Chapter 2

Scaling, polymers and spins

2.1 Scaling theory

This chapter is concerned with some of the nonrigorous work on the self-avoiding walk: the scaling theory which leads to the scaling relations stated in Section 1.4, the connection with polymers and the derivation of the Flory values for the critical exponent ν, and finally, the interpretation of the self-avoiding walk as a "zero-component" ferromagnet.

We begin in this section with a discussion of scaling theory, giving heuristic derivations of Fisher's scaling relation (1.4.9) and of the hyperscaling relations (1.4.14) and (1.4.24). There are a variety of approaches which can be used to derive these relations, and here we content ourselves with giving a representative sample of the types of arguments which are frequently used. Although the sort of arguments we will describe are part of the standard lore of theoretical physics (applicable to a wide variety of models), from a mathematical point of view they may appear to be on rather shaky ground. There will be no rigorous results in this section, and we will make frequent use of the symbol $\approx$, which implies a leap of faith.

Our starting point will be an assumption about the behaviour of the two-point function in the limit as both $z \nearrow z_c$ and $x \to \infty$. The correlation length $\xi(z)$ is to be interpreted as the important length scale of the system. For $x \to \infty$ at fixed $z < z_c$, the two-point function is believed to obey the *Ornstein-Zernike* decay

$$G_z(0,x) \sim C_z |x|^{-(d-1)/2} e^{-|x|/\xi(z)}, \tag{2.1.1}$$

N. Madras and G. Slade, *The Self-Avoiding Walk*,
Modern Birkhäuser Classics, DOI 10.1007/978-1-4614-6025-1_2,

where C_z depends only on z, and strictly speaking the norm on the right side is equivalent to but not equal to the Euclidean norm. A proof of (2.1.1) for x on a coordinate axis will be given in Theorem 4.4.7. However the Ornstein-Zernike decay describes the behaviour of the two-point function on a length scale $|x| \gg \xi(z)$, and is not believed to be accurate on the important length scale where x is of the order of $\xi(z)$. Instead, the decay of the *critical* two-point function is considered to be fundamental on the scale of the correlation length, and we define a function $h(z;x)$ by

$$G_z(0,x) = \frac{1}{|x|^{d-2+\eta}} h(z;x). \tag{2.1.2}$$

The assumption now is that the important contribution to $h(z;x)$ will come from x of the order of the correlation length $\xi(z)$, and that we can write

$$G_z(0,x) \approx \frac{1}{|x|^{d-2+\eta}} g(|x|/\xi(z)), \tag{2.1.3}$$

for some universal function g of a single variable. The function g will be assumed to decay at infinity sufficiently rapidly that its product with any power of x is integrable, for example an exponential function.

Given (2.1.3), the following argument can be put forth in support of Fisher's relation $\gamma = (2-\eta)\nu$. We assume that the main contribution to the susceptibility

$$\chi(z) = \sum_x G_z(0,x)$$

is due to x of the order of the correlation length, and that we may therefore substitute (2.1.3) into the sum over x. By definition of $\bar{\gamma}$ we then have

$$\begin{aligned} (z_c - z)^{-\bar{\gamma}} &\approx \sum_x \frac{g(|x|/\xi)}{|x|^{d-2+\eta}} \approx \int_0^\infty r^{1-\eta} g(r/\xi) dr \\ &= \text{const.}\xi^{2-\eta} \sim \text{const.}(z_c - z)^{-(2-\eta)\bar{\nu}}. \end{aligned} \tag{2.1.4}$$

This gives

$$\bar{\gamma} = (2-\eta)\bar{\nu}. \tag{2.1.5}$$

It has already been argued in Section 1.3 that $\bar{\gamma} = \gamma$, and we will shortly argue that $\bar{\nu} = \nu$, which then gives $\gamma = (2-\eta)\nu$.

The following is an alternate derivation of Fisher's relation which does not rely on (2.1.3). In the sum

$$G_{z_c}(0,x) = \sum_{N=0}^{\infty} c_N(0,x)\mu^{-N} \tag{2.1.6}$$

we assume that $c_N(0,x)$ is significant only when $|x|$ is of the order of N^ν. There are about $N^{d\nu}$ such sites x, and we assume that an N-step self-avoiding walk is equally likely to end at any one of them, or in other words,

$$c_N(0,x) \approx c_N N^{-d\nu} \approx \mu^N N^{\gamma-1-d\nu}. \tag{2.1.7}$$

Restricting the sum in (2.1.6) to N between $c_1|x|^{1/\nu}$ and $c_2|x|^{1/\nu}$, for some positive constants c_1 and c_2, and then using (2.1.7), gives

$$G_{z_c}(0,x) \approx \sum_{N=c_1|x|^{1/\nu}}^{c_2|x|^{1/\nu}} N^{\gamma-1-d\nu} \approx |x|^{(\gamma-d\nu)/\nu}. \tag{2.1.8}$$

This implies $-(d-2+\eta) = (\gamma - d\nu)/\nu$, which can be rewritten as $\gamma = (2-\eta)\nu$.

Continuing in the spirit of the calculation leading to (2.1.5), we now argue that $\bar\nu = \nu_p = \nu$. For any $p \in (0,\infty)$,

$$\begin{aligned}\sum_x |x|^p G_z(0,x) &\approx \sum_x |x|^p \frac{g(|x|/\xi)}{|x|^{d-2+\eta}} \approx \int_0^\infty r^{p+1-\eta} g(r/\xi)dr \\ &= \text{const.}\xi^{p+2-\eta} \sim \text{const.}(z_c - z)^{-(p+2-\eta)\bar\nu}. \end{aligned} \tag{2.1.9}$$

Using the definition of ξ_p and (2.1.5), this gives

$$\xi_p(z)^p \approx (z_c - z)^{\bar\gamma-(2-\eta)\bar\nu-p\bar\nu} = (z_c - z)^{-p\bar\nu} \tag{2.1.10}$$

and hence $\bar\nu = \nu_p$. To show that $\bar\nu = \nu$, we first observe that by (1.3.20),

$$\sum_x |x|^2 G_z(0,x) \approx (z_c - z)^{-(2\nu+\gamma)}. \tag{2.1.11}$$

Comparing with (2.1.9), with $p = 2$, gives

$$(4-\eta)\bar\nu = 2\nu + \gamma. \tag{2.1.12}$$

Now by (2.1.5) and the equality of $\bar\gamma$ and γ we conclude that $\bar\nu = \nu$.

We now turn to the hyperscaling relations (1.4.14) and (1.4.24). These cannot be derived from the scaling hypothesis (2.1.3) and require additional assumptions. Less numerical testing has been done of the hyperscaling relations than on the calculation of γ and ν, but both the Monte Carlo and series extrapolation computations which have been done are consistent with them.

The hyperscaling relation involving α_{sing} may at first glance seem somewhat surprising. It would perhaps seem natural to assume that the probability that an N-step self-avoiding walk ends at x would be proportional to

the characteristic volume $N^{-d\nu}$ in the limit as $N \to \infty$, as is the case for simple random walk. However this leads to the conclusion $\alpha_{sing} - \gamma - 1 = -d\nu$ rather than to the hyperscaling relation $\alpha_{sing} - 2 = -d\nu$. This incorrect argument fails to take into account the fact that for fixed x it is difficult for a long self-avoiding walk to return near to its starting point at the origin, and as the length of the walk goes to infinity x must be regarded as being close to the origin. The argument should be reasonable for x of the order of the typical length scale N^{ν}, and indeed this is what we assumed in the second derivation of Fisher's relation given above.

To obtain the hyperscaling relation $\alpha_{sing} - 2 = -d\nu$, we proceed as follows. First we assume that $c_N(0,x)$ will have the same scaling behaviour for any fixed x as $N \to \infty$, and consider the case $x = e$, with e a nearest neighbour of the origin. This assumption is mild in comparison with the assumptions we will make next. By adding an extra step to a walk ending at e we obtain a closed self-avoiding loop. Let $n = (N+1)/2$. Then by summing over the position of the walk after n steps, and using symmetry, we have

$$c_N(0,e) = (2d)^{-1} \sum_x \sum_{\substack{\omega^{(1)} : \omega^{(1)}(n) = x \\ \omega^{(2)} : \omega^{(2)}(n) = x}} I[\omega^{(1)} \cap \omega^{(2)} = \{0,x\}]. \quad (2.1.13)$$

Here both of the self-avoiding walks $\omega^{(i)}$ begin at the origin and consist of n steps. We now make three assumptions. First, we assume that the main contribution to the above sum will be from x of the order of n^{ν}. Thus there are of the order of $n^{\nu d}$ relevant terms in the sum. Second, we assume that the effect of the avoidance constraint between $\omega^{(1)}$ and $\omega^{(2)}$ can be incorporated by replacing the quantity being summed over x by $c_N(0,x)^2$ multiplied by the square of the probability that two n-step self-avoiding walks beginning at the same point avoid each other; this probability is $c_{2n}/c_n^2 \sim 2^{\gamma-1}A^{-1}n^{1-\gamma}$. Here we use the square of this probability to account for the avoidance both near 0 and near x. Third, we assume that for x of the order of n^{ν} the probability that an n-step walk ends at x is of the order of the inverse of the characteristic volume $n^{\nu d}$, so that $c_n(0,x)$ is of the order of $\mu^n n^{\gamma-1} n^{-\nu d}$. With these three assumptions we have from (2.1.13) that

$$c_N(0,e) \approx n^{\nu d}[\mu^n n^{\gamma-1-\nu d}]^2[n^{1-\gamma}]^2 = \mu^{2n} n^{-\nu d}. \quad (2.1.14)$$

Comparison with the definition of α_{sing} gives $\alpha_{sing} - 2 = -d\nu$.

We next turn to the hyperscaling relation (1.4.24) for Δ_4, which is believed to hold only for $d \leq 4$. For simplicity, we take $N_1 = N_2 = n$ in the definition (1.4.21) of Δ_4. To begin, we assume that since a self-avoiding

walk of length n goes a distance of about n^ν in each direction, such a self-avoiding walk will primarily lie in a hypercube of side n^ν and volume $n^{\nu d}$.

Given $0 \le D \le d$, consider a subset of the hypercubic lattice such that the cube of side R contains of the order of R^D points as $R \to \infty$. Such a subset can in some sense be thought of as being D-dimensional. From this point of view a long self-avoiding walk, which consists of n points in volume approximately equal to $n^{\nu d}$, is a $1/\nu$-dimensional set. Typically two D-dimensional subsets will intersect in d dimensions if $2D \ge d$, but otherwise will not. This suggests that two n-step self-avoiding walks with a common point will typically have additional intersections if $d\nu \le 2$, and typically will not if $d\nu > 2$.

Consider first the case of $d\nu > 2$, for which we have already seen in (1.4.26) that the hyperscaling relation fails. According to the values of ν given in Table 1.2, this inequality says $d > 4$. Two n-step self-avoiding walks lying in the cube of volume $n^{\nu d}$ will typically not intersect each other, and so there should be no overcounting in writing

$$c_{n,n} \approx c_n^2 n^2 \sim A^2 \mu^{2n} n^{2\gamma}. \tag{2.1.15}$$

Here the factor n^2 comes from choosing a point on each walk at which the two walks can be joined. By definition of Δ_4, this gives $2\Delta_4 + \gamma - 2 = 2\gamma$, or $\Delta_4 = 1 + \gamma/2$. Using the mean-field value $\gamma = 1$ known to be correct for $d \ge 5$, we obtain $\Delta_4 = 3/2$ (which of course is consistent with the rigorous result for $d \ge 6$ obtained in Section 1.5).

We next consider the case $d\nu \le 2$, which corresponds to $d \le 4$. Here the factor of n^2 in (2.1.15) would overcount. Given one n-step walk, which will lie roughly within a cube of volume $n^{\nu d}$, a second n-step walk will typically intersect the first if it is started at any one of the $n^{\nu d}$ points in the cube. This leads to

$$c_{n,n} \approx c_n^2 n^{\nu d} \sim A^2 \mu^{2n} n^{2\gamma - 2 + \nu d}. \tag{2.1.16}$$

By definition of Δ_4, this gives $2\Delta_4 + \gamma - 2 = 2\gamma - 2 + \nu d$, which simplifies to the hyperscaling relation $d\nu - 2\Delta_4 + \gamma = 0$.

2.2 Polymers

One of the most important applications of the self-avoiding walk is as a model for linear polymer molecules in chemical physics. In this section we shall briefly describe some aspects of this role, including a nonrigorous derivation of the "Flory values" for the critical exponent ν.

A *polymer* is a molecule that consists of many "monomers" (groups of atoms) joined together by chemical bonds. The *functionality* of a monomer

is the number of available chemical bonds that it has, i.e. the number of other monomers with which it must bond. If each monomer has functionality two, then a *linear* polymer is formed. If we denote a monomer by (A), then a linear polymer may be represented schematically as

$$\cdots\text{—(A)—(A)—(A)—(A)—(A)—}\cdots.$$

One simple example is polyethylene, where each monomer is CH_2 (one carbon atom and two hydrogen atoms). The pattern terminates either by bonding with a monomer of functionality one, such as CH_3, at each end, or else by closing on itself to form a "ring polymer". When we speak of linear polymers, we shall be referring to the former. By way of contrast, if a polymer includes monomers of functionality three or more, then a *branched polymer* is formed; these are often modelled by lattice trees or lattice animals (see Section 5.5.1).

The preceding paragraph deals only with the topological structure of a polymer. Properties of its spatial configuration are no less important. Polymers can be very large; some linear polymers consist of more than 10^5 monomers. Thus the length scale of the entire polymer is macroscopic with respect to the length scale of the individual monomers. Consider a linear polymer consisting of $N+1$ monomers, and label the monomers $0, 1, \ldots, N$ from one end to the other. Let $x(i) \in \mathbf{R}^3$ denote the location of the i-th monomer. Then the i-th (monomer-monomer) bond may be represented by the line segment joining $x(i-1)$ to $x(i)$. Typically, the length of each bond is essentially constant throughout the chain, as is the angle between each pair of consecutive monomer-monomer bonds. However, there is some rotational freedom for the i-th bond around the axis determined by the $(i-1)$-th bond. In some cases, a reasonably good approximation may be obtained by allowing the rotational angle of the i-th bond around the $(i-1)$-th bond to take on three different values, say $0°$ and $\pm 120°$, perhaps with different probabilities (an angle of $0°$ means that the i-th, $(i-1)$-th, and $(i-2)$-th bonds all lie in one plane). These angles correspond to local configurations of minimal energy, and depend on the details of the monomers.

We see that one possible model for the spatial configuration of a linear polymer is simply a random walk in $\mathbf{R}^3$, and in fact this model is known as the *ideal polymer chain*. Alternatively, one can work with a lattice approximation, say a random walk on $\mathbf{Z}^3$. The model can be embellished by turning it into a Markov chain (or random walk with some finite memory), and it works reasonably well in some situations. However, there is a fundamental limitation of the ideal polymer chain, namely the *excluded volume* effect.

Two monomers cannot occupy the same position in space: the presence of a monomer at position x prohibits any other part of the polymer from getting too close to x, that is, other monomers are excluded from a certain volume of space. This is the excluded volume effect. When we take this effect into account, it becomes apparent that a self-avoiding walk is a more appropriate model for a linear polymer than is a random walk. The self-avoiding walk model is best for the case of a dilute polymer solution (where polymers are far apart, so that there is little interaction between distinct molecules) and a good solvent (which minimizes attractive forces between monomers).

We remark that there are some situations in which polymers really do behave ideally on large length scales, even though excluded volume effects are present. One is in a dense system (or "melt") of many polymers, where monomers fill three-dimensional space uniformly and a given polymer interacts with many other monomers besides its own. Another is at certain values of temperature and solvent quality where roughly speaking the attractive forces between monomers exactly balance the excluded volume repulsion (the "Θ point"). For more details, see the general polymer references listed in the Notes at the end of the chapter.

For the remainder of this section, we shall only discuss linear polymers in dilute solutions with good solvents. These are believed to be in the same "universality class" as the self-avoiding walk, which means in particular that they have the same critical exponents. For example, consider the *radius of gyration* of a polymer, which is the average distance of the monomers from the centre of mass of the polymer. The radius of gyration of polymers can be determined experimentally, for example from light scattering properties. For a polymer consisting of N monomers, the radius of gyration is expected to be asymptotic to DN^{ν} as $N \to \infty$, where D and ν are constants. The exponent ν is believed to be *universal*: it should be the same for all linear polymers (in dilute solution with good solvents), and for the self-avoiding walk as well. Moreover the exponent ν for the radius of gyration is believed to be the same as the critical exponent ν defined in (1.1.5) for the mean-square displacement, since polymers are expected to have only one macroscopic length scale. In contrast, the amplitude D is non-universal: it depends on microscopic details of the monomers and the solvent molecules.

The chemist Paul J. Flory developed an effective (but nonrigorous) method for computing the exponent ν [Flory (1949)]. We give a brief description of this method in general dimension d; for simplicity, we ignore all multiplicative constants. (A more probabilistic description of the method will be given afterward.) Fix N and consider a linear polymer with $N+1$ monomers, represented by an N-step walk $\omega = (\omega(0), \ldots, \omega(N))$ in $\mathbf{Z}^d$ (not

necessarily self-avoiding). Let L be the radius of gyration of ω, or any other "effective radius" of the walk. Then ω consists of $N+1$ monomers (sites) spread through a box of volume L^d. Assuming uniformity, this gives a density of

$$\rho = \frac{N}{L^d} \tag{2.2.1}$$

monomers per unit volume. The repulsive energy per unit volume depends on the number of pairs of monomers per unit volume, which we approximate by ρ^2. This is a "mean-field" approximation: it uses the assumption of uniformity very heavily, ignoring the strong correlations in the locations of consecutive monomers along the polymer. If we accept this approximation, then the total repulsive energy of the polymer is given by

$$E_{rep} = L^d \rho^2 = \frac{N^2}{L^d}. \tag{2.2.2}$$

Naturally the repulsive energy is lower for highly extended chains, i.e. large values of L.

Now consider the free energy F of the polymer of radius L, in the absence of the repulsion. This is given (up to constants) by (-1) times the entropy[1], and the entropy in turn is just the logarithm of the number of walks of radius L. Without repulsions, this can be found from the Gaussian behaviour of the ideal chain, as follows. Taking L now to denote the end-to-end distance and fixing $\omega(0) = 0$, we have

$$\Pr\{\omega(N) = x\} \approx N^{-d/2} \exp(-|x|^2/N) \tag{2.2.3}$$

for every $x \in \mathbf{Z}^d$, and hence

$$\Pr\{|\omega(N)| = L\} \approx \frac{L^{d-1}}{N^{d/2}} \exp(-L^2/N). \tag{2.2.4}$$

The total number of N-step walks is $(2d)^N$ in the nearest-neighbour case, so the free energy is

$$\begin{aligned} F &= -\log[(2d)^N \Pr\{|\omega(N)| = L\}] \\ &= -(d-1)\log L + \frac{L^2}{N} + \text{terms independent of } L. \end{aligned} \tag{2.2.5}$$

The term F may also be viewed as an "elastic energy" term, which prevents L from getting too large. The total energy of the polymer is now given by

[1] In thermodynamics we have $F = U - TS$, where U is internal energy, T is temperature, and S is entropy. Here U depends on the number of monomers but not on L, so for our purposes it is constant and hence we ignore it.

the sum of the two energy terms (2.2.2) and (2.2.5):

$$E_{rep} + F = \frac{N^2}{L^d} + \frac{L^2}{N} - (d-1)\log L + K, \tag{2.2.6}$$

where K is independent of L. Now put $L = N^\nu$. Then the total energy (2.2.6) becomes

$$E_{rep} + F = N^{2-d\nu} + N^{2\nu-1} - \nu(d-1)\log N + K. \tag{2.2.7}$$

The value of ν that minimizes the energy (2.2.7) may be found by first equating the first two powers of N: solving $2 - d\nu = 2\nu - 1$ gives

$$\nu = \frac{3}{d+2}. \tag{2.2.8}$$

Substituting this back into (2.2.7), the first two terms become $N^{(4-d)/(d+2)}$, and these are the dominant terms if and only if $d < 4$. Therefore this argument predicts that (2.2.8) gives the correct value of ν whenever $d < 4$. When $d = 4$, this argument also predicts $\nu = 3/(4+2) = 1/2$ since this is the only value for which the first two terms of (2.2.7) remain bounded. However, when $d > 4$, any value of ν in the interval $[2/d, 1/2]$ keeps the first two terms of (2.2.7) bounded. Pushing this argument further suggests that we should take the largest value in this interval so as to minimize the $-\nu(d-1)\log N$ term in (2.2.7), obtaining $\nu = 1/2$ for $d > 4$. This answer makes sense: since ν equals $1/2$ in the ideal case, the addition of a repulsive energy term should not decrease ν below $1/2$, and so we conclude that $\nu = 1/2$ whenever $d > 4$.

To summarize, the above argument makes the following predictions for ν:

$$\nu_{Flory} = \begin{cases} 1 & \text{if } d = 1 \\ 3/4 & \text{if } d = 2 \\ 3/5 & \text{if } d = 3 \\ 1/2 & \text{if } d \geq 4. \end{cases} \tag{2.2.9}$$

These predictions are known as the *Flory values* for ν. As described in Section 1.1, they are known to be correct for $d = 1$ and $d \geq 5$, and they are believed to be correct for $d = 2$ and $d = 4$ as well. The Flory value for $d = 3$ is generally believed to be slightly too large: numerical and field theory calculations indicate that the actual value is probably close to 0.59 (some references are given in the Notes for Section 1.1). The success of Flory's argument is all the more remarkable when one realizes that it benefits greatly from the cancellation of two errors: both E_{rep} and F are greatly overestimated (see p. 46 of de Gennes (1979) for a brief discussion).

To conclude this section, we shall recast the Flory argument in a more probabilistic language. In (2.2.4) we calculated the probability that an N-step random walk ω (starting at the origin) has $|\omega(N)| = L$. Now let us estimate the probability that ω is self-avoiding given that $|\omega(N)| = L$. We shall write $L = N^\nu$ and choose ν to maximize this probability. As above, we assume that the $N+1$ sites of ω are spread uniformly through a box of volume L^d. Given that $\omega(0), \ldots, \omega(k-1)$ are all distinct, the probability that $\omega(k)$ does not coincide with any one of the previous k sites is approximately $1 - kL^{-d}$ (this is the "mean-field" approximation). Hence the probability that ω is self-avoiding given that $|\omega(N)| = L$ is approximately

$$\prod_{k=1}^{N}(1 - kL^{-d}) \approx \exp\left(-\sum_{k=1}^{N} kL^{-d}\right) \approx \exp(-N^2/L^d). \tag{2.2.10}$$

Multiplying (2.2.10) by (2.2.4) yields

$$\Pr\{\omega \text{ is self-avoiding and } |\omega(N)| = L\} \approx \frac{L^{d-1}}{N^{d/2}} \exp\left[-\frac{L^2}{N} - \frac{N^2}{L^d}\right]. \tag{2.2.11}$$

To find the most likely value of L, we maximize the above probability for fixed N. Since the logarithm of this probability is just the negative of the total energy (2.2.6), we are again led to the Flory exponents.

2.3 The $N \to 0$ limit

In this section we describe a connection, discovered by de Gennes, between the self-avoiding walk and the spin systems of classical statistical mechanics: the self-avoiding walk can be considered to be a "zero-component" ferromagnet. Although this connection has not yet provided methods for obtaining rigorous results for the self-avoiding walk, it has been an important tool for physicists and has been used for example to compute the values for the critical exponents γ and ν for $d = 2, 3, 4$ given in (1.1.11) - (1.1.14). To make the discussion more self-contained, we first describe very briefly the basic set-up of spin models. The prototype of these models is the Ising model, and we begin with this fundamental model of ferromagnetism.

For simplicity we restrict attention to the hypercubic lattice $\mathbf{Z}^d$, although this is not essential. Let Λ denote the sites in $\mathbf{Z}^d$ which are in the cube $[-L, L]^d$, for $L \geq 1$. Eventually we will want to take the limit as $L \to \infty$. In the Ising model, a spin variable $S^{(x)}$ taking the value plus one or minus one is associated to each site $x \in \Lambda$. These spin variables interact

via a Hamiltonian

$$\mathcal{H} = -\sum_{(x,y)} S^{(x)} S^{(y)}, \tag{2.3.1}$$

where the sum represents the sum over all nearest-neighbour pairs of sites in Λ. The Hamiltonian represents the energy of a spin configuration (choice of ± 1 for each spin), and is lowest when neighbouring spins agree. The expected value of any function F of the spins in Λ is then given by

$$\langle F \rangle = \frac{1}{Z} E\left(F e^{-\beta \mathcal{H}}\right), \tag{2.3.2}$$

where the expectation E on the right side is with respect to the product of the Bernoulli measures assigning probability one-half to each of the possible values ± 1 for the spin variables, and the *partition function*

$$Z = E\left(e^{-\beta \mathcal{H}}\right) \tag{2.3.3}$$

is a normalization factor. The nonnegative parameter β corresponds to inverse temperature. The partition function and expectations depend on the volume Λ, but to simplify the notation no subscripts Λ will be used to keep track of this.

An important example is $F = S^{(0)} S^{(x)}$, the product of the values of the spins at the origin and at x. For any finite volume Λ it follows from the symmetry of the Hamiltonian under the global spin flip, in which each spin is multiplied by minus one, that $\langle S^{(y)} \rangle = 0$ for any site $y \in \Lambda$. Hence the *two-point function* $\langle S^{(0)} S^{(x)} \rangle$ represents the correlation between the spins at the origin and at x. It follows from the fact that the two-point function lies in the compact interval $[0, 1]$ that there is a subsequence of volumes tending to infinity such that the limit of the two-point function exists along the subsequence. (The same subsequence can be used for all x by a diagonal argument.) In fact it can be shown using correlation inequalities that the infinite volume limit of the two-point function exists, without recourse to subsequences. The infinite volume limit is often referred to as the *thermodynamic limit*. Here we are using *free boundary conditions*, in which spins on the inside boundary of Λ interact only with their nearest neighbours inside Λ. It is known that in the thermodynamic limit, for high temperatures (or in other words for low β) the two-point function decays exponentially as $|x| \to \infty$. The inverse of the decay rate defines a correlation length $\xi(\beta)$. For dimensions $d \geq 2$ there is a critical value β_c (corresponding to the Curie point) such that the correlation length diverges to infinity as $\beta \nearrow \beta_c$. This corresponds to the onset of long range order.

Associated with the critical point β_c, a number of critical exponents can be defined which are analogous to the exponents defined for the self-avoiding walk. For example a critical exponent, known as ν as for the

self-avoiding walk, defines the power law according to which the correlation length diverges:

$$\xi(\beta) \sim \text{const.}(\beta_c - \beta)^{-\nu} \quad \text{as } \beta \nearrow \beta_c. \tag{2.3.4}$$

The susceptibility is defined by

$$\chi(\beta) = d^{-1} \sum_x \langle S^{(0)} \cdot S^{(x)} \rangle, \tag{2.3.5}$$

for the infinite volume theory with $\beta < \beta_c$. The susceptibility diverges as $\beta \nearrow \beta_c$ and the power law at which the divergence takes place defines a critical exponent γ:

$$\chi(\beta) \sim \text{const.}(\beta_c - \beta)^{-\gamma} \quad \text{as } \beta \nearrow \beta_c. \tag{2.3.6}$$

These qualitative analogies between the critical behaviours of spin systems and the self-avoiding walk can be made more quantitative. For this we need to introduce a generalization of the Ising model, known as the N-vector or $O(N)$ model. In this generalization the Ising spins are replaced by spins taking values in the N-dimensional sphere of radius $\sqrt{N}$, for some positive integer N, and the Hamiltonian becomes

$$\mathcal{H} = -\sum_{\langle x,y \rangle} S^{(x)} \cdot S^{(y)}, \tag{2.3.7}$$

where the dot product is the usual Euclidean one. The two-point function for the N-vector model is then defined in finite volume as for the Ising model, with the change that now the single spin distribution is the uniform measure on $\mathcal{S}(N, \sqrt{N})$, where

$$\mathcal{S}(n, r) = \{(a_1, \ldots, a_n) \in \mathbf{R}^n : a_1^2 + \cdots + a_n^2 = r^2\} \tag{2.3.8}$$

is the sphere of radius r in $\mathbf{R}^n$. For $N = 1$ this is just the Ising model. For $N \geq 2$ the N-vector model also has a critical point, and shares many common features with the Ising model (although the change from discrete to continuous symmetry group introduces new elements). Critical exponents can be defined, which will in general depend on N as well as on the dimension d. In a manner to be described in more detail below, the N-vector model can be defined in the limit as $N \to 0$, and this limit gives the self-avoiding walk. The N-vector model can be analyzed, at least non-rigorously, using renormalization methods, and this analysis yields values for the critical exponents in which N appears as a parameter which may be assigned values other than positive integers. Taking $N = 0$ in the expression for the critical exponents then gives values which are believed to

correspond to the self-avoiding walk exponents, and indeed the values given for two dimensions in (1.1.11) and (1.1.12) were obtained by Nienhuis in this way. The self-avoiding walk exponents for three dimensions and the logarithmic corrections for $d = 4$ can be arrived at similarly.

To take the $N \to 0$ limit, consider a fixed finite volume Λ with free boundary conditions (i.e. we consider only sites in Λ in the sum defining the Hamiltonian, and do not take the infinite volume limit). Our aim is to show that the two-point function for the N-vector model converges to that of the self-avoiding walk, i.e. for any $\beta \geq 0$ and for any fixed i, j and sites x, y,

$$\lim_{N \to 0} \langle S_i^{(x)} S_j^{(y)} \rangle = \delta_{i,j} \sum_{\omega : x \to y} \beta^{|\omega|} = \delta_{i,j} G_\beta(x, y), \tag{2.3.9}$$

where the subscript i denotes the i-th spin component and the sum is over all self-avoiding walks (in Λ) of any length, from x to y. In the process, it will be necessary to define what is meant by the limit on the left side of (2.3.9), since the N-vector model two-point function has only been defined when N is a positive integer. To begin with some notation, for a function F of the spins in Λ and for N a positive integer, we write

$$\langle F \rangle = \frac{1}{Z} E\left(F e^{-\beta \mathcal{H}}\right) = \frac{1}{Z} \int F e^{-\beta \mathcal{H}} d\mu_N, \tag{2.3.10}$$

where $d\mu_N$ denotes the product over the spins of uniform measures on $\mathcal{S}(N, \sqrt{N})$, and the partition function Z is the normalization

$$Z = E(e^{-\beta \mathcal{H}}) = \int e^{-\beta \mathcal{H}} d\mu_N. \tag{2.3.11}$$

To obtain (2.3.9) it will be argued that

$$\lim_{N \to 0} Z = \lim_{N \to 0} E(e^{-\beta \mathcal{H}}) = 1 \tag{2.3.12}$$

and

$$\lim_{N \to 0} E\left(S_i^{(x)} S_j^{(y)} e^{-\beta \mathcal{H}}\right) = \delta_{i,j} \sum_{\omega : x \to y} \beta^{|\omega|}. \tag{2.3.13}$$

The analysis will not proceed by extending the definitions of the expectations whose limits are being taken in the above two equations to positive real values of N, and then taking the limit in the strict mathematical sense. Rather, we will show that a certain plausible interpretation of the limit leads to (2.3.12) and (2.3.13); thus our arguments do not lead to these equations as rigorous mathematical statements.

Both (2.3.12) and (2.3.13) will be obtained in the same way. The first step is to expand the exponential in a power series:

$$e^{-\beta\mathcal{H}} = \prod_{\langle x,y\rangle} \exp[\beta S^{(x)} \cdot S^{(y)}] = \prod_{\langle x,y\rangle} \sum_{m_{xy}=0}^{\infty} \frac{\beta^{m_{xy}}}{m_{xy}!} (S^{(x)} \cdot S^{(y)})^{m_{xy}}. \quad (2.3.14)$$

Then we label the nearest-neighbour (undirected) bonds of Λ by $b_1, \ldots b_B$, and for each bond b_α label one of its endpoints b_α^- and the other b_α^+. In this notation (2.3.14) can be written

$$e^{-\beta\mathcal{H}} = \sum_{m_1,\ldots,m_B=0}^{\infty} \frac{\beta^{\Sigma_\alpha m_\alpha}}{\prod_\alpha m_\alpha!} \prod_\alpha (S^{(b_\alpha^-)} \cdot S^{(b_\alpha^+)})^{m_\alpha}. \quad (2.3.15)$$

Hence Z or the two-point function be computed in terms of expectations of products of powers of spin components. Such expectations can be evaluated using the following lemma.

Lemma 2.3.1 *Fix an integer $N \geq 1$. Let $S = (S_1, \ldots, S_N)$ denote a vector which is uniformly distributed on $\mathcal{S}(N, \sqrt{N})$. Given nonnegative integers $k_1, \ldots, k_N$,*

$$E(S_1^{k_1} \cdots S_N^{k_N}) = \begin{cases} \frac{2\Gamma(\frac{N+2}{2}) \prod_{l=1}^N \Gamma(\frac{k_l+1}{2})}{\pi^{N/2}\Gamma(\frac{k_1+\cdots+k_N+N}{2})} N^{(k_1+\cdots+k_N-2)/2} & \text{all } k_l \text{ even} \\ 0 & \text{otherwise,} \end{cases}$$

where Γ denotes the Gamma function.

Proof. The lemma is clearly true for $N = 1$, so we fix $N \geq 2$. Suppose $U = (U_1, \ldots, U_N)$ is uniformly distributed on the sphere $\mathcal{S}(N, 1)$. Using the fact that $\Gamma(\frac{N+2}{2}) = \frac{N}{2}\Gamma(\frac{N}{2})$, it suffices to show for all $k = 1, \ldots, N$ that if the integers m_l are all even then

$$E(U_1^{m_1} U_2^{m_2} \cdots U_k^{m_k}) = \frac{\Gamma\left(\frac{N}{2}\right) \prod_{l=1}^k \Gamma\left(\frac{m_l+1}{2}\right)}{\pi^{k/2}\Gamma\left(\frac{m_1+\cdots+m_k+N}{2}\right)}, \quad (2.3.16)$$

and that this expectation is equal to 0 if any m_i is an odd integer. The latter follows by symmetry. We will prove the former by induction on k.

For $k = 1$, we use the fact [proved in Watson (1983), p.44] that the marginal density of U_i is

$$\frac{\Gamma\left(\frac{N}{2}\right)}{\pi^{1/2}\Gamma\left(\frac{N-1}{2}\right)} (1-a^2)^{(N-3)/2}, \qquad -1 \leq a \leq 1. \quad (2.3.17)$$

It then follows from the identity

$$\int_0^1 t^c(1-t)^d\,dt = \frac{\Gamma(c+1)\Gamma(d+1)}{\Gamma(c+d+2)} \tag{2.3.18}$$

(for $c, d > -1$) that

$$E(U_1^m) = 0 \quad \text{if } m \text{ is odd, and} \tag{2.3.19}$$

$$E(U_1^m) = \frac{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{m+1}{2}\right)}{\pi^{1/2}\Gamma\left(\frac{N+m}{2}\right)} \quad \text{if } m \text{ is even} \tag{2.3.20}$$

which gives (2.3.16) for $k = 1$.

Suppose now that $k > 1$ and assume that (2.3.16) is true for $k-1$. Conditioned on $U_1 = a$, the distribution of $(U_2, \ldots, U_N)$ is uniform on the set $\mathcal{S}(N-1, (1-a^2)^{1/2})$. The inductive hypothesis then gives the conditional expectation

$$E(U_2^{m_2}\cdots U_k^{m_k}|U_1 = a) \tag{2.3.21}$$
$$= (1-a^2)^{(m_2+\cdots+m_k)/2}\frac{\Gamma\left(\frac{N-1}{2}\right)\prod_{l=2}^k \Gamma\left(\frac{m_l+1}{2}\right)}{\pi^{(k-1)/2}\Gamma\left(\frac{m_2+\cdots+m_k+N-1}{2}\right)}.$$

Inserting this into

$$E(U_1^{m_1}U_2^{m_2}\cdots U_k^{m_k})$$
$$= \int_{-1}^{+1} a^{m_1} E(U_2^{m_2}\cdots U_k^{m_k}|U_1 = a)\frac{\Gamma\left(\frac{N}{2}\right)}{\pi^{1/2}\Gamma\left(\frac{N-1}{2}\right)}(1-a^2)^{(N-3)/2}\,da$$

then gives the desired result (2.3.16). □

We now use Lemma 2.3.1 to define what we mean by the limit as $N \to 0$ of expectations like those in the statement of the lemma. It follows from the lemma that for any positive integer N and any index i,

$$E(S_i^2) = 1;$$

in fact this can be seen more easily by symmetry and the fact that $E(S_1^2 + \cdots + S_N^2) = N$. We will therefore assert, by way of definition, that

$$\lim_{N\to 0} E(S_i^2) = 1. \tag{2.3.22}$$

Also, we will define the limit as $N \to 0$ of any expectation as in the statement of the lemma to be zero if $k_1 + \ldots + k_N > 2$. This is consistent with the result of the lemma; e.g. if two k_l's equal 2 and the others are 0, then

$$E(S_i^2 S_j^2) = \frac{2\Gamma(\frac{N+2}{2})\left(\frac{1}{2}\right)^2}{\Gamma(\frac{N+4}{2})}N, \tag{2.3.23}$$

which converges to zero as $N \to 0$. According to this definition,

$$\lim_{N\to 0} E(S_1^{k_1} \cdots S_N^{k_N}) = \begin{cases} 1 & \text{all } k_l = 0, \text{ or one } k_l = 2 \text{ and } k_j = 0,\ j \neq l \\ 0 & \text{otherwise.} \end{cases} \tag{2.3.24}$$

Consider now the partition function

$$Z = \sum_{m_1,\ldots,m_B=0}^{\infty} \frac{\beta^{\Sigma_\alpha m_\alpha}}{\prod_\alpha m_\alpha!} E\left(\prod_\alpha (S^{(b_\alpha^-)} \cdot S^{(b_\alpha^+)})^{m_\alpha}\right). \tag{2.3.25}$$

Equation (2.3.24) provides a means of extending Z to $N = 0$, by taking the limit as $N \to 0$ termwise in the above sum. A graphical interpretation of the sum in (2.3.25) can be obtained by associating to each term in the sum a graph whose edges are given by m_i undirected edges joining the endpoints of the bond b_i. It then follows from (2.3.24) that any term whose corresponding graph has a vertex from which other than two or zero edges emanate will approach zero in the limit as $N \to 0$. This can be seen by considering a specific example. Consider the graph consisting of the four nearest-neighbour edges $\{z,x\}, \{z,x\}, \{z,y\}, \{z,w\}$. The expectation arising from the corresponding term in (2.3.25) is

$$\sum_{i,j,k,l} E\left(S_i^{(y)} S_i^{(z)} S_j^{(w)} S_j^{(z)} S_k^{(x)} S_k^{(z)} S_l^{(x)} S_l^{(z)}\right), \tag{2.3.26}$$

where the sum is over spin components. Since spins at different sites are independent, the expectation in the above sum factors into a product of four expectations. The factor corresponding to the site z is

$$E\left(S_i^{(z)} S_j^{(z)} S_k^{(z)} S_l^{(z)}\right), \tag{2.3.27}$$

which will go to zero in the limit as $N \to 0$, for any choice of i,j,k,l, by (2.3.24). There is further N-dependence arising from the number of terms in the sum over spin components i,j,k,l, but this will be interpreted as only helping to drive the limit to zero.

The relevant graphs in the limit are therefore the graphs consisting of a finite number (possibly zero) of nonintersecting self-avoiding polygons in Λ, where the degenerate polygons consisting of two edges linking a pair of nearest-neighbour sites are allowed possibilities. The graph with no edges corresponds to the term in the sum with all $m_i = 0$, and contributes an amount 1. A two-edge polygon with nearest-neighbour vertices x, y contributes an amount

$$\frac{\beta^2}{2} E\left([S^{(x)} \cdot S^{(y)}]^2\right). \tag{2.3.28}$$

Since an expectation involving an odd power of $S_i^{(x)}$ is zero, this is equal to

$$\frac{\beta^2}{2} \sum_{i=1}^{N} E\left([S_i^{(x)} S_i^{(y)}]^2\right) = \frac{\beta^2}{2} N \tag{2.3.29}$$

and hence does not contribute in the limit.

A nondegenerate polygon, in other words a polygon consisting of at least four bonds, contributes an amount

$$\beta^k E\left((S^{(y_1)} \cdot S^{(y_2)})(S^{(y_2)} \cdot S^{(y_3)}) \cdots (S^{(y_r)} \cdot S^{(y_1)})\right) \tag{2.3.30}$$

where y_l is a neighbour of y_{l+1} for each $l = 1, \ldots, r-1$, y_r is a neighbour of y_1, and $y_1, \ldots, y_r$ are distinct. This expression is equal to

$$\beta^k \sum_{i=1}^{N} E\left((S_i^{(y_1)} S_i^{(y_2)} \cdots S_i^{(y_r)})^2\right) = \beta^k N, \tag{2.3.31}$$

which also converges to 0 as $N \to 0$. We are thus led to conclude that

$$\lim_{N \to 0} Z = 1. \tag{2.3.32}$$

For the two-point function the analysis is similar. We would like to compute the limit as $N \to 0$ of the expectation

$$\sum_{m_1, \ldots, m_B = 0}^{\infty} \frac{\beta^{\sum_\alpha m_\alpha}}{\prod_\alpha m_\alpha!} E\left(S_i^{(x)} S_j^{(y)} \prod_\alpha (S^{(b_\alpha^-)} \cdot S^{(b_\alpha^+)})^{m_\alpha}\right). \tag{2.3.33}$$

For $x = y$ the limit of the above expression is equal to 1 by an analysis similar to that used to analyze the partition function. Suppose now that $x \neq y$. Again there is a correspondence between the terms in the sum and graphs on Λ, but now there can be a nonzero contribution to the limit only from those graphs in which exactly one edge emanates from each of the vertices x and y, and either two or zero edges emanate from every other vertex. Such a graph must consist of a self-avoiding walk from x to y together with a finite number of (possibly degenerate) self-avoiding polygons. The contribution to the limit from the polygons is equal to zero as it is for the partition function. The contribution due to the self-avoiding walk with vertices $(x, v_1, \ldots, v_{k-1}, y)$ is

$$\beta^k E\left(S_i^{(x)} (S^{(x)} \cdot S^{(v_1)})(S^{(v_1)} \cdot S^{(v_2)}) \cdots (S^{(v_{k-1})} \cdot S^{(y)}) S_j^{(y)}\right). \tag{2.3.34}$$

This expression is equal to $\beta^k \delta_{i,j}$, since upon expanding the dot products everything has expectation 0 except $(S_i^{(x)} S_i^{(v_1)} \cdots S_i^{(v_{k-1})} S_i^{(y)})^2$. Since there is one such term for every self-avoiding walk from x to y,

$$\lim_{N \to 0} \langle S_i^{(x)} S_j^{(y)} \rangle = \delta_{i,j} \sum_{\omega : x \to y} \beta^{|\omega|} = \delta_{i,j} G_\beta(x, y). \tag{2.3.35}$$

This correspondence of two-point functions is responsible for the general belief that the critical exponents γ and ν also correspond in the $N \to 0$ limit.

We end this section with a nonrigorous discussion of the equality in the $N \to 0$ limit of the self-avoiding walk critical exponent α_{sing}, defined in (1.4.13), and the critical exponent for the singular part of the specific heat of the N-vector model. To distinguish between these two exponents we shall denote the latter by α_s.

To define the specific heat we first introduce the expected energy per unit volume, which is given by

$$\mathcal{E}(\beta) = \langle \sum_{\langle 0,x \rangle} S^{(0)} \cdot S^{(x)} \rangle = 2dN \langle S_i^{(0)} S_i^{(e)} \rangle, \tag{2.3.36}$$

for any fixed i and any nearest neighbour e of the origin. The prefactor $2dN$ is irrelevant as far as the behaviour of $\mathcal{E}(\beta)$ near the critical point is concerned, so we introduce

$$u(\beta) = \langle S_i^{(0)} S_i^{(e)} \rangle. \tag{2.3.37}$$

The specific heat is defined as the rate of change of the energy with respect to temperature β^{-1}, i.e.

$$C(\beta) = -\beta^2 \frac{du}{d\beta}. \tag{2.3.38}$$

Typically the specific heat either diverges as β increases to β_c, or there is a nonnegative integer M such that it has M but not $M+1$ derivatives at β_c^-, with

$$C(\beta) \approx \sum_{j=0}^{M} \frac{C^{(j)}(\beta_c)}{j!} (\beta - \beta_c)^j + (\beta_c - \beta)^{-\alpha_s} \tag{2.3.39}$$

for some exponent $-\alpha_s \in (M, M+1]$. In principle both α_s and M can depend on N. Here we are using the symbol $\approx$ which indicates a crude correspondence between the right and left sides; in particular in the correction term we are dropping sign and constant factors, and also possible logarithmic factors which can be expected to be present when $-\alpha_s$ is an integer.

We also allow $M = -1$ in (2.3.39), with the empty sum interpreted as zero, to deal simultaneously with both finite and infinite $C(\beta_c)$. For $M = -1$, requiring that the energy $u(\beta)$ remains bounded as $\beta \nearrow \beta_c$ implies that $-\alpha_s \in (-1, 0]$.

In view of (2.3.38), the behaviour (2.3.39) of the specific heat suggests that

$$u(\beta) \approx \sum_{j=0}^{M+1} \frac{u^{(j)}(\beta_c)}{j!}(\beta - \beta_c)^j + (\beta_c - \beta)^{-\alpha_s+1}. \tag{2.3.40}$$

Assuming that the form of the above relation persists in the limit as $N \to 0$, from (2.3.37) and (2.3.35) we obtain

$$G_\beta(0,e) = \sum_{n=1}^{\infty} c_n(0,e)\beta^n \approx \sum_{j=0}^{M+1} g_j(\beta - \beta_c)^j + (\beta_c - \beta)^{-\alpha_s+1}, \tag{2.3.41}$$

where

$$g_j = \frac{1}{j!}\frac{d^j}{d\beta^j}G_\beta(0,e)\Big|_{\beta_c} = \sum_{n=1}^{\infty} c_n(0,e)\binom{n}{j}\beta_c^{n-j}. \tag{2.3.42}$$

The exponent α_s in (2.3.41) is interpreted as the $N \to 0$ limit of an N-dependent exponent. Our goal now is to argue that this limiting value of α_s is equal to α_{sing}. We further assume that in (2.3.41) M is the largest integer such that g_{M+1} is finite, so that $g_{M+2} = \infty$. Assuming now that $c_n(0,e) \approx n^{\alpha_{sing}-2}\mu^n = n^{\alpha_{sing}-2}\beta_c^{-n}$, g_j will be finite if and only if $\alpha_{sing} - 2 + j < -1$. Thus we have $-\alpha_{sing} \in (M, M+1]$, which is consistent with the restriction $-\alpha_s \in (M, M+1]$ in (2.3.39).

Writing $\beta = \beta_c e^{-t}$, so that $\beta_c - \beta \sim \beta_c t$, we have

$$G_\beta(0,e) - \sum_{j=0}^{M+1} g_j(\beta - \beta_c)^j \approx \sum_{n=1}^{\infty} n^{\alpha_{sing}-2}\left[e^{-nt} - \sum_{j=0}^{M+1}\binom{n}{j}(-t)^j\right]. \tag{2.3.43}$$

Approximating the sum over n on the right side by an integral and then making the change of variables $y = xt$, the right side of (2.3.43) is given approximately by

$$\int_1^\infty x^{\alpha_{sing}-2}\left[e^{-xt} - \sum_{j=0}^{M+1}\binom{x}{j}(-t)^j\right]dx$$

$$= t^{-\alpha_{sing}+1}\int_t^\infty y^{\alpha_{sing}-2}\left[e^{-y} - \sum_{j=0}^{M+1}\binom{y/t}{j}(-t)^j\right]dy.$$

As $t \to 0$, the right side behaves like

$$t^{-\alpha_{sing}+1} \int_t^\infty y^{\alpha_{sing}-2} \left[e^{-y} - \sum_{j=0}^{M+1} \frac{1}{j!}(-y)^j \right] dy. \tag{2.3.44}$$

In view of the fact that $-\alpha_{sing} \in (M, M+1]$, the above integral is convergent both for large y and for $y \to 0$ (apart from a logarithmic divergence as $y \to 0$ when $-\alpha_{sing} = M+1$). The integral is clearly nonzero, since the quantity in square brackets in the integrand is of the same sign for all positive y, by Taylor's Theorem with remainder. Hence the overall behaviour is $t^{-\alpha_{sing}+1}$. Comparing now with (2.3.41), we conclude that $\alpha_s = \alpha_{sing}$.

2.4 Notes

Section 2.1. Scaling theory is discussed in many theoretical physics texts on critical phenomena, for example Amit (1984), and we shall make no attempt here to refer to the original literature.

Section 2.2. Some general references on polymers which elaborate on the topics mentioned here include Flory (1971), de Gennes (1979), Doi and Edwards (1986), and des Cloiseaux and Jannink (1990). A readable survey is given in Flory's 1974 Nobel lecture [Flory (1976)]. Whittington (1982) discusses several additional topics in the statistical mechanics of polymers and self-avoiding walks.

Flory (1949) originally discussed only the three-dimensional case of the argument presented in this section. The extension to other dimensions was first observed by Fisher (1969). There are many other arguments which derive the Flory exponents; for example, see Edwards (1965), Freed (1981), and Bouchaud and Georges (1989).

Section 2.3. For a general introduction to rigorous results for spin systems, see for example Fernández, Fröhlich and Sokal (1992), Thompson (1988), Glimm and Jaffe (1987), Ellis (1985), Ruelle (1969). More physics-oriented accounts are given in for example Itzykson and Drouffe (1989), Parisi (1988), Amit (1984).

The fact that the $N \to 0$ limit of the N-vector model gives the self-avoiding walk was first observed in de Gennes (1972); see also de Gennes (1979). The use of Lemma 2.3.1 in deriving the $N \to 0$ limit appears to be new. Other approaches can be found in Aragão de Carvalho, Caracciolo and Fröhlich (1983), Halley and Dasgupta (1983), Domb (1976), and Bowers and McKerrell (1973). The calculation of critical exponents for the

two-dimensional N-vector model, and corresponding identification of the exponents for $N = 0$, was carried out in Nienhuis (1982); see also Nienhuis (1984) and Nienhuis (1987). For $d = 3$, the $N = 0$ critical exponents are calculated in Le Guillou and Zinn-Justin (1989). Logarithmic corrections in four dimensions were computed in Larkin and Khmel'Nitskii (1969), Wegner and Riedel (1973) and Brezin, Le Guillou and Zinn-Justin (1973).

There is an intimate relation between spin systems and interacting random walks of various types (going far beyond the $N \to 0$ limit). This was emphasized by Symanzik (1969), and developed further in Brydges, Fröhlich and Spencer (1982). A detailed account of the random walk representations of spin systems is given in Fernández, Fröhlich and Sokal (1992).