Chapter 7

Pattern theorems

7.1 Patterns

In this chapter we shall prove a useful theorem due to Kesten (1963) about the occurrence of patterns on self-avoiding walks, and investigate a number of its applications. Briefly, a *pattern* is a (short) self-avoiding walk that occurs as part of a longer self-avoiding walk. Kesten's Pattern Theorem says that if a given pattern can possibly occur several times on a self-avoiding walk, then it must occur at least aN times on almost all N-step self-avoiding walks, for some $a > 0$ (in this context, "almost all" means "except for an exponentially small fraction"). This can be viewed as a weak analogue of classical "large deviations" estimates for the strong law of large numbers, which say that certain events have exponentially small probabilities [see for example Chapter 1 of Ellis (1985)].

Another statistic of interest regarding patterns is the frequency of occurrence of a particular pattern at the beginning of self-avoiding walks. In general dimension d, it is an open problem to prove that the fraction of N-step self-avoiding walks that begin with a given pattern converges as N tends to infinity. This has been done in certain special cases: for $d \geq 5$ (see Section 6.7), and for bridges in every dimension (see Section 8.3). The existence of such a limit would provide a natural definition of a probability measure for infinite self-avoiding walks. We can only prove the following weaker results in the general case: if P is a pattern that can occur at the beginning of an arbitrarily long self-avoiding walk, then the fraction of N-step self-avoiding walks beginning with this pattern is bounded away from zero as N tends to infinity; also, the ratio of these fractions for N and $N+2$ converges to one. These results and some extensions will be discussed in

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Section 7.4. The proofs of these results rely heavily upon Kesten's original pattern theorem.

Kesten originally applied his pattern theorem to prove the following *ratio limit theorems*:

$$\lim_{N\to\infty} \frac{c_{N+2}}{c_N} = \mu^2, \tag{7.1.1}$$

$$\lim_{N\to\infty} \frac{q_{2N+2}}{q_{2N}} = \mu^2, \tag{7.1.2}$$

$$\lim_{N\to\infty} \frac{b_{N+1}}{b_N} = \mu. \tag{7.1.3}$$

We shall prove these results in Section 7.3. Unfortunately, the same methods do not allow us to prove

$$\lim_{N\to\infty} \frac{c_{N+1}}{c_N} = \mu. \tag{7.1.4}$$

Equation (7.1.4) in $\mathbf{Z}^d$ has only been proven for $d \geq 5$ (see Theorem 6.1.1); finding a proof for $d = 2, 3, 4$ remains an open problem. To get a feeling for why (7.1.1) is easier to prove than (7.1.4), consider the following easy exercise: prove that

$$c_{N+2} \geq c_N \qquad \text{for every } N. \tag{7.1.5}$$

The idea of the proof is given in Figure 7.1. (In detail: Given an N-step

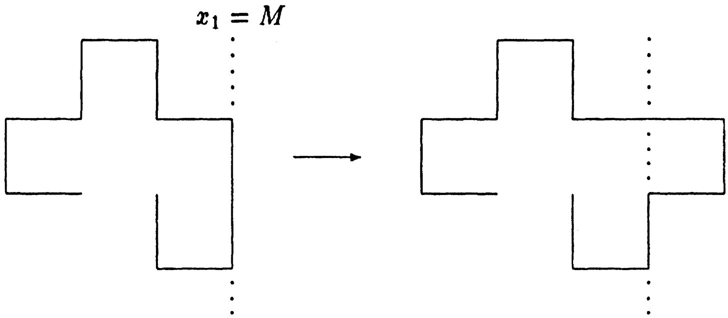

Figure 7.1: The idea behind the proof that $c_{N+2} \geq c_N$: increasing the length of a self-avoiding walk by 2.

self-avoiding walk ω, let $M = \max\{\omega_1(i) : 0 \leq i \leq N\}$. On the one hand, if a step of ω joins two points u and v in the hyperplane $x_1 = M$, then replace that step by three steps: u to $u + (1, 0, \ldots, 0)$ to $v + (1, 0, \ldots, 0)$ to v. On the other hand, if this hyperplane does not contain a step of ω, then it must contain an endpoint [$\omega(0)$ or $\omega(N)$]; in this case, add two steps to the end of the walk in the $+x_1$ direction. In either case we get an $(N+2)$-step self-avoiding walk, from which ω can be determined unambiguously.) Now try the following exercise: prove that

$$c_{N+1} \geq c_N \qquad \text{for every } N. \tag{7.1.6}$$

It is much harder to construct a one-to-one mapping from the set of N-step walks to the set of $(N+1)$-step walks, but it can in fact be done; for the lengthy details, see O'Brien (1990). Finally, we observe that (7.1.6) is easy to prove on the triangular lattice and other lattices which are not bipartite (i.e. that contain self-avoiding polygons with an odd number of steps). On such lattices, it turns out that (7.1.4) can be proven by the methods of this chapter; see the Remark preceding Theorem 7.3.2.

Pattern theorems have found several other applications, including: evaluating the ergodicity properties of certain Monte Carlo algorithms (see Sections 9.4.1 and 9.4.2), investigating self-avoiding walks restricted to subsets of $\mathbf{Z}^d$ (see Section 8.2), and establishing the frequency of knots in three-dimensional self-avoiding polygons (see Section 8.4).

It is now time to make precise definitions about patterns and their occurrence. To begin with, we can take the word "pattern" to be a synonym for "self-avoiding walk".

Definition 7.1.1 *A pattern $P = (p(0), \ldots, p(n))$ is said to* occur at the j-th step of the self-avoiding walk $\omega = (\omega(0), \ldots, \omega(N))$ *if there exists a vector v in $\mathbf{Z}^d$ such that $\omega(j+k) = p(k) + v$ for every $k = 0, \ldots, n$. (Evidently, v must be $\omega(j) - p(0)$.)*

Definition 7.1.2 *Let S_N denote the set of N-step self-avoiding walks ω such that $\omega(0) = 0$. For $k \geq 0$ and P a pattern, let $c_N[k, P]$ denote the number of walks in S_N for which P occurs at no more than k different steps. Let $\mathcal{F}_N[P]$ denote the subset of walks in S_N for which P occurs at the 0-th step. We say that P is a* proper front pattern *if $\mathcal{F}_N[P]$ is non-empty for all sufficiently large N. We say that P is a* proper internal pattern *if for every k there is a self-avoiding walk on which P occurs at k or more different steps.*

Kesten's Pattern Theorem tells us that if P is a proper internal pattern, then there exists an $a > 0$ such that

$$\limsup_{N \to \infty} (c_N[aN, P])^{1/N} < \mu. \tag{7.1.7}$$

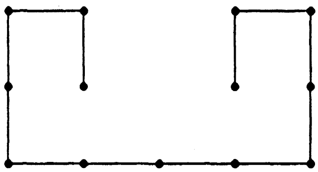

Figure 7.2: This pattern can occur twice on a self-avoiding walk in $\mathbf{Z}^2$, but not three times.

The theorem actually tells us a bit more; see Theorem 7.2.3.

The basic results about "front patterns" say that if P is a proper front pattern, then

$$\liminf_{N\to\infty} \frac{|\mathcal{F}_N[P]|}{c_N} > 0 \tag{7.1.8}$$

(where $|\cdot|$ denotes cardinality) and

$$\lim_{n\to\infty} \frac{|\mathcal{F}_{N+2}[P]|}{|\mathcal{F}_N[P]|} = \mu^2. \tag{7.1.9}$$

Further results about front patterns appear in Section 7.4.

We take this opportunity to note some equivalent characterizations of proper internal patterns.

Proposition 7.1.3 *Let P be a pattern. The following are equivalent:*
(a) P is a proper internal pattern;
(b) There exists a cube $Q = \{x : 0 \le x_i \le b\}$ and a self-avoiding walk ϕ such that: P occurs at some step of ϕ, ϕ is contained in Q, and the two endpoints of ϕ are corners of Q;
(c) There exists a self-avoiding walk ω such that P occurs at three or more steps of ω.

We remark that if (b) above holds for P, then it is always possible to take

$$b = 2 + \max\{||u - v||_\infty : \ u \text{ and } v \text{ are sites of } P\}.$$

The proof of this proposition is straightforward, except for showing that (c) implies the other assertions. This implication is proven in Hammersley and Whittington (1985). Although we shall not require part (c) in this book, it is worth noting that the proposition is false if we change "three" to "two" in part (c), since there exist patterns which can occur at the beginning and end of a self-avoiding walk but never in the middle; an example in $\mathbf{Z}^2$ is the pattern $NWS^2E^4N^2WS$ shown in Figure 7.2. (In this notation, N denotes a step in the direction $(0,1)$ ["North"], etc.)

7.2 Kesten's Pattern Theorem

In this section, we shall formulate and prove Kesten's Pattern Theorem in its full generality, following the structure of Kesten's original proof. The general version of the theorem is a bit stronger than (7.1.7): in addition to specifying a pattern, one may also require that a certain amount of space around the pattern be unoccupied. The precise generalization is as follows.

Definition 7.2.1 *A* cube *is any set of the form*

$$Q = \{x \in \mathbf{Z}^d : a_i \leq x_i \leq a_i + b \text{ for all } i = 1, \ldots, d\},$$

where $a_1, \ldots, a_d$, and b are integers, and $b > 0$. Each cube has 2^d corners *(extreme points of the convex hull). If Q is a cube as above, then let $\overline{Q}$ denote the cube which is two units larger in all directions:*

$$\overline{Q} = \{x \in \mathbf{Z}^d : a_i - 2 \leq x_i \leq a_i + b + 2 \text{ for all } i = 1, \ldots, d\};$$

and let ∂Q denote the set of points in $\overline{Q}$ but not in Q (a kind of "external boundary" of Q),

$$\partial Q = \overline{Q} \setminus Q.$$

An outer point *of ∂Q is a point of ∂Q which has at least one nearest neighbour that is not in $\overline{Q}$.*

Definition 7.2.2 *Suppose that Q is a cube and P is an n-step pattern such that $p(0)$ and $p(n)$ are corners of Q, and $p(i) \in Q$ for every $i = 0, \ldots, n$ (in particular, P is a proper internal pattern; see Proposition 7.1.3). We say that (P, Q)* occurs at the j-th step of the self-avoiding walk ω *if there exists a v in $\mathbf{Z}^d$ such that $\omega(j + k) = p(k) + v$ for every $k = 0, \ldots, n$, and $\omega(i)$ is not in $Q + v$ for every $i < j$ and every $i > j + n$. For every $k \geq 0$, let $c_N[k, (P, Q)]$ denote the number of self-avoiding walks in S_N for which (P, Q) occurs at no more than k different steps.*

Theorem 7.2.3 *(a) Let Q be a cube and P be a pattern as in Definition 7.2.2. Then there exists an $a > 0$ such that*

$$\limsup_{N \to \infty} (c_N[aN, (P, Q)])^{1/N} < \mu. \tag{7.2.1}$$

(b) For any proper internal pattern P, there exists an $a > 0$ such that

$$\limsup_{N \to \infty} (c_N[aN, P])^{1/N} < \mu. \tag{7.2.2}$$

Before proceeding, we shall show that part (b) of the theorem [which is (7.1.7)] follows from part (a). Let P be a proper internal pattern, and choose ϕ and Q as in Proposition 7.1.3(b). Since P occurs on ϕ, any walk on which (ϕ, Q) occurs at m different steps must have P occurring at m or more different steps. Therefore

$$c_N[k, P] \leq c_N[k, (\phi, Q)] \quad \text{for every } k \geq 0,$$

from which we see that part (a) of Theorem 7.2.3 is indeed stronger than part (a). Thus it suffices to prove part (a).

The first ingredient in the proof of Theorem 7.2.3(a) is the following basic geometrical lemma. Part (a) of the lemma will construct a pattern that exactly fills a cube. Part (b) will show that we can splice a proper internal pattern onto a self-avoiding walk if we erase the part of the walk that occupies the corresponding enlarged cube $\overline{Q}$.

Lemma 7.2.4 *(a) Let Q be a cube in $\mathbf{Z}^d$. Then there exists a self-avoiding walk ω, whose endpoints are corners of Q, which is entirely contained in Q and visits every point of Q. (In particular, the number of steps in ω is one less than the number of points in Q.)*
(b) Let $P = (p(0), \ldots, p(k))$ be a pattern contained in the cube Q, whose endpoints are corners of Q. Let x and y be two distinct outer points of ∂Q. Then there exists a self-avoiding walk ω' with the following properties: its initial point is x and its last point is y; it is entirely contained in $\overline{Q}$; there exists a j such that $\omega'(j+i) = p(i)$ for every $i = 0, \ldots, k$; and $\omega'(i) \in \partial Q$ whenever $i < j$ or $i > j+k$. In particular, (P, Q) occurs at the j-th step of ω'.

Proof. (a) This is proven by induction on the dimension. It is obvious in one dimension. Assume that it has been proven for dimension $d-1$. For simplicity, assume

$$Q = \{x \in \mathbf{Z}^d : 0 \leq x_i \leq b, i = 1, \ldots, d\}.$$

The intersection of Q with each of the hyperplanes $x_d = l$ $(l = 0, \ldots, b)$ is a $(d-1)$-dimensional cube embedded in $\mathbf{Z}^d$; call it Q^l. By the inductive hypothesis, there is a self-avoiding walk that starts at the origin and fills up Q^0 while staying inside Q^0, and whose last point is a corner of Q^0. Since every corner of Q^l is a nearest neighbour of a corner of Q^{l+1}, it is clear that we can find the desired walk for Q by filling up each of the $(d-1)$-dimensional cubes $Q^0, \ldots, Q^d$ in turn.

(b) First choose a self-avoiding walk $\omega^{[1]}$ from x to $p(0)$ which does not touch y and contains only outer points of ∂Q (except necessarily for the last two points of the walk). Then one can find a self-avoiding walk $\omega^{[2]}$ from $p(k)$ to y which stays in ∂Q and never touches $\omega^{[1]}$ (to do this, simply avoid outer points until the very end). Then the desired walk ω' is the concatenation of $\omega^{[1]}$, P, and $\omega^{[2]}$. □

Now we must add to our stockpile of notation. We fix a positive integer r which will be the "radius" of the cube Q of interest that occurs in the statement of the Pattern Theorem. For a given N-step self-avoiding walk ω, we extend Definition 7.2.1 by specifying cubes centred at points of ω: for $j = 0, \ldots, N$, let

$$\begin{aligned} Q(j) &= \{x \in \mathbf{Z}^d : |x_i - \omega_i(j)| \le r \text{ for every } i = 1, \ldots, d\}, \\ \overline{Q}(j) &= \{x \in \mathbf{Z}^d : |x_i - \omega_i(j)| \le r + 2 \text{ for every } i = 1, \ldots, d\}, \\ \partial Q(j) &= \overline{Q}(j) \setminus Q(j). \end{aligned}$$

We say that E^* *occurs at the j-th step of ω* if $Q(j)$ is completely covered by ω [i.e. for every v in $Q(j)$ there exists an i such that $v = \omega(i)$]. See Figure 7.3. For every $k \ge 1$, we say that E_k *occurs at the j-th step of ω* if at least k points of $\overline{Q}(j)$ are covered by ω; and we say that $\tilde{E}_k$ *occurs at the j-th step of ω* if E^* or E_k (or both) occur there.

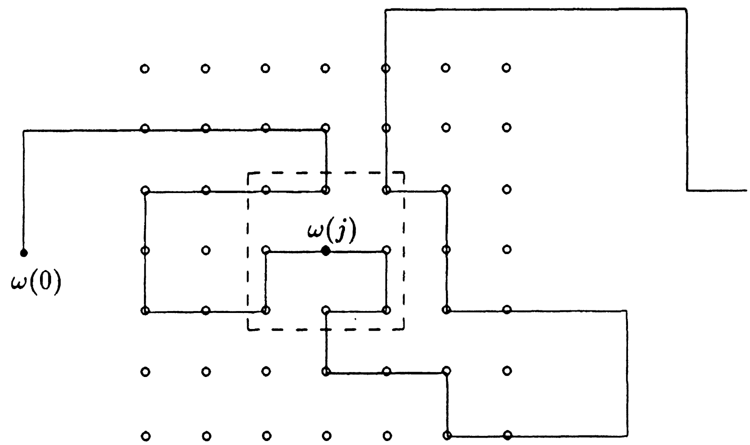

Figure 7.3: An example in $\mathbf{Z}^2$ with $r = 1$: The nine sites inside the dashed box comprise $Q(j)$. The sites of $\overline{Q}(j)$ are marked with o. Both E^* and E_{29} occur at the j-th step of this walk.

In the following, we will use E to denote any of E^*, E_k, or $\tilde{E}_k$. If m is a positive integer, we say that $E(m)$ *occurs at the j-th step of ω* if E

occurs at the m-th step of the $2m$-step walk $(\omega(j-m),\ldots,\omega(j+m))$. [If $j-m<0$ or $j+m>N$, then an obvious modification must be made in this definition: for $j-m<0$, it means that E occurs at the j-th step of $(\omega(0),\ldots,\omega(j+m))$; for $j+m>N$, it means that E occurs at the m-th step of $(\omega(j-m),\ldots,\omega(N))$.] In particular, if $E(m)$ occurs at the j-th step of ω, then E occurs at the j-th step of ω; this would not necessarily be true if we replaced E by (P,Q). For every $k\geq 0$, let $c_N[k,E]$ (respectively, $c_N[k,E(m)]$) denote the number of self-avoiding walks in $\mathcal{S}_N$ for which E (respectively, $E(m)$) occurs at no more than k different steps. Observe that $c_N[k,E(m)]$ is non-increasing in m for fixed N and k because occurrences of $E(m)$ are more frequent as m increases.

The next lemma says that if E occurs on almost all walks, then (for some m) $E(m)$ must occur on almost all walks (in fact, it must occur *often* on almost all walks). Thus, if a self-avoiding walk is likely to fill a cube, then it is also likely to fill a cube within some bounded number of steps.

Lemma 7.2.5 *If*

$$\liminf_{N\to\infty}(c_N[0,E])^{1/N}<\mu, \tag{7.2.3}$$

then there exists an $a_1>0$ and an integer m such that

$$\limsup_{N\to\infty}(c_N[a_1N,E(m)])^{1/N}<\mu. \tag{7.2.4}$$

Proof. Since $c_N[0,E]=c_N[0,E(N)]$, it follows that there exist $\epsilon>0$ and an integer m such that

$$c_m[0,E(m)]<(\mu(1-\epsilon))^m$$

and

$$c_m<(\mu(1+\epsilon))^m.$$

Consider an N-step self-avoiding walk ω, and let $M=\lfloor N/m\rfloor$. If $E(m)$ occurs at most k times in ω, then $E(m)$ occurs in at most k of the M m-step subwalks

$$(\omega((i-1)m),\omega((i-1)m+1),\ldots,\omega(im)) \qquad (i=1,\ldots,M).$$

Counting the number of ways in which k or fewer of these subwalks can contain an occurrence of $E(m)$ (and remembering to count the last $N-Mm$ steps of ω), we are led to the bound

$$\begin{aligned} c_N[k,E(m)] &\leq \sum_{j=0}^{k}\binom{M}{j}(c_m)^j(c_m[0,E(m)])^{M-j}c_{N-Mm} \qquad (7.2.5)\\ &\leq \mu^{mM}c_{N-Mm}\sum_{j=0}^{k}\binom{M}{j}(1+\epsilon)^{jm}(1-\epsilon)^{Mm-jm}. \end{aligned}$$

It suffices to show that there is a $\rho > 0$ and a $t < 1$ such that

$$c_N[\rho M, E(m)]^{1/M} < t\mu^m \tag{7.2.6}$$

for all sufficiently large M, since this gives (7.2.4) whenever $0 < a_1 < \rho/m$. But if ρ is a small positive number, then

$$\begin{aligned} &\sum_{j=0}^{\rho M} \binom{M}{j} (1+\epsilon)^{jm}(1-\epsilon)^{Mm-jm} \\ &\leq \quad (\rho M + 1) \binom{M}{\rho M} \left(\frac{1+\epsilon}{1-\epsilon}\right)^{\rho M m} (1-\epsilon)^{Mm}. \end{aligned} \tag{7.2.7}$$

(For readability, we often write ρM instead of $\lfloor \rho M \rfloor$.) As $M \to \infty$, the M-th root of the right-hand side of (7.2.7) converges to

$$\frac{1}{\rho^\rho (1-\rho)^{1-\rho}} \left(\frac{1+\epsilon}{1-\epsilon}\right)^{\rho m} (1-\epsilon)^m,$$

which is less than 1 whenever $0 < \rho < \rho_0$, for some sufficiently small ρ_0. Combining this with (7.2.5), we see that (7.2.6) holds if $0 < \rho < \rho_0$ and M is sufficiently large. □

Remark. Although we will not need this fact, it is worth pointing out that the lim inf in (7.2.3) is in fact a limit. This follows from $c_{N+M}[0, E] \leq c_N[0, E] c_M[0, E]$ and Lemma 1.2.2.

The next lemma is the heart of the proof of the Pattern Theorem. It says that almost all walks fill some cube (of the fixed radius r). The starting point of the proof is the observation that all walks cover at least $r+3$ points of the cube of radius $r+2$ centred at the origin; so if the lemma were false, then there would exist a K such that almost all walks cover K points of some cube (and in fact many cubes), but almost never cover $K+1$ points of any cube. This is used to obtain a contradiction.

Lemma 7.2.6 $\liminf_{N\to\infty} c_N[0, E^*]^{1/N} < \mu$.

Proof. Assume that the lemma is false, i.e. assume that

$$\lim_{N\to\infty} c_N[0, E^*]^{1/N} = \mu. \tag{7.2.8}$$

We make three observations: First, $c_N[0, \tilde{E}_k]$ is a nondecreasing function of k. Secondly, if E^* does not occur on a given walk then $E_{(2r+5)^d}$ cannot occur; therefore

$$c_N[0, E^*] \leq c_N[0, \tilde{E}_{(2r+5)^d}] \leq c_N, \tag{7.2.9}$$

and hence (7.2.8) implies that

$$\lim_{N\to\infty} c_N[0, \tilde{E}_{(2r+5)^d}]^{1/N} = \mu. \tag{7.2.10}$$

Thirdly, $c_N[0, \tilde{E}_{r+3}] = 0$ for all $N \geq r+2$ [since the first $r+3$ points of any walk ω must be in $\overline{Q}(0)$]. We conclude from these observations that there exists a K [with $r+3 \leq K < (2r+5)^d$] such that

$$\liminf_{N\to\infty} c_N[0, \tilde{E}_K]^{1/N} < \mu \tag{7.2.11}$$

and

$$\liminf_{N\to\infty} c_N[0, \tilde{E}_{K+1}]^{1/N} = \mu. \tag{7.2.12}$$

By (7.2.11) and Lemma 7.2.5, there exist an $a_1 > 0$ and an integer m such that

$$\limsup_{N\to\infty} c_N[a_1 N, \tilde{E}_K(m)]^{1/N} < \mu. \tag{7.2.13}$$

Define the set of self-avoiding walks

$$T_N = \{\omega \in \mathcal{S}_N : \tilde{E}_{K+1} \text{ never occurs; } E_K(m) \text{ occurs at least } a_1 N \text{ times}\}. \tag{7.2.14}$$

Observe that replacing $E_K(m)$ by $\tilde{E}_K(m)$ in (7.2.14) does not change anything, since the condition that $\tilde{E}_{K+1}$ never occurs ensures that $E^*(m)$ never occurs. The cardinality of T_N satisfies

$$|T_N| \geq c_N[0, \tilde{E}_{K+1}] - c_N[a_1 N, \tilde{E}_K(m)], \tag{7.2.15}$$

and therefore, by (7.2.12) and (7.2.13),

$$\lim_{N\to\infty} |T_N|^{1/N} = \mu. \tag{7.2.16}$$

Thus, there is a number K such that it is not unusual to find lots of cubes with exactly K points occupied and *no* cubes with more than K points occupied. The rest of the proof is simply a matter of counting. The main idea is the following. Given such a walk $\omega \in T_N$, consider the collection of all cubes that have exactly K points covered. Remove the pieces of ω that cover a particular (small) subcollection of these cubes, and consider all possible ways of replacing them with pieces that entirely fill the same cubes. This is not a one-to-one transformation, and the length of the resulting walk is a bit different, but we can still arrange it so that the number of resulting walks is larger than $|T_N|$ by an exponential factor, and this will contradict (7.2.16).

Suppose that ω is an N-step self-avoiding walk such that $\tilde{E}_{K+1}$ never occurs on ω and $E_K(m)$ occurs at the j_1-th, j_2-th,..., j_s-th steps of ω (and perhaps at other steps as well). Suppose in addition that

$$0 < j_1 - m, j_s + m < N, \text{ and } j_l + m < j_{l+1} - m \text{ for all } l = 1, \ldots, s-1 \tag{7.2.17}$$

and

$$\overline{Q}(j_1), \ldots, \overline{Q}(j_s) \text{ are pairwise disjoint.} \tag{7.2.18}$$

For $l = 1, \ldots, s$, let

$$\sigma_l = \min\{i : \omega(i) \in \overline{Q}(j_l)\} \quad \text{and} \quad \tau_l = \max\{i : \omega(i) \in \overline{Q}(j_l)\}.$$

Since $E_K(m)$ occurs at the j_l-th step and E_{K+1} does not occur at the j_l-th step, there must be exactly K points of $\overline{Q}(j_l)$ that are occupied by points of ω, and those points must lie between $\omega(j_l - m)$ and $\omega(j_l + m)$ on the walk. Therefore $j_l - m \leq \sigma_l < j_l < \tau_l \leq j_l + m$ for every l. Consider all possible ways of replacing each subwalk $(\omega(\sigma_l), \ldots, \omega(\tau_l))$ by a subwalk that stays inside $\overline{Q}(j_l)$ and completely covers $Q(j_l)$ [such subwalks exist by Lemma 7.2.4; we can do this operation simultaneously for all subwalks because we have ensured that there is no overlap amongst the subwalks nor amongst the cubes $\overline{Q}(j_l)$]. The result is always a self-avoiding walk ψ on which E^* occurs at least s times, and whose length N' satisfies

$$N' < N + s(2r+5)^d. \tag{7.2.19}$$

Now consider all triples (ω, ψ, J) where: ω is a self-avoiding walk in T_N; $J = \{j_1, \ldots, j_s\}$ is a subset of $\{1, \ldots, N\}$ such that (7.2.17) and (7.2.18) hold, $E_K(m)$ occurs at each j_l in J, and $s = \lfloor \delta N \rfloor$ (here δ is a small positive number that will be specified at the end of the proof); and ψ is a self-avoiding walk that can be obtained from ω and J by the procedure of the preceding paragraph. We shall estimate the number of such triples both from above and below to obtain a contradiction. For both estimates, we shall use the observation that each cube $\overline{Q}(j)$ intersects exactly $V \equiv (4r+9)^d$ cubes of "radius" $r+2$ [this is because $\overline{Q}(j)$ intersects the cube of radius $r+2$ centred at x if and only if $||\omega(j) - x||_\infty \leq 2(r+2)$].

First, the number of such triples is at least the cardinality of T_N times the minimum number of possible choices of J for walks ω in T_N. Each ω in T_N contains at least $a_1 N$ occurrences of $E_K(m)$, and so we can find $h_1 < \ldots < h_u$, where $u = \lfloor a_1 N/((2m+2)V) \rfloor - 2$, such that (*i*) $E_K(m)$ occurs at the h_l-th step of ω for every $l = 1, \ldots, u$, (*ii*) $0 < h_1 - m$, $h_u + m < N$, and $h_l + m < h_{l+1} - m$ for every $l = 1, \ldots, u-1$, and (*iii*) the cubes $\overline{Q}(h_1), \ldots, \overline{Q}(h_u)$ are pairwise disjoint. Clearly, any subset of

$\{h_1, \ldots, h_u\}$ that has cardinality $\lfloor \delta N \rfloor$ is a possible choice for J. So if we set $\rho = a_1/((2m+2)V)$, then (dropping $\lfloor \cdot \rfloor$ from the notation)

$$\text{number of triples} \geq |T_N| \binom{\rho N - 2}{\delta N}. \tag{7.2.20}$$

For an upper bound, consider a triple (ω, ψ, J). Observe that E^* occurs at least $|J| = \lfloor \delta N \rfloor$ times on ψ; it may occur more than $|J|$ times because making a change in a cube $\overline{Q}(j_l)$ can produce occurrences of E^* in some of the cubes of radius $r+2$ that intersect $\overline{Q}(j_l)$. However, since E^* never occurs on ω, we infer that E^* occurs no more than $V|J|$ times on ω. Therefore, given ψ, there are at most $\binom{V\delta N}{\delta N}$ possibilities for the locations of the cubes $\overline{Q}(j_l)$, $l = 1, \ldots, |J|$. Given ψ and the locations of these $|J|$ cubes, each cube $\overline{Q}(j_l)$ determines a subwalk of ψ that replaced some subwalk of ω. Since each of the replaced subwalks of ω had length $2m$ or less, there are at most $(\sum_{i=0}^{2m} c_i)^{\delta N}$ possibilities for ω if we know both ψ and the locations of the $|J|$ cubes. Finally, if we know ω and the locations of the cubes, then J is uniquely determined. So if we define $Z = \sum_{i=0}^{2m} c_i$, then using $\binom{V\delta N}{\delta N} \leq 2^{V\delta N}$ and (7.2.19) we see that

$$\text{number of triples} \leq 2^{V\delta N} Z^{\delta N} \sum_{i=0}^{N+(2r+5)^d \delta N} c_i. \tag{7.2.21}$$

We now combine (7.2.20) and (7.2.21), take N-th roots, and let $N \to \infty$; by (7.2.16), we obtain

$$\mu \frac{\rho^\rho}{\delta^\delta (\rho - \delta)^{\rho - \delta}} \leq 2^{V\delta} \mu^{1+(2r+5)^d \delta} Z^\delta.$$

Setting $Y = 2^V \mu^{(2r+5)^d} Z$ and $t = \delta/\rho$ gives

$$1 \leq (t^t (1-t)^{1-t} Y^t)^\rho.$$

To obtain a contradiction, then, it suffices to show that the function $f(t) = t^t(1-t)^{1-t}Y^t$ is less than 1 for sufficiently small $t > 0$; this is true because $\lim_{t \searrow 0} f(t) = 1$ and $\lim_{t \searrow 0} f'(t) = -\infty$. This contradiction proves the lemma. □

We are now ready to prove Kesten's Pattern Theorem. The ideas for this proof are really the same as those already used in the proof of Lemma 7.2.6.

Proof of Theorem 7.2.3. First, assume without loss of generality that the cube in the statement of the theorem is

$$Q = \{x \in \mathbf{Z}^d : |x_i| \leq r, i = 1, \ldots, d\}.$$

Assume that the theorem is false; then for every $a > 0$,

$$\limsup_{N\to\infty} c_N[aN,(P,Q)]^{1/N} = \mu. \tag{7.2.22}$$

We shall say that E^{**} *occurs at the j-th step of* ω if the cube $\overline{Q}(j)$ is completely covered by ω. By Lemmas 7.2.6 and 7.2.5, there exist $a' > 0$ and m' such that

$$\limsup_{N\to\infty} c_N[a'N, E^{**}(m')]^{1/N} < \mu. \tag{7.2.23}$$

Let $a > 0$ be a small unspecified number, and let H_N denote the following set of walks:

$$\begin{aligned} H_N &= \{\omega \in \mathcal{S}_N : (P,Q) \text{ occurs at most } aN \text{ times on } \omega; \\ &\qquad E^{**}(m') \text{ occurs at least } a'N \text{ times } \}. \end{aligned}$$

The cardinality of H_N satisfies

$$|H_N| \geq c_N[aN,(P,Q)] - c_N[a'N, E^{**}(m)],$$

and therefore, by (7.2.22) and (7.2.23),

$$\lim_{N\to\infty} |H_N|^{1/N} = \mu. \tag{7.2.24}$$

Let δ be a small positive number, to be specified at the end of the proof. Consider all triples (ω, v, J) such that: ω is in H_N; $J = \{j_1, \ldots, j_s\}$ is a subset of $\{1, \ldots, N\}$ such that $E^{**}(m')$ occurs at each j_l, (7.2.17) holds with m replaced by m', and $s = \lfloor \delta N \rfloor$; and v is a self-avoiding walk obtained by replacing the occurrence of $E^{**}(m')$ at each j_l by an occurrence of (P,Q), analogously to the method described in the proof of Lemma 7.2.6 [σ_l and τ_l are defined in the same way, and we use part (b) of Lemma 7.2.4]. We remark that the occurrences of $E^{**}(m')$ guarantee that (7.2.18) holds. Arguing as we did for (7.2.20), we see that

$$\text{number of triples} \geq |H_N| \binom{\rho N - 2}{\delta N}, \tag{7.2.25}$$

where now $\rho = a'/(2m'+2)$. For the upper bound, we use the fact that v has at most $aN + 2m'V\delta N$ occurrences of (P,Q). [This allows for (i) at most aN occurrences of (P,Q) on ω, and (ii) the possibility that changing a single occurrence of $E^{**}(m')$ to a (P,Q) may create several other occurrences of (P,Q) either by creating additional occurrences of P or by vacating sites of other cubes.] Also, note that v has at most N steps. Therefore the analogue of (7.2.21) here is

$$\text{number of triples} \leq 2^{aN+2m'V\delta N} Z'^{\delta N} \sum_{i=0}^{N} c_i, \tag{7.2.26}$$

where $Z' = \sum_{i=0}^{2m'} c_i$. We now combine (7.2.25) and (7.2.26), put $a = \delta$, take N-th roots, and let $N \to \infty$; by (7.2.24), we obtain

$$\mu \frac{\rho^\rho}{\delta^\delta (\rho - \delta)^{\rho - \delta}} \leq 2^{\delta + 2m'V\delta} Z'^{\delta} \mu.$$

As in the proof of Lemma 7.2.6, this leads to a contradiction for sufficiently small δ, and so the theorem is proven. □

7.3 The main ratio limit theorem

The principal task of this section is to prove Equations (7.1.1), (7.1.2), and (7.1.3). The proof of each will be based on Lemma 7.3.1 and Theorem 7.3.2, which will also be used in the next section as the basis for analogous results for walks with specified end patterns.

Lemma 7.3.1 gives three conditions which together are sufficient for the ratio limit theorems to hold. The first two conditions will be relatively easy to verify in our cases of interest; the third will follow from Theorem 7.3.2 below.

Lemma 7.3.1 *Let $\{a_N\}$ be a sequence of positive numbers and let $\phi_N = a_{N+2}/a_N$. Assume that*
(i) $\lim_{N\to\infty} a_N^{1/N} = \mu$,
(ii) $\liminf_{N\to\infty} \phi_N > 0$, *and*
(iii) there exists a constant $D > 0$ such that

$$\phi_N \phi_{N+2} \geq (\phi_N)^2 - \frac{D}{N} \tag{7.3.1}$$

for all sufficiently large N. Then

$$\lim_{N\to\infty} \phi_N = \mu^2. \tag{7.3.2}$$

Proof. First observe that (ii) and (iii) imply that there exists a constant $B > 0$ such that

$$\phi_{N+2} \geq \phi_N - \frac{B}{N} \qquad \text{for all sufficiently large } N. \tag{7.3.3}$$

Let $\sigma_N = \phi_N - \mu^2$. To prove the lemma, we shall show (by contradiction) that the lim sup of σ_N cannot be strictly positive, nor can the lim inf be strictly negative.

First assume that $\limsup_{N\to\infty} \sigma_N > 0$. Then there exists an $\epsilon > 0$ (possibly $\epsilon = +\infty$) and a sequence $N(1) < N(2) < \ldots$ such that $\lim_{j\to\infty} \sigma_{N(j)} = \epsilon$. For each $j \geq 1$, define

$$M(j) = \left\lfloor \frac{N(j)\sigma_{N(j)}}{2B} \right\rfloor;$$

note that $M(j) \to \infty$ as $j \to \infty$. For sufficiently large j and every $0 \leq k < M(j)$, (7.3.3) implies that

$$\begin{aligned} \phi_{N(j)+2k} &\geq \phi_{N(j)} - \frac{kB}{N(j)} \\ &\geq \mu^2 + \sigma_{N(j)} - \frac{M(j)B}{N(j)} \\ &\geq \mu^2 + \frac{\sigma_{N(j)}}{2}. \end{aligned}$$

Therefore

$$\frac{a_{N(j)+2M(j)}}{a_{N(j)}} = \prod_{k=0}^{M(j)-1} \phi_{N(j)+2k} \geq \left(\mu^2 + \frac{\sigma_{N(j)}}{2}\right)^{M(j)}$$

Take $M(j)$-th roots of this inequality, and let $j \to \infty$, obtaining $\mu^2 \geq \mu^2 + \epsilon/2$, which is a contradiction. Therefore $\limsup_{N\to\infty} \sigma_N \leq 0$.

Next, assume that $\liminf_{N\to\infty} \sigma_N < 0$. The procedure is similar to that of the preceding paragraph. Since σ_N is bounded below, there exists an $\epsilon > 0$ and a sequence $N(1) < N(2) < \ldots$ such that $\lim_{j\to\infty} \sigma_{N(j)} = -\epsilon$, and such that $\sigma_{N(j)} < 0$ for every j. Without loss of generality, we can assume that the constant B of (7.3.3) satisfies $B \geq \mu^2$. For each $j \geq 1$, define

$$M(j) = \left\lfloor \frac{N(j)|\sigma_{N(j)}|}{4B} \right\rfloor;$$

since $-\mu^2 < \sigma_{N(j)} < 0$, it follows that

$$M(j) \leq \frac{N(j)\mu^2}{4B} \leq \frac{N(j)}{4}.$$

For sufficiently large j and every $0 < k \leq M(j)$, (7.3.3) implies that

$$\phi_{N(j)-2k} \leq \phi_{N(j)} + \frac{kB}{N(j) - 2k}$$

$$\begin{aligned} &\le \mu^2 - |\sigma_{N(j)}| + \frac{N(j)|\sigma_{N(j)}|}{4(N(j) - N(j)/2)} \\ &= \mu^2 - \frac{|\sigma_{N(j)}|}{2}. \end{aligned}$$

As before, we obtain

$$\frac{a_{N(j)}}{a_{N(j)-2M(j)}} = \prod_{k=1}^{M(j)} \phi_{N(j)-2k} \le \left(\mu^2 - \frac{|\sigma_{N(j)}|}{2}\right)^{M(j)}$$

We take $M(j)$-th roots of this inequality and let $j \to \infty$, obtaining the contradiction $\mu^2 \le \mu^2 - \epsilon/2$. Therefore $\liminf_{N\to\infty} \sigma_N \ge 0$. □

Remark. It is apparent that Lemma 7.3.2 remains true if we replace $N + 2$ by $N + 1$ everywhere. Our inability to prove that $c_{N+1}/c_N \to \mu$ in $\mathbf{Z}^d$ ($d = 2, 3, 4$) is due to the failure of our proof of the corresponding analogue of the next theorem. As will become clear during the course of the proof, the reason for this failure can be seen most simply in Figure 7.4: there does not exist a pair of patterns U and V in $\mathbf{Z}^d$ having the same endpoints whose lengths differ by 1. However on a lattice where such a pair of patterns exists, for example the triangular lattice, we can modify our argument easily to show that $c_{N+1}/c_N \to \mu$ on that lattice.

Theorem 7.3.2 *There exists a constant $D > 0$ such that*

$$\phi_N \phi_{N+2} \ge (\phi_N)^2 - \frac{D}{N} \qquad \textit{for all sufficiently large } N, \qquad (7.3.4)$$

where ϕ_N is defined according to any one of the following:
(a) $\phi_N = c_{N+2}/c_N$ for every N;
(b) $\phi_N = b_{N+2}/b_N$ for every N; or
(c) $\phi_N = c_{N+2}(0,x)/c_N(0,x)$ for all N of the same parity as $||x||_1$, where x is a given point of $\mathbf{Z}^d$.

Proof. First we define two patterns, $U = (u(0), \ldots, u(9))$ and $V = (v(0), \ldots, v(11))$. Each begins at the origin and lies in the (x_1, x_2)-plane (i.e. $u_i(j) = 0 = v_i(j)$ for all $i = 3, \ldots, d$ and every j). The steps in the (x_1, x_2)-plane are $\mathrm{N^3E^3S^3}$ for U, and $\mathrm{N^3ESENES^3}$ for V (see Figure 7.4). Let Q be the cube

$$Q = \{x \in \mathbf{Z}^d : 0 \le x_i \le 3 \text{ for every } i = 1, \ldots, d\},$$

so that U and V are both contained in Q, and their endpoints are corners of Q. The main idea is that (U, Q) and (V, Q) must both occur many times on almost all self-avoiding walks, and changing a U to a V increases the length of a walk by two; this gives us a way to transform N-step walks into $(N+2)$-step walks, and $(N+2)$-step walks into $(N+4)$-step walks. We will then do some counting based on all possibilities for these transformations.

As usual, $\mathcal{S}_N$ is the set of N-step self-avoiding walks whose initial point is the origin. If we are in case (a) of the theorem, let W_N be $\mathcal{S}_N$; if we are in case (b), let W_N be the set of all bridges in $\mathcal{S}_N$; and if we are in case (c), let W_N be the set of all walks ω in $\mathcal{S}_N$ such that $\omega(N) = x$. Let w_N denote the cardinality of W_N. Then

$$\lim_{N\to\infty} w_N^{1/N} = \mu \tag{7.3.5}$$

[where we have used Equation (3.1.10) for part (b) and Corollary 3.2.6 for part (c)]. For integers $N > 0$, $i \geq 0$, and $j \geq 0$, let $W_N(i,j)$ be the set of all walks in W_N on which (U,Q) occurs at precisely i different steps and (V,Q) occurs at precisely j different steps. Let $w_N(i,j)$ denote the cardinality of $W_N(i,j)$. For integers $a, b \geq 0$, define

$$w_N(\geq a, \geq b) = \sum_{i\geq a, j\geq b} w_N(i,j).$$

In particular, $w_N(\geq 0, \geq 0) = w_N$.

Consider the collection of all pairs (ω, ω') such that $\omega \in W_N(i,j)$ and ω' can be obtained from ω by changing one occurrence of (U,Q) to an occurrence of (V,Q). In other words, (ω,ω') is an allowed pair if there exists a k such that (U,Q) occurs at the k-th step of ω, (V,Q) occurs at the k-th step of ω', $\omega(l) = \omega'(l)$ for all $l = 0,\ldots,k$, and $\omega(l) = \omega'(l+2)$ for all $l = k+9,\ldots,N$. In particular, $\omega' \in W_{N+2}(i-1, j+1)$. Counting the number of allowed pairs in two ways, we see that

$$\text{Number of pairs} = i w_N(i,j) = (j+1) w_{N+2}(i-1, j+1).$$

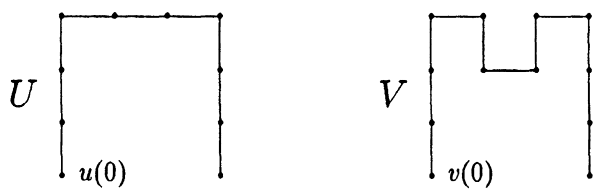

Figure 7.4: The patterns U and V in the (x_1, x_2)-plane.

Therefore

$$w_{N+2}(\geq 0, \geq 1) = \sum_{i\geq 1, j\geq 0} w_{N+2}(i-1, j+1) = \sum_{i\geq 1, j\geq 0} \frac{i w_N(i,j)}{j+1} \tag{7.3.6}$$

and

$$w_{N+4}(\geq 0, \geq 2) = \sum_{i\geq 2, j\geq 0} w_{N+4}(i-2, j+2) = \sum_{i\geq 2, j\geq 0} \frac{i(i-1) w_N(i,j)}{(j+1)(j+2)}. \tag{7.3.7}$$

The Schwarz inequality tells us that

$$\left(\sum_{i\geq 1, j\geq 0} \frac{i w_N(i,j)}{j+1}\right)^2 \leq \left(\sum_{i\geq 1, j\geq 0} w_N(i,j)\right)\left(\sum_{i\geq 1, j\geq 0} \frac{i^2 w_N(i,j)}{(j+1)^2}\right); \tag{7.3.8}$$

combining this with (7.3.6) implies that

$$[w_{N+2}(\geq 0, \geq 1)]^2 \leq w_N \left(\sum_{i\geq 1, j\geq 0} \frac{i^2 w_N(i,j)}{(j+1)^2)}\right). \tag{7.3.9}$$

For $N \geq 1$, define

$$\Xi_N = \frac{w_{N+4}(\geq 0, \geq 2)}{w_N} - \left(\frac{w_{N+2}(\geq 0, \geq 1)}{w_N}\right)^2 \tag{7.3.10}$$

and

$$\hat{\Xi}_N = \phi_N \phi_{N+2} - (\phi_N)^2 - \Xi_N. \tag{7.3.11}$$

The error term $\hat{\Xi}_N$ is easy to bound:

$$\begin{aligned} |\hat{\Xi}_N| &\leq \left|\frac{w_{N+4} - w_{N+4}(\geq 0, \geq 2)}{w_N}\right| + \left|\frac{w_{N+2}^2 - [w_{N+2}(\geq 0, \geq 1)]^2}{w_N^2}\right| \\ &\leq \frac{c_{N+4}[1, (V,Q)]}{w_N} + \frac{2 w_{N+2} c_{N+2}[0, (V,Q)]}{w_N^2}, \end{aligned}$$

and hence Theorem 7.2.3 and Equation (7.3.5) imply that $\hat{\Xi}_N$ decays to 0 exponentially fast. Therefore to prove the theorem it suffices to find a lower bound for Ξ_N of the form $-\text{const.}/N$. By Theorem 7.2.3 and Equation (7.3.5), there exists an $a > 0$ such that

$$\limsup_{N\to\infty} \left(1 - \frac{w_N(\geq 0, \geq aN)}{w_N}\right)^{1/N} < 1. \tag{7.3.12}$$

Using (7.3.7) and (7.3.9),

$$\begin{aligned} \Xi_N &\geq \left(\sum_{i\geq 0, j\geq 0} \frac{i(i-1)w_N(i,j)}{(j+1)(j+2)} - \sum_{i\geq 0, j\geq 0} \frac{i^2 w_N(i,j)}{(j+1)^2} \right) \frac{1}{w_N} \\ &= \left(\sum_{i\geq 0, j\geq 0} \frac{(-i^2 - ij - i)w_N(i,j)}{(j+1)^2(j+2)} \right) \frac{1}{w_N}. \end{aligned}$$

Since $w_N(i,j) = 0$ if $i > N$ or $j > N$, we can bound the factor $-i^2 - ij - i$ below by $-3N^2$. Splitting the sum over j into $aN \leq j \leq N$ and $0 \leq j < aN$, we then obtain

$$\Xi_N \geq \frac{-3N^2 w_N(\geq 0, \geq aN)}{(aN)^3 w_N} + (-3N^2)\left(1 - \frac{w_N(\geq 0, \geq aN)}{w_N}\right).$$

By (7.3.12), the second term in the last line above decays to 0 exponentially fast, and the first term is asymptotic to $-3/a^3 N$. Thus the theorem is proven. □

Before we proceed with the proofs of the main ratio limit theorems, we prove a lemma that will be needed to prove assumption (*ii*) of Lemma 7.3.1 in the fixed-endpoint case.

Lemma 7.3.3 *Let x be a nonzero point of $\mathbf{Z}^d$. Then $c_{N+2}(0,x) \geq c_N(0,x)$ for all sufficiently large N having the same parity as $||x||_1$.*

Proof. The idea is similar to the proof that $c_{N+2} \geq c_N$ as depicted in Figure 7.1, but now we must not touch the endpoints. Fix an integer $A > ||x||_\infty$. Suppose $N > (2A+1)^d$, and let ω be an N-step self-avoiding walk with $\omega(0) = 0$ and $\omega(N) = x$. Then at least one point of ω must lie outside the cube $\{y \in \mathbf{Z}^d : ||y||_\infty \leq A\}$; notice that the endpoints of ω lie inside this cube. Let $M = \max\{||\omega(i)||_\infty : 0 \leq i \leq N\}$. Observe that $M > A$. Then there exists $j \in \{0, \ldots, N\}$ and $i \in \{1, \ldots, d\}$ such that $|\omega_i(j)| = M$. Choose j as small as possible; then, since $\omega(j)$ is not an endpoint of ω, we must have $\omega_i(j) = \omega_i(j+1)$. Let v be the vector whose coordinates are all 0 except the i-th, which is $+1$ if $\omega_i(j) = M$ and is -1 if $\omega_i(j) = -M$. Thus, v is the unit outer normal vector to the cube $\{y : ||y||_\infty \leq M\}$ at the point $\omega(j)$. Define the new $(N+2)$-step walk ω^* by

$$\omega^*(k) = \begin{cases} \omega(k), & k = 0, \ldots, j; \\ \omega(j) + v, & k = j+1; \\ \omega(j+1) + v, & k = j+2; \\ \omega(k-2), & k = j+3, \ldots, N+2. \end{cases}$$

(Thus we replace the step from $\omega(j)$ to $\omega(j+1)$ by three steps.) Then ω^* is self-avoiding, and has the same endpoints as ω.

No two ω's can give rise to the same ω^*, because the the two added points have larger norm $||\cdot||_\infty$ than any other points of ω^* and hence are unambiguously determined. This proves the lemma. □

We are now ready to prove the main ratio limit theorem.

Theorem 7.3.4 *(a)* $\lim_{N\to\infty} c_{N+2}/c_N = \mu^2$.
(b) For every fixed nonzero x in $\mathbf{Z}^d$, $\lim_{N\to\infty} c_{N+2}(0,x)/c_N(0,x) = \mu^2$ *(here, N is restricted to having the same parity as $||x||_1$).*
(c) $\lim_{N\to\infty} q_{2N+2}/q_{2N} = \mu^2$.
(d) $\lim_{N\to\infty} b_{N+1}/b_N = \mu$.

Proof. Part (a) follows immediately from Lemma 7.3.1, Theorem 7.3.2(a), and (7.1.5) (which implies $\phi_N \geq 1$). Similarly, part (b) follows from Lemma 7.3.1, Corollary 3.2.6, Theorem 7.3.2(c), and Lemma 7.3.3. Part (c) is a direct consequence of part (b) and the basic relation (3.2.1).

Part (d) requires some additional work. First we apply Lemma 7.3.1 with $a_N = b_N$ [the hypotheses of the lemma follow from Corollary 3.1.6, Theorem 7.3.2(b), and the inequality $b_{N+2}/b_N \geq 1$, which is a consequence of (1.2.15)] to obtain

$$\lim_{N\to\infty} \frac{b_{N+2}}{b_N} = \mu^2. \tag{7.3.13}$$

For every integer j, define

$$L_j = \liminf_{N\to\infty} \frac{b_{N-j}}{b_N};$$

we want to show that $L_1 = \mu^{-1}$ and that the lim inf is in fact a limit.

By (7.3.13), $L_{j+2} = \mu^{-2} L_j$ for every j. Therefore

$$L_j = \mu^{-j} \text{ for all even } j, \text{ and } L_j = \mu^{1-j} L_1 \text{ for all odd } j.$$

From (4.2.2), we see that for every j and every $N > j$,

$$\frac{b_{N-j}}{b_N} = \sum_{s=1}^{N-j} \lambda_s \frac{b_{N-j-s}}{b_N}.$$

Applying Fatou's Lemma to the above equation gives

$$L_j \geq \sum_{s=1}^{\infty} \lambda_s L_{j+s}. \tag{7.3.14}$$

Define

$$\Sigma_o = \sum_{s\geq 1, s \text{ odd}} \lambda_s \mu^{-s} \quad \text{and} \quad \Sigma_e = \sum_{s\geq 1, s \text{ even}} \lambda_s \mu^{-s}.$$

By (4.2.4), $\Sigma_o + \Sigma_e = 1$. Applying (7.3.14) with $j = 0$ yields

$$1 \geq L_1 \mu \Sigma_o + \Sigma_e,$$

which implies that $L_1\mu \leq 1$. Next, applying (7.3.14) with $j = 1$ gives

$$L_1 \geq \mu^{-1}\Sigma_o + L_1\Sigma_e,$$

which implies that $L_1\mu \geq 1$. Therefore $L_1 = \mu^{-1}$, i.e.

$$\limsup_{N\to\infty} \frac{b_{N+1}}{b_N} = \mu.$$

Combining this with (7.3.13), we finally obtain

$$\liminf_{N\to\infty} \frac{b_{N+1}}{b_N} = \liminf_{N\to\infty} \frac{b_{N+1}}{b_{N-1}} \frac{b_{N-1}}{b_N} = \mu^2 L_1 = \mu,$$

and so part (d) is proven. □

7.4 End patterns

In this section, we shall prove Equations (7.1.8) and (7.1.9), as well as various extensions of these results. To begin, we extend the notion of *front patterns* from Definition 7.1.2 to the analogous notion of *tail patterns.*

Definition 7.4.1 *Let $P = (p(0), \ldots, p(n))$ and $R = (r(0), \ldots, r(m))$ be patterns. Let $\mathcal{T}_N[R]$ denote the subset of walks in $\mathcal{S}_N$ for which R occurs at the $(N-m)$-th step. We say that R is a* proper tail pattern *if $\mathcal{T}_N[R]$ is non-empty for all sufficiently large N. Let $\mathcal{S}_N[P,R]$ denote the intersection of $\mathcal{F}_N[P]$ with $\mathcal{T}_N[R]$. For every x in $\mathbf{Z}^d$, let $\mathcal{S}_N[x;P,R]$ denote the set of all walks in $\mathcal{S}_N[P,R]$ whose last point $\omega(N)$ is x.*

Consideration of front patterns and tail patterns together leads to results such as (7.4.7), which is used to analyze the behaviour of the "slithering-snake" Monte Carlo algorithm in Section 9.4.2, and Proposition 7.4.4, a lower bound for $c_N(0,x)$ which is stronger than the earlier bound (3.2.11).

In Section 6.7, we saw how the lace expansion is used to prove the existence of $\lim_{N\to\infty} |\mathcal{F}_N[P]|/c_N$ in high dimensions. (That section used the notation $P_{n,N}(P)$ to denote $\mathcal{F}_N[P]/c_N$ where $n = |P|$.) This limit is believed to exist in every dimension, but this remains unproven in 2, 3, or 4 dimensions, where the best results are Theorem 7.4.5 below and the following theorem.

Theorem 7.4.2 *If P is a proper front pattern and R is a proper tail pattern, then*

$$\liminf_{N\to\infty} \frac{|\mathcal{F}_N[P]|}{c_N} > 0 \tag{7.4.1}$$

and

$$\liminf_{N\to\infty} \frac{|\mathcal{T}_N[R]|}{c_N} > 0. \tag{7.4.2}$$

Proof. It suffices to prove (7.4.1), since (7.4.2) then follows by considering walks with reversed steps. Suppose $P = (p(0), \ldots, p(n))$. Since P is a proper front pattern, there must be a cube Q and a self-avoiding walk ω^P of length n' with the following properties: ω^P is entirely contained in Q; $\omega^P(n')$ is a corner of Q; and P occurs at the 0-th step of ω^P (see Figure 7.5). By

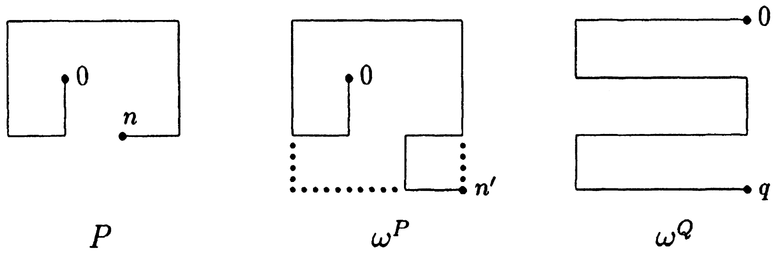

Figure 7.5: Proof of Theorem 7.4.2: the proper front patterns P, ω^P, and ω^Q. The dotted lines in the centre picture denote the boundary of Q.

Lemma 7.2.4(a), there exists a self-avoiding walk ω^Q whose last point equals $\omega^P(n')$, whose first point is another corner of Q, which is entirely contained in Q and visits every point of Q. Let q denote the number of steps in ω^Q. Evidently, $q \geq n'$.

Our first observation is that for every $N \geq n'$

$$|\mathcal{F}_N[P]| \geq |\mathcal{F}_N[\omega^P]| \geq |\mathcal{F}_{N+q-n'}[\omega^Q]|. \tag{7.4.3}$$

The first inequality is obvious, since $\mathcal{F}_N[\omega^P] \subset \mathcal{F}_N[P]$. For the second, we can define a one-to-one transformation $\omega \mapsto \omega^*$ from $\mathcal{F}_{N+q-n'}[\omega^Q]$ to $\mathcal{F}_N[\omega^P]$ as follows: for each ω in $\mathcal{F}_{N+q-n'}[\omega^Q]$, let ω^* be the (unique) N-step walk that has ω^P occurring at its 0-th step and whose last $N-n'$ steps are identical to those of ω, and translated so that $\omega^*(0) = 0$ (see Figure 7.6). Then ω^* is self-avoiding, hence it must be in $\mathcal{F}_N[\omega^P]$. Now, because of the observation (7.4.3), and because $c_N \leq c_{N+q-n'}$ [by (7.1.6)], it suffices

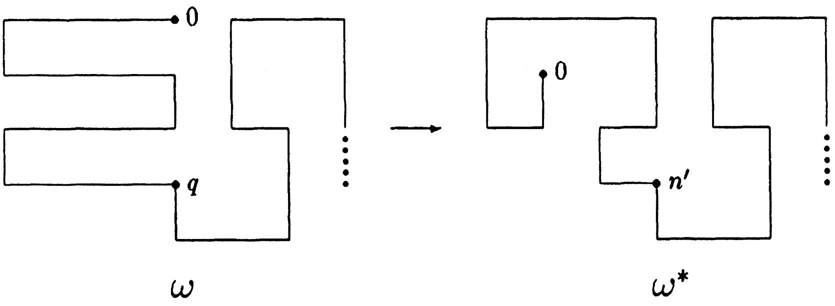

Figure 7.6: Proof of Theorem 7.4.2: the transformation of ω in $\mathcal{F}_{N+q-n'}[\omega^Q]$ to ω^* in $\mathcal{F}_N[\omega^P]$.

to show that

$$\liminf_{M\to\infty} \frac{|\mathcal{F}_M[\omega^Q]|}{c_M} > 0. \quad (7.4.4)$$

[We remark that it is not necessary to invoke (7.1.6) here; we could instead use (7.1.5) and $c_N \leq 2dc_{N-1}$ to conclude that $c_N \leq 2dc_{N+q-n'}$, which suffices for (7.4.4).]

Since ω^Q is a proper internal pattern (see Proposition 7.1.3), Theorem 7.2.3 says that there exists an $\epsilon > 0$ and an even integer k such that

$$c_k[0,\omega^Q] \leq (\mu(1-\epsilon))^k. \quad (7.4.5)$$

For all integers $l \geq j \geq 0$, let $\mathcal{G}_{l,j}$ be the set of walks ω in $\mathcal{S}_l$ such that ω^Q occurs at the j-th step of ω, and let $\mathcal{H}_{l,j} = \cup_{i=0}^{j}\mathcal{G}_{l,i}$. Thus $\mathcal{H}_{l,j}$ is the set of l-step self-avoiding walks starting at the origin on which ω^Q occurs at one of the first j steps. Then by (7.4.5),

$$\begin{aligned} |\mathcal{S}_{M+k} \setminus \mathcal{H}_{M+k,k}| &\leq c_k[0,\omega^Q]c_M \\ &\leq \mu^k(1-\epsilon)^k c_M. \end{aligned}$$

Therefore

$$\begin{aligned} c_{M+k} - c_M\mu^k(1-\epsilon)^k &\leq |\mathcal{H}_{M+k,k}| \\ &\leq \sum_{j=0}^{k} |\mathcal{G}_{M+k,j}| \\ &\leq \sum_{j=0}^{k} c_j|\mathcal{F}_M[\omega^Q]|c_{k-j}. \end{aligned}$$

We divide this by $\mu^k c_M$, obtaining

$$\frac{c_{M+k}}{\mu^k c_M} - (1-\epsilon)^k \leq \left(\sum_{j=0}^{k} \frac{c_j c_{k-j}}{\mu^k} \right) \frac{|\mathcal{F}_M[\omega^Q]|}{c_M}. \tag{7.4.6}$$

Since k is a fixed even number, Theorem 7.3.4(a) implies that c_{M+k}/c_M converges to μ^k, and hence the left side of (7.4.6) has a strictly positive limit as $M \to \infty$. This proves (7.4.4) and the theorem. □

An extension of the preceding proof allows one to prove the stronger statement

$$\liminf_{N\to\infty} \frac{|\mathcal{S}_N[P,R]|}{c_N} > 0 \tag{7.4.7}$$

whenever P and R are proper front and tail patterns, respectively. For details, see Madras (1988).

We shall now consider the occurrence of end patterns on walks with specified endpoints.

Proposition 7.4.3 *Let e be a point in $\mathbf{Z}^d$ with $|e| = 1$. Let P and R be patterns such that $\mathcal{S}_N[e;P,R]$ is non-empty for all sufficiently large odd N. Then*

$$\liminf_{N\to\infty, N \text{ odd}} \frac{|\mathcal{S}_N[e;P,R]|}{c_N(0,e)} > 0.$$

Proof. Assume $P = (p(0),\ldots,p(n))$ and $R = (r(0),\ldots,r(m))$ with $p(0) = 0$ and $r(m) = e$. Let $P' = (r(0),\ldots,r(m),p(0),\ldots,p(n))$. Since the pattern P' can occur on arbitrarily large self-avoiding polygons, P' must be a proper internal pattern. Therefore, by Theorem 7.2.3 there exists an $\epsilon > 0$ such that

$$\limsup_{N\to\infty} (c_N[\epsilon N, P'])^{1/N} < \mu. \tag{7.4.8}$$

Let $\mathcal{S}_N(e)$ be the set of walks in $\mathcal{S}_N$ having $\omega(N) = e$, and let $\mathcal{S}'_N(e)$ be the set of walks in $\mathcal{S}_N(e)$ on which P' occurs at more than ϵN different steps. By (7.4.8) and Corollary 3.2.6,

$$|\mathcal{S}'_N(e)| \geq c_N(0,e) - c_N[\epsilon N, P'] \geq \frac{1}{2} c_N(0,e) \tag{7.4.9}$$

for all sufficiently large (odd) N. If ω is in $\mathcal{S}_N(e)$ and P' occurs at the j-th step of ω, then $(\omega(j+m+1),\ldots,\omega(N),\omega(0),\ldots,\omega(j+m))$ is a translation of a self-avoiding walk ψ in $\mathcal{S}_N[e;P,R]$. Consider all pairs (ω,ψ) such that $\omega \in \mathcal{S}'_N(e)$ and ψ can be obtained from ω in this way. On the one hand, since each ω gives rise to at least ϵN different ψ's, the number of such pairs is bounded below by $\epsilon N|\mathcal{S}'_N(e)|$. On the other hand, each ψ can be obtained from no more than N different ω's, and so the number of pairs is bounded above by $N|\mathcal{S}_N[e; P, R]|$. Therefore

$$N|\mathcal{S}_N[e; P, R]| \geq \epsilon N|\mathcal{S}'_N(e)| \geq \frac{\epsilon N}{2} c_N(0, e)$$

for all sufficiently large N [the second inequality is given by (7.4.9)]. The theorem follows. □

One would like to prove the analogue of Proposition 7.4.3 when e is replaced by any given point x in $\mathbf{Z}^d$ (and N is restricted to having the same parity as $||x||_1$). However, the best known result is the following. Let e be a nearest neighbour of the origin, and let x be a non-zero point in $\mathbf{Z}^d$. Assume that $\mathcal{S}_N[x; P, R]$ is non-empty for all sufficiently large N with the same parity as $||x||_1$. Then for odd $||x||_1$ we have

$$\liminf_{N\to\infty, N \text{ odd}} \frac{|\mathcal{S}_N[x; P, R]|}{c_N(0, e)} > 0, \tag{7.4.10}$$

and for even $||x||_1$ (7.4.10) holds after we replace N by $N+1$ in the numerator. For the proof, see Madras (1988). If we knew that $c_N(0, e)/c_{N'}(0, x)$ (where N' equals N or $N+1$, according to whether $||x||_1$ is odd or even) had a positive lower bound for sufficiently large N, then we could immediately deduce the desired analogue of Proposition 7.4.3. Unfortunately, it remains an open problem to prove this lower bound, which is a particular case of Conjecture 1.4.1. We can however use Proposition 7.4.3 to prove a corresponding upper bound. This does not help to generalize Proposition 7.4.3, but it does prove a special case of Conjecture 1.4.1.

Proposition 7.4.4 *Let e and x be non-zero points of $\mathbf{Z}^d$, with $||e||_2 = 1$. Then there exists a positive constant A and an integer N_A (both depending on x) such that*

$$c_N(0, e) \leq A c_N(0, x) \qquad \textit{for all } N \geq N_A$$

if $||x||_1$ is odd, and

$$c_N(0, e) \leq A c_{N+1}(0, x) \qquad \textit{for all } N \geq N_A$$

if $||x||_1$ is even.

Proof. Let $(r(0), \ldots, r(m+1))$ be a proper internal pattern having $r(0) = x$, $r(m) = e$, and $r(m+1) = 0$. Then m has the opposite parity to $||x||_1$. Let $R = (r(0), \ldots, r(m))$ and let P be the 0-step pattern (0). Then

Proposition 7.4.3 holds for this P and R, so there exists a $\kappa > 0$ such that $|\mathcal{S}_N[e; P, R]| \geq \kappa c_N(0, e)$ for all sufficiently large N. The first $N - m$ steps of a walk in $\mathcal{S}_N[e; P, R]$ is a self-avoiding walk from 0 to x, and so $|\mathcal{S}_N[e; P, R]| \leq c_{N-m}(0, x)$. The proposition now follows from these two inequalities and Lemma 7.3.3. □

We are now ready to prove ratio limit theorems for the number of walks with specified end patterns. The procedure is the same as in Section 7.3.

Theorem 7.4.5 *Let P be a proper front pattern and let R be a proper tail pattern. Then:*
(a) $\lim_{N\to\infty} |\mathcal{F}_{N+2}[P]|/|\mathcal{F}_N[P]| = \mu^2$.
(b) $\lim_{N\to\infty} |\mathcal{S}_{N+2}[P, R]|/|\mathcal{S}_N[P, R]| = \mu^2$.
(c) Suppose in addition that x is a fixed nonzero point of $\mathbf{Z}^d$ and that $\mathcal{S}_N[x; P, R]$ is non-empty for all sufficiently large N having the same parity as $||x||_1$. Then $\lim_{N\to\infty} |\mathcal{S}_{N+2}[x; P, R]|/|\mathcal{S}_N[x; P, R]| = \mu^2$ (where N is restricted to having the same parity as $||x||_1$).

Proof. We apply Lemma 7.3.1 in each case. Beginning with the most substantial hypothesis of the lemma, we observe that the analogue of Theorem 7.3.2 holds in each of the three present cases. In fact, the same proof works, with the following modifications:

1. Let W_N be $\mathcal{F}_N[P]$ in case (a), $\mathcal{S}_N[P, R]$ in case (b), and $\mathcal{S}_N[x; P, R]$ in case (c).

2. In the definition of $W_N(i, j)$, count only those occurrences of (U, Q) and (V, Q) which do not touch the end patterns; i.e. only count occurrences after the $|P|$-th step, and no later than the $(N - |R| - 9)$-th step for (U, Q) and the $(N - |R| - 11)$-th step for (V, Q).

3. $\lim_{N\to\infty} w_N^{1/N} = \mu$ by Theorem 7.4.2 for case (a), Equation (7.4.7) for case (b), and Equation (7.4.10) and Corollary 3.2.6 for case (c).

Now we verify that the hypotheses of Lemma 7.3.1 all hold. The previous paragraph shows that assumption (*iii*) of Lemma 7.3.1 holds in each of the present three cases. Also, assumption (*i*) holds in each case by point 3 in the preceding paragraph. So it only remains to check the second assumption of Lemma 7.3.1 in each case.

For case (a), Theorem 7.3.4(a) and Theorem 7.4.2 imply that

$$\liminf_{N\to\infty} \frac{|\mathcal{F}_{N+2}[P]|}{|\mathcal{F}_N[P]|} \geq \liminf_{N\to\infty} \frac{|\mathcal{F}_{N+2}[P]|}{c_N} = \liminf_{N\to\infty} \frac{|\mathcal{F}_{N+2}[P]|}{c_{N+2}} \mu^2 > 0.$$

Case (b) is similar, using (7.4.7). For case (c), we use the inequality $|\mathcal{S}_{N+2}[x;P,R]| \geq |\mathcal{S}_N[x;P,R]|$ for all sufficiently large N (the proof is exactly as the same as for Lemma 7.3.3, except that A must be taken large enough so that $\omega(j)$ is not on either end pattern; $A = \max\{|P|, ||x||_1 + |R|\}$ suffices). □

7.5 Notes

Sections 7.2 and 7.3. The results of these sections are due to Kesten (1963). In that paper Kesten also proved the following bounds on the convergence rates in the ratio limit theorems: for all sufficiently large N,

$$\left|\frac{c_{N+2}}{c_N} - \mu^2\right| \leq KN^{-1/3}, \tag{7.5.1}$$

$$-KN^{-1/3} \leq \frac{c_{N+2}(0,x)}{c_N(0,x)} - \mu^2 \leq KN^{-1/4}, \tag{7.5.2}$$

where K is a constant [and N has the same parity as $||x||_1$ in (7.5.2)].

We conjecture that the following strengthening of the Pattern Theorem is true: for every proper internal pattern P, there exists a $t = t(P) > 0$ such that for any $\epsilon > 0$ only exponentially few N-step walks have fewer than $(t-\epsilon)N$ or more than $(t+\epsilon)N$ occurrences of P. A more modest open problem is to prove that the expected number of occurrences of a proper internal pattern P on an N-step walk is asymptotic to tN as $N \to \infty$, for some $t = t(P) > 0$ (where expectation is with respect to the uniform probability measure on $\mathcal{S}_N$).

The proof of part (d) of Theorem 7.3.4 is essentially a special case of a ratio limit theorem in renewal theory; see Proposition 1.2 in Chapter 3 of Orey (1971).

Section 7.4. The results of this section are due to Madras (1988). That paper also showed that the convergence rate of (7.5.1) holds in Theorem 7.4.5(a,b), and the rate of (7.5.2) holds in Theorem 7.4.5(c).