Chapter 4
Existence Theory of Swirling Flow

4.1 Leray's Theory

We will consider the stochastic Navier–Stokes equation for the swirling flow (1.23), see Sect. 1.4, in the next three sections. Similar results hold for the stochastic Navier–Stokes equation (1.65) describing fully developed turbulence. However, to emphasize that (1.23) and (1.65) are not the same equations we will set the coefficients c_k to $c_k = h_k$ in (1.23) below. The h_ks can then be large but decay with increasing k. In this section we will first explain the probabilistic setting and prove some a priori estimates.

We let $(\Omega, \mathscr{F}, \mathbb{P})$, Ω is a set (of events) and $\mathscr{F}$ a σ-algebra on Ω, denote a probability space with $\mathbb{P}$ the probability measure of Brownian motion and $\mathscr{F}_t$ a filtration generated by all the Brownian motions b_t^k on $[t, \infty)$. If $f : \Omega \to H$ is a random variable, mapping Ω into a Hilbert space H, for example, $H = L^2(\mathbb{T}^3)$, then $L^2(\Omega, \mathscr{F}, \mathbb{P}; H)$ is a Hilbert space with norm:

$$\|f\|^2_{L^2(\Omega,\mathscr{F},\mathbb{P};H)} = E(|f(\omega)|_2^2) = \int_\Omega |f(\omega)|_2^2 \mathbb{P}(\mathrm{d}\omega) = \int_H |x|^2 f_\# \mathbb{P}(\mathrm{d}x),$$

where E denotes the expectation with respect to $\mathbb{P}$ and $f_\#\mathbb{P}$ denotes the pull back of the measure $\mathbb{P}$ to H. A stochastic process f_t in $\mathscr{L}^2 = L^2([0,T]; L^2(\Omega, \mathscr{F}, \mathbb{P}; H))$ has the norm

$$\|f_t\|^2_{\mathscr{L}^2} = \int_0^T E(|f(t,\omega)|_2^2)\mathrm{d}t$$

and f_t has the following properties; see [51].

Definition 4.1.

1. $f(t,\omega) : \mathbb{R}^+ \times \Omega \to \mathbb{R}$ is measurable with respect to $\mathscr{B} \times \mathscr{F}$ where $\mathscr{B}$ is the σ-algebra of the Borel sets on $[0,\infty)$, $\omega \in \Omega$.
2. $f(t,\omega)$ is adapted to the filtration $\mathscr{F}_t$.

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$$E\left(\int_0^T f^2(t,\omega)\mathrm{d}t\right) < \infty.$$

We are mostly interested in the Hilbert spaces $H = H^m(\mathbb{T}^3) = W^{(m,2)}$ that are the Sobolev spaces based on L^2 with the Sobolev norm

$$\|u\|_m^2 = |(1-\Delta^2)^{m/2}u|_2^2.$$

The corresponding norm on $\mathscr{L}_m^2 = L^2([0,T];L^2(\Omega,\mathscr{F},\mathbb{P};H^m(\mathbb{T}^3)))$ is

$$\|u\|_{\mathscr{L}_m^2} = \left[\int_0^T E(\|u\|_m^2)\mathrm{d}t\right]^{1/2}$$

more information about Sobolev spaces can be found in [1]. We will abuse notation slightly in this section by writing u instead of U; see Sect. 1.4. This is done for future reference and an easier comparison with Leray's classical estimates.

Let $\langle\cdot,\cdot\rangle$ denote the inner product on $L^2(\mathbb{T}^3)$. The following a priori estimates provide the foundation of the probabilistic version of Leray's theory.

Lemma 4.1. *The L^2 norms $|u|_2(\omega,t)$ and $|\nabla u|_2(\omega,t)$ satisfy the identity*

$$\mathrm{d}|u|_2^2 + 2\nu|\nabla u|_2^2\mathrm{d}t = 2\sum_{k\neq 0}\langle u, h_k^{1/2}e_k\rangle \mathrm{d}b_t^k + \sum_{k\neq 0} h_k \mathrm{d}t \tag{4.1}$$

and the bounds

$$|u|_2^2(\omega,t) \le |u|_2^2(0)\mathrm{e}^{-2\nu\lambda_1 t} + 2\sum_{k\neq 0}\int_0^t \mathrm{e}^{-2\nu\lambda_1(t-s)}\langle u, h_k^{1/2}e_k\rangle \mathrm{d}b_s^k + \frac{1-\mathrm{e}^{-2\nu\lambda_1 t}}{2\nu\lambda_1}\sum_{k\neq 0} h_k, \tag{4.2}$$

$$\int_0^t |\nabla u|_2^2(\omega,s)\mathrm{d}s \le \frac{1}{2\nu}(|u|_2^2(0) - |\mathbf{U}|^2) + \frac{1}{\nu}\sum_{k\neq 0}\int_0^t \langle u, h_k^{1/2}e_k\rangle \mathrm{d}b_s^k + \frac{t}{2\nu}\sum_{k\neq 0} h_k, \tag{4.3}$$

where λ_1 is the smallest eigenvalue of $-\Delta$ with vanishing boundary conditions on the box $[0,1]^3$ and $h_k = |h_k^{1/2}|^2$. **U** *is the velocity vector from Sect. 1.4. The expectations of these norms are also bounded:*

$$E(|u|_2^2)(t) \le E(|u|_2^2(0))\mathrm{e}^{-2\nu\lambda_1 t} + \frac{1-\mathrm{e}^{-2\nu\lambda_1 t}}{2\nu\lambda_1}\sum_{k\neq 0} h_k, \tag{4.4}$$

$$E\left(\int_0^t |\nabla u|_2^2(s)\mathrm{d}s\right) \le \frac{1}{2\nu}[E(|u|_2^2(0)) - |\mathbf{U}|^2] + \frac{t}{2\nu}\sum_{k\neq 0} h_k. \tag{4.5}$$

Proof. The identity (4.1) follows from Leray's theory and Ito's lemma. We apply Ito's lemma to the L^2 norm of u squared:

$$d\int_{\mathbb{T}^3}|u|^2 dx = 2\int_{\mathbb{T}^3}\frac{\partial u}{\partial t}\cdot u dx dt + 2\sum_{k\neq 0}\int_{\mathbb{T}^3} u\cdot h_k^{1/2} e_k dx db_t^k + \sum_{k\neq 0} h_k \int_{\mathbb{T}^3} dx dt, \tag{4.6}$$

where $k \in \mathbb{Z}^3$ and $h_k^{1/2} \in \mathbb{R}^3$. Now by use of the Navier–Stokes equation (1.21)

$$\begin{aligned} d|u|_2^2 &= 2\int_{\mathbb{T}^3} \nu\Delta u\cdot u + (-u\cdot\nabla u + \nabla\Delta^{-1}(\text{trace}(\nabla u)^2)\cdot u dx dt \\ &\quad +2\sum_{k\neq 0}\int_{\mathbb{T}^3} u\cdot h_k^{1/2} e_k dx db_t^k + \sum_{k\neq 0} h_k dt \\ &= -2\nu|\nabla u|_2^2 dt + 2\sum_{k\neq 0}\int_{\mathbb{T}^3} u\cdot h_k^{1/2} e_k dx db_t^k + \sum_{k\neq 0} h_k dt \end{aligned}$$

since the divergent-free vector u is orthogonal both to the gradient $\nabla\Delta^{-1}(\text{trace}(\nabla u)^2)$ and $u\cdot\nabla u$ by the divergence theorem. Notice that the inner product (average) of u and the stirring force f in (1.21) vanish, $\langle u, f\rangle = \bar{u}\cdot f = 0$, so f can be omitted in the computation. The first term in the last expression is obtained by integration by parts. This is the identity (4.1). The inequality (4.2) is obtained by applying Poincaré's inequality

$$\lambda_1 |u|_2^2 \leq |\nabla u|_2^2, \tag{4.7}$$

where λ_1 is the smallest eigenvalue of $-\Delta$ with vanishing boundary conditions on the cube $[0,1]^3$.[1] By Poincaré's inequality

$$d|u|_2^2 + 2\nu\lambda_1|u|_2^2 dt \leq d|u|_2^2 + 2\nu|\nabla u|_2^2 dt$$

$$= 2\sum_{k\neq 0}\langle u, h_k^{1/2} e_k\rangle db_t^k + \sum_{k\neq 0} h_k dt.$$

Solving the inequality gives (4.2). Equation (4.3) is obtained by integrating (4.1)

$$|u|_2^2(t) + 2\nu\int_0^t |\nabla u|_2^2(s) ds = |u|_2^2(0) + 2\sum_{k\neq 0}\int_0^t \langle u, h_k^{1/2} e_k\rangle db_s^k + t\sum_{k\neq 0} h_k$$

and dropping $|u - \mathbf{U}|_2^2(t) > 0$, by use of (1.37).

Finally we take the expectations of (4.2) and (4.3) to obtain, respectively, (4.4) and (4.5), using that the function $\langle u, h_k^{1/2} e_k\rangle(\omega, t)$ is adapted to the filtration $\mathscr{F}_t$.

The following amplification of Leray's a priori estimates will play an important role in the a priori estimates of the solution of the stochastic Navier–Stokes equation below.

[1] We should subtract the mean from u in Poincaré's inequality because of the periodic boundary conditions, but the mean just washes out in the estimates.

Lemma 4.2. *Let $u_{\frac{1}{2B}} = u(x,t+\frac{1}{2B})$ denote the translation of u in time by the number $\frac{1}{2B}$. Then the L^2 norms of the differences $|u-u_{\frac{1}{2B}}|_2(\omega,t)$ and $|\nabla u - \nabla u_{\frac{1}{2B}}|_2(\omega,t)$ satisfy the identity*

$$\mathrm{d}|u-u_{\frac{1}{2B}}|_2^2 + 2\nu|\nabla u-\nabla u_{\frac{1}{2B}}|_2^2\mathrm{d}t = 2\sum_{k\neq 0}\langle u-u_{\frac{1}{2B}}, h_k^{1/2}e_k\rangle \mathrm{d}(b_t^k - b_{t+\frac{1}{2B}}^k) \tag{4.8}$$

and the bounds

$$\begin{aligned}|u-u_{\frac{1}{2B}}|_2^2(\omega,t) \leq\ & |u-u_{\frac{1}{2B}}|_2^2(0)\mathrm{e}^{-2\nu\lambda_1 t}\\ &+2\sum_{k\neq 0}\int_0^t \mathrm{e}^{-2\nu\lambda_1(t-s)}\langle u-u_{\frac{1}{2B}}, h_k^{1/2}e_k\rangle \mathrm{d}(b_s^k - b_{t+\frac{1}{2B}}^k)\end{aligned} \tag{4.9}$$

$$\begin{aligned}\int_0^t |\nabla u-\nabla u_{\frac{1}{2B}}|_2^2(\omega,s)\mathrm{d}s \leq\ & \frac{1}{2\nu}|u-u_{\frac{1}{2B}}|_2^2(0)\\ &+\frac{1}{\nu}\sum_{k\neq 0}\int_0^t \langle u-u_{\frac{1}{2B}}, h_k^{1/2}e_k\rangle \mathrm{d}(b_s^k - b_{t+\frac{1}{2B}}^k),\end{aligned} \tag{4.10}$$

where λ_1 is the smallest eigenvalue of $-\Delta$ with vanishing boundary conditions on the box $[0,1]^3$ and $h_k = |h_k^{1/2}|^2$. The expectations of these norms are also bounded

$$E(|u-\nabla u_{\frac{1}{2B}}|_2^2)(t) \leq E(|u-\nabla u_{\frac{1}{2B}}|_2^2(0))\mathrm{e}^{-2\nu\lambda_1 t} \tag{4.11}$$

$$E\left(\int_0^t |\nabla u-\nabla u_{\frac{1}{2B}}|_2^2(s)\mathrm{d}s\right) \leq \frac{1}{2\nu}E(|u-\nabla u_{\frac{1}{2B}}|_2^2(0)) \tag{4.12}$$

by the expectations of the initial data of the differences.

The proof of this lemma is analogous to the proof of Lemma 4.1 and can be found in [17].

Remark 4.1. Notice that in the notation of Sect. 1.4 $|U-U_{\frac{1}{2B}}|_2^2 = |u-u_{\frac{1}{2B}}|_2^2$ because the constant velocity $\mathbf{U}$ cancels out.

4.2 The A Priori Estimate of the Turbulent Solutions

The mechanism of the turbulence production are fast oscillations driving large turbulent noise that was initially seeded by small white noise, as explained in the previous section. These fast oscillations are generated by the fast constant flow $U = U_1$, where we have dropped the subscript 1, and the flow is rotating with amplitude A and angular velocity Ω. The frequency of these oscillations increases with U and $A\Omega$. The bigger U and $A\Omega$ are the more efficient this turbulence production mechanism becomes.

In this section we will establish an a priori estimate on the norm of the turbulent solution that allows us to extend the local existence and uniqueness to the whole real-time axis. Thus the a priori estimates suffice to give global existence and uniqueness. We recall the oscillatory kernel (1.34) from Sect. 1.4:

$$\sum_{k\neq 0} h_k^{1/2} \int_0^t \mathrm{e}^{-(4\pi^2|k|^2+2\pi \mathrm{i} U_1 k_1)(t-s)-2\pi \mathrm{i} A(k_2,k_3)[\sin(\Omega t+\theta)-\sin(\Omega s+\theta)]} \mathrm{d}b_s^k e_k(x). \quad (4.13)$$

The imaginary part of the argument of the exponential creates oscillations and as U_1 and $A\Omega$ become larger these oscillations become faster. We take advantage of this mechanism to produce the a priori estimates.

Next lemma plays a key role in the proof of the useful estimate of the turbulent solution. It is a version of the Riemann–Lebesgue lemma which captures the averaging effect (mixing) of the oscillations.

Lemma 4.3. *Let the Fourier transform in time be*

$$\tilde{w} = \int_0^T w(s)\mathrm{e}^{-2\pi \mathrm{i}(k_1 U + A(k_2,k_3)\Omega)s} \mathrm{d}s,$$

where $A(k_2,k_3) = A\sqrt{k_2^2+k_3^2}$ *and* $w = w(k,t)$, $k = (k_1,k_2,k_3)$, *is a vector with three components. If* T *is an even integer multiple of* $\frac{1}{k_1 U + A(k_2,k_3)\Omega}$, *then*

$$\tilde{w} = \widetilde{\#w}, \quad (4.14)$$

where

$$\#w = \frac{1}{2}\left[w(s) - w\left(s + \frac{1}{2[k_1 U + A(k_2,k_3)\Omega]}\right)\right] = \frac{1}{2}\int_{s+\frac{1}{2[k_1 U + A(k_2,k_3)\Omega]}}^{s} \frac{\partial w}{\partial r} \mathrm{d}r \quad (4.15)$$

and $\#w$ *satisfies the estimate*

$$|\#w| \leq \frac{1}{4|k_1 U + A(k_2,k_3)\Omega|} \operatorname{ess\,sup}_{[s,s+\frac{1}{2(k_1 U_1 + A(k_2,k_3)\Omega)}]} \left|\frac{\partial w}{\partial s}\right|. \quad (4.16)$$

Proof. The proof is similar to the proof of the Riemann–Lebesgue lemma for the Fourier transform in time, let $B(k) = k_1 U + A(k_2,k_3)\Omega$:

$$\begin{aligned}
\tilde{w}(k) &= \int_0^T w(s)\mathrm{e}^{-2\pi \mathrm{i} B s} \mathrm{d}s \\
&= -\int_0^T w(s)\mathrm{e}^{-2\pi \mathrm{i} B (s-\frac{1}{2B})} \mathrm{d}s \\
&= -\int_0^T w\left(s+\frac{1}{2B}\right)\mathrm{e}^{-2\pi \mathrm{i} B s} \mathrm{d}s,
\end{aligned}$$

where we have used in the last step that w is a periodic function on the interval $[0,T]$. Taking the average of the first and the last expression we get

$$\tilde{w} = \frac{1}{2}\int_0^T \left(w(s) - w\left(s+\frac{1}{2B}\right)\right) \mathrm{e}^{-2\pi \mathrm{i} Bs}\mathrm{d}s = \widetilde{\#w}.$$

Now

$$\begin{aligned} |\#w| &= \frac{1}{2}\left|\left(w(s) - w\left(s+\frac{1}{2B}\right)\right)\right| \\ &\leq \frac{1}{2}\int_s^{s+\frac{1}{2B}} \left|\frac{\partial w}{\partial r}\right| \mathrm{d}r \\ &\leq \frac{1}{4|B|}\text{ess sup}_{[s,s+\frac{1}{2B}]}\left|\frac{\partial w}{\partial s}\right| \end{aligned}$$

by the mean-value theorem.

Corollary 4.1. *If T is not an even integer multiple of $\frac{1}{B(k)} = \frac{1}{k_1 U + A(k_2,k_3)\Omega}$, then*

$$\tilde{w} = \widetilde{\#w} - \frac{1}{2}\int_{-\frac{1}{2B}}^{0} w\left(s+\frac{1}{2B}\right)\mathrm{e}^{-2\pi \mathrm{i} Bs}\mathrm{d}s + \frac{1}{2}\int_{T-\frac{1}{2B}}^{T} w\left(s+\frac{1}{2B}\right)\mathrm{e}^{-2\pi \mathrm{i} Bs}\mathrm{d}s, \quad (4.17)$$

where $\tilde{w}$ satisfies the estimate

$$|\tilde{w}| \leq |\widetilde{\#w}| + \frac{1}{|B|}\text{ess sup}_{[-\frac{1}{2B},0]\cap[T-\frac{1}{2B},T]}\left|w\left(s+\frac{1}{2B}\right)\right|. \quad (4.18)$$

Proof. The proof is the same as of the lemma except for the step

$$\begin{aligned} \tilde{w}(k) &= \int_0^T w(s)\mathrm{e}^{-2\pi \mathrm{i} Bs}\mathrm{d}s = -\int_0^T w(s)\mathrm{e}^{-2\pi \mathrm{i} B(s-\frac{1}{2B})}\mathrm{d}s \\ &= -\int_0^T w(s+\frac{1}{2B})\mathrm{e}^{-2\pi \mathrm{i} Bs}\mathrm{d}s - \int_{-\frac{1}{2B}}^{0} w\left(s+\frac{1}{2B}\right)\mathrm{e}^{-2\pi \mathrm{i} Bs}\mathrm{d}s \\ &\quad + \int_{T-\frac{1}{2B}}^{T} w\left(s+\frac{1}{2B}\right)\mathrm{e}^{-2\pi \mathrm{i} Bs}\mathrm{d}s. \end{aligned}$$

The lemma allows us to estimate the Fourier transform (in t) of w in terms of the time derivative of w, with a gain of $(k_1 U + A(k_2,k_3)\Omega)^{-1}$. Below we will use it in an estimate showing that the limit of $\#w$ is zero when $|B(k)| = |(k_1 U + A(k_2,k_3)\Omega)| \to \infty$.

Lemma 4.4. *The integral*

$$\int_0^t (2\pi|k|)^p \mathrm{e}^{-(4\pi^2 \nu |k|^2 + 2\pi \mathrm{i}[B(k)(t-s)+g])}\mathrm{d}s,$$

where $B(k) = k_1 U + A(k_2, k_3)\Omega$, is bounded by

$$(2\pi)^p \int_0^t |k|^p \mathrm{e}^{-4\pi^2 \nu |k|^2 (t-s)} \mathrm{d}s \le C\, t^{1-\frac{p}{2}} \tag{4.19}$$

for $0 \le p < 2$, where C is a constant. In particular,

$$\int_{t-\delta}^t (2\pi|k|)^p \mathrm{e}^{-(4\pi^2 \nu |k|^2 + 2\pi \mathrm{i}[B(k)(t-s)+g])} \mathrm{d}s \le C\, \delta^{1-\frac{p}{2}}. \tag{4.20}$$

Proof. We estimate the integral

$$\begin{aligned}\int_0^t |k|^p \mathrm{e}^{-4\pi^2 \nu |k|^2 (t-s)} \mathrm{d}s &= \int_0^t |k|^p \mathrm{e}^{-4\pi^2 \nu |k|^2 r} \mathrm{d}r \\ &\le \left(\frac{p}{4\pi^2}\right)^{\frac{p}{2}} \mathrm{e}^{-p} \int_0^t r^{-\frac{p}{2}} \mathrm{d}r = C t^{1-\frac{p}{2}},\end{aligned}$$

where

$$k = \frac{1}{2\pi}\sqrt{\frac{p}{r}}$$

is the value of k where the integrand achieves its maximum.

The rotation can resonate with the uniform (linear) flow due to the nonlinearities in the Navier–Stokes equation. The following lemma restricts the values of velocity coefficients so that no resonance occurs.

Lemma 4.5. *Suppose that for $k_1 < 0$ and $\frac{\sqrt{k_2^2+k_3^2}}{|k_1|} \neq 0$ or ∞, the constants U, A, and Ω satisfy the non-resonance condition*

$$\left| \frac{U}{A\Omega} + \frac{\sqrt{k_2^2 + k_3^2}}{k_1} \right| \ge \frac{C}{|k_1|^r}, \tag{4.21}$$

where C is a constant and $0 < r < 1$; then for all $k = (k_1, k_2, k_2) \neq 0$,

$$|U k_1 + A\Omega \sqrt{k_2^2 + k_3^2}| \neq 0 \tag{4.22}$$

and

$$\lim_{|k| \to \infty} |U k_1 + A\Omega \sqrt{k_2^2 + k_3^2}| = \infty. \tag{4.23}$$

Moreover,

$$|U k_1 + A\Omega \sqrt{k_2^2 + k_3^2}| \ge B = \min(U, A\Omega, CA\Omega). \tag{4.24}$$

Proof. If $k_1 > 1$, then

$$\left|Uk_1 + A\Omega\sqrt{k_2^2+k_3^2}\right| = U|k_1| + A\Omega\sqrt{k_2^2+k_3^2} > 0$$

so (4.22) and (4.23) hold. If $k_1 < 0$, then by (4.21)

$$\left|Uk_1 + A\Omega\sqrt{k_2^2+k_3^2}\right| \geq C\ \Omega A|k_1|^{1-r} > 0$$

and

$$\lim_{|k|\to\infty}\left|Uk_1 + A\Omega\sqrt{k_2^2+k_3^2}\right| \geq C\ \Omega A \lim_{|k_1|\to\infty}|k_1|^{1-r} = \infty$$

if $|k_1| \to \infty$. If on the other hand $|k_1| < \infty$ when $|k| \to \infty$ then (4.23) also holds. When $k_1 = 0$, (4.22) and (4.23) are obvious and also if $k_2 = k_3 = 0$.

The lower bound (4.24) is read of

$$\left|Uk_1 + A\Omega\sqrt{k_2^2+k_3^2}\right|$$

when $k_1 \geq 1$. Then it is either U or $A\Omega$. When $k_1 = 0$ then it is $A\Omega$ and by (4.21), when $k_1 \leq -1$, it is greater than or equal $CA\Omega$.

The next question to ask is in which space do the turbulent solutions live? This was pointed out by Onsager in 1945 [53]. He pointed out that if the solutions satisfy the Kolmogorov scaling down to the smallest scales, they must be Hölder continuous function with Hölder exponent 1/3. In three dimensions this means that they live in the Sobolev space $H^{\frac{11}{6}+\varepsilon}$ based on $L^2(\mathbb{T}^3)$.

If $\frac{q}{p}$ is a rational number let $\frac{q}{p}^+$ denote any real number $s > \frac{q}{p}$.

Theorem 4.1. *Let the velocity $U = U_1$ of the mean flow and the product $A\Omega$ of the amplitude A and the frequency Ω of the rotation be sufficiently large, in the uniform rotating flow (1.19), with U, $A\Omega$ also satisfying the non-resonance conditions (4.21). Then the solution of the integral equation (1.32) is uniformly bounded in $\mathscr{L}^2_{\frac{11}{6}^+}$,*

$$\text{ess sup}_{t\in[0,\infty)} E(\|u\|^2_{\frac{11}{6}^+})(t) \leq \left(1 - C\left(\frac{1}{B^2} + \delta^{\frac{1}{6}^-}\right)\right)^{-1}\left[\sum_{k\neq 0}\frac{3(1+(2\pi|k|)^{\frac{11}{3}^+})}{8\pi^2\nu|k|^2}h_k + \frac{C'}{B}\right], \tag{4.25}$$

where $B = \min(|U|, A\Omega, CA\Omega)$ is large, δ small, and C and C' are constants.

Corollary 4.2 (Onsager's Observation). *The solutions of the integral equation (1.32) are Hölder continuous with exponent* $1/3$.

Remark 4.2. The estimate (4.25) provides the answer to the question we posed in Sect. 1.4 how fast the coefficients $h_k^{1/2}$ had to decay in Fourier space. They have to decay sufficiently fast for the expectation of the $H^{\frac{11}{6}^+} = W^{(\frac{11}{6}^+,\,2)}$ Sobolev norm of the initial function u_0, to be finite. This expectation appear on the right-hand side of (4.25). In other words the $\mathscr{L}^2_{\frac{11}{6}+}$ norm of the initial function u_0 has to be finite.

The proof of the theorem involves long estimates and can be found in [17]. An outline of the proof is given in Appendix A.

We consider the integral equation

$$u(x,t) = \sum_{k\neq 0}\left[h_k^{1/2}A_t^k - \int_0^t \mathrm{e}^{-[4\pi^2\nu|k|^2+2\pi\mathrm{i}B(k)](t-s)-2\pi\mathrm{i}g(k,t,s)} \times \left(\widehat{u\cdot\nabla u} + \frac{\mathrm{i}k}{2\pi|k|^2}(\widehat{\mathrm{tr}(\nabla u)^2})\right)(k,s)\mathrm{d}s\right] e_k(x),$$

where $B(k) = Uk_1 + A(k_2,k_3)\Omega$.

Lemma 4.6. *The initial condition* $(u - u_{\frac{1}{2B}})(0)$ *satisfies the estimate*

$$|u-u_{\frac{1}{2B}}|_2^2(0) \leq 2\sum_{j\neq 0}|A^j_{\frac{1}{2B(k)}}|^2 + \frac{C}{|B(k)|^2}\text{ess sup}_{t\in[0,\frac{1}{2B}]}\|u\|^2_{\frac{11}{6}+}. \tag{4.26}$$

Proof. We use the integral equation

$$\begin{aligned} u - u_{\frac{1}{2B}} = \sum_{k\neq 0}\Bigg[& h_k^{1/2}(A_t^k - A^k_{t+\frac{1}{2B}}) \\ & - \left(\int_0^t \mathrm{e}^{-[4\pi^2\nu|k|^2+2\pi\mathrm{i}B(k)](t-s)-2\pi\mathrm{i}g(k,t,s)}\right) \\ & \times\left(\widehat{u\cdot\nabla u} + \frac{\mathrm{i}k}{2\pi|k|^2}(\widehat{\mathrm{tr}(\nabla u)^2})\right)(k,s)\mathrm{d}s \\ & - \int_0^{t+\frac{1}{2B}} \mathrm{e}^{-[4\pi^2\nu|k|^2+2\pi\mathrm{i}B(k)](t+\frac{1}{2B}-s)-2\pi\mathrm{i}g(k,t+\frac{1}{2B},s)} \\ & \times\left(\widehat{u\cdot\nabla u} + \frac{\mathrm{i}k}{2\pi|k|^2}(\widehat{\mathrm{tr}(\nabla u)^2})\right)(k,s)\mathrm{d}s\Bigg] e_k(x), \end{aligned}$$

where $B(k) = Uk_1 + A(k_2,k_3)\Omega$. At $t = 0$,

$$|u-u_{\frac{1}{2B}}|^2(0) = |u_{\frac{1}{2B}}|^2(0) = 2\sum_{j\neq 0}h_j|A^j_{\frac{1}{2B}}|^2 + \frac{C}{|B(k)|^2}\text{ess sup}_{t\in[0,\frac{1}{2B}]}\|u\|^2_{\frac{11}{6}+}$$

by the same estimates as above.

Lemma 4.7. *The identity (4.1) in Lemma 4.1 can be modified for* $a > 0$

$$\mathrm{d}(\mathrm{e}^{\nu a t}|u|_2^2) + 2\nu \mathrm{e}^{\nu a t}|\nabla u|_2^2 \mathrm{d}t = \nu a \mathrm{e}^{\nu a t}|u|_2^2 \mathrm{d}t + 2\mathrm{e}^{\nu a t}\sum_{k\neq 0}\langle u, h_k^{1/2} e_k\rangle \mathrm{d}b_t^k + \mathrm{e}^{\nu a t}\sum_{k\neq 0} h_k \mathrm{d}t \tag{4.27}$$

and produces the estimates

$$\begin{aligned}|u|_2^2(t) \leq |u|_2^2(0)\left(\mathrm{e}^{-\nu a t} + \frac{a\mathrm{e}^{-2\nu\lambda_1 t}}{(a-2\lambda_1)}\right) + 2\sum_{k\neq 0}\int_0^t \mathrm{e}^{-\nu a(t-s)}\langle u, h_k^{1/2} e_k\rangle \mathrm{d}b_s^k \\ +2\sum_{k\neq 0}\int_0^t \mathrm{e}^{-\nu a(t-s)}\int_0^s \mathrm{e}^{-2\nu\lambda_1(s-r)}\langle u, h_k^{1/2} e_k\rangle \mathrm{d}b_r^k \mathrm{d}s + \frac{1}{\nu}\left(\frac{1}{a} + \frac{1}{2\lambda_1}\right)\sum_{k\neq 0} h_k\end{aligned} \tag{4.28}$$

and

$$\begin{aligned}\int_0^t \mathrm{e}^{-\nu a(t-s)}|\nabla u|_2^2(s)\mathrm{d}s \leq \frac{1}{2\nu}(|u|_2^2(0) - |\mathbf{U}|^2)\left(\mathrm{e}^{-\nu a t} + \frac{a\mathrm{e}^{-2\nu\lambda_1 t}}{(a-2\lambda_1)}\right) \\ +\frac{1}{\nu}\sum_{k\neq 0}\int_0^t \mathrm{e}^{-\nu a(t-s)}\langle u, h_k^{1/2} e_k\rangle \mathrm{d}b_s^k \\ +\frac{1}{\nu}\sum_{k\neq 0}\int_0^t \mathrm{e}^{-\nu a(t-s)}\int_0^s \mathrm{e}^{-2\nu\lambda_1(s-r)}\langle u, h_k^{1/2} e_k\rangle \mathrm{d}b_r^k \mathrm{d}s \\ +\frac{1}{2\nu^2}\left(\frac{1}{a} + \frac{1}{2\lambda_1}\right)\sum_{k\neq 0} h_k,\end{aligned} \tag{4.29}$$

where λ_1 *is the smallest eigenvalue of* $-\Delta$ *with vanishing boundary conditions on the box* $[0,1]^3$ *and* $h_k = |h_k^{1/2}|^2$.

Proof. We multiply the identity (4.1) in Lemma 4.1 by $\mathrm{e}^{\nu a t}$ to get (4.27). Then integration gives the equality

$$\begin{aligned}|u|_2^2(t) + 2\nu\int_0^t \mathrm{e}^{-\nu a(t-s)}|\nabla u|_2^2(s)\mathrm{d}s = |u|_2^2(0)\mathrm{e}^{-\nu a t} + \nu a\int_0^t \mathrm{e}^{-\nu a(t-s)}|u|_2^2(s)\mathrm{d}s \\ +2\sum_{k\neq 0}\int_0^t \mathrm{e}^{-\nu a(t-s)}\langle u, h_k^{1/2} e_k\rangle \mathrm{d}b_s^k \\ +\frac{(1-\mathrm{e}^{-\nu a(t-s)})}{\nu a}\sum_{k\neq 0} h_k.\end{aligned}$$

Now substituting the estimate (4.2), from Lemma 4.1, for $|u|_2^2$ on the right-hand side gives the two inequalities (4.28) and (4.29) as in Lemma 4.1.

Lemma 4.8. *The functions H, K, and L in the proof of Theorem 4.1 satisfy the estimate*

$$E(H+K+L) \leq \frac{C}{|B(k)|^2} E(\text{ess sup}_{t\in[0,\frac{1}{2B}]} \|u\|^2_{\frac{11}{6}+}) + \frac{C'}{B} \qquad (4.30)$$

with $B = \min(U, A\Omega, CA\Omega)$.

The proof of the lemma involves long formulas for H, K, and L and can be found in [17].

Remark 4.3. Corollary 4.2 is the resolution of a famous question in turbulence, for the swirling flows: *Is turbulence always caused by the blow up of the velocity u?* The answer according to Theorem 4.1 is *no*; the solutions are not singular. However, they are not smooth either, contrary to the belief, stemming from Leray's theory [42], that if solutions are not singular then they are smooth. By Corollary 4.2 the solutions are Hölder continuous with exponent $1/3$ in three dimensions. This confirms an observation made by Onsager [54] in 1945. In particular the gradient ∇u and vorticity $\nabla \times u$ are not continuous in general as discussed in Sect. 3.7.

Remark 4.4. U and $A\Omega$ do not have to be made very large for the estimate (4.25) to be satisfied, because $B(k) \to \infty$ as $|k| \to \infty$. How big U and $A\Omega$ have to be for (4.25) to hold is probably best answered by a numerical simulation.

We can now prove that ess $\sup_{t\in[0,\infty)} \|u(t)\|^2_{\frac{11}{6}+}$ is bounded with probability close to one.

Lemma 4.9. *For all $\varepsilon > 0$ there exists an R such that*

$$\mathbb{P}(\text{ess sup}_{t\in[0,\infty)} \|u(t)\|^2_{\frac{11}{6}+} < R) > 1-\varepsilon. \qquad (4.31)$$

Proof. By Chebyshev's inequality and the estimate (4.25) we get that

$$\mathbb{P}(\text{ess sup}_{t\in[0,\infty)} \|u(t)\|^2_{\frac{11}{6}+} \geq R) < \frac{C}{R} < \varepsilon$$

for R sufficiently large.

4.3 Existence Theory of the Stochastic Navier–Stokes Equation

In this section we prove the existence of the turbulent solutions of the initial value problem (1.23). The following theorem states the existence of turbulent solutions in three dimensions. First we write the initial value problem (1.23) as the integral equation (4.32)

$$u(x,t) = u_0(x,t) - \int_0^t \mathrm{e}^{K(t-s)} * [u \cdot \nabla u - \nabla \Delta^{-1} \mathrm{tr}(\nabla u)^2] \mathrm{d}s. \tag{4.32}$$

Here e^{Kt} is the oscillatory heat kernel (1.33) and

$$u_0(x,t) = \sum_{k \neq 0} h_k^{1/2} A_t^k e_k(x)$$

the A_t^ks being the oscillatory Ornstein–Uhlenbeck-type processes from (1.34).

Theorem 4.2. *If the uniform flow U and product of the amplitude and frequency $A\Omega$, of the rotation, are sufficiently large, $B = \min(|U|, A\Omega, CA\Omega)$, δ is small and the non-resonance conditions (4.21) are satisfied, so that the a priori bound (4.25) holds, then the integral equation (4.32) has unique global solution $u(x,t)$ in the space $C([0,\infty); L^2(\Omega, \mathscr{F}, \mathbb{P}; H^{\frac{11}{6}^+}))$, u is adapted to the filtration generated by the stochastic process*

$$u_0(x,t) = \sum_{k \neq 0} h_k^{1/2} A_t^k e_k$$

and

$$E\left(\int_0^t \|u\|^2_{\frac{11}{6}^+} \mathrm{d}s\right) \leq \left(1 - C\left(\frac{1}{B^2} + \delta^{\frac{1}{6}^-}\right)\right)^{-1} \left[\sum_{k \neq 0} \frac{3(1 + (2\pi|k|)^{\frac{11}{3}^+})}{8\pi^2 \nu |k|^2} h_k + \frac{C'}{B}\right] t. \tag{4.33}$$

This theorem is a standard application of the contraction mapping principle to prove global existence and uniqueness. Then the unique local solution is extended to the whole positive time axis by use of the a priori bound (4.25). A detailed proof can be found in [17].

We now add the initial condition $u(x,0) = u^0(x)$, with mean zero, to the integral equation (4.32).

Theorem 4.3. *If the uniform flow U and the product of the amplitude $A\Omega$ and frequency of the rotation, $B = \min(|U|, A\Omega, CA\Omega)$, are sufficiently large, δ small, and the non-resonance conditions (4.21) are satisfied, so that the a priori bound (4.25) holds, then the integral equation*

$$u(x,t) = \mathrm{e}^{Kt} * u^0(x) + u_0(x,t) - \int_0^t \mathrm{e}^{K(t-s)} * (u \cdot \nabla u - \nabla \Delta^{-1} (\nabla u)^2) \, \mathrm{d}s, \tag{4.34}$$

where e^{Kt} is the oscillating kernel in (1.33), has unique global solution $u(x,t)$ in the space $C([0,\infty); L^2(\Omega, \mathscr{F}, \mathbb{P}; H^{\frac{11}{6}^+}))$, u is adapted to the filtration generated by the stochastic process

$$u_0(x,t) = \sum_{k \neq 0} h_k^{1/2} A_t^k e_k$$

and

$$E\left(\int_0^t \|u\|^2_{\frac{11}{6}^+} \mathrm{d}s\right) \leq \left(1 - C\left(\frac{1}{B^2} + \delta^{\frac{1}{6}^-}\right)\right)^{-1} \left[\sum_{k\neq 0} \frac{(1+(2\pi|k|)^{\frac{11}{3}^+})}{2\pi^2\nu|k|^2} h_k + \frac{C'}{B}\right] t. \tag{4.35}$$

The proof of the theorem is exactly the same as the proof of Theorem 4.2 once the a priori bound (4.25) is established. A proof can be found in [17].

Corollary 4.3. *For any initial data $u^0 \in \dot{L}^2(\mathbb{T}^3)$, the L^2 space with mean zero, and any $t_0 > 0$, there exists a mean flow U, an amplitude and angular velocity $A\Omega$, and δ small, such that (4.34) has a unique solution in $C([t_0,\infty); L^2(\Omega,\mathscr{F},\mathbb{P}; H^{\frac{11}{6}^+}))$.*

Proof. For $t > 0$, $\mathrm{e}^{Kt} * u^0(x)$ is smooth. Now apply Theorem 4.3.

Next we prove a Gronwall estimate that can be use to prove local (in t) stability and irreducibility; see [17].

Lemma 4.10. *Let u be a solution of (4.32) with an initial function $u_0(x,t) = \sum_{k\neq 0} h_k^{1/2} A_t^k e_k$ and initial condition $u^0(x)$ and y a solution of*

$$y_t + \mathbf{U}\cdot\nabla y = \nu\Delta y - y\cdot\nabla y + \nabla\Delta^{-1}\mathrm{tr}(\nabla y)^2 + f \tag{4.36}$$

with initial condition $y^0(x)$, then

$$\begin{aligned}\|u-y\|^2_{\frac{11}{6}^+}(t) \leq{}& [3\|u^0 - y^0\|^2_{\frac{11}{6}^+} + 3\|\sum_{k\neq 0} h_k^{1/2} A_t^k e_k - \mathrm{e}^{Kt} * f\|^2_{\frac{11}{6}^+} \\ &+ \delta^2 C_1 \operatorname{ess\,sup}_{s\in[t-\delta,t]}(\|u\|^2_{\frac{11}{6}^+} + \|y\|^2_{\frac{11}{6}^+})]\mathrm{e}^{C_2\int_0^{t-\delta}(1+\|u\|^2_{\frac{11}{6}^+} + \|y\|^2_{\frac{11}{6}^+})\mathrm{d}s},\end{aligned} \tag{4.37}$$

where C_1 and C_2 are constants and δ can be made arbitrarily small. The A_t^ks are the oscillatory Ornstein–Uhlenbeck-type processes (1.35) and e^{Kt} is the oscillatory kernel in (1.33).

Proof. We subtract the integral equation for y from that of u:

$$\begin{aligned} u &= u^0 + \sum_{k\neq 0} h_k^{1/2} A_t^k e_k + \mathrm{e}^{Kt} * (-u\cdot\nabla u + \nabla\Delta^{-1}\mathrm{tr}(\nabla u)^2), \\ y &= y^0 + \mathrm{e}^{Kt} * f + \mathrm{e}^{Kt} * (-y\cdot\nabla y + \nabla\Delta^{-1}\mathrm{tr}(\nabla y)^2). \end{aligned}$$

Thus

$$\begin{aligned}\|u-y\|^2_{\frac{11}{6}^+}(t) \leq{}& [3\|u^0 - y^0\|^2_{\frac{11}{6}^+} + 3\|\sum_{k\neq 0} h_k^{1/2} A_t^k e_k - \mathrm{e}^{Kt} * f\|^2_{\frac{11}{6}^+} \\ &+ 3\|\mathrm{e}^{Kt} * (-w\nabla u - y\nabla w + \nabla\Delta^{-1}\mathrm{tr}\nabla\alpha\cdot\nabla w)\|^2_{\frac{11}{6}^+}],\end{aligned}$$

where $w = u - y$ and $\alpha = u + y$. Now the same estimates as in Theorem 4.1 give

$$\begin{aligned}\|u-y\|^2_{\frac{11}{6}+}(t) \le\ & 3\|u^0-y^0\|^2_{\frac{11}{6}+} + 3\|\sum_{k\neq 0} h_k^{1/2} A_t^k e_k - \mathrm{e}^{Kt} * f\|^2_{\frac{11}{6}+} \\ & + C_1 \delta^2 \text{ess sup}_{s\in[t-\delta,t]}(\|u\|^2_{\frac{11}{6}+} + \|y\|^2_{\frac{11}{6}+}) \\ & + C_2 \int_0^{t-\delta} (1+\|u\|^2_{\frac{11}{6}+} + \|y\|^2_{\frac{11}{6}+})(\|u-y\|^2_{\frac{11}{6}+})\mathrm{d}s.\end{aligned}$$

Then Grönwall's inequality gives (4.37).