Chapter 1

Introduction

1.1 The basic questions

Imagine that you are standing at an intersection in the centre of a large city whose streets are laid out in a square grid. You choose a street at random and begin walking away from your starting point, and at each intersection you reach you choose to continue straight ahead or to turn left or right. There is only one rule: you must not return to any intersection already visited in your journey. In other words, your path should be self-avoiding. It is possible that you will lead yourself into a trap, reaching an intersection whose neighbours have all been visited already, but barring this disaster you continue walking until you have walked some large number N of blocks. There are two basic questions:

- How many possible paths could you have followed?
- Assuming that any one path is just as likely as any other, how far will you be on the average from your starting point?

These questions are straightforward enough, but the answers are only known for small values of N. It is widely accepted that a search for general exact formulas is an enormously difficult problem which lies beyond the reach of current methods. A less difficult question would be to ask for the asymptotic behaviour of the answers as N becomes very large, but this too is very hard. Physicists and chemists who are interested in this and related problems have applied a variety of methods and have produced many intriguing results, but a great deal of work is still needed to settle these issues in a mathematically rigorous way. In this book we will state some of the

N. Madras and G. Slade, *The Self-Avoiding Walk*,
Modern Birkhäuser Classics, DOI 10.1007/978-1-4614-6025-1_1,

results of nonrigorous work in the field, and describe the rigorous work in some detail.

At first glance one might expect that the easiest way to answer the above questions, at least approximately, would be to use a computer. Much numerical work has been done in this direction, and in Chapter 9 some of it will be discussed. Here too, however, the situation is not so easy: exact enumeration of all possible routes has been done to date only for $N \leq 34$, with further enumerations made difficult because of the exponential growth in the number of paths as N increases. Larger values of N can be studied by extrapolation of the exact enumeration data, or by Monte Carlo simulations.

There is no need to restrict the walk to a two-dimensional grid, and it is easy to generalize the above questions to general dimension d. It is also possible to generalize the problem by changing from a rectangular to a triangular or other type of grid. There is at least one case where the above questions can be easily answered, and this is the case of a one-dimensional walk. A self-avoiding walker in one dimension has no alternative but to continue travelling in the direction initially chosen, so there are exactly two paths for every value of N and the distance travelled is exactly N blocks. That was easy, but not very interesting. Higher dimensions provide a vastly richer structure.

In general, a self-avoiding walk takes place on a graph. A graph (more precisely, an undirected graph) is a collection of points, together with a collection of pairs of points known as *edges.* The basic example that will concern us most is the *d-dimensional hypercubic lattice* $\mathbf{Z}^d$. The points of this graph are the points of the d-dimensional Euclidean space $\mathbf{R}^d$ whose components are all integers, and the edges are given by the set of all unit line segments joining neighbouring points. The points will be referred to as *sites*, and the unit line segments as *nearest-neighbour bonds.* Sites will typically be denoted by letters such as u, v, x, y, and their components by subscripts: $x = (x_1, x_2, \ldots, x_d)$. The usual Euclidean dot product on $\mathbf{Z}^d$ will be written $x \cdot y = \sum_{i=1}^d x_i y_i$, and the Euclidean norm will be written $|x| = \sqrt{x \cdot x}$. We will also use the notation $||x||_p = (\sum_{i=1}^d x_i^p)^{1/p}$, and $||x||_\infty = \max\{|x_i|: i = 1, \ldots, d\}$.

An N-step self-avoiding walk ω on $\mathbf{Z}^d$, beginning at the site x, is defined as a sequence of sites $(\omega(0), \omega(1), \ldots, \omega(N))$ with $\omega(0) = x$, satisfying $|\omega(j+1) - \omega(j)| = 1$, and $\omega(i) \neq \omega(j)$ for all $i \neq j$. We write $|\omega| = N$ to denote the length of ω, and we denote the components of $\omega(j)$ by $\omega_i(j)$ $(i = 1, \ldots, d)$. Let c_N denote the number of N-step self-avoiding walks beginning at the origin. By convention, $c_0 = 1$. Then the first of our basic questions above is asking for the value of c_N. More modestly, we could ask

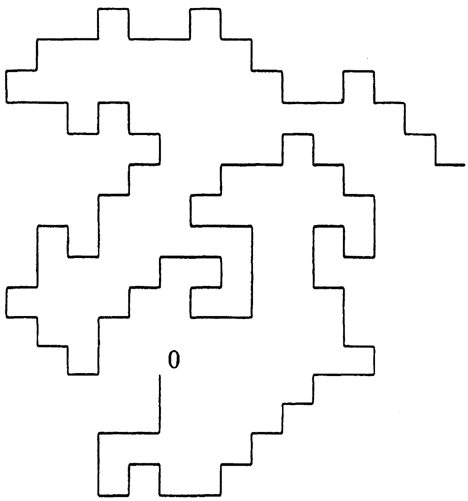


Figure 1.1: A two-dimensional self-avoiding walk with 115 steps.

for the asymptotic form of c_N as $N \to \infty$. It is easy to find the exact values of c_N (as a function of d) for very small values of N, for example $c_1 = 2d$, $c_2 = 2d(2d-1)$, $c_3 = 2d(2d-1)^2$, and $c_4 = 2d(2d-1)^3 - 2d(2d-2)$ (for c_4 the second term subtracts the contribution of squares to the first term). However, the combinatorics quickly become difficult as N increases and then soon become intractable. Tables in Appendix C give enumerations of c_N for dimensions two through six.

The simplest bounds on the behaviour of c_N are obtained as follows. An upper bound on c_N is given by the number of walks which have no immediate reversals, or in other words which never visit the same site at times i and $i+2$. Avoiding immediate reversals allows $2d$ choices for the initial step, and $2d-1$ choices for the $N-1$ remaining steps, for a total of $2d(2d-1)^{N-1}$. For a lower bound we simply count the number of walks in which each step is in one of the d positive coordinate directions. Such walks are necessarily self-avoiding. Thus we have

$$d^N \leq c_N \leq 2d(2d-1)^{N-1}. \qquad (1.1.1)$$

To discuss the average distance from the origin after N steps, we need to introduce a probability measure on N-step self-avoiding walks. The measure that we shall use throughout this book is the uniform measure, which assigns equal weight c_N^{-1} to each N-step self-avoiding walk. It is worth noting that although we originally introduced the self-avoiding walk in terms of a walker moving in time, the uniform measure is a measure on paths of length N and does not define a stochastic process evolving in time (for example, a walk may be trapped and impossible to extend without introducing a self-intersection).

Denoting expectation with respect to the uniform measure by angular brackets, the average distance (squared) from the origin after N steps is then given by the *mean-square displacement*

$$\langle |\omega(N)|^2 \rangle = \frac{1}{c_N} \sum_{\omega:|\omega|=N} |\omega(N)|^2. \tag{1.1.2}$$

The sum over ω is the sum over all N-step self-avoiding walks beginning at the origin. Like c_N, the mean-square displacement can also be calculated by hand for very small values of N, but the combinatorics quickly become intractable as N increases. Enumerations are tabulated in Appendix C.

It is instructive to compare the behaviour of the self-avoiding walk with that of the simple random walk. An N-step simple random walk on $\mathbf{Z}^d$, starting at the origin, is a sequence $\omega = (\omega(0), \omega(1), \ldots, \omega(N))$ of sites with $\omega(0) = 0$ and $|\omega(j+1) - \omega(j)| = 1$, with the uniform measure on the set of all such walks. Without the self-avoidance constraint the situation is rather easy. Indeed, since each site has $2d$ nearest neighbours, the number of N-step simple random walks is exactly $(2d)^N$. To analyse the mean-square displacement, we represent the simple random walk in the following way. Let $\{X^{(i)}\}$ be independent and identically distributed random variables with $X^{(i)}$ uniformly distributed over the $2d$ (positive and negative) unit vectors. Then the position after N steps can be represented as the sum $S_N = X^{(1)} + X^{(2)} + \ldots + X^{(N)}$. Expanding $|S_N|^2$, the mean-square displacement is given by

$$\langle |S_N|^2 \rangle = \sum_{i,j=1}^{N} \langle X^{(i)} \cdot X^{(j)} \rangle. \tag{1.1.3}$$

For $i \neq j$, $\langle X^{(i)} . X^{(j)} \rangle = 0$, using independence and the fact that $\langle X^{(i)} \rangle = 0$. Since $\langle X^{(i)} . X^{(i)} \rangle = 1$, it follows that the mean-square displacement is equal to N. Similarly, if we consider a random walk in $\mathbf{Z}^d$ in which steps lie in a symmetric finite set $\Omega \subset \mathbf{Z}^d$ of cardinality $|\Omega|$, with each possible step equally likely, then the number of N-step walks is $|\Omega|^N$ and the mean-square displacement is $N\sigma^2$, where σ^2 is the mean-square displacement of a single step.

For the self-avoiding walk it is believed that there is exponential growth of c_N with power law corrections, unlike the pure exponential growth of the simple random walk. It is also believed that the mean-square displacement will not always be linear in the number of steps, in contrast to the diffusive behaviour of the simple random walk. These beliefs are in harmony with known properties of other models of statistical mechanics, and are supported by numerical and nonrigorous calculations. The conjectured behaviour of c_N and $\langle|\omega(N)|^2\rangle$ is thus

$$c_N \sim A\mu^N N^{\gamma-1} \tag{1.1.4}$$

and

$$\langle|\omega(N)|^2\rangle \sim DN^{2\nu}, \tag{1.1.5}$$

where A, D, μ, γ and ν are dimension-dependent positive constants. We shall refer to μ as the *connective constant*, and γ and ν are examples of *critical exponents*. In four dimensions the above two relations should be modified by logarithmic factors; see (1.1.13) and (1.1.14) below. Here $f(N) \sim g(N)$ means that f is asymptotic to g as $N \to \infty$:

$$\lim_{N\to\infty} \frac{f(N)}{g(N)} = 1.$$

For ordinary random walk (1.1.4) and (1.1.5) hold with $\gamma = 1$ and $\nu = 1/2$, both for the nearest-neighbour and more general walks.

In the next section the existence of the limit

$$\mu = \lim_{N\to\infty} c_N^{1/N} \tag{1.1.6}$$

will be proven, which is the first step in justifying (1.1.4). The simple bounds of (1.1.1) then immediately imply that

$$d \le \mu \le 2d - 1. \tag{1.1.7}$$

The exact value of μ is not known for the hypercubic lattice in any dimension $d \ge 2$, although for the honeycomb lattice in two dimensions there is nonrigorous evidence that $\mu = \sqrt{2+\sqrt{2}}$. Improvements to (1.1.7) will be discussed in the next section. For high dimensions it is known that as $d \to \infty$

$$\mu = 2d - 1 - \frac{1}{2d} - \frac{3}{(2d)^2} + O\left(\frac{1}{(2d)^3}\right); \tag{1.1.8}$$

references are given in the Notes. In fact Fisher and Sykes (1959) established the coefficients in the $1/d$ expansion up to and including order d^{-4}, although there is no rigorous control of their error term. Intuitively (1.1.8) says that in high dimensions the principal effect of the self-avoidance constraint is to rule out immediate reversals.

Concerning γ, we will show in Section 1.2 that $c_N \geq \mu^N$ and hence $\gamma \geq 1$ in all dimensions. There is still no proof, however, that γ is finite in two, three or four dimensions, where the best bounds are

$$c_N \leq \begin{cases} \mu^N \exp[KN^{1/2}] & d = 2 \\ \mu^N \exp[KN^{2/(2+d)} \log N] & d = 3, 4 \end{cases} \tag{1.1.9}$$

for a positive constant K; these bounds will be discussed in Sections 3.1 and 3.3. In Chapter 6 we will describe a proof that (1.1.4) holds with $\gamma = 1$ for $d \geq 5$. In addition to characterizing the asymptotic behaviour of c_N, the exponent γ provides a measure of the probability that two N-step self-avoiding walks starting at the same point do not intersect. In fact, this probability is equal to c_{2N}/c_N^2, and assuming (1.1.4) we have

$$\frac{c_{2N}}{c_N^2} \sim \frac{2^{\gamma-1}}{A} N^{1-\gamma}. \tag{1.1.10}$$

If $\gamma > 1$ then this probability goes to zero as $N \to \infty$, while if $\gamma = 1$ it remains positive. For the simple random walk the analogous probability is known to remain positive as $N \to \infty$ for $d > 4$, and roughly speaking to go to zero like $(\log N)^{-1/2}$ for $d = 4$ and as an inverse power of N for $d = 2, 3$. A survey of the simple random walk results is given in Section 10.3.

Intuitively it is to be expected that the repulsive interaction of the self-avoiding walk will tend to drive the endpoint of the walk away from the origin faster than for simple random walk, or in other words that $\nu \geq 1/2$. However it is still an open question to prove that this "obvious" inequality $\langle |\omega(N)|^2 \rangle \geq CN$ holds in all dimensions. On the other hand, bounding $|\omega(N)|^2$ above by N^2 in (1.1.2) gives the upper bound $\langle |\omega(N)|^2 \rangle \leq N^2$, or $\nu \leq 1$. This bound is optimal in one dimension, but seems far from optimal in two or more dimensions. No upper bound of the form $CN^{2-\epsilon}$ $(C, \epsilon > 0)$, or in other words $\nu < 1$, has been proven for dimensions two, three or four, however. For $d \geq 5$ it has been proved that $\nu = 1/2$; this proof will be described in Chapter 6. It will also be shown that for high dimensions the diffusion constant D is strictly greater than the simple random walk value of 1. Thus in high dimensions the self-avoiding walk does move away from the origin more quickly than the simple random walk, but only at the level of the diffusion constant and not at the level of the exponent ν. The tendency of the self-avoiding walk to move away from the origin more quickly than the simple random walk should become less pronounced as the dimension increases, and hence it is to be expected that ν is a nonincreasing function of the dimension.

The critical exponents γ and ν are believed to be dimension dependent, but independent of the type of allowed steps (as long as there are only finitely many possible steps and the allowed steps are symmetric) or even of the type of lattice—the exponents are believed, for example, to be the same for the square and triangular lattices. This lack of dependence on the detailed definition of the model is known as ***universality***, and models with the same exponents are said to be in the same ***universality class***. The ***connective constant*** μ appearing in (1.1.4) represents the effective coordination number of the lattice and is not universal—it depends on the details of the allowed steps and the underlying lattice, as well as the dimension d.

It seems clear that in high dimensions the self-avoiding walk should be closer to the simple random walk than in low dimensions, since a simple random walk is less likely to intersect itself in high dimensions. Four dimensions plays a special role: for simple random walk the expected time of the first return to the origin, conditioned on the event that this return occurs, is finite for $d > 4$; this suggests that above four dimensions self-avoidance is a short-range effect rather than a long-range one, and hence that it will not affect the critical exponents. In addition, as mentioned above, the probability that two independent simple random walks of length N do not intersect remains bounded away from zero as $N \to \infty$ for $d > 4$, but not for $d \leq 4$.

The conjectured values of γ and ν are as follows:

$$\gamma = \begin{cases} \frac{43}{32} & d = 2 \\ 1.162... & d = 3 \\ 1 \text{ with logarithmic corrections} & d = 4 \\ 1 & d \geq 5 \end{cases} \tag{1.1.11}$$

$$\nu = \begin{cases} \frac{3}{4} & d = 2 \\ 0.59... & d = 3 \\ \frac{1}{2} \text{ with logarithmic corrections} & d = 4 \\ \frac{1}{2} & d \geq 5 \end{cases} \tag{1.1.12}$$

Currently the only rigorous results which prove power law behaviour and confirm the conjectured values of γ and ν are for $d \geq 5$. These are discussed in detail in Chapter 6. The conjectured logarithmic corrections to γ and ν in four dimensions, predicted by the renormalization group, are given by:

$$c_N \sim A\mu^N[\log N]^{1/4}, \quad d = 4 \tag{1.1.13}$$

$$\langle|\omega(N)|^2\rangle \sim DN[\log N]^{1/4}, \quad d = 4. \tag{1.1.14}$$

Equations (1.1.11) to (1.1.14) are typical of what is found for other statistical mechanical models, such as the Ising model or percolation. A common feature is the existence of a certain dimension, the so-called *upper critical* *dimension*, at which there are logarithmic corrections to critical exponents and above which all critical exponents are dimension independent and are given by the corresponding critical exponents for a simpler model, known as the *mean-field*[1] model. For the self-avoiding walk the mean-field model is the simple random walk and the simple random walk critical exponents are sometimes referred to as the mean-field exponents.

The rational values for two dimensions given in (1.1.11) and (1.1.12) come from a nonrigorous exact solution of the $O(N)$ spin model which includes the self-avoiding walk as the special case $N = 0$ (see Section 2.3). This remarkable work exploits a connection between the $O(N)$ model and the Coulomb gas and uses the renormalization group. From a different approach, nonrigorous conformal invariance arguments reproduce the same rational values. There is no analogous exact solution in three dimensions, and the $d = 3$ values given in (1.1.11) and (1.1.12) are from numerical results and field-theoretic calculations using the ϵ-expansion. References for these topics are given in the Notes.

An early conjecture for the values of ν was made by Flory, and will be discussed in Section 2.2. The Flory exponents are given by $\nu_{Flory} = 3/(2+d)$ for $d \leq 4$ and $\nu_{Flory} = 1/2$ for $d > 4$. This agrees with Equation (1.1.12) for $d = 2$ and $d \geq 4$ (apart from the logarithmic correction when $d = 4$), and comes very close for $d = 3$. The exact Flory value $\nu_{Flory} = 3/5$ in three dimensions has been ruled out by numerical work, however.

1.2 The connective constant

If (1.1.4) correctly represents the behaviour of c_N for large N, then the limit

$$\mu = \lim_{N\to\infty} c_N^{1/N} \tag{1.2.1}$$

must exist. One purpose of this section is to prove the existence of this limit as a simple consequence of a subadditive property of $\log c_N$. It then follows immediately from (1.1.1) that

$$d \leq \mu \leq 2d - 1. \tag{1.2.2}$$

The proof involves the notion of concatenation of two self-avoiding walks.

[1]This terminology has its origin in the Ising model. For the Ising model the upper critical dimension is also four, and above four dimensions critical exponents are given by the exactly solvable model in which a spin interacts with the *average* of all the other spins. References are given in the Notes.

Definition 1.2.1 *The* concatenation $\omega^{(1)} \circ \omega^{(2)}$ *of an M-step self-avoiding walk $\omega^{(2)}$ to an N-step self-avoiding walk $\omega^{(1)}$ is the $(N+M)$-step walk ω, which in general need not be self-avoiding, given by*

$$\begin{aligned} \omega(k) &= \omega^{(1)}(k), & k &= 0, \ldots, N \\ \omega(k) &= \omega^{(1)}(N) + \omega^{(2)}(k-N) - \omega^{(2)}(0), & k &= N+1, \ldots, N+M. \end{aligned}$$

The product $c_N c_M$ is equal to the cardinality of the set of $(N+M)$-step simple random walks which are self-avoiding for the initial N steps and the final M steps, but which may not be completely self-avoiding. This can be seen by concatenations of M-step walks to N-step walks, and implies that

$$c_{N+M} \leq c_N c_M. \tag{1.2.3}$$

In fact equality holds in (1.2.3) only if N or M is zero, since otherwise there will be at least one M-step walk whose concatenation with a given N-step walk fails to be self-avoiding. Taking logarithms in (1.2.3) shows that the sequence $\{\log c_n\}$ is *subadditive:*

$$\log c_{N+M} \leq \log c_N + \log c_M. \tag{1.2.4}$$

The existence of the limit (1.2.1) is a consequence of (1.2.4) and the following standard result; this was first observed by Hammersley and Morton (1954).

Lemma 1.2.2 *Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers which is subadditive, i.e., $a_{n+m} \leq a_n + a_m$. Then the limit $\lim_{n\to\infty} n^{-1} a_n$ exists in $[-\infty, \infty)$ and is equal to*

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\geq 1} \frac{a_n}{n}. \tag{1.2.5}$$

Proof. It suffices to show that

$$\limsup_{n\to\infty} \frac{a_n}{n} \leq \frac{a_k}{k} \tag{1.2.6}$$

for every k, since taking the $\liminf_{k\to\infty}$ in (1.2.6) gives existence of the limit, and then (1.2.5) can be seen by taking the $\inf_{k\geq 1}$ in (1.2.6).

To prove (1.2.6), we fix k and let

$$A_k = \max_{1\leq r\leq k} a_r. \tag{1.2.7}$$

Given a positive integer n we let j denote the largest integer which is strictly less than n/k. Then $n = jk + r$ for some integer r with $1 \leq r \leq k$. Using subadditivity, we have

$$a_n \leq j a_k + a_r \leq \frac{n}{k} a_k + A_k. \tag{1.2.8}$$

Dividing by n and taking the $\limsup_{n\to\infty}$ then gives (1.2.6).

Equation (1.2.5) shows that $\lim_{n\to\infty} n^{-1}a_n < \infty$. In general, the possibility that the limit equals $-\infty$ cannot be excluded, as is illustrated by the example of $a_n = -n^2$. For many applications, however, this is ruled out by an *a priori* bound such as $a_n \geq 0$. □

Together with (1.2.4), Lemma 1.2.2 implies the existence of the limit $\log\mu \equiv \lim_{N\to\infty} N^{-1}\log c_N$, and hence gives (1.2.1). In fact (1.2.5) shows more:

$$\log\mu = \inf_{N\geq 1} N^{-1}\log c_N, \tag{1.2.9}$$

and hence

$$\mu^N \leq c_N, \quad N \geq 1. \tag{1.2.10}$$

This inequality can be summarized by the statement $\gamma \geq 1$, where γ is as introduced in (1.1.4), although strictly speaking we do not know that γ exists. Equation (1.2.10) also yields $\mu \leq c_N^{1/N}$. This gives a sequence of upper bounds for μ, but they converge to μ very slowly. A better bound is

$$\mu \leq \left(\frac{c_N}{c_1}\right)^{1/(N-1)}, \quad N \geq 2. \tag{1.2.11}$$

References for this and other improvements are given in the Notes.

Another sequence of upper bounds for μ can be obtained by considering walks which are self-avoiding only over a finite time scale or *memory* τ. We define $c_{N,\tau}$ to be the number of N-step walks ω beginning at the origin, for which $\omega(i) \neq \omega(j)$ whenever $0 < |i-j| \leq \tau$. Self-intersections occurring after an interval of more than τ steps are permitted. For example, $c_{N,2} = 2d(2d-1)^{N-1}$ for $N \geq 1$, since memory $\tau = 2$ simply rules out immediate reversals. For $\tau \geq N$, $c_{N,\tau} = c_N$. Memory $\tau = 0$ corresponds to the simple random walk.

The sequence $\{\log c_{N,\tau}\}_{N=1}^{\infty}$ is subadditive for every τ (for the same reason that $\{\log c_N\}_{N=1}^{\infty}$ is), and hence by Lemma 1.2.2 there is a μ_τ such that

$$\mu_\tau = \lim_{N\to\infty} c_{N,\tau}^{1/N} = \inf_{N\geq 1} c_{N,\tau}^{1/N}. \tag{1.2.12}$$

Since $c_{N,\tau} \geq c_N$, μ_τ provides an upper bound for μ. The next lemma shows that this sequence of upper bounds converges monotonically to μ.

Lemma 1.2.3 *$\mu_\tau \searrow \mu$ as $\tau \to \infty$.*

Proof. For $\sigma \leq \tau$, $c_{N,\sigma} \geq c_{N,\tau}$ and hence $\mu_\sigma \geq \mu_\tau$. By (1.2.12), $\mu_\tau \leq c_{N,\tau}^{1/N}$ for all N, τ. Taking $N = \tau$ gives

$$\mu \leq \mu_\tau \leq c_{\tau,\tau}^{1/\tau} = c_\tau^{1/\tau}. \tag{1.2.13}$$

Taking the limit $\tau \to \infty$ and using (1.2.1) gives the desired result. □

The connective constant for the walk with memory $\tau = 4$ was shown in Fisher and Sykes (1959) to be given by the largest root of the cubic equation

$$\theta^3 - 2(d-1)\theta^2 - 2(d-1)\theta - 1 = 0. \tag{1.2.14}$$

For $d = 2$ this gives $\mu_4(2) = 2.8312$, where we have made the dimension dependence explicit by writing $\mu_\tau(d)$.

A number of investigations into the self-avoiding walk have approached the problem via the limit of finite memory walks as the memory goes to infinity. This approach was used in particular by Brydges and Spencer (1985) in applying their lace expansion to study weakly self-avoiding walk for $d > 4$, and will be adopted in Section 6.8 to obtain an upper bound in high dimensions on $c_N(0, x)$, the number of N-step self-avoiding walks which begin at the origin and end at x.

A lower bound on μ can be obtained in terms of *bridges*.

Definition 1.2.4 *An N-step* bridge *is defined to be an N-step self-avoiding walk ω whose first components satisfy the inequality*

$$\omega_1(0) < \omega_1(i) \leq \omega_1(N)$$

for $1 \leq i \leq N$. The number of N-step bridges starting at the origin is denoted b_N. By convention, $b_0 = 1$.

The concatenation of two bridges will always yield another bridge, so

$$b_M b_N \leq b_{M+N}. \tag{1.2.15}$$

Hence $\{-\log b_n\}$ is subadditive and so by Lemma 1.2.2 the limit

$$\mu_{Bridge} \equiv \lim_{n\to\infty} b_n^{1/n} = \sup_{n\geq 1} b_n^{1/n} \tag{1.2.16}$$

exists. Clearly $b_n \leq c_n$. Therefore $\mu_{Bridge} \leq \mu$, and so by (1.2.16)

$$b_N^{1/N} \leq \mu_{Bridge} \leq \mu. \tag{1.2.17}$$

In Section 3.1 it will be shown that in fact $\mu_{Bridge} = \mu$. Although the lower bound (1.2.17) is very slowly convergent, a more sophisticated use of bridges leads to better lower bounds. References can be found in the Notes at the end of this chapter.

We conclude this section with a table showing the current best rigorous upper and lower bounds on μ, together with estimates of the precise value, for the hypercubic lattice in dimensions $d = 2, 3, 4, 5, 6$.

d	lower bound	estimate	upper bound
2	2.61987^a	$2.6381585 \pm 0.0000010^d$	2.69576^b
3	4.43733^c	4.6839066 ± 0.0002^e	4.756^b
4	6.71800^c	6.7720 ± 0.0005^f	6.832^b
5	8.82128^c	8.83861^g	8.881^b
6	10.871199^c	10.87879^g	10.903^b

Table 1.1: Current best rigorous upper and lower bounds on the hypercubic lattice connective constant μ, together with estimates of actual values.
a) Conway and Guttmann (to be published), b) Alm (1992), c) Hara and Slade (1992b), d) Guttmann and Enting (1988), e) Guttmann (1987), f) Guttmann (1978), g) Guttmann (1981).

1.3 Generating functions

A common tool for understanding the behaviour of a sequence is its generating function. The generating function of the sequence $\{c_N\}$ is defined by

$$\chi(z) = \sum_{N=0}^{\infty} c_N z^N = \sum_{\omega} z^{|\omega|}. \tag{1.3.1}$$

The sum over ω is the sum over all self-avoiding walks, of arbitrary length $|\omega|$, which begin at the origin. The parameter z is known as the *activity*. Physically the activity occurs in the study of a canonical ensemble of polymers of variable length, and in this context is nonnegative. From a mathematical point of view, however, it will sometimes be useful to consider χ to be an analytic function of complex z.

Given two sites x and y, let $c_N(x,y)$ be the number of N-step self-avoiding walks ω with $\omega(0) = x$ and $\omega(N) = y$. The *two-point function* is the generating function for the sequence $c_N(x,y)$, i.e.,

$$G_z(x,y) = \sum_{N=0}^{\infty} c_N(x,y) z^N = \sum_{\omega : x \to y} z^{|\omega|}. \tag{1.3.2}$$

On the right side, the sum over ω is the sum over all self-avoiding walks, of arbitrary length, which begin at x and end at y. This is clearly translation invariant, so $G_z(x,y) = G_z(0, y-x)$. The two-point function is the self-avoiding walk analogue of the simple random walk Green function with killing rate $1 - 2dz$:

$$C_z(x,y) = \sum_{N=0}^{\infty} p_N(x,y)(2dz)^N, \tag{1.3.3}$$

where $p_N(x,y)$ is the probability that an N-step simple random walk beginning at x ends at y.

The generating function for c_N can be written in terms of the two-point function as

$$\chi(z) = \sum_{x \in Z^d} G_z(0,x). \tag{1.3.4}$$

In analogy with spin systems (see Section 2.3) we will refer to the generating function $\chi(z)$ as the *susceptibility*. The power series defining the susceptibility has radius of convergence

$$z_c \equiv \left[\lim_{N\to\infty} c_N^{1/N}\right]^{-1} = \frac{1}{\mu}, \tag{1.3.5}$$

and hence defines an analytic function in the *complex* parameter z if $|z| < z_c$. Since $c_N(0,x) \leq c_N$, the two-point function has radius of convergence at least z_c. It will be shown in Section 3.2 that in fact the radius of convergence is equal to z_c, for all $x \neq 0$. We will refer to z_c as the *critical point*, since it plays a role analogous to the critical point in statistical mechanical systems such as the Ising model or percolation.

It follows from (1.2.10) that

$$\chi(z) \geq \sum_{N=0}^{\infty} (\mu z)^N = \frac{1}{1-\mu z} \tag{1.3.6}$$

and hence χ is "continuous" at the critical point, in the sense that $\chi(z) \to \infty$ as $z \nearrow z_c$. The manner of divergence of $\chi(z)$ at the critical point is related to the behaviour of the coefficients c_N for large N. To see this, we proceed as follows.

First we introduce the notation

$$f(x) \simeq g(x) \quad \text{as } x \to x_0 \tag{1.3.7}$$

to mean that there are positive constants C_1 and C_2 such that

$$C_1 g(x) \leq f(x) \leq C_2 g(x) \tag{1.3.8}$$

uniformly for x near its limiting value. Assuming that there is a γ such that

$$c_N \simeq \mu^N N^{\gamma-1} \quad \text{as } N \to \infty, \tag{1.3.9}$$

it can be concluded that

$$\chi(z) \simeq (z_c - z)^{-\gamma} \quad \text{as } z \nearrow z_c, \tag{1.3.10}$$

as follows. We write $z = \mu^{-1}e^{-t}$, so that $t \simeq z_c - z$. By the definition of $\chi(z)$,

$$\chi(z) \simeq \sum_{N=1}^{\infty} N^{\gamma-1}e^{-tN} \simeq \int_1^{\infty} x^{\gamma-1}e^{-tx}dx$$
$$= t^{-\gamma}\int_t^{\infty} y^{\gamma-1}e^{-y}dy \simeq t^{-\gamma}.$$

In the above the sum can be replaced by the integral using Riemann sum approximations. The second integral converges as $t \searrow 0$, since by (1.2.10) $\gamma \geq 1$. Thus it is conjectured that

$$\chi(z) \sim A'(z_c - z)^{-\bar{\gamma}} \quad \text{as } z \nearrow z_c, \tag{1.3.11}$$

with $\bar{\gamma} = \gamma$.

As for the converse, it does not follow directly from (1.3.10) that (1.3.9) holds, without further assumptions. In general, the problem of extracting the large-n asymptotics of a sequence from the manner of divergence of its generating function is a Tauberian problem. An example of a Tauberian theorem providing a converse to the above argument will be given in Lemma 6.3.4.

Power law behaviour such as (1.3.10) is also observed for spin systems and percolation, and is characteristic of critical phenomena. It follows from (1.3.6) that $\bar{\gamma} \geq 1$, assuming that $\bar{\gamma}$ exists. In four dimensions, where it is believed that $c_N \sim A\mu^N(\log N)^{1/4}$, we expect similarly that $\chi(z) \sim A'(z_c - z)^{-1}|\log(z_c - z)|^{1/4}$.

The analogue $\chi_0(z)$ of $\chi(z)$ for simple random walk can be calculated explicitly:

$$\chi_0(z) = \sum_{N=0}^{\infty}(2dz)^N = \frac{1}{1-2dz}.$$

Thus the mean-field value of $\bar{\gamma}$ is 1, which not surprisingly is equal to the mean-field value for γ. The inequality $\bar{\gamma} \geq 1$ is an example of a *mean-field bound.* There is a sufficient condition for the opposite bound $\bar{\gamma} \leq 1$ known as the bubble condition, which is known to hold for $d \geq 5$ (and is believed not to hold for $d \leq 4$), and which will be discussed in detail in Section 1.5. There are many examples in critical phenomena of rigorous mean-field bounds, but, as mentioned in Section 1.1, no general proof is known of the mean-field bound $\nu \geq 1/2$.

We now turn our attention to the long distance behaviour of the two-point function. Below the critical point the two-point function decays exponentially. To see this, we note that $c_N(0,x) = 0$ for $N < \|x\|_\infty$, and hence

$$G_z(0,x) = \sum_{N=\|x\|_\infty}^{\infty} c_N(0,x) z^N \le \sum_{N=\|x\|_\infty}^{\infty} c_N z^N. \tag{1.3.12}$$

Since $c_N^{1/N} \to \mu$ by (1.2.1), for any $\epsilon > 0$ there is a positive K_ϵ such that

$$c_N \le K_\epsilon(\mu+\epsilon)^N \tag{1.3.13}$$

for all $N \ge 1$. Given a positive $z < z_c = \mu^{-1}$, we choose $\epsilon(z) > 0$ such that $\theta_z \equiv (\mu + \epsilon(z))z < 1$. Then substitution of (1.3.13) into (1.3.12) gives

$$G_z(0,x) \le C_z \exp[-|\log\theta_z|\,\|x\|_\infty], \tag{1.3.14}$$

with $C_z = K_{\epsilon(z)}(1-\theta_z)^{-1}$. This shows the desired exponential decay of the subcritical two-point function.

We define the *mass* $m(z)$ to be the rate of exponential decay of the two-point function along a coordinate axis:

$$m(z) = \liminf_{n\to\infty} \frac{-\log G_z(0,(n,0,\ldots,0))}{n}. \tag{1.3.15}$$

In Theorem 4.1.3 it will be shown that in fact the lim inf in the definition of m can be replaced by the limit. The *correlation length* $\xi(z)$, defined by $\xi(z) = m(z)^{-1}$, provides a characteristic length scale for the model.

The mass $m(z)$ is clearly not infinite for $0 < z < z_c$: considering only the shortest self-avoiding walk from 0 to $(n,0,\ldots,0)$ gives

$$G_z(0,(n,0,\ldots,0)) \ge z^n, \tag{1.3.16}$$

and hence $m(z) \le -\log z$. By (1.3.14), $m(z) > 0$ for $z \in (0, z_c)$. By definition the mass is a nonincreasing function of positive z, and in Section 4.1 it will be shown that $m(z) \searrow 0$ as $z \nearrow z_c$. Given that the radius of convergence of $G_z(0,x)$ is z_c for all $x \ne 0$, it follows that $m(z) = -\infty$ for $z > z_c$. It has been proven that $m(z_c) = 0$ for $d \ge 5$; see Corollary 6.1.7. Although this is believed to be true in all dimensions, a negative mass at the critical point has not yet been ruled out rigorously in dimensions 2, 3 or 4.

Since the mass $m(z)$ approaches zero as $z \nearrow z_c$, it follows that the correlation length $\xi(z) = m(z)^{-1}$ diverges as $z \nearrow z_c$. It is believed that the manner of divergence of $\xi(z)$ is via a power law of the form

$$\xi(z) \sim \text{const.}(z_c - z)^{-\nu} \quad \text{as } z \nearrow z_c. \tag{1.3.17}$$

Formal scaling theory predicts that $\bar{\nu} = \nu$; this will be discussed in Section 2.1. The equality of these two critical exponents is part of a general belief that all length scales for the self-avoiding walk should be governed by the same critical exponent. The same belief generally applies to other statistical mechanical models as well.

Another correlation length, ξ_p, known as the *correlation length of order p*, is defined for each $p > 0$ by

$$\xi_p(z) = \left[\frac{\sum_\omega |\omega(|\omega|)|^p z^{|\omega|}}{\sum_\omega z^{|\omega|}}\right]^{1/p} = \left[\frac{\sum_x |x|^p G_z(0,x)}{\sum_x G_z(0,x)}\right]^{1/p}. \tag{1.3.18}$$

By Hölder's inequality ξ_p is increasing in p. A formal argument similar to that showing $\bar{\nu} = \nu$ gives

$$\xi_p(z) \sim \text{const.}(z_c - z)^{-\nu_p} \text{ as } z \nearrow z_c,$$

with $\nu_p = \nu$ for all p.

For $p = 2$ there is no need to appeal to scaling theory to argue that $\nu_2 = \nu$. Instead we can argue as we did for the equality of γ and $\bar{\gamma}$, in the following way. We will assume that there exist exponents γ and ν such that $c_N \simeq \mu^N N^{\gamma-1}$ and $\langle|\omega|^2\rangle \simeq N^{2\nu}$, and show that this implies that $\xi_2(z) \simeq (z_c - z)^{-\nu}$. Given the assumptions, we have

$$\begin{aligned} \sum_x |x|^2 G_z(0,x) &= \sum_N z^N \sum_{\omega:|\omega|=N} |\omega(N)|^2 \qquad (1.3.19) \\ &\simeq \sum_N z^N N^{2\nu} c_N \simeq \sum_N z^N N^{2\nu+\gamma-1}\mu^N. \end{aligned}$$

Again writing $z = \mu^{-1}e^{-t}$, we obtain

$$\begin{aligned} \sum_x |x|^2 G_z(0,x) &\simeq \sum_N N^{2\nu+\gamma-1}e^{-tN} \\ &\simeq \int_1^\infty x^{2\nu+\gamma-1}e^{-tx}dx \simeq t^{-2\nu-\gamma}. \qquad (1.3.20) \end{aligned}$$

This implies that

$$\xi_2(z)^2 \simeq \frac{t^{-2\nu-\gamma}}{\chi(z)} \simeq t^{-2\nu-\gamma+\bar{\gamma}}. \tag{1.3.21}$$

Using $\bar{\gamma} = \gamma$ it follows that $\xi_2(z) \simeq (z_c - z)^{-\nu}$, so $\nu_2 = \nu$.

1.4 Critical exponents

So far we have introduced the five critical exponents $\gamma, \bar{\gamma}, \nu, \bar{\nu}, \nu_p$. It was shown in Section 1.3 that if γ exists then $\gamma = \bar{\gamma}$. Heuristic arguments that $\nu = \bar{\nu} = \nu_p$ (for $0 < p < \infty$) will be given in the Section 2.1. The exponents were defined as follows:

$$c_N \sim A\mu^N N^{\gamma-1} \quad (1.4.1)$$

$$\chi(z) \sim A'(z_c - z)^{-\bar{\gamma}} \quad (1.4.2)$$

$$\langle|\omega(N)|^2\rangle \sim DN^{2\nu} \quad (1.4.3)$$

$$\xi(z) \sim B(z_c - z)^{-\bar{\nu}} \quad (1.4.4)$$

$$\xi_p(z) \sim B_p(z_c - z)^{-\nu_p}. \quad (1.4.5)$$

We have written the above relations as if the various quantities involved are *asymptotically* given by power laws. This is consistent with the existing rigorous results, but some authors prefer a more conservative definition of the exponents. For example, one could require only that $c_N \simeq \mu^N N^{\gamma-1}$ [see (1.3.7)], with corresponding statements for the other exponents. A weaker definition, appearing sometimes in the literature, is to define the exponents by equations such as

$$\bar{\gamma} = - \lim_{z \nearrow z_c} \frac{\log \chi(z)}{\log(z_c - z)}, \quad (1.4.6)$$

but we will not need this definition. We shall take the optimistic view that the power law behaviour is asymptotic, although none of (1.4.1)–(1.4.5) has been proven in dimensions 2, 3, or 4 for any of these definitions of the exponents.

We will use the notation

$$f(x) \approx g(x) \quad (1.4.7)$$

in informal (nonrigorous) discussions to mean that $f(x)$ and $g(x)$ appear to have the same asymptotic behaviour in some sense which we will not attempt to specify.

In this section three additional critical exponents η, α_{sing}, and Δ_4 will be introduced. All these critical exponents are believed to be universal in the sense that they depend only on the dimension d of the lattice. In particular, the exponents are believed to be the same for the nearest-neighbour self-avoiding walk on $\mathbf{Z}^d$ as for a self-avoiding walk on $\mathbf{Z}^d$ in which steps can be within a fixed finite set $\Omega \subset \mathbf{Z}^d$ which is symmetric with respect to the symmetries of the lattice. Moreover the exponents ought even to be the same for a self-avoiding walk which can take unboundedly long steps, provided the weight of a step decays rapidly enough with its length (e.g., exponentially). This independence of the step set Ω is partially borne out in the rigorous results in high dimensions in Chapter 6.

We begin with the exponent η, which describes the conjectured long-distance behaviour of the two-point function at the critical point. Given that $m(z) \to 0$ as $z \nearrow z_c$, and given the belief that $m(z_c) = 0$ in all dimensions, it might be expected that the two-point function decays via a power law at the critical point. For simple random walk (with $d > 2$) the mass is certainly zero at the critical point, as it is well-known that the critical simple random walk two-point function $C_{1/2d}(0, x)$ decays like $|x|^{2-d}$ at large distances [see for example Lawler (1991)]. The conjectured large distance behaviour of the critical self-avoiding walk two-point function is

$$G_{z_c}(0, x) \sim \frac{C}{|x|^{d-2+\eta}} \quad \text{as } |x| \to \infty, \tag{1.4.8}$$

where C is a constant. This is believed to hold in all dimensions $d \geq 2$, including $d = 2$. Comparison with the simple random walk decay yields the mean field value 0 for η. Unfortunately it has not yet been proved rigorously that $G_{z_c}(0, x)$ is even finite for $d = 2, 3$ or 4, for any value of $x \neq 0$. For $d \geq 5$ somewhat weaker decay than (1.4.8) has been proved; see Theorem 6.1.6.

Assuming that (1.4.8) does provide the correct behaviour, it follows from the fact that the susceptibility is infinite at the critical point that $\eta \leq 2$. The value of η is believed to be determined from the values of γ and ν according to Fisher's scaling relation

$$\gamma = (2 - \eta)\nu. \tag{1.4.9}$$

The hypotheses leading to (1.4.9) will be discussed in Section 2.1. Inserting the conjectured values for γ and ν given in (1.1.11) and (1.1.12) into (1.4.9) gives the values for η appearing in Table 1.2. In contrast to γ and ν, the renormalization group predicts no logarithmic corrections to η in four dimensions. Logarithmic corrections are however expected in higher order terms in the asymptotic expansion of the critical two-point function in four dimensions.

One way to gain insight into the long distance behaviour of the critical two-point function is to examine the behaviour of its Fourier transform near the origin. In general, given a function $f(x)$ on the lattice whose absolute value is summable, we define its Fourier transform by

$$\hat{f}(k) = \sum_{x \in Z^d} f(x) e^{ik \cdot x}, \quad k \in [-\pi, \pi]^d. \tag{1.4.10}$$

d	2	3	≥ 4
γ	$\frac{43}{32}$	$1.162\ldots$	1
ν	$\frac{3}{4}$	$0.59\ldots$	$\frac{1}{2}$
η	$\frac{5}{24}$	$0.03\ldots$	0

Table 1.2: Conjectured values of γ, ν, η.

It is generally expected in critical phenomena that (1.4.8) is associated with behaviour of the form

$$\hat{G}_{z_c}(k) \sim \frac{C'}{k^{2-\eta}} \quad \text{as } k \to 0 \tag{1.4.11}$$

for some constant C'. [However (1.4.8) and (1.4.11) are not mathematically equivalent — an example of a function satisfying (1.4.11) but not (1.4.8) is given in the Notes at the end of the chapter.] Equation (1.4.11) has been established for $d \geq 5$ with $\eta = 0$ (see Theorem 6.1.6), but not yet for $d = 2, 3$ or 4. The conjectured values of η are all nonnegative. It is thus suggestive to conjecture the *infrared bound*

$$\hat{G}_z(k) \leq \frac{C}{k^2}, \tag{1.4.12}$$

with C independent of $k \in [-\pi, \pi]^d$ and $z \leq z_c$. For the nearest-neighbour Ising model and other reflection-positive spin systems the infrared bound is known rigorously to hold and was of considerable importance in the proof of mean-field behaviour of such models above four dimensions. For the self-avoiding walk it is still an open problem to prove the infrared bound in dimensions $2, 3$ or 4, but in higher dimensions it has been proved (see Theorem 6.1.6). It is worth noting that the infrared bound is believed by some to be false for percolation and lattice animals below dimensions six and eight respectively (see Section 5.5 for more details about these models).

The exponent α_{sing} describes the behaviour of the number $c_N(0, x)$ of N-step self-avoiding walks which begin at the origin and end at x, as $N \to \infty$ with x fixed. For x equal to a nearest-neighbour e of the origin, $c_N(0, e)$ is closely related to the number of self-avoiding polygons. Self-avoiding polygons will be studied in detail in Section 3.2. It will be shown in Corollary 3.2.6 that the leading asymptotic behaviour of $c_N(0, x)$ as $N \to \infty$ is μ^N. As is the case for c_N, this leading behaviour is believed to have a power law correction of the form

$$c_N(0,x) \sim B\mu^N N^{\alpha_{sing}-2}. \tag{1.4.13}$$

Here x is fixed and nonzero, and $N \to \infty$ through a sequence of values with the same parity as $||x||_1$. It is believed that α_{sing} is independent of x, and we formalize this conjecture for future reference as follows.

Conjecture 1.4.1 *For every pair of nonzero points x and y in $\mathbf{Z}^d$, there exist positive constants A_1 and A_2 and an integer N_0 (all depending on x and y) such that*

$$A_1 c_N(0,y) \le c_N(0,x) \le A_2 c_N(0,y) \qquad \text{for all } N \ge N_0$$

if $||x-y||_1$ is even, and

$$A_1 c_{N+1}(0,y) \le c_N(0,x) \le A_2 c_{N+1}(0,y) \qquad \text{for all } N \ge N_0$$

if $||x-y||_1$ is odd.

A special case of this conjecture is proven in Proposition 7.4.4. The value of B is also believed to be independent of x (as it is for simple random walk). For simple random walk the local central limit theorem states that the probability $p_N(0,x)$ that a simple random walk starting at 0 ends after N steps at x is given asymptotically by const.$N^{-d/2}\exp[-d|x|^2/2N] \sim$ const.$N^{-d/2}$, as $N \to \infty$. Hence the mean-field value of $\alpha_{sing} - 2$ is $-d/2$. The value of α_{sing} is believed to be determined from the value of ν and the dimension d via the hyperscaling relation

$$\alpha_{sing} - 2 = -d\nu. \tag{1.4.14}$$

This hyperscaling relation will be discussed in Section 2.1. If (1.4.14) and the values given for ν in Table 1.2 are true, then it would follow that $\alpha_{sing} - 2 < -1$ in all dimensions and hence that the critical two-point function $G_{z_c}(0,x) = \sum_N c_N(0,x)\mu^{-N}$ is finite in all dimensions, including $d = 2$. This is in contrast to the situation for simple random walk, where in two dimensions the Green function is infinite at the critical point.

The strongest bounds on $c_N(0,x)$ are for high dimensions. It is proved in Theorem 6.1.3 that for d sufficiently large, or for $d > 4$ for a walk allowed to take long enough steps, that

$$c_N(0,x) \le B\mu^N N^{-d/2} \tag{1.4.15}$$

for some constant B. Although this bound has not yet been extended to all $d \ge 5$ for the nearest-neighbour model, the weaker result that for all

$a < -1 + d/2$

$$\sup_x \sum_{N=0}^{\infty} N^a c_N(0,x)\mu^{-N} < \infty \tag{1.4.16}$$

has been proved for all $d \geq 5$; see Theorem 6.1.4. Either of (1.4.15) or (1.4.16) could be summarized by the inequality $\alpha_{sing} - 2 \leq -d/2$. For dimensions 2, 3 and 4, the best results are for x a nearest neighbour of the origin. These results are described in Section 8.1 and can be summarized by the inequalities

$$\alpha_{sing} \leq \frac{5}{2} \quad (d = 2) \tag{1.4.17}$$

$$\alpha_{sing} \leq 2 \quad (d = 3) \tag{1.4.18}$$

$$\alpha_{sing} < 2 \quad (d \geq 4). \tag{1.4.19}$$

Finally, we introduce the critical exponent Δ_4. Let $c_{N_1,N_2}(x)$ denote the number of pairs of self-avoiding walks of lengths N_1 and N_2 and respective starting points 0 and x which intersect each other, and let

$$c_{N_1,N_2} = \sum_x c_{N_1,N_2}(x). \tag{1.4.20}$$

This quantity occurs in the study of interacting polymer chains. The asymptotic behaviour of c_{N_1,N_2} is believed to be given by

$$c_{N_1,N_2} \sim \text{const.}\mu^{N_1+N_2} N_1^{2\Delta_4+\gamma-2} f(N_1/N_2) \quad \text{as } N_1, N_2 \to \infty \tag{1.4.21}$$

for some critical exponent Δ_4 and universal scaling function f. The quantity

$$g(z) = \xi(z)^{-d}\chi(z)^{-2} \sum_{N_1,N_2=0}^{\infty} c_{N_1,N_2} z^{N_1+N_2} \tag{1.4.22}$$

represents a kind of average intersection probability. In quantum field theory, an analogue of $g(z)$ is referred to as the renormalized coupling constant. An informal calculation in which (1.4.21) is substituted into (1.4.22) leads to

$$g(z) \sim \text{const.}(z_c - z)^{d\nu-2\Delta_4+\gamma} \quad \text{as } z \nearrow z_c. \tag{1.4.23}$$

For simple random walk it is known that the analogue of $g(z)$ satisfies (1.4.23), with $d/2 - 2\Delta_4 + 1 = 0$ for $d = 2, 3, 4$ (with a logarithmic correction in four dimensions), and $\Delta_4 = 3/2$ for $d \geq 5$; see Section 10.3. Similar behaviour is believed to hold for the self-avoiding walk. In particular, for the self-avoiding walk it is believed that in dimensions 2, 3 and 4 the hyperscaling relation

$$d\nu - 2\Delta_4 + \gamma = 0 \tag{1.4.24}$$

is satisfied. Heuristic arguments in support of this hyperscaling relation will be given in Section 2.1. It has been proved that $\Delta_4 = 3/2$ for the self-avoiding walk in dimensions $d \geq 6$ (see Theorem 1.5.5 and the Remark following its statement); it is believed that $\Delta_4 = 3/2$ for all $d > 4$.

Elementary bounds on Δ_4 can be obtained as follows. Consider all pairs of N-step self-avoiding walks $\omega^{(1)}$ and $\omega^{(2)}$ which intersect somewhere, with $\omega^{(1)}$ beginning at the origin and $\omega^{(2)}$ beginning anywhere. There are $c_{N,N}$ such pairs. Since there are $N+1$ possible sites on each of $\omega^{(1)}$ and $\omega^{(2)}$ where an intersection can occur, $c_{N,N} \leq (N+1)^2 c_N^2$. On the other hand if we count only those pairs for which $\omega^{(2)}(0) = \omega^{(1)}(j)$ for some $j = 0, \ldots, n$, we obtain $c_{N,N} \geq (N+1)c_N^2$. Together these bounds give

$$\frac{\gamma+1}{2} \leq \Delta_4 \leq \frac{\gamma+2}{2}. \tag{1.4.25}$$

This can be rewritten as

$$1 \leq 2\Delta_4 - \gamma \leq 2. \tag{1.4.26}$$

The upper bound implies that the hyperscaling relation (1.4.24) fails if $d\nu > 2$. Since it is known that $\nu = 1/2$ for $d \geq 5$ (see Section 6.1), this implies failure of hyperscaling for $d > 4$.

1.5 The bubble condition

The lower bound on the susceptibility (1.3.6) can be rewritten in terms of $z_c = \mu^{-1}$ as

$$\chi(z) \geq \frac{z_c}{z_c - z} \tag{1.5.1}$$

for $0 \leq z < z_c$. The bubble condition is a sufficient condition for the complementary bound

$$\chi(z) \leq \frac{C}{z_c - z} \tag{1.5.2}$$

for some constant C and for $0 \leq z < z_c$. Thus the bubble condition implies that $\bar{\gamma} = 1$ in the sense that

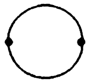
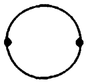
$$\chi(z) \simeq (z_c - z)^{-1} \quad \text{as } z \nearrow z_c. \tag{1.5.3}$$

The bubble condition was proven to hold in five or more dimensions in Hara and Slade (1992b) (see Section 6.1), and is believed not to hold for $d \leq 4$.

To state the bubble condition we first introduce the *bubble diagram*

$$\mathsf{B}(z) = \sum_{x \in Z^d} G_z(0,x)^2. \tag{1.5.4}$$

The name "bubble diagram" comes from a Feynman diagram notation in which the two-point function or *propagator* evaluated at sites x and y is denoted by a line terminating at x and y. In this notation

$\mathsf{B}(z) = \sum_x$ 0  x = 

where in the diagram on the right it is implicit that one vertex is fixed at the origin and the other is summed over the lattice. The bubble diagram can be rewritten in terms of the Fourier transform of the two-point function, using (1.5.4) and the Parseval relation, as

$$\mathsf{B}(z) = ||G_z(0,\cdot)||_2^2 = ||\hat{G}_z||_2^2 = \int_{[-\pi,\pi]^d} \hat{G}_z(k)^2 \frac{d^dk}{(2\pi)^d}. \qquad (1.5.5)$$

Definition 1.5.1 *The* bubble condition *states that the bubble diagram is finite at the critical point, i.e.*

$$\mathsf{B}(z_c) < \infty.$$

In view of the definition of η in (1.4.8) or (1.4.11), it follows from (1.5.5) that the bubble condition is satisfied provided $\eta > (4-d)/2$. Hence the bubble condition for $d > 4$ is implied by the infrared bound $\eta \geq 0$. If the values for η given in Table 1.2 are correct, then the bubble condition will not hold in dimensions 2, 3 or 4, with the divergence of the bubble diagram being only logarithmic in four dimensions.

The next lemma provides the principal step in proving that the bubble condition implies (1.5.2) and hence implies (1.5.3).

Lemma 1.5.2 *For any $z \in [0, z_c)$, the derivative of the susceptibility satisfies*

$$\frac{\chi(z)^2}{\mathsf{B}(z)} - \chi(z) \leq z\chi'(z) \leq \chi(z)^2 - \chi(z). \qquad (1.5.6)$$

Proof. Below the critical point the derivative of χ can be obtained by term by term differentiation:

$$z\chi'(z) = \sum_\omega |\omega| z^{|\omega|} = \sum_\omega (|\omega|+1) z^{|\omega|} - \chi(z), \qquad (1.5.7)$$

where the sums are over self-avoiding walks of arbitrary length which begin at the origin. The summation on the right side can be written

$$\sum_y \sum_{\omega:0\to y} \sum_x I[\omega(j) = x \text{ for some } j] z^{|\omega|}$$

$$= \sum_{x,y} \sum_{\substack{\omega^{(1)}:0\to x \\ \omega^{(2)}:x\to y}} z^{|\omega^{(1)}|+|\omega^{(2)}|} I[\omega^{(1)}\cap\omega^{(2)} = \{x\}]$$
$$\equiv Q(z), \qquad (1.5.8)$$

where I denotes the indicator function and the last summation is over self-avoiding walks $\omega^{(1)}$ and $\omega^{(2)}$ of arbitrary length and having the prescribed endpoints. Then

$$z\chi'(z) = Q(z) - \chi(z). \qquad (1.5.9)$$

The upper bound in (1.5.6) then follows since the indicator function in the middle member of (1.5.8) is bounded above by one.

The first step toward obtaining the lower bound is to use the inclusion-exclusion relation in the form

$$I[\omega^{(1)}\cap\omega^{(2)} = \{x\}] = 1 - I[\omega^{(1)}\cap\omega^{(2)} \neq \{x\}].$$

This gives

$$Q(z) = \chi(z)^2 - \sum_{x,y} \sum_{\substack{\omega^{(1)}:0\to x \\ \omega^{(2)}:x\to y}} z^{|\omega^{(1)}|+|\omega^{(2)}|} I[\omega^{(1)}\cap\omega^{(2)} \neq \{x\}]. \qquad (1.5.10)$$

In the last term on the right side of (1.5.10), let $w = \omega^{(2)}(l)$ be the site of the last intersection of $\omega^{(2)}$ with $\omega^{(1)}$, where time is measured along $\omega^{(2)}$ beginning at its starting point x. Then the portion of $\omega^{(2)}$ corresponding to times greater than l must avoid all of $\omega^{(1)}$. Relaxing the restrictions that this portion of $\omega^{(2)}$ avoid both the remainder of $\omega^{(2)}$ and the part of $\omega^{(1)}$ linking w to x gives the upper bound

$$\sum_{x,y} \sum_{\substack{\omega^{(1)}:0\to x \\ \omega^{(2)}:x\to y}} z^{|\omega^{(1)}|+|\omega^{(2)}|} I[\omega^{(1)}\cap\omega^{(2)} \neq \{x\}] \leq Q(z)[\mathsf{B}(z) - 1]. \qquad (1.5.11)$$

Here the factor $\mathsf{B}(z) - 1$ arises from the two paths joining w and x. The upper bound involves $\mathsf{B}(z) - 1$ rather than $\mathsf{B}(z)$ since there will be no contribution here from the $x = 0$ term in (1.5.4). This type of distinction will be crucial in similar bounds on the lace expansion used in Chapter 6.

Combining (1.5.10) and (1.5.11) gives

$$Q(z) \geq \chi(z)^2 - Q(z)[\mathsf{B}(z) - 1]. \qquad (1.5.12)$$

This inequality is illustrated in Figure 1.2. Solving for $Q(z)$ gives

$$Q(z) \geq \frac{\chi(z)^2}{\mathsf{B}(z)}.$$

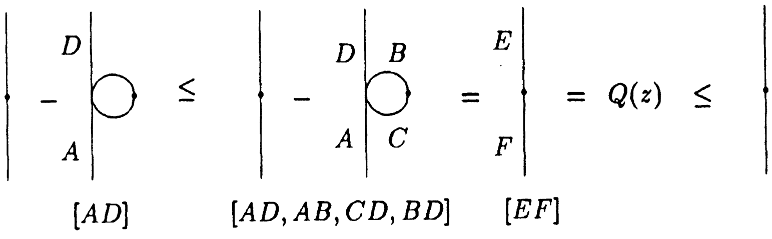


Figure 1.2: A diagrammatic representation of the inequality $\chi(z)^2 - Q(z)[\mathsf{B}(z) - 1] \leq Q(z) \leq \chi(z)^2$ occurring in the proof of Lemma 1.5.2. The list of pairs of lines indicates interactions between the propagators, in the sense that the corresponding walks must avoid each other.

Combining this inequality with (1.5.9) completes the proof of the lemma. □

The quantity $\chi(z)^{-2}Q(z)$ can be interpreted as the probability that two self-avoiding walks of arbitrary length, which start at the origin, do not intersect. Lemma 1.5.2 can be restated as saying that this probability lies in the interval $[\mathsf{B}(z)^{-1}, 1]$, and hence remains strictly positive at the critical point if the bubble condition is satisfied.

In the next theorem it is shown that the lower bound of Lemma 1.5.2 implies that if the bubble condition is satisfied, then (1.5.3) holds.

Theorem 1.5.3 *If the bubble condition is satisfied, and hence in particular if the infrared bound holds and $d > 4$, then there is a positive function $\epsilon(z)$ with $\lim_{z \nearrow z_c} \epsilon(z) = 0$ such that for z less than but near z_c*

$$\frac{z_c}{z_c - z} \leq \chi(z) \leq \frac{z_c[\mathsf{B}(z_c) + \epsilon(z)]}{z_c - z}.$$

Hence if in fact there is a constant A such that $\chi(z) \sim Az_c(z_c - z)^{-1}$, then $1 \leq A \leq \mathsf{B}(z_c)$.

Proof. The lower bound in the statement of the theorem is just (1.5.1). For the upper bound, let $z_1 \in (0, z_c)$. It follows from the lower bound in (1.5.6) that for $z \in [z_1, z_c)$

$$\begin{aligned} z\left(-\frac{d\chi^{-1}}{dz}\right) &\geq \frac{1}{\mathsf{B}(z)} - \frac{1}{\chi(z)} \\ &\geq \frac{1}{\mathsf{B}(z_c)} - \frac{1}{\chi(z_1)}. \end{aligned} \tag{1.5.13}$$

We bound the factor of z on the left side by z_c and then integrate from z_1 to z_c. Using the fact that $\chi(z_c)^{-1} = 0$ by (1.5.1), this gives

$$z_c\chi(z_1)^{-1} \geq [\mathsf{B}(z_c)^{-1} - \chi(z_1)^{-1}](z_c - z_1). \tag{1.5.14}$$

Rewriting gives

$$\chi(z_1) \leq \frac{\mathsf{B}(z_c)}{1 - \mathsf{B}(z_c)\chi(z_1)^{-1}} \frac{z_c}{z_c - z_1}. \tag{1.5.15}$$

This gives the desired upper bound on the susceptibility, since by (1.5.1) the inverse susceptibility on the right side can be made arbitrarily small by taking z_1 sufficiently close to z_c. □

Although the bubble condition is expected not to hold in four dimensions, it is nevertheless possible to draw some conclusions from the lower bound of Lemma 1.5.2 if we assume the infrared bound (1.4.12). While not sharp compared to the expected behaviour

$$\chi(z) \sim \frac{A}{z_c - z}|\log(z_c - z)|^{1/4},$$

the upper bound that we obtain on χ shows that the deviation from mean-field behaviour is at worst logarithmic in four dimensions, if the infrared bound is satisfied.

Theorem 1.5.4 *Let $d = 4$. If the infrared bound (1.4.12) is satisfied then for z less than but near z_c,*

$$\frac{z_c}{z_c - z} \leq \chi(z) \leq C\frac{|\log(z_c - z)|}{z_c - z}$$

for some constant C which does not depend on z.

Proof. The lower bound in the statement of the theorem is just (1.5.1), which holds in all dimensions. It remains to prove the upper bound. In the following, C represents a constant whose value may change from one occurrence to another.

Let $0 < z < z_c$. Since

$$\chi(z) = \hat{G}_z(0) \geq |\hat{G}_z(k)|$$

for all k, it follows from the infrared bound that

$$|\hat{G}_z(k)| \leq \frac{2}{|\hat{G}_z(k)|^{-1} + \chi(z)^{-1}} \leq \frac{2C}{k^2 + C\chi(z)^{-1}}. \tag{1.5.16}$$

Using the fact that

$$\mathsf{B}(z) = \int_{[-\pi,\pi]^4} \hat{G}_z(k)^2 \frac{d^4k}{(2\pi)^4},$$

a routine calculation using (1.5.16) gives the bound

$$\mathsf{B}(z) \leq C[1 + \log\chi(z)]. \tag{1.5.17}$$

By (1.5.6), (1.5.1) and (1.5.17), for z sufficiently close to z_c we have

$$\begin{aligned} -z\frac{d\chi^{-1}}{dz} &\geq \frac{1}{\mathsf{B}(z)} - \frac{1}{\chi(z)} \\ &\geq \frac{1}{2\mathsf{B}(z)} \\ &\geq \frac{C}{1+\log\chi(z)} \end{aligned}$$

and therefore

$$-[1+\log\chi(z)]\frac{d\chi^{-1}}{dz} \geq C. \tag{1.5.18}$$

The left side of (1.5.18) is the derivative of $-\chi(z)^{-1}[2 + \log\chi(z)]$. Hence for z close to z_c integration of (1.5.18) over the interval (z, z_c) gives

$$\chi(z)^{-1}[2 + \log\chi(z)] \geq C(z_c - z),$$

where we used (1.5.1) to see that the contribution from the upper limit of integration on the left side is zero. Decreasing C slightly we obtain

$$\frac{1}{C(z_c - z)} \geq \chi(z)[\log\chi(z)]^{-1}. \tag{1.5.19}$$

Taking logarithms, and taking z sufficiently close to z_c, gives

$$C|\log(z_c - z)| \geq \log\chi(z) - \log\log\chi(z) \geq \frac{1}{2}\log\chi(z). \tag{1.5.20}$$

Inserting the lower bound for $[\log\chi(z)]^{-1}$ given by (1.5.20) into (1.5.19) gives

$$C(z_c - z)^{-1} \geq \chi(z)|\log(z_c - z)|^{-1}.$$

This gives the upper bound on χ in the statement of the theorem. □

Finally we turn to a connection between the bubble diagram and the critical exponent Δ_4 for the renormalized coupling constant $g(z)$, which was defined in (1.4.22) by

$$g(z) = \xi(z)^{-d}\chi(z)^{-2} \sum_{N_1,N_2=0}^{\infty} c_{N_1,N_2} z^{N_1+N_2}. \tag{1.5.21}$$

Here c_{N_1,N_2} is the sum over sites x of the number of intersecting pairs of self-avoiding walks of length N_1 and N_2 starting at 0 and x respectively. The critical behaviour of $g(z)$ is believed to be of the form $(z_c - z)^{d\nu - 2\Delta_4 + \gamma}$.

The next theorem gives sufficient conditions for Δ_4 to take its mean-field value 3/2. The theorem is most efficiently stated in terms of the *repulsive* bubble diagram $\mathsf{R}(z) < \mathsf{B}(z)$, which is defined by taking only those contributions to the bubble from pairs of walks which are mutually avoiding apart from their common endpoints:

$$\mathsf{R}(z) = \sum_{x \in Z^d} \sum_{\substack{\omega^{(1)}: 0 \to x \\ \omega^{(2)}: 0 \to x}} z^{|\omega^{(1)}| + |\omega^{(2)}|} I[\omega^{(1)} \cap \omega^{(2)} = \{0, x\}]. \qquad (1.5.22)$$

Theorem 1.5.5 *If* $\mathsf{B}(z_c) < \infty$ *and in addition* $\mathsf{R}(z_c) - 1 < 1/4$, *then* $g(z) \simeq \xi(z)^{-d}(z_c - z)^{-2}$. *If also* $\xi(z) \simeq (z_c - z)^{-\nu}$, *then* $\Delta_4 = 3/2$ *in the sense that*

$$g(z) \simeq (z_c - z)^{d\nu - 3 + \gamma} = (z_c - z)^{d\nu - 2}$$

(assuming that the exponent γ *for* c_n *is equal to the exponent for the susceptibility).*

Remark. The best current bound on $\mathsf{R}(z_c) - 1$ in five dimensions is $0.434636 > 0.25$ [Hara and Slade (1991b)]. For $d = 6$ the same reference reports $\mathsf{B}(z_c) - 1 \leq 0.25974$. However the repulsive bubble in six dimensions satisfies $\mathsf{R}(z_c) - 1 \leq 0.2343 < 0.25$, and is smaller still in more than six dimensions [Hara and Slade (unpublished)]. Together with Theorem 1.5.5 and the result of Hara and Slade (1991a) that for $d \geq 5$ the correlation length exhibits the mean-field behaviour $\xi(z) \sim \text{const.}(z_c - z)^{-1/2}$ (and that the exponent for c_n is $\gamma = 1$), this implies that

$$g(z) \simeq (z_c - z)^{(d-4)/2} \qquad (1.5.23)$$

for $d \geq 6$. Although the same conclusion cannot yet be made for $d = 5$, it will be shown in Chapter 6 (see Theorem 6.2.5 and the remark preceding it, and Theorem 6.1.5) that for a sufficiently spread-out self-avoiding walk in more than four dimensions, $\mathsf{B}(z_c) - 1 < 1/4$ and $\xi(z) \sim \text{const.}(z_c - z)^{-1/2}$, and hence $g(z) \simeq (z_c - z)^{(d-4)/2}$.

Proof of Theorem 1.5.5. By Theorem 1.5.3, the bubble condition implies that $\chi(z) \simeq (z_c - z)^{-1}$. Hence to prove the theorem it suffices to show that

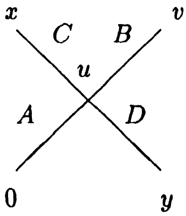
$$\sum_{N_1, N_2 = 0}^{\infty} c_{N_1,N_2} z^{N_1 + N_2} \simeq (z_c - z)^{-4}. \qquad (1.5.24)$$

The left side is equal to

$$\sum_{x,y,v} \sum_{\substack{\omega : 0 \to v \\ \rho : x \to y}} z^{|\omega|+|\rho|} I[\omega \cap \rho \neq \phi]. \tag{1.5.25}$$

In a nonzero contribution to this sum, let u be the first site along ω where ω and ρ intersect. Then the portion of ω before u avoids ρ as well as the latter part of ω, while the latter part of ω avoids only the former part of ω and may intersect ρ. This gives the following diagrammatic interpretation of the left side of (1.5.24) (in which the list of pairs indicates mutually avoiding walks):

$$\sum_{u,x,y,v}$$



$$[AB, CD, AC, AD] \tag{1.5.26}$$

Neglecting all mutual avoidance between the four lines of the diagram gives the upper bound $\chi^4 \leq \text{const.}(z_c - z)^{-4}$ for the left side of (1.5.24).

For a lower bound on (1.5.26) we apply inclusion-exclusion, as follows. The indicator function for the event that the various mutual avoidances shown in (1.5.26) occur can be written as one minus the event that at least one of the required mutual avoidances is violated. This leads to the lower bound

$$\sum_{u,v,x,y} \sum_{\substack{\omega^{(1)} : 0 \to u \\ \omega^{(2)} : u \to v}} \sum_{\substack{\rho^{(1)} : x \to u \\ \rho^{(2)} : u \to y}} z^{|\omega^{(1)}|+|\omega^{(2)}|+|\rho^{(1)}|+|\rho^{(2)}|}$$
$$\times \Big\{ 1 - I[\omega^{(1)} \cap \omega^{(2)} \neq \{u\}] - I[\omega^{(1)} \cap \rho^{(1)} \neq \{u\}]$$
$$- I[\omega^{(1)} \cap \rho^{(2)} \neq \{u\}] - I[\rho^{(1)} \cap \rho^{(2)} \neq \{u\}] \Big\}. \tag{1.5.27}$$

This bound is equal to

$$\chi^4 - 4\chi^2 \sum_{u,x} \sum_{\substack{\gamma^{(1)} : 0 \to u \\ \gamma^{(2)} : u \to x}} z^{|\gamma^{(1)}|+|\gamma^{(2)}|} I[\gamma^{(1)} \cap \gamma^{(2)} \neq \{u\}]. \tag{1.5.28}$$

We now argue as in (1.5.11), but this time we let w be the site of the *first* intersection (measured along $\gamma^{(2)}$) of $\gamma^{(2)}$ with $\gamma^{(1)}$. This gives the lower bound

$$\chi^4 - 4\chi^4[\mathsf{R}(z) - 1] \geq \text{const.}\chi^4 \tag{1.5.29}$$

for (1.5.26), assuming that $\mathsf{R}(z_c) - 1 < 1/4$. □

1.6 Notes

Section 1.1. Existence of the connective constant $\mu = \lim_{N\to\infty} c_N^{1/N}$ was first proven in Hammersley and Morton (1954); this paper essentially marks the beginning of rigorous results for the self-avoiding walk. The (nonrigorous) derivation of $\mu = \sqrt{2+\sqrt{2}}$ for the honeycomb lattice is due to Nienhuis (1982); see also Nienhuis (1984) and Nienhuis (1987). For high dimensions, it was shown in Kesten (1964) that $\mu = 2d-1-(2d)^{-1}+O(d^{-2})$, and this has recently been improved to $\mu = 2d-1-(2d)^{-1}-3(2d)^{-2}+O(d^{-3})$ using the lace expansion [Hara and Slade (unpublished)].

The conjectured values for γ and ν in two dimensions arise from an exact solution which is described in the articles by Nienhuis cited above. An alternate approach, based on conformal invariance, is discussed in Duplantier (1989), Duplantier (1990), and references therein. A rigorous argument leading to these two-dimensional critical exponents remains an open problem of major importance, and a solution would likely have far-reaching implications. For $d = 3$, field theoretic computations of the critical exponents are given in Le Guillou and Zinn-Justin (1989). Monte Carlo computations of the exponents are given for example in Madras and Sokal (1988), and numerical computations using extrapolation of exact enumerations are given in Guttmann and Wang (1991). The logarithmic corrections in four dimensions are obtained in Larkin and Khmel'Nitskii (1969), Wegner and Riedel (1973) and Brezin, Le Guillou and Zinn-Justin (1973). For recent progress on rigorous results in four dimensions, see Brydges, Evans and Imbrie (1992) and Arnaudon, Iagolnitzer and Magnen (1991). Existence of critical exponents for $d \geq 5$ is proven in Hara and Slade (1992a,1992b).

A necessary and sufficient condition for a bound of the form $c_N \leq \text{const.}\mu^N N^H$ for some finite H, i.e. for the finiteness of the critical exponent γ, is given in Hammersley (1991, 1992). We note the presence of a minor error in Hammersley (1991): the right side of (30) does not follow from the inequality that precedes it. This is easy to fix, however, as follows. In Hammersley's notation, the bound $f(m) \leq Gm^H\mu^m$ implies that $f(m,r) \leq \sum G^r n_1^H n_2^H \cdots n_r^H \mu^m$, where the sum is over all $n_1, \ldots, n_r \geq 1$ that sum to m. By the arithmetic-geometric inequality we have $n_1 n_2 \cdots n_r \leq (m/r)^r$, which implies $f(m,r) \leq \binom{m-1}{r-1} G^r (m/r)^{rH} \mu^m$. This gives us (30) with $Hu \log H$ replaced by $-Hu \log u$, and Hammersley's Equation (3) follows.

A rigorous understanding of the self-avoiding walk on finitely ramified fractals has recently emerged; see Hattori (1992) for a review.

For the Ising model (and also for φ^4 field theory), the following references prove results concerning mean-field behaviour above four dimensions: Sokal (1979), Aizenman (1982), Fröhlich (1982), Aizenman and Fernández (1986), Fernández, Fröhlich and Sokal (1992).

Section 1.2. The bound $(c_N/c_1)^{1/(N-1)}$ (for all $N \geq 2$) is attributed to Alm in Ahlberg and Janson (1980). The latter reference obtains an improvement when $c_N/c_{N-1} > c_1 - 2$: they show that μ is bounded above by the unique positive root of the polynomial

$$c_1 x^{N-1} = [c_N - (c_1 - 2)c_{N-1}]x + (c_1 - 2)[(c_1 - 1)c_{N-1} - c_N] \quad (1.6.1)$$

(for all $N \geq 3$). Currently the best upper bounds available are due to Alm (1992).

A method for obtaining lower bounds on μ using bridges was given in Guttmann (1983). The current best lower bound in two dimensions, due to Conway and Guttmann (to be published), also uses bridges. For $d \geq 3$, the best lower bounds are due to Hara and Slade (1992b), who use a different approach involving loop erasure.

The numerical estimates for μ cited in Table 1.1 are from exact enumeration data.

Section 1.3. Exponential decay of the subcritical two-point function was proven in Fisher (1966), as part of a study of the form of the distribution of $c_N(0,x)$.

Section 1.4. We make no attempt here to refer to the original literature on critical exponents; the ideas in this section are part of the standard physics picture of critical phenomena.

The infrared bound was proven for reflection-positive spin systems in all dimensions in Fröhlich, Simon and Spencer (1976). For branched polymers and for percolation, there are arguments that the infrared bound does not hold below eight and six dimensions respectively; see Bovier, Fröhlich and Glaus (1986) and Adler (1984) respectively.

For $d \geq 5$ it has been proven that $\hat{G}_{z_c}(k) \sim \text{const.} k^{-2}$ as $k \to 0$, but although it is believed that $G_{z_c}(x)$ is asymptotic to a multiple of $|x|^{2-d}$, this has not yet been proven (see Theorem 6.1.6 for a weaker result). It is thus of interest to know under what conditions behaviour of the form

$k^{-2+\eta}$ for a Fourier transform $\hat{g}(k)$ implies behaviour of the form $|x|^{2-d-\eta}$ for $g(x)$. For the case $\eta = 0$, the following sufficient condition was pointed out to us by S. Kotani (private communication); we omit the proof.

Theorem 1.6.1 *Let $d \geq 3$, and let $\mathbf{T}^d \equiv (\mathbf{R}/2\pi\mathbf{Z})^d$. Let $\hat{g}$ be a function in $C^{d-2}(\mathbf{T}^d\backslash\{0\})$, let $\hat{h}(k) = k^2\hat{g}(k)$, and for $x \in \mathbf{Z}^d$ let $g(x) = (2\pi)^{-d}\int_{T^d}\hat{g}(k)e^{-ik\cdot x}d^dk$. Suppose that there is a neighourhood $U \subset \mathbf{T}^d$ of 0 such that*

$$\hat{h} \in \begin{cases} C^{d-1}(U) & \text{if} \quad d = 3, 4 \\ C^{d-2}(U) & \text{if} \quad d \geq 5. \end{cases}$$

Then as $|x| \to \infty$,

$$g(x) \sim \hat{h}(0)\frac{\Gamma(d/2)}{2(d-2)\pi^{d/2}}|x|^{-(d-2)}.$$

The following shows that in general the hypothesis of existence of $d-2$ derivatives for $\hat{h}$ cannot be relaxed: we give an example[2] of a function $\hat{g}$ on $\mathbf{T}^d$, for $d \geq 3$, which is asymptotic to a multiple of k^{-2} as $k \to 0$, with $\hat{h}(k) = k^2\hat{g}(k)$ having $d-3$ but not $d-2$ derivatives in a neighbourhood of $k = 0$, but for which $g(x)$ is not bounded above by a multiple of $|x|^{2-d}$ for large x.

Example 1.6.2 Let $d \geq 3$, and let $C(x)$ be the critical simple random walk two-point function (or in other words the Green function) studied in Appendix A. Then $C(x)$ is asymptotic to a multiple of $|x|^{2-d}$ for large x [see, e.g., Lawler (1991)]. Also, $\hat{C}(k)^{-1} = 1-d^{-1}\sum_{\mu=1}^{d}\cos k_\mu$ is asymptotic to $(2d)^{-1}k^2$ as $k \to 0$. Fix q such that $d-3 < q < d-2$, and for $-\pi \leq t \leq \pi$ define

$$\hat{f}(t) = \sum_{m=-\infty}^{\infty} 2^{-q|m|}\exp[it(\text{sgn}\, m)2^{|m|}], \tag{1.6.2}$$

where sgn $m = +1$ if $m > 0$; $= 0$ if $m = 0$; $= -1$ if $m < 0$. For $k \in [-\pi, \pi]^d$, let

$$\hat{F}(k) = \epsilon\prod_{\mu=1}^{d}\hat{f}(k_\mu) \tag{1.6.3}$$

where ϵ is chosen small enough that $1 + \hat{C}(k)^{-1}\hat{F}(k)$ is strictly positive uniformly in all $k \in [-\pi, \pi]^d$. (This is possible since $\hat{C}(k)^{-1}$ and the product in (1.6.3) are both bounded uniformly in k.) Observe that $\hat{F} \in C^s(\mathbf{T}^d)$ for $s < q$, but that for $s > q$, $\partial_\mu^s\hat{F}(0)$ does not exist. Now let

$$\hat{g}(k) = \hat{C}(k) + \hat{F}(k) = \hat{C}(k)[1 + \hat{C}(k)^{-1}\hat{F}(k)] \tag{1.6.4}$$

[2] The example was arrived at in conversation with T. Hara.

and

$$\hat{h}(k) = k^2 \hat{g}(k). \tag{1.6.5}$$

Then $\hat{g}(k)$ is asymptotic to $(2d)k^{-2}$ as $k \to 0$, and $\hat{h}(k) \in C^{d-3}(\mathbf{T}^d)$. However $g(x)$ is not bounded above by a multiple of $|x|^{2-d}$ for large x, because $F(x) = \epsilon|x|^{-q}$ for x having one component of the form $\pm 2^{|m|}$ (for any integer m) and all other components zero.

See Appendix A of Sokal (1982) for a discussion of some related issues.

Section 1.5. For reflection positive spin systems the infrared bound was proven in Fröhlich, Simon and Spencer (1976). As a consequence the bubble diagram for such systems is finite at the critical point above four dimensions, and diverges logarithmically in four dimensions. This was used to prove mean-field behaviour for spin systems for dimensions greater than four in Aizenman (1982) and Fröhlich (1982). In Bovier, Felder and Fröhlich (1984) Theorem 1.5.3 was proved, although at that time for the self-avoiding walk neither the infrared bound nor the bubble condition were known to hold in any dimension. In the same paper it was observed that if the infrared bound holds in four dimensions then the deviation from mean-field behaviour for the susceptibility is at most logarithmic. Our proof of Theorem 1.5.4 yields this conclusion in a slightly stronger form, following the methods used for spin systems in Aizenman and Graham (1983). Results analogous to Theorem 1.5.5 were obtained for spin systems in Aizenman (1982) and Fröhlich (1982). The proof that $\Delta_4 = 3/2$ for $d \geq 6$ is new, and is due to Hara and Slade (unpublished).

For percolation and branched polymers (lattice trees and lattice animals) the role of the bubble diagram is played by the triangle and the square diagram respectively; see Section 5.5. For percolation see Aizenman and Newman (1984), Nguyen (1987), Barsky and Aizenman (1991), Hara and Slade (1990a) and Nguyen and Yang (1991). For lattice trees and lattice animals see Bovier, Fröhlich and Glaus (1986), Tasaki and Hara (1987) and Hara and Slade (1990b).