Chapter 10

Related topics

10.1 Weak self-avoidance and the Edwards model

The weakly self-avoiding walk, known also as the self-repellent walk and as the Domb-Joyce model [Domb and Joyce (1972)], is a measure on ordinary random walks in which self-intersections are discouraged but not forbidden. The measure associates to an n-step simple random walk ω the weight

$$Q_n^{\lambda}(\omega) = \frac{1}{Z_n(\lambda)} \prod_{0 \le s < t \le n} [1 - \lambda v_{st}(\omega)], \tag{10.1.1}$$

where $0 < \lambda \le 1$, $Z_n(\lambda)$ is a normalization constant, the product is over pairs of integers s and t, and $v_{st}(\omega)$ is 1 if $\omega(s) = \omega(t)$ and otherwise is 0. Taking $\lambda = 1$ gives the uniform measure on n-step self-avoiding walks, while $0 < \lambda < 1$ gives a measure in which self-intersections diminish the probability of a walk. Setting $\lambda = 0$ just gives simple random walk. An alternate parametrization of the interaction which appears frequently is to take

$$\lambda = 1 - e^{-\beta}. \tag{10.1.2}$$

Then in terms of β,

$$Q_n^{\lambda}(\omega) = \frac{1}{\tilde{Z}_n(\beta)} \prod_{0 \le s < t \le n} e^{-\beta v_{st}(\omega)}, \tag{10.1.3}$$

where $\tilde{Z}_n(\beta)$ is a normalization constant. Here it is $\beta = \infty$ which corresponds to the strictly self-avoiding walk.

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It is a (nonrigorous) prediction of the renormalization group method that the weakly self-avoiding walk, for any $\lambda > 0$, is in the same universality class as the strictly self-avoiding walk. This is borne out in the existing rigorous results. For $d = 1$ it was shown in Bolthausen (1990), using large deviation techniques, that there is a $\lambda_0 > 0$ such that for $\lambda \in (0, \lambda_0]$ there is a $c > 0$ such that

$$\lim_{n\to\infty} Q_n^\lambda\{\omega : c \leq n^{-1}|\omega(n)| \leq c^{-1}\} = 1. \tag{10.1.4}$$

In fact the same conclusion was obtained in a more general setting than just the nearest-neighbour walk. This shows that if λ is sufficiently small then the one-dimensional weakly self-avoiding walk behaves like the strictly self-avoiding walk, in the sense that $\nu = 1$. For $d > 4$, Brydges and Spencer (1985) used the lace expansion to show that for λ sufficiently close to zero the mean-square displacement of the model defined by (10.1.1) is linear in the number of steps, and the scaling limit of the endpoint of the walk is Gaussian. This could be extended to cover all $\lambda < 1$ for $d \geq 5$, using the methods that handled the $\lambda = 1$ case; see Chapter 6. So also above four dimensions the weakly self-avoiding walk has the same scaling behaviour as simple random walk. There are no rigorous results in two and three dimensions; four dimensions will be discussed below.

The Edwards model is a continuous space and time analogue of the weakly self-avoiding walk, introduced in Edwards (1965). Its relation to the weakly self-avoiding walk is similar to that of Brownian motion to simple random walk. The strength of the self-avoidance interaction for the Edwards model is analogous to the parameter β of (10.1.2). It could be hoped that as this interaction strength goes to infinity the Edwards model would approach a limit corresponding to a continuum limit of the self-avoiding walk; however methods allowing for such a limit to be carried out rigorously remain to be found.

The Edwards model is defined formally as a measure on d-dimensional continuous paths on an interval $[0, T]$, by multiplying the Wiener measure on such paths by a factor suppressing self-intersections. Specifically, if we denote the Wiener measure by dW^T and a typical path by $r(t)$, then the Edwards model is defined by the measure

$$d\mu^T = \frac{1}{Z_T} e^{-gJ} dW^T \tag{10.1.5}$$

where Z_T is a normalization factor, g is a positive parameter measuring the strength of the interaction, and J is a functional on paths defined by

$$J = J(r) = \int_0^T \int_0^T \delta(r(s) - r(t))ds\,dt. \tag{10.1.6}$$

The quantity $J(r)$ can be interpreted as the amount of time spent by the path r at its double points, and serves in the measure μ^T to suppress self-intersections.

Rigorous sense can be made of the measure μ^T by first replacing the delta function in (10.1.6) by a regularized delta function δ_ϵ, yielding a well-defined interaction J_ϵ and corresponding measure μ_ϵ^T, and then taking the limit removing the regularization ϵ. This procedure can be carried out literally when $d = 1$, but for higher dimensions it is not so simple and a renormalization is required. In fact the situation is quite similar to the construction of the φ^4 quantum field theory [see Glimm and Jaffe (1987)], and methods used to construct these theories in two and three dimensions provide a basis from which to approach the Edwards model. The required renormalization is simplest in two dimensions, and the Edwards model in $d = 2$ was first constructed in Varadhan (1969). In two dimensions the Edwards measure is absolutely continuous with respect to the Wiener measure. For $d = 3$ the construction was carried out in Westwater (1980,1982) for all T and g; here the measure is singular with respect to the Wiener measure, in a dramatic departure from the formal expression (10.1.5). For small g an alternate construction of the three-dimensional measure was given in Bolthausen (1991), which made use of some simplifications in the constructive field theory technology [Brydges, Fröhlich and Spencer (1983), Bovier, Felder and Fröhlich (1984)].

Although for both two and three dimensions the Edwards model has been constructed for all times T and all $g \geq 0$, there is insufficient control to compute the limiting behaviour of the expected value of $r(T)^2$ as $T \to \infty$, and critical exponents such as ν are not currently accessible. The construction of the Edwards model for any finite T can be considered a construction of a subcritical model, and to obtain control of critical exponents a control of the critical $T = \infty$ model is required. However in one dimension a proof has been given that $|r(T)|$ behaves like a multiple of T as $T \to \infty$; see Westwater (1985). It is believed that the Edwards model is in the same universality class as the self-avoiding walk, i.e. that the critical exponents will be the same.

An alternate regularization of the Edwards measure (10.1.5) is to consider a version of the model in discrete time and space. In Bovier, Felder and Fröhlich (1984), such a regularization was given, and the necessary renormalization was performed to construct the continuum limit of Green functions such as the two-point function in two and three dimensions; the continuum measure itself was however not constructed. A natural discretization of the Edwards model is to replace the delta function in the interaction by a discrete version. Specifically, we discretize $r(t)$ to $n^{-1/2}\omega(\lfloor nt \rfloor)$, with ω

a simple random walk, and define

$$\delta_n(x) = \begin{cases} n^{d/2} & \text{if } \|x\|_\infty \le \frac{1}{2}n^{-1/2} \\ 0 & \text{otherwise.} \end{cases} \tag{10.1.7}$$

Then we replace J of (10.1.6) by

$$\begin{aligned} J_n &= \frac{1}{n^2}\sum_{s=0}^{n}\sum_{t=0}^{n}\delta_n(n^{-1/2}[\omega(s)-\omega(t)]) \\ &= n^{(d-4)/2}\sum_{s,t=0}^{n} v_{st}(\omega). \end{aligned} \tag{10.1.8}$$

This gives the measure (10.1.3) on simple random walks, with interaction strength $\beta = 2gn^{(d-4)/2}$. From this relation it is clear that in dimensions two and three the discrete Edwards model interaction is weaker than that of the weakly self-avoiding walk. In Stoll (1989) the two dimensional Edwards measure was constructed by taking the continuum limit of this discrete model; this has not yet been carried out in three dimensions.

In four dimensions the discrete Edwards model and the weakly self-avoiding walk are identical, apart from a factor of two in the coupling constants. As this book is being written, rigorous results in four dimensions are beginning to appear. Brydges, Evans and Imbrie (1992) have considered a model of weakly self-avoiding walk on a four dimensional hierarchical lattice, and have proved that a quantity closely related to the critical two-point function decays asymptotically as a multiple of $|x|^{-2}$ if the interaction is sufficiently weak. This work uses an identity to write the two-point function of the model as the two-point function of a quantum field theory, and then performs a renormalization group analysis of the quantum field theory. Arnaudon, Iagolnitzer and Magnen (1991) have announced a proof that the critical two-point function of a continuum four-dimensional Edwards model with fixed ultraviolet cutoff (a regularization analogous to discretization) and sufficiently weak interaction behaves asymptotically like a multiple of $|x|^{-2}$, with $\log|x|$ and $\log\log|x|$ corrections, using constructive field theory methods.

10.2 Loop-erased random walk

During the 1980s considerable progress was made in the study of the loop-erased self-avoiding random walk, which is a model of self-avoiding walk different from the one studied in this book. In this section we give a brief definition of the loop-erased random walk, and state the principal rigorous results which have been obtained for it. Most of the rigorous work is due to Lawler, and is described in his book [Lawler (1991)].

There are two equivalent formulations of the model. The first, from which the name is derived, can be described as follows. Consider the path of an infinite ordinary simple random walk, for the moment in at least three dimensions. We associate to this walk an infinite self-avoiding walk by erasing loops from the path chronologically. In more detail, we begin by looking for the first time that the walk intersects itself and then erase the portion of the walk (the loop) between the (first) two visits to the site where the intersection occurs. Then we erase the first loop from the resulting path, and continue inductively. This leads to an infinite self-avoiding walk. To define a measure on the set of all n-step self-avoiding walks, we assign to each n-step self-avoiding walk ω a weight equal to the probability that the first n steps of the loop-erased walk agree with ω. This family of measures is *consistent*, in the sense of (6.7.2). In particular, a walk which cannot be extended by a single step and remain self-avoiding is assigned weight zero in this measure, and hence the loop erased walk does not define the uniform measure on the set of n-step self-avoiding walks.

The above procedure works in dimensions $d \geq 3$, where simple random walk is transient, but for the recurrent case $d = 2$ more care is needed. In two dimensions it is necessary to use a limiting process to define the model. Roughly speaking a measure is defined on n-step self-avoiding walks by first performing loop erasure as above on simple random walk paths which lie in a finite box of side length $N \geq n$, thereby obtaining an N-dependent measure, and then the limit is taken of this measure as N goes to infinity.

A second (equivalent) formulation of the model, which goes by the name Laplacian self-avoiding walk, provides a description as a "kinetically-growing" walk, i.e. as a stochastic process defined by transition probabilities. To avoid the special difficulties associated with two dimensions, we consider here only $d \geq 3$. The Laplacian walk is defined to be the process whose transition probabilities are as follows. Given an n-step self-avoiding walk ω, the probability that the next step is to a neighbour x of $\omega(n)$ is proportional to the probability that simple random walk starting from x will never intersect ω. To state this more precisely we introduce the following definition. Given a site $x \in \mathbf{Z}^d$ and a set $A \subset \mathbf{Z}^d$, we define $Q_A(x)$ to be the probability that an infinite simple random walk beginning at x never enters the set A. The transition probabilities of the Laplacian walk are then given, for $\omega = \{\omega(0), \ldots \omega(n)\}$ and x a neighbour of $\omega(n)$, by

$$P(\omega(n+1) = x|\omega) = \frac{Q_\omega(x)}{\sum_{y:|y-\omega(n)|=1} Q_\omega(y)}. \qquad (10.2.1)$$

A proof that this is equivalent to the loop-erased walk is given in Lawler

(1991). The name Laplacian self-avoiding walk derives from the fact that $Q_A(x)$ is a harmonic function on the complement of A, with boundary conditions zero on A and one at infinity.

It is now known that the loop-erased self-avoiding walk has upper critical dimension equal to four, and that if (1.1.12) and (1.1.14) accurately represent the behaviour of the mean-square displacement of the self-avoiding walk for dimensions three and four, then the loop-erased self-avoiding walk is in a different universality class (i.e. has different critical exponents) than the self-avoiding walk defined using the uniform measure. We end this section with a statement of the rigorous results, beginning with high dimensions, where the results are strongest.

Theorem 10.2.1 *(a) [Lawler (1980)] Let $d \geq 5$ and let $\hat{S}(n)$ denote the loop-erased walk after n steps. There is a constant b, depending only on the dimension, such that the process $\hat{X}_n(t) = (bn)^{-1/2}\hat{S}(\lfloor nt \rfloor])$ converges in distribution to the Wiener process [normalized as in (6.6.3)]. Moreover, the mean-square displacement of the loop-erased walk is asymptotic to b times the number of steps.*
(b) [Lawler (1986)] Let $d = 4$. There is a sequence b_n such that the process $\hat{X}_n(t) = (b_n n)^{-1/2}\hat{S}([nt])$ converges in distribution to the Wiener process. The sequence b_n satisfies

$$\frac{1}{3} \leq \liminf_{n\to\infty} \frac{\log b_n}{\log\log n} \leq \limsup_{n\to\infty} \frac{\log b_n}{\log\log n} \leq \frac{1}{2}$$

and the mean-square displacement satisfies

$$\frac{1}{3} \leq \liminf_{n\to\infty} \frac{\log[n^{-1}E(|\hat{S}(n)|^2)]}{\log\log n} \leq \limsup_{n\to\infty} \frac{\log[n^{-1}E(|\hat{S}(n)|^2)]}{\log\log n} \leq \frac{1}{2}.$$

It is conjectured in Lawler (1986) that in (b) of the above theorem $\lim_{n\to\infty} \log b_n / \log\log n = 1/3$. This is different behaviour than the correction $(\log n)^{1/4}$ to the mean-square displacement that is predicted by the renormalization group for the self-avoiding walk. In two and three dimensions Monte-Carlo results suggest a more dramatic discrepancy between the loop-erased walk and the self-avoiding walk, namely $\nu = 4/5$ in two dimensions and $\nu \approx 0.616$ in three dimensions [Guttman and Bursill (1990)]. The following theorem proves that in three dimensions the mean-square displacement of the loop-erased walk behaves differently than the $n^{1.18}$ behaviour expected for the self-avoiding walk.

Theorem 10.2.2 *[Lawler (1988)] For every $\epsilon > 0$ there is a positive constant K such that*

$$E(|\hat{S}(n)|^2) \geq Kn^{3/2-\epsilon} \quad \text{for } d = 2,$$

and

$$E(|\hat{S}(n)|^2) \geq Kn^{6/5-\epsilon} \quad \textit{for } d = 3.$$

These results for the loop-erased self-avoiding random walk are proved using probabilistic methods quite unlike the methods used in this book.

10.3 Intersections of random walks

The critical exponents γ and Δ_4 for the self-avoiding walk are closely related to intersection probabilities for self-avoiding walks. To be specific, assuming that c_n has the asymptotic behaviour specified in (1.1.11) and (1.1.13) [in fact we know (1.1.11) does hold for $d \geq 5$], then the probability that two n-step self-avoiding walks beginning at the origin do not intersect is given by

$$\frac{c_{2n}}{c_n^2} \sim \begin{cases} A^{-1}2^{\gamma-1}n^{1-\gamma} & d \neq 4 \\ A^{-1}(\log n)^{-1/4} & d = 4. \end{cases} \tag{10.3.1}$$

The critical exponent Δ_4 is relevant for intersection probabilities of self-avoiding walks beginning at different sites. A measure of this is the renormalized coupling constant, defined in (1.4.22), which is believed to satisfy

$$g(z) \sim \text{const.}(z_c - z)^{d\nu-2\Delta_4+\gamma} \quad \text{as } z \nearrow z_c, \tag{10.3.2}$$

with Δ_4 obeying the hyperscaling relation $d\nu - 2\Delta_4 + \gamma = 0$ in dimensions $2, 3, 4$ (with a logarithmic correction in four dimensions) and $\Delta_4 = 3/2$ for $d \geq 5$ (this is proved for $d \geq 6$; see Theorem 1.5.5 and the Remark below its statement).

While the above conjectures remain unproven for the self-avoiding walk in low dimensions, it is natural to ask if corresponding statements for simple random walk can be proven. In the remainder of this section we give a brief summary of some of the results which have been obtained in this direction.

To discuss the analogue of (10.3.1) for simple random walk, we denote by $f(n)$ the probability that the paths of two n-step simple random walks beginning at the origin do not intersect (apart from the fact that they have a common initial point). For the statement of the next theorem we introduce the notation $f(n) \cong g(n)$ to mean that $\log f(n) \sim \log g(n)$.

Theorem 10.3.1 *For $d > 4$, $f(n) \sim$ const. as $n \to \infty$, for some constant strictly between 0 and 1 which depends only on the dimension. For $d = 4$, $f(n) \cong (\log n)^{-1/2}$. For $d = 2$ or 3 there is an exponent ζ such that $f(n) \cong n^{-\zeta}$, with*

$$\tfrac{1}{2} + \tfrac{1}{8\pi} \leq \zeta < \tfrac{3}{4} \qquad d = 2$$

$$\tfrac{1}{4} \leq \zeta < \tfrac{1}{2} \qquad d = 3.$$

For $d > 4$ this was proved in Lawler (1980). For $d = 4$ the proof is given in Lawler (1982,1985a,1991), and for $d = 2, 3$ the proof is given in Burdzy and Lawler (1990a,1990b). Nonrigorous conformal field theory arguments predict that in two dimensions $\zeta = 5/8$; see Duplantier and Kwon (1988). Monte-Carlo computations are consistent with this prediction, and also give a value near 0.29 for ζ in three dimensions [Burdzy, Lawler and Polaski (1989), Duplantier and Kwon (1988), Li and Sokal (1990)]. Results for generating functions related to the above theorem are given in Park (1989).

The logarithmic behaviour in four dimensions is the hallmark of the critical nature of four dimensions for random walk intersections. Heuristically this can be seen from the fact that Brownian motion paths have Hausdorff dimension two, and hence four dimensions is marginal for the intersection of two Brownian paths. By the same argument three dimensions is critical for triple points of three paths (two two-dimensional paths in three dimensions will typically intersect in a one dimensional set, and the intersection of this set with a third two-dimensional path will be marginal in three dimensions). Bounds on intersection probabilities of three random walks in three dimensions are obtained in Lawler (1985b,1991), and using rigorous renormalization group methods in Felder and Fröhlich (1985). On a nonrigorous level, results of this type have been considerably generalized using renormalization methods; see Duplantier (1988).

We denote the analogue for simple random walk of the renormalized coupling constant $g(z)$ by $g_0(z)$. For $g_0(z)$ the following theorem gives $\Delta_4 = 3/2$ for $d > 4$, and hyperscaling for $d \leq 4$ (with a logarithmic correction in four dimensions).

Theorem 10.3.2 *Let $t = (2d)^{-1} - z$. Then there are positive constants $c_1, c_2, c_3, c_1', c_2', c_3'$ such that for $z < (2d)^{-1}$*

$$\left. \begin{array}{r} c_1 \\ c_2 |\log t|^{-1} \\ c_3 t^{(4-d)/2} \end{array} \right\} \leq g_0(z) \leq \left\{ \begin{array}{ll} c_1' & d < 4 \\ c_2' |\log t|^{-1} & d = 4 \\ c_3' t^{(4-d)/2} & d > 4. \end{array} \right.$$

Results for the probability of intersection of walks of fixed length n, one beginning at the origin and the other at $x \simeq \sqrt{n}$, are given in Lawler (1982,1991). These results effectively yield more detailed information than Theorem 10.3.2. In (and near) four dimensions the above result was proved in Felder and Fröhlich (1985) using a rigorous renormalization group argument; see also Aizenman (1985) for related work on the intersection of Brownian paths. A proof of Theorem 10.3.2 using inclusion-exclusion methods is given in Park (1989).

10.4 The "myopic" or "true" self-avoiding walk

The model of self-avoiding walk discussed in this book is not a random walk in the usual sense, being defined via a measure on paths rather than via transition probabilities as a stochastic process. One model of self-avoiding walk which *is* defined by transition probabilities is the so-called "true" self-avoiding walk; this model is essentially described by the MSAW algorithm[1] of Section 9.1. The epithet "true" is a misnomer, as the paths of this model need not in general be self-avoiding, nor is it the model of self-avoiding walk which is most commonly studied. In Lawler (1991) this model is referred to as the "myopic" self-avoiding walk; this name emphasizes the short-sightedness of the walk in its effort to be self-avoiding. Although the myopic self-avoiding walk has played a relatively minor role in applications [see however Family and Daoud (1984) for an application to polymers under certain conditions], it is interesting to see how it compares to the usual self-avoiding walk.

The transition probabilities for the myopic self-avoiding walk are defined as follows. Consider a walker on the hypercubic lattice $\mathbf{Z}^d$, beginning at the origin and taking nearest-neighbour steps. The first step is to a nearest neighbour of the origin, each neighbour being chosen with equal probability $(2d)^{-1}$. In subsequent steps, if there are neighbours of the current position which have not yet been visited, the next site is chosen uniformly from the neighbours not yet visited. If all neighbours have already been visited (i.e. if the walk is trapped) then the next site is chosen uniformly from among those neighbours which have been visited least often in the past. This leads to paths with self-intersections — looking just one step ahead cannot prevent the walk from becoming trapped, and a step must always be taken to some neighbour. A simple example demonstrates the computation of weights assigned to paths by the myopic self-avoiding walk: the myopic self-avoiding walk assigns to the walk ENWN in two dimensions the weight $\frac{1}{4}\frac{1}{3}\frac{1}{3}\frac{1}{2} = \frac{1}{72}$; for comparison the self-avoiding walk assigns to the same path $c_4^{-1} = \frac{1}{100}$. It is worth noting that the weights associated to the myopic self-avoiding walk are not symmetric with respect to time-reversal: in two dimensions the walk ENWN has weight $\frac{1}{72}$ whereas the time-reversed walk SESW has weight $\frac{1}{108}$.

The above description defines a walk which is prohibited from stepping to neighbours which were visited most often in the past. A less restrictive

[1] There is a slight difference between the MSAW algorithm and the model treated in this section, as the former assigns nonzero weight only to self-avoiding walks, unlike the latter.

self-avoidance constraint would merely discourage such steps. This leads us to consider a nearest-neighbour walk, starting at the origin, with transition probabilities

$$P(\omega(n+1) = x + a | \omega(n) = x) = \frac{e^{-\lambda N_{x+a}}}{\sum_{b:|b|=1} e^{-\lambda N_{x+b}}} \qquad (10.4.1)$$

where $|a| = 1$, $\lambda \geq 0$ represents the strength of the repulsion, and N_u denotes the number of visits to the site u up to time n. The case $\lambda = \infty$ then corresponds to the prohibitive model introduced in the previous paragraph.

There are as yet no rigorous results concerning the critical behaviour of the myopic self-avoiding walk. However the nonrigorous results indicate that the myopic self-avoiding walk behaves quite differently from the self-avoiding walk. Both field theoretic methods [Amit, Parisi and Peliti (1983)] and calculations related to those leading to the Flory exponents for the self-avoiding walk [Pietronero (1983)] point to an upper critical dimension of two. The diffusive behaviour $\nu = 1/2$ is expected above two dimensions, logarithmic corrections to diffusive behaviour are obtained in two dimensions, and the exponent $\nu = 2/3$ is found in one dimension. The claim that $\nu = 2/3$ in one dimension clearly does not apply when $\lambda = \infty$, for which the myopic walk behaves ballistically and $\nu = 1$. This indicates that the $\lambda = \infty$ walk belongs to a different universality class than the finite λ version; however the nonrigorous results appear to claim that the upper critical dimension is two for all $\lambda \leq \infty$. A survey, with references to numerical calculations, is given in Peliti and Pietronero (1987).