# Chapter 1
# The Sheffer A-Type 0 Orthogonal Polynomial Sequences and Related Results

In this chapter, we present a rigorous development of I. M. Sheffer's characterization of the *A-Type 0* orthogonal polynomial sequences. We first develop the results that led to the main theorem that characterizes the general *A-Type 0* polynomial sequences via a linear generating function. From there, we develop the additional theory that Sheffer utilized in order to determine which *A-Type 0* polynomial sequences are also orthogonal. We then address Sheffer's additional characterizations of *B-Type* and *C-Type*, as well as E.D. Rainville's $\sigma$-*Type* classification. Lastly, we cover J. Meixner's approach to the same characterization problem studied by Sheffer and then discuss an extension of Meixner's analysis by W.A. Al-Salam. Portions of the analysis addressed throughout this chapter are supplemented with informative concrete examples.

## 1.1 Preliminaries

Throughout this chapter, we make use of each of the following definitions, terminologies and notations.

**Definition 1.1.** We always assume that a ***set*** of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ is such that each $P_n(x)$ has degree exactly $n$, which we write as $\deg(P_n(x)) = n$.

**Definition 1.2.** A set of polynomials $\{Q_n(x)\}_{n=0}^{\infty}$ is ***monic*** if $Q_n(x) - x^n$ is of degree at most $n-1$ or equivalently if the leading coefficient of each $Q_n(x)$ is unitary.

**Definition 1.3.** We shall define a ***generating function*** for a polynomial sequence $\{P_n(x)\}_{n=0}^{\infty}$ as follows:

$$\sum_{\Lambda} \zeta_n P_n(x) t^n = F(x,t),$$

with $\Lambda \subseteq \{0,1,2,\ldots\}$ and $\{\zeta_n\}_{n=0}^{\infty}$ a sequence in $n$ that is independent of $x$ and $t$. Moreover, we say that the function $F(x,t)$ ***generates*** the set $\{P_n(x)\}_{n=0}^{\infty}$.

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It is important to mention that a generating function need not converge, as in general, several relationships can be derived when $F(x,t)$ is divergent.

**Definition 1.4.** In this chapter, the term ***orthogonal polynomials*** refers to a set of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ that satisfies one of the two weighted inner products below:

$$\text{Continuous}: \ \langle P_m(x), P_n(x)\rangle = \int_{\Omega_1} P_m(x)P_n(x)w(x)dx = \alpha_n \delta_{m,n}, \tag{1.1}$$

$$\text{Discrete}: \ \langle P_m(x), P_n(x)\rangle = \sum_{\Omega_2} P_m(x)P_n(x)w(x) = \beta_n \delta_{m,n}, \tag{1.2}$$

where $\Omega_1 \subseteq \mathbb{R}$, $\Omega_2 \subseteq \mathbb{W}$, $\delta_{m,n}$ denotes the Kronecker delta and $w(x) > 0$ is entitled the ***weight function***.

For example, the Laguerre, Hermite, and Meixner–Pollaczek polynomials satisfy a continuous orthogonality relation of the form (1.1). On the other hand, the Charlier, Meixner, and Krawtchouk polynomials satisfy a discrete orthogonality relation of the form (1.2) (cf. [6]).

Now, it is well-known that a necessary and sufficient condition for a set of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ to be orthogonal is that it satisfies a three-term recurrence relation (see [8]), which can be written in different (equivalent) forms. In particular, we utilize the following two forms in this chapter and adhere to the nomenclature used in [2].

**Definition 1.5 (The Three-Term Recurrence Relations).** It is a necessary and sufficient condition that an orthogonal polynomial sequence $\{P_n(x)\}_{n=0}^{\infty}$ satisfies an ***unrestricted three-term recurrence relation*** of the form

$$\begin{aligned} &P_{n+1}(x) = (A_n x + B_n)P_n(x) - C_n P_{n-1}(x), \quad A_n A_{n-1} C_n > 0 \\ &\text{where } \ P_{-1}(x) = 0 \ \text{ and } \ P_0(x) = 1. \end{aligned} \tag{1.3}$$

If $Q_n(x)$ represents the monic form of $P_n(x)$, then it is a necessary and sufficient condition that $\{Q_n(x)\}_{n=0}^{\infty}$ satisfies the following ***monic three-term recurrence relation***:

$$\begin{aligned} &Q_{n+1}(x) = (x + b_n)Q_n(x) - c_n Q_{n-1}(x), \quad c_n > 0 \\ &\text{where } \ Q_{-1}(x) = 0 \ \text{ and } \ Q_0(x) = 1. \end{aligned} \tag{1.4}$$

We entitle the conditions $A_n A_{n-1} C_n > 0$ and $c_n > 0$ above ***positivity conditions***.

Lastly, we mention that all of the power series in this chapter are *formal* power series, i.e., they may or may not converge. In [9], Sheffer used the symbol '$\cong$' to denote formal series. For simplicity, we will use the equal sign throughout our present work and it will be tacitly assumed that each power series is nonetheless formal.

## 1.2 Sheffer's Analysis of the Type 0 Polynomial Sequences

In this section, we discuss each of the theorems of I.M. Sheffer's work [9] that were necessary in characterizing all of the *Type 0* orthogonal sets. With respect to space constraints, we write each proof, and some examples as well, with as much detail as possible. To begin, we consider the very well-studied Appell polynomial sets $\{P_n(x)\}_{n=0}^{\infty}$, which are defined as

$$A(t)\mathrm{e}^{xt} = \sum_{n=0}^{\infty} P_n(x)t^n, \quad A(t) = \sum_{n=0}^{\infty} a_n t^n, \quad a_0 = 1. \tag{1.5}$$

An example of an Appell set is $\{x^n/n!\}_{n=0}^{\infty}$, which is clear since

$$\mathrm{e}^{xt} = \sum_{n=0}^{\infty} \frac{x^n}{n!} t^n.$$

Now, we differentiate (1.5) with respect to $x$. The left-hand side becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[A(t)\mathrm{e}^{xt}\right] = tA(t)\mathrm{e}^{xt} = \sum_{n=0}^{\infty} P_n(x)t^{n+1} = \sum_{n=1}^{\infty} P_{n-1}(x)t^n$$

and the right-hand side becomes

$$\sum_{n=1}^{\infty} P_n'(x)t^n.$$

Therefore, after comparing coefficients of $t^n$ in the results above, we achieve the equivalent characterization of Appell sets

$$P_n'(x) = P_{n-1}(x), \quad n = 0, 1, 2, \ldots.$$

Next, we consider the set of Newton polynomials $\{N_n(x)\}_{n=0}^{\infty}$, which is *not* an Appell set:

$$N_0(x) := 1, \quad N_n(x) := \frac{x(x-1)\cdots(x-n+1)}{n!}, \quad n = 1, 2, \ldots.$$

For the difference operator defined by $\Delta f(x) := f(x+1) - f(x)$, it can be shown that

$$\Delta N_n(x) = N_n(x+1) - N_n(x) = N_{n-1}(x)$$

and

$$(1+t)^x = \mathrm{e}^{x\ln(1+t)} = \sum_{n=0}^{\infty} N_n(x)t^n.$$

We observe that the operator $\Delta$ functions as $\mathrm{d}/\mathrm{d}x$ does on the Appell polynomials and that the generating function above is in a *more general* form than Eq. (1.5), i.e., the $t$ in the exponent of Eq. (1.5) is replaced by $H(t) = \ln(t+1)$. Due to this analysis, Sheffer was motivated to define a class of *difference polynomial sets* that satisfy

$$J\left[P_n(x)\right] = P_{n-1}(x), \quad n = 0, 1, 2, \ldots$$

with $J$ a general degree-lowering operator.

Thus, we now continue with a result regarding such a general degree-lowering operator $J$, which is an essential structure in all of the *Type 0* analysis that follows.

**Lemma 1.1.** *Assume that $J$ is a linear operator that acts on the set of monomials $\{x^n\}_{n=0}^{\infty}$ such that* $\deg(J[x^n]) \leq n$. *Then, $J$ has the following structure:*

$$J[y(x)] = \sum_{n=0}^{\infty} L_n(x) \frac{\mathrm{d}^n}{\mathrm{d}x^n} y(x), \tag{1.6}$$

*which is valid for all polynomials $y(x)$, with* $\deg(L_n(x)) \leq n$.

*Proof.* We first note that since $J$ is assumed to be a linear operator that acts on the set of monomials $\{x^n\}_{n=0}^{\infty}$, it can act on any polynomial. Therefore, if we show that Eq. (1.6) holds for $y(x) = x^n$, we have proven the theorem. Using the fact that

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k} x^n = n(n-1)(n-2)\cdots(n-k+1)x^{n-k}$$

we can then recursively define $L_n(x)$ by the following:

$$J[x^n] = \sum_{k=0}^{n} L_k(x)[n(n-1)(n-2)\cdots(n-k+1)x^{n-k}], \quad n = 0, 1, 2, \ldots. \tag{1.7}$$

Since for each $n = 0, 1, 2, \ldots$ we assumed that $\deg(J[x^n]) \leq n$, we must require that $L_k(x)[n(n-1)(n-2)\cdots(n-k+1)x^{n-k}]$ be of degree at most $n$ for $k = 0, 1, \ldots, n$. This will occur if and only if $\deg(L_k(x)) \leq k$, since for any polynomials $P_m(x)$ and $Q_n(x)$, $\deg(P_m(x)Q_n(x)) = \deg(P_m(x)) + \deg(Q_n(x))$. □

In Lemma 1.1, we determined the structure that $J$ must adhere to in order for $\deg(J[x^n]) \leq n$. Next, we determine the form that $J$ must have in order for $\deg(J[x^n]) = n - 1$. As we shall see, this will amount to restrictions on $L_n(x)$ in Eq. (1.6). Also, in order to naturally generalize our degree-lowering operator $J$, we additionally require that $J[c] = 0$ for all constants $c$, analogous to $\frac{\mathrm{d}}{\mathrm{d}x}[c] = 0$.

**Lemma 1.2.** *Necessary and sufficient conditions for $J$ as defined in Eq.* (1.6) *to exist such that* $\deg(J[x^n]) = n - 1$ *are as follows:*

$$L_0(x) = 0, \quad L_n(x) = l_{n,0} + l_{n,1}x + \cdots + l_{n,n-1}x^{n-1}, \quad n = 1, 2, \ldots \tag{1.8}$$

*and*

$$\lambda_n := nl_{1,0} + n(n-1)l_{2,1} + \cdots + n!l_{n,n-1} \neq 0, \quad n = 1,2,\ldots. \tag{1.9}$$

*Proof.* ($\Rightarrow$) We initially assume that $J[1] = 0$ and $\deg(J[x^n]) = n-1$ for $n = 1,2,\ldots$ and that $L_n(x)$ takes on the form

$$L_n(x) = l_{n,0} + l_{n,1}x + \cdots + l_{n,n}x^n,$$

from which we show that Eq. (1.8) and Eq. (1.9) necessarily follow. We begin by finding the coefficients of $x^n$ and $x^{n-1}$ in Eq. (1.7). Namely, we analyze the summand in Eq. (1.7):

$$L_k(x)[n(n-1)(n-2)\cdots(n-k+1)x^{n-k}] \tag{1.10}$$

for $k = 0,1,2,\ldots,n$ and determine the leading coefficient in each case and subsequently add the results, thus obtaining the coefficient of $x^n$, which will be valid for $n = 0,1,2,\ldots$. We follow the same procedure for achieving the coefficient of $x^{n-1}$.

For $k = 0$, we observe that Eq. (1.10) becomes $L_0(x)x^n = l_{0,0}x^n$, which clearly has a leading coefficient of $l_{0,0}$. For $k = 1$, we see that Eq. (1.10) turns out to be $L_1(x)nx^{n-1} = (l_{1,0} + l_{1,1}x)nx^{n-1}$ and therefore, the leading coefficient is $nl_{1,1}$. With $k = 2$, Eq. (1.10) becomes $L_2(x)n(n-1)x^{n-2} = (l_{2,0} + l_{2,1}x + l_{2,2}x^2)n(n-1)x^{n-2}$, which yields $n(n-1)l_{2,2}$ as the leading coefficient. Continuing in this fashion, we realize that for $k = n$ the leading coefficient of Eq. (1.10) is $n!l_{n,n}$. So, the coefficient of $x^n$ is

$$l_{0,0} + nl_{1,1} + n(n-1)l_{2,2} + \cdots + n!l_{n,n}, \quad n = 0,1,2,\ldots. \tag{1.11}$$

We next successively compare (1.11) against $J[x^n]$ for $n = 0,1,2,\ldots$. For $n = 0$, Eq. (1.11) becomes $l_{0,0}$ and it must be that $l_{0,0} = 0$, since $J[1] = 0$, and thus $L_0(x) = 0$. With $n = 1$, Eq. (1.11) turns out to be $l_{1,1}$, which must be equal to zero, as $J[x] = \text{const}$. Continuing in this manner, it follows that $l_{j,j} = 0$ for $j = 0,1,2,\ldots$, thus establishing (1.8).

Then, using the same logic that was used to determine the coefficient of $x^n$, we achieve the coefficient of $x^{n-1}$ in Eq. (1.10), which we call $\lambda_n$, to be

$$\lambda_n := nl_{1,0} + n(n-1)l_{2,1} + \cdots + n!l_{n,n-1}, \quad n = 0,1,2,\ldots.$$

Since we have already shown that the coefficient of $x^n$ is zero, in order to have $\deg(J[x^n]) = n-1$ we must also require that $\lambda_n \neq 0$, thus proving the necessity of the statement.

($\Leftarrow$) From substituting Eq. (1.8) with the restriction (1.9) into Eq. (1.7), the sufficiency of the statement is immediate.

□

Due to Lemma 1.2, we can now modify the structure of Eq. (1.6), since our primary concern is when $\deg(J[x^n]) = n-1$. We have

$$J[y(x)] = \sum_{n=1}^{\infty} [l_{n,0} + l_{n,1}x + \cdots + l_{n,n-1}x^{n-1}] \frac{\mathrm{d}^n}{\mathrm{d}x^n} y(x), \quad \lambda_n \neq 0, \quad n = 1,2,\ldots. \tag{1.12}$$

The summation above starts at 1 via $L_0(x) = 0$.

Next, given a set of polynomials $\mathcal{S} = \{P_n(x)\}_{n=0}^{\infty}$, we want to determine how many operators $J$ exist, such that $J$ transforms each polynomial $P_k(x) \in \mathcal{S}$ to the polynomial immediately preceding it in the sequence, i.e., to $P_{k-1}(x) \in \mathcal{S}$. As it turns out, there is exactly one such operator.

**Theorem 1.1.** *For a given polynomial set $\mathcal{S} = \{P_n(x)\}_{n=0}^{\infty}$, there exists a unique operator $J$ such that*

$$J[P_n(x)] = P_{n-1}(x), \qquad n = 1,2,\ldots \tag{1.13}$$

*with $J[P_0(x)] := 0$.*

*Proof.* To show the existence and uniqueness of $J$, we substitute $y(x) = P_n(x) \in \mathcal{S}$ into Eq. (1.12), which yields

$$J[P_n(x)] = \sum_{k=1}^{n} [l_{k,0} + l_{k,1}x + \cdots + l_{k,k-1}x^{k-1}] \frac{\mathrm{d}^k}{\mathrm{d}x^k} P_n(x).$$

Moreover, from Eq. (1.13), we require

$$\sum_{k=1}^{n} [l_{k,0} + l_{k,1}x + \cdots + l_{k,k-1}x^{k-1}] \frac{\mathrm{d}^k}{\mathrm{d}x^k} P_n(x) = P_{n-1}(x). \tag{1.14}$$

Therefore, upon successively comparing the coefficients of Eq. (1.14) for $n = 1,2,\ldots$ we see that the sequence $\{l_{i,j}\}$ is uniquely determined, thus establishing the uniqueness of $J$ given $\mathcal{S}$. □

We say that the set $\{P_n(x)\}_{n=0}^{\infty}$ ***corresponds*** to the operator $J$ if Eq. (1.13) is satisfied.

*Example 1.1.* To concretely demonstrate how the sequence $\{l_{i,j}\}$ is uniquely constructed, we consider $J$ as in Eq. (1.12) acting on the Appell set

$$\mathcal{S} = \{x^n/n!\}_{n=0}^{\infty}.$$

First, for $n = 0$, we see that $J[1] = 0$ gives us no information. For $n = 1$, we require $J[x] = 1$. Therefore,

$$J[x] = \sum_{k=1}^{1} [l_{k,0} + l_{k,1}x + \cdots + l_{k,k-1}x^{k-1}] \frac{\mathrm{d}^k}{\mathrm{d}x^k}[x] = l_{1,0} \cdot 1,$$

which implies that $l_{1,0} = 1$. Then, for $n = 2$, we must have $J[x^2/2!] = x$ and thus

$$\begin{aligned} J\left[\frac{x^2}{2!}\right] &= \sum_{k=1}^{2} [l_{k,0} + l_{k,1}x + \cdots + l_{k,k-1}x^{k-1}] \frac{\mathrm{d}^k}{\mathrm{d}x^k}\left[\frac{x^2}{2!}\right] \\ &= l_{1,0} \cdot x + (l_{2,0} + l_{2,1}x) \cdot 1 \\ &= l_{2,0} + (1 + l_{2,1})x. \end{aligned}$$

Therefore, $l_{2,0} = l_{2,1} = 0$. Next, for $n = 3$, we see that $J[x^3/3!] = x^2/2!$ must hold. Hence,

$$\begin{aligned} J\left[\frac{x^3}{3!}\right] &= \sum_{k=1}^{3} [l_{k,0} + l_{k,1}x + \cdots + l_{k,k-1}x^{k-1}] \frac{\mathrm{d}^k}{\mathrm{d}x^k}\left[\frac{x^3}{3!}\right] \\ &= l_{1,0} \cdot \frac{x^2}{2!} + (l_{2,0} + l_{2,1}x) \cdot x + (l_{3,0} + l_{3,1}x + l_{3,2}x^2) \cdot 1 \\ &= l_{3,0} + l_{3,1}x + \left(\frac{1}{2!} + l_{3,2}\right)x^2. \end{aligned}$$

Therefore, it must be that $l_{3,0} = l_{3,1} = l_{3,2} = 0$.

In fact, continuing in this fashion, the interested reader can readily show that all of the $l$-values in this sequence will be uniquely determined to be zero, except $l_{1,0} = 1$. This is certainly clear since

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x^n}{n!}\right] = \frac{x^{n-1}}{(n-1)!}.$$

We next prove the converse of Theorem 1.1.

**Theorem 1.2.** *Associated to each operator $J$ are infinitely many sets $\{P_n(x)\}_{n=0}^{\infty}$ such that Eq. (1.13) holds. More specifically, exactly one of these sets $\{B_n(x)\}_{n=0}^{\infty}$, entitled the* **basic set***, is such that*

$$B_0(x) = 1 \quad \textit{and} \quad B_n(0) = 0, \qquad n = 0, 1, 2, \ldots.$$

*Proof.* Let $Q_m(x)$ be a polynomial such that $\deg(Q_m) = m$. Then, via Eq. (1.12), we can construct a polynomial, say $P_{m+1}(x)$, such that $J[P_{m+1}(x)] = Q_m(x)$, where $\deg(P_{m+1}(x)) = m+1$. However, since $J[\text{const}] = 0$, the polynomial $P_{m+1}(x)$ is unique only up to an additive constant. This proves the infinitude of sets $\{P_n(x)\}_{n=0}^{\infty}$ that correspond to a given $J$.

By assigning $B_0(x) := 1$ and assuming $B_n(0) = 0$ for $n > 0$, one can successively and uniquely determine the set $\{B_n(x)\}_{n=0}^{\infty}$ such that $J[B_n(x)] = B_{n-1}(x)$ and $\deg(B_n(x)) = n$.

□

The next result states that for $\{P_n(x)\}_{n=0}^{\infty}$ to be a set corresponding to $J$, it must be expressed as a linear combination of polynomials from the basic set. However, the scalers in this linear combination appear in a special way and play a key role in the later characterizations.

**Theorem 1.3.** *A necessary and sufficient condition that $\{P_n(x)\}_{n=0}^{\infty}$ be a set corresponding to $J$ is that there exist a sequence of constants $\{a_k\}$ such that*

$$P_n(x) = a_0 B_n(x) + a_1 B_{n-1}(x) + \cdots + a_n B_0(x), \qquad a_0 \neq 0. \tag{1.15}$$

*Proof.* ($\Rightarrow$) Assume that $\{P_n(x)\}_{n=0}^{\infty}$ satisfies (1.15). Therefore, $\deg(P_n(x)) = n$ and by the linearity of $J$ we have

$$J[P_n(x)] = J\left[\sum_{i=0}^{n} a_i B_{n-i}(x)\right] = \sum_{i=0}^{n} a_i J[B_{n-i}(x)] = \sum_{i=0}^{n-1} a_i B_{n-i-1}(x) = P_{n-1}(x),$$

which follows since $J[B_0(x)] = 0$ and $J[B_n(x)] = B_{n-1}(x)$ for $n > 0$ via Theorem 1.2. Thus $J[P_n(x)] = P_{n-1}(x)$.
($\Leftarrow$) We now assume that $\{P_n(x)\}_{n=0}^{\infty}$ corresponds to $J$. Since $\deg(B_n(x)) = n$, given $P_n(x)$, there must exist constants $\{a_{n,k}\}_{k=0}^{n}$ such that

$$P_n(x) = a_{n,0} B_n(x) + a_{n,1} B_{n-1}(x) + \cdots + a_{n,n} B_0(x), \qquad a_{n,0} \neq 0.$$

We act on this relation with $J$ and see that the left-hand side becomes

$$J[P_n(x)] = P_{n-1}(x) = a_{n-1,0} B_{n-1}(x) + a_{n-1,1} B_{n-2}(x) + \cdots + a_{n-1,n-1} B_0(x)$$

and since $J[B_0(x)] = 0$, we see that the right-hand side turns into

$$a_{n,0} B_{n-1}(x) + a_{n,1} B_{n-2}(x) + \cdots + a_{n,n-1} B_0(x).$$

Therefore, from comparing coefficients of the results directly above, we infer that

$$a_{n,k} = a_{n-1,k}, \qquad k = 0, 1, \ldots, n-1.$$

Next, we momentarily fix $k$. Then, the relation immediately above implies that for all $n \geq k$ each $a_{n,k}$ is equal to $a_{n-1,k}$. Thus, the first index in the series $\{a_{n,k}\}$ is superfluous and thus can be omitted. We conclude that $\{a_k\}$ exist such that Eq. (1.15) is satisfied. □

It may at first appear counterintuitive that the elements of the sequence $\{a_k\}$ appear as they do in Eq. (1.15). However, the proof of Theorem 1.3 shows why this is the case. For emphasis, consider expressing a polynomial $P_n(x)$ corresponding to $J$ as a linear combination of basic polynomials in the following "natural" way:

$$P_n(x) = a_n B_n(x) + a_{n-1} B_{n-1}(x) + \cdots + a_0 B_0(x).$$

Then,

$$J[P_n(x)] = J\left[\sum_{i=0}^{n} a_i B_i(x)\right] = \sum_{i=0}^{n} a_i J[B_i(x)] = \sum_{i=0}^{n} a_i B_{i-1}(x) \neq P_{n-1}(x).$$

We next wish to determine what conditions are needed for a set $\{Q_n(x)\}_{n=0}^{\infty}$ to correspond to $J$, given that a set $\{P_n(x)\}_{n=0}^{\infty}$ corresponds to $J$. As it turns out, $Q_n(x)$ must be written as a linear combination of $\{P_k(x)\}_{k=0}^{n}$.

**Corollary 1.1.** *Given that $\{P_n(x)\}_{n=0}^{\infty}$ is a set corresponding to $J$, a necessary and sufficient condition that $\{Q_n(x)\}_{n=0}^{\infty}$ also corresponds to $J$ is that constants $\{b_k\}$ exist such that*

$$Q_n(x) = b_0 P_n(x) + b_1 P_{n-1}(x) + \cdots + b_n P_0(x), \qquad b_0 \neq 0.$$

*Proof.* The proof is similar to that of Theorem 1.3 and is left as an exercise for the reader. □

In light of the preceding theorems, we now state the definition of Sheffer *Type k*.

**Definition 1.6.** Let the set $\mathcal{S} := \{P_n(x)\}_{n=0}^{\infty}$ correspond to the unique operator $J$. Then, $\mathcal{S}$ is of ***Sheffer Type k***, or simply ***Type k***, if the coefficients $\{L_j(x)\}_{j=0}^{\infty}$ in Eq. (1.12) are such that $\deg(L_j(x)) \leq k$ for all $j$ and there exists at least one $L_i(x) \in \{L_j(x)\}_{j=0}^{\infty}$ such that $\deg(L_i(x)) = k$. If $\{L_j(x)\}_{j=0}^{\infty}$ is unbounded, then $\mathcal{S}$ is of ***Type*** $\infty$.

With this definition, we have the following result.

**Theorem 1.4.** *There exist infinitely many sets for each Sheffer* Type k *(k finite or infinite).*

*Proof.* We know from Theorem 1.2 that associated to each operator $J$ are infinitely many sets $\{P_n(x)\}_{n=0}^{\infty}$ such that Eq. (1.13) holds. This result is entirely independent of the degrees of the coefficients $\{L_j(x)\}_{j=0}^{\infty}$ in Eq. (1.12) and therefore the *Type*. Hence, Theorem 1.2 holds for all $\{L_j(x)\}_{j=0}^{\infty}$, even if it is unbounded, so there are infinitely many sets of every *Type* (finite or infinite). □

We now consider what effect replacing $\mathcal{S} := \{P_n(x)\}_{n=0}^{\infty}$ with $\{c_n P_n(x)\}_{n=0}^{\infty}$ has on the *Type* classification of $\mathcal{S}$. Assuming that $\mathcal{S}$ corresponds to $J$, we immediately observe that

$$J[c_n P_n(x)] = c_n J[P_n(x)] = c_n P_{n-1}(x) \neq c_{n-1} P_{n-1}(x).$$

This simple manipulation tells us that the *Type* is not necessarily preserved since we may need a new operator, say $\breve{J}$, such that $\breve{J}[c_n P_n(x)] = c_{n-1}P_{n-1}(x)$. We demonstrate this concretely in the following examples.

*Example 1.2.* In Example 1.1, we analyzed the Appell set $\mathcal{S} := \{x^n/n!\}_{n=0}^{\infty}$ and showed that $l_{1,0} = 1$ and that every other $l$-value was zero by utilizing (1.12). Moreover, we actually showed that $L_1(x) = 1$ and $L_j(x) \equiv 0$ for $j = 2, 3, \ldots$ and thus, that $\mathcal{S}$ is a *Type 0* set, since $k = 0$ in Definition 1.6. We next consider the set $\breve{\mathcal{S}} := \{c_n x^n/n!\}_{n=0}^{\infty}$, where $c_i \neq 0$ and each $c_i$ is distinct.

For $n = 1$, we require $J[c_1 x] = c_0$. Therefore,

$$J[c_1 x] = \sum_{k=1}^{1} [l_{k,0} + l_{k,1}x + \cdots + l_{k,k-1}x^{k-1}] \frac{\mathrm{d}^k}{\mathrm{d}x^k}[c_1 x] = l_{1,0} \cdot c_1,$$

which implies that $l_{1,0} = c_0/c_1$ and therefore $L_1(x) = c_0/c_1$. Then, for $n = 2$, we must have $J[c_2 x^2/2!] = c_1 x$ and thus

$$\begin{aligned} J\left[\frac{c_2 x^2}{2!}\right] &= \sum_{k=1}^{2} [l_{k,0} + l_{k,1}x + \cdots + l_{k,k-1}x^{k-1}] \frac{\mathrm{d}^k}{\mathrm{d}x^k} \left[\frac{c_2 x^2}{2!}\right] \\ &= l_{1,0} \cdot c_2 x + (l_{2,0} + l_{2,1}x) \cdot c_2 \\ &= l_{2,0}c_2 + (c_0 c_2/c_1 + l_{2,1}c_2)x. \end{aligned}$$

So, $l_{2,0} = 0$ and $l_{2,1} = (c_1^2 - c_0 c_2)/(c_1 c_2)$ giving $L_1(x) = \frac{(c_1^2 - c_0 c_2)}{c_1 c_2} x$. Therefore, we already see that $\breve{\mathcal{S}}$ is *not* a *Type 0* set. In fact, the interested reader can show that $\breve{\mathcal{S}}$ is actually *Type* $\infty$.

*Example 1.3.* We next consider a very important *Type 0* set, the importance of which will become most evident upon the completion of Sect. 1.3. This set is defined as $\mathcal{H}_n(x) := 2^{-n}H_n(x)/n!$, where

$$H_n(x) := 2^n n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k x^{n-2k}}{2^{2k} k!(n-2k)!}$$

are the Hermite polynomials. For convenience, we write out the first four polynomials from the set $\{\mathcal{H}_n(x)\}_{n=0}^{\infty}$:

$$\mathcal{H}_0(x) = 1, \quad \mathcal{H}_1(x) = x, \quad \mathcal{H}_2(x) = \frac{1}{2}x^2 - \frac{1}{4} \quad \text{and} \quad \mathcal{H}_3(x) = \frac{1}{6}x^3 - \frac{1}{4}x.$$

We initially see that $J[\mathcal{H}_1(x)] = \mathcal{H}_0(x)$ implies

$$J[\mathcal{H}_1(x)] = \sum_{k=1}^{1} [l_{k,0} + l_{k,1}x + \cdots + l_{k,k-1}x^{k-1}] \frac{\mathrm{d}^k}{\mathrm{d}x^k} [\mathcal{H}_1(x)] = l_{1,0} = 1$$

and therefore, $L_1(x) = 1$. Then, $J[\mathcal{H}_2(x)] = \mathcal{H}_1(x)$ yields

$$J[\mathcal{H}_2(x)] = \sum_{k=1}^{2} [l_{k,0} + l_{k,1}x + \cdots + l_{k,k-1}x^{k-1}] \frac{\mathrm{d}^k}{\mathrm{d}x^k} [\mathcal{H}_2(x)]$$
$$= l_{2,0} + (1 + l_{2,1}x)x = x$$

and it must be that $l_{2,0} = l_{2,1} = 0$, i.e., $L_2(x) = 0$. Continuing, one concludes that $L_1(x) = 1$ and $L_j(x) = 0$ for $j = 2, 3, \ldots$ and thus $\{\mathcal{H}_n(x)\}_{n=0}^{\infty}$ is a *Type 0* set. Moreover, writing (1.12) in the operator form

$$J = \sum_{k=1}^{\infty} L_k(x) \frac{\mathrm{d}^k}{\mathrm{d}x^k}, \tag{1.16}$$

we find the unique operator $J$ for our current set to be $J = \mathrm{d}/\mathrm{d}x$.

*Example 1.4.* Now consider the set $\mathrm{H}_n(x) := H_n(x)/(n!)^2$, where $\{H_n(x)\}_{n=0}^{\infty}$ are the Hermite polynomials as defined in the last example. The first four polynomials from the set $\{\mathrm{H}_n(x)\}_{n=0}^{\infty}$ are

$$\mathrm{H}_0(x) = 1, \quad \mathrm{H}_1(x) = 2x, \quad \mathrm{H}_2(x) = x^2 - \frac{1}{2} \quad \text{and} \quad \mathrm{H}_3(x) = \frac{2}{9}x^3 - \frac{1}{3}x.$$

Using the same procedure as in Examples 1.2 and 1.3, one can show that $L_1(x) = 1/2$, $L_2(x) = \frac{1}{2}x$, $L_3(x) = -1/4$, and $L_j(x) = 0$ for $j = 4, 5, \ldots$. Thus, $\{\mathrm{H}_n(x)\}_{n=0}^{\infty}$ is a *Type 1* set and Eq. (1.16) becomes

$$J = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dx}} + \frac{1}{2}x\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{1}{4}\frac{\mathrm{d}^3}{\mathrm{d}x^3}.$$

Next, we notice that if $\{P_n(x)\}_{n=0}^{\infty}$ is a *Type 0* set, then each $L_n(x)$ must be a constant and we can therefore restate Definition 1.6 specifically for *Type 0* sets as follows.

**Definition 1.7.** $\{P_n(x)\}_{n=0}^{\infty}$ is a ***Type 0*** set if Eq. (1.13) holds with $J$ defined by

$$J[y(x)] := \sum_{n=1}^{\infty} c_n y^{(n)}(x), \qquad c_1 \neq 0. \tag{1.17}$$

We emphasize that as we have seen in Examples 1.2–1.4, Eq. (1.17) may or may not terminate, i.e., it may be finite *Type k* or *Type* $\infty$. We also have the following definition.

**Definition 1.8.** Let $J(t)$ be the formal power series

$$J(t) := \sum_{n=1}^{\infty} c_n t^n, \qquad c_1 \neq 0,$$

which we entitle the **generating function for *J***, with $J$ as in Eq. (1.17).

Now, let the formal power series inverse of $J(t)$ be

$$H(t) := \sum_{n=1}^{\infty} s_n t^n, \quad s_1 = c_1^{-1} \neq 0. \tag{1.18}$$

This is a valid definition because if $J(t)$ is formally substituted for $t$ in (1.18) and the coefficients are collected to form a single power series in $t$, then the coefficient of each $t^n$ is a polynomial in $c_1, c_2, \ldots, c_n, s_1, s_2, \ldots, s_n$. Therefore, we can choose each $s_n$ recursively and uniquely as a function of $c_1, c_2, \ldots, c_n, s_1, s_2, \ldots, s_{n-1}$ so the series has a single term $t$, i.e.,

$$J(H(t)) = H(J(t)) = t.$$

In considering $\exp(xH(t))$, we see that each coefficient of $t^n$ in the formal power series expansion only comprises $s_1, s_2, \ldots, s_n$. Upon multiplying $\exp(xH(t))$ by

$$A(t) := \sum_{n=0}^{\infty} a_n t^n, \ \ a_0 \neq 0,$$

we achieve a series in $t$ where the coefficient of each $t^n$ involves elements of the sequences $a_1, a_2, \ldots, a_n$ and $s_1, s_2, \ldots, s_n$, such that each coefficient of $t^n$ is a polynomial in $x$ of degree exactly $n$, adhering to Definition 1.1. This leads to the main result of this section.

**Theorem 1.5.** *The set $\{P_n(x)\}_{n=0}^{\infty}$ corresponds to the operator $J$ and is of Sheffer* Type 0 *if and only if the sequence $\{a_n\}_{n=0}^{\infty}$ exists such that*

$$A(t)\mathrm{e}^{xH(t)} = \sum_{n=0}^{\infty} P_n(x) t^n, \tag{1.19}$$

*where*

$$A(t) := \sum_{n=0}^{\infty} a_n t^n, \ \ a_0 = 1 \quad \text{and} \quad H(t) := \sum_{n=1}^{\infty} s_n t^n, \ \ s_1 = 1. \tag{1.20}$$

*Proof.* We show that both necessity and sufficiency will follow if we prove that the basic set $\{B_n(x)\}_{n=0}^{\infty}$ corresponding to $J$ in Eq. (1.13) has the following generating function:

$$\mathrm{e}^{xH(t)} = \sum_{n=0}^{\infty} B_n(x) t^n. \tag{1.21}$$

Since $\exp[xH(t)] = \sum_{n=0}^{\infty} [H^n(t) x^n / n!]$, we let this expansion have the form

$$e^{xH(t)} = \sum_{n=0}^{\infty} C_n(x)t^n. \tag{1.22}$$

Then, $\{C_n(x)\}_{n=0}^{\infty}$ must be such that $\deg(C_n(x)) = n$. We show that $\{C_n(x)\}_{n=0}^{\infty}$ is the basic set.

Upon setting $x = 0$ in Eq. (1.22) and comparing coefficients, we see immediately that $C_0(0) = 1$, and therefore $C_0(x) = 1$, and that $C_n(0) = 0$ for $n > 0$. Thus, $\{C_n(x)\}_{n=0}^{\infty}$ satisfies the initial conditions of the basic set. We clearly have $J[C_0(x)] = 0$ and next show that $J[C_n(x)] = C_{n-1}(x)$ for $n = 1,2,3,\ldots$. We apply $J$ to Eq. (1.22):

$$\begin{aligned}
J\left[\sum_{n=0}^{\infty} C_n(x)t^n\right] &= \sum_{n=0}^{\infty} J[C_n(x)]t^n = J\left[e^{xH(t)}\right] = \sum_{n=0}^{\infty} c_n \frac{d^n}{dx^n}\left[e^{xH(t)}\right] \\
&= c_1 \frac{d}{dx}\left[e^{xH(t)}\right] + c_2 \frac{d^2}{dx^2}\left[e^{xH(t)}\right] + c_3 \frac{d^3}{dx^3}\left[e^{xH(t)}\right] + \cdots \\
&= \left(c_1 H(t) + c_2 H^2(t) + c_3 H^3(t) + \cdots\right) e^{xH(t)} \\
&= J(H(t))e^{xH(t)} = te^{xH(t)} = \sum_{n=0}^{\infty} C_n(x)t^{n+1} = \sum_{n=0}^{\infty} C_{n-1}(x)t^n
\end{aligned}$$

(with $C_{-1}(x) := 0$), which follows since $H(t)$ and $J(t)$ are formal inverses of one another. By comparing coefficients of $t^n$ in the relation directly above, we achieve $J[C_n(x)] = C_{n-1}(x)$ for $n = 1,2,3,\ldots$ and hence, $C_n(x) \equiv B_n(x)$ and Eq. (1.21) is established. Lastly, we now multiply Eq. (1.21) by $A(t)$ as in Eq. (1.20), which yields

$$A(t)e^{xH(t)} = \sum_{n=0}^{\infty} a_n t^n \sum_{n=0}^{\infty} B_n(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k B_{n-k}(x)t^n = \sum_{n=0}^{\infty} P_n(x)t^n, \tag{1.23}$$

as a result of Theorem 1.3. □

We also have the following.

**Theorem 1.6.** *The sequence of $a_k$-terms in Eq.* (1.19) *is exactly the same as those in Eq.* (1.15).

*Proof.* This statement follows immediately from the proof of Theorem 1.5, i.e., Eq. (1.23). □

The next result plays an integral part in determining all of the Sheffer *Type 0* orthogonal polynomials in the next section. This characterization is also interesting unto itself, as it is expressed entirely in terms of elements of the generating function (1.19).

**Corollary 1.2.** $\{P_n(x)\}_{n=0}^{\infty}$ *is of* Type 0 *if and only if sequences* $\{q_{k,0}\}$ *and* $\{q_{k,1}\}$ *exist such that*

$$\sum_{k=1}^{\infty}(q_{k,0}+q_{k,1}x)P_{n-k}(x)=nP_n(x), \quad n=1,2,\ldots, \tag{1.24}$$

*where*

$$\left.\begin{array}{l}\sum_{k=0}^{\infty} q_{k+1,0}t^k = A'(t)/A(t)\\ \sum_{k=0}^{\infty} q_{k+1,1}t^k = H'(t)\end{array}\right\} \tag{1.25}$$

*with* $A(t)$ *and* $H(t)$ *as defined in Eq.* (1.20).

*Proof.* We assume that $\{P_n(x)\}_{n=0}^{\infty}$ is a *Type 0* set. We first consider the right-hand side of Eq. (1.24) and multiply it by $t^n$ and sum for $n=0,1,2,\ldots$. This yields

$$\sum_{n=0}^{\infty} nP_n(x)t^n = t\sum_{n=0}^{\infty} nP_n(x)t^{n-1} = t\frac{\mathrm{d}}{\mathrm{d}t}\left(A(t)\mathrm{e}^{xH(t)}\right) = t\mathrm{e}^{xH(t)}\left(xA(t)H'(t)+A'(t)\right). \tag{1.26}$$

Since $J^k[P_n(x)] = P_{n-k}(x)$, we see that

$$\sum_{n=0}^{\infty} J^k[P_n(x)]t^n = t^k\sum_{n=k}^{\infty} P_{n-k}(x)t^{n-k} = t^k\sum_{m=0}^{\infty} P_m(x)t^m = t^kA(t)\mathrm{e}^{xH(t)}.$$

Therefore, we now multiply the left-hand side of Eq. (1.24) by $t^n$, sum for $n=0,1,2,\ldots$, and utilize (1.20) to obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(q_{k,0}+q_{k,1}x)P_{n-k}(x)t^n &= \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(q_{k,0}+q_{k,1}x)J^k[P_n(x)]t^n\\ &= \sum_{k=1}^{\infty}(q_{k,0}+q_{k,1}x)\sum_{n=0}^{\infty}J^k[P_n(x)]t^n = \sum_{k=1}^{\infty}(q_{k,0}+q_{k,1}x)t^kA(t)\mathrm{e}^{xH(t)}\\ &= \left(t\sum_{k=0}^{\infty}q_{k+1,0}t^k + tx\sum_{k=0}^{\infty}q_{k+1,1}t^k\right)A(t)\mathrm{e}^{xH(t)} = t\mathrm{e}^{xH(t)}\left(xA(t)H'(t)+A'(t)\right).\end{aligned} \tag{1.27}$$

Hence, we see that both Eqs. (1.26) and (1.27) are formally equal and from comparing coefficients of $t^n$ we obtain Eq. (1.24). □

## 1.3 The Type 0 Orthogonal Polynomials

We now determine which Sheffer *Type 0* sets are also orthogonal. We first assume that $\{Q_n(x)\}_{n=0}^{\infty}$ is an orthogonal set that satisfies the monic three-term relation of the form Eq. (1.4), which we write using the notation in [9]:

$$Q_n(x) = (x+\lambda_n)Q_{n-1}(x) + \mu_n Q_{n-2}(x), \quad \mu_n \neq 0, \quad n = 1,2,\ldots. \tag{1.28}$$

In our first theorem of this section, we determine a necessary and sufficient form that the recursion coefficients $\lambda_n$ and $\mu_n$ in Eq. (1.28) must have in order to characterize a *Type 0* orthogonal set. First, we note that since $\{Q_n(x)\}_{n=0}^{\infty}$ is an orthogonal set, then so is $\{c_n Q_n(x)\}_{n=0}^{\infty}$. Now, we want to determine for which sets $\{Q_n(x)\}_{n=0}^{\infty}$ satisfying (1.28) do there exist a sequence of constants $\{c_n\}_{n=0}^{\infty}$ such that

$$P_n(x) = c_n Q_n(x), \qquad n = 0,1,2,\ldots \tag{1.29}$$

is a *Type 0* set. To accomplish this, we simultaneously use Eqs. (1.24) and (1.28), i.e., a characterization of a *Type 0* set and a characterization of an orthogonal set.

**Theorem 1.7.** *A necessary and sufficient condition that an orthogonal polynomial set $\{Q_n(x)\}_{n=0}^{\infty}$ satisfying (1.28) be such that $P_n(x) = c_n Q_n(x)$ is of* Type 0 *for some $c_n \neq 0$ is that $\lambda_n$ and $\mu_n$ have the form*

$$\lambda_n = \alpha + bn \quad \text{and} \quad \mu_n = (n-1)(c+dn) \tag{1.30}$$

*with $c + dn \neq 0$ for $n > 1$.*

*Proof.* $(\Rightarrow)$ We assume that Eq. (1.29) holds and show that $\lambda_n$ and $\mu_n$ are as in Eq. (1.30). To begin, replace $n$ with $n-k+1$ in Eq. (1.28) and rewrite it in the following way:

$$xQ_{n-k}(x) = Q_{n-k+1}(x) - \lambda_{n-k+1}Q_{n-k}(x) - \mu_{n-k+1}Q_{n-k-1}(x). \tag{1.31}$$

Then, substituting the right-hand side of Eq. (1.29) into Eq. (1.24) and using Eq. (1.31), we see that

$$\begin{aligned} nc_nQ_n(x) &= \sum_{k=1}^{\infty} q_{k,0}c_{n-k}Q_{n-k}(x) + \sum_{k=1}^{\infty} q_{k,1}c_{n-k}[xQ_{n-k}(x)] \\ &= \sum_{k=1}^{\infty} q_{k,0}c_{n-k}Q_{n-k}(x) + \sum_{k=1}^{\infty} q_{k,1}c_{n-k}Q_{n-k+1}(x) \\ &\quad - \sum_{k=1}^{\infty} q_{k,1}c_{n-k}\lambda_{n-k+1}Q_{n-k}(x) - \sum_{k=1}^{\infty} q_{k,1}c_{n-k}\mu_{n-k+1}Q_{n-k-1}(x). \end{aligned} \tag{1.32}$$

We next compare coefficients of $Q_j(x)$ for $j = n, n-1, n-2$ using Eq. (1.32) to obtain relationships involving the $q$'s and in turn develop expressions for the recursion coefficients $\lambda_n$ and $\mu_n$. First, by comparing the coefficients of $Q_n(x)$ we see that $nc_n = c_{n-1}q_{1,1}$ and iterating this relationship, we obtain

$$c_n = c_0 q_{1,1}^n / n!. \tag{1.33}$$

After comparing the coefficients of $Q_{n-1}(x)$, we realize that

$$0 = q_{1,0}c_{n-1} + q_{2,1}c_{n-2} - q_{1,1}c_{n-1}\lambda_n.$$

Dividing both sides of this equation by $c_{n-2}$ and using Eq. (1.33) yield

$$\lambda_n = \left(q_{1,0}q_{1,1} + q_{2,1}(n-1)\right)/q_{1,1}^2. \tag{1.34}$$

Lastly, upon comparing the coefficients of $Q_{n-2}(x)$, we achieve

$$0 = q_{2,0}c_{n-2} + q_{3,1}c_{n-3} - q_{2,1}c_{n-2}\lambda_{n-1} - q_{1,1}c_{n-1}\mu_n.$$

We also divide both sides of this equation by $c_{n-2}$, use Eq. (1.33), and also call upon Eq. (1.34) to obtain

$$\mu_n = (n-1)\left(q_{2,0}q_{1,1}^2 - q_{1,0}q_{1,1}q_{2,1} - (q_{1,1}^2 - q_{1,1}q_{3,1})(n-2)\right)/q_{1,1}^4. \tag{1.35}$$

Thus, we see that $\lambda_n$ is at most linear in $n$ and that $\mu_n$ is at most quadratic in $n$ with a factor of $(n-1)$, i.e., $\lambda_n$ and $\mu_n$ satisfy (1.30).
($\Leftarrow$) We assume that $\lambda_n$ and $\mu_n$ agree with Eq. (1.30). We show that Eq. (1.29) is of *Type 0*. Now, if in Eq. (1.28) we replace $Q_n(x)$ with $c_n d_n Q_n(x) = d_n P_n(x)$ where $c_n := \alpha^n/n!$ $(\alpha \neq 0)$ and $d_n := c_n^{-1}$ then our three-term recurrence relation (1.28), with $\lambda_n$ and $\mu_n$ as in Eq. (1.30), remains unaltered and we can therefore write

$$d_n P_n(x) = (x + a + bn)d_{n-1}P_{n-1}(x) + (n-1)(c + dn)d_{n-2}P_{n-2}(x).$$

Dividing both sides of this relation by $d_n$ defined above gives a relation of the form

$$nP_n(x) = (\alpha x + \beta + \gamma n)P_{n-1}(x) + (\delta + \varepsilon n)P_{n-2}(x),$$

where $\alpha \neq 0$, $\delta + \varepsilon n \neq 0$ for $n > 1$ and with $\beta = a\alpha^2$, $\gamma = b\alpha$, $\delta = c\alpha^2$, and $\varepsilon = d\alpha^2$. This, of course, can be written in the form

$$xP_{n-1}(x) = \alpha^{-1}nP_n(x) - \alpha^{-1}(\beta + \gamma n)P_{n-1}(x) - \alpha^{-1}(\delta + \varepsilon n)P_{n-2}(x).$$

That is, we have expressed $xP_{n-1}(x)$ as a linear combination of $P_n(x)$, $P_{n-1}(x)$, and $P_{n-2}(x)$. Therefore, sequences $\{q_{k,0}\}$ and $\{q_{k,1}\}$ exist such that

$$T_n := (q_{1,0} + q_{1,1}x)P_{n-1}(x) + (q_{2,0} + q_{2,1}x)P_{n-2}(x) + \cdots = nP_n(x).$$

Furthermore, it can be shown that these $q$'s are related in the following way:

$$q_{k+2,1} = \gamma q_{k+1,1} + \varepsilon q_{k,1}; \quad q_{1,1} = \alpha, \;\; q_{2,1} = \alpha\gamma, \tag{1.36}$$

$$q_{k+1,0} = \frac{1}{\alpha}\left(q_{k,1}(\delta-(k-1)\varepsilon)+q_{k+1,1}(\beta-\gamma k)+q_{k+2,1}(k+1)\right). \quad (1.37)$$

Thus, we have shown that $\{P_n(x)\}_{n=0}^{\infty}$ satisfies Corollary 1.2 and is therefore a *Type 0* set. □

As it turns out, we need the relations (1.36) and (1.37) to determine all of the *Type 0* orthogonal sets. Now, if Eq. (1.36) is known, then Eq. (1.37) can be determined. Therefore, we solve (1.36) by considering the characteristic equation

$$u^2-\gamma u-\varepsilon=0, \quad (1.38)$$

which has the solution

$$u_{1,2} = \left(\gamma \pm \sqrt{\gamma^2+4\varepsilon}\right)/2.$$

We analyze the different cases of the discriminant $\gamma^2+4\varepsilon$ and determine the value of $q_{k,1}$, and therefore $q_{k+1,0}$, in each case. Thus, in each case, this leads to expressions for $A(t)$ and $H(t)$ in Eq. (1.20) and therefore the generating function (1.19), which yields the corresponding orthogonal set. For the repeated root cases $(\gamma^2+4\varepsilon=0)$ we use the general solution structure $q_{k,1}=Au_1^k+Bku_1^k$ and for the distinct root cases $(\gamma^2+4\varepsilon\neq 0)$, we use the general solution structure $q_{k,1}=Au_1^k+Bu_2^k$, where $u_1$ and $u_2$ are the roots listed above. There are a total of four cases, which we label as Ia, Ib, IIa and IIb. We work out the rigorous details of Case Ia and summarize the remaining cases.

*Case Ia.* $\gamma^2+4\varepsilon=0$ and $\gamma\neq 0$
In this case, we see that Eq. (1.38) yields $u_1=u_2=\gamma/2$ as a solution, which implies

$$q_{k,1}=A(\gamma/2)^k+Bk(\gamma/2)^k.$$

Using our initial conditions in Eq. (1.36), we have the following system:

$$q_{1,1}=\frac{1}{2}A\gamma+\frac{1}{2}B\gamma=\alpha,$$
$$q_{2,1}=\frac{1}{4}A\gamma^2+\frac{1}{4}B\gamma^2=\alpha\gamma,$$

which has the solutions $A=0$ and $B=2\alpha/\gamma$ and thus,

$$q_{k,1}=k\alpha(\gamma/2)^{k-1}. \quad (1.39)$$

By substituting the right-hand side of Eq. (1.36) into Eq. (1.37) and then using Eq. (1.39) accordingly, after some algebraic manipulations, we obtain the following:

$$q_{k+1,0}=\left(\frac{\gamma}{2}\right)^{k-1}\left(\frac{1}{2}(\beta\gamma+2\delta)k+\frac{1}{2}(\beta+\gamma)\gamma+\frac{1}{2}(\gamma^2+4\varepsilon)k\right).$$

Then, recalling that the discriminant is zero in this case, we obtain

$$q_{k+1,0} = \left(\frac{\gamma}{2}\right)^{k-1}\left(\frac{1}{2}(\beta\gamma+2\delta)k+\frac{1}{2}(\beta+\gamma)\gamma\right). \tag{1.40}$$

For this case, we can now determine the series $H(t)$ and $A(t)$ as in Eq. (1.20) using Eq. (1.25). We first substitute (1.39) into our $H'(t)$ expression of Eq. (1.25) and then integrate, which leads to the geometric series

$$H(t) = \alpha t\sum_{k=0}^{\infty}\left(\frac{\gamma t}{2}\right)^{k} = \frac{2\alpha t}{2-\gamma t}.$$

It is also worth mentioning that the formal inverse of $H(t)$ above is therefore readily determined to be

$$J(t) = \frac{2t}{2\alpha+\gamma t}.$$

For $A(t)$, we first notice that we have the following first-order differential equation:

$$A'(t) - \sum_{k=0}^{\infty} q_{k+1,0}t^{k}A(t) = 0,$$

which, via integrating factor, has the solution

$$A(t) = \mu\exp\left[\int\sum_{k=0}^{\infty} q_{k+1,0}t^{k}\mathrm{d}t\right].$$

We then substitute (1.40) into the result directly above, evaluate the sum and integrate, which eventually yields

$$A(t) = \mu\left(\frac{2-\gamma t}{2}\right)^{-2\delta(\gamma-2)/\gamma^2}\exp\left(\frac{2t(\beta\gamma+2\delta)}{\gamma(2-\gamma t)}\right),$$

with $(-2\delta(\gamma-2)/\gamma^2)$ not equal to a nonnegative integer, so that $\mu_n \neq 0$ is satisfied.

*Case Ib.* $\gamma^2+4\varepsilon=0$ and $\gamma=0$

In this case, we also have $\varepsilon=0$. This gives

$$q_{1,1}=\alpha,\quad q_{k,1}=0\quad(k>1),\quad q_{1,0}=\beta,\quad q_{2,0}=\delta\quad\text{and}\quad q_{k,0}=0\quad(k>2)$$

and thus,

$$H(t)=\alpha t,\quad J(t)=t/\alpha\quad\text{and}\quad A(t)=\mu\exp\left(\beta t+\frac{1}{2}\delta t^2\right),$$

with $\delta \neq 0$, so that $\mu_n \neq 0$ holds.

For Cases IIa and IIb that follow ($\gamma^2 + 4\varepsilon \neq 0$), we first derive the general $q_{k,1}$ and $q_{k+1,0}$ terms. We have

$$q_{k,1} = \frac{\alpha(u_1^k - u_2^k)}{\sqrt{\gamma^2 + 4\varepsilon}} \quad \text{and} \quad q_{k+1,0} = \frac{\lambda u_1^k - \sigma u_2^k}{\sqrt{\gamma^2 + 4\varepsilon}} \tag{1.41}$$

with

$$\lambda := \delta + 2\varepsilon + (\beta + \gamma)u_1 \quad \text{and} \quad \sigma := \delta + 2\varepsilon + (\beta + \gamma)u_2.$$

*Case IIa.* $\gamma^2 + 4\varepsilon \neq 0$ and $\varepsilon = 0$
In this case, we consequently have $\gamma \neq 0$. Via Eq. (1.41), we observe that

$$H(t) = \alpha \int_0^t \frac{\mathrm{d}\tau}{1 - \gamma\tau} = -\frac{\alpha}{\gamma} \ln(1 - \gamma t), \quad J(t) = \frac{1}{\gamma}\left(1 - \exp\left(-\frac{\gamma t}{\alpha}\right)\right),$$
$$A(t) = \mu(1 - \gamma t)^{-(\beta\gamma + \gamma^2 + \delta)/\gamma^2} \exp\left(-\frac{\delta t}{\gamma}\right),$$

with $\delta \neq 0$ so that $\mu_n \neq 0$.

*Case IIb.* $\gamma^2 + 4\varepsilon \neq 0$ and $\varepsilon \neq 0$
Here, from Eq. (1.41), one can achieve

$$H(t) = \alpha \int_0^t \frac{\mathrm{d}\tau}{1 - \gamma\tau - \varepsilon\tau^2} = \alpha \frac{\ln((u_2 t - 1)/(u_1 t - 1))}{\sqrt{\gamma^2 + 4\varepsilon}},$$
$$J(t) = \frac{\exp\left(\frac{t}{\alpha}\sqrt{\gamma^2 + 4\varepsilon}\right) - 1}{u_1 \exp\left(\frac{t}{\alpha}\sqrt{\gamma^2 + 4\varepsilon}\right) - u_2}, \qquad A(t) = \mu \frac{(1 - u_2 t)^{h_2}}{(1 - u_1 t)^{h_1}},$$

with

$$h_i = \frac{u_i(\beta + \gamma) + (\delta + 2\varepsilon)}{u_i\sqrt{\gamma^2 + 4\varepsilon}}, \quad i = 1, 2.$$

We next state the main result of this section and for simplicity, we redefine the parameters involved in each of our $H(t)$ and $A(t)$ expressions in the same manner as Sheffer. This result displays the *general forms* of each of the generating functions for the orthogonal *Type 0* sets.

**Theorem 1.8.** *A polynomial set* $\{P_n(x)\}_{n=0}^{\infty}$ *is* Type 0 *and orthogonal if and only if* $A(t)\mathrm{e}^{xH(t)}$ *is of one of the following forms:*

$$A(t)e^{xH(t)} = \mu(1-bt)^c \exp\left(\frac{dt+atx}{1-bt}\right), \quad abc\mu \neq 0, \tag{1.42}$$

$$A(t)e^{xH(t)} = \mu \exp\left(t(b+ax)+ct^2\right), \quad ac\mu \neq 0, \tag{1.43}$$

$$A(t)e^{xH(t)} = \mu e^{ct}(1-bt)^{d+ax}, \quad abc\mu \neq 0, \tag{1.44}$$

$$A(t)e^{xH(t)} = \mu(1-t/c)^{d_1+x/a}(1-t/b)^{d_2-x/a}, \quad abc\mu \neq 0, \quad b \neq c. \tag{1.45}$$

*Proof.* This statement follows from the above analysis. □

By judiciously choosing each of the parameters in Eqs. (1.42)–(1.45) we can achieve all of the Sheffer *Type 0* orthogonal polynomials previously discussed. For emphasis, we write each of these parameter selections below and then display the corresponding generating function as it appears in contemporary literature.

**The Laguerre Polynomials** In Eq. (1.42), we select the parameters as $\mu = 1$, $a = -1$, $b = 1$, $c = -(\alpha+1)$, and $d = 0$ to obtain

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n = (1-t)^{-(\alpha+1)} \exp\left(\frac{xt}{t-1}\right).$$

**The Hermite Polynomials** With the assignments $\mu = 1$, $a = 2$, $b = 0$, and $c = -1$ in Eq. (1.43), we have

$$\sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!} = \exp(2xt - t^2).$$

**The Charlier Polynomials** If in Eq. (1.44) we choose $\mu = 1$, $a = 1$, $b = 1/a$, $c = 1$, and $d = 0$, then we achieve

$$\sum_{n=0}^{\infty} \frac{C_n(x;a)t^n}{n!} = e^t \left(1 - \frac{t}{a}\right)^x.$$

**The Meixner Polynomials** In Eq. (1.45), we select $\mu = 1$, $a = 1$, $b = 1$, $c =$ arbitrary constant, $d_1 = 0$, and $d_2 = -\beta$ leading to

$$\sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} M(x;\beta,c)t^n = \left(1 - \frac{t}{c}\right)^x (1-t)^{-(x+\beta)}.$$

**The Meixner–Pollaczek Polynomials** Taking $\mu = 1$, $a = -i$, $b = e^{i\phi}$, $c = e^{-i\phi}$, and $d_1 = d_2 = -\lambda$ in Eq. (1.45) leads to

$$\sum_{n=0}^{\infty} P_n^{(\lambda)}(x;\phi)t^n = (1 - e^{i\phi}t)^{-\lambda+ix}(1 - e^{-i\phi}t)^{-\lambda-ix}.$$

**The Krawtchouk Polynomials** Lastly, selecting $\mu = 1, a = 1, b = -1, c = p/(1-p)$, $d_1 = 0$, and $d_2 = N$ in Eq. (1.45) yields

$$\sum_{n=0}^{N} \mathrm{C}(N,n)K_n(x;p,N)t^n = \left(1 - \frac{1-p}{p}t\right)^x (1+t)^{N-x},$$

for $x = 0, 1, 2, \ldots, N$, where $\mathrm{C}(N,n)$ denotes the binomial coefficient.

Refer to [6] and the references therein for definitions and characterizations of each of these *A-Type 0* sets. For additional analyses regarding these orthogonal sets, also consider [3–5, 8, 10].

## 1.4 An Overview of the Classification of Type

We now discuss the essential details of the three kinds of characterizations that Sheffer developed in [9], which are entitled *A-Type*, *B-Type* and *C-Type*. The definition of *Type* is dependent on which characterization of *Type 0* that is to be generalized. That is, each of the Sheffer Types generalizes a certain *Type 0* characterizing structure.

### *1.4.1 The Sheffer A-Type Classification*

We deem all of the characterizations up to this juncture as ***A-Type k*** and therefore restate Definition 1.6 accordingly.

**Definition 1.9.** Let the set $\mathcal{S} := \{P_n(x)\}_{n=0}^{\infty}$ correspond to the unique operator $J$. Then, $\mathcal{S}$ is of ***A-Type k*** if the coefficients $\left\{L_j(x)\right\}_{j=0}^{\infty}$, as defined in Eq. (1.12), are such that $\deg(L_j(x)) \le k$ for all $i$ and there exists at least one $L_i(x) \in \left\{L_j(x)\right\}_{j=0}^{\infty}$ such that $\deg(L_i(x)) = k$. If $\left\{L_j(x)\right\}_{j=0}^{\infty}$ is unbounded, then $\mathcal{S}$ is of ***A-Type*** $\infty$.

We also will make use of the following result.

**Theorem 1.9.** *The set* $\{P_n(x)\}_{n=0}^{\infty}$ *is of* A-Type 0 *if and only if a sequence of constants* $\{s_n\}$ *exist such that*

$$P_n'(x) = s_1 P_{n-1}(x) + s_2 P_{n-2}(x) + \cdots + s_n P_0(x), \qquad n = 1, 2, 3, \ldots, \tag{1.46}$$

*where the elements of the sequence* $\{s_n\}$ *are the same as those in Eq.* (1.18).

*Proof.* We differentiate (1.19) with respect to $x$ and see that

$$\sum_{n=0}^{\infty} P_n'(x)t^n = H(t)A(t)\mathrm{e}^{xH(t)} = \sum_{n=0}^{\infty} P_n(x)H(t)t^n.$$

Then, from comparing coefficients of $t^n$ in

$$\sum_{n=0}^{\infty} P_n'(x)t^n = \sum_{n=0}^{\infty} P_n(x)[s_1t + s_2t^2 + s_3t^3 + \cdots]t^n,$$

we obtain our result. It is readily seen that this argument is both necessary and sufficient. □

## 1.4.2 The Sheffer B-Type Classification

To begin, we state that Theorem 1.9 can be extended in a natural way. Consider the structure

$$T_0(x)P_{n-1}(x) + T_1(x)P_{n-2}(x) + \cdots + T_{n-1}(x)P_0(x), \qquad n = 1,2,3,\ldots.$$

Then, by successively setting $n = 1,2,\ldots$, we observe that given a set $\{P_n(x)\}_{n=0}^{\infty}$, a unique sequence of polynomials $\{T_n(x)\}$ exists such that

$$P_n'(x) = T_0(x)P_{n-1}(x) + T_1(x)P_{n-2}(x) + \cdots + T_{n-1}(x)P_0(x), \quad n = 1,2,\ldots \quad (1.47)$$

with $T_0(x) \neq 0$. However, $\deg(T_n(x)) \leq n$ since, e.g., Eq. (1.47) readily reduces to Eq. (1.46) if $\{P_n(x)\}_{n=0}^{\infty}$ is of *A-Type 0*. The following statement immediately follows.

**Theorem 1.10.** *For every set $\{P_n(x)\}_{n=0}^{\infty}$ there exist a unique sequence of polynomials $\{T_n(x)\}$, with $\deg(T_n(x)) \leq n$, such that Eq. (1.47) holds.*

The converse of this theorem does not hold, as we have the following result.

**Theorem 1.11.** *Given a sequence of polynomials $\{T_n(x)\}$, with $\deg(T_n(x)) \leq n$, there exist infinitely many sets $\{P_n(x)\}_{n=0}^{\infty}$ such that Eq. (1.47) is satisfied.*

*Proof.* This result is immediate, since the constant term of $P_n(x)$ in Eq. (1.47) can be made arbitrary. □

Based on the above results, we have the following definition.

**Definition 1.10.** A set $\{P_n(x)\}_{n=0}^{\infty}$ is of ***B-Type k*** if the maximum degree of the polynomials $\{T_n(x)\}$ in Eq. (1.47) is $k$. Otherwise, $\{P_n(x)\}_{n=0}^{\infty}$ is classified as ***B-Type*** $\infty$.

We can now establish the following statement.

**Theorem 1.12.** *A set $\{P_n(x)\}_{n=0}^{\infty}$ is of* B-Type 0 *if and only if it is of* A-Type 0.

*Proof.* ($\Rightarrow$) If $\{P_n(x)\}_{n=0}^{\infty}$ is of *B-Type 0*, then $\{T_n(x)\}$ in Eq. (1.47) must be a sequence of constants. Moreover, Eq. (1.47) is reduced to Eq. (1.46) and thus $\{P_n(x)\}_{n=0}^{\infty}$ is an *A-Type 0* set.
($\Leftarrow$) If $\{P_n(x)\}_{n=0}^{\infty}$ is of *A-Type 0*, then Eq. (1.46) holds, which is Eq. (1.47) with constant coefficients. Thus, $\{P_n(x)\}_{n=0}^{\infty}$ is of *B-Type 0*. □

We also have the classification of *B-Type k* sets below.

**Theorem 1.13.** *The characterization* (1.47) *is equivalent to the relation*

$$H(x,t) = A(t)\exp\left[t\int_0^x T(\xi,t)\mathrm{d}\xi\right], \tag{1.48}$$

*with $A(t)$ as in Eq.* (1.20).

*Proof.* ($\Rightarrow$) First, we define

$$H := H(x,t) = \sum_{k=0}^{\infty} P_k(x)t^k, \tag{1.49}$$

$$T := T(x,t) = \sum_{k=0}^{\infty} T_k(x)t^k. \tag{1.50}$$

We now consider

$$tHT = t\sum_{k=0}^{\infty} P_k(x)t^k \sum_{k=0}^{\infty} T_k(x)t^k. \tag{1.51}$$

We can find the coefficient of $t^n$ in the right-hand side of Eq. (1.51) by considering the following structure, which is $t$ multiplied by the general term in each of the sums of Eq. (1.51):

$$tP_{k_0}t^{k_0}T_{k_1}t^{k_1}$$

and finding all nonnegative integer solutions to the equation $1+k_0+k_1 = n$, i.e., the sum of the $t$-exponents. For the solution $k_0 = n-1$ and $k_1 = 0$, we obtain $T_0(x)P_{n-1}(x)$ and for the solution $k_0 = n-2$ and $k_1 = 1$, we achieve $T_1(x)P_{n-2}(x)$. Continuing in this fashion, we see that the coefficient of $t^n$ turns out to be

$$T_0(x)P_{n-1}(x) + T_1(x)P_{n-2}(x) + \cdots + T_{n-1}(x)P_0(x) = P_n'(x)$$

via Eq. (1.47). Thus, the right-hand side of Eq. (1.51) is $\sum_{n=0}^{\infty} P_n'(x)t^n = \partial H(x,t)/\partial x$ and we have constructed the following first-order differential equation, as Eq. (1.51) becomes

$$\frac{\partial}{\partial x}H - tTH = 0; \qquad H(0,t) = \sum_{n=0}^{\infty} P_n(0)t^n = A(t), \tag{1.52}$$

with $A(t)$ as in Eq. (1.20). Furthermore, we have shown that this relation is equivalent to Eq. (1.47). The general solution to Eq. (1.52) can be determined via the integrating factor $\exp\left[-\int_0^x tT(\xi,t)\mathrm{d}\xi\right]$ to be

$$H(x,t) = A(t)\exp\left[t\int_0^x T(\xi,t)\mathrm{d}\xi\right],$$

where $A(t)$ is an arbitrary power series with a nonzero constant term. However, in considering the initial condition of Eq. (1.52), we see that $A(t)$ must be as in Eq. (1.20). This argument is certainly both necessary and sufficient. □

Next, we note that if we write $T(x,t)$ in Eq. (1.50) as a power series in $x$, as opposed to $t$, the coefficients of each of the $x^n$-terms are power series in $t$. To facilitate this, momentarily let $T_n(x) = c_{n,n}x^n + c_{n,n-1}x^{n-1} + \cdots + c_{n,0}$, and then we have

$$T(x,t) = (c_{0,0} + c_{1,0}t + c_{2,0}t^2 + \cdots) + (c_{1,1}t + c_{2,1}t^2 + c_{3,1}t^3 + \cdots)x + (c_{2,2}t^2 + c_{3,2}t^3 + c_{4,2}t^4 + \cdots)x^2 + \cdots.$$

Letting $x = \xi$ above, integrating this result with respect to $\xi$ from 0 to $x$ and multiplying by $t$, we see that

$$t\int_0^x T(\xi,t)\mathrm{d}\xi = xH_1(t) + x^2H_2(t) + x^3H_3(t) + \cdots,$$

where each $H_i(t)$ is a power series in $t$ that starts with the $t^i$-term (or higher if the coefficient of the $t^i$-term is zero), i.e., $H_i(t) = h_{i,i}t^i + h_{i,i+1}t^{i+1} + h_{i,i+2}t^{i+2} + \cdots$. However, $H_1(t)$ must begin with a linear term in $t$, since by Eq. (1.47) $T_0(x) \neq 0$ and $h_{1,1} = T_0(x)$ by construction. Thus, we can write (1.48) in the following way:

$$H(x,t) = A(t)\exp\left[xH_1(t) + x^2H_2(t) + x^3H_3(t) + \cdots\right], \quad h_{1,1} \neq 0. \tag{1.53}$$

Now, for $\{P_n(x)\}_{n=0}^{\infty}$ to be of *B-Type k*, it is necessary and sufficient that $T(x,t)$ is a polynomial in $x$ of degree $k$. This restriction is equivalent to terminating the sum $\left[xH_1(t) + x^2H_2(t) + x^3H_3(t) + \cdots\right]$ in Eq. (1.53) at $k+1$ via the presence of the integral in Eq. (1.48). Thus, we obtain (1.54) in Theorem 1.14 below, which is the culmination of the above analysis.

**Theorem 1.14.** *A set is of* B-Type k *if and only if*

$$H(x,t) = A(t)\exp\left[xH_1(t) + \cdots + x^{k+1}H_{k+1}(t)\right],$$
$$\text{with } H_i(t) = h_{i,i}t^i + h_{i,i+1}t^{i+1} + \cdots, \quad h_{1,1} \neq 0, \quad i = 1,2,\ldots,k+1. \tag{1.54}$$

### 1.4.3 The Sheffer C-Type Classification

The last classification that Sheffer developed is entitled *C-Type*. As was done in the previous section, we establish all of the theorems necessary for understanding this classification, beginning with the following.

**Theorem 1.15.** *For each set* $\{P_n(x)\}_{n=0}^{\infty}$*, there exist a unique sequence of polynomials* $\{U_n(x)\}_{n=0}^{\infty}$*, with* $\deg(U_n(x)) \leq n$*, such that*

$$nP_n(x) = U_1(x)P_{n-1}(x) + U_2(x)P_{n-2}(x) + \cdots + U_n(x)P_0(x), \quad n = 1,2,\ldots. \tag{1.55}$$

*Proof.* Analogous to establishing (1.47) in Theorem 1.10, we set $n = 1,2,\ldots$ in Eq. (1.55) and then successively and uniquely determine the set $\{U_n(x)\}_{n=0}^{\infty}$. Through this process, it becomes clear that no polynomial $U_n(x)$ can exceed the degree of its subscript. The result therefore follows. □

This leads us to the following definition.

**Definition 1.11.** A set $\{P_n(x)\}_{n=0}^{\infty}$ is classified as *C-Type* $k$ if the maximum degree of the polynomials $\{U_n(x)\}$ in Eq. (1.55) is $k+1$. Else, $\{P_n(x)\}_{n=0}^{\infty}$ is classified as *C-Type* $\infty$.

We note that we use $k+1$ in the above definition, as opposed to $k$ as in our previous *Type* definitions, since otherwise, Eq. (1.55) would not be satisfied. As an example, consider the case when each $U_i(x)$ is constant.

Now, we define

$$U := U(x,t) = \sum_{n=0}^{\infty} U_{n+1}(x)t^n.$$

Then, calling upon Eq. (1.55) and using the same methodology that was used to construct (1.52) in the previous section, we derive the following first-order differential equation:

$$\frac{\partial}{\partial t}H - UH = 0; \qquad H(x,0) = P_0(x) = a_0$$

with $H := H(x,t)$ as in Eq. (1.49). This can easily be solved using the integrating factor $\exp\left[-\int_0^t U(x,\tau)\mathrm{d}\tau\right]$, yielding the solution

$$H(x,t) = a_0 \exp\left[\int_0^t U(x,\tau)\mathrm{d}\tau\right], \tag{1.56}$$

after incorporating our initial condition. Then, from comparing (1.56) with Eq. (1.48), we have

$$a_0 \exp\left[\int_0^t U(x,\tau)\mathrm{d}\tau\right] = A(t)\exp\left[t\int_0^x T(\xi,t)\mathrm{d}\xi\right]$$

and taking the natural logarithm of both sides of this relation leads to

$$\ln a_0 + \int_0^t U(x,\tau)\mathrm{d}\tau = \ln A(t) + t\int_0^x T(\xi,t)\mathrm{d}\xi. \tag{1.57}$$

Next, differentiate (1.57) with respect to $t$ in order to obtain

$$U(x,t) = \frac{A'(t)}{A(t)} + \int_0^x \frac{\partial}{\partial t}\left(tT(\xi,t)\right)\mathrm{d}\xi \tag{1.58}$$

and also differentiate (1.57) with respect to $x$, which leads to

$$tT(x,t) = \int_0^t \frac{\partial}{\partial x} U(x,\tau)\mathrm{d}\tau. \tag{1.59}$$

We now substitute (1.59) into Eq. (1.58) and then put this new expression for $U(x,t)$ into Eq. (1.56), which leads to

$$H(x,t) = A(t)\exp\left[\int_0^t \int_0^x \frac{\partial}{\partial \xi} U(\xi,\tau)\xi \mathrm{d}\tau\right]. \tag{1.60}$$

We see that the exponent in Eq. (1.60) is a polynomial in $x$ of degree $k+1$ if and only if the polynomial of maximal degree in $\{U_n(x)\}_{n=1}^{\infty}$ is of degree $k+1$, i.e., if and only if the set $\{P_n(x)\}_{n=0}^{\infty}$ for which the sequence $\{U_n(x)\}_{n=1}^{\infty}$ corresponds to is of *C-Type k*. Under this assumption, the exponent in Eq. (1.60) has the following structure:

$$\tilde{U}_1(x)t + \tilde{U}_2(x)t^2 + \cdots + \tilde{U}_{k+1}(x)t^{k+1},$$

which is the same form as the exponent in Eq. (1.48) when it corresponds to a *B-Type 0* set. Thus, Eq. (1.60) can therefore be reduced to Eq. (1.54) using the same type of manipulation that was used in the previous section. Hence, we have proven the following theorem.

**Theorem 1.16.** *A set* $\{P_n(x)\}_{n=0}^{\infty}$ *is of* C-Type k *if and only if it is of* B-Type k.

### *1.4.4 A Summary of the Rainville σ-Type Classification*

To complete our summary of *Type*, we conclude with a natural extension of Sheffer's classification, entitled *σ-Type*, which was constructed by E.D. Rainville and originally appeared in [8]. Now, we have seen that various results were ascertained from the generating function (1.19), which we now know characterizes *A-Type 0* sets. Moreover, we know that for $y = \exp[xH(t)]$ with $D := \mathrm{d}/\mathrm{d}x$ we have

$$Dy = H(t)y.$$

We intend to then modify the operator $D$ so that it behaves in a similar fashion and also obtain a generating relation for the most basic type of sets classified by this modified operator. That is, if we replace $D$ above by another differential operator, say $\sigma$, and the exponential function $\exp(z)$ by another function, e.g., $F(z)$, then we want to find $\sigma$ such that

$$\sigma F(z) = F(z).$$

From there, we can construct an analogue of Eq. (1.19) in Theorem 1.5. This leads to the following definition of our differential operator $\sigma$ (using the same notation as in [8]).

**Definition 1.12.** We define the differential operator $\sigma$ as follows:

$$\sigma := D\prod_{i=1}^{q}(\theta + b_i - 1), \quad D := \frac{\mathrm{d}}{\mathrm{d}x} \text{ and } \theta := xD,$$

with $b_i \neq 0$ and $b_i$ not equal to a negative integer.

In fact, this operator is a degree-lowering operator. To facilitate its application on polynomials and its degree-lowering nature, consider acting on a monomial, like $x^2$, with $q = 1$ and $b_1 = 1$. The function of the $q$ and $b_i$-terms will become more transparent in Theorem 1.17 at the end of this section.

We now define $\sigma$-*Type*.

**Definition 1.13.** Let $\{P_n(x)\}_{n=0}^{\infty}$ be a set such that

$$J_\sigma[P_n(x)] = \sum_{k=0}^{\infty} T_k(x)\sigma^{k+1}P_n(x) = P_{n-1}(x),$$

with $\deg(T_j(x)) \leq k$. If the polynomial of maximal degree in the set of coefficients $\{T_j(x)\}_{j=0}^{\infty}$ is of degree $k$, then we classify $\{P_n(x)\}_{n=0}^{\infty}$ as ***σ-Type k***. If the coefficients $\{T_j(x)\}_{j=0}^{\infty}$ are unbounded, then $\{P_n(x)\}_{n=0}^{\infty}$ is ***σ-Type*** $\infty$.

Based on this definition, we see that polynomials of $\sigma$-*Type 0* satisfy the following form:

$$J_\sigma[P_n(x)] = \sum_{k=0}^{\infty} c_k\sigma^{k+1}P_n(x) = P_{n-1}(x),$$

where each $c_k$ is a constant and $c_0 \neq 0$. Since each of the $c_k$'s is a constant, analogous to our *A-Type 0* analysis, there exists a generating function for $J_\sigma$, which we call $J_\sigma(t)$, and a corresponding inverse $H_\sigma(t)$ such that $J_\sigma(H_\sigma(t)) = H_\sigma(J_\sigma(t)) = t$:

$$J_\sigma(t) = \sum_{n=0}^{\infty} c_n t^{n+1}, \ c_0 \neq 0 \quad \text{and} \quad H_\sigma(t) = \sum_{n=0}^{\infty} h_n t^{n+1}, \ h_0 \neq 0.$$

This leads us to the main characterization theorem for $\sigma$-*Type 0* sets.

**Theorem 1.17.** *A set is of* $\sigma$-Type 0*, with* $\sigma$ *as defined in Definition 1.12, if and only if* $\{P_n(x)\}_{n=0}^{\infty}$ *satisfies the generating function*

$$A(t)_0F_q(-;b_1,b_2,\cdots,b_q;xH(t)) = \sum_{n=0}^{\infty} P_n(x)t^n$$

*with* $H(t)$ *as defined above and* $A(t)$ *as in Eq.* (1.20).

In Theorem 1.17, we wrote the generating function in the ***generalized hypergeometric form***, which is defined as

$$_rF_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix}\middle|\, x\right) = \sum_{k=0}^{\infty} \frac{(a_1,\ldots,a_r)_k}{(b_1,\ldots,b_s)_k}\frac{x^k}{k!}, \tag{1.61}$$

where the ***Pochhammer symbol*** $(a)_k$ is

$$(a)_k := a(a+1)(a+2)\cdots(a+k-1), \quad (a)_0 := 1, \tag{1.62}$$

and

$$(a_1,\ldots,a_j)_k := (a_1)_k \ldots (a_j)_k.$$

We now see that the selection of $q$ is dependent on the number of denominator parameters (the $b_i$'s) in the generating function of Theorem 1.17. For a proof of Theorem 1.17, the interested reader can refer to [8].

## 1.5 A Brief Discussion of Meixner's Analysis

In 1934, J. Meixner published [7] (written in German) in which he considered the generating relation (1.19) (with the same assumptions on the $A(t)$ and $H(t)$ as in Sheffer's work [9]) to be the *definition* of a certain class of polynomials. From there, he determined all sets that satisfy this relation that were also orthogonal and reached the same conclusions as Sheffer did in [9]. In other words, Meixner determined all orthogonal sets $\{P_n(x)\}_{n=0}^{\infty}$ that satisfy

$$f(t)\mathrm{e}^{xu(t)} = \sum_{n=0}^{\infty} \frac{P_n(x)}{n!}t^n; \quad f(0)=1, \ u(0)=0, \ \frac{\mathrm{d}}{\mathrm{d}t}u(0)=1, \tag{1.63}$$

which we have written using Meixner's notation, which we essentially adhere to throughout this section.

In a similar manner as Sheffer, Meixner defined a general degree-lowering, linear, differential operator of infinite order, which we call $J$ as opposed to his "$t$" to avoid confusion with $t$-parameters, which satisfies a certain commutation relation with the differential operator $D$. Moreover, $J(t)$ is a formal power series without a constant term and with a unitary linear coefficient. The formal power series inverse of $J(t)$ was defined to be $u(t)$, i.e., $u(J(t)) = J(u(t)) = t$.

Throughout his work, Meixner considered $\{P_n(x)\}_{n=0}^{\infty}$ to be a set of monic polynomials. With this convention, he utilized the following monic three-term recurrence relation:

$$P_{n+1}(x) = (x + l_{n+1})P_n(x) + k_{n+1}P_{n-1}(x), \quad n = 0, 1, 2, \dots \tag{1.64}$$

with $l_{n+1} \in \mathbb{R}$ and $k_{n+1} \in \mathbb{R}^-$ and demonstrated that

$$J(D)P_n(x) = nP_{n-1}(x). \tag{1.65}$$

Now, we act on Eq. (1.64) with $J(D)$ as in Eq. (1.65) and obtain

$$(n+1)P_n(x) = (x + l_{n+1})nP_{n-1}(x) + J'(D)P_n(x) + k_{n+1}(n-1)P_{n-2}(x), \tag{1.66}$$

where $J'(D)$ is of course the derivative of the formal power series $J(D)$. Then, we replace $n$ with $n-1$ in Eq. (1.64) and multiply both sides by $n$ to obtain

$$nP_n(x) = (x + l_n)nP_{n-1}(x) + k_n nP_{n-2}(x). \tag{1.67}$$

Next, we subtract (1.67) from Eq. (1.66) which yields

$$\left(1 - J'(D)\right) P_n(x) = (l_{n+1} - l_n)nP_{n-1}(x) + \left(\frac{k_{n+1}}{n} - \frac{k_n}{n-1}\right) n(n-1)P_{n-2}(x).$$

We then replace $n$ with $n+1$ in the recursion coefficients above leading to

$$\left(1 - J'(D)\right) P_n(x) = (l_{n+2} - l_{n+1})nP_{n-1}(x) + \left(\frac{k_{n+2}}{n+1} - \frac{k_{n+1}}{n}\right) n(n-1)P_{n-2}(x).$$

We assign

$$\lambda := l_{n+1} - l_n, \quad n = 1, 2, \dots, \tag{1.68}$$

$$\kappa := \frac{k_{n+1}}{n} - \frac{k_n}{n-1}, \quad n = 2, 3, \dots \tag{1.69}$$

giving

$$\left(1 - J'(D)\right) P_n(x) = \lambda nP_{n-1}(x) + \kappa n(n-1)P_{n-2}(x), \quad n = 0, 1, 2, \dots$$

and via Eq. (1.65) we see that this recurrence can be written as

$$\left(1 - J'(D)\right) P_n(x) = \lambda J(D)P_n(x) + \kappa J^2(D)P_n(x), \quad n = 0, 1, 2, \dots \tag{1.70}$$

From rewriting Eqs. (1.68) and (1.69) as

$$l_{n+1} = l_n + \lambda \quad \text{and} \quad k_{n+1} = n\left(\frac{k_n}{n-1} + \kappa\right)$$

and iterating, we see that $l_{n+1} = l_1 + n\lambda$ and $k_{n+1} = k_2 + (n-1)\kappa$. We then substitute these recursion coefficients into Eq. (1.64) to obtain the following three-term recurrence relation:

$$P_{n+1}(x) = (x + l_1 + n\lambda)P_n(x) + n(k_2 + (n-1)\kappa)P_{n-1}(x), \quad n = 0, 1, 2, \ldots \tag{1.71}$$

with $k_2 < 0$ and $\kappa \leq 0$ from the original restrictions imposed upon Eq. (1.64).

Now, from Eq. (1.70), it follows that

$$J'(u(t)) = 1 - \lambda t - \kappa t^2.$$

Differentiating both sides of the relation $J(u(t)) = t$ tells us that $J'(u(t)) = 1/u'(t)$. Thus, our relation directly above becomes

$$J'(u(t)) = 1 - \lambda t - \kappa t^2 = \frac{1}{u'(t)}. \tag{1.72}$$

By setting $x = 0$, we note that the generating function (1.63) turns into

$$f(t) = \sum_{n=0}^{\infty} \frac{P_n(0)t^n}{n!}.$$

Thus, multiplying both sides of Eq. (1.71) by $t^n/n!$, setting $x = 0$ and summing for $n = 0, 1, 2, \ldots$ lead to the differential equation

$$\frac{f'(t)}{f(t)} = \frac{k_2 t}{1 - \lambda t - \kappa t^2}.$$

Factoring $1 - \lambda t - \kappa t^2$ as $(1 - \alpha t)(1 - \beta t)$ gives

$$\frac{f'(t)}{f(t)} = \frac{k_2 t}{(1 - \alpha t)(1 - \beta t)}, \qquad \alpha, \beta \in \mathbb{C}. \tag{1.73}$$

We can now exhaust every possible combination of $\alpha$ and $\beta$ (and incorporate $\lambda$ and $\kappa$ as well), substitute each of them into Eqs. (1.72) and (1.73), and solve the resulting differential equations. In each case, the solution to Eq. (1.72) will yield an expression for $u(t)$ and the solution to Eq. (1.73) will yield an expression for $f(t)$. In substituting these into Eq. (1.63), we achieve a generating function for each orthogonal set.

Below, we write the results for each of these aforementioned cases. In each case, part $(i)$ denotes the solutions to Eqs. (1.72) and (1.73), which respectively yield our expressions for $u(t)$ and $f(t)$. In part $(ii)$, we write each of the resulting generating functions in their rescaled form, so they appear as they do in the contemporary literature.

*Case I.* The Hermite Polynomials: $\alpha = \beta = 0$ $(\lambda = \kappa = 0)$

$$(i)\;\; u(t) = t \quad \text{and} \quad f(t) = \exp\left(k_2 t^2/2\right)$$

$$(ii)\;\; \exp\left(2xt - t^2\right) = \sum_{n=0}^{\infty} \frac{H(x)}{n!} t^n$$

*Case II.* The Laguerre Polynomials: $\alpha = \beta \neq 0$

$$(i)\;\; u(t) = \frac{t}{1-\alpha t} \quad \text{and} \quad f(t) = (1-\alpha t)^{k_2/\alpha^2} \exp\left(\frac{t}{1-\alpha t}\frac{k_2}{\alpha}\right)$$

$$(ii)\;\; (1-t)^{-(\alpha+1)} \exp\left(\frac{xt}{t-1}\right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n$$

*Case III.* The Charlier Polynomials: $\alpha \neq 0$ and $\beta = 0$ $(\kappa = 0)$

$$(i)\;\; u(t) = -\frac{1}{\alpha}\ln(1-\alpha t) \quad \text{and} \quad f(t) = (1-\alpha t)^{-k_2/\alpha^2} \mathrm{e}^{-k_2 t/\alpha}$$

$$(ii)\;\; \mathrm{e}^t \left(1 - \frac{t}{\alpha}\right)^x = \sum_{n=0}^{\infty} \frac{C_n(x;\alpha)}{n!} t^n$$

*Case IV.* Meixner determined *two* orthogonal sets that stem from this case. The general $u(t)$ and $f(t)$ are as follows:

$$u(t) = \frac{1}{\alpha-\beta} \ln\left(\frac{1-\beta t}{1-\alpha t}\right) \quad \text{and} \quad f(t) = \left(\frac{(1-\beta t)^{1/\beta}}{(1-\alpha t)^{1/\alpha}}\right)^{k_2/(\alpha-\beta)}.$$

(a) **The Meixner Polynomials**: $\alpha \neq \beta$ and $\alpha, \beta \in \mathbb{R}$ $(\kappa \neq 0)$

$$\left(1 - \frac{t}{c}\right)^x (1-t)^{-(x+\beta)} = \sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} M(x;\beta,c) t^n.$$

(b) **The Meixner–Pollaczek Polynomials**: $\alpha \neq \beta$, $\alpha$ and $\beta$ complex conjugates $(\kappa \neq 0)$

$$(1-e^{i\phi}t)^{-\lambda+ix}(1-e^{-i\phi}t)^{-\lambda-ix} = \sum_{n=0}^{\infty} P_n^{(\lambda)}(x;\phi)t^n.$$

*Remark 1.1.* The Krawtchouk polynomials are the *third* orthogonal set that comes from Case IV above. These polynomials were not included in either Meixner's or Sheffer's work.

*Example 1.5.* As a simple example of the scaling process, we see that in Case I above, $u(t) = t$ and $f(t) = \exp\left(k_2t^2/2\right)$, so that

$$f(t)e^{xu(t)} = \exp\left(xt + \frac{1}{2}k_2t^2\right).$$

Thus, we can obtain the generating relation in Case I by simply choosing $k_2 = -1/2$ and rescaling $t$ via $t \to 2t$.

### 1.5.1 Al-Salam's Extension of Meixner's Characterization

To briefly supplement our discussion of Meixner's analysis, we state that W. A. Al-Salam extended the results of Meixner, and therefore Sheffer, in [1]. Namely, he showed that the left-hand side of Eq. (1.63) can be replaced with $\exp(Q(x,t))$, where $Q(x,t)$ is a polynomial in $x$ with coefficients that are functions of $t$, as seen below:

$$\exp(Q(x,t)) = \sum_{n=0}^{\infty} P_n(x)\frac{t^n}{n!},$$

$$Q(x,t) = \sum_{j=0}^{k} x^j a^{(j)}(t), \quad k \geq 1, \quad a^{(j)}(t) = \sum_{r=0}^{\infty} a_r^{(j)} t^r, \quad j = 0,1,2,\ldots,k$$

and that the resulting orthogonal polynomials $\{P_n(x)\}_{n=0}^{\infty}$ will be the *same* as those achieved by Meixner and Sheffer. This showed that the conditions on the generating function (1.63) can be weakened without yielding new orthogonal sets.

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