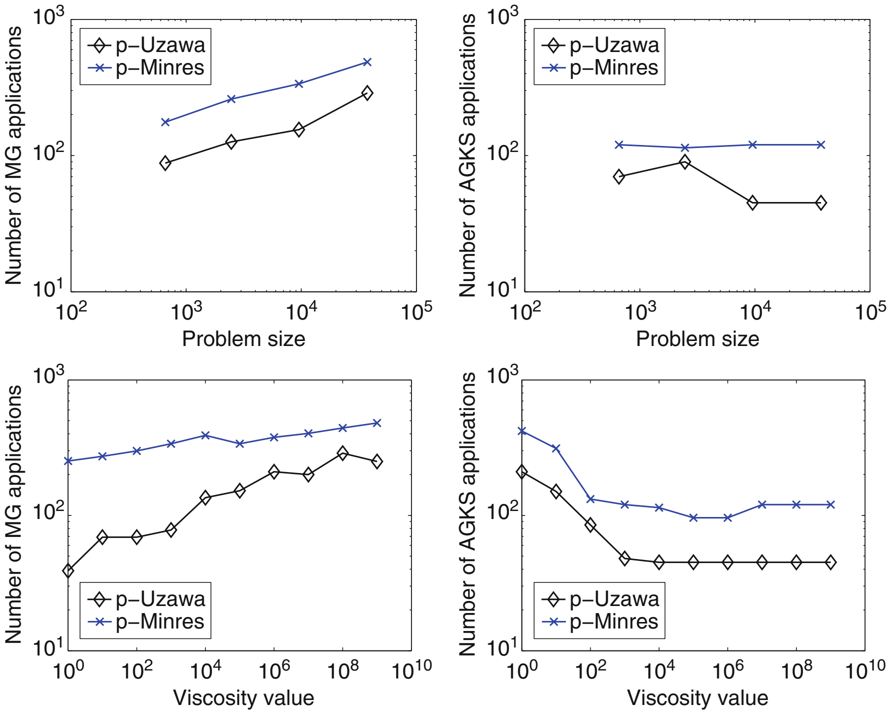

Fig. 15.3: The plot of the number of (*top-left*) MG applications versus problem size for fixed viscosity value $m = 10^8$, (*bottom-left*) MG applications versus viscosity value for fixed level = 4, (*top-right*) AGKS applications versus problem size for fixed viscosity value $m = 10^8$, (*bottom-right*) AGKS applications versus viscosity value for fixed level = 4

pCG and p-Minres solution steps are called the inner- and outer-solve, respectively. Our approach for the S-system is similar to the one we take in the p-Uzawa solver. But, notice that now the inner solver requires more accuracy in order to guarantee a convergent p-Minres solver.

As in the p-Uzawa case, the effectiveness of the AGKS preconditioner has been confirmed as it maintains both the m- and h-robustness whereas MG suffers from the loss of both; see Tables 15.7–15.12. Furthermore, we observe that the choice of $\hat{K}^{-1}$—an application of either MG or AGKS—in the inner-solve dramatically affects the performance of inner-solve. Specifically, the scaled PMM preconditioner is m-robust, but not h-robust for the inner-solve with MG, whereas it is both m- and h-robust for inner-solve with AGKS regardless of the mesh aspect ratio or island configuration.

Table 15.7: Number of iterations and average reduction factors for p-Minres, $Q2$-$Q1$, rectangular mesh, and MG

$N \backslash m$	10^0	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9
					Outer-solve					
659	*15*, 0.546	*15*, 0.594	*18*, 0.517	*19*, 0.517	*19*, 0.506	*19*, 0.596	*20*, 0.589	*20*, 0.588	*23*, 0.588	*24*, 0.589
2467	*20*, 0.516	*19*, 0.531	*21*, 0.552	*24*, 0.456	*23*, 0.456	*24*, 0.455	*25*, 0.455	*28*, 0.455	*29*, 0.456	*30*, 0.460
9539	*21*, 0.345	*19*, 0.460	*24*, 0.487	*24*, 0.491	*24*, 0.492	*24*, 0.491	*25*, 0.683	*26*, 0.677	*28*, 0.678	*32*, 0.698
37507	*21*, 0.371	*21*, 0.476	*23*, 0.509	*26*, 0.508	*30*, 0.503	*26*, 0.502	*29*, 0.500	*31*, 0.499	*34*, 0.800	*36*,0.825
					Inner-solve					
81	*6*, 0.497	*7*, 0.623	*7*, 0.655	*7*, 0.666	*7*, 0.649	*7*, 0.659	*7*, 0.659	*7*, 0.659	*7*, 0.660	*7*, 0.661
289	*8*, 0.699	*9*, 0.703	*9*, 0.713	*9*, 0.720	*9*, 0.715	*9*, 0.717	*9*, 0.713	*9*, 0.721	*9*, 0.735	*9*, 0.735
1089	*9*, 0.747	*11*, 0.752	*11*, 0.744	*11*, 0.749	*11*, 0.750	*11*, 0.760	*11*, 0.759	*11*, 0.761	*11*, 0.762	*11*, 0.760
4225	*12*, 0.795	*13*, 0.801	*13*, 0.813	*13*, 0.810	*13*, 0.811	*13*, 0.801	*13*, 0.803	*13*, 0.805	*13*, 0.808	*13*, 0.811

Table 15.8: Number of iterations and average reduction factors for p-Minres, $Q2$-$Q1$, rectangular mesh, and AGKS

$N\backslash m$	10^0	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9
					Outer-solve					
659	*29*, 0.546	*23*, 0.394	*18*, 0.417	*16*, 0.417	*18*, 0.406	*16*, 0.396	*16*, 0.389	*18*, 0.388	*20*, 0.388	*20*, 0.389
2467	*40*, 0.316	*30*, 0.431	*17*, 0.452	*17*, 0.456	*16*, 0.456	*16*, 0.455	*16*, 0.455	*19*, 0.455	*19*, 0.456	*19*, 0.460
9539	*50*, 0.745	*45*, 0.660	*20*, 0.487	*20*, 0.491	*19*, 0.492	*16*, 0.491	*16*, 0.483	*20*, 0.477	*20*, 0.478	*20*, 0.608
37507	*70*, 0.871	*52*, 0.476	*22*, 0.509	*20*, 0.508	*19*, 0.503	*16*, 0.502	*16*, 0.500	*20*, 0.499	*20*, 0.500	*20*,0.525
					Inner-solve					
81	*20*, 0.797	*20*, 0.703	*5*, 0.585	*5*, 0.576	*5*, 0.579	*5*, 0.529	*5*, 0.569	*5*, 0.548	*5*, 0.567	*5*, 0.554
289	*20*, 0.763	*20*, 0.694	*5*, 0.580	*5*, 0.572	*5*, 0.582	*5*, 0.532	*5*, 0.556	*5*, 0.552	*5*, 0.571	*5*, 0.548
1089	*20*, 0.773	*20*, 0.714	*5*, 0.580	*5*, 0.575	*5*, 0.578	*5*, 0.532	*5*, 0.561	*5*, 0.555	*5*, 0.562	*5*, 0.556
4225	*20*, 0.768	*20*, 0.701	*5*, 0.583	*5*, 0.573	*5*, 0.576	*5*, 0.530	*5*, 0.561	*5*, 0.550	*5*, 0.548	*5*, 0.552

Table 15.9: Number of iterations and average reduction factors for p-Minres, $Q2$-$Q1$, skewed mesh, and MG

$N\backslash m$	10^0	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9
Outer-solve										
659	*18*, 0.547	*19*, 0.578	*19*, 0.582	*28*, 0.657	*43*, 0.767	*88*, 0.897	∗, 0.999	∗, 0.999	∗,0.998	∗, 0.999
2467	*20*, 0.556	*21*, 0.581	*26*, 0.622	*32*, 0.706	*46*, 0.786	*92*, 0.905	∗, 0.942	∗, 0.971	∗, 0.989	∗, 0.995
9539	*20*, 0.545	*25*, 0.578	*29*, 0.677	*43*, 0.757	*57*, 0.817	*97*, 0.917	∗, 0.978	∗, 0.987	∗, 0.988	∗, 0.998
37507	*20*, 0.561	*27*, 0.657	*31*, 0.679	*49*, 0.783	*70*, 0.813	*119*, 0.883	∗, 0.991	∗, 0.998	∗, 0.998	∗,0.999
Inner-solve										
81	*6*, 0.499	*9*, 0.633	*9*, 0.675	*9*, 0.686	*9*, 0.679	*9*, 0.679	*9*, 0.679	*9*, 0.679	*9*, 0.680	*9*, 0.681
289	*8*, 0.729	*10*, 0.753	*10*, 0.753	*10*, 0.760	*10*, 0.755	*10*, 0.757	*10*, 0.753	*10*, 0.761	*10*, 0.775	*10*, 0.775
1089	*9*, 0.787	*13*, 0.792	*13*, 0.784	*13*, 0.789	*13*, 0.790	*13*, 0.810	*13*, 0.819	*13*, 0.811	*13*, 0.822	*13*, 0.820
4225	*12*, 0.825	*15*, 0.851	*15*, 0.865	*15*, 0.860	*15*, 0.861	*15*, 0.851	*15*, 0.853	*15*, 0.855	*15*, 0.858	*15*, 0.861

Table 15.10: Number of iterations and average reduction factors for p-Minres, $Q2$-$Q1$, skewed mesh, and AGKS

$N\backslash m$	10^0	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9
					Outer-solve					
659	*33*, 0.694	*26*, 0.647	*26*, 0.644	*27*, 0.649	*27*, 0.654	*29*, 0.663	*29*, 0.665	*31*, 0.679	*39*, 0.753	*38*, 0.785
2467	*59*, 0.816	*33*, 0.713	*29*, 0.674	*29*, 0.690	*37*, 0.733	*36*, 0.740	*36*, 0.761	*46*, 0.773	*49*, 0.792	*48*, 0.783
9539	*75*, 0.875	*44*, 0.772	*34*, 0.702	*34*, 0.711	*49*, 0.782	*58*, 0.814	*60*, 0.820	*60*, 0.840	*70*, 0.893	*72*, 0.902
37507	*90*, 0.851	*55*, 0.807	*38*, 0.729	*39*, 0.733	*50*, 0.783	*61*, 0.823	*68*, 0.840	*68*, 0.858	*75*, 0.903	*72*, 0.902
					Inner-solve					
81	*25*, 0.817	*23*, 0.813	*15*, 0.787	*15*, 0.776	*15*, 0.779	*15*, 0.729	*15*, 0.769	*15*, 0.748	*15*, 0.767	*15*, 0.754
289	*20*, 0.763	*20*, 0.694	*15*, 0.780	*15*, 0.772	*15*, 0.782	*15*, 0.732	*15*, 0.776	*15*, 0.772	*15*, 0.771	*15*, 0.748
1089	*20*, 0.773	*20*, 0.714	*15*, 0.780	*15*, 0.777	*15*, 0.778	*15*, 0.732	*15*, 0.761	*15*, 0.755	*15*, 0.762	*15*, 0.756
4225	*20*, 0.768	*20*, 0.701	*15*, 0.783	*15*, 0.773	*15*, 0.776	*15*, 0.730	*15*, 0.761	*16*, 0.770	*15*, 0.748	*15*, 0.772

Table 15.11: Number of iterations and average reduction factors for p-Minres, $Q2$-$Q1$, rectangular mesh, L-shaped island, and MG

$N\backslash m$	10^0	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9
					Outer-solve					
659	*12*, 0.377	*16*, 0.478	*27*, 0.617	*67*, 0.717	*97*, 0.817	$*$, 0.976	$*$, 0.969	$*$, 0.980	$*$,0.988	$*$, 0.989
2467	*13*, 0.316	*17*, 0.431	*17*, 0.452	*20*, 0.546	*80*, 0.836	$*$, 0.951	$*$, 0.965	$*$, 0.962	$*$, 0.976	$*$, 0.992
9539	*18*, 0.525	*24*, 0.575	*25*, 0.607	*29*, 0.657	*87*, 0.887	$*$, 0.981	$*$, 0.978	$*$, 0.987	$*$, 0.988	$*$, 0.998
37507	*18*, 0.491	*27*, 0.667	*27*, 0.649	*49*, 0.778	*50*, 0.793	$*$, 0.962	$*$, 0.995	$*$, 0.993	$*$, 0.998	$*$,0.985
					Inner-solve					
81	*6*, 0.497	*7*, 0.625	*7*, 0.655	*7*, 0.666	*7*, 0.649	*7*, 0.659	*7*, 0.659	*8*, 0.669	*8*, 0.670	*8*, 0.671
289	*8*, 0.699	*9*, 0.703	*9*, 0.713	*9*, 0.720	*9*, 0.715	*9*, 0.717	*9*, 0.713	*10*, 0.741	*10*, 0.745	*10*, 0.745
1089	*9*, 0.747	*11*, 0.752	*11*, 0.744	*11*, 0.749	*11*, 0.750	*11*, 0.760	*11*, 0.759	*12*, 0.765	*12*, 0.769	*12*, 0.770
4225	*12*, 0.795	*13*, 0.801	*13*, 0.813	*13*, 0.810	*13*, 0.811	*13*, 0.801	*13*, 0.803	*14*, 0.807	*14*, 0.818	*14*, 0.819

Table 15.12: Number of iterations and average reduction factors for p-Minres, $Q2$-$Q1$, rectangular mesh, L-shaped island, and AGKS

$N\backslash m$	10^0	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9
					Outer-solve					
659	*39*, 0.746	*33*, 0.694	*32*, 0.697	*32*, 0.677	*32*, 0.706	*32*, 0.696	*32*, 0.689	*34*, 0.718	*34*, 0.748	*32*, 0.819
2467	*54*, 0.816	*41*, 0.731	*24*, 0.752	*20*, 0.656	*18*, 0.556	*18*, 0.555	*20*, 0.585	*21*, 0.595	*23*, 0.606	*22*, 0.760
9539	*60*, 0.845	*51*, 0.760	*31*, 0.687	*23*, 0.591	*21*, 0.492	*20*, 0.591	*22*, 0.583	*23*, 0.577	*24*, 0.578	*23*, 0.608
37507	*82*, 0.871	*64*, 0.476	*35*, 0.509	*27*, 0.508	*23*, 0.503	*21*, 0.502	*23*, 0.500	*22*, 0.499	*23*, 0.500	*21*,0.525
					Inner-solve					
81	*20*, 0.797	*20*, 0.703	*6*, 0.588	*5*, 0.577	*6*, 0.581	*6*, 0.532	*6*, 0.575	*6*, 0.553	*6*, 0.569	*6*, 0.565
289	*20*, 0.763	*20*, 0.694	*6*, 0.582	*6*, 0.576	*6*, 0.584	*6*, 0.536	*6*, 0.569	*6*, 0.565	*6*, 0.574	*6*, 0.549
1089	*20*, 0.773	*20*, 0.714	*6*, 0.583	*6*, 0.577	*6*, 0.579	*6*, 0.535	*6*, 0.563	*6*, 0.667	*6*, 0.565	*6*, 0.568
4225	*20*, 0.768	*20*, 0.701	*6*, 0.585	*6*, 0.576	*6*, 0.578	*6*, 0.532	*6*, 0.560	*6*, 0.561	*6*, 0.554	*6*, 0.561

15.4 Conclusion

We provide several concluding remarks on the performance of the AGKS preconditioner under two different solvers. For p-Uzawa and p-Minres solvers, we report numerical results for only $Q2$-$Q1$ discretization on a rectangular or skewed mesh with a single square-shaped or L-shaped island.

The p-Uzawa solver turns out to be the best choice since AGKS preserves both m- and h-robustness regardless of the discretization type or deterioration in the aspect ratio of the mesh. The change in one of the above only causes increase in the number of iterations, but qualitatively m- and h-robustness are maintained. Moreover, we observe that the asymptotic regime of the p-Uzawa solver starts with the m value 10^3; see left-bottom in Fig. 15.3. As island configuration changes, the number of iterations of both K- and S-solve slightly increases. In addition to that, as the discretization changes, the m-robustness of PMM for S-solve is lost. The asymptotic regime of the p-Uzawa solver becomes $m \geq 10^7$; see Tables 15.4 and 15.6.

The AGKS preconditioner under the p-Minres solver also maintains both m- and h-robustness as the discretization, the aspect ratio of the mesh, or the island configuration change. However, the number of iterations in the p-Minres solver increases dramatically when the mesh is skewed. Compared to p-Uzawa, one needs a more accurate inner-solve for a convergent p-Minres. In addition, the asymptotic regime of p-Minres solver is $m \geq 10^7$. Combining these three features, p-Minres becomes less desirable compared to p-Uzawa; see bottom-right and top-right in Fig. 15.3. However, this solver is potentially useful for large-size problems as the AGKS preconditioner maintains h-robustness.

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Chapter 16
Extension of Karmarkar's Algorithm for Solving an Optimization Problem

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Abstract In this paper, we propose an algorithm of an interior point method to solve a linear complementarity problem (LCP). The study is based on the transformation of a LCP into a convex quadratic problem; then we use the linearization approach to obtain the simplified problem of Karmarkar. Theoretical results deduct of those are established later, we show that this algorithm enjoys the best theoretical polynomial complexity, namely, $O(n+m+1)L$, iteration bound. The numerical tests confirm that the algorithm is robust.

16.1 Introduction

Let us consider the linear complementarity problem (LCP): find vectors x and y in real space $\Re^n$ that satisfy the following conditions:

$$x \geq 0,\ y = Mx + q \geq 0 \text{ and } x^t y = 0,$$

where q is a given vector in $\Re^n$ and M is a given $n \times n$ real matrix. LCP has important applications in mathematical programming and various areas of engineering [1, 6, 7]. Primal-dual path-following is the most attractive method among interior point methods to solve a large wide of optimization problems because of their polynomial complexity and their numerical efficiency [2, 4, 5, 8, 16–18]. Since Karmarkar's seminal paper [19] a number of various interior point algorithms were proposed and analyzed. For these the reader refers to [13, 19–21]. The primal-dual

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interior point methods ($IPMs$) for LO problems were first introduced by Kojima et al. [11] and Megiddo [12]. They have shown their powers in solving large classes of optimization problems.

The principal idea of this method is to replace a LCP by a convex quadratic program. After the appearance of the Karmarkar's algorithm [9], the researchers introduced extensions for the convex quadratic programming [3, 10, 14, 15, 22]. We propose in this paper an interior point method of type projective to solve a more general problem where the objective function is not inevitably linear. We combine the approach of linearization with ingredients brought by Karmarkar.

The paper is organized as follows. In the next section, the statement of theproblem is presented; we deal with the preparation of the algorithm and the description of the algorithm. In Sect. 16.3, we state its polynomial complexity. In Sect. 16.4, its numerical implementation is stated. In Sect. 16.5, a conclusion and remarks are given.

We use the classical notation. In particular, $\Re^{n+m}$ denotes the $(n+m)$-dimensional Euclidean space. Given $u, v \in \Re^{n+m}$, $u^t v = \sum_{i=1}^{n+m} u_i v_i$ is their inner product, and $\|u\| = \sqrt{u^t u}$ is the Euclidean norm. Given a vector $z \in \Re^{n+m}$, $D = diag(z)$ is the $(n+m) \times (n+m)$ diagonal matrix. I is the identity matrix and e is the identity vector.

16.2 Statement of the Problem

We consider the convex nonlinear programming (CNP) in a standard form as follows:

$$\min \{f(x) : Ax = b,\ x \geq 0\}, \qquad (CNP)$$

where $f : \Re^n \to \Re$ is a convex nonlinear function, $A \in \Re^{m\times n}$, $\text{rank.}(A) = m,\ b \in \Re^m$ and its dual problem

$$\max \left\{L(x,y,s) : A^t y + s = -\nabla f(x),\ s \geq 0,\ y \in \Re^m\right\},$$

where $L(x,y,s)$ is the Lagrangian function.

The LCP associated with the convex nonlinear programming (CNP) is written as follows:

$$\text{find } z \in \Re^{n+m} \text{ such that } z^t w = 0,\ w = Mz + q,\ (w,\ z) \geq 0, \qquad (LCP)$$

where $w \in \Re^{n+m},\ z = (x,y) \in \Re^{n+m},\ M = \begin{pmatrix} \nabla^2 f(x) & A^t \\ -A & 0 \end{pmatrix} \in \Re^{(n+m)\times(n+m)}$ is a matrix, $q \in \Re^{n+m}$.

Remark 16.1. In general, we cannot transform an arbitrary LCP in a convex quadratic program unless the matrix M is positive semi-definite.

Theorem 16.2. *[20] A LCP is equivalent to the following convex quadratic program:*

$$\min\left\{z^t(Mz+q): Mz+q \geq 0,\ z \geq 0\right\}, \tag{16.1}$$

where (z^*, Mz^*+q) is a solution of the LCP if and only if z^* is a optimal solution of the problem (16.1) with $(z^*)^t(Mz^*+q)=0$.

In the next section we have introduced Karmarkar's algorithm for solving the *LCP*.

16.2.1 Preparation of the Algorithm

We can write the problem (16.1) under the following simplified Karmarkar's form:

$$\min\{g(t): Bt=0,\ t \in S_{n+m+1}\}, \tag{16.2}$$

where $g: \Re^{n+m+1} \to \Re$ is a linear, convex, and differentiable function, B is a matrix, t is a vector, and $S_{n+m+1} = \left\{t \in \Re^{n+m+1} : e^t_{n+m+1}t = 1, t \geq 0\right\}$ is the simplex of dimension $(n+m)$ and of the center $a_i = \frac{1}{n+m+1}$, $i=1,\ldots,\ n+m+1$.

We introduce the projective Karmarkar's transformation defined by

$$T_k : \Re^{n+m} \to S_{n+m+1}$$

$$z \to t,$$

where

$$\begin{cases} t_i = \dfrac{z_i/z_i^k}{1+\sum\limits_{i=1}^{n+m} z_i/z_i^k},\ i=1,\ldots,\ n+m \\ t_{n+m+1} = 1-\sum\limits_{i=1}^{n+m} t_i, \end{cases}$$

and we have

$$z = T_k^{-1}(t) = \frac{D_k t[n+m]}{t_{n+m+1}},$$

where

$$t[n+m] = (D_k^{-1}z)t_{n+m+1} = (z_i)_{i=1}^{n+m},\ D_k = diag(z_k).$$

Thus the problem

$$\min\left\{\ f(z) = z^t(Mz+q): Mz+q \geq 0\ \right\} \Leftrightarrow \min\left\{\ f(z) = z^t(Mz+q): Mz = l\ \right\}$$

is transformed as follows:

$$\min\left\{f(T_k^{-1}(t)): M\frac{D_k t[n+m]}{t_{n+m+1}} = l,\ \sum_{i=1}^{n+m+1} t_i = 1,\ t[n+m] \geq 0,\ t_{n+m+1} \geq 0\right\}; \tag{16.3}$$

hence, it is advisable to write (16.3) under the equivalent form

$$\min\{g(t) = t_{n+m+1} f(D_k t[n+m]) : M_k t = 0,\ t \in S_{n+m+1}\}, \tag{16.4}$$

where

$$M_k = [MD_k,\ -l],\ t = \begin{bmatrix} t[n+m] \\ t_{n+m+1} \end{bmatrix}.$$

Note that the optimal value of g is *zero* and the center of the simplex is feasible for (16.4); also note that the function g is convex on the set $\{t \in S_{n+m+1} : M_k t = 0\}$.

Applying the linearization of the function g in the neighborhood of the center of the simplex a_i and introducing a ball of center a considered as a neighborhood of a, we have $g(t) = g(a) + \langle \nabla g(a), t-a \rangle$, for all $t \in \left\{ t \in \Re^{n+m+1} : \|t-a\|^2 \leq \beta^2 \right\}$.

Then we have the following subproblem:

$$\min \left\{ \nabla g(a)^t t : M_k t = 0, e^t_{n+m+1} t = 1, \|t-a\|^2 \leq \beta^2 \right\}. \tag{16.5}$$

Lemma 16.3. *The optimal solution of the problem (16.5) is explicitly given by*

$$t^k = a - \beta d^k,$$

where $d^k = \frac{P^k}{\|P^k\|}$, $P^k = p_{B_k} \nabla g(a)$, $B_k = \begin{bmatrix} M_k \\ e^t_{n+m+1} \end{bmatrix}$.

Proof. We put $z = t - a$, then we have $B_k z = \begin{bmatrix} M_k \\ e^t_{n+m+1} \end{bmatrix} (t-a) = 0$, and the subproblem (16.5) is equivalent to

$$\min \left\{ \nabla g(a)^t z : B_k z = 0, \|z\|^2 \leq \beta^2 \right\}; \tag{16.6}$$

z^* is a solution of (16.6) if and only if $\exists \lambda \in \Re^{n+m+1}$, $\exists \mu \geq 0$ such that

$$\nabla g(a) + B^t_k \lambda + \mu z^* = 0. \tag{16.7}$$

Multiplying both members of (16.7) by B_k we obtain

$$\begin{aligned} B_k \nabla g(a) + B_k B^t_k \lambda + \mu B_k z^* &= 0 \\ &\Leftrightarrow B_k \nabla g(a) + B_k B^t_k \lambda = 0. \end{aligned}$$

Then we have

$$\lambda = -(B_k B^t_k)^{-1} (B_k \nabla g(a));$$

by substituting in (16.7) we obtain

$$z^* = -\frac{1}{\mu} P^k, \text{where } P^k = \left[I - B^t_k (B_k B^t_k)^{-1} B_k \right] \nabla g(a),$$

$$\|z^*\| = \frac{1}{\mu}\left\|P^k\right\| = \beta \Longrightarrow z^* = -\beta\frac{P^k}{\|P^k\|} = -\beta d^k,$$

and we have

$$t^k = t^* = a + z^* = a - \beta d^k.$$

□

16.2.2 Description of the Algorithm

In this subsection, we describe the generic algorithm for our extension of *LCP*:

Begin algorithm

Step(1)

Initialization: $\varepsilon > 0, 0 < \beta < 1$, z^0 : is a strictly feasible point.

Step(2)

While$(f(z^k) - f(z^*) \geq \varepsilon)$ **do**

Compute the matrices:

- $D_k = diag(z^k)$
- $M_k = [MD_k, \ -l]$
- $B_k = \begin{bmatrix} M_k \\ e^t_{n+m+1} \end{bmatrix}$

Compute:

- $P^k = p_{B_k}\nabla g(a) = \left[I - B_k^t(B_kB_k^t)^{-1}B_k\right]\nabla g(a)$
- $d^k = \frac{P^k}{\|P^k\|}$
- $t^k = a - \beta d^k$

Take:

$z^{k+1} = T_k^{-1}(t^k)$. Let $k = k+1$ and go back to **Step (2)**

End While

End algorithm.

16.3 Convergence of Algorithm

In order to establish the convergence of our algorithm, we introduce a potential function associated with problem (16.1) defined by

$$F(z) = (n+m+1)\log(f(z) - f(z^*)) - \sum_{i=1}^{n+m}\log(z_i).$$

We have the following lemma.

Lemma 16.4. *For each iteration, we obtain a reduction of the function g, i.e.,*

$$g(t^k) \leq g(a).$$

Proof. We have

$$g(z^k) = g(a) + \left\langle \nabla g(a),\, z^k - a \right\rangle \text{and } t^k = a - \beta \frac{P^k}{\|P^k\|}.$$

Then, we get

$$\begin{aligned} g(z^k) - g(a) &= \left\langle \nabla g(a),\ -\beta \frac{P^k}{\|P^k\|} \right\rangle \\ &= -\frac{\beta}{\|P^k\|} \left\langle \nabla g(a),\, P^k \right\rangle \\ &= -\frac{\beta}{\|P^k\|} \left\| P^k \right\|^2 < 0, \end{aligned}$$

whence the result.□

Theorem 16.5. *In every iteration of our algorithm, potential function is reduced of a constant value such that*

$$F(z^{k+1}) < F(z^k) - \delta.$$

Proof. We have

$$\begin{aligned} F(z^{k+1}) - F(z^k) &= (n+m+1)\log\left[\frac{f(z^{k+1}) - f(z^*)}{f(z^k) - f(z^*)}\right] - \sum_{i=1}^{n+m} \log\left(\frac{z_i^{k+1}}{z_i^k}\right), \\ &= (n+m+1)\log\frac{g(t^k)}{g(a)} - \sum_{i=1}^{n+m} \log\left(t_i^k\right), \\ &\leq (n+m+1)\log\left(1 - \frac{\beta}{n+m+1} + \frac{\beta^2}{2(1-\beta)^2}\right), \\ &\leq -\beta + \frac{\beta^2}{2(1-\beta)^2}. \end{aligned}$$

If we use the following result of Karmarkar [9]

$$-\sum_{i=1}^{n+m} \log\left(t_i^k\right) \leq \frac{\beta^2}{2(1-\beta)^2},$$

then we have

$$F(z^{k+1}) < F(z^k) - \delta \text{ where } \delta = \beta - \frac{\beta^2}{2(1-\beta)^2},$$

which completes the proof. □

Consider the following assumptions:

Assumption 1. The initial solution z^0verifies $z^0 \geq 2^{-2L} e_{n+m+1}$.

Assumption 2. The optimal solution z^* verifies $z^* \leq 2^{2L} e_{n+m+1}$; for any solution z we have $-2^{3L} \leq f(z^*) \leq 2^{3L}$.

In the following theorem, we study the complexity analysis of our algorithm.

Theorem 16.6. *For each iteration, the algorithm finds the optimal solution after $O((n+m+1)L)$ iterations.*

Proof. We have

$$\frac{f(z^k) - f(z^*)}{f(z^0) - f(z^*)} = \eta(z^k) \exp\left[\frac{F(z^k) - F(z^0)}{n+m+1}\right].$$

Under the assumptions (16.1) and (16.2), we have $\eta(z^k) \leq 2^{2L}$; then

$$\begin{aligned} f(z^k) - f(z^*) &\leq 2^{2L}(f(z^0) - f(z^*)) \exp\left[\frac{F(z^k) - F(z^0)}{n+m+1}\right] \\ &\leq 2^{2L} 2^{3L} \exp\left(\frac{-k\delta}{n+m+1}\right). \end{aligned}$$

Hence, we get

$$k \geq \xi(n+m+1)L, \text{ where } \xi \in \Re^*_+,$$

which gives the result. □

16.4 Numerical Implementation

In this section, we deal with the numerical implementation of our algorithm applied to some problems of monotone *LCP*. Here we use $(z^0)^t = ((x^0)^t, (y^0)^t)^t$ to denote the feasible starting solution of the algorithm, z^* the optimal solution of *LCP*, and **Iter** means the iterations number produced by the algorithm. The implementation is manipulated in DEV C++. Our tolerance is 10^{-6}.

Problem 1:

$$M = \begin{pmatrix} 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ -2 & -1 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \; q = \begin{pmatrix} -4 & -5 & 8 & 7 & 3 \end{pmatrix}^t.$$

The feasible starting solution is

$z^0 = \begin{pmatrix} 2 & 2 & 2 & 2 & 2 \end{pmatrix}^t$.

The optimal solution is

$z^* = \begin{pmatrix} 2 & 3 & 2 & 1 & 1 \end{pmatrix}^t$.

Iter: 6.

Problem 2:

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 3 & 0.8 & 0.32 & 1.128 & 0.0512 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.8 & 0.32 & 1.128 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.8 & 0.32 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.32 & -0.8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.28 & -0.32 & -0.8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.0512 & -1.28 & -0.32 & -0.8 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$q = \begin{pmatrix} -0.0256 & -0.064 & -0.16 & -0.4 & -1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^t.$

The feasible starting solution is

$z^0 = \begin{pmatrix} 0.18 & 0.18 & 0.18 & 0.18 & 0.25 & 3 & 4 & 5 & 6 & 9 \end{pmatrix}^t.$

The optimal solution is

$z^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^t.$

Iter: 11.

16.5 Concluding Remarks

In this paper we have extended results from Karmarkar [9] and also proved that the polynomial complexity of the algorithm for solving *LCP* is no more than $O((n+m+1)L)$. Our numerical results are acceptable whereas getting a starting feasible solution for our algorithm. Finally, the numerical tests are interesting for investigating the behavior of the algorithm so as to be compared with other approaches.

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Chapter 17
State-Dependent Sweeping Process with Perturbation

Tahar Haddad and Touma Haddad

Abstract We prove, via a new projection algorithm, the existence of solutions for differential inclusion generated by sweeping process with closed convex sets depending on state.

17.1 Introduction

The existence of solutions for the first-order differential inclusion governed by state-dependent sweeping process

$$\begin{cases} -\dot{u}(t) \in N_{C(t,u(t))}(u(t)) \text{ a.e. on } [0,T]; \\ u(t) \in C(t,u(t)), \text{ for all } t \in [0,T] \\ u(0) = u_0 \in C(0,u_0), \end{cases} \tag{17.1}$$

where $N_{C(t,u(t))}(\cdot)$ denotes the normal cone to $C(t,u(t))$, has been studied when the sets $C(t,u(t))$ are convex by Kunze and Monteiro Marques for the first time in Hilbert space H; see [7]. They used an implicit projection algorithm based on the fixed point theorem (implicit discretization). Recently, in [1], the authors treated the problem (17.1) in uniformly convex and uniformly smooth Banach spaces when the sets $C(t,u(t))$ are convex.

In this chapter we are interested by the new variant of state-dependent sweeping process

$$\begin{cases} -\dot{u}(t) \in N_{C(u(t))}(u(t)) + Au(t) + f(t) \text{ a.e. on } [0,T]; \\ u(t) \in C(u(t)), \text{ for all } t \in [0,T] \\ u(0) = u_0 \in C(u_0), \end{cases} \tag{17.2}$$

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where the linear operator A is bounded, f be a continuous and uniformly bounded function and the constraints $C(u) \in H$ are convex. Problem (17.2) includes as a special case the following evolution quasi-variational inequality:

Find $u : I \to H, u(0) = u_0 \in C(u_0)$, such that $u(t) \in C(u(t))$ for all $t \in [0,T]$, and

$$\langle l(t), w - u(t)\rangle \leq \langle \dot{u}(t), w - u(t)\rangle + a(u(t), w - u(t)) \quad \text{a.e. on } [0,T] \tag{17.3}$$

for all $w \in C(u(t))$. Here $a(\cdot,\cdot)$ is a real bilinear, symmetric, bounded, and elliptic form on $H \times H$, $l \in W^{1,2}((0,T);H)$, and $K(u) \subset H$ is a set of constraints. The quasi-variational inequality of type (17.3) arises in superconductivity model (see Duvaut and Lions [6]). By using a new projection algorithm (explicit discretization) and techniques from nonsmooth analysis, we give a new proof of the variant of state-dependent sweeping process described by (17.2) which improves the ones given in [1, 7].

This chapter is organized as follows. Section 17.2 contains some definitions, notations, and important results needed in the chapter. In Sect. 17.3, we prove an existence result for (17.2) when $C(u)$ is a convex set of the Hilbert space H moving in a Lipschitz continuous way. In Sect. 17.4, we state an application to the quasi-variational inequality (17.3).

17.2 Notation and Preliminaries

In the sequel, H denotes a real separable Hilbert space. Let S be a closed subset of H. We denote by $\mathbb{B}$ the closed unit ball of H and by $d_S(\cdot)$ the usual distance function associated with S, i.e. $d(x,S) := \inf_{u\in S} \|x-u\|$ $(x \in H.)$ We need first to recall some notations and definitions needed in the chapter.

Let $\varphi : H \to \mathbb{R} \cup +\infty$ be a convex lower semicontinuous (l.s.c) function and let x be any point where φ is finite. We recall that the *subdifferential* $\partial\varphi(x)$ (in the sense of convex analysis) is the set of all $\xi \in H$ such that

$$\langle \xi, x' - x\rangle \leq \varphi(x') - \varphi(x)$$

for all $x^{'} \in H$. By convention we set $\partial\varphi(x) = \emptyset$ if $\varphi(x)$ is not finite. Let S be a nonempty closed subset of H and x be a point in S. The convex normal cone of S at x is defined by (see for instance [4])

$$N_S(x) = \{\xi \in H |\ \langle \xi, x' - x\rangle \leq 0 \quad \text{for all } x^{'} \in S\}.$$

It is well known (see for example [4]) that $N_S(x)$ the normal cone of a closed convex set S at $x \in H$ can be defined in terms of projection operator $\text{Proj}_S(.)$ as follows:

$$N_S(x) = \{\xi \in H |\ \text{there exists } \ r > 0 \text{ such that } x \in \text{Proj}_S(x + r\xi)\}.$$

Let us recall the two following results. For their proofs we refer to [2, 8], respectively.

Proposition 17.1. *Let S be a nonempty closed subset of H and $x \in S$. Then*

$$\partial d_S(x) = N_S(x) \cap \mathbb{B}.$$

Proposition 17.2. *Let $C : H \rightrightarrows H$ be a Hausdorff-continuous set-valued mapping with nonempty closed convex values. Then the mapping*

$$(x,y) \mapsto \partial d_{C(x)}(y)$$

has closed convex values and satisfying the following upper semicontinuity property: Let (x_n) be a sequence in H converging to $x \in H$, and (y_n) be a sequence in H with $y_n \in C(x_n)$ for all n, converging to $y \in C(x)$, then for any $\xi \in H$, we have

$$\limsup_n \sigma(\partial d_{C(x_n)}(y_n), \xi)) \leq \sigma(\partial d_{C(x)}(y), \xi),$$

where

$$\sigma(\partial d_{C(x)}(y), \xi) := \sup_{p \in \partial d_{C(x)}(y)} \langle p, \xi \rangle$$

stands for the support function of $\partial d_{C(x)}(y)$ at ξ.

Let now B be a bounded set of a normed space E. Then the Kuratowski measure of noncompactness of B, $\alpha(B)$, is defined by

$$\alpha(B) = \inf\{d > 0 |\ B = \bigcup_{i=1}^{m} B_i \text{ for some } m \text{ and } B_i \text{ with } \operatorname{diam}(B_i) \leq d\}$$

Here $\operatorname{diam}(A)$ stands for the diameter of A given by

$$\operatorname{diam} A := \sup_{x,y \in A} \|x - y\|.$$

In the following lemma we recall (see for instance Proposition 9.1 in [5]) some useful properties for the measure of noncompactness α.

Lemma 17.3. *Let H be an infinite dimensional real Banach space and D_1, D_2 be two bounded subsets of H.*

1. $\alpha(D_1) = 0 \Leftrightarrow D_1$ *is relatively compact.*
2. $\alpha(\lambda D_1) = |\lambda| \alpha(D_1)$ *for all* $\lambda \in \mathbb{R}$.
3. $D_1 \subset D_2 \Rightarrow \alpha(D_1) \leq \alpha(D_2)$.
4. $\alpha(D_1 + D_2) \leq \alpha(D_1) + \alpha(D_2)$.
5. *if* $x_0 \in H$ *and* r *is a positive real number, then* $\alpha(x_0 + r\mathbb{B}) = 2r$.

17.3 Main Result

The following existence theorem establishes our main result in this chapter.

Theorem 17.4. *Let H be a separable Hilbert space and let $C : H \to H$ be a set-valued mapping with nonempty closed convex values satisfying the following assumptions:*
$(\mathscr{H}_1)$ *C is Lipschitz continuous with constant $0 < L < 1$, i.e. for all $x, u, v \in H$, we have*

$$|d_{C(u)}(x) - d_{C(v)}(x)| \le L\|u - v\|;$$

$(\mathscr{H}_2)$ *there exists a strongly compact set S such that $C(u) \subset S$ for all $u \in H$. Let $A : H \to H$ be a linear bounded operator. Assume also that $f : [0,T] \to H$ is continuous and uniformly bounded, that is, there exists $\beta > 0$ such that $\|f(t)\| \le \beta$ for all $t \in [0,T]$. Then for any $u_0 \in C(u_0)$, there exists at least one Lipschitz solution of (17.2).*

Proof. Let $\rho > 0$ such that $C(u) \subset S \subset \rho\mathbb{B}$ for all $u \in H$. For each $n \in N$, we consider the following partition of the interval $I := [0,T]$

$$I^n_{i+1} :=]t^n_i, t^n_{i+1}], \quad t^n_i := i\mu_n, \quad 0 \le i \le n-1, \quad I^n_0 := \{t^n_0\}.$$

Algorithm 1. Put $\mu_n := \frac{T}{n}$. Fix $n \ge 2$. We define by induction

- $u^n_0 = u_0 \in C(u_0)$, and $f^n_0 = f(t^n_0)$
- $0 \le i \le n-1 : u^n_{i+1} = \text{Proj}_{C(u^n_i)}(u^n_i - \mu_n A u^n_i - \mu_n f^n_i)$
- $f^n_{i+1} = f(t^n_{i+1})$

The existence of the projection is ensured since C has closed convex values, and so the Algorithm 1 is well defined. Using the sequences (u^n_i) and (f^n_i) to construct sequences of mapping u_n and f_n from $[0,T]$ to H by defining their restrictions to each interval I^n_i as follows:

For $t \in I^n_0$ set $f_n(t) = f^n_0$ and $u_n(t) = u_0$; for $t \in I^n_{i+1}$ $(0 \le i \le n-1)$ set $f_n(t) = f^n_i$, and

$$u_n(t) = u^n_i + (u^n_{i+1} - u^n_i)\frac{(t - t^n_i)}{\mu_n} \tag{17.4}$$

Clearly u_n is continuous on $[0,T]$ and differentiable on $[0,T]\backslash\{t^n_i\}$ with

$$\dot{u}_n(t) = \frac{u^n_{i+1} - u^n_i}{\mu_n}, \ \forall t \in [0,T]\backslash\{t^n_i\}. \tag{17.5}$$

By Algorithm 1, we have

$$u^n_{i+1} = \text{Proj}_{C(u^n_i)}(u^n_i - \mu_n A u^n_i - \mu_n f^n_i).$$

Using the characterization of the normal cone in terms of projection operator, we can write for a.e. $t \in [0,T]$

$$u^n_i - u^n_{i+1} - \mu_n A u^n_i - \mu_n f^n_i \in N_{C(u^n_i)}(u^n_{i+1}),$$

or

$$-\frac{u_{i+1}^n - u_i^n}{\mu_n} - Au_i^n - f_i^n \in N_{C(u_i^n)}(u_{i+1}^n). \tag{17.6}$$

Let us find an upper bound estimate for the expression $\|-\frac{u_{i+1}^n - u_i^n}{\mu_n} - Au_i^n - f_i^n\|$. By Algorithm 1, $\|f_i^n\| \leq \beta$ and $u_{i+1}^n \in C(u_i^n) \subset \rho\mathbb{B}$, that is, $\|u_i^n\| \leq \rho$, for all $i \geq 0$. Therefore the Lipschitz property of C ensures that

$$\begin{aligned}\|u_i^n - u_{i+1}^n - \mu_n Au_i^n - \mu_n f_i^n\| &= d_{C(u_i^n)}(u_i^n - \mu_n Au_i^n - \mu_n f_i^n)\\ &\leq d_{C(u_i^n)}(u_i^n) - d_{C(u_{i-1}^n)}(u_i^n) + \mu_n\|A\|\|u_i^n\| + \mu_n\|f_i^n\|\\ &\leq L\|\|u_i^n - u_{i-1}^n\| + \mu_n(\rho\|A\| + \beta).\end{aligned} \tag{17.7}$$

By construction we have

$$\begin{aligned}\|u_i^n - u_{i-1}^n\| &= \|u_i^n - u_{i-1}^n + \mu_n Au_{i-1}^n + \mu_n f_{i-1}^n - \mu_n Au_{i-1}^n - \mu_n f_{i-1}^n\|\\ &\leq \|u_{i-1}^n - u_i^n - \mu_n Au_{i-1}^n - \mu_n f_{i-1}^n\| + \mu_n\|A\|\|u_{i-1}^n\| + \mu_n\|f_{i-1}^n\|\\ &= d_{C(u_{i-1}^n)}(u_{i-1}^n - \mu_n Au_{i-1}^n - \mu_n f_{i-1}^n) + \mu_n\|A\|\|u_{i-1}^n\| + \mu_n\|f_{i-1}^n\|\\ &\leq d_{C(u_{i-1}^n)}(u_{i-1}^n) - d_{C(u_{i-2}^n)}(u_{i-1}^n) + 2\mu_n\|A\|\|u_{i-1}^n\| + 2\mu_n\|f_{i-1}^n\|\\ &\leq L\|u_{i-1}^n - u_{i-2}^n\| + 2\mu_n(\rho\|A\| + \beta).\end{aligned}$$

By induction we obtain

$$\begin{aligned}\|u_i^n - u_{i-1}^n\| &\leq 2\mu_n(\rho\|A\| + \beta) + L\Big(2\mu_n\|A\|\rho + 2\mu_n\beta + L\|u_{i-2}^n - u_{i-3}^n\|\Big)\\ &= 2\mu_n(\rho\|A\| + \beta)\big(1 + L\big) + L^2\|u_{i-2}^n - u_{i-3}^n\|\\ &\cdots \qquad\qquad\qquad \cdots\\ &\leq 2\mu_n(\rho\|A\| + \beta)\big(1 + L + L^2 + \cdots + L^{i-2}\big) + L^{i-1}\|u_1^n - u_0^n\|.\end{aligned}$$

The initial condition $u_0 \in C(u_0)$ entails

$$\begin{aligned}\|u_1^n - u_0^n\| &\leq \|u_0^n - u_1^n - \mu_n Au_0^n - \mu_n f_0^n\| + \mu_n\|A\|\|u_0^n\| + \mu_n\|f_0^n\|\\ &\leq d_{C(u_0^n)}(u_0^n - \mu_n Au_0^n - \mu_n f_0^n) + \mu_n\|A\|\rho + \mu_n\beta\\ &\leq d_{C(u_0^n)}(u_0^n) + 2\mu_n\|A\|\rho + 2\mu_n\beta\\ &= 2\mu_n(\rho\|A\| + \beta).\end{aligned}$$

So

$$\|u_i^n - u_{i-1}^n\| \leq 2\mu_n(\rho\|A\| + \beta)\big(1 + L + L^2 + \cdots + L^{i-1}\big). \tag{17.8}$$

Hence (17.7) and (17.8) imply that

$$\|u_i^n - u_{i+1}^n - \mu_n Au_i^n - \mu_n f_i^n\| \leq \mu_n(2\rho\|A\| + 2\beta)\big(1 + L + L^2 + \cdots + L^i\big).$$

Using the fact that $L < 1$, we get

$$\begin{aligned}\|u_i^n - u_{i+1}^n - \mu_n Au_i^n - \mu_n f_i^n\| &\leq \mu_n(2\rho\|A\| + 2\beta)\big(\tfrac{1-L^{i+1}}{1-L}\big)\\ &\leq \left(\frac{2\|A\|\rho + 2\beta}{1-L}\right)\mu_n,\end{aligned}$$

or

$$\| - \frac{u_{i+1}^n - u_i^n}{\mu_n} - Au_i^n - f_i^n \| \leq \left(\frac{2\|A\|\rho + 2\beta}{1-L}\right). \tag{17.9}$$

The inclusion (17.6) and Proposition 17.1 give

$$-\frac{u_{i+1}^n - u_i^n}{\mu_n} - Au_i^n - f_i^n \in \left(\frac{2\|A\|\rho + 2\beta}{1-L}\right) \partial d_{C(u_i^n)}(u_{i+1}^n). \tag{17.10}$$

Now let us define the step functions from $[0,T]$ to $[0,T]$ by

$$\begin{array}{ll} \theta_n(t) = t_i^n; & t \in I_{i+1}^n, \\ \eta_n(t) = t_{i+1}^n; & t \in I_{i+1}^n, \\ \theta_n(0) = \eta_n(0) = 0. & \end{array} \tag{17.11}$$

Then (17.4), (17.5), (17.10), and (17.11) yield that

$$-\dot{u}_n(t) - Au_n(\theta_n(t)) - f_n(t) \in \left(\frac{2\|A\|\rho + 2\beta}{1-L}\right) \partial d_{C(u_n(\theta_n(t)))}(u_n(\eta_n(t))) \text{ a.e. on } [0,T]. \tag{17.12}$$

As $\lim_{n\to+\infty} \theta_n(t) = \lim_{n\to+\infty} \eta_n(t) = t$, we can write by the continuity of f $\lim_{n\to+\infty} f(\theta_n(t)) = \lim_{n\to+\infty} f_n(t) = f(t)$, uniformly on $[0,T]$. Let us prove that the sequence (u_n) has a convergent subsequence. By (17.5) and (17.9)

$$\|\dot{u}_n(t)\| \leq \left(\frac{2\|A\|\rho + 2\beta}{1-L}\right) + \|A\|\rho + \beta := \gamma, \tag{17.13}$$

and it is clear that the sequence $(u_n(t))$ is equi-Lipschitz with constant γ. Now we show that the set $\mathscr{X}(t) = \{u_n(t) | n \geq 2\}$ is relatively compact in H for every $t \in [0,T]$. From the definition of (u_n) we have for all $t \in [0,T]$ and all $n \geq 2$, $u_n(\eta_n(t)) \in C(u_n(\theta_n(t))) \subset S$. Then the set $\{u_n(\eta_n(t)) | n \geq 2\}$ is relatively compact in H for all $t \in [0,T]$, and so by Lemma 17.3 we get

$$\alpha(u_n(\eta_n(t)) | n \geq 2\}) = 0.$$

We have $\mathscr{X}(t) = \{u_n(t) | n \geq 2\} = \{u_n(t) - u_n(\eta_n(t)) + u_n(\eta_n(t)) | n \geq 2\}$ for all $t \in [0,T]$. Then by Lemma 17.3 we obtain that

$$\begin{aligned} \alpha(\mathscr{X}(t)) &\leq \alpha(\{u_n(t) - u_n(\eta_n(t)) | n \geq 2\}) + \alpha(\{u_n(\eta_n(t)) | n \geq 2\}) \\ &\leq \alpha\left(\left\{\int_t^{\eta_n(t)} \dot{u}_n(s) ds | n \geq 2\right\}\right) + 0 \\ &\leq \alpha\left(B\left(0, \frac{T}{n}\gamma\right)\right) \\ &= 2\gamma\frac{T}{n} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Here by Lemma 17.3 the set $\mathscr{X}(t)$ is relatively strongly compact in H for all $t \in [0,T]$.

Then the all assumptions of Arzela–Ascoli theorem are satisfied and hence there exists a Lipschitz mapping $u : [0,T] \to H$ with ratio γ such that

- (u_n) converges uniformly to u on $[0,T]$, that is, $\lim_{n\to+\infty} \max_{t\in[0,T]} \|u_n(t) - u(t)\| = 0$;
- $(\dot{u}_n)$ weakly converges to $\dot{u}$ in $L^1([0,T],H)$.

Since $\lim_{n\to+\infty} \theta_n(t) = \lim_{n\to+\infty} \eta_n(t) = t$, we have

$$\begin{aligned}\lim_{n\to+\infty} u_n(\theta_n(t)) &= \lim_{n\to+\infty} u_n(\eta_n(t))\\ &= \lim_{n\to+\infty} u_n(t) = u(t)\end{aligned}$$

uniformly on $[0,T]$. Using now the Lipschitz property of C and the fact that $u_n(\eta_n(t)) \in C(u_n(\theta_n(t)))$, $\forall t \in [0,T]$ and for all $n \geq 2$, we get

$$\begin{aligned}d(u(t),C(u(t))) &= d_{C(u(t))}(u(t)) - d_{C(u_n(\theta_n(t)))}(u_n(\eta_n(t)))\\ &\leq \|u_n(\eta_n(t)) - u(t)\| + L\|u_n(\eta_n(t)) - u(t)\|\\ &\leq (1+L)\|u_n - u\|_\infty \to 0 \text{ as } n \to \infty,\end{aligned}$$

and so the closeness of the set $C(u(t))$ ensures that $u(t) \in C(u(t))$ for all $t \in [0,T]$. We proceed now to prove that

$$-\dot{u}(t) \in N_{C(u(t))}(u(t)) + Au(u(t)) + f(t) \quad \text{for almost all } t \in [0,T].$$

Applying Castaing techniques (see for instance [3]), the uniform convergence of u_n to u, the weak convergence of $\dot{u}_n$ to $\dot{u}$ in $L^1([0,T],H)$, the uniform convergence of f_n to f , and Mazur's lemma entail

$$-\dot{u}(t) - Au(t) - f(t) \in \bigcap_n \overline{\text{co}}\{-\dot{u}_k(t) - A(u_k(t)) - f_k(t) | k \geq n\}$$

for almost all $t \in [0,T]$. Hence $\overline{\text{co}}$ denotes the closed convex hull.

Fix any such $t \in [0,T]$ and consider any $\xi \in H$. The last relation above yields

$$\langle \xi, -\dot{u}(t) - Au(t) - f(t)\rangle \leq \inf_n \sup_{k\geq n} \langle \xi, -\dot{u}_k(t) - A(u_k(t)) - f_k(t)\rangle.$$

According to (17.12) we obtain that

$$\begin{aligned}\langle \xi, -\dot{u}(t) - Au(t) - f(t)\rangle &\leq \limsup_n \sigma\left(\left(\frac{2\|A\|\rho + 2\beta}{1-L}\right)\partial d_{C(u_n(\theta_n(t)))}(u_n(\eta_n(t))), \xi\right)\\ &\leq \sigma\left(\left(\frac{2\|A\|\rho + 2\beta}{1-L}\right)\partial^P d_{C(t,u(t))}(u(t)), \xi\right),\end{aligned}$$

where the last inequality follows from the upper semicontinuity property given in Proposition 17.2 and because of $\theta_n(t) \to t$ and $\eta_n(t) \to t$, $u_n(\eta_n(t)) \to u(t)$, $u_n(\theta_n(t)) \to u(t)$, strongly. Since the set $\partial d_{C(u(t))}(u(t))$ is closed convex (see Proposition 17.2 and $u(t) \in C(u(t))$, we obtain that

$$-\dot{u}(t) - Au(t) - f(t) \in \Big(\frac{2\|A\|\rho + 2\beta}{1-L}\Big)\partial d_{C(u(t))}(u(t)) \subset N_{C(u(t))}(u(t))$$

and so

$$-\dot{u}(t) \in N_{C(u(t))}(u(t)) + Au(t) + f(t) \quad \text{for a.e. } t \in [0,T].$$

This completes the proof of the theorem. □

17.4 Application

As a direct application of our main result we obtain an existence result for the evolution quasi-variational inequality:
Find $u : I \to H, u(0) = u_0 \in C(u_0)$, such that $u(t) \in C(u(t))$ for all $t \in [0,T]$, and

$$\langle l(t), w - u(t)\rangle \leq \langle \dot{u}(t), w - u(t)\rangle + a(u(t), w - u(t)) \quad \text{a.e. on } [0,T] \tag{17.14}$$

for all $w \in C(u(t))$.

Here $a(\cdot,\cdot)$ is a real bilinear, symmetric, bounded, and elliptic form on $H \times H$, $l \in W^{1,2}((0,T);H)$ and $K(u) \subset H$ is a set of constraints. The differential variational inequality of type (17.3) arises in superconductivity model (see Duvaut and Lions [6])

Proposition 17.5. *Assume that $C : H \rightrightarrows H$ is Lipschitz continuous with ratio $0 < L < 1$ and convex values such that $C(u) \subset S$ for all $u \in H$ for some strongly compact set $S \subset H$. Assume that l is uniformly bounded, that is, there exists $\beta > 0$ such that $\|l(t)\| \leq \beta$ for all $t \in [0,T]$. Then, for every $u_0 \in C(u_0)$, there exists at least one Lipschitz solution of (17.14).*

Proof. Let A be a linear and bounded operator on H associated with $a(\cdot,\cdot)$, that is, $a(u,v) = \langle Au, v\rangle$ for all $u, v \in H$ and put $f(t) = -l(t)$, for all $t \in [0,T]$. Since C has convex values, the evolution quasi-variational inequality of type (17.14) can be rewritten in the form of (17.2) as follows:

$$-\dot{u}(t) \in N_{C(t,u(t))}(u(t)) + Au(t) + f(t) \quad \text{a.e. on } [0,T],$$

with $u(0) = u_0 \in C(u_0)$. By the Sobolev embedding theorem, $W^{1,2}((0,T);H) \subset C((0,T);H)$, we conclude that f is continuous. Thus all assumptions of Theorem 17.4 are satisfied and so the proof is complete. □

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Chapter 18
Boundary Value Problems for Impulsive Fractional Differential Equations with Nonlocal Conditions

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Abstract In this study, we discuss some existence results for the solutions to impulsive fractional differential equations with nonlocal conditions by using contraction mapping principle and Krasnoselskii's fixed point theorem.

18.1 Introduction

This work is concerned with the existence and uniqueness of the solutions to the boundary value problem (BVP for short), for the following impulsive fractional differential equation with nonlocal conditions:

$$\begin{cases} {}^{C}D^{\alpha}y(t) = f(t,y(t)), \ \ t \in J := [0,T], t \neq t_k, \ 1 < \alpha \leq 2 \\ \Delta y(t_k) = I_k(y(t_k^-)), \ \ \Delta y^{'}(t_k) = I_k^*(y(t_k^-)), \ k = 1,2,\ldots,p \\ \quad ay(0) + by(T) = g_1(y), \ \ cy^{'}(0) + dy^{'}(T) = g_2(y), \end{cases} \tag{18.1}$$

where ${}^{C}D^{\alpha}$ is the Caputo fractional derivative, $f \in C(J \times R, R)$, $I_k, I_k^* \in C(R,R)$, $g_1, g_2 : PC(J,R) \to R$ ($PC(J,R)$will be defined later), $\Delta y(t_k) = y(t_k^+) - y(t_k^-)$ with $y(t_k^+) = \lim_{h \to 0^+} y(t_k + h)$, $y(t_k^-) = \lim_{h \to 0^-} y(t_k + h)$, and $\Delta y'(t_k)$ has a similar meaning for $y'(t)$, $0 = t_0 < t_1 < t_2 < \ldots < t_p < t_{p+1} = T$, a, b, c, and d are real constants with $a + b \neq 0$, $c + d \neq 0$.

The subject of fractional differential equations has been recently addressed by several authors and it is gaining much importance. This is due to the fact that the fractional derivatives serve an excellent tool for the description of hereditary properties of different materials and processes. Actually, fractional differential equations arise in many engineering and scientific disciplines such as physics, chemistry, biology, electrochemistry, electromagnetic, control theory, economics, signal and

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image processing, aerodynamics, and porous media (see [12–15, 17, 20, 22, 23, 25, 27] and references therein). On the other hand, theory of impulsive differential equations for integer order has become important and found its extensive applications in mathematical modeling of phenomena and practical situations in both physical and social sciences in recent years. One can see a remarkable development in impulsive theory. For instance, for the general theory and applications of impulsive differential equations we refer the readers to [11, 19, 24, 28].

Boundary value problems take place in the studies of fractional differential equations differently many times (see [1–4, 6, 16, 21, 26, 31] and the relevant references therein). More precisely, nonlocal conditions were initiated by Byszewski [9] when he proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems. As pointed out in [8, 10], nonlocal boundary conditions can be more useful than standard conditions to describe physical phenomena. For instance, $g(y)$ may be given by

$$g(y) = \sum_{i=1}^{m} \eta_i y(\xi_i)$$

where $\eta_i,\ i = 1,2,\ldots,m$ are given constants and $0 < \xi_1 < \xi_2 < \ldots < \xi_m < T$.

That is why, nonlocal BVPs for fractional differential equations have received considerable attention (see [5, 8, 32]). However, to the best of our knowledge, there are few studies considering BVPs for impulsive fractional differential equations with nonlocal conditions (see[7, 30]).

Motivated by the mentioned recent work above, in this study, we investigate the existence and uniqueness of solutions to the nonlocal BVP for fractional differential equation with impulses. Throughout this chapter, in Sect. 18.2, we present some notations and preliminary results about fractional calculus and differential equations to be used in the following sections. In Sect. 18.3, we discuss some existence and uniqueness results for solutions of BVP (18.1), namely, the first one is based on Banach's fixed point theorem, and the second one is based on the Krasnoselskii's fixed point theorem. At the end, we give an illustrative example for our results.

18.2 Preliminaries

Set $J_0 = [0,t_1],\ J_1 = (t_1,t_2],\ldots,J_{k-1} = (t_{k-1},t_k],\ J_k = (t_k,t_{k+1}],\ J' := [0,T]\backslash \{t_1,t_2,\ldots,t_p\}$ and define the set of functions:

$PC(J,R) = \{y : J \to R : y \in C((t_k,t_{k+1}],R),\ k = 0,1,2,\ldots,p$ and there exist $y(t_k^+)$ and $y(t_k^-),\ k = 1,2,\ldots,p$ with $y(t_k^-) = y(t_k)\}$ and

$PC^1(J,R) = \{y \in PC(J,R),\ y' \in C((t_k,t_{k+1}],R),\ k = 0,1,2,\ldots,p,$ and there exist $y'(t_k^+)$ and $y'(t_k^-),\ k = 1,2,\ldots,p$ with $y'(t_k^-) = y'(t_k)\}$ which is a Banach space with the norm $\|y\| = \sup_{t\in J}\left\{\|y\|_{PC},\left\|y'\right\|_{PC}\right\}$ where $\|y\|_{PC} := \sup\{|y(t)| : t \in J\}$.

Definition 18.1. ([17, 22]) The fractional (arbitrary) order integral of the function $h \in L^1(J,R_+)$ of order $\alpha \in R_+$ is defined by

$$I_{0^+}^{\alpha}h(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds$$

where $\Gamma(.)$ is the Euler gamma function.

Definition 18.2. ([17, 22]) For a function h given on the interval J, Caputo fractional derivative of order $\alpha > 0$ is defined by

$$^{C}D_{0^+}^{\alpha}h(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}h^{(n)}(s)ds,\ n = [\alpha]+1$$

where the function $h(t)$ has absolutely continuous derivatives up to order $(n-1)$.

Lemma 18.3. *([17, 31]) Let $\alpha > 0$, then the differential equation*

$$^{C}D^{\alpha}h(t) = 0$$

has solution

$$h(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1}t^{n-1},\ \ c_i \in R,\ i = 0,1,2,\ldots,n-1,\ n = [\alpha]+1.$$

Lemma 18.4. *([17, 31]) Let $\alpha > 0$, then*

$$I^{\alpha}\, {}^{C}D^{\alpha}h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1}t^{n-1}$$

for some $c_i \in R,\ i = 0,1,2,\ldots,n-1,\ n = [\alpha]+1.$

Theorem 18.5. *([18])(Krasnoselskii's fixed point theorem) Let M be a closed convex and nonempty subset of a Banach space X. Let A,B be the operators such that (i) $Ax+By \in M$ whenever $x,y \in M$, (ii) A is compact and continuous, (iii) B is a contraction mapping. Then, there exists $z \in M$ such that $z = Az + Bz$.*

Theorem 18.6. *([29])(Banach's fixed point theorem) Let S be a nonempty closed subset of a Banach space X, then any contraction mapping T of S into itself has a unique fixed point.*

Lemma 18.7. *Let $1 < \alpha \leq 2$ and $\sigma : J \to R$ be continuous. A function $y(t)$ is a solution of the fractional integral equation*

$$y(t) = \begin{cases} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds + m_0 + m_1 t,\ if\ t \in J_0, \\ \int_{t_k}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds + \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds \\ +\sum_{i=1}^{k}(t-t_k)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds + \sum_{i=1}^{k-1}(t_k-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds \\ +\sum_{i=1}^{k}I_i(y(t_i)) + \sum_{i=1}^{k-1}(t-t_i)I_i^*(y(t_i)) + I_k^*(y(t_k)) + m_0 + m_1 t,\ if\ t \in J_k \end{cases} \tag{18.2}$$

if and only if $y(t)$ is a solution of the fractional BVP

$$\begin{cases} {}^{C}D^{\alpha}y(t) = \sigma(t), \quad t \in J' \\ \Delta y(t_k) = I_k(y(t_k^-)), \ \Delta y'(t_k) = I_k^*(y(t_k^-)), \\ ay(0) + by(T) = g_1(y), cy'(0) + dy'(T) = g_2(y) \end{cases} \tag{18.3}$$

where $k = 1, 2, \dots, p$ *and*

$$\begin{aligned} m_0 = {} & \frac{g_1(y)}{a+b} - \frac{b}{a+b}\Bigg[\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds + \sum_{i=1}^{k}(T-t_k)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds \\ & + \sum_{i=1}^{k-1}(t_k-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds + \int_{t_k}^{T}\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds + \sum_{i=1}^{k}I_i(y(t_i)) \\ & + \sum_{i=1}^{k-1}(T-t_i)I_i^*(y(t_i)) + I_k^*(y(t_k))\Bigg] \\ & + \frac{bT}{(a+b)(c+d)}\left[-g_2(y) + d\int_{t_k}^{T}\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds + \sum_{i=1}^{k-1}d\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds\right], \end{aligned}$$

$$m_1 = \frac{g_2(y)}{c+d} - \frac{d}{c+d}\left[\int_{t_k}^{T}\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds + \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds\right].$$

Proof. Let y be the solution of (18.3). If $t \in J_0$, then Lemma 18.4 implies that

$$y(t) = I^{\alpha}\sigma(t) - c_0 - c_1 t = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds - c_0 - c_1 t,$$

$$y'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds - c_1$$

for some $c_0, c_1 \in R$.

If $t \in J_1$, then Lemma 18.4 implies that

$$y(t) = \int_{t_1}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds - d_0 - d_1(t-t_1),$$

$$y'(t) = \int_{t_1}^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds - d_1$$

for some $d_0, d_1 \in R$. Thus, we have

$$y(t_1^-) = \int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds - c_0 - c_1 t_1, \ y(t_1^+) = -d_0,$$

$$y'(t_1^-) = \int_0^{t_1}\frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds - c_1, \ y'(t_1^+) = -d_1.$$

In view of $\Delta y(t_1) = y(t_1^+) - y(t_1^-) = I_1(y(t_1^-))$ and $\Delta y'(t_1) = y'(t_1^+) - y'(t_1^-) = I_1^*(y(t_1^-))$, we have

$$-d_0 = \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds - c_0 - c_1t_1 + I_1(y(t_1^-)),$$
$$-d_1 = \int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds - c_1 + I_1^*(y(t_1^-)),$$

hence, for $t \in J_1$,

$$\begin{aligned} y(t) &= \int_{t_1}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds + \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds \\ &\quad +(t-t_1)\int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds + I_1(y(t_1^-)) \\ &\quad +(t-t_1)I_1^*(y(t_1^-)) - c_0 - c_1t, \\ y'(t) &= \int_{t_1}^{t} \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds + \int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds + I_1^*(y(t_1^-)) - c_1. \end{aligned}$$

If $t \in J_2$, then Lemma 18.4 implies that

$$y(t) = \int_{t_2}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds - e_0 - e_1(t-t_2),$$
$$y'(t) = \int_{t_2}^{t} \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds - e_1$$

for some $e_0, e_1 \in R$. Thus we have

$$\begin{aligned} y(t_2^-) &= \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds + \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds \\ &\quad +(t_2-t_1)\int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds + I_1(y(t_1^-)) \\ &\quad +(t_2-t_1)I_1^*(y(t_1^-)) - c_0 - c_1t_2, \\ y(t_2^+) &= -e_0, \end{aligned}$$

$$\begin{aligned} y'(t_2^-) &= \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds + \int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds + I_1^*(y(t_1^-)) - c_1, \\ y'(t_2^+) &= -e_1. \end{aligned}$$

In view of $\Delta y(t_2) = y(t_2^+) - y(t_2^-) = I_2(y(t_2^-))$ and $\Delta y'(t_2) = y'(t_2^+) - y'(t_2^-) = I_2^*(y(t_2^-))$, we have

$$-e_0 = y(t_2^-) + I_2(y(t_2^-)),$$
$$-e_1 = y'(t_2^-) + I_2^*(y(t_2^-)),$$

hence, for $t \in J_2$,

$$\begin{aligned} y(t) &= \int_{t_2}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds + \int_{0}^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds \\ &\quad + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds + (t_2-t_1)\int_{0}^{t_1} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds \\ &\quad + (t-t_2)\left[\int_{0}^{t_1} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds\right] \\ &\quad + I_1(y(t_1^-)) + I_2(y(t_2^-)) + (t-t_1)I_1^*(y(t_1^-)) + I_2^*(y(t_2^-)) - c_0 - c_1 t. \end{aligned}$$

By repeating the same process, if $t \in J_k$, then again from Lemma 18.4, we get

$$y(t) = \begin{cases} \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds + \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds \\ + \sum_{i=1}^{k}(t-t_k)\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds + \sum_{i=1}^{k-1}(t_k-t_i)\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\sigma(s)ds \\ + \sum_{i=1}^{k} I_i(y(t_i^-)) + \sum_{i=1}^{k-1}(t-t_i)I_i^*(y(t_i^-)) + I_k^*(y(t_k^-)) - c_0 - c_1 t. \end{cases}$$

Applying the conditions $ay(0)+by(T) = g_1(y)$, $cy'(0)+dy'(T) = g_2(y)$ and replacing $-c_0$ and $-c_1$ with m_0 and m_1, respectively, we obtain (18.2).

Conversely, assume that y satisfies the impulsive fractional integral equation (18.2), then by direct computation, it can be seen that the solution given by (18.2) satisfies (18.3). The proof is complete.

18.3 Main Results

Definition 18.8. A function $y \in PC^1(J,R)$ with its α-derivative existing on J' is said to be a solution of (18.1) if y satisfies the equation ${}^C D^{\alpha} y(t) = f(t,y(t))$ on J' and satisfies the conditions

$$\Delta y(t_k) = I_k(y(t_k^-)),\ \Delta y'(t_k) = I_k^*(y(t_k^-)),$$
$$ay(0)+by(T) = g_1(y), cy'(0)+dy'(T) = g_2(y).$$

The following are the main results of this chapter.

Theorem 18.9. *Assume that*

(A1) The function $f : J \times R \to R$ is continuous and there exists a constant $L_1 > 0$ such that $\|f(t,u)-f(t,v)\| \le L_1\|u-v\|$, $\forall t \in J$, and $u,v \in R$,
(A2) $I_k, I_k^* : R \to R$ are continuous and there exist constants $L_2 > 0$, $L_3 > 0$, $M_1 > 0$ and $M_2 > 0$ such that $\|I_k(u)-I_k(v)\| \le L_2\|u-v\|$, $\|I_k^*(u)-I_k^*(v)\| \le L_3\|u-v\|$, $\|I_k(u)\| \le M_1$, $\|I_k^*(u)\| \le M_2$ for each $u,v \in R$ and $k = 1,2,\ldots,p$,

(A3) There exist constants $q_i > 0,\ G_i > 0$ *and* $g_i : PC(J,R) \to R$ *are continuous functions such that* $\|g_i(u) - g_i(v)\| \le q_i \|u - v\|, \|g_i(u)\| \le G_i,\ i = 1,2.$

Moreover,

$$\left(1+\frac{|b|}{|a+b|}\right)\left(\frac{L_1 T^{\alpha}}{\Gamma(\alpha+1)}\left(1+p+2p\alpha+\frac{\alpha(1+p)|d|}{|c+d|}\right)\right.$$
$$\left.+\,(pL_2+pL_3T+L_3)\right)+\frac{q_1}{|a+b|}+\frac{q_2T}{|c+d|}+\frac{|b|\,q_2T}{|(a+b)(c+d)|}$$
$$:=\Omega_{a,b,c,d,p,T,L_1,L_2,L_3,q_1,q_2}<1 \tag{18.4}$$

with

$$L_1 \le \frac{1}{2}\left[\frac{T^{\alpha}}{\Gamma(\alpha+1)}\left(1+\frac{|b|}{|a+b|}\right)\left(1+p+2p\alpha+\frac{\alpha(1+p)|d|}{|c+d|}\right)\right]^{-1}.$$

Then, BVP(̃18.1) has a unique solution on J.

Proof. Define an operator $F : PC^1(J,R) \to PC^1(J,R)$ by

$$(Fy)(t)=\begin{cases} \int_{t_k}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,y(s))ds+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,y(s))ds \\ \quad +\sum_{i=1}^{k}(t-t_k)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,y(s))ds \\ \quad +\sum_{i=1}^{k-1}(t_k-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,y(s))ds \\ +\sum_{i=1}^{k}I_i(y(t_i^-))+\sum_{i=1}^{k-1}(t-t_i)I_i^*(y(t_i^-))+I_k^*(y(t_k^-))+C_0+C_1t,\ if\ t\in J_k \end{cases}$$

where

$$C_0 = \frac{g_1(y)}{a+b}-\frac{b}{a+b}\left[\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,y(s))ds\right.$$
$$+\sum_{i=1}^{k}(T-t_k)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,y(s))ds+\sum_{i=1}^{k-1}(t_k-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,y(s))ds$$
$$\left.+\int_{t_k}^{T}\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,y(s))ds+\sum_{i=1}^{k}I_i(y(t_i^-))+\sum_{i=1}^{k-1}(T-t_i)I_i^*(y(t_i^-))+I_k^*(y(t_k^-))\right]$$
$$+\frac{bT}{(a+b)(c+d)}\left[-g_2(y)+d\int_{t_k}^{T}\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,y(s))ds\right.$$
$$\left.+\sum_{i=1}^{k-1}d\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,y(s))ds\right],$$

$$C_1 = \frac{g_2(y)}{c+d}-\frac{d}{c+d}\left[\int_{t_k}^{T}\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,y(s))ds+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,y(s))ds\right]$$

with $\sup_{t\in J}\|f(t,0)\| = K$. Choosing

$$\frac{r}{2} \geq \left(1+\frac{|b|}{|a+b|}\right)\left[\frac{KT^{\alpha}}{\Gamma(\alpha+1)}\left(1+p+2p\alpha+\frac{\alpha(1+p)|d|}{|c+d|}\right)\right.$$
$$\left.+(pM_1+pM_2T+M_2)\right]+\frac{G_1}{|a+b|}+\frac{G_2T}{|c+d|}+\frac{|b|\,G_2T}{|(a+b)(c+d)|},$$

we show that $FB_r \subset B_r$, where $B_r = \{y \in PC(J,R) : \|y\| \leq r\}$. For $y \in B_r$, we have

$$\begin{aligned}
\|(Fy)(t)\| \leq & \int_{t_k}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (\|f(s,y(s))-f(s,0)\|+\|f(s,0)\|)\,ds \\
& +\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} (\|f(s,y(s))-f(s,0)\|+\|f(s,0)\|)\,ds \\
& +\sum_{i=1}^{k}(t-t_k)\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} (\|f(s,y(s))-f(s,0)\|+\|f(s,0)\|)\,ds \\
& +\sum_{i=1}^{k-1}(t_k-t_i)\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} (\|f(s,y(s))-f(s,0)\|+\|f(s,0)\|)\,ds \\
& +\sum_{i=1}^{k}\left\|I_i(y(t_i^-))\right\| + \sum_{i=1}^{k-1}(t-t_i)\left\|I_i^*(y(t_i^-))\right\| + \left\|I_k^*(y(t_k^-))\right\| + \frac{\|g_1(y)\|}{|a+b|} \\
& +\frac{|b|}{|a+b|}\left[\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)} (\|f(s,y(s))-f(s,0)\|+\|f(s,0)\|)\,ds\right. \\
& +\sum_{i=1}^{k}(T-t_k)\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} (\|f(s,y(s))-f(s,0)\|+\|f(s,0)\|)\,ds \\
& +\sum_{i=1}^{k-1}(t_k-t_i)\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} (\|f(s,y(s))-f(s,0)\|+\|f(s,0)\|)\,ds \\
& +\int_{t_k}^{T} \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} (\|f(s,y(s))-f(s,0)\|+\|f(s,0)\|)\,ds \\
& \left.+\sum_{i=1}^{k}\left\|I_i(y(t_i^-))\right\| + \sum_{i=1}^{k-1}(T-t_i)\left\|I_i^*(y(t_i^-))\right\| + \left\|I_k^*(y(t_k^-))\right\|\right] \\
& +\frac{|b|\,T}{|(a+b)(c+d)|}\left[\|g_2(y)\|+|d|\int_{t_k}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} (\|f(s,y(s))\right. \\
& -\ f(s,0)\|+\|f(s,0)\|)\,ds \\
& \left.+\sum_{i=1}^{k-1}|d|\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} (\|f(s,y(s))-f(s,0)\|+\|f(s,0)\|)\,ds\right] \\
& +\frac{\|g_2(y)\|\,t}{|c+d|}+\frac{|d|\,t}{|c+d|}\left[\int_{t_k}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} (\|f(s,y(s))-f(s,0)\|+\|f(s,0)\|)\,ds\right. \\
& \left.+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} (\|f(s,y(s))-f(s,0)\|+\|f(s,0)\|)\,ds\right]
\end{aligned}$$

$$
\begin{aligned}
\|(Fy)(t)\| \leq{}& (L_1 r+K)\Bigg[\int_{t_k}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}ds\\
&+\sum_{i=1}^{k}|t-t_k|\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds+\sum_{i=1}^{k-1}(t_k-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds\\
&+\frac{|b|}{|a+b|}\left(\int_{t_k}^{T}\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}ds+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right.\\
&\left.+\sum_{i=1}^{k}(T-t_k)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds+\sum_{i=1}^{k-1}(t_k-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds\right)\\
&+\frac{|bd|T}{|(a+b)(c+d)|}\left(\int_{t_k}^{T}\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds\right)\\
&+\frac{|d|t}{|c+d|}\left(\int_{t_k}^{T}\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds\right)\Bigg]\\
&+\frac{\|g_1(y)\|}{|a+b|}+\frac{\|g_2(y)\|t}{|c+d|}+\frac{|b|\,\|g_2(y)\|T}{|(a+b)(c+d)|}\\
&+\sum_{i=1}^{k}\left\|I_i(y(t_i^-))\right\|+\sum_{i=1}^{k-1}|t-t_i|\left\|I_i^*(y(t_i^-))\right\|+\left\|I_k^*(y(t_k^-))\right\|\\
&+\frac{|b|}{|a+b|}\left[\sum_{i=1}^{k}\left\|I_i(y(t_i^-))\right\|+\sum_{i=1}^{k-1}(T-t_i)\left\|I_i^*(y(t_i^-))\right\|+\left\|I_k^*(y(t_k^-))\right\|\right]\\
\leq{}& (L_1 r+K)\left[\frac{T^\alpha}{\Gamma(\alpha+1)}+\frac{pT^\alpha}{\Gamma(\alpha+1)}+\frac{2pT^\alpha}{\Gamma(\alpha)}\right.\\
&+\frac{|b|}{|a+b|}\left(\frac{T^\alpha}{\Gamma(\alpha+1)}+\frac{pT^\alpha}{\Gamma(\alpha+1)}+\frac{2pT^\alpha}{\Gamma(\alpha)}\right)\\
&\left.+\left(\frac{|bd|}{|(a+b)(c+d)|}+\frac{|d|}{|c+d|}\right)\frac{(1+p)T^\alpha}{\Gamma(\alpha)}\right]\\
&+\frac{G_1}{|a+b|}+\frac{G_2T}{|c+d|}+\frac{|b|G_2T}{|(a+b)(c+d)|}+pM_1+pM_2T+M_2\\
&+\frac{|b|}{|a+b|}(pM_1+pM_2T+M_2)\\
\leq{}& (L_1 r+K)\frac{T^\alpha}{\Gamma(\alpha+1)}\left(1+\frac{|b|}{|a+b|}\right)\left(1+p+2p\alpha+\frac{\alpha(1+p)|d|}{|c+d|}\right)\\
&+\left(1+\frac{|b|}{|a+b|}\right)(pM_1+pM_2T+M_2)\\
&+\frac{G_1}{|a+b|}+\frac{G_2T}{|c+d|}+\frac{|b|G_2T}{|(a+b)(c+d)|}
\end{aligned}
$$

$$\leq L_1 \frac{T^\alpha}{\Gamma(\alpha+1)}\left(1+\frac{|b|}{|a+b|}\right)\left(1+p+2p\alpha+\frac{\alpha(1+p)|d|}{|c+d|}\right)r$$
$$+\left(1+\frac{|b|}{|a+b|}\right)\left[\frac{KT^\alpha}{\Gamma(\alpha+1)}\left(1+p+2p\alpha+\frac{\alpha(1+p)|d|}{|c+d|}\right)\right.$$
$$\left.+(pM_1+pM_2T+M_2)\right]+\frac{G_1}{|a+b|}+\frac{G_2T}{|c+d|}+\frac{|b|G_2T}{|(a+b)(c+d)|}.$$

Now, for $x,y\in PC(J,R)$ and for each $t\in J$, we obtain

$$\|(Fx)(t)-(Fy)(t)\|$$
$$\leq \int_{t_k}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|f(s,x(s))-f(s,y(s))\|\,ds$$
$$+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}\|f(s,x(s))-f(s,y(s))\|\,ds$$
$$+\sum_{i=1}^{k}|t-t_k|\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\|f(s,x(s))-f(s,y(s))\|\,ds$$
$$+\sum_{i=1}^{k-1}(t_k-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\|f(s,x(s))-f(s,y(s))\|\,ds$$
$$+\sum_{i=1}^{k}\left\|I_i(x(t_i^-))-I_i(y(t_i^-))\right\|+\sum_{i=1}^{k-1}|t-t_i|\left\|I_i^*(x(t_i^-))-I_i^*(y(t_i^-))\right\|$$
$$+\left\|I_k^*(x(t_k^-))-I_k^*(y(t_k^-))\right\|+\frac{\|g_1(x)-g_1(y)\|}{|a+b|}$$
$$+\frac{|b|}{|a+b|}\left[\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}\|f(s,x(s))-f(s,y(s))\|\,ds\right.$$
$$+\sum_{i=1}^{k}(T-t_k)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\|f(s,x(s))-f(s,y(s))\|\,ds$$
$$+\sum_{i=1}^{k-1}(t_k-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\|f(s,x(s))-f(s,y(s))\|\,ds$$
$$+\int_{t_k}^{T}\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\|f(s,x(s))-f(s,y(s))\|\,ds+\sum_{i=1}^{k}\left\|I_i(x(t_i^-))-I_i(y(t_i^-))\right\|$$
$$\left.+\sum_{i=1}^{k-1}(T-t_i)\left\|I_i^*(x(t_i^-))-I_i^*(y(t_i^-))\right\|+\left\|I_k^*(x(t_k^-))-I_k^*(y(t_k^-))\right\|\right]$$
$$+\frac{|b|T}{|(a+b)(c+d)|}\left[\|g_2(x)-g_2(y)\|+|d|\int_{t_k}^{T}\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\|f(s,x(s))-f(s,y(s))\|\,ds\right.$$

$$+\sum_{i=1}^{k-1}|d|\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\|f(s,x(s))-f(s,y(s))\|\,ds\Bigg]$$
$$+\frac{\|g_2(x)-g_2(y)\|\,t}{|c+d|}+\frac{|d|\,t}{|c+d|}\left[\int_{t_k}^{T}\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\|f(s,x(s))-f(s,y(s))\|\,ds\right.$$
$$\left.+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}\|f(s,x(s))-f(s,y(s))\|\,ds\right].$$

Then, we have

$$\begin{aligned}&\|(Fx)(t)-(Fy)(t)\|\\ \leq&\left[\left(1+\frac{|b|}{|a+b|}\right)\left(\frac{L_1T^{\alpha}}{\Gamma(\alpha+1)}\left(1+p+2p\alpha+\frac{\alpha(1+p)|d|}{|c+d|}\right)\right.\right.\\ &\left.\left.+\ (pL_2+pL_3T+L_3)\right)+\frac{q_1}{|a+b|}+\frac{q_2T}{|c+d|}+\frac{|b|\,q_2T}{|(a+b)(c+d)|}\right]\|x(s)-y(s)\|\\ \leq&\ \Omega_{a,b,c,d,p,T,L_1,L_2,L_3,q_1,q_2}\|x(s)-y(s)\|.\end{aligned}$$

Therefore, by (18.4) and thanks to Theorem 18.6, the operator F is contraction mapping. Consequently, BVP (18.1) has a unique solution.

Theorem 18.10. *Assume that(A1)–(A3) hold with*
(A4) $\|f(t,x)\|\leq\gamma(t),\ \forall(t,x)\in J\times R$, *where* $\gamma\in L^1(J,R)$.
Then the BVP has at least one solution on J.

Proof. Let us fix

$$\rho\geq\left[\left(1+\frac{|b|}{|a+b|}\right)\left\{\frac{\|\gamma\|_{L_1}T^{\alpha}}{\Gamma(\alpha+1)}\left(1+p+2p\alpha+\frac{\alpha(1+p)|d|}{|c+d|}\right)\right.\right.$$
$$\left.\left.+(pM_1+pM_2T+M_2)\right\}+\frac{G_1}{|a+b|}+\frac{G_2T}{|c+d|}+\frac{|b|\,G_2T}{|(a+b)(c+d)|}\right]$$

and consider $B_\rho=\{y\in PC(J,R):\|y\|_\infty\leq\rho\}$. We define the operators ϕ and ψ on B_ρ by

$$\begin{aligned}(\phi y)(t)=&\int_{t_k}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,y(s))ds+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,y(s))ds\\ &+\sum_{i=1}^{k}(t-t_k)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,y(s))ds\\ &+\sum_{i=1}^{k-1}(t_k-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,y(s))ds\\ &-\frac{b}{a+b}\left[\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,y(s))ds\right.\end{aligned}$$

$$+\sum_{i=1}^{k}(T-t_k)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,y(s))ds$$
$$\left.+\sum_{i=1}^{k-1}(t_k-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,y(s))ds+\int_{t_k}^{T}\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,y(s))ds\right]$$
$$+\frac{bdT}{(a+b)(c+d)}\left[\int_{t_k}^{T}\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,y(s))ds\right.$$
$$\left.+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,y(s))ds\right]$$
$$-\frac{dt}{c+d}\left[\int_{t_k}^{T}\frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,y(s))ds+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,y(s))ds\right],$$

$$\begin{aligned}(\psi y)(t) &= \sum_{i=1}^{k}I_i(y(t_i^-))+\sum_{i=1}^{k-1}(t-t_i)I_i^*(y(t_i^-))+I_k^*(y(t_k^-))\\ &\quad -\frac{b}{a+b}\left[\sum_{i=1}^{k}I_i(y(t_i^-))+\sum_{i=1}^{k-1}(T-t_i)I_i^*(y(t_i^-))+I_k^*(y(t_k^-))\right]\\ &\quad +\frac{g_1(y)}{a+b}-\frac{g_2(y)bT}{(a+b)(c+d)}+\frac{g_2(y)t}{c+d}.\end{aligned}$$

Now, one can observe that if $x,y\in B_\rho$, then $\phi x+\psi y\in B_\rho$ checking the inequality

$$\|\phi x+\psi y\|\le\rho.$$

It is obvious that ψ is contraction mapping for

$$\left(1+\frac{|b|}{|a+b|}\right)(pL_2+pL_3T+L_3)+\frac{q_1}{|a+b|}+\frac{q_2T}{|c+d|}+\frac{|b|\,q_2T}{|(a+b)(c+d)|}<1.$$

Continuity of f implies the operator ϕ is continuous. Also, the inequality

$$\|(\phi y)(t)\|\le\frac{\|\gamma\|_{L_1}T^\alpha}{\Gamma(\alpha+1)}\left(1+\frac{|b|}{|a+b|}\right)\left(1+p+2p\alpha+\frac{\alpha\,(1+p)\,|d|}{|c+d|}\right)$$

implies that ϕ is uniformly bounded on B_ρ.

Now, in order to prove the compactness of the operator ϕ, equicontinuity of $(\phi y)(t)$ is left. Letting $(t,y)\in J\times B_\rho$, and using the fact that f is bounded on the compact set $J\times B_\rho$, we define $\sup_{t\in J\times R}\|f(t,y)\|=f_{\max}<\infty$. Then, for $\tau_1,\tau_2\in J_k$ with $\tau_1<\tau_2,\ 0\le k\le p$, we have

$$|(\phi y)(\tau_2)-(\phi y)(\tau_1)|\le\int_{\tau_1}^{\tau_2}\left|(\phi y)'(s)\right|ds\le L(\tau_2-\tau_1)$$

where

$$
\begin{aligned}
\left|(\phi y)'(t)\right| \leq & \int_{t_k}^{t} \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,y(s))|\, ds + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,y(s))|\, ds \\
& + \frac{|d|}{|c+d|} \left[\int_{t_k}^{T} \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,y(s))|\, ds \right. \\
& \left. + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,y(s))|\, ds \right] \\
\leq & f_{\max} T^{\alpha-1} \left[\frac{(1+p)}{\Gamma(\alpha)} \left(1 + \frac{|d|}{|c+d|} \right) \right] := L \text{ for any } t \in J_k.
\end{aligned}
$$

This implies that ϕ is equicontinuous on all the subintervals $J_k,\ k = 0,1,2,\ldots,p$. Therefore, ϕ is relatively compact on B_ρ. By the Arzela–Ascoli Theorem, ϕ is compact on B_ρ. Consequently, we conclude the result of our theorem dependent upon the Krasnoselskii's theorem.

18.4 An Example

Consider the following impulsive fractional BVP

$$
\begin{aligned}
{}^{C}D^{\frac{3}{2}} y(t) &= \frac{\sin 5t\, |y(t)|}{(t+5)^3 (1+|y(t)|)},\ t \in [0,1],\ t \neq \frac{1}{2}, \\
\Delta y(\frac{1}{2}) &= \frac{\left|y(\frac{1}{2}^-)\right|}{5+\left|y(\frac{1}{2}^-)\right|},\ \Delta y'(\frac{1}{2}) = \frac{\left|y'(\frac{1}{2}^-)\right|}{20+\left|y'(\frac{1}{2}^-)\right|} \qquad (18.5) \\
2y(0)+3y(1) &= \sum_{i=1}^{m} \eta_i y(\xi_i),\ \ y'(0)+5y'(1) = \sum_{j=1}^{m} \widetilde{\eta}_j \widetilde{y}(\xi_i)
\end{aligned}
$$

where $0 < \eta_1 < \eta_2 < \ldots < 1$, $0 < \widetilde{\eta}_1 < \widetilde{\eta}_2 < \ldots < 1$, and η_i, $\widetilde{\eta}_j$ are given positive constants with $\sum_{i=1}^{m} \eta_i < \frac{2}{15}$ and $\sum_{j=1}^{m} \widetilde{\eta}_j < \frac{3}{15}$.

Here, $a = 2,\ b = 3,\ c = 1,\ d = 5,\ \alpha = \frac{3}{2},\ T = 1,\ p = 1$. Obviously, $L_1 = \frac{1}{125}$, $L_2 = \frac{1}{5}$, $L_3 = \frac{1}{20}$, $q_1 = \frac{2}{15}$, $q_2 = \frac{3}{15}$ and by (18.4), it can be find that

$$
\Omega_{a,b,c,d,p,T,L_1,L_2,L_3,q_1,q_2} = \frac{16}{125\sqrt{\pi}} + \frac{14}{25} = 0.63222 < 1.
$$

Therefore, due to fact that all the assumptions of Theorem 18.9 hold, the BVP (18.5) has a unique solution. Besides, one can easily check the result of Theorem 18.10 for the BVP (18.5).

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Chapter 19
The Construction of Particular Solutions of the Nonlinear Equation of Schrodinger Type

K.R. Yesmakhanova and Zh.R. Myrzakulova

Abstract Using the method of $\bar{\partial}$-problem, based on the nonlocal $\bar{\partial}$-problem, partial solutions for $2+1$-dimensional nonlinear equation of Schrodinger type are constructed.

19.1 Introduction

In connection with the intensive development of soliton theory, the investigation of multidimensional nonlinear integrable equations has now become an urgent task. In the work of Ablowitz M.J., Kaup D.J., Newell A.C., Segur H., Kruskal M.D., Shabat A.B., Zakharov V.E., Dubrovin B.A., Matveev V.B., Novikov S.P., Manakov S., and others, various methods for finding exact solutions of these equations have been used. One such method is the $\bar{\partial}$—dressing method. This method allows to simultaneously construct a nonlinear equation and its Lax representation and the exact solutions. Adapting the method of $\bar{\partial}$ to the specific problems of differential equations is one of the most pressing challenges facing the nonlinear mathematical physics [1–3]. The needs of mathematical physics and its applications necessitate the construction of new classes of integrable systems and their research. In this study the relevance of multidimensional, in particular, the 2+1-dimensional integrable nonlinear equations is beyond doubt. In this chapter, using the method of $\bar{\partial}$-problem, we construct the particular solutions, namely, the soliton-like solutions of 2+1-dimensional nonlinear equation of Schrodinger type. Method of $\bar{\partial}$-problem originates with the work of Zakharov and Shabat, which proposed a scheme for construction of the integrable equations and calculating the time of their solutions.

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19.2 Statement of the Problem

We consider 2+1-dimensional nonlinear Schrodinger equation type

$$iq_t + M_1 q + vq = 0, \quad ir_t - M_1 r - vr = 0, \quad M_2 v = -2M_1(rq), \tag{19.1}$$

where q, r, and v $(v = 2(U_1 - U_2))$ are some complex functions. The operators M_1 and M_2 are defined by

$$M_1 = 4\,(a_2 - 2ab - b)\,\partial^2_{xx} + 4\alpha\,(b-a)\,\partial^2_{xy} + \alpha^2\partial^2_{yy}, \tag{19.2}$$

$$M_2 = 4a\,(a+1)\,\partial^2_{xx} - 2\alpha\,(2a+1)\,\partial^2_{xy} + \alpha^2\partial^2_{yy}, \tag{19.3}$$

where a, b are arbitrary real constants and α is a complex constant.

It also arises in the theory of multidimensional integrable systems. The solution of equation (19.1) satisfies the boundary conditions: $q \to 0$, $r \to 0$, $v \to 0$ for $x,y \to \pm\infty$. To construct solutions of (19.1), following the method proposed in [1], it is necessary to solve the matrix integral equation of the form

$$W(\lambda,\bar{\lambda}) = V(\lambda,\bar{\lambda}) + \frac{1}{2\pi i}\iint\limits_G \frac{d\lambda' \wedge d\bar{\lambda}'}{\lambda' - \lambda}\iint\limits_G W(\mu,\bar{\mu})R(\mu,\bar{\mu};\lambda',\bar{\lambda}')d\mu \wedge d\bar{\mu} \tag{19.4}$$

for $W(\lambda,\bar{\lambda})$ norm of $V \equiv 1$ and $G = E$. The equation (19.4) is Fredholm integral equation of the second kind, we believe that the kernel $R(\mu,\bar{\mu},\lambda,\bar{\lambda})$ must have a weak singularity. We construct exact solutions for nonlinear Schrodinger type. B_0 and $\tilde{U}$ are given by

$$B_0 = -2i[B_1, W_{-1}], \quad U = i(W_{-1})_{diag}. \tag{19.5}$$

It follows that

$$q = -2i(W_{-1})_{12}, \quad r = 2i(W_{-1})_{21}, \quad U = i(W_{-1})_{diag} = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}. \tag{19.6}$$

19.3 Construction of Particular Solutions of 2+1-Dimensional Nonlinear Equation of Schrodinger Type

For this we take the kernel of R in the formula (19.4) in the form

$$R(\mu,\bar{\mu};\lambda,\bar{\lambda};x,y,t) = e^{F(\mu,x,y,t)}R_0(\mu,\bar{\mu};\lambda,\bar{\lambda})e^{-F(\lambda,x,y,t)}, \tag{19.7}$$

where R_0 is an arbitrary 2×2 matrix function and

$$F(\mu,x,y,t) = i\mu I x + \frac{2i\mu}{\alpha}B_1 y - 4i\mu^2 C_2 t. \tag{19.8}$$

Here, the diagonal and the constant 2×2 matrices B_1, C_2, and I are given as

$$B_1 = \begin{pmatrix} a+1 & 0 \\ 0 & a \end{pmatrix}, \quad C_2 = \begin{pmatrix} b+1 & 0 \\ 0 & b \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{19.9}$$

Theorem 19.1. *If one has a kernel of R in the form (19.7), then the solutions of the nonlinear Schrodinger equation of* (19.1) *are given by*

$$U_1(x,y,t) = -\frac{1}{2\pi}\iint_E d\lambda\wedge d\bar{\lambda}\iint_E R_{011}(\mu,\bar{\mu};\lambda,\bar{\lambda})\cdot$$
$$\cdot\exp\left(i(\mu-\lambda)x+\frac{2i}{\alpha}(\mu-\lambda)(a+1)y-4i(\mu^2-\lambda^2)(b+1)t\right)d\mu\wedge d\bar{\mu}, \tag{19.10}$$
$$q(x,y,t) = \frac{1}{\pi}\iint_E d\lambda'\wedge d\bar{\lambda}'\iint_E R_{012}(\mu,\bar{\mu};\lambda,\bar{\lambda})\cdot$$
$$\cdot\exp\left(i(\mu-\lambda)x+\frac{2i}{\alpha}(\mu-\lambda)ay+\frac{2i}{\alpha}\mu y-4i(\mu^2-\lambda^2)bt-4i\mu^2t\right)d\mu\wedge d\bar{\mu}, \tag{19.11}$$
$$r(x,y,t) = -\frac{1}{\pi}\iint_E d\lambda\wedge d\bar{\lambda}\iint_E R_{021}(\mu,\bar{\mu};\lambda,\bar{\lambda})\cdot$$
$$\cdot\exp\left(i(\mu-\lambda)x+\frac{2i}{\alpha}(\mu-\lambda)ay-\frac{2i}{\alpha}\lambda y-4i(\mu^2-\lambda^2)bt+4i\lambda^2t\right)d\mu\wedge d\bar{\mu}, \tag{19.12}$$
$$U_2(x,y,t) = -\frac{1}{2\pi}\iint_E d\lambda\wedge d\bar{\lambda}\iint_E R_{022}(\mu,\bar{\mu};\lambda,\bar{\lambda})\cdot$$
$$\cdot\exp\left(i(\mu-\lambda)x+\frac{2i}{\alpha}(\mu-\lambda)ay-4i(\mu^2-\lambda^2)bt\right)d\mu\wedge d\bar{\mu}. \tag{19.13}$$

Proof. Let's start with the matrix $\bar{\partial}$-problem. Consider the equation (19.4) with the additional condition $W\to 1$ in $|\lambda|\to\infty$. In this case the function W can be expanded in the neighborhood of $\lambda=\infty$ in a series of negative powers of λ:

$$W = 1+\lambda^{-1}W_{-1}+\lambda^{-2}W_{-2}+\lambda^{-3}W_{-3}+\ldots \tag{19.14}$$

one expands the integrand $\frac{1}{\lambda'-\lambda}$ in the integral equation (19.4) in powers of λ^k

$$\frac{1}{\lambda'-\lambda} = \frac{1}{\lambda'}\sum_{k=0}^{\infty}\left(\frac{\lambda}{\lambda'}\right)^k. \tag{19.15}$$

Substituting (19.14), (19.15) with $\lambda\to\infty$ in (19.4), we obtain the expression

$$\lambda^0: \qquad 1 = 1, \tag{19.16}$$

$$\lambda^{-1}: \quad W_{-1} = -\frac{1}{2\pi i}\iint_E d\lambda'\wedge d\bar{\lambda}'\iint_E R(\mu,\bar{\mu};\lambda,\bar{\lambda})d\mu\wedge d\bar{\mu}, \tag{19.17}$$

$$\lambda^{-2}: \quad W_{-2} = -\frac{1}{2\pi i}\iint\limits_E \lambda' d\lambda' \wedge d\bar{\lambda}' \iint\limits_E R(\mu,\bar{\mu};\lambda,\bar{\lambda}) d\mu \wedge d\bar{\mu} -$$

$$-\frac{1}{2\pi i}\iint\limits_E \lambda' d\lambda' \wedge d\bar{\lambda}' \iint\limits_E W_{-1} R(\mu,\bar{\mu};\lambda,\bar{\lambda}) d\mu \wedge d\bar{\mu}, \tag{19.18}$$

$$\ldots$$

from (19.17) using the condition (19.6), obtain the solutions

$$q(x,y,t) = \frac{1}{\pi}\iint\limits_E d\lambda' \wedge d\bar{\lambda}' \iint\limits_E (W(\mu,\bar{\mu})R(\mu,\bar{\mu};\lambda,\bar{\lambda}))_{12} d\mu \wedge d\bar{\mu}, \tag{19.19}$$

$$r(x,y,t) = -\frac{1}{\pi}\iint\limits_E d\lambda' \wedge d\bar{\lambda}' \iint\limits_E (W(\mu,\bar{\mu})R(\mu,\bar{\mu};\lambda,\bar{\lambda}))_{21} d\mu \wedge d\bar{\mu}, \tag{19.20}$$

$$U(x,y,t) = \frac{1}{2\pi}\iint\limits_E d\lambda' \wedge d\bar{\lambda}' \iint\limits_E (W(\mu,\bar{\mu})R(\mu,\bar{\mu};\lambda,\bar{\lambda}))_{diag} d\mu \wedge d\bar{\mu}. \tag{19.21}$$

For a given nucleus with a weak singularity R with $W(\infty) = 1$, therefore, have

$$q(x,y,t) = \frac{1}{\pi}\iint\limits_E d\lambda' \wedge d\bar{\lambda}' \iint\limits_E (R)_{12}(\mu,\bar{\mu};\lambda,\bar{\lambda}) d\mu \wedge d\bar{\mu}, \tag{19.22}$$

$$r(x,y,t) = -\frac{1}{\pi}\iint\limits_E d\lambda' \wedge d\bar{\lambda}' \iint\limits_E (R)_{21}(\mu,\bar{\mu};\lambda,\bar{\lambda}) d\mu \wedge d\bar{\mu}. \tag{19.23}$$

Substituting (19.6) in (19.21) one obtains

$$U_1(x,y,t) = -\frac{1}{2\pi}\iint\limits_E d\lambda' \wedge d\bar{\lambda}' \iint\limits_E (R)_{11}(\mu,\bar{\mu};\lambda,\bar{\lambda}) d\mu \wedge d\bar{\mu}, \tag{19.24}$$

$$U_2(x,y,t) = -\frac{1}{2\pi}\iint\limits_E d\lambda' \wedge d\bar{\lambda}' \iint\limits_E (R)_{22}(\mu,\bar{\mu};\lambda,\bar{\lambda}) d\mu \wedge d\bar{\mu}. \tag{19.25}$$

Assume that R_0 is an arbitrary 2×2 matrix. Then the solutions have the form

$$q(x,y,t) = \frac{1}{\pi}\iint\limits_E d\lambda' \wedge d\bar{\lambda}' \iint\limits_E R_{012}(\mu,\bar{\mu};\lambda,\bar{\lambda}) \cdot$$

$$\cdot \exp\left(i(\mu-\lambda)x + \frac{2i}{\alpha}(\mu-\lambda)ay + \frac{2i}{\alpha}\mu y - 4i(\mu^2-\lambda^2)bt - 4i\mu^2 t\right) d\mu \wedge d\bar{\mu},$$

$$r(x,y,t) = -\frac{1}{\pi}\iint\limits_E d\lambda \wedge d\bar{\lambda} \iint\limits_E R_{021}(\mu,\bar{\mu};\lambda,\bar{\lambda}) \cdot \tag{19.26}$$

$$\cdot \exp\left(i(\mu-\lambda)x + \frac{2i}{\alpha}(\mu-\lambda)ay - \frac{2i}{\alpha}\lambda y - 4i(\mu^2-\lambda^2)bt + 4i\lambda^2 t \right) d\mu \wedge d\bar{\mu}, \tag{19.27}$$

and

$$U_1(x,y,t) = -\frac{1}{2\pi} \iint_E d\lambda \wedge d\bar{\lambda} \iint_E R_{011}(\mu,\bar{\mu};\lambda,\bar{\lambda}) \cdot$$

$$\cdot \exp\left(i(\mu-\lambda)x + \frac{2i}{\alpha}(\mu-\lambda)(a+1)y - 4i(\mu^2-\lambda^2)(b+1)t \right) d\mu \wedge d\bar{\mu}, \tag{19.28}$$

$$U_2(x,y,t) = -\frac{1}{2\pi} \iint_E d\lambda \wedge d\bar{\lambda} \iint_E R_{022}(\mu,\bar{\mu};\lambda,\bar{\lambda}) \cdot$$

$$\cdot \exp\left(i(\mu-\lambda)x + \frac{2i}{\alpha}(\mu-\lambda)ay - 4i(\mu^2-\lambda^2)bt \right) d\mu \wedge d\bar{\mu}. \tag{19.29}$$

Thus Theorem 19.1 is proved. □

Now consider the degenerate kernel R_0, which has the form

$$R_0(\mu,\bar{\mu};\lambda,\bar{\lambda}) = \sum_{k=1}^{N} f_{0k}(\mu,\bar{\mu}) g_{0k}(\lambda,\bar{\lambda}), \tag{19.30}$$

where f_{0k} and g_{0k} are linearly independent arbitrary 2×2 matrix-valued functions and N is an arbitrary integer. Substituting the expression for R_0 (19.30) in the formula (19.7) obtain the kernel of the form:

$$R(\mu,\bar{\mu};\lambda,\bar{\lambda},x,y,t) = e^{F(\mu,x,y,t)} \sum_{k=1}^{N} f_{0k}(\mu,\bar{\mu}) g_{0k}(\lambda,\bar{\lambda}) e^{-F(\lambda,x,y,t)}. \tag{19.31}$$

Theorem 19.2. *If the kernel R is given in the form* (19.31)*, then the solutions of the nonlinear Schrodinger equation of* (19.1) *are given by*

$$U_1(x,y,t) = -\frac{1}{2\pi} \sum_{k,l=1}^{N} (\xi_k (I-A)^{-1}_{kl} \eta_l)_{11}, \tag{19.32}$$

$$q(x,y,t) = \frac{1}{\pi} \sum_{k,l=1}^{N} (\xi_k (I-A)^{-1}_{kl} \eta_l)_{12}, \tag{19.33}$$

$$r(x,y,t) = -\frac{1}{\pi} \sum_{k,l=1}^{N} (\xi_k (I-A)^{-1}_{kl} \eta_l)_{21}, \tag{19.34}$$

$$U_2(x,y,t) = -\frac{1}{2\pi} \sum_{k,l=1}^{N} (\xi_k (I-A)^{-1}_{kl} \eta_l)_{22}, \tag{19.35}$$

where

$$\xi_k(x,y,t) = \iint\limits_E e^{i\lambda x + \frac{2i\lambda}{\alpha} B_1 y - 4i\lambda^2 C_2 t} f_{0k}(\lambda,\bar{\lambda}) d\lambda \wedge d\bar{\lambda}, \tag{19.36}$$

$$\eta_l(x,y,t) = \iint\limits_E g_{0l}(\lambda,\bar{\lambda}) e^{-i\lambda Ix - \frac{2i\lambda}{\alpha} B_1 y + 4i\lambda^2 C_2 t} d\lambda \wedge d\bar{\lambda}, \tag{19.37}$$

$$A_{kl}(x,y,t) = \frac{1}{2\pi i} \iint\limits_E d\mu \wedge d\bar{\mu} \iint\limits_E \frac{1}{\lambda - \mu} g_{0l}(\mu,\bar{\mu}) \cdot$$

$$\cdot \exp\left(i(\mu - \lambda) Ix + \frac{2i}{\alpha}(\mu - \lambda) B_1 y - 4i(\mu^2 - \lambda^2) C_2 t \right) f_{0k}(\lambda,\bar{\lambda}) d\lambda \wedge d\bar{\lambda}. \tag{19.38}$$

We begin the proof of Theorem 19.2 with the trivial case $N = 1$.

Proof. In this case, the kernel of R has the form

$$R(\mu,\bar{\mu};\lambda,\bar{\lambda},x,y,t) = e^{F(\mu,x,y,t)} f_{01}(\mu,\bar{\mu}) g_{01}(\lambda,\bar{\lambda}) e^{-F(\lambda,x,y,t)}, \tag{19.39}$$

where F is given by (19.8). Substituting in the formula (19.4) instead of $R(\mu,\bar{\mu};\lambda,\bar{\lambda},x,y,t)$ its expression (19.39), we reduce (19.4) to the form

$$W(\lambda,\bar{\lambda}) = V(\lambda,\bar{\lambda}) + \frac{1}{2\pi i} \iint\limits_G \frac{d\lambda' \wedge d\bar{\lambda}'}{\lambda' - \lambda} \cdot$$

$$\cdot \iint\limits_G W(\mu,\bar{\mu}) e^{F(\mu,x,y,t)} f_{01}(\mu,\bar{\mu}) g_{01}(\lambda',\bar{\lambda}'^{-F(\lambda',x,y,t)} d\mu \wedge d\bar{\mu}. \tag{19.40}$$

Assume that equation (19.40) has a solution. Introduce the notation

$$\iint\limits_G W(\mu,\bar{\mu}) e^{F(\mu,x,y,t)} f_{01}(\mu,\bar{\mu}) d\mu \wedge d\bar{\mu} = h_1. \tag{19.41}$$

Then (19.40) takes the form

$$W(\lambda,\bar{\lambda}) = V(\lambda,\bar{\lambda}) + \frac{1}{2\pi i} \iint\limits_G \frac{h_1 g_{01}(\lambda',\bar{\lambda}'^{-F(\lambda',x,y,t)}}{\lambda' - \lambda} d\lambda' \wedge d\bar{\lambda}'. \tag{19.42}$$

To find the $W(\lambda,\bar{\lambda})$, one must calculate h_1. The equation for h_1 follows from (19.40). Indeed, multiplying the integral equation (19.40) on the $e^{F(\lambda,x,y,t)} f_{01}(\lambda,\bar{\lambda})$ on the left and integrating over λ twice, we obtain the following equation:

$$h_1 = \xi_1 + h_1 A_{11}. \tag{19.43}$$

It follows that

$$h_1 = \xi_1 (I - A_{11})^{-1}, \tag{19.44}$$

where h_1 is set to (19.41) and

$$\xi_1 = \iint_G V(\lambda,\bar{\lambda})e^{F(\lambda,x,y,t)} f_{01}(\lambda,\bar{\lambda})d\lambda \wedge d\bar{\lambda}, \quad (19.45)$$

$$A_{11} = \frac{1}{2\pi i}\iint_G d\lambda' \wedge d\bar{\lambda}' \iint_G \frac{g_{01}(\lambda',\bar{\lambda}'^{F(\lambda,x,y,t)-F(\lambda',x,y,t)} f_{01}(\lambda,\bar{\lambda})}{\lambda'-\lambda} d\lambda \wedge d\bar{\lambda}. \quad (19.46)$$

Thus, the integral matrix equation (19.4) with a degenerate kernel (19.39) reduces to equation (19.43). If (19.43) is not solvable, then, obviously, (19.4) is also solvable. Assume that equation (19.43) has a solution h_1. Substitute h_1 formula (19.44) in equation (19.42) (with $A_{11} \neq 0$). Finally, we obtain

$$W(\lambda,\bar{\lambda}) = V(\lambda,\bar{\lambda}) + \frac{1}{2\pi i}\iint_G \frac{\xi_1(1-A_{11})^{-1} g_{01}(\lambda',\bar{\lambda}'^{-F(\lambda',x,y,t)}}{\lambda'-\lambda} d\lambda' \wedge d\bar{\lambda}'. \quad (19.47)$$

Formula (19.47) gives an explicit solution of the integral matrix equation (19.4), which is parametrized by two arbitrary matrix functions $f_{01}(\lambda,\bar{\lambda})$ and $g_{01}(\lambda,\bar{\lambda})$. These solutions represent the most wide class of exact solutions of $\bar{\partial}$-problem (19.4). From the foregoing it is clear that the integral matrix equation (19.4) and linear algebraic equations (19.43) are equivalent. In view of formulas (19.14) and (19.15), we have

$$1 + \frac{W_{-1}}{\lambda} + \frac{W_{-2}}{\lambda^2} + \ldots = 1 +$$

$$+ \frac{1}{2\pi i}\iint_G \left(-\frac{1}{\lambda} - \frac{\lambda'}{\lambda^2} - \ldots\right)\xi_1(1-A_{11})^{-1} g_{01}(\lambda',\bar{\lambda}'^{-F(\lambda',x,y,t)} d\lambda' \wedge d\bar{\lambda}'. \quad (19.48)$$

Now we equate coefficients of powers of λ:

$$\lambda^0: \qquad 1 = 1, \quad (19.49)$$

$$\lambda^{-1}: \quad W_{-1} = -\frac{1}{2\pi i}\iint_G \xi_1(1-A_{11})^{-1} g_{01}(\lambda',\bar{\lambda}'^{-F(\lambda',x,y,t)} d\lambda' \wedge d\bar{\lambda}', \quad (19.50)$$

$$\lambda^{-2}: \quad W_{-2} = -\frac{1}{2\pi i}\iint_G \lambda'\xi_1(1-A_{11})^{-1} g_{01}(\lambda',\bar{\lambda}'^{-F(\lambda',x,y,t)} d\lambda' \wedge d\bar{\lambda}', \quad (19.51)$$

$$\ldots$$

In the formula (19.50) introduce the notation

$$W_{-1} = -\frac{1}{2\pi i}\iint_G \xi_1(1-A_{11})^{-1} g_{01}(\lambda',\bar{\lambda}'^{-F(\lambda',x,y,t)} d\lambda' \wedge d\bar{\lambda}' = -\frac{1}{2\pi i}\xi_1(1-A_{11})^{-1}\eta_1. \quad (19.52)$$

In view of (19.6), obtain the solution of (19.1) for the case $N = 1$ as follows:

$$U_1(x,y,t) = \frac{1}{2\pi}(\xi_1(1-A_{11})^{-1}\eta_1)_{11}, \tag{19.53}$$

$$q(x,y,t) = \frac{1}{\pi}(\xi_1(1-A)_{11}^{-1}\eta_1)_{12}, \tag{19.54}$$

$$r(x,y,t) = -\frac{1}{\pi}(\xi_1(1-A)_{11}^{-1}\eta_1)_{21}, \tag{19.55}$$

$$U_2(x,y,t) = \frac{1}{2\pi}(\xi_1(1-A_{11})^{-1}\eta_1)_{22}, \tag{19.56}$$

where

$$\xi_1(x,y,t) = \iint_E e^{i\lambda Ix + \frac{2i\lambda}{\alpha}B_1y - 4i\lambda^2 C_2 t} f_{01}(\lambda,\bar{\lambda})d\lambda \wedge d\bar{\lambda}, \tag{19.57}$$

$$\eta_1(x,y,t) = \iint_E g_{01}(\lambda,\bar{\lambda})e^{-i\lambda Ix - \frac{2i\lambda}{\alpha}B_1y + 4i\lambda^2 C_2 t} d\lambda \wedge d\bar{\lambda}, \tag{19.58}$$

and

$$A_{11}(x,y,t) = \frac{1}{2\pi i}\iint_E d\mu \wedge d\bar{\mu} \iint_E \frac{g_{01}(\mu,\bar{\mu})}{\lambda-\mu}\cdot$$

$$\cdot g\left(i(\mu-\lambda)Ix + \frac{2i}{\alpha}(\mu-\lambda)B_1y - 4i(\mu^2-\lambda^2)C_2t\right) f_{01}(\lambda,\bar{\lambda})d\lambda \wedge d\bar{\lambda}. \tag{19.59}$$

Consider the case $N = 2$. In this case, the kernel of R is given by expression

$$R(\mu,\bar{\mu};\lambda,\bar{\lambda},x,y,t) = e^{F(\mu,x,y,t)}[f_{01}(\mu,\bar{\mu})g_{01}(\lambda,\bar{\lambda}) + f_{02}(\mu,\bar{\mu})g_{02}(\lambda,\bar{\lambda})]e^{-F(\lambda,x,y,t)}. \tag{19.60}$$

Substituting (19.60) in (19.4), one gets

$$W(\lambda,\bar{\lambda}) = V(\lambda,\bar{\lambda}) + \frac{1}{2\pi i}\iint_G \frac{d\lambda' \wedge d\bar{\lambda}'}{\lambda'-\lambda}\cdot$$

$$\cdot \iint_G W(\mu,\bar{\mu})e^{F(\mu,x,y,t)}[f_{01}(\mu,\bar{\mu})g_{01}(\lambda',\bar{\lambda}') + f_{02}(\mu,\bar{\mu})g_{02}(\lambda',\bar{\lambda}'^{F(-\lambda',x,y,t)} d\mu \wedge d\bar{\mu}. \tag{19.61}$$

hence

$$W(\lambda,\bar{\lambda}) = V(\lambda,\bar{\lambda}) + \frac{1}{2\pi i}\iint_G \frac{[h_1 g_{01}(\lambda',\bar{\lambda}') + h_2 g_{02}(\lambda',\bar{\lambda}'^{-F(\lambda',x,y,t)}}{\lambda'-\lambda} d\lambda' \wedge d\bar{\lambda}', \tag{19.62}$$

where

$$h_i = \iint_G W(\mu,\bar{\mu})e^{F(\mu,x,y,t)} f_{0i}(\mu,\bar{\mu})d\mu \wedge d\bar{\mu}, \ \ (i = 1,2). \tag{19.63}$$

To find the $W(\lambda,\bar{\lambda})$, we must calculate all the h_i. The system of equations for the h_i follows from (19.62). Multiplying (19.62) on $e^{F(\lambda,x,y,t)}f_{01}(\lambda,\bar{\lambda})$ on the left and integrating over λ, one gets

$$\iint_G W(\lambda,\bar{\lambda})e^{F(\lambda,x,y,t)}f_{01}(\lambda,\bar{\lambda})d\lambda\wedge d\bar{\lambda} = \iint_G V(\lambda,\bar{\lambda})e^{F(\lambda,x,y,t)}f_{01}(\lambda,\bar{\lambda})d\lambda\wedge d\bar{\lambda}+$$

$$+\frac{h_1}{2\pi i}\iint_G \frac{d\lambda'\wedge d\bar{\lambda}'}{\lambda'-\lambda}\iint_G g_{01}(\lambda',\bar{\lambda}'^{F(\lambda,x,y,t)-F(\lambda',x,y,t)}f_{01}(\lambda,\bar{\lambda})d\lambda\wedge d\bar{\lambda}+$$

$$+\frac{h_2}{2\pi i}\iint_G \frac{d\lambda'\wedge d\bar{\lambda}'}{\lambda'-\lambda}\iint_G g_{02}(\lambda',\bar{\lambda}'^{F(\lambda,x,y,t)-F(\lambda',x,y,t)}f_{01}(\lambda,\bar{\lambda})d\lambda\wedge d\bar{\lambda}. \quad (19.64)$$

Similarly, multiplying (19.62) on the $e^{F(\lambda,x,y,t)}f_{02}(\lambda,\bar{\lambda})$ on the left and integrating over λ, we get

$$\iint_G W(\lambda,\bar{\lambda})e^{F(\lambda,x,y,t)}f_{02}(\lambda,\bar{\lambda})d\lambda\wedge d\bar{\lambda} = \iint_G V(\lambda,\bar{\lambda})e^{F(\lambda,x,y,t)}f_{02}(\lambda,\bar{\lambda})d\lambda\wedge d\bar{\lambda}+$$

$$+\frac{h_1}{2\pi i}\iint_G \frac{d\lambda'\wedge d\bar{\lambda}'}{\lambda'-\lambda}\iint_G g_{01}(\lambda',\bar{\lambda}'^{F(\lambda,x,y,t)-F(\lambda',x,y,t)}f_{02}(\lambda,\bar{\lambda})d\lambda\wedge d\bar{\lambda}+$$

$$+\frac{h_2}{2\pi i}\iint_G \frac{d\lambda'\wedge d\bar{\lambda}'}{\lambda'-\lambda}\iint_G g_{02}(\lambda',\bar{\lambda}'^{F(\lambda,x,y,t)-F(\lambda',x,y,t)}f_{02}(\lambda,\bar{\lambda})d\lambda\wedge d\bar{\lambda}. \quad (19.65)$$

This implies a linear algebraic system

$$h_1 = \xi_1 + h_1A_{11} + h_2A_{21},$$
$$h_2 = \xi_2 + h_1A_{12} + h_2A_{22}. \quad (19.66)$$

Further

$$h_1(1-A_{11}) - h_2A_{21} = \xi_1,$$
$$-h_1A_{12} + h_2(1-A_{22}) = \xi_2. \quad (19.67)$$

It follows from (19.66) shows that $I-A = \begin{pmatrix} 1-A_{11} & -A_{21} \\ -A_{12} & 1-A_{22} \end{pmatrix}$. Assume that $\det(I-A)\neq 0$. Then one can find an expression for h_k $(k=1,2)$:

$$h_k = \sum_{k,l=1}^{2} \xi_k (I-A)^{-1}_{kl}, \quad (19.68)$$

where

$$\xi_k = \iint_G V(\lambda,\bar{\lambda})e^{F(\lambda,x,y,t)} f_{0k}(\lambda,\bar{\lambda})d\lambda \wedge d\bar{\lambda}, \tag{19.69}$$

$$A_{kl} = \frac{1}{2\pi i}\iint_G d\lambda' \wedge d\bar{\lambda}' \iint_G \frac{g_{0k}(\lambda',\bar{\lambda}'^{F(\lambda,x,y,t)-F(\lambda',x,y,t)} f_{0l}(\lambda,\bar{\lambda})}{\lambda'-\lambda} d\lambda \wedge d\bar{\lambda}. \tag{19.70}$$

Thus, the linear integral equation (19.4) with a degenerate kernel (19.60) reduces to the system (19.66). Assume that the system (19.66) has a solution h_1 and h_2. Substitute the expression (19.68) in (19.62). Finally, we obtain

$$W(\lambda,\bar{\lambda}) = V(\lambda,\bar{\lambda}) + \frac{1}{2\pi i}\iint_G \frac{\sum_{k,l=1}^{2} \xi_k (I-A)^{-1}_{kl} g_{0l}(\lambda',\bar{\lambda}'^{-F(\lambda',x,y,t)}}{\lambda'-\lambda} d\lambda' \wedge d\bar{\lambda}'. \tag{19.71}$$

Formula (19.71) gives the solution of the integral matrix equation (19.4), parameterized with four arbitrary matrix functions $f_{0k}(\lambda,\bar{\lambda})$ and $g_{0k}(\lambda,\bar{\lambda})$, $k=1,2$. These solutions represent the most wide class of solutions of (19.4). Using linear algebraic systems (19.66) and (19.6), we obtain the solutions

$$U_1(x,y,t) = -\frac{1}{2\pi}\sum_{k,l=1}^{2} (\xi_k (I-A_{kl})^{-1}\eta_l)_{11}, \tag{19.72}$$

$$q(x,y,t) = \frac{1}{\pi}\sum_{k,l=1}^{2} (\xi_k (I-A)^{-1}_{kl}\eta_l)_{12}, \tag{19.73}$$

$$r(x,y,t) = -\frac{1}{\pi}\sum_{k,l=1}^{2} (\xi_k (I-A)^{-1}_{kl}\eta_l)_{21}, \tag{19.74}$$

$$U_2(x,y,t) = -\frac{1}{2\pi}\sum_{k,l=1}^{2} (\xi_k (I-A_{kl})^{-1}\eta_l)_{22}. \tag{19.75}$$

Here

$$\xi_k(x,y,t) = \iint_E e^{i\lambda Ix + \frac{2i\lambda}{\alpha}B_1 y - 4i\lambda^2 C_2 t} f_{0k}(\lambda,\bar{\lambda})d\lambda \wedge d\bar{\lambda}, \tag{19.76}$$

$$\eta_l(x,y,t) = \iint_E g_{0l}(\lambda,\bar{\lambda})e^{-i\lambda Ix - \frac{2i\lambda}{\alpha}B_1 y + 4i\lambda^2 C_2 t} d\lambda \wedge d\bar{\lambda}, \tag{19.77}$$

and

$$A_{kl}(x,y,t) = \frac{1}{2\pi i}\iint_E d\mu \wedge d\bar{\mu} \iint_E \frac{g_{0l}(\mu,\bar{\mu})}{\lambda-\mu} \cdot$$

$$\cdot \exp\left(i(\mu-\lambda)Ix + \frac{2i}{\alpha}(\mu-\lambda)C_1 y - 4i(\mu^2-\lambda^2)C_2 t\right) f_{0k}(\lambda,\bar{\lambda})d\lambda \wedge d\bar{\lambda}. \tag{19.78}$$

In the case of N, R_0 has the form

$$R_0(\mu,\bar{\mu};\lambda,\bar{\lambda},x,y,t) = \sum_{k=1}^{N} f_{0k}(\mu,\bar{\mu})g_{0k}(\lambda,\bar{\lambda}), \tag{19.79}$$

respectively, degenerate kernel R has the form

$$R(\mu,\bar{\mu};\lambda,\bar{\lambda},x,y,t) = e^{F(\mu,x,y,t)} \sum_{k=1}^{N} [f_{0k}(\mu,\bar{\mu})g_{0k}(\lambda,\bar{\lambda})]e^{-F(\lambda,x,y,t)}. \tag{19.80}$$

Substituting (19.80) in (19.4), we obtain

$$W(\lambda,\bar{\lambda}) = V(\lambda,\bar{\lambda}) + \frac{1}{2\pi i}\iint_G \frac{d\lambda' \wedge d\bar{\lambda}'}{\lambda' - \lambda}\cdot$$

$$\cdot \iint_G W(\mu,\bar{\mu})e^{F(\mu,x,y,t)} \sum_{k=1}^{N} [f_{0k}(\mu,\bar{\mu})g_{0k}(\lambda',\bar{\lambda}'^{F(-\lambda',x,y,t)} d\mu \wedge d\bar{\mu}. \tag{19.81}$$

We introduce the notation

$$W(\lambda,\bar{\lambda}) = V(\lambda,\bar{\lambda}) + \frac{1}{2\pi i}\iint_G \frac{\sum_{l=1}^{N} h_l g_{0l}(\lambda',\bar{\lambda}'^{-F(\lambda',x,y,t)}}{\lambda' - \lambda} d\lambda' \wedge d\bar{\lambda}', \tag{19.82}$$

where

$$h_l = \iint_G W(\mu,\bar{\mu})e^{F(\mu,x,y,t)} f_{0k}(\mu,\bar{\mu})d\mu \wedge d\bar{\mu}. \tag{19.83}$$

To find the $W(\lambda,\bar{\lambda})$, one must compute all h_k. The system of equations for h_k follows from (19.82). Indeed, multiplying the integral equation (19.82) on the $e^{F(\lambda,x,y,t)} f_{0k}(\lambda,\bar{\lambda})$ on the left and integrating over λ twice, we get

$$\iint_G W(\lambda,\bar{\lambda})e^{F(\lambda,x,y,t)} f_{0k}(\lambda,\bar{\lambda})d\lambda \wedge d\bar{\lambda} = \iint_G V(\lambda,\bar{\lambda})e^{F(\lambda,x,y,t)} f_{0k}(\lambda,\bar{\lambda})d\lambda \wedge d\bar{\lambda} +$$

$$+ \frac{1}{2\pi i}\iint_G \frac{1}{\lambda' - \lambda}\iint_G g_{0l}(\lambda',\bar{\lambda}'^{-F(\lambda',x,y,t)} \sum_{l=1}^{N} h_l e^{F(\lambda,x,y,t)} f_{0k}(\lambda,\bar{\lambda})d\lambda \wedge d\bar{\lambda}. \tag{19.84}$$

Hence we obtain the following system:

$$h_k = \xi_k + \sum_{l=1}^{N} h_l A_{lk}, \quad k = 1,\ldots,N, \tag{19.85}$$

where

$$h_k = \iint_G W(\mu,\bar{\mu})e^{F(\mu,x,y,t)} f_{0k}(\mu,\bar{\mu})d\mu \wedge d\bar{\mu}, \tag{19.86}$$

$$\xi_k = \iint_G V(\lambda,\bar{\lambda})e^{F(\lambda,x,y,t)} f_{0k}(\lambda,\bar{\lambda})d\lambda \wedge d\bar{\lambda}, \tag{19.87}$$

$$A_{lk} = \frac{1}{2\pi i}\iint_G d\lambda' \wedge d\bar{\lambda}' \iint_G \frac{g_{0l}(\lambda',\bar{\lambda}'^{F(\lambda,x,y,t)-F(\lambda',x,y,t)} f_{0k}(\lambda,\bar{\lambda})}{\lambda'-\lambda} d\lambda \wedge d\bar{\lambda}. \tag{19.88}$$

Here, $k,l = 1,...,N$. Thus, equation (19.4) with a degenerate kernel (19.80) reduces to the system (19.85). If the system (19.85) is solvable, it is obvious that $\bar{\partial}$-problem (19.4) is also solvable. Assume that (19.85) has a solution h_1, $h_2,...,h_N$:

$$h_k = \xi_k(I-A)^{-1}_{kl}, \tag{19.89}$$

where $I-A$ is $N\times N$ matrix with elements A_{lk}, given in the form (assume that $\det(I-A)\neq 0$):

$$I-A = \begin{pmatrix} 1-A_{11} & -A_{21} & ... & -A_{1N} \\ -A_{12} & 1-A_{22} & ... & -A_{2N} \\ --.----.-- & & ... & --.-- \\ --.----.-- & & ... & --.-- \\ -A_{N1} & -A_{N2} & ... & 1-A_{NN} \end{pmatrix}. \tag{19.90}$$

Substitute (19.89) in (19.82). Finally, we obtain

$$W(\lambda,\bar{\lambda}) = V(\lambda,\bar{\lambda}) + \frac{1}{2\pi i}\iint_G \frac{\sum_{k,l=1}^{N} \xi_k(I-A)^{-1}_{kl} g_{0l}(\lambda',\bar{\lambda}'^{-F(\lambda',x,y,t)}}{\lambda'-\lambda} d\lambda' \wedge d\bar{\lambda}', \tag{19.91}$$

Formula (19.91) gives an explicit solution W, associated with the integral equation (19.4), which is parametrized by $2N$ matrix of arbitrary functions $f_{0k}(\lambda,\bar{\lambda})$ and $g_{0k}(\lambda,\bar{\lambda})$. Now, if $V = 1$, using the formula (19.14) and (19.15), we obtain solutions of the nonlinear Schrodinger-type equations for the case of N in the form (19.31)–(19.14) xi_k, eta_l and A_{kl} given by (19.35)–(19.37). □

Soliton-like solutions. To construct a soliton-like solutions, we consider the degenerate singular kernel R as follows:

$$R(\mu,\bar{\mu};\lambda,\bar{\lambda};x,y,t) = e^{F(\mu,x,y,t)} \sum_1^N f_{0k}\delta(\mu-\mu_k)g_{0k}\delta(\lambda-\lambda_k)e^{-F(\lambda,x,y,t)}. \tag{19.92}$$

Formulate the following theorem:

Theorem 19.3. *If the kernel R is given in the form of* (19.92)*, then N-soliton-like solutions of nonlinear Schrodinger-type equations* (19.1) *have the form*

$$U_1(x,y,t) = -\frac{1}{2\pi}\sum_{k,l=1}^{N}(\xi_k(I-A)^{-1}_{kl}\eta_l)_{11}, \tag{19.93}$$

$$q(x,y,t) = \frac{1}{\pi}\sum_{k,l=1}^{N}(\xi_k(I-A)^{-1}_{kl}\eta_l)_{12}, \tag{19.94}$$

$$r(x,y,t) = -\frac{1}{\pi}\sum_{k,l=1}^{N}(\xi_k(I-A)^{-1}_{kl}\eta_l)_{21}, \tag{19.95}$$

$$U_2(x,y,t) = -\frac{1}{2\pi}\sum_{k,l=1}^{N}(\xi_k(I-A)^{-1}_{kl}\eta_l)_{22}, \tag{19.96}$$

where

$$\xi_k = -2ie^{i\lambda_k Ix+\frac{2i\lambda_k}{\alpha}B_1y-4i\lambda_k^2C_2t}f_{0k}, \quad k=1,2,...,N, \tag{19.97}$$

$$\eta_l = -2ig_{0l}e^{-i\lambda_l' Ix-\frac{2i\lambda_l'}{\alpha}B_1y+4i\lambda_l'^2C_2t}, \quad l=1,2,...,N, \tag{19.98}$$

$$A_{lk} = \frac{2i}{\pi}\frac{g_{0l}e^{i(\mu_k-\lambda_l)Ix+\frac{2i}{\alpha}(\mu_k-\lambda_l)B_1y-4i(\mu_k^2-\lambda_l^2)C_2t}f_{0k}}{\lambda_l-\mu_k}, \quad \lambda_l \neq \mu_k, \forall k,l=1,2,...,N. \tag{19.99}$$

Proof. Let's start with the one-soliton-like the case.

Soliton-like solutions. Find the soliton-like solution of (19.1). Suppose that the kernel has the form

$$R_0(\mu,\bar{\mu};\lambda,\bar{\lambda}) = f_{01}g_{01}\delta(\mu-\mu_1)\delta(\lambda-\lambda_1), \tag{19.100}$$

where f_{01}, g_{01}, μ_1, λ_1 are arbitrary complex constants and $\delta(\mu-\mu_1)$, $\delta(\lambda-\lambda_1)$ is the Dirac delta function. Then, R has the following form:

$$R(\mu,\bar{\mu};\lambda,\bar{\lambda};x,y,t) = e^{F(\mu,x,y,t)}f_{01}\delta(\mu-\mu_1)g_{01}e^{-F(\lambda,x,y,t)}\delta(\lambda-\lambda_1), \tag{19.101}$$

where $F(\lambda)$ is given by the formulas (19.8). Substituting (19.100) in the (19.4) obtain

$$W(\lambda,\bar{\lambda}) = V(\lambda,\bar{\lambda}) + \frac{1}{2\pi i}\iint_G \frac{d\lambda'\wedge d\bar{\lambda}'}{\lambda'-\lambda}\cdot$$

$$\cdot\iint_G W(\mu,\bar{\mu})e^{F(\mu,x,y,t)}f_{01}\delta(\mu-\mu_1)g_{01}e^{-F(\lambda',x,y,t)}\delta(\lambda'-\lambda_1)d\mu\wedge d\bar{\mu}. \tag{19.102}$$

Believe that

$$h_1 = \iint_G W(\mu,\bar{\mu})e^{F(\mu,x,y,t)}f_{01}\delta(\mu-\mu_1)d\mu\wedge d\bar{\mu}. \tag{19.103}$$

Recall that, by definition, Dirac δ functions satisfy

$$-\frac{1}{2i}\iint_G \delta(\mu-\mu_1)W(\mu,\bar{\mu})d\mu\wedge d\bar{\mu} = W(\mu_1,\bar{\mu}_1). \tag{19.104}$$

Using this property of δ functions, one finds from (19.86)

$$h_1 = -2iW(\mu_1,\bar{\mu}_1)e^{F(\mu_1,x,y,t)}f_{01}. \tag{19.105}$$

Then, from (19.102), we get

$$W(\lambda,\bar{\lambda}) = V(\lambda,\bar{\lambda}) + \frac{h_1}{2\pi i}\iint_G \frac{g_{01}e^{-F(\lambda',x,y,t)}\delta(\lambda'-\lambda_1)}{\lambda'-\lambda}d\lambda'\wedge d\bar{\lambda}'. \tag{19.106}$$

To find the W to compute h_1. Indeed, multiplying (19.106) on the $e^{F(\lambda)}f_{01}\delta(\lambda-\lambda_1)$ on the left and integrating over λ, obtain

$$\iint_G W(\lambda,\bar{\lambda})e^{F(\lambda,x,y,t)}f_{01}\delta(\lambda-\lambda_1)d\lambda\wedge d\bar{\lambda} = \iint_G V(\lambda,\bar{\lambda})e^{F(\lambda,x,y,t)}f_{01}\delta(\lambda-\lambda_1)d\lambda\wedge d\bar{\lambda}+$$

$$+\frac{h_1}{2\pi i}\iint_G \frac{d\lambda'\wedge d\bar{\lambda}'}{\lambda'-\lambda}\iint_G g_{01}\delta(\lambda'^{F(\mu,x,y,t)-F(\lambda',x,y,t)}f_{01}\delta(\mu-\bar{\mu}_1)d\mu\wedge d\bar{\mu}. \tag{19.107}$$

Hence, we obtain (19.43). Here, ξ_k, η_l and A_{kl} have the form

$$\xi_1 = -2iV(\lambda_1,\bar{\lambda}_1)e^{i\lambda_1 Ix+\frac{2i\lambda_1}{\alpha}B_1 y-4i\lambda_1^2 C_2 t}f_{01}, \tag{19.108}$$

$$A_{11} = -\frac{2i}{\pi}\frac{g_{01}e^{i(\mu_1-\lambda_1)Ix+\frac{2i}{\alpha}(\mu_1-\lambda_1)B_1 y-4i(\mu_1^2-\lambda_1^2)C_2 t}f_{01}}{\mu_1-\lambda_1} \quad \lambda_1\neq\mu_1. \tag{19.109}$$

Thus, the formula (19.4) with (19.100) reduces to the solution of linear algebraic equations for the coefficients of h_1 (19.43). Substituting (19.44) in (19.106) and using (19.104), we have

$$W(\lambda,\bar{\lambda}) = V(\lambda,\bar{\lambda}) - \frac{1}{\pi}\frac{\xi_1(1-A_{11})^{-1}g_{01}e^{F(\lambda_1,x,y,t)}}{\lambda_1-\lambda}. \tag{19.110}$$

This formula is the solution of the integral equation (19.4). The corresponding solution of 2+1-dimensional nonlinear Schrodinger equation of (19.1) is given by (19.53)–(19.56) and (19.108), (19.109), but with η_1 of the form

$$\eta_1 = -2ig_{01}e^{-i\lambda_1' Ix-\frac{2i\lambda_1'}{\alpha}B_1 y+4i\lambda_1'^2 C_2 t}. \tag{19.111}$$

Two-soliton-like solutions. Now we find the two-soliton-like solution of equation (19.1). In this case, the kernel of the integral matrix equation (19.4) is given by

$$R_0(\mu,\bar{\mu};\lambda,\bar{\lambda}) = f_{01}g_{01}\delta(\mu-\mu_1)\delta(\lambda-\lambda_1)+f_{02}g_{02}\delta(\mu-\mu_2)\delta(\lambda-\lambda_2), \tag{19.112}$$

where f_{01},f_{02},g_{01},g_{02} and λ_1,λ_2,μ_1,μ_2 are arbitrary complex constants. As in the case $N=1$, soliton-like solutions reduce to two algebraic equations

$$h_1 = \xi_1 + h_1 A_{11} + h_2 A_{21}, \quad h_2 = \xi_2 + h_1 A_{12} + h_2 A_{22}, \tag{19.113}$$

where

$$h_i = \iint_G W(\mu,\bar{\mu}) e^{F(\mu,x,y,t)} f_{0i} \delta(\mu - \mu_i) d\mu \wedge d\bar{\mu} = -2iW(\mu_i,\bar{\mu}_i) e^{F(\mu_i,x,y,t)} f_{0i},\ i = 1,2. \tag{19.114}$$

We find the h_i, $i = 1,2$, and then, as in the case of two-soliton-like solutions, we solve equation of (19.1) in the form (19.72)–(19.75) with

$$\xi_k = -2ie^{i\lambda_k Ix + \frac{2i\lambda_k}{\alpha} B_1 y - 4i\lambda_k^2 C_2 t} f_{0k}, \quad k = 1,2, \tag{19.115}$$

$$\eta_l = -2ig_l e^{-i\lambda_l' Ix - \frac{2i\lambda_l'}{\alpha} B_1 y + 4i\lambda_l'^2 C_2 t}, \quad l = 1,2, \tag{19.116}$$

$$A_{lk} = \frac{2i}{\pi} \frac{g_{0l} e^{i(\mu_k - \lambda_l)Ix + \frac{2i}{\alpha}(\mu_k - \lambda_l) H_2 y - 4i(\mu_k^2 - \lambda_l^2) H_2 t} f_{0k}}{\lambda_l - \mu_k}, \quad \lambda_l \neq \mu_k,\ \forall k,l = 1,2. \tag{19.117}$$

N-soliton-like solutions. We define N discrete points on the complex plane: $\mu_k \in G$, $\lambda_l \in G$, $\mu_k \neq \lambda_k$ for $\forall j,k$. If in the formula (19.80) the function $f_{0k}(\mu,\bar{\mu})$ and $g_{0k}(\mu,\bar{\mu})$ are

$$f_{0k}(\mu,\bar{\mu}) = f_k \delta(\mu - \mu_k), \quad g_{0k}(\mu,\bar{\mu}) = g_k \delta(\lambda - \lambda_k), \quad k = 1,2,...,N, \tag{19.118}$$

then the singular degenerate kernel R has the form

$$R(\mu,\bar{\mu},\lambda,\bar{\lambda};x,y,t) = \sum_{k=1}^{N} e^{F(\mu,x,y,t)} f_k \delta(\mu - \mu_k) g_k \delta(\lambda - \lambda_k) e^{-F(\lambda,x,y,t)}. \tag{19.119}$$

Hence, one has

$$W(\lambda,\bar{\lambda}) = V(\lambda,\bar{\lambda}) + \frac{1}{2\pi i} \iint_G \frac{d\lambda' \wedge d\bar{\lambda}'}{\lambda' - \lambda} \cdot$$
$$\cdot \iint_G W(\mu,\bar{\mu}) e^{F(\mu,x,y,t)} \sum_{k=1}^{N} [f_{0k} \delta(\mu - \mu_k) g_{0k} \delta(\lambda' - \lambda_k)] e^{F(-\lambda',x,y,t)} d\mu \wedge d\bar{\mu}. \tag{19.120}$$

Introduce the notation

$$W(\lambda,\bar{\lambda}) = V(\lambda,\bar{\lambda}) + \frac{h_l}{2\pi i} \iint_G \frac{1}{\lambda' - \lambda} \sum_{l=1}^{N} g_{0l} \delta(\lambda' - \lambda_l) e^{-F(\lambda',x,y,t)} d\lambda' \wedge d\bar{\lambda}', \tag{19.121}$$

where

$$h_k = \iint_G W(\mu,\bar{\mu}) e^{F(\mu,x,y,t)} f_{0k} \delta(\mu - \mu_k) d\mu \wedge d\bar{\mu}, \qquad k = 1,2,...,N. \tag{19.122}$$

To find the $W(\lambda,\bar{\lambda})$, we compute all h_k. After some intermediate calculations we obtain the following algebraic system:

$$h_k = \xi_k + \sum_{l=1}^{N} h_l A_{lk}, \quad k = 1,...,N, \tag{19.123}$$

where

$$h_k = \iint_G W(\mu,\bar{\mu}) e^{F(\mu,x,y,t)} f_{0k}(\mu,\bar{\mu}) d\mu \wedge d\bar{\mu}, \tag{19.124}$$

$$\xi_k = -2ie^{i\lambda_k Ix + \frac{2i\lambda_k}{\alpha} B_1 y - 4i\lambda_k^2 C_2 t} f_{0k}, \quad k = 1,2,...,N, \tag{19.125}$$

$$A_{lk} = \frac{2i}{\pi} \frac{g_{0l} e^{i(\mu_k-\lambda_l)Ix + \frac{2i}{\alpha}(\mu_k-\lambda_l)B_1 y - 4i(\mu_k^2-\lambda_l^2)C_2 t} f_{0k}}{\lambda_l - \mu_k}, \quad \lambda_l \neq \mu_k,\ \forall k,l = 1,2,...,N. \tag{19.126}$$

Here, $k,l = 1,...,N$. Thus, the linear integral equation (19.4) with a singular degenerate kernel (19.119) is reduced a linear algebraic system (19.123). The expressions obtained from equation (19.123) for h_k are substituted in (19.121). Let us, obtain the solution of the integral matrix equation (19.4) as follows:

$$W(\lambda,\bar{\lambda}) = V(\lambda,\bar{\lambda}) + \frac{1}{2\pi i} \iint_G \frac{\sum_{k,l=1}^{N} \xi_k (I-A)^{-1}_{kl} g_{0l} \delta(\lambda'-\lambda_l) e^{-F(\lambda',x,y,t)}}{\lambda'-\lambda} d\lambda' \wedge d\bar{\lambda}'. \tag{19.127}$$

In view of formula (19.104) from (19.127) obtain the solution of (19.4) in the form

$$W(\lambda,\bar{\lambda}) = V(\lambda,\bar{\lambda}) + \frac{\sum_{k,l=1}^{N} \xi_k (I-A)^{-1}_{kl} g_{0l} e^{-F(\lambda_k,x,y,t)}}{\pi(\lambda_l-\lambda)}. \tag{19.128}$$

Hence, using the formula (19.14) and (19.6) in the formula (19.128), we get N-soliton-like solutions in the form (19.32)–(19.35), where xi_k and A_{kl} are given by (19.125), (19.126), and

$$\eta_l = -2ig_{0l} e^{-i\lambda_l Ix - \frac{2i\lambda_l}{\alpha} B_1 y + 4i\lambda_l^2 C_2 t}, \quad l = 1,2,...,N. \tag{19.129}$$

□

Here present two theorems without proof.

Theorem 19.4. *If the kernel R is given by*

$$R(\mu,\bar{\mu};\lambda,\bar{\lambda};x,y,t) = e^{F(\mu,x,y,t)} \sum_{k=1}^{N} f_{0k} \delta(\mu-\mu_k) g_{0k} \delta^{(n,0)}(\lambda-\lambda_k) e^{-F(\lambda,x,y,t)}, \tag{19.130}$$

than the exact N-soliton-like solutions of nonlinear Schrodinger-type equations (19.1) have the form

$$U_1(x,y,t) = -\frac{i}{\pi}\sum_{k,l=1}^{N}(\xi_k(I+A)^{-1}_{kl}\eta_l)_{11}, \tag{19.131}$$

$$q(x,y,t) = \frac{2i}{\pi}\sum_{k,l=1}^{N}(\xi_k(I+A)^{-1}_{kl}\eta_l)_{12}, \tag{19.132}$$

$$r(x,y,t) = -\frac{2i}{\pi}\sum_{k,l=1}^{N}(\xi_k(I+A)^{-1}_{kl}\eta_l)_{21}, \tag{19.133}$$

$$U_2(x,y,t) = -\frac{i}{\pi}\sum_{k,l=1}^{N}(\xi_k(I+A)^{-1}_{kl}\eta_l)_{22}, \tag{19.134}$$

where $\xi_k,: \eta_l$ *and* A_{lk} *are given by*

$$\xi_k = -2iV(\mu_k,\bar{\mu}_k)e^{F(\mu_k,x,y,t)}f_{0k}, \quad k=1,2,...,N, \tag{19.135}$$

$$\eta_l = \frac{(-1)^n g_{0l}}{(\lambda_l-\lambda)^n}\frac{\partial^n e^{-F(\lambda,x,y,t)}}{\partial\lambda^n}|_{\lambda=\lambda_l}, \quad l=1,2,...,N, \tag{19.136}$$

$$A_{lk} = \frac{1}{\pi}\iint_G \frac{g_{0l}e^{-F(\lambda',x,y,t)}\delta^{(n,0)}(\lambda'-\lambda_l)e^{F(\mu_k,x,y,t)}f_{0k}}{\mu_k-\lambda'}d\lambda'\wedge d\bar{\lambda}', \tag{19.137}$$

here $\lambda_l \neq \mu_k$, $\forall k,l=1,2,...,N$.

Similarly, we formulate a theorem.

Theorem 19.5. *If the kernel* $R(\mu,\bar{\mu};\lambda,\bar{\lambda};x,y,t)$ *defined by*

$$R(\mu,\bar{\mu};\lambda,\bar{\lambda};x,y,t) = e^{F(\mu,x,y,t)}\sum_{k=1}^{N}f_{0k}\delta^{(0,m)}(\mu-\mu_k)g_{0k}\delta(\lambda-\lambda_k)e^{-F(\lambda,x,y,t)}, \tag{19.138}$$

then the exact N-soliton-like solution of nonlinear Schrodinger-type equations (19.1) have the form

$$U_1(x,y,t) = -\frac{i}{\pi}\sum_{k,l=1}^{N}(\xi_k(I+A)^{-1}_{kl}\eta_l)_{11}, \tag{19.139}$$

$$q(x,y,t) = \frac{2i}{\pi}\sum_{k,l=1}^{N}(\xi_k(I+A)^{-1}_{kl}\eta_l)_{12}, \tag{19.140}$$

$$r(x,y,t) = -\frac{2i}{\pi}\sum_{k,l=1}^{N}(\xi_k(I+A)^{-1}_{kl}\eta_l)_{21}, \tag{19.141}$$

$$U_2(x,y,t) = -\frac{i}{\pi}\sum_{k,l=1}^{N}(\xi_k(I+A)^{-1}_{kl}\eta_l)_{22}, \tag{19.142}$$

where

$$\xi_k = (-1)^m f_{0k}\frac{\partial^m(Ve^{F(\lambda,x,y,t)})}{\partial\bar{\lambda}^m}, \quad k = 1,2,...,N, \tag{19.143}$$

$$\eta_l = \frac{g_{0l}e^{-F(\lambda_l,x,y,t)}}{\lambda_l - \lambda}, \quad l = 1,2,...,N, \tag{19.144}$$

$$A_{lk} = \frac{(-1)^m}{\pi}\iint_G \frac{g_{0l}e^{-F(\lambda',x,y,t)}\delta(\lambda'-\lambda_1)e^{F(\mu_k,x,y,t)}f_{0k}}{(\mu_k-\lambda'^m}d\lambda'\wedge d\bar{\lambda}', \tag{19.145}$$

here $\lambda' \neq \mu_k$ *è* k,l *varies from* 1 *to* N.

Proofs of Theorems 19.4 and 19.5 are similar to the previous Theorems 19.1−19.3. Thus, in this study we received partial solutions, i.e., N-soliton-like solutions of 2+1-dimensional nonlinear Schrodinger equation of the method of $\bar{\partial}$-problem.

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Chapter 20
A Method of Solution for Integro-Differential Parabolic Equation with Purely Integral Conditions

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Abstract The objective of this paper is to prove existence, uniqueness, and continuous dependence upon the data of solution to integro-differential parabolic equation with purely integral conditions. The proofs are based on a priory estimates and Laplace transform method. Finally, we obtain the solution by using a numerical technique for inverting the Laplace transforms.

20.1 Introduction

In this paper we are concerned with the following parabolic Integro-differential equation,

$$\frac{\partial v}{\partial t}(x,t) - \frac{\partial^2 v}{\partial x^2}(x,t) = g(x,t) + \int_0^t a(t-s)\, v(x,s)\, ds,\ 0 < x < 1,\ 0 < t \leq T, \quad (20.1)$$

subject to the initial condition

$$v(x,0) = \Phi(x),\ \ 0 < x < 1, \quad (20.2)$$

and the integral conditions

$$\int_0^1 v(x,t)\, dx = r(t),\ \ 0 < t \leq T, \quad (20.3)$$

$$\int_0^1 xv(x,t)\, dx = q(t),\ \ 0 < t \leq T, \quad (20.4)$$

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where v is an unknown function, r, q, and $\Phi(x)$ are given functions supposed to be sufficiently regular, a is suitably defined function satisfying certain conditions to be specified later, and T is a positive constant. Certain problems of modern physics and technology can be effectively described in terms of nonlocal problems for partial differential equations [3–7, 9–13, 15, 16, 20, 21, 23–27]. Ang [2] has considered a one-dimensional heat equation with nonlocal (integral) conditions. The author has taken the Laplace transform of the problem and then used numerical technique for the inverse Laplace transform to obtain the numerical solution.

This paper is organized as follows. In Sect. 20.2, we begin introducing certain function spaces which are used in the next sections, and we reduce the posed problem to one with homogeneous integral conditions. In Sect. 20.3, we first establish the existence of solution by the Laplace transform. In Sect. 20.4, we establish a priory estimates, which give the uniqueness and continuous dependence upon the data.

20.2 Statement of the Problem and Notation

Since integral conditions are inhomogeneous, it is convenient to convert problem (20.1)–(20.2) to an equivalent problem with homogenous integral conditions. For this, we introduce a new function $u(x,t)$ representing the deviation of the function $v(x,t)$ from the function

$$u(x,t) = v(x,t) - u_1(x,t), \quad 0 < x < 1,\ 0 < t \le T, \tag{20.5}$$

where

$$u_1(x,t) = 6(2q(t) - r(t))x - 2(3q(t) - 2r(t)). \tag{20.6}$$

Problem (20.1)–(20.2) with inhomogeneous integral conditions (20.3), (20.4) can be equivalently reduced to the problem of finding a function u satisfying

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = f(x,t) + \int_0^t a(t-s)\,u(x,s)\,ds,\ 0 < x < 1,\ 0 < t \le T, \tag{20.7}$$

$$u(x,0) = \varphi(x),\ \ 0 < x < 1, \tag{20.8}$$

$$\int_0^1 u(x,t)\,dx = 0,\ \ 0 < t \le T, \tag{20.9}$$

$$\int_0^1 xu(x,t)\,dx = 0,\ \ 0 < t \le T \tag{20.10}$$

where

$$f(x,t) = g(x,t) - \left(\frac{\partial u_1}{\partial t}(x,t) - \frac{\partial^2 u_1}{\partial x^2}(x,t) - \int_0^t a(t-s)\,u_1(x,s)\,ds\right) \tag{20.11}$$

and

$$\varphi(x) = \Phi(x) - u_1(x,0) \tag{20.12}$$

Hence, instead of solving for v, we simply look for u. The solution of problem (20.1)–(20.4) will be obtained by the relation (20.5), (20.6). We introduce the appropriate function spaces that will be used in the rest of the note. Let H be a Hilbert space with a norm $\|.\|_H$.

Let $L^2(0,1)$ be the standard function space.

Definition 20.1. (i) Denote by $L^2(0,T,H)$ the set of all measurable abstract functions $u(.,t)$ from $(0,T)$ into H equipped with the norm

$$\|u\|_{L^2(0,T,H)} = \left(\int_0^T \|u(.,t)\|_H^2 \, dt \right)^{1/2} < \infty \tag{20.13}$$

(ii) Let $C(0,T,H)$ be the set of all continuous functions $u(.,t) : (0,T) \longrightarrow H$ with

$$\|u\|_{C(0,T,H)} = \max_{0 \le t \le T} \|u(.,t)\|_H < \infty \tag{20.14}$$

(iii) We denote by $C_0(0,1)$ the vector space of continuous functions with compact support in $(0,1)$. Since such function are Lebesgue integrable with respect to dx, we can define on $C_0(0,1)$ the bilinear form given by

$$((u,w)) = \int_0^1 J_x^m u . J_x^m w dx, \quad m \ge 1 \tag{20.15}$$

where

$$J_x^m u = \int_0^x \frac{(x-\zeta)^{m-1}}{(m-1)!} u(\zeta,t)\, d\zeta; \quad \text{for } m \ge 1 \tag{20.16}$$

The bilinear form (20.15) is considered as a scalar product on $C_0(0,1)$ is not complete.

Definition 20.2. Denote by $B_2^m(0,1)$, the completion of $C_0(0,1)$ for the scalar product (20.15), which is denoted $(.,.)_{B_2^m(0,1)}$, introduced by [5]. By the norm of function u from $B_2^m(0,1)$, $m \ge 1$, we understand the nonnegative number:

$$\|u\|_{B_2^m(0,1)} = \left(\int_0^1 (J_x^m u)^2 \, dx \right)^{1/2} = \|J_x^m u\|; \text{ for } m \ge 1 \tag{20.17}$$

Lemma 20.3. *For all $m \in \mathbb{N}^*$, the following inequality holds:*

$$\|u\|^2_{B_2^m(0,1)} \le \frac{1}{2} \|u\|^2_{B_2^{m-1}(0,1)} . \tag{20.18}$$

Proof. See [5]. □

Corollary 20.4. *For all $m \in \mathbb{N}^*$, we have the elementary inequality*

$$\|u\|^2_{B_2^m(0,1)} \leq \left(\frac{1}{2}\right)^m \|u\|^2_{L^2(0,1)}. \tag{20.19}$$

Definition 20.5. We denote by $L^2(0,T;B_2^m(0,1))$ the space of functions which are square integrable in the Bochner sense, with the scalar product

$$(u,w)_{L^2(0,T;B_2^m(0,1))} = \int_0^T (u(.,t),w(.,t))_{B_2^m(0,1)}\,dt. \tag{20.20}$$

Since the space $B_2^m(0,1)$ is a Hilbert space, it can be shown that $L^2(0,T;B_2^m(0,1))$ is a Hilbert space as well. The set of all continuous abstract functions in $[0,T]$ equipped with the norm

$$\sup_{0\leq t\leq T} \|u(.,t)\|_{B_2^m(0,1)}$$

is denoted $C(0,T;B_2^m(0,1))$.

Corollary 20.6. *For every $u \in L^2(0,1)$, from which we deduce the continuity of the imbedding $L^2(0,1) \longrightarrow B_2^m(0,1)$, for $m \geq 1$.*

Lemma 20.7. (*Gronwall Lemma*) *Let $f_1(t)$, $f_2(t) \geq 0$ be two integrable functions on $[0,T]$, $f_2(t)$ is nondecreasing. If*

$$f_1(\tau) \leq f_2(\tau) + c\int_0^\tau f_1(t)\,dt, \qquad \forall \tau \in [0,T], \tag{20.21}$$

where $c \in \mathbb{R}^+$, then

$$f_1(t) \leq f_2(t)\exp(ct), \qquad \forall t \in [0,T]. \tag{20.22}$$

Proof. The proof is the same as that of Lemma 1.3.19 in [19]. □

20.3 Existence of the Solution

In this section we shall apply the Laplace transform technique to find solutions of partial differential equations; we have the Laplace transform

$$V(x,s) = \mathscr{L}\{v(x,t);t \longrightarrow s\} = \int_0^\infty v(x,t)\exp(-st)\,dt, \tag{20.23}$$

where s is positive reel parameter. Taking the Laplace transforms on both sides of (20.1), we have

$$(s - A(s))V(x,s) - \frac{d^2}{dx^2}V(x,s) = G(x,s) + s\Phi(x), \tag{20.24}$$

where $G(x,s) = \mathscr{L}\{g(x,t); t \longrightarrow s\}$. Similarly, we have

$$\int_0^1 V(x,s)\,dx = R(s), \tag{20.25}$$

$$\int_0^1 xV(x,s)\,dx = Q(s), \tag{20.26}$$

where

$$R(s) = \mathscr{L}\{r(t); t \longrightarrow s\}$$

and

$$Q(s) = \mathscr{L}\{q(t); t \longrightarrow s\}.$$

Now, we have the following cases:
Case 1: If $s - A(s) > 0$
Case 2: If $s - A(s) < 0$
Case 3: If $s - A(s) = 0$
We only consider cases 2 and 3, as case 1 can be dealt with similarly as in [2]. For $(s - A(s)) = 0$, we have

$$\frac{d^2}{dx^2}V(x,s) = -G(x,s) - s\Phi(x), \tag{20.27}$$

The general solution for case 3 is given by

$$V(x,s) = -\int_0^x \int_0^y [G(x,s) + s\Phi(x)]\,dzdy + C_1(s)x + C_2(s), \tag{20.28}$$

Putting the integral conditions $(3.3), (3.4)$ in (3.6) we get

$$\begin{aligned}&\frac{1}{2}C_1(s) + C_2(s)\\ &= \int_0^1 \int_0^x \int_0^y [G(x,s) + s\Phi(x)]\,dzdy + R(s),\end{aligned} \tag{20.29}$$

$$\begin{aligned}&\frac{1}{3}C_1(s) + \frac{1}{2}C_2(s)\\ &= \int_0^1 \int_0^x \int_0^y x[G(x,s) + s\Phi(x)]\,dzdy + Q(s),\end{aligned} \tag{20.30}$$

where

$$\begin{aligned}C_1(s) = {} & 12\int_0^1 \int_0^x \int_0^y x[G(x,s) + s\Phi(x)]\,dzdy - \\ & 6\int_0^1 \int_0^x \int_0^y [G(x,s) + s\Phi(x)]\,dzdy + \\ & 12Q(s) - 6R(s),\end{aligned} \tag{20.31}$$

$$C_2(s) = 4\int_0^1\int_0^x\int_0^y [G(x,s)+s\Phi(x)]\,dzdy - 6\int_0^1\int_0^x\int_0^y x[G(x,s)+s\Phi(x)]\,dzdy - 6Q(s)+4R(s). \tag{20.32}$$

For case 2, that is, $(s-A(s))<0$,using the method of variation of parameter, we have the general solution as

$$V(x,s) = \frac{1}{\sqrt{A(s)-s}}\int_0^x (G(x,s)+s\Phi(x))\sin\left(\sqrt{A(s)-s}\right)(x-\tau)\,d\tau + d_1(s)\cos\sqrt{(A(s)-s)}x + d_2(s)\sin\sqrt{(A(s)-s)}x \tag{20.33}$$

From the integral conditions (20.25), (20.26) we get

$$d_1(s)\int_0^1 \cos\sqrt{(A(s)-s)}xdx + d_2(s)\int_0^1 \sin\sqrt{(A(s)-s)}xdx = R(s) - \frac{1}{\sqrt{A(s)-s}}\int_0^1\int_0^x [(G(x,s)+s\Phi(x)) \sin\left(\sqrt{A(s)-s}\right)(x-\tau)\Big]\,d\tau dx, \tag{20.34}$$

$$d_1(s)\int_0^1 x\cos\sqrt{(A(s)-s)}xdx + d_2(s)\int_0^1 x\sin\sqrt{(A(s)-s)}xdx = Q(s) - \frac{1}{\sqrt{A(s)-s}}\int_0^1\int_0^x [x(G(x,s)+s\Phi(x)) \sin\left(\sqrt{A(s)-s}\right)(x-\tau)\Big]\,d\tau dx. \tag{20.35}$$

Thus d_1, d_2 are given by

$$\begin{pmatrix} d_1(s) \\ d_2(s) \end{pmatrix} = \begin{pmatrix} a_{11}(s) & a_{12}(s) \\ a_{21}(s) & a_{22}(s) \end{pmatrix}^{-1} \times \begin{pmatrix} b_1(s) \\ b_2(s) \end{pmatrix}, \tag{20.36}$$

and

$$a_{11}(s) = \int_0^1 \cos\sqrt{(A(s)-s)}xdx,$$
$$a_{12}(s) = \int_0^1 \sin\sqrt{(A(s)-s)}xdx,$$
$$a_{21}(s) = \int_0^1 x\cos\sqrt{(A(s)-s)}xdx,$$
$$a_{22}(s) = \int_0^1 x\sin\sqrt{(A(s)-s)}xdx,$$

$$
\begin{aligned}
b_1(s) &= R(s) - \frac{1}{\sqrt{A(s)-s}} \int_0^1 \int_0^x (G(x,s) + s\Phi(x)) \\
&\quad \times \sin\left(\sqrt{A(s)-s}\right)(x-\tau)\, d\tau dx, \\
b_2(s) &= Q(s) - \frac{1}{\sqrt{A(s)-s}} \int_0^1 \int_0^x [x(G(x,s) + s\Phi(x)) \\
&\quad \sin\left(\sqrt{A(s)-s}\right)(x-\tau)\Big] d\tau dx.
\end{aligned} \tag{20.37}
$$

If it is not possible to calculate the integrals directly, then we calculate it numerically. We approximate similarly as given in [2]. If the Laplace inversion is possible directly for (20.28) and (20.33), in this case we shall get our solution. In another case we use the suitable approximate method and then use the numerical inversion of the Laplace transform. Considering $A(s) - s = k(s)$ and using Gauss's formula given in [1] we have the following approximations of the integrals:

$$
\begin{aligned}
&\int_0^1 \begin{pmatrix} 1 \\ x \end{pmatrix} \cos\sqrt{k(s)}x dx \\
&\simeq \frac{1}{2}\sum_{i=1}^{N} w_i \begin{pmatrix} 1 \\ \frac{1}{2}[x_i+1] \end{pmatrix} \cos\left(\sqrt{k(s)}\frac{1}{2}[x_i+1]\right),
\end{aligned} \tag{20.38}
$$

$$
\begin{aligned}
&\int_0^1 \begin{pmatrix} 1 \\ x \end{pmatrix} \sin\sqrt{k(s)}x dx \\
&\simeq \frac{1}{2}\sum_{i=1}^{N} w_i \begin{pmatrix} 1 \\ \frac{1}{2}[x_i+1] \end{pmatrix} \sin\left(\sqrt{k(s)}\frac{1}{2}[x_i+1]\right),
\end{aligned} \tag{20.39}
$$

$$
\begin{aligned}
&\int_0^x (G(x,s) + s\Phi(x)) \sin\left(\sqrt{k(s)}\right)(x-\tau)\, d\tau \\
&\simeq \frac{x}{2}\sum_{i=1}^{N} w_i \left[G\left(\frac{x}{2}[x_i+1];s\right) + s\Phi\left(\frac{x}{2}[x_i+1]\right)\right] \\
&\quad \times \sin\left(\sqrt{k(s)}\left[x - \frac{x}{2}[x_i+1]\right]\right),
\end{aligned} \tag{20.40}
$$

$$
\begin{aligned}
&\int_0^1 \left[[G(\tau,s) + s\Phi(\tau)] \int_\tau^1 \begin{pmatrix} 1 \\ x \end{pmatrix} \sin\left(\sqrt{k(s)}\right)(x-\tau)\, dx\right] d\tau \\
&\simeq \frac{1}{2}\sum_{i=1}^{N} w_i \left[G\left(\frac{1}{2}[x_i+1];s\right) + s\Phi\left(\frac{1}{2}[x_i+1]\right)\right] \times \\
&\left(\frac{1-\frac{1}{2}[x_i+1]}{2}\right) \times \sum_{i=1}^{N} w_j \left(\begin{matrix} 1 \\ \frac{1-\frac{1}{2}[x_i+1]}{2}x_j + \frac{1-\frac{1}{2}[x_i+1]}{2} \end{matrix}\right) \times
\end{aligned}
$$

$$\sin\left[\sqrt{k(s)}\times\left(\frac{1-\frac{1}{2}[x_i+1]}{2}x_j+\frac{1+\frac{1}{2}[x_i+1]}{2}-\frac{1}{2}(x_i+1)\right)\right], \qquad (20.41)$$

where x_i and w_i are the abscissa and weights, defined as

$$x_i : i^{th} \text{ zero of } P_n(x), \qquad \omega_i = 2/\left(1-x_i^2\right)\left[P_n'(x)\right]^2.$$

Their tabulated values can be found in [1] for different values of N.

20.3.1 Numerical Inversion of Laplace Transform

Sometimes, an analytical inversion of a Laplace domain solution is difficult to obtain [28]; therefore, a numerical inversion method must be used. A nice comparison of four frequently used numerical Laplace inversion algorithms is given by Hassan Hassanzadeh, Mehran Pooladi-Darvish [18]. In this work we use the Stehfest's algorithm [29] that is easy to implement. This numerical technique was first introduced by Graver [17] and its algorithm then offered by [29]. Stehfest's algorithm approximates the time domain solution as

$$v(x,t) \approx \frac{\ln 2}{t}\sum_{n=1}^{2m}\beta_n V\left(x;\frac{n\ln 2}{t}\right), \qquad (20.42)$$

where, m is the positive integer,

$$\beta_n = (-1)^{n+m}\sum_{k=\left[\frac{n+1}{2}\right]}^{\min(n,m)}\frac{k^m(2k)!}{(m-k)!k!(k-1)!(n-k)!(2k-n)!}, \qquad (20.43)$$

and $[q]$ denotes the integer part of the real number q.

20.4 Uniqueness and Continuous Dependence of the Solution

We establish an a priori estimate; the uniqueness and continuous dependence of the solution with respect to the data are immediate consequences.

Theorem 20.8. *If* $u(x,t)$ *is a solution of problem* (20.7)–(20.10) *and* $f \in C\left(\overline{D}\right)$, *then we have a priory estimates:*

$$\|u(.,\tau)\|^2_{L^2(0,1)} \leq c_1\left(\|f(.,t)\|^2_{L^2\left(0,T;\,B_2^1(0,1)\right)}+\|\varphi\|^2_{L^2(0,1)}\right) \qquad (20.44)$$

$$\left\| \frac{\partial u(.,\tau)}{\partial t} \right\|^2_{L^2\left(0,T;\,B_2^1(0,1)\right)}$$
$$\le c_2 \left(\|f(.,t)\|^2_{L^2\left(0,T;\,B_2^1(0,1)\right)} + \|\varphi\|^2_{L^2(0,1)} \right) \tag{20.45}$$

where $c_1 = \exp(a_0 T)$, $c_2 = \frac{\exp(a_0 T)}{1-a_0}$, $1 < a(x,t) < a_0$, *and* $0 \le \tau \le T$.

Proof. Taking the scalar product in $B_2^1(0,1)$ of equation (20.7) and $\frac{\partial u}{\partial t}$ and integrating over $(0,\tau)$, we have

$$\int_0^\tau \left(\frac{\partial u(.,t)}{\partial t}, \frac{\partial u(.,t)}{\partial t} \right)_{B_2^1(0,1)} dt -$$
$$\int_0^\tau \left(\frac{\partial^2 u(.,t)}{\partial x^2}, \frac{\partial u(.,t)}{\partial t} \right)_{B_2^1(0,1)} dt$$
$$= \int_0^\tau \left(f(.,t), \frac{\partial u(.,t)}{\partial t} \right)_{B_2^1(0,1)} dt +$$
$$\int_0^\tau \left(\int_0^t a(t-s)\, u(x,s)\, ds, \frac{\partial u(.,t)}{\partial t} \right)_{B_2^1(0,1)} dt \tag{20.46}$$

By integrating by parts, the first and second terms in the left-hand side of (20.46) we obtain

$$\left\| \frac{\partial u(.,t)}{\partial t} \right\|^2_{L^2\left(0,T;\,B_2^1(0,1)\right)} +$$
$$\frac{1}{2} \|u(.,\tau)\|^2_{L^2(0,1)} - \frac{1}{2} \|\varphi\|^2_{L^2(0,1)}$$
$$= \int_0^\tau \left(f(.,t), \frac{\partial u(.,t)}{\partial t} \right)_{B_2^1(0,1)} dt +$$
$$\int_0^\tau \left(\int_0^t a(t-s)\, u(x,s)\, ds, \frac{\partial u(.,t)}{\partial t} \right)_{B_2^1(0,1)} dt \tag{20.47}$$

By the **Cauchy inequality**, the first term in the right-hand side of (20.46) is bounded by

$$\frac{1}{2} \|f(.,t)\|^2_{L^2\left(0,T;\,B_2^1(0,1)\right)} + \frac{1}{2} \left\| \frac{\partial u(.,t)}{\partial t} \right\|^2_{L^2\left(0,T;\,B_2^1(0,1)\right)} \tag{20.48}$$

and second term in the right-hand side of (20.46) is bounded by

$$\frac{a_0}{2} \int_0^t \|u(x,s)\|^2_{L^2\left(0,T;\,B_2^1(0,1)\right)} ds + \frac{a_0}{2} \left\| \frac{\partial u(.,t)}{\partial t} \right\|^2_{L^2\left(0,T;\,B_2^1(0,1)\right)} \tag{20.49}$$

Substitution of (20.48), (20.49) into (20.47) yields

$$(1-a_0)\left\|\frac{\partial u(.,t)}{\partial t}\right\|^2_{L^2\left(0,T;\,B_2^1(0,1)\right)}+\|u(.,\tau)\|^2_{L^2(0,1)}\leq$$

$$\left(\|f(.,t)\|^2_{L^2\left(0,T;\,B_2^1(0,1)\right)}+\|\varphi\|^2_{L^2(0,1)}\right)+\frac{a_0}{2}\int_0^t\|u(x,s)\|^2_{L^2\left(0,T;\,B_2^1(0,1)\right)}ds. \quad (20.50)$$

By Gronwall Lemma, we have

$$(1-a_0)\left\|\frac{\partial u(.,t)}{\partial t}\right\|^2_{L^2\left(0,T;\,B_2^1(0,1)\right)}+\|u(.,\tau)\|^2_{L^2(0,1)}$$
$$\leq \exp(a_0T)\left(\|f(.,t)\|^2_{L^2\left(0,T;\,B_2^1(0,1)\right)}+\|\varphi\|^2_{L^2(0,1)}\right). \quad (20.51)$$

From (20.51), we obtain estimates (20.44) and (20.45). □

Corollary 20.9. *If problem* (20.7)–(20.10) *has a solution, then this solution is unique and depends continuously on* (f,φ).

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Chapter 21
A Better Error Estimation On Szász–Baskakov–Durrmeyer Operators

Neha Bhardwaj and Naokant Deo

Abstract In this paper, we study a modified sequence of mixed summation–integral type operators; by this modification we give approximation properties and better approximation for these operators. Then we study the rate of convergence, Voronovskaya results and Korovkin theorem.

21.1 Introduction

We consider a sequence of mixed summation–integral type operators having Szász–Mirakjan basis function in summation and weight function of Baskakov operators in integration as follows:

$$(S_n f)(x) = (n-1) \sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} b_{n,k}(t) f(t) dt, \;\; x \in [0,\infty) \tag{21.1}$$

where $s_{n,k}(x) = \frac{e^{-nx}(nx)^k}{k!}$, $b_{n,k}(x) = \begin{pmatrix} n+k-1 \\ k \end{pmatrix} \frac{t^k}{(1+t)^{n+k}}$ and $f \in C[0,\infty)$ such that $|f(t)| \leq M(1+t)^{\gamma}$ for some $M > 0, \gamma > 0$.

Some approximation properties of modified form of Szász–Mirakjan operators as well as Baskakov operators were studied by Deo and Singh [2], Duman et al. [5], Gupta and Deo [7], Heilmann et al. [8], Kasana et al. [9] and Sahai and Prasad [15].

Now we need the following lemmas to study the properties of King [10] type modified mixed summation–integral operators.

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Lemma 21.1. *Let $e_i(t)=t^i$, $i=0,1,2,3,4$, then for $x \geq 0$, $n \in N$ and $n>5$, we have*

(i) $S_n(e_0;x)=1$

(ii) $S_n(e_1;x)=\dfrac{nx+1}{n-2}$

(iii) $S_n(e_2;x)=\dfrac{n^2x^2+4nx+2}{(n-2)(n-3)}$

(iv) $S_n(e_3;x)=\dfrac{n^3x^3+9n^2x^2+18nx+6}{(n-2)(n-3)(n-4)}$

(v) $S_n(e_4;x)=\dfrac{n^4x^4+16n^3x^3+72n^2x^2+96nx+24}{(n-2)(n-3)(n-4)(n-5)}$

Lemma 21.2. *Let $\varphi_x^i(t)=(t-x)^i$, $i=1,2,3$, then for $x \geq 0$, $n \in N$ and $n>4$, we have*

(i) $S_n(\varphi_x;x)=\dfrac{2x+1}{n-2}$

(ii) $S_n(\varphi_x^2;x)=\dfrac{(n+6)x^2+2(n+3)x+2}{(n-2)(n-3)}$

(iii) $S_n(\varphi_x^3;x)=\dfrac{2x^3(5n+12)+9x^2(3n+4)+12x(n+2)+6}{(n-2)(n-3)(n-4)}$

Several mathematicians (see [4, 6, 9, 11, 13, 14]) had studied this type of modification for different operators; now we consider same modification for mixed summation–integral operators.

In this paper, we deal with the approximation properties of King-type modified Szász–Baskakov–Durrmeyer operators and obtain better error estimation, rate of convergence, Voronovskaya result as well as Korovkin theorem.

21.2 Construction of Operators and Auxiliary Results

In this section we construct the operators and give necessary basic results.

We assume that $\{r_n(x)\}$ is a sequence of real-valued continuous functions defined on $[0,\infty)$ with $0 \leq r_n(x) \leq x < \infty$, for $x \in [0,\infty)$, $n \in N$ then we have

$$(\hat{S}_n f)(x)=(n-1)\sum_{k=0}^{\infty} e^{-nr_n(x)}\frac{(nr_n(x))^k}{k!}\int_0^{\infty}\binom{n+k-1}{k}\frac{t^k}{(1+t)^{n+k}}f(t)dt, \tag{21.2}$$

where $r_n(x)=\frac{(n-2)x-1}{n}$ with

$$f \in E=\left\{h \in C[0,\infty): \lim_{x\to+\infty}\frac{h(x)}{1+x^2}\ \ is\ \ finite\right\}. \tag{21.3}$$

The Banach lattice E equipped with the norm $\|f\|_* = \sup_{x\in[0,+\infty)} \frac{|f(x)|}{1+x^2}$ is isomorphic to $C[0,1]$ and the set $\{e_0, e_1, e_2\}$ is a K_+-subset of E.

The classical Peetre's K_2-functional and the second modulus of smoothness of a function $f \in C_B[0,\infty)$ are defined, respectively, by

$$K_2(f,\delta) = \inf\left\{\|f-g\| + \delta\left\|g''\right\| : g \in W_\infty^2\right\}, \quad \delta > 0$$

where $W_\infty^2 = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. From [3], there exists a positive constant C such that

$$K_2(f,\delta) \le C\omega_2\left(f,\sqrt{\delta}\right) \tag{21.4}$$

and

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h\le\delta}\ \sup_{x\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|.$$

Now we consider the Lipschitz type space

$$Lip_M^*(\gamma) = \left\{ f \in C[0,\infty) : |f(t)-f(x)| \le M\frac{|t-x|^\gamma}{(t+x)^{\gamma/2}}; x,t \in (0,\infty), \right\}$$

where M is any positive constant and $0 < \gamma \le 1$.

Now from Lemmas 21.1 and 21.2, we obtain the following results at once.

Lemma 21.3. *Let $e_i(x) = x^i$, $i = 0,1,2,3,4$, then for each $x \ge 0$ and $n > 5$, we have*

(i) $\hat{S}_n(e_0;x) = 1$
(ii) $\hat{S}_n(e_1;x) = x$
(iii) $\hat{S}_n(e_2;x) = \dfrac{(n-2)^2x^2 + 2(n-2)x - 1}{(n-2)(n-3)}$
(iv) $\hat{S}_n(e_3;x) = \dfrac{(n-2)^3x^3 + 4(n-2)^2x^2 + 3(n-2)x - 4}{(n-2)(n-3)(n-4)}$
(v) $\hat{S}_n(e_4;x) = \dfrac{x\left[(n-2)^3x^3 + 12(n-2)^2x^2 + 30(n-2)x - 91\right]}{(n-3)(n-4)(n-5)}$

Lemma 21.4. *For $x \in [0,\infty)$, $n \in N$, $n > 3$ and $\varphi_x(t) = e_1 - e_0x$, we have*

(i) $\hat{S}_n(\varphi_x;x) = 0$
(ii) $\hat{S}_n(\varphi_x^2;x) = \frac{(n-2)x^2+2(n-2)x-1}{(n-2)(n-3)}$
(iii) $\hat{S}_n(\varphi_x^m;x) = O\left(n^{-\left[\frac{m+1}{2}\right]}\right)$

The operators $\hat{S}_n$ preserve the linear functions, i.e., for $h(t) = at + b$, where a, b any real constants, we obtain $\hat{S}_n(h;x) = h(x)$.

21.3 Voronovskaya-Type Results

In this section first we establish a direct local approximation theorem for the modified operators $\hat{S}_n$ in ordinary approximation then compute the rate of convergence and Voronovskaya-type result of these operators (21.2).

Theorem 21.5. *Let* $f \in C_B[0,\infty)$, *then for every* $x \in [0,\infty)$ *and for* $C > 0$, $n > 3$, *we have*

$$\left|\left(\hat{S}_n f\right)(x) - f(x)\right| \leq C\omega_2\left(f, \sqrt{\frac{(n-2)x^2 + 2(n-2)x - 1}{(n-2)(n-3)}}\right). \tag{21.5}$$

Proof. Let $g \in W_\infty^2$. Using Taylor's expansion

$$g(y) = g(x) + g'(x)(y-x) + \int_x^y (y-u)g''(u)du.$$

From Lemma 21.4, we have

$$\left(\hat{S}_n g\right)(x) - g(x) = \left(\hat{S}_n \int_x^y (y-u)g''(u)du\right)(x).$$

We know that

$$\left|\int_x^y (y-u)g''(u)du\right| \leq (y-u)^2 \left\|g''\right\|.$$

Therefore

$$\left|\left(\hat{S}_n g\right)(x) - g(x)\right| \leq \left(\hat{S}_n (y-u)^2\right)(x) \left\|g''\right\| = \frac{(n-2)x^2 + 2(n-2)x - 1}{(n-2)(n-3)}.$$

By Lemma 21.3, we have

$$\left|\left(\hat{S}_n f\right)(x)\right| \leq (n-1)\sum_{k=0}^{\infty} s_{n,k}(r_n(x)) \int_0^\infty b_{n,k}(t) f(t)dt \leq \|f\|.$$

Hence

$$\begin{aligned}\left|\left(\hat{S}_n f\right)(x) - f(x)\right| &\leq \left|\left(\hat{S}_n (f-g)\right)(x) - (f-g)(x)\right| + \left|\left(\hat{S}_n g\right)(x) - g(x)\right| \\ &\leq 2\|f-g\| + \left(\frac{(n-2)x^2 + 2(n-2)x - 1}{(n-2)(n-3)}\right)\left\|g''\right\|\end{aligned}$$

taking the infimum on the right side over all $g \in W_\infty^2$ and using (21.4), we get the required result. □

Remark 21.6. Under the same conditions of Theorem 21.5, we obtain

$$\left|(S_n f)(x) - f(x)\right| \leq C\omega_2\left(f, \sqrt{\frac{(n+6)x^2 + 2(n+3)x + 2}{(n-2)(n-3)}}\right). \tag{21.6}$$

Theorem 21.7. *If a function f is such that its first and second derivative are bounded in $[0,\infty)$, then we get*

$$\left(\hat{S}_n f\right)(x) - f(x) = \frac{1}{2}\left(\frac{(n-2)x^2 + 2(n-2)x - 1}{(n-2)(n-3)}\right) f''(x) + I, \tag{21.7}$$

for $n > 3$ where $I \to 0$ as $n \to \infty$.

Proof. Applying Taylor's theorem we write that

$$f(t) - f(x) = (t-x)f'(x) + \frac{(t-x)^2}{2!} f''(x) + \frac{(t-x)^2}{2!}\xi(t,x), \tag{21.8}$$

where $\xi(t,x)$ is a bounded function $\forall\, t,x$ and $\lim_{t\to x}\xi(t,x) = 0$

Using (21.2) and (21.8), we obtain

$$\left(\hat{S}_n f\right)(x) - f(x) = f'(x)\hat{S}_n(\varphi_x, x) + \frac{f''(x)}{2}\hat{S}_n\left(\varphi_x^2, x\right) + \frac{1}{2}\hat{S}_n\left(\varphi_x^2, x\right)\xi(t,x).$$

From Lemma 21.4, we get

$$\left(\hat{S}_n f\right)(x) - f(x) = \frac{f''(x)}{2}\left(\frac{(n-2)x^2 + 2(n-2)x - 1}{(n-2)(n-3)}\right) + \frac{1}{2}\hat{S}_n\left(\varphi_x^2, x\right)\xi(t,x).$$

Now, we have to show that as $n \to \infty$, the value of $I = \frac{1}{2}\hat{S}_n\left(\varphi_x^2, x\right)\xi(t,x) \to 0$. Let $\varepsilon > 0$ be given since $\xi(t,x) \to 0$ as $t \to x$, then there exists $\delta > 0$ such that when $|t-x| < \delta$, we have $|\xi(t,x)| < \varepsilon$ and when $|t-x| \geq \delta$, we write

$$|\xi(t,x)| \leq C < C\frac{(t-x)^2}{\delta^2}.$$

Thus, for all $t,x \in [0,\infty)$

$$|\xi(t,x)| \leq \varepsilon + C\frac{(t-x)^2}{\delta^2}$$

and

$$I \leq \left(\hat{S}_n \varphi_x^2\left(\varepsilon + \frac{C\varphi_x^2}{\delta^2}\right)\right)(x) \leq \varepsilon\left(\hat{S}_n\varphi_x^2\right)(x) + \frac{C}{\delta^2}\left(\hat{S}_n\varphi_x^4\right)(x)$$

By Lemma 21.4, we obtain

$$I \to 0 \text{ as } n \to \infty.$$

This leads to (21.7). □

Remark 21.8. Under the same conditions of Theorem 21.7, we obtain

$$(S_n f)(x) - f(x) = \left(\frac{2x+1}{n-2}\right) f'(x) + \frac{\left[(n+6)x^2 + (n+3)x^2 + 2\right]}{(n-2)(n-3)}\frac{f''(x)}{2} + R, \tag{21.9}$$

where $R = \frac{1}{2}S_n\left(\varphi_x^2, x\right)\xi(t,x) \to 0$ as $n \to \infty$.

Theorem 21.9. *If $g \in C_B^2[0,\infty)$ then we have for $n > 3$,*

$$\left|\left(\hat{S}_n g\right)(x) - g(x)\right| \leq \frac{1}{2}\left(\frac{(n-2)x^2 + 2(n-2)x - 1}{(n-2)(n-3)}\right)\|g\|_{C_B^2}. \tag{21.10}$$

Proof. We have

$$g(t) - g(x) = (t-x)g'(x) + \frac{1}{2}(t-x)^2 g''(\zeta) \tag{21.11}$$

where $t \leq \zeta \leq x$. From Lemma 21.4 and (21.11), we get

$$\begin{aligned}\left|\left(\hat{S}_n g\right)(x) - g(x)\right| &\leq \|g'\|\left|\left(\hat{S}_n \varphi_x\right)(x)\right| + \frac{1}{2}\|g''\|\left|\left(\hat{S}_n \varphi_x^2\right)(x)\right| \\ &\leq \frac{1}{2}\left(\frac{(n-2)x^2 + 2(n-2)x - 1}{(n-2)(n-3)}\right)\|g''\| \\ &= \frac{1}{2}\left(\frac{(n-2)x^2 + 2(n-2)x - 1}{(n-2)(n-3)}\right)\|g\|_{C_B^2}.\end{aligned}$$

□

Remark 21.10. Under the same conditions of Theorem 21.9, we obtain

$$|(S_n g)(x) - g(x)| \leq \frac{(n+6)x^2 + 2(n+3)x + 2}{2(n-2)(n-3)}\|g\|_{C_B^2}. \tag{21.12}$$

Theorem 21.11. *For $f \in C_B[0,\infty)$, we obtain*

$$\begin{aligned}\left|\left(\hat{S}_n f\right)(x) - f(x)\right| \leq A\Bigg\{&\omega_2\left(f, \frac{1}{2}\sqrt{\frac{1}{2}\left(\frac{(n-2)x^2 + 2(n-2)x - 1}{(n-2)(n-3)}\right)}\right) \\ &+ \min\left(1, \frac{1}{4}\left(\frac{(n-2)x^2 + 2(n-2)x - 1}{(n-2)(n-3)}\right)\right)\|f\|_{C_B}\Bigg\},\end{aligned} \tag{21.13}$$

where constant A depends on f & $\left\{\frac{1}{2}\left(\frac{(n-2)x^2 + 2(n-2)x - 1}{(n-2)(n-3)}\right)\right\}$.

Proof. For $f \in C_B[0,\infty)$ and $g \in C_B^2[0,\infty)$ we write

$$\left(\hat{S}_n f\right)(x) - f(x) = \left(\hat{S}_n f\right)(x) - \left(\hat{S}_n g\right)(x) + \left(\hat{S}_n g\right)(x) - g(x) + g(x) - f(x)$$

From (21.10) and Peetre K_2-functions, we get

$$\begin{aligned}\left|\left(\hat{S}_n f\right)(x) - f(x)\right| &= \left|\left(\hat{S}_n f\right)(x) - \left(\hat{S}_n g\right)(x)\right| + \left|\left(\hat{S}_n g\right)(x) - g(x)\right| + |g(x) - f(x)| \\ &\leq \left\|\hat{S}_n f\right\|\|f-g\| + \frac{1}{2}\left(\frac{(n-2)x^2+2(n-2)x-1}{(n-2)(n-3)}\right)\|g\|_{C_B^2} + \|f-g\| \\ &\leq 2\|f-g\| + \frac{1}{2}\left(\frac{(n-2)x^2 + 2(n-2)x - 1}{(n-2)(n-3)}\right)\|g\|_{C_B^2}\end{aligned}$$

$$
\begin{aligned}
&= 2\left\{\|f-g\| + \frac{1}{4}\left(\frac{(n-2)x^2+2(n-2)x-1}{(n-2)(n-3)}\right)\|g\|_{C_B^2}\right\} \\
&\leq 2K_2\left\{f, \frac{1}{4}\left(\frac{(n-2)x^2+2(n-2)x-1}{(n-2)(n-3)}\right)\right\} \\
&\leq 2A\left\{\omega_2\left(f, \frac{1}{2}\sqrt{\left(\frac{(n-2)x^2+2(n-2)x-1}{(n-2)(n-3)}\right)}\right)\right. \\
&\quad \left. + \min\left(1, \frac{1}{4}\left(\frac{(n-2)x^2+2(n-2)x-1}{(n-2)(n-3)}\right)\right)\|f\|_{C_B}\right\}.
\end{aligned}
$$

This completes the proof. □

Remark 21.12. By the same conditions of Theorem 21.11, we get

$$
\begin{aligned}
|(S_nf)(x) - f(x)| \leq 2A\left\{\omega_2\left(f, \frac{1}{2}\sqrt{\frac{(n+6)x^2+2(n+3)x+2}{(n-2)(n-3)}}\right)\right. \\
\left. + \min\left(1, \frac{(n+6)x^2+2(n+3)x+2}{4(n-2)(n-3)}\right)\|f\|_{C_B}\right\}.
\end{aligned}
\tag{21.14}
$$

Theorem 21.13. *For every $f \in C[0,\infty)$, $x \in [0,\infty)$, we obtain*

$$
\left|\left(\hat{S}_nf\right)(x) - f(x)\right| \leq 2\omega(f, \delta_x) \tag{21.15}
$$

where

$$
\delta_x = \sqrt{\frac{(n-2)x^2+2(n-2)x-1}{(n-2)(n-3)}},
$$

and $\omega(f, \delta_x)$ is the modulus of continuity of f.

Proof. Let $f \in C[0,\infty)$ and $x \in [0,\infty)$. Using linearity and monotonicity of $\hat{S}_n$, we obtain, for every $\delta > 0$, $n \in N$ and $n > 3$, that

$$
\left|\left(\hat{S}_nf\right)(x) - f(x)\right| \leq \omega(f,\delta)\left\{1 + \frac{1}{\delta}\sqrt{\hat{S}_n\left(\varphi_x^2, x\right)}\right\}.
$$

By using Lemma 21.4 and choosing $\delta = \delta_x$ this completes the proof. □

Remark 21.14. For the original operator S_n defined in, we may write that, for every $f \in C[0,\infty)$

$$
|(S_nf)(x) - f(x)| \leq 2\omega(f, \phi_x) \tag{21.16}
$$

where

$$
\phi_x = \sqrt{\frac{(n+6)x^2+2(n+3)x+2}{(n-2)(n-3)}}
$$

and $\omega(f,\phi_x)$ is the modulus of continuity of f. The error estimate in Theorem 21.13 is better than that of (21.16); for $f \in C[0,\infty)$ and $x \in [0,\infty)$, we get $\delta_x \leq \phi_x$.

Finally we compute rate of convergence of these operators by means of the Lipschitz class $Lip_M(\gamma)$, $(0<\gamma\leq 1)$. As usual, we say that $f \in C_B[0,\infty)$ belongs to $Lip_M(\gamma)$ if the inequality

$$|f(t)-f(x)| \leq M|t-x|^{\gamma} \tag{21.17}$$

holds.

Theorem 21.15. *If $f \in Lip_M(\gamma)$ and $x \in [0,\infty)$ then we have for $n>3$,*

$$\left|\left(\hat{S}_n f\right)(x)-f(x)\right| \leq M\left[\frac{(n-2)x^2+2(n-2)x-1}{(n-2)(n-3)}\right]^{\gamma/2}.$$

Proof. For $f \in Lip_M(\gamma)$ and $x \geq 0$, from inequality (21.17) and using the Hölder inequality with $p=\frac{2}{\gamma}, q=\frac{2}{2-\gamma}$, we get

$$\begin{aligned}\left|\left(\hat{S}_n f\right)(x)-f(x)\right| &\leq \left(\hat{S}_n|f(t)-f(x)|\right)(x) \leq M\left(\hat{S}_n|t-x|^{\gamma}\right)(x) \leq M\left\{\left(\hat{S}_n\varphi_x^2\right)(x)\right\}^{\gamma/2}\\ &\leq M\left[\frac{(n-2)x^2+2(n-2)x-1}{(n-2)(n-3)}\right]^{\gamma/2}\end{aligned}$$

This leads to the result. □

Remark 21.16. From Lemma 21.1, for the original operator S_n, then we have the following result

$$|(S_n f)(x)-f(x)| \leq M\left\{\frac{(n+6)x^2+2(n+3)x+2}{2(n-2)(n-3)}\right\}^{\gamma/2}$$

for every $f \in Lip_M(\gamma)$, $x \geq 0$.

21.4 Korovkin-Type Approximation Theorem

Ozarslan and Aktuglu [12] proved Korovkin-type approximation theorem for Szász–Mirakian Beta operators. In this section we give the proof of this theorem for modified operators $\hat{S}_n$. For this we have the following lemma, which proves that $\hat{S}_n$ maps E into itself.

Lemma 21.17. *There exists a constant M such that, for $\alpha(x)=(1+x^2)^{-1}$, we have*

$$\alpha(x)\hat{S}_n\left(\frac{1}{\alpha};x\right) \leq M$$

holds for all $x \in [0,\infty)$ *,* $n \in N$ *and* $n > 3$*. Furthermore, for all* $f \in E$*, we have*

$$\left\|\hat{S}_n(f)\right\|_* \leq M\|f\|_*$$

Proof. From Lemma 21.3 and (21.3), we have

$$\alpha(x)\hat{S}_n\left(\frac{1}{\alpha};x\right) = \frac{1}{1+x^2}\hat{S}_n\left(1+t^2;x\right) = \frac{1}{1+x^2}\left[\hat{S}_n\left(e_0;x\right)+\hat{S}_n\left(e_2;x\right)\right]$$
$$= \frac{1}{1+x^2}\left[1+\frac{(n-2)^2x^2+2\,(n-2)\,x-1}{(n-2)\,(n-3)}\right] \leq M.$$

Also,

$$\alpha(x)\hat{S}_n\left(f;x\right) = \alpha\left(x\right)\left|\hat{S}_n\left(\alpha\frac{f}{\alpha};x\right)\right| \leq \|f\|_*\alpha\left(x\right)\hat{S}_n\left(\frac{1}{\alpha};x\right) \leq M\|f\|_*.$$

Taking the supremum over $x \in [0,\infty)$ in the above inequality, gives the result. □

Theorem 21.18. *For all* $f \in E, \hat{S}_n\left(f;x\right)$ *converges uniformly to* f *on* $[0,b]$ *if and only if* $\lim_{n\to\infty} r_n\left(x\right) = x$ *uniformly on* $[0,b]$.

Proof. Using Theorem 4.14(vi) of [1], the universal Korovkin-type property with respect to monotone operators, results similar to Theorem 3 [12] can be obtained.
□

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Chapter 22
About New Class of Volterra-Type Integral Equations with Boundary Singularity in Kernels

Nusrat Rajabov

Abstract In this work, we investigate one class of Volterra type integral equation, in model and non model case, when kernels have first order singularity and logarithmic singularity. In depend of the signs parameters solution to this integral equation can contain two arbitrary constants, one constant and may be have unique solution. In the case, when general solution of integral equation contains arbitrary constant, we stand and investigate different boundary value problems when conditions is given in singular point. For considered integral equation, the solution found can represented in generalized power series.

22.1 Introduction

Let $\Gamma = \{x\colon a < x < b\}$ be the set of point on real axis and let us consider an integral equation

$$\varphi(x) + \int_0^x \left[K_1(x,t) + K_2(x,t)\ln\left(\frac{x-a}{t-a}\right)\right]\frac{\varphi(t)}{t-a}dt = f(x), \qquad (22.1)$$

where $K_1(x,t) and K_2(x,t)$ are given functions on the rectangle $\overline{R}$ with R defined as the set $\{a < x < b, a < t < b\}$ and $f(x)$ is a given function in $\overline{\Gamma}$ and $\varphi(x)$ to be found. The theory of the above integral equation at $K_2(x,t) = 0$ has been constructed in [1–5]. In this work based on the roots of the algebraic equation

$$\lambda^2 + K_1(a,a)\lambda + K_2(a,a) = 0,$$

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signs $K_1(a,a)$ and $K_2(a,a)$, the general solution of the model integral equation in explicit form is obtained. Moreover, using the method similar to regularization method [1–6] in theory one-dimensional singular integral equation [2], the problem of finding general solution of the integral equation stated above is reduced to the problem of finding general solution of integral equation with weak singularity. The solution to this equation is sough in the class of functions $\varphi(x) \in C[a,b]$ vanishing at the singular point $x = a$ i.e $\varphi(x) = o[(x-a)^{\varepsilon}]$ $\varepsilon > 0$ and $x \to a$.

22.2 Modelling of Integral Equation

We investigate the following integral equation (in the case of $K_1(x,t) = p =$ constant and $K_2(x,t) = q =$ constant in (1.1)) :

$$\varphi(x) + \int_0^x \left[p + q \ln\left(\frac{x-a}{t-a}\right)\right] \frac{\varphi(t)}{t-a} dt = f(x), \tag{22.2}$$

where p, q are given constants. Support that the solution of the characteristic equation (22.2) exists and belongs to $C(\Gamma_0)$. Also, assume $f(x) \in C''(\Gamma_0)$. Then differentiating both sides of (22.2) twice arrives at an ordinary differential equation of the second order with left singular point. Writing out solution obtained ordinary differential equation according to [7] and returning to conversely, we find solution integral equation (22.2). For (22.2) the following confirmation is obtained:

Theorem 22.1. *Let in integral equation* (22.2)*,* $p < 0,\ \ q > 0,\ \ D = p^2 - 4q > 0$, $f(x) \in C[a,b]$, $f(a) = 0$ *with the following asymptotic behavior:* $f(x) = [(x-a)^{\delta_1}]$, $\delta_1 > \lambda_1,\ \ \lambda_1 = \dfrac{|p| + \sqrt{D}}{2}$ *at* $x \to a$*. Then the integral equation* (22.2) *in class of function* $\varphi(x) \in C[a,b]$ *vanishing in point* $x = a$ *is always solvability and its solution is given by the following formula:*

$$\varphi(x) = (x-a)^{\lambda_1} C_1 + (x-a)^{\lambda_2} C_2 + f(x) - \frac{1}{\sqrt{p^2 - 4q}}$$

$$\times \int_a^x \left[\lambda_2^2 \left(\frac{x-a}{t-a}\right)^{\lambda_2} - \lambda_1^2 \left(\frac{x-a}{t-a}\right)^{\lambda_1}\right] \frac{f(t)}{t-a} dt \equiv K_1^-[C_1, C_2, f(x)] \tag{22.3}$$

where $\lambda_2 = \dfrac{|p| - \sqrt{D}}{2}$ *and* C_1, C_2 *are arbitrary constants.*

Theorem 22.2. *Let in integral equation* (22.2)*,* $p > 0,\ \ q < 0,\ \ p^2 > 4q$*. function* $f(x) \in C[a,b]$, $f(a) = 0$ *with following asymptotic behavior:*

$$f(x) = O[(x-a)^{\delta_2}],\ \delta_2 > \lambda_1^1,\ \lambda_1^1 = \frac{\sqrt{p^2 - 4q} - p}{2} \ \ at\ \ x \to a. \tag{22.4}$$

Then the integral equation (22.2) in class of function $\varphi(x) \in C[a,b]$ vanishing in point $x = a$ is always solvability; its general solution contains one arbitrary constant and is given by the formula

$$\varphi(x) = (x-a)^{\lambda_1^1} C_3 + f(x) - \frac{1}{\sqrt{p^2+4[|q|]}}$$

$$\times \int_a^x \left[(\lambda_2^1)^2 \left(\frac{t-a}{x-a}\right)^{|\lambda_2^1|} - (\lambda_1^1)^2 \left(\frac{x-a}{t-a}\right)^{\lambda_1^1} \right] \frac{f(t)}{t-a} dt \equiv K_2^-[C_3, f(x)], \quad (22.5)$$

where $\lambda_2^1 = \dfrac{-p-\sqrt{p^2+4|q|}}{2}$ *and C_3 is arbitrary constant.*

Theorem 22.3. *Let in integral equation (22.2), $p < 0, \quad q < 0, \quad p^2 > 4q$. Assume that a function $f(x) \in C[a,b]$, $f(a) = 0$ with the following asymptotic behavior:*

$$f(x) = O[(x-a)^{\delta_3}], \, \delta_3 > \lambda_1^2, \, \lambda_1^2 = \frac{\sqrt{p^2-4q}+|p|}{2} \;\; at \;\; x \to a. \quad (22.6)$$

Then the integral equation (22.2) in class of function $\varphi(x) \in C[a,b]$ vanishing in point $x = a$ is always solvability; its general solution contains one arbitrary constant and is given by the formula

$$\varphi(x) = (x-a)^{\lambda_1^2} C_4 + f(x) - \frac{1}{\sqrt{p^2+4[|q|]}}$$

$$\times \int_a^x \left[(\lambda_2^2)^2 \left(\frac{t-a}{x-a}\right)^{|\lambda_2^2|} - (\lambda_1^2)^2 \left(\frac{x-a}{t-a}\right)^{\lambda_1^2} \right] \frac{f(t)}{t-a} dt \equiv K_2^-[C_4, f(x)], \quad (22.7)$$

where $\lambda_2^1 = \dfrac{|p|-\sqrt{p^2+4|q|}}{2} < 0$ *and C_4 is arbitrary constant.*

Theorem 22.4. *Let in integral equation (22.2), $p > 0, \quad q > 0, \quad p^2 > 4q$. Assume that a function $f(x) \in C[a,b]$, $f(a) = 0$ with the following asymptotic behavior:*

$$f(x) = O[(x-a)^{\varepsilon}], \, \varepsilon > 0 \;\; at \;\; x \to a. \quad (22.8)$$

Then the integral equation (22.2) in class of function $\varphi(x) \in C[a,b]$ vanishing in point $x = a$ has a unique solution which is given by formula

$$\varphi(x) = f(x) - \frac{1}{\sqrt{p^2 - 4q}}$$

$$\times \int_a^x \left[\lambda_2^2 \left(\frac{t-a}{x-a} \right)^{\lambda_2} - \lambda_1^2 \left(\frac{t-a}{x-a} \right)^{\lambda_1} \right] \frac{f(t)}{t-a} dt \equiv K_4^- [f(x)], \quad (22.9)$$

where $\lambda_1 = \dfrac{-p + \sqrt{p^2 - 4q}}{2}$ *and* $\lambda_2 = \dfrac{-p - \sqrt{p^2 - 4q}}{2}$.

Theorem 22.5. *Let in integral equation (22.2), $p < 0$, $p^2 = 4q$. Assume that a function $f(x) \in C[a,b]$, $f(a) = 0$ with the following asymptotic behavior:*

$$f(x) = O[(x-a)^{\varepsilon}], \varepsilon > 0 \ \ at \ \ x \to a. \quad (22.10)$$

Then, the integral equation (22.2) in class of function $\varphi(x) \in C[a,b]$ vanishing in point $x = a$ is always solvability; its general solution contains two arbitrary constants and is given by the following formula:

$$\varphi(x) = (x-a)^{|p|/2}[C_5 + \ln(x-a)C_6] + f(x)$$

$$+ \frac{|p|}{2} \int_a^x \left(\frac{x-a}{t-a} \right)^{|p|/2} \left[2 + \frac{|p|}{2} \ln \left(\frac{x-a}{t-a} \right) \right] \frac{f(t)}{t-a} dt \equiv K_5^- [C_5, C_7, f(x)] \quad (22.11)$$

where C_5, C_6 are arbitrary constants.

Theorem 22.6. *Let in integral equation (22.2), $p > 0$, $p^2 = 4q$. Assume that a function $f(x) \in C[a,b]$, $f(a) = 0$ with asymptotic behavior (22.8). Then the integral equation (22.2) in class of function $\varphi(x) \in C[a,b]$ vanishing in point $x = a$ has a unique solution, which is given by the formula*

$$\varphi(x) = f(x) - \frac{p}{2} \int_a^x \left(\frac{t-a}{x-a} \right)^{p/2} \left[2 - \frac{p}{2} \ln \left(\frac{x-a}{t-a} \right) \right] \frac{f(t)}{t-a} dt \equiv K_6^- [f(x)]. \quad (22.12)$$

Theorem 22.7. *Let in integral equation (22.2), $p < 0$, $p^2 < 4q$. Assume that a function $f(x) \in C[a,b]$, $f(a) = 0$. with the following asymptotic behavior:*

$$f(x) = O[(x-a)^{\delta_5}], \delta_5 > \frac{|p|}{2} \ \ at \ \ x \to a$$

Then the integral equation (22.2) in class of function $\varphi(x) \in C[a,b]$ vanishing in point $x = a$ is always solvability; its general solution contains two arbitrary constants and is given by the following formula:

$$\varphi(x) = (x-a))^{|p|/2}\left\{\cos\left[\frac{\sqrt{4q-p^2}}{2}\ln(x-a)\right]C_7+\sin\left[\frac{\sqrt{4q-p^2}}{2}\ln(x-a)\right]C_8\right\}$$

$$+f(x)+\frac{1}{\sqrt{4q-p^2}}\int_a^x\left(\frac{x-a}{t-a}\right)^{|p|/2}\left[(p^2-4q)\sin\left[\frac{\sqrt{4q-p^2}}{2}\ln\left(\frac{x-a}{t-a}\right)\right]\right.$$

$$\left.-p\sqrt{4q-p^2}\cos\left[\frac{\sqrt{4q-p^2}}{2}\ln\left(\frac{x-a}{t-a}\right)\right]\right]\frac{f(t)}{t-a}dt \equiv K_7^-[C_7,C_8,f(x)], \quad (22.13)$$

where C_7, C_8 are arbitrary constants.

Theorem 22.8. *Let in integral equation (22.2), $p>0$, $p^2-4q<0$. Function $f(x)\in C[a,b]$, $f(a)=0$ with asymptotic behavior (22.8). Then the integral equation (22.2) in class of function $\varphi(x)\in C[a,b]$ vanishing in point $x=a$ has a unique solution, which is given by the formula*

$$\varphi(x) = f(x)+\frac{1}{\sqrt{4q-p^2}}\int_a^x\left(\frac{t-a}{x-a}\right)^{p/2}\left[(p^2-4q)\sin\left[\frac{\sqrt{4q-p^2}}{2}\ln\left(\frac{x-a}{t-a}\right)\right]\right.$$

$$\left.-p\sqrt{4q-p^2}\cos\left[\frac{\sqrt{4q-p^2}}{2}\ln\left(\frac{x-a}{t-a}\right)\right]\right]\frac{f(t)}{t-a}dt \equiv K_8^-[f(x)]. \quad (22.14)$$

Theorems 22.1–22.8 are proved using the relation of the integral equation (22.2) with corresponding ordinary differential equation and method developed in [1–5].

Corollary 22.9. *If $q=0$ in integral equation (22.2), then from (22.2) to (22.3) it follows the solution of the equation*

$$\varphi(x)+p\int_a^x\frac{\varphi(t)}{t-a}dt = f(x),$$

at $p<0$ given by the formula

$$\varphi(x) = (x-a)^{|p|}[C_1+f(x)-p\int_a^x\left(\frac{x-a}{t-a}\right)^{|p|}\frac{f(t)}{t-a}dt,$$

that is, in this case obtained solution integral equation (22.2) coincides with formula (22.10) from [4] or with formula (22.11) from [5]. At $p>0$ and $q=0$ we have

$$\varphi(x) = f(x)-p\int_a^x\left(\frac{x-a}{t-a}\right)^{p}\frac{f(t)}{t-a}dt$$

that is, in this case, obtained solution coincides with formula (22.12) from [4] or with formula (22.13) from [5].

Corollary 22.10. *If* $p = 0$, $q > 0$ *in integral equation* (22.2)*, then* (22.2) *admits the following form:*

$$\varphi(x) + q \int_a^x \ln\left(\frac{x-a}{t-a}\right) \frac{\varphi(t)}{t-a} dt = f(x). \tag{22.15}$$

According to formula (22.14) *at* $q > 0$ *the solution for this equation is given by the formula*

$$\varphi(x) = f(x) - \sqrt{q} \int_a^x \sin\sqrt{q}\left[\ln\left(\frac{x-a}{t-a}\right)\right] \frac{f(t)}{t-a} dt. \tag{22.16}$$

Theorem 22.11. *Let in integral equation* (22.15)*,* $q > 0$, $f(x) \in C[a,b]$, $f(a) = 0$ *with asymptotic behavior* (22.10)*. Then the integral equation* (22.15) *in class of function* $\varphi(x) \in C[a,b]$ *vanishing in point* $x = a$ *has a unique solution which is given by formula* (22.16)*.*

If $q < 0$ in integral equation (22.15), then from formula (22.3), it follows that the solution of the integral equation (22.15) is given by the formula

$$\varphi(x) = (x-a)^{\sqrt{|q|}} C_9 + f(x)$$

$$- \frac{q^2}{2\sqrt{|q|}} \int_a^x \left[\left(\frac{t-a}{x-a}\right)^{\sqrt{|q|}} - \left(\frac{x-a}{t-a}\right)^{\sqrt{|q|}}\right] \frac{f(t)}{t-a} dt \equiv K_9^-[C_9, f(x)]. \tag{22.17}$$

So, in this case, we have the following confirmation:

Theorem 22.12. *Let in integral equation* (22.15)*,* $q < 0$, $f(x) \in C[a,b]$, $f(a) = 0$ *with asymptotic behavior*

$$f(x) = o\left[(x-a)^{\delta_6}\right], \delta_6 > \sqrt{|q|} \ \ at \ \ x \to a.$$

Then, the integral equation (22.15)*, in class of function* $\varphi(x) \in C[a,b]$ *vanishing in point* $x = a$ *is always solvability; its general solution contains one arbitrary constant and is given by formula* (22.17) *where* C_9 *is arbitrary constant.*

22.3 General Case

Let us rewrite (22.1) as follows:

$$\varphi(x) + \int_a^x \left[K_1(a,a) + K_2(a,a)\ln\left(\frac{x-a}{t-a}\right)\right] \frac{\varphi(t)}{t-a} dt = F(x), \tag{22.18}$$

where

$$F(x) = f(x) - \int_a^x \left[K_1(x,t) - K_1(a,a) + (K_2(x,t) - K_2(a,a)) \ln\left(\frac{x-a}{t-a}\right)\right] \frac{\varphi(t)}{t-a} dt, \tag{22.19}$$

Assuming for a moment that $F(x)$ is known, we can find a general solution to (22.1). Let $K_1(a,a) < 0$, $K_2(a,a) > 0$, *and* $(K_1(a,a))^2 - 4K_2(a,a) > 0$, and let functions $f(x)$, $K_1(x,t)$, $K_2(x,t)$, and $\varphi(x)$ be such that $F(x) \in C(\overline{R})$, $F(a) = 0$ with following asymptotic behavior:

$$F(x) = [(x-a)^{\gamma_1}],\ \gamma_1 > \delta_1,\ \delta_1 = \frac{|K_1(a,a)| + \sqrt{D}}{2},$$

$$D^2 = (K_1(a,a))^2 - 4K_2(a,a), \text{ at } x \to a.$$

Then according to Theorem 22.1 general solution of nonhomogeneous integral equation (22.1) is

$$\varphi(x) = (x-a)^{\delta_1} C_1 + (x-a)^{\delta_2} C_2 + F(x)$$

$$- \frac{1}{\sqrt{D}} \int_a^x \left[\delta_2^2 \left(\frac{x-a}{t-a}\right)^{\delta_2} - \delta_1^2 \left(\frac{x-a}{t-a}\right)^{\delta_1}\right] \frac{F(t)}{t-a} dt \equiv K_1^-[C_1, C_2, F(x)], \tag{22.20}$$

where $\delta_2 = \dfrac{|K_1(a,a)| - \sqrt{D}}{2}$ and C_1, C_2 are arbitrary constants.

Substituting for $F(x)$ from formula (22.19) we arrive at the solution of the following integral equation:

$$\varphi(x) + \int_a^x \frac{M(x,t)}{t-a} \varphi(t) dt = K_1[C_1, C_2, f(x)], \tag{22.21}$$

where

$$M(x,t) = K_1(x,t) - K_1(a,a) + (K_2(x,t) - K_2(a,a)) \ln\left(\frac{x-a}{t-a}\right)$$

$$- \frac{1}{\sqrt{D}} \int_t^x \left[\delta_2^2 \left(\frac{x-a}{t-a}\right)^{\delta_2} - \delta_1^2 \left(\frac{x-a}{t-a}\right)^{\delta_1}\right]$$

$$\times \left[K_1(\tau,t) - K_1(a,a) + (K_2(\tau,t) - K_2(a,a)) \ln\left(\frac{\tau-a}{t-a}\right)\right] \frac{d\tau}{\tau-a}. \tag{22.22}$$

If the kernels $K_1(x,t)$, $K_2(x,t)$ in (22.1) are such that for any $K(x,t) \in C(\overline{R})$ $(x,t) \to (a,a)$ satisfies the conditions

$$K_1(x,t) - K_1(a,a)$$

$$= o[(x-a)^{\alpha_1}(t-a)^{\alpha_2}],\ \alpha_1 > \delta_1,\ \alpha_2 > \delta_1 \text{ at } (x,t) \to (a,a). \tag{22.23}$$

$$K_2(\tau,t) - K_2(a,a)$$

$$= o[(x-a)^{\alpha_3}(t-a)^{\alpha_4}],\ \alpha_3 > \delta_1,\ \alpha_4 > \delta_1 \text{ at } (x,t) \to (a,a); \tag{22.24}$$

then $M(x,t)$ satisfies the following inequality:

$$|M(x,t)| \leq D_1(x-a)^{\alpha_1}(t-a)^{\alpha_2} + D_2(x-a)^{\alpha_3}(t-a)^{\alpha_4}\ln\left(\frac{x-a}{t-a}\right)$$

$$D_3(x-a)^{\alpha_3}(t-a)^{\alpha_3+\alpha_4-\delta_1}\cdot\ln\left(\frac{x-a}{t-a}\right) + D_4(x-a)^{\delta_3}(t-a)^{\alpha_1+\alpha_2-\delta_2}$$

$$+D_5(x-a)^{\delta_1}(t-a)^{\alpha_1+\alpha_2-\delta_1} + D_6(x-a)^{\alpha_3}(t-a)^{\alpha_4}$$

$$+D_7(x-a)^{\delta_1}(t-a)^{\alpha_3+\alpha_4-\delta_1} + D_8(x-a)^{\delta_2}(t-a)^{\alpha_3+\alpha_4-\delta_2}, \qquad (22.25)$$

where D_j $(1 \leq j \leq 8)$ are given constants. From inequality (22.25) we see that if $\alpha_1+\alpha_2 > \delta_2$, $\alpha_3+\alpha_4 > \delta_2$ then $M(a,a) = 0$. In other words, the kernel $M_1(x,t) = (t-a)^{-1}M(x,t)$ has a weak singularity at $t = a$.

Let function $f(x) \in C[a,b]$, $f(a) = 0$ with asymptotic behavior

$$f(x) = [(x-a)^{\alpha_5}],\ \alpha_5 > \delta_1,\ \delta_1 = \frac{|K_1(a,a)| + \sqrt{D}}{2} \text{ at } x \to a. \qquad (22.26)$$

Then, the integral equation (22.21), as second kind Volterra-type integral equation with weak singularity, has a unique solution, which is given by formula

$$\varphi(x) = K_1^-[C_1, C_2, f(x)] - \int_a^x \Gamma(x,t)K_1^-[C_1, C_2, f(t)]dt, \qquad (22.27)$$

where $\Gamma(x,t)$ is a resolvent of the integral equation (22.21) and C_1, C_2 are arbitrary constants.

Thus, from the preceding discussion the theorem follows.

Theorem 22.13. *Let in (22.1), $K_1(x,t) \in C(\overline{R})$, $K_2(x,t) \in C(\overline{R})$, functions $K_1(x,t)$, $K_2(x,t)$ in neighborhood point $(x,t) = (a,a)$ satisfy condition (22.23), (22.24), and let $K_1(a,a) < 0$, $K_2(a,a) > 0$, $D = (K_1(a,a))^2 - 4K_2(a,a) > 0$, $f(x) \in C[a,b]$, $f(a) = 0$ with asymptotic behavior (22.10), $\alpha_1+\alpha_2 > \delta_2$, $\alpha_3+\alpha_4 > \delta_2$. Then the integral equation (22.2) in class of function $\varphi(x) \in C[a,b]$ vanishing in point $x = a$ is always solvability and its solution is given by formula (22.27), where C_1, C_2 are arbitrary constants.*

Let $K_1(a,a) > 0$, $K_2(a,a) < 0$, $D = (K_1(a,a))^2 - 4K_2(a,a) > 0$, functions $f(x)$, $K_1(x,t)$, $K_2(x,t)$, and $\varphi(x)$ be such that function $F(x) \in C(\overline{R})$, $F(a) = 0$ with following asymptotic behavior :

$$F(x) = O[(x-a)^{\alpha_6}],\ \alpha_6 > \lambda_1^1,\ \lambda_1^1 = \frac{\sqrt{D} - K_1(a,a)}{2} \text{ at } x \to a. \qquad (22.28)$$

Then, the integral equation (22.1) in class of function $\varphi(x) \in C[a,b]$ vanishing in point $x = a$ is always solvability;, its general solution contains one arbitrary constant and is given by the formula

$$\varphi(x) = (x-a)^{\lambda_1^1} C_3 + F(x)$$

$$-\frac{1}{\sqrt{p^2+4[|q|]}} \int_a^x \left[(\lambda_2^1)^2 \left(\frac{t-a}{x-a}\right)^{|\lambda_2^1|} - (\lambda_1^1)^2 \left(\frac{x-a}{t-a}\right)^{\lambda_1^1} \right] \frac{F(t)}{t-a} dt \equiv K_2^-[C_3, F(x)], \tag{22.29}$$

where $\lambda_2^1 = \dfrac{-K_1(a,a) - \sqrt{D}}{2}$ andC_3 is arbitrary constant.

Substituting for $F(x)$ formula (22.19) we arrive at solution of the following integral equation:

$$\varphi(x) + \int_a^x \frac{M_1(x,t)}{t-a} \varphi(t) dt = K_2^-[C_3, f(x)], \tag{22.30}$$

where

$$M_1(x,t) = K_1(x,t) - K_1(a,a) + (K_2(x,t) - K_2(a,a)) \ln\left(\frac{x-a}{t-a}\right)$$

$$-\frac{1}{\sqrt{D}} \int_t^x \left[\lambda_2^1 \left(\frac{\tau-a}{x-a}\right)^{|\lambda_2^1|} - \lambda_2^2 \left(\frac{x-a}{t-a}\right)^{\lambda_2^2} \right]$$

$$\times \left[K_1(\tau,t) - K_1(a,a) + (K_2(\tau,t) - K_2(a,a)) \ln\left(\frac{\tau-a}{t-a}\right) \right] \frac{d\tau}{\tau-a}. \tag{22.31}$$

Let in integral equation (22.1) the functions $K_1(x,t), K_2(x,t)$ in neighborhood point $(x,t) = (a,a)$ satisfy the conditions (22.23), (22.24). Then we have

$$|M_1(x,t)| \leq T_1(x-a)^{\alpha_1}(t-a)^{\alpha_2} + T_2(x-a)^{\alpha_3}(t-a)^{\alpha_4} \ln\left(\frac{x-a}{t-a}\right)$$

$$+T_3(x-a)^{-|\lambda_2^1|}(t-a)^{|\lambda_2^1|+\alpha_1+\alpha_2} + T_4(x-a)^{\lambda_2^2}(t-a)^{\alpha_1+\alpha_2-\lambda_2^2}$$

$$+T_5(x-a)^{\alpha_3-|\lambda_2^1|}(t-a)^{\alpha_4} + T_6(x-a)^{-|\lambda_2^1|}(t-a)^{|\lambda_2^1|+\alpha_3+\alpha_4}, \tag{22.32}$$

where $T_j (1 \leq j \leq 6)$ is known constant. Multiplying both sides of (22.30) to $(x-a)^{|\lambda_2^1|}$ and introducing $(x-a)^{|\lambda_2^1|}\varphi(x) = \psi(x)$ we obtain a new integral equation

$$\psi(x) + \int_a^x \frac{M_2(x,t)}{t-a} \psi(t) dt = (x-a)^{|\lambda_2^1|} K_2^-[C_3, f(x)], \tag{22.33}$$

where

$$M_2(x,t) = \left(\frac{x-a}{t-a}\right)^{|\lambda_2^1|} M_1(x,t),$$

For $M_2(x,t)$ we have the following inequality:

$$|M_2(x,t)| \leq T_1(x-a)^{\alpha_1+|\lambda_2^1|}(t-a)^{\alpha_2-|\lambda_2^1|} + T_2(x-a)^{\alpha_3+|\lambda_2^1|}(t-a)^{\alpha_4-|\lambda_2^1|}\ln\left(\frac{x-a}{t-a}\right)$$

$$+T_3(t-a)^{\alpha_1+\alpha_2} + T_4(x-a)^{\lambda_2^2+|\lambda_2^1|}(t-a)^{\alpha_1+\alpha_2-|\lambda_2^1|-\lambda_2^2}$$

$$+T_5(x-a)^{\alpha_3}(t-a)^{\alpha_4-|\lambda_2^1|} + T_6(t-a)^{\alpha_3+\alpha_4}. \qquad (22.34)$$

From here follows, if $\alpha_j > |\lambda_2^1|(j=1,2,4)$, $\alpha_1+\alpha_2 > |\lambda_2^1|+\lambda_2^2$, then

$$|M_2(0,t)| \leq T_3(t-a)^{\alpha_1+\alpha_2} + T_{6)}(t-a)^{\alpha_3+\alpha_4}, \;\; M_2(x,0) = 0.$$

Thus, if fulfilling the following conditions: $\alpha_1+\alpha_2 > |\lambda_2^1|+\lambda_2^2$, $\alpha_j > |\lambda_2^1|(j = 1,2,4)$, then kernel integral equation (22.33) has a weak singularity. If $f(a) = 0$ with asymptotic behavior (22.26), then right part of integral equation (22.33) vanishes in point $x = a$. Consequently the integral equation (22.33) has only one solution:

$$\psi(x) = (x-a)^{|\lambda_2^1|}K_2^-[C_3, f(x)] - \int_a^x \Gamma(x,t)(t-a)^{|\lambda_2^1|}K_2^-[C_3, f(t)]dt,$$

where $\Gamma(x,t)$ is a resolvent of the integral equation (22.33). Then $\varphi(x)$ the solution of the integral equation (22.1) is given by the following formula:

$$\varphi(x) = K_2^-[C_3, f(x)] - \int_a^x \left(\frac{t-a}{x-a}\right)^{|\lambda_2^1|}\Gamma(x,t)K_2^-[C_3, f(t)]dt, \qquad (22.35)$$

where C_3 is an arbitrary constant. So, we proved the following confirmation:

Theorem 22.14. *Let in* (22.1)*,* $K_1(x,t) \in C(\overline{R})$, $K_2(x,t) \in C(\overline{R})$, *and let functions* $K_1(x,t)$, $K_2(x,t)$ *in neighborhood point* $(x,t) = (a,a)$ *satisfy conditions* (22.23) *and* (22.24) *at* $\alpha_1+\alpha_2 > |\lambda_2^1|$, $\alpha_j > |\lambda_2^1|(j=1,2,4)$, $K_1(a,a) > 0$, $K_2(a,a) < 0$, $D = (K_1(a,a))^2 - 4K_2(a,a) > 0$, $f(x) \in C[a,b]$, $f(a) = 0$ *with asymptotic behavior* (22.26)*. Then the integral equation* (22.1) *in class of function* $\varphi(x) \in C[a,b]$ *vanishing in point* $x = a$ *is always solvability and its solution is given by formula* (22.35)*, where* C_1, C_3 *are arbitrary constants.*

Remark 22.15. Confirmation is similar to Corollary 22.9 obtained in the case $K_1(a,a) < 0, K_2(a,a) < 0, D > 0$.

Now let $K_1(a,a) > 0, K_2(a,a) > 0, D > 0, f(x) \in C[a,b], f(a) = 0$ with asymptotic behavior

$$f(x) = [(x-a)^{\varepsilon}], \varepsilon > 0 \;\text{ at }\; x \to a. \qquad (22.36)$$

In this case, if corresponding solution to the model of integral equation exists, then this is given by the formula

$$\varphi(x) = F(x) - \frac{1}{\sqrt{p^2-4q}}$$

$$\times \int_a^x \left[\lambda_2^2 \left(\frac{t-a}{x-a}\right)^{\lambda_2} - \lambda_1^2 \left(\frac{t-a}{x-a}\right)^{\lambda_1}\right] \frac{F(t)}{t-a} dt \equiv K_4^-[F(x)], \tag{22.37}$$

where $\lambda_1 = \frac{-K_1(a,a)+\sqrt{D}}{2}$, $\lambda_2 = \frac{-K_1(a,a)-\sqrt{D}}{2}$, $(\lambda_1 < 0, \lambda_2 < 0, \lambda_1 > \lambda_2)$.

Substituting for $F(x)$ from formula (22.37) we arrive at solution of the following integral equation:

$$\varphi(x) + \int_a^x \frac{M_3(x,t)}{t-a} \varphi(t) dt = K_4^-[f(x)], \tag{22.38}$$

where

$$M_3(x,t) = K_1(x,t) - K_1(a,a) + (K_2(x,t) - K_2(a,a)) \ln\left(\frac{x-a}{t-a}\right)$$

$$-\frac{1}{\sqrt{D}} \int_t^x \left[(\lambda_1)^2 \left(\frac{\tau-a}{x-a}\right)^{|\lambda_1|} - (\lambda_2)^2 \left(\frac{\tau-a}{x-a}\right)^{|\lambda_2|}\right]$$

$$\times \left[K_1(\tau,t) - K_1(a,a) + (K_2(\tau,t) - K_2(a,a)) \ln\left(\frac{\tau-a}{t-a}\right)\right] \frac{d\tau}{\tau-a}. \tag{22.39}$$

Multiplying both sides of (22.38) to $(x-a)^{|\lambda_2|}$ and introducing $(x-a)^{|\lambda_2|}\varphi(x) = \psi(x)$, we obtain new integral equation

$$\psi(x) + \int_a^x \frac{N_2(x,t)}{t-a} \psi(t) dt = (x-a)^{|\lambda_2|} K_4^-[f(x)], \tag{22.40}$$

where

$$N_2(x,t) = \left(\frac{x-a}{t-a}\right)^{|\lambda_2|} M_3(x,t).$$

For $N_2(x,t)$, we have the following inequality:

$$|N_2(x,t)| \le E_1(x-a)^{\alpha_1+|\lambda_2|}(t-a)^{\alpha_2-|\lambda_2|} + E_2(x-a)^{\alpha_3+|\lambda_2|}(t-a)^{\alpha_4-|\lambda_2|} \ln\left(\frac{x-a}{t-a}\right)$$

$$+E_3(t-a)^{\alpha_1+\alpha_2} + E_4(x-a)^{|\lambda_2|-|\lambda_1|}(t-a)^{\alpha_3+\alpha_4+|\lambda_1|-|\lambda_2|}$$

$$+E_5(x-a)^{\alpha_3+|\lambda_2^{|}}(t-a)^{\alpha_4-|\lambda_2|} + E_6(t-a)^{\alpha_3+\alpha_4}. \tag{22.41}$$

From here follows, if $\alpha_j > |\lambda_2| (j=2,4), \alpha_3+\alpha_4+|\lambda_1|-|\lambda_2| > 0$, then

$$|N_2(0,t)| \le E_3(t-a)^{\alpha_1+\alpha_2} + E_6(t-a)^{\alpha_3+\alpha_4}, N_2(x,0) = 0.$$

Thus, if fulfilling the following conditions: $\alpha_j > |\lambda_2|(j=2,4), \alpha_3+\alpha_4+|\lambda_1|-|\lambda_2|>0$, then kernel integral equation (22.40) has weak singularity. If $f(a)=0$ with asymptotic behavior (22.36), then right part of integral equation (22.40) vanishes in point $x=a$. Then the integral equation (22.40) has a unique solution, which is given by the formula

$$\psi(x)=(x-a)^{|\lambda_2|}K_4^-[f(x)]-\int_a^x \Gamma_1(x,t)(t-a)^{|\lambda_2|}K_4^-[f(t)]dt,$$

where $\Gamma_1(x,t)$ is resolvent of the integral equation (22.40). Then we determine from the formula that

$$\varphi(x)=K_4^-[f(x)]-\int_a^x \left(\frac{t-a}{x-a}\right)^{|\lambda_2|}\Gamma_1(x,t)K_2^-[f(t)]dt. \tag{22.42}$$

So, we proved the following confirmation:

Theorem 22.16. *Let in* (22.1)*,* $K_1(x,t)\in C(\overline{R})$*,* $K_2(x,t)\in C(\overline{R})$*, functions* $K_1(x,t)$*,* $K_2(x,t)$ *in neighborhood point* $(x,t)=(a,a)$ *satisfy condition (22.23),* (22.24) *at* $\alpha_j>|\lambda_2|(j=2,4), \alpha_3+\alpha_4+|\lambda_1|-|\lambda_2|>0, K_1(a,a)>0$*,* $K_2(a,a)>0$*,* $D=K_1(a,a))^2-4K_2(a,a)>0, f(x)\in C[a,b], f(a)=0$ *with asymptotic behavior* (22.36)*. Then the integral equation* (22.1) *in class of function* $\varphi(x)\in C[a,b]$ *vanishing in point* $x=a$ *has a unique solution which is given by formula* (22.42).

Remark 22.17. Confirmation is similar to Corollaries 22.9–22.11 obtained in the following cases: $K_1(a,a)<0, D=0$; $K_1(a,a)>0, D=0$; $K_1(a,a)<0, D<0$; and $K_1(a,a)>0, D<0$.

22.4 Property of the Solution

Let there be a fulfillment in any condition of Theorem 22.1. Differentiating the solution of the type (22.3), immediate verification, we can easily convince to correctness of the following equality:

$$D_x^a(\varphi(x))=\lambda_1(x-a)^{\lambda_1}C_1+\lambda_2(x-a)^{\lambda_2}C_2+D_x^a(f(x))+|p|f(x)$$
$$-\frac{1}{\sqrt{p^2-4q}}\int_a^x\left[\lambda_2^3\left(\frac{x-a}{t-a}\right)^{\lambda_2}-\lambda_1^3\left(\frac{x-a}{t-a}\right)^{\lambda_1}\right]\frac{f(t)}{t-a}dt, \tag{22.43}$$

where $D_x^a(\varphi(x))=(x-a)\dfrac{d\varphi(x)}{dx}$.

From equality (22.3) and (22.43) we find

$$C_1=\frac{1}{\lambda_2-\lambda_1}\left\{(x-a)^{-\lambda_1}[\lambda_2\varphi(x)-D_x^a(\varphi(x))]\right\}_{x=a}, \tag{22.44}$$

$$C_2=-\frac{1}{\lambda_2-\lambda_1}\left\{(x-a)^{-\lambda_2}[\lambda_1\varphi(x)-D_x^a(\varphi(x))]\right\}_{x=a}. \tag{22.45}$$

From integral representation (22.5) it follows that if all conditions of Theorem 22.2 are fulfilled, then the solution of the type (22.5) has the properties

$$[(x-a)^{-\lambda_1}\varphi(x)]_{x=a} = C_3. \tag{22.46}$$

From integral representation (22.7) it follows that if parameters *pandq* and function $f(x)$ in (22.2) satisfy all condition of Theorem 22.3, then the solution of the type (22.7) has the properties

$$[(x-a)^{-\lambda_1^2}\varphi(x)]_{x=a} = C_4. \tag{22.47}$$

From integral representation (22.11) it follows that

$$D_x^a(\varphi(x)) = (x-a)^{|p|/2}[C_5 + (1+\ln(x-a))C_6] + D_x^a f(x) + |p| f(x)$$
$$+\frac{|p|}{2}\int_a^x \left(\frac{x-a}{t-a}\right)^{|p|/2}\left[2+\frac{|p|}{2}+\frac{|p|}{2}\ln\left(\frac{x-a}{t-a}\right)\right]\frac{f(t)}{t-a}dt. \tag{22.48}$$

Using the formulas (22.11) and (22.48), we easily see that when fulfilling any condition of Theorem 22.5, then solution of the type (22.11) has the following properties:

$$\Big[(x-a)^{-|p|/2}[(1+\ln(x-a))\varphi(x) - \ln(x-a)D_x^a\varphi(x)]\Big]_{x=a} = C_5, \tag{22.49}$$

$$\Big[(x-a)^{-|p|/2}[D_x^a\varphi(x) - \varphi(x)]\Big]_{x=a} = C_6, \tag{22.50}$$

fulfillment of any condition in Theorem 22.7, then from integral representation (22.13) we have

$$D_x^a(\varphi(x)) = (x-a)^{|p|/2}$$
$$\times\left\{\left[\frac{|p|}{2}\cos\left[\frac{\sqrt{4q-p^2}}{2}\ln(x-a)\right] - \frac{\sqrt{4q-p^2}}{2}\sin\left[\frac{\sqrt{4q-p^2}}{2}\ln(x-a)\right]\right]C_7\right.$$
$$\left.\left[\frac{|p|}{2}\sin\left[\frac{\sqrt{4q-p^2}}{2}\ln(x-a)\right] + \frac{\sqrt{4q-p^2}}{2}\cos\left[\frac{\sqrt{4q-p^2}}{2}\ln(x-a)\right]\right]\right\}$$
$$+D_x^a(f(x)) - f(x)$$
$$\frac{1}{\sqrt{4q-p^2}}\int_a^x\left(\frac{x-a}{t-a}\right)^{|p|/2}\left\{\left[p\sqrt{4q-p^2}+\frac{|p|}{2}(p^2-2q)\right]\sin\left[\frac{\sqrt{4q-p^2}}{2}\ln\left(\frac{x-a}{t-a}\right)\right]\right.$$
$$\left.+\frac{\sqrt{4q-p^2}}{2}\left[\frac{3p^2}{2}-2q\right]\cos\left[\frac{\sqrt{4q-p^2}}{2}\ln\left(\frac{x-a}{t-a}\right)\right]\cdot\frac{f(t)}{t-a}dt\right\}. \tag{22.51}$$

Using the formulas (22.13) and (22.51), we observe that, when fulfilling any condition of Theorem 22.7, the solution of the type (22.13) has the following properties:

$$C_7 = \lim_{x\to a} W_3(\varphi),\ C_8 = \lim_{x\to a} W_4(\varphi), \tag{22.52}$$

where

$$W_3(\varphi) = (x-a)^{-|p|/2}\left\{\cos\left[\frac{\sqrt{4q-p^2}}{2}\ln(x-a)\right]\varphi(x)\right.$$

$$\left. -\frac{2}{\sqrt{p^2-4q}}\sin\left[\frac{\sqrt{4q-p^2}}{2}\ln(x-a)\right][D_x^a(\varphi(x))-\varphi(x)]\right\},$$

$$W_4(\varphi) = (x-a)^{-|p|/2}\left\{\sin\left[\frac{\sqrt{4q-p^2}}{2}\ln(x-a)\right]\varphi(x)\right.$$

$$\left. +\frac{2}{\sqrt{p^2-4q}}\cos\left[\frac{\sqrt{4q-p^2}}{2}\ln(x-a)\right]\left[\left[\frac{\sqrt{4q-p^2}}{2}\ln(x-a)\right][D_x^a(\varphi(x))-\varphi(x)]\right]\right\}.$$

22.5 Boundary Value Problems

When the general solution contains arbitrary constants, higher mentioned properties of the solution of the integral equation (22.1) give possibility for integral equation (22.1) and we investigate the following boundary value problems:

Problem N_1. Is required the solution of integral equation (22.1) from class $C[a,b]$ at $p<0,\ \ q>0,\ p^2-4q>0$ by boundary conditions

$$\begin{cases} \left[(x-a)^{-\lambda_1}[\lambda_2\varphi(x)-D_x^a(\varphi(x))]\right]_{x=a} = A \\ \left[(x-a)^{-\lambda_2}[-\lambda_1\varphi(x)+D_x^a(\varphi(x))]\right]_{x=a} = B \end{cases} \tag{22.53}$$

where A,B are given constants.

Problem N_2. Is required the solution of integral equation (22.1) from class $C[a,b]$ at $p>0,\ q<0,\ p^2-4q>0$ by boundary conditions

$$\left[(x-a)^{-\lambda_1^1}\varphi(x)\right]_{x=a} = A_1, \tag{22.54}$$

where A_1 is a given constant and $\lambda_1^1 = \dfrac{-p+\sqrt{p^2+4|q|}}{2}$.

Problem N_3. Is required the solution of integral equation (22.1) from class $C[a,b]$ at $p<0,\ q<0$ by boundary conditions

$$\left[(x-a)^{-\lambda_1^2}\varphi(x)\right]_{x=a} = B_1, \tag{22.55}$$

where B_1 is a given constant and $\lambda_1^2 = \frac{|p|+\sqrt{p^2+4|q|}}{2}$.

Problem N_4. Is required the solution of integral equation (22.1) from class $C[a,b]$ at $p<0,\ p^2=4q$ by boundary conditions

$$\left[(x-a)^{|p|/2}\left[(1+\ln(x-a))\varphi(x)-\ln(x-a)D_x^a\varphi(x)\right]\right]_{x=a}=A_2, \quad (22.56)$$

$$\left[(x-a)^{|p|/2}\left[D_x^a\varphi(x)-\varphi(x)\right]\right]_{x=a}=B_2, \quad (22.57)$$

where A_2, B_2 are given constants.

Problem N_5. Is required the solution of integral equation (22.1) from class $C[a,b]$ at $p<0$, $p^2<4q$ by boundary conditions

$$[W_3(\varphi)]_{x=a}=A_3,\ [W_4(\varphi)]_{x=a}=B_3, \quad (22.58)$$

where A_3, B_3 are given constants.

Solution to Problem N_1. Let there be a fulfillment in any condition of Theorem 22.1. Then using the solution of the type (22.3) and its properties (22.44) and (22.45) and condition (22.53) we have

$$C_1=\frac{A}{\lambda_2-\lambda_1},\ C_2=\frac{A}{\lambda_1-\lambda_2}.$$

Substituting obtained valued C_1 and C_2 in formula (22.3) we find the solution of Problem N_1 in the form

$$\varphi(x)=K_1^-\left[\frac{A}{\lambda_2-\lambda_1},\frac{A}{\lambda_1-\lambda_2},f(x)\right] \quad (22.59)$$

So, the proof is completed.

Theorem 22.18. *Let in integral equation* (22.2) *parameters* p *and* q*and function* $f(x)$ *satisfy any condition of Theorem 22.1. Then, Problem* N_1 *has a unique solution which is given by formula* (22.59).

Solution to Problem N_2. Let there be a fulfillment in any condition of Theorem 22.2. Then using the solution of the type (22.5) and its properties (22.46) and condition (22.54) we have $C_3=A_1$. Substituting this value C_3 in formula (22.5), we find the solution of Problem N_2 in the form

$$\varphi(x)=K_2^-[A_1,f(x)] \quad (22.60)$$

So, we prove it.

Theorem 22.19. *Let in integral equation* (22.2) *parameters* p *and* q*, function* $f(x)$ *satisfy any condition of Theorem 22.2. Then Problem* N_2 *has a unique solution which is given by formula* (22.60).

Solution to Problem N_3. Let there be a fulfillment in any condition of Theorem 22.3. Then, using the solution of the type (22.7) and its properties (22.47) and condition (22.55), we have $C_4=B_1$. Substituting this value C_4 in formula (22.7), we find the solution of Problem N_3 in the form

$$\varphi(x) = K_3^-[B_1, f(x)], \tag{22.61}$$

whence the result.

Theorem 22.20. *Let in integral equation* (22.2) *parameters p and q, function $f(x)$ satisfy any condition of Theorem* 22.3. *Then, Problem N_3 has a unique solution which is given by formula* (22.61).

Solution to Problem N_4. Let there be a fulfillment in any condition of Theorem 22.5. Then, using the solution of the type (22.11) and its properties (22.49),(22.50) and conditions (22.56), (22.57), we have $C_5 = A_2, C_6 = B_2$. Substituting these values C_5, C_6 in formula (22.11), we find the solution of Problem N_4 in the form

$$\varphi(x) = K_5^-[A_2, B_2, f(x)]. \tag{22.62}$$

Theorem 22.21. *Let in integral equation* (22.2) *parameters p and q, function $f(x)$ satisfy any condition of Theorem* 22.5. *Then, Problem N_4 has a unique solution which is given by formula* (22.62).

Solution to Problem N_5. Let there be a fulfillment in any condition of Theorem 22.7. Then using the solution of the type (22.13), its properties (22.52), and conditions (22.58), we have $C_7 = A_3, C_8 = B_3$. Substituting the values C_7 and C_8 in formula (22.13), we find the solution of Problem N_5 in the form

$$\varphi(x) = K_7^-[A_3, B_3, f(x)]. \tag{22.63}$$

Theorem 22.22. *Let in integral equation* (22.2) *parameters p and q, function $f(x)$ satisfy any condition of Theorem* 22.7. *Then Problem N_5 has a unique solution which is given by formula (22.63).*

22.6 Presentation the Solution of the Integral Equation (22.2) in the Generalized Power Series

Suppose that $f(x)$ has a uniformly convergent power series expansion on Γ :

$$f(x) = \sum_{k=0}^{\infty} (x-a)^{k+\gamma} f_k, \tag{22.64}$$

where $\gamma = constant > 0$ and $f_k, k = 0, 1, 2, \cdots$, are given constants. We attempt to find a solution of (22.2) in the form

$$\varphi(x) = \sum_{k=0}^{\infty} (x-a)^{k+\gamma} \varphi_k, \tag{22.65}$$

where the coefficients $\varphi_k (k = 0, 1, 2, \cdots)$ are unknown.

Substituting power series representations of $f(x)$ and $\varphi(x)$ into (22.2), equating the coefficients of the corresponding functions, and solving for φ_k, we obtain

$$\varphi_k = \frac{(k+\gamma)^2}{(k+\gamma)^2 + p(k+\gamma) + q} f_k, k = 0, 1, 2, \dots \tag{22.66}$$

If $(k+\gamma)^2 + p(k+\gamma) + q \neq 0$ for in all $k = 0, 1, 2, 3, \cdots$ putting the found coefficients back into (22.65), we arrive at the particular solution of (22.2)

$$\varphi(x) = \sum_{k=0}^{\infty} (x-a)^{k+\gamma} \varphi_k = \sum_{k=0}^{\infty} (x-a)^{k+\gamma} \frac{(k+\gamma)^2}{(k+\gamma)^2 + p(k+\gamma) + q} f_k. \tag{22.67}$$

If, for some values $k = k_1$ and $k = k_2$, constants γ, p, q satisfy

$$k_1 = -\gamma + \frac{-p + \sqrt{p^2 - 4q}}{2},$$

$$k_1 = -\gamma - \frac{p + \sqrt{p^2 - 4q}}{2}.$$

then the solution to integral equation (22.2) can be represented in form (22.64); it is necessary and sufficient that $f_{k_j} = 0,\ j = 1, 2$, that is, it is necessary and sufficient that function $f(x)$ in point $x = a$ satisfies the following two solvability conditions:

$$\left[[(x-a)^{-\gamma} f(x)]^{(k_j)} \right]_{x=a} = 0, j = 1, 2. \tag{22.68}$$

In this case the solution of the integral equation (22.1) in the class of function can be represented in form (22.2) is given by the formula

$$\begin{aligned}\varphi(x) = &\sum_{k=0}^{k_1-1} (x-a)^{k+\gamma} \frac{(k+\gamma)^2}{(k+\gamma)^2 + p(k+\gamma) + q} f_k \\ &+ \sum_{k=k_1+1}^{k_2-1} (x-a)^{k+\gamma} \frac{(k+\gamma)^2}{(k+\gamma)^2 + p(k+\gamma) + q} f_k \\ &+ \sum_{k=k_2+1}^{\infty} (x-a)^{k+\gamma} \frac{(k+\gamma)^2}{(k+\gamma)^2 + p(k+\gamma) + q} f_k \\ &+ \varphi_{k_1} (x-a)^{k_1} + \varphi_{k_2} (x-a)^{k_2},\end{aligned} \tag{22.69}$$

where $\varphi_{k_1}, n\ \varphi_{k_2}$ are arbitrary constants.

Immediately testing it we see that if convergence radius of the series (22.64) is defined by formula $R = \frac{1}{l}, l = \lim_{n\to\infty} \frac{|f_{n+1}|}{|f_n|}$, then convergence radius of the series (22.67) and (22.69) is also defined by this formula. So, we prove the next result.

Theorem 22.23. *Let in integral equation* (22.2)*, function* $f(x)$ *represents in form uniformly converges generalized power-series type (22.64)) and* $(k+\gamma)^2+p(k+\gamma)+q\neq 0$ *for* $k=0,1,2,3,\cdots$*. Then, the integral equation* (22.2) *in class of function* $\varphi(x)$ *represented in form* (22.65) *has a unique solution, which is given by formula* (6.5)*. For values* $k=k_j$*,* $j=1,2,(k_j+\gamma)^2+p(k_j+\gamma)+q=0$*, the existence of the solution of* (22.2) *can be represented in form* (22.65)*; it is necessary and a sufficient fulfillment of two solvability condition types* (22.69)*. In this case integral equation (22.2) in class function represented in form* (22.65) *is always solvability and its general solution contains two arbitrary constants and is given by formula* (22.69)*.*

22.7 Conjugate Integral Equation

Integral equation type

$$\psi(x)+\frac{1}{x-a}\int_x^b\left[p+q\ln\left(\frac{t-a}{x-a}\right)\right]\psi(t)dt=g(x), \tag{22.70}$$

where $g(x)$ – are given function, will be conjugate integral equation for (22.1). For (22.70) we have the following confirmation:

Theorem 22.24. *Let in integral equation* (22.70)*,* $p<0$*,* $q>0$*,* $p^2>4q$*, and let* $g(x)\in C[a,b]$*. Then the integral equation* (22.70) *in class of function* $\psi(x)\in C(a,b)$ *has a unique solution, which is given by the formula*

$$\psi(x)=g(x)-\frac{1}{\sqrt{p^2-4q}(x-a)}\int_x^b\left[\mu_1^2\left(\frac{t-a}{x-a}\right)^{|\mu_1|}-\mu_2^2\left(\frac{t-a}{x-a}\right)^{|\mu_2|}\right]g(t)dt$$

$$\equiv K_1^-[f(x)], \tag{22.71}$$

where $\mu_1=\dfrac{-|p|+\sqrt{p^2-4q}}{2}$*,* $\mu_2=\dfrac{-|p|-\sqrt{p^2-4q}}{2}$*. Moreover this solution in point* $x=a$ *turns into infinity with following asymptotic behavior:*

$$\psi(x)=O\left[(x-a)^{-(|\mu_2|+1)}\right] \text{ at } x\to a. \tag{22.72}$$

Let in integral equation (22.70), $p>0,q<0,p^2>4q$. In this case, at $x\to a$ the integral in right part of formula (22.71)) converges if $g(a)=0$ with following asymptotic behavior:

$$g(x)=o\left[(x-a)^{\delta_7}\right], \delta_7>\mu_1-1 \text{ at } x\to a. \tag{22.73}$$

Then, $\mu_1 = \dfrac{p+\sqrt{p^2+4|q|}}{2} > 1,\quad \mu_2 = \dfrac{p-\sqrt{p^2+4|q|}}{2} < 0$, and solution of the type (22.71) may be written in the form

$$\psi(x) = g(x) - \frac{1}{\sqrt{p^2-4q}(x-a)} \int_x^b \left[\mu_1^2 \left(\frac{x-a}{t-a}\right)^{|\mu_1|} - \mu_2^2 \left(\frac{t-a}{x-a}\right)^{|\mu_2|} \right] g(t)dt. \tag{22.74}$$

Then it follows that $\psi(a) = 0$ with following asymptotic behavior:

$$\psi(x) = o\left[(x-a)^{(|\mu_2|-1)}\right] \text{ at } x \to a. \tag{22.75}$$

If in (22.70) $p < 0,\ q < 0,\ p^2 + 4|q| > 0$, and if the solution of the integral equation (22.70) exists, then it may be represented in form (22.74), where $\mu_1 = \dfrac{-|p|+\sqrt{p^2+4|q|}}{2} > 0,\quad \mu_2 = -\left(\dfrac{|p|+\sqrt{p^2+4|q|}}{2}\right) < 0$. The integral in right part of expression (22.74) converges if $g(a) = 0$ with asymptotic behavior (22.73). In this case, $\psi(a) = \infty$, with asymptotic behavior (22.75).

If $p > 0, q > 0, p^2 - 4q > 0$, and if the solution of (22.70) exists, then it may be represented in the form

$$\psi(x) = g(x) - \frac{1}{\sqrt{p^2-4q}(x-a)}$$

$$\times \int_x^b \left[\mu_1^2 \left(\frac{x-a}{t-a}\right)^{\mu_1} - \mu_2^2 \left(\frac{x-a}{t-a}\right)^{\mu_2} \right] g(t)dt,\ \mu_1 > \mu_2, \tag{22.76}$$

where $\mu_1 = \dfrac{p+\sqrt{p^2-4q}}{2} > 0,\ \ \mu_2 = \dfrac{p-\sqrt{p^2-4q}}{2} > 0$, and $\mu_1 > \mu_2$. The integral in right part of expression (22.76) converges if $g(a) = 0$ with asymptotic behavior

$$g(x) = o\left[(x-a)^{\delta_8}\right],\ \delta_8 > \mu_2 - 1 \text{ at } x \to a. \tag{22.77}$$

In this case $\psi(a) = \infty$ with asymptotic behavior

$$\psi(x) = O\left[(x-a)^{-(\mu_2+1)}\right] \text{ at } x \to a.$$

Remark 22.25. For conjugate equation (22.70), confirmation is similar to Theorem 22.7 obtained in the case $p^2 - 4q = 0$ and $p^2 - 4q < 0$.

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Chapter 23
Fractional Integration of the Product of Two Multivariables H-Function and a General Class of Polynomials

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Abstract A significantly large number of earlier works on the subject of fractional calculus give interesting account of the theory and applications of fractional calculus operators in many different areas of mathematical analysis (such as ordinary and partial differential equations, integral equations, special functions, summation of series, etc.). The main object of the present paper is to study and develop the Saigo operators. First, we establish two results that give the images of the product of two multivariables H-function and a general class of polynomials in Saigo operators. On account of the general nature of the Saigo operators, multivariable H-functions and a general class of polynomials a large number of new and Known Images involving Riemann-Liouville and Erde'lyi-Kober fractional integral operators and several special functions notably generalized Wright hypergeometric function, Mittag-Leffler function, Whittaker function follow as special cases of our main findings. Results given by Kilbas, Kilbas and Sebastian, Saxena et al. and Gupta et al., follow as special cases of our findings.

23.1 Introduction

The fractional integral operator involving various special functions has found significant importance and applications in various subfields of applicable mathematical analysis. Since last four decades, a number of workers like Love [13], McBride [15], Kalla [3, 4], Kalla and Saxena [5, 6], Saxena et al. [22], Saigo [18–20], Kilbas [7], Kilbas and Sebastian [9] and Kiryakova [11, 12] have studied in depth the properties, applications and different extensions of various hypergeometric operators of fractional integration. A detailed account of such operators along

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with their properties and applications can be found in the research monographs by Smako, Kilbas and Marichev [21], Miller and Ross[16]; Kiryakova [11, 12], Kilbas, Srivastava and Trujillo [10] and Debnath and Bhatta [1].

A useful generalization of the hypergeometric fractional integrals, including the Saigo operators [18–20], has been introduced by Marichev [14] (see details in Samko et al. [21] and also see Kilbas and Saigo[8])as follows:

Let α, β, η be complex numbers and x>0, then the generalized fractional integral operators (the Saigo operators [18]) involving Gaussian hypergeometric function are defined by the following equations:

$$\left(I_{0^+}^{\alpha,\beta,\eta} f\right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\, {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x}\right) f(t)\,dt, \quad (Re(\alpha)>0) \tag{23.1}$$

and

$$\left(I_{-}^{\alpha,\beta,\eta} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta}\, {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t}\right) f(t)\,dt, \quad (Re(\alpha)>0), \tag{23.2}$$

where ${}_2F_1(.)$ is the Gaussian hypergeometric function defined by:

$${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}. \tag{23.3}$$

When $\beta = -\alpha$, equations (23.1) and (23.2) reduce to the following classical Riemann–Liouville fractional integral operator (see Samko et al. [21], p. 94, (5.1), (5.3)):

$$\left(I_{0^+}^{\alpha,-\alpha,\eta} f\right)(x) = \left(I_{0^+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\,dt, \ (x>0) \tag{23.4}$$

and

$$\left(I_{-}^{\alpha,-\alpha,\eta} f\right)(x) = \left(I_{-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t)\,dt, \ (x>0). \tag{23.5}$$

Again, if $\beta = 0$, Equations (23.1) and (23.2) reduce to the following Erde'lyi–Kober fractional integral operator (see Samko et al. [21], p.322, Eqns. (18.5), (18.6)):

$$\left(I_{0^+}^{\alpha,0,\eta} f\right)(x) = \left(I_{\eta,\alpha}^{+} f\right)(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^{\eta} f(t)\,dt, \ (x>0) \tag{23.6}$$

and

$$\left(I_{-}^{\alpha,0,\eta} f\right)(x) = \left(K_{\eta,\alpha}^{-} f\right)(x) = \frac{x^{\eta}}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\eta} f(t)\,dt, \ (x>0) \tag{23.7}$$

Recently, Gupta et al. [2] have obtained the images of the product of two H-functions in Saigo operator given by (23.1) and (23.2) and thereby generalized

several important results obtained earlier by Kilbas, Kilbas and Sebastian and Saxena et al. as mentioned in this paper cited above. It has recently become a subject of interest for many researchers in the field of fractional calculus and its applications. Motivated by these avenues of applications, a number of workers have made use of the fractional calculus operators to obtain the image formulas. The aim of this paper is to obtain two results that give the images of the product of two multivariables H-function and a general class of polynomials in Saigo operators.
The H-function of several variables is defined and represented as follows (Srivastava et al. [23], pp. 251–252, (C.1)– (C.3)):

$$H[z_1,\ldots,z_r] \equiv H^{0,N:M_1,N_1;\ldots;M_r,N_r}_{P,Q:P_1,Q_1;\ldots;P_r,Q_r}\left[\begin{array}{c|c} z_1 & (a_j;\alpha'_j,\ldots,\alpha^{(r)}_j)_{1,p}:(c'_j,\gamma'_j)_{1,p_1};\ldots;(c^{(r)}_j,\gamma^{(r)}_j)_{1,p_r} \\ \vdots & \\ z_r & (b_j;\beta'_j,\ldots,\beta^{(r)}_j)_{1,q}:(d'_j,\delta'_j)_{1,q_1};\ldots;(f^{(r)}_j,F^{(r)}_j)_{1,q_r} \end{array}\right]$$

$$= \left(\frac{1}{2\pi i}\right)^r \int_{L_1}\cdots\int_{L_r} \phi_1(\xi_1)\ldots\phi_r(\xi_r)\,\psi(\xi_1,\ldots\xi_r)\,z_1^{\xi_1}\ldots z_r^{\xi_r}\,d\xi_1\ldots d\xi_r, \tag{23.8}$$

where

$$\phi_i(\xi_i) = \frac{\prod_{j=1}^{M_i}\Gamma(d^{(i)}_j-\delta^{(i)}_j\xi_i)\prod_{j=1}^{N_i}\Gamma(1-c^{(i)}_j+\gamma^{(i)}_j\xi_i)}{\prod_{j=M_i+1}^{Q_i}\Gamma(1-d^{(i)}_j+\delta^{(i)}_j\xi_i)\prod_{j=N_i+1}^{P_i}\Gamma(c^{(i)}_j-\gamma^{(i)}_j\xi_i)}, \tag{23.9}$$

$$\psi(\xi_1,\ldots,\xi_r) = \frac{\prod_{j=1}^{N}\Gamma(1-a_j+\sum_{i=1}^{r}\alpha^{(i)}_j\xi_i)}{\prod_{j=1}^{Q}\Gamma(1-b_j+\sum_{i=1}^{r}\beta^{(i)}_j\xi_i)\prod_{j=N+1}^{P}\Gamma(a_j-\sum_{i=1}^{r}\alpha^{(i)}_j\xi_i)}, \quad (\forall i\in\{1,\ldots,r\}). \tag{23.10}$$

It is assumed that the various H-functions of several variables occurring in this paper always satisfy the appropriate existence and convergence conditions corresponding appropriately to those recorded in the book by Srivastava et al. [23, pp. 251–253, (C.4)–(C.6)]. In case $r=2$, (23.8) reduces to the H-function of two variables (Srivastava et al.) ([23], p. 82, (6.1.1)).

Also, $S^m_n[x]$ occurring in the sequel denotes the general class of polynomials introduced by Srivastava ([24], p. 1, (1)):

$$S^m_n[x] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!}A_{n,k}x^k, n=0,1,2,\ldots, \tag{23.11}$$

where m is an arbitrary positive integer and the coefficients $A_{n,k}\,(n,k\geq 0)$ are arbitrary constants, real or complex. On suitably specializing the coefficients $A_{n,k}$, $S^m_n[x]$ yields a number of known polynomials as its special cases. These include, among others, the Hermite polynomials, the Jacobi polynomials, the Laguerre polynomials, the Bessel polynomials, the Gould–Hopper polynomials and the Brafman polynomials; (see Srivastava and Singh ([25], pp. 158–161)).

23.2 Preliminary Lemmas

The following lemmas will be required to establish our main results:

Lemma 23.1 (Kilbas and Sebastain [9], p. 871, (15)–(18)). *Let* $\alpha,\beta,\eta\in\mathbb{C}$ *be such that* $Re(\alpha)>0$ *and* $Re(\mu)>\max\{0,Re(\beta-\eta)\}$*); then, there holds the following relation:*

$$\left(I_{0^+}^{\alpha,\beta,\eta}t^{\mu-1}\right)(x)=\frac{\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu+\alpha+\eta)\Gamma(\mu-\beta)}x^{\mu-\beta-1}. \tag{23.12}$$

In particular, if $\beta=-\alpha$ *and* $\beta=0$ *in* (23.12)*, we have*

$$\left(I_{0^+}^{\alpha}t^{\mu-1}\right)(x)=\frac{\Gamma(\mu)}{\Gamma(\mu+\alpha)}x^{\mu+\alpha-1},\ Re(\alpha)>0,\ Re(\mu)>0, \tag{23.13}$$

$$\left(I_{\eta,\alpha}^{+}t^{\mu-1}\right)(x)=\frac{\Gamma(\mu+\eta)}{\Gamma(\mu+\alpha+\eta)}x^{\mu-1},\ Re(\alpha)>0, Re(\mu)>-Re(\eta). \tag{23.14}$$

Lemma 23.2 (Kilbas and Sebastain [9], p. 872, (21)–(24)). *Let* $\alpha,\beta,\eta\in\mathbb{C}$ *be such that* $Re(\alpha)>0)$ *and* $Re(\mu)<1+\min\{Re(\beta),Re(\eta)\}$*); then, there holds the following relation:*

$$\left(I_{-}^{\alpha,\beta,\eta}t^{\mu-1}\right)(x)=\frac{\Gamma(\beta-\mu+1)\Gamma(\eta-\mu+1)}{\Gamma(1-\mu)\Gamma(\alpha+\beta+\eta-\mu+1)}x^{\mu-\beta-1}. \tag{23.15}$$

In particular, if $\beta=-\alpha$ *and* $\beta=0$ *in* (23.15)*, author has*

$$\left(I_{-}^{\alpha}t^{\mu-1}\right)(x)=\frac{\Gamma(1-\alpha-\mu)}{\Gamma(1-\mu)}x^{\mu+\alpha-1},\ 1-Re(\mu)>Re(\alpha)>0, \tag{23.16}$$

$$\left(K_{\eta,\alpha}^{-}t^{\mu-1}\right)(x)=\frac{\Gamma(\eta-\mu+1)}{\Gamma(1-\mu+\alpha+\eta)}x^{\mu-1},\ Re(\mu)<1+Re(\eta). \tag{23.17}$$

23.3 Main Results

Image 1:

$$\begin{aligned}&\Big\{I_{0^+}^{\alpha,\beta,\eta}\Big(t^{\mu-1}(b-at)^{-\nu}\textstyle\prod_{j=1}^{s}S_{n_j}^{m_j}\left[c_jt^{\lambda_j}(b-at)^{-\delta_j}\right]\\&\times H\left[z_1t^{\sigma_1}(b-at)^{-\omega_1}\dots z_rt^{\sigma_r}(b-at)^{-\omega_r}\right]\\&\times H\left[z_1't^{\sigma_1'}(b-at)^{-\omega_1'}\dots z_l't^{\sigma_l'}(b-at)^{-\omega_l'}\right]\Big)\Big\}(x)\end{aligned}$$

$$= b^{-\nu} x^{\mu-\beta-1} \sum_{k_1=0}^{[n_1/m_1]} \dots \sum_{k_s=0}^{[n_s/m_s]} \frac{(-n_1)_{m_1 k_1} \dots (-n_s)_{m_s k_s}}{k_1! \dots k_s!}$$
$$\times A'_{n_1,m_1} \dots A^{(s)}_{n_s,m_s} c_1^{k_1} \dots c_s^{k_s} (b)^{-\sum_{j=1}^{s} \delta_j k_j} (x)^{\sum_{j=1}^{s} \lambda_j k_j}$$
$$\times H^{0,N+N'+3: \; M_1,N_1;\dots;M_r,N_r;M'_1,N'_1;\dots;M'_l,N'_l;1,0}_{P+P'+3,Q+Q'+3: \; P_1,Q_1;\dots;P_r,Q_r;P'_1,Q'_1;\dots;P'_l,Q'_l;0,1} \left[\begin{array}{c|c} z_1 \frac{x^{\sigma_1}}{b^{\omega_1}} & \\ \vdots & \\ z_r \frac{x^{\sigma_r}}{b^{\omega_r}} & A : C \\ z_1 \frac{x^{\sigma'_1}}{b^{\omega'_1}} & B : D \\ \vdots & \\ z_r \frac{x^{\sigma'_l}}{b^{\omega'_l}} & \\ -\frac{a}{b} x & \end{array} \right] \tag{23.18}$$

where

$$A = \left(1 - \nu - \sum_{j=1}^{s} \delta_j k_j; \omega_1, \dots, \omega_r, \omega'_1, \dots, \omega'_l, 1\right),$$
$$\left(1 - \mu - \sum_{j=1}^{s} \lambda_j k_j; \sigma_1, \dots, \sigma_r, \sigma'_1, \dots, \sigma'_l, 1\right),$$
$$\left(1 - \mu - \eta + \beta - \sum_{j=1}^{s} \lambda_j k_j; \sigma_1, \dots, \sigma_r, \sigma'_1, \dots, \sigma'_l, 1\right),$$
$$(a_j; \alpha'_j, \dots, \alpha_j^{(r)}, \underbrace{0, \dots, 0}_{l}, 0)_{1,P}, \; (A_j; \underbrace{0, \dots, 0}_{r} B'_j, \dots, B_j^{(l)}, 0)_{1,P'}$$
$$B = \left(1 - \nu - \sum_{j=1}^{s} \delta_j k_j; \omega_1, \dots, \omega_r, \omega'_1, \dots, \omega'_l, 0\right), \tag{23.19}$$
$$\left(1 - \mu + \beta - \sum_{j=1}^{s} \lambda_j k_j; \sigma_1, \dots, \sigma_r, \sigma'_1, \dots, \sigma'_l, 1\right)$$
$$(C_j; \underbrace{0, \dots, 0}_{r}, D'_j, \dots, D_j^{(l)}, 0)_{1,Q'}$$
$$C = (c'_j, \gamma'_j)_{1,P_1}; \dots; (c_j^{(r)}, \gamma_j^{(r)})_{1,P_r}; \; (C'_j, E'_j)_{1,P'_1}; \dots; \; (C_j^{(l)}, E_j^{(l)})_{1,P'_l}; --$$
$$D = (d'_j, \delta'_j)_{1,Q_1}; \dots; (d_j^{(r)}, \delta_j^{(r)})_{1,Q_r};$$
$$(G'_j, H'_j)_{1,Q'_1}; \dots; (G_j^{(l)}, H_j^{(l)})_{1,Q'_l}; (0,1)$$

The sufficient conditions of validity of (23.18) are the following:

(i) $\alpha, \beta, \eta, \mu, \nu, \delta_j, \omega_i, \omega'_k, a, b, c, z_i, z'_k \in C$ and $\lambda_j, \sigma_i, \sigma'_k > 0 \; \forall i \in \{1, \dots, r\}, k \in \{1, \dots, l\}$ and $j \in \{1, \dots, s\}$.

(ii) $|\arg z_i| < \frac{1}{2}\Omega_i \pi$ and $\Omega_i > 0; |\arg z_i| < \frac{1}{2}\Omega'_i \pi$ and $\Omega'_i > 0$, where $\Omega_i = -\sum_{j=N+1}^{P} \alpha_j^{(i)} - \sum_{j=1}^{Q} \beta_j^{(i)} + \sum_{j=1}^{N_i} \gamma_j^{(i)} - \sum_{j=N_i+1}^{P_i} \gamma_j^{(i)} + \sum_{j=1}^{M_i} \delta_j^{(i)} - \sum_{j=M_i+1}^{Q_i} \delta_j^{(i)}$; $\forall i \in \{1,\ldots,r\}$ Ω'_i defined as similar to Ω_i.

(iii) $Re(\alpha) > 0$ and

$$Re(\mu) + \sum_{i=1}^{r} \sigma_i \min_{1\le j\le M_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) + \sum_{k=1}^{l} \sigma'_k \min_{1\le j\le M'_k} Re\left(\frac{G_j^{(k)}}{H_j^{(k)}}\right) > \max\{0, Re(\beta-\eta)\}$$

$$Re(\nu) + \sum_{i=1}^{r} \omega_i \min_{1\le j\le M_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) + \sum_{k=1}^{l} \omega'_k \min_{1\le j\le M'_k} Re\left(\frac{G_j^{(k)}}{H_j^{(k)}}\right) > \max\{0, Re(\beta-\eta)\}.$$

(iv) $\left|\frac{a}{b}x\right| < 1$.

Proof. In order to prove (23.18), we first express the product of a general class of polynomials occurring on its left-hand side in the series form given by (23.11), replace both multivariable H-functions occurring therein by its well-known Mellin–Barnes contour integral given by (23.8), interchange the order of summations $(\xi_1,\ldots,\xi_r)$ and $\left(\xi'_1,\ldots,\xi'_l\right)$ integrals, respectively, and taking the fractional integral operator inside (which is permissible under the conditions stated) and make a little simplification. Next, we express the terms $(b-ax)^{-(\nu+\sum_{j=1}^{s}\delta_j k_j+\sum_{i=1}^{r}\omega_i\xi_i+\sum_{k=1}^{l}\omega'_k\xi'_k)}$ in the terms of Mellin–Barnes contour integral (Srivastava et al. [23], 94 p. 18, (2.6.3); p. 10, (2.1.1)) and it takes the following form (Say I) after a little simplification:

$$\begin{aligned} I = (b)^{-\nu} &\sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_s=0}^{[n_s/m_s]} \frac{(-n_1)_{m_1k_1}\ldots(-n_s)_{m_sk_s}}{k_1!\ldots k_s!} \\ &\times A'_{n_1,m_1}\ldots A^{(s)}_{n_s,m_s} {c_1}^{k_1}\ldots {c_s}^{k_s}(b)^{-\sum_{j=1}^{s}\delta_j k_j} \\ &\frac{1}{(2\pi i)^{r+l+1}} \int_{L_1}\cdots\int_{L_r}\int_{L'_1}\cdots\int_{L'_l} \psi(\xi_1,\ldots,\xi_r)\psi'\left({\xi'}_1,\ldots,{\xi'}_l\right) \\ &\times \prod_{i=1}^{r}\phi_i(\xi_i) z_i^{\xi_i} \prod_{k=1}^{l} {\phi'}_k\left({\xi'}_k\right) {z'_k}^{{\xi'}_k}(b)^{-(\sum_{i=1}^{r}\omega_i\xi_i+\sum_{k=1}^{l}{\omega'}_k{\xi'}_k)} \\ &\times \int_L \frac{\Gamma\left(\nu+\sum_{j=1}^{s}\delta_j k_j+\sum_{i=1}^{r}\omega_i\xi_i+\sum_{k=1}^{l}{\omega'}_k{\xi'}_k+\xi\right)}{\Gamma\left(\nu+\sum_{j=1}^{s}\delta_j k_j+\sum_{i=1}^{r}\omega_i\xi_i+\sum_{k=1}^{l}{\omega'}_k{\xi'}_k\right)\Gamma(1+\xi)}\left(-\frac{a}{b}\right)^{\xi} \\ &\times d\xi \left(I_{0^+}^{\alpha,\beta,\eta} t^{\mu+\sum_{j=1}^{s}\lambda_j k_j+\sum_{i=1}^{r}\sigma_i\xi_i+\sum_{k=1}^{l}{\sigma'}_k{\xi'}_k+\xi-1}\right)(x). \end{aligned} \tag{23.20}$$

Finally, applying the Lemma 23.1 and reinterpreting the Mellin–Barnes contour integral thus obtain in terms of the multivariable H-function defined by (23.8), we arrive at the right-hand side of (23.18) after a little simplification. □

If we put $\beta = -\alpha$ in Image 1, we arrive at the following new and interesting corollary concerning Riemann–Liouville fractional integral operator defined by (23.4) and using (23.13):

Corollary 23.3.

$$\Bigg\{ I_{0+}^{\alpha}\Bigg(t^{\mu-1}(b-at)^{-\nu}\prod_{j=1}^{s} S_{n_j}^{m_j}\left[c_j t^{\lambda_j}(b-at)^{-\delta_j}\right]$$
$$\times H\left[z_1 t^{\sigma_1}(b-at)^{-\omega_1}\ldots z_r t^{\sigma_r}(b-at)^{-\omega_r}\right]$$
$$\times H\left[z'_1 t^{\sigma'_1}(b-at)^{-\omega'_1}\ldots z'_l t^{\sigma'_l}(b-at)^{-\omega'_l}\right]\Bigg)\Bigg\}(x)$$
$$= b^{-\nu}x^{\mu+\alpha-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}\ldots(-n_s)_{m_sk_s}}{k_1!\ldots k_s!}$$
$$\times A'_{n_1,m_1}\ldots A^{(s)}_{n_s,m_s} c_1^{k_1}\ldots c_s^{k_s}(b)^{-\sum_{j=1}^{s}\delta_j k_j}(x)^{\sum_{j=1}^{s}\lambda_j k_j} \tag{23.21}$$
$$H^{0,N+N'+2:\;M_1,N_1;\ldots;M_r,N_r;M'_1,N'_1;\ldots;M'_l,N'_l;1,0}_{P+P'+2,Q+Q'+2:\;P_1,Q_1;\ldots;P_r,Q_r;P'_1,Q'_1;\ldots;P'_l,Q'_l;0,1}\left[\begin{array}{c|c} z_1\frac{x^{\sigma_1}}{b^{\omega_1}} & \\ \vdots & \\ z_r\frac{x^{\sigma_r}}{b^{\omega_r}} & A':C \\ z_1\frac{x^{\sigma'_1}}{b^{\omega'_1}} & B':D \\ \vdots & \\ z_r\frac{x^{\sigma'_l}}{b^{\omega'_l}} & \\ -\frac{a}{b}x & \end{array}\right]$$

where

$$A' = \left(1-\nu-\textstyle\sum_{j=1}^{s}\delta_j k_j;\omega_1,\ldots,\omega_r,\omega'_1,\ldots,\omega'_l,1\right),$$
$$\left(1-\mu-\textstyle\sum_{j=1}^{s}\lambda_j k_j;\sigma_1,\ldots,\sigma_r,\sigma'_1,\ldots,\sigma'_l,1\right),$$
$$(a_j;\alpha'_j,\ldots,\alpha_j^{(r)},\underbrace{0,\ldots,0}_{l},0)_{1,P},(A_j;\underbrace{0,\ldots,0}_{r}B'_j,\ldots,B_j^{(l)},0)_{1,P'}$$
$$B' = \left(1-\nu-\textstyle\sum_{j=1}^{s}\delta_j k_j;\omega_1,\ldots,\omega_r,\omega'_1,\ldots,\omega'_l,0\right),$$
$$\left(1-\mu-\alpha-\textstyle\sum_{j=1}^{s}\lambda_j k_j;\sigma_1,\ldots,\sigma_r,\sigma'_1,\ldots,\sigma'_l,1\right), \tag{23.22}$$
$$\left(1-\mu-\alpha-\eta-\textstyle\sum_{j=1}^{s}\lambda_j k_j;\sigma_1,\ldots,\sigma_r,\sigma'_1,\ldots,\sigma'_l,1\right),$$
$$(b_j;\beta'_j,\ldots,\beta_j^{(r)})_{1,Q}(b_j;\beta'_j,\ldots,\beta_j^{(r)},\underbrace{0,\ldots,0}_{l},0)_{1,Q},$$
$$(C_j;\underbrace{0,\ldots,0}_{r},D'_j,\ldots,D_j^{(l)},0)_{1,Q'}$$

where C and D are same as given in (23.19) *and the conditions of existence of the above corollary follow easily with the help of Image* 1.

Again, if we put $\beta = 0$ in Image 1, we get the following result which is also beloved to be new and pertains to Erde'lyi–Kober fractional integral operators defined by (23.6) and using (23.14).

Corollary 23.4.

$$\left\{I^{+}_{\eta,\alpha}\left(t^{\mu-1}(b-at)^{-\nu}\prod_{j=1}^{s}S^{m_j}_{n_j}\left[c_jt^{\lambda_j}(b-at)^{-\delta_j}\right]\right.\right.$$
$$\left.\left.H\left[z_1t^{\sigma_1}(b-at)^{-\omega_1}\dots z_rt^{\sigma_r}(b-at)^{-\omega_r}\right]H\left[z'_1t^{\sigma'_1}(b-at)^{-\omega'_1}\dots z'_lt^{\sigma'_l}(b-at)^{-\omega'_l}\right]\right)\right\}(x)$$

$$= b^{-\nu}x^{\mu-1}\sum_{k_1=0}^{[n_1/m_1]}\dots\sum_{k_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}\dots(-n_s)_{m_sk_s}}{k_1!\dots k_s!}$$
$$\times A'_{n_1,m_1}\dots A^{(s)}_{n_s,m_s}c_1^{k_1}\dots c_s^{k_s}(b)^{-\sum_{j=1}^{s}\delta_jk_j}(x)^{\sum_{j=1}^{s}\lambda_jk_j}$$
$$\times H^{0,N+N'+2:\ M_1,N_1;\dots;M_r,N_r;M'_1,N'_1;\dots;M'_l,N'_l;1,0}_{P+P'+2,Q+Q'+2:P_1,Q_1;\dots;P_r,Q_r;P'_1,Q'_1;\dots;P'_l,Q'_l;0,1}\left[\begin{array}{c|c} z_1\frac{x^{\sigma_1}}{b^{\omega_1}} & \\ \vdots & \\ z_r\frac{x^{\sigma_r}}{b^{\omega_r}} & \\ z_1\frac{x^{\sigma'_1}}{b^{\omega'_1}} & A'':C \\ & B'':D \\ \vdots & \\ z_r\frac{x^{\sigma'_l}}{b^{\omega'_l}} & \\ -\frac{a}{b}x & \end{array}\right], \tag{23.23}$$

where C and D are same as given in (23.19) *and*

$$\begin{aligned} A'' &= \left(1-\nu-\sum_{j=1}^{s}\delta_jk_j;\omega_1,\dots,\omega_r,\omega'_1,\dots,\omega'_l,1\right), \\ &\left(1-\mu-\eta-\sum_{j=1}^{s}\lambda_jk_j;\sigma_1,\dots,\sigma_r,\sigma'_1,\dots,\sigma'_l,1\right), \\ &(a_j;\alpha'_j,\dots,\alpha^{(r)}_j,\underbrace{0,\dots,0}_{l},0)_{1,P},(A_j;\underbrace{0,\dots,0}_{r}B'_j,\dots,B^{(l)}_j,0)_{1,P'} \\ B'' &= \left(1-\nu-\sum_{j=1}^{s}\delta_jk_j;\omega_1,\dots,\omega_r,\omega'_1,\dots,\omega'_l,0\right), \\ &\left(1-\mu-\alpha-\eta-\sum_{j=1}^{s}\lambda_jk_j;\sigma_1,\dots,\sigma_r,\sigma'_1,\dots,\sigma'_l,1\right), \\ &(b_j;\beta'_j,\dots,\beta^{(r)}_j)_{1,Q}(b_j;\beta'_j,\dots,\beta^{(r)}_j,\underbrace{0,\dots,0}_{l},0)_{1,Q}, \\ &(C_j;\underbrace{0,\dots,0}_{r},D'_j,\dots,D^{(l)}_j,0)_{1,Q'}. \end{aligned} \tag{23.24}$$

The sufficient conditions of validity of (23.23) *are:*

(*i*) $Re(\alpha) > 0$ *and*

$$Re(\mu)+\sum_{i=1}^{r}\sigma_i\min_{1\le j\le M_i}Re\left(\frac{d^{(i)}_j}{\delta^{(i)}_j}\right)+\sum_{k=1}^{l}\sigma'_k\min_{1\le j\le M'_k}Re\left(\frac{G^{(k)}_j}{H^{(k)}_j}\right) > -Re(\eta)$$

$$Re\,(\nu)+\sum_{i=1}^{r}\omega_i\min_{1\le j\le M_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)+\sum_{k=1}^{l}\omega'_k\min_{1\le j\le M'_k}Re\left(\frac{G_j^{(k)}}{H_j^{(k)}}\right)>-Re\,(\eta)$$

and the conditions $(i),(ii)$and (iv) in Image 1 are also satisfied.

Image 2:

$$\begin{aligned}
&\Big\{I_{-}^{\alpha,\beta,\eta}\Big(t^{\mu-1}(b-at)^{-\nu}\prod_{j=1}^{s}S_{n_j}^{m_j}\left[c_j t^{\lambda_j}(b-at)^{-\delta_j}\right]\\
&\times H\left[z_1 t^{\sigma_1}(b-at)^{-\omega_1}\dots z_r t^{\sigma_r}(b-at)^{-\omega_r}\right]\\
&\times H\left[z'_1 t^{\sigma'_1}(b-at)^{-\omega'_1}\dots z'_l t^{\sigma'_l}(b-at)^{-\omega'_l}\right]\Big)\Big\}(x)\\
&=b^{-\nu}x^{\mu-\beta-1}\sum_{k_1=0}^{[n_1/m_1]}\dots\sum_{k_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}\dots(-n_s)_{m_sk_s}}{k_1!\dots k_s!}\\
&\times A'_{n_1,m_1}\dots A_{n_s,m_s}^{(s)}c_1{}^{k_1}\dots c_s{}^{k_s}(b)^{-\sum_{j=1}^{s}\delta_j k_j}(x)^{\sum_{j=1}^{s}\lambda_j k_j}\\
&\times H^{0,N+N'+3:\;M_1,N_1;\dots;M_r,N_r;M'_1,N'_1;\dots;M'_l,N'_l;1,0}_{P+P'+3,Q+Q'+3:P_1,Q_1;\dots;P_r,Q_r;P'_1,Q'_1;\dots;P'_l,Q'_l;0,1}\left[\begin{array}{c|c} z_1\frac{x^{\sigma_1}}{b^{\omega_1}} & \\ \vdots & \\ z_r\frac{x^{\sigma_r}}{b^{\omega_r}} & A^*:C \\ z_1\frac{x^{\sigma'_1}}{b^{\omega'_1}} & B^*:D \\ \vdots & \\ z_r\frac{x^{\sigma'_l}}{b^{\omega'_l}} & \\ -\frac{a}{b}x & \end{array}\right]
\end{aligned}\tag{23.25}$$

where C and D are given by (23.19) and

$$\begin{aligned}
A^*=&\left(1-\nu-\textstyle\sum_{j=1}^{s}\delta_j k_j;\omega_1,\dots,\omega_r,\omega'_1,\dots,\omega'_l,1\right),\\
&\left(\mu-\beta+\textstyle\sum_{j=1}^{s}\lambda_j k_j;\sigma_1,\dots,\sigma_r,\sigma'_1,\dots,\sigma'_l,1\right),\\
&\left(\mu-\eta+\textstyle\sum_{j=1}^{s}\lambda_j k_j;\sigma_1,\dots,\sigma_r,\sigma'_1,\dots,\sigma'_l,1\right),\\
&(a_j;\alpha'_j,\dots,\alpha_j^{(r)},\underbrace{0,\dots,0}_{l},0)_{1,P},(A_j;\underbrace{0,\dots,0}_{r}B'_j,\dots,B_j^{(l)},0)_{1,P'},\\
B^*=&\left(1-\nu-\textstyle\sum_{j=1}^{s}\delta_j k_j;\omega_1,\dots,\omega_r,\omega'_1,\dots,\omega'_l,0\right),\\
&\left(\mu+\textstyle\sum_{j=1}^{s}\lambda_j k_j;\sigma_1,\dots,\sigma_r,\sigma'_1,\dots,\sigma'_l,1\right),\\
&\left(\mu-\alpha-\beta-\eta+\textstyle\sum_{j=1}^{s}\lambda_j k_j;\sigma_1,\dots,\sigma_r,\sigma'_1,\dots,\sigma'_l,1\right),\\
&(b_j;\beta'_j,\dots,\beta_j^{(r)})_{1,Q}(b_j;\beta'_j,\dots,\beta_j^{(r)},\underbrace{0,\dots,0}_{l},0)_{1,Q},\\
&(C_j;\underbrace{0,\dots,0}_{r},D'_j,\dots,D_j^{(l)},0)_{1,Q'}.
\end{aligned}\tag{23.26}$$

The sufficient conditions of validity of (23.25) are as follows:

(i) $Re\,(\alpha)>0$ and

$$Re(\mu)-\sum_{i=1}^{r}\sigma_i \min_{1\le j\le M_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)-\sum_{k=1}^{l}\sigma'_k \min_{1\le j\le M'_k} Re\left(\frac{G_j^{(k)}}{H_j^{(k)}}\right)<1+\min\{Re(\beta),Re(\eta)\}$$
$$Re(\nu)-\sum_{i=1}^{r}\omega_i \min_{1\le j\le M_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)-\sum_{k=1}^{l}\omega'_k \min_{1\le j\le M'_k} Re\left(\frac{G_j^{(k)}}{H_j^{(k)}}\right)<1+\min\{Re(\beta),Re(\eta)\}$$

and the conditions $(i),(ii)$ and (iv) in Image 1are also satisfied.

Proof. We easily obtain the Image 2 after a little simplification on making use of similar lines as adopted in Image 1 and using Lemma 23.2. □

If we put $\beta=-\alpha$ and $\beta=0$ in Image 2 and using (23.16) and (23.17), in succession we shall easily arrive at the corresponding corollaries concerning Riemann–Liouville and Erde'lyi–Kober fractional integral operators, respectively.

Corollary 23.5.

$$\Big\{I_{-}^{\alpha}\Big(t^{\mu-1}(b-at)^{-\nu}\prod_{j=1}^{s}S_{n_j}^{m_j}\left[c_j t^{\lambda_j}(b-at)^{-\delta_j}\right]$$
$$\times H\left[z_1 t^{\sigma_1}(b-at)^{-\omega_1}\ldots z_r t^{\sigma_r}(b-at)^{-\omega_r}\right]$$
$$H\left[z'_1 t^{\sigma'_1}(b-at)^{-\omega'_1}\ldots z'_l t^{\sigma'_l}(b-at)^{-\omega'_l}\right]\Big)\Big\}(x)$$

$$=b^{-\nu}x^{\mu+\alpha-1}\sum_{k_1=0}^{[n_1/m_1]}\ldots\sum_{k_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}\ldots(-n_s)_{m_sk_s}}{k_1!\ldots k_s!}$$
$$\times A'_{n_1,m_1}\ldots A^{(s)}_{n_s,m_s}c_1^{k_1}\ldots c_s^{k_s}(b)^{-\sum_{j=1}^{s}\delta_j k_j}(x)^{\sum_{j=1}^{s}\lambda_j k_j}$$
$$\times H^{0,N+N'+2:\;M_1,N_1;\ldots;M_r,N_r;M'_1,N'_1;\ldots;M'_l,N'_l;1,0}_{P+P'+2,Q+Q'+2:P_1,Q_1;\ldots;P_r,Q_r;P'_1,Q'_1;\ldots;P'_l,Q'_l;0,1}\left[\begin{array}{c|c} z_1\frac{x^{\sigma_1}}{b^{\omega_1}} & \\ \vdots & \\ z_r\frac{x^{\sigma_r}}{b^{\omega_r}} & A^{**}:C \\ z_1\frac{x^{\sigma'_1}}{b^{\omega'_1}} & B^{**}:D \\ \vdots & \\ z_r\frac{x^{\sigma'_l}}{b^{\omega'_l}} & \\ -\frac{a}{b}x & \end{array}\right] \quad (23.27)$$

where C and D are given by (23.19) and conditions of validity are same as (23.25) and

$$A^{**}=\left(1-\nu-\sum_{j=1}^{s}\delta_j k_j;\omega_1,\ldots,\omega_r,\omega'_1,\ldots,\omega'_l,1\right),$$
$$\left(\alpha+\mu+\sum_{j=1}^{s}\lambda_j k_j;\sigma_1,\ldots,\sigma_r,\sigma'_1,\ldots,\sigma'_l,1\right)$$
$$(a_j;\alpha'_j,\ldots,\alpha_j^{(r)},\underbrace{0,\ldots,0}_{l},0)_{1,P},(A_j;\underbrace{0,\ldots,0}_{r}B'_j,\ldots,B_j^{(l)},0)_{1,P'}$$
$$B^{**}=\left(1-\nu-\sum_{j=1}^{s}\delta_j k_j;\omega_1,\ldots,\omega_r,\omega'_1,\ldots,\omega'_l,0\right), \quad (23.28)$$
$$\left(\mu+\sum_{j=1}^{s}\lambda_j k_j;\sigma_1,\ldots,\sigma_r,\sigma'_1,\ldots,\sigma'_l,1\right),$$
$$(b_j;\beta'_j,\ldots,\beta_j^{(r)})_{1,Q}(b_j;\beta'_j,\ldots,\beta_j^{(r)},\underbrace{0,\ldots,0}_{l},0)_{1,Q},$$
$$(C_j;\underbrace{0,\ldots,0}_{r},D'_j,\ldots,D_j^{(l)},0)_{1,Q'}.$$

Corollary 23.6.

$$\Big\{K_{\eta,\alpha}^{-}\Big(t^{\mu-1}(b-at)^{-\nu}\prod_{j=1}^{s}S_{n_j}^{m_j}\left[c_j t^{\lambda_j}(b-at)^{-\delta_j}\right]$$
$$\times H\left[z_1 t^{\sigma_1}(b-at)^{-\omega_1}\ldots z_r t^{\sigma_r}(b-at)^{-\omega_r}\right]$$
$$\times H\left[z'_1 t^{\sigma'_1}(b-at)^{-\omega'_1}\ldots z'_l t^{\sigma'_l}(b-at)^{-\omega'_l}\right]\Big)\Big\}(x)$$
$$= b^{-\nu}x^{\mu-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}\ldots(-n_s)_{m_sk_s}}{k_1!\ldots k_s!}$$
$$\times A'_{n_1,m_1}\ldots A^{(s)}_{n_s,m_s}c_1{}^{k_1}\ldots c_s{}^{k_s}(b)^{-\sum_{j=1}^{s}\delta_jk_j}(x)^{\sum_{j=1}^{s}\lambda_jk_j}$$
$$\times H^{0,N+N'+2:\;M_1,N_1;\ldots;M_r,N_r;M'_1,N'_1;\ldots;M'_l,N'_l;1,0}_{P+P'+2,Q+Q'+2:P_1,Q_1;\ldots;P_r,Q_r;P'_1,Q'_1;\ldots;P'_l,Q'_l;0,1}\left[\begin{array}{c|c} z_1\frac{x^{\sigma_1}}{b^{\omega_1}} & \\ \vdots & \\ z_r\frac{x^{\sigma_r}}{b^{\omega_r}} & A^{***}:C \\ z_1\frac{x^{\sigma'_1}}{b^{\omega'_1}} & B^{***}:D \\ \vdots & \\ z_r\frac{x^{\sigma'_l}}{b^{\omega'_l}} & \\ -\frac{a}{b}x & \end{array}\right] \tag{23.29}$$

where C and D are given by (23.19) *and*

$$\begin{aligned}
A^{***} &= \Big(1-\nu-\textstyle\sum_{j=1}^{s}\delta_jk_j;\omega_1,\ldots,\omega_r,\omega'_1,\ldots,\omega'_l,1\Big),\\
&\Big(\mu-\eta+\textstyle\sum_{j=1}^{s}\lambda_jk_j;\sigma_1,\ldots,\sigma_r,\sigma'_1,\ldots,\sigma'_l,1\Big),\\
&(a_j;\alpha'_j,\ldots,\alpha_j^{(r)},\underbrace{0,\ldots,0}_{l},0)_{1,P},(A_j;\underbrace{0,\ldots,0}_{r}B'_j,\ldots,B_j^{(l)},0)_{1,P'},\\
B^{***} &= \Big(1-\nu-\textstyle\sum_{j=1}^{s}\delta_jk_j;\omega_1,\ldots,\omega_r,\omega'_1,\ldots,\omega'_l,0\Big),\\
&\Big(\mu-\alpha-\eta+\textstyle\sum_{j=1}^{s}\lambda_jk_j;\sigma_1,\ldots,\sigma_r,\sigma'_1,\ldots,\sigma'_l,1\Big),\\
&(b_j;\beta'_j,\ldots,\beta_j^{(r)})_{1,Q}(b_j;\beta'_j,\ldots,\beta_j^{(r)},\underbrace{0,\ldots,0}_{l},0)_{1,Q},\\
&(C_j;\underbrace{0,\ldots,0}_{r},D'_j,\ldots,D_j^{(l)},0)_{1,Q'}
\end{aligned} \tag{23.30}$$

The conditions of validity of the above results follow easily from the conditions given with Image 2.

23.4 Special Cases and Applications

The generalized fractional integral operator Images 1 and 2 established here are unified in nature and act as key formulae. Thus the product of general class of polynomials involved in Images 1 and 2 reduces to a large spectrum of polynomials listed by Srivastava and Singh ([25], pp. 158–161), and so from Images 1 and 2 we can further obtain various fractional integral results involving a number of simpler polynomials. Again, the multivariable H-function occurring in these images can be suitably specialized to a remarkably wide variety of useful functions (or product of several such functions) which are expressible in terms of generalized Wright hypergeometric function, generalized Mittag–Leffler function and Bessel functions of one variable. For example

1. If we reduce the multivariable H-function in to the Fox H-functions in Image 1 and then reduce one H-function to the exponential function by taking $\sigma_1 = 1$, $\omega_1 \to 0$, we get the following result after a little simplification which is believe to be new:

$$\{I_{0+}^{\alpha,\beta,\eta} t^{\mu-1}(b-at)^{-\nu} \prod_{j=1}^{s} S_{n_j}^{m_j}\left[c_j t^{\lambda_j}(b-at)^{-\delta_j}\right] e^{-z_1 t} H_{P_2,Q_2}^{M_2,N_2}\left[z_2 t^{\sigma_2}(b-at)^{-\omega_2}\ \begin{matrix}(c_j,\gamma_j)_{1,P_2}\\(d_j,\delta_j)_{1,Q_2}\end{matrix}\right]\Bigg\}(x)$$

$$= b^{-\nu} x^{\mu-\beta-1} \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_s=0}^{[n_s/m_s]} \frac{(-n_1)_{m_1k_1}\cdots(-n_s)_{m_sk_s}}{k_1!\ldots k_s!}
\times A'_{n_1,m_1}\ldots A^{(s)}_{n_s,m_s} c_1^{k_1}\ldots c_s^{k_s}(b)^{-\sum_{j=1}^{s}\delta_j k_j}(x)^{\sum_{j=1}^{s}\lambda_j k_j}$$

$$H_{3,3:\,0,1;P_2,Q_2;0,1}^{0,3:\,1,0;M_2,N_2;1,0}\left[\begin{matrix} z_1 x\\ z_2\frac{x^{\sigma_2}}{b^{\omega_2}}\\ -\frac{a}{b}x\end{matrix}\ \middle|\ \begin{matrix}\{\left(1-\nu-\sum_{j=1}^{s}\delta_j k_j;1,\omega_2,1\right),\\ \left(1-\mu-\sum_{j=1}^{s}\lambda_j k_j;1,\sigma_2,1\right),\\ \left(1-\mu-\eta+\beta-\sum_{j=1}^{s}\lambda_j k_j;1,\sigma_2,1\right)\}:\\ \{\left(1-\nu-\sum_{j=1}^{s}\delta_j k_j;1,\omega_2,0\right),\\ \left(1-\mu+\beta-\sum_{j=1}^{s}\lambda_j k_j;1,\sigma_2,1\right),\\ \left(1-\mu-\alpha-\eta-\sum_{j=1}^{s}\lambda_j k_j;1,\sigma_2,1\right)\}:\end{matrix}\ \begin{matrix}-;(c_j,\gamma_j)_{1,P_2};-\\(0,1);(d_j,\delta_j)_{1,Q_2};(0,1)\end{matrix}\right] \tag{23.31}$$

The conditions of validity of the above result easily follow from (23.19).

- If we put $\beta = -\alpha$ and $\nu, \omega_2 = 0$ and $S_{n_j}^{m_j} = 1$ and make suitable adjustment in the parameters in (23.31), we arrive at the known result (see Kilbas and Saigo [18], p. 52, (2.7.9)).

- If we put $\nu, \omega_2, z_1 = 0$ and $S_{n_j}^{m_j} = 1$ and make suitable adjustment in the parameters in (23.31), we arrive at the known result (see Gupta et al. [2] p. 209, (25)).

2. If we reduce the H-function of one variable to generalized Wright hypergeometric function ([23], p.19, (2.6.11)) in the result given by (23.31), we get the following new and interesting result after little simplification:

$$\{I_{0^+}^{\alpha,\beta,\eta} t^{\mu-1} (b-at)^{-\nu} \prod_{j=1}^{s} S_{n_j}^{m_j} \left[c_j t^{\lambda_j} (b-at)^{-\delta_j} \right] e^{-z_1 t} {}_{P_2}\Psi_{Q_2}$$
$$\times \left[-z_2 t^{\sigma_2} (b-at)^{-\omega_2} \left| \begin{matrix} (1-c_j, \gamma_j)_{1,P_2} \\ (0,1), (1-d_j, \delta_j)_{1,Q_2} \end{matrix} \right. \right] \Bigg\} (x)$$
$$= b^{-\nu} x^{\mu-\beta-1} \sum_{k_1=0}^{[n_1/m_1]} \ldots \sum_{k_s=0}^{[n_s/m_s]} \frac{(-n_1)_{m_1 k_1} \ldots (-n_s)_{m_s k_s}}{k_1! \ldots k_s!}$$
$$\times A'_{n_1,m_1} \ldots A^{(s)}_{n_s,m_s} c_1^{k_1} \ldots c_s^{k_s} (b)^{-\sum_{j=1}^{s} \delta_j k_j} (x)^{\sum_{j=1}^{s} \lambda_j k_j}$$

$$H^{0,3:1,0;1,P_2;1,0}_{3,3:0,1;P_2,Q_2;0,1} \left[\begin{matrix} z_1 x \\ -z_2 \frac{x^{\sigma_2}}{b^{\omega_2}} \\ -\frac{a}{b} x \end{matrix} \left| \begin{matrix} \{\left(1-\nu-\sum_{j=1}^{s} \delta_j k_j; 1, \omega_2, 1\right), \\ \left(1-\mu-\sum_{j=1}^{s} \lambda_j k_j; 1, \sigma_2, 1\right), \\ \left(1-\mu-\eta+\beta-\sum_{j=1}^{s} \lambda_j k_j; 1, \sigma_2, 1\right)\} : \\ \{\left(1-\nu-\sum_{j=1}^{s} \delta_j k_j; 1, \omega_2, 0\right), \\ \left(1-\mu+\beta-\sum_{j=1}^{s} \lambda_j k_j; 1, \sigma_2, 1\right), \\ \left(1-\mu-\alpha-\eta-\sum_{j=1}^{s} \lambda_j k_j; 1, \sigma_2, 1\right)\} : \end{matrix} \right. \right.$$
$$\left. \begin{matrix} -; (c_j, \gamma_j)_{1,P_2}; - \\ (0,1); (d_j, \delta_j)_{1,Q_2}; (0,1) \end{matrix} \right] \quad (23.32)$$

The conditions of validity of the above result easily follow from (23.19).

- If we put $\beta = -\alpha$ and $\nu, \omega_2 = 0$ and $S_{n_j}^{m_j} = 1$ and make suitable adjustment in the parameters in (23.32), we arrive at the known result [see [7], p. 117, (11)].
- If we put $\nu, \omega_2, z_1 = 0$ and $S_{n_j}^{m_j} = 1$ and make suitable adjustment in the parameters in (23.32), we arrive at the known result [see Gupta et al. [2], p. 210, (27)].
- If we take $z_2, \sigma_2 = 1$, and $\omega_2 = 0$ in (23.31) and reduce the H-function of one variable occurring therein to generalized Mittag–Laffler function (Prabhakar) ([17], p. 19, (2.6.11)), we easily get after little simplification the following new and interesting result:

$$\left\{ I_{0^+}^{\alpha,\beta,\eta} \left(t^{\mu-1}(b-at)^{-\nu} \prod_{j=1}^{s} S_{n_j}^{m_j} \left[c_j t^{\lambda_j} (b-at)^{-\delta_j} \right] e^{-z_1 t} E_{M_2,N_2}^{\rho} [t] \right\} (x)$$
$$= \frac{b^{-\nu}}{\Gamma(\rho)} x^{\mu-\beta-1} \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_s=0}^{[n_s/m_s]} \frac{(-n_1)_{m_1k_1} \ldots (-n_s)_{m_sk_s}}{k_1! \ldots k_s!}$$
$$\times A'_{n_1,m_1} \ldots A_{n_s,m_s}^{(s)} c_1{}^{k_1} \ldots c_s{}^{k_s} (b)^{-\sum\limits_{j=1}^{s} \delta_j k_j} (-x)^{\sum\limits_{j=1}^{s} \lambda_j k_j} H_{3,2:\,0,1;1,3;0,1}^{0,3:1,0;1,1;1,0}$$
$$\left[\begin{array}{c|l} & \{ \left(1-\nu-\sum_{j=1}^{s}\delta_j k_j;1,0,1\right), \\ & \left(1-\mu-\sum_{j=1}^{s}\lambda_j k_j;1,1,1\right), \\ z_1 x & \left(1-\mu-\eta+\beta-\sum_{j=1}^{s}\lambda_j k_j;1,1,1\right)\} : \\ x & \\ -\frac{a}{b}x & \{ \left(1-\nu-\sum_{j=1}^{s}\delta_j k_j;1,0,0\right), \\ & \left(1-\mu+\beta-\sum_{j=1}^{s}\lambda_j k_j;1,1,1\right), \\ & \left(1-\mu-\alpha-\eta-\sum_{j=1}^{s}\lambda_j k_j;1,1,1\right)\} : \end{array} \right.$$
$$\left. \begin{array}{l} -;(1-\rho,1);- \\ (0,1);(0,1),(1-\nu;o),(1-N_2;M_2);(0,1) \end{array} \right] \quad (23.33)$$

The conditions of validity of the above result can be easily followed directly from those given with (23.19).

- If we put $\beta=-\alpha$ and $\nu,\omega_2=0$ and $S_{n_j}^{m_j}=1$ and make suitable adjustment in the parameters in (23.33), we arrive at the known result (see Saxena et al. [22], p. 168, (2.1)).
- If we put $\nu=0$ and $S_{n_j}^{m_j}=1$, and make suitable adjustment in the parameters in (23.33), we arrive at the known result (see Gupta et al. [2], p. 210, (29)).
- If we take $\beta=-\alpha$ and $\nu,\omega_2=0$ and $S_{n_j}^{m_j}=1$, $z_2=\frac{1}{4}$, $\sigma_2=2$ and reduce the H-function to the Bessel function of first kind in (23.31), we also get known result (see Kilbas and Sebastain [9] 3, p. 873, (25) to (29)).

A number of other special cases of Images 1 and 2 can also be obtained, but we do not mention them here on account of lack of space.

23.5 Conclusion

In this paper, we have obtained the images of the generalized fractional integral operators given by Saigo. The images have been developed in terms of the product of the two multivariables H-function and a general class of polynomials in a compact and elegant form with the help of Saigo operators. Most of the results obtained in this paper are useful in deriving certain composition formulas involving Riemann–Liouville, Erde'lyi–Kober fractional calculus operators and multivariable

H-functions. The findings of this paper provide an extension of the results given earlier by Kilbas, Kilbas and Saigo, Kilbas and Sebastain, Saxena et al. and Gupta et al. as mentioned earlier.

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Chapter 24
Non-asymptotic Norm Estimates for the q-Bernstein Operators

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Abstract The aim of this paper is to present new non-asymptotic norm estimates in $C[0,1]$ for the q-Bernstein operators $B_{n,q}$ in the case $q > 1$. While for $0 < q \leq 1$, $\|B_{n,q}\| = 1$ for all $n \in \mathbb{N}$, in the case $q > 1$, the norm $\|B_{n,q}\|$ grows rather rapidly as $n \to +\infty$ and $q \to +\infty$. Both theoretical and numerical comparisons of the new estimates with the previously available ones are carried out. The conditions are determined under which the new estimates are better than the known ones.

24.1 Introduction

Prior to presenting the subject of this paper, let us recall some notions of the q-calculus (see, e.g., [1], Chap. 10). Given $q > 0$, for any nonnegative integer k, the *q-integer* $[k]_q$ is defined by

$$[k]_q := 1 + q + \cdots + q^{k-1} \ (k = 1, 2, \dots), \ [0]_q := 0;$$

and the *q-factorial* $[k]_q!$ by

$$[k]_q! := [1]_q [2]_q \dots [k]_q \ (k = 1, 2, \dots), \ [0]_q! := 1.$$

For integers k and n with $0 \leq k \leq n$, the *q-binomial* coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is defined by:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

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In addition, the following standard notations will be employed:

$$(a;q)_0 := 1, \quad (a;q)_k := \prod_{s=0}^{k-1}(1-aq^s), \ (a;q)_\infty := \prod_{s=0}^{\infty}(1-aq^s).$$

The space of the continuous functions on $[0,1]$ equipped with the uniform norm $\|.\|$ is denoted by $C[0,1]$.

Definition 24.1 ([12]). Let $f \in C[0,1]$. The *q-Bernstein polynomial* of f is

$$B_{n,q}(f;x) := \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) p_{nk}(q;x), \ n = 1,2,\ldots,$$

where the *q-Bernstein basic polynomials* $p_{nk}(q;x)$ are given by:

$$p_{nk}(q;x) := \begin{bmatrix} n \\ k \end{bmatrix}_q x^k (x;q)_{n-k}, \ k = 0,1,\ldots n. \tag{24.1}$$

Note that for $q = 1$, $B_{n,q}(f;x)$ is the classical Bernstein polynomial. Conventionally, the name *q-Bernstein polynomials* is reserved for $q \neq 1$.

Definition 24.2. The *q-Bernstein operator* on $C[0,1]$ is given by:

$$B_{n,q} : f \mapsto B_{n,q}(f;.).$$

A detailed review of the results on the q-Bernstein polynomials along with the extensive bibliography has been provided in [9]. The popularity of the q-Bernstein polynomials is attributed to the fact that they are closely related to the q-binomial and the q-deformed Poisson probability distributions (cf. [3, 17]). The q-binomial distribution plays an important role in the q-boson theory, providing a q-deformation for the quantum harmonic formalism. More specifically, it has been used to construct the binomial state for the q-boson. Meanwhile, its limit form called the q-deformed Poisson distribution defines the distribution of energy in a q-analogue of the coherent state [2, 5]. Consequently, the properties of the q-deformed binomial distribution and related q-Bernstein basis (24.1) are essential for applications in physics, analysis, and approximation theory.

Similar to the classical Bernstein polynomials, the q-Bernstein polynomials have the end-point interpolation property, possess the divided differences representation, and exhibit the saturation phenomena. This is while the q-Bernstein operators have linear functions as their fixed points (see [6, 9, 12, 14, 16]).

Nevertheless, the striking differences in between the properties of the q-Bernstein polynomials and those of the classical ones appear in their convergence properties. What is more, in terms of convergence, the cases $0 < q < 1$ and $q > 1$ are not similar to each other, as shown in [4, 8]. This is because, for $0 < q < 1$, $B_{n,q}$ are *positive* linear operators on $C[0,1]$, whereas, for $q > 1$, no positivity occurs. In addition, the case $q > 1$ is aggravated by the rather irregular behavior of basic polynomials (24.1), which, in this case, combine the fast increase in magnitude with the sign oscillations. For details see [15].

In this paper, new results are presented on the bounds of the norms of the q-Bernstein operators in the case when $q > 1$ varies. Generally speaking, the norm of a linear operator characterizes its modulus of continuity. For $q > 1$, the erratic behavior of the q-Bernstein polynomials can be explained to a certain degree by the fact that the continuity of the q-Bernstein operators deteriorates in a relatively rapid manner as n and/or q increase. The asymptotic estimates of the norms have been provided in [10, 15], where it is shown that

$$\|B_{n,q}\| \sim \frac{2}{e} \cdot \frac{q^{n(n-1)/2}}{n} \quad \text{as } n \to \infty, q \to +\infty.$$

In distinction to these results, this paper deals with non-asymptotic estimates valid for all $q > 1$. Here, it should be stated that knowledge concerning the rate of growth for a sequence of the approximating operators is very important since such rate affects the construction of the corresponding algorithms in the theory of regularizability of inverse linear operators (see [11]). Also, studies on the norms of various projection operators play a significant role in the structure theory of L_p spaces (see [13]). The authors would like to mention that I. Novikov in [7] has investigated the asymptotic properties of a particular sequence of Bernstein polynomials from a different point of view.

Finally, it must be pointed out that all the numerical results have been obtained in a Maple 8 environment using 500 decimal digits of mantissa in computations with floating point representation.

24.2 Lower Estimates

In this section, we obtain direct estimates from below for the norm $\|B_{n,q}\|$ with any $q > 1$. The case $n = 2$ is rather straightforward as

$$\|B_{2,q}\| = \frac{q^2+1}{2q}.$$

Therefore, we have to obtain estimates only for $n \geq 3$.

Theorem 24.3. *For all $q > 1, n \geq 3$, we have*

$$\|B_{n,q}\| \geq K(n;q) := \max\left\{1, \frac{1}{2^{n-1}} \cdot \left(\frac{q^2-1}{q^2}\right)^n \cdot q^{n(n-1)/2}\right\} \tag{24.2}$$

Proof. Since $\|B_{n,q}\| = \max_{x\in[0,1]} \sum_{k=0}^{n} |p_{nk}(q;x)|$, one can write $\|B_{n,q}\| \geq \sum_{k=0}^{n} |p_{nk}(q;x)|$ for any $x \in [0,1]$. Let $x_0 \in (1/q, 1)$. Then, for $k = 0, 1, \ldots, n-2$,

$$|p_{nk}(q;x_0)| = \begin{bmatrix} n \\ k \end{bmatrix}_q x_0^k (1-x_0)(qx_0-1)\dots(q^{n-k-1}x_0-1)$$

$$= q^{n(n-1)/2-k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_{\frac{1}{q}} x_0^{n-1}(1-x_0)\left(\frac{1}{qx_0};\frac{1}{q}\right)_{n-k-1}$$

$$\geq q^{n(n-1)/2-k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_{\frac{1}{q}} x_0^{n-1}(1-x_0)\left(\frac{1}{qx_0};\frac{1}{q}\right)_{n-1}.$$

Meanwhile,

$$|p_{n,n-1}(q;x_0)| = \begin{bmatrix} n \\ n-1 \end{bmatrix}_q x_0^{n-1}(1-x_0)$$

$$= q^{n(n-1)/2-(n-1)(n-2)/2} \begin{bmatrix} n \\ n-1 \end{bmatrix}_{\frac{1}{q}} x_0^{n-1}(1-x_0)$$

$$\geq q^{n(n-1)/2-(n-1)(n-2)/2} \begin{bmatrix} n \\ n-1 \end{bmatrix}_{\frac{1}{q}} x_0^{n-1}(1-x_0)\left(\frac{1}{qx_0};\frac{1}{q}\right)_{n-1},$$

and

$$p_{nn}(q;x_0) = x_0^n = x_0^{n-1}(1-x_0) + x_0^{n-1}(2x_0-1)$$

$$\geq x_0^{n-1}(1-x_0)\left(\frac{1}{qx_0};\frac{1}{q}\right)_{n-1} + x_0^{n-1}(2x_0-1).$$

Therefore, for any $x_0 \in (1/q, 1)$,

$$\sum_{k=0}^{n} |p_{nk}(q;x_0)| \geq q^{n(n-1)/2} x_0^{n-1}(1-x_0)\left(\frac{1}{qx_0};\frac{1}{q}\right)_{n-1} \cdot \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{\frac{1}{q}} q^{-k(k-1)/2} + x_0^{n-1}(2x_0-1).$$

By virtue of the Rothe identity (cf. [1], Chap. 10, Corollary 10.2.2),

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{\frac{1}{q}} q^{-k(k-1)/2} = \left(-1;\frac{1}{q}\right)_n = 2\left(-\frac{1}{q};\frac{1}{q}\right)_{n-1}.$$

Setting $x_0 = \frac{q+1}{2q}$, one obtains:

$$\left(1-\frac{1}{q^j x_0}\right) = 1 - \frac{2}{q^j+q^{j-1}} \geq 1 - \frac{2}{q^j+1} = \frac{q^j-1}{q^j+1}, \quad j = 1,\dots n-1,$$

whence

$$\left(\frac{1}{qx_0};\frac{1}{q}\right)_{n-1}\left(-\frac{1}{q};\frac{1}{q}\right)_{n-1} \geq \left(\frac{1}{q};\frac{1}{q}\right)_{n-1} \geq \left(1-\frac{1}{q}\right)\left(1-\frac{1}{q^2}\right)^{n-2}.$$

Then it follows that

$$\sum_{k=0}^{n}\left|p_{nk}\left(q;\frac{q+1}{2q}\right)\right| \geqslant 2q^{n(n-1)/2}\cdot\left(\frac{q+1}{2q}\right)^{n-1}\cdot\frac{(q-1)^2}{2q^2}\left(\frac{q^2-1}{q^2}\right)^{n-2}+\frac{1}{q}\cdot\left(\frac{q+1}{2q}\right)^{n-1}$$

$$\geq \frac{1}{2^{n-1}}\cdot q^{n(n-1)/2}\cdot\left(\frac{q^2-1}{q^2}\right)^{n}+\frac{1}{q}\cdot\left(\frac{q+1}{2q}\right)^{n-1}.$$

This completes the proof. □

Now, we compare the derived estimate with the previously known ones from [10], namely,

$$\|B_{n,q}\| \geq L(n;q) := \max\left\{1, \frac{1}{2^{2n-1}}\cdot q^{n(n-1)/2}\right\}. \tag{24.3}$$

and, for $q \geq 3$,

$$\|B_{n,q}\| \geq M(n;q) := \frac{2}{3\sqrt{3}ne}q^{n(n-1)/2}. \tag{24.4}$$

It is not difficult to see that for $n = 3,4$, and 5, estimate (24.2) is the best one for all $q > 1$. As to $n \geq 6$, the best estimate depends on the interval of q. Table 24.1 exhibits the optimal lower bounds for $\|B_{n,q}\|$ for different values of n as a function of q. The value q_0 is the positive solution of $\left(1-1/q^2\right)^6 2^{-5} = 1/\left(9\sqrt{3}e\right)$ whence $q_0 \approx 4.67673$.

n	$q \in \left(1,\sqrt{2}\right)$	$q \in \left(\sqrt{2},3\right)$	$q \in (3,q_0)$	$q \in (q_0,\infty)$
$3,4,5$	$K(n,q)$	$K(n,q)$	$K(n,q)$	$K(n,q)$
6	$K(n,q)$	$K(n,q)$	$M(n,q)$	$K(n,q)$
7	$K(n,q)$	$K(n,q)$	$M(n,q)$	$M(n,q)$
8	$K(n,q)$	$K(n,q)$	$M(n,q)$	$M(n,q)$
$\geqslant 9$	$L(n,q)$	$K(n,q)$	$M(n,q)$	$M(n,q)$

Table 24.1: Optimal lower bounds for $\|B_{n,q}\|$

For $n = 3$ and $n = 9$, the relations among the estimates are illustrated by Figs. 24.1 and 24.2.

24.3 Upper Estimates

Theorem 24.4. *The following estimate holds for all $n \geq 3$ and all $q > 1$:*

$$\|B_{n,q}\| \leq H(n;q) := 1+\frac{2^n}{n+1}\cdot q^{n(n-1)/2}. \tag{24.5}$$

Proof. For $k = 0,1,\ldots,n-1$ and $x \in [0,1]$, one has:

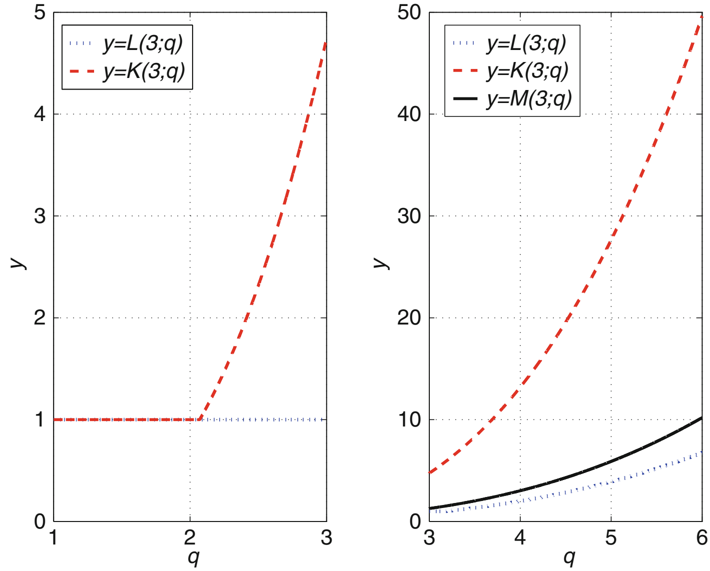

Fig. 24.1: Graphs of $y = K(3;q)$, $y = L(3;q)$, and $y = M(3;q)$

$$|p_{nk}(q;x)| = \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(n-k)(n-k-1)/2} x^k (1-x) \prod_{j=1}^{n-k-1} \left(x - \frac{1}{q^j} \right)$$
$$\leq \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(n-k)(n-k-1)/2} x^k (1-x),$$

while $|p_{nn}(q;x)| = x^n(1-x) + x^{n+1}$. Using the Rothe identity, we obtain:

$$\sum_{k=0}^{n} |p_{nk}(q;x)| \leq q^{n(n-1)/2}(1-x) \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} \left(\frac{x}{q^{n-1}} \right)^k + x^{n+1}$$

$$= q^{n(n-1)/2}(1-x)(-x;1/q)_n + x^{n+1} \leq q^{n(n-1)/2}(1-x)(1+x)^n + x^{n+1},\ x \in [0,1].$$

Clearly,

$$\max_{x\in[0,1]} (1-x)(1+x)^n = \frac{2^{n+1}}{n+1} \left(1 - \frac{1}{n+1} \right)^n.$$

Since the sequence $\left\{ \left(1 - \frac{1}{n+1}\right)^n \right\}$ is decreasing in n, it follows that

$$\max_{x\in[0,1]} (1-x)(1+x)^n \leq \frac{2^n}{n+1} \text{ for } n \geq 2,$$

leading to estimate (24.5). □

Next, we compare estimate (24.5) with the two previously known upper estimates for the norm $\|B_{n,q}\|$ from [10], which are:

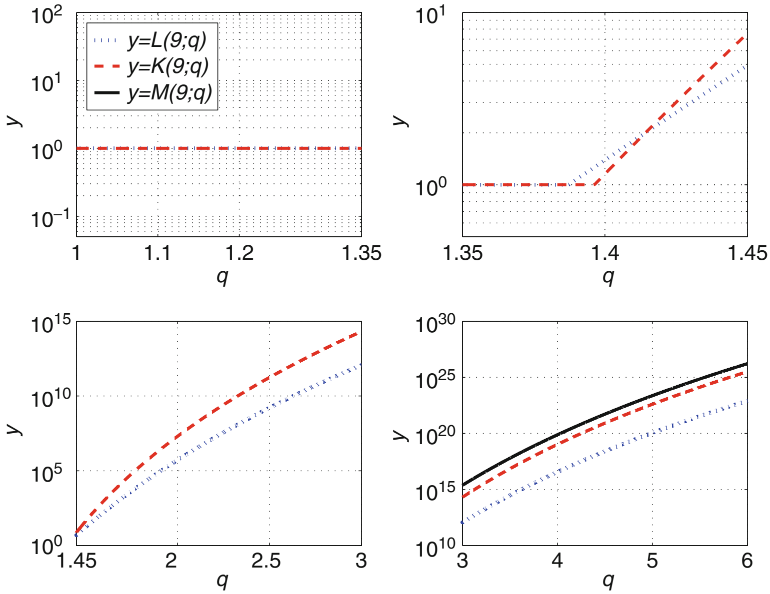

Fig. 24.2: Graphs of $y = K(9;q)$, $y = L(9;q)$, and $y = M(9;q)$

$$(i) \quad \|B_{n,q}\| \leq F(n;q) := 1 + \frac{q-1}{q} \cdot 3^{n-1} \cdot (n-1) q^{n(n-1)/2} \tag{24.6}$$

and

$$(ii) \quad \|B_{n,q}\| \leq G(n;q) := 1 + \frac{1}{4} e^{\frac{1}{q-1}} \cdot q^{n(n-1)/2}. \tag{24.7}$$

Clearly,

$$\|B_{n,q}\| \leq \min\{F(n;q), G(n;q), H(n;q)\}.$$

It is obvious that estimate (24.6) is exact for $q = 1$ and, as such, is better than (24.5) and (24.7) in a right neighborhood of 1. On the other hand, (24.7) provides a better upper bound for $\|B_{n,q}\|$ than the others for large values of q. At this stage, we prove that estimate (24.5) is an optimal one in a certain interval $[q_1, q_2]$, where q_1 and q_2 depend on n.

Theorem 24.5. *For any $n \geq 3$, there exists an interval $[q_1, q_2]$ with $q_1 = q_1(n)$ and $q_2 = q_2(n)$, such that*

$$\min\{F(n;q), G(n;q), H(n;q)\} = H(n;q) \text{ for } q \in [q_1, q_2].$$

Proof. Let $n \geq 3$. For $q > 1$, consider the functions: $f(q) = \frac{q-1}{q} 3^{n-1} (n-1)$, $g(q) = \frac{1}{4} \exp(1/(q-1))$, and $h(q) = \frac{2^n}{n+1}$. Clearly, both equations $f(q) = h(q)$ and $g(q) = h(q)$ have unique solutions q_1 and q_2, respectively. Henceforth, the theorem will be proved if we show that $q_1 < q_2$ for all $n \geq 3$.

Indeed, $f(q) = h(q)$ for $q = q_1 = \frac{3^{n-1}\left(n^2-1\right)}{3^{n-1}\left(n^2-1\right)-2^n}$, while $g(q) = h(q)$ for $q = q_2 = 1 + 1/\left(\ln\left(\frac{2^{n+2}}{n+1}\right)\right)$. Obviously, for $n \geq 3$

$$(n+1) < \frac{1}{3}\left(\frac{3}{2}\right)^n \left(n^2-1\right).$$

Hence,

$$n < \frac{1}{3}\left(\frac{3}{2}\right)^n \left(n^2-1\right) - 1.$$

In addition, for $n \geq 3$,

$$\ln\left(\frac{2^{n+2}}{n+1}\right) < \ln 2^n < n.$$

Combining the last two inequalities, one can see that

$$\ln\left(\frac{2^{n+2}}{n+1}\right) < \frac{1}{3}\left(\frac{3}{2}\right)^n \left(n^2-1\right) - 1.$$

Equivalently,

$$\frac{2^n}{3^{n-1}\left(n^2-1\right)-2^n} < \frac{1}{\ln\left(\frac{2^{n+2}}{n+1}\right)},$$

whence

$$1 + \frac{2^n}{3^{n-1}\left(n^2-1\right)-2^n} < 1 + \frac{1}{\ln\left(\frac{2^{n+2}}{n+1}\right)}$$

which proves that $q_1 < q_2$ for $n \geq 3$. □

Table 24.2 includes the intervals $[q_1, q_2]$ for some values of n. Moreover, the relations among the upper estimates are illustrated by Fig. 24.3 for $n = 3$ and $n = 4$.

n	Intervals on which $H(n;q)$ is the minimum
3	$[1+0.125, 1+0.48090]$
4	$\left[1+4.1131\times10^{-2}, 1+0.39224\right]$
5	$\left[1+1.6736\times10^{-2}, 1+0.32677\right]$
10	$\left[1+5.2578\times10^{-4}, 1+0.16892\right]$
25	$\left[1+1.9039\times10^{-7}, 1+6.4696\times10^{-2}\right]$
50	$\left[1+1.8827\times10^{-12}, 1+3.1141\times10^{-2}\right]$
100	$\left[1+7.3797\times10^{-22}, 1+1.5132\times10^{-2}\right]$

Table 24.2: n values and intervals on which $H(n;q)$ is the minimum

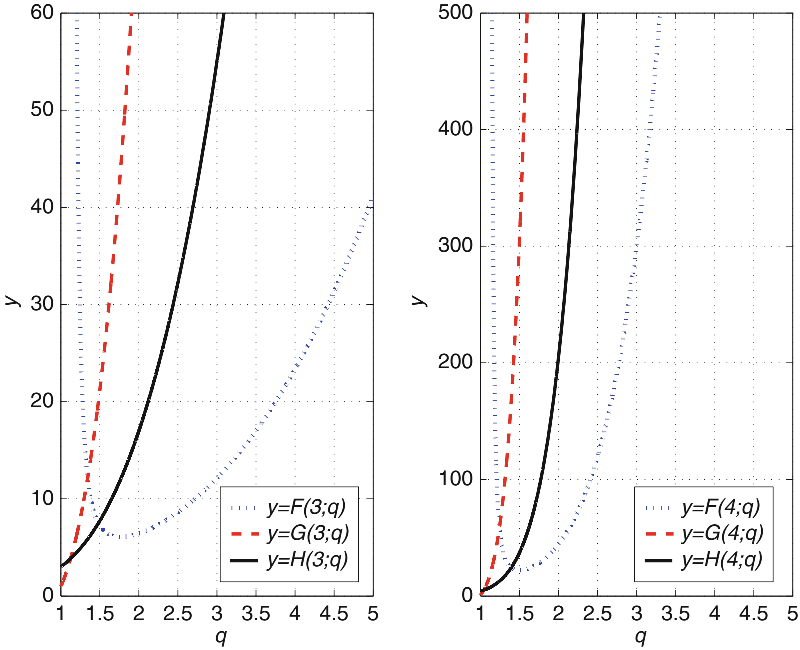

Fig. 24.3: Graphs of $y = F(n;q)$, $y = G(n;q)$, and $y = H(n;q)$ for $n = 3, 4$

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Chapter 25
Approximation Techniques in Impulsive Control Problems for the Tubes of Solutions of Uncertain Differential Systems

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Abstract The paper deals with the control problems for the system described by differential equations containing impulsive terms (or measures). The problem is studied under uncertainty conditions with set-membership description of uncertain variables, which are taken to be unknown but bounded with given bounds (e.g., the model may contain unpredictable errors without their statistical description). The main problem is to find external and internal estimates for set-valued states of nonlinear dynamical impulsive control systems and related nonlinear differential inclusions with uncertain initial state. Basing on the techniques of approximation of the generalized trajectory tubes by the solutions of usual differential systems without measure terms and using the techniques of ellipsoidal calculus we present here a new state estimation algorithms for the studied impulsive control problem. The examples of construction of such ellipsoidal estimates of reachable sets and trajectory tubes of impulsive control systems are given.

25.1 Introduction

Consider a dynamic system described by a differential equation

$$dx(t) = f(t,x(t),u(t))dt + B(t,x(t),u(t))dv(t),\ x \in R^n,\ t_0 \le t \le T, \tag{25.1}$$

with unknown but bounded initial condition

$$x(t_0 - 0) = x^0,\ \ x^0 \in X^0 \subset R^n. \tag{25.2}$$

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Here $u(t)$ is a usual (measurable) control with constraint

$$u(t) \in U, \quad U \subset R^m,$$

and $v(t)$ is an impulsive control (a control measure) which is continuous from the right, with bounded variation

$$\mathrm{Var}_{t \in [t_0, T]}\, v(t) \leq \mu \;\; (\mu > 0).$$

We assume $f(t,x,u)$ and $n \times k$-matrix $B(t,x,u)$ to be continuous in their variables.

The dynamical problems with impulsive control inputs arise in various applications such as finance, mechanics, hybrid systems, chaotic communications systems and nano-electronics, renewable resource management, or aerospace navigation, where the solution is contained in the set of control processes with trajectories of bounded variation. This in turn gives a strong impetus to the rapid development of the theory of such systems and numerical schemes implementing the control strategies.

Therefore, impulsive systems arise naturally from a wide variety of applications and can be used as an appropriate description of these phenomena of abrupt qualitative dynamical changes of essentially continuous time systems. Significant progress has been made in the theory of impulsive differential equations in recent decades. Among the long list of publications devoted to impulsive control problems, we specifically mention the results most closely related to this study [1–3, 5, 15, 18, 19, 23, 24, 27]. However, the corresponding theory for uncertain impulsive systems has not yet been fully developed.

In this paper the impulsive control problem for a dynamic system (25.1) with unknown but bounded initial states (25.2) is studied. Using the ideas of the guaranteed state estimation approach [12–14, 16–18] and the techniques of differential inclusions theory [6, 20, 26] we study the set-valued solutions (trajectory tubes) of the related differential inclusion of impulsive type. We present the modified state estimation approaches which use the special nonlinear structure of the impulsive control system. Examples and numerical results related to procedures of set-valued approximations of trajectory tubes and reachable sets are also presented.

25.2 Problem Statement

Let R^n be the n-dimensional Euclidean space and $x'y$ be the usual inner product of $x, y \in R^n$ with the prime as a transpose and with $\| x \| = (x'x)^{1/2}$. Denote *comp* R^n to be the variety of all compact subsets $A \subset R^n$ and *conv* R^n to be the variety of all compact convex subsets $A \subset R^n$. We denote as $B(a,r)$ the ball in R^n, $B(a,r) = \{x \in R^n : \| x - a \| \leq r\}$, and I is the identity $n \times n$-matrix. Denote by $E(a,Q)$ the ellipsoid in R^n, $E(a,Q) = \{x \in R^n : \; (x-a)'Q^{-1}(x-a) \leq 1\}$, with center $a \in R^n$ and symmetric positive definite $n \times n$-matrix Q. For any $n \times n$-matrix $M = \{m_{ij}\}$ denote $Tr(M) = \sum_{i=1}^{i=n} m_{ii}$.

We consider here a nonlinear dynamical system of a simpler type described by a differential equation with a measure

$$dx(t) = f(t, x(t), u(t))dt + B(t)dv(t), \quad x \in R^n, \quad t_0 \le t \le T, \tag{25.3}$$

with unknown but bounded initial condition

$$x(t_0 - 0) = x^0, \quad x^0 \in X^0. \tag{25.4}$$

Here $X^0 \in comp\, R^n$, $u(t)$ is a usual (measurable) control with constraint

$$u(t) \in U, \quad U \in comp\, R^m, \tag{25.5}$$

and $v(t)$ is a scalar impulsive control function which is continuous from the right, with constrained variation ($\mu > 0$ is fixed)

$$\mathrm{Var}_{t \in [t_0, T]}\, v(t) \le \mu, \tag{25.6}$$

$$\mathrm{Var}_{t \in [t_0, T]}\, v(t) = \sup_{\{t_i | t_0 \le t_1 \le \ldots \le t_k = T\}} \{\sum_{i=1}^{k} |v(t_i) - v(t_{i-1})|\} \le \mu.$$

We assume that n-vector functions $f(t, x, u)$ and $B(t)$ are continuous in their variables.

The guaranteed estimation problem consists in describing the trajectory tube [16]

$$X(\cdot, t_0, X^0) = \bigcup_{\{u(\cdot), v(\cdot)\}} \{x[\cdot] \,|\, x[t] = x(t, t_0, x^0, u, v), x^0 \in X^0\} \tag{25.7}$$

of solutions $x[t] = x(t, t_0, x^0, u, v)$ to the system (25.3)–(25.4) under constraints (25.5)–(25.6). Note that the set $X(t, t_0, X^0)$ coincides with the reachable set of the system (25.3)–(25.4) at the instant t and $X(t_0, t_0, X^0) = X^0$.

It should be noted also that the exact description of reachable sets of a control system is a difficult problem even in the case of linear dynamics. The estimation theory and related algorithms basing on ideas of construction outer and inner set-valued estimates of reachable sets have been developed in [4, 17] for linear control systems and in [7–11] for some classes of nonlinear systems.

In this paper, the modified state estimation approaches which use the special quadratic structure of nonlinearity of the studied impulsive control system and use also the advantages of ellipsoidal calculus are presented.

The main approach to the solution of the problem under consideration is based on the sequence of following steps:

- Use the reparametrization procedure to reformulate the impulsive control problem as a conventional auxiliary problem which does not contain impulsive terms.
- Apply existing results to this problem.
- Express the obtained solution in terms of the original problem.

The estimation algorithm basing on combination of discrete-time versions of evolution funnel equations [20, 26] and ellipsoidal calculus [4, 17] is given. Examples and numerical results related to procedures of set-valued approximations of trajectory tubes and reachable sets are also presented. The applications of the problems studied in this paper are in guaranteed state estimation for nonlinear systems with unknown but bounded errors and in nonlinear control theory.

25.3 Preliminary Results

In this section we present some auxiliary results needed for the implementation of a three-stage procedure for solving the basic problem outlined above.

25.3.1 Reformulation of the Problem with the Appropriate Differential Inclusion

Consider a differential inclusion related to (25.3)–(25.4)

$$dx(t) \in F(t, x(t))dt + B(t)dv(t), \tag{25.8}$$

with the initial condition

$$x(t_0 - 0) = x^0, \;\; x^0 \in X^0. \tag{25.9}$$

Here we use the notation

$$F(t,x) = f(t,x,U) = \cup\{\, f(t,x,u) \mid u \in U \,\}.$$

Definition 25.1. [21] A function $x[t] = x(t,t_0,x^0)$ $(x^0 \in X^0, t \in [t_0,T])$ will be called a solution (a trajectory) of the differential inclusion (25.8) if for all $t \in [t_0,T]$ we have

$$x[t] = x^0 + \int_{t_0}^{t} \psi(t)dt + \int_{t_0}^{t} B(t)dv(t), \tag{25.10}$$

where $\psi(\cdot) \in L_1^n[t_0,T]$ is a selector of F, i.e., $\psi(t) \in F(t,x[t])$ a.e. The last integral in (25.10) is taken as the Riemann–Stieltjes integral.

Following the scheme of the proof of the well-known Caratheodory theorem one can prove the existence of solutions $x[\cdot] = x(\cdot,t_0,x^0) \in BV^n[t_0,T]$ for all $x^0 \in X^0$ where $BV^n[t_0,T]$ is the space of n-vector functions with bounded variation at $[t_0,T]$.

25.3.2 Discontinuous Replacement of Time

Let us introduce a new time variable [19, 22, 27]:

$$\eta(t) = t + \int_{t_0}^{t} dv(t),$$

and a new state coordinate $\tau(\eta) = \inf\{\, t \mid \eta(t) \geq \eta \,\}$.

Consider the following auxiliary differential inclusion of a classical type which no longer has measures or impulses [7]

$$\frac{d}{d\eta}\begin{pmatrix} z \\ \tau \end{pmatrix} \in G(\tau, z), \quad t_0 \leq \eta \leq T + \mu, \tag{25.11}$$

with the initial condition

$$z(t_0) = x^0, \quad \tau(t_0) = t_0.$$

Here

$$G(\tau, z) = \bigcup_{0 \leq \nu \leq 1} \left\{ (1-\nu)\begin{pmatrix} F(\tau, z) \\ 1 \end{pmatrix} + \nu \begin{pmatrix} B(\tau) \\ 0 \end{pmatrix} \right\}. \tag{25.12}$$

Denote $w = \{z, \tau\} = w(\eta, t_0, w^0)$ $(w^0 = \{z^0, t_0\})$ the extended state vector of the system (25.11) and consider the trajectory tube $W[\cdot]$ of this differential inclusion (25.11):

$$W[\eta] = \bigcup_{w^0 \in X^0 \times \{t_0\}} w(\eta, t_0, w^0), \quad t_0 \leq \eta \leq T + \mu.$$

The next lemma explains the construction of the auxiliary differential inclusion (25.11).

Lemma 25.2 ([7]). *The set $X[T] = X(T, t_0, X^0)$ is the projection of $W[T+\mu]$ at the subspace of variables z:*

$$X[T] = \pi_z W[T + \mu].$$

Remark 25.3. It follows from results of [7] that the set-valued function $G(\tau, z)$ in the auxiliary differential inclusion (25.11) has convex and compact values and is Lipschitz continuous in both variables $\{\tau, z\}$.

25.3.3 Estimation Results for Uncertain Nonlinear Systems

In [8, 9, 11] we studied the uncertain control systems described by ordinary differential equations with uncertain parameters and presented techniques of constructing the external and internal ellipsoidal estimates of trajectory tubes $X(\cdot,t_0,X^0)$. The techniques were based on a combination of ellipsoidal calculus [4, 17] and the techniques of evolution funnel equations [20]. Let us recall some basic results.

Consider the differential inclusion generated by a nonlinear control system with classical controls only (impulsive control terms are absent here), namely, we consider the differential inclusion of the following type:

$$\dot{x} \in Ax + \tilde{f}(x)d + P(t), \; x^0 \in X^0, \; t_0 \le t \le T, \tag{25.13}$$

where $x \in R^n$, $\|x\| \le K$, $X^0 = E(a_0,Q_0)$, $P(t) = E(\hat{a},\hat{Q})$, $d, a_0, \hat{a}$ are given n-vectors, a scalar function $\tilde{f}(x)$ has a form $\tilde{f}(x) = x'Bx$, and matrices B, Q_0, and $\hat{Q}$ are symmetric and positive definite (more complicated cases with different quadratic forms $f_i(x)$ included in the right-hand side of differential inclusion (25.13) were also studied in [8]).

Denote as $x(\cdot,t_0,x^0)$ the absolutely continuous solution to (25.13) with the initial condition $x(t_0) = x^0$ and recall the following definition.

Definition 25.4. The set

$$X(\cdot) = X(\cdot,t_0,X^0) = \bigcup_{x^0 \in X^0} \{x(\cdot,t_0,x^0)\} \tag{25.14}$$

is called a trajectory tube to system (25.13) with initial state $\{t_0,X^0\}$, $t \in [t_0,T]$. The cross-section $X(t) = X(t,t_0,X^0)$ of trajectory tube $\mathscr{X}(\cdot,t_0,X^0)$ at instant $t \ge t_0$ is called a reachable set to system (25.13) with $X(t_0) = X(t_0,t_0,X^0) = X^0$.

Let k_0^- and k_0^+ be positive numbers such that the following two inclusions hold

$$E(a_0,(k_0^-)^2B^{-1}) \subseteq E(a_0,Q_0) \subseteq E(a_0,(k_0^+)^2B^{-1}). \tag{25.15}$$

We assume that k_0^- is maximal and k_0^+ is minimal for which the inclusions (25.15) are true.

Theorem 25.5 ([10]). *The inclusions hold*

$$E(a^-(t),r^-(t)B^{-1}) \subseteq X(t,t_0,X^0) \subseteq E(a^+(t),r^+(t)B^{-1}), \; t_0 \le t \le T, \tag{25.16}$$

where functions $a^+(t)$, $r^+(t)$ are the solutions of the following system of ordinary differential equations

$$\begin{aligned} \dot{a}^+(t) &= Aa^+(t) + ((a^+(t))'Ba^+(t) + r^+(t))d + \hat{a}, \; t_0 \le t \le T, \\ \dot{r}^+(t) &= \max_{\|l\|=1} \{ l' (2r^+(t)(B^{1/2}AB^{-1/2} + 2B^{1/2}d\,(a^+(t))'B^{1/2} + \\ & q^{-1}(r^+(t))B^{1/2}\hat{Q}B^{1/2})l\} + q(r^+(t))r^+(t), \;\; q(r) = ((nr)^{-1}Tr(B\hat{Q}))^{1/2}, \end{aligned} \tag{25.17}$$

with initial condition

$$a^+(t_0) = a_0, \;\; r^+(t_0) = (k_0^+)^2, \tag{25.18}$$

and where functions $a^-(t)$, $r^-(t)$ are the solutions of the following system of ordinary differential equations

$$\begin{aligned}\dot{a}^-(t) =& Aa^-(t) + ((a^-(t))'Ba^-(t) + r^-(t))d + \hat{a},\ t_0 \le t \le T,\\ \dot{r}^-(t) =& 2\min_{\|l\|=1}\{l^{'}(r^-(t)(B^{1/2}AB^{-1/2}+\\ & 2B^{1/2}d(a^-(t))'B^{1/2}) + (r^-(t))^{1/2}(B^{1/2}\hat{Q}B^{1/2})^{1/2})l\},\end{aligned} \tag{25.19}$$

with

$$a^-(t_0) = a_0,\ \ r^-(t_0) = (k_0^-)^2. \tag{25.20}$$

Remark 25.6. The inclusions (25.5) give two ellipsoidal estimates for the trajectory tube $X(t)$, the internal one $(E(a^-(t), r^-(t)B^{-1}))$ with respect to inclusion operation and the external one $(E(a^+(t), r^+(t)B^{-1}))$. Parameters of both ellipsoids are easily computable, for example, using technical computing software (such as MATLAB, Mathematica, and Mathcad).

Example 25.7. Consider the following control system

$$\begin{cases}\dot{x}_1 = 2x_1 + u_1,\\ \dot{x}_2 = 2x_2 + x_1^2 + x_2^2 + u_2, \quad x^0 \in X^0,\ \ 0 \le t \le T.\end{cases} \tag{25.21}$$

Here, we take $t_0 = 0$, $T = 0.4$, $X^0 = B(0,1)$, $P(t) = B(0,r)$, $r = 0.01$. In this case we have $A = 2I$, $B = I$, $d_1 = 0$, $d_2 = 1$.

The trajectory tube $X(t)$ with its external ellipsoidal tube $E^+(t) = E(a^+(t), Q^+(t))$ and its internal ellipsoidal tube $E^-(t) = E(a^-(t), Q^-(t))$ found by Theorem 25.5 are shown as 3Dgraphs in Fig. 25.1. We see there that the reachable set $X(t)$ lies inside

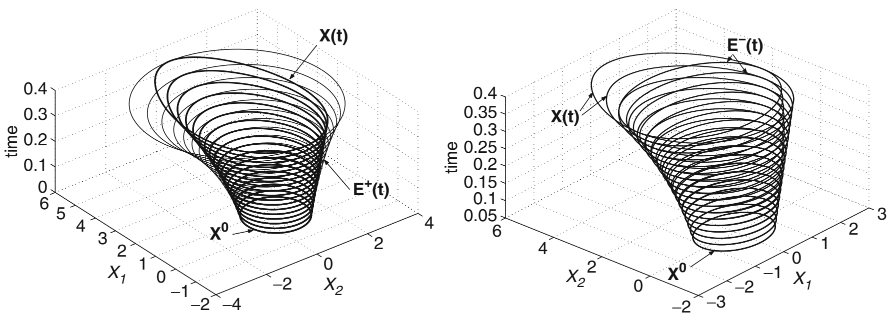

Fig. 25.1: Trajectory tube $X(t)$ and its estimating ellipsoidal tubes $E^+(t)$ (*left picture*) and $E^-(t)$ (*right picture*)

the ellipsoidal estimate $E^+(t)$ and contains the ellipsoidal estimates $E^-(t)$. Both ellipsoids touch the set $X(t)$ at some points so the estimating sets $E^+(t)$ and $E^-(t)$ really produce related bounds for $X(t)$ which are enough accurate in some sense.

25.4 Main Results

Based on the above techniques of approximation of the generalized trajectory tubes by the solutions of usual differential systems without measure terms and using the techniques of ellipsoidal calculus we present here new state estimation algorithms for more complicated dynamics defined by impulsive control problems.

25.4.1 State Estimates for Nonlinear Impulsive Systems

Consider the following impulsive control system

$$dx(t) = (Ax + f(x)d + u(t))dt + Gdv(t),\ t_0 \le t \le T, \tag{25.22}$$

$$x(t_0 - 0) = x^0,\ \ x^0 \in X^0 = E(a, k^2B^{-1})\ (k \neq 0). \tag{25.23}$$

Here A is a constant $n \times n$-matrix and $d, G \in R^n$,

$$f(x) = x'Bx, \tag{25.24}$$

where B is a symmetric positive definite $n \times n$-matrix, $u(t) \in U$, $U = E(\hat{a}, \hat{Q})$, $\mathrm{Var}_{t\in[t_0,T]}\ v(t) \le \mu$.

Following the idea of the previous section we introduce the nonlinear differential inclusion

$$\frac{d}{d\eta}\begin{pmatrix} z \\ \tau \end{pmatrix} \in H(\tau, z), \quad t_0 \le \eta \le T + \mu, \tag{25.25}$$

with initial condition

$$z(t_0) = x^0 \in X^0 = E(a, k^2B^{-1}), \ \ \tau(t_0) = t_0,$$

where

$$H(\tau, z) = \bigcup_{0\le\ \nu\ \le 1} \left\{ (1-\nu)\begin{pmatrix} Az + f(z)d + E(\hat{a}, \hat{Q}) \\ 1 \end{pmatrix} + \nu \begin{pmatrix} G \\ 0 \end{pmatrix} \right\}. \tag{25.26}$$

Let $W(t; t_0, X^0 \times \{t_0\})$ be a trajectory tube of the inclusion (25.25)–(25.26).

Theorem 25.8. *For any* $\sigma > 0$ *the following inclusion is true:*

$$W(t_0 + \sigma) \subseteq \bigcup_{0\le\ \nu\ \le 1} \begin{pmatrix} E(a^+(\sigma,\nu), Q^+(\sigma,\nu)) \\ \\ t_0 + \sigma(1-\nu) \end{pmatrix} + o(\sigma)B_*(0,1). \tag{25.27}$$

Here, $B_*(0,1)$ *is a unit ball in* R^{n+1}, $\lim_{\sigma\to+0} \sigma^{-1}o(\sigma) = 0$ *and*

$$\begin{aligned} a^+(\sigma,\nu) &= a(\sigma,\nu) + \sigma(1-\nu)\hat{a} + \sigma\nu G, \\ Q^+(\sigma,\nu) &= (p^{-1}+1)Q(\sigma,\nu) + (p+1)\sigma^2(1-\nu)^2\hat{Q}, \end{aligned} \tag{25.28}$$

$p = p(\sigma, \nu)$ is the unique positive solution of the equation

$$\sum_{i=1}^{n} \frac{1}{p+\lambda_i} = \frac{n}{p(p+1)},$$

numbers $\lambda_i = \lambda_i(\sigma, \nu) \geq 0$ satisfy the equation

$$|Q(\sigma, \nu) - \lambda \sigma^2 (1-\nu)^2 \hat{Q}| = 0$$

and the following relations hold:

$$\begin{aligned} a(\sigma, \nu) &= a + \sigma(1-\nu)(Aa + a'^2 d), \\ Q(\sigma, \nu) &= k^2 (I + \sigma R) B^{-1} (I + \sigma R)', \\ R &= (1-\nu)(A + 2da'B). \end{aligned} \tag{25.29}$$

Proof. The proof follows directly from Theorem 25.5. Parameters of estimating set in (25.27) are calculated based on formulas (25.25)–(25.26). □

25.4.2 Algorithm for External Estimation

Now we describe the algorithm which follows directly from Theorem 25.8 and may be used in theoretical modeling and applied calculations.

Subdivide the time segment $[t_0, T+\mu]$ into subsegments $\{[t_i, t_{i+1}]\}$ where $t_i = t_0 + ih$ $(i = 1, \ldots, m)$, $h = (T + \mu - t_0)/m$, $t_m = T + \mu$. Define also the partition $\{[\nu_i, \nu_{i+1}]\}$ of $[0,1]$ where $\nu_i = ih_*$ $(i = 1, \ldots, m)$, $h_* = 1/m$, $\nu_m = 1$. The algorithm is based on the consequent repetition of the following five steps. So

1. Given $X_0 = E(a, k_0^2 B^{-1})$ $(k_0 \neq 0)$, find m ellipsoids $E(a_1^i, Q_1^i)$ from Theorem 25.8 for $a_1^i = a^+(\sigma, \nu_i)$, $Q_1^i = Q^+(\sigma, \nu_i)$, $\sigma = h$ $(i = 1, \ldots, m)$.
2. Next, find in R^{n+1} the ellipsoid $E(w_1(\sigma), O_1(\sigma))$ such that for $i = 1, \ldots, m$ we have (see also the algorithm in [25])

$$W(\sigma, \nu_i) = \begin{pmatrix} E(a^+(\sigma, \nu_i), Q^+(\sigma, \nu_i)) \\ t_0 + \sigma(1-\nu_i) \end{pmatrix} \subseteq E(w_1(\sigma), O_1(\sigma)).$$

3. Apply Lemma 25.2 and find the ellipsoid $E(a_1, Q_1) = \pi_z\, E(w_1(\sigma), O_1(\sigma))$.
4. Find the smallest constant $k_1 > 0$ such that $E(a_1, Q_1) \subset E(a_1, k_1^2 B^{-1})$, and it is not difficult to prove that k_1^2 is the maximal eigenvalue of the matrix $B^{1/2} Q_1 B^{1/2}$.
5. Consider the system on the next subsegment $[t_1, t_2]$ with $E(a_1, k_1^2 B^{-1})$ as the initial ellipsoid at instant t_1 and go to the first step.

At the end of the process we will get the external ellipsoidal estimate $\tilde{E}(T) = E(a^+(T), Q^+(T))$ of the reachable set $X(T)$ with accuracy tending to zero when $m \to \infty$.

Example 25.9. Consider the following impulsive control system

$$\begin{cases} dx_1(t) &= x_2(t)dt, \\ dx_2(t) &= du(t), \ \ 0 \le t \le T. \end{cases} \tag{25.30}$$

The impulsive controls $u(t)$ are continuous from the right, with bounded variation $\mathrm{Var}_{t\in[0,T]}u(t) \le 1$. To simplify calculations we assume also that every control $u(t)$ is increasing on $[0,T]$.

The initial states x_0 of the impulsive control system are assumed to be unknown but bounded, with given ellipsoidal bound,

$$x_0 \in X_0 = E(0,R), \qquad R = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}.$$

We apply the algorithm proposed above and find the external ellipsoidal estimate $\tilde{E}(T) = E(a^+(T), Q^+(T))$ of the exact reachable set $X(T) = X(t,t_0,X_0)$. Both sets $E(a^+(T), Q^+(T))$ and $X(T)$ are shown in Fig. 25.2 for $T = 1$.

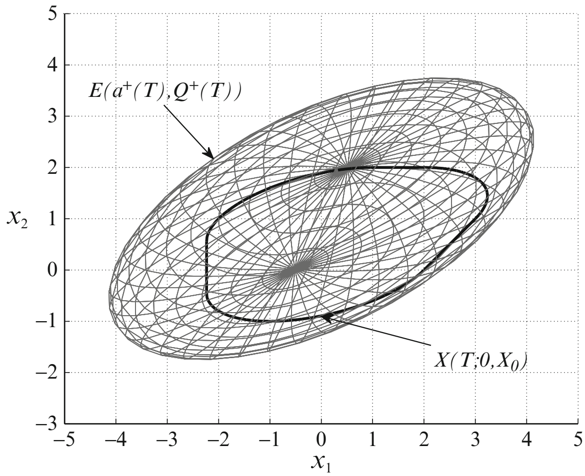

Fig. 25.2: Ellipsoidal estimate $E(a^+(T), Q^+(T))$ of the set $X(T)$ for $T = 1$

Remark 25.10. It should be noted that the external ellipsoidal estimates of reachable sets of impulsive systems, obtained in this paper, are less precise than the estimates of reachable sets of dynamic systems with classical control (e.g., compare simulation results in examples with Figs. 25.1 and 25.2). The reason is that in impulsive control problems the estimation algorithm is more complicated and contains, in particular, an additional operation of projection onto the subspace of state variables.

Remark 25.11. The construction of internal ellipsoidal estimates of reachable sets is much more difficult for impulsive nonlinear systems and is still an open problem for such systems.

25.5 Conclusions

We considered the problems of state estimation for dynamical impulsive control systems with unknown but bounded initial state.

The solution to the differential system is studied through the techniques of trajectory tubes with their cross-sections $X(t)$ being the reachable sets.

Basing on results of ellipsoidal calculus developed for uncertain dynamical systems with classical (measurable) controls we present the modified state estimation approach and related numerical algorithm which use the special structure of the impulsive control system.

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Chapter 26
A New Viewpoint to Fourier Analysis in Fractal Space

Mengke Liao, Xiaojun Yang and Qin Yan

Abstract Fractional analysis is an important method for mathematics and engineering, and fractional differentiation inequalities are great mathematical topic for research. In this paper we point out a new viewpoint to Fourier analysis in fractal space based on the local fractional calculus and propose the local fractional Fourier analysis. Based on the generalized Hilbert space, we obtain the generalization of local fractional Fourier series via the local fractional calculus. An example is given to elucidate the signal process and reliable result.

26.1 Introduction

Fractional calculus has been used in describing physical phenomena such as viscoelasticity [1–3], continuum mechanics [4–6], quantum mechanics [7–9], diffusion and wave phenomena [10–16] and other branches of applied mathematics [17–21] and nonlinear dynamics [22–24] have been studied.

As is well known, fractal curves are everywhere continuous but nowhere differentiable, and we cannot employ fractional calculus to describe the motions in cantor time–space [25, 26]. Recently, a modified Riemann–Liouville derivative [27–32] and local fractional derivative [33–54] has been proposed to deal with the non-differential functions. Local fractional calculus is revealed to deal with everywhere continuous but nowhere differentiable functions on cantor sets. For these merits, local fractional calculus was successfully applied in the local fractional

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Laplace transform (also called the Yang–Laplace transform) [48–51], the local fractional Fourier transform (also called the Yang–Fourier transform) [48, 49, 52–55], the Hölder inequality in fractal space [56], the local fractional short time transform [48, 49], the local fractional wavelet transform [48, 49], the fractal signals [54, 57], the discrete Yang–Fourier transform [58] and the fast Yang–Fourier transform [55].

In this paper we investigate the local fractional calculus of real functions, the fractional-order complex mathematics and the generalized Hilbert space, and we focus on local fractional Fourier analysis based on local fractional calculus. The paper is organized as follows. In Sect. 26.2 the local fractional calculus of the real functions is discussed; in Sect. 26.3 we investigate the fractional-order complex mathematics and the complex Mittag–Leffler functions; in Sect. 26.4 we prove the generalization of local fractional Fourier series in generalized Hilbert space; in Sect. 26.5 we propose the local fractional Fourier analysis; in Sect. 26.6 we give an example of the expansion of local fractional Fourier series with the complex Mittag–Leffler functions, and conclusions are in Sect. 26.7.

26.2 Local Fractional Calculus of Real Functions

26.2.1 Local Fractional Continuity

Definition 26.1. If there exists [48, 49]

$$|f(x) - f(x_0)| < \varepsilon^{\alpha} \tag{26.1}$$

with $|x - x_0| < \delta$,for$\varepsilon, \delta > 0$ and $\varepsilon, \delta \in$ R, now $f(x)$ is called local fractional continuous at $x = x_0$, denoted by $\lim_{x \to x_0} f(x) = f(x_0)$. Then$f(x)$ is called local fractional continuous on the interval (a,b), denoted by

$$f(x) \in C_{\alpha}(a,b). \tag{26.2}$$

Definition 26.2. A function $f(x)$ is called a non-differentiable function of exponent α, $0 < \alpha \leq 1$, which satisfies Hölder function of exponent α, then for $x, y \in X$ such that [48, 49, 54]

$$|f(x) - f(y)| \leq C|x - y|^{\alpha}. \tag{26.3}$$

Definition 26.3. A function $f(x)$is called to be continuous of order α,$0 < \alpha \leq 1$, or shortly α continuous, when we have that [48, 49, 54]

$$f(x) - f(x_0) = o\left((x - x_0)^{\alpha}\right). \tag{26.4}$$

Remark 26.4. Compared with (26.4), (26.1) is standard definition of local fractional continuity. Here (26.3) is unified local fractional continuity.

26.2.2 Local Fractional Calculus

Definition 26.5. Let $f(x) \in C_\alpha(a,b)$. Local fractional derivative of $f(x)$ of order α at $x = x_0$ is defined as[48–58]

$$f^{(\alpha)}(x_0) = \frac{d^\alpha f(x)}{dx^\alpha}\Big|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^\alpha (f(x) - f(x_0))}{(x - x_0)^\alpha}, \tag{26.5}$$

where$\Delta^\alpha (f(x) - f(x_0)) \cong \Gamma(1+\alpha)\Delta(f(x) - f(x_0))$. For any$x \in (a,b)$, there exists

$$f^{(\alpha)}(x) = D_x^{(\alpha)} f(x),$$

denoted by

$$f(x) \in D_x^{(\alpha)}(a,b).$$

Definition 26.6. Let $f(x) \in C_\alpha(a,b)$. Local fractional integral of $f(x)$ of order α in the interval $[a,b]$ is given [48–58]

$${}_aI_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^\alpha = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{j=N-1} f(t_j)(\Delta t_j)^\alpha, \tag{26.6}$$

where $\Delta t_j = t_{j+1} - t_j$,$\Delta t = \max\{\Delta t_1, \Delta t_2, \Delta t_j, \ldots\}$and $[t_j, t_{j+1}]$, $j = 0, \ldots, N-1$,$t_0 = a$, $t_N = b$, is a partition of the interval$[a,b]$.

For convenience, we assume that
${}_aI_a^{(\alpha)} f(x) = 0$ if $a = b$ and ${}_aI_b^{(\alpha)} f(x) = -{}_bI_a^{(\alpha)} f(x)$ if $a < b$.
For any $x \in (a,b)$, we get

$${}_aI_x^{(\alpha)} f(x), \tag{26.7}$$

denoted by

$$f(x) \in I_x^{(\alpha)}(a,b).$$

Remark 26.7. If $f(x) \in D_x^{(\alpha)}(a,b)$, or $I_x^{(\alpha)}(a,b)$, we have that

$$f(x) \in C_\alpha(a,b). \tag{26.8}$$

Remark 26.8. The following relations hold:

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b E_\alpha(x^\alpha)(dx)^\alpha = E_\alpha(b^\alpha) - E_\alpha(a^\alpha) \tag{26.9}$$

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b \sin_\alpha x^\alpha (dx)^\alpha = \cos_\alpha a^\alpha - \cos_\alpha b^\alpha \tag{26.10}$$

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b \sin_\alpha x^\alpha (dx)^\alpha = \cos_\alpha a^\alpha - \cos_\alpha b^\alpha \tag{26.11}$$

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b x^{k\alpha}(dx)^\alpha = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)}\left(b^{(k+1)\alpha}-a^{(k+1)\alpha}\right) \tag{26.12}$$

26.3 Fractional-Order Complex Mathematics

Definition 26.9. Fractional-order complex number is defined by [48, 49]

$$I^\alpha = x^\alpha + i^\alpha y^\alpha, x, y \in \mathrm{R}, 0 < \alpha \leq 1, \tag{26.13}$$

where its conjugate of complex number shows that

$$\overline{I^\alpha} = x^\alpha - i^\alpha y^\alpha_{,} \tag{26.14}$$

and where the fractional modulus is derived as

$$|I^\alpha| = I^\alpha \overline{I^\alpha} = \overline{I^\alpha} I^\alpha = \sqrt{x^{2\alpha}+y^{2\alpha}}. \tag{26.15}$$

Definition 26.10. Complex Mittag–Leffler function in fractal space is defined by [48, 49]

$$E_\alpha(z^\alpha) := \sum_{k=0}^{\infty} \frac{z^{\alpha k}}{\Gamma(1+k\alpha)}, \tag{26.16}$$

for $z \in C$(complex number set) and $0 < \alpha \leq 1$.

The following rules hold:

$$E_\alpha(z_1^\alpha)E_\alpha(z_2^\alpha) = E_\alpha\left((z_1+z_2)^\alpha\right) \tag{26.17}$$

$$E_\alpha(z_1^\alpha)E_\alpha(-z_2^\alpha) = E_\alpha\left((z_1-z_2)^\alpha\right) \tag{26.18}$$

$$E_\alpha(i^\alpha z_1^\alpha)E_\alpha(i^\alpha z_2^\alpha) = E_\alpha\left(i^\alpha(z_1^\alpha+z_2^\alpha)^\alpha\right) \tag{26.19}$$

When $z^\alpha = i^\alpha x^\alpha$, the complex Mittag–Leffler function is computed by

$$E_\alpha(i^\alpha x^\alpha) = \cos_\alpha x^\alpha + i^\alpha \sin_\alpha x^\alpha \tag{26.20}$$

with $\cos_\alpha x^\alpha := \sum\limits_{k=0}^{\infty}(-1)^k \frac{x^{2\alpha k}}{\Gamma(1+2\alpha k)}$ and $\sin_\alpha x^\alpha := \sum\limits_{k=0}^{\infty}(-1)^k \frac{x^{\alpha(2k+1)}}{\Gamma[1+\alpha(2k+1)]}$, for $x \in \mathrm{R}$ and $0 < \alpha \leq 1$, we have that

$$E_\alpha(i^\alpha x^\alpha)E_\alpha(i^\alpha y^\alpha) = E_\alpha\left(i^\alpha(x+y)^\alpha\right) \tag{26.21}$$

and

$$E_\alpha(i^\alpha x^\alpha)E_\alpha(-i^\alpha y^\alpha) = E_\alpha\left(i^\alpha(x-y)^\alpha\right). \tag{26.22}$$

26.4 Generalization of Local Fractional Fourier Series in Generalized Hilbert Space

26.4.1 Generalized Inner Product Space

Definition 26.11. Let V be a complex or real vector space. A generalized inner product on a vector space V is a function$\langle x^{\alpha}, y^{\alpha}\rangle_{\alpha}$ on pairs (x^{α}, y^{α}) of vectors in $V \times V$ taking values satisfying the following properties [48, 49]:

(1) $\langle x^{\alpha}, x^{\alpha}\rangle_{\alpha} \geq 0$ for all $x^{\alpha} \in V$ and $\langle x^{\alpha}, x^{\alpha}\rangle_{\alpha} = 0$ only if $x = 0$
(2) $\langle x^{\alpha}, y^{\alpha}\rangle_{\alpha} = \overline{\langle y^{\alpha}, x^{\alpha}\rangle_{\alpha}}$ for all $x^{\alpha}, y^{\alpha} \in V$
(3) For all $x^{\alpha}, y^{\alpha}, z^{\alpha} \in V$ and scalars $a, b \in \mathrm{R}$,

$$\langle a^{\alpha}x^{\alpha} + b^{\alpha}y^{\alpha}, z^{\alpha}\rangle_{\alpha} = a^{\alpha}\langle x^{\alpha}, z^{\alpha}\rangle_{\alpha} + b^{\alpha}\langle y^{\alpha}, z^{\alpha}\rangle_{\alpha} \tag{26.23}$$

A generalized inner product space is a generalized vector space with an inner product.

Given a generalized inner product space, the following definition provides a norm:

$$\|x^{\alpha}\|_{\alpha} = \langle x^{\alpha}, x^{\alpha}\rangle_{\alpha}^{\frac{1}{2}} = \sqrt{\sum_{k=1}^{\infty} \left|x_k^{\alpha}\right|^2}. \tag{26.24}$$

Now we can define a scalar (or dot) product of two T-periodic functions $f(t)$ and $g(t)$ as

$$\langle f, g\rangle_{\alpha} = \int_0^T f(t)\,\overline{g(t)}\,(dt)^{\alpha}. \tag{26.25}$$

For more materials, we see [48, 49].

26.4.2 Generalized Hilbert Space

Definition 26.12. A generalized Hilbert space is a complete generalized inner product space [48, 49].

Suppose $\{e_n^{\alpha}\}$ is an orthonormal system in an inner product space X. The following are equivalent [48, 49]:

1. *span* $\left\{e_1^{\alpha}, \ldots, e_n^{\alpha}\right\} = X$, i.e., $\{e_n^{\alpha}\}$ is a basis.
2. (*Pythagorean theorem in fractal space*)
   The equation

$$\sum_{k=1}^{\infty} |a_k^{\alpha}|^2 = \|f\|_{\alpha}^2 \tag{26.26}$$

   for all $f \in X$, where $a_k^{\alpha} = \left\langle f, e_k^{\alpha}\right\rangle_{\alpha}$.

3. *(Generalized Pythagorean theorem in fractal space)*
   Generalized equation

$$\langle f,g\rangle = \sum_{k=1}^{n} a_k^\alpha \overline{b_k^\alpha} \tag{26.27}$$

   for all $f,g \in X$, where $a_k^\alpha = \langle f, e_n^\alpha\rangle_\alpha$ and $b_k^\alpha = \left\langle g, e_k^\alpha\right\rangle_\alpha$.
4. $f = \sum_{k=1}^{n} a_k^\alpha e_k^\alpha$ with sum convergent in X for all $f \in X$.

For more details, see [48, 49].

Here we can take any sequence of T-periodic fractal functions φ_k, $k = 0,1,\ldots$ that are

1. Orthogonal:

$$\left\langle \varphi_k, \varphi_j\right\rangle_\alpha = \int_0^T \varphi_k(t)\overline{\varphi_j(t)}(dt)^\alpha = 0 (if k \neq j) \tag{26.28}$$

2. Normalized:

$$\langle \varphi_k, \varphi_k\rangle_\alpha = \int_0^T \varphi_k^2(t)(dt)^\alpha = 1 \tag{26.29}$$

3. Complete: If a function $x(t)$ is such that

$$\langle x, \varphi_k\rangle_\alpha = \int_0^T x(t)\varphi_k(t)(dt)^\alpha = 0 \tag{26.30}$$

for all i, then $x(t) \equiv 0$.

26.4.3 Generalization of Local Fractional Fourier Series in Generalized Hilbert Space

26.4.3.1 Generalization of Local Fractional Fourier Series in Generalized Hilbert Space

Definition 26.13. Let $\{\varphi_k(t)\}_{k=1}^{\infty}$ be a complete, orthonormal set of functions. Then any T-periodic fractal signal $f(t)$ can be uniquely represented as an infinite series

$$f(t) = \sum_{k=0}^{\infty} \phi_k \varphi_k(t) \tag{26.31}$$

This is called the local fractional Fourier series representation of $f(t)$ in the generalized Hilbert space. The scalars ϕ_i are called the local fractional Fourier coefficients of $f(t)$.

26.4.3.2 Local Fractional Fourier Coefficients

To derive the formula for ϕ_k, write

$$f(t)\,\varphi_k(t) = \sum_{i=0}^{\infty} \phi_j \varphi_j(t)\varphi_k(t)\,, \tag{26.32}$$

and integrate over one period by using the generalized Pythagorean theorem in fractal space

$$\begin{aligned}
&\langle f, \varphi_k \rangle_\alpha \\
&= \int_0^T f(t)\,\varphi_k(t)\,(dt)^\alpha \\
&= \int_0^T \sum_{j=0}^{\infty} \phi_j \varphi_j(t)\varphi_k(t)\,(dt)^\alpha \\
&= \sum_{j=0}^{\infty} \left(\phi_j \left(\int_0^T \varphi_j(t)\,\varphi_k(t)\,(dt)^\alpha\right)\right) \\
&= \sum_{j=0}^{\infty} \phi_j \left\langle \varphi_j, \varphi_k \right\rangle_\alpha \\
&= \phi_k
\end{aligned} \tag{26.33}$$

Because the functions $\varphi_k(t)$ form a complete orthonormal system, the partial sums of the local fractional Fourier series

$$f(t) = \sum_{k=0}^{\infty} \phi_k \varphi_k(t) \tag{26.34}$$

converge to $f(t)$ in the following sense:

$$\lim_{N\to\infty}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^T \left(f(t) - \sum_{k=1}^{\infty} \phi_k \varphi_k(t)\right)\overline{\left(f(t) - \sum_{k=1}^{\infty} \phi_k \varphi_k(t)\right)}\,(dt)^\alpha\right) = 0. \tag{26.35}$$

Therefore, we can use the partial sums

$$f_N(t) = \sum_{k=1}^{N} \phi_k \varphi_k(t) \tag{26.36}$$

to approximate $f(t)$.

Meanwhile, we have that

$$\int_0^T f^2(t)\,(dt)^\alpha = \sum_{k=1}^{\infty} \phi_k^2. \tag{26.37}$$

The sequence of T-periodic functions in fractal space $\{\varphi_k(t)\}_{k=0}^{\infty}$ defined by

$$\varphi_0(t) = \left(\frac{1}{T}\right)^{\frac{\alpha}{2}} and \varphi_k(t) = \begin{cases} \left(\frac{2}{T}\right)^{\frac{\alpha}{2}} \sin_\alpha\left(k^\alpha \omega_0^\alpha t^\alpha\right), if k \geq 1 is odd \\ \left(\frac{2}{T}\right)^{\frac{\alpha}{2}} \cos_\alpha\left(k^\alpha \omega_0^\alpha t^\alpha\right), if k > 1 is even \end{cases} \tag{26.38}$$

is complete and orthonormal, where $\omega_0 = \frac{2\pi}{T}$.

A more common way of writing down the local fractional trigonometric Fourier series of $f(t)$ is this

$$f(t) = a_0 + \sum_{i=1}^{\infty} a_k \sin_\alpha(k^\alpha \omega_0^\alpha t^\alpha) + \sum_{i=1}^{\infty} b_k \cos_\alpha(k^\alpha \omega_0^\alpha t^\alpha) \tag{26.39}$$

Then the local fractional Fourier coefficients can be computed by

$$\begin{cases} a_0 = \frac{1}{T^\alpha} \int_0^T f(t)(dt)^\alpha, \\ a_k = \left(\frac{2}{T}\right)^\alpha \int_0^T f(t) \sin_\alpha\left(k^\alpha \omega_0^\alpha t^\alpha\right)(dt)^\alpha, \\ b_k = \left(\frac{2}{T}\right)^\alpha \int_0^T f(t) \cos_\alpha\left(k^\alpha \omega_0^\alpha t^\alpha\right)(dt)^\alpha. \end{cases} \tag{26.40}$$

This result is equivalent to results [48, 49, 53, 54].

Another useful complete orthonormal set is furnished by the Mittag–Leffler functions:

$$\varphi_k(t) = \sqrt{\frac{1}{T^\alpha}} E_\alpha(i^\alpha k^\alpha \omega_0^\alpha t^\alpha), k = 0, \pm 1, \pm 2, \ldots \tag{26.41}$$

where $\omega_0 = \frac{2\pi}{T}$.

26.5 Local Fractional Fourier Analysis

Any periodic fractal function $f(t)$ can be represented with a set of Mittag–Leffler functions as shown below.

$$\begin{aligned} f(t) &= \sum_{k=-\infty}^{\infty} F_k E_\alpha(i^\alpha k^\alpha \omega_0^\alpha t^\alpha) \\ &= F_0 + F_1 E_\alpha(i^\alpha \omega_0^\alpha t^\alpha) + F_2 E_\alpha(2^\alpha i^\alpha \omega_0^\alpha t^\alpha) + \ldots + F_n E_\alpha(n^\alpha i^\alpha \omega_0^\alpha t^\alpha) \\ &+ \ldots + F_{-1} E_\alpha(-i^\alpha \omega_0^\alpha t^\alpha) + F_{-2} E_\alpha(-2^\alpha i^\alpha \omega_0^\alpha t^\alpha) + \ldots + F_{-n} E_\alpha(-n^\alpha i^\alpha \omega_0^\alpha t^\alpha) + \ldots, \end{aligned} \tag{26.42}$$

where $\omega_0 = \frac{2\pi}{T}$.

Representing a fractal function in terms of its local fractional Fourier series components with the Mittag–Leffler functions in fractal space is called the local fractional Fourier analysis. Here the Mittag–Leffler function terms are orthogonal to each other since

$$\frac{1}{T^\alpha} \int_0^T E_\alpha(i^\alpha m^\alpha \omega_0^\alpha t^\alpha) \overline{E_\alpha\left(i^\alpha n^\alpha \omega_0^\alpha t^\alpha\right)} (dt)^\alpha = 0, \quad m \neq n, \tag{26.43}$$

and the energy of these fractal signals is unity because

$$\frac{1}{T^\alpha} \int_0^T E_\alpha(i^\alpha m^\alpha \omega_0^\alpha t^\alpha) \overline{E_\alpha\left(i^\alpha n^\alpha \omega_0^\alpha t^\alpha\right)} (dt)^\alpha = 1, \quad m = n. \tag{26.44}$$

Now this process also shows that

$$\begin{aligned}
&\left\langle f(t), E_\alpha\left(i^\alpha j^\alpha \omega_0^\alpha t^\alpha\right)\right\rangle_\alpha \\
&= \int_0^T f(t) E_\alpha\left(-i^\alpha j^\alpha \omega_0^\alpha t^\alpha\right)(dt)^\alpha \\
&= \int_0^T \left(\sum_{k=-\infty}^{\infty} F_k E_\alpha\left(i^\alpha k^\alpha \omega_0^\alpha t^\alpha\right)\right) E_\alpha\left(-i^\alpha j^\alpha \omega_0^\alpha t^\alpha\right)(dt)^\alpha \\
&= \int_0^T \left(\sum_{k=-\infty}^{\infty} F_k E_\alpha\left(i^\alpha k^\alpha \omega_0^\alpha t^\alpha\right) E_\alpha\left(-i^\alpha j^\alpha \alpha \omega_0^\alpha t^\alpha\right)\right)(dt)^\alpha \\
&= \sum_{k=-\infty}^{\infty} F_k \left(\int_0^T E_\alpha\left(i^\alpha k^\alpha \omega_0^\alpha t^\alpha\right) E_\alpha\left(-i^\alpha j^\alpha \omega_0^\alpha t^\alpha\right)(dt)^\alpha\right) \\
&= \sum_{k=-\infty}^{\infty} F_k \left(\int_0^T E_\alpha\left(i^\alpha (k-j)^\alpha \omega_0^\alpha t^\alpha\right)(dt)^\alpha\right) \\
&= T^\alpha \sum_{k=-\infty}^{\infty} F_k \left\langle E_\alpha\left(i^\alpha k^\alpha \omega_0^\alpha t^\alpha\right), E_\alpha\left(i^\alpha j^\alpha \omega_0^\alpha t^\alpha\right)\right\rangle_\alpha \\
&= F_j T^\alpha
\end{aligned} \tag{26.45}$$

Hence we get the local fractional Fourier coefficient as follows:

$$F_k = \frac{1}{T^\alpha} \int_0^T f(t) E_\alpha(-i^\alpha k^\alpha \omega_0^\alpha t^\alpha)(dt)^\alpha. \tag{26.46}$$

In like manner, we derive F_k as

$$\langle f(t), E_\alpha(-i^\alpha k^\alpha \omega_0^\alpha t^\alpha)\rangle_\alpha = \overline{\left\langle f(t), E_\alpha\left(i^\alpha k^\alpha \omega_0^\alpha t^\alpha\right)\right\rangle_\alpha} = \overline{F_k}.$$

The weights of the Mittag–Leffler functions are computed by

$$\begin{aligned}
F_k &= \frac{\frac{1}{\Gamma(1+\alpha)} \int_0^T f(t) \overline{E_\alpha\left(i^\alpha k^\alpha \omega_0^\alpha t^\alpha\right)}(dt)^\alpha}{\frac{1}{\Gamma(1+\alpha)} \int_0^T E_\alpha\left(i^\alpha k^\alpha \omega_0^\alpha t^\alpha\right) \overline{E_\alpha\left(i^\alpha k^\alpha \omega_0^\alpha t^\alpha\right)}(dt)^\alpha} \\
&= \frac{1}{T^\alpha} \int_0^T f(t) \overline{E_\alpha\left(i^\alpha k^\alpha \omega_0^\alpha t^\alpha\right)}(dt)^\alpha \\
&= \frac{1}{T^\alpha} \int_0^T f(t) E_\alpha\left(-i^\alpha k^\alpha \omega_0^\alpha t^\alpha\right)(dt)^\alpha.
\end{aligned} \tag{26.47}$$

For any interval $[t_0, t_0+T]$, we show that

$$\begin{aligned}
F_k &= \frac{\frac{1}{\Gamma(1+\alpha)} \int_{t_0}^{t_0+T} f(t) \overline{E_\alpha\left(i^\alpha k^\alpha \omega_0^\alpha t^\alpha\right)}(dt)^\alpha}{\frac{1}{\Gamma(1+\alpha)} \int_{t_0}^{t_0+T} E_\alpha\left(i^\alpha k^\alpha \omega_0^\alpha t^\alpha\right) \overline{E_\alpha\left(i^\alpha k^\alpha \omega_0^\alpha t^\alpha\right)}(dt)^\alpha} \\
&= \frac{1}{T^\alpha} \int_{t_0}^{t_0+T} f(t) \overline{E_\alpha\left(i^\alpha k^\alpha \omega_0^\alpha t^\alpha\right)}(dt)^\alpha \\
&= \frac{1}{T^\alpha} \int_{t_0}^{t_0+T} f(t) E_\alpha\left(-i^\alpha k^\alpha \omega_0^\alpha t^\alpha\right)(dt)^\alpha
\end{aligned} \tag{26.48}$$

When $T \to \infty$ and $\omega_0 \to 0$, the sum becomes a local fractional integral and ω_0^α becomes local fractional continuous. Therefore, the resulting representation is termed as the analysis equation $f_\omega^{F,\alpha}(\omega)$, given by [48, 49, 52–54]

$$f_\omega^{F,\alpha}(\omega) = \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} E_\alpha(-i^\alpha \omega^\alpha x^\alpha) f(x)(dx)^\alpha. \tag{26.49}$$

The function $f(t)$ can recovered from $f_{\omega}^{F,\alpha}(\omega)$ as

$$f(x)=\frac{1}{(2\pi)^{\alpha}}\int_{-\infty}^{\infty}E_{\alpha}\left(i^{\alpha}\omega^{\alpha}x^{\alpha}\right)f_{\omega}^{F,\alpha}(\omega)\,(d\omega)^{\alpha}. \tag{26.50}$$

26.6 An Illustrative Example

Expand a l-period fractal signal $X(t)=A\left(0<t\leq\frac{l}{2}\right)$ in local fractional Fourier series.

Since the local fractional Fourier coefficients can be derived as

$$F_0=\frac{1}{l^{\alpha}}\int_0^{\frac{l}{2}}f(t)\,(dt)^{\alpha}=\frac{1}{l^{\alpha}}\int_0^{\frac{l}{2}}A\,(dt)^{\alpha}=\frac{A}{2^{\alpha}}, \tag{26.51}$$

$$\begin{aligned}&F_k\\&=\tfrac{1}{l^{\alpha}}\textstyle\int_0^{\frac{l}{2}}f(t)\,E_{\alpha}\left(-i^{\alpha}k^{\alpha}\left(\frac{2\pi}{l}\right)^{\alpha}t^{\alpha}\right)(dt)^{\alpha}\\&=\tfrac{1}{l^{\alpha}}\textstyle\int_0^{\frac{l}{2}}AE_{\alpha}\left(-i^{\alpha}k^{\alpha}\left(\frac{2\pi}{l}\right)^{\alpha}t^{\alpha}\right)(dt)^{\alpha}\\&=\tfrac{A\Gamma(1+\alpha)}{(2\pi)^{\alpha}}\left(1-E_{\alpha}\left(-i^{\alpha}k^{\alpha}\pi^{\alpha}\right)\right)\\&=\tfrac{A\Gamma(1+\alpha)}{(2\pi)^{\alpha}}\left(1-E_{\alpha}\left(-i^{\alpha}k^{\alpha}\pi^{\alpha}\right)\right)\end{aligned} \tag{26.52}$$

$$F_{-k}=\overline{F_{-k}}=\frac{A\Gamma(1+\alpha)}{(2\pi)^{\alpha}}\left(1-E_{\alpha}\left(i^{\alpha}k^{\alpha}\pi^{\alpha}\right)\right) \tag{26.53}$$

Hence, for $0<t\leq\frac{l}{2}$, the fractal signal is presented as

$$\begin{aligned}&X(t)\\&=\sum_{k=-\infty}^{\infty}F_kE_{\alpha}\left(i^{\alpha}k^{\alpha}\omega_0^{\alpha}t^{\alpha}\right)\\&=\tfrac{A}{2^{\alpha}}+\sum_{k=1}^{\infty}\left[\tfrac{A\Gamma(1+\alpha)}{(2\pi)^{\alpha}}\left(1-E_{\alpha}\left(-i^{\alpha}k^{\alpha}\pi^{\alpha}\right)\right)\right]E_{\alpha}\left(\tfrac{i^{\alpha}k^{\alpha}(2\pi)^{\alpha}t^{\alpha}}{l^{\alpha}}\right)\\&+\sum_{k=1}^{\infty}\left[\tfrac{A\Gamma(1+\alpha)}{(2\pi)^{\alpha}}\left(1-E_{\alpha}\left(i^{\alpha}k^{\alpha}\pi^{\alpha}\right)\right)\right]E_{\alpha}\left(-\tfrac{i^{\alpha}k^{\alpha}(2\pi)^{\alpha}t^{\alpha}}{l^{\alpha}}\right)\\&=\tfrac{A}{2^{\alpha}}+\sum_{k=1}^{\infty}\left[\tfrac{A\Gamma(1+\alpha)}{(2\pi)^{\alpha}}\left(1-(-1)^k\right)\right]E_{\alpha}\left(\tfrac{i^{\alpha}k^{\alpha}(2\pi)^{\alpha}t^{\alpha}}{l^{\alpha}}\right)\\&+\sum_{k=1}^{\infty}\left[\tfrac{A\Gamma(1+\alpha)}{(2\pi)^{\alpha}}\left(1-(-1)^k\right)\right]E_{\alpha}\left(-\tfrac{i^{\alpha}k^{\alpha}(2\pi)^{\alpha}t^{\alpha}}{l^{\alpha}}\right)\end{aligned} \tag{26.54}$$

26.7 Conclusions

In this paper, the local fractional Fourier series in generalized Hilbert space is investigated, and the local fractional Fourier analysis is proposed based on the Mittag–Leffler functions. Particular attention is devoted to the analytical technique of the local fractional Fourier analysis for treating these local fractional continuous functions in a way accessible to applied scientists. There is an efficient example, which is given to elucidate the signal process and reliable result. It is shown that local fractional Fourier analysis is the convenient Fourier analysis [59] when fractal dimension α is equal to 1.

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Chapter 27
Non-solvability of Balakrishnan–Taylor Equation with Memory Term in $\mathbb{R}^N$

Abderrahmane Zaraï and Nasser-eddine Tatar

Abstract We establish a nonexistence result for a viscoelastic problem with Balakrishnan-Taylor damping and a nonlinear source in the whole space. The nonexistence result is based on the test function method developed by Mitidieri and Pohozaev. We establish some necessary conditions for local existence and global existence as well.

27.1 Introduction

In the last 45 years or so, blow up in finite time and nonexistence of solutions for partial differential equations and systems have received an increasing attention. One can find a rather extensive bibliography on works concerning parabolic and hyperbolic equations and systems on bounded domains.

On the whole space $\mathbb{R}^N$, the pioneering work for the heat equation with a power-type nonlinearity is due to Fujita [4] in (1966). For the wave equation we can quote John [7] (1979), see also Glassey [5, 6] and Kirane and Tatar [8]. Their works have been extended and generalized to different degenerate and singular equations and inequalities and on different unbounded domains (like exterior domains and cones).

The question of non-solvability of evolution equations has been treated and discussed from different angles using different methods and techniques. The basic idea in most of these works is to compare solutions with sub-solutions that blow up in finite time.

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Our concern, in this paper, is a viscoelastic problem with a power-type source as an external force on the whole space $\mathbb{R}^N$, $N \geq 1$. Here we study the case where the kernel h decays polynomially just to fix ideas, but the result remains valid for many other types of kernels such as exponentially decaying functions. Namely, we are concerned with the following initial-boundary value problem

$$\begin{cases} u_{tt} - \left(\xi_0 + \xi_1 \|\nabla u(t)\|_2^2 + \sigma(\nabla u(t), \nabla u_t(t))\right) \Delta u \\ + \int_0^t h(t-s)\Delta u ds + \delta u_t = |u|^p \quad in \;\; \mathbb{R}^N \times [0,+\infty) \\ \qquad u(0,x) = u_0(x) \in L^1_{loc}\left(\mathbb{R}^N\right) \\ \qquad u_t(0,x) = u_1(x) \in L^1_{loc}\left(\mathbb{R}^N\right), \end{cases} \tag{27.1}$$

where $p > 1$ and $u_0(x)$ and $u_1(x)$ are given initial data. Here h represents the kernel of the memory term. All the parameters ξ_0, ξ_1 and σ are assumed to be positive constants. The model in hand in a bounded domain Ω of $\mathbb{R}^N$, with Balakrishnan–Taylor damping ($\sigma > 0$) and $h = 0$, was initially proposed by Balakrishnan and Taylor in 1989 [1] and Bass and Zes [2]. It is related to the panel flutter equation and to the spillover problem. So far it has been studied by Y. You [13], H. R. Clark [3] and N.e. Tatar and A. Zaraï [10–12] and several results on exponential decay and blow up in finite time have been obtained.

In the present work, we are interested in conditions for non-solvability of (27.1). The method we use is the so-called test function method developed by Mitidieri and Pohozaev [9]. Our proof is based on an argument by contradiction, which involves a priori estimates for the weak solutions of (27.1) and a careful choice of a special test functions and a scaling argument.

The main goal of this paper is to find a range of values for p for which we have nonexistence under minimal assumptions on h.

The plan of the remaining part of the paper is as follows: in the next section we present the notation and what we mean by a (weak) solution to our problem. Section 27.3 contains our result on nonexistence of solutions. In Sect. 27.4 we present some necessary conditions for local existence and global existence of solutions.

27.2 Preliminaries

We shall denote by Q_T the set $Q_T := (0,T) \times \mathbb{R}^N$ and $Q := Q_\infty$.

We next make it clear what we mean by a weak solution of problem (27.1).

Definition 27.1. A weak solution of problem (27.1) is a continuous function u: $\mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$ such that

$$\begin{aligned} &\int_{Q_T} |u|^p \varphi dxdt + \int_{\mathbb{R}^N} u_1(x)\varphi(0,x)dx + \delta \int_{\mathbb{R}^N} u_0(x)\varphi(0,x)dx \\ &= \int_{Q_T} u\varphi_{tt} dxdt - \int_{Q_T} M(t)u\Delta\varphi dxdt - \delta \int_{Q_T} u\varphi_t dxdt \\ &+ \int_{Q_T} u(s,x)\left(\int_s^T h(t-s)\Delta\varphi(t)dt\right) dsdx \end{aligned} \tag{27.2}$$

holds for any $\varphi \in C_0^2(Q_T), \varphi \geq 0$ and satisfying

$$\varphi(T,x) = \varphi_t(T,x) = \varphi_t(0,x) = 0,$$

where

$$M(t) = \xi_0 + \xi_1 \|\nabla u(t)\|_2^2 + \sigma(\nabla u(t), \nabla u_t(t)).$$

By $\varphi \in C_0^2(Q_T)$ we mean a function φ in $C_{t,x}^{2,2}$ and with compact support.

Now, we state the hypothesis

(H) $h : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded C^1-function satisfying

$$h(t) \leq \frac{K}{(1+t)^\rho},\ t \geq 0,$$

for some constant $K > 0$ and $\rho \in (2, \infty)$.

27.3 Nonexistence Result

In this section we prove our result. It provides a whole range of values for p for which no weak solutions can exist globally in time.

Theorem 27.2. *Suppose that* $\int_{\mathbb{R}^N} u_1(x)\,dx + \delta \int_{\mathbb{R}^N} u_0(x)\,dx > 0$ *and* (**H**) *hold. Assume that* θ*,* N *and* $\tilde{p}$ *are as in the following table:*

$N = 1$	$\theta = 2, \quad \tilde{p} = 2$
$N = 2$	$\theta = 2, \quad \tilde{p} = 1$
$N = 3$	$\theta = 1, \quad \tilde{p} = \frac{1}{3}$

Then, there does not exist any global weak solution to (27.1) for all $1 < p < 1 + \tilde{p}$.

Proof. The proof is by contradiction. Assume that a weak solution of (27.1) exists globally in time. We introduce the test function

$$\varphi(t,x) := \phi\left(\frac{|x|}{R}\right)\mu\left(\frac{t}{R^\theta}\right) \tag{27.3}$$

with $\phi \in C_0^\infty(\mathbb{R}^N)$, $\phi \geq 0$, $\mu \in C^2(\mathbb{R}^+)$, $\mu \geq 0$ such that

$$\phi(w), \mu(w) = \begin{cases} 1, & |w| \leq 1 \\ 0, & |w| > 2 \end{cases}$$

and μ satisfies $-C \leq \mu'(t) \leq 0$, $\mu'(2R^\theta) = 0$ for $R >> 1$. The function $\varphi(t,x)$ is supposed to have bounded second-order partial derivatives. Moreover, we assume without loss of generality that

$$\begin{aligned} &\int_{supp\Delta\varphi} M(t)\,|\Delta\varphi|^q(\varphi)^{1-q}dxdt \\ &+ \int_{supp\varphi_{tt}} |\varphi_{tt}|^q(\varphi)^{1-q}dxdt + \int_{supp\varphi_t} |\varphi_t|^q(\varphi)^{1-q}dxdt < \infty, \end{aligned} \tag{27.4}$$

where q is the conjugate exponent of p. If this condition is not satisfied for our function $\varphi(t,x)$, then we pick $\varphi^{\lambda}(t,x)$ with some sufficiently large $\lambda>0$. We choose this test function in the definition of a weak solution and start estimating the different terms in this definition. By multiplying and dividing by $\varphi^{1/p}$, then applying the ε-Young inequality, we see that

$$\begin{aligned}\int_Q u\varphi_{tt}dtdx &\le \int_{supp\varphi_{tt}} u\varphi^{1/p}\varphi^{-1/p}\varphi_{tt}dtdx\\ &\le \varepsilon\int_{supp\varphi_{tt}}|u|^p\varphi dtdx + C_\varepsilon\int_{supp\varphi_{tt}}\varphi^{-q/p}|\varphi_{tt}|^q dtdx.\end{aligned}\tag{27.5}$$

Likewise, we find

$$\begin{aligned}&-\int_Q M(t)u\Delta\varphi dxdt\\ &\le \varepsilon\int_{supp\Delta\varphi}|u|^p\varphi dxdt + C_\varepsilon\int_{supp\Delta\varphi}|M(t)|^q(\varphi)^{-q/p}|\Delta\varphi|^q dxdt,\end{aligned}\tag{27.6}$$

$$\begin{aligned}&-\delta\int_Q u\varphi_t dxdt\\ &\le \varepsilon\int_{supp\varphi_t}|u|^p\varphi dxdt + C_\varepsilon\int_{supp\varphi_t}\delta^q(\varphi)^{-q/p}|\varphi_t|^q dxdt\end{aligned}\tag{27.7}$$

and

$$\begin{aligned}&\int_Q u\left(\int_s^{+\infty}h(t-s)\Delta\varphi(t)dt\right)dsdx\\ &\le \varepsilon\int_{supp\Delta\varphi}|u|^p\varphi dsdx + C_\varepsilon\int_{supp\Delta\varphi}(\varphi)^{-q/p}\left|\int_s^{+\infty}h(t-s)\Delta\varphi(t)dt\right|^q dsdx.\end{aligned}\tag{27.8}$$

Taking into account the last three estimates (27.5)–(27.8) in (27.2) we see that for small ε (for instance, $\varepsilon\le 1/5$)

$$\begin{aligned}&\frac{1}{5}\int_Q|u|^p\varphi dxdt + \int_{\mathbb{R}^N}u_1(x)\varphi(0,x)dx + \delta\int_{\mathbb{R}^N}u_0(x)\varphi(0,x)dx\\ &\le C_{1/5}\int_{supp\varphi}(\varphi)^{-q/p}\Big[|\varphi_{tt}|^q + |M(t)|^q|\Delta\varphi|^q\\ &\quad+\delta^q|\varphi_t|^q + \left|\int_s^{+\infty}h(t-s)\Delta\varphi(t)dt\right|^q\Big]dsdx.\end{aligned}\tag{27.9}$$

Let us now consider the following scaling: $t=R^\theta\tau$ and $x=Ry$. Then, it is clear that

$$\int_{supp\varphi_{tt}}(\varphi)^{-q/p}|\varphi_{tt}|^q dtdx \le CR^{N+\theta-2\theta q}\int_\Omega(\varphi)^{-q/p}|\varphi_{\tau\tau}|^q d\tau dy,\tag{27.10}$$

$$\begin{aligned}&\int_{supp\Delta\varphi}(\varphi)^{-q/p}|M(t)|^q|\Delta\varphi|^q dtdx\\ &\le CR^{N+\theta-2q}\int_\Omega(\varphi)^{-q/p}|\Delta\varphi|^q\Big\{\xi_0+\xi_1R^{N-2}\int_{\mathbb{R}^N}|\nabla_*u|^2dy\\ &+R^{N-\theta-2}\frac{\sigma d}{2d\tau}\int_{\mathbb{R}^N}|\nabla_*u|^2dy\Big\}^q d\tau dy\\ &\le CR^{(q+1)N+\theta-4q}\int_\Omega(\varphi)^{-q/p}|\Delta\varphi|^q\Big\{\xi_0+\xi_1\int_{\mathbb{R}^N}|\nabla_*u|^2dy\\ &+\frac{\sigma d}{2d\tau}\int_{\mathbb{R}^N}|\nabla_*u|^2dy\Big\}^q d\tau dy,\end{aligned}\tag{27.11}$$

where $\nabla_* u = \sum_{i=1}^{N} \frac{\partial u}{\partial y_i}$, and

$$\int_{supp\varphi_t} \delta^q (\varphi)^{-q/p} |\varphi_t|^q dxdt \le C\delta^q R^{N+\theta-\theta q} \int_\Omega (\varphi)^{-q/p} |\varphi_\tau|^q d\tau dy. \tag{27.12}$$

Here and in the rest of the proof C is a positive constant which may be different at different occurrences. For the term containing the memory let us rewrite it as

$$\int_{supp\Delta\varphi} (\varphi)^{-q/p} \left| \int_t^{+\infty} h(\nu-t)\Delta\varphi(\nu)d\nu \right|^q dtdx$$

and use the scaling to get

$$\begin{aligned} &\int_{supp\Delta\varphi} (\varphi)^{-q/p} \left| \int_t^{+\infty} h(\nu-t)\Delta\varphi(\nu)d\nu \right|^q dtdx \\ &= \int_{D_R} |\Delta\phi|^q \phi^{-q/p} \int_0^{2R} (\mu)^{-q/p} \left| \int_t^{+\infty} h(\nu-t)\mu(\nu)d\nu \right|^q dtdx \\ &\le CR^{N+\theta-2q} \int_\Omega |\Delta\phi|^q \varphi^{-\frac{q}{p}} \left| \int_{R^\theta\tau}^{+\infty} h(\nu-R^\theta\tau)\mu(\nu)d\nu \right|^q d\tau dy, \end{aligned} \tag{27.13}$$

where $\Omega := \{(\tau,y) : 1 \le \tau, |y| \le 2\}$ and $D_R := \{x \in \mathbb{R}^N : R < |x| < 2R\}$. In virtue of the assumption **(H)** and by the change of variable $1+\nu-R^\theta\tau = \eta$ and the fact that μ is non increasing we see that

$$\int_{R^\theta\tau}^{+\infty} h(\nu-R^\theta\tau)\mu(\nu)d\nu \le K\int_1^{+\infty} \frac{\mu(\eta+R^\theta\tau-1)}{\eta^\rho}d\eta$$

as $R^\theta\tau \ge 1$ and as $\mu(\eta)=0$ for $\eta \ge 2$ and $\mu(\eta) \le 1$ we have

$$\int_{R^\theta\tau}^{+\infty} h(\nu-R^\theta\tau)\mu(\nu)d\nu \le K\int_1^2 \frac{1}{\eta^\rho}d\eta \le C$$

and therefore

$$\begin{aligned} &\int_{supp\Delta\varphi} (\varphi)^{-q/p} \left| \int_t^{+\infty} h(\nu-t)\Delta\varphi(\nu)d\nu \right|^q dtdx \\ &\le CR^{N+\theta-2q} \int_\Omega |\Delta\varphi|^q (\varphi)^{-q/p} d\tau dy. \end{aligned} \tag{27.14}$$

The relations (27.9) and (27.4) together with the estimates (27.10)–(27.14) yield

$$\begin{aligned} &\frac{1}{5}\int_Q |u|^p \varphi dxdt + \int_{\mathbb{R}^N} u_1(x)\varphi(0,x)dx + \delta\int_{\mathbb{R}^N} u_0(x)\varphi(0,x)dx \\ &\le C\Big\{ R^{N+\theta-2\theta q} + R^{(q+1)N+\theta-4q} \int_\Omega (\varphi)^{-q/p} |\Delta\varphi|^q \tilde{M}^q(\tau)d\tau dy \\ &\qquad + R^{N+\theta-\theta q} + R^{N+\theta-2q} \Big\}, \end{aligned} \tag{27.15}$$

where

$$\tilde{M}(\tau) = \xi_0 + \xi_1 \int_{\mathbb{R}^N} |\nabla_* u|^2 dy + \frac{\sigma d}{2d\tau} \int_{\mathbb{R}^N} |\nabla_* u|^2 dy.$$

Now, if $1 < p < 1+\tilde{p}$, where $\tilde{p}$ is as in the table, then from (27.15) we infer

$$\lim_{R\to\infty}\left[\frac{1}{5}\int_{Q}|u|^{p}\,\varphi dxdt+\int_{\mathbb{R}^N}u_1(x)\,\varphi(0,x)\,dx+\delta\int_{\mathbb{R}^N}u_0(x)\,\varphi(0,x)\,dx\right]\le 0.$$

In fact, the parameter θ has been chosen, after some simple computations, as large as possible so that the four exponents in (27.15) be nonpositive. That is,

$$\begin{cases} N+\theta-2\theta q<0\\ (q+1)N+\theta-4q<0\\ N+\theta-\theta q<0\\ N+\theta-2q<0.\end{cases}$$

On the other hand, the left hand side of (27.15) is equal to $\frac{1}{5}\int_Q|u|^p\,dxdt+\int_{\mathbb{R}^N}u_1(x)\,dx+\delta\int_{\mathbb{R}^N}u_0(x)\,dx$. This is a contradiction since we have assumed that $\int_{\mathbb{R}^N}u_1(x)\,dx+\delta\int_{\mathbb{R}^N}u_0(x)>0$. Hence the theorem is proved. □

27.4 Necessary Conditions for Local and Global Solutions

Theorem 27.3. *Let u be a local solution to (27.1) where $T<+\infty$ and $p>1$. Then, there exist constants α and β such that*

$$\liminf_{|x|\to\infty}(u_1(x)+\delta u_0(x))\le C_{1/5}T^{1-q}\left(\frac{\alpha}{T^q}+\beta\right).$$

Proof. By the definition of a weak solution, for any $\varphi\in C_0^\infty(Q_T)$, $\varphi\ge 0$ we have

$$\begin{aligned}&\textstyle\int_{Q_T}|u|^p\,\varphi dxdt+\int_{\mathbb{R}^N}(u_1(x)+\delta u_0(x))\,\varphi(0,x)\,dx\\ &\le\textstyle\int_{Q_T}|u|\,|\varphi_{tt}|\,dxdt+\int_{Q_T}|M(t)|\,|u|\,|\Delta\varphi|\,dxdt+\delta\int_{Q_T}|u|\,|\varphi_t|\,dxdt\\ &+\textstyle\int_{Q_T}|u(s,x)|\left|\int_s^T h(t-s)\Delta\varphi(t)dt\right|dsdx.\end{aligned}\tag{27.16}$$

Using the ε-Young inequality we can estimate all the terms in the right hand side of (27.16). Indeed, writing $|u|\,|\varphi_{tt}|=|u|\,\varphi^{1/p}\varphi^{-1/p}\,|\varphi_{tt}|$, we find for $\varepsilon>0$

$$\int_{Q_T}|u|\,|\varphi_{tt}|\,dtdx\le\varepsilon\int_{Q_T}|u|^p\,\varphi dtdx+C_\varepsilon\int_{Q_T}(\varphi)^{-q/p}\,|\varphi_{tt}|^q\,dtdx.\tag{27.17}$$

In a similar manner, we find

$$\begin{aligned}&\int_{Q_T}|M(t)|\,|u|\,|\Delta\varphi|\,dtdx\\ &\le\varepsilon\int_{Q_T}|u|^p\,\varphi dxdt+C_\varepsilon\int_{Q_T}|M(t)|^q\,(\varphi)^{-q/p}\,|\Delta\varphi|^q\,dxdt,\end{aligned}\tag{27.18}$$

$$\delta \int_{Q_T} |u|\,|\varphi_t|\,dtdx$$
$$\leq \varepsilon \int_{Q_T} |u|^p\,\varphi dxdt + C_\varepsilon \delta^q \int_{Q_T} (\varphi)^{-q/p}\,|\varphi_t|^q\,dxdt \tag{27.19}$$

and

$$\begin{aligned} &\int_{Q_T} |u(s,x)|\left|\int_s^T h(t-s)\Delta\varphi(t)dt\right| dsdx \\ \leq\ &\varepsilon \int_{Q_T} |u|^p\,\varphi dsdx + C_\varepsilon \int_{Q_T} (\varphi)^{-q/p}\left|\int_s^T h(t-s)\Delta\varphi(t)dt\right|^q dsdx. \end{aligned} \tag{27.20}$$

Taking $\varepsilon \leq 1/5$, we deduce from (27.17)–(27.20) and (27.16) that

$$J := \int_{\mathbb{R}^N} (u_1(x) + \delta u_0(x))\,\varphi(0,x)\,dx$$
$$\begin{aligned} &\leq C_{1/5} \int_{Q_T} \Big(|\varphi_{tt}|^q + |M(t)|^q\,(\varphi)^{-q/p}\,|\Delta\varphi|^q \\ &\quad + |\varphi_t|^q + \left|\int_s^T h(t-s)\Delta\varphi(t)dt\right|^q \Big)\,(\varphi)^{-q/p}. \end{aligned} \tag{27.21}$$

We choose the test function

$$\varphi(t,x) := \phi\left(\frac{|x|}{R}\right)\mu\left(\frac{t}{T}\right),$$

where $\phi \in C_0^\infty(\mathbb{R}^N)$, $\phi \geq 0$, $supp\phi \subset \left\{x \in \mathbb{R}^N : 1 < |x| < 2\right\}$, $|\Delta\phi| \leq k\phi$, and we take

$$\mu\left(\frac{t}{T}\right) := \begin{cases} 1,\ 0 \leqslant t \leqslant T/2 \\ 1 - \frac{(t-T/2)^3}{(T/2)^3},\ T/2 \leqslant t \leqslant T \\ 0,\ t \geqslant T. \end{cases}.$$

Next, we estimate the for terms in the right hand side of (27.16). By making the change of variables $t = \tau T$ and using the assumption on φ, we find,

$$\int_{Q_T} (\varphi)^{-q/p}\,|\varphi_{tt}|^q \leq \alpha T^{1-2q} \int_{\mathbb{R}^N} \phi, \tag{27.22}$$

$$\int_{Q_T} |M(t)|^q\,(\varphi)^{-q/p}\,|\Delta\varphi|^q \leq \frac{3}{4} M^q k^q R^{-2q} T \int_{\mathbb{R}^N} \phi, \tag{27.23}$$

$$\delta^q \int_{Q_T} (\varphi)^{-q/p}\,|\varphi_t|^q \leq \beta T^{1-q} \int_{\mathbb{R}^N} \phi \tag{27.24}$$

and

$$\int_{Q_T} (\varphi)^{-q/p}\left|\int_s^T h(t-s)\Delta\varphi(t)dt\right|^q \leq Ck^q R^{-2q} T^2 \left(\int_0^\infty h^p(t)dt\right)^{q/p} \int_{\mathbb{R}^N} \phi. \tag{27.25}$$

Gathering the relations (27.21)–(27.25), we infer that

$$\inf_{|x|>R}(u_1(x)+\delta u_0(x))\int_{\mathbb{R}^N}\phi$$

$$\leq C_{1/5}\left[\alpha T^{1-2q}+\frac{3}{4}M^q k^q R^{-2q}T+\beta T^{1-q}+Ck^q R^{-2q}T^2\right]\int_{\mathbb{R}^N}\phi. \tag{27.26}$$

Taking the sup with respect to t of both sides of (27.26), then, letting $R\to+\infty$, we obtain

$$\liminf_{|x|\to\infty}(u_1(x)+\delta u_0(x))\leq C_{1/5}\left[\alpha T^{1-2q}+\beta T^{1-q}\right]. \tag{27.27}$$

Hence the theorem is proved. □

We can immediately deduce the following result

Corollary 27.4. *Suppose that* $p>1$ *and* $u_1(x)+\delta u_0(x)\geq 0$*. If (27.1) admits a global weak solution, then*

$$\liminf_{|x|\to\infty}(u_1(x)+\delta u_0(x))=0.$$

Proof. Suppose that (27.1) has a global weak solution and that

$$S:=\liminf_{|x|\to\infty}(u_1(x)+\delta u_0(x))>0.$$

Then from (27.27), it appears that

$$T\leq\max\left\{\left(\frac{\alpha+\beta}{S}C_{1/5}\right)^{1/(q-1)},\left(\frac{\alpha+\beta}{S}C_{1/5}\right)^{1/(2q-1)}\right\}.$$

This is a contradiction. □

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Chapter 28
Study of Third-Order Three-Point Boundary Value Problem with Dependence on the First-Order Derivative

A. Guezane-Lakoud and L. Zenkoufi

Abstract Under certain conditions on the nonlinearity f and by using Leray–Schauder nonlinear alternative and the Banach contraction theorem, we prove the existence and uniqueness of nontrivial solution of the following third-order three-point boundary value problem (BVP1):

$$\begin{cases} u''' + f(t, u(t), u'(t)) = 0, & t \in (0,1) \\ \alpha u'(1) = \beta u(\eta), & u(0) = u'(0) = 0 \end{cases}$$

where $\beta,\ \alpha \in \mathbb{R}_+^*,\ 0 < \eta < 1;$

then we study the positivity by applying the well-known Guo–Krasnosel'skii fixed-point theorem. The interesting point lies in the fact that the nonlinear term is allowed to depend on the first-order derivative u'.

28.1 Introduction

The study of boundary value problems for certain linear ordinary differential equations was initiated by Il'in and Moiseev [12]. Since then more general boundary value problems for certain nonlinear ordinary differential equations been extensively studied by many authors, see [7, 9–11, 13]. Recently, the study of existence of

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positive solution to third-order boundary value problems has gained much attention and is a rapidly growing field; see [1, 3–6]. However, the approaches used in the literature are usually topological degree theory and fixed-point theorems in cone.

By using the Leray–Schauder nonlinear alternative, the Banach contraction theorem and Guo–Krasnosel'skii theorem we discuss the existence, uniqueness and positivity of solution to the third-order three-point nonhomogeneous boundary value problem

$$u''' + f\left(t, u(t), u'(t)\right) = 0, \quad t \in (0,1) \tag{28.1}$$

$$\alpha u'(1) = \beta u(\eta), \;\; u(0) = u'(0) = 0 \tag{28.2}$$

Throughout this paper we make the following assumptions:

$(\mathbf{I}_1)$: $\beta,\, \alpha \in \mathbb{R}_+^*$, $0 < \eta < 1$ and $f \in C((0,1) \times [0;\infty) \times [0;\infty); [0;\infty))$.

$(\mathbf{I}_2)$: We will use the classical Banach spaces, $C[0,1]$, $C^1[0,1]$, $L^1[0,1]$. We also use the Banach space $X = \left\{u \in C^1[0,1] / u \in C[0,1],\ u' \in C[0,1]\right\}$, equipped with the norm

$\|u\|_X = \max\{\|u\|_\infty,\ \|u'\|_\infty\}$, where $\|u\|_\infty = \max\limits_{t\in[0,1]} |u(t)|$.

28.2 Preliminary Lemmas

In this section, we present several important preliminary lemmas.

Lemma 28.1. *Let* $2\alpha \neq \beta\eta^2$ *and* $y \in L^1[0,1]$, *then the problem*

$$u''' + y(t) = 0, \quad 0 < t < 1 \tag{28.3}$$

$$\alpha u'(1) = \beta u(\eta), \;\; u(0) = u'(0) = 0 \tag{28.4}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)\, y(s)\, ds + \frac{\beta t^2}{2\alpha - \beta\eta^2} \int_0^1 G(\eta,s)\, y(s)\, ds, \tag{28.5}$$

where

$$G(t,s) = \frac{1}{2} \begin{cases} (1-s)t^2, & t \le s \\ \left(-s + 2t - t^2\right)s, & s \le t. \end{cases} \tag{28.6}$$

Proof. Integrating (28.3) over the interval $[0,t]$ for $t \in [0,1]$, we have

$$u'(t) = -\int_0^t (t-s)\, y(s)\, ds + C_1 t + C_2$$

$$u(t) = -\frac{1}{2}\int_0^t (t-s)^2\, y(s)\, ds + \frac{1}{2}C_1 t^2 + C_2 t + C_3.$$

(1) From $u(0) = u'(0) = 0$ we get $C_3 = C_2 = 0$.
(2) From $\alpha u'(1) = \beta u(\eta)$, we deduce

$$\frac{-\beta}{2}\int_0^\eta (\eta - s)^2 y(s)\,ds + \alpha\int_0^1 (1-s)\,y(s)\,ds = \left(\frac{2\alpha - \beta\eta^2}{2}\right) C_1$$

$$C_1 = \frac{-\beta}{2\alpha - \beta\eta^2}\int_0^\eta (\eta - s)^2 y(s)\,ds + \frac{2\alpha}{2\alpha - \beta\eta^2}\int_0^1 (1-s)\,y(s)\,ds$$

and

$$u(t) = -\frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds \\ +\frac{t^2}{2}\left(\frac{-\beta}{2\alpha - \beta\eta^2}\int_0^\eta (\eta - s)^2 y(s)\,ds + \frac{2\alpha}{2\alpha - \beta\eta^2}\int_0^1 (1-s)\,y(s)\,ds\right),$$

so

$$u(t) = -\frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds + \frac{t^2}{2}\int_0^1 (1-s)\,y(s)\,ds \\ +\frac{\beta t^2}{2\left[2\alpha - \beta\eta^2\right]}\left(-\int_0^\eta (\eta - s)^2 y(s)\,ds + \eta^2\int_0^\eta (1-s)\,y(s)\,ds \right. \\ \left. +\eta^2\int_\eta^1 (1-s)\,y(s)\,ds\right).$$

Elementary operations give

$$u(t) = \int_0^1 G(t,s)\,y(s)\,ds \\ +\frac{\beta t^2}{2\alpha - \beta\eta^2}\int_0^1 G(\eta,s)\,y(s)\,f ds,$$

which implies the Lemma 28.1. □

We need some properties of functions $G(t,s)$.

Lemma 28.2. *For all* $(t,s) \in [0,1]\times[0,1]$, *we have*

$$0 \le \frac{\partial G(t,s)}{\partial t} = \begin{cases} (1-s)\,t, & t \le s \\ (1-t)\,s, & s \le t \end{cases} = G^*(t,s) \le 2G(1,s).$$

Lemma 28.3. *For all* $(t,s) \in [\tau,1]\times[0,1]$, *we have*

$$\tau^2 G(1,s) \le G(t,s) \le G(1,s) = \frac{1}{2}(1-s)\,s.$$

Proof. For all $t,s \in [0,1]$, if $s \le t$, it follows from (28.6) that

$$G(t,s) = \frac{1}{2}\left(2t - t^2 - s\right)s = \frac{1}{2}\left[1 - s - \left(1 - t^2\right)\right]s$$
$$\leq \frac{1}{2}(1-s)s = G(1,s),$$

and

$$G(t,s) = \frac{1}{2}\left(2t - t^2 - s\right)s$$
$$= \frac{1}{2}st^2(1-s) + \frac{1}{2}(1-t)\left[(t-s) + (1-s)t\right]s$$
$$\geq t^2 G(1,s).$$

If $t \leq s$, it follows from (28.6) that

$$\frac{1}{2}t^2(1-s)s \leq G(t,s) = \frac{1}{2}t^2(1-s) \leq G(1,s).$$

Thus

$$t^2 G(1,s) \leq G(t,s) \leq G(1,s), \quad \forall (t,s) \in [0,1] \times [0,1].$$

Therefore

$$\tau^2 G(1,s) \leq G(t,s) \leq G(1,s), \quad \forall (t,s) \in [\tau,1] \times [0,1]$$

which implies Lemma 28.3. □

Definition 28.4. We define an operator T by

$$Tu(t) = \int_0^1 G(t,s) f\left(s, u(s), u'(s)\right) ds + \frac{\beta t^2}{2\alpha - \beta\eta^2}\int_0^1 G(\eta,s) f\left(s,u(s),u'(s)\right) ds. \tag{28.7}$$

The function $u \in E$ is a solution of the BVP (28.1)–(28.2) if and only if $Tu(t) = u(t)$; $(u$ is a fixed point of $T)$.

28.3 Existence Results

Now we give some existence results for the BVP (28.1)–(28.2).

Theorem 28.5. *Assume that* $u \in X$, $2\alpha \neq \beta\eta^2$ *and there exists a nonnegative function* $k, h \in L^1([0,1], \mathbb{R}_+)$, *such that*

$$|f(t,x,y) - f(t,u,v)| \leq k(t)|x-u| + h(t)|y-v|, \ \forall x,y,u,v \in \mathbb{R}, \ t \in [0,1]$$

and

$$\int_0^1 G(1,s)(k(s)+h(s))\,ds < \frac{\left|2\alpha-\beta\eta^2\right|}{2\left(\left|2\alpha-\beta\eta^2\right|+\beta\right)},$$

then the BVP (28.1)–(28.2) *has a unique solution in* X.

Proof. Since we have

$$Tu(t) = \int_0^1 G(t,s)\,f\left(s,u(s),u'(s)\right)ds + \frac{\beta t^2}{2\alpha-\beta\eta^2}\int_0^1 G(\eta,s)\,f\left(s,u(s),u'(s)\right)ds,$$

we shall prove that T is a contraction. Let $u,v \in X$. Then,

$$|Tu(t)-Tv(t)| \le \int_0^1 G(1,s)\left|f\left(s,u(s),u'(s)\right)-f\left(s,v(s),v'(s)\right)\right|ds + \frac{\beta}{|2\alpha-\beta\eta^2|}\int_0^1 G(1,s)\left|f\left(s,u(s),u'(s)\right)-f\left(s,v(s),v'(s)\right)\right|.$$

So, we can obtain

$$|Tu(t)-Tv(t)| \le \left(1+\frac{\beta}{|2\alpha-\beta\eta^2|}\right)\times \int_0^1 G(1,s)\left(k(s)\,|u(s)-v(s)|+h(s)\left|u'(s)-v'(s)\right|\right)ds,$$

and so

$$\begin{aligned}|Tu(t)-Tv(t)| &\le \left(1+\frac{\beta}{|2\alpha-\beta\eta^2|}\right)\times \int_0^1 G(1,s)(k(s)+h(s))\,ds \max_{0\le t\le 1}\left\{\|u-v\|_\infty,\left\|u'-v'\right\|_\infty\right\}\\ &\le \|u-v\|_X .\end{aligned}$$

We have

$$T'u(t) = \int_0^1 G^*(t,s)\,f\left(s,u(s),u'(s)\right)ds + \frac{2\beta t}{2\alpha-\beta\eta^2}\int_0^1 G(\eta,s)\,f\left(s,u(s),u'(s)\right)ds.$$

Similarly, we have

$$\begin{aligned}\left|T'u(t)-T'v(t)\right| &\le 2\left(1+\frac{\beta}{|2\alpha-\beta\eta^2|}\right)\times \max\left\{\|u-v\|_\infty,\left\|u'-v'\right\|_\infty\right\}\int_0^1 G(1,s)(k(s)+h(s))\,ds\\ &\le \|u-v\|_X .\end{aligned}$$

From this we get

$$\max\left\{\|Tu-Tv\|_{\infty},\left\|T'u-T'v\right\|_{\infty}\right\}\leq\|u-v\|_{X}\ .$$

Obviously, we have,

$$\|Tu-Tv\|_{X}\leq\|u-v\|_{X}\,.$$

Then T is a contraction, so it has a unique fixed point which is the unique solution of BVP (28.1)–(28.2). □

We will employ the following Leray–Schauder nonlinear alternative [2].

Lemma 28.6. *Let F be Banach space and Ω be a bounded open subset of F, $0\in\Omega$. $T:\overline{\Omega}\to F$ be a completely continuous operator. Then, either there exists $x\in\partial\Omega$, $\lambda>1$ such that $T(x)=\lambda x$ or there exists a fixed point $x^*\in\overline{\Omega}$.*

Theorem 28.7. *We assume that $f(t,0,0)\neq 0, 2\alpha\neq\beta\eta^2$ and there exist nonnegative functions $k,l,h\in L^1[0,1]$ such that*

$$|f(t,u,v)|\leq k(t)|u|+l(t)|v|+h(t),\ \ (t,x)\in[0,1]\times\mathbb{R},$$

$$2\left(1+\frac{\beta}{|2\alpha-\beta\eta^2|}\right)\int_0^1 G(1,s)(k(s)+l(s))\,ds<1.$$

Then the BVP (28.1)–(28.2) has at least one nontrivial solution $u^*\in X$.

Proof. Setting

$$F=2\left(1+\frac{\beta}{|2\alpha-\beta\eta^2|}\right)\int_0^1 G(1,s)(k(s)+l(s))\,ds,$$

$$G=2\left(1+\frac{\beta}{|2\alpha-\beta\eta^2|}\right)\int_0^1 G(1,s)h(s)\,ds,$$

we prove that T is completely continuous operator on Ω.

1) T is continuous. Let $2\alpha\neq\beta\eta^2$ and $(u_k)_{k\in\mathbb{N}}$ a convergent sequence to u in X. We can get

$$\begin{aligned}|Tu_k(t)-Tu(t)|&\leq\left(1+\frac{\beta}{|2\alpha-\beta\eta^2|}\right)\times\\&\int_0^1 G(1,s)\left|f\left(s,u_k(s),u_k'(s)\right)-f\left(s,u(s),u'(s)\right)\right|ds\\&\leq\left(1+\frac{\beta}{|2\alpha-\beta\eta^2|}\right)\times\\&\int_0^1 G(1,s)\left(k(s)|u_k(s)-u(s)|+h(s)\left|u_k'(s)-u'(s)\right|\right)ds,\end{aligned}$$

and so

$$|Tu_k(t) - Tu(t)| \leq \|u_k - u\|_X \left(1 + \frac{\beta}{|2\alpha - \beta\eta^2|}\right) \int_0^1 G(1,s)(k(s) + h(s))\,ds.$$

Similarly, we have

$$\left|T'u_k(t) - T'u(t)\right| \leq 2\|u_k - u\|_X \left(1 + \frac{\beta}{|2\alpha - \beta\eta^2|}\right) \int_0^1 G(1,s)(k(s) + h(s))\,ds.$$

Then,

$$\|Tu_k - Tu\|_X \leq \|u_k - u\|_X\,,$$

which implies that $\|Tu(t) - Tv(t)\| \underset{n\to\infty}{\longrightarrow} 0.$

2) Let $B_r = \{u \in X : \|u\|_X \leq r\}$ a bounded subset. We will prove that $T(\Omega \cap B_r)$ is relatively compact.

(*i*) $T(\Omega \cap B_r)$ is uniformly bounded. For some $u \in \Omega \cap B_r$, we have

$$|Tu(t)| \leq \left(1 + \frac{\beta}{|2\alpha - \beta\eta^2|}\right) \int_0^1 G(1,s)\left|f\left(s, u(s), u'(s)\right)\right| ds$$

and

$$\left|T'u(t)\right| \leq 2\left(1 + \frac{\beta}{|2\alpha - \beta\eta^2|}\right) \int_0^1 G(1,s)\left|f\left(s, u(s), u'(s)\right)\right| ds.$$

From the above inequalities we have

$$\|Tu\|_X \leq F\|u\|_X + G \leq Fr + G.$$

Then, $T(\Omega \cap B_r)$ is uniformly bounded.

(*ii*) $T(\Omega \cap B_r)$ is equicontinuous. $\forall\, t_1, t_2 \in [0,1]\,;\, u \in \Omega$, we have

$$|Tu(t_1) - Tu(t_2)| = \left| \int_0^1 (G(t_1,s) - G(t_2,s))\, f\left(s, u(s), u'(s)\right) ds \right. \\ \left. + \frac{\beta\left(t_1^2 - t_2^2\right)}{|2\alpha - \beta\eta^2|} \int_0^1 G^*(\eta_i, s)\, f\left(s, u(s), u'(s)\right) ds \right|.$$

$$|Tu(t_1) - Tu(t_2)| \leq L\left[\int_0^{t_1} |G(t_1,s) - G(t_2,s)|\,ds + \int_{t_1}^{t_2} |G(t_1,s) - G(t_2,s)|\,ds \right. \\ \left. + \int_{t_2}^1 |G(t_1,s) - G(t_2,s)|\,ds\right] \\ + \frac{L\beta\left|t_1^2 - t_2^2\right|}{|2\alpha - \beta\eta^2|} \int_0^1 G^*(\eta_i, s)\,ds,$$

where $L = \max\limits_{0<s<1} |f(s, u(s), u'(s))|$. Hence,

$$|Tu(t_1) - Tu(t_2)| \le L(t_2 - t_1)\left[\int_0^{t_1} |-2s + s(t_1 + t_2)|\,ds + \int_{t_2}^1 |(1-s)(t_1+t_2)|\,ds \right.$$
$$\left. + \frac{(t_1+t_2)\beta}{|2\alpha - \beta\eta^2|}\int_0^1 G^*(\eta_i, s)\,ds\right]$$
$$+ L\int_{t_1}^{t_2} \left|\left(t_1^2 - st_2 + s^2\right) + \left(t_1^2 - t_2^2\right)s\right| ds.$$

Then

$$|Tu(t_1) - Tu(t_2)| \le L(t_2 - t_1)\left[1 - t_2^2 + t_1(t_1 - t_2 + 3)\right.$$
$$\left. + \frac{(t_1+t_2)\beta}{|2\alpha - \beta\eta^2|}\int_0^1 G^*(\eta_i, s)\,ds\right]$$

Similarly we have

$$\left|T'u(t_1) - T'u(t_2)\right| = \left|\int_0^1 \left(G^*(t_1, s) - G^*(t_2, s)\right) f\left(s, u(s), u'(s)\right) ds\right.$$
$$\left. + \tfrac{2\beta(t_1 - t_2)}{|2\alpha - \beta\eta^2|}\int_0^1 G^*(\eta_i, s) f(s, u(s), u'(s))\,ds\right|,$$

and so

$$\left|T'u(t_1) - T'u(t_2)\right| \le L(t_2 - t_1)\left[\int_0^{t_1} s\,ds + \int_{t_2}^1 |s-1|\,ds\right.$$
$$\left. + \frac{2\beta}{|2\alpha - \beta\eta^2|}\int_0^1 G^*(\eta_i, s)\,ds\right]$$
$$+ L\int_{t_1}^{t_2} |(t_1 - s) + (t_2 - t_1)s|\,ds.$$

Then

$$\left|T'u(t_1) - T'u(t_2)\right| \le L(t_2 - t_1)\left[1 + (t_1 - t_2) + \frac{1}{2}(3t_2 - 5t_1)\right.$$
$$\left. + \frac{2\beta}{|2\alpha - \beta\eta^2|}\int_0^1 G^*(\eta_i, s)\,ds\right],$$

and $|Tu(t_1) - Tu(t_2)| \underset{t_1 \to t_2}{\longrightarrow} 0$ as $|T'u(t_1) - T'u(t_2)| \underset{t_1 \to t_2}{\longrightarrow} 0$. Consequently $T(\Omega \cap B_r)$ is equicontinuous. From Arzela–Ascoli theorem, we deduce that T is a completely continuous operator. Remarking that $F < 1$. $f(t,0,0) \neq 0$ and $G > 0$, then there exists an interval $[\sigma, \tau] \subset [0,1]$ such that $\min\limits_{\sigma \le t \le r} |f(t,0,0)| > 0$ and $h(t) \ge |f(t,0,0)|$, $t \in [0,1]$.

Let $m = G(1-F)^{-1}$, $\Omega = \{u \in X : ||u|| < m\}$. We assume that $u \in \partial\Omega$, $\lambda > 1$ such that $Tu = \lambda u$, then

$$\lambda m = \lambda \|u\| = \|Tu\|_X = \max \left\{ \|Tu\|_\infty , \left\|T'u\right\|_\infty \right\}.$$

We have

$$\begin{aligned} |Tu(t)| \leq & \sup_{0\leq t\leq 1} \int_0^1 G(t,s) \left| f\left(s,u(s),u'(s)\right) \right| ds \\ & + \sup_{0\leq t\leq 1} \frac{\beta t^2}{|2\alpha - \beta\eta^2|} \int_0^1 G(\eta,s) \left| f\left(s,u(s),u'(s)\right) \right| ds. \end{aligned}$$

$$\leq \left(1 + \frac{\beta}{|2\alpha - \beta\eta^2|}\right) \int_0^1 G(1,s) \left(k(s)|u(s)| + l(s)\left|u'(s)\right| + h(s)\right) ds.$$

We also have

$$\begin{aligned} |Tu(t)| \leq & \left(1 + \frac{\beta}{|2\alpha - \beta\eta^2|}\right) \max\left\{\|u\|_\infty , \left\|u'\right\|_\infty\right\} \int_0^1 G(1,s)(k(s) + l(s))\, ds. \\ & + \left(1 + \frac{\beta}{|2\alpha - \beta\eta^2|}\right) \int_0^1 G(1,s) h(s)\, ds \\ \leq & \|u\|_X \left(1 + \frac{\beta}{|2\alpha - \beta\eta^2|}\right) \int_0^1 G(1,s)(k(s) + l(s))\, ds \\ & + \left(1 + \frac{\beta}{|2\alpha - \beta\eta^2|}\right) \int_0^1 G(1,s) h(s)\, ds \end{aligned}$$

and

$$\begin{aligned} \left|T'u(t)\right| \leq & \sup_{0\leq t\leq 1} \int_0^1 G^*(t,s) \left| f\left(s,u(s),u'(s)\right) \right| ds \\ & + \sup_{0\leq t\leq 1} \frac{2\beta t}{|2\alpha - \beta\eta^2|} \int_0^1 G(\eta,s) \left| f\left(s,u(s),u'(s)\right) \right| ds. \end{aligned}$$

$$\begin{aligned} \left|T'u(t)\right| \leq & 2\|u\|_X \left(1 + \frac{\beta}{|2\alpha - \beta\eta^2|}\right) \int_0^1 G(1,s)(k(s) + l(s))\, ds \\ & +2\left(1 + \frac{\beta}{|2\alpha - \beta\eta^2|}\right) \int_0^1 G(1,s) h(s)\, ds. \end{aligned}$$

This shows that

$$\lambda m = \|Tu\|_X \leq F\,||u||_X + G = Fm + G.$$

From this we get

$$\lambda \leq F + \frac{G}{m} = F + \frac{G}{G(1-F)^{-1}} = F + (1-F) = 1,$$

which contradicts $\lambda > 1$. By applying Lemma 28.6, T has a fixed-point $u^* \in \overline{\Omega}$ and then the BVP (1.1)–(1.2) has a nontrivial solution $u^* \in X$. □

28.4 Positive Results

In this section, we discuss the existence of positive solution of BVP (28.1)–(28.2). We make the following additional assumptions:

(*Q1*) $f(t,u,v) = a(t)f_1(u,v)$ where $a \in C((0,1),\mathbb{R}_+)$ and $f_1 \in C(\mathbb{R}_+ \times \mathbb{R},\mathbb{R}_+)$.

(*Q2*) $\int_{\tau_1}^{\tau_2} G(1,s)\,a(s)\,f_1(u(s),u'(s))\,ds > 0$, $\frac{1}{2} \le \tau_1 \le s,t \le \tau_2 \le 1$

We need some properties of functions $G(t,s)$.

Lemma 28.8. *For all* $0 \le s,t \le 1$, *we have*

$$G^*(t,s) \le 2G(1,s),$$
$$G(t,s) \le G(1,s).$$

Lemma 28.9. *For all* $\frac{1}{2} \le \tau_1 \le s,t \le \tau_2 \le 1$, *we have*

$$\tau_1^2 G(1,s) \le G(t,s),$$
$$2(1-\tau_2)\,G(1,s) \le G^*(t,s).$$

Proof. It is easy to see that if $t \le s$, then $G^*(t,s) = (1-s)t = (1-s)s\frac{t}{s} \ge (1-s)s\tau_1 \ge 2(1-\tau_2)G(1,s)$. If $s \le t$, then $-t \le -s$, and hence $G^*(t,s) = (1-t)s = \frac{1-t}{1-s}(1-s)s \ge \frac{1}{s}(1-\tau_2)(1-s)s \ge 2(1-\tau_2)G(1,s)$. □

Lemma 28.10. *Let* $u \in X$ *and assume that* $2\alpha > \beta\eta^2$, *then the unique solution* u *of the* BVP (28.1)–(28.2) *is nonnegative and satisfies*

$$\min_{t\in[\tau_1,\tau_2]} \left(u(t) + u'(t)\right) \ge \gamma \|u\|_X,$$

where $\gamma = \min_{t\in[\tau_1,\tau_2]} \left(\tau_1^2, (1-\tau_2)\right) \frac{\int_{\tau_1}^{\tau_2} G(1,s)a(s)f_1\left(u(s),u'(s)\right)ds}{\left(1+\frac{\beta}{2\alpha-\beta\eta^2}\right) w \int_0^1 G(1,s)a(s)f_1(u(s),u'(s))ds}$.

Proof. Let $u \in X$, it is obvious that $u(t)$ is nonnegative. For a $t \in [0,1]$, by (28.5) and Lemma 28.8, it follows that

$$u(t) = \int_0^1 G(t,s)\,f\left(s,u(s),u'(s)\right)ds + \frac{\beta t^2}{2\alpha-\beta\eta^2}\int_0^1 G(\eta,s)\,a(s)\,f_1\left(u(s),u'(s)\right)ds.$$

Then

$$u(t) \le \left(1 + \frac{\beta}{2\alpha-\beta\eta^2}\right)\int_0^1 G(1,s)\,a(s)\,f_1\left(u(s),u'(s)\right)ds,$$

and so

$$\|u\|_\infty \le \left(1 + \frac{\beta}{2\alpha-\beta\eta^2}\right)\int_0^1 G(1,s)\,a(s)\,f_1\left(u(s),u'(s)\right)ds.$$

On the other hand, (28.5) and Lemma 28.9 imply that, for any $t \in [\tau_1, \tau_2]$, we have

$$u(t) \geq \tau_1^2 \int_{\tau_1}^{\tau_2} G(1,s)\, a(s)\, f_1\left(u(s), u'(s)\right) ds,$$

$$u(t) \geq \frac{\tau_1^2 \int_{\tau_1}^{\tau_2} G(1,s)\, a(s)\, f_1(u(s), u'(s))\, ds}{\left(1 + \frac{\beta}{2\alpha - \beta\eta^2}\right) \int_0^1 G(1,s)\, a(s)\, f_1(u(s), u'(s))\, ds} \|u\|_\infty .$$

Therefore, we have

$$\min_{t \in [\tau_1, \tau_2]} u(t) \geq \gamma_1 \|u\|_\infty ,$$

where $\gamma_1 = \frac{\tau_1^2 \int_{\tau_1}^{\tau_2} G(1,s)a(s)f_1\left(u(s),u'(s)\right)ds}{\left(1+\frac{\beta}{2\alpha-\beta\eta^2}\right)\int_0^1 G(1,s)a(s)f_1(u(s),u'(s))ds}$. Similarly, we get

$$\begin{aligned} u'(t) = &\int_0^1 G^*(t,s)\, a(s)\, f_1\left(u(s), u'(s)\right) ds \\ &+ \frac{2\beta t}{2\alpha - \beta\eta^2} \int_0^1 G(\eta,s)\, a(s)\, f_1\left(u(s), u'(s)\right) ds \end{aligned}$$

$$\begin{aligned} u'(t) \leq &\int_0^1 2G(1,s)\, a(s)\, f_1\left(u(s), u'(s)\right) ds \\ &+ \frac{2\beta}{2\alpha - \beta\eta^2} \int_0^1 G(1,s)\, a(s)\, f_1\left(u(s), u'(s)\right) ds, \end{aligned}$$

and hence

$$\left\|u'\right\|_\infty \leq 2\left(1 + \frac{\beta}{2\alpha - \beta\eta^2}\right) \int_0^1 G(1,s)\, a(s)\, f_1\left(u(s), u'(s)\right) ds.$$

On the other hand, for $\frac{1}{2} \leq \tau_1 \leq s,t \leq \tau_2 < 1$

$$\begin{aligned} u'(t) = &\int_0^1 G^*(t,s)\, f\left(s, x(s), x'(s)\right) ds \\ &+ \frac{2\beta t}{2\alpha - \beta\eta^2} \int_0^1 G(\eta,s)\, f\left(s, x(s), x'(s)\right) ds, \end{aligned}$$

which implies that

$$\begin{aligned} u'(t) &\geq \int_{\tau_1}^{\tau_2} G^*(t,s)\, a(s)\, f_1\left(u(s), u'(s)\right) ds \\ &\geq \int_{\tau_1}^{\tau_2} 2(1-\tau_2)\, G(1,s)\, a(s)\, f_1\left(u(s), u'(s)\right) ds \\ &\geq \frac{(1-\tau_2) \int_{\tau_1}^{\tau_2} G(1,s)\, a(s)\, f_1(u(s), u'(s))\, ds}{\left(1 + \frac{\beta}{2\alpha - \beta\eta^2}\right) \int_0^1 G(1,s)\, a(s)\, f_1(u(s), u'(s))\, ds} \left\|u'\right\|_\infty . \end{aligned}$$

Therefore,

$$\min_{t\in[\tau_1,\tau_2]} u'(t) \geq \gamma_2 \left\|u'\right\|_\infty,$$

where $\gamma_2 = \frac{\int_{\tau_1}^{\tau_2}(1-\tau_2)G(1,s)a(s)f_1\left(u(s),u'(s)\right)ds}{\left(1+\frac{\beta}{2\alpha-\beta\eta^2}\right)\int_0^1 G(1,s)a(s)f_1(u(s),u'(s))ds}$. Finally,

$$\min_{t\in[\tau_1,\tau_2]} \left(u(t)+u'(t)\right) \geq \gamma\|u\|_X$$

where $\gamma = \min_{t\in[\tau_1,\tau_2]}(\gamma_1,\gamma_2)$. The proof is complete. □

Definition 28.11. We define the cone K by

$$X^+ = \{u \in X : u(t) \geq 0, \ 0 < \tau_1 \leq t \leq \tau_2 < 1\}$$

$$K = \left\{u \in X^+ : \min_{t\in[\tau_1,\tau_2]}\left(u(t)+u'(t)\right) \geq \gamma\|u\|_X\right\}$$

K is a nonempty closed and convex subset of X.

Lemma 28.12. *The operator defined in (28.7) is completely continuous and satisfies* $T(K) \subseteq K$.

Proof. Now let us prove that T is completely continuous.

1) T is continuous. Let $(u_k)_{k\in\mathbb{N}}$ a convergent sequence to u in X. From $f_1 \in C(\mathbb{R}_+\times\mathbb{R},\mathbb{R}_+)$: $\forall A>0\ \exists\eta>0$ such that $\left|\left(u_k(t),u_k'(t)\right)-(u(t),u'(t))\right|<\eta$ $\left|f_1\left(u_k(s),u_k'(s)\right)-f_1\left(u(s),u'(s)\right)\right|<A$, we have

$$|Tu_k(t)-Tu(t)| \leq \left(1+\frac{\beta}{|2\alpha-\beta\eta^2|}\right)\times \int_0^1 G(1,s)\,a(s)\max_{0<s<1}\left|f_1\left(u_k(s),u_k'(s)\right)-f_1\left(u(s),u'(s)\right)\right|ds.$$

So,

$$|Tu_k(t)-Tu(t)| \leq \left(1+\frac{\beta}{|2\alpha-\beta\eta^2|}\right)A\int_0^1 G(1,s)\,a(s)\,ds;$$
$$A = \max_{0<s<1}\left|f_1\left(u_k(s),u_k'(s)\right)-f_1\left(u(s),u'(s)\right)\right|.$$

Similarly, we have

$$\left|T'u_k(t)-T'u(t)\right| \leq 2A\left(1+\frac{\beta}{|2\alpha-\beta\eta^2|}\right)\int_0^1 G(1,s)\,a(s)\,ds.$$

Then, $\|Tu_k(t)-Tu(t)\| \underset{k\to\infty}{\longrightarrow} 0$.

2) Let $B_r = \{u \in X : \|u\|_X \leq r\}$ a bounded subset and Ω a bounded open subset of a Banach space X, such that $T : \overline{\Omega} \to X$. We will prove that $T(\Omega \cap B_r)$ is relatively compact :

(i) $T(\Omega \cap B_r)$ is uniformly bounded. For some $u \in \Omega \cap B_r$, since f_1 and a are continuous, there exists a positive constant L such $L = \max_{t\in[0,1]} |a(t) f_1(u(t), u'(t))|$ then,

$$|Tu(t)| \leq L\left(1 + \frac{\beta}{|2\alpha - \beta\eta^2|}\right)\int_0^1 G(1,s)\,a(s)\,ds$$

and

$$|T'u(t)| \leq 2L\left(1 + \frac{\beta}{|2\alpha - \beta\eta^2|}\right)\int_0^1 G(1,s)\,a(s)\,ds.$$

From the above inequalities we deduce

$$\|Tu\|_X \leq 3L\left(1 + \frac{\beta}{|2\alpha - \beta\eta^2|}\right)\int_0^1 G(1,s)\,a(s)\,ds.$$

Then, $T(\Omega \cap B_r)$ is uniformly bounded.

(ii) $T(\Omega \cap B_r)$ is equicontinuous. Indeed, $\forall\, t_1,\, t_2 \in [0,1]$, $u \in B_r$, we have

$$\begin{aligned}|Tu(t_1) - Tu(t_2)| = \Bigg| &\int_0^1 (G(t_1,s) - G(t_2,s))\,a(s)\,f_1\left(u(s), u'(s)\right) ds \\ &+ \frac{\beta\left(t_1^2 - t_2^2\right)}{|2\alpha - \beta\eta^2|}\int_0^1 G^*(\eta_i,s)\,a(s)\,f_1\left(u(s), u'(s)\right) ds\Bigg|,\end{aligned}$$

which gives that

$$\begin{aligned}|Tu(t_1) - Tu(t_2)| \leq L\Bigg[&\int_0^{t_1} |G(t_1,s) - G(t_2,s)|\,ds + \int_{t_1}^{t_2} |G(t_1,s) - G(t_2,s)|\,ds \\ &+ \int_{t_2}^1 |G(t_1,s) - G(t_2,s)|\,ds\Bigg] \\ &+ \frac{L\beta\left|t_1^2 - t_2^2\right|}{|2\alpha - \beta\eta^2|}\int_0^1 G^*(\eta_i,s)\,ds,\end{aligned}$$

and so

$$\begin{aligned}|Tu(t_1) - Tu(t_2)| \leq L(t_2 - t_1)\Bigg[&\int_0^{t_1} |-2s + s(t_1 + t_2)|\,ds \\ &+ \int_{t_2}^1 |(1-s)(t_1+t_2)|\,ds \\ &+ \frac{\beta(t_1+t_2)}{|2\alpha - \beta\eta^2|}\int_0^1 G^*(\eta_i,s)\,ds\Bigg] \\ &+ L\int_{t_1}^{t_2} \left|\left(t_1^2 - st_2 + s^2\right) + \left(t_1^2 - t_2^2\right)s\right| ds.\end{aligned}$$

Thus

$$|Tu(t_1) - Tu(t_2)| \leq L(t_2 - t_1)\left[1 - t_2^2 + t_1(t_1 - t_2 + 3) + \frac{\beta(t_1 + t_2)}{|2\alpha - \beta\eta^2|}\int_0^1 G^*(\eta_i, s)\,ds\right].$$

Similarly, we have

$$\left|T'u(t_1) - T'u(t_2)\right| = \left|\int_0^1 (G^*(t_1, s) - G^*(t_2, s))\, f\left(s, u(s), u'(s)\right) ds + \tfrac{2\beta(t_1 - t_2)}{|2\alpha - \beta\eta^2|}\int_0^1 G^*(\eta_i, s)\, f(s, u(s), u'(s))\,ds\right|,$$

and

$$\left|T'u(t_1) - T'u(t_2)\right| \leq L(t_2 - t_1)\left[\int_0^{t_1} s\,ds + \int_{t_2}^1 |s - 1|\,ds + \frac{2\beta}{|2\alpha - \beta\eta^2|}\int_0^1 G^*(\eta_i, s)\,ds\right] + L\int_{t_1}^{t_2} |(t_1 - s) + (t_2 - t_1)s|\,ds,$$

which yield

$$\left|T'u(t_1) - T'u(t_2)\right| \leq L(t_2 - t_1)\left[1 + (t_1 - t_2) + \frac{1}{2}(3t_2 - 5t_1) + \frac{2\beta}{|2\alpha - \beta\eta^2|}\int_0^1 G^*(\eta_i, s)\,ds\right].$$

Then $|Tu(t_1) - Tu(t_2)| \to 0$ and $|T'u(t_1) - T'u(t_2)| \to 0$, as $t_1 \to t_2$; consequently $T(\Omega \cap B_r)$ is equicontinuous. From Arzela–Ascoli theorem, we deduce that T is completely continuous mapping. Now let us prove that $TK \subset K$. In fact for any $u \in K$, $\forall\, t \in [0,1]$ we have

$$\|Tu\| \leq \left(1 + \frac{\beta}{|2\alpha - \beta\eta^2|}\right)\int_0^1 G(1, s)\, a(s)\, f_1\left(u(s), u'(s)\right) ds.$$

Lemma 28.9 implies that $\forall\, t \in [\tau_1, \tau_2]$ we have

$$\begin{aligned} Tu(t) &\geq \int_0^1 G(t, s)\, a(s)\, f_1\left(u(s), u'(s)\right) ds \\ &\geq \tau_1^2 \int_{\tau_1}^{\tau_2} G(1, s)\, a(s)\, f_1\left(u(s), u'(s)\right) ds. \end{aligned}$$

Consequently

$$Tu(t) \geq \frac{\tau_1^2 \int_{\tau_1}^{\tau_2} G(1,s)\,a(s)\,f_1(u(s),u'(s))\,ds}{\left(1+\frac{\beta}{|2\alpha-\beta\eta^2|}\right)\int_0^1 G(1,s)\,a(s)\,f_1(u(s),u'(s))\,ds}\|Tu\|_\infty .$$

Similarly, we have

$$\|T'u\|_\infty \leq 2\left(1+\frac{\beta}{|2\alpha-\beta\eta^2|}\right)\int_0^1 G(1,s)\,a(s)\,f_1\left(u(s),u'(s)\right)ds.$$

Therefore

$$\begin{aligned} T'u(t) &\geq \int_0^1 G^*(t,s)\,a(s)\,f_1\left(u(s),u'(s)\right)ds \\ &\geq \int_{\tau_1}^{\tau_2} G^*(t,s)\,a(s)\,f_1\left(u(s),u'(s)\right)ds \\ &\geq \int_{\tau_1}^{\tau_2} 2(1-\tau_2)\,G(1,s)\,a(s)\,f_1\left(u(s),u'(s)\right)ds \end{aligned}$$

and

$$T'u(t) \geq \frac{(1-\tau_2)\int_{\tau_1}^{\tau_2} G(1,s)\,a(s)\,f_1(u(s),u'(s))\,ds}{\left(1+\frac{\beta}{|2\alpha-\beta\eta^2|}\right)\int_0^1 G(1,s)\,a(s)\,f_1(u(s),u'(s))\,ds}\|T'u\|_\infty .$$

Consequently,

$$\min_{t\in[\tau_1,\tau_2]}\left(Tu(t)+T'u(t)\right) \geq \gamma\|Tu\|_X .$$

Then, it is obvious that $\forall\, u \in K \Longrightarrow TK \subset K$.

□

To establish the existence of positive solutions of BVP (28.1)–(28.2), we will employ the following Guo–Krasnosel'skii fixed-point theorem. [8]

Theorem 28.13. *Let E be a Banach space, and let $K \subset E$,be a cone. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, and let*

$$\mathscr{A} : K \cap \left(\overline{\Omega_2}\backslash\Omega_1\right) \to K$$

be a completely continuous operator. In addition suppose either:

(i) $||\mathscr{A}u|| \leq ||u||,\ u \in K\cap\partial\Omega_1$, *and* $||\mathscr{A}u|| \geq ||u||,\ u \in K\cap\partial\Omega_2$; *or*
(ii) $||\mathscr{A}u|| \geq ||u||,\ u \in K\cap\partial\Omega_1$, *and* $||\mathscr{A}u|| \leq ||u||,\ u \in K\cap\partial\Omega_2$

holds. Then $\mathscr{A}$ has a fixed point in $K \cap \left(\overline{\Omega_2}\backslash\Omega_1\right)$.

The main result of this section is the following:

Theorem 28.14. *Let* (I_1) *and* (I_2) *hold,* $2\alpha > \beta\eta^2$ *and assume that*

$$f_0 = \lim_{(|u|+|v|)\to 0} \frac{f_1(u,v)}{|u|+|v|}, \qquad f_\infty = \lim_{(|u|+|v|)\to\infty} \frac{f_1(u,v)}{|u|+|v|}.$$

Then the problem $(BVP1)$ *has at least one positive solution in the case:*

(i) $f_0 = 0$ *and* $f_\infty = \infty$ $(superlinear)$ *or*
(ii) $f_0 = \infty$ *and* $f_\infty = 0$ $(sublinear)$

Proof. We shall prove that the problem BVP (28.1)–(28.2) has at least one positive solution in both the superlinear and sublinear cases. For this we use Theorem 28.13. We prove the superlinear case. Since $f_0 = 0$, then for any $\varepsilon > 0$, $\exists\delta_1 > 0$, such that $f_1(u,v) \le \varepsilon(|u|+|v|)$, for $|u|+|v| < \delta_1$. Let Ω_1 be an open set in X defined by

$$\Omega_1 = \{y \in X/\,||y|| < \delta_1\}.$$

Then, for any $u \in K\cap\partial\Omega_1$, it yields

$$Tu(t) \le \left(1+\frac{\beta}{2\alpha-\beta\eta^2}\right)\int_0^1 G(1,s)\,a(s)\,f_1\left(u(s),u'(s)\right)ds.$$

Therefore

$$\|Tu(t)\|_\infty \le \varepsilon\|u\|_X\left(1+\frac{\beta}{2\alpha-\beta\eta^2}\right)\int_0^1 G(1,s)\,a(s)\,ds$$

and

$$T'u(t) \le 2\left(1+\frac{\beta}{2\alpha-\beta\eta^2}\right)\int_0^1 G(1,s)\,a(s)\,f_1\left(u(s),u'(s)\right)ds.$$

So

$$\left\|T'u(t)\right\|_\infty \le 2\varepsilon\|u\|_X\left(1+\frac{\beta}{2\alpha-\beta\eta^2}\right)\int_0^1 G(1,s)\,a(s)\,ds.$$

If we choose $\varepsilon = \left[2\left(1+\frac{\beta}{2\alpha-\beta\eta^2}\right)\int_0^1 G(1,s)\,a(s)\,ds\right]^{-1}$, then it yields

$$\|Tu\| \le \|u\|, \;\; \forall u \in K\cap\partial\Omega_1.$$

Now from $f_\infty = \infty$, then $\forall M > 0$, $\exists H > 0$, such that $f_1(u,v) \ge M(|u|+|v|)$ for $|u|+|v| \ge H$. Let $H_1 = \max\left\{2\delta_1, \frac{H}{\gamma}\right\}$. Denote by Ω_2 the open set $\Omega_2 = \{y \in X/\,\|y\| < H_1\}$. If $u \in K\cap\partial\Omega_2$, then

$$\min_{t\in[\tau_1,\tau_2]}\left\{u(t),u'(t)\right\} \ge \gamma\|u\|_X = \gamma H_1 \ge H.$$

Let $u \in K \cap \partial\Omega_2$, then

$$Tu(t) \geq \frac{\tau_1^2 \int_{\tau_1}^{\tau_2} G(1,s)a(s)f_1\left(u(s),u'(s)\right)ds}{\left(1+\frac{\beta}{2\alpha-\beta\eta^2}\right)\int_0^1 G(1,s)a(s)f_1(u(s),u'(s))ds} \times$$
$$\left(1+\frac{\beta}{2\alpha-\beta\eta^2}\right)\int_0^1 G(1,s)a(s)f_1\left(u(s),u'(s)\right)ds$$
$$\geq M\gamma\left(1+\frac{\beta}{2\alpha-\beta\eta^2}\right)\int_0^1 G(1,s)a(s)ds\,\|u\|_X$$

and

$$T'u(t) \geq \frac{(1-\tau_2)\int_{\tau_1}^{\tau_2} G(1,s)a(s)f_1\left(u(s),u'(s)\right)ds}{\left(1+\frac{\beta}{2\alpha-\beta\eta^2}\right)\int_0^1 G(1,s)a(s)f_1(u(s),u'(s))ds} \times$$
$$\left(1+\frac{\beta}{2\alpha-\beta\eta^2}\right)\int_0^1 G(1,s)a(s)f_1\left(u(s),u'(s)\right)ds$$
$$T'u(t) \geq M\gamma\left(1+\frac{\beta}{2\alpha-\beta\eta^2}\right)\int_0^1 G(1,s)a(s)ds\,\|u\|_X.$$

Choosing $M=\left[\gamma\left(1+\frac{\beta}{2\alpha-\beta\eta^2}\right)\int_0^1 G(1,s)a(s)ds\right]^{-1}$, we get $\|Tu\|_X \geq \|u\|_X$, $\forall u \in K\cap\partial\Omega$. By the first part of Theorem 28.13, T has at least one fixed point in $K\cap\left(\bar{\Omega}_2\diagdown\Omega_1\right)$ such that $H \leq ||y|| \leq H_1$. This completes the superlinear case of the Theorem 28.14.

Case II. Now we assume that $f_0=\infty$ and $f_\infty=0$ (sublinear case). Proceeding as above and by the second part of Theorem 28.13, we proof the sublinear case. This achieves the proof of Theorem 28.14. □

28.5 Examples

Example 28.15. Consider the following boundary value problem:

$$\begin{cases} u'''+tu+t^2u'=0, \quad 0<t<1 \\ u(0)=u'(0)=0, \ \ \alpha u'(1)=\beta u(\eta). \end{cases} \tag{28.8}$$

Set

$$\alpha=\frac{1}{2},\ \beta=\frac{1}{3},\ \eta=\frac{1}{4},$$

and

$$f(t,u,v)=tu+t^2v$$

One can choose

$$\begin{cases} k(t)=t \\ h(t)=t^2 \end{cases},\ \ t\in[0,1],$$

where k, $l \in L^1[0,1]$ are nonnegative functions, and

$$\begin{aligned} |f(t,x,y) - f(t,u,v)| &\leq t|x-u| + t^2|y-v|, \\ &\leq k(t)|x-u| + h(t)|y-v|, \end{aligned}$$

and

$$\int_0^1 G(1,s)(k(s)+h(s))\,ds < \frac{|2\alpha - \beta\eta^2|}{2(|2\alpha-\beta\eta^2|+\beta)}.$$

Hence, by Theorem 28.5, the boundary value problem (28.8) has a unique solution in X.

Now, if we estimate f as

$$\begin{aligned} |f(t,u,v)| &\leq t|u| + t^2|v|, \\ &\leq k(t)|u| + l(t)|v| + h(t), \end{aligned}$$

then one can choose

$$\begin{cases} k(t) = \frac{2}{3}t \\ l(t) = \frac{(t+1)}{5} \\ h(t) = 0 \end{cases},\ \ t \in [0,1],$$

and

$$2\left(1 + \frac{\beta}{|2\alpha - \beta\eta^2|}\right)\int_0^1 G(1,s)(k(t)+l(t))\,ds < 1,$$

where k, l and $h \in L^1[0,1]$ are nonnegative functions. Hence, by Theorem 28.7, the boundary value problem (28.8) has at least one nontrivial solution, $u^* \in X$.

Example 28.16. Consider the following boundary value problem:

$$\begin{cases} u''' + t^2u^2 + t^2(u')^2 = 0, & 0 < t < 1 \\ u(0) = u'(0) = 0, & \alpha u'(1) = \beta u(\eta), \end{cases} \tag{28.9}$$

where

$$\begin{aligned} f(t,u,v) &= t^2\left(u^2 + \frac{1}{7}v^2\right) \\ &= a(t)f_1(u,v), \end{aligned}$$

where $a(t) = t^2 \in C((0,1),\mathbb{R}_+)$, $f_1(u,v) \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$. If we put $u = r\cos\varphi$ and $v = r\sin\varphi$, when $(|u|+|v|) \to 0 \Longrightarrow r \to 0$ and when $(|u|+|v|) \to \infty \Longrightarrow r \to \infty$, then

$$\begin{aligned} f_0 &= \lim_{(|u|+|v|)\to 0} \frac{f_1(u,v)}{|u|+|v|} = 0, \\ f_\infty &= \lim_{(|u|+|v|)\to 0} \frac{f_1(u,v)}{|u|+|v|} = \infty. \end{aligned}$$

By Theorem 28.14 (i), the BVP (28.9) has at least one positive solution.

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Chapter 29
Reverse and Forward Fractional Integral Inequalities

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Abstract Here we present reverse Lp fractional integral inequalities for left and right Riemann-Liouville, generalized Riemann-Liouville, Hadamard, Erdelyi-Kober and multivariate Riemann-Liouville fractional integrals. Then we derive reverse Lp fractional inequalities regarding the left Riemann-Liouville, the left and right Caputo and the left and right Canavati type fractional derivatives. We finish the article with general forward fractional integral inequalities regarding Erdelyi-Kober and multivariate Riemann-Liouville fractional integrals by involving convexity.

29.1 Introduction

We start with some facts about fractional integrals needed in the sequel; for more details, see for instance [1, 11].

Let $a<b$, $a,b\in\mathbb{R}$. By $C^N([a,b])$, we denote the space of all functions on $[a,b]$ which have continuous derivatives up to order N, and $AC([a,b])$ is the space of all absolutely continuous functions on $[a,b]$. By $AC^N([a,b])$, we denote the space of all functions g with $g^{(N-1)}\in AC([a,b])$. For any $\alpha\in\mathbb{R}$, we denote by $[\alpha]$ the integral part of α (the integer k satisfying $k\le\alpha<k+1$), and $\lceil\alpha\rceil$ is the ceiling of α ($\min\{n\in\mathbb{N}, n\ge\alpha\}$). By $L_1(a,b)$, we denote the space of all functions integrable on the interval (a,b), and by $L_\infty(a,b)$ the set of all functions measurable and essentially bounded on (a,b). Clearly, $L_\infty(a,b)\subset L_1(a,b)$.

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We start with the definition of the Riemann–Liouville fractional integrals, see [14]. Let $[a,b]$, $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$. The Riemann–Liouville fractional integrals $I_{a+}^{\alpha} f$ and $I_{b-}^{\alpha} f$ of order $\alpha > 0$ are defined by

$$\left(I_{a+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1} dt, \quad (x > a), \tag{29.1}$$

$$\left(I_{b-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b f(t)(t-x)^{\alpha-1} dt, \quad (x < b), \tag{29.2}$$

respectively. Here $\Gamma(\alpha)$ is the gamma function. These integrals are called the left-sided and the right-sided fractional integrals. We mention some properties of the operators $I_{a+}^{\alpha} f$ and $I_{b-}^{\alpha} f$ of order $\alpha > 0$; see also [16]. The first result yields that the fractional integral operators $I_{a+}^{\alpha} f$ and $I_{b-}^{\alpha} f$ are bounded in $L_p(a,b)$, $1 \leq p \leq \infty$, that is,

$$\left\|I_{a+}^{\alpha} f\right\|_p \leq K \|f\|_p \quad , \quad \left\|I_{b-}^{\alpha} f\right\|_p \leq K \|f\|_p, \tag{29.3}$$

where

$$K = \frac{(b-a)^{\alpha}}{\alpha \Gamma(\alpha)}. \tag{29.4}$$

Inequality (29.3), that is, the result involving the left-sided fractional integral, was proved by H. G. Hardy in one of his first papers; see [12].

In this article, we prove reverse and forward Hardy-type fractional Inequalities and we are motivated by [5, 6, 12, 13].

29.2 Main Results

We present our first result.

Theorem 29.1. *Let $0 < p < 1, q < 0$ such that $\frac{1}{p} + \frac{1}{q} = 1; \alpha_i > 0, i = 1, \ldots, m$. Let $f_i : (a,b) \to \mathbb{R}, i = 1, \ldots, m$, of fixed strict sign a.e., which are Lebesgue measurable functions, so that $\left\| \prod_{i=1}^m \left(I_{a+}^{\alpha_i} f_i\right) \right\|_p$, $\prod_{i=1}^m \|f_i\|_q$ are finite. Then*

$$\left\| \prod_{i=1}^m \left(I_{a+}^{\alpha_i} f_i\right) \right\|_p \geq \frac{(b-a)^{\sum_{i=1}^m \alpha_i + m\left(\frac{1}{p}-1\right) + \frac{1}{p}}}{\left[\left(p \sum_{i=1}^m \alpha_i + m(1-p) + 1 \right)^{\frac{1}{p}} \left(\prod_{i=1}^m \Gamma(\alpha_i) \left(p(\alpha_i - 1) + 1\right)^{\frac{1}{p}} \right) \right]}$$

$$\cdot \left(\prod_{i=1}^m \|f_i\|_q \right). \tag{29.5}$$

Proof. By (29.1) we have

$$\left(I_{a+}^{\alpha_i} f_i\right)(x) = \frac{1}{\Gamma(\alpha_i)} \int_a^x (x-t)^{\alpha_i - 1} f_i(t)\, dt, \tag{29.6}$$

$x > a, i = 1, \ldots, m$. We have that

$$\left|\left(I_{a+}^{\alpha_i} f_i\right)(x)\right| = \frac{1}{\Gamma(\alpha_i)} \int_a^x (x-t)^{\alpha_i - 1} |f_i(t)|\, dt, \tag{29.7}$$

$x > a, i = 1, \ldots, m$. By reverse Hölder's inequality we get

$$\left|\left(I_{a+}^{\alpha_i} f_i\right)(x)\right| \geq \frac{1}{\Gamma(\alpha_i)} \left(\int_a^x (x-t)^{p(\alpha_i - 1)} dt\right)^{\frac{1}{p}} \left(\int_a^x |f_i(t)|^q dt\right)^{\frac{1}{q}}$$

$$\geq \frac{1}{\Gamma(\alpha_i)} \frac{(x-a)^{(\alpha_i - 1) + \frac{1}{p}}}{(p(\alpha_i - 1) + 1)^{\frac{1}{p}}} \left(\int_a^b |f_i(t)|^q dt\right)^{\frac{1}{q}}, \tag{29.8}$$

$x > a, i = 1, \ldots, m$. Therefore

$$\prod_{i=1}^m \left|\left(I_{a+}^{\alpha_i} f_i\right)(x)\right|^p \geq \frac{1}{\left(\prod\limits_{i=1}^m \Gamma(\alpha_i)\right)^p} \frac{(x-a)^{p \sum\limits_{i=1}^m \alpha_i + m(1-p)}}{\prod\limits_{i=1}^m (p(\alpha_i - 1) + 1)} \left(\prod_{i=1}^m \int_a^b |f_i(t)|^q dt\right)^{\frac{p}{q}}, \tag{29.9}$$

$x \in (a, b)$. Consequently we get

$$\int_a^b \left(\prod_{i=1}^m \left|\left(I_{a+}^{\alpha_i} f_i\right)(x)\right|^p\right) dx \geq \left(\frac{1}{\prod\limits_{i=1}^m \left(\Gamma(\alpha_i)^p (p(\alpha_i - 1) + 1)\right)}\right)$$

$$\cdot \left(\int_a^b (x-a)^{p \sum\limits_{i=1}^m \alpha_i + m(1-p)} dx\right) \left(\prod_{i=1}^m \int_a^b |f_i(t)|^q dt\right)^{\frac{p}{q}} \tag{29.10}$$

$$= \frac{(b-a)^{p \sum\limits_{i=1}^m \alpha_i + m(1-p) + 1} \left(\prod\limits_{i=1}^m \int_a^b |f_i(t)|^q dt\right)^{\frac{p}{q}}}{\left[\left(p \sum\limits_{i=1}^m \alpha_i + m(1-p) + 1\right) \left(\prod\limits_{i=1}^m \left(\Gamma(\alpha_i)^p (p(\alpha_i - 1) + 1)\right)\right)\right]}, \tag{29.11}$$

proving the claim. □

We give also the following general variant in:

Theorem 29.2. *Let $0 < p < 1, q < 0$ such that $\frac{1}{p} + \frac{1}{q} = 1, r > 0; \alpha_i > 0, i = 1, \ldots, m$. Let $f_i : (a, b) \to \mathbb{R}, i = 1, \ldots, m$, of fixed strict sign a.e., which are Lebesgue measurable functions, so that $\left\| \prod\limits_{i=1}^m \left(I_{a+}^{\alpha_i} f_i\right) \right\|_r$, $\prod\limits_{i=1}^m \|f_i\|_q$ are finite. Then*

$$\left\|\prod_{i=1}^{m}\left(I_{a+}^{\alpha_i}f_i\right)\right\|_r \geq \frac{(b-a)^{\sum_{i=1}^{m}\alpha_i - m + \frac{m}{p} + \frac{1}{r}}}{\left[\left(r\left(\sum_{i=1}^{m}\alpha_i - m + \frac{m}{p}\right)+1\right)^{\frac{1}{r}}\left(\prod_{i=1}^{m}\Gamma(\alpha_i)\left(p(\alpha_i-1)+1\right)^{\frac{1}{p}}\right)\right]}$$

$$\cdot\left(\prod_{i=1}^{m}\|f_i\|_q\right). \tag{29.12}$$

Proof. Using $r > 0$ and (29.8) we get

$$\left|\left(I_{a+}^{\alpha_i}f_i\right)(x)\right|^r \geq \frac{1}{\Gamma(\alpha_i)^r}\frac{(x-a)^{r\left((\alpha_i-1)+\frac{1}{p}\right)}}{(p(\alpha_i-1)+1)^{\frac{r}{p}}}\left(\int_a^b |f_i(t)|^q\,dt\right)^{\frac{r}{q}}, \tag{29.13}$$

and

$$\prod_{i=1}^{m}\left|\left(I_{a+}^{\alpha_i}f_i\right)(x)\right|^r \geq \frac{1}{\prod_{i=1}^{m}\Gamma(\alpha_i)^r}\frac{(x-a)^{r\left(\sum_{i=1}^{m}\alpha_i - m + \frac{m}{p}\right)}}{\left(\prod_{i=1}^{m}(p(\alpha_i-1)+1)\right)^{\frac{r}{p}}}\left(\prod_{i=1}^{m}\left(\int_a^b |f_i(t)|^q\,dt\right)^{\frac{1}{q}}\right)^r. \tag{29.14}$$

Consequently

$$\int_a^b\left(\prod_{i=1}^{m}\left|\left(I_{a+}^{\alpha_i}f_i\right)(x)\right|^r\right)dx \geq \frac{\left(\int_a^b (x-a)^{r\left(\sum_{i=1}^{m}\alpha_i - m + \frac{m}{p}\right)}\,dx\right)}{\left(\prod_{i=1}^{m}\Gamma(\alpha_i)^r\right)\left(\prod_{i=1}^{m}(p(\alpha_i-1)+1)\right)^{\frac{r}{p}}}$$

$$\cdot\left(\prod_{i=1}^{m}\left(\int_a^b |f_i(t)|^q\,dt\right)^{\frac{1}{q}}\right)^r \tag{29.15}$$

$$= \frac{(b-a)^{r\left(\sum_{i=1}^{m}\alpha_i - m + \frac{m}{p}\right)+1}}{\left(r\left(\sum_{i=1}^{m}\alpha_i - m + \frac{m}{p}\right)+1\right)\left(\prod_{i=1}^{m}\Gamma(\alpha_i)\left(p(\alpha_i-1)+1\right)^{\frac{1}{p}}\right)^r}, \tag{29.16}$$

$$\cdot\left(\prod_{i=1}^{m}\left(\int_a^b |f_i(t)|^q\,dt\right)^{\frac{1}{q}}\right)^r.$$

The claim is proved. □

We continue with

Theorem 29.3. *Let* $0 < p < 1, q < 0$ *such that* $\frac{1}{p}+\frac{1}{q}=1; \alpha_i > 0, i = 1,\ldots,m$. *Let* $f_i : (a,b) \to \mathbb{R}, i = 1,\ldots,m$, *of fixed strict sign a.e., which are Lebesgue measurable functions, so that* $\left\|\prod_{i=1}^{m}\left(I_{b-}^{\alpha_i}f_i\right)\right\|_p$, $\prod_{i=1}^{m}\|f_i\|_q$ *are finite. Then*

$$\left\| \prod_{i=1}^{m} \left(I_{b-}^{\alpha_i} f_i \right) \right\|_p \geq \frac{(b-a)^{\sum_{i=1}^{m} \alpha_i + m\left(\frac{1}{p}-1\right)+\frac{1}{p}}}{\left[\left(p \sum_{i=1}^{m} \alpha_i + m(1-p) + 1 \right)^{\frac{1}{p}} \left(\prod_{i=1}^{m} \Gamma(\alpha_i) \left(p(\alpha_i - 1) + 1 \right)^{\frac{1}{p}} \right) \right]}$$

$$\cdot \left(\prod_{i=1}^{m} \| f_i \|_q \right). \tag{29.17}$$

Proof. By (29.2) we have

$$\left(I_{b-}^{\alpha_i} f_i \right)(x) = \frac{1}{\Gamma(\alpha_i)} \int_x^b (t-x)^{\alpha_i - 1} f_i(t)\, dt, \tag{29.18}$$

$x < b, i = 1, \ldots, m$. We have that

$$\left| \left(I_{b-}^{\alpha_i} f_i \right)(x) \right| = \frac{1}{\Gamma(\alpha_i)} \int_x^b (t-x)^{\alpha_i - 1} |f_i(t)|\, dt, \tag{29.19}$$

$x < b, i = 1, \ldots, m$. By reverse Hölder's inequality we get

$$\left| \left(I_{b-}^{\alpha_i} f_i \right)(x) \right| \geq \frac{1}{\Gamma(\alpha_i)} \left(\int_x^b (t-x)^{p(\alpha_i - 1)}\, dt \right)^{\frac{1}{p}} \left(\int_x^b |f_i(t)|^q\, dt \right)^{\frac{1}{q}} \tag{29.20}$$

$$\geq \frac{1}{\Gamma(\alpha_i)} \frac{(b-x)^{\alpha_i - 1 + \frac{1}{p}}}{\left(p(\alpha_i - 1) + 1 \right)^{\frac{1}{p}}} \left(\int_a^b |f_i(t)|^q\, dt \right)^{\frac{1}{q}}, \tag{29.21}$$

$x < b, i = 1, \ldots, m$. Therefore

$$\prod_{i=1}^{m} \left| \left(I_{b-}^{\alpha_i} f_i \right)(x) \right|^p \geq \frac{1}{\left(\prod_{i=1}^{m} \Gamma(\alpha_i) \right)^p} \frac{(b-x)^{p \sum_{i=1}^{m} \alpha_i + m(1-p)}}{\prod_{i=1}^{m} \left(p(\alpha_i - 1) + 1 \right)} \left(\prod_{i=1}^{m} \int_a^b |f_i(t)|^q\, dt \right)^{\frac{p}{q}}, \tag{29.22}$$

$x \in (a, b)$. Consequently we get

$$\int_a^b \left(\prod_{i=1}^{m} \left| \left(I_{b-}^{\alpha_i} f_i \right)(x) \right|^p \right) dx \geq \left(\frac{1}{\left(\prod_{i=1}^{m} \Gamma(\alpha_i) \right)^p \left(\prod_{i=1}^{m} \left(p(\alpha_i - 1) + 1 \right) \right)} \right)$$

$$\cdot \left(\int_a^b (b-x)^{p \sum_{i=1}^{m} \alpha_i + m(1-p)}\, dx \right) \left(\prod_{i=1}^{m} \int_a^b |f_i(t)|^q\, dt \right)^{\frac{p}{q}} \tag{29.23}$$

$$= \frac{(b-a)^{p\sum_{i=1}^{m}\alpha_i+m(1-p)+1}\left(\prod_{i=1}^{m}\int_a^b |f_i(t)|^q dt\right)^{\frac{p}{q}}}{\left[\left(p\sum_{i=1}^{m}\alpha_i+m(1-p)+1\right)\left(\prod_{i=1}^{m}\left(\Gamma(\alpha_i)^p\left(p(\alpha_i-1)+1\right)\right)\right)\right]}, \tag{29.24}$$

proving the claim. □

It follows

Theorem 29.4. *Let* $0 < p < 1, q < 0$ *such that* $\frac{1}{p}+\frac{1}{q}=1, r>0; \alpha_i>0, i=1,\ldots,m.$ *Let* $f_i : (a,b) \to \mathbb{R}, i=1,\ldots,m,$ *of fixed strict sign a.e., which are Lebesgue measurable functions, so that* $\left\|\prod_{i=1}^{m}\left(I_{b-}^{\alpha_i}f_i\right)\right\|_r, \prod_{i=1}^{m}\|f_i\|_q$ *are finite. Then*

$$\left\|\prod_{i=1}^{m}\left(I_{b-}^{\alpha_i}f_i\right)\right\|_r \geq \frac{(b-a)^{\sum_{i=1}^{m}\alpha_i-m+\frac{m}{p}+\frac{1}{r}}}{\left[\left(r\left(\sum_{i=1}^{m}\alpha_i-m+\frac{m}{p}\right)+1\right)^{\frac{1}{r}}\left(\prod_{i=1}^{m}\Gamma(\alpha_i)\left(p(\alpha_i-1)+1\right)^{\frac{1}{p}}\right)\right]} \cdot \left(\prod_{i=1}^{m}\|f_i\|_q\right). \tag{29.25}$$

Proof. Using $r>0$ and (29.21) we get

$$\left|\left(I_{b-}^{\alpha_i}f_i\right)(x)\right|^r \geq \frac{1}{\Gamma(\alpha_i)^r}\frac{(b-x)^{r\left((\alpha_i-1)+\frac{1}{p}\right)}}{(p(\alpha_i-1)+1)^{\frac{r}{p}}}\left(\int_a^b |f_i(t)|^q dt\right)^{\frac{r}{q}}, \tag{29.26}$$

and

$$\prod_{i=1}^{m}\left|\left(I_{b-}^{\alpha_i}f_i\right)(x)\right|^r \geq \frac{1}{\prod_{i=1}^{m}\Gamma(\alpha_i)^r}\frac{(b-x)^{r\left(\sum_{i=1}^{m}\alpha_i-m+\frac{m}{p}\right)}}{\left(\prod_{i=1}^{m}(p(\alpha_i-1)+1)\right)^{\frac{r}{p}}}\left(\prod_{i=1}^{m}\left(\int_a^b |f_i(t)|^q dt\right)^{\frac{1}{q}}\right)^r. \tag{29.27}$$

Consequently it holds

$$\int_a^b\left(\prod_{i=1}^{m}\left|\left(I_{b-}^{\alpha_i}f_i\right)(x)\right|^r\right)dx \geq \frac{\left(\int_a^b (b-x)^{r\left(\sum_{i=1}^{m}\alpha_i-m+\frac{m}{p}\right)}dx\right)}{\left(\prod_{i=1}^{m}\Gamma(\alpha_i)^r\right)\left(\prod_{i=1}^{m}(p(\alpha_i-1)+1)\right)^{\frac{r}{p}}} \cdot\left(\prod_{i=1}^{m}\left(\int_a^b |f_i(t)|^q dt\right)^{\frac{1}{q}}\right)^r \tag{29.28}$$

$$= \frac{(b-a)^{r\left(\sum\limits_{i=1}^{m}\alpha_i - m + \frac{m}{p}\right)+1}}{\left(r\left(\sum\limits_{i=1}^{m}\alpha_i - m + \frac{m}{p}\right)+1\right)\left(\prod\limits_{i=1}^{m}\Gamma(\alpha_i)\left(p(\alpha_i-1)+1\right)^{\frac{1}{p}}\right)^r}, \tag{29.29}$$

$$\cdot\left(\prod_{i=1}^{m}\left(\int_a^b |f_i(t)|^q\,dt\right)^{\frac{1}{q}}\right)^r.$$

The claim is proved. □

We need

Definition 29.5. ([14, p. 99]) The fractional integrals of a function f with respect to given function g are defined as follows:

Let $a,b \in \mathbb{R}$, $a < b$, $\alpha > 0$. Here g is an increasing function on $[a,b]$ and $g \in C^1([a,b])$. The left- and right-sided fractional integrals of a function f with respect to another function g in $[a,b]$ are given by

$$\left(I_{a+;g}^{\alpha}f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{g'(t)\,f(t)\,dt}{(g(x)-g(t))^{1-\alpha}},\ x > a, \tag{29.30}$$

$$\left(I_{b-;g}^{\alpha}f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \frac{g'(t)\,f(t)\,dt}{(g(t)-g(x))^{1-\alpha}},\ x < b, \tag{29.31}$$

respectively.

We present

Theorem 29.6. *Let* $0 < p < 1, q < 0$ *such that* $\frac{1}{p}+\frac{1}{q} = 1; \alpha_i > 0, i = 1,\ldots,m$. *Here* $a,b \in \mathbb{R}$ *and strictly increasing* g *with* $I_{a+;g}^{\alpha_i}$ *as in Definition 29.5; see (29.30). Let* $f_i : (a,b) \to \mathbb{R}, i = 1,\ldots,m$, *of fixed strict sign a.e., which are Lebesgue measurable functions, so that* $\left\|\prod\limits_{i=1}^{m}\left(I_{a+;g}^{\alpha_i}f_i\right)\right\|_{L_p(g)}$, $\prod\limits_{i=1}^{m}\|f_i\|_{L_q(g)}$ *are finite. Then*

$$\left\|\prod_{i=1}^{m}\left(I_{a+;g}^{\alpha_i}f_i\right)\right\|_{L_p(g)} \geq \frac{(g(b)-g(a))^{\sum\limits_{i=1}^{m}\alpha_i + m\left(\frac{1}{p}-1\right)+\frac{1}{p}}}{\left[\left(p\sum\limits_{i=1}^{m}\alpha_i + m(1-p)+1\right)^{\frac{1}{p}}\left(\prod\limits_{i=1}^{m}\Gamma(\alpha_i)\left(p(\alpha_i-1)+1\right)^{\frac{1}{p}}\right)\right]}$$

$$\cdot\left(\prod_{i=1}^{m}\|f_i\|_{L_q(g)}\right). \tag{29.32}$$

Proof. By (29.30) we have

$$\left(I_{a+;g}^{\alpha_i}f_i\right)(x) = \frac{1}{\Gamma(\alpha_i)}\int_a^x \frac{g'(t)f_i(t)}{(g(x)-g(t))^{1-\alpha_i}}dt, \tag{29.33}$$

$x > a, i = 1, \ldots, m$. We have that

$$\left|\left(I_{a+;g}^{\alpha_i} f_i\right)(x)\right| = \frac{1}{\Gamma(\alpha_i)} \int_a^x (g(x) - g(t))^{\alpha_i - 1} g'(t) |f_i(t)| dt$$

$$= \frac{1}{\Gamma(\alpha_i)} \int_a^x (g(x) - g(t))^{\alpha_i - 1} |f_i(t)| dg(t), \tag{29.34}$$

$x > a, i = 1, \ldots, m$. By reverse Hölder's inequality we get

$$\left|\left(I_{a+;g}^{\alpha_i} f_i\right)(x)\right| \geq \frac{1}{\Gamma(\alpha_i)} \left(\int_a^x (g(x) - g(t))^{p(\alpha_i - 1)} dg(t)\right)^{\frac{1}{p}} \left(\int_a^x |f_i(t)|^q dg(t)\right)^{\frac{1}{q}}$$

$$\geq \frac{1}{\Gamma(\alpha_i)} \frac{(g(x) - g(a))^{\alpha_i - 1 + \frac{1}{p}}}{(p(\alpha_i - 1) + 1)^{\frac{1}{p}}} \left(\int_a^b |f_i(t)|^q dg(t)\right)^{\frac{1}{q}} \tag{29.35}$$

$$= \frac{1}{\Gamma(\alpha_i)} \frac{(g(x) - g(a))^{\alpha_i - 1 + \frac{1}{p}}}{(p(\alpha_i - 1) + 1)^{\frac{1}{p}}} \|f_i\|_{L_q(g)}, \tag{29.36}$$

$x > a, i = 1, \ldots, m$. So we got

$$\left|\left(I_{a+;g}^{\alpha_i} f_i\right)(x)\right| \geq \frac{(g(x) - g(a))^{\alpha_i - 1 + \frac{1}{p}}}{\Gamma(\alpha_i)(p(\alpha_i - 1) + 1)^{\frac{1}{p}}} \|f_i\|_{L_q(g)}, \tag{29.37}$$

$x > a, i = 1, \ldots, m$. Hence

$$\prod_{i=1}^m \left|\left(I_{a+;g}^{\alpha_i} f_i\right)(x)\right|^p \geq \frac{(g(x) - g(a))^{p \sum_{i=1}^m \alpha_i + m(1-p)}}{\prod_{i=1}^m (\Gamma(\alpha_i)^p (p(\alpha_i - 1) + 1))} \prod_{i=1}^m \|f_i\|_{L_q(g)}^p, \tag{29.38}$$

$x \in (a, b)$. Consequently, we obtain

$$\int_a^b \left(\prod_{i=1}^m \left|\left(I_{a+;g}^{\alpha_i} f_i\right)(x)\right|^p\right) dg(x) \geq \frac{\prod_{i=1}^m \|f_i\|_{L_q(g)}^p \int_a^b (g(x) - g(a))^{p \sum_{i=1}^m \alpha_i + m(1-p)} dg(x)}{\prod_{i=1}^m (\Gamma(\alpha_i)^p (p(\alpha_i - 1) + 1))}$$

$$= \prod_{i=1}^m \left[\frac{\|f_i\|_{L_q(g)}^p}{(\Gamma(\alpha_i)^p (p(\alpha_i - 1) + 1))}\right] \frac{(g(b) - g(a))^{p \sum_{i=1}^m \alpha_i + m(1-p) + 1}}{\left(p \sum_{i=1}^m \alpha_i + m(1-p) + 1\right)}, \tag{29.39}$$

proving the claim. □

We also give

Theorem 29.7. *Let* $0 < p < 1, q < 0$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$; $\alpha_i > 0, i = 1, \ldots, m$; $r > 0$. *Here* $a, b \in \mathbb{R}$ *and strictly increasing* g *with* $I_{a+;g}^{\alpha_i}$ *as in Definition 29.5; see (29.30). Let* $f_i : (a,b) \to \mathbb{R}, i = 1, \ldots, m$, *of fixed strict sign a.e., which are Lebesgue measurable functions, and* $\left\| \prod_{i=1}^{m} \left(I_{a+;g}^{\alpha_i} f_i \right) \right\|_{L_r(g)}$, $\prod_{i=1}^{m} \|f_i\|_{L_q(g)}$ *are finite. Then*

$$\left\| \prod_{i=1}^{m} \left(I_{a+;g}^{\alpha_i} f_i \right) \right\|_{L_r(g)} \geq \frac{(g(b)-g(a))^{\sum_{i=1}^{m} \alpha_i - m + \frac{m}{p} + \frac{1}{r}}}{\left[\left(r \left(\sum_{i=1}^{m} \alpha_i - m + \frac{m}{p} \right) + 1 \right)^{\frac{1}{r}} \left(\prod_{i=1}^{m} \Gamma(\alpha_i) \left(p(\alpha_i - 1) + 1 \right)^{\frac{1}{p}} \right) \right]} \cdot \left(\prod_{i=1}^{m} \|f_i\|_{L_q(g)} \right). \tag{29.40}$$

Proof. Using $r > 0$ and (29.37) we get

$$\left| \left(I_{a+;g}^{\alpha_i} f_i \right)(x) \right|^r \geq \frac{(g(x)-g(a))^{r\left(\alpha_i - 1 + \frac{1}{p}\right)}}{\Gamma(\alpha_i)^r \left(p(\alpha_i - 1) + 1 \right)^{\frac{r}{p}}} \|f_i\|_{L_q(g)}^r, \tag{29.41}$$

and

$$\prod_{i=1}^{m} \left| \left(I_{a+;g}^{\alpha_i} f_i \right)(x) \right|^r \geq \frac{(g(x)-g(a))^{r\left(\sum_{i=1}^{m} \alpha_i - m + \frac{m}{p}\right)}}{\left(\prod_{i=1}^{m} \Gamma(\alpha_i) \left(p(\alpha_i - 1) + 1 \right)^{\frac{1}{p}} \right)^r} \left(\prod_{i=1}^{m} \|f_i\|_{L_q(g)} \right)^r, \tag{29.42}$$

$x \in (a,b)$. Consequently, it holds

$$\int_a^b \prod_{i=1}^{m} \left| \left(I_{a+;g}^{\alpha_i} f_i \right)(x) \right|^r dg(x) \geq \frac{\left(\int_a^b (g(x)-g(a))^{r\left(\sum_{i=1}^{m} \alpha_i - m + \frac{m}{p}\right)} dg(x) \right)}{\left(\prod_{i=1}^{m} \left(\Gamma(\alpha_i) \left(p(\alpha_i - 1) + 1 \right)^{\frac{1}{p}} \right) \right)^r} \cdot \left(\prod_{i=1}^{m} \|f_i\|_{L_q(g)} \right)^r \tag{29.43}$$

$$= \frac{(g(b)-g(a))^{r\left(\sum_{i=1}^{m} \alpha_i - m + \frac{m}{p}\right)+1} \left(\prod_{i=1}^{m} \|f_i\|_{L_q(g)} \right)^r}{\left(r \left(\sum_{i=1}^{m} \alpha_i - m + \frac{m}{p} \right) + 1 \right) \left(\prod_{i=1}^{m} \left(\Gamma(\alpha_i) \left(p(\alpha_i - 1) + 1 \right)^{\frac{1}{p}} \right) \right)^r}. \tag{29.44}$$

The claim is proved. □

We continue with

Theorem 29.8. *Let $0 < p < 1, q < 0$ such that $\frac{1}{p} + \frac{1}{q} = 1; \alpha_i > 0, i = 1, \ldots, m$. Here $a, b \in \mathbb{R}$ and strictly increasing g with $I_{b-;g}^{\alpha_i}$ as in Definition 29.5; see (29.31). Let $f_i : (a,b) \to \mathbb{R}, i = 1, \ldots, m$, of fixed strict sign a.e., which are Lebesgue measurable functions, and $\left\| \prod_{i=1}^{m} \left(I_{b-;g}^{\alpha_i} f_i \right) \right\|_{L_p(g)}$, $\prod_{i=1}^{m} \| f_i \|_{L_q(g)}$ are finite. Then*

$$\left\| \prod_{i=1}^{m} \left(I_{b-;g}^{\alpha_i} f_i \right) \right\|_{L_p(g)} \geq \frac{(g(b) - g(a))^{\sum_{i=1}^{m} \alpha_i + m\left(\frac{1}{p} - 1\right) + \frac{1}{p}}}{\left[\left(p \sum_{i=1}^{m} \alpha_i + m(1-p) + 1 \right)^{\frac{1}{p}} \left(\prod_{i=1}^{m} \Gamma(\alpha_i) (p(\alpha_i - 1) + 1)^{\frac{1}{p}} \right) \right]} \cdot \left(\prod_{i=1}^{m} \| f_i \|_{L_q(g)} \right). \tag{29.45}$$

Proof. By (29.31) we have

$$\left(I_{b-;g}^{\alpha_i} f_i \right)(x) = \frac{1}{\Gamma(\alpha_i)} \int_x^b \frac{g'(t) f_i(t)}{(g(t) - g(x))^{1-\alpha_i}} dt, \tag{29.46}$$

$x < b, i = 1, \ldots, m$. We have that

$$\left| \left(I_{b-;g}^{\alpha_i} f_i \right)(x) \right| = \frac{1}{\Gamma(\alpha_i)} \int_x^b (g(t) - g(x))^{\alpha_i - 1} g'(t) |f_i(t)| dt$$

$$= \frac{1}{\Gamma(\alpha_i)} \int_x^b (g(t) - g(x))^{\alpha_i - 1} |f_i(t)| dg(t), \tag{29.47}$$

$x < b, i = 1, \ldots, m$. By reverse Hölder's inequality we get

$$\left| \left(I_{b-;g}^{\alpha_i} f_i \right)(x) \right| \geq \frac{1}{\Gamma(\alpha_i)} \left(\int_x^b (g(t) - g(x))^{p(\alpha_i - 1)} dg(t) \right)^{\frac{1}{p}} \left(\int_x^b |f_i(t)|^q dg(t) \right)^{\frac{1}{q}}$$

$$\geq \frac{1}{\Gamma(\alpha_i)} \frac{(g(b) - g(x))^{\alpha_i - 1 + \frac{1}{p}}}{(p(\alpha_i - 1) + 1)^{\frac{1}{p}}} \left(\int_a^b |f_i(t)|^q dg(t) \right)^{\frac{1}{q}} \tag{29.48}$$

$$= \frac{1}{\Gamma(\alpha_i)} \frac{(g(b) - g(x))^{\alpha_i - 1 + \frac{1}{p}}}{(p(\alpha_i - 1) + 1)^{\frac{1}{p}}} \| f_i \|_{L_q(g)}, \tag{29.49}$$

$x < b, i = 1, \ldots, m$. So we got

$$\left| \left(I_{b-;g}^{\alpha_i} f_i \right)(x) \right| \geq \frac{(g(b) - g(x))^{\alpha_i - 1 + \frac{1}{p}}}{\Gamma(\alpha_i)(p(\alpha_i - 1) + 1)^{\frac{1}{p}}} \| f_i \|_{L_q(g)}, \tag{29.50}$$

$x < b, i = 1, \ldots, m$. Hence

$$\prod_{i=1}^{m}\left|\left(I_{b-;g}^{\alpha_i}f_i\right)(x)\right|^p \geq \frac{(g(b)-g(x))^{p\sum\limits_{i=1}^{m}\alpha_i+m(1-p)}}{\prod\limits_{i=1}^{m}\left(\Gamma(\alpha_i)^p(p(\alpha_i-1)+1)\right)}\prod_{i=1}^{m}\|f_i\|_{L_q(g)}^p, \tag{29.51}$$

$x \in (a,b)$. Consequently, we obtain

$$\int_a^b\left(\prod_{i=1}^{m}\left|\left(I_{b-;g}^{\alpha_i}f_i\right)(x)\right|^p\right)dg(x) \geq \frac{\prod\limits_{i=1}^{m}\|f_i\|_{L_q(g)}^p\left(\int_a^b(g(b)-g(x))^{p\sum\limits_{i=1}^{m}\alpha_i+m(1-p)}dg(x)\right)}{\prod\limits_{i=1}^{m}\left(\Gamma(\alpha_i)^p(p(\alpha_i-1)+1)\right)}$$

$$=\prod_{i=1}^{m}\left[\frac{\|f_i\|_{L_q(g)}^p}{\left(\Gamma(\alpha_i)^p(p(\alpha_i-1)+1)\right)}\right]\frac{(g(b)-g(a))^{p\sum\limits_{i=1}^{m}\alpha_i+m(1-p)+1}}{\left(p\sum\limits_{i=1}^{m}\alpha_i+m(1-p)+1\right)}, \tag{29.52}$$

proving the claim. □

We also give

Theorem 29.9. *Let $0<p<1$, $q<0$ such that $\frac{1}{p}+\frac{1}{q}=1$; $\alpha_i>0, i=1,\ldots,m$, $r>0$. Here $a,b\in\mathbb{R}$ and strictly increasing g with $I_{b-;g}^{\alpha_i}$ as in Definition 29.5; see (29.31). Let $f_i:(a,b)\to\mathbb{R}, i=1,\ldots,m$, of fixed strict sign a.e., which are Lebesgue measurable functions, and $\left\|\prod\limits_{i=1}^{m}\left(I_{b-;g}^{\alpha_i}f_i\right)\right\|_{L_r(g)}$, $\prod\limits_{i=1}^{m}\|f_i\|_{L_q(g)}$ are finite. Then*

$$\left\|\prod_{i=1}^{m}\left(I_{b-;g}^{\alpha_i}f_i\right)\right\|_{L_r(g)} \geq \frac{(g(b)-g(a))^{\sum\limits_{i=1}^{m}\alpha_i-m+\frac{m}{p}+\frac{1}{r}}}{\left[\left(r\left(\sum\limits_{i=1}^{m}\alpha_i-m+\frac{m}{p}\right)+1\right)^{\frac{1}{r}}\left(\prod\limits_{i=1}^{m}\Gamma(\alpha_i)(p(\alpha_i-1)+1)^{\frac{1}{p}}\right)\right]}$$

$$\cdot\left(\prod_{i=1}^{m}\|f_i\|_{L_q(g)}\right). \tag{29.53}$$

Proof. Using $r>0$ and (29.50) we get

$$\left|\left(I_{b-;g}^{\alpha_i}f_i\right)(x)\right|^r \geq \frac{(g(b)-g(x))^{r\left(\alpha_i-1+\frac{1}{p}\right)}}{\Gamma(\alpha_i)^r(p(\alpha_i-1)+1)^{\frac{r}{p}}}\|f_i\|_{L_q(g)}^r, \tag{29.54}$$

and

$$\prod_{i=1}^{m}\left|\left(I_{b-;g}^{\alpha_i}f_i\right)(x)\right|^r \geq \frac{(g(b)-g(x))^{r\left(\sum\limits_{i=1}^{m}\alpha_i-m+\frac{m}{p}\right)}}{\prod\limits_{i=1}^{m}\left(\Gamma(\alpha_i)(p(\alpha_i-1)+1)^{\frac{1}{p}}\right)^r}\left(\prod_{i=1}^{m}\|f_i\|_{L_q(g)}\right)^r, \tag{29.55}$$

$x \in (a,b)$. Consequently, it holds

$$\int_a^b \prod_{i=1}^m \left| \left(I_{b-;g}^{\alpha_i} f_i \right)(x) \right|^r dg(x) \geq \frac{\left(\int_a^b (g(b)-g(x))^{r\left(\sum_{i=1}^m \alpha_i - m + \frac{m}{p}\right)} dg(x) \right)}{\left(\prod_{i=1}^m \left(\Gamma(\alpha_i)(p(\alpha_i-1)+1)^{\frac{1}{p}} \right) \right)^r}$$

$$\cdot \left(\prod_{i=1}^m \|f_i\|_{L_q(g)} \right)^r \tag{29.56}$$

$$= \frac{(g(b)-g(a))^{r\left(\sum_{i=1}^m \alpha_i - m + \frac{m}{p}\right)+1} \left(\prod_{i=1}^m \|f_i\|_{L_q(g)} \right)^r}{\left(r\left(\sum_{i=1}^m \alpha_i - m + \frac{m}{p} \right) + 1 \right) \left(\prod_{i=1}^m \left(\Gamma(\alpha_i)(p(\alpha_i-1)+1)^{\frac{1}{p}} \right) \right)^r}. \tag{29.57}$$

The claim is proved. □

We need

Definition 29.10. ([13]). Let $0 < a < b < \infty$, $\alpha > 0$. The left- and right-sided Hadamard fractional integrals of order α are given by

$$\left(J_{a+}^{\alpha} f \right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{y} \right)^{\alpha-1} \frac{f(y)}{y} dy, \quad x > a, \tag{29.58}$$

and

$$\left(J_{b-}^{\alpha} f \right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{y}{x} \right)^{\alpha-1} \frac{f(y)}{y} dy, \quad x < b, \tag{29.59}$$

respectively.

Notice that the Hadamard fractional integrals of order α are special cases of left- and right-sided fractional integrals of a function f with respect to another function, here $g(x) = \ln x$ on $[a,b]$, $0 < a < b < \infty$.

Above f is a Lebesgue measurable function from (a,b) into $\mathbb{R}$, such that $\left(J_{a+}^{\alpha}(|f|) \right)(x)$ and/or $\left(J_{b-}^{\alpha}(|f|) \right)(x) \in \mathbb{R}$, $\forall\, x \in (a,b)$.

We present

Theorem 29.11. *Let $0 < p < 1, q < 0$ such that $\frac{1}{p} + \frac{1}{q} = 1; \alpha_i > 0, i = 1,\ldots,m$. Here $0 < a < b < \infty$, and $J_{a+}^{\alpha_i}$ as in Definition 29.10; see (29.58). Let $f_i : (a,b) \to \mathbb{R}, i = 1,\ldots,m$, of fixed strict sign a.e., which are Lebesgue measurable functions, and $\left\| \prod_{i=1}^m \left(J_{a+}^{\alpha_i} f_i \right) \right\|_{L_p(ln)}$, $\prod_{i=1}^m \|f_i\|_{L_q(ln)}$ are finite. Then*

$$\left\|\prod_{i=1}^{m}\left(J_{a+}^{\alpha_i} f_i\right)\right\|_{L_p(ln)} \geq \frac{\left(\ln\left(\frac{b}{a}\right)\right)^{\sum_{i=1}^{m}\alpha_i+m\left(\frac{1}{p}-1\right)+\frac{1}{p}}}{\left(p\sum_{i=1}^{m}\alpha_i+m(1-p)+1\right)^{\frac{1}{p}}\left(\prod_{i=1}^{m}\left(\Gamma(\alpha_i)\left(p(\alpha_i-1)+1\right)^{\frac{1}{p}}\right)\right)} \cdot \left(\prod_{i=1}^{m}\|f_i\|_{L_q(ln)}\right). \tag{29.60}$$

Proof. By Theorem 29.6, for $g(x)=\ln x$. □

We also have

Theorem 29.12. *Let $0<p<1, q<0$ such that $\frac{1}{p}+\frac{1}{q}=1; \alpha_i>0, i=1,\dots,m; r>0$. Here $0<a<b<\infty$, and $J_{a+}^{\alpha_i}$ as in Definition 29.10; see (29.58). Let $f_i:(a,b)\to\mathbb{R}, i=1,\dots,m$, of fixed strict sign a.e., which are Lebesgue measurable functions, and $\left\|\prod_{i=1}^{m}\left(J_{a+}^{\alpha_i} f_i\right)\right\|_{L_r(ln)}$, $\prod_{i=1}^{m}\|f_i\|_{L_q(ln)}$ are finite. Then*

$$\left\|\prod_{i=1}^{m}\left(J_{a+}^{\alpha_i} f_i\right)\right\|_{L_r(ln)} \geq \frac{\left(\ln\left(\frac{b}{a}\right)\right)^{\sum_{i=1}^{m}\alpha_i-m+\frac{m}{p}+\frac{1}{r}}}{\left(r\left(\sum_{i=1}^{m}\alpha_i-m+\frac{m}{p}\right)+1\right)^{\frac{1}{r}}\left(\prod_{i=1}^{m}\left(\Gamma(\alpha_i)\left(p(\alpha_i-1)+1\right)^{\frac{1}{p}}\right)\right)} \cdot \left(\prod_{i=1}^{m}\|f_i\|_{L_q(ln)}\right). \tag{29.61}$$

Proof. By Theorem 29.7, for $g(x)=\ln x$. □

We continue with

Theorem 29.13. *Let $0<p<1, q<0$ such that $\frac{1}{p}+\frac{1}{q}=1; \alpha_i>0, i=1,\dots,m$. Here $0<a<b<\infty$, and $J_{b-}^{\alpha_i}$ as in Definition 29.10; see (29.59). Let $f_i:(a,b)\to\mathbb{R}, i=1,\dots,m$, of fixed strict sign a.e., which are Lebesgue measurable functions, and $\left\|\prod_{i=1}^{m}\left(J_{b-}^{\alpha_i} f_i\right)\right\|_{L_p(ln)}$, $\prod_{i=1}^{m}\|f_i\|_{L_q(ln)}$ are finite. Then*

$$\left\|\prod_{i=1}^{m}\left(J_{b-}^{\alpha_i} f_i\right)\right\|_{L_p(ln)} \geq \frac{\left(\ln\left(\frac{b}{a}\right)\right)^{\sum_{i=1}^{m}\alpha_i+m\left(\frac{1}{p}-1\right)+\frac{1}{p}}}{\left(p\sum_{i=1}^{m}\alpha_i+m(1-p)+1\right)^{\frac{1}{p}}\left(\prod_{i=1}^{m}\left(\Gamma(\alpha_i)\left(p(\alpha_i-1)+1\right)^{\frac{1}{p}}\right)\right)} \cdot \left(\prod_{i=1}^{m}\|f_i\|_{L_q(ln)}\right). \tag{29.62}$$

Proof. By Theorem 29.8, for $g(x) = \ln x$. □

We also have

Theorem 29.14. *Let $0 < p < 1, q < 0$ such that $\frac{1}{p} + \frac{1}{q} = 1; \alpha_i > 0, i = 1, \ldots, m; r > 0$. Here $0 < a < b < \infty$, and $J_{b-}^{\alpha_i}$ as in Definition 29.10; see (29.59). Let $f_i : (a,b) \to \mathbb{R}, i = 1, \ldots, m$, of fixed strict sign a.e., which are Lebesgue measurable functions, and $\left\| \prod_{i=1}^{m} \left(J_{b-}^{\alpha_i} f_i \right) \right\|_{L_r(ln)}$, $\prod_{i=1}^{m} \|f_i\|_{L_q(ln)}$ are finite. Then*

$$\left\| \prod_{i=1}^{m} \left(J_{b-}^{\alpha_i} f_i \right) \right\|_{L_r(ln)} \geq \frac{\left(\ln\left(\frac{b}{a}\right)\right)^{\sum_{i=1}^{m} \alpha_i - m + \frac{m}{p} + \frac{1}{r}}}{\left(r \left(\sum_{i=1}^{m} \alpha_i - m + \frac{m}{p} \right) + 1 \right)^{\frac{1}{r}} \left(\prod_{i=1}^{m} \left(\Gamma(\alpha_i) \left(p(\alpha_i - 1) + 1 \right)^{\frac{1}{p}} \right) \right)}$$

$$\cdot \left(\prod_{i=1}^{m} \|f_i\|_{L_q(ln)} \right). \tag{29.63}$$

Proof. By Theorem 29.9, for $g(x) = \ln x$. □

We need

Definition 29.15. ([16]) Let (a,b), $0 \leq a < b < \infty$; $\alpha, \sigma > 0$. We consider the left- and right-sided fractional integrals of order α as follows:

1) For $\eta > -1$, we define

$$\left(I_{a+;\sigma,\eta}^{\alpha} f \right)(x) = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_a^x \frac{t^{\sigma\eta+\sigma-1} f(t)\, dt}{\left(x^\sigma - t^\sigma \right)^{1-\alpha}} \tag{29.64}$$

2) For $\eta > 0$, we define

$$\left(I_{b-;\sigma,\eta}^{\alpha} f \right)(x) = \frac{\sigma x^{\sigma\eta}}{\Gamma(\alpha)} \int_x^b \frac{t^{\sigma(1-\eta-\alpha)-1} f(t)\, dt}{\left(t^\sigma - x^\sigma \right)^{1-\alpha}} \tag{29.65}$$

These are the Erdélyi-Kober-type fractional integrals.

We present

Theorem 29.16. *Let $0 < p < 1, q < 0$ such that $\frac{1}{p} + \frac{1}{q} = 1; \alpha_i > 0, i = 1, \ldots, m$. Here $0 \leq a < b < \infty$, $\sigma > 0$, $\eta > -1$, and $I_{a+;\sigma,\eta}^{\alpha_i}$ is as in Definition 29.15; see (29.64). Let $f_i : (a,b) \to \mathbb{R}$, $i = 1, \ldots, m$, of fixed strict sign a.e., which are Lebesgue measurable functions, and $\left\| \prod_{i=1}^{m} \left(x^{\sigma(\alpha_i+\eta)} \left(I_{a+;\sigma,\eta}^{\alpha_i} f_i \right)(x) \right) \right\|_{L_p(x^\sigma)}$, $\prod_{i=1}^{m} \|x^{\sigma\eta} f_i(x)\|_{L_q(x^\sigma)}$ are finite. Then*

$$\left\| \prod_{i=1}^{m} \left(x^{\sigma(\alpha_i+\eta)} \left(I_{a+;\sigma,\eta}^{\alpha_i} f_i \right)(x) \right) \right\|_{L_p(x^\sigma)} \geq \frac{(b^\sigma - a^\sigma)^{\sum_{i=1}^{m} \alpha_i + m\left(\frac{1}{p}-1\right)+\frac{1}{p}}}{\left(p \sum_{i=1}^{m} \alpha_i + m(1-p) + 1 \right)^{\frac{1}{p}}}$$

$$\cdot \frac{1}{\left(\prod_{i=1}^{m} \left(\Gamma(\alpha_i) \left(p(\alpha_i - 1) + 1 \right)^{\frac{1}{p}} \right) \right)} \left(\prod_{i=1}^{m} \| x^{\sigma\eta} f_i(x) \|_{L_q(x^\sigma)} \right). \tag{29.66}$$

Proof. By Definition 29.15 (see (29.64)) we have

$$\left(I_{a+;\sigma,\eta}^{\alpha_i} f_i \right)(x) = \frac{\sigma x^{-\sigma(\alpha_i+\eta)}}{\Gamma(\alpha_i)} \int_a^x \frac{t^{\sigma\eta+\sigma-1} f_i(t)\, dt}{(x^\sigma - t^\sigma)^{1-\alpha_i}}, \tag{29.67}$$

$x > a$. We rewrite (29.67) as follows:

$$L_1(f_i)(x) := x^{\sigma(\alpha_i+\eta)} \left(I_{a+;\sigma,\eta}^{\alpha_i} f_i \right)(x)$$

$$= \frac{1}{\Gamma(\alpha_i)} \int_a^x (x^\sigma - t^\sigma)^{\alpha_i - 1} (t^{\sigma\eta} f_i(t))\, dt^\sigma, \tag{29.68}$$

and by calling $F_{1i}(t) = t^{\sigma\eta} f_i(t)$, we have

$$L_1(f_i)(x) = \frac{1}{\Gamma(\alpha_i)} \int_a^x (x^\sigma - t^\sigma)^{\alpha_i - 1} F_{1i}(t) dt^\sigma, \tag{29.69}$$

$i = 1, \ldots, m$, $x > a$. Furthermore we notice that

$$|L_1(f_i)(x)| = \frac{1}{\Gamma(\alpha_i)} \int_a^x (x^\sigma - t^\sigma)^{\alpha_i - 1} |F_{1i}(t)|\, dt^\sigma, \tag{29.70}$$

$i = 1, \ldots, m$, $x > a$. So that now we can act as in the proof of Theorem 29.6. □

We continue with

Theorem 29.17. *Let* $0 < p < 1$, $q < 0$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$; $\alpha_i > 0$, $i=1,\ldots,m$, $r>0$. *Here* $0 \leq a < b < \infty$, $\sigma > 0$, $\eta > -1$, *and* $I_{a+;\sigma,\eta}^{\alpha_i}$ *is as in Definition 29.15; see (29.64). Let* $f_i : (a,b) \to \mathbb{R}$, $i = 1, \ldots, m$, *of fixed strict sign a.e., which are Lebesgue measurable functions, and* $\left\| \prod_{i=1}^{m} \left(x^{\sigma(\alpha_i+\eta)} \left(I_{a+;\sigma,\eta}^{\alpha_i} f_i \right)(x) \right) \right\|_{L_r(x^\sigma)}$, $\prod_{i=1}^{m} \| x^{\sigma\eta} f_i(x) \|_{L_q(x^\sigma)}$ *are finite. Then*

$$\left\| \prod_{i=1}^{m} \left(x^{\sigma(\alpha_i+\eta)} \left(I_{a+;\sigma,\eta}^{\alpha_i} f_i \right)(x) \right) \right\|_{L_r(x^\sigma)} \geq \frac{(b^\sigma - a^\sigma)^{\sum_{i=1}^{m} \alpha_i - m + \frac{m}{p} + \frac{1}{r}}}{\left(r \left(\sum_{i=1}^{m} \alpha_i - m + \frac{m}{p} \right) + 1 \right)^{\frac{1}{r}}}$$

$$\cdot \frac{1}{\left(\prod_{i=1}^{m}\left(\Gamma(\alpha_i)\left(p(\alpha_i-1)+1\right)^{\frac{1}{p}}\right)\right)}\left(\prod_{i=1}^{m}\left\|x^{\sigma\eta}f_i(x)\right\|_{L_q(x^\sigma)}\right). \tag{29.71}$$

Proof. Based on the proof of Theorem 29.16 and similarly acting as in the proof of Theorem 29.7. □

We also have

Theorem 29.18. *Let* $0<p<1$, $q<0$ *such that* $\frac{1}{p}+\frac{1}{q}=1$; $\alpha_i>0$, $i=1,\ldots,m$. *Here* $0\le a<b<\infty$, $\sigma>0$, $\eta>0$, *and* $I_{b-;\sigma,\eta}^{\alpha_i}$ *is as in Definition 29.15; see (29.65). Let* $f_i:(a,b)\to\mathbb{R}$, $i=1,\ldots,m$, *of fixed strict sign a.e., which are Lebesgue measurable functions, and* $\left\|\prod_{i=1}^{m}\left(x^{-\sigma\eta}\left(I_{b-;\sigma,\eta}^{\alpha_i}f_i\right)(x)\right)\right\|_{L_p(x^\sigma)}$, $\prod_{i=1}^{m}\left\|x^{-\sigma(\eta+\alpha_i)}f_i(x)\right\|_{L_q(x^\sigma)}$ *are finite. Then*

$$\left\|\prod_{i=1}^{m}\left(x^{-\sigma\eta}\left(I_{b-;\sigma,\eta}^{\alpha_i}f_i\right)(x)\right)\right\|_{L_p(x^\sigma)} \ge \frac{(b^\sigma-a^\sigma)^{\sum_{i=1}^{m}\alpha_i+m\left(\frac{1}{p}-1\right)+\frac{1}{p}}}{\left(p\sum_{i=1}^{m}\alpha_i+m(1-p)+1\right)^{\frac{1}{p}}}$$

$$\cdot \frac{1}{\left(\prod_{i=1}^{m}\left(\Gamma(\alpha_i)\left(p(\alpha_i-1)+1\right)^{\frac{1}{p}}\right)\right)}\left(\prod_{i=1}^{m}\left\|x^{-\sigma(\eta+\alpha_i)}f_i(x)\right\|_{L_q(x^\sigma)}\right). \tag{29.72}$$

Proof. By Definition 29.15 (see (29.65)) we have

$$\left(I_{b-;\sigma,\eta}^{\alpha_i}f_i\right)(x)=\frac{\sigma x^{\sigma\eta}}{\Gamma(\alpha_i)}\int_x^b \frac{t^{\sigma(1-\eta-\alpha_i)-1}f_i(t)\,dt}{(t^\sigma-x^\sigma)^{1-\alpha_i}}, \tag{29.73}$$

$x<b$. We rewrite (29.73) as follows:

$$L_2(f_i)(x):=x^{-\sigma\eta}\left(I_{b-;\sigma,\eta}^{\alpha_i}f_i\right)(x)$$

$$=\frac{1}{\Gamma(\alpha_i)}\int_x^b (t^\sigma-x^\sigma)^{\alpha_i-1}\left(t^{-\sigma(\eta+\alpha_i)}f_i(t)\right)dt^\sigma, \tag{29.74}$$

and by calling $F_{2i}(t)=t^{-\sigma(\eta+\alpha_i)}f_i(t)$, we have

$$L_2(f_i)(x)=\frac{1}{\Gamma(\alpha_i)}\int_x^b (t^\sigma-x^\sigma)^{\alpha_i-1}F_{2i}(t)dt^\sigma, \tag{29.75}$$

$i=1,\ldots,m$, $x<b$. Furthermore we notice that

$$|L_2(f_i)(x)|=\frac{1}{\Gamma(\alpha_i)}\int_x^b (t^\sigma-x^\sigma)^{\alpha_i-1}|F_{2i}(t)|\,dt^\sigma, \tag{29.76}$$

$i=1,\ldots,m$, $x<b$. So that now we can act as in the proof of Theorem 29.8. □

We continue with

Theorem 29.19. *Let* $0<p<1, q<0$ *such that* $\frac{1}{p}+\frac{1}{q}=1$; $\alpha_i>0, i=1,\dots,m, r>0$. *Here* $0\leq a<b<\infty$, $\sigma>0$, $\eta>0$, *and* $I_{b-;\sigma,\eta}^{\alpha_i}$ *is as in Definition 29.15; see (29.65). Let* $f_i:(a,b)\to\mathbb{R}$, $i=1,\dots,m$, *of fixed strict sign a.e., which are Lebesgue measurable functions, and* $\left\|\prod_{i=1}^{m}\left(x^{-\sigma\eta}\left(I_{b-;\sigma,\eta}^{\alpha_i}f_i\right)(x)\right)\right\|_{L_r(x^{\sigma})}$, $\prod_{i=1}^{m}\left\|x^{-\sigma(\eta+\alpha_i)}f_i(x)\right\|_{L_q(x^{\sigma})}$ *are finite. Then*

$$\left\|\prod_{i=1}^{m}\left(x^{-\sigma\eta}\left(I_{b-;\sigma,\eta}^{\alpha_i}f_i\right)(x)\right)\right\|_{L_r(x^{\sigma})}\geq\frac{(b^{\sigma}-a^{\sigma})^{\sum_{i=1}^{m}\alpha_i-m+\frac{m}{p}+\frac{1}{r}}}{\left(r\left(\sum_{i=1}^{m}\alpha_i-m+\frac{m}{p}\right)+1\right)^{\frac{1}{r}}}$$

$$\cdot\frac{1}{\left(\prod_{i=1}^{m}\left(\Gamma(\alpha_i)(p(\alpha_i-1)+1)^{\frac{1}{p}}\right)\right)}\left(\prod_{i=1}^{m}\left\|x^{-\sigma(\eta+\alpha_i)}f_i(x)\right\|_{L_q(x^{\sigma})}\right). \quad (29.77)$$

Proof. Based on the proof of Theorem 29.18 and acting similarly as in the proof of Theorem 29.9. □

We make

Definition 29.20. Let $\prod_{i=1}^{N}(a_i,b_i)\subset\mathbb{R}^N$, $N>1$, $a_i<b_i$, $a_i,b_i\in\mathbb{R}$. Let $\alpha_i>0$, $i=1,\dots,N$; $f\in L_1\left(\prod_{i=1}^{N}(a_i,b_i)\right)$, and set $a=(a_1,\dots,a_N)$, $b=(b_1,\dots,b_N)$, $\alpha=(\alpha_1,\dots,\alpha_N)$, $x=(x_1,\dots,x_N)$, $t=(t_1,\dots,t_N)$. We define the left mixed Riemann–Liouville fractional multiple integral of order α (see also [15]):

$$\left(I_{a+}^{\alpha}f\right)(x):=\frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)}\int_{a_1}^{x_1}\dots\int_{a_N}^{x_N}\prod_{i=1}^{N}(x_i-t_i)^{\alpha_i-1}f(t_1,\dots,t_N)\,dt_1\dots dt_N, \quad (29.78)$$

with $x_i>a_i$, $i=1,\dots,N$. We also define the right mixed Riemann–Liouville fractional multiple integral of order α (see also [13]):

$$\left(I_{b-}^{\alpha}f\right)(x):=\frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)}\int_{x_1}^{b_1}\dots\int_{x_N}^{b_N}\prod_{i=1}^{N}(t_i-x_i)^{\alpha_i-1}f(t_1,\dots,t_N)\,dt_1\dots dt_N, \quad (29.79)$$

with $x_i<b_i$, $i=1,\dots,N$.

Notice $I_{a+}^{\alpha}(|f|)$, $I_{b-}^{\alpha}(|f|)$ are finite if $f\in L_{\infty}\left(\prod_{i=1}^{N}(a_i,b_i)\right)$. We present

Theorem 29.21. *Let $0 < p < 1, q < 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Here all as in Definition 29.20, and (29.78) for I_{a+}^{α}. Let $f_j : \prod_{i=1}^{N}(a_i,b_i) \to \mathbb{R}$, $j = 1,\ldots,m$, of fixed strict sign a.e., $f_j \in L_1\left(\prod_{i=1}^{N}(a_i,b_i)\right)$. We assume that $\left\|\prod_{j=1}^{m} I_{a+}^{\alpha} f_j\right\|_{p,\prod_{i=1}^{N}(a_i,b_i)}$, $\prod_{j=1}^{m}\|f_j\|_{q,\prod_{i=1}^{N}(a_i,b_i)}$ are finite. Then it holds*

$$\left\|\prod_{j=1}^{m} I_{a+}^{\alpha} f_j\right\|_{p,\prod_{i=1}^{N}(a_i,b_i)} \geq \prod_{i=1}^{N}\left(\frac{(b_i-a_i)^{\left(m\left((\alpha_i-1)+\frac{1}{p}\right)+\frac{1}{p}\right)}}{(m(p(\alpha_i-1)+1)+1)^{\frac{1}{p}}\left(\Gamma(\alpha_i)(p(\alpha_i-1)+1)^{\frac{1}{p}}\right)^m}\right)$$
$$\cdot\left(\prod_{j=1}^{m}\|f_j\|_{q,\prod_{i=1}^{N}(a_i,b_i)}\right). \tag{29.80}$$

Proof. By Definition 29.20 (see (29.78)) we have

$$\left(I_{a+}^{\alpha} f_j\right)(x) = \frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)}\int_{a_1}^{x_1}\cdots\int_{a_N}^{x_N}\prod_{i=1}^{N}(x_i-t_i)^{\alpha_i-1} f_j(t_1,\ldots,t_N)\,dt_1\ldots dt_N, \tag{29.81}$$

furthermore it holds

$$\left|\left(I_{a+}^{\alpha} f_j\right)(x)\right| = \frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)}\int_{a_1}^{x_1}\cdots\int_{a_N}^{x_N}\prod_{i=1}^{N}(x_i-t_i)^{\alpha_i-1} \left|f_j(t_1,\ldots,t_N)\right|dt_1\ldots dt_N, \tag{29.82}$$

$j = 1,\ldots,m$, $x \in \prod_{i=1}^{N}(a_i,b_i)$. By reverse Hölder's inequality we get

$$\left|\left(I_{a+}^{\alpha} f_j\right)(x)\right| \geq \frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)}\left(\int_{a_1}^{x_1}\cdots\int_{a_N}^{x_N}\prod_{i=1}^{N}(x_i-t_i)^{p(\alpha_i-1)}dt_1\ldots dt_N\right)^{\frac{1}{p}}$$
$$\cdot\left(\int_{a_1}^{x_1}\cdots\int_{a_N}^{x_N}\left|f_j(t_1,\ldots,t_N)\right|^q dt_1\ldots dt_N\right)^{\frac{1}{q}} \tag{29.83}$$

$$\geq \frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)}\left(\prod_{i=1}^{N}\left(\int_{a_i}^{x_i}(x_i-t_i)^{p(\alpha_i-1)}dt_i\right)^{\frac{1}{p}}\right)\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}\left|f_j(t)\right|^q dt\right)^{\frac{1}{q}} \tag{29.84}$$

$$= \frac{1}{\prod_{i=1}^{N} \Gamma(\alpha_i)} \left(\prod_{i=1}^{N} \left(\frac{(x_i - a_i)^{(\alpha_i - 1) + \frac{1}{p}}}{(p(\alpha_i - 1) + 1)^{\frac{1}{p}}} \right) \right) \left(\int_{\prod_{i=1}^{N}(a_i,b_i)} |f_j(t)|^q \, dt \right)^{\frac{1}{q}}. \quad (29.85)$$

Hence

$$\prod_{j=1}^{m} \left| \left(I_{a+}^{\alpha} f_j \right)(x) \right|^p \geq \frac{1}{\left(\prod_{i=1}^{N} \Gamma(\alpha_i) \right)^{mp}} \left(\prod_{i=1}^{N} \frac{(x_i - a_i)^{(\alpha_i - 1) + \frac{1}{p}}}{(p(\alpha_i - 1) + 1)^{\frac{1}{p}}} \right)^{mp}$$

$$\cdot \prod_{j=1}^{m} \left(\int_{\prod_{i=1}^{N}(a_i,b_i)} |f_j(t)|^q \, dt \right)^{\frac{p}{q}}, \quad (29.86)$$

for $x \in \prod_{i=1}^{N}(a_i,b_i)$. Consequently, we get

$$\int_{\prod_{i=1}^{N}(a_i,b_i)} \prod_{j=1}^{m} \left| \left(I_{a+}^{\alpha} f_j \right)(x) \right|^p dx \geq \frac{\left(\prod_{j=1}^{m} \left(\int_{\prod_{i=1}^{N}(a_i,b_i)} |f_j(t)|^q \, dt \right)^{\frac{p}{q}} \right)}{\left(\prod_{i=1}^{N} \Gamma(\alpha_i) \right)^{mp} \left(\prod_{i=1}^{N} (p(\alpha_i - 1) + 1)^m \right)}$$

$$\cdot \left(\int_{\prod_{i=1}^{N}(a_i,b_i)} \prod_{i=1}^{N} (x_i - a_i)^{m(p(\alpha_i - 1) + 1)} \, dx_1 \ldots dx_N \right) \quad (29.87)$$

$$= \prod_{i=1}^{N} \left(\frac{(b_i - a_i)^{m(p(\alpha_i - 1) + 1) + 1}}{(m(p(\alpha_i - 1) + 1) + 1)(\Gamma(\alpha_i))^p (p(\alpha_i - 1) + 1))^m} \right)$$

$$\cdot \left(\prod_{j=1}^{m} \left(\int_{\prod_{i=1}^{N}(a_i,b_i)} |f_j(t)|^q \, dt \right)^{\frac{p}{q}} \right), \quad (29.88)$$

proving the claim. □

We have

Theorem 29.22. *Let* $0 < p < 1, q < 0$ *such that* $\frac{1}{p} + \frac{1}{q} = 1; r > 0$. *Here all as in Definition 29.20, and (29.78) for* I_{a+}^{α}. *Let* $f_j : \prod_{i=1}^{N}(a_i,b_i) \to \mathbb{R}$, $j = 1, \ldots, m$, *of fixed strict sign a.e.,* $f_j \in L_1\left(\prod_{i=1}^{N}(a_i,b_i) \right)$. *We assume that* $\left\| \prod_{j=1}^{m} I_{a+}^{\alpha} f_j \right\|_{r, \prod_{i=1}^{N}(a_i,b_i)}$,

$\prod_{j=1}^{m} \left\| f_j \right\|_{q, \prod_{i=1}^{N} (a_i,b_i)}$ *are finite. Then*

$$\left\| \prod_{j=1}^{m} I_{a+}^{\alpha} f_j \right\|_{r, \prod_{i=1}^{N} (a_i,b_i)} \geq \prod_{i=1}^{N} \left(\frac{(b_i - a_i)^{\left(m\left((\alpha_i - 1) + \frac{1}{p}\right) + \frac{1}{r}\right)}}{\left(mr\left((\alpha_i - 1) + \frac{1}{p}\right) + 1\right)^{\frac{1}{r}} \Gamma(\alpha_i)^m (p(\alpha_i - 1) + 1)^{\frac{m}{p}}} \right)$$

$$\cdot \left(\prod_{j=1}^{m} \left\| f_j \right\|_{q, \prod_{i=1}^{N} (a_i,b_i)} \right). \quad (29.89)$$

Proof. We have

$$\left(I_{a+}^{\alpha} f_j\right)(x) = \frac{1}{\prod_{i=1}^{N} \Gamma(\alpha_i)} \int_{a_1}^{x_1} \ldots \int_{a_N}^{x_N} \prod_{i=1}^{N} (x_i - t_i)^{\alpha_i - 1} f_j(t_1, \ldots, t_N)\, dt_1 \ldots dt_N, \quad (29.90)$$

furthermore it holds

$$\left|\left(I_{a+}^{\alpha} f_j\right)(x)\right| = \frac{1}{\prod_{i=1}^{N} \Gamma(\alpha_i)} \int_{a_1}^{x_1} \ldots \int_{a_N}^{x_N} \prod_{i=1}^{N} (x_i - t_i)^{\alpha_i - 1} \left|f_j(t_1, \ldots, t_N)\right| dt_1 \ldots dt_N, \quad (29.91)$$

$j = 1, \ldots, m$, $x \in \prod_{i=1}^{N} (a_i, b_i)$. By using (29.85) of the proof of Theorem 29.21 and $r > 0$ we get

$$\prod_{j=1}^{m} \left|\left(I_{a+}^{\alpha} f_j\right)(x)\right|^r \geq \frac{1}{\left(\prod_{i=1}^{N} \Gamma(\alpha_i)\right)^{mr}} \left(\prod_{i=1}^{N} \left(\frac{(x_i - a_i)^{(\alpha_i - 1) + \frac{1}{p}}}{(p(\alpha_i - 1) + 1)^{\frac{1}{p}}} \right) \right)^{mr}$$

$$\cdot \prod_{j=1}^{m} \left(\int_{\prod_{i=1}^{N} (a_i,b_i)} \left|f_j(t)\right|^q dt \right)^{\frac{r}{q}}, \quad (29.92)$$

for $x \in \prod_{i=1}^{N} (a_i, b_i)$. Consequently, we get

$$\int_{\prod_{i=1}^{N} (a_i,b_i)} \prod_{j=1}^{m} \left|\left(I_{a+}^{\alpha} f_j\right)(x)\right|^r dx \geq \frac{1}{\left(\prod_{i=1}^{N} \Gamma(\alpha_i)\right)^{mr}} \frac{\left(\prod_{j=1}^{m} \left(\int_{\prod_{i=1}^{N} (a_i,b_i)} \left|f_j(t)\right|^q dt \right)^{\frac{1}{q}} \right)^r}{\left(\prod_{i=1}^{N} (p(\alpha_i - 1) + 1)^{\frac{mr}{p}} \right)}$$

$$\cdot \left(\int_{\prod_{i=1}^{N} (a_i, b_i)} \prod_{i=1}^{N} (x_i - a_i)^{mr\left((\alpha_i - 1) + \frac{1}{p} \right)} dx \right) \tag{29.93}$$

$$= \prod_{i=1}^{N} \left(\frac{(b_i - a_i)^{mr\left((\alpha_i - 1) + \frac{1}{p} \right) + 1}}{\left(mr \left((\alpha_i - 1) + \frac{1}{p} \right) + 1 \right) \Gamma(\alpha_i)^{mr} (p(\alpha_i - 1) + 1)^{\frac{mr}{p}}} \right)$$

$$\cdot \left(\prod_{j=1}^{m} \|f_j\|_{q, \prod_{i=1}^{N} (a_i, b_i)} \right)^r, \tag{29.94}$$

proving the claim. □

We also give

Theorem 29.23. *Let* $0 < p < 1, q < 0$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Here all as in Definition* 29.20, *and* (29.79) *for* I_{b-}^{α}. *Let* $f_j : \prod_{i=1}^{N} (a_i, b_i) \to \mathbb{R}$, $j = 1, \ldots, m$, *of fixed strict sign a.e.,* $f_j \in L_1 \left(\prod_{i=1}^{N} (a_i, b_i) \right)$. *We assume that* $\left\| \prod_{j=1}^{m} I_{b-}^{\alpha} f_j \right\|_{p, \prod_{i=1}^{N} (a_i, b_i)}$, $\prod_{j=1}^{m} \|f_j\|_{q, \prod_{i=1}^{N} (a_i, b_i)}$ *are finite. Then it holds*

$$\left\| \prod_{j=1}^{m} I_{b-}^{\alpha} f_j \right\|_{p, \prod_{i=1}^{N} (a_i, b_i)} \geq \prod_{i=1}^{N} \left(\frac{(b_i - a_i)^{\left(m\left((\alpha_i - 1) + \frac{1}{p} \right) + \frac{1}{p} \right)}}{(m(p(\alpha_i - 1) + 1) + 1)^{\frac{1}{p}} \left(\Gamma(\alpha_i) (p(\alpha_i - 1) + 1)^{\frac{1}{p}} \right)^m} \right)$$

$$\cdot \left(\prod_{j=1}^{m} \|f_j\|_{q, \prod_{i=1}^{N} (a_i, b_i)} \right). \tag{29.95}$$

Proof. By Definition 29.20 (see (29.79)) we have

$$\left(I_{b-}^{\alpha} f_j \right)(x) = \frac{1}{\prod_{i=1}^{N} \Gamma(\alpha_i)} \int_{x_1}^{b_1} \ldots \int_{x_N}^{b_N} \prod_{i=1}^{N} (t_i - x_i)^{\alpha_i - 1} f_j(t_1, \ldots, t_N) \, dt_1 \ldots dt_N, \tag{29.96}$$

furthermore it holds

$$\left| \left(I_{b-}^{\alpha} f_j \right)(x) \right| = \frac{1}{\prod_{i=1}^{N} \Gamma(\alpha_i)} \int_{x_1}^{b_1} \ldots \int_{x_N}^{b_N} \prod_{i=1}^{N} (t_i - x_i)^{\alpha_i - 1} \left| f_j(t_1, \ldots, t_N) \right| dt_1 \ldots dt_N, \tag{29.97}$$

$j=1,\ldots,m$, $x\in\prod_{i=1}^{N}(a_i,b_i)$. By reverse Hölder's inequality we get

$$\left|\left(I_{b-}^{\alpha}f_j\right)(x)\right|\geq\frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)}\left(\int_{x_1}^{b_1}\ldots\int_{x_N}^{b_N}\prod_{i=1}^{N}(t_i-x_i)^{p(\alpha_i-1)}dt_1\ldots dt_N\right)^{\frac{1}{p}}$$

$$\cdot\left(\int_{x_1}^{b_1}\ldots\int_{x_N}^{b_N}\left|f_j(t_1,\ldots,t_N)\right|^q dt_1\ldots dt_N\right)^{\frac{1}{q}} \tag{29.98}$$

$$\geq\frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)}\left(\prod_{i=1}^{N}\left(\int_{x_i}^{b_i}(t_i-x_i)^{p(\alpha_i-1)}dt_i\right)^{\frac{1}{p}}\right)$$

$$\cdot\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}\left|f_j(t)\right|^q dt\right)^{\frac{1}{q}} \tag{29.99}$$

$$=\frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)}\left(\prod_{i=1}^{N}\left(\frac{(b_i-x_i)^{(\alpha_i-1)+\frac{1}{p}}}{(p(\alpha_i-1)+1)^{\frac{1}{p}}}\right)\right)$$

$$\cdot\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}\left|f_j(t)\right|^q dt\right)^{\frac{1}{q}}. \tag{29.100}$$

Hence

$$\prod_{j=1}^{m}\left|\left(I_{b-}^{\alpha}f_j\right)(x)\right|^p\geq\frac{1}{\left(\prod_{i=1}^{N}\Gamma(\alpha_i)\right)^{mp}}\left(\prod_{i=1}^{N}\frac{(b_i-x_i)^{(\alpha_i-1)+\frac{1}{p}}}{(p(\alpha_i-1)+1)^{\frac{1}{p}}}\right)^{mp}$$

$$\cdot\prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}\left|f_j(t)\right|^q dt\right)^{\frac{p}{q}}, \tag{29.101}$$

for $x\in\prod_{i=1}^{N}(a_i,b_i)$. Consequently, we get

$$\int_{\prod_{i=1}^{N}(a_i,b_i)}\prod_{j=1}^{m}\left|\left(I_{b-}^{\alpha}f_j\right)(x)\right|^p dx\geq\frac{\left(\prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}\left|f_j(t)\right|^q dt\right)^{\frac{p}{q}}\right)}{\left(\prod_{i=1}^{N}\Gamma(\alpha_i)\right)^{mp}\left(\prod_{i=1}^{N}(p(\alpha_i-1)+1)^m\right)}$$

$$\cdot \left(\int_{\prod_{i=1}^{N}(a_i,b_i)} \prod_{i=1}^{N} (b_i - x_i)^{m(p(\alpha_i-1)+1)} dx_1 \ldots dx_N \right) \tag{29.102}$$

$$= \prod_{i=1}^{N} \left(\frac{(b_i - a_i)^{m(p(\alpha_i-1)+1)+1}}{(m(p(\alpha_i - 1) + 1) + 1)((\Gamma(\alpha_i))^p (p(\alpha_i - 1) + 1))^m} \right)$$

$$\cdot \left(\prod_{j=1}^{m} \left(\int_{\prod_{i=1}^{N}(a_i,b_i)} |f_j(t)|^q dt \right)^{\frac{p}{q}} \right), \tag{29.103}$$

proving the claim. □

We have

Theorem 29.24. *Let* $0 < p < 1, q < 0$ *such that* $\frac{1}{p} + \frac{1}{q} = 1; r > 0$. *Here all as in Definition 29.20, and (29.79) for* I_{b-}^{α}. *Let* $f_j : \prod_{i=1}^{N}(a_i,b_i) \to \mathbb{R}$, $j = 1,\ldots,m$, *of fixed strict sign a.e.,* $f_j \in L_1\left(\prod_{i=1}^{N}(a_i,b_i)\right)$. *We assume that* $\left\| \prod_{j=1}^{m} I_{b-}^{\alpha} f_j \right\|_{r,\prod_{i=1}^{N}(a_i,b_i)}$, $\prod_{j=1}^{m} \|f_j\|_{q,\prod_{i=1}^{N}(a_i,b_i)}$ *are finite. Then*

$$\left\| \prod_{j=1}^{m} I_{b-}^{\alpha} f_j \right\|_{r,\prod_{i=1}^{N}(a_i,b_i)} \geq \prod_{i=1}^{N} \left(\frac{(b_i - a_i)^{\left(m\left((\alpha_i-1)+\frac{1}{p}\right)+\frac{1}{r}\right)}}{\left(mr\left((\alpha_i - 1) + \frac{1}{p}\right) + 1\right)^{\frac{1}{r}} \Gamma(\alpha_i)^m (p(\alpha_i - 1) + 1)^{\frac{m}{p}}} \right)$$

$$\cdot \left(\prod_{j=1}^{m} \|f_j\|_{q,\prod_{i=1}^{N}(a_i,b_i)} \right). \tag{29.104}$$

Proof. We have

$$\left(I_{b-}^{\alpha} f_j\right)(x) = \frac{1}{\prod_{i=1}^{N} \Gamma(\alpha_i)} \int_{x_1}^{b_1} \ldots \int_{x_N}^{b_N} \prod_{i=1}^{N} (t_i - x_i)^{\alpha_i - 1} f_j(t_1,\ldots,t_N) dt_1 \ldots dt_N, \tag{29.105}$$

furthermore it holds

$$\left|\left(I_{b-}^{\alpha} f_j\right)(x)\right| = \frac{1}{\prod_{i=1}^{N} \Gamma(\alpha_i)} \int_{x_1}^{b_1} \ldots \int_{x_N}^{b_N} \prod_{i=1}^{N} (t_i - x_i)^{\alpha_i - 1} \left|f_j(t_1,\ldots,t_N)\right| dt_1 \ldots dt_N, \tag{29.106}$$

$j=1,\ldots,m$, $x \in \prod_{i=1}^{N}(a_i,b_i)$. By using (29.100) of the proof of Theorem 29.23 and $r>0$ we get

$$\prod_{j=1}^{m}\left|\left(I_{b-}^{\alpha}f_j\right)(x)\right|^r \geq \frac{1}{\left(\prod_{i=1}^{N}\Gamma(\alpha_i)\right)^{mr}}\left(\prod_{i=1}^{N}\left(\frac{(b_i-x_i)^{(\alpha_i-1)+\frac{1}{p}}}{(p(\alpha_i-1)+1)^{\frac{1}{p}}}\right)\right)^{mr}$$

$$\cdot\prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}\left|f_j(t)\right|^q dt\right)^{\frac{r}{q}}, \tag{29.107}$$

for $x \in \prod_{i=1}^{N}(a_i,b_i)$. Consequently, we get

$$\int_{\prod_{i=1}^{N}(a_i,b_i)}\prod_{j=1}^{m}\left|\left(I_{b-}^{\alpha}f_j\right)(x)\right|^r dx \geq \frac{1}{\left(\prod_{i=1}^{N}\Gamma(\alpha_i)\right)^{mr}}\frac{\left(\prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}\left|f_j(t)\right|^q dt\right)^{\frac{1}{q}}\right)^r}{\left(\prod_{i=1}^{N}(p(\alpha_i-1)+1)^{\frac{mr}{p}}\right)}$$

$$\cdot\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}\prod_{i=1}^{N}(b_i-x_i)^{mr\left((\alpha_i-1)+\frac{1}{p}\right)}dx\right) \tag{29.108}$$

$$=\prod_{i=1}^{N}\left(\frac{(b_i-a_i)^{mr\left((\alpha_i-1)+\frac{1}{p}\right)+1}}{\left(mr\left((\alpha_i-1)+\frac{1}{p}\right)+1\right)\Gamma(\alpha_i)^{mr}(p(\alpha_i-1)+1)^{\frac{mr}{p}}}\right)$$

$$\cdot\left(\prod_{j=1}^{m}\left\|f_j\right\|_{q,\prod_{i=1}^{N}(a_i,b_i)}\right)^r, \tag{29.109}$$

proving the claim. □

Definition 29.25. ([1], p. 448). The left generalized Riemann–Liouville fractional derivative of f of order $\beta>0$ is given by

$$D_a^{\beta}f(x)=\frac{1}{\Gamma(n-\beta)}\left(\frac{d}{dx}\right)^n\int_a^x(x-y)^{n-\beta-1}f(y)\,dy, \tag{29.110}$$

where $n=[\beta]+1$, $x\in[a,b]$. For $a,b\in\mathbb{R}$, we say that $f\in L_1(a,b)$ has an L_∞ fractional derivative $D_a^{\beta}f$ ($\beta>0$) in $[a,b]$, if and only if:

(1) $D_a^{\beta-k}f\in C([a,b])$, $k=2,\ldots,n=[\beta]+1$
(2) $D_a^{\beta-1}f\in AC([a,b])$

(3) $D_a^{\beta} f \in L_\infty(a,b)$.
Above we define $D_a^0 f := f$ and $D_a^{-\delta} f := I_{a+}^{\delta} f$, if $0 < \delta \le 1$.

From [1, p. 449] and [11] we mention and use

Lemma 29.26. *Let $\beta > \alpha \ge 0$ and let $f \in L_1(a,b)$ have an L_∞ fractional derivative $D_a^{\beta} f$ in $[a,b]$ and let $D_a^{\beta-k} f(a) = 0$, $k = 1,\ldots,[\beta]+1$, then*

$$D_a^{\alpha} f(x) = \frac{1}{\Gamma(\beta-\alpha)} \int_a^x (x-y)^{\beta-\alpha-1} D_a^{\beta} f(y)\, dy, \tag{29.111}$$

for all $a \le x \le b$. Here $D_a^{\alpha} f \in AC([a,b])$ for $\beta-\alpha \ge 1$, and $D_a^{\alpha} f \in C([a,b])$ for $\beta-\alpha \in (0,1)$.

Notice here that

$$D_a^{\alpha} f(x) = \left(I_{a+}^{\beta-\alpha}\left(D_a^{\beta} f\right)\right)(x), \quad a \le x \le b. \tag{29.112}$$

We present

Theorem 29.27. *Let $0 < p < 1, q < 0$ such that $\frac{1}{p}+\frac{1}{q}=1$; $\beta_i > \alpha_i \ge 0$, $i = 1,\ldots,m$. Let $f_i \in L_1(a,b)$ have an L_∞ fractional derivative $D_a^{\beta_i} f_i$ in $[a,b]$ and let $D_a^{\beta_i-k_i} f_i(a) = 0$, $k_i = 1,\ldots,[\beta_i]+1$, so that $D_a^{\beta_i} f_i$ are functions of fixed strict sign a.e. Then*

$$\left\|\prod_{i=1}^m \left(D_a^{\alpha_i} f_i\right)\right\|_p \ge \frac{(b-a)^{\sum_{i=1}^m(\beta_i-\alpha_i)+m\left(\frac{1}{p}-1\right)+\frac{1}{p}}}{\left(p\sum_{i=1}^m(\beta_i-\alpha_i)+m(1-p)+1\right)^{\frac{1}{p}}}$$

$$\cdot \frac{1}{\left(\prod_{i=1}^m \Gamma(\beta_i-\alpha_i)\left(p(\beta_i-\alpha_i-1)+1\right)^{\frac{1}{p}}\right)} \left(\prod_{i=1}^m \left\|D_a^{\beta_i} f_i\right\|_q\right). \tag{29.113}$$

Proof. Using Theorem 29.1, see (29.5), and Lemma 29.26, see (29.112). □

We also give

Theorem 29.28. *Let $0 < p < 1, q < 0$ such that $\frac{1}{p}+\frac{1}{q}=1$; $r > 0$, $\beta_i > \alpha_i \ge 0$, $i = 1,\ldots,m$. Let $f_i \in L_1(a,b)$ have an L_∞ fractional derivative $D_a^{\beta_i} f_i$ in $[a,b]$ and let $D_a^{\beta_i-k_i} f_i(a) = 0$, $k_i = 1,\ldots,[\beta_i]+1$, so that $D_a^{\beta_i} f_i$ are functions of fixed strict sign a.e. Then*

$$\left\| \prod_{i=1}^{m} \left(D_{a}^{\alpha_i} f_i \right) \right\|_r \geq \frac{(b-a)^{\sum_{i=1}^{m}(\beta_i-\alpha_i)-m+\frac{m}{p}+\frac{1}{r}}}{\left(r\left(\sum_{i=1}^{m} (\beta_i - \alpha_i) - m + \frac{m}{p} \right) + 1 \right)^{\frac{1}{r}}}$$

$$\cdot \frac{1}{\left(\prod_{i=1}^{m} \Gamma(\beta_i - \alpha_i) \left(p(\beta_i - \alpha_i - 1) + 1 \right)^{\frac{1}{p}} \right)} \left(\prod_{i=1}^{m} \left\| D_{a}^{\beta_i} f_i \right\|_q \right). \tag{29.114}$$

Proof. Using Theorem 29.2, see (29.12), and Lemma 29.26, see (29.112). □

We need

Definition 29.29. ([8], p. 50, [1], p. 449) Let $\nu \geq 0$, $n := \lceil \nu \rceil$, $f \in AC^n([a,b])$. Then the left Caputo fractional derivative is given by

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\, dt$$

$$= \left(I_{a+}^{n-\nu} f^{(n)} \right)(x), \tag{29.115}$$

and it exists almost everywhere for $x \in [a,b]$, in fact $D_{*a}^{\nu} f \in L_1(a,b)$, ([1], p. 394).

We have $D_{*a}^{n} f = f^{(n)}$, $n \in \mathbb{Z}_+$.
We also need

Theorem 29.30. *([4]). Let $\nu \geq \rho + 1$, $\rho > 0$, $\nu, \rho \notin \mathbb{N}$. Call $n := \lceil \nu \rceil$, $m^* := \lceil \rho \rceil$. Assume $f \in AC^n([a,b])$, such that $f^{(k)}(a) = 0$, $k = m^*, m^*+1, \ldots, n-1$, and $D_{*a}^{\nu} f \in L_\infty(a,b)$. Then $D_{*a}^{\rho} f \in AC([a,b])$ (where $D_{*a}^{\rho} f = \left(I_{a+}^{m^*-\rho} f^{(m^*)} \right)(x)$), and*

$$D_{*a}^{\rho} f(x) = \frac{1}{\Gamma(\nu-\rho)} \int_a^x (x-t)^{\nu-\rho-1} D_{*a}^{\nu} f(t)\, dt$$

$$= \left(I_{a+}^{\nu-\rho} \left(D_{*a}^{\nu} f \right) \right)(x), \tag{29.116}$$

$\forall\, x \in [a,b]$.

We present

Theorem 29.31. *Let $0 < p < 1, q < 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$; and let $\nu_i \geq \rho_i + 1$, $\rho_i > 0$, $\nu_i, \rho_i \notin \mathbb{N}$, $i = 1, \ldots, m$. Call $n_i := \lceil \nu_i \rceil$, $m_i^* := \lceil \rho_i \rceil$. Suppose $f_i \in AC^{n_i}([a,b])$, such that $f_i^{(k_i)}(a) = 0$, $k_i = m_i^*, m_i^*+1, \ldots, n_i - 1$, and $D_{*a}^{\nu_i} f_i \in L_\infty(a,b)$. Assume $D_{*a}^{\nu_i} f_i$, $i = 1, \ldots, m$, are functions of fixed strict sign a.e. Then*

$$\left\| \prod_{i=1}^{m} \left(D_{*a}^{\rho_i} f_i \right) \right\|_p \geq \frac{(b-a)^{\sum_{i=1}^{m}(\nu_i-\rho_i)+m\left(\frac{1}{p}-1\right)+\frac{1}{p}}}{\left(p \sum_{i=1}^{m} (\nu_i - \rho_i) + m(1-p) + 1 \right)^{\frac{1}{p}}}$$

$$\cdot \frac{1}{\left(\prod_{i=1}^{m} \Gamma\left(\nu_i - \rho_i\right)\left(p(\nu_i - \rho_i - 1) + 1\right)^{\frac{1}{p}}\right)} \left(\prod_{i=1}^{m} \left\| D_{*a}^{\nu_i} f_i \right\|_q\right). \tag{29.117}$$

Proof. Using Theorem 29.1, see (29.5), and Theorem 29.30, see (29.116). □

We also give

Theorem 29.32. *Let $0 < p < 1, q < 0$ such that $\frac{1}{p} + \frac{1}{q} = 1, r > 0$; and let $\nu_i \geq \rho_i + 1, \rho_i > 0, \nu_i, \rho_i \notin \mathbb{N}, i = 1, \ldots, m$. Call $n_i := \lceil \nu_i \rceil$, $m_i^* := \lceil \rho_i \rceil$. Suppose $f_i \in AC^{n_i}([a,b])$, such that $f_i^{(k_i)}(a) = 0$, $k_i = m_i^*, m_i^* + 1, \ldots, n_i - 1$, and $D_{*a}^{\nu_i} f_i \in L_\infty(a,b)$. Assume $D_{*a}^{\nu_i} f_i$, $i = 1, \ldots, m$, are functions of fixed strict sign a.e. Then*

$$\left\| \prod_{i=1}^{m} \left(D_{*a}^{\rho_i} f_i\right) \right\|_r \geq \frac{(b-a)^{\sum_{i=1}^{m}(\nu_i - \rho_i) - m + \frac{m}{p} + \frac{1}{r}}}{\left(r\left(\sum_{i=1}^{m}(\nu_i - \rho_i) - m + \frac{m}{p}\right) + 1\right)^{\frac{1}{r}}}$$

$$\cdot \frac{1}{\left(\prod_{i=1}^{m} \Gamma\left(\nu_i - \rho_i\right)\left(p(\nu_i - \rho_i - 1) + 1\right)^{\frac{1}{p}}\right)} \left(\prod_{i=1}^{m} \left\| D_{*a}^{\nu_i} f_i \right\|_q\right). \tag{29.118}$$

Proof. Using Theorem 29.2, see (29.12), and Theorem 29.30, see (29.116). □

We need

Definition 29.33. ([2, 9, 10]) Let $\alpha \geq 0$, $n := \lceil \alpha \rceil$, $f \in AC^n([a,b])$. We define the right Caputo fractional derivative of order $\alpha \geq 0$ by

$$\overline{D}_{b-}^{\alpha} f(x) := (-1)^n I_{b-}^{n-\alpha} f^{(n)}(x), \tag{29.119}$$

we set $\overline{D}_{-}^{0} f := f$, i.e.,

$$\overline{D}_{b-}^{\alpha} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (J - x)^{n-\alpha-1} f^{(n)}(J)\, dJ. \tag{29.120}$$

Notice that $\overline{D}_{b-}^{n} f = (-1)^n f^{(n)}$, $n \in \mathbb{N}$.

In [3] we introduced a balanced fractional derivative combining both right and left fractional Caputo derivatives.

We need

Theorem 29.34. *([4]) Let $f \in AC^n([a,b])$, $\alpha > 0$, $n \in \mathbb{N}$, $n := \lceil \alpha \rceil$, $\alpha \geq \rho + 1$, $\rho > 0$, $r = \lceil \rho \rceil$, $\alpha, \rho \notin \mathbb{N}$. Assume $f^{(k)}(b) = 0$, $k = r, r+1, \ldots, n-1$, and $\overline{D}_{b-}^{\alpha} f \in L_\infty([a,b])$. Then*

$$\overline{D}_{b-}^{\rho} f(x) = \left(I_{b-}^{\alpha-\rho}\left(\overline{D}_{b-}^{\alpha} f\right)\right)(x) \in AC([a,b]), \tag{29.121}$$

that is,

$$\overline{D}_{b-}^{\rho} f(x) = \frac{1}{\Gamma(\alpha-\rho)} \int_x^b (t-x)^{\alpha-\rho-1} \left(\overline{D}_{b-}^{\alpha} f \right)(t)\, dt, \tag{29.122}$$

$\forall\, x \in [a,b]$.

We present

Theorem 29.35. *Let* $0 < p < 1, q < 0$ *such that* $\frac{1}{p} + \frac{1}{q} = 1; \alpha_i \geq \rho_i + 1, \rho_i > 0, i = 1,\ldots,m$. *Suppose* $f_i \in AC^{n_i}([a,b])$, $n_i \in \mathbb{N}$, $n_i := \lceil \alpha_i \rceil$, $r_i = \lceil \rho_i \rceil$, $\alpha_i, \rho_i \notin \mathbb{N}$, *and* $f_i^{(k_i)}(b) = 0$, $k_i = r_i, r_i+1,\ldots,n_i-1$, *and* $\overline{D}_{b-}^{\alpha_i} f_i \in L_\infty([a,b])$, $i = 1,\ldots,m$. *Assume* $\overline{D}_{b-}^{\alpha_i} f_i$, $i = 1,\ldots,m$, *are functions of fixed strict sign a.e. Then*

$$\left\| \prod_{i=1}^m \left(\overline{D}_{b-}^{\rho_i} f_i \right) \right\|_p \geq \frac{(b-a)^{\sum_{i=1}^m (\alpha_i-\rho_i) + m\left(\frac{1}{p}-1\right) + \frac{1}{p}}}{\left(p \sum_{i=1}^m (\alpha_i-\rho_i) + m(1-p) + 1 \right)^{\frac{1}{p}}}$$

$$\cdot \frac{1}{\left(\prod_{i=1}^m \Gamma(\alpha_i-\rho_i)\left(p(\alpha_i-\rho_i-1)+1\right)^{\frac{1}{p}} \right)} \left(\prod_{i=1}^m \left\| \overline{D}_{b-}^{\alpha_i} f_i \right\|_q \right). \tag{29.123}$$

Proof. Using Theorem 29.3, see (29.17), and Theorem 29.34, see (29.121). □

We also give

Theorem 29.36. *Let* $0 < p < 1, q < 0$ *such that* $\frac{1}{p} + \frac{1}{q} = 1, r > 0$; $\alpha_i \geq \rho_i + 1, \rho_i > 0$, $i = 1,\ldots,m$. *Suppose* $f_i \in AC^{n_i}([a,b])$, $n_i \in \mathbb{N}$, $n_i := \lceil \alpha_i \rceil$, $r_i = \lceil \rho_i \rceil$, $\alpha_i, \rho_i \notin \mathbb{N}$, *and* $f_i^{(k_i)}(b) = 0$, $k_i = r_i, r_i+1,\ldots,n_i-1$, *and* $\overline{D}_{b-}^{\alpha_i} f_i \in L_\infty([a,b])$, $i = 1,\ldots,m$. *Assume* $\overline{D}_{b-}^{\alpha_i} f_i$, $i = 1,\ldots,m$, *are functions of fixed strict sign a.e. Then*

$$\left\| \prod_{i=1}^m \left(\overline{D}_{b-}^{\rho_i} f_i \right) \right\|_r \geq \frac{(b-a)^{\sum_{i=1}^m (\alpha_i-\rho_i) - m + \frac{m}{p} + \frac{1}{r}}}{\left(r \left(\sum_{i=1}^m (\alpha_i-\rho_i) - m + \frac{m}{p} \right) + 1 \right)^{\frac{1}{r}}}$$

$$\cdot \frac{1}{\left(\prod_{i=1}^m \Gamma(\alpha_i-\rho_i)\left(p(\alpha_i-\rho_i-1)+1\right)^{\frac{1}{p}} \right)} \left(\prod_{i=1}^m \left\| \overline{D}_{b-}^{\alpha_i} f_i \right\|_q \right). \tag{29.124}$$

Proof. Using Theorem 29.4, see (29.25), and Theorem 29.34, see (29.121). □

We need

Definition 29.37. Let $\nu > 0$, $n := [\nu]$, $\alpha := \nu - n$ $(0 \leq \alpha < 1)$. Let $a, b \in \mathbb{R}$, $a \leq x \leq b$, $f \in C([a,b])$. We consider $C_a^\nu([a,b]) := \{f \in C^n([a,b]) : I_{a+}^{1-\alpha} f^{(n)} \in C^1([a,b])\}$. For $f \in C_a^\nu([a,b])$, we define the left generalized ν-fractional derivative of f over $[a,b]$ as

$$\Delta_a^\nu f := \left(I_{a+}^{1-\alpha} f^{(n)}\right)', \tag{29.125}$$

see [1], p. 24, and Canavati derivative in [7].

Notice here $\Delta_a^\nu f \in C([a,b])$. So that

$$(\Delta_a^\nu f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{-\alpha} f^{(n)}(t)\, dt, \tag{29.126}$$

$\forall\, x \in [a,b]$. Notice here that

$$\Delta_a^n f = f^{(n)}, \quad n \in \mathbb{Z}_+. \tag{29.127}$$

We need

Theorem 29.38. *([4]) Let $f \in C_a^\nu([a,b])$, $n = [\nu]$, such that $f^{(i)}(a) = 0$, $i = r, r+1, \ldots, n-1$, where $r := [\rho]$, with $0 < \rho < \nu$. Then*

$$(\Delta_a^\rho f)(x) = \frac{1}{\Gamma(\nu-\rho)} \int_a^x (x-t)^{\nu-\rho-1} (\Delta_a^\nu f)(t)\, dt, \tag{29.128}$$

i.e.,

$$(\Delta_a^\rho f) = I_{a+}^{\nu-\rho} (\Delta_a^\nu f) \in C([a,b]). \tag{29.129}$$

Thus $f \in C_a^\rho([a,b])$.

We present

Theorem 29.39. *Let $0 < p < 1$, $q < 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$; $\nu_i > \rho_i > 0$, $i = 1, \ldots, m$. Let $f_i \in C_a^{\nu_i}([a,b])$, $n_i = [\nu_i]$, such that $f_i^{(k_i)}(a) = 0$, $k_i = r_i, r_i + 1, \ldots, n_i - 1$, where $r_i := [\rho_i]$, $i = 1, \ldots, m$. Assume $\Delta_a^{\nu_i} f_i$, $i = 1, \ldots, m$, are functions of fixed strict sign a.e. Then*

$$\left\| \prod_{i=1}^m (\Delta_a^{\rho_i} f_i) \right\|_p \geq \frac{(b-a)^{\sum_{i=1}^m (\nu_i - \rho_i) + m\left(\frac{1}{p} - 1\right) + \frac{1}{p}}}{\left(p \sum_{i=1}^m (\nu_i - \rho_i) + m(1-p) + 1\right)^{\frac{1}{p}}}$$

$$\cdot \frac{1}{\left(\prod_{i=1}^m \Gamma(\nu_i - \rho_i)\,(p(\nu_i - \rho_i - 1) + 1)^{\frac{1}{p}}\right)} \left(\prod_{i=1}^m \|\Delta_a^{\nu_i} f_i\|_q\right). \tag{29.130}$$

Proof. Using Theorem 29.1, see (29.5), and Theorem 29.38, see (29.129). $\square$

We also give

Theorem 29.40. *Let* $0 < p < 1$, $q < 0$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$, $r > 0$; $\nu_i > \rho_i > 0$, $i = 1, \ldots, m$. *Let* $f_i \in C_a^{\nu_i}([a,b])$, $n_i = [\nu_i]$, *such that* $f_i^{(k_i)}(a) = 0$, $k_i = r_i, r_i + 1, \ldots, n_i - 1$, *where* $r_i := [\rho_i]$, $i = 1, \ldots, m$. *Assume* $\Delta_a^{\nu_i} f_i$, $i = 1, \ldots, m$, *are functions of fixed strict sign a.e. Then*

$$\left\| \prod_{i=1}^{m} \left(\Delta_a^{\rho_i} f_i \right) \right\|_r \geq \frac{(b-a)^{\sum_{i=1}^{m} (\nu_i - \rho_i) - m + \frac{m}{p} + \frac{1}{r}}}{\left(r \left(\sum_{i=1}^{m} (\nu_i - \rho_i) - m + \frac{m}{p} \right) + 1 \right)^{\frac{1}{r}}}$$

$$\cdot \frac{1}{\left(\prod_{i=1}^{m} \Gamma(\nu_i - \rho_i) \left(p(\nu_i - \rho_i - 1) + 1 \right)^{\frac{1}{p}} \right)} \left(\prod_{i=1}^{m} \| \Delta_a^{\nu_i} f_i \|_q \right). \tag{29.131}$$

Proof. Using Theorem 29.2, see (29.12), and Theorem 29.38, see (29.129). □

We need

Definition 29.41. ([2]) Let $\nu > 0$, $n := [\nu]$, $\alpha = \nu - n$, $0 < \alpha < 1$, $f \in C([a,b])$. Consider

$$C_{b-}^{\nu}([a,b]) := \{ f \in C^n([a,b]) : I_{b-}^{1-\alpha} f^{(n)} \in C^1([a,b]) \}. \tag{29.132}$$

Define the right generalized ν-fractional derivative of f over $[a,b]$ by

$$\Delta_{b-}^{\nu} f := (-1)^{n-1} \left(I_{b-}^{1-\alpha} f^{(n)} \right)'. \tag{29.133}$$

We set $\Delta_{b-}^{0} f = f$. Notice that

$$\left(\Delta_{b-}^{\nu} f \right)(x) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b (J - x)^{-\alpha} f^{(n)}(J) \, dJ, \tag{29.134}$$

and $\Delta_{b-}^{\nu} f \in C([a,b])$.

We also need

Theorem 29.42. *([4]) Let* $f \in C_{b-}^{\nu}([a,b])$, $0 < \rho < \nu$. *Assume* $f^{(i)}(b) = 0$, $i = r, r+1, \ldots, n-1$, *where* $r := [\rho]$, $n := [\nu]$. *Then*

$$\Delta_{b-}^{\rho} f(x) = \frac{1}{\Gamma(\nu - \rho)} \int_x^b (J - x)^{\nu - \rho - 1} \left(\Delta_{b-}^{\nu} f \right)(J) \, dJ, \tag{29.135}$$

$\forall \, x \in [a,b]$, *i.e.*

$$\Delta_{b-}^{\rho} f = I_{b-}^{\nu - \rho} \left(\Delta_{b-}^{\nu} f \right) \in C([a,b]), \tag{29.136}$$

and $f \in C_{b-}^{\rho}([a,b])$.

We present

Theorem 29.43. *Let* $0 < p < 1, q < 0$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$; $\nu_i > \rho_i > 0$, $i = 1, \ldots, m$. *Let* $f_i \in C_{b-}^{\nu_i}([a,b])$ *such that* $f_i^{(k_i)}(b) = 0$, $k_i = r_i, r_i + 1, \ldots, n_i - 1$, *where* $r_i := [\rho_i]$, $n_i := [\nu_i]$, $i = 1, \ldots, m$. *Assume* $\Delta_{b-}^{\nu_i} f_i$, $i = 1, \ldots, m$, *are functions of fixed strict sign a.e. Then*

$$\left\| \prod_{i=1}^{m} \left(\Delta_{b-}^{\rho_i} f_i \right) \right\|_p \geq \frac{(b-a)^{\sum_{i=1}^{m} (\nu_i - \rho_i) + m\left(\frac{1}{p} - 1\right) + \frac{1}{p}}}{\left(p \sum_{i=1}^{m} (\nu_i - \rho_i) + m(1-p) + 1 \right)^{\frac{1}{p}}}$$

$$\cdot \frac{1}{\left(\prod_{i=1}^{m} \Gamma(\nu_i - \rho_i) (p(\nu_i - \rho_i - 1) + 1)^{\frac{1}{p}} \right)} \left(\prod_{i=1}^{m} \left\| \Delta_{b-}^{\nu_i} f_i \right\|_q \right). \tag{29.137}$$

Proof. Using Theorem 29.3, see (29.17), and Theorem 29.42, see (29.136). □

We also give

Theorem 29.44. *Let* $0 < p < 1, q < 0$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$, $r > 0$; $\nu_i > \rho_i > 0$, $i = 1, \ldots, m$. *Let* $f_i \in C_{b-}^{\nu_i}([a,b])$ *such that* $f_i^{(k_i)}(b) = 0$, $k_i = r_i, r_i + 1, \ldots, n_i - 1$, *where* $r_i := [\rho_i]$, $n_i := [\nu_i]$, $i = 1, \ldots, m$. *Assume* $\Delta_{b-}^{\nu_i} f_i$, $i = 1, \ldots, m$, *are functions of fixed strict sign a.e. Then*

$$\left\| \prod_{i=1}^{m} \left(\Delta_{b-}^{\rho_i} f_i \right) \right\|_r \geq \frac{(b-a)^{\sum_{i=1}^{m} (\nu_i - \rho_i) - m + \frac{m}{p} + \frac{1}{r}}}{\left(r \left(\sum_{i=1}^{m} (\nu_i - \rho_i) - m + \frac{m}{p} \right) + 1 \right)^{\frac{1}{r}}}$$

$$\cdot \frac{1}{\left(\prod_{i=1}^{m} \Gamma(\nu_i - \rho_i) (p(\nu_i - \rho_i - 1) + 1)^{\frac{1}{p}} \right)} \left(\prod_{i=1}^{m} \left\| \Delta_{b-}^{\nu_i} f_i \right\|_q \right). \tag{29.138}$$

Proof. Using Theorem 29.4, see (29.25), and Theorem 29.42, see (29.136). □

We continue with

Terminology 29.45. Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive σ-finite measures, and let $k_i : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be nonnegative measurable functions, $k_i(x, \cdot)$ measurable on Ω_2, and

$$K_i(x) = \int_{\Omega_2} k_i(x,y) \, d\mu_2(y), \quad \text{for any } x \in \Omega_1, \tag{29.139}$$

$i = 1, \ldots, m$. We assume that $K_i(x) > 0$ a.e. on Ω_1, and the weight functions are nonnegative measurable functions on the related set.

We consider measurable functions $g_i : \Omega_1 \to \mathbb{R}$ with the representation

$$g_i(x) = \int_{\Omega_2} k_i(x,y) f_i(y) \, d\mu_2(y), \tag{29.140}$$

where $f_i : \Omega_2 \to \mathbb{R}$ are measurable functions, $i = 1, \ldots, m$.
Here u stands for a weight function on Ω_1.

For $m \in \mathbb{N}$, the first author in [5] proved the following general result:

Theorem 29.46. *Let $j \in \{1, \ldots, m\}$ be fixed. Assume that the function $x \mapsto \left(\frac{u(x) \prod_{i=1}^{m} k_i(x,y)}{\prod_{i=1}^{m} K_i(x)} \right)$ is integrable on Ω_1, for each $y \in \Omega_2$. Define λ_m on Ω_2 by*

$$\lambda_m(y) := \int_{\Omega_1} \left(\frac{u(x) \prod_{i=1}^{m} k_i(x,y)}{\prod_{i=1}^{m} K_i(x)} \right) d\mu_1(x) < \infty. \tag{29.141}$$

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, \ldots, m$, are convex and increasing functions. Then

$$\int_{\Omega_1} u(x) \prod_{i=1}^{m} \Phi_i\left(\left| \frac{g_i(x)}{K_i(x)} \right| \right) d\mu_1(x) \tag{29.142}$$

$$\leq \left(\prod_{\substack{i=1 \\ i \neq j \\ \emptyset}}^{m} \int_{\Omega_2} \Phi_i(|f_i(y)|) \, d\mu_2(y) \right) \left(\int_{\Omega_2} \Phi_j\left(\left| f_j(y) \right| \right) \lambda_m(y) \, d\mu_2(y) \right),$$

true for all measurable functions, $i = 1, \ldots, m$, $f_i : \Omega_2 \to \mathbb{R}$ such that:

(i) f_i, $\Phi_i(|f_i|)$, are both $k_i(x,y)\,d\mu_2(y)$ -integrable, μ_1 -a.e. in $x \in \Omega_1$,
(ii) $\lambda_m \Phi_j\left(\left|f_j\right|\right); \Phi_1(|f_1|), \Phi_2(|f_2|), \Phi_3(|f_3|), \ldots, \widehat{\Phi_j\left(\left|f_j\right|\right)}, \ldots, \Phi_m(|f_m|)$, are all μ_2 -integrable,
and for all corresponding functions g_i given by (29.140). Above $\widehat{\Phi_j\left(\left|f_j\right|\right)}$ means missing item.

We make

Remark 29.47. We remind the beta function

$$B(x,y) := \int_0^1 t^{x-1} (1-t)^{y-1} \, dt, \tag{29.143}$$

for $Re(x), Re(y) > 0$, and the incomplete beta function

$$B(x;\alpha,\beta)=\int_0^x t^{\alpha-1}(1-t)^{\beta-1}\,dt, \tag{29.144}$$

where $0<x\le 1$; $\alpha,\beta>0$.

For $I^{\alpha_i}_{a+;\sigma,\eta}$ Erdelyi–Kober fractional integral, $\alpha_i>0$, $i=1,\dots,m$, by [6] the corresponding

$$k_i(x,y)=\frac{\sigma x^{-\sigma(\alpha_i+\eta)}}{\Gamma(\alpha_i)}\chi_{(a,x]}(y)\frac{y^{\sigma\eta+\sigma-1}}{(x^\sigma-y^\sigma)^{1-\alpha_i}}, \tag{29.145}$$

$x,y\in(a,b)$, where χ stands for the characteristic function.

Also from [6] we get

$$K_i(x)=\left(I^{\alpha_i}_{a+;\sigma;\eta}(1)\right)(x) \tag{29.146}$$

$$=\frac{B(\eta+1,\alpha_i)-B\left(\left(\frac{a}{x}\right)^\sigma;\eta+1,\alpha_i\right)}{\Gamma(\alpha_i)}, \tag{29.147}$$

$i=1,\dots,m$.

We also make

Remark 29.48. For $I^{\alpha_i}_{b-;\sigma,\eta}$ Erdelyi–Kober fractional integral, $\alpha_i>0$, $i=1,\dots,m$, by [6] the corresponding

$$k_i(x,y)=\frac{\sigma x^{\sigma\eta}}{\Gamma(\alpha_i)}\chi_{[x,b)}(y)\frac{y^{\sigma(1-\eta-\alpha_i)-1}}{(y^\sigma-x^\sigma)^{1-\alpha_i}}, \tag{29.148}$$

$x,y\in(a,b)$. Furthermore, by [6] we have

$$K_i(x)=\left(I^{\alpha_i}_{b-;\sigma;\eta}(1)\right)(x) \tag{29.149}$$

$$=\frac{\left(B(\eta,\alpha_i)-B\left(\left(\frac{x}{b}\right)^\sigma;\eta,\alpha_i\right)\right)}{\Gamma(\alpha_i)}, \tag{29.150}$$

$i=1,\dots,m$.

We give

Theorem 29.49. *Here* $k_i(x,y)$ *and* $\left(I^{\alpha_i}_{a+;\sigma;\eta}(1)\right)(x)$ *are as in Remark 29.47, for* $I^{\alpha_i}_{a+;\sigma,\eta}$ *Erdelyi–Kober fractional integral. Let* $j\in\{1,\dots,m\}$ *be fixed. Assume that the function* $x\mapsto\left(\frac{u(x)\prod_{i=1}^m k_i(x,y)}{\prod_{i=1}^m\left(I^{\alpha_i}_{a+;\sigma;\eta}(1)\right)(x)}\right)$ *is integrable on* (a,b), *for each* $y\in(a,b)$. *Define* λ_m^+ *on* (a,b) *by*

$$\lambda_m^+(y) := \int_a^b \left(\frac{u(x) \prod_{i=1}^m k_i(x,y)}{\prod_{i=1}^m \left(I_{a+;\sigma;\eta}^{\alpha_i}(1)\right)(x)} \right) dx < \infty. \tag{29.151}$$

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, \ldots, m$, *are convex and increasing functions. Then*

$$\int_a^b u(x) \prod_{i=1}^m \Phi_i \left(\left| \frac{I_{a+;\sigma,\eta}^{\alpha_i} f_i(x)}{\left(I_{a+;\sigma;\eta}^{\alpha_i}(1)\right)(x)} \right| \right) dx \tag{29.152}$$

$$\leq \left(\prod_{\substack{i=1 \\ i \neq j}}^m \int_a^b \Phi_i(|f_i(y)|)\, dy \right) \left(\int_a^b \Phi_j\left(\left|f_j(y)\right|\right) \lambda_m^+(y)\, dy \right),$$

true for all measurable functions, $i = 1, \ldots, m$, $f_i : (a,b) \to \mathbb{R}$ *such that:*

(i) f_i, $\Phi_i(|f_i|)$, *are both* $k_i(x,y)\,dy$ *-integrable, a.e. in* $x \in (a,b)$.
(ii) $\lambda_m^+ \Phi_j\left(\left|f_j\right|\right); \Phi_1(|f_1|), \Phi_2(|f_2|), \Phi_3(|f_3|), \ldots, \widehat{\Phi_j\left(\left|f_j\right|\right)}, \ldots, \Phi_m(|f_m|)$, *are all integrable. Above* $\widehat{\Phi_j\left(\left|f_j\right|\right)}$ *means missing item.*

Proof. Direct application of Theorem 29.46. □

We also give

Theorem 29.50. *Here* $k_i(x,y)$ *and* $\left(I_{b-;\sigma;\eta}^{\alpha_i}(1)\right)(x)$ *are as in Remark 29.48, for* $I_{b-;\sigma,\eta}^{\alpha_i}$ *Erdelyi–Kober fractional integral. Let* $j \in \{1, \ldots, m\}$ *be fixed. Assume that the function* $x \mapsto \left(\frac{u(x) \prod_{i=1}^m k_i(x,y)}{\prod_{i=1}^m \left(I_{b-;\sigma;\eta}^{\alpha_i}(1)\right)(x)} \right)$ *is integrable on* (a,b), *for each* $y \in (a,b)$. *Define* λ_m^- *on* (a,b) *by*

$$\lambda_m^-(y) := \int_a^b \left(\frac{u(x) \prod_{i=1}^m k_i(x,y)}{\prod_{i=1}^m \left(I_{b-;\sigma;\eta}^{\alpha_i}(1)\right)(x)} \right) dx < \infty. \tag{29.153}$$

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, \ldots, m$, *are convex and increasing functions. Then*

$$\int_a^b u(x) \prod_{i=1}^m \Phi_i \left(\left| \frac{I_{b-;\sigma,\eta}^{\alpha_i} f_i(x)}{\left(I_{b-;\sigma;\eta}^{\alpha_i}(1)\right)(x)} \right| \right) dx \tag{29.154}$$

$$\leq \left(\prod_{\substack{i=1 \\ i \neq j}}^m \int_a^b \Phi_i(|f_i(y)|)\, dy \right) \left(\int_a^b \Phi_j\left(\left|f_j(y)\right|\right) \lambda_m^-(y)\, dy \right),$$

true for all measurable functions, $i = 1, \ldots, m$, $f_i : (a,b) \to \mathbb{R}$ *such that:*

(i) f_i, $\Phi_i(|f_i|)$, *are both* $k_i(x,y)\,dy$ *-integrable, a.e. in* $x \in (a,b)$.
(ii) $\lambda_m^- \Phi_j(|f_j|)\,; \Phi_1(|f_1|), \Phi_2(|f_2|), \Phi_3(|f_3|), \ldots, \widehat{\Phi_j(|f_j|)}, \ldots, \Phi_m(|f_m|)$, *are all integrable. Above* $\widehat{\Phi_j(|f_j|)}$ *means missing item.*

Proof. Direct application of Theorem 29.46. □

When $k(x,y) = k_1(x,y) = k_2(x,y) = \ldots = k_m(x,y)$, then $K(x) := K_1(x) = K_2(x) = \ldots = K_m(x)$. Then from Corollary 5, of [5], we get

Proposition 29.51. *Assume that the function* $x \mapsto \left(\frac{u(x)k^m(x,y)}{K^m(x)}\right)$ *is integrable on* Ω_1, *for each* $y \in \Omega_2$. *Define* U_m *on* Ω_2 *by*

$$U_m(y) := \int_{\Omega_1} \left(\frac{u(x)\,k^m(x,y)}{K^m(x)}\right) d\mu_1(x) < \infty. \tag{29.155}$$

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, \ldots, m$, *are convex and increasing functions. Then*

$$\int_{\Omega_1} u(x) \prod_{i=1}^{m} \Phi_i\left(\left|\frac{g_i(x)}{K(x)}\right|\right) d\mu_1(x) \tag{29.156}$$

$$\leq \left(\prod_{i=2}^{m} \int_{\Omega_2} \Phi_i(|f_i(y)|)\,d\mu_2(y)\right)\left(\int_{\Omega_2} \Phi_1(|f_1(y)|)\,U_m(y)\,d\mu_2(y)\right),$$

true for all measurable functions, $i = 1, \ldots, m$, $f_i : \Omega_2 \to \mathbb{R}$ *such that:*

(i) f_i, $\Phi_i(|f_i|)$, *are both* $k(x,y)\,d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$,
(ii) $U_m \Phi_1(|f_1|), \Phi_2(|f_2|), \Phi_3(|f_3|), \ldots, \Phi_m(|f_m|)$, *are all* μ_2 *-integrable, and for all corresponding functions* g_i *given by (29.140).*

Remark 29.52. For I_{a+}^{α} left mixed Riemann–Liouville fractional multiple integral of order α the corresponding $k(x,y)$ is

$$k_{a+}(x,y) = \frac{1}{\prod_{i=1}^{N} \Gamma(\alpha_i)} \chi_{\prod_{i=1}^{N}(a_i,x_i]}(y) \prod_{i=1}^{N} (x_i - y_i)^{\alpha_i - 1}, \tag{29.157}$$

$\forall\, x, y \in \prod_{i=1}^{N} (a_i, b_i)$ and the corresponding $K(x)$ is

$$K_{a+}(x) = (I_{a+}^{\alpha} 1)(x) = \prod_{i=1}^{N} \frac{(x_i - a_i)^{\alpha_i}}{\Gamma(\alpha_i + 1)}, \tag{29.158}$$

$\forall\, x \in \prod_{i=1}^{N} (a_i, b_i)$, by [6].

We also make

Remark 29.53. For I_{b-}^{α} right mixed Riemann–Liouville fractional multiple integral of order α the corresponding $k(x,y)$ is

$$k_{b-}(x,y) = \frac{1}{\prod_{i=1}^{N} \Gamma(\alpha_i)} \chi_{\prod_{i=1}^{N} [x_i,b_i)}(y) \prod_{i=1}^{N} (y_i - x_i)^{\alpha_i - 1}, \tag{29.159}$$

$\forall\, x,y \in \prod_{i=1}^{N} (a_i,b_i)$ and the corresponding $K(x)$ is

$$K_{b-}(x) = (I_{b-}^{\alpha} 1)(x) = \prod_{i=1}^{N} \frac{(b_i - x_i)^{\alpha_i}}{\Gamma(\alpha_i + 1)}, \tag{29.160}$$

$\forall\, x \in \prod_{i=1}^{N} (a_i,b_i)$, by [6].

We give

Proposition 29.54. *Here we follow Remark 29.52. Assume that the function* $x \mapsto \left(\frac{u(x) k_{a+}^{m}(x,y)}{\left[(I_{a+}^{\alpha} 1)(x) \right]^m} \right)$ *is integrable on* $\prod_{i=1}^{N} (a_i,b_i)$*, for each* $y \in \prod_{i=1}^{N} (a_i,b_i)$*. Define* U_m^{+} *on* $\prod_{i=1}^{N} (a_i,b_i)$ *by*

$$U_m^{+}(y) := \int_{\prod_{i=1}^{N} (a_i,b_i)} \left(\frac{u(x) k_{a+}^{m}(x,y)}{\left[(I_{a+}^{\alpha} 1)(x) \right]^m} \right) dx < \infty. \tag{29.161}$$

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$*,* $i = 1, \ldots, m$*, are convex and increasing functions. Then*

$$\int_{\prod_{i=1}^{N} (a_i,b_i)} u(x) \prod_{i=1}^{m} \Phi_i \left(\left| \frac{(I_{a+}^{\alpha} f_i)(x)}{(I_{a+}^{\alpha} 1)(x)} \right| \right) dx \tag{29.162}$$

$$\leq \left(\prod_{i=2}^{m} \int_{\prod_{i=1}^{N} (a_i,b_i)} \Phi_i(|f_i(y)|)\, dy \right) \left(\int_{\prod_{i=1}^{N} (a_i,b_i)} \Phi_1(|f_1(y)|)\, U_m^{+}(y)\, dy \right),$$

true for all measurable functions, $i = 1, \ldots, m$*,* $f_i : \prod_{i=1}^{N} (a_i,b_i) \to \mathbb{R}$ *such that:*

(i) f_i*,* $\Phi_i(|f_i|)$*, are both* $k_{a+}(x,y)\, dy$ *-integrable, a.e. in* $x \in \prod_{i=1}^{N} (a_i,b_i)$*.*
(ii) $U_m^{+} \Phi_1(|f_1|), \Phi_2(|f_2|), \Phi_3(|f_3|), \ldots, \Phi_m(|f_m|)$*, are all integrable.*

We finish this article with

Proposition 29.55. *Here we follow Remark 29.53. Assume that the function* $x \mapsto \left(\frac{u(x)k_{b-}^{m}(x,y)}{\left[(I_{b-}^{\alpha}1)(x)\right]^{m}}\right)$ *is integrable on* $\prod_{i=1}^{N}(a_i,b_i)$, *for each* $y \in \prod_{i=1}^{N}(a_i,b_i)$. *Define* U_m^- *on* $\prod_{i=1}^{N}(a_i,b_i)$ *by*

$$U_m^-(y) := \int_{\prod_{i=1}^{N}(a_i,b_i)} \left(\frac{u(x)\,k_{b-}^{m}(x,y)}{\left[(I_{b-}^{\alpha}1)(x)\right]^{m}}\right) dx < \infty. \tag{29.163}$$

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1,\ldots,m$, *are convex and increasing functions. Then*

$$\int_{\prod_{i=1}^{N}(a_i,b_i)} u(x) \prod_{i=1}^{m} \Phi_i\left(\left|\frac{(I_{b-}^{\alpha}f_i)(x)}{(I_{b-}^{\alpha}1)(x)}\right|\right) dx \tag{29.164}$$

$$\leq \left(\prod_{i=2}^{m} \int_{\prod_{i=1}^{N}(a_i,b_i)} \Phi_i(|f_i(y)|)\,dy\right)\left(\int_{\prod_{i=1}^{N}(a_i,b_i)} \Phi_1(|f_1(y)|)\,U_m^-(y)\,dy\right),$$

true for all measurable functions, $i = 1,\ldots,m$, $f_i : \prod_{i=1}^{N}(a_i,b_i) \to \mathbb{R}$ *such that:*

(i) f_i, $\Phi_i(|f_i|)$, *are both* $k_{b-}(x,y)\,dy$ *-integrable, a.e. in* $x \in \prod_{i=1}^{N}(a_i,b_i)$.
(ii) $U_m^-\Phi_1(|f_1|), \Phi_2(|f_2|), \Phi_3(|f_3|), \ldots, \Phi_m(|f_m|)$, *are all integrable.*

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