Chapter 6

Above four dimensions

6.1 Overview of the results

The lace expansion has been used to resolve many of the issues concerning the self-avoiding walk in five or more dimensions. Proving convergence of the lace expansion for $d = 5$ involves a myriad of major technical difficulties, due to the fact that the best bound on the small parameter responsible for convergence of the expansion, namely $||H_{z_c}||_2^2 = \mathsf{B}(z_c) - 1$, is 0.493. However many of these technical difficulties are not present if the small parameter can be taken to be arbitrarily small, and it is in the context of an arbitrarily small parameter that the proof becomes most transparent. For this reason, in this chapter we give the proof of convergence of the lace expansion and its consequences for the critical behaviour in two contexts: for the nearest-neighbour model with large d, and for the "spread-out" self-avoiding walk with steps (x, y) satisfying $0 < ||x - y||_\infty \leq L$, for $d > 4$ and large L.

For each of these two models we will use Ω to denote the coordination number, i.e. $\Omega = 2d$ for the nearest-neighbour model and $\Omega = (2L + 1)^d - 1$ for the spread-out model. It will be shown that in either case the behaviour of $||H_{z_c}||_2^2$ is governed by the contribution to the corresponding ordinary random walk critical ($z = \Omega^{-1}$) bubble diagram due to the Ω terms in which two single step walks end at the same site, i.e. $\Omega/\Omega^2 = \Omega^{-1}$. Hence the small parameter can be made arbitrarily small by increasing Ω.

In the remainder of this section we summarize the results that will be obtained in this chapter. We discuss both the nearest-neighbour model and the spread-out model simultaneously, combining the statements that d is sufficiently large for the nearest-neighbour model, and L is sufficiently large for the spread-out model, into the single statement that Ω is sufficiently

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large. We emphasize that all of the results stated in this section, with the exception of Theorem 6.1.3, have been proven in Hara and Slade (1992a,b) for the nearest-neighbour model for $d \geq 5$.

Asymptotic formulas for c_n and the mean-square displacement are given in the following theorem, whose proof can be found in Section 6.4.2.

Theorem 6.1.1 *There is an Ω_0 such that for $\Omega \geq \Omega_0$ there are positive A, D such that the following hold (assuming $d > 4$ for the spread-out model).*

(a) $c_n = A\mu^n[1 + O(n^{-\epsilon})]$ *as $n \to \infty$, for any $\epsilon < \min\{(d-4)/2, 1\}$.*

(b) $\langle |\omega(n)|^2 \rangle = Dn[1 + O(n^{-\epsilon})]$ *as $n \to \infty$, for any $\epsilon < \min\{(d-4)/4, 1\}$.*

Remark. Bounds on the constants A and D will be given in Section 6.2.3. In particular, for the nearest-neighbour model in high dimensions D is strictly greater than one, indicating that the self-avoiding walk does move away from the origin more quickly than ordinary random walk, although only at the level of the diffusion constant. For the nearest-neighbour model in five dimensions the current best bounds are given in Hara and Slade (1992b) to be $1 \leq A \leq 1.493$ and $1.098 \leq D \leq 1.803$.

A corollary of (a) is that $\lim_{n\to\infty} c_{n+1}/c_n = \mu$ [cf. Equation (7.1.4)]. This is believed to be true in all dimensions, but remains unproved for $d = 2, 3, 4$. Theorem 6.1.1 is proven via a Tauberian-type theorem, after first controlling the susceptibility and correlation length of order two. The results for χ and ξ_2 are stated in the next theorem, which is proved in Section 6.2.3. [The notation $f(z) \sim g(z)$ means $\lim_{z \nearrow z_c} f(z)/g(z) = 1$.]

Theorem 6.1.2 *There is an Ω_0 such that for $\Omega \geq \Omega_0$ (assuming $d > 4$ for the spread-out model)*

$$\chi(z) \sim \frac{Az_c}{z_c - z}$$

and

$$\xi_2(z) \sim \left(\frac{Dz_c}{z_c - z}\right)^{1/2},$$

where the constants A, D are the same as in Theorem 6.1.1.

For $c_n(0, x)$ we will prove the following theorem, which gives the hyperscaling inequality $\alpha_{sing} - 2 \leq -d/2$. In fact this inequality is believed to be an equality; see Section 2.1. Theorem 6.1.3 is the only result stated in this section which has not been proved for the nearest-neighbour model for all $d \geq 5$.

Theorem 6.1.3 *There is an Ω_0 such that for $\Omega \geq \Omega_0$ (assuming $d > 4$ for the spread-out model) there is a constant B such that*

$$\sup_{x \in Z^d} c_n(0,x) \leq B\mu^n n^{-d/2}.$$

This theorem is proved in Section 6.8. An immediate consequence of Theorem 6.1.3 is the following result, which is a weaker version of the statement that $\alpha_{sing} - 2 \leq -d/2$. This weaker statement has been proven for the nearest-neighbour model for all $d \geq 5$; we comment briefly on the method of proof in the Notes for this chapter.

Corollary 6.1.4 *There is an Ω_0 such that for $\Omega \geq \Omega_0$ (assuming $d > 4$ for the spread-out model)*

$$\sup_{x \in Z^d} \sum_{n=0}^{\infty} n^a c_n(0,x)\mu^{-n} < \infty$$

for any $a < (d-2)/2$.

For the correlation length $\xi(z) = 1/m(z)$ [see (1.3.15)] we have the following result, which is proved in Section 6.5.1.

Theorem 6.1.5 *There is an Ω_0 such that for $\Omega \geq \Omega_0$ (assuming $d > 4$ for the spread-out model)*

$$\xi(z) \sim \sqrt{\frac{D}{2d}} \left(\frac{z_c}{z_c - z}\right)^{1/2},$$

with the same constant D as in Theorem 6.1.1.

By Theorems 6.1.1, 6.1.2 and 6.1.5, the length scales defined by the mean square displacement, the correlation length of order two, and the correlation length are as expected all governed by the same critical exponent $\nu = 1/2$.

Using Theorem 6.1.5 it can be shown that the renormalized coupling constant $g(z)$ of (1.4.22) obeys

$$g(z) \simeq (z_c - z)^{(d-4)/2} \quad \text{as } z \nearrow z_c, \tag{6.1.1}$$

for the spread-out model with Ω sufficiently large and for the nearest-neighbour model for $d \geq 6$. Unfortunately (6.1.1) remains unproven for $d = 5$. Further details are given in the Remark under Theorem 1.5.5.

The results for the critical two-point function are stronger in k-space than in x-space, and are summarized in the following theorem, whose proof can be found in Section 6.5.2. The upper bound on $G_{z_c}(0,x)$ in the theorem, for $p < (d-2)/2$, follows immediately from Corollary 6.1.4 and the fact that $|x|^p c_n(0,x) \leq n^p c_n(0,x)$. The k-space result provides a strong infrared bound.

Theorem 6.1.6 *There is an Ω_0 such that for $\Omega \geq \Omega_0$ (assuming $d > 4$ for the spread-out model) the following hold. For any p satisfying $p < (d-2)/2$ or $p \leq 2$, there is a constant $C(p)$ such that for all x, $G_{z_c}(0,x) \leq C(p)|x|^{-p}$. There is a positive constant such that the Fourier transform satisfies $\hat{G}_{z_c}(k) = const.[k^2 + O(k^{2+\epsilon})]^{-1}$ as $k \to 0$, for any $\epsilon < \min\{(d-4)/2, 1\}$. In addition, there is a positive constant such that $0 \leq \hat{G}_{z_c}(k) \leq const.k^{-2}$ for all $k \in [-\pi,\pi]^d$.*

Corollary 6.1.7 *There is an Ω_0 such that for $\Omega \geq \Omega_0$ (assuming $d > 4$ for the spread-out model)*

$$m(z_c) = 0.$$

Proof. The bound on $\hat{G}_{z_c}(k)$ of Theorem 6.1.6 implies that the critical bubble diagram $\mathsf{B}(z_c) = (2\pi)^{-d}\int_{[-\pi,\pi]^d} \hat{G}_{z_c}(k)^2 d^d k$ is finite (see Section 1.5). It then follows from Theorem 4.1.6 that $m(z_c) = 0$. □

To discuss the scaling limit, we first introduce some notation. Let $C_d[0,1]$ denote the continuous $\mathbf{R}^d$-valued functions on $[0,1]$, equipped with the supremum norm. Given an n-step self-avoiding walk ω, we define $X_n \in C_d[0,1]$ by setting $X_n(k/n) = (Dn)^{-1/2}\omega(k)$ for $k = 0,1,2,\ldots,n$, and taking $X_n(t)$ to be the linear interpolation of this. We denote by dW the Wiener measure on $C_d[0,1]$. Expectation with respect to the uniform measure on the n-step self-avoiding walks is denoted by $\langle\cdot\rangle_n$. The following theorem is proved in Section 6.6.

Theorem 6.1.8 *There is an Ω_0 such that for $\Omega \geq \Omega_0$ (assuming $d > 4$ for the spread-out model), the scaled self-avoiding walk converges in distribution to Brownian motion. In other words for any bounded continuous function f on $C_d[0,1]$,*

$$\lim_{n\to\infty}\langle f(X_n)\rangle_n = \int f dW.$$

The next result concerns the existence of a measure on infinitely long self-avoiding walks. We defer the precise definition of this measure until Section 6.7, where the following theorem will be proved.

Theorem 6.1.9 *There is an Ω_0 such that for $\Omega \geq \Omega_0$ (assuming $d > 4$ for the spread-out model) the infinite self-avoiding walk exists.*

The key ingredient in the proofs of the above theorems is the convergence of the lace expansion, which is proved in the next section.

6.2 Convergence of the lace expansion

This section is divided into three parts. The first part proves a lemma which encapsulates the basic structure of the proof of convergence of the lace expansion, and also gives a number of properties of simple random walk which will be needed in the convergence proof. The second part gives the proof of convergence of the lace expansion, and states a number of consequences. The last part gives the proof of Theorem 6.1.2, i.e. existence of and mean-field values for the critical exponents for the susceptibility and the correlation length of order two.

6.2.1 Preliminaries

The following elementary lemma will be used to prove convergence of the lace expansion. It states that under an appropriate continuity assumption, if a set of inequalities implies a stronger set of inequalities, then in fact the stronger inequalities must hold.

Lemma 6.2.1 *Let $f_1, \ldots, f_n$ be nonnegative functions defined on the interval $[0, p_1)$, and let $p_0 \in [0, p_1)$ and $a < 1$ be given. Suppose that*

1. *f_i is continuous on the interval $[0, p_1)$, for $i = 1, \ldots, n$,*
2. *$f_i(p) \leq a$ for $0 \leq p \leq p_0$, for $i = 1, \ldots, n$,*
3. *for each $p \in [p_0, p_1)$, if $f_i(p) \leq 1$ for all $i = 1, \ldots, n$, then in fact $f_i(p) \leq a$ for all $i = 1, \ldots, n$. (In other words a set of inequalities implies a stronger set of inequalities.)*

Then $f_i(p) \leq a$ for all $p \in [0, p_1)$ and all $i = 1, \ldots, n$.

Proof. Define $f_{max}(p) = \max_{1 \leq i \leq n} f_i(p)$. By the second assumption, it suffices to show that $f_{max}(p) \leq a$ for $p \in [p_0, p_1)$. By the third assumption $f_{max}(p) \notin (a, 1]$ for all $p \in [p_0, p_1)$. By the first assumption $f_{max}(p)$ is continuous in $p \in [0, p_1)$. Since $f_{max}(p_0) \leq a$ by the second assumption, the above two facts imply that $f_{max}(p)$ cannot enter the forbidden interval $(a, 1]$ when $p \in [p_0, p_1)$ and hence $f_{max}(p) \leq a$ for all $p \in [0, p_1)$. □

Before defining the functions f_i that we will use, we need to introduce two models of ordinary random walk corresponding to the two models of self-avoiding walk discussed in the previous section. For the usual nearest-neighbour simple random walk we denote the coordination number by $\Omega = 2d$, and also use Ω to denote the set of sites which are nearest neighbours of the origin. The critical ($z = \Omega^{-1}$) two-point function for this model is shown in (A.8) to be given by

$$C^{(0)}(0,x) = \int_{[-\pi,\pi]^d} \frac{e^{-ik\cdot x}}{1-\hat{D}_0(k)} \frac{d^dk}{(2\pi)^d}, \tag{6.2.1}$$

where

$$\hat{D}_0(k) = \frac{1}{\Omega}\sum_{x\in\Omega} e^{ik\cdot x} = d^{-1}\sum_{\mu=1}^{d} \cos k_\mu. \tag{6.2.2}$$

Let $L \geq 1$ be an integer. For the ordinary "spread-out" random walk in $\mathbf{Z}^d$ whose steps (x, y) satisfy $0 < ||x-y||_\infty \leq L$, we will use Ω to denote the set of $x \in \mathbf{Z}^d$ with $0 < ||x||_\infty \leq L$, and also write Ω for the cardinality of this set, i.e. $\Omega = (2L+1)^d - 1$. For $x \in \mathbf{Z}^d$, let $C^{(L)}(0,x)$ denote the critical spread-out ordinary random walk two-point function. This is given in (A.8) by

$$C^{(L)}(0,x) = \int_{[-\pi,\pi]^d} \frac{e^{-ik\cdot x}}{1-\hat{D}_L(k)} \frac{d^dk}{(2\pi)^d}, \tag{6.2.3}$$

where

$$\hat{D}_L(k) = \frac{1}{\Omega}\sum_{x\in\Omega} e^{ik\cdot x} = \frac{1}{\Omega}\sum_{x\in\Omega} \cos(k\cdot x). \tag{6.2.4}$$

We write simply $C(0,x)$ and $\hat{D}(k)$ when we wish to discuss both the spread-out and nearest-neighbour models simultaneously. The following lemma is a combination of the statements of Lemmas A.3 and A.5, in which some bounds have been degraded for a unified statement.

Lemma 6.2.2 *For any $d \geq 1$ there is an Ω_0 such that for any $k \in [-\pi,\pi]^d$ and $\Omega \geq \Omega_0$,*

$$1 - \hat{D}(k) \geq \frac{k^2}{2\pi^2 d}. \tag{6.2.5}$$

For any $d \geq 1$ there is an Ω_0 such that for all $\Omega \geq \Omega_0$

$$\sup_{n\geq 0} n^{d/2}||\hat{D}^n||_1 < \infty. \tag{6.2.6}$$

Let s denote a fixed small positive number for the spread-out model, and let $s = 0$ for the nearest-neighbour model. There is a K such that for all

Ω (assuming $d > 4$ for the spread-out model and $d \geq 5$ for the nearest-neighbour model)

$$||\hat{C}||_2^2 - 1 = \left|\left|\frac{1}{1-\hat{D}}\right|\right|_2^2 - 1 \leq K\Omega^{-1+s}, \tag{6.2.7}$$

and

$$\left|\left|\frac{\partial_\mu^2 \hat{D}}{[1-\hat{D}]^2}\right|\right|_1 + 2\left|\left|\frac{(\partial_\mu \hat{D})^2}{[1-\hat{D}]^3}\right|\right|_1 \leq K\Omega^{-1+s+2/d} \tag{6.2.8}$$

(the $2/d$ in the exponent can be omitted for the nearest-neighbour model). The above norms are all L^p norms on $[-\pi,\pi]^d$ with measure $(2\pi)^{-d}d^dk$. The constant K depends on the dimension (but not on L) for the spread-out model, and is a universal constant for the nearest-neighbour model.

In the following we will maintain the convention that K and Ω_0 depend on the dimension when a statement is applied to the spread-out model, but are universal constants when the same statement is applied to the nearest-neighbour model.

6.2.2 The convergence proof

To prove convergence of the lace expansion, we will use Lemma 6.2.1 with $n = 2$, $p_0 = \Omega^{-1}$, $p_1 = z_c$, $a = 2/3$,

$$f_1(p) = \frac{||H_p||_2^2}{3K\Omega^{-1+s}} \quad \text{and} \quad f_2(p) = \frac{||\, x_\mu^2 G_p||_\infty}{3K\Omega^{-1+s+2/d}}, \tag{6.2.9}$$

with K the constant of Lemma 6.2.2. Here s is as in the statement of Lemma 6.2.2, and the $2/d$ can be omitted from the exponent in the definition of f_2 for the nearest-neighbour model.

The following three results confirm that the hypotheses of Lemma 6.2.1 are satisfied, either for the nearest-neighbour model in sufficiently high dimensions, or for the spread-out model in more than four dimensions with Ω sufficiently large. It will then follow from the lemma that $||H_p||_2^2$ and $||\, x_\mu^2 G_p||_\infty$ are both small (for large Ω) uniformly in $p \in [0, z_c)$. This will give good bounds on the lace expansion, when combined with Theorem 5.4.4.

For simplicity we deal explicitly only with the strictly self-avoiding walk, although the results of this section also hold for all finite memories $2 \leq \tau < \infty$, subject to the replacement of z_c by the finite memory critical point $z_c(0;\tau)$. In particular, the constants of Corollaries 6.2.6 and 6.2.7 and Theorem 6.2.9 are independent of τ. Finite memory is used only to prove the bound on $c_n(0,x)$ of Theorem 6.1.3.

Lemma 6.2.3 *The above functions f_1 and f_2 are continuous on the interval $[0, z_c)$.*

Proof. We begin with f_1. Since the subcritical two-point function decays exponentially by (1.3.14), $||H_p||_2^2$ is finite for $p < z_c$. This norm can be rewritten as a power series in p with positive coefficients, which therefore must have radius of convergence at least z_c. Hence it is continuous in $p \in [0, z_c)$.

For f_2, we fix $r \in [0, z_c)$. Arguing as in the derivation of (1.3.14), there is a constant M, depending on r but not on x, such that for any $p \in [0, r]$ and any x,

$$\frac{d}{dp} x_\mu^2 G_p(0, x) \leq M. \tag{6.2.10}$$

Hence for $p_1 < p_2 \leq r$ we have

$$\begin{aligned} 0 &\leq f_2(p_2) - f_2(p_1) \\ &\leq (3K)^{-1}\Omega^{1-s-2/d} \sup_x x_\mu^2 [G_{p_2}(0, x) - G_{p_1}(0, x)] \\ &\leq (3K)^{-1}\Omega^{1-s-2/d} M(p_2 - p_1). \end{aligned}$$

This implies continuity of f_2 for $p < r$, and hence for $p < z_c$ since r is arbitrary. □

Lemma 6.2.4 *For $p \in [0, \Omega^{-1}]$, $f_i(p) \leq 1/3$ for $i = 1, 2$.*

Proof. For $p \in [0, \Omega^{-1}]$, $G_p(0, x) \leq G_{1/\Omega}(0, x)$. Since in general the self-avoiding walk two-point function is bounded above by the ordinary random walk two-point function having the same activity, $G_p(0, x) \leq G_{1/\Omega}(0, x) \leq C(0, x)$. Now $H_p(0, x) = G_p(0, x) - \delta_{0,x}$, so $||H_p||_2^2 = ||G_p||_2^2 - 1 \leq ||C||_2^2 - 1$. Hence by the Parseval relation $||H_p||_2^2 \leq ||\hat{C}||_2^2 - 1$, and the desired bound on f_1 follows from (6.2.7). For f_2 we use the Fourier transform to write

$$\begin{aligned} x_\mu^2 G_p(0, x) &\leq x_\mu^2 C(0, x) = -\int_{[-\pi,\pi]^d} \partial_\mu^2 \hat{C}(k) e^{-ik\cdot x} \frac{d^d k}{(2\pi)^d} \\ &= -\int_{[-\pi,\pi]^d} \left[\frac{\partial_\mu^2 \hat{D}}{(1-\hat{D})^2} + \frac{2(\partial_\mu \hat{D})^2}{(1-\hat{D})^3} \right] e^{-ik\cdot x} \frac{d^d k}{(2\pi)^d}. \end{aligned} \tag{6.2.11}$$

The desired bound then follows from (6.2.8). □

This leaves the last and most substantial assumption of Lemma 6.2.1 to be shown. The following result confirms that the final hypothesis of

Lemma 6.2.1 is satisfied for f_1 and f_2 of (6.2.9), i.e. that for each $p \in [\Omega^{-1}, z_c)$, if $f_i(p) \leq 1$ $(i = 1, 2)$ then in fact $f_i(p) \leq 2/3$ $(i = 1, 2)$.

Remark. The next theorem states that a pair of inequalities implies a stronger pair. In conjunction with Lemmas 6.2.1, 6.2.3 and 6.2.4, this means that in fact the stronger pair of inequalities holds. Hence the weaker inequalities also hold, and any consequences of the weaker inequalities used in the course of the proof [such as the infrared bound (6.2.19)] will have been shown to hold, once the theorem is proved.

Theorem 6.2.5 *There is an Ω_0 such that for $\Omega \geq \Omega_0$ (with $d > 4$ for the spread-out model) the following implication holds. For any $p \in [\Omega^{-1}, z_c)$, if*

$$||H_p||_2^2 \leq 3K\Omega^{-1+s} \quad \text{and} \quad ||\, x_\mu^2 G_p||_\infty \leq 3K\Omega^{-1+s+2/d} \tag{6.2.12}$$

then in fact

$$||H_p||_2^2 \leq 2K\Omega^{-1+s} \quad \text{and} \quad ||\, x_\mu^2 G_p||_\infty \leq 2K\Omega^{-1+s+2/d}. \tag{6.2.13}$$

Here s is as in the statement of Lemma 6.2.2, and the $2/d$ in the exponent in the bound on $||x_\mu^2 G_p||_\infty$ can be omitted for the nearest-neighbour model.

Proof. We assume the weaker pair of bounds, and prove the stronger pair. For the proof we will work with Fourier transforms. As will be described in more detail below [in the paragraph containing (6.2.27)], the assumed bounds (6.2.12), together with Theorem 5.4.4, imply (absolute) convergence of the lace expansion. Hence by (5.2.18),

$$\hat{F}_p(k) \equiv \hat{G}_p(k)^{-1} = 1 - p\Omega\hat{D}(k) - \hat{\Pi}_p(k). \tag{6.2.14}$$

Since $F_p(0) = \chi(p)^{-1} > 0$ for $p < z_c$, it follows by adding and subtracting $\hat{F}_p(0)$ to $\hat{F}_p(k)$ that for $p \geq \Omega^{-1}$

$$\begin{aligned} \hat{F}_p(k) &= \hat{F}_p(0) + p\Omega[1 - \hat{D}(k)] + \hat{\Pi}_p(0) - \hat{\Pi}_p(k) \\ &\geq [1 - \hat{D}(k)] + [\hat{\Pi}_p(0) - \hat{\Pi}_p(k)]. \end{aligned} \tag{6.2.15}$$

The basic idea of the proof is that the assumed bounds imply that the second term on the right side is a small perturbation of the first, which in turn implies that $\hat{G}_p = 1/\hat{F}_p(k)$ is bounded above by a small perturbation of its ordinary random walk counterpart, and hence by Lemma 6.2.2 the improved bounds hold.

We now bound the difference $\hat{\Pi}_p(0) - \hat{\Pi}_p(k)$, using Theorem 5.4.4. It follows from (5.4.1), (5.2.16), symmetry, and (5.4.21) that

$$\begin{aligned}\hat{\Pi}_p(0) - \hat{\Pi}_p(k) &\geq -\sum_{j=1}^{\infty}[\hat{\Pi}_p^{(2j+1)}(0) - \hat{\Pi}_p^{(2j+1)}(k)] \\ &\geq -\sum_{\mu=1}^{d}(1-\cos k_\mu)||x_1^2 H_p||_\infty \\ &\quad \times \sum_{j=1}^{\infty}(j+1)^2||H_p||_2^{2j+1}||G_p||_2^{2j-1}. \end{aligned} \quad (6.2.16)$$

For the norm $||G_p||_2$, we note that by definition $H_p(0,x) = G_p(0,x) - \delta_{0,x}$, and hence using (6.2.12) we have

$$||G_p||_2^2 = ||H_p||_2^2 + 1 \leq 2 \quad (6.2.17)$$

for sufficiently large Ω. The right side of (6.2.16) is dominated for large Ω by the $j = 1$ term, and hence by (6.2.5) and (6.2.12) we have

$$\hat{\Pi}_p(0) - \hat{\Pi}_p(k) \geq -K_1\Omega^{-u}[1 - \hat{D}(k)], \quad (6.2.18)$$

where $u = 3/2$ for the nearest-neighbour model and $u = 5/2 - 5s/2 - 2/d$ for the spread-out model, and K_1 is a constant which is independent of L for the spread-out model and independent of d for the nearest-neighbour model. We will use K_1 as a "variable constant" in what follows, to denote various constants which are independent of L or d as in (6.2.18) and whose precise values are irrelevant. Substituting (6.2.18) into (6.2.15) gives the infrared bound

$$\hat{F}_p(k) \geq [1 - K_1\Omega^{-u}][1 - \hat{D}(k)]. \quad (6.2.19)$$

We are now in a position to obtain the improved bound on $||H_p||_2^2$. By the Parseval relation and (6.2.17),

$$||H_p||_2^2 = ||\hat{G}_p||_2^2 - 1,$$

where the norm on the right side denotes the L^2 norm on $[-\pi,\pi]^d$ with measure $(2\pi)^{-d}d^dk$. Hence by (6.2.19) we have

$$\begin{aligned}||H_p||_2^2 &= \left|\left|\frac{1}{\hat{F}_p}\right|\right|_2^2 - 1 \\ &\leq (1 + K_1\Omega^{-u})\left|\left|\frac{1}{1-\hat{D}}\right|\right|_2^2 - 1. \end{aligned} \quad (6.2.20)$$

Applying (6.2.7), this gives

$$||H_p||_2^2 \leq (1 + K_1\Omega^{-u+1-s})K\Omega^{-1+s}. \tag{6.2.21}$$

For the nearest-neighbour model $-u+1-s = -1/2$, while for the spread-out model with $d > 4$, $-u+1-s = -3/2 + 3s/2 + 2/d < 0$. This gives the desired result that for Ω sufficiently large

$$||H_p||_2^2 \leq 2K\Omega^{-1+s}. \tag{6.2.22}$$

We turn now to the bound on $||x_\mu^2 G_p||_\infty$. We give the proof with the $2/d$ present in the exponent, but for the nearest-neighbour model this can be omitted by following the same proof. [The significant difference between the two models occurs in (6.2.31).]

In terms of the Fourier transform we can write

$$x_\mu^2 G_p(0,x) = -\int \partial_\mu^2 \hat{G}_p(k) e^{-ik\cdot x} \frac{d^dk}{(2\pi)^d}. \tag{6.2.23}$$

Explicit computation of the derivative on the right side gives the following expression, in which we have simplified the notation by dropping arguments and denoting partial differentiation with respect to k_μ by the subscript μ.

$$\hat{G}_{\mu,\mu} = p\Omega\frac{\hat{D}_{\mu,\mu}}{\hat{F}^2} + 2(p\Omega)^2\frac{\hat{D}_\mu^2}{\hat{F}^3} + \frac{\hat{\Pi}_{\mu,\mu}}{\hat{F}^2} + 4p\Omega\frac{\hat{D}_\mu\hat{\Pi}_\mu}{\hat{F}^3} + 2\frac{\hat{\Pi}_\mu^2}{\hat{F}^3}. \tag{6.2.24}$$

We insert (6.2.24) into (6.2.23), and take absolute values inside the integral and the sum of five terms. Applying (6.2.19) to bound $\hat{F}$ from below gives $\hat{F}^{-j} \leq (1+K_1\Omega^{-u})(1-\hat{D})^{-j}$ for $j \geq 1$. Applying (6.2.8) and using $1 \leq p\Omega$ then yields

$$\begin{aligned} x_\mu^2 G_p(0,x) \leq\ & (1+K_1\Omega^{-u})(p\Omega)^2 \\ & \times \left[K\Omega^{-1+s+2/d} + ||\hat{\Pi}_{\mu,\mu}||_\infty \left|\left| \frac{1}{1-\hat{D}} \right|\right|_2^2 \right. \\ & \left. + 4\left|\left| \frac{\hat{D}_\mu\hat{\Pi}_\mu}{(1-\hat{D})^3} \right|\right|_1 + 2\left|\left| \frac{\hat{\Pi}_\mu^2}{(1-\hat{D})^3} \right|\right|_1 \right]. \end{aligned} \tag{6.2.25}$$

The last three terms on the right side are error terms. Before bounding these, we first bound the factor $p\Omega$. By (6.2.12),

$$||H_p||_\infty \leq ||x_\mu^2 G_p||_\infty \leq 3K\Omega^{-1+s+2/d}; \tag{6.2.26}$$

this follows from the facts that $H_p(0,0) = 0$, and for $x \neq 0$, $H_p(0,x) = G_p(0,x)$ and $1 \leq x_\mu^2$ for some μ. (This bound on $||H_p||_\infty$ is inefficient for the spread-out model, for which the factor $\Omega^{2/d}$ on the right side should not be necessary, but it is adequate for our needs.) Applying (6.2.26) and (6.2.12) to (5.4.18) and (5.4.20), we see that for sufficiently large Ω the lace expansion converges and

$$|\hat{\Pi}_p(k)| \leq p\Omega K_1 \Omega^{-1+s+2/d}. \tag{6.2.27}$$

Since $\chi(p)^{-1} = 1 - p\Omega - \hat{\Pi}_p(0) > 0$,

$$p\Omega \leq 1 - \hat{\Pi}_p(0) \leq 1 + p\Omega K_1 \Omega^{-1+s+2/d},$$

so that for Ω sufficiently large

$$p \leq \Omega^{-1}[1 + K_1 \Omega^{-1+s+2/d}]. \tag{6.2.28}$$

Since $-u \leq -1 + s + 2/d$, the factor $(1 + K_1\Omega^{-u})(p\Omega)^2$ in (6.2.25) can be replaced by $1 + K_1\Omega^{-1+s+2/d}$, for Ω large.

We next consider bounds on the derivatives of $\hat{\Pi}_p$ appearing in (6.2.25). It follows from (6.2.12) and (5.4.20) that

$$|\partial_\mu^2 \hat{\Pi}_p(k)| \leq K_1 \Omega^{-2+2s+2/d} \tag{6.2.29}$$

(the $N = 2$ loop term dominates). We also will need a bound on $\partial_\mu \hat{\Pi}_p(k)$. Since by symmetry this derivative is zero whenever $k_\mu = 0$, it follows from Taylor's Theorem and the above bound on the second derivative that

$$|\partial_\mu \hat{\Pi}_p(k)| \leq K_1 \Omega^{-2+2s+2/d} |k_\mu|. \tag{6.2.30}$$

Similarly,

$$|\partial_\mu D(k)| \leq K_1 \Omega^{2/d} |k_\mu|. \tag{6.2.31}$$

Turning now to the three error terms in (6.2.25), for the first we use (6.2.29) and (6.2.7) to bound it above by $K_1\Omega^{-2+2s+2/d}$. For the other two terms we first note that by symmetry, (6.2.5) and (6.2.7),

$$\left\| \frac{k_\mu^2}{(1-\hat{D})^3} \right\|_1 \leq K_1. \tag{6.2.32}$$

Hence by (6.2.30) and (6.2.31) the second error term is bounded above by

$$K_1 \Omega^{2/d} \Omega^{-2+2s+2/d} \left\| \frac{k_\mu^2}{(1-\hat{D})^3} \right\|_1 \leq K_1 \Omega^{-2+2s+4/d}. \tag{6.2.33}$$

Finally, the last error term can be bounded above by $K_1\Omega^{-4+4s+4/d}$, using (6.2.30) and then (6.2.32). Taking Ω sufficiently large then gives the desired result

$$x_\mu^2 G_p(0,x) \leq 2K\Omega^{-1+s+2/d}. \tag{6.2.34}$$

□

The following results, which follow relatively easily from Theorem 6.2.5, will be fundamental in the rest of the chapter.

Corollary 6.2.6 *For $\Omega \geq \Omega_0$ (with $d > 4$ for the spread-out model),*

$$\begin{aligned} \|H_z\|_\infty &\leq 2K\Omega^{-1+s+2/d}, \\ \|x_\mu^2 G_z\|_\infty &\leq 2K\Omega^{-1+s+2/d}, \end{aligned}$$

and

$$\|H_z\|_2^2 \leq 2K\Omega^{-1+s}$$

for all complex z in the closed disk $|z| \leq z_c$. Here s is as in the statement of Lemma 6.2.2, and for the nearest-neighbour model the $2/d$ can be omitted from the exponent in the first two inequalities.

Proof. Since the left sides are largest at $z = z_c$, we can restrict attention to this case. The left sides are monotone increasing in real positive z, and satisfy the above bounds uniformly in $z < z_c$ by Theorem 6.2.5 (see (6.2.26) and the Remark preceding Theorem 6.2.5). Therefore the same bounds hold at $z = z_c$ by the monotone convergence theorem. □

Corollary 6.2.7 *For $\Omega \geq \Omega_0$ (with $d > 4$ for the spread-out model), there is a constant K_1 such that the following bounds hold uniformly in $k \in [-\pi, \pi]^d$ and $|z| \leq z_c$:*

$$\begin{aligned} |\hat{\Pi}_z(k)| &\leq K_1\Omega^{-1+s+2/d} \\ |\partial_\mu \hat{\Pi}_z(k)| &\leq K_1\Omega^{-2+2s+2/d}|k_\mu| \\ |\partial_\mu^2 \hat{\Pi}_z(k)| &\leq K_1\Omega^{-2+2s+2/d}. \end{aligned}$$

In fact the series representations of these quantities are bounded absolutely (absolute values inside sums over x, N) and uniformly by the right sides. The critical point obeys

$$\Omega^{-1} \leq z_c \leq \Omega^{-1}[1 + K_1\Omega^{-1+s+2/d}].$$

Also,

$$1 - z_c\Omega - \hat{\Pi}_{z_c}(0) = 0.$$

For any $p \in [0, z_c]$

$$\hat{\Pi}_p(0) - \hat{\Pi}_p(k) \geq -K_1 \Omega^{-u}[1 - \hat{D}(k)]$$

and for any $p \in [\Omega^{-1}, z_c]$

$$\hat{F}_p(k) \geq [1 - K_1 \Omega^{-u}][1 - \hat{D}(k)].$$

Here s *is as in the statement of Lemma 6.2.2, and for the nearest-neighbour model the* $2/d$ *can be omitted from the exponents in the first four inequalities. The exponent* u *is equal to* $3/2$ *for the nearest-neighbour model, while for the spread-out model* $u = 5/2 - 5s/2 - 2/d$.

Proof. Given Corollary 6.2.6, the first four inequalities follow exactly as in the proof of Theorem 6.2.5. It then follows from the dominated convergence theorem that for $u \in \{0, 1, 2\}$, $\partial_\mu^u \hat{\Pi}_z(k)$ is continuous on the closed disk $|z| \leq z_c$. Since $\chi(p) \to \infty$ as $p \nearrow z_c$ by (1.3.6), we have

$$\chi(p)^{-1} = \hat{F}_p(0) \to 1 - z_c\Omega - \hat{\Pi}_{z_c}(0) = 0.$$

The last two bounds of the corollary follow from (6.2.18) and (6.2.19) for $p < z_c$, and then follow at z_c by taking the limit. □

By Corollary 6.2.7 and the fact that $-\nabla_k^2 \hat{D}(0) \geq 1$, there is a constant C_4 such that for Ω sufficiently large and $p \in [\Omega^{-1}, z_c]$,

$$\begin{aligned} \nabla_k^2 \hat{F}_p(0) &= -p\Omega\nabla_k^2 \hat{D}(0) - \nabla_k^2 \hat{\Pi}_p(0) \\ &\geq C_4 > 0. \end{aligned} \tag{6.2.35}$$

The following lemma will allow for bounds on $\partial_z \hat{\Pi}_z(k)$ in the closed disk $|z| \leq z_c$.

Lemma 6.2.8 *For any* $p \in (0, z_c]$ *and* $m = 1, 2, 3, \ldots$,

$$\partial_p^m G_p(0, x) \leq m!\, p^{-m}\, H_p * \cdots * H_p * G_p(x), \tag{6.2.36}$$

where there are m *factors of* H_p *in the convolution.*

Proof. By definition,

$$\partial_p^m G_p(0, x) = m! p^{-m} \sum_{\substack{\omega : 0 \to x \\ |\omega| \geq m}} \binom{|\omega|}{m} p^{|\omega|},$$

where the sum is over all self-avoiding walks from 0 to x. The binomial coefficient on the right side counts the number of ways to choose $0 < i_1 <$

$i_2 < \cdots < i_m \leq |\omega|$, so it is also the number of ways to break ω into $m+1$ pieces such that the first m pieces each consist of at least one step. The upper bound then follows by neglecting the mutual avoidance between these pieces. □

Theorem 6.2.9 *For Ω sufficiently large (with $d > 4$ for the spread-out model),*

$$|\partial_z \hat{\Pi}_z(k)| \leq 7K\Omega^{s+2/d} \tag{6.2.37}$$

uniformly in $k \in [-\pi, \pi]^d$ and $|z| \leq z_c$. In fact the series representation of the left side is bounded absolutely (absolute values inside sums over x and N) and uniformly by the right side. Here K is the constant of Lemma 6.2.2, s is as in the statement of Lemma 6.2.2, and the $2/d$ can be omitted from the exponent for the nearest-neighbour model. Hence for Ω sufficiently large there is a positive constant C_3 such that for any $p \in (0, z_c]$

$$-\partial_z \hat{F}_p(0) = \Omega + \partial_z \hat{\Pi}_p(0) \geq C_3 > 0 \tag{6.2.38}$$

Proof. The bound (6.2.38) clearly follows from (6.2.37), so it suffices to obtain (6.2.37). But by (5.4.18), (5.4.19), Corollary 6.2.6 and the upper bound on $z_c\Omega$ of Corollary 6.2.7, to prove (6.2.37) it suffices to show that

$$||\partial_z|_{z=|z|} H_z||_\infty \leq 4K\Omega^{s+2/d}. \tag{6.2.39}$$

Since $H_p(0,x)$ is a power series with nonnegative coefficients, it suffices to obtain (6.2.39) at $z = z_c$. By Lemma 6.2.8 and the fact that $G_z(0,x) = H_z(0,x) + \delta_{0,x}$,

$$\begin{aligned}\partial_z H_{z_c}(0,x) = \partial_z G_{z_c}(0,x) &\leq z_c^{-1} H_{z_c} * H_{z_c}(x) + z_c^{-1} H_{z_c}(0,x) \\ &\leq z_c^{-1} ||H_{z_c}||_2^2 + z_c^{-1} H_{z_c}(0,x).\end{aligned}$$

The desired result now follows from Corollary 6.2.6 and the fact that z_c is bounded below by Ω^{-1}. □

We conclude this section with an upper bound on the susceptibility, which in particular implies that it is finite in the closed disk $|z| \leq z_c$ everywhere except at the critical point itself.

Theorem 6.2.10 *For Ω sufficiently large (with $d > 4$ for the spread-out model), the inverse susceptibility $\hat{F}_z(0) = 1 - z\Omega - \hat{\Pi}_z(0)$ satisfies*

$$|\hat{F}_z(0)| \geq \frac{\Omega}{2}|z_c - z| \tag{6.2.40}$$

for all z with $|z| \leq z_c$.

Proof. Let $|z| \leq z_c$. By Corollary 6.2.7 $\hat{F}_{z_c}(0) = 0$ and hence

$$\begin{aligned} |\hat{F}_z(0)| &= \left| \int_{z_c}^{z} \partial_z \hat{F}_z(0) dz \right| \\ &= |z_c - z| \left| \Omega + \int_0^1 \partial_z \hat{\Pi}_{(1-t)z_c + tz}(0) dt \right|. \end{aligned} \quad (6.2.41)$$

The lemma then follows, using Theorem 6.2.9. □

6.2.3 Proof of Theorem 6.1.2

The critical bubble diagram $\mathsf{B}(z_c) = ||G_{z_c}||_2^2 = 1 + ||H_{z_c}||_2^2$ is finite by Corollary 6.2.6. It follows from Theorem 1.5.3 that $\bar{\gamma} = 1$, in the sense that there are positive constants c_1 and c_2 such that for all $p < z_c$,

$$c_1(z_c - p)^{-1} \leq \chi(p) \leq c_2(z_c - p)^{-1}. \quad (6.2.42)$$

To obtain the stronger *asymptotic* behaviour stated in Theorem 6.1.2, we observe that since $\hat{F}_{z_c}(0) = 1 - z_c\Omega - \hat{\Pi}_{z_c}(0) = 0$ by Corollary 6.2.7,

$$\begin{aligned} \chi(z) &= \frac{1}{\hat{F}_z(0) - \hat{F}_{z_c}(0)} \\ &= \left(\frac{1}{z_c - z}\right) \left(\Omega + \frac{\hat{\Pi}_{z_c}(0) - \hat{\Pi}_z(0)}{z_c - z}\right)^{-1}. \end{aligned} \quad (6.2.43)$$

It then follows from Theorem 6.2.9 that as $z \nearrow z_c$

$$\chi(z) \sim [\Omega + \partial_z \hat{\Pi}_{z_c}(0)]^{-1} (z_c - z)^{-1}. \quad (6.2.44)$$

Defining

$$A = z_c^{-1} [\Omega + \partial_z \hat{\Pi}_{z_c}(0)]^{-1} \quad (6.2.45)$$

gives the statement of Theorem 6.1.2 for the susceptibility.

For the correlation length of order 2, we note that by symmetry and direct calculation,

$$\xi_2(z)^2 = \frac{-\nabla_k^2 \hat{G}_z(0)}{\hat{G}_z(0)} = [-z\Omega \nabla_k^2 \hat{D}(0) - \nabla_k^2 \hat{\Pi}_z(0)] \chi(z). \quad (6.2.46)$$

The desired asymptotic behaviour of $\xi_2(z)$ now follows from the asymptotic behaviour of $\chi(z)$ and (6.2.35), if we define

$$D = A[-z_c \Omega \nabla_k^2 \hat{D}(0) - \nabla_k^2 \hat{\Pi}_{z_c}(0)]. \quad (6.2.47)$$

[Continuity at z_c of $\nabla_k^2 \hat{\Pi}_z(0)$ is discussed in the proof of Corollary 6.2.7.]
□

We end this section with bounds on the constants A and D, for simplicity restricting the discussion to the nearest-neighbour model in high dimensions.

Proposition 6.2.11 *For the nearest-neighbour model with d sufficiently large, there are positive universal constants c_1, c_2, c_3 such that*

$$1 \leq A \leq 1 + c_1 d^{-1} \quad \textit{and} \quad 1 + c_2 d^{-1} \leq D \leq 1 + c_3 d^{-1}.$$

In particular D is strictly greater than 1.

Proof. For the first bound we conclude from Theorem 1.5.3 that $1 \leq A \leq \mathsf{B}(z_c)$. But by Corollary 6.2.6, $\mathsf{B}(z_c) \leq 1 + c_1 d^{-1}$ for some constant c_1.

For the bound on the diffusion constant D, we have from (6.2.47) and (6.2.45) that

$$D = \frac{1 - (2dz_c)^{-1} \nabla_k^2 \hat{\Pi}_{z_c}(0)}{1 + (2d)^{-1} \partial_z \hat{\Pi}_{z_c}(0)}. \tag{6.2.48}$$

It suffices to show that there are positive constants a_i such that

$$-a_1 d^{-3/2} \leq -(2dz_c)^{-1} \nabla_k^2 \hat{\Pi}_{z_c}(0) \leq a_2 d^{-1} \tag{6.2.49}$$

and

$$-a_3 d^{-1} \leq (2d)^{-1} \partial_z \hat{\Pi}_{z_c}(0) \leq -a_4 d^{-1}. \tag{6.2.50}$$

Beginning with (6.2.49), it follows from Corollary 6.2.7 and the fact that $2dz_c \geq 1$ that

$$|(2dz_c)^{-1} \nabla_k^2 \hat{\Pi}_{z_c}(0)| \leq a_2 d^{-1}. \tag{6.2.51}$$

This gives the upper bound of (6.2.49). For the lower bound, by symmetry it can be concluded that for fixed μ

$$-\nabla_k^2 \hat{\Pi}_{z_c}(0) \geq -d \sum_{j=1}^{\infty} \sum_x x_\mu^2 \Pi_{z_c}^{(2j+1)}(0, x). \tag{6.2.52}$$

By (5.4.20) and Corollary 6.2.6 the right side is bounded below by a multiple of $-d^{-3/2}$.

Turning now to (6.2.50), the lower bound follows immediately from (6.2.37). For the upper bound, we write

$$\partial_z \hat{\Pi}_{z_c}(0) = -\partial_z \hat{\Pi}_{z_c}^{(1)}(0) + \sum_{N=2}^{\infty} (-1)^N \partial_z \hat{\Pi}_{z_c}^{(N)}(0). \tag{6.2.53}$$

The first term on the right side (with its minus sign) is bounded above by the contribution due to the walk which steps to a neighbour of the origin and then back to the origin, which is $-\partial_z(2dz^2) = -4dz_c \leq -2$. Thus it suffices to show that the second term on the right side is bounded in absolute value by a multiple of d^{-1}. This follows from Corollary 6.2.6 and the bound $||\partial_z H_{z_c}||_\infty \leq K_1$ of (6.2.39), together with (5.4.19). □

6.3 Fractional derivatives

In this section we describe some elementary properties of what we term fractional derivatives. This terminology is somewhat inaccurate, but is useful in a suggestive sense in the analysis of the large-n asymptotics of power series coefficients. Given a power series $f(z) = \sum_{n=0}^\infty a_n z^n$ and $\epsilon \geq 0$, we define the fractional derivative

$$\delta_z^\epsilon f(z) = \sum_{n=0}^\infty n^\epsilon a_n z^n. \tag{6.3.1}$$

Note that for ϵ equal to a positive integer, δ_z^ϵ does not give the usual derivative. We will use (6.3.1) with $\epsilon \in (0,1)$. Allowing ϵ to take on arbitrary negative values defines a relative of the antiderivative, as follows. For $\alpha > 0$ we define

$$\delta_z^{-\alpha} f(z) = \sum_{n=1}^\infty n^{-\alpha} a_n z^n. \tag{6.3.2}$$

Both of the above quantities will be finite at least strictly within the circle of convergence of $f(z)$.

The following lemma provides formulas which are convenient for estimating fractional derivatives.

Lemma 6.3.1 *Let $f(z) = \sum_{n=0}^\infty a_n z^n$ have radius of convergence R. Then for any z with $|z| < R$, and for any $\alpha > 0$,*

$$\delta_z^{-\alpha} f(z) = C_\alpha \int_0^\infty [f(ze^{-\lambda^{1/\alpha}}) - f(0)] d\lambda, \tag{6.3.3}$$

where $C_\alpha = [\alpha\Gamma(\alpha)]^{-1}$. In addition, for any z with $|z| < R$ and for any $\epsilon \in (0,1)$,

$$\delta_z^\epsilon f(z) = C_{1-\epsilon} z \int_0^\infty f'(ze^{-\lambda^{1/(1-\epsilon)}}) e^{-\lambda^{1/(1-\epsilon)}} d\lambda. \tag{6.3.4}$$

The identities (6.3.3) and (6.3.4) also hold for $z = R$, if $a_n \geq 0$.

Proof. Let $|z| < R$. We first note that for any $\alpha > 0$,

$$n^{-\alpha} = \frac{1}{\alpha\Gamma(\alpha)} \int_0^\infty e^{-n\lambda^{1/\alpha}} d\lambda, \qquad (6.3.5)$$

as can be seen by making the substitution $y = n\lambda^{1/\alpha}$ in the integral on the right side. Therefore

$$\sum_{n=1}^\infty n^{-\alpha} a_n z^n = C_\alpha \sum_{n=1}^\infty a_n \int_0^\infty (ze^{-\lambda^{1/\alpha}})^n d\lambda. \qquad (6.3.6)$$

Since the right side converges absolutely the order of integration and summation can be interchanged to yield (6.3.3).

For (6.3.4), we write $n^\epsilon = n^{-(1-\epsilon)}n$ and use (6.3.5) with $\alpha = 1-\epsilon$ to obtain

$$\sum_{n=0}^\infty n^\epsilon a_n z^n = C_{1-\epsilon} z \sum_{n=1}^\infty n a_n \int_0^\infty (ze^{-\lambda^{1/(1-\epsilon)}})^{n-1} e^{-\lambda^{1/(1-\epsilon)}} d\lambda. \qquad (6.3.7)$$

Since the right side converges absolutely we can interchange the order of summation and integration to obtain

$$\sum_{n=0}^\infty n^\epsilon a_n z^n = C_{1-\epsilon} z \int_0^\infty f'(ze^{-\lambda^{1/(1-\epsilon)}}) e^{-\lambda^{1/(1-\epsilon)}} d\lambda. \qquad (6.3.8)$$

Now suppose that $a_n \geq 0$ and take $z = R$. Then the above interchanges of sum and integral are justified by Fubini's Theorem. □

The following lemma provides an error estimate analogous to the error estimate in Taylor's theorem. In applications of the lemma, R will be the radius of convergence of f.

Lemma 6.3.2 *Let $\epsilon \in (0,1)$ and let $f(z) = \sum_{n=0}^\infty a_n z^n$. Let $R > 0$ and suppose that $A_\epsilon \equiv \sum_{n=0}^\infty n^\epsilon |a_n| R^{n-\epsilon} < \infty$, so in particular $f(z)$ converges for $|z| \leq R$. Then for any z with $|z| \leq R$,*

$$|f(z) - f(R)| \leq 2^{1-\epsilon} A_\epsilon |R - z|^\epsilon. \qquad (6.3.9)$$

Suppose that $B_\epsilon \equiv \sum_{n=1}^\infty n^{1+\epsilon} |a_n| R^{n-1-\epsilon} < \infty$, so in particular $f'(z) = \sum_{n=1}^\infty n a_n z^{n-1}$ converges for $|z| \leq R$. Then for any z with $|z| \leq R$,

$$|f(z) - f(R) - f'(R)(z-R)| \leq \frac{2^{1-\epsilon}}{1+\epsilon} B_\epsilon |R - z|^{1+\epsilon}. \qquad (6.3.10)$$

Proof. We just give the proof of (6.3.10). The proof of (6.3.9) is similar and simpler. By definition,

$$f(z)-f(R)-f'(R)(z-R) = (z-R)\sum_{n=1}^{\infty} a_n R^{n-1} \sum_{j=0}^{n-1}\left[\left(\frac{z}{R}\right)^j - 1\right]. \quad (6.3.11)$$

But in general

$$\begin{aligned} w^j - 1 &= (w-1)^\epsilon \left[\frac{w^j-1}{w-1}\right]^\epsilon (w^j-1)^{1-\epsilon} \\ &= (w-1)^\epsilon \left[\sum_{m=0}^{j-1} w^m\right]^\epsilon (w^j-1)^{1-\epsilon}. \end{aligned} \quad (6.3.12)$$

Taking absolute values in (6.3.12) and using $w = z/R$ and $|w| \leq 1$ gives

$$\left|\left(\frac{z}{R}\right)^j - 1\right| \leq |z-R|^\epsilon j^\epsilon 2^{1-\epsilon} R^{-\epsilon}. \quad (6.3.13)$$

Since $\sum_{j=0}^{n-1} j^\epsilon \leq (1+\epsilon)^{-1} n^{1+\epsilon}$, (6.3.10) follows from (6.3.11) and (6.3.13). □

The intuition behind the following lemma is that if a power series with radius of convergence R behaves like $|R-z|^{-b}$ near $z = R$, for some $b \geq 1$, then roughly speaking it should have coefficient of z^n not much worse than order $R^{-n} n^{b-1}$.

Lemma 6.3.3 *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence greater than or equal to $R > 0$.*
(i) Suppose that for $|z| < R$, $|f(z)| \leq const.|R-z|^{-b}$ for some $b \geq 1$. Then $|a_n| \leq O(R^{-n} n^\alpha)$, for any $\alpha > b-1$.
(ii) If for some $b \geq 1$ a bound on the derivative of the form $|f'(z)| \leq const.|R-z|^{-b}$ holds for every $|z| < R$, then $|a_n| \leq O(R^{-n} n^{-\alpha})$ for any $\alpha < 2-b$.

Proof. (*i*) Fix $b \geq 1$ and let $\alpha > b-1$. Since $n^{-\alpha} a_n$ is the coefficient of z^n in the fractional antiderivative $\delta_z^{-\alpha} f(z)$,

$$n^{-\alpha} a_n = \frac{1}{2\pi i} \oint \delta_z^{-\alpha} f(z) \frac{dz}{z^{n+1}}, \quad (6.3.14)$$

where the integral is around a circle of radius $r < R$ centred at the origin. By Lemma 6.3.1,

$$n^{-\alpha}|a_n| \leq \text{const.} r^{-n} \int_{-\pi}^{\pi} d\theta \int_0^\infty d\lambda |f(re^{i\theta} e^{-\lambda^{1/\alpha}}) - f(0)|. \quad (6.3.15)$$

Since $f(z) - f(0) = O(|z|)$ for z near zero, the contribution to the integral with respect to λ due to $\lambda \in [1, \infty)$ is finite. Using the assumed bound on $f(z)$, we thus have

$$n^{-\alpha}|a_n| \leq \text{const.} r^{-n} \left[1 + \int_{-\pi}^{\pi} d\theta \int_0^1 d\lambda |R - re^{i\theta}e^{-\lambda^{1/\alpha}}|^{-b}\right].$$

Replacing the R on the right side by r gives an upper bound. Taking the limit $r \to R$ in the upper bound leads to

$$n^{-\alpha}|a_n| \leq \text{const.} R^{-n-b} \left[1 + \int_{-\pi}^{\pi} d\theta \int_0^1 d\lambda |1 - e^{i\theta}e^{-\lambda^{1/\alpha}}|^{-b}\right]. \quad (6.3.16)$$

To check that the integral on the right side is finite, it suffices to show that the corresponding quantity with limits of integration $\theta = \pm 1$ is finite (or any other small finite interval containing $\theta = 0$). Thus it suffices to verify that

$$\int_0^1 d\theta \int_0^1 d\lambda |1 - e^{i\theta}e^{-\lambda^{1/\alpha}}|^{-b} < \infty. \quad (6.3.17)$$

As we now show, it is an exercise in calculus to see that the left side is bounded for $\alpha > b - 1 \geq 0$.

Making the substitution $u = \lambda^{1/\alpha}$ and writing the absolute value on the right side as the square root of the sum of the squares of its real and imaginary parts leads to an upper bound for (6.3.17) of the form

$$\int_0^1 d\theta \int_0^1 du u^{\alpha-1}[(1 - e^{-u})^2 + e^{-2u}\theta^2]^{-b/2}. \quad (6.3.18)$$

The change of variables $\theta_1 = \theta e^{-u}/(1 - e^{-u})$ in (6.3.18) gives

$$\int_0^1 du u^{\alpha-1} \frac{1 - e^{-u}}{e^{-u}} (1 - e^{-u})^{-b} \int_0^{e^{-u}/(1-e^{-u})} d\theta_1 [1 + \theta_1^2]^{-b/2}. \quad (6.3.19)$$

The θ_1-integral is bounded uniformly in u if $b > 1$, while if $b = 1$ it is finite for u near 1 and $O(|\log u|)$ for u near 0. Hence for $b \geq 1$, (6.3.19) is bounded above by a multiple of

$$\int_0^1 du\, u^{\alpha-b} |\log u|, \quad (6.3.20)$$

which is finite for $\alpha > b - 1$.

(*ii*) Given the bound on the derivative, it follows from (*i*) that $|na_n| \leq O(R^{-n}n^p)$ for any $p > b - 1$. Therefore $|a_n| \leq O(R^{-n}n^{p-1})$ for any $\alpha \equiv 1 - p < 2 - b$. □

Remark. The hypothesis $b \geq 1$ in Lemma 6.3.3(i) is not artificial. For example, let $f(z) = \sum_{n=1}^{\infty} n^{-2} z^{2^n}$. Then $f(z)$ is finite for $|z| \leq 1$ so in particular $|f(z)| \leq \text{const.}|1-z|^{-b}$ for any $b \in [0,1)$. However $a_N = [\log_2 N]^{-2}$ for $N = 2^n$, so $a_n \neq O(n^{b-1+\epsilon})$ for $\epsilon \in (0, 1-b)$.

The following lemma is a kind of Tauberian theorem, in which information more detailed than merely the asymptotic form of a power series near its singularity provides information about the large-n asymptotics of the coefficients of the power series.

Lemma 6.3.4 *Let*

$$f(z) = \frac{1}{\varphi(z)} = \sum_{n=0}^{\infty} b_n z^n,$$

where $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$. *Suppose that for some* $\epsilon \in (0,1)$

$$\sum_{n=0}^{\infty} n^{1+\epsilon} |a_n| R^n < \infty,$$

so in particular $\varphi(z)$ *and* $\varphi'(z)$ *are finite when* $|z| = R$. *Assume in addition that* $\varphi'(R) \neq 0$. *Suppose that* $\varphi(R) = 0$ *and that* $\varphi(z) \neq 0$ *for* $|z| \leq R, z \neq R$. *Then*

$$f(z) = \frac{1}{-\varphi'(R)} \frac{1}{R-z} + O(|R-z|^{\epsilon-1}) \tag{6.3.21}$$

uniformly in $|z| \leq R$, *and*

$$b_n = R^{-n-1} \left[\frac{1}{-\varphi'(R)} + O(n^{-\alpha}) \right] \quad \text{as } n \to \infty, \tag{6.3.22}$$

for every $\alpha < \epsilon$.

Proof. Since $\varphi(R) = 0$,

$$\begin{aligned} f(z) &= -\frac{1}{\varphi(R) - \varphi(z)} \qquad (6.3.23) \\ &= -\frac{1}{\varphi'(R)(R-z) + [\varphi(R) - \varphi(z) - \varphi'(R)(R-z)]}. \end{aligned}$$

Let

$$h(z) = \frac{\varphi(R) - \varphi(z) - \varphi'(R)(R-z)}{R-z} \tag{6.3.24}$$

and

$$\psi(z) = -\frac{h(z)}{\varphi'(R) + h(z)} = -1 - \frac{\varphi'(R)}{\varphi(z)}(R-z). \tag{6.3.25}$$

Then ψ is analytic in $|z| < R$. Also,

$$f(z) = -\frac{1}{\varphi'(R)} \frac{1}{R-z}[1+\psi(z)]. \tag{6.3.26}$$

Since $h(z) = O(|R-z|^\epsilon)$ uniformly in $|z| \leq R$ by Lemma 6.3.2, it is also the case that $\psi(z) = O(|R-z|^\epsilon)$ uniformly in $|z| \leq R$. This proves (6.3.21).

Let C_r be the circle of radius r centred at the origin and oriented counterclockwise. The coefficient b_n is given by the contour integral

$$b_n = \frac{1}{2\pi i} \int_{C_{R/2}} \frac{f(z)}{z^{n+1}} dz, \tag{6.3.27}$$

so by (6.3.26)

$$b_n = -\frac{1}{\varphi'(R)} \left[\frac{1}{R^{n+1}} + \frac{1}{2\pi i} \int_{C_{R/2}} \frac{\psi(z)}{(R-z)z^{n+1}} dz \right]. \tag{6.3.28}$$

It remains to show that the second term in (6.3.28) gives a correction of the desired size.

We use statement (ii) of Lemma 6.3.3 for the correction term, as follows. A straightforward calculation using the bound on the $(1+\epsilon)$-derivative of φ assumed in the statement of the lemma, together with Lemma 6.3.2, gives

$$\left| \frac{d}{dz} \frac{\psi(z)}{R-z} \right| \leq O(|R-z|^{\epsilon-2}) \tag{6.3.29}$$

uniformly in $|z| \leq R$. Hence the coefficient of z^n of $(R-z)^{-1}\psi(z)$ is bounded above by $O(R^{-n}n^{-\alpha})$, for every $\alpha < \epsilon$, by Lemma 6.3.3(ii). This gives the required bound on the second term of (6.3.28). □

6.4 c_n and the mean-square displacement

This section consists of two parts. In the first part we obtain bounds on fractional derivatives involving $\hat{\Pi}_z(k)$, and then in the second part these bounds are used in conjunction with the results of Section 6.3 to prove Theorem 6.1.1.

6.4.1 Fractional derivatives of the two-point function

We begin by obtaining bounds on norms of fractional derivatives of the two-point function. Bounds on fractional derivatives of $\hat{\Pi}_z(k)$ are then obtained, using a generalization of Theorem 5.4.4 involving fractional z-derivatives.

The results of this section hold for finite or infinite memory, subject to the replacement of z_c by the finite memory critical point $z_c(0;\tau)$ in all occurrences. We use K_2 and c in this section to denote constants which may depend on Ω, and which may change from one occurrence to the next. They are however independent of the memory.

For $\lambda \geq 0$ we define

$$p_\lambda = z_c e^{-\lambda^{1/(1-\epsilon)}}, \tag{6.4.1}$$

and as usual we write

$$\hat{F}_z(k) = \frac{1}{\hat{G}_z(k)} = 1 - z\Omega\hat{D}(k) - \hat{\Pi}_z(k). \tag{6.4.2}$$

The following lemma will be used to bound norms of fractional derivatives of the two-point function.

Lemma 6.4.1 *For Ω sufficiently large (with $d > 4$ for the spread-out model), there is a positive constant c such that for any k or λ*

$$\hat{F}_{p_\lambda}(k) \geq c[1 - e^{-\lambda^{1/(1-\epsilon)}}\hat{D}(k)]. \tag{6.4.3}$$

Proof. Since $\hat{F}_{z_c}(0) = 0$,

$$\begin{aligned} \hat{F}_{p_\lambda}(k) &= [\hat{F}_{p_\lambda}(k) - \hat{F}_{p_\lambda}(0)] + [\hat{F}_{p_\lambda}(0) - \hat{F}_{z_c}(0)] \\ &= p_\lambda\Omega[1 - \hat{D}(k)] + [\hat{\Pi}_{p_\lambda}(0) - \hat{\Pi}_{p_\lambda}(k)] + \int_{p_\lambda}^{z_c}[-\partial_p\hat{F}_p(0)]dp. \end{aligned} \tag{6.4.4}$$

By Theorem 6.2.9,

$$\int_{p_\lambda}^{z_c}[-\partial_p\hat{F}_p(0)]dp \geq C_3(z_c - p_\lambda). \tag{6.4.5}$$

Also, by Corollary 6.2.7,

$$\hat{\Pi}_{p_\lambda}(0) - \hat{\Pi}_{p_\lambda}(k) \geq -K_1\Omega^{-1/2}[1 - \hat{D}(k)]. \tag{6.4.6}$$

Take $\Omega \geq \max\{4C_3/3, 64K_1^2\}$, and consider first the case of λ bounded away from infinity in such a way that $p_\lambda \geq 4K_1\Omega^{-3/2}$. Then by (6.4.4)–(6.4.6)

$$\begin{aligned} \hat{F}_{p_\lambda}(k) &\geq \frac{3p_\lambda\Omega}{4}[1 - \hat{D}(k)] + C_3(z_c - p_\lambda) \\ &\geq C_3 z_c[1 - e^{-\lambda^{1/(1-\epsilon)}}\hat{D}(k)], \end{aligned} \tag{6.4.7}$$

which gives (6.4.3) for this range of λ. For λ such that $p_\lambda \leq 4K_1\Omega^{-3/2}$, we have $p_\lambda \leq (2\Omega)^{-1}$, and so we use

$$\hat{G}_{p_\lambda}(k) \leq \hat{G}_{p_\lambda}(0)$$

and bound the right side by the ordinary random walk susceptibility at $p = (2\Omega)^{-1}$, which is finite. Therefore $\hat{F}_{p_\lambda}(k)$ is bounded below by a constant, and so (6.4.3) holds (decreasing c if necessary). □

We are now able to obtain bounds on fractional derivatives of the two-point function.

Theorem 6.4.2 *For Ω sufficiently large (with $d > 4$ for the spread-out model) there is a positive constant K_2 (which may depend on ϵ and Ω) such that for any $p \in [0, z_c]$,*

$$||\delta_p^\epsilon \partial_p G_p||_\infty \leq K_2 \quad \textit{if} \quad 0 < \epsilon < \min\{(d-4)/2, 1\}, \tag{6.4.8}$$

$$||\delta_p^\epsilon G_p||_2 \leq K_2 \quad \textit{if} \quad 0 < \epsilon < \min\{(d-4)/4, 1\}, \tag{6.4.9}$$

and

$$||x_\mu^2 \delta_p^\epsilon G_p||_\infty \leq K_2 \quad \textit{if} \quad 0 < \epsilon < \min\{(d-4)/2, 1\}. \tag{6.4.10}$$

Proof. Let $\epsilon \in (0,1)$. For an upper bound, we take $p = z_c$. We define p_λ as in (6.4.1). The proof of each of these three inequalities is similar, and we focus mainly on the first one. By Lemma 6.3.1 [using the fact that $G_p(0,x)$ has nonnegative coefficients $c_n(0,x)$], we have

$$\delta_p^\epsilon \partial_p G_p(0,x) = C_{1-\epsilon} z_c \int_0^\infty \partial_p^2 G_p(0,x)\big|_{p=p_\lambda} e^{-\lambda^{1/(1-\epsilon)}} d\lambda. \tag{6.4.11}$$

Using Lemma 6.2.8 to bound the derivative of G_p in the integrand, and then going to the Fourier transform, we can bound the right side as

$$\delta_p^\epsilon \partial_p G_p(0,x) \leq 2\int \frac{d^dk}{(2\pi)^d} C_{1-\epsilon} z_c \int_0^\infty d\lambda e^{-\lambda^{1/(1-\epsilon)}} p_\lambda^{-2} \left|\hat{H}_{p_\lambda}(k)^2 \hat{G}_{p_\lambda}(k)\right|. \tag{6.4.12}$$

Using the fact that $\hat{H}_z(k) = [z\Omega\hat{D}(k) + \hat{\Pi}_z(k)]\hat{F}_z(k)^{-1}$, and then using (6.2.37) to bound $z^{-1}\hat{\Pi}_z(k)$, it can be shown that there is a constant K_4 (depending on Ω but not on λ) such that

$$p_\lambda^{-1}|\hat{H}_{p_\lambda}(k)| \leq K_4 \hat{F}_{p_\lambda}(k)^{-1}. \tag{6.4.13}$$

We bound the right side of (6.4.12), using (6.4.13) and (6.4.3), by

$$\begin{aligned} &2K_4^2 \int \frac{d^dk}{(2\pi)^d} C_{1-\epsilon} z_c \int_0^\infty d\lambda e^{-\lambda^{1/(1-\epsilon)}} \hat{F}_{p_\lambda}(k)^{-3} \\ &\leq \text{const.} \int \frac{d^dk}{(2\pi)^d} \int_0^\infty d\lambda e^{-\lambda^{1/(1-\epsilon)}} [1 - e^{-\lambda^{1/(1-\epsilon)}} \hat{D}(k)]^{-3}. \end{aligned}$$

Now by (6.3.4), the right side of the above inequality is equal to

$$\text{const.} \int \frac{d^d k}{(2\pi)^d} \frac{1}{\hat{D}(k)} \delta_p^\epsilon \left[(1 - p\hat{D}(k))^{-2}\right]\Big|_{p=1}$$
$$= \text{const.} \int \frac{d^d k}{(2\pi)^d} \sum_{n=2}^{\infty} n(n-1)^\epsilon \hat{D}(k)^{n-2}.$$

By (6.2.6), the right side is finite for $1+\epsilon-(d/2) < -1$. This proves (6.4.8).

For (6.4.9), we proceed in a similar fashion. Using Lemmas 6.3.1 and 6.2.8 and the Parseval relation gives

$$\|\delta_p^\epsilon G_p\|_2 \leq \left\| C_{1-\epsilon} z_c \int_0^\infty d\lambda e^{-\lambda^{1/(1-\epsilon)}} p_\lambda^{-1} \hat{H}_{p_\lambda}(k) \hat{G}_{p_\lambda}(k) \right\|_2,$$

where the norm on the left is with respect to normalized Lebesgue measure on $[-\pi, \pi]^d$. Arguing as above and using the triangle inequality for $\|\cdot\|_2$ gives

$$\|\delta_p^\epsilon G_p\|_2 \leq \text{const.} \sum_{n=1}^{\infty} n^\epsilon \|\hat{D}(k)^{n-1}\|_2.$$

The desired bound now follows from the fact that $\|\hat{D}(k)^n\|_2 \leq O(n^{-d/4})$, by (6.2.6).

For (6.4.10), by Lemma 6.3.1 we have

$$x_\mu^2 \delta_p^\epsilon G_p(0,x) = C_{1-\epsilon} z_c \int_0^\infty d\lambda e^{-\lambda^{1/(1-\epsilon)}} x_\mu^2 \partial_p G_{p_\lambda}(0,x). \tag{6.4.14}$$

It follows from (6.2.10) that $x_\mu^2 \partial_p G_{p_\lambda}(0,x)$ is bounded uniformly in x and $\lambda \geq \lambda_0$, for any fixed positive λ_0. Taking for simplicity $\lambda_0 = 1$, it suffices to bound

$$\int_0^1 d\lambda e^{-\lambda^{1/(1-\epsilon)}} x_\mu^2 \partial_p G_{p_\lambda}(0,x). \tag{6.4.15}$$

Applying Lemma 6.2.8 and the Fourier transform, and noting that p_λ is bounded below by p_1 for $\lambda \in [0,1]$, the above integral is bounded above by

$$\text{const.} \int_0^1 d\lambda e^{-\lambda^{1/(1-\epsilon)}} \int \frac{d^d k}{(2\pi)^d} \left| \partial_\mu^2 [\hat{H}_{p_\lambda}(k) \hat{G}_{p_\lambda}(k)] \right|. \tag{6.4.16}$$

It follows from Corollary 6.2.7 that $|\partial_\mu^2 \hat{F}_{p_\lambda}(k)|$ is bounded (uniformly in λ). Also, it follows from Taylor's theorem and the bound on $|\partial_\mu \hat{\Pi}_z(k)|$ of Corollary 6.2.7, together with (6.2.5), that

$$\sum_{\mu=1}^{d} [\partial_\mu \hat{F}_{p_\lambda}(k)]^2 \leq \text{const.} k^2 \leq \text{const.} [1 - e^{-\lambda^{1/(1-\epsilon)}} \hat{D}(k)].$$

It then follows from direct computation of the second derivative of

$$[\hat{F}_{p\lambda}(k)^{-1} - 1]\hat{F}_{p\lambda}(k)^{-1}$$

occurring in (6.4.16), together with (6.4.3) and symmetry, that

$$x_\mu^2 \delta_p^\epsilon G_p(0,x) \leq \text{const.}[1 + \int_0^\infty d\lambda e^{-\lambda^{1/(1-\epsilon)}} \int \frac{d^dk}{(2\pi)^d} \hat{F}_{p\lambda}(k)^{-3}].$$

Now the discussion below (6.4.13) can be applied. □

The following corollary of Theorem 6.4.2 will be used to prove Theorem 6.1.1.

Corollary 6.4.3 *For Ω sufficiently large (with $d > 4$ for the spread-out model), there is a K_2 (which may depend on ϵ and Ω) such that for any $k \in [-\pi,\pi]^d$ and $|z| \leq z_c$,*

$$|\delta_z^\epsilon \partial_z \hat{\Pi}_z(k)|, \;\; |\delta_z^\epsilon \partial_\mu^u \hat{\Pi}_z(k)| \leq K_2, \tag{6.4.17}$$

for $u = 0, 1, 2$, where the first bound holds for any nonnegative $\epsilon < \min\{(d-4)/2, 1\}$ and the second for any nonnegative $\epsilon < \min\{(d-4)/4, 1\}$. In fact the series representations of the left side are bounded absolutely by K_2.

Proof. We write the left sides as sums over sites x and number of loops N. For upper bounds, we take absolute values inside sums over both x and N, and consider $z = z_c$. For the first bound, the derivatives bring down a factor $|\omega|^{1+\epsilon}$. This can be distributed among the subwalks of the N loop diagram, using Hölder's inequality in the form

$$|\omega|^{1+\epsilon} = \left[\sum_{j=0}^{2N-1} |\omega_j|\right]^{1+\epsilon} \leq (2N-1)^\epsilon \sum_{j=0}^{2N-1} |\omega_j|^{1+\epsilon}.$$

The resulting diagrams can then be bounded using Lemma 5.4.3, with the subwalk weighted by $|\omega|^{1+\epsilon}$ bounded with the L^∞ norm. Convergence then follows using an extension of Theorem 5.4.4 for fractional derivatives, together with Corollary 6.2.6 and (6.4.8).

Similarly, for the second bound, the derivatives bring down factors $|\omega|^\epsilon$ and $|x_\mu^u|$. As in the proof of Theorem 5.4.4, these can be distributed among the subwalks in a diagram, with each factor on a distinct subwalk (the one-loop diagram does not contribute). The subwalk weighted with $|x_\mu^u|$ is bounded using the L^∞ norm, and all other subwalks are bounded using the L^2 norm, using (6.4.9) for the subwalk weighted with $|\omega|^\epsilon$. Convergence follows using Corollary 6.2.6. □

6.4.2 Proof of Theorem 6.1.1

In this section we give the proof of Theorem 6.1.1. We begin with c_n.

Proof of Theorem 6.1.1(a). The susceptibility is given by

$$\chi(z) = \frac{1}{\hat{F}_z(0)} = \frac{1}{1 - z\Omega - \hat{\Pi}_z(0)}. \tag{6.4.18}$$

Fix $\epsilon < \min\{(d-4)/2, 1\}$. By Corollary 6.4.3, for any $\epsilon' < \min\{(d-4)/2, 1\}$, $\sum_n n^{1+\epsilon'}|\pi_n|z_c^n < \infty$, where π_n is the coefficient of z^n in the power series representation of $\hat{\Pi}_z(0)$. Moreover by Theorem 6.2.9 $\partial_z \hat{F}_{z_c}(0) \neq 0$, and by Theorem 6.2.10 the only singularity of $\chi(z)$ on the circle $|z| = z_c$ is at $z = z_c$. It then follows immediately from Lemma 6.3.4 that

$$\begin{aligned} c_n &= z_c^{-n-1}\left[\frac{1}{-\partial_z \hat{F}_{z_c}(0)} + O(n^{-\epsilon})\right] \\ &= A\mu^n[1 + O(n^{-\epsilon})], \end{aligned} \tag{6.4.19}$$

where in agreement with (6.2.45)

$$A = \frac{1}{-z_c\partial_z \hat{F}_{z_c}(0)}. \tag{6.4.20}$$

□

We now turn to the mean-square displacement.

Proof of Theorem 6.1.1(b). By definition of the Fourier transform,

$$\langle|\omega(n)|^2\rangle_n = \frac{-\nabla_k^2 \hat{c}_n(0)}{c_n}. \tag{6.4.21}$$

The asymptotic behaviour of the denominator on the right side was obtained in (6.4.19), and we now proceed to analyze the numerator.

Since $\hat{c}_n(k)$ is the coefficient of z^n in $\hat{G}_z(k)$,

$$-\nabla_k^2 \hat{c}_n(0) = -\frac{1}{2\pi i}\oint \nabla_k^2 \hat{G}_z(0)\frac{dz}{z^{n+1}} = \frac{1}{2\pi i}\oint \frac{\nabla_k^2 \hat{F}_z(0)}{\hat{F}_z(0)^2}\frac{dz}{z^{n+1}}, \tag{6.4.22}$$

where the integrals are performed around a small circle centred at the origin. We define an error term $E(z)$ by

$$\frac{\nabla_k^2 \hat{F}_z(0)}{\hat{F}_z(0)^2} = \frac{\nabla_k^2 \hat{F}_{z_c}(0)}{[\partial_z \hat{F}_{z_c}(0)]^2(z_c - z)^2} + E(z). \tag{6.4.23}$$

Inserting the right side of (6.4.23) into the right side of (6.4.22), the integral corresponding to the first term can be performed exactly to give

$$-\nabla_k^2 \hat{c}_n(0) = \frac{\nabla_k^2 \hat{F}_{z_c}(0)}{[\partial_z \hat{F}_{z_c}(0)]^2}(n+1)z_c^{-(n+2)} + \frac{1}{2\pi i}\oint E(z)\frac{dz}{z^{n+1}}. \qquad (6.4.24)$$

The remaining task is to bound the last term in (6.4.24). This is done using Lemma 6.3.3. Let $\epsilon < \min\{(d-4)/4, 1\}$. It follows from Lemma 6.3.3(i) that if it can be shown that $|E(z)| \leq \text{const.}|z_c - z|^{-2+\epsilon}$ for all $|z| \leq z_c$, then the second term on the right side of (6.4.24) is $O(z_c^{-n} n^\alpha)$ for every $\alpha > 1-\epsilon$. Assuming for the moment this bound on the error term and using (6.4.19), we then have the desired result

$$\langle |\omega(n)|^2 \rangle_n = Dn + O(n^\alpha), \qquad (6.4.25)$$

with

$$D = \frac{\nabla_k^2 \hat{F}_{z_c}(0)}{z_c[-\partial_z \hat{F}_{z_c}(0)]}. \qquad (6.4.26)$$

We now establish the upper bound on $|E(z)|$ used in the previous paragraph. We first use (6.4.23) to write $E(z)$ as a difference of two fractions, and then write this difference over a common denominator and add and subtract $\nabla_k^2 \hat{F}_z(0)\hat{F}_z(0)^2$ in the numerator. This leads to

$$E(z) = T_1 + T_2 \qquad (6.4.27)$$

with

$$T_1 = [\partial_z \hat{F}_{z_c}(0)]^{-2} \frac{\nabla_k^2 \hat{F}_z(0) - \nabla_k^2 \hat{F}_{z_c}(0)}{(z_c - z)^2} \qquad (6.4.28)$$

and

$$T_2 = \frac{-\nabla_k^2 \hat{F}_z(0)[\hat{F}_z(0)^2 - [\partial_z \hat{F}_{z_c}(0)]^2 (z_c - z)^2]}{[\partial_z \hat{F}_{z_c}(0)]^2 \hat{F}_z(0)^2 (z_c - z)^2}. \qquad (6.4.29)$$

For T_1, we use existence of an ϵ-derivative in the numerator by Corollary 6.4.3, together with the Taylor theorem type bound of (6.3.9) to conclude that

$$|T_1| \leq O(|z_c - z|^{\epsilon - 2}). \qquad (6.4.30)$$

For T_2 we factor the difference of squares in the numerator and bound the denominator using Theorem 6.2.10, obtaining

$$|T_2| \leq \text{const.}(z_c - z)^{-4}[\hat{F}_z(0) + \partial_z \hat{F}_{z_c}(0)(z_c - z)][\hat{F}_z(0) - \partial_z \hat{F}_{z_c}(0)(z_c - z)]. \qquad (6.4.31)$$

The middle factor on the right side is $O(|z_c - z|^{1+\epsilon})$, by Corollary 6.4.3 and Lemma 6.3.2. For the last factor, the second term is clearly $O(|z_c - z|)$, as is the first, by virtue of the bound on the middle factor. Therefore $|T_2| \leq O(|z_c - z|^{\epsilon - 2})$. □

6.5 Correlation length and infrared bound

6.5.1 The correlation length

In this section we prove Theorem 6.1.5, which states that for sufficiently large Ω,

$$\xi(z) \sim \sqrt{\frac{D}{2d}}\left(\frac{z_c}{z_c - z}\right)^{1/2} \quad \text{as } z \nearrow z_c. \tag{6.5.1}$$

We work with the fully self-avoiding walk, with positive activity $p < z_c$, and as usual write $m(p) = \xi(p)^{-1}$.

By Proposition 4.1.1(b) and Theorem 4.1.6 [and the fact that $\mathsf{B}(z_c) < \infty$ by Corollary 6.2.6], $m(p)$ is strictly positive and finite for $p < z_c$, and $m(p) \searrow 0$ as $p \nearrow z_c$. For any function f defined on $\mathbf{Z}^d$, and $m \in \mathbf{R}$, we define

$$f^{(m)}(x) = f(x)e^{mx_1}. \tag{6.5.2}$$

The following lemma, whose proof is deferred to the end of this section, is a key ingredient in the proof of (6.5.1).

Lemma 6.5.1 *For Ω sufficiently large (with $d > 4$ for the spread-out model), there is a $\delta > 0$ (which may depend on Ω) such that for $p \in [z_c - \delta, z_c)$ and $m < m(p)$,*

$$||H_p^{(m)}||_2^2 \equiv \sum_x [H_p^{(m)}(0,x)e^{mx_1}]^2 \leq 2K\Omega^{-1+s},$$

where K and s are as in the statement of Lemma 6.2.2.

Lemma 6.5.1 leads to the following result.

Corollary 6.5.2 *Let Ω be sufficiently large (with $d > 4$ for the spread-out model). There is a positive constant K_3 which is independent of p and m (but may depend on ϵ and Ω) such that*

$$\sum_x |x_1|^{2+\epsilon}|\Pi_p^{(m)}(0,x)| \leq K_3$$

for all $p \in [z_c - \delta, z_c)$, $m < m(p)$ and $\epsilon < \min\{(d-4)/2, 1\}$.

Proof. The sum on the left side involves diagrams having two or more loops, weighted with both $|x_1|^{2+\epsilon}$ and e^{mx_1}. We split the former among subwalks along one side of the diagram using Hölder's inequality, and factor the latter along subwalks on the other side of the diagram. We then bound the resulting diagrams using Lemma 5.4.3. The subwalk weighted with $|x_1|^{2+\epsilon}$ is bounded using the infinity norm, as follows:

$$\begin{aligned}\sup_x |x_1|^{2+\epsilon} \sum_n c_n(0,x)p^n &\leq \sup_x |x_1|^2 \sum_n n^\epsilon c_n(0,x)p^n \\ &= \|x_1^2 \delta_p^\epsilon G_p\|_\infty. \end{aligned} \qquad (6.5.3)$$

The right side is finite for ϵ as in the statement of the corollary, by Theorem 6.4.2. All other subwalks are bounded as in Lemma 5.4.3 using the L^2 norm, yielding factors of $\|H_p\|_2$, $\|G_p\|_2 = 1 + \|H_p\|_2$, $\|H_p^{(m)}\|_2$ and $\|G_p^{(m)}\|_2 = 1 + \|H_p^{(m)}\|_2$. The sum of all diagrams is then bounded above by a geometric series with an m-dependent ratio. The geometric series converges for Ω sufficiently large by Lemma 6.5.1 and Corollary 6.2.6, uniformly for p and m as in the statement of the lemma. □

Proof of Theorem 6.1.5. The proof is modelled on the corresponding random walk result in Theorem A.2(b). Let $p \in [z_c - \delta, z_c)$. For $m < m(p)$, let $\chi^{(m)}(p) = \sum_x G_p^{(m)}(0,x)$. Because $G_p(0,x)$ decays exponentially with decay rate $m(p)$ by Lemma 4.1.5, $\chi^{(m)}(p)$ is finite if $m < m(p)$. By multiplying (5.2.17) by e^{mx_1} and then taking the Fourier transform, we obtain

$$\hat{G}_p^{(m)}(k) = \frac{1}{1 - p\Omega\hat{D}^{(m)}(k) - \hat{\Pi}_p^{(m)}(k)}. \qquad (6.5.4)$$

[The Fourier transform $\hat{\Pi}_p^{(m)}$ exists for $p \in [z_c - \delta, z_c)$ and $m < m(p)$, by Corollary 6.5.2.] The function $\hat{D}^{(m)}(k)$ is defined by

$$\hat{D}^{(m)}(k) = \frac{1}{\Omega} \sum_{y \in \Omega} e^{my_1} e^{ik\cdot y}. \qquad (6.5.5)$$

Using (6.5.4) and (6.5.5),

$$\chi(p)^{-1} - \chi^{(m)}(p)^{-1} = p \sum_{y \in \Omega} [\cosh my_1 - 1] + \hat{\Pi}_p^{(m)}(0) - \hat{\Pi}_p(0). \qquad (6.5.6)$$

We intend to take the limit as $m \nearrow m(p)$ in (6.5.6). As a first observation, we show that for any $p < z_c$,

$$\lim_{m \nearrow m(p)} \chi^{(m)}(p) = \infty. \qquad (6.5.7)$$

This can be seen as follows. For simplicity we deal in the remainder of this paragraph only with the nearest-neighbour model; a modified argument applies to the spread-out model. Let $B_R = \{y \in \mathbf{Z}^d : ||y||_\infty \leq R\}$ and $\partial B_R = \{y \in \mathbf{Z}^d : ||y||_\infty = R\}$. For $y \in \partial B_R$ let $G_z^R(0,y) = \sum z^{|\omega|}$ where the sum is over all nearest-neighbour self-avoiding walks from 0 to y which hit ∂B_R for the first and only time at y. Then for $x \notin B_R$, we have the following Lieb-Simon type inequality:

$$G_p(0,x) \leq \sum_{y \in \partial B_R} G_p^R(0,y) G_p(y,x) \leq \sum_{y \in \partial B_R} G_p(0,y) G_p(y,x). \tag{6.5.8}$$

Multiplying this inequality by $e^{m(p)x_1}$, it follows from Lemma A.1 that if $\chi^{(m(p))}(p)$ were finite then $G^{(m(p))}(0,x)$ would decay exponentially, contradicting the definition of $m(p)$. It then follows from the monotone convergence theorem, and the fact that $\chi^{(m)}(p)$ is finite if $m < m(p)$, that (6.5.7) holds.

The remainder of the proof is concerned with showing that the limit of the right side of (6.5.6), as $m \nearrow m(p)$, is a multiple of $m(p)^2$ plus a higher order correction. Together with Theorem 6.1.2, which states that $\chi(p) \sim \text{const.}(z_c - p)^{-1}$, this will show that $m(p)^2$ is asymptotic to a multiple of $z_c - p$ as $p \nearrow z_c$.

By definition and symmetry,

$$\begin{aligned} \hat{\Pi}_p^{(m)}(0) - \hat{\Pi}_p(0) &= \sum_x [\cosh m x_1 - 1] \Pi_p(0,x) \\ &= \frac{m^2}{2d} \sum_x |x|^2 \Pi_p(0,x) \\ &\quad + \sum_x \left[\cosh m x_1 - 1 - \frac{m^2 x_1^2}{2}\right] \Pi_p(0,x). \end{aligned} \tag{6.5.9}$$

Fix $\epsilon < \min\{(d-4)/2, 1\}$. There is a positive constant C such that

$$0 \leq \cosh m x_1 - 1 - \frac{m^2 x_1^2}{2} \leq C m^{2+\epsilon} |x_1|^{2+\epsilon} \cosh m x_1. \tag{6.5.10}$$

Hence by Corollary 6.5.2,

$$\begin{aligned} \left| \sum_x \left[\cosh m x_1 - 1 - \frac{m^2 x_1^2}{2}\right] \Pi_p(0,x) \right| &\leq C m^{2+\epsilon} \sum_x |x_1|^{2+\epsilon} |\Pi_p^{(m)}(0,x)| \\ &\leq C K_3 m^{2+\epsilon}, \end{aligned}$$

uniformly in $m < m(p)$ and $p \in [z_c - \delta, z_c)$. Hence the limit of the left side as $m \nearrow m(p)$ is $O[m(p)^{2+\epsilon}]$.

From this fact, together with (6.5.7) and (6.5.9), we conclude that taking the limit as $m \nearrow m(p)$ in (6.5.6) gives

$$\begin{aligned}\chi(p)^{-1} &= p\sum_{y\in\Omega}[\cosh(m(p)y_1)-1] \\ &\quad +\frac{m(p)^2}{2d}\sum_x |x|^2\Pi_p(0,x)+O(m(p)^{2+\epsilon}). \qquad (6.5.11)\end{aligned}$$

Since as noted at the beginning of this section $m(p)\to 0$ as $p\nearrow z_c$, the right side of (6.5.11) is asymptotic to

$$m(p)^2\left[\frac{z_c}{2}\sum_{y\in\Omega}y_1^2+\frac{1}{2d}\sum_x|x|^2\Pi_{z_c}(0,x)\right]=\frac{m(p)^2}{2d}\nabla_k^2\hat{F}_{z_c}(0)$$

as $p\nearrow z_c$. The right side is positive, by (6.2.35). Also, by Theorem 6.1.2 the left side of (6.5.11) is asymptotic to $(Az_c)^{-1}(z_c-p)$. Thus by (6.4.20) and (6.4.26) we have

$$m(p)^2\sim\frac{2d}{A\nabla_k^2\hat{F}_{z_c}(0)}\frac{z_c-p}{z_c}=\frac{2d}{D}\frac{z_c-p}{z_c}, \qquad (6.5.12)$$

which proves Theorem 6.1.5. □

We now complete the remaining step of the proof.

Proof of Lemma 6.5.1. The proof uses Lemma 6.2.1 with m playing the role of p, $n=1$, $a=2/3$, $p_0=0$, $p_1=m(p)$ and

$$f_1(m)=\frac{||H_p^{(m)}||_2^2}{3K\Omega^{-1+s}}.$$

We begin by considering the hypotheses of Lemma 6.2.1 in this context. First, for any $p<z_c$, $f_1(m)$ is continuous in $m\in[0,m(p))$. This follows from the fact that if $p<z_c$ and $m<m(p)$ then $||H_p^{(m)}||_2^2<\infty$ (by Lemma 4.1.5), together with the monotone convergence theorem. Next, by Corollary 6.2.6, $f_1(0)\le 2/3$ for all $p<z_c$, if Ω is sufficiently large.

It thus suffices to show that the remaining, and substantial, hypothesis of Lemma 6.2.1 is satisfied, namely that there is a $\delta>0$ such that given $p\in[z_c-\delta,z_c)$, $m<m(p)$ and Ω sufficiently large (independently of p,m), if $||H_p^{(m)}||_2^2\le 3K\Omega^{-1+s}$ then in fact $||H_p^{(m)}||_2^2\le 2K\Omega^{-1+s}$. The remainder of the proof is concerned with showing the existence of such a δ.

Denoting the reciprocal of $\hat{G}_p^{(m)}(k)$ by $\hat{F}_p^{(m)}(k)$, we have

$$|\hat{G}_p^{(m)}(k)| \le \frac{1}{|\mathrm{Re}\hat{F}_p^{(m)}(k)|}. \tag{6.5.13}$$

Now

$$\begin{aligned}\mathrm{Re}\hat{F}_p^{(m)}(k) &\ge \mathrm{Re}\hat{F}_p^{(m)}(k) - \hat{F}_p^{(m)}(0) \\ &= p\Omega\mathrm{Re}[\hat{D}^{(m)}(0) - \hat{D}^{(m)}(k)] + \mathrm{Re}[\hat{\Pi}_p^{(m)}(0) - \hat{\Pi}_p^{(m)}(k)].\end{aligned}$$

But for $p \ge \Omega^{-1}$,

$$\begin{aligned}p\Omega\mathrm{Re}[\hat{D}^{(m)}(0) - \hat{D}^{(m)}(k)] &= p\sum_{y\in\Omega} e^{my_1}[1-\cos k\cdot y] \\ &= p\sum_{y\in\Omega} \cosh my_1[1-\cos k\cdot y] \\ &\ge 1 - \hat{D}(k).\end{aligned} \tag{6.5.14}$$

Also, by Corollary 6.2.7 we have

$$\begin{aligned}&\mathrm{Re}[\hat{\Pi}_p^{(m)}(0) - \hat{\Pi}_p^{(m)}(k)] \\ &= [\hat{\Pi}_p(0) - \hat{\Pi}_p(k)] + \mathrm{Re}\left[\left(\hat{\Pi}_p^{(m)}(0) - \hat{\Pi}_p^{(m)}(k)\right) - \left(\hat{\Pi}_p(0) - \hat{\Pi}_p(k)\right)\right] \\ &\ge -K_1\Omega^{-u}[1-\hat{D}(k)] + \sum_x [\cosh mx_1 - 1]\Pi_p(0,x)[1-\cos k\cdot x]\end{aligned}$$

(with $u = 3/2$ for the nearest-neighbour model and $u = 5/2 - 5s/2 - 2/d$ for the spread-out model). Therefore

$$\begin{aligned}\mathrm{Re}\hat{F}_p^{(m)}(k) &\ge [1-K_1\Omega^{-u}][1-\hat{D}(k)] \\ &\quad + \sum_x [\cosh mx_1 - 1]\Pi_p(0,x)[1-\cos k\cdot x].\end{aligned}$$

Since

$$0 \le \cosh t - 1 \le \mathrm{const.}|t|^\epsilon \cosh t$$

for $0 \le \epsilon \le 2$, and since [by (6.2.5)]

$$0 \le 1 - \cos k\cdot x \le \frac{(k\cdot x)^2}{2} \le \pi^2 d|x|^2[1-\hat{D}(k)],$$

we have

$$\begin{aligned}&\left|\sum_x [\cosh mx_1 - 1]\Pi_p(0,x)[1-\cos k\cdot x]\right| \\ &\qquad \le c_1 dm^\epsilon[1-\hat{D}(k)]\sum_x |x|^2|x_1|^\epsilon|\Pi_p^{(m)}(0,x)|,\end{aligned}$$

where c_1 is a universal constant. Now the proof of Corollary 6.5.2 goes through equally well assuming $||H_p^{(m)}||_2^2 < 3K\Omega^{-1+s}$ rather than $||H_p^{(m)}||_2^2 < 2K\Omega^{-1+s}$, so under this assumption we have a bound on the sum over x in the right side of the above inequality by a constant independent of p and $m < m(p)$ (but possibly depending on ϵ and Ω). Changing the value of K_1, we then have

$$|\hat{G}_p^{(m)}(k)| \leq [1 + K_1\Omega^{-u} + Cm^\epsilon]\frac{1}{[1 - \hat{D}(k)]}, \qquad (6.5.15)$$

where C is independent of p but may depend on Ω.

Using (6.5.15), the Parseval relation, the fact that $||H_p^{(m)}||_2^2 = ||G_p^{(m)}||_2^2 - 1$, and the ordinary random walk bound of (6.2.7) gives

$$\begin{aligned} ||H_p^{(m)}||_2^2 &\leq [1 + K_1\Omega^{-u} + Cm^\epsilon]\left|\left|\frac{1}{1-\hat{D}}\right|\right|_2^2 - 1 \\ &\leq K\Omega^{-1+s}[1 + K_1\Omega^{-u} + Cm^\epsilon] + K_1\Omega^{-u} + Cm^\epsilon. \end{aligned}$$

For $d > 4$, $-u < -1 + s$, so if Ω is sufficiently large, say $\Omega \geq \Omega_1$ (not depending on m, p), then

$$K_1\Omega^{-u} \leq \min\{1/4, K\Omega^{-1+s}/4\}.$$

For fixed $\Omega \geq \Omega_1$ we then choose δ sufficiently small that

$$Cm(p)^\epsilon \leq \min\{1/4, K\Omega^{-1+s}/4\}$$

for $p \in [z_c - \delta, z_c)$; this is possible because $m(p) \to 0$ as $p \nearrow z_c$. Then for $p \in [z_c - \delta, z_c)$ and $m < m(p)$ we have

$$\begin{aligned} ||H_p^{(m)}||_2^2 &\leq K\Omega^{-1+s}\left[1 + \frac{1}{4} + \frac{1}{4}\right] + \frac{1}{4}K\Omega^{-1+s} + \frac{1}{4}K\Omega^{-1+s} \\ &= 2K\Omega^{-1+s}. \end{aligned}$$

□

6.5.2 The infrared bound

In this section we give the proof of Theorem 6.1.6, apart from the bound $G_{z_c}(0,x) \leq C(p)|x|^{-p}$ for $p < (d-2)/2$, which is a consequence of Corollary 6.1.4. The bound $G_{z_c}(0,x) \leq \text{const.}|x|^{-2}$ has already been established in Corollary 6.2.6, and the upper bound on $\hat{G}_{z_c}(k)$ follows from the last inequality of Corollary 6.2.7. Let $\epsilon < \min\{(d-4)/2, 1\}$. Here we show that as $k \to 0$,

$$\hat{G}_{z_c}(k) = \frac{2d}{\nabla_k^2 \hat{F}_{z_c}(0)} \frac{1}{k^2 + O(k^{2+\epsilon})}. \tag{6.5.16}$$

Since $\hat{G}_{z_c}(0)^{-1} = 0$ by Corollary 6.2.7,

$$\begin{aligned} \hat{G}_{z_c}(k)^{-1} &= \hat{G}_{z_c}(k)^{-1} - \hat{G}_{z_c}(0)^{-1} \\ &= z_c\Omega[1 - \hat{D}(k)] + \hat{\Pi}_{z_c}(0) - \hat{\Pi}_{z_c}(k). \end{aligned} \tag{6.5.17}$$

The last two terms on the right side can be written as

$$\frac{k^2}{2d} \sum_x |x|^2 \Pi_{z_c}(0, x) + \sum_x \left[1 - \cos k \cdot x - \frac{(k \cdot x)^2}{2} \right] \Pi_{z_c}(0, x).$$

The quantity in square brackets can be bounded above by $c_2 k^{2+\epsilon} |x|^{2+\epsilon}$, for some universal constant c_2. The sum over x can then be bounded as in Corollary 6.5.2. The result is

$$\hat{G}_{z_c}(k)^{-1} = k^2 [C^{-1} + O(k^\epsilon)], \tag{6.5.18}$$

where

$$C^{-1} = \frac{1}{2d} \left[-z_c \Omega \nabla_k^2 \hat{D}(0) + \sum_x |x|^2 \Pi_{z_c}(0, x) \right] = \frac{1}{2d} \nabla_k^2 \hat{F}_{z_c}(0). \tag{6.5.19}$$

6.6 Convergence to Brownian motion

Given an n-step self-avoiding walk ω, we define

$$X_n(k/n) = (Dn)^{-1/2} \omega(k), \quad k = 0, 1, \ldots, n \tag{6.6.1}$$

where D is the diffusion constant given in (6.4.26). We then obtain a continuous function X_n on the interval $[0, 1]$, taking values in $\mathbf{R}^d$, by defining $X_n(t)$ to be the linear interpolation of $X_n(k/n)$. In this section we prove Theorem 6.1.8, which states that if Ω is sufficiently large (with $d > 4$ for the spread-out model) then $X_n(t)$ converges in distribution to Brownian motion, or in other words that

$$\lim_{n \to \infty} \langle f(X_n) \rangle_n = \int f dW \tag{6.6.2}$$

for every bounded continuous function f on $C_d[0,1]$ (the latter denotes the $\mathbf{R}^d$-valued continuous functions on $[0,1]$, equipped with the supremum norm). Here W is the Wiener measure, normalized such that

$$\int e^{ik\cdot B_t} dW = e^{-k^2 t/2d}, \tag{6.6.3}$$

and the angular brackets on the left side of (6.6.2) denote expectation with respect to the uniform measure on the set of n-step self-avoiding walks beginning at the origin.

To prove this result it suffices to show both convergence of the finite-dimensional distributions to Gaussian distributions and tightness [see e.g. Billingsley (1968)]. Tightness follows readily from Theorem 6.1.1 and will be discussed at the end of this section.

For convergence of the finite-dimensional distributions, we need to prove that for any positive integer N, any $0 < t_1 < t_2 < \ldots < t_N \leq 1$ and any bounded, continuous function g on $\mathbf{R}^{dN}$,

$$\lim_{n\to\infty} \langle g(X_n(t_1),\ldots,X_n(t_N))\rangle_n = \int g(B_{t_1},\ldots,B_{t_N}) dW. \tag{6.6.4}$$

Since weak convergence of probability measures on $\mathbf{R}^m$ is implied by convergence of the corresponding characteristic functions, it suffices to consider only

$$g(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{6.6.5}$$

where $\mathbf{x} = (x^{(1)},\ldots,x^{(N)})$ with each $x^{(i)} \in \mathbf{R}^d$, and similarly for $\mathbf{k}$, and $\mathbf{k}\cdot\mathbf{x} = \sum_{i=1}^N k^{(i)}\cdot x^{(i)}$. Equivalently, we can replace this g by

$$g(\mathbf{x}) = \exp\left[i\sum_{j=1}^{N} k^{(j)}\cdot(x^{(j)} - x^{(j-1)})\right], \tag{6.6.6}$$

which will be better suited to take into account the "effective independence" of the self-avoiding walk on distinct intervals $[t_i, t_{i+1}]$.

Let $\mathbf{a} = (a_1,\ldots,a_N)$, with each a_i a nonnegative integer, and let

$$\Delta\omega(\mathbf{a}) = (\omega(a_1), \omega(a_2) - \omega(a_1), \ldots, \omega(a_N) - \omega(a_{N-1})). \tag{6.6.7}$$

We define

$$M(\mathbf{k},\mathbf{a}) = \sum_{\omega:|\omega|=a_N} e^{i\mathbf{k}\cdot\Delta\omega(\mathbf{a})} K[0,a_N], \tag{6.6.8}$$

where the sum over ω is a sum over simple random walks, and $K[a,b]$ was defined in (5.2.6). (We work in this section with infinite memory, i.e., with the fully self-avoiding walk.) Inserting (6.6.6) into (6.6.4) and using the above notation, we see that for convergence of the finite-dimensional distributions it suffices to show that for $N = 1, 2, 3, \ldots$,

$$\lim_{n\to\infty} c_{nt_N}^{-1} M(\mathbf{k}/\sqrt{Dn}, nt) = \exp\left[-\frac{1}{2d}\sum_{i=1}^{N}\left(k^{(i)}\right)^2 (t_i - t_{i-1})\right]. \quad (6.6.9)$$

[The nt and nt_N on the left are to be interpreted as $(\lfloor nt_1\rfloor, \ldots, \lfloor nt_N\rfloor)$ and $\lfloor nt_n\rfloor$ respectively; similar shorthand is used throughout this section. Also, $(k^{(i)})^2$ denotes the square of the Euclidean norm of the vector $k^{(i)}$. Finally, there is no difficulty in the replacement of $X_n(t)$ by $(Dn)^{-1/2}\omega(\lfloor nt\rfloor)$ that has been made in the left side of (6.6.9).]

We obtain (6.6.9) for $N = 1$ in Section 6.6.1, and prove (6.6.9) for $N \geq 2$ by induction on N in Section 6.6.2.

6.6.1 The scaling limit of the endpoint

In this section we prove (6.6.9) for $N = 1$. In fact, a minor generalization of (6.6.9) will be needed to take the induction step, and we prove the generalization here.

Theorem 6.6.1 *Let Ω be sufficiently large (with $d > 4$ for the spread-out model). Let h_n be any fixed nonnegative sequence with $\lim_{n\to\infty} h_n = 0$, and let $g = \{g_n\}$ be any real sequence with $|g_n| \leq h_n$ for all n. Fix $t > 0$ and let $T = t(1 - g_n)$. Then for any $k \in \mathbf{R}^d$,*

$$\lim_{n\to\infty} \frac{\hat{c}_{nT}(k/\sqrt{Dn})}{c_{nT}} = \exp\left[-k^2 t/2d\right], \quad (6.6.10)$$

uniformly in g.

Proof. Fix any $\epsilon < \min\{(d-4)/4, 1\}$. By (6.4.19), the denominator of (6.6.10) can be written

$$c_{nT} = z_c^{-nT-1}\{[-\partial_z \hat{F}_{z_c}(0)]^{-1} + O(n^{-\epsilon})\}, \quad (6.6.11)$$

uniformly in g. The numerator of (6.6.10) is the coefficient of z^{nT} in $\hat{G}_z(k/\sqrt{Dn})$, and hence is given by

$$\hat{c}_{nT}(k/\sqrt{Dn}) = \frac{1}{2\pi i}\oint \frac{1}{\hat{F}_z(k/\sqrt{Dn})}\frac{dz}{z^{nT+1}}, \quad (6.6.12)$$

where the integration contour is a small circle centred at the origin. The task now is to obtain the asymptotic form of the integral on the right side.

We extract the leading contribution to the right side of (6.6.12) as follows. We subtract $\hat{F}_{z_c}(0) = 0$ from $\hat{F}_z(k/\sqrt{Dn})$, and then add and subtract

$$\partial_z \hat{\Pi}_{z_c}(0)(z_c - z) + \frac{1}{2d}\nabla_k^2 \hat{\Pi}_{z_c}(0)\frac{k^2}{Dn}.$$

The result can be written

$$\hat{F}_z(k/\sqrt{Dn}) = \alpha - \beta z + E, \tag{6.6.13}$$

where

$$\alpha = \alpha(k/\sqrt{Dn}) = -z_c \partial_z \hat{F}_{z_c}(0) - \frac{1}{2d}\nabla_k^2 \hat{\Pi}_{z_c}(0)\frac{k^2}{Dn}, \tag{6.6.14}$$

$$\beta = \beta(k/\sqrt{Dn}) = -\partial_z \hat{F}_{z_c}(0) - \Omega[1 - \hat{D}(k/\sqrt{Dn})], \tag{6.6.15}$$

and

$$E = -\hat{\Pi}_z(k/\sqrt{Dn}) - \partial_z \hat{\Pi}_{z_c}(0)(z_c - z) + \hat{\Pi}_{z_c}(0) + \frac{k^2}{2dDn}\nabla_k^2 \hat{\Pi}_{z_c}(0). \tag{6.6.16}$$

The error term can be written

$$E = E_1 + E_2 + E_3, \tag{6.6.17}$$

where

$$E_1 = \hat{\Pi}_{z_c}(k/\sqrt{Dn}) - \hat{\Pi}_z(k/\sqrt{Dn}) - \partial_z \hat{\Pi}_{z_c}(k/\sqrt{Dn})(z_c - z), \tag{6.6.18}$$

$$E_2 = [\partial_z \hat{\Pi}_{z_c}(k/\sqrt{Dn}) - \partial_z \hat{\Pi}_{z_c}(0)](z_c - z), \tag{6.6.19}$$

and

$$E_3 = \hat{\Pi}_{z_c}(0) - \hat{\Pi}_{z_c}(k/\sqrt{Dn}) + \frac{k^2}{2dDn}\nabla_k^2 \hat{\Pi}_{z_c}(0). \tag{6.6.20}$$

By (6.6.13),

$$\frac{1}{\hat{F}_z(k/\sqrt{Dn})} = \frac{1}{\alpha - \beta z} - \frac{E}{(\alpha - \beta z)\hat{F}_z(k/\sqrt{Dn})}. \tag{6.6.21}$$

We now insert (6.6.21) into (6.6.12). The integral corresponding to the first term on the right side of (6.6.21) is $\beta^{nT}\alpha^{-(nT+1)}$. A straightforward calculation using the definition of D in (6.4.26) and the fact that $1 - \hat{D}(u) \sim -(1/2d)u^2\nabla^2\hat{D}(0)$ shows that

$$\begin{aligned} \frac{\beta^{nT}}{\alpha^{nT+1}} &\sim \frac{1}{-\partial_z \hat{F}_{z_c}(0) z_c^{nT+1}}\left[1 - \frac{k^2}{2dn}\right]^{nT} \\ &\sim \frac{1}{-\partial_z \hat{F}_{z_c}(0) z_c^{nT+1}} \exp[-k^2 t/2d], \end{aligned} \tag{6.6.22}$$

uniformly in g. Comparing (6.6.11), the theorem follows from (6.6.22) if it can be shown that

$$\frac{1}{2\pi i}\oint \frac{E}{(\alpha-\beta z)\hat{F}_z(k/\sqrt{Dn})}\frac{dz}{z^{nT+1}} = o(z_c^{-nT}) \qquad (6.6.23)$$

uniformly in g. We show that (6.6.23) holds by using Lemma 6.3.3.

The first step is to obtain lower bounds on the two factors in the denominator of the integrand of (6.6.23). We begin with $|\alpha - \beta z| = \beta|\alpha/\beta - z|$. For large n, β is bounded away from zero by Theorem 6.2.9. Also, it can be seen from (6.6.22) that $\alpha/\beta \geq z_c$ for n large. Hence there is a positive constant such that for large n and $|z| \leq z_c$,

$$|\alpha - \beta z| \geq \text{const.}|z_c - z|. \qquad (6.6.24)$$

For a lower bound on $|\hat{F}_z(k/\sqrt{Dn})|$, we write

$$\hat{F}_z(k/\sqrt{Dn}) = \hat{F}_z(k/\sqrt{Dn}) - \hat{F}_{z_c}(k/\sqrt{Dn}) + \hat{F}_{z_c}(k/\sqrt{Dn}). \qquad (6.6.25)$$

By Corollary 6.4.3 and Lemma 6.3.2, the first two terms on the right side combine to give $-\partial_z \hat{F}_{z_c}(k/\sqrt{Dn})(z_c - z) + O(|z_c - z|^{1+\epsilon})$. By the dominated convergence theorem the derivative appearing here is continuous in k, and hence differs from its value at $k = 0$ by $o(1)$. Thus we have

$$\hat{F}_z(k/\sqrt{Dn}) = [-\partial_z \hat{F}_{z_c}(0) + O(|z_c - z|^{\epsilon}) + o(1)](z_c - z) + \hat{F}_{z_c}(k/\sqrt{Dn}). \qquad (6.6.26)$$

By Corollary 6.2.7, the last term on the right side is nonnegative. Since the first term in square brackets on the right side is also nonnegative by Theorem 6.2.9, it follows that for z in a small neighbourhood of z_c (inside the closed disk of radius z_c), the right side of (6.6.26) is bounded below by const.$|z_c - z|$ (for large n). Outside of this neighbourhood, by Lemma 6.2.10 there is a constant $c > 0$ such that $|\hat{F}_z(0)| \geq c$. Hence $|\hat{F}_z(k/\sqrt{Dn})| \geq c/2$ if n is sufficiently large, since by Corollary 6.2.7 there is a bound on $|\nabla_k \hat{F}_z(k)|$ which is uniform in k and $|z| \leq z_c$. Therefore for $|z| \leq z_c$ we have

$$|\hat{F}_z(k/\sqrt{Dn})| \geq \text{const.}|z_c - z|. \qquad (6.6.27)$$

We now turn to upper bounds on E_i for $i = 1, 2, 3$. Beginning with E_1, it follows from (6.6.24), (6.6.27), and a straightforward calculation using Corollary 6.4.3 and Lemma 6.3.2 that

$$\left|\frac{d}{dz}\frac{E_1}{(\alpha-\beta z)(\hat{F}_z(k/\sqrt{Dn}))}\right| \leq O(|z_c - z|^{\epsilon-2}), \qquad (6.6.28)$$

and hence by Lemma 6.3.3(ii) (6.6.23) is satisfied if E is replaced by E_1.

For E_2, we show

$$|E_2| \leq n^{-\epsilon/2}|z_c - z|, \tag{6.6.29}$$

which suffices by Lemma 6.3.3(i). To do so we write $\pi_m(x)$ for the coefficient of z^m in $\Pi_z(0,x)$, so that

$$\partial_z \hat{\Pi}_{z_c}(k/\sqrt{Dn}) - \partial_z \hat{\Pi}_{z_c}(0) = -\sum_{x,m} m\pi_m(x) z_c^{m-1}[1 - \cos(k \cdot x/\sqrt{Dn})]. \tag{6.6.30}$$

Since $|1-\cos t| \leq O(t^\epsilon)$ for small $\epsilon \leq 2$, and since $|x|^\epsilon|\pi_m(x)| \leq m^\epsilon|\pi_m(x)|$, the right side of (6.6.30) is $O(n^{-\epsilon/2})$ by Corollary 6.4.3, which gives (6.6.29).

Finally, for E_3 we use symmetry to write

$$E_3 = \sum_{x,m} \pi_m(x) z_c^m \left[1 - \cos\frac{k \cdot x}{\sqrt{Dn}} - \frac{(k \cdot x)^2}{2Dn}\right]. \tag{6.6.31}$$

For small positive ϵ, $|1 - \cos t - t^2/2| \leq O(|t|^{2+\epsilon})$. Since $|x|^{2+\epsilon}|\pi_m(x)| \leq m^\epsilon|x|^2|\pi_m(x)|$, it follows from Corollary 6.4.3 that $|E_3| \leq O(n^{-1-\epsilon/2})$. Then Lemma 6.3.3($i$) gives (6.6.23) for E replaced by E_3. □

6.6.2 The finite-dimensional distributions

In this section we complete the proof of Theorem 6.1.8 by showing that (6.6.9) holds for $N \geq 2$. The proof of (6.6.9) is by induction on N, with the case $N = 1$ having been treated in the previous section. Lemma 5.2.5 is a basic element of the induction argument.

To perform the induction step, some flexibility is needed in the number of steps in the walk. Let $g = \{g_n\}$ be any sequence satisfying $0 \leq g_n \leq n^{-1/2}$, and let $\mathbf{T} = (t_1, t_2, \ldots, t_{N-1}, T)$, where $T = t_N(1 - g_n)$. It suffices to prove the following theorem.

Theorem 6.6.2 *Let Ω be sufficiently large (with $d > 4$ for the spread-out model) and let $N \geq 2$. Suppose that*

$$\lim_{n\to\infty} c_{n\mathbf{T}}^{-1} M(\mathbf{k}/\sqrt{Dn}, n\mathbf{T}) = \exp\left[-\frac{1}{2d}\sum_{i=1}^{N}\left(k^{(i)}\right)^2(t_i - t_{i-1})\right] \tag{6.6.32}$$

holds uniformly in g, when N is replaced by $N-1$. Then in fact (6.6.32) holds as stated, uniformly in g.

Proof. To simplify the notation we write $\kappa = (\kappa_1, \ldots, \kappa_N) = \mathbf{k}/\sqrt{Dn}$. By (6.6.8), and Lemma 5.2.5 with $m = nt_{N-1}$,

$$M(\kappa, n\mathbf{T}) = \sum_{I \ni nt_{N-1}} \sum_{\omega:|\omega|=nT} e^{i\kappa\cdot\Delta\omega(n\mathbf{T})} K[0, I_1] J[I_1, I_2] K[I_2, nT]. \tag{6.6.33}$$

The sum over I is the sum over intervals $[I_1, I_2]$ of integers with either $0 \le I_1 < nt_{N-1} < I_2 \le nT$ or $I_1 = I_2 = nt_{N-1}$.

In (6.6.33) we factor the walk ω into three independent subwalks on the subintervals $[0, I_1]$, $I = [I_1, I_2]$ and $[I_2, nT]$. We fix a sequence b_n with $\lim_{n\to\infty} b_n = \infty$ and $b_n = o(n^{1/2})$, for example $b_n = n^{1/4}$. It will become apparent that the significant contribution to the right side of (6.6.33) is due to intervals I with $|I| \le b_n$. We take n sufficiently large that for such I, $nt_{N-2} < I_1 \le nt_{N-1} \le I_2 < nT$.

Denote by $M^{\le}_{\mathbf{k}}$ and $M^{>}_{\mathbf{k}}$ respectively the contributions to the right side of (6.6.33) due to $|I| \le b_n$ and $|I| > b_n$. By factoring the exponential we can resum to obtain

$$M^{\le}_{\mathbf{k}} \equiv \sum_{\substack{I \ni nt_{N-1} \\ |I| \le b_n}} M(\kappa_1, \ldots, \kappa_{N-1}; nt_1, \ldots, nt_{N-2}, I_1) \times \sum_{\omega:|\omega|=|I|} E_1(\omega, I) J[0, |I|] \hat{c}_{nT-I_2}(\kappa_N), \tag{6.6.34}$$

where

$$\begin{aligned} E_1(\omega, I) &= \exp[i\kappa_{N-1}\cdot\omega(nt_{N-1} - I_1) + i\kappa_N\cdot\omega(I_2 - nt_{N-1})] \\ &= 1 + O(b_n n^{-1/2}) \end{aligned} \tag{6.6.35}$$

uniformly in ω and $|I| \le b_n$. For $|I| \le b_n$ and n sufficiently large, $I_1 \in [nt_{N-1}(1 - n^{-1/2}), nt_{N-1}]$. Hence by the induction hypothesis,

$$M(\kappa_1, \ldots, \kappa_{N-1}; nt_1, \ldots, nt_{N-2}, I_1) = c_{I_1}\left[\exp\left[-\frac{1}{2d}\sum_{i=1}^{N-1}\left(k^{(i)}\right)^2 (t_i - t_{i-1})\right] + E_2(I)\right], \tag{6.6.36}$$

where $|E_2(I)| = o(1)$ uniformly in $|I| \le b_n$. Similarly it follows from Theorem 6.6.1 that for $|I| \le b_n$,

$$\hat{c}_{nT-I_2}(\kappa_n) = c_{nT-I_2}\left[\exp\left[-\frac{1}{2d}\left(k^{(N)}\right)^2 (t_N - t_{N-1})\right] + E_3(I)\right], \tag{6.6.37}$$

where $|E_3(I)| = o(1)$ uniformly in $|I| \le b_n$.

Substituting (6.6.35)-(6.6.37) into (6.6.34) leads to

$$M_{\mathbf{k}}^{\le} = \exp\left[-\frac{1}{2d}\sum_{i=1}^{N}\left(k^{(i)}\right)^2 (t_i - t_{i-1})\right] M_{\mathbf{0}}^{\le} + A, \tag{6.6.38}$$

where

$$|A| \le o(1) \sum_{\substack{I \ni nt_{N-1} \\ |I| \le b_n}} c_{I_1} \sum_{|\omega|=|I|} |J[0,|I|]| c_{nT-I_2}. \tag{6.6.39}$$

Since $M(0, n\mathbf{T}) = c_{nT}$, we have $M_{\mathbf{0}}^{\le} = c_{nT} - M_{\mathbf{0}}^{>}$. Hence

$$\begin{aligned} c_{nT}^{-1} M(\kappa, n\mathbf{T}) &= \exp\left[-\frac{1}{2d}\sum_{i=1}^{N}\left(k^{(i)}\right)^2 (t_i - t_{i-1})\right] \left[1 - c_{nT}^{-1} M_{\mathbf{0}}^{>}\right] \\ &\quad + c_{nT}^{-1} A + c_{nT}^{-1} M_{\mathbf{k}}^{>}. \end{aligned} \tag{6.6.40}$$

Now by Theorem 6.1.1(a) and (6.6.39),

$$c_{nT}^{-1}|A| \le o(1) \sum_{|I|=1}^{b_n} |I| \sum_{|\omega|=|I|} |J[0,|I|]| z_c^{|I|} \tag{6.6.41}$$

(here $|I|$ is merely a summation index and the sum is no longer a sum over intervals). In (6.6.41) the factor $|I|$ counts the number of possibilities for $nt_{N-1} \in I$. Extending the summation over $|I|$ on the right side to infinity, it follows from the (absolute) bound on $\partial_z \hat{\Pi}$ of Theorem 6.2.9 that

$$c_{nT}^{-1}|A| \le o(1). \tag{6.6.42}$$

It suffices now to show that $c_{nT}^{-1} M_{\mathbf{k}}^{>} = o(1)$ as $n \to \infty$. Arguing as for $M_{\mathbf{k}}^{\le}$,

$$c_{nT}^{-1}|M_{\mathbf{k}}^{>}| \le O(1) \sum_{|I|=b_n+1}^{\infty} |I| \sum_{|\omega|=|I|} |J[0,|I|]| z_c^{|I|}. \tag{6.6.43}$$

The right side goes to zero as $n \to \infty$ since by Theorem 6.2.9

$$\sum_{|I|=1}^{\infty} |I| \sum_{\omega:|\omega|=|I|} |J[0,|I|]| z_c^{|I|} < \infty.$$

□

6.6.3 Tightness

Tightness is proved via the following lemma. Although not stated explicitly in Billingsley (1968), the lemma follows in a straightforward manner from Theorem 8.4 and results on pages 87-89 [both references to Billingsley (1968)]. For the statement of the lemma, we define a process closely related to $X_n(t)$ by

$$Y_n(t) = (Dn)^{-1/2}\omega(\lfloor nt \rfloor). \tag{6.6.44}$$

Lemma 6.6.3 *The sequence $\{X_n\}$ is tight if there exist constants $K \geq 0$ and $a > 1/2$ such that for $0 \leq t_1 < t_2 < t_3 \leq 1$ and for all n,*

$$\langle |Y_n(t_2) - Y_n(t_1)|^{2a} |Y_n(t_3) - Y_n(t_2)|^{2a} \rangle_n \leq K|t_2 - t_1|^a |t_3 - t_2|^a, \tag{6.6.45}$$

where the angular brackets denote expectation with respect to the uniform measure on the set of n-step self-avoiding walks.

We will use Theorem 6.1.1 to show that (6.6.45) holds with $a = 1$. With $a = 1$ the left side of (6.6.45) is equal to

$$\frac{1}{D^2 n^2 c_n} \sum_{|\omega|=n} |\omega(nt_2) - \omega(nt_1)|^2 |\omega(nt_3) - \omega(nt_2)|^2 K[0,n], \tag{6.6.46}$$

where the sum on the right side is over all n-step ordinary random walks, and brackets indicating integer part have been omitted to simplify the notation. The inequality

$$K[0,n] \leq K[0,nt_1]K[nt_1,nt_2]K[nt_2,nt_3]K[nt_3,n] \tag{6.6.47}$$

allows for the replacement of the sum over ω by sums over independent subwalks on the intervals $[0,nt_1]$, $[nt_1,nt_2]$, $[nt_2,nt_3]$, $[nt_3,n]$, for an upper bound on (6.6.46). Also, by Theorem 6.1.1(a),

$$c_n^{-1} \leq \text{const.} c_{nt_1}^{-1} c_{nt_2-nt_1}^{-1} c_{nt_3-nt_2}^{-1} c_{n-nt_3}^{-1}. \tag{6.6.48}$$

Using the above two inequalities in (6.6.46) gives

$$\begin{aligned} &\langle |Y_n(t_2) - Y_n(t_1)|^2 |Y_n(t_3) - Y_n(t_2)|^2 \rangle_n \\ &\leq \text{const.} n^{-2} c_{nt_2-nt_1}^{-1} \sum_{|\omega|=nt_2-nt_1} |\omega(nt_2 - nt_1)|^2 K[0, nt_2 - nt_1] \\ &\quad \times c_{nt_3-nt_2}^{-1} \sum_{|\omega|=nt_3-nt_2} |\omega(nt_3 - nt_2)|^2 K[0, nt_3 - nt_2] \\ &= \text{const.} n^{-2} \langle |\omega(nt_2 - nt_1)|^2 \rangle_{nt_2-nt_1} \langle |\omega(nt_3 - nt_2)|^2 \rangle_{nt_3-nt_2}. \end{aligned}$$

But by Theorem 6.1.1(b), the expectations on the right side are bounded above by a multiple of $(nt_2 - nt_1)(nt_3 - nt_2)$, which gives (6.6.45).

6.7 The infinite self-avoiding walk

In this section we give the proof of Theorem 6.1.9. Throughout the section we work only with the fully self-avoiding walk, i.e. $\tau = \infty$.

We begin by defining the infinite self-avoiding walk. Given $n \geq m$ and an m-step self-avoiding walk ω, we let $P_{m,n}(\omega)$ denote the fraction of n-step walks whose first m steps are given by ω. In other words, $P_{m,n}(\omega)$ is the fraction of n-step self-avoiding walks which *extend* ω. Then we define

$$P_m(\omega) = \lim_{n \to \infty} P_{m,n}(\omega) \tag{6.7.1}$$

if the limit exists. If the limit does exist, then the probability measures P_m on m-step walks will be *consistent* in the sense that for each $n \geq m$ and each m-step self-avoiding walk ω,

$$P_m(\omega) = \sum_{\rho > \omega} P_n(\rho), \tag{6.7.2}$$

where the sum is over all n-step self-avoiding walks ρ which extend ω. This consistency property allows for the definition via cylinder sets of a measure P_∞ on the set of all infinite self-avoiding walks. The measure P_∞ is the *infinite self-avoiding walk.*

Although the limit (6.7.1) is believed to exist in all dimensions, the closest results to existence of the limit in general dimensions are Theorems 7.4.2 and 7.4.5(a). These state that for any m-step self-avoiding walk ω which can be extended to an infinite self-avoiding walk, $\liminf_{n\to\infty} P_{m,n}(\omega) > 0$ and $\lim_{n\to\infty} P_{m,n+2}(\omega)/P_{m,n}(\omega) = 1$. (For bridges the situation is easier, and existence of the infinite bridge in all dimensions is proven in Section 8.3.) The remainder of this section is devoted to a proof that the limit in (6.7.1) exists if Ω is sufficiently large (with $d > 4$ for the spread-out model).

Given a nonnegative integer m, let $\mathbf{k} = (k^{(1)}, \ldots, k^{(m)})$, where $k^{(i)} \in [-\pi, \pi]^d$. Given $n \geq m$ and an n-step self-avoiding walk ω, let ω_m be the first m steps of ω, and

$$\mathbf{k} \cdot \omega_m = \sum_{i=1}^{m} k^{(i)} \cdot \omega(i).$$

Let

$$\bar{\varphi}_{m,n}(\mathbf{k}) = \sum_{|\omega|=n} e^{i\mathbf{k}\cdot\omega_m} K[0,n],$$

where the sum is over all n-step ordinary random walks and K was introduced in (5.2.6). We also define

$$\varphi_{m,n}(\mathbf{k}) = \sum_{|\omega|=m} e^{i\mathbf{k}\cdot\omega} P_{m,n}(\omega) = \frac{1}{c_n}\bar{\varphi}_{m,n}(\mathbf{k}),$$

where the sum is over all m-step self-avoiding walks (walks which do have self-intersections would make no contribution so the sum can also be considered to be over ordinary random walks). Since $\{P_{m,n}\}_n$ is clearly tight, a standard convergence theorem [see Billingsley (1968), p. 46] guarantees that existence of the limit (6.7.1) follows from existence of the limit

$$\varphi_m(\mathbf{k}) = \lim_{n\to\infty} \varphi_{m,n}(\mathbf{k}), \tag{6.7.3}$$

for all $\mathbf{k} \in [-\pi, \pi]^{md}$.

We now recall some notation and a lemma from Section 5.2. For $m \geq 0$, we defined a quantity similar to the Fourier transform $\hat{G}_z(k)$ of the two-point function by

$$\Gamma_z(\mathbf{k}, m) = \sum_{n=m}^{\infty} \bar{\varphi}_{m,n}(\mathbf{k}) z^n.$$

Since $|\Gamma_z(\mathbf{k}, m)| \leq \chi(|z|)$, this power series converges for $|z| < z_c$. We also defined a quantity similar to $\hat{\Pi}_z(k)$, again for $m \geq 0$, by

$$\Psi_z(\mathbf{k}, m) = \sum_{s=m}^{\infty} z^s \sum_{|\omega|=s} e^{i\mathbf{k}\cdot\omega_m} J[0, s], \tag{6.7.4}$$

where the sum is over ordinary random walks. It follows from the absolute bound on the lace expansion of Theorem 6.2.9 that for $v = 0, 1$, $\partial_z^v \Psi_z(\mathbf{k}, m)$ is bounded by a finite constant uniformly in $\mathbf{k}$ and $|z| \leq z_c$. For $j < m$ we define $\bar{\mathbf{k}}_j = (k^{(j+1)}, \ldots, k^{(m)})$. In Lemma 5.2.6, it was shown that for $m \geq 1$

$$\begin{aligned}\Gamma_z(\mathbf{k}, m) &= z\Omega\hat{D}\Big(\sum_{j=1}^{m} k^{(j)}\Big)\Gamma_z(\bar{\mathbf{k}}_1, m-1) \\ &+ \sum_{s=2}^{m-1} z^s \sum_{|\omega|=s} \exp\Big[i\sum_{j=1}^{m} k^{(j)} \cdot \omega(\min\{j, s\})\Big] J[0, s]\Gamma_z(\bar{\mathbf{k}}_s, m-s) \\ &+ \Psi_z(\mathbf{k}, m)\chi(z).\end{aligned}$$

Let $N_z(\mathbf{k}, m) = \chi(z)^{-1}\Gamma_z(\mathbf{k},m)$. The above identity and induction on m then can be used to argue that for $v = 0, 1$, $\partial_z^v N_z(\mathbf{k}, m)$ is uniformly bounded in $\mathbf{k}$ and $|z| \leq z_c$.

To prove existence of the limit (6.7.3), we proceed as follows. By definition of N_z,

$$\begin{aligned}\bar{\varphi}_{m,n}(\mathbf{k}) &= \frac{1}{2\pi i}\oint N_z(\mathbf{k}, m)\chi(z)\frac{dz}{z^{n+1}} \qquad (6.7.5) \\ &= N_{z_c}(\mathbf{k}, m)c_n + \frac{1}{2\pi i}\oint [N_z(\mathbf{k}, m) - N_{z_c}(\mathbf{k}, m)]\chi(z)\frac{dz}{z^{n+1}},\end{aligned}$$

where the contour is a small circle centred at the origin. It suffices to show that the second term on the right side is $o(c_n)$, which by Theorem 6.1.1(a) is equivalent to $o(z_c^{-n})$. Hence by Lemma 6.3.3(*ii*) it suffices to show that for $|z| \leq z_c$,

$$\left| \frac{d}{dz}[N_z(\mathbf{k},m) - N_{z_c}(\mathbf{k},m)]\chi(z) \right| \leq O(|z_c - z|^{-1}), \tag{6.7.6}$$

for this would imply that the second term on the right side of (6.7.5) is $O(z_c^{-n} n^{-\alpha})$, for every $\alpha < 1$.

Now since $|\partial_z N_z|$ is uniformly bounded for $|z| \leq z_c$,

$$\frac{d}{dz}[N_z(\mathbf{k},m) - N_{z_c}(\mathbf{k},m)]\chi(z) = O(1)\chi(z) + O(|z_c - z|)\frac{d}{dz}\chi(z).$$

The first term on the right side is $O(|z_c - z|^{-1})$ by Theorem 6.2.10. It follows easily from Theorem 6.2.9 and Theorem 6.2.10 that the second term on the right side is also $O(|z_c - z|^{-1})$. Thus we have (6.7.6), and the proof of Theorem 6.1.9 is complete.

6.8 The bound on $c_n(0,x)$

In this section we prove Theorem 6.1.3, which states that for the nearest-neighbour model in sufficiently high dimensions or for the spread-out model with $d > 4$ and L sufficiently large, there is a positive constant B such that

$$\sup_{x \in Z^d} c_n(0,x) \leq B\mu^n n^{-d/2}. \tag{6.8.1}$$

This shows that if as believed $c_n(0,x) \sim \text{const.}\mu^n n^{\alpha_{sing}-2}$ then $\alpha_{sing} - 2 \leq -d/2$, i.e. we have proved an inequality corresponding to the conjectured hyperscaling relation $\alpha_{sing} - 2 = -d\nu$. This is the only result stated in Section 6.1 which has not yet been proved for the nearest-neighbour model for all $d \geq 5$. We assume in this section that n and $||x||_1$ have the same parity; otherwise $c_n(0,x) = 0$.

For reasons to be discussed momentarily, with some reluctance we reintroduce a finite memory as in Section 5.2; recall that the estimates of Section 6.2 are uniform in the memory. Let $c_{n,\tau}(0,x)$ denote the number of n-step walks ending at x which are self-avoiding with memory τ. For any $\tau \in [0,\infty]$ and for any x,

$$c_n(0,x) \leq c_{n,\tau}(0,x) = \int_{[-\pi,\pi]^d} \frac{d^dk}{(2\pi)^d} \hat{c}_{n,\tau}(k) e^{-ik\cdot x}. \tag{6.8.2}$$

Denote by $\vec{\pi}$ the d-dimensional vector whose components are all π. Then since n and $||x||_1$ have the same parity,

$$\hat{c}_{n,\tau}(k-\vec{\pi}) = (-1)^n \hat{c}_{n,\tau}(k). \tag{6.8.3}$$

Using this fact, together with periodicity, we can then conclude that the integral on the right side of (6.8.2) is twice the integral over $[-\pi/2, \pi/2] \times [-\pi, \pi]^{d-1}$, and hence

$$c_n(0,x) \leq \frac{2}{(2\pi)^d} \int_{-\pi/2}^{\pi/2} dk_1 \int_{[-\pi,\pi]^{d-1}} dk_2 \ldots dk_d \, \hat{c}_{n,\tau}(k) e^{-ik\cdot x}. \tag{6.8.4}$$

To estimate the integral on the right side, we will first use contour integration to estimate the integrand. In Theorem 6.6.1 we have already obtained good control of $\hat{c}_n(k)$ for k of order $n^{-1/2}$, but now we require estimates valid for all k.

In (5.2.5), we introduced $z_c(k;\tau)$ as the radius of convergence of $\hat{G}_z(k;\tau)$. Thus $z_c(k;\tau)$ is the zero of $\hat{F}_z(k;\tau) = \hat{G}_z(k;\tau)^{-1}$ which is closest to the origin. Our contour integration method requires us to track this zero as a function of k, and because it will occur frequently we abbreviate the notation by writing $z_\tau(k) \equiv z_c(k;\tau)$ and $z_\tau \equiv z_c(0;\tau)$. In general zeroes of $\hat{F}_z(k;\tau)$ occur in complex conjugate pairs since $\overline{\hat{G}_z(k;\tau)} = \hat{G}_{\bar{z}}(k;\tau)$, but it will be shown that for small k there is a unique, and hence real, zero near z_τ. Clearly $z_\tau(k) \geq z_\tau$, since $|\hat{G}_z(k;\tau)| \leq \hat{G}_z(0;\tau)$. Similarly, $z_\tau \leq z_c$. Without introducing a finite memory we are unable to control $\hat{\Pi}_z$ beyond z_c and therefore are unable to analyze $z_\infty(k)$ for k bounded away from 0. However with a memory we will see that there is an analytic continuation of $\hat{G}_z(k;\tau)$ beyond z_τ, which permits us to control the integral in (6.8.4).

For $\epsilon > 0$ and $\tau \geq 4$ we define

$$D_\tau(\epsilon) \equiv \{z : |z| \leq z_\tau[1 + \epsilon\tau^{-1}\log\tau]\}. \tag{6.8.5}$$

We wish to show that there is an analytic continuation of $\hat{G}_z(k;\tau)$ to the disk $D_\tau(\epsilon)$, for some positive ϵ. As a first step we have the following consequence of Theorem 6.4.2.

Theorem 6.8.1 *Let Ω be sufficiently large (with $d > 4$ for the spread-out model). There is a positive constant ϵ_0 (which may depend on d, Ω but not on τ) such that for any $\tau \leq \infty$, $\epsilon < \epsilon_0$ and $p \in (0, z_\tau]$,*

$$||\delta_p^\epsilon \partial_p G_p(0,\cdot;\tau)||_\infty \leq 5K\Omega^{s+2/d}, \tag{6.8.6}$$

$$||\delta_p^\epsilon G_p(0,\cdot;\tau)||_2^2 \leq 3K\Omega^{-1+s}, \tag{6.8.7}$$

and

$$|| \, |x|^2 \delta_p^\epsilon G_p(0,\cdot;\tau)||_\infty \leq 3K\Omega^{-1+s+2/d}. \tag{6.8.8}$$

Here K is the constant of Lemma 6.2.2, $s = 0$ for the nearest-neighbour model, $s = 1/20$ for the spread-out model, and the $2/d$ in the exponents can be omitted for the nearest-neighbour model.

Proof. Theorem 6.4.2 (with finite memory) immediately gives finite bounds (uniform in τ) for the left sides, for any $\epsilon < \min\{(d-4)/4, 1\}$. In fact, if we take $\epsilon < \min\{(d-4)/8, 1/2\}$ (say), then Theorem 6.4.2 can be used to give finite bounds (again uniform in τ) on the derivatives with respect to ϵ of the norms on the left sides. Hence the left sides can be made as close as desired to their $\epsilon = 0$ values by taking ϵ sufficiently small, independent of τ. The theorem then follows from the $\epsilon = 0$ bounds given in Corollary 6.2.6 and (6.2.39). □

This leads to the following corollary, which extends Corollary 6.2.7 and Theorem 6.2.9. The corollary in particular provides an analytic continuation of $\hat{\Pi}_z(k;\tau)$ to $D_\tau(\epsilon_0)$, and hence a meromorphic continuation of $\hat{G}_z(k;\tau)$ to the same disk.

Corollary 6.8.2 *Let Ω be sufficiently large (with $d > 4$ for the spread-out model). There is a positive constant K_1 which is independent of Ω such that for any $\epsilon \leq \epsilon_0$, any τ, k, and any $z \in D_\tau(\epsilon_0)$,*

$$|\hat{\Pi}_z(k;\tau)| \leq K_1\Omega^{-1+s+2/d}, \tag{6.8.9}$$

$$|\partial_z\hat{\Pi}_z(k;\tau)| \leq K_1\Omega^{s+2/d}, \tag{6.8.10}$$

$$|\partial_\mu\hat{\Pi}_z(k;\tau)| \leq K_1\Omega^{-2+2s+2/d}|k_\mu|, \tag{6.8.11}$$

$$|\partial_\mu^2\hat{\Pi}_z(k;\tau)| \leq K_1\Omega^{-2+2s+2/d}, \tag{6.8.12}$$

$$|\hat{\Pi}_z(0;\tau) - \hat{\Pi}_z(k;\tau)| \leq K_1\Omega^{-2+2s+2/d}k^2. \tag{6.8.13}$$

Proof. By Theorem 5.4.4 and the second Remark below Theorem 5.4.2, the quantities on the left sides of (6.8.10)–(6.8.13) can be bounded in terms of various norms involving

$$H_z(0,x;\tau) = \sum_{n=1}^{\tau} c_{n,\tau}(0,x)z^n. \tag{6.8.14}$$

Here we have used the fact that for finite memory, all diagram subwalks have length at most τ, and hence the sum in the above equation can be truncated at $n = \tau$.

For example, the left side of the second inequality can be bounded in terms of the norms

$$||\partial_z H_{|z|}(0,\cdot\,;\tau)||_\infty \text{ and } ||H_{|z|}(0,\cdot\,;\tau)||_2.$$

Each of these norms can be bounded using Theorem 6.8.1. To see this, we note that for $z \in D_\tau(\epsilon)$ and $v \in \{0,1\}$,

$$\partial_z^v H_{|z|}(0,x;\tau) \leq \sum_{n=1}^{\tau} n^v c_{n,\tau}(0,x) z_\tau^{n-v}(1+\epsilon\tau^{-1}\log\tau)^{n-v}. \tag{6.8.15}$$

For $1 \leq n \leq \tau$,

$$(1+\epsilon\tau^{-1}\log\tau)^n \leq (1+\epsilon\tau^{-1}\log\tau)n^\epsilon. \tag{6.8.16}$$

Taking $v = 0$, using (6.8.16) in (6.8.15), and extending the summation to all $n \geq 1$ gives

$$||H_{|z|}(0,\cdot\,;\tau)||_\infty \leq (1+\epsilon\tau^{-1}\log\tau)||\delta_z^\epsilon G_{z_\tau}(0,x;\tau)||_\infty. \tag{6.8.17}$$

Similarly,

$$||H_{|z|}(0,\cdot\,;\tau)||_2 \leq (1+\epsilon\tau^{-1}\log\tau)||\delta_z^\epsilon G_{z_\tau}(0,\cdot\,;\tau)||_2. \tag{6.8.18}$$

The case of $v = 1$ is slightly more involved:

$$\partial_z H_{|z|}(0,x;\tau) \leq (1+\epsilon\tau^{-1}\log\tau)\sum_{n=1}^{\tau} n^{1+\epsilon} c_{n,\tau}(0,x) z_\tau^{n-1}. \tag{6.8.19}$$

Since

$$\delta_z^\epsilon \partial_z G_z(0,x;\tau) = \sum_{n=2}^{\infty}(n-1)^\epsilon n c_{n,\tau}(0,x) z^{n-1}, \tag{6.8.20}$$

it follows from (6.8.19) that

$$||\partial_z H_{|z|}(0,\cdot\,;\tau)||_\infty \leq (1+\epsilon\tau^{-1}\log\tau)\left(2^\epsilon||\delta_z^\epsilon\partial_z G_{z_\tau}(0,\cdot\,;\tau)||_\infty + 1\right). \tag{6.8.21}$$

The second bound of the corollary then follows just as in Section 6.2, using Theorem 6.8.1.

The other bounds are similar, apart from the last one, which follows by integration of the third bound. □

By (5.2.18),

$$\hat{F}_z(k;\tau) = 1 - z\Omega\hat{D}(k) - \hat{\Pi}_z(k;\tau).$$

Also, by Corollary 6.2.7, $\hat{F}_{z_\tau}(0;\tau) = 0$. The next lemma provides an extension of Theorem 6.2.10 for finite memory.

Lemma 6.8.3 *Let Ω be sufficiently large (with $d > 4$ for the spread-out model). Then*

$$|\hat{F}_z(0;\tau)| \geq \frac{\Omega}{2}|z_\tau - z| \tag{6.8.22}$$

for all τ, and for all $z \in D_\tau(\epsilon_0)$. In particular, z_τ is the unique zero of $\hat{F}_z(0;\tau)$ in $D_\tau(\epsilon_0)$.

Proof. The proof is identical to that of Theorem 6.2.10, using Corollary 6.8.2. □

The next lemma gives bounds which allow for the estimation of $\hat{c}_{n,\tau}(k)$ by contour integration. There are separate estimates for two distinct sets of k values.

Lemma 6.8.4 *Let Ω be sufficiently large (with $d > 4$ for the spread-out model). There is a positive constant c_1 (depending on d, Ω but not on τ) such that the following hold.*
(a) For $k \in [-\pi/2, \pi/2] \times [-\pi, \pi]^{d-1}$ with $k^2 \leq c_1\tau^{-1}\log\tau$, $\hat{F}_z(k;\tau)$ has a unique zero $z_\tau(k)$ in $D_\tau(\epsilon_0)$, which is in fact located inside $D_\tau(\epsilon_0/2)$. Moreover, $z_\tau(k)$ is a simple zero and

$$|\hat{F}_z(k;\tau)| \geq \frac{\Omega}{2}|z - z_\tau(k)| \tag{6.8.23}$$

for all τ and all $z \in D_\tau(\epsilon_0)$.
(b) There is a positive $\epsilon_1 < \epsilon_0$ (depending on d, Ω but not on τ) such that for $k \in [-\pi/2, \pi/2] \times [-\pi, \pi]^{d-1}$ with $k^2 \geq c_1\tau^{-1}\log\tau$, $\hat{F}_z(k;\tau)$ has no zero in $D_\tau(\epsilon_1)$. Moreover there is a constant c_2 (depending on d, Ω but not on τ) such that for these values of k

$$|\hat{F}_z(k;\tau)| \geq c_2[\tau^{-1}\log\tau + |z\arg z|] \tag{6.8.24}$$

for all τ and all $z \in D_\tau(\epsilon_1)$. (Here $\arg z \in [-\pi, \pi]$.*)*

Proof. (a) Consider k near the origin. By Lemma 6.8.3 and Rouché's Theorem, to see that there is a unique zero in $D_\tau(\epsilon_0)$ it suffices to show that $|\hat{F}_z(k;\tau) - \hat{F}_z(0;\tau)| < |\hat{F}_z(0;\tau)|$ on the boundary of $D_\tau(\epsilon_0)$. But the left side is equal to

$$\left| z\Omega[1 - \hat{D}(k)] + \hat{\Pi}_z(0;\tau) - \hat{\Pi}_z(k;\tau) \right| \leq c'k^2 \tag{6.8.25}$$

by (6.8.13) and the fact that $1 - \hat{D}(k)$ is order k^2, where c' is independent of τ. By Lemma 6.8.3, on the boundary of $D_\tau(\epsilon_0)$

$$|\hat{F}_z(0;\tau)| \geq \frac{z_\tau\Omega\epsilon_0}{2}\tau^{-1}\log\tau. \tag{6.8.26}$$

Since $z_\tau \geq z_0 = 1/\Omega$, uniqueness of the zero follows if

$$k^2 < \frac{\epsilon_0}{2c'}\tau^{-1}\log\tau. \tag{6.8.27}$$

Thus we take $c_1 < \epsilon_0/(2c')$.

To see that the zero is located in $D_\tau(\epsilon_0/2)$, we proceed as follows. Differentiation of the equation $\hat{F}_{z_\tau(k)}(k;\tau) = 0$ with respect to k_μ, together with Corollary 6.8.2, gives

$$|\partial_\mu z_\tau(k)| = \left|\frac{\partial_\mu \hat{F}}{\partial_z \hat{F}}\right| \leq \text{const.}|k_\mu| \tag{6.8.28}$$

for small k. Therefore

$$z_\tau(k) = z_\tau(0) + \int_0^1 \frac{d}{dt} z_\tau(tk)dt \leq z_\tau(0) + \text{const.}k^2, \tag{6.8.29}$$

with the constant independent of τ. This gives the desired result, if we take c_1 sufficiently small.

For the lower bound on $\hat{F}$, we have

$$\begin{aligned} |\hat{F}_z(k;\tau)| &= |\hat{F}_z(k;\tau) - \hat{F}_{z_\tau(k)}(k;\tau)| \\ &= |z_\tau(k) - z|\,|\Omega D(k) + \int_0^1 \partial_z \hat{\Pi}_{z_\tau + t(z_\tau(k)-z)}(k;\tau)dt| \\ &\geq \frac{\Omega}{2}|z - z_\tau(k)|, \end{aligned} \tag{6.8.30}$$

for k near zero and Ω sufficiently large, by Corollary 6.8.2.

(b) It suffices to prove the lower bound (6.8.24). For this we add and subtract $\hat{\Pi}_z(0;\tau)$ to $\hat{F}_z(k;\tau)$, and then subtract $\hat{F}_{z_\tau}(0;\tau) = 0$, obtaining

$$\hat{F}_z(k;\tau) = \Omega(z_\tau - z\hat{D}(k)) + \hat{\Pi}_{z_\tau}(0;\tau) - \hat{\Pi}_z(0;\tau) + \hat{\Pi}_z(0;\tau) - \hat{\Pi}_z(k;\tau). \tag{6.8.31}$$

Using Corollary 6.8.2 then gives

$$|\hat{F}_z(k;\tau)| \geq \Omega|z_\tau - z\hat{D}(k)| - K_1\Omega^{s+2/d}|z_\tau - z| - K_1k^2\Omega^{-2+2s+2/d}. \tag{6.8.32}$$

For the middle term on the right side, we use

$$|z_\tau - z| \leq |z_\tau - z\hat{D}(k)| + |z|\,|1 - \hat{D}(k)|. \tag{6.8.33}$$

It follows from the fact that $|1 - \cos t| \leq t^2/2$ for all $t \in \mathbf{R}$ that $|1 - \hat{D}(k)|$ is bounded above by $k^2/(2d)$ for the nearest-neighbour model and by $(k^2\Omega^{2/d})/2$ for the spread-out model. We write this upper bound as $a(\Omega)k^2$. Then using $|z| \leq 2z_\tau$ we have

$$\begin{aligned}|\hat{F}_z(k;\tau)| &\geq (\Omega - K_1\Omega^{s+2/d})|z_\tau - z\hat{D}(k)| \\ &\quad - [2z_\tau a(\Omega)K_1\Omega^{s+2/d} + K_1\Omega^{-2+2s+2/d}]k^2.\end{aligned}$$

Next we write $z = |z|e^{i\theta}$, and use the inequality $|a| + |b| \leq \sqrt{2}|a + ib|$ to obtain

$$|z_\tau - z\hat{D}(k)| \geq \frac{1}{\sqrt{2}}\left[|z_\tau - |z|\hat{D}(k)\cos\theta| + |z\hat{D}(k)\sin\theta|\right]. \qquad (6.8.34)$$

Now for $z \in D_\tau(\epsilon_1)$, and for k^2 large as stated in the lemma but within a small sphere of radius $O(1)$ centred at the origin [so that $\hat{D}(k)$ is bounded away from zero], it follows from (6.2.5) that

$$\begin{aligned}|z\hat{D}(k)\cos\theta| &\leq z_\tau(1 + \epsilon_1\tau^{-1}\log\tau)\left(1 - \frac{k^2}{2\pi^2 d}\right) \\ &\leq z_\tau\left(1 - \frac{k^2}{4\pi^2 d}\right),\end{aligned}$$

if we choose ϵ_1 sufficiently small (independent of τ). For this range of k and for Ω sufficiently large we then have

$$|\hat{F}_z(k;\tau)| \geq c_2[k^2 + |z||\arg z|] \qquad (6.8.35)$$

for some constant c_2 depending on d, Ω, but not on τ.

For k^2 at least $O(1)$ from the origin, $\hat{D}(k)$ is bounded away from 1. Hence by (6.8.9) and the fact that $z_\tau\Omega = 1 - \hat{\Pi}_{z_\tau}(0;\tau)$, for Ω sufficiently large there are $\beta, \beta' \in (0,1)$ such that

$$\begin{aligned}|\hat{F}_z(k;\tau)| &\geq 1 - |z\Omega\hat{D}(k)| - |\hat{\Pi}_z(k;\tau)| \\ &\geq 1 - \beta(1 + o(1))(1 + \epsilon_1\tau^{-1}\log\tau) - o(1) \\ &> \beta'.\end{aligned}$$

[Here $o(1)$ denotes a quantity which goes to zero as Ω increases, uniformly in τ.]

This completes the proof. □

We are now in a position to estimate $\hat{c}_{n,\tau}(k)$.

Lemma 6.8.5 *Let Ω be sufficiently large (with $d > 4$ for the spread-out model).*

(a) For $k \in [-\pi/2, \pi, 2] \times [-\pi, \pi]^{d-1}$ *with* $k^2 \leq c_1 \tau^{-1} \log \tau$,

$$\begin{aligned} \hat{c}_{n,\tau}(k) &= z_\tau(k)^{-(n+1)} \Big[-\partial_z \hat{F}_{z_\tau(k)}(k;\tau)^{-1} \\ &\quad + O[(1+(\epsilon_0/3)\tau^{-1}\log\tau)^{-(n+1)} \log\tau] \Big] . \end{aligned} \tag{6.8.36}$$

(b) For $k \in [-\pi/2, \pi, 2] \times [-\pi, \pi]^{d-1}$ *with* $k^2 \geq c_1 \tau^{-1} \log \tau$,

$$\hat{c}_{n,\tau}(k) = z_\tau^{-(n+1)} (1 + \epsilon_1 \tau^{-1} \log \tau)^{-(n+1)} O[\log \tau]. \tag{6.8.37}$$

Here $O[f(n,\tau)]$ *denotes a quantity which is bounded in absolute value by const.*$|f(n,\tau)|$, *with the constant independent of* n, τ.

Proof. (a) Let C be the circle of radius $z_\tau/2$ centred at the origin, oriented clockwise. Then

$$\hat{c}_{n,\tau}(k) = \frac{1}{2\pi i} \oint_C \hat{G}_z(k;\tau) \frac{dz}{z^{n+1}}. \tag{6.8.38}$$

Since $|z_\tau(k)| \geq z_\tau$, $z_\tau(k)$ is not inside C. By Lemma 6.8.4 and the Residue Theorem, deforming the contour from C to the boundary $\partial D_\tau(\epsilon_0)$ of $D_\tau(\epsilon_0)$ gives

$$\begin{aligned} \hat{c}_{n,\tau}(k) &= z_\tau(k)^{-(n+1)} [-\partial_z \hat{F}_{z_\tau(k)}(k;\tau)^{-1} \\ &\quad + \frac{1}{2\pi i} \oint_{\partial D_\tau(\epsilon_0)} \hat{G}_z(k;\tau) \left(\frac{z_\tau(k)}{z} \right)^{n+1} dz]. \end{aligned}$$

To bound the integral on the right side we first note that since $z_\tau(k) \in D_\tau(\epsilon_0/2)$, for $z \in \partial D_\tau(\epsilon_0)$ we have

$$\left| \frac{z_\tau(k)}{z} \right|^{n+1} \leq O[(1+(\epsilon_0/3)\tau^{-1}\log\tau)^{-(n+1)}]. \tag{6.8.39}$$

Also, by (6.8.23)

$$\begin{aligned} \oint_{\partial D_\tau(\epsilon_0)} |\hat{G}_z(k;\tau)||dz| &\leq O[|\log|z_\tau(1+\epsilon_0\tau^{-1}\log\tau) - z_\tau(k)|,|] \\ &\leq O[\log\tau]. \end{aligned} \tag{6.8.40}$$

This proves (6.8.36).

(b) We perform contour integration as in part (a), deforming the contour to $\partial D_\tau(\epsilon_1)$. Here there is no singularity inside $\partial D_\tau(\epsilon_1)$, and only the error term contributes. Explicitly,

$$\hat{c}_{n,\tau}(k) = \frac{1}{2\pi i} \oint_{\partial D_\tau(\epsilon_1)} \frac{1}{\hat{F}_z(k;\tau) z^{n+1}} dz. \tag{6.8.41}$$

The factor $z^{-(n+1)}$ in the integrand is responsible for everything in the upper bound stated in the lemma except for the $\log \tau$, which comes from integrating $1/\hat{F}$ using the lower bound of Lemma 6.8.4(b). □

We need one more lemma before proving Theorem 6.1.3. Recall from Lemma 1.2.3 that as the memory τ goes to infinity, z_τ converges to z_c. The next lemma gives a bound on the rate of this convergence.

Lemma 6.8.6 *Let Ω be sufficiently large (with $d > 4$ for the spread-out model) and $\epsilon < \min\{(d-4)/2, 1\}$. There is a K_1 (possibly depending on d, ϵ, Ω but not on τ) such that for all τ,*

$$0 \leq z_c - z_\tau \leq K_1 \tau^{-(1+\epsilon)}. \tag{6.8.42}$$

Proof. Consider two memories $\sigma < \tau$. Then $z_\sigma \leq z_\tau$. Since $\hat{F}_{z_\tau}(0;\tau) = \hat{F}_{z_\sigma}(0;\sigma) = 0$,

$$(z_\tau - z_\sigma)\Omega = [\hat{\Pi}_{z_\sigma}(0;\sigma) - \hat{\Pi}_{z_\sigma}(0;\tau)] + [\hat{\Pi}_{z_\sigma}(0;\tau) - \hat{\Pi}_{z_\tau}(0;\tau)]. \tag{6.8.43}$$

Using (6.8.10) to estimate the second term on the right side, for some constant C we have

$$0 \leq C\Omega(z_\tau - z_\sigma) \leq \hat{\Pi}_{z_\sigma}(0;\sigma) - \hat{\Pi}_{z_\sigma}(0;\tau). \tag{6.8.44}$$

By Lemma 5.4.5 and Corollary 6.2.6, the right side is bounded by a multiple of

$$\|\sum_{n=\sigma/6}^{\infty} c_{n,\sigma}(0,\cdot) z_\sigma^n\|_\infty, \tag{6.8.45}$$

Inserting $1 \leq [6n/\sigma]^{1+\epsilon}$ into the summation, the above norm is bounded above by a multiple of

$$\sigma^{-(1+\epsilon)}\, \delta_z^\epsilon \partial_z G_z(0,x;\sigma)|_{z=z_\sigma}. \tag{6.8.46}$$

The bound on the fractional derivative of (6.8.46) given in Theorem 6.4.2, together with (6.8.44), then gives

$$0 \leq z_\tau - z_\sigma \leq K_1 \sigma^{-(1+\epsilon)}, \tag{6.8.47}$$

for some constant K_1. Letting $\tau \to \infty$, we obtain (6.8.42). □

Proof of Theorem 6.1.3. We now take $\tau = n^{1/b}$ with $b \in (1, 1+\epsilon)$, and use (6.8.4). As a first observation, by Lemma 6.8.6 we have

$$\lim_{n\to\infty} \left(\frac{z_{n^{1/b}}}{z_c}\right)^n = \lim_{n\to\infty} [1 - O(n^{-(1+\epsilon)/b})]^n = 1. \tag{6.8.48}$$

To estimate the integral of $|\hat{c}_{n,n^{1/b}}(k)|$, we divide the integral over k into two parts as in Lemma 6.8.5. For k as in part (b) of the lemma, we have from (6.8.48) and the lemma that

$$|\hat{c}_{n,n^{1/b}}(k)| \leq C\mu^n n^{-C' n^{1-1/b}} \log n \leq \mu^n O(n^{-d/2}). \tag{6.8.49}$$

Integrating this over k as in part (b) of the lemma gives a bound of the desired form (in fact the decay is much better than required).

We now bound $\hat{c}_{n,n^{1/b}}$ for k as in part (a) of the lemma. The quantity in square brackets on the right side of (6.8.36) is bounded above uniformly in k and $\tau = n^{1/b}$, so we just concentrate on the factor $z_\tau(k)^{-n}$. By (6.8.48), for $\tau = n^{1/b}$ we have

$$z_\tau(k)^{-n} = [1 + o(1)]\mu^n \left(\frac{z_\tau}{z_\tau(k)}\right)^n. \tag{6.8.50}$$

By (6.8.29),

$$z_\tau(k) = z_\tau \left[1 + z_\tau^{-1} \sum_{\mu=1}^{d} \int_0^1 \partial_\mu z_\tau(tk) k_\mu dt\right]. \tag{6.8.51}$$

It suffices to show that the right side is bounded below by $z_\tau[1 + C_1 k^2]$, for some constant $C_1 > 0$ which is independent of both τ and small k. In fact, given such a bound we would have

$$|\hat{c}_{n,n^{1/b}}(k)| \leq C\mu^n [1 - C_2 k^2]^n \leq C\mu^n e^{-nC_2 k^2}, \tag{6.8.52}$$

and extending the integration domain to $\mathbf{R}^d$ then gives an upper bound of the required form $\mu^n n^{-d/2}$.

We now complete the proof by obtaining such a lower bound on the right side of (6.8.51). As in (6.8.28), we have

$$\partial_\mu z_\tau(l) = \frac{\partial_\mu \hat{F}_{z_\tau(l)}(l;\tau)}{-\partial_z \hat{F}_{z_\tau(l)}(l;\tau)}. \tag{6.8.53}$$

The denominator on the right side is positive and bounded above uniformly in τ and small l, by Corollary 6.8.2. For the numerator, we use $z_\tau(l) \geq \Omega^{-1}$ and Corollary 6.8.2 to obtain

$$\partial_\mu \hat{F}_{z_\tau(l)}(l;\tau) \geq \Omega^{-1} \sum_{x \in \Omega} x_\mu \sin l \cdot x - K_1 \Omega^{-2+2s+2/d} |l_\mu|. \tag{6.8.54}$$

Expanding $\sin l \cdot x$ and using symmetry, the first term on the right side is given by $\Omega^{-1} \sum_{x \in \Omega} x_\mu^2 l_\mu$ plus a term of order l^3 which is negligible for l going to zero depending on n. Thus the first term dominates the second, and we have the desired lower bound. This completes the proof. □

6.9 Notes

Section 6.1. The results stated in Section 6.1 for sufficiently large Ω have all been proven in Hara and Slade (1992a,1992b) for the nearest-neighbour model with $d \geq 5$, except for Theorem 6.1.4. Their proof relies on the fortuitous fact that the critical bubble diagram is not too large in five dimensions. Since the critical bubble diagram can be expected to diverge as $d \rightarrow 4^+$ (with any reasonable definition for noninteger dimensions), the proof is not entirely natural. A more natural proof (which we hope will one day be forthcoming) would rely on the fact that the bubble is finite rather than small. The methods used for the nearest-neighbour model with $d \geq 5$ are quite similar to those of Chapter 6, except for the proof of convergence of the lace expansion. The latter, while similar in spirit to the proof given in Section 6.2, is enormously more complex (and in fact is computer assisted) due to the fact that the small parameter cannot be taken to be arbitrarily small.

For the nearest-neighbour model with $d \geq 5$, Corollary 6.1.4 cannot be deduced from Theorem 6.1.3 since the latter has not yet been proven in this context. Instead, in Hara and Slade (1992a) (6.4.8) is extended to show that the supremum norm of the a-th derivative of the critical two-point function is finite [rather than just the $(1+\epsilon)$-th derivative].

Section 6.2. The first proof of convergence of the lace expansion was for weakly self-avoiding walk with $d > 4$, in Brydges and Spencer (1985). The proof began by considering walks which are self-avoiding with finite memory τ, and then used an intricate induction on the memory. Later, in Yang and Klein (1988), the same methods were applied to a weakly self-avoiding walk taking steps of arbitrary length m parallel to the coordinate axes in $\mathbf{Z}^d$ with probability proportional to m^{-2}, and it was shown that if the self-avoidance is sufficiently weak then the scaling limit of the endpoint has a Cauchy distribution if $d > 2$ (which is the distribution of the scaling limit in any dimension for the corresponding random walk without the self-avoidance constraint). An alternate proof of Brydges and Spencer's results, which uses the lace expansion and an induction on finite memory but avoids the use of generating functions, is given in Golowich and Imbrie (1992).

In Slade (1987) and Slade (1989) it was proven that $\nu = 1/2$ and $\gamma = 1$ for the strictly self-avoiding walk in sufficiently high dimensions. Convergence of the expansion was proven using Lemma 6.2.1. No estimate was given of how high the dimension had to be.

The convergence proof used in Section 6.2 is the prototype for the proofs of convergence of the lace expansions for lattice trees and animals and for percolation, used to prove Theorems 5.5.1 and 5.5.2.

Section 6.3. The fractional derivative analysis is taken from Hara and Slade (1992a).

Section 6.4. This section follows the methods of Hara and Slade (1992a).

Section 6.5. The proof of mean-field behaviour for the correlation length in Theorem 6.1.5 is modelled on the proof of the analogous result for percolation in Hara (1990), and follows Hara and Slade (1992a).

Section 6.6. It was proven in Slade (1988, 1989) that the scaling limit of the self-avoiding walk is Gaussian in sufficiently high dimensions, using a finite-memory cut-off. Hara and Slade (1992a) used the fractional derivative argument presented here.

Section 6.7. The definition used here for the infinite self-avoiding walk was introduced in Lawler (1980). Lawler (1989) constructed the infinite self-avoiding walk in sufficiently high dimensions.

Section 6.8. The bound on $c_n(0,x)$ obtained in Section 6.8 is new. The method of proof takes its inspiration from Brydges and Spencer (1985). The proof is unsatisfactory in its use of finite memory; it should be possible to improve Theorem 6.1.4 to prove that $c_n(0,x)$ is asymptotic to a multiple of $n^{-d/2}$ (for fixed x, as $n \to \infty$), without making use of finite memory. To do so remains an open problem. No estimate has been made of how high the dimension need be for the proof to work for the nearest-neighbour model, but we would guess something on the order of $d \geq 10$.