Chapter 1
Problem-Posing Research in Mathematics Education: Some Answered and Unanswered Questions

Jinfa Cai, Stephen Hwang, Chunlian Jiang, and Steven Silber

Abstract This chapter synthesizes the current state of knowledge in problem-posing research and suggests questions and directions for future study. We discuss ten questions representing rich areas for problem-posing research: (a) Why is problem posing important in school mathematics? (b) Are teachers and students capable of posing important mathematical problems? (c) Can students and teachers be effectively trained to pose high-quality problems? (d) What do we know about the cognitive processes of problem posing? (e) How are problem-posing skills related to problem-solving skills? (f) Is it feasible to use problem posing as a measure of creativity and mathematical learning outcomes? (g) How are problem-posing activities included in mathematics curricula? (h) What does a classroom look like when students engage in problem-posing activities? (i) How can technology be used in problem-posing activities? (j) What do we know about the impact of engaging in problem-posing activities on student outcomes?

Contents

J. Cai (✉) • S. Hwang • S. Silber
Department of Mathematical Sciences, University of Delaware, Newark, DE, USA
e-mail: jcai@udel.edu; hwangste@udel.edu; spsilber@udel.edu

C. Jiang
Faculty of Education, University of Macau, Macau, China
e-mail: cljiang@umac.mo

F.M. Singer et al. (eds.), *Mathematical Problem Posing*,
Research in Mathematics Education, DOI 10.1007/978-1-4614-6258-3_1

Introduction

There is a long history of integrating mathematical problem solving into school curricula (Stanic & Kilpatrick, 1988). In the past several decades, there have been significant advances in the understanding of the affective, cognitive, and metacognitive aspects of problem solving in mathematics and other disciplines (e.g., Cai, 2003; Frensch & Funke, 1995; Lester, 1994; McLeod & Adams, 1989; Schoenfeld, 1985, 1992; Silver, 1985). In contrast, problem-posing research is a relatively new endeavor (Brown & Walter, 1993; Kilpatrick, 1987; Silver, 1994). Nevertheless, there have been efforts to incorporate problem posing into school mathematics at different educational levels around the world (e.g., Chinese National Ministry of Education, Office of School Teaching Materials and Institute of Curriculum and Teaching Materials, 1986; Hashimoto, 1987; Healy, 1993; Keil, 1964/1967; Ministry of Education of China, 2011; National Council of Teachers of Mathematics (NCTM), 1989; van den Brink, 1987). These efforts indicate interest among many practitioners in making problem posing a more prominent feature of classroom instruction.

Despite the interest in integrating mathematical problem posing into classroom practice, our knowledge remains relatively limited about the cognitive processes involved when solvers generate their own problems, the instructional strategies that can effectively promote productive problem posing, and the effectiveness of engaging students in problem-posing activities. In the discussion below, we synthesize the current state of knowledge in problem-posing research and suggest some directions for future study. In particular, we discuss the following questions:

1. Why is problem posing important in school mathematics?
2. Are teachers and students capable of posing important mathematical problems?
3. Can students and teachers be effectively trained to pose high-quality problems?
4. What do we know about the cognitive processes of problem posing?
5. How are problem-posing skills related to problem-solving skills?
6. Is it feasible to use problem posing as a measure of creativity and mathematical learning outcomes?
7. How are problem-posing activities included in mathematics curricula?
8. What does a classroom look like when students engage in problem-posing activities?
9. How can technology be used in problem-posing activities?
10. What do we know about the impact of engaging students in problem-posing activities on student outcomes?

Each of these questions represents a rich area for problem-posing research. As we explore each question, we begin by examining the work that has been done and by summarizing what we know as a field. We then consider, for each overarching question, some related questions that remain unanswered and which we feel merit further attention from the research community.

Why is Problem Posing Important in School Mathematics?

Problem posing has long been recognized as a critically important intellectual activity in scientific investigation. According to Einstein, the formulation of an interesting problem is often more important than its solution (Einstein & Infeld, 1938). However, whereas the case for problem solving in school mathematics has seemed relatively clear, the importance of problem posing in school mathematics has required slightly more explanation. As we noted above, problem solving has long been a fundamental part of mathematics education (Stanic & Kilpatrick, 1988). Although 30 years ago Getzels (1979) lamented that, compared to problem solving, problem posing was a neglected area of research, in recent years both educators and researchers have begun to give problem posing concerted attention.

Kilpatrick (1987) observed that in real life, problems must often be created or discovered by the solver. Thus, the onus of noticing a problem and subsequently framing it in a productive way is squarely on the solver. Indeed, in his analysis of invention in mathematics, the mathematician Jacques Hadamard (1945) considered the identification and posing of good problems to be an important part of doing high-quality mathematics. Thus, if a goal of education is to prepare students for the kinds of thinking they will need, it seems reasonable that problem posing should be an important part of the curriculum. Moreover, approaches to mathematics instruction that attempt to engage students in experiences that are more authentic to inquiry within the discipline of mathematics should provide students with opportunities to explore, make conjectures, and pose meaningful problems (Bonotto, 2013).

Problem posing is also a critical aspect of the work of teachers, both in posing problems for students and in helping students develop into better problem posers (Crespo, 2003; Olson & Knott, 2013). Teachers regularly must formulate and pose worthwhile problems for their students, even when they are working with problems given in curriculum materials (NCTM, 1991). The problems that a teacher poses can shape the mathematical learning in their classes and "further their mathematical goals for the class" (NCTM, 2000, p. 53). In addition, teachers can use problem-posing tasks to gain greater insight into their students' understandings of mathematics (Cai et al., in press; Kotsopoulos & Cordy, 2009; Leung, 2013; Silver, 1994).

As we will discuss in greater depth below, the theoretical arguments supporting the importance of problem posing in school mathematics are bolstered by a growing body of empirical evidence. Researchers are actively exploring links between problem posing and other aspects of mathematical ability including conceptual understanding, problem solving, and creativity (e.g., Cai et al., in press; Cai & Hwang, 2002; Ellerton, 1986; Silver & Cai, 1996; Singer & Moscovici, 2008; Van Harpen & Sriraman, 2013). Given its potential to enhance the teaching and learning of mathematics, it is clear that problem posing is an important part of research and practice in school mathematics.

Are Teachers and Students Capable of Posing Important Mathematical Problems?

If we recognize problem posing as an important intellectual activity in school mathematics, then we must determine if teachers and students are capable of posing important and worthwhile mathematical problems. In fact, a fundamental line of research in problem posing has been exploring the kinds of problems that teachers and students can pose. In this line of research, researchers typically design a problem situation and ask subjects to pose problems which can be solved using the information given in the situation. Different types of problem situations have been used, some of which are knowledge-free and others of which are knowledge-rich (see Figure 1.1 for four examples of such problem situations). Some situations are quite structured (Situation 1), whereas others are relatively open (Situation 3). Stoyanova and Ellerton (1996) classified the degree of structure in problem situations as free, semi-structured, and structured.

Mathematical problem-posing research has explored the performance of school students, prospective teachers, and in-service teachers (e.g., Cai, 1998; Cai et al., in press; Cai & Hwang, 2002; Crespo, 2003; English, 1998; L. Ma, 1999; Silver & Cai, 2005; Stickles, 2011). In general, the findings have supported the claim that both

Situation 1. Children were to pose problems based on the following statements about Rufus the dog: Rufus managed to get into the Bradley house one afternoon. He chewed up four of Amy's shoes, three of her toys, and six of her socks. He also chewed up five of Brad's shoes, seven of his toys, and two of his socks. Mrs. Smith baked two dozen biscuits. Rufus made off with twelve biscuits. He buried eight of them before Mrs. Smith discovered him. (This situation was used for elementary school students in English, 1998.)

Situation 2. Ann has 34 marbles, Billy has 27 marbles, and Chris has 23 marbles. Write and solve as many problems as you can that use this information. (This situation was used for middle school students in Silver & Cai, 2005.)

Situation 3.

1

3 5

7 9 11

13 15 17 19

21 23 25 27 29

The pattern continues. I wanted to make up some problems that used this pattern for a group of high students/college freshmen. Help me by writing as many problems as you can in the space below. (This situation was used for prospective secondary mathematics teachers in Cai, 2012.)

Situation 4. Imagine that you are teaching division with fractions. To make this meaningful for kids, something that many teachers try to do is relate mathematics to other things. Sometimes they try to come up with real-world situations or story problems to show the application of some particular piece of content. What would you say would be a good story or model for $1\frac{3}{4} \div \frac{1}{2}$? (This situation was used for in-service elementary teachers in L. Ma, 1999.)

Figure 1.1. Four sample problem situations used in research on problem posing.

students and teachers are capable of posing interesting and important mathematical problems. For example, for Situation 2 in Figure 1.1, middle school students were able to pose problems such as the following (Silver & Cai, 2005):

How many marbles do they have altogether?
How many more marbles does Billy have than Chris?
How many more marbles would they need to have together to have as many marbles as Sammy, who has 103?
Can Ann give marbles to Billy and Chris so that they all have the same number? If so, how can this be done?

Suppose Billy gives some marbles to Chris. How many marbles should he give Chris in order for them to have the same number of marbles?
Suppose Ann gives some marbles to Chris. How many marbles should she give Chris in order for them to have the same number of marbles?

For Situation 3 in Figure 1.1, prospective secondary teachers were able to pose problems such as these (Cai, 2012):

What is the first number on the *n*th row?
What is the number on the *i*th row and *j*th column?
What is the last number on the *n*th row?
What is the sum of the numbers in the *n*th row?
How many numbers are there in the *n*th row?
What is the sum of the numbers in the first *n* rows?
What is the pattern of each of the numbers in each diagonal line?
What is the sum of $1^3+2^3+3^3+4^3+\cdots+(n-1)^3+n^3$?
What is the middle number in an odd row?

And, L. Ma (1999) found that in-service elementary teachers could pose problems in response to Situation 4 in Figure 1.1, such as:

Cut an apple into four pieces evenly. Get three pieces and put them together with a whole apple. Given that ½ apple will be a serving, how many servings can we get from the 1¾ apples?
A train goes back and forth between two stations. From Station A to Station B is uphill and from Station B back to Station A is downhill. The train takes 1¾ hours going from Station B to Station A. It is only ½ time of that from Station A to Station B. How long does the train take going from Station A to Station B?
Given that we paid 1¾ Yuan to buy ½ of a cake, how much would a whole cake cost?
We know that the area of a rectangle is the product of length and width. Let's say that the area of a rectangle board is 1¾ square meters, its width is ½ meters, what is its length?

However, the ability to pose valid problems appears to be connected to other factors. For example, in her comparison of US and Chinese elementary teachers' understanding of elementary mathematics, L. Ma (1999) found that the teachers' abilities to pose problems like the ones cited above for the given fraction division was associated with their understanding of the meaning of fraction division. The US teachers in her study were unable to produce appropriate problems, and their difficulties were rooted in their inadequate conceptions of fraction division. In contrast, the Chinese teachers were generally able to pose at least one problem for the given fraction division based on one of three understandings of the concept (measurement model, partitive model, factors, and product). Stickles (2011) also found that secondary and middle school teachers were capable of posing problems, but that their success was partial and was related to experience and background. Specifically, the teachers in her study were prolific problem posers when presented with a given set of information, but struggled with crafting valid problems. The teachers were more successful when reformulating problems that were given to them.

Unanswered Question 1

Even though research has shown that students and teachers are capable of posing interesting and important mathematical problems, researchers have also found that some students and teachers pose nonmathematical problems, unsolvable problems, and irrelevant problems (e.g., Cai & Hwang, 2002; Silver & Cai, 1996; Silver, Mamona-Downs, Leung, & Kenney, 1996). For example, Silver and Cai (1996) found that nearly 30% of problems posed by middle school students were either nonmathematical problems or simply nonproblem statements (even though the directions clearly asked for problems). This suggests the following question: Why do students pose nonmathematical, trivial, or otherwise suboptimal problems or statements? Crespo and Sinclair (2008) hypothesized that these difficulties might be related to a lack of opportunity for students to explore a problem situation adequately before and during the posing process. There is clearly a need to investigate how students and teachers interpret and parse problem situations when engaging in problem posing.

Unanswered Question 2

Researchers have used many different types of problem situations to investigate problem posing, ranging from simply deleting a question from a textbook problem to very open-ended problem situations. With respect to mathematical problem-solving research in the past several decades, researchers have explored the effects of various task variables on students' problem solving. For example, several classifications of task variables related to problem solving are considered in Goldin and McClintock (1984): syntax variables, content and context variables, structure variables, and heuristic behavior variables. Syntax variables are factors dealing with how problem statements are written. These are factors that may contribute to ease or difficulty in reading comprehension, such as problem length and numerical and symbolic forms within the problem. Content variables refer to the semantic elements of the problem, such as the mathematical topic or the field of application, whereas context variables refer to the problem representation and the format of information in the problem. Structure variables refer to factors involved in the solution process, such as problem complexity and factors relating to specific algorithms or solution strategies. Finally, heuristic process variables refer to the interactions between the mental operations of the problem solver and the task. Considering heuristic variables separately from subject variables (factors that differ between the individuals solving the problem) is difficult, as heuristic processes involve the problem solver interacting with the task. However, the interaction between heuristic processes and the other task variables can have a significant impact on problem-solving ability.

Less is known about how problem situations influence students' problem-posing responses. How do different characteristics of problem situations affect subjects' problem posing? Leung and Silver (1997) developed and analyzed a Test of Arithmetic Problem Posing (TAPP), which they then used to examine how the presence of numerical information impacted preservice teachers' problem-posing abilities. Results from the TAPP indicated that the preservice teachers performed better on problem-posing tasks that included specific numerical information than on tasks without specific numerical information. This result provides some insight into how task variables can impact problem posing, yet more research must be done on the impact of various other variables. Adapting the TAPP to examine how different characteristics of problem situations affects subjects' problem posing could offer a way to study the effect of other task variables.

Can Students and Teachers Be Effectively Trained to Pose High-Quality Problems?

Although students and teachers are able to pose problems, even when those problems are mathematically sound they are not always of high quality. Thus, some studies have addressed the question of how to improve the abilities of teachers and students to pose better problems. Researchers have noted the importance of opportunities for exploration of mathematical situations in developing students' problem-posing abilities. Crespo and Sinclair (2008) suggested that without the opportunity to explore the limits of the mathematical situation in which students are working, the students are limited in the types of problems they can pose. Similarly, Koichu and Kontorovich (2013) found that the successful prospective teachers in their study posed the most interesting problems when blending exploration and problem solving with their problem posing. It would appear that students are able to improve the breadth and level of challenge of the problems they pose when they have experience solving such problems, and are prompted by informal contexts such as pictures, which may leave more room for exploration, instead of formal symbolic contexts (Crespo, 2003; English, 1998).

Indeed, with respect to formal symbolic contexts, Isik and Kar (2012) identified several types of difficulties experienced by prospective elementary teachers when posing problems related to daily life situations that could be solved using given linear equations or systems of two linear equations. These included conceptual difficulties, such as incorrectly translating the meaning of mathematical operations in the equations into corresponding verbal problem statements or posing separate problems for each equation in a system, contextual difficulties, such as assigning unrealistic values to the unknowns, and violations of the conventions of word problems, such as using symbolic representations in the problems posed. These difficulties suggest that, in order to pose high-quality problems that are based in formal symbolic contexts, teachers will need to build their conceptual understanding of the underlying mathematics (L. Ma, 1999) and their pedagogical understandings.

Some researchers have explored the characteristics of practice in the discipline of mathematics in order to identify and propose various collections of strategies to facilitate high-quality problem posing. Contreras (2007) discussed how to use the "fundamental mathematical processes" (p. 16) of proving, reversing, specializing, generalizing, and extending to pose new problems from a given problem. Moore-Russo and Weiss (2011) similarly described how to apply five "generative moves" that mathematicians use in determining what could be done next to spawn new, related geometry problems from an existing problem under consideration. The five generative moves (strengthening/weakening hypothesis, strengthening/weakening conclusion, generalize, specialize, consider converse) are consonant with the processes described by Contreras.

Unanswered Question 3

It would appear to be feasible to improve the quality of problems that students and teachers pose. Existing research suggests that strategies matter in how we train students and teachers to pose problems. However, it is not clear which strategies are most effective for teaching problem posing, nor is it clear which strategies are best for problem posers to use in particular problem situations. Further exploration of these strategies and their productiveness for problem posing in different mathematical situations is warranted. What strategies and ways of thinking are most productive for posing problems, and under what types of mathematical situations are different strategies effective?

What Do We Know About the Cognitive Processes of Problem Posing?

There are many potential processes involved in posing problems, and they may vary depending on the type of problem posing under consideration. These can involve techniques for reformulating existing problems, heuristics, or strategies for generating problems from given situations, and processes for exploring a mathematical context and testing its boundaries to develop a "feel" for the kinds of questions that can be asked. Researchers have worked to gain better understandings of these processes and to document the kinds of strategies that are used in problem posing.

In their study of middle school students' problem posing, Silver and Cai (1996) found that many students produced responses that consisted of a series of related problems, often generated by varying a single element, and that the complexity of the problems tended to increase within a series. Their results suggested that there were distinct processes that guided (and perhaps constrained) the students' problem posing. English (1998) observed that third graders' ability to pose multiple problems appeared limited to tinkering with the contexts of an original problem. Cai and Hwang (2002) suggested a potential parallel between students' thinking when posing and solving problems. Specifically, they observed that the sequence of pattern-based

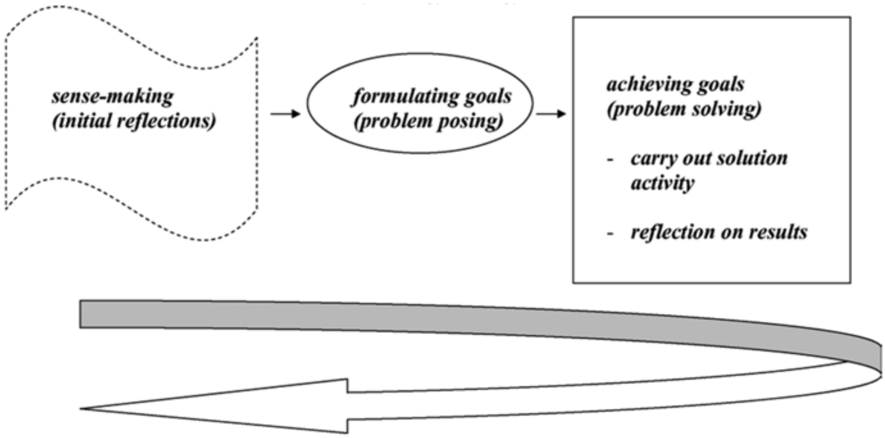

Figure 1.2. The recursive process of problem posing and solving proposed by Cifarelli and Cai (2005).

problems posed by students appeared to reflect a common sequence of thought when solving pattern problems (gathering data, analyzing the data for trends, making predictions). Thus, students might have a solution process in mind when thinking about posing problems.

Cai and Cifarelli (2005; Cifarelli & Cai, 2005) further refined this link between problem solving and problem posing, describing a recursive process of chains of solving and posing (Figure 1.2). Cai and Cifarelli (2005) examined how two college students posed and solved their own problems in an open-ended computer simulation task that involved the path of a billiard ball. They identified two different levels of reasoning strategies—hypothesis-driven and data-driven—that students appeared to incorporate in their posing and solving processes. They observed that problem solvers' self-generated questions reframed the problems they were working on and significantly changed the strategies they were using. Therefore, Cai and Cifarelli considered the posing and solving process to be mathematical exploration. Indeed, in a follow-up study, Cifarelli and Cai (2005) described mathematical exploration as structured by this recursive process. This cycling and entwining of posing and solving corresponds with the observations of Christou, Mousoulides, Pittalis, and Pitta-Pantazi (2005) about prospective teachers' use of dynamic geometry software to solve problems. Christou et al. found that the dynamic geometry software acted as a mediation tool that supported the processes of modelling, conjecturing, experimenting, and generalizing. In using the software to explore problem situations and extract meaning from them, the prospective teachers generated new problems as part of their problem-solving processes. For example, in their explorations of the figure formed by the bisectors of the interior angles of a parallelogram, the prospective teachers engaged in problem posing through experimenting with special cases (e.g., a rectangle) and making and checking conjectures based on the evidence they were gathering.

Pittalis, Christou, Mousoulides, and Pitta-Pantazi (2004) proposed a model of cognitive processes involved in problem posing. The model encompasses four processes: filtering quantitative information, translating quantitative information from one form to another, comprehending, and organizing quantitative information by giving it meaning or creating relations between provided information, and editing quantitative information from the given stimuli. Based on empirical testing, Pittalis et al. asserted that these processes correspond to different types of problem-posing tasks, and that the filtering and editing processes were most important in posing problems.

Christou, Mousoulides, Pittalis, Pitta-Pantazi, and Sriraman (2005) built on this model to develop a taxonomy of problem-posing processes related to different types of tasks. Tested with 143 sixth graders from Cyprus, their taxonomy also includes four processes. Tasks that involve posing problems from situations without restrictions involve the process of editing quantitative information. Tasks that involve posing problems that have specified answers involve the process of selecting quantitative information. Tasks that require students to pose problems corresponding to given equations or computations involve the process of comprehending and organizing quantitative information. And, tasks that involve posing problems from given graphs, diagrams, or tables involve the process of translating quantitative information from one form to another. Based on this model, the researchers found that students were more successful when first posing problems involving comprehending, then translation, and finally editing and selecting.

Although theories of the cognitive processes of problem posing are relatively new, there is a longer history of attention to strategies that may be useful in posing problems. Building on Polya's "looking back" stage in problem solving, Brown and Walter (1990) proposed the well-known "What if not" strategy. Along the same lines, Abu-Elwan (2002) and Cai and Brook (2006) suggested posing problems through a process of extending or generalizing an already-solved problem. Indeed, Gonzales (1998) even referred to this process as a fifth step to Polya's four-step method. Lavy and Bershadsky (2003) described the use of the "What if not" strategy for mathematical problem posing, dividing the activity into two stages. In the first stage, all the attributes included in the statement of the original problem are listed. In the second stage, each of the listed attributes is negated by asking "what if not attribute *k*?" and alternatives are proposed. Each of the offered alternatives creates a new problem situation.

Unanswered Question 4

Although we know that students and teachers are capable of posing mathematical problems, we have a considerably less fine-grained understanding of how they go about posing those mathematical problems in any given situation. Some researchers have identified general strategies students may use to pose problems. Others have explored some of the variables that may have an impact on students' problem posing. However, there is not yet a general problem-posing analogue to well-established general frameworks for problem solving such as Polya's (1957) four steps. Much more research is needed to develop a broadly applicable understanding of the fundamental processes and strategies of problem posing.

Unanswered Question 5

A better understanding of the cognitive processes of problem posing can also inform teaching. Ideally, the more that teachers know about their students' thinking, the better equipped they are to help their students develop (Cai, 2005). However, there is much work needed to connect research-based understandings of student cognition to teachers' practice. Much as Cognitively Guided Instruction (CGI) has provided a theoretical and empirical framework that has helped teachers understand their students' mathematical thinking and problem solving (Carpenter, Fennema, & Franke, 1996; Fennema et al., 1996), research that illuminates cognitive models of students' problem posing has the potential to improve teaching. In that vein, we ask the following question: How can an understanding of students' problem-posing cognition help teachers to improve student learning?

How Are Problem-Posing Skills Related to Problem-Solving Skills?

One important direction for research on problem posing is probing the links between problem posing and problem solving (see, e.g., Cai, 1998; Cai & Hwang, 2002; Ellerton, 1986; Kilpatrick, 1987; Silver & Cai, 1996). Kilpatrick (1987) provided a theoretical argument that the quality of the problems subjects pose might serve as an index of how well they can solve problems. In addition to this theoretical argument, several researchers have conducted empirical studies examining potential connections between problem posing and problem solving. Ellerton (1986) compared the mathematical problems generated by eight high-ability young children with those generated by eight low-ability young children, asking each to pose a mathematical problem that would be quite difficult for his or her friends to solve. Ellerton reported that the more able students posed problems that were more complex than those posed by less able students.

Silver and Cai (1996) analyzed the responses of more than 500 middle school students to a task that asked them to pose three questions based on a driving situation. The student-posed problems were analyzed according to their type, solvability, and complexity. In addition, Silver and Cai used eight open-ended tasks to measure the students' mathematical problem-solving performance. They found that problem-solving performance was highly correlated with problem-posing performance. Compared to less successful problem solvers, good problem solvers generated more, and more complex, mathematical problems.

Silver and Cai (1996) measured problem-solving performance using tasks that were rarely related to the problem-posing tasks. In other studies, Cai and his associates (Cai, 1998; Cai & Hwang, 2002) examined Chinese and US students' problem-solving and problem-posing performances using closely related problem-posing and problem-solving tasks. Cai and Hwang (2002) found differential relationships between posing and solving for US and Chinese sixth-grade students. There was a stronger link between problem solving and problem posing for the Chinese sample, whereas the link was much weaker for the US sample. Posing a variety of problem types appeared to be strongly associated with abstract strategy use in the Chinese sample. Cai and Hwang indicated that the differential nature of the relationships for US and Chinese students should not be interpreted as implying a lack of generality in the link between problem solving and problem posing. Rather, the stronger link between the variety of posed problems and problem-solving success for the Chinese sample could be attributable to the fact that the US students almost never used abstract strategies. Indeed, in a follow-up analysis that included data from seventh-grade US students, Cai and Hwang (2003) identified a corresponding link between the students' use of abstract problem-solving strategies and their ability to pose problems that extended beyond the given information.

Unanswered Question 6

Cross-national and cross-regional comparative studies provide unique opportunities to understand students' mathematical thinking and reasoning. Although there is a large body of cross-national studies of mathematical problem solving, there have been few attempts to use problem posing in such cross-national studies (e.g., Cai, 1998; Cai & Hwang, 2002; Yuan & Sriraman, 2011). How do students in different countries and regions pose mathematical problems? Observations of differences in problem posing across regions, such as in the study of Cai and Hwang, may provide fertile ground for further research. Analyzing, for example, differences in the magnitude of the relationship between problem solving and problem posing for students from different regions, may offer insights into the nature of the relationship. Indeed, in their analysis of problem posing among students from the United States and from two distinct regions of China, Van Harpen and Sriraman (2013) have found differences that suggest a strong link between mathematical knowledge and problem-posing success. In the future, we hope that more researchers around the world will engage in mathematical problem-posing research in cross-cultural contexts.

Is it Feasible to Use Problem Posing as a Measure of Creativity and Mathematical Learning Outcomes?

Student outcomes in mathematics classes are typically assessed by having the students solve problems. However, as noted above, researchers have found that students' success in problem solving is associated with their problem-posing abilities

(Cai et al., in press; Cai & Hwang, 2002; Silver & Cai, 1996). Moreover, there is some evidence that asking students to pose problems may provide additional useful insights into what mathematics students have learned and what students have learned about doing mathematics. For example, in a study of prospective elementary teachers' conceptual understanding of fractions, Tichá and Hošpesová (2013) used problem posing as a diagnostic tool to gauge the prospective teachers' understanding. By analyzing the problems that the prospective teachers posed, Tichá and Hošpesová were able to identify conceptual flaws and confusion that needed to be addressed. Similarly, Kotsopoulos and Cordy (2009) made use of their seventh-grade students' journal records of problem posing as a type of formative assessment to gauge the progress the students were making. This allowed these teacher-researchers to determine whether they "were on-track with our learning objectives for the four experiments" (p. 272).

As part of a large-scale study, Cai et al. (in press) investigated the feasibility of using problem posing to measure curricular effects on student learning. In particular, they compared the effects of a *Standards*-based middle school mathematics curriculum with those of more traditional curricula on students' algebra learning. Using parallel problem-solving and problem-posing tasks, they confirmed the association between students' abilities to solve and pose problems, and found that this relationship held for students using both types of curriculum. In addition, by using qualitative rubrics to assess different characteristics of students' responses, Cai and his colleagues found that students whose posed problems exhibited positive characteristics (such as reflecting the linearity of a given graph in their posed problem or embedding their posed problems in real-life contexts) were also strong problem solvers. However, student performance in general was poor on the problem-posing tasks in this study, suggesting that the students might need more experience with problem posing in order to have broader success on posing-oriented measures.

Given the generative qualities of problem posing, one might expect that problem-posing activities might be valid measures of students' creativity. Indeed, Silver (1997) has proposed a relationship between engaging students in problem posing and their development of creative fluency, flexibility, and novelty. Studying elementary children in Taiwan, Leung (1997) developed an 18-task instrument that was useful in measuring the students' general problem-posing competence as well as in highlighting their creative problem posing. Similarly, Van Harpen and Sriraman (2013) used a problem-posing test to examine US and Chinese high school students' problem-posing creativity along the three dimensions of fluency, flexibility, and novelty. Generally, performance on such tests has revealed weaknesses in problem posing. However, Voica and Singer (2012) have suggested that there are important nuances in the relationship between problem posing and creativity. Specifically, in their study of fourth to sixth graders' modifications to problems, they found that students who stayed close to the given problem's context displayed deeper understanding of the mathematics than those who posed modified problems that were ostensibly more creative because they strayed further from the original. Nevertheless, Voica and Singer (2013) have found that, with sufficiently careful analysis of students' cognition during problem modification, problem posing can provide useful evidence of students' cognitive flexibility.

Despite the theoretical feasibility of using problem-posing tasks as measures of student outcomes, it seems that students will need further experiences and preparation in order for problem-posing measures to provide the most useful information. The low levels of success students display may be due to a general lack of experience with problem-posing tasks. In addition, Crespo and Sinclair (2008) emphasize the need for students to develop aesthetic criteria for judging the mathematical quality of posed problems. The development of such criteria and the disposition to apply them may also be part of the experiences prerequisite for problem posing to be practically feasible as an outcome measure.

Unanswered Question 7

Given the potential for problem-posing tasks to be used as measures of creativity and other mathematical learning outcomes, it is incumbent on the mathematics education research community to develop and validate suitable problem-posing instruments. What kinds of problem-posing tasks best reveal students' creativity and their mathematical understandings and misunderstandings? Given the results of the LieCal problem-posing assessment (Cai et al., in press), in order for problem-posing measures to provide useful information, it will also be important for researchers to investigate the kinds of preparation students will need to perform adequately on them.

How Are Problem-Posing Activities Included in Mathematics Curricula?

If problem-posing activities are to play a more central role in classrooms, they must be more prominently represented in curricula. As noted above, researchers have adapted several kinds of materials in order to generate problem-posing situations for research purposes. Similarly, if teachers are to engage students in problem posing in the classroom, they must have sources for problem-posing activities. Such sources may be supplements to curricula, as in the case of the materials developed by Lu and Wang (2006). Lu and Wang and their associates (Lu & Wang, 2006; Wang & Lu, 2000) launched a project focused on developing and implementing a set of teaching materials about mathematical situations and problem-posing tasks. The teaching materials, including mathematical situations and problem-posing tasks, were not intended to replace textbooks; instead, they were used to supplement regular textbook problems. By 2006, more than 300 schools in ten provinces in China had participated in the project. Teachers received training to use mathematical situations and problem-posing tasks along with their regular curriculum.

However, education reform movements have also recommended that problem-posing activities be included in mathematics curricula themselves. Internationally, school mathematics reforms have recommended that students be able to “formulate interesting problems based on a wide variety of situations, both within and outside of mathematics” (NCTM, 2000) and that instructional activities should emphasize learning problem-posing skills. In the United States, the NCTM Principles and Standards for School Mathematics (2000) emphasized the use of problem generation activities, where problems are “posed out of a situation or experience” (Stickles, 2011).

Similarly, reforms to curriculum standards in China have increased the prominence of problem posing. The 9-year compulsory education mathematics curriculum standards call for providing students opportunities to pose problems, understand problems, and apply the knowledge and skills learned to solve real-life problems (Basic Education Curriculum Material Development Center, Chinese Ministry of Education, 2003). Similarly, the curriculum standards for senior high school mathematics also call for developing students’ abilities to pose, analyze, and solve problems from mathematics and real life (Basic Education Curriculum Material Development Center, Chinese Ministry of Education, 2003). Indeed, in the reform standards, students are encouraged to discover and pose problems in order to prepare them to think independently and be inquirers.

However, the implications for the inclusion of problem posing in the curriculum are not necessarily clear. Ellerton (2013) has pointed out that although the Common Core State Standards—currently the most widely accepted US standards—call for problem-posing activities to be included in mathematics curricula, primarily the emphasis has been on problem-solving activities. In the Common Core State Standards, problem-posing activities are explicitly mentioned once (National Governors Association Center for Best Practices, 2010, p. 7), whereas problem solving is explicitly stated throughout the standards. The Common Core State Standards do recommend emphasizing the ability to “recognize and describe situations” for third-, fifth-, sixth-, and seventh-grade mathematics, which can be interpreted as problem posing (National Governors Association Center for Best Practices, 2010), but do not provide any recommendations on how to incorporate such activities into teaching plans (Ellerton, 2013).

This ambivalence is reflected in the available research on problem posing and curricula. Although reform movements have called for problem-posing activities to be included in mathematics curricula, there has not yet been a substantial body of research examining whether and how curricula incorporate problem posing. There is some evidence that more recent versions of textbooks emphasize problem posing more than previous versions. For example, an analysis of all problem-posing tasks in two editions of the Chinese elementary mathematics textbook series published by the People’s Education Press found that between the 1994 edition and the 2004 edition, there was an increase in the percentage of problem-posing tasks (Cai, Jiang, Hwang, Nie, & Hu, in press). Notably, this problem-posing increase appears to have been related to an accompanying increase in curricular focus on data and statistics.

Unanswered Question 8

The lack of a robust body of research in this area leads us to call for greater attention to the textbooks that students and teachers actually use, not merely to the curriculum frameworks on which those textbooks are based (Cai et al., in press). How do the actual textbooks include problem posing? There are many ways to include problem posing, and it is not clear what choices textbook writers and curriculum developers have made in creating the existing materials. Given the emphasis on mathematical modelling in current curriculum frameworks, it would be helpful in particular to know what role problem posing might play in mathematical modelling tasks in textbooks.

Unanswered Question 9

If curriculum designers intend to integrate problem posing into textbooks and teaching materials, what are the best ways to do so? In the analysis of Chinese elementary mathematics textbooks mentioned above, Cai and his colleagues gave special attention to three types of problem-posing tasks: tasks which included a sample problem, tasks that required students to pose problems corresponding to given operations, and tasks that required students to pose problems based on data charts (Cai et al., in press). They found significant differences with respect to these types of tasks between the 1994 and 2004 editions of textbooks. However, it is not clear whether these shifts are reflective of an attempt to utilize problem posing more effectively in the curriculum, and if so, what criteria were used to make those judgments. Further work is needed to understand the effectiveness of different ways of building problem posing into curricula.

What Does a Classroom Look Like When Students Engage in Problem-Posing Activities?

Even when problem posing is included in textbooks and curriculum materials, there remains the significant work of implementation in actual classrooms. Classrooms are complex by their very nature, with students and teachers establishing patterns of practice and norms that can influence student learning (Boaler, 2003; Yackel & Cobb, 1996). Indeed, Crespo and Sinclair (2008) have pointed out that classroom activity around problem posing will involve the negotiation of sociomathematical norms, such as in determining criteria for what counts as a mathematically interesting problem. Researchers must therefore consider how the intended curriculum is realized by teachers and students and what factors influence implementation (Ball & Cohen, 1996; National Research Council, 2004). With respect to understanding how problem posing can be enacted in classrooms, there is a need for both theoretical frameworks and careful analyses of practice.

To that end, Ellerton (2013) has proposed an Active Learning Framework that situates the processes of problem posing in the broader processes of mathematics classrooms. Arranged along a spectrum from passive student processes to active student processes, Ellerton's framework suggests that classrooms that do not include problem posing, stopping instead at problem solving, cut short students' mathematical experiences. In particular, students are deprived of opportunities to reflect, critique, and question. Thus, this framework portrays problem posing in classrooms as a capstone activity that allows students to consolidate and think critically about the knowledge they have gained.

Although not specifically an analysis of problem posing in classrooms, Singer and Moscovici (2008) have described a learning cycle in constructivist instruction that includes problem posing as an extension and application of problem solving. In an example of instruction with ninth graders, Singer and Moscovici describe three phases of inquiry: immersion, structuring, and applying. In the third, applying phase, students use the patterns they have developed in earlier phases in related and unrelated situations and create new situations that need solving. Parallel to the role of problem posing in Ellerton's (2013) framework, Singer and Moscovici characterize the role of problem posing in a constructivist approach to instruction as that of consolidating and extending what they have learned.

Looking more specifically at the collective activities of students in classrooms, Kontorovich, Koichu, Leikin, and Berman (2012) have proposed a theoretical framework to help researchers handle the complexity of students' mathematical problem posing in small groups. This framework integrates five facets: task organization, students' knowledge base, problem-posing heuristics and schemes, group dynamics and interactions, and individual considerations of aptness. The last facet refers to the posers' comprehensions of implicit requirements of a problem-posing task and reflects their assumptions about the relative importance of these requirements. Kontorovich et al. applied their framework to analyze the problem-posing processes and decision making of two groups of high school students with similar backgrounds who were given the same problem-posing task.

In implementing the supplementary problem-posing curriculum materials designed in their project, Lu and Wang and their associates aimed to help teachers learn how to develop mathematical situations and to pose problems (Lu & Wang, 2006). As supplementary material for the regular mathematics curriculum, a series of teaching cases was developed by mathematics educators across grade levels and across content areas. Figure 1.3 presents a sample teaching case for *Making a Billboard* from Lu and Wang (2006, p. 359). The teaching materials given to teachers included different problem situations together with examples of problems which students might be expected to pose. Figure 1.3 shows a problem situation with six such sample problems. These sample problems were given to teachers as guidelines in much the same way as worked examples might be given in textbooks. When students were given the problem situations, they were encouraged to pose as many problems as they could.

After students had posed several problems, the teacher would show them how to solve some of the posed problems. Figure 1.4 shows a sample solution to problem 3.

Mathematics content: Linear equation with one unknown (for junior high school students).

Situation: A factory is planning to make a billboard. A master worker and his apprentice are employed to do the job. It will take 4 days by the master worker alone to complete the job, but it takes 6 days for the apprentice alone to complete the job.

Students' Task: Please create problems based on the situation. Students may add conditions for problems they create.

Problem 1. How many days will it take the two workers to complete the job together?

Problem 2. If the master joins the work after the apprentice has worked for 1 day, how many additional days will it take the master and the apprentice to complete the job together?

Problem 3. After the master has worked for 2 days, the apprentice joins the master to complete the job. How many days in total will the master have to work to complete the job?

Problem 4. If the master has to leave for other business after the two workers have worked together on the job for 1 day, how many additional days will it take the apprentice to complete the remaining part of the job?

Problem 5. If the apprentice has to leave for other business after the two workers have worked together for 1 day, how many additional days will it take the master to complete the remaining part of the job?

Problems 6. The master and the apprentice are paid 450 Yuan after they completed the job. How much should the master and the apprentice each receive if each worker's payment is determined by the proportion of the job the worker completed?

Figure 1.3. Sample teaching case and examples of problems posed by students in response to the task.

Suppose the two workers worked together for x days, the master worker did $(x+2)$ days.

$\frac{1}{4}(x+2)+\frac{1}{6}x=1$, and $x=\frac{6}{5}$;

So the master worked: $x+2=2+\frac{6}{5}=\frac{16}{5}$ days.

Figure 1.4. Solution presented by a teacher to posed problem 3 in Figure 1.3.

Once students had solved each of the posed problems, they were encouraged to pose new problems. Additional problems posed by students are shown in Figure 1.5. The teacher would then show students how to solve these problems.

Cai (2012) provided another example of problem posing in classroom instruction in a study of 14 preservice teachers engaging in the problem-posing activity shown in Situation 3 of Figure 1.1. The preservice teachers were divided into four groups and given 30 min to pose as many problems as they could. Then the class used another 70 min to solve the posed problems. During the process of solving the posed

Problem 7. The apprentice started the work by himself for 1 day, and then the master joined the effort, and they completed the remaining part of the job together. Finally, they received 490 Yuan in total for completing the job. How much should the master and the apprentice each receive if each worker's payment is determined by the proportion of the job the worker completed?

Problem 8. The master started the work by himself for 1 day, and then the apprentice joined the effort, and they completed the remaining part of the job together. Finally, they received 450 Yuan in total for completing the job. How much should the master and the apprentice each receive if each worker's payment is determined by the proportion of the job the worker completed?

Figure 1.5. Additional problems posed by students.

problems, each preservice teacher could pose additional problems. The preservice teachers posed a total of nine different mathematical problems after the first 30 min. Two groups posed the same question, "What is the sum of the numbers in the first n rows?", and the ensuing discussion produced an unanticipated result.

The first group of students answered the question based on the fact that the sum of the numbers in the first n rows is the "sum of the sum" of the numbers in each of the first n rows. Since the sum of the numbers in the nth row is n^3, the sum of the numbers in the first n rows should be $1^3+2^3+3^3+4^3+\cdots+(n-1)^3+n^3$. Then they posed the following question: What is the sum of $1^3+2^3+3^3+4^3+\cdots+(n-1)^3+n^3$?

The second group used a different approach to answer the original question. After some observations, students realized that the first row has one odd number which is 1, the second row has two odd numbers which are 3 and 5, the third row has three odd numbers which are 7, 9, 11, and so on. The nth row should have n odd numbers. Therefore, the sum of the numbers in the first n rows of the pattern should be the sum of the first $(1+2+3+4+\cdots+n)$ odd numbers. Since $1+3+5+\cdots+(2m-1)=m^2$, the sum of the numbers in the first n rows in the pattern should be $(1+2+3+4+\cdots+n)^2$.

After the two groups of students presented their answers to the class, they integrated their findings and realized that $1^3+2^3+3^3+4^3+\cdots+(n-1)^3+n^3=[n(n+1)/2]^2$ because $1+2+3+4+\cdots+n=n(n+1)/2$. This was not a result that the students had expected, nor was its development from this activity anticipated by the instructor beforehand. This example from empirical research showed that collective problem posing in the classroom context could lead to surprising results. Classrooms that include problem-posing activities may therefore allow students' voices to become relevant in the development of the mathematics they are learning and provide spaces to foster creativity and mathematical power.

Unanswered Question 10

Although we have discussed a few examples of classroom instruction involving problem posing, few researchers have tried to describe carefully the dynamics of classroom instruction where students are engaged in problem-posing activities.

Because classroom instruction is generally complex, with many salient features that can be investigated, researchers will need to identify those features that are most relevant for problem posing and which may be most influenced by the introduction of problem-posing activities. This leads to our tenth unanswered question: What are the key features of effective problem-posing and problem-posing instruction in classrooms?

Unanswered Question 11

In addition to identifying and describing the distinctive features of classrooms in which students engage in problem posing, it is also important to consider how teachers might change their practice and their classroom cultures to make problem posing an accepted practice (Leung, 2013). Indeed, the prevailing norms that shape school mathematics teaching are rooted in both teachers' and students' understandings of what is expected of them (Brousseau, 1984, 1997; Herbst, 2002) and in the practical rationality (Herbst & Chazan, 2003) that guides teachers' judgments about what actions are appropriate in the classroom. Moore-Russo and Weiss (2011) point out the potential difficulty in challenging and altering these norms and expectations, asking "Is it normative to encourage students to modify a problem or to introduce their own assumptions when solving problems?" and "Do teachers commonly encourage students to pose their own problems?" Thus, it is important to investigate the practical questions of whether and how problem posing can fit into the obligations teachers feel in their practice. What are the dynamics of negotiating a classroom culture in which posing is an expected behavior, and what supports do teachers need to be able to reposition themselves and their students for problem posing?

How Can Technology Be Used in Problem-Posing Activities?

The use of technology in the teaching and learning of mathematics has been a topic of interest for researchers in mathematics education. In particular, the flexibility of computer-based technologies for facilitating exploration and experimentation seems relevant to problem posing. Indeed, NCTM (1991) highlighted the promise of technology for problem posing (and solving) "in activities that permit students to design their own explorations and create their own mathematics" (p. 134). For example, Cai and Cifarelli (2005) made use of a computer microworld to allow students to explore a mathematical situation involving the motion of a billiard ball. The microworld provided the students with relative autonomy and freedom in exploring the relationships and boundaries of the mathematical situation. These explorations facilitated the students' generation of multiple questions and conjectures. Thus, by increasing opportunities for students to explore a problem situation and test its boundaries (Crespo & Sinclair, 2008), computer-based technologies may ultimately help students to extend given problems by posing related questions (Santos-Trigo & Diaz-Barriga, 2000) and to pose higher quality problems overall.

Computer-based systems have been particularly well suited to providing students with opportunities to explore dynamic visualizations of geometric situations. Christou, Mousoulides, Pittalis, and Pitta-Pantazi (2005) found that the use of dynamic geometry software facilitated the generation of new problems during the problem-solving process. Students were able to use the dynamic features of the software, and "dragging" in particular, to make and check conjectures, experiment, and generalize. Similarly, Chazan (1990) described how teachers could use the *Geometric Supposers* to increase student exploration and develop students' inquiry skills: verifying, conjecturing, generalizing, communicating, proving, and making connections. The *Supposers* are software programs that facilitate geometric constructions which can then be recorded and repeated with new initial conditions. Chazan found that the use of these programs could help students to pose very good problems by drawing auxiliary lines or systematically varying aspects of problems.

Although geometric situations appear to be particularly well suited to the dynamic visualization power of computer-based tools to aid in problem posing, some researchers have also investigated technological tools in other mathematical contexts. For example, Abramovich and Norton (2006) described the use of graphing software to explore the behavior of quadratic functions, in particular using the locus approach to investigate questions about quadratics with varying parameters. They posit that the use of graphing technology allows for the posing of problems that would be too difficult or abstract for prospective secondary teachers to formulate or solve purely algebraically. Abramovich and Cho (2006) further extended the range of technological tools for problem posing, investigating the use of spreadsheet-based environments to enable elementary preservice teachers and students to pose and solve money sharing and money changing problems. As with the geometric environments, the spreadsheet allows problem posers to explore the consequences of varying parameters of the problem situation. In addition, Abramovich and Cho noted that the spreadsheet tool helped the poser to generate data that ensured the solvability of the posed problems.

Taking advantage of the power of computers to engage students in games, Chang, Wu, Weng, and Sung (2012) implemented a problem-posing system that asked students to pose and refine problems which would then be presented in one of six computer game contexts. The mathematical focus of this project was on elementary word problems. By engaging students in this problem-posing game system, the researchers sought to improve the students' problem-posing and problem-solving skills as well as their flow experience. In particular, Chang et al. found that students using the technology-based activity were more engaged and challenged than students receiving traditional problem-posing instruction in the control group, who became tired of the tasks.

The recent rise of sophisticated web-based technologies has also had an impact on mathematical problem posing. Researchers have begun to investigate how web-based environments can facilitate the work of students and teachers to pose problems, discuss the solutions, and evaluate and improve the problems and solutions. For example, Beal and Cohen (2012) used a web-based content-authoring and sharing system in which middle school students posed mathematics and science problems and solved problems authored by their peers. The system included social media aspects, in which students could compliment or criticize their peers' problems. Beal and Cohen found that students were able to create problems successfully, generating four problems each on average. However, the students engaged in problem-solving activities much more often than authoring new problems, despite being given more points for posing than for solving problems. Nevertheless, both students and teachers responded positively to the activity.

Lan and Lin (2011) developed a web-based Question-Posing Indicators Service (QPIS) system which they used with first year college students in a programming course. Analogous to the social elements of the system used by Beal and Cohen (2012), the QPIS system has a question-posing module where students can pose questions on course content or for reflective thinking, a tool module where students can search problems posed by their peers and give comments to their peers, and an assessment module where students/teachers can evaluate the question-posing abilities of individual students. In particular, the quality of the posed questions was evaluated in a number of ways, such as:

- Content usefulness (whether a question helps students increase their understanding and/or learning)
- Content richness (multimedia content is taken as richer than text-based mode)
- Level of thinking skills reflected by question type (lower order such as true/false questions, intermediate order such as multiple choice questions, and higher order such as matching and short answer questions)
- Self and peer assessment modules
- Expert assessment modules

Lan and Lin found that the QPIS system could serve as both a learning and assessment tool in higher education by encouraging students to carry out active learning, constructive criticism, and knowledge sharing.

Unanswered Question 12

The rapid evolution of technology means that new tools are always becoming available. For purposes of improving educational outcomes, it can be difficult to keep pace with these developments. Of particular concern is the tendency in education to adopt technologies without having a clear picture of their impacts and effectiveness. This raises a key and persistent unanswered question. Are particular technological tools effective, and how do they affect students' problem posing? Some of the studies mentioned above (e.g., Chang et al., 2012; Lan & Lin, 2011) have measured changes in students' problem solving and problem posing. However, this question is not simply about looking for improved performance on existing tasks. As Abramovich and Norton (2006) point out, technological tools can not only enhance the curriculum, but also change it. Thus, studies of the effects of introducing technological tools for problem posing must consider how these tools may change the tasks and the learning goals of mathematics instruction.

What Do We Know About the Impact of Engaging in Problem-Posing Activities on Student Outcomes?

The ultimate goal of educational research is to improve students' learning. Research on problem posing is no exception. The NCTM *Principles and Standards for School Mathematics* (NCTM, 2000) suggest that problem-posing activities should be beneficial for both students and teachers, with students learning to pose problems in both school and out-of-school contexts (Bonotto, 2013), and teachers using problem posing to promote and challenge students' thinking (Stickles, 2011). Indeed, there are at least two reasons to expect that engaging students in problem-posing activities should have a positive impact on their learning. First, problem-posing activities are usually cognitively demanding tasks with the potential to provide intellectual contexts for students' rich mathematical development. Doyle (1983) argued that tasks with different cognitive demands are likely to induce different kinds of learning. Cognitively demanding problem-posing activities can promote students' conceptual understanding, foster their ability to reason and communicate mathematically, and capture their interest and curiosity (NCTM, 1991). Indeed, researchers (e.g., Silver, 1994) have suggested that student-posed problems are more likely to connect mathematics to students' own interests, something that is often not the case with traditional textbook problems. Second, problem-solving processes often involve the generation and solution of subsidiary problems (Polya, 1957). Previous studies (e.g., Cai & Hwang, 2002) have suggested that the ability to pose complex problems might be associated with more robust problem-solving abilities. Thus, encouraging students to generate problems is not only likely to foster student understanding of problem situations, but also to nurture the development of more advanced problem-solving strategies.

Even though theoretical arguments suggest that engaging students in problem-posing activities in classrooms should have a positive impact on students' learning and problem posing, there are relatively few empirical studies that systematically document this effect. English (1997) developed a problem-posing program and found in her post-interview that fifth graders in the problem-posing program did, in fact, pose quantitatively more, as well as more complex, problems. Similarly, Crespo (2003) examined the changes in the problem-posing strategies of a group of elementary preservice teachers as they posed problems to students. She found that, after teachers had engaged in problem-posing activities, they were able to pose more problems with multiple approaches and solutions, as well as pose problems that were more open-ended, exploratory, and cognitively complex.

Given the documented association between students' problem-solving and problem-posing abilities (e.g., Ellerton, 1986; Silver & Cai, 1996), some researchers have specifically investigated the effects of engaging in problem-posing activities on problem-solving performance. Traylor (2005) used a pretest–posttest design to compare the posing and solving performance of eighth-grade algebra students who engaged in both types of activities for the first 9 weeks of the school year to that of students in control classes who had not engaged in posing activities.

The results were mixed, with no clear benefit to engaging in problem posing. However, Traylor suggested that these results may have been influenced by the participants' comparative lack of effort on the posttest, which she attributed to the fact that the test did not "count" toward the students' grades and that, 9 weeks into the school year, students were no longer so eager to please their teachers.

Other researchers have found somewhat more positive effects of problem posing. Abu-Elwan (2002) conducted an experiment with 50 student–teachers, half of whom were given opportunities to pose problems as an extension of Polya's (1957) fourth problem-solving step. The experimental instruction was based on the suggestion of Gonzales (1994) to extend Polya's four steps to include a fifth stage in which students posed related problems. The control group received instruction based only on Polya's original four steps. Abu-Elwan found that the experimental group performed significantly better than the control group in both problem solving and problem posing.

In a study of the effects of problem-posing instruction on Turkish 10th graders' learning of probability, Demir (2005) found that students who had been taught using a problem-posing approach performed significantly better on a probability achievement test. Moreover, Demir documented significant positive effects on affect. Specifically, students who had experienced problem-posing instruction developed more positive attitudes toward probability and mathematics.

Similarly, researchers have investigated the effect of problem posing on various mathematics outcomes for prospective teachers. Positive impacts have been documented of problem posing on the prospective teachers' mathematical knowledge and understanding with respect to fraction concepts (Toluk-Uçar, 2009) and concepts from geometry (Lavy & Shriki, 2010). In addition, problem posing has been found to have positive impacts on other types of mathematics outcomes. For example, Toluk-Uçar found that problem posing had a positive effect on prospective teachers' views of understanding in mathematics, and Akay and Boz (2010) found that instruction integrated with problem posing had resulted in more positive attitudes toward mathematics and greater mathematics self-efficacy in prospective elementary mathematics teachers. Lavy and Shriki noted that, in addition to the prospective teachers' gains in geometric knowledge, problem posing was associated with gains in metamathematical knowledge about definitions, argumentation, and proof.

Unanswered Question 13

Over a decade ago, English (1997) observed that, as a field, we knew little about the relationship between students' problem-posing abilities and their competence in other areas of mathematics. It is clear that progress has been made on this front. Although some of the studies described above have focused specifically on students' problem-posing behavior after engaging in problem-posing activities, others have begun to explore connections between problem posing and broader student outcomes. However, no large-scale validation or efficacy studies have been carried out to examine the effect of engaging problem-posing activities more generally on students' learning of mathematics. Thus, the next unanswered question we raise is: What is the impact of engaging in problem-posing activities on students' mathematics achievement?

Research in reading has shown that engaging students in problem posing can lead to significant gains in reading comprehension. The results from one meta-analysis showed that the effect sizes were .36 using standardized tests and .86 using researcher-developed tests (Rosenshine, Meister, & Chapman, 1996). Although it is theoretically sound to engage students in problem-posing activities in an attempt to understand and improve their learning, more empirical studies are needed to demonstrate any actual effects on mathematics learning. The research in reading can serve as a model for systematically investigating the effect of mathematical exploration in general and problem-posing activities in particular on students' learning of mathematics.

Unanswered Question 14

Engaging in problem posing has the potential to influence more than just the mathematics that students learn, but also their dispositions toward mathematics. Silver (1994) argued that problem posing could influence students' attitudes, affect, and beliefs about mathematics. However, Silver carefully pointed out that, although studies did not typically report negative student reactions to problem posing, the influence of problem posing could be either positive or negative. The findings of Akay and Boz (2010) and Demir (2005) do provide some evidence that problem-posing activities may foster positive views of mathematics and greater self-efficacy. These affective gains may also be reinforced by the use of innovative technologies to stimulate student engagement, as in the work of Chang et al. (2012) and Beal and Cohen (2012). Given that many students suffer from anxiety that interferes with their achievement when solving mathematics problems (X. Ma, 1999; McLeod, 1992), problem posing may therefore offer a more approachable path to problem solving. Yet, the research basis for such a claim remains thin, and the question remains. How does problem posing influence affective aspects of students' mathematics learning? Systematic studies of the effects of problem posing on students' attitudes, affect, and beliefs about mathematics are needed.

Looking to the Future

Although research on mathematical problem posing is comparatively new in the field of mathematics education, researchers have gained some key footholds. Current curriculum frameworks and the curriculum materials that are built on those frameworks include problem posing, if somewhat peripherally to problem solving. We know that students and teachers are capable of posing problems. We have recognized that problem posing offers potential benefits for what mathematics students learn and what students learn about the practice of mathematics. Although students likely need more experiences and preparation with problem posing, it seems reasonable to assert that problem-posing tasks can provide useful measures of various student outcomes. Problem posing has found its way into some curriculum materials and some mathematics classrooms, though much work remains to understand how to encourage this process and produce the best results. And, there are encouraging signs that students who engage in mathematical problem posing seem to develop positive outcomes with respect to their mathematical understandings and dispositions.

Acknowledging both the work that has been done and the many questions that remain unanswered, we conclude our survey of the state of research on mathematical problem posing with a final, very broad unanswered question: How might we understand problem posing? This area of research, though comparatively new within mathematics education, has produced a number of empirical results. Yet, it remains ripe for theoretical work that will provide a cohesive framework for understanding these empirical results and the overall phenomenon of problem posing. This is not necessarily a call for a single, overarching theory of problem posing. Indeed, researchers have focused on many potentially distinct forms of problem posing, such as the kind of problem posing teachers do for their students and the kind of problem posing individuals do when reflecting, and we may need multiple frameworks to understand problem posing in all its guises. Nevertheless, there is a clear need for more robust theory-building so that we may better understand problem posing. A journey of a thousand miles begins with a single step. The journey of problem-posing research in mathematics has taken its first step, but many more steps need to follow.

References

Abramovich, S., & Cho, E. (2006). Technology as a medium for elementary preteachers' problem-posing experience in mathematics. *Journal of Computers in Mathematics and Science Teaching, 25*(4), 309–323.

Abramovich, S., & Norton, A. (2006). Equations with parameters: A locus approach. *Journal of Computers in Mathematics and Science Teaching, 25*(1), 5–28.

Abu-Elwan, R. (2002). Effectiveness of problem posing strategies on prospective mathematics teachers' problem solving performance. *Journal of Science and Mathematics Education in Southeast Asia, 25*(1), 56–69.

Akay, H., & Boz, N. (2010). The effect of problem posing oriented analyses-II course on the attitudes toward mathematics and mathematics self-efficacy of elementary prospective mathematics teachers. *Australian Journal of Teacher Education*, *35*(1), 6. Retrieved from http://ro.ecu.edu.au/ajte/vol35/iss1/6

Ball, D. L., & Cohen, D. K. (1996). Reform by the book: What is—or might be—the role of curriculum materials in teacher learning and instructional reform? *Educational Researcher, 25*(9), 6–8, 14.

Basic Education Curriculum Material Development Center, Chinese Ministry of Education. (2003). *National mathematics curriculum standards at the compulsory education level* (draft for consultation) [in Chinese]. Beijing, China: Beijing Normal University.

Beal, C. R., & Cohen, P. R. (2012). Teach ourselves: Technology to support problem posing in the STEM classroom. *Creative Education, 3*, 513–519.

Boaler, J. (2003). *Studying and capturing the complexity of practice—The case of the "dance of agency."* Paper presented at the 27th International Group for the Psychology of Mathematics Education Conference held jointly with the 25th PME-NA Conference, Honolulu, HI.

Bonotto, C. (2013). Artifacts as sources for problem-posing activities. *Educational Studies in Mathematics, 83*, 37–55.

Brousseau, G. (1984). The crucial role of the didactical contract in the analysis and construction of situations in teaching and learning mathematics. In H. G. Steiner (Ed.), *Theory of mathematics education* (pp. 110–119). Occasional paper 54. Bielifeld, Germany: Institut fur Didaktik der Mathematik.

Brousseau, G. (1997). *Theory of didactical situations in mathematics* (N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield, Eds. & Trans.). Dordrecht, The Netherlands: Kluwer Academic.

Brown, S. I., & Walter, M. I. (1990). *The art of problem posing* (2nd ed.). Hillsdale, NJ: Erlbaum.

Brown, S. I., & Walter, M. I. (1993). Problem posing in mathematics education. In S. I. Brown & M. I. Walter (Eds.), *Problem posing: Reflections and application* (pp. 16–27). Hillsdale, NJ: Erlbaum.

Cai, J. (1998). An investigation of U.S. and Chinese students' mathematical problem posing and problem solving. *Mathematics Education Research Journal, 10*, 37–50.

Cai, J. (2003). Singaporean students' mathematical thinking in problem solving and problem posing: An exploratory study. *International Journal of Mathematical Education in Science and Technology, 34*(5), 719–737.

Cai, J. (2005). U.S. and Chinese teachers' knowing, evaluating, and constructing representations in mathematics instruction. *Mathematical Thinking and Learning: An International Journal, 7*(2), 135–169.

Cai, J. (2012). *Problem posing as lenses of improving students' learning in classroom*. Newark, DE: University of Delaware.

Cai, J., & Brook, M. (2006). New perspectives of looking back in mathematical problem solving. *Mathematics Teaching, 196*, 42–45.

Cai, J., & Cifarelli, V. V. (2005). Exploring mathematical exploration: How two college students formulated and solved their own mathematical problems. *Focus on Learning Problems in Mathematics, 27*(3), 43–72.

Cai, J., & Hwang, S. (2002). Generalized and generative thinking in U.S. and Chinese students' mathematical problem solving and problem posing. *The Journal of Mathematical Behavior, 21*, 401–421.

Cai, J., & Hwang, S. (2003). A perspective for examining the link between problem posing and problem solving. In N. A. Pateman, B. J. Dougherty, & J. T. Zilliox (Eds.), *Proceedings of the 2003 Joint Meeting of PME and PMENA* (Vol. 3, pp. 103–110).

Cai, J., Moyer, J. C., Wang, N., Hwang, S., Nie, B., & Garber, T. (2013). Mathematical problem posing as a measure of curricular effect on students' learning. *Educational Studies in Mathematics, 83*, 57–69.

Cai, J., Jiang, C., Hwang, S., Nie, B., & Hu, D. (in press). Mathematical problem-posing tasks in U.S. and Chinese standards-based textbooks. In P. Felmer, J. Kilpatrick, & E. Pehkonen (Eds.), Problem solving in mathematics education: New advances and perspectives. Springer.

Carpenter, T. P., Fennema, E., & Franke, M. L. (1996). Cognitively guided instruction: A knowledge base for reform in primary mathematics instruction. *The Elementary School Journal, 97*, 3–20.

Chang, K. E., Wu, L. J., Weng, S. E., & Sung, Y. T. (2012). Embedding game-based problem-solving phase into problem-posing system for mathematics learning. *Computers & Education, 58*(2), 775–786. doi:10.1016/j.compedu.2011.10.002.

Chazan, D. (1990). Students' microcomputer-aided exploration in geometry. *Mathematics Teacher, 83*, 628–635.

Chinese National Ministry of Education, Office of School Teaching Materials and Institute of Curriculum and Teaching Materials. (1986). *A collection of mathematical syllabuses (1949–1985)*. Beijing, China: Author.

Christou, C., Mousoulides, N., Pittalis, M., & Pitta-Pantazi, D. (2005). Problem solving and problem posing in a dynamic geometry environment. *The Montana Mathematics Enthusiast, 2*, 125–143.

Christou, C., Mousoulides, N., Pittalis, M., Pitta-Pantazi, D., & Sriraman, B. (2005). An empirical taxonomy of problem posing processes. *Zentralbatt für Didaktik der Mathematik (ZDM), 37*, 149–158.

Cifarelli, V. V., & Cai, J. (2005). The evolution of mathematical explorations in open-ended problem-solving situations. *The Journal of Mathematical Behavior, 24*, 302–324.

Contreras, J. (2007). Unraveling the mystery of the origin of mathematical problems: Using a problem-posing framework with prospective mathematics teachers. *The Mathematics Educator, 17*(2), 15–23.

Crespo, S. (2003). Learning to pose mathematical problems: Exploring changes in preservice teachers' practices. *Educational Studies in Mathematics, 52*, 243–270.

Crespo, S., & Sinclair, N. (2008). What makes a problem mathematically interesting? Inviting prospective teachers to pose better problems. *Journal of Mathematics Teacher Education, 11*, 395–415.

Demir, B. B. (2005). *The effect of instruction with problem posing on tenth grade students' probability achievement and attitudes toward probability* (Unpublished master's thesis). Middle East Technical University, Ankara, Turkey.

Doyle, W. (1983). Academic work. *Review of Educational Research, 53*, 159–199.

Einstein, A., & Infeld, L. (1938). *The evolution of physics*. New York, NY: Simon & Schuster.

Ellerton, N. F. (1986). Children's made-up mathematics problems—A new perspective on talented mathematicians. *Educational Studies in Mathematics, 17*, 261–271.

Ellerton, N. F. (2013). Engaging pre-service middle-school teacher-education students in mathematical problem posing: Development of an active learning framework. *Educational Studies in Mathematics, 83*, 87–101.

English, L. D. (1997). The development of fifth-grade children's problem-posing abilities. *Educational Studies in Mathematics, 34*, 183–217.

English, L. D. (1998). Children's problem posing within formal and informal contexts. *Journal for Research in Mathematics Education, 29*(1), 83–106.

Fennema, E., Carpenter, T. P., Franke, M. L., Levi, L., Jacobs, V. R., & Empson, S. B. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education, 27*, 403–434.

Frensch, P., & Funke, J. (1995). *Complex problem solving: The European perspective*. Hillsdale, NJ: Erlbaum.

Getzels, J. W. (1979). Problem finding: A theoretical note. *Cognitive Science, 3*, 167–172.

Goldin, G. A., & McClintock, C. E. (Eds.). (1984). *Task variables in mathematical problem solving*. Philadelphia, PA: Franklin Institute Press.

Gonzales, N. A. (1994). Problem posing: A neglected component in mathematics courses for prospective elementary and middle school teachers. *School Science and Mathematics, 94*(2), 78–84.

Gonzales, N. A. (1998). A blueprint for problem posing. *School Science and Mathematics, 98*(8), 448–456.

Hadamard, J. W. (1945). *Essay on the psychology of invention in the mathematical field*. Princeton, NJ: Princeton University Press.

Hashimoto, Y. (1987). Classroom practice of problem solving in Japanese elementary schools. In J. P. Becker, & T. Miwa (Eds.), *Proceedings of the U.S.-Japan Seminar on Mathematical Problem Solving* (pp. 94–119). Carbondale, IL: Southern Illinois University.

Healy, C. C. (1993). *Creating miracles: A story of student discovery*. Berkeley, CA: Key Curriculum Press.

Herbst, P. G. (2002). Engaging students in proving: A double bind on the teacher. *Journal for Research in Mathematics Education, 33*, 176–203.

Herbst, P., & Chazan, D. (2003). Exploring the practical rationality of mathematics teaching through conversations about videotaped episodes: The case of engaging students in proving. *For the Learning of Mathematics, 23*(1), 2–14.

Isik, C., & Kar, T. (2012). The analysis of the problems the pre-service teachers experience in posing problems about equations. *Australian Journal of Teacher Education, 37*(9). Retrieved from http://ro.ecu.edu.au/cgi/viewcontent.cgi?article=1568&context=ajte

Keil, G. E. (1964/1967). Writing and solving original problems as a means of improving verbal arithmetic problem solving ability (Doctoral dissertation, Indiana University, 1964). *Dissertation Abstracts International, 25*(12), 7109.

Kilpatrick, J. (1987). Problem formulating: Where do good problems come from? In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 123–147). Hillsdale, NJ: Erlbaum.

Koichu, B., & Kontorovich, I. (2013). Dissecting success stories on mathematical problem posing: A case of the Billiard Task. *Educational Studies in Mathematics, 83*, 71–86.

Kontorovich, I., Koichu, B., Leikin, R., & Berman, A. (2012). An exploratory framework for handling the complexity of mathematical problem posing in small groups. *The Journal of Mathematical Behavior, 31*(1), 149–161. doi:10.1016/j.jmathb.2011.11.002.

Kotsopoulos, D., & Cordy, M. (2009). Investigating imagination as a cognitive space for learning mathematics. *Educational Studies in Mathematics, 70*, 259–274.

Lan, Y. F., & Lin, P. C. (2011). Evaluation and improvement of student's question-posing ability in a web-based learning environment. *Australasian Journal of Educational Technology, 27*(4), 581–599.

Lavy, I., & Bershadsky, I. (2003). Problem posing via "what if not?" strategy in solid geometry—A case study. *The Journal of Mathematical Behavior, 22*(4), 369–387.

Lavy, I., & Shriki, A. (2010). Engaging in problem posing activities in a dynamic geometry setting and the development of prospective teachers' mathematical knowledge. *The Journal of Mathematical Behavior, 29*(1), 11–24.

Lester, F. K. (1994). Musing about mathematical problem-solving research: 1970–1994. *Journal for Research in Mathematics Education, 25*(6), 660–675.

Leung, S. S. (1997). On the role of creative thinking in problem posing. *Zentralbatt für Didaktik der Mathematik (ZDM), 29*, 81–85.

Leung, S. S. (2013). Teachers implementing mathematical problem posing in the classroom: Challenges and strategies. *Educational Studies in Mathematics, 83*(1), 103–116. doi:10.1007/s10649-012-9436-4.

Leung, S. S., & Silver, E. A. (1997). The role of task format, mathematics knowledge, and creative thinking on the arithmetic problem posing of prospective elementary school teachers. *Mathematics Education Research Journal, 9*(1), 5–24.

Lu, C., & Wang, B. (2006). *Research on mathematics teaching through using mathematical situations and posing problem in high school and primary school* [in Chinese]. Guiyang, China: Guizhou People's Publishing House.

Ma, L. (1999). *Knowing and teaching elementary mathematics*. Mahwah, NJ: Lawrence Erlbaum.

Ma, X. (1999). A meta-analysis of the relationship between anxiety toward mathematics and achievement in mathematics. *Journal for Research in Mathematics Education, 30*, 520–540.

McLeod, D. B. (1992). Research on affect in mathematics education: A reconceptualization. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 575–596). New York, NY: Macmillan.

McLeod, D., & Adams, V. (Eds.). (1989). *Affect and mathematical problem solving: A new perspective*. New York, NY: Springer.

Ministry of Education of China. (2011). *Mathematics curriculum standard of compulsory education (2011 version)* [in Chinese]. Beijing, China: Beijing Normal University Press. Retrieved from http://www.moe.gov.cn/publicfiles/business/htmlfiles/moe/moe_711/201201/xxgk_129268.html

Moore-Russo, D., & Weiss, M. (2011). Practical rationality, the disciplinary obligation, and authentic mathematical work: A look at geometry. *The Mathematics Enthusiast, 8*(3), 463–482.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Governors Association Center for Best Practices. (2010). *Common core state standards for mathematics*. Washington, DC: National Governors Association Center for Best Practices, Council of Chief State School Officers.

National Research Council. (2004). *On evaluating curricular effects: Judging the quality of K-12 mathematics education*. In J. Confrey & V. Stohl (Eds.). Washington, DC: National Academy Press.

Olson, J. C., & Knott, L. (2013). When a problem is more than a teacher's question. *Educational Studies in Mathematics, 83*, 27–36.

Pittalis, M., Christou, C., Mousoulides, N., & Pitta-Pantazi, D. (2004). A structural model for problem posing. In M. J. Hoines & A. B. Fuglestad (Eds.), *Proceedings of the 28th Annual Meeting of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 49–56). Bergen, Norway: Bergen University College.

Polya, G. (1957). *How to solve it*. Princeton, NJ: Princeton University Press.

Rosenshine, B., Meister, C., & Chapman, S. (1996). Teaching students to generate questions: A review of the intervention studies. *Review of Educational Research, 66*(2), 181–221.

Santos-Trigo, M., & Diaz-Barriga, E. (2000). Posing questions from proposed problems: Using technology to enhance mathematical problem solving. *Mathematics Teacher, 93*(7), 578–580.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334–370). New York, NY: Macmillan.

Silver, E. A. (1985). *Teaching and learning mathematical problem solving: Multiple research perspectives*. Mahwah, NJ: Erlbaum.

Silver, E. A. (1994). On mathematical problem posing. *For the Learning of Mathematics, 14*(1), 19–28.

Silver, E. A. (1997). Fostering creativity through instruction rich in mathematical problem solving and problem posing. *Zentralbatt für Didaktik der Mathematik (ZDM), 97*(3), 75–80.

Silver, E. A., & Cai, J. (1996). An analysis of arithmetic problem posing by middle school students. *Journal for Research in Mathematics Education, 27*(5), 521–539.

Silver, E. A., & Cai, J. (2005). Assessing students' mathematical problem posing. *Teaching Children Mathematics, 12*(3), 129–135.

Silver, E. A., Mamona-Downs, J., Leung, S. S., & Kenney, P. A. (1996). Posing mathematical problems: An exploratory study. *Journal for Research in Mathematics Education, 27*, 293–309.

Singer, F. M., & Moscovici, H. (2008). Teaching and learning cycles in a constructivist approach to instruction. *Teaching and Teacher Education, 24*(6), 1613–1634.

Stanic, G. M. A., & Kilpatrick, J. (1988). Historical perspectives on problem solving in mathematics curriculum. In R. I. Charles & E. A. Silver (Eds.), *Research agenda for mathematics education: The teaching and assessing of mathematical problem solving* (pp. 1–22). Reston, VA: National Council of Teachers of Mathematics.

Stickles, P. (2011). An analysis of secondary and middle school teachers' mathematical problem posing. *Investigations in Mathematics Learning, 3*(2), 1–34.

Stoyanova, E., & Ellerton, N. F. (1996). A framework for research into students' problem posing in school mathematics. In P. C. Clarkson (Ed.), *Technology in mathematics education* (pp. 518–525). Melbourne, Australia: Mathematics Education Research Group of Australasia.

Tichá, M., & Hošpesová, A. (2013). Developing teachers' subject didactic competence through problem posing. *Educational Studies in Mathematics, 83*(1), 133–143.

Toluk-Uçar, Z. (2009). Developing pre-service teachers understanding of fractions through problem posing. *Teaching and Teacher Education, 25*(1), 166–175.

Traylor, K. C. (2005). Effect of problem posing on eighth grade students' mathematical problem solving skills. *Instructional Technology Monographs, 2*(2). Retrieved March 11, 2013, from http://itm.coe.uga.edu/archives/fall2005/ktraylor.htm

van den Brink, J. F. (1987). Children as arithmetic book authors. *For the Learning of Mathematics, 7*, 44–48.

Van Harpen, X. Y., & Sriraman, B. (2013). Creativity and mathematical problem posing: An analysis of high school students' mathematical problem posing in China and USA. *Educational Studies in Mathematics, 82*(2), 201–221.

Voica, C., & Singer, F. M. (2012). *Problem modification as an indicator of deep understanding*. Paper presented at Topic Study Group 3, Activities and Programs for Gifted Students, ICME-12, Seoul, Korea.

Voica, C., & Singer, F. M. (2013). Problem modification as a tool for detecting cognitive flexibility in school children. *Zentralbatt für Didaktik der Mathematik (ZDM), 45*, 267–279.

Wang, B., & Lu, C. (2000). Innovation and mathematical education of the primary and middle school [in Chinese]. *Journal of Mathematics Education, 9*(4), 34–37.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27*, 458–477.

Yuan, X., & Sriraman, B. (2011). An exploratory study of relationships between students' creativity and mathematical problem posing abilities. In B. Sriraman & K. Lee (Eds.), *The elements of creativity and giftedness in mathematics* (pp. 5–28). Rotterdam, The Netherlands: Sense.