Chapter 18
Problem Posing for and Through Investigations in a Dynamic Geometry Environment

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Abstract This chapter analyzes three different types of problem posing associated with geometry investigations in school mathematics, namely (a) problem posing through proving; (b) problem posing for investigation; and (c) problem posing through investigation. Mathematical investigations and problem posing which are central for activities of professional mathematicians, when integrated in school mathematics, allow teachers and students to experience meaningful mathematical activities, including the discovery of new mathematical facts when posing mathematical problems. A dynamic geometry environment (DGE) plays a special role in mathematical problem posing. I describe different types of problem posing associated with geometry investigations by using examples from a course with prospective mathematics teachers. Starting from one simple problem I invite the readers to track one particular mathematical activity in which participants arrive at least at 25 new problems through investigation in a DGE and through proving.

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Background

Mathematical Inquiry in the Mathematics Classroom

Inquiry and investigations are basic characteristics of the development of mathematics, science, and technology. According to Wells (1999), inquiry is a way of teaching and learning which integrates wonderment and puzzlement and arouses interest and motivation in learners. Investigation activities are associated with seeking knowledge, information, or truth through questioning.

Mathematical investigations are central to the activity of any research mathematician. In the past two decades mathematical investigations have become an integral part of mathematics teaching and learning in school (Da Ponte, 2007; Leikin, 2004, 2012; Silver, 1994; Yerushalmy, Chazan, & Gordon, 1990). Investigation tasks in mathematics classrooms are usually challenging, cognitively demanding, and enable highly motivated work by students (e.g., Yerushalmy et al., 1990). Borba and Villarreal (2005) stressed that "the experimental approach gains more power with the use of technology" (p. 75) by providing learners with the opportunity to propose and test conjectures using multiple examples, obtain quick feedback, use multiple representations, and become involved in the modeling process.

Both problem-posing and investigation problems in a broad range of types of mathematical tasks are called "open problems" (Pehkonen, 1995). This chapter focuses on problem posing associated with investigations in geometry. Yerushalmy et al. (1990) suggested to consider investigations in geometry as activities that include experimenting to arrive at a conjecture, conjecturing, testing the conjecture, and proving or refuting it. The conjectures raised by the students and teachers become new proof problems.

Investigations in geometry are naturally associated with the use of dynamic geometry environments (DGEs) (Mariotti, 2002; Schwartz, Yerushalmy, & Wilson, 1993; Yerushalmy et al., 1990). Numerous studies have explored the role of DGEs in the instructional process, specifically in concept acquisition, geometric constructions, proofs, and measurements (e.g., Chazan & Yerushalmy, 1998; Hölzl, 1996; Jones, 2000; Mariotti, 2002; Yerushalmy & Chazan, 1993). In this chapter, these problems will be referred to as *problems posed through investigation*.

Teachers Devolve Mathematical Investigations to the Classroom

Teachers' roles in integration of investigation tasks in teaching and learning cannot be overestimated. Teachers' knowledge, skills, and beliefs determine whether and how they implement mathematical investigations in their classes. To make systematic use of mathematical investigations in school, several potential pitfalls have to be overcome.

First, the majority of teachers of mathematics in school nowadays do not have personal experience in learning mathematics through mathematical investigations, while many teachers have limited experience in the use of dynamic software for mathematical investigations. Geometry investigations using DGEs require teachers to rethink teaching: they have to deal with unfamiliar or even new mathematical practices, and "take a more prominent role in designing learning activities for their students" (Healy & Lagrange, 2010, p. 288). When they are challenged by new (for them) teaching approaches, the teachers are often unenthusiastic and reluctant to adopt these practices and express preferences for the teaching methods used by their own teachers before them (e.g., Lampert & Ball, 1998; Leikin, 2008).

Second, implementation of investigation problems requires devolving investigation problems to the class (e.g., Da Ponte, 2007; Yerushalmy et al., 1990). Yet, often teachers cannot even find investigation problems in regular instructional materials. Thus, integration of mathematical investigations in the classroom means that teachers have to create investigation problems for their students.

Third, usually investigations in geometry are supported by DGEs that frequently lead to technological difficulties with the environment, or with classroom equipment, as well as other issues (Healy & Lagrange, 2010). Additionally, navigation of a lesson that engages students in investigation activities requires the teacher to possess diverse didactical skills, technological knowledge, and profound mathematical knowledge since these activities lead to unpredicted mathematical conjectures that sometimes require complex proving.

Da Ponte and Henriques (2013) and Ellerton (2013) stressed the importance of the integration of problem posing and investigation activities in teacher education programs. They demonstrated the effectiveness of these activities in the development of teachers' conceptions about the importance of problem-posing and investigation activities in school mathematics and the development of teachers' knowledge. When teachers themselves are involved in investigation activities, their thinking processes are stimulated so that they experience mathematical processes themselves (Da Ponte & Henriques, 2013). Teachers have to be educated for the generation of investigation tasks, for the classroom use of mathematical investigations, and for fluent management of mathematical lessons.

This chapter describes the integration of these activities in a geometry course for prospective secondary school mathematics teachers.

The Context

This chapter presents reflective insight from a long-term study conducted using design research methodology. As a design experiment it was a formative research study to examine and refine educational design (Collins, Joseph, & Bielaczyc, 2004). The setting was directed towards promoting learning, producing useful knowledge as well as modeling learning and teaching advancement (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). That is, the present study had both a pragmatic and a theoretical orientation. The design experiment was performed in the context of a geometry course (within the teacher certificate program) aimed at the advancement of problem-solving and problem-posing expertise, by employing Multiple Proof Tasks (Leikin, 2008) and Mathematical Investigations (Leikin, accepted). The data in this chapter were collected from 22 prospective mathematics teachers aged 22–40, all of whom had a B.Sc. degree in mathematics prior to their participation in the program.

In this context an *Investigation Task* was defined as a complex task that includes:

1. Solving a proof problem in several ways;
2. Transforming the proof problem into an investigation problem;
3. Investigating the geometry object (from the proof problem) in a DGE for additional properties (experimenting and conjecturing); and
4. Proving or refuting conjectures.

The collected data included students' written work and protocols of group discussions and group interviews. In this chapter, similar to the exploration of investigation activities in calculus performed by Da Ponte and Henriques (2013), I provide theoretical analysis of problem-posing types associated with geometry investigations.

Investigation tasks of this type lead to three types of problem posing:

1. Problem posing through proving;
2. Problem posing for investigation; and
3. Problem posing through investigation, including problem posing through construction.

Types of Problem Posing Associated with Geometry Investigations: Definitions

Problem Posing Through Proving

Proving is an integral part of investigation activities in geometry. Through proving, one can also realize new and unforeseen properties of a given object that are proven at one of the proof stages. Then, proving each of such properties encompasses a new geometry problem. Problem posing through proving, if taken as a problem-posing strategy, is similar to the "chaining" strategy described by Hoehn (1993).

De Villiers (2012) analyzed the "looking back" discovery function of proof using specific advanced geometry examples. He noticed that it is "possible to design learning activities for younger students in the junior secondary school that allow acquainting students with the idea that a deductive argument can provide additional insight, and some form of novel discovery" (p. 1133). I provide such an example later in this chapter.

Problem Posing for Investigation

Several studies consider problem transformation (also called reformulation) as an instance of problem-posing activity (Stoyanova, 1998, with reference to Duncker, 1945; Leikin & Grossman, 2013; Mamona-Downs, 1993; Silver, 1994). Transformation of a proof problem into an investigation problem is considered herein as *problem posing for investigation*.

Problem posing related to problem transformation is explored by researchers focusing on systematic transformations of a given problem involving variations in goals and givens. The "what if not?" scheme is the most well-known problem-posing strategy (Brown & Walter, 1993, 2005). The "what if not?" strategy, which is based on changes in givens, leads to making room for conjecturing and producing new insights about problem outcomes. Leikin and Grossman (2013) pointed out an additional type of problem posing which they called the "what if yes?" strategy, which is based on the addition of properties to the given object (e.g., considering a special case of a square for a given parallelogram).

Leikin and Grossman (2013) classified problem transformations either as static or dynamic—with respect to the dynamic behavior of geometric figures in DGEs—as follows: *Dynamic changes* are those that can be obtained by dragging within a DGE, while *static changes* are those that cannot be obtained by dragging. Dragging (and thus dynamic change) does not change any of the critical properties of the figure constructed in the DGE (see distinction between figure and drawing by Laborde, 1992). For example, by dragging a rectangle, it can be transformed into a square ("what if yes?" strategy) but cannot be transformed into a parallelogram ("what if not?" strategy), which is not a rectangle. Static changes in a DGE usually require additional construction without changing the given figure, or constructing a new figure.

Problem transformations can also be obtained by the "goal manipulation" strategy (Silver, Mamona-Downs, Leung, & Kenny, 1996), in which the givens remain unchanged and only the goal is changed, or by the "symmetry" strategy (Hoehn, 1993; Silver et al., 1996) that leads to the creation of a problem in which the givens and the goals have been interchanged.

Leikin and Grossman (2013) found that *investigation* problems posed by teachers can be of *discovery* and *verification* types, depending on the degree of their openness. *Verification problem*s do not require conjecturing but do ask for checking a proposition that needed to be proved. On the contrary, *discovery problem*s are open problems that require conjecturing, analyzing conjectures, and proving. The problems posed by the teachers presented in this chapter are analyzed in terms of their openness.

Problem Posing Through Investigation in a DGE

Problem posing through investigation is usually associated with dragging and constructions in a DGE. Dragging is a critical feature of DGEs, which makes investigation possible. The two main functions of dragging are *testing* and *searching* (Hölzl, 2001):

- Testing verifies that a figure constructed in the process of experimentation satisfies all the conditions given in the task.
- Searching is aimed at finding new properties of a given figure and recognizing unforeseen regularities, relationships, and invariants.

In this context the distinction within DGEs between *drawing* and *figure* that was introduced by Parzysz (1988) and further developed by Laborde (1992) is especially important. Drawings and figures are visual images of geometric objects. Figures (rigorous constructions) are images of geometric objects constructed in such a way that all the necessary properties of the object are present. For example, if users drag any corner of a figure representing a square, the figure changes its size but remains a square. In this sense, a "figure does not refer to one object but to an infinity of objects" (Laborde, 1992, p. 128), which continuously preserve all critical properties under dragging. By contrast, drawings resemble the indented geometric object, with all its properties, but in a DGE they do not pass the drag test. In this way a corrected soft construction in a DGE is a drawing. Soft constructions have only part of the properties of a given object, and naturally—when corrected—do not pass the drag test. For example, when a drawing of a square is dragged it loses some of its properties and becomes some type of quadrilateral, i.e., a rectangle.

Based on the distinction between figures and drawings in a DGE, I suggested differentiation between two types of dragging: *figure dragging* and *correction dragging* (Leikin, 2012) that facilitate posing problems through two corresponding types of investigations in DGEs—a *figure investigation* and a *correction investigation*. Table 18.1 (based on Leikin, 2012) summarizes the differences between the two types of investigations.

Table 18.1
Distinctions Between Figure and Correction Investigations

	Investigation type	
Features	Figure investigation	Correction investigation
Dragging	Investigation dragging of the figure which is continuous and arbitrary	Correction dragging of the drawing (to achieve given conditions) which is discrete and purposeful
PP strategy	Searching for properties which are immune to dragging	"What if yes?" strategy Searching for properties that repeatedly occur in the corrected objects
Measurement	Exact	Approximate

Note here that figure investigations in DGEs cannot be performed using "what if not?" or "what if yes" schemes. "What if not?" is impossible since robust construction presumes that no properties of the figure can be "reduced" (Brown & Walter, 1993, 2005). "What if yes?" is impossible since adding properties to the constructed figure can only be done by means of soft constructions. "What if yes?" problem-posing strategies can be performed by means of correction investigations. Investigations in DGEs can also be performed based on static changes performed on the figure accompanied by subsequent dragging. Namely, investigations in a DGE can include performing auxiliary constructions. These constructions themselves can lead to unpredicted results. In this sense problem posing through investigation includes problem posing by construction.

In the next section I exemplify these findings through a reflective account of one particular mathematical activity when the participants arrived at least 25 new problems through investigation within a DGE and through proving. Most of the posed problems remain without proof, and the readers are invited to prove the problems, further perform geometry investigations and pose new problems related to the given mathematical object.

Tracking Geometry Investigation Through the Lens of Problem Posing

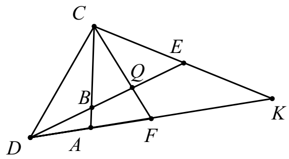
Problem Posing Through Proving

Prospective secondary school mathematics teachers (PMTs) were asked to produce at least two different proofs to Problem 1 (see Figure 18.1). As a rule, this part of the task was performed as homework with the subsequent classroom discussion focused on presentation of the solutions, analysis of similarities, and differences between the proofs and views on the elegance of the proofs and their level of difficulty. PMTs—as a group—produced two different solutions (Figure 18.1). As described below, one of these solutions appeared to be a source for a new problem.

In the discussion that took place during the lesson, PMTs regarded Proof 1.1 (Figure 18.1) as being easier than Proof 1.2 for two reasons: (a) In Proof 1.1 the auxiliary construction is performed "within the given figure" whereas in Proof 1.2 auxiliary construction is "outside the given figure"; and (b) Proof 1.1 is based on the problem givens and properties of the midline in the triangle and Thales theorem, whereas Proof 1.2 is based on the similarity of triangles.

At the same time, PMTs shared the opinion that "Proof 1.2 is *more interesting* since it shows additional properties of the given figure." They argued that Proof 1.2 leads to posing a new problem (Problem 2 shown in Figure 18.2). A statement in Problem 2 follows from Proof 1.2 that includes two facts: $CD \parallel GF$ and $DC = GF$.

Problem 1



Given:
Δ DCK,
DE median in Δ DCK
CF median in Δ DCK
CA median in Δ DCF

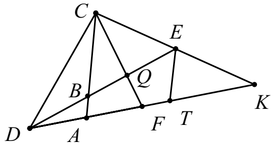
Prove that

$\frac{DE}{DB} = \frac{5}{2}$

in at least 2 different ways

Proof 1.1

Auxilary construction $ET||CA$



$DK = 20x \Rightarrow DF = 10x$, $DA = 5x \Rightarrow AK = 15x$

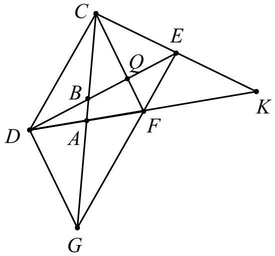
$CE = EK \,\&\, ET||CA \Rightarrow ET$ midline in ΔACK

$\Rightarrow AT = \frac{1}{2} AK = 7.5x \Rightarrow DT = 12.5x$

$\frac{DE}{DB} = \frac{DT}{DA}; \frac{DE}{DB} = \frac{12.5x}{5x} \Rightarrow \frac{DE}{DB} = \frac{5}{2}$

Proof 1.2

Auxilary construction *EG* through *F* (*G* on *CA*)



EF – midline $\Rightarrow DC = 2EF$, $DC||EF$

$DA = AF, DC||GF \Rightarrow \Delta DCA \cong \Delta FGA \Rightarrow DC = GF$

$EF = x, DC = GF = 2x \Rightarrow GE = 3x$

$\Delta DCB \sim \Delta EGB \Rightarrow DB = 2y, BE = 3y \Rightarrow DE = 5y$

$\frac{DE}{DB} = \frac{5}{2}$

Figure 18.1. Two proofs for Problem 1.

Problem 2

Given: ΔDCK,

DE median in ΔDCK

BF median in ΔDCK

CA median in ΔDCF

EG passes through F, G on CA

Prove:

$DCFG$ - parallelogram

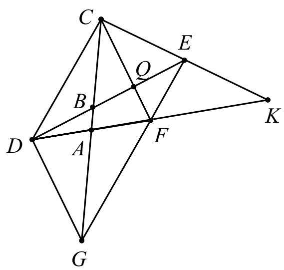
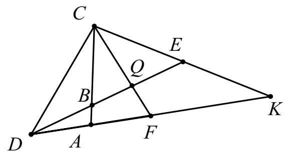


Figure 18.2. Problem posed through Proof 1.2.

Problem Posing for Investigation

At the second stage of coping with Problem 1, participants were required to transform the proof problem into an investigation problem. Figure 18.3 demonstrates two of the investigation problems (3A and 3B) created by PMTs. Problem 3A exemplifies a *verification problem* since it does not require conjecturing but only checking a proposition that had to be proved. Problem 3B illustrates a *discovery problem*, as it is formulated as an open problem that requires conjecturing, analyzing conjectures, and proving (see additional examples of discovery problems in Figure 18.5).

Problem 3



Given:

ΔDCK,

DE median in ΔDCK

BF median in ΔDCK

CA median in ΔDCF

3A. Verification problem:
Is it true that $\frac{DE}{DB} = \frac{5}{2}$?

3B. Discovery problem:
Find different relations between the elements in the given figure.

Figure 18.3. Transforming Problem 1 into new investigation-oriented problems.

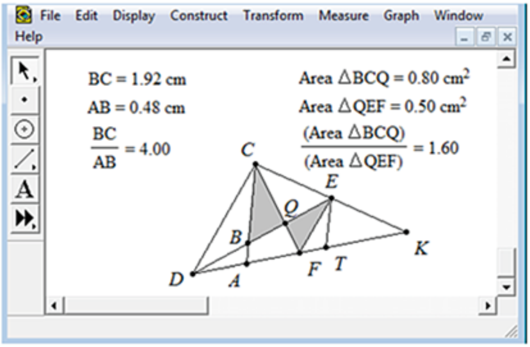
Problem 3B allowed participants to search for all possible relationships between elements in the given figure and other figures that can be achieved by auxiliary constructions from the given figure. The investigation and the constructions were performed in different DGEs (e.g., Geo-Gebra, Geometry Sketchpad or Geometry Investigator) according to the PMTs' preferences. The PMTs were allowed to perform investigations with robust as well as soft construction. Investigations were mostly directed at searching for those relationships and properties of a robust construction which are immune to dragging in DGE.

Problem Posing Through Investigation

Overall PMTs discovered more than 20 properties related to the geometrical object from Problem 1. Figures 18.4, 18.5, and 18.6 depict examples from the collective problem-posing space related to the properties discovered by PMTs. Figure 18.4 demonstrates properties discovered with auxiliary constructions "inside" the given geometry object. In contrast, Figure 18.5 depicts properties which are based on the auxiliary constructions "outside" the given geometry object. Thus, properties in Figure 18.5 are considered as requiring more advanced thinking. Discovery of properties presented in both Figures 18.4 and 18.5 was based on the *figure investigation* that included carrying out auxiliary constructions, measurements, and search for the invariants (properties which are immune to dragging).

Problem 4: Given: ΔDCK, DE median in ΔDCK, BF median in ΔDCK, CA median in ΔDCF

Auxilary construction "within the given figure" EF, $ET \| CA$



Prove that

4a. $\frac{BC}{AB} = 4$ 4c. $\frac{A(BCQ)}{A(EQF)} = \frac{8}{5}$ 4e. $\frac{AT}{DK} = \frac{3}{8}$ 4g. $\frac{DA}{DT} = \frac{2}{5}$

4b. $\frac{CQ}{QF} = 2$ 4d. $\frac{A(DCQ)}{A(EQF)} = 4$ 4f. $\frac{QE}{BQ} = \frac{5}{4}$ 4h. $\frac{A(DCK)}{A(ETK)} = \frac{16}{3}$

The new posed problems ask to prove discovered properties.

4*b*, 4*d* are trivial discoveries: Q is intersection of medians

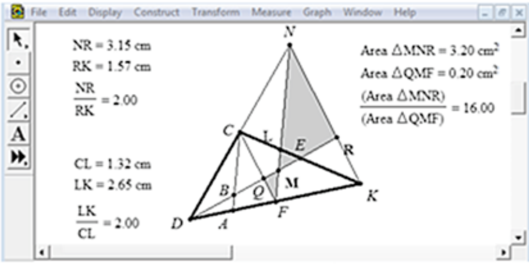
Figure 18.4. Posing a problem through investigation: Looking within the figure.

The whole group discussion focused on the *newness of the discovered properties* and the connections between the properties. Some of the discovered properties were evaluated as *trivial* ones since PMTs ought to know these properties without investigation. Properties 4*b, d, g* are trivial for different reasons: $\frac{CQ}{QF} = 2$ since Q is the point of intersection of medians in triangle *DCK*. For the same reason $\frac{A(DCQ)}{A(EQF)} = 4$ is associated with similarity of the triangles with a coefficient of similarity equal to 2. Property $\frac{DA}{DT} = \frac{2}{5}$ follows immediately from the property proven in Problem 1. Note that at advanced stages of the course, trivial discoveries were given a negative evaluation as an indicator of a lack of basic geometry knowledge and an absence of PMTs' critical reasoning.

Properties 4*a, c, e, f, h* are *nontrivial* since they do not constitute geometric theorems from the geometry course, and they do require proving in several stages. The PMTs were asked to prove properties that were nontrivial. I invite the readers also to perform these proofs.

Problem 5: Given: ΔDCK, DE median in ΔDCK, BF median in ΔDCK, CA median in ΔDCF

Auxilary construction A: $CN = DC$ (N on continuation of DC); NK, NF, L, M, R



5a. $\frac{NR}{RK} = 2$

5c. $\frac{A(MNR)}{A(QMF)} = 16$

5e. $\frac{A(MNR)}{A(CFK)} = \frac{32}{30}$

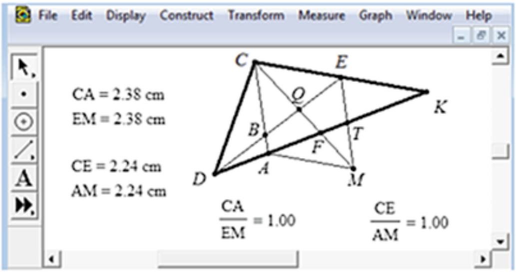
Prove that *(The new posed problems ask to prove discovered properties.)*

5b. $\frac{A(MNR)}{A(BCQ)} = 4$

5d. $\frac{A(MLE)}{A(QMF)} = 1$

5f. $\frac{A(MNR)}{A(CLF)} = \frac{16}{5}$

Auxilary construction B: EM through T, $M = ET \cap CQ$



5h. $CAME$ is a parallelogram

5i. $\frac{A(DCK)}{A(ACEM)} = \frac{4}{3}$

Figure 18.5. Posing a problem through investigation: Looking beyond the figure.

As noted above, discovery of additional nontrivial properties is associated with auxiliary constructions "outside the triangle" (Figure 18.5). The participants agreed that most of these properties were surprising and that *surprise is one of the special characteristics of a nontrivial discovery*. The PMTs found property 5*h*: *CAME* is a parallelogram, to be the most surprising. They were asked to prove all the discovered nontrivial properties (see Appendix A for proof that *CAME* is a parallelogram). Note here that problem 5*h* can be considered as *posed through construction* since property "*CAME* is a parallelogram" was discovered accidentally when line *ET* was drawn (Auxilliary construction B in Figure 18.5).

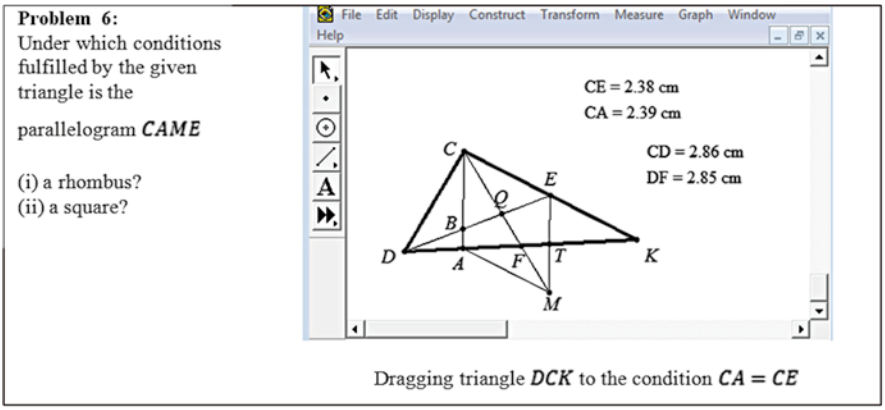


Figure 18.6. Transforming Problem 5*h* into a discovery problem.

Overall, about 20 nontrivial properties were discovered by the participants; thus, in this way about 20 new *problems were posed through investigations*. The richness of the collective spaces of the nontrivial discovered properties and thus of the problem-posing space was almost shocking for the participants. While they doubted that each participant alone can discover such a rich collection of properties however, they arrived at the conclusion that "collaborative work is essential in order to discover many properties" and that the collective space of discovered properties serves as a source for the development of their problem-posing and problem-solving expertise.

Back to Problem Posing for Investigation

The discovery that *CAME* is a parallelogram (property 5*h* in Figure 18.5) led one of the PMTs to pose a new investigation problem (Problem 6 in Figure 18.6).

This investigation problem differs significantly from Problem 3B (Figure 18.3) posed for investigation previously. While both problems are open and belong to the category of discovery problems, Problem 3B is unfocused and allows solvers to search for all possible invariants. In contrast, Problem 6 directs solvers to discover special conditions of the given figure that are sufficient for the nearly constructed parallelogram to be a rhombus (or a square). Problem 6 is posed based on Problem 5*h* by the combination of two problem-transformation strategies: symmetry changes (when goals and givens are interchanged) and the "what if yes?" strategy (Leikin & Grossman, 2013). Last but not least important, Problem 6 requires *correction investigation*.

Back to Problem Posing Through Investigation

In contrast to figure investigation performed in a DGE for Problem 3B that was directed at searching for robust constructions (Figures. 18.4 and 18.5), the investigation related to Problem 6 was performed by correction strategy, in which triangle *DCK* was dragged to obtain the drawing of a rhombus (a square) from the parallelogram *ACEM*. In this way, by dragging the triangle to a state in which in parallelogram *ACEM* sides *CA* and *CE* are equal (*ACEM* becomes a rhombus), the participants conjectured that $CD=DE$; in other words $DK=2CD$ (Figure 18.6) is based on the repeated observation of the properties in "corrected drawing." This strategy did not allow for "exact" measuring but did allow for raising the conjecture based on the repeating properties in the corrected situations.

Investigation related to Problem 6 was also performed (with the instructor's guidance) with robust constructions by searching for properties that are immune to dragging. One of the robust constructions started out with the construction of a rhombus/a square and the consequent construction of the triangle *DCK* so that segments *CF* and *CA* will be medians in triangle *DCK* and triangle *DCK* respectively (see the diagrams for Problems 7*A* and 7*B* in Figure 18.7). In this way participants posed Problem 7a: "If rhombus *CAME* is given and triangle *DCK* is constructed so that *DK* intersects *EM* at the midpoint *T* on *EM*, *F* (intersection of *DK* and *CM*) is a midpoint on *DK*, *A* is a midpoint on *DF*, then $DK=2DC$." When the rhombus is a square (Problem 7b) then angle *CDA* is 36.87°.

Problems 7*a* and 7*b* are nontrivial ones with complex proofs (see Appendix B). These problems and the investigations (Figure 18.7) are associated with necessary

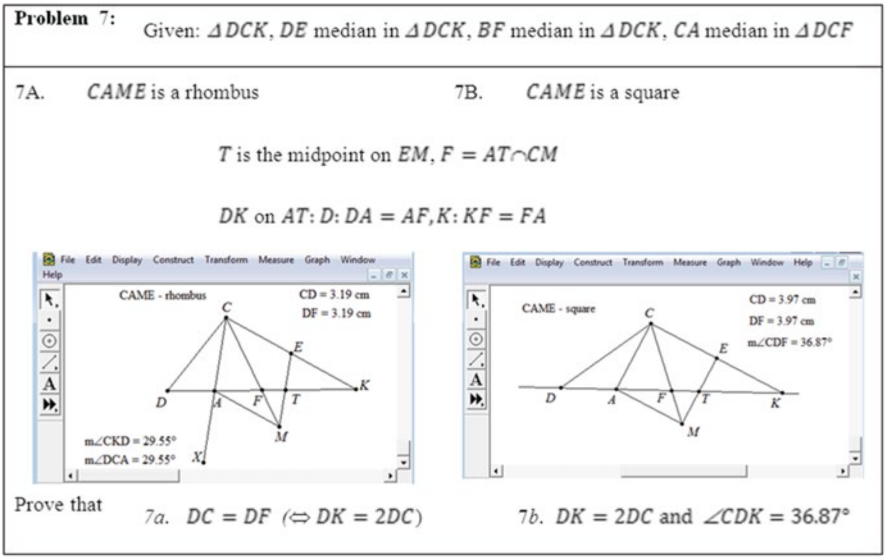
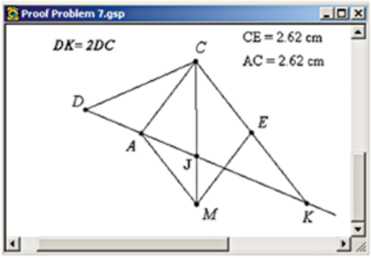


Figure 18.7. Problem posing through investigation: Focusing on new givens and goals.

Problem 8: Given:

ΔDCK, $DK = 2DC = 4DA$, $CE = EK$, $ACEM$ – parallelogram, $J = DK \cap CM$



8*a* What can be said about quadrilateral $ACEM$?

8*b* Prove: $DJ = JK$

Figure 18.8. Problem 8*a* is an inverse problem to Problem 7*a*.

conditions that the triangle should satisfy for *CAME* to be either a rhombus or a square. As an alternative, PMTs suggested investigating Problem 8, which was an inverse problem to Problem 7*a*. In this case the construction started with a triangle *DCK* in which $DK = 2DC$ and resulted with a verification that *ACEM* is a rhombus (Figure 18.8). Interestingly, the PMTs found this problem better connected to Problem 1 "since the triangle in this problem is given and the proof focuses on the properties of the quadrilateral."

Concluding Comments

In this chapter I have demonstrated the power of investigations in DGEs as an effective problem-posing tool. Problem posing in mathematics is one of the central mathematical tasks directed at the development of mathematical knowledge and creativity. Not less importantly, problem posing is an important pedagogical skill that enhances teachers' proficiency and makes teaching more flexible. This chapter has presented three types of problem-posing acts associated with geometry investigations: (a) problem posing through proving; (b) problem posing for investigation; and (c) problem posing through investigation. These three types of problem posing are mutually dependent and interrelated (see Figure 18.9).

The PMTs who participated in the activity described in this chapter were encouraged to perform geometry investigations of this type during a 56-hour course. Throughout the course their competencies developed gradually, and by the end of the course PMTs were able to design activities of this kind for their peers (see Appendix C "PMTs' posed problems" in support of this finding). The participants expressed their willingness to "teach their students in a similar way," though

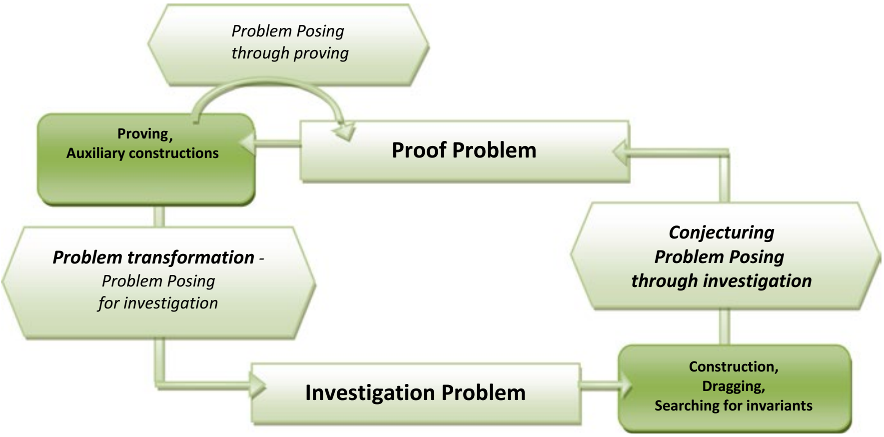


Figure 18.9. Problem-posing types associated with investigations in DGE.

(not surprisingly) they were skeptical whether, under the pressure of meeting the demands of the school mathematics curriculum, these activities could be implemented systematically in a regular mathematics classroom. The contrast between PMTs' enjoyment from coping with investigation problems, problem posing for and through investigation and their uncertainty with regard to the usefulness of similar activities in the classroom setting is rooted in the stable nature of teachers' beliefs (Cooney, Shealy, & Arvold, 1998) and the "conviction loop": "To implement new pedagogical approaches, teachers must be convinced of the suitability of those approaches in their work with students and, at the same time, to be convinced of the suitability of those approaches they have to implement them in school" (Leikin, 2008, p. 80). I suggest that, in order to break the conviction loop, PMTs should be assigned to implement geometry investigations with individual students or with classes during their school practicum.

In my view, the majority of proof problems from school textbooks, when opened for investigations and formulated as discovery problems, lead to doing mathematics rich in surprises, discoveries, and proofs. At the same time, finding sufficiently rich examples to support the emergence of a variety of ways of problem posing is critical for effective work with PMTs and school students. Therefore, teacher educators and mathematics teachers should execute a critical choice of the tasks for their learners.

The PMTs were astonished by the number of new problems formulated during the session described in this chapter. This type of activities led them to the conclusion that "through investigations in a DGE, a teacher can solve multiple problems related to one particular geometric object and prepare more interesting lessons for his/her students." Students and teachers involved in the real doing of mathematics find that they enjoy mathematical discovery at the level which is appropriate to their own abilities.

Appendix A

Proof for Problem *5h* (Figure 18.5)

(1) $ET \parallel CA$ thus triangles *CBQ* and *MEQ* are similar;

(2) $\frac{QE}{BQ} = \frac{5}{4}$ (2*b*, Figure 18.4);

(3) From (1) and (2) $\frac{EM}{BC} = \frac{5}{4}$;

(4) $\frac{AC}{BC} = \frac{5}{4}$ (*1b*, Figure 18.4) thus $CA = EM$.

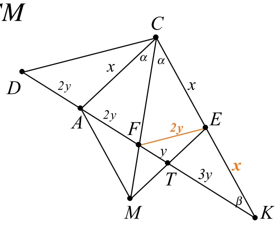
(5) Hence $EM \parallel CA$ and $EM = CA$; that is *CEMA* is a parallelogram.

Appendix B

Proof for Problems 7*a*, 7*b* (Figure 18.7)

Construction outline:

CAME is a rhombus, *T*-midpoint on *EM*, $F = AT \cap CM$
DK on $AT : D : DA = AF$, $K : KF = FA$



Prove:
7a. $DC = DF \left(\Leftrightarrow DK = 2DC\right)$
7b. *CAME* is a square when $\angle CDK = 36.9°$
7c. K' on *CE*: $K'E = EC \Leftrightarrow K'$ coincides with *K*

Proof

1. According to the construction: $AC = CE = EM = AM = x$,
   $ET = TM = \frac{1}{2}x$, $FA = AD = 2y$, $FK = DF = 4y \Rightarrow DK = 8y$;
2. *CAME* is a rhombus $\Rightarrow \Delta CAF \cong \Delta CEF \Rightarrow FE = 2y$;
3. *MF*-bisects angle AMT, $\mathrm{AM} = 2\mathrm{MT} \Rightarrow \mathrm{AF} = \mathrm{FT}$; $\mathrm{FT} = \mathrm{y}$; $\mathrm{TK} = 3\mathrm{y}$
4. $\Delta TEK \cong \Delta TMA$; AT = TK, ME $\parallel$ CA $\Rightarrow TE$ midline on $\Delta ACK \Rightarrow EK = CE$ (18.7c)
5. *TE* midline on $\Delta ACK \Rightarrow CD = 2EF \Rightarrow DC = 4y$
   $\Rightarrow DC = AF \left(\Leftrightarrow DK = 2DC\right)$ (18.7a)
6. $\Delta FEK \sim \Delta DCK \sim \Delta DAC \Rightarrow \angle DCA = \angle CKD$
7. If *CAME* is a square $\alpha = 45°$; $\tan\beta = \frac{1}{2} \Leftrightarrow \beta = 26.57° \Leftrightarrow \angle CDK$
   $= 71.57° \angle CDK = 36.9°$ (18.7b)

Appendix C

Problems posed for and through investigations by a PMT who participated in the study

Rasha's Problem

Initial problem: *Midline in a triangle*
The segment connecting the midpoints of two sides of a triangle is parallel to the third side and is half as long. (Given: $\Delta ABC,\ AE = EB,\ AP = PC$; Prove: $EP \parallel BC,\ EP = \frac{1}{2}BC$)

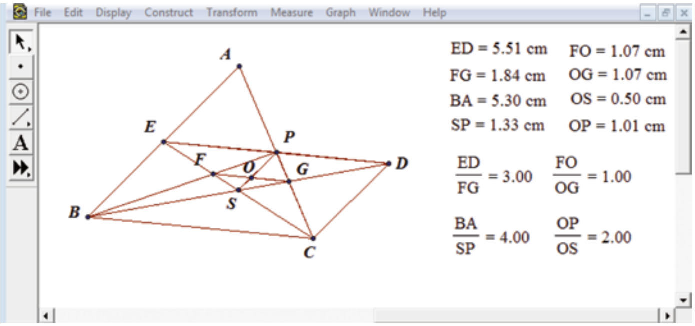
Posed problems:
Given:

$$\Delta ABC,\ AE = EB,\ AP = PC$$

PD is a continuation of EP, $ED = 2EP,\ F = EC \cap BP,\ G = BD \cap AC,$
$S = EC \cap BD,\ O = FG \cap SP$

Prove:

$$\frac{ED}{FG} = 3;\ \frac{BA}{SP} = 4,\ FO = OG,\ OP = 2OS$$



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