Chapter 11
Optimal Control for Distributed Linear Systems Subjected to Null Controllability with Constraints on the State

Michelle Mercan

Introduction

Let $d \in \mathbb{N}^*$ and Ω be a bounded open subset of $\mathbb{R}^d$ with boundary Γ of class C^2, $T > 0$. Let also ω be an open nonempty subset of Ω. Set $Q = \Omega \times (0,T)$, $\Sigma = \Gamma \times (0,T)$, and $G = \omega \times (0,T)$. We consider the parabolic evolution equation

$$\begin{cases} y' - \Delta y + a_0 y = h + k\chi_\omega & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = 0 & \text{in } \Omega, \end{cases} \tag{1}$$

where $(.)'$ is the partial derivative with respect to time t, $a_0 \in L^\infty(Q)$, $(h,k) \in L^2(Q) \times L^2(G)$, and χ_ω denotes the characteristic function of the control set ω. It is well known that problem (1) admits a unique solution y in the following Hilbert space

$$\Xi^{1,2}(Q) = H^1((0,T);L^2(\Omega)) \cap L^2((0,T);H^2(\Omega) \cap H_0^1(\Omega)).$$

Let $\{e_i,\ 1 \leq i \leq M\}$ be a set of functions of $L^2(Q)$ such that

$$e_i\chi_\omega \; 1 \leq i \leq M \text{ are linearly independent.} \tag{2}$$

From now on, we use the notation

$$y = y(h,k)$$

to mean that each source term h and k plays a particular role. More precisely, we would like to choose the control pair (h,k) in order to achieve two objectives that we present under the form (in the cascade sense) of two problems.

Michelle Mercan (✉)
Laboratoire CEREGMIA, Université des Antilles et de La Guyane,
Campus Fouillole 97159 Pointe-à-Pitre Guadeloupe (FWI), France
e-mail: Michelle.Mercan@etu.univ-ag.fr

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Problem 1. Let $H \subset L^2(Q)$ be a Hilbert space and $\{e_i,\ 1 \le i \le M\}$ be a set of functions of $L^2(Q)$ and assume that (2) holds. Fix $h \in H$. Then the Follower's problem can be stated as follows: Given $a_0 \in L^\infty(Q)$, find a control $k \in L^2(G)$ such that if y is solution of

$$\begin{cases} y' - \Delta y + a_0 y = h + k\chi_\omega & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = 0 & \text{in } \Omega, \end{cases} \tag{3}$$

then,

$$\int_Q y e_i \, dxdt = 0, \tag{4}$$

and

$$y(T) = 0, \text{ in } \Omega. \tag{5}$$

The role of k is to insure the null-controllability property (5) in the presence of the forcing term h and under the constraint (4).

In the sequel, we introduce a suitable nonnegative weight function θ, which will be defined below, and consider the Hilbert space

$$H = \{h | h \in L^2(Q), \theta h \in L^2(Q)\} \tag{6}$$

endowed with the scalar product and the norm

$$(h,l)_\theta = \int_Q \theta^2 h l \, dxdt, \quad \|h\|_H = \|\theta h\|_{L^2(Q)}.$$

For fixed $h \in H$, we will see that there exists several controls k such that (3), (4), and (5) are satisfied. Thus, we need to add some criteria to select k. More precisely, we will see that k is of the form $k = k_0(h) + v$. We consider then the maps $\mathscr{F}$ and $\mathscr{F}_1$ defined, respectively, by

$$\begin{aligned} \mathscr{F} : H &\to L^2(G) \\ h &\mapsto v = \mathscr{F}(h) \end{aligned} \tag{7}$$

and

$$\begin{aligned} \mathscr{F}_1 : H &\to L^2(G) \\ h &\mapsto \mathscr{F}_1(h) = k_0(h). \end{aligned} \tag{8}$$

We will see below (see section "Optimal Strategy for the Leader") that these maps are linear and continuous from H into $L^2(G)$.

In addition to the null-controllability problem (5) subject to the constraint (4), the second objective is to choose the forcing term h such that

$$y(h,k) \text{ is not too far from } z_d$$

where z_d is given in $L^2(Q)$.

In order to achieve this objective, we introduce the cost function J defined by

$$J(h) = \frac{1}{2}\|y(h,k) - z_d\|^2_{L^2(Q)} + \frac{N}{2}\|h\|^2_H$$

where $z_d \in L^2(Q)$ and $\mathscr{U}_{ad}$ is a nonempty closed convex subset of H. Then, we consider the following minimization problem:

Problem 2. Find $\hat{h} \in \mathscr{U}_{ad}$ such that

$$J(\hat{h}) = \min_{h \in \mathscr{U}_{ad}} J(h). \tag{9}$$

Problem 1 is a null-controllability problem with state constraints. Few results are known for such problem. Indeed, recently O. Nakoulima [6] gave a result of null controllability for the linear heat equation with constraints on a distributed control. His result was based on an observability inequality adapted to the constraint. In [4], G. Mophou and O. Nakoulima proved the existence of sentinels with given sensitivity by solving a problem of null controllability with constraint on the control. In [3], O. Nakoulima and G. Mophou studied a null controllability with constraints on the state for a semilinear heat equation by proving that the considering problem was equivalent to null controllability with constraint on the control. G. Mophou [5] generalizes these results to the case where the nonlinear term contains gradient terms.

Problem 2 is an optimal control problem. Such problem has been widely studied by J.L. Lions [2].

In this paper, we extend the works of G. Mophou and O. Nakoulima [3, 4] to a problem of control with two controls that we have to determine successively under some constraints. This is done by solving the combination of Problems 1 and 2, called Stackelberg problem. In this case, the controls h and k are, respectively, called Leader and Follower.

The main results of this paper are the following theorems.

Theorem 1. *Existence, uniqueness, and characterization of the Follower.*
Let Ω be a bounded open subset of $\mathbb{R}^n$ with boundary Γ of class C^2, and let H be the Hilbert space defined by (6). Then, for every $e_i \in L^2(Q)$, $1 \leq i \leq M$ verifying (2) and every $h \in H$, there exists a unique control $k = k(h) \in L^2(Q)$ such that the solution $y = y(h, k(h))$ of (3) satisfies (4) and (5). Moreover, the control k can be selected such that

$$\|k\| \leq C\|h\|_H \tag{10}$$

where $C = C\left(\Omega, \omega, a_0, T, \sum_{i=1}^{M} \|e_i\|_{L^2(Q)}\right) > 0.$

Theorem 2. *Existence, uniqueness, and characterization of the Leader.*
Let Ω be a bounded open subset of $\mathbb{R}^n$ with boundary Γ of class C^2. Let also θ be defined as previously, and $\mathscr{F}$ and $\mathscr{F}_1$ be the linear and continuous maps, respectively,

defined by (7) *and* (8)*. Then, the minimization problem* (9) *admits a unique solution* $\hat{h}$ *characterized by the following optimality condition*

$$\left(\Lambda^{-1}(\frac{1}{\theta}I+\mathscr{F}_1^*+\mathscr{F})(p)+N\hat{h},h-\hat{h}\right)_H \geq 0,\ \forall h \in \mathscr{U}_{ad} \tag{11}$$

where Λ^{-1} *is the isometric isomorphism from* H' *into* H*,* I *is the identity operator of* $L^2(Q)$*, and* p *is solution of*

$$\begin{cases} -p'-\Delta p+a_0 p = y(\hat{h},k)-z_d & \text{in } Q, \\ p=0 & \text{on } \Sigma, \\ p(T)=0 & \text{in } \Omega. \end{cases}$$

The rest of this paper is organized as follows. Section "Equivalence Between the Null-Controllability Problem with Constraints on the State and a Null- Controllability Problem with Constraint on the Control" is devoted to proving the equivalence between the null-controllability problem with constraints on the state and a null-controllability problem with constraint on the control. In section "Optimal Strategy for the Follower", we solve the null-controllability problem with constraint on the control. Finally, in section "Optimal Strategy for the Leader", we solve the Leader's problem.

Equivalence Between the Null-Controllability Problem with Constraints on the State and a Null-Controllability Problem with Constraint on the Control

Proposition 1. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *with boundary* Γ *of class* C^2*. Then, there exists a positive real weight function* θ *(a precise definition of* θ *will be given later on), two finite dimensional subspaces* $\mathscr{M}$ *and* $\mathscr{M}_\theta$ *such that for any* $h \in H$*, there exists* $k_0 = k_0(h) \in \mathscr{M}_\theta$ *such that the null-controllability problem with constraints on the state* (3)*,* (4)*, and* (5) *is equivalent to the following null-controllability problem with constraint on the control: Given* $a_0 \in L^\infty(Q)$ *and* $k_0 \in \mathscr{M}_\theta$*, find* $v \in L^2(G)$ *such that*

$$v \in \mathscr{M}^\perp \tag{12}$$

$$k = k_0 + v \tag{13}$$

and the solution y *of*

$$\begin{cases} y'-\Delta y+a_0 y = h+(k_0+v)\chi_\omega & \text{in } Q, \\ y=0 & \text{on } \Sigma, \\ y(0)=0 & \text{in } \Omega, \end{cases} \tag{14}$$

satisfies

$$y(T) = 0 \text{ in } Q. \tag{15}$$

Proof. We interpret the constraint (4) by using the adjoint state. More precisely, for any $e_i,\ 1 \le i \le M$, we consider the adjoint system

$$\begin{cases} -q_i' - \Delta q_i + a_0 q_i = e_i \text{ in } Q, \\ q_i = 0 \text{ on } \Sigma, \\ q_i(T) = 0 \text{ in } \Omega. \end{cases} \tag{16}$$

Since $a_0 \in L^\infty(Q)$ and $e_i \in L^2(Q)$, problem (16) admits a unique solution

$$q_i = q_i(z) \in \Xi^{1,2}(Q).$$

We multiply both sides of the differential equation (3) by q_i solution of (16) and we integrate over Q. By applying the Green formula, we obtain

$$\int_Q y e_i \, dxdt = \int_Q (h + k\chi_\omega) q_i \, dxdt.$$

From (4), we have

$$0 = \int_Q (h + k\chi_\omega) q_i \, dxdt.$$

Thus,

$$\int_G k q_i \, dxdt = -\int_Q h q_i \, dxdt. \tag{17}$$

Let

$$\mathscr{M} = Span\{q_i \chi_\omega,\ 1 \le i \le M\}$$

be the vector subspace of $L^2(G)$ generated by the M functions $q_i\chi_\omega,\ 1 \le i \le M$. We denote $\mathscr{M}^\perp$ the orthogonal of $\mathscr{M}$ in $L^2(G)$. We set

$$\mathscr{M}_\theta = \frac{1}{\theta}\mathscr{M}$$

the vector subspace of $L^2(G)$ generated by the M functions $\dfrac{1}{\theta} q_i \chi_\omega,\ 1 \le i \le M$.

Since the matrix $\left(\displaystyle\int_0^T \int_\omega \frac{1}{\theta} q_i q_j \, dxdt \right)_{1 \le i,j \le M}$ is symmetric positive definite (cf. Lemma 3), there exists a unique $k_0 = k_0(h) \in \mathscr{M}_\theta$ such that

$$\int_G k_0 q_i \, dxdt = -\int_Q h q_i \, dxdt,\ 1 \le i \le M. \tag{18}$$

Thus, combining (17) and (18), we deduce that

$$\int_G (k - k_0) q_i \, dxdt = 0 \quad 1 \le i \le M.$$

Consequently

$$k - k_0 \in \mathscr{M}^{\perp}.$$

Then $k = k_0 + v$ with $v \in \mathscr{M}^{\perp}$. Therefore, replacing $k\chi_\omega$ by $(k_0 + v)\chi_\omega$ in (3), we obtain (14).

Conversely, fix $h \in L^2(Q)$. For every $e_i \in L^2(Q)$, $1 \le i \le M$, assume that (v, y) is the solution of (12), (13), (14), and (15). Then, by solving (16), we obtain the functions q_i, $1 \le i \le M$. Let $\mathscr{M}$ and $\mathscr{M}_\theta$ be defined as previously. Let also $\mathscr{M}^{\perp}$ be the orthogonal of $\mathscr{M}$ in $L^2(G)$, $v \in \mathscr{M}^{\perp}$ and k_0 verifying (18).

Multiplying both sides of Eq. (14) by q_i and integrating by parts over Q, we obtain

$$\int_Q y' q_i\, dxdt - \int_Q \Delta y q_i\, dxdt + \int_Q a_0 q_i\, dxdt = \int_Q [h + (k_0 + v)\chi_\omega]\, q_i\, dxdt,$$

i.e.,

$$-\int_Q h q_i\, dxdt + \int_Q y e_i\, dxdt = \int_Q (k_0 + v)\chi_\omega q_i\, dxdt.$$

Since $v \in \mathscr{M}^{\perp}$ and k_0 verifies (18), the previous identity is reduced to (4). Thus, (k, y) is solution of (3), (4), and (5). □

Lemma 1. *Assume that* (2) *holds. Then, the functions* $q_i\chi_\omega$, $1 \le i \le M$ *are linearly independent. Moreover, the functions* $\dfrac{1}{\theta} q_i \chi_\omega$, $1 \le i \le M$ *are also linearly independent.*

Proof.

For $\gamma_i \in \mathbb{R}$, $1 \le i \le M$, let $\tilde{k} = \sum_{i=1}^{M} \gamma_i q_i$ on Q such that $\tilde{k}_{|G} = 0$. Since q_i is solution of (16), we have

$$-\frac{\partial \tilde{k}}{\partial t} - \Delta \tilde{k} + a_0 \tilde{k} = \sum_{i=1}^{M} \gamma_i e_i, \text{ in } Q, \tag{19}$$

$$\tilde{k} = 0, \text{ on } \Sigma. \tag{20}$$

Then, $\tilde{k}$ being identically zero on G, we deduce that $\tilde{k} = 0$ in Q. This means that $\sum_{i=1}^{M} \gamma_i e_i = 0$ in Q. Thus,

$$\sum_{i=1}^{M} \gamma_i e_i = 0, \quad \text{in } G.$$

Since the functions $e_i\chi_\omega$, $i \in \{1, \ldots, M\}$ satisfy (2), we conclude that $\gamma_i = 0$, $1 \le i \le M$.

The second assertion of the lemma follows immediately. □

In order to obtain a priori estimates on $k_0(h)$, we need the following result which is proved in [3].

Lemma 2. *Let q_i be defined by (16) and θ be a positive function defined below by relation (31). Let also $A_\theta = \left(\int_G \frac{1}{\theta} q_i q_j \, dxdt\right)_{i,j}$, $1 \leq i, j \leq M$. Then, there exists $\delta > 0$ such that*

$$(A_\theta X, X)_{\mathbb{R}^M} \geq \delta \|X\|^2_{\mathbb{R}^M}$$

where

$$(A_\theta X, X)_{\mathbb{R}^M} = \int_G \frac{1}{\theta} \left(\sum_{i=1}^{M} X_i p_i\right) \left(\sum_{j=1}^{M} X_j p_j\right) dxdt$$

and

$$X = (X_1, \ldots, X_M) \in \mathbb{R}^M.$$

Proposition 2. *Let θ be defined below by relation (31) and h be in H. Let also q_i and $k_0(h)$ be defined, respectively, by (16) and (18). Then, there exists $C = C(\Omega, a_0, T, \sum_{i=1}^{M} \|e_i\|_{L^2(Q)}) > 0$ such that*

$$\|\theta k_0(h)\|_{L^2(G)} \leq C\|h\|_H \tag{21}$$

$$\|k_0(h)\|_{L^2(G)} \leq C\|h\|_H. \tag{22}$$

Proof. From (18), we have

$$\int_G k_0(h) q_i \, dxdt = -\int_Q h q_i \, dxdt, \; 1 \leq i \leq M. \tag{23}$$

Since $k_0(h) \in Span\{\frac{1}{\theta} q_1 \chi_\omega, \ldots, \frac{1}{\theta} q_M \chi_\omega\}$, there exists

$$\alpha = (\alpha_1, \ldots, \alpha_M) \in \mathbb{R}^M$$

such that

$$k_0(h) = \sum_{j=1}^{M} \alpha_j \frac{1}{\theta} q_j \chi_\omega. \tag{24}$$

Thus, replacing $k_0(h)$ by $\sum_{j=1}^{M} \alpha_j \frac{1}{\theta} q_j \chi_\omega$ in (23), we obtain

$$\int_G \sum_{j=1}^{M} \alpha_j \frac{1}{\theta} q_j q_i \, dxdt = -\int_Q h q_i \, dxdt, \; 1 \leq i \leq M$$

and consequently,

$$\int_G \sum_{j=1}^{M} \alpha_j \frac{1}{\theta} q_j \chi_\omega \sum_{i=1}^{M} \alpha_i q_i \, dxdt = -\int_Q \theta h \sum_{i=1}^{M} \alpha_i \frac{1}{\theta} q_i \, dxdt.$$

Applying to this latter identity Lemma 2 with $X = \alpha$ to the left-hand side and to the right-hand side and using Cauchy–Schwartz inequality, we obtain

$$\delta\|\alpha\|^2 \leq \|h\|_H \sum_{i=1}^{M} |\alpha_i| \|q_i\|_{L^2(Q)}. \tag{25}$$

From the energy inequality for q_i solution of (16), it follows that for $1 \leq i \leq M$,

$$\|q_i\|_{L^2(Q)} \leq C(\Omega, a_0, T)\|e_i\|_{L^2(Q)}$$

which, combined with (25) and the fact that $\delta > 0$ gives

$$\|\alpha\|^2 \leq \delta^{-1} C(\Omega, a_0, T)\|h\|_H \|\alpha\|_{\mathbb{R}^M} \sqrt{\sum_{i=1}^{M} \|e_i\|^2_{L^2(Q)}},$$

i.e.,

$$\|\alpha\| \leq \delta^{-1} C(\Omega, a_0, T)\|h\|_H \sqrt{\sum_{i=1}^{M} \|e_i\|^2_{L^2(Q)}}. \tag{26}$$

Finally, from (24), we have

$$\begin{aligned}
\|\theta k_0(h)\|_{L^2(G)} &\leq \sum_{j=1}^{M} |\alpha_j| \|q_j\|_{L^2(G)}, \\
&\leq C(\Omega, a_0, T) \sum_{j=1}^{M} |\alpha_j| \|e_j\|_{L^2(Q)}, \\
&\leq C(\Omega, a_0, T)\|\alpha\| \left(\sum_{i=1}^{M} \|e_i\|_{L^2(Q)} \right)^{\frac{1}{2}},
\end{aligned}$$

and

$$\|k_0(h)\|_{L^2(G)} \leq C(\Omega, a_0, T)\|\alpha\| \left(\sum_{i=1}^{M} \|e_i\|_{L^2(Q)} \right)^{\frac{1}{2}}.$$

Hence, using (26) and the fact that $\dfrac{1}{\theta}$ is bounded in $L^\infty(Q)$, and setting

$$C = C(\Omega, a_0, T, \sum_{i=1}^{M} \|e_i\|_{L^2(Q)}) = \delta^{-1} C(\Omega, a_0, T)^2 \sum_{i=1}^{M} \|e_i\|^2_{L^2(Q)},$$

we deduce (21) and (22). □

Optimal Strategy for the Follower

Controllability Problem with Constraint on the Control

We consider a auxiliary function $\psi \in C^2(\overline{\Omega})$ which satisfies the following conditions:

$$\begin{aligned} \psi(x) &> 0 \ \forall x \in \Omega, \\ \psi(x) &= 0 \ \forall x \in \Gamma, \\ |\psi(x)| &\neq 0 \ \forall x \in \overline{\Omega - \omega}. \end{aligned} \tag{27}$$

Such a function exists according to A. Fursikov and O. Yu. Imanuvilov [1]. Then, for any $\lambda > 0$, we define the following weight functions:

$$\varphi(x,t) = \frac{e^{\lambda(\psi(x)+m_1)}}{t(T-t)}, \tag{28}$$

$$\eta(x,t) = \frac{e^{\lambda(|\psi(x)|_\infty+m_2)} + e^{\lambda(\psi(x)+m_1)}}{t(T-t)}, \tag{29}$$

for $(x,t) \in Q$ and $m > 1$ and we adopt the following notations:

$$\begin{aligned} L &= \frac{\partial}{\partial t} - \Delta + a_0 I, \\ L^* &= -\frac{\partial}{\partial t} - \Delta + a_0 I, \\ \mathscr{V} &= \{\rho \in C^\infty(\overline{Q}) \mid \rho = 0 \text{ on } \Sigma\}. \end{aligned}$$

Then, we have the following Carleman inequality (see [1, 3]).

Proposition 3. *Let ψ, φ, and η be defined by (27), (28), and (29). Then, there exists $\lambda_0 = \lambda_0(\Omega, \omega, a_0)$, $s_0 = s_0(\Omega, \omega, a_0, T)$ and $C = C(\Omega, \omega, a_0, T)$ such that for any $\lambda \geq \lambda_0$, any $s \geq s_0$ and any $\rho \in \mathscr{V}$, we have*

$$\begin{aligned} &\int_Q \frac{e^{-2s\eta}}{s\varphi}\left(|\frac{\partial \rho}{\partial t}|^2 + |\Delta\rho|^2\right) dxdt + s\lambda^2 \int_Q \varphi e^{-2s\eta} |\nabla\rho|^2 \, dxdt + \\ &s^3\lambda^4 \int_Q \varphi^3 e^{-2s\eta} |\rho|^2 \, dxdt \leq C\left(\int_Q e^{-2s\eta} |L^*\rho|^2 \, dxdt + s^3\lambda^4 \int_G \varphi^3 e^{-2s\eta} |\rho|^2 \, dxdt\right). \end{aligned} \tag{30}$$

As φ does not vanish over Q, we set

$$\theta = \varphi^{-\frac{3}{2}} e^{s\eta}. \tag{31}$$

From the definition of φ and η given by (28) and (29), the function θ is positive and $\dfrac{1}{\theta}$ is bounded. Since $\dfrac{1}{\varphi}$ is also bounded, taking $\lambda \geq \lambda_0 > 1$ and $s \geq s_0 > 1$, we obtain the following observability inequality:

$$\int_Q \frac{1}{\theta^2}|\rho|^2\,dxdt \leq C\left(\int_Q |L^*\rho|^2\,dxdt + \int_G |\rho|^2\,dxdt\right),\ \forall \rho \in \mathscr{V}. \tag{32}$$

Denote by:

- P the orthogonal projection operator from $L^2(G)$ into $\mathscr{M}$.
- $P\rho$ the orthogonal projection of $\rho\chi_\omega$ for $\rho \in L^2(Q)$.

From (32), we derive the following adapted Carleman estimate which is proved in [3, 4, 6].

Proposition 4. *Assume that* (2) *holds. Let θ be defined by* (31)*. Then, there exists $\lambda_0 = \lambda_0(\Omega,\omega,a_0) > 1$, $s_0 = s_0(\Omega,\omega,a_0,T) > 1$ and $C = C(\Omega,\omega,a_0,T) > 0$ such that for any $\lambda \geq \lambda_0$ and $s \geq s_0$ and for any $\rho \in \mathscr{V}$, we have*

$$\int_Q \frac{1}{\theta^2}|\rho|^2\,dxdt \leq C\left(\int_Q |L^*\rho|^2\,dxdt + \int_G |\rho - P\rho|^2\,dxdt\right). \tag{33}$$

Now, we consider the following symmetric bilinear form:

$$a(\rho,\hat{\rho}) = \int_Q L^*\rho L^*\hat{\rho}\,dxdt + \int_G (\rho - P\rho)(\hat{\rho} - P\hat{\rho})\,dxdt. \tag{34}$$

According to Proposition 4, this symmetric bilinear form is a scalar product over $\mathscr{V}$. Let $V = \overline{\mathscr{V}}$ the completion of $\mathscr{V}$ with respect to the norm

$$\rho \mapsto \|\rho\|_V = \sqrt{a(\rho,\rho)}. \tag{35}$$

Then, V is a Hilbert space.

Assume that (2) holds. Let H be a Hilbert space defined by (6) and $h \in H$. Let also θ and $k_0(h)$ be, respectively, defined by (31) and (18). Then, thanks to the estimation of $\theta k_0(h)$ given by (21) and the Cauchy–Schwartz inequality, the linear form defined on V by

$$\rho \mapsto \int_Q h\rho\,dxdt + \int_G k_0(h)\rho\,dxdt$$

is continuous on V. Thus, Lax–Milgram theorem allows us to say that for any $h \in H$, there exists a unique $\rho_\theta = \rho_\theta(h) \in V$ solution of the variational equation

$$a(\rho_\theta,\rho) = \int_Q L^*\rho_\theta L^*\rho\,dxdt + \int_G (\rho_\theta - P\rho_\theta)(\rho - P\rho)\,dxdt,\ \forall \rho \in V,$$
$$a(\rho_\theta,\rho) = \int_Q (h + k_0(h)\chi_\omega)\rho\,dxdt,\ \forall \rho \in V. \tag{36}$$

Proposition 5. *Assume that (2) holds. Let $h \in H$, and let ρ_θ be the unique solution of* (36)*. Set*

$$v_\theta = -(\rho_\theta \chi_\omega - P\rho_\theta) \tag{37}$$

and

$$y_\theta = L^* \rho_\theta. \tag{38}$$

Then, the pair (v_θ, y_θ) *is such that* (12)–(15) *hold.*

Moreover, there exists $C = C(\Omega, \omega, a_0, T, \sum_{i=1}^{M} \|e_i\|_{L^2(Q)}) > 0$ *such that*

$$\|\rho_\theta\|_V \leq C\|h\|_H, \tag{39}$$

$$\|v_\theta\|_{L^2(G)} \leq C\|h\|_H, \tag{40}$$

$$\|y_\theta\|_{\Xi^{1,2}(Q)} \leq C\|h\|_H. \tag{41}$$

Proof. We proceed in two steps.

Step 1. We prove that (v_θ, y_θ) is solution of (12)–(15).

Since $\rho_\theta \in V$, then $v_\theta = -(\rho_\theta \chi_\omega - P\rho_\theta) \in L^2(G)$ and $y_\theta \in L^2(Q)$. As $P\rho_\theta \in \mathscr{M}$, the function $v_\theta \in \mathscr{M}^\perp$. Replacing $-(\rho_\theta \chi_\omega - P\rho_\theta)$ by v_θ and $L^*\rho_\theta$ by y_θ in (36), we have

$$\int_Q y_\theta L^* \rho \, dxdt + \int_G (\rho_\theta - P\rho_\theta)(\rho - P\rho) \, dxdt = \int_Q (h + k_0 \chi_\omega) \rho \, dxdt.$$

As $P\rho \in \mathscr{M}$, then

$$\int_Q y_\theta L^* \rho \, dxdt \int_G (\rho_\theta - P\rho_\theta) \rho \, dxdt = \int_Q (h + k_0 \chi_\omega) \rho \, dxdt \; \forall \rho \in V.$$

This means that

$$\int_Q y_\theta L^* \rho \, dxdt = \int_Q (h + k_0 \chi_\omega) \rho \, dxdt + \int_G v_\theta \rho \, dxdt, \; \forall \rho \in V. \tag{42}$$

Actually, y_θ is the weak solution of a heat equation. Indeed, for $\phi \in L^2(Q)$, let $\mathfrak{p}$ be the solution of

$$\begin{cases} -\mathfrak{p}' - \Delta \mathfrak{p} + a_0 \mathfrak{p} = \phi \text{ in } Q, \\ \mathfrak{p} = 0 \text{ on } \Sigma, \\ \mathfrak{p}(0) = 0 \text{ in } \Omega. \end{cases}$$

Thus, $\mathfrak{p} \in V$, and replacing ρ in (42) by $\mathfrak{p}$, we obtain

$$\int_Q y_\theta \phi \, dxdt = \int_Q (h + k_0 \chi_\omega) \mathfrak{p} \, dxdt + \int_G v_\theta \mathfrak{p} \, dxdt.$$

Consequently, y_θ is the weak solution, by transposition of the system (14) with $k = v_\theta$ (see [2]). And we know that the solution of this equation is in $\Xi^{1,2}(Q)$. Hence, $y_\theta \in C([0,T], L^2(\Omega))$. Then, multiplying the first equation of (3) by $\varphi \in \mathscr{V}$ and integrating by parts over Q, it follows that for any $\varphi \in \mathscr{V}$,

$$\int_\Omega y_\theta(T)\varphi(T)\,dx - \int_\Omega y_\theta(0)\varphi(0)\,dx + \int_Q y_\theta L^*\varphi\,dxdt = \int_Q (h+k_0\chi_\omega)\varphi\,dxdt + \int_G v_\theta\varphi\,dxdt.$$

As $\varphi \in \mathscr{V}$, we deduce from (42) that

$$\int_\Omega y_\theta(T)\varphi(T)\,dx = 0,\ \forall\varphi\in\mathscr{V}.$$

Therefore, $y_\theta(T) = 0$ in Ω. Consequently, the pair (v_θ, y_θ) is solution of the problem (12)–(15).

Step 2. Let us prove the estimates (39)–(41).

Replacing φ by ρ_θ in (36), it follows from (33) and (21) that

$$\begin{aligned} a(\rho_\theta,\rho_\theta) &= \|y_\theta\|^2_{L^2(Q)} + \|v_\theta\|^2_{L^2(G)}, \\ &\le \|\theta(h+k_0)\|_{L^2(Q)}\|\frac{1}{\theta}\rho_\theta\|_{L^2(Q)}, \\ &\le C\|h\|_H\|\rho_\theta\|_V. \end{aligned}$$

From the definition of the norm on V given by (35), we obtain (39) and then (40). Finally, (41) is a consequence of (40) and the classic properties of heat equations.□

Proposition 6. *Assume that the assumptions of Proposition 5 hold. Then there exists a unique control v such that*

$$v = \min_{\tilde v\in\mathscr{E}}\|\tilde v\| \tag{43}$$

where $\mathscr{E} = \{\tilde v \in \mathscr{M}^\perp \mid (\tilde v,\tilde y)$ *satisfies* $(12)-(15)\}$.

Furthermore, there exists $C = C(\Omega,\omega,a_0,T,\sum_{i=1}^M\|e_i\|_{L^2(Q)}) > 0$ *such that*

$$\|v\|_{L^2(G)} \le C\|h\|_H. \tag{44}$$

Proof. According to Proposition 5, the pair (v_θ, y_θ) satisfies (12)–(15). Consequently, $\mathscr{E}$ is nonempty. Since $\mathscr{E}$ is also a closed convex subset of $L^2(G)$, we deduce that there exists a unique control v of minimal norm in $L^2(G)$. Particularly,

$$\|v\|_{L^2(G)} \le \|v_\theta\|_{L^2(G)}.$$

Hence, using (40), we obtain (44). □

From now on, we denote by $v = \mathscr{F}(h)$ the optimal control verifying (43) and by $y(h,k(h))$ the optimal state with $k(h) = k_0(h) + \mathscr{F}(h)$.

Penalization Method

In this subsection, we characterize the optimal solution. To this end, we use a penalization method of Lions (see [2]).
Let

$$\begin{cases} u \in \mathscr{M}^{\perp},\ z \in L^2(Q), \\ z' - \Delta z \in L^2(Q),\ z = 0 \text{ on } \Sigma, \\ z(0) = 0,\ z(T) = 0. \end{cases} \tag{45}$$

We define for any $h \in H$ and for any (u,z) verifying (45),

$$I_\varepsilon(u,z) = \frac{1}{2}\|u\|^2_{L^2(G)} + \frac{1}{2\varepsilon}\|Lz - h - k_0 - u\chi_\omega\|^2_{L^2(Q)} \tag{46}$$

and we consider the following problem

$$\inf\{I_\varepsilon(u,z),\ (u,z) \text{ verifying (45)}\}. \tag{47}$$

Since I_ε is coercive, continuous, and strictly convex, Problem (47) admits a unique solution $(v_\varepsilon = v_\varepsilon(h), y_\varepsilon = y_\varepsilon(h))$, i.e.,

$$I_\varepsilon(v_\varepsilon, y_\varepsilon) \le I_\varepsilon(u,z).$$

We give now the optimality system verified by $(v_\varepsilon, y_\varepsilon)$.

Proposition 7. *Assume that the assumptions of Proposition 5 hold. Then, the following assertions are equivalent:*

(i) $(v_\varepsilon, y_\varepsilon) \in \mathscr{M}^{\perp} \times \Xi^{1,2}(Q)$ *is an optimal solution of Problem* (47).
(ii) *There exists* $\rho_\varepsilon \in V$ *such that the triplet* $(v_\varepsilon, y_\varepsilon, \rho_\varepsilon)$ *is solution of the following optimality system:*

$$v_\varepsilon = -(\rho_\varepsilon\chi_\omega - P\rho_\varepsilon) \in \mathscr{M}^{\perp} \tag{48}$$

$$\begin{cases} y'_\varepsilon - \Delta y_\varepsilon + a_0 y_\varepsilon = h + k_0\chi_\omega + v_\varepsilon\chi_\omega - \varepsilon\rho_\varepsilon & \text{in } Q, \\ y_\varepsilon = 0 & \text{on } \Sigma, \\ y_\varepsilon(0) = 0 & \text{on } \Omega, \end{cases} \tag{49}$$

$$y_\varepsilon(T) = 0 \text{ in } \Omega, \tag{50}$$

$$\begin{cases} -\rho'_\varepsilon - \Delta\rho_\varepsilon + a_0\rho_\varepsilon = 0 \text{ in } Q, \\ \rho_\varepsilon = 0 \text{ on } \Sigma. \end{cases} \tag{51}$$

Proof. We express the Euler–Lagrange optimality conditions which characterize $(v_\varepsilon, y_\varepsilon)$.

$$\begin{cases} \dfrac{d}{d\lambda} I_\varepsilon(v_\varepsilon, y_\varepsilon + \lambda\varphi)_{|_{\lambda=0}} = 0,\ \forall \varphi \in C^\infty(\overline{Q}) \text{ such that} \\ \varphi = 0 \text{ on } \Sigma,\ \varphi(0) = \varphi(T) = 0 \text{ in } \Omega, \end{cases}$$

$$\frac{d}{d\lambda} I_\varepsilon(v_\varepsilon + \lambda v, y_\varepsilon)_{|_{\lambda=0}} = 0,\ \forall v \in \mathscr{M}^{\perp}.$$

After calculations, we have

$$\begin{cases} \displaystyle\int_Q \frac{1}{\varepsilon}(Ly_\varepsilon - h - k_0\chi_\omega - v_\varepsilon\chi_\omega)L\varphi\,dxdt = 0, \\ \forall\varphi \in C^\infty(\overline{Q}) \ \text{ such that },\ \varphi = 0 \text{ on } \Sigma,\ \varphi(0) = \varphi(T) = 0 \text{ in } \Omega \end{cases} \tag{52}$$

and

$$\int_G v_\varepsilon v\,dxdt - \int_Q \frac{1}{\varepsilon}(Ly_\varepsilon - h - k_0\chi_\omega - v_\varepsilon\chi_\omega)v\,dxdt = 0,\ \forall v \in \mathscr{M}^\perp. \tag{53}$$

Then we define the adjoint state

$$\rho_\varepsilon = \rho_\varepsilon(h) = -\frac{1}{\varepsilon}(Ly_\varepsilon - h - k_0\chi_\omega - v_\varepsilon\chi_\omega). \tag{54}$$

Hence, we deduce that $Ly_\varepsilon = h + k_0\chi_\omega + v_\varepsilon\chi_\omega - \varepsilon\rho_\varepsilon \in L^2(Q)$. And, since $(v_\varepsilon, y_\varepsilon)$ verifies (45), we have $y_\varepsilon = 0$ on Σ, $y_\varepsilon(0) = 0$ in Ω, and $y_\varepsilon(T) = 0$ in Ω. Thus, $(v_\varepsilon, y_\varepsilon, \rho_\varepsilon)$ is such that (49)–(50) hold. Since $h + k_0\chi_\omega + v_\varepsilon - \varepsilon\rho_\varepsilon \in L^2(Q)$, we obtain that $y_\varepsilon \in \Xi^{1,2}(Q)$. Now, replacing $-\dfrac{1}{\varepsilon}(Ly_\varepsilon - h - k_0\chi_\omega - v_\varepsilon\chi_\omega)$ by ρ_ε, in (52) and (53), we, respectively, obtain

$$\begin{cases} \displaystyle\int_Q \rho_\varepsilon L\varphi\,dxdt = 0, \\ \forall\varphi \in C^\infty(\overline{Q}) \ \text{ such that },\ \varphi = 0 \text{ on } \Sigma,\ \varphi(0) = \varphi(T) = 0 \text{ in } \Omega \end{cases} \tag{55}$$

and

$$\int_G v_\varepsilon v\,dxdt + \int_Q \rho_\varepsilon v\,dxdt = 0,\ \forall v \in \mathscr{M}^\perp. \tag{56}$$

Therefore, from (55), we derive

$$L^*\rho_\varepsilon = -\rho_\varepsilon' - \Delta\rho_\varepsilon + a_0\rho_\varepsilon = 0 \text{ in } Q.$$

Thus, $\rho_\varepsilon \in L^2(Q)$ and $L^*\rho_\varepsilon \in L^2(Q)$. Consequently, we can define ρ_ε on Σ and show that $\rho_\varepsilon = 0$ on Σ.
From (56), we have

$$\int_G (v_\varepsilon + \rho_\varepsilon\chi_\omega)v\,dxdt = 0,\ \forall v \in \mathscr{M}^\perp.$$

Hence, $v_\varepsilon + \rho_\varepsilon\chi_\omega \in \mathscr{M}^\perp$. Since $v_\varepsilon \in \mathscr{M}^\perp$, we have $v_\varepsilon + \rho_\varepsilon\chi_\omega = P(v_\varepsilon + \rho_\varepsilon\chi_\omega) = P\rho_\varepsilon$. Thus, $v_\varepsilon = -(\rho_\varepsilon\chi_\omega - P\rho_\varepsilon) \in \mathscr{M}^\perp$. □

Furthermore, we have the following estimates:

Proposition 8. *Let $(v_\varepsilon, y_\varepsilon, \rho_\varepsilon)$ be defined as in Proposition 7. Then, there exists a positive constant C, independent on ε such that*

$$\|v_\varepsilon\|_{L^2(G)} \le C, \tag{57}$$

$$\|y_\varepsilon\|_{\Xi^{1,2}(Q)} \leq C, \tag{58}$$
$$\|\rho_\varepsilon \chi_\omega\|_{L^2(G)} \leq C, \tag{59}$$
$$\|\rho_\varepsilon\|_V \leq C. \tag{60}$$

Proof. The structure of I_ε, on the one hand, and the existence of (v_θ, y_θ) on the other hand show that

$$0 \leq I_\varepsilon(v_\varepsilon, y_\varepsilon) \leq I_\varepsilon(v_\theta, y_\theta) = \frac{1}{2}\|v_\theta\|^2_{L^2(Q)} \leq C.$$

Thus, we have (57) and

$$\|Ly_\varepsilon - h - k_0\chi_\omega - v_\varepsilon \chi_\omega\|_{L^2(Q)} \leq C\sqrt{\varepsilon}. \tag{61}$$

Consequently, (54) and (61) give $\|\varepsilon\rho_\varepsilon\|_{L^2(Q)} \leq C\sqrt{\varepsilon}$, and y_ε being solution of (49), we obtain (58), thanks to the regularity properties of heat equations.

Furthermore, since $L^*\rho_\varepsilon = 0$ and (57) holds, using the definition of the norm on V given by (35), we obtain (60).

On the other hand, since $\rho_\varepsilon \in V$, applying the observability inequality (33) to ρ_ε, we have $\|\frac{1}{\theta}\rho_\varepsilon\|_{L^2(G)} \leq C$. Thus, using (48), (57), and the fact that $\frac{1}{\theta} \in L^\infty(Q)$, we deduce that $\|\frac{1}{\theta}P\rho_\varepsilon\|_{L^2(G)} \leq C$. Since $P\rho_\varepsilon \in \mathscr{M}$ which is finite dimensional, we have $\|P\rho_\varepsilon\|_{L^2(G)} \leq C$. Hence, using again (48) and (57), we obtain estimate (59). $\square$

Now, we can pass to the limit when ε tends to zero to obtain the singular optimality system associated to Problem 1.

Proposition 9. *Let $v = \mathscr{F}(h)$ be the unique solution of (43). Let also P be the orthogonal projection operator from $L^2(G)$ into $\mathscr{M}$. Then*

$$\mathscr{F}(h) = -(\rho\chi_\omega - P\rho) \tag{62}$$

where $\rho \in V$ is solution of

$$L^*\rho = 0 \text{ in } Q, \tag{63}$$
$$\rho = 0 \text{ on } \Sigma. \tag{64}$$

Proof. We proceed in three steps.
Step 1. We study the convergence of $(v_\varepsilon, y_\varepsilon)$.

According to (57) and (58), we can extract two subsequences, still denoted $(v_\varepsilon)_\varepsilon$ and $(y_\varepsilon)_\varepsilon$ such that

$$v_\varepsilon \rightharpoonup v_0(h) \text{ weakly in } L^2(G), \tag{65}$$
$$y_\varepsilon \rightharpoonup y_0(h) \text{ weakly in } \Xi^{1,2}(Q). \tag{66}$$

And, as $v_\varepsilon \in \mathscr{M}^\perp$ which is a closed vector subspace of $L^2(G)$, we have

$$v_0(h) \in \mathscr{M}^\perp. \tag{67}$$

Since the injection of $\Xi^{1,2}(Q)$ into $L^2(Q)$ is compact, the pair $(v_0 = v_0(h), y_0 = y_0(h))$ is such that

$$\begin{cases} y_0' - \Delta y_0 + a_0 y_0 = h + k_0\chi_\omega + v_0\chi_\omega & \text{in } Q, \\ y_0 = 0 & \text{on } \Sigma, \\ y_0(0) = 0 & \text{in } \Omega. \end{cases} \tag{68}$$

$$y_0(T) = 0 \text{ in } \Omega. \tag{69}$$

Step 2. We show that $(v_0,\ y_0) = (\mathscr{F}(h), y(h,k(h)))$.
From the expression of I_ε given by (46), we can write

$$\frac{1}{2}\|v_\varepsilon\|^2_{L^2(G)} \leq I_\varepsilon(v_\varepsilon, y_\varepsilon).$$

Since $(\mathscr{F}(h), y(h,k(h)))$ satisfies (12)–(15) and (43), this latter inequality becomes

$$\frac{1}{2}\|v_\varepsilon\|^2_{L^2(G)} \leq I_\varepsilon(v_\varepsilon, y_\varepsilon) \leq \frac{1}{2}\|\mathscr{F}(h)\|^2_{L^2(G)}. \tag{70}$$

Then, using (65) while passing to the limit in (70), we obtain

$$\frac{1}{2}\|v_0\|^2_{L^2(G)} \leq \liminf_{\varepsilon\to 0} I_\varepsilon(v_\varepsilon, y_\varepsilon) \leq \frac{1}{2}\|\mathscr{F}(h)\|^2_{L^2(G)}.$$

Consequently

$$\|v_0\|_{L^2(G)} \leq \|\mathscr{F}(h)\|_{L^2(G)},$$

and thus,

$$\|v_0\|_{L^2(G)} = \|\mathscr{F}(h)\|_{L^2(G)}.$$

Hence,

$$v_0 = \mathscr{F}(h), \tag{71}$$

and since (68) admits a unique solution, it follows that $y_0 = y(h,k(h))$.

Remark 1. *Note that $\|\mathscr{F}(h)\|_{L^2(G)} \leq C\|h\|_H$. Indeed, as (v_θ, y_θ) satisfies (45), we can write*

$$I_\varepsilon(v_\varepsilon, y_\varepsilon) \leq I_\varepsilon(v_\theta, y_\theta) = \frac{1}{2}\|v_\theta\|_{L^2(G)}.$$

Threfore, using the fact that v_θ verifies (40) and the definition of I_ε given by (46), we obtain that $\|v_\varepsilon\|_{L^2(G)} \leq C\|h\|_H$. Hence, in view of (65) and (71), we have $\|\mathscr{F}(h)\|_{L^2(G)} \leq C\|h\|_H$.

Step 3. According to estimates (59) and (60), we can extract a subsequence, still denoted $(\rho_\varepsilon)_\varepsilon$ such that

$$\rho_\varepsilon \chi_\omega \rightharpoonup \rho(h)\chi_\omega \text{ weakly in } L^2(G), \tag{72}$$

$$\rho_\varepsilon \chi_\omega \rightharpoonup \rho(h)\chi_\omega \text{ weakly in } V, \tag{73}$$

and it follows from (51) that $\rho(h)$ is solution of

$$\begin{cases} L^*\rho = 0 \text{ in } Q, \\ \quad \rho = 0 \text{ on } \Sigma. \end{cases}$$

As P is a compact operator, we deduce from (72) that

$$P\rho_\varepsilon \to P\rho(h) \text{ strongly in } L^2(G). \tag{74}$$

Therefore, combining (72) and (74), we obtain

$$v_\varepsilon = -(\rho_\varepsilon \chi_\omega - P\rho_\varepsilon) \rightharpoonup \mathscr{F}(h) = -(\rho(h)\chi_\omega - P\rho(h)) \text{ weakly in } L^2(G).$$

Thus, we have showed that for any $h \in H$, the unique pair $(\mathscr{F}(h), y(h,k(h)))$ satisfies (12)–(15) where $\mathscr{F}(h) = -(\rho(h)\chi_\omega - P\rho(h))$ and $\rho = \rho(h)$ is solution of (63). □

Proof of Theorem 1

We have proven that there exists a unique control $v = v(h) \in \mathscr{M}^\perp$ solution of (43) such that the pair (v,y) verifies (14) and (15). Therefore, Proposition 1 allows us to say that the control $k = k(h) = (k_0(h) + v(h))$ with $k_0 \in \mathscr{M}_\theta$ is such that $(k, y(k))$ satisfies the null-controllability problem with constraints on the state (3), (4), and (5). Therefore, using (22) and (44), we deduce (10).

Optimal Strategy for the Leader

Properties of $\mathscr{F}$

Lemma 3. *For any $h \in H$, let $\rho = \rho(h)$ be the solution of (63). Then, the map $\mathscr{F}$ defined by*

$$\mathscr{F}(h) = -(\rho - P\rho)\chi_\omega \tag{75}$$

is linear and continuous from H into $L^2(G)$.

Proof. Consider the vector subspace V_0 from V defined by

$$V_0 = \{\varphi \in V | L^*\varphi = 0\}.$$

Since $\mathscr{F}(h)$ is solution of problem (12)–(15) and verifies (62), we multiply the first equation of (14) by $\varphi \in V_0$ and we integrate by parts. Then, we obtain

$$\int_Q h\varphi\, dxdt + \int_Q k_0(h)\varphi\, dxdt + \int_Q v\chi_\omega \varphi\, dxdt = 0\ \forall \varphi \in V_0,$$

i.e.,

$$\int_Q h\varphi\, dxdt + \int_Q k_0(h)\varphi\, dxdt - \int_Q (\rho - P\rho)\chi_\omega \varphi\, dxdt = 0\ \forall \varphi \in V_0,$$

or equivalently,

$$\int_Q h\varphi\, dxdt + \int_Q k_0(h)\varphi\, dxdt + \int_Q \mathscr{F}(h)\chi_\omega \varphi\, dxdt = 0\ \forall \varphi \in V_0.$$

Using the fact that the map $\varphi \mapsto \int_Q h\varphi\, dxdt + \int_Q k_0(h)\varphi\, dxdt$ is linear and continuous on V and

$$\begin{aligned} -\int_G \mathscr{F}(h)\varphi\, dxdt &= \int_G (\rho(h) - P\rho(h))\varphi\, dxdt, \\ &= \int_G (\rho(h) - P\rho(h))(\varphi - P\varphi)\, dxdt, \\ &= a(\rho(h), \varphi), \end{aligned}$$

we deduce that $\rho = \rho(h)$ is solution of the variational problem

$$a(\rho, \varphi) = \int_Q h\varphi\, dxdt + \int_Q k_0(h)\varphi\, dxdt\ \forall \varphi \in V_0. \tag{76}$$

Hence, the map $h \mapsto \rho = \rho(h)\chi_\omega$ is linear from H to $L^2(G)$. And since the projection operator $I - P$ which is defined from $L^2(G)$ to $\mathscr{M}^\perp \subset L^2(G)$ is also linear, we deduce that the map $\mathscr{F}$ is linear from H to $L^2(G)$. Hence, it follows from Remark 1 that $\mathscr{F}$ is continuous on H since $\|\mathscr{F}(h)\|_{L^2(G)} \leq C\|h\|_H$. □

Remark 2. *Let k_0 be defined as in (18) . Then*

1. *$k_0 \in H$. Indeed, since $k_0 \in \mathscr{M}_\theta$, we have on the one hand, $k_0 \in L^2(G)$, and on the other hand, $\theta k_0 \in \mathscr{M} \subset L^2(G)$.*
2. *In view of (18), the map $\mathscr{F}_1 : h \mapsto k_0(h)$ is linear, and since (22) holds, this map is continuous on H.*

From now on, we denote $k_0(h) = \mathscr{F}_1(h)$.

Proof of Theorem 2

We consider the cost function J defined by

$$J(h) = \frac{1}{2}\|y(h, k(h)) - z_d\|^2_{L^2(Q)} + \frac{N}{2}\|h\|^2_H \tag{77}$$

from which we associate the minimization problem

$$\inf_{h \in \mathscr{U}_{ad}} J(h) \tag{78}$$

where $\mathscr{U}_{ad}$ is a nonempty closed convex subspace of $L^2(Q)$.
Using the properties of the maps $\mathscr{F}$ given by Lemma 3 and $\mathscr{F}_1$ given by Remark 1, we have that J is strictly convex, continuous, and coercive. Thus, we have the following classic result:

Proposition 10. *Problem (78) has a unique control $\hat{h} \in \mathscr{U}_{ad}$.*

Observing that $k(h) = k_0(h) + v(h) = \mathscr{F}_1(h) + \mathscr{F}(h)$, we will denote, now and in the sequel, by $\hat{y} = y(\hat{h}, \hat{k} = \hat{k}(\hat{h}))$ the state associated to the optimal control $\hat{h}$. Let us characterize $\hat{h}$.

Writing the Euler–Lagrange condition, we obtain

$$\frac{d}{d\lambda} J(\hat{h} + \lambda(h - \hat{h}))_{|\lambda=0} \geq 0, \; \forall h \in \mathscr{U}_{ad}$$

which after calculations gives

$$\frac{d}{d\lambda} J(\hat{h} + \lambda(h - \hat{h}))_{|\lambda=0} = (\hat{y} - z_d, y(h - \hat{h}, k(h - \hat{h}))_{L^2(Q)} + (N\hat{h}, h - \hat{h})_H.$$

Thus,

$$(\hat{y} - z_d, y(h - \hat{h}, k(h - \hat{h}))_{L^2(Q)} + (N\hat{h}, h - \hat{h})_H \geq 0, \; \forall h \in \mathscr{U}_{ad}.$$

We interpret this condition using the adjoint state notion. To make our calculations easier, we set $w = h - \hat{h}$ and we denote $y = y(w, k(w))$. Let p be the solution of the following system:

$$\begin{cases} -p' - \Delta p + a_0 p = \hat{y} - z_d & \text{in } Q, \\ p = 0 & \text{on } \Sigma, \\ p(T) = 0 & \text{in } \Omega. \end{cases} \tag{79}$$

Since $\hat{y} - z_d \in L^2(Q)$, we know that $p \in \Xi^{1,2}(Q)$. Multiply the first equation of (79) by y and integrate by parts over Q, we obtain

$$\int_Q p(w + (\mathscr{F}_1(w) + F(w))\chi_\omega)\, dxdt = \int_Q y(\hat{y} - z_d)\, dxdt.$$

This means that,

$$\int_Q pw\, dxdt + \int_Q p\mathscr{F}_1(w)\chi_\omega\, dxdt + \int_Q p\mathscr{F}(w)\chi_\omega\, dxdt = \int_Q y(\hat{y} - z_d)\, dxdt.$$

Let H' be the dual of the Hilbert space H. Let also Λ^{-1} be the isometric isomorphism from H' to H. Observing on the one hand that $\mathscr{F} = \mathscr{F}^*$ because of the symmetry of the operator $a(.,.)$, and on the other hand that we can write

$$\int_Q pw\,dxdt = \int_Q \frac{1}{\theta} p\theta w = \langle \frac{1}{\theta} p, w\rangle_{H',H},$$

$$\int_G p\mathscr{F}_1(w)\,dxdt = \langle \mathscr{F}_1^*(p), w\rangle_{H',H},$$

and

$$\int_G p\mathscr{F}(w)\,dxdt = \langle \mathscr{F}^*(p), w\rangle_{H',H},$$

we have

$$\int_Q pw\,dxdt = (\Lambda^{-1}(\frac{1}{\theta} p), w)_H,$$

$$\int_G p\mathscr{F}_1(w)\,dxdt = (\Lambda^{-1}\mathscr{F}_1^*(p), w)_H,$$

and

$$\int_G p\mathscr{F}(w)\,dxdt = (\Lambda^{-1}\mathscr{F}(p), w)_H.$$

Therefore, the Euler–Lagrange condition gives

$$\left(\Lambda^{-1}\frac{1}{\theta}(p) + \Lambda^{-1}\mathscr{F}_1^*(p) + \Lambda^{-1}\mathscr{F}(p) + N\hat{h}, h - \hat{h}\right)_H \geq 0, \ \forall h \in \mathscr{U}_{ad}$$

or

$$\left(\Lambda^{-1}(\frac{1}{\theta}I + \mathscr{F}_1^* + \mathscr{F})(p) + N\hat{h}, h - \hat{h}\right)_H \geq 0, \ \forall h \in \mathscr{U}_{ad}$$

where I is the identity operator of $L^2(Q)$.

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