Chapter 22
A Malliavin Calculus Approach to General Stochastic Differential Games with Partial Information

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Abstract In this article, we consider stochastic differential game where the state process is governed by a controlled Itô–Lévy process and the information available to the controllers is possibly less than the general information. All the system coefficients and the objective performance functional are assumed to be random. We use Malliavin calculus to derive a maximum principle for the optimal control of such problem. The results are applied to solve a worst-case scenario portfolio problem in finance.

1 Introduction

Suppose the dynamics of a state process $X(t) = X^{(u_0,u_1)}(t,\omega)$; $t \geq 0$, $\omega \in \Omega$, is a controlled Itô–Lévy process in $\mathbb{R}$ of the form

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$$\begin{cases} dX(t) = b(t, X(t), u_0(t), \omega)dt + \sigma(t, X(t), u_0(t), \omega)dB(t) \\ \qquad\quad + \int_{\mathbb{R}_0} \gamma(t, X(t^-), u_0(t^-), u_1(t^-, z), z, \omega)\tilde{N}(dt, dz); \\ X(0) \;= x \in \mathbb{R}, \end{cases} \tag{22.1}$$

where the coefficients $b : [0, T] \times \mathbb{R} \times U \times \Omega \to \mathbb{R}$, $\sigma : [0, T] \times \mathbb{R} \times U \times \Omega \to \mathbb{R}$ and $\gamma : [0, T] \times \mathbb{R} \times U \times K \times \mathbb{R}_0 \times \Omega$ are all continuously differentiable (C^1) with respect to $x \in \mathbb{R}$ and $u_0 \in U$, $u_1 \in K$ for each $t \in [0, T]$ and a.a. $\omega \in \Omega$; U, K are given open convex subsets of $\mathbb{R}^2$ and $\mathbb{R} \times \mathbb{R}_0$, respectively. Here $\mathbb{R}_0 = \mathbb{R} - \{0\}$, $B(t) = B(t, \omega)$ and $\eta(t) = \eta(t, \omega)$, given by

$$\eta(t) = \int_0^t \int_{\mathbb{R}_0} z\tilde{N}(ds, dz); \; t \geq 0, \; \omega \in \Omega, \tag{22.2}$$

are a one-dimensional Brownian motion and an independent pure jump Lévy martingale, respectively, on a given filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$. Thus

$$\tilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt \tag{22.3}$$

is the *compensated Poisson jump measure* of $\eta(\cdot)$, where $N(dt, dz)$ is the *Poisson jump measure* and $\nu(dz)$ is the *Lévy measure* of the pure jump Lévy process $\eta(\cdot)$. For simplicity, we assume that

$$\int_{\mathbb{R}_0} z^2 \nu(dz) < \infty. \tag{22.4}$$

The processes $u_0(t)$ and $u_1(t, z)$ are the control processes and have values in a given open convex set U and K, respectively, for a.a. $t \in [0, T]$, $z \in \mathbb{R}_0$ for a given fixed $T > 0$. Also, $u_0(\cdot)$ and $u_1(\cdot)$ are càdlàg and adapted to a given filtration $\{\mathcal{E}_t\}_{t \geq 0}$, where

$$\mathcal{E}_t \subseteq \mathcal{F}_t, \qquad t \in [0, T].$$

$\{\mathcal{E}_t\}_{t \geq 0}$ represents the information available to the controller at time t. For example, we could have

$$\mathcal{E}_t = \mathcal{F}_{(t-\delta)+}; \quad t \in [0, T], \; \delta > 0 \text{ is a constant,}$$

meaning that the controller gets a delayed information compared to $\mathcal{F}_t$. We refer to [15, 12] for more information about stochastic control of Itô diffusions and jump diffusions, respectively, and to [2], [4], [8], [9], [14] for other papers dealing with optimal control under partial information/observation.

Let $f : [0, T] \times \mathbb{R} \times U \times K \times \Omega \to \mathbb{R}$ and $g : \mathbb{R} \times \Omega \to \mathbb{R}$ are given continuously differentiable (C^1) with respect to $x \in \mathbb{R}$ and $u_0 \in U$, $u_1 \in K$. Suppose there are two players in the stochastic differential game and the given performance functionals for players are as follows:

$$J_i(u_0, u_1) = \mathbb{E}^x \left[\int_0^T \int_{\mathbb{R}_0} f_i(t, X(t), u_0(t), u_1(t, z), z, \omega)\mu(dz)dt + g_i(X(T), \omega) \right],$$

$$i = 1, 2,$$

where μ is a measure on the given measurable space $(\Omega, \mathcal{F})$ and $\mathbb{E}^x = \mathbb{E}^x_P$ denotes the expectation with respect to P given that $X(0) = x$. Suppose that the controls $u_0(t)$ and $u_1(t, z)$ have the form

$$u_0(t) = (\pi_0(t), \theta_0(t)); \quad t \in [0, T]; \tag{22.5}$$

$$u_1(t, z) = (\pi_1(t, z), \theta_1(t, z)); \quad (t, z) \in [0, T] \times \mathbb{R}_0. \tag{22.6}$$

Let $\mathscr{A}_\Pi$ and $\mathscr{A}_\Theta$ denote the given family of controls $\pi = (\pi_0, \pi_1)$ and $\theta = (\theta_0, \theta_1)$ such that they are contained in the set of càdlàg $\mathscr{E}_t$-adapted controls, Eq. (22.1) has a unique strong solution up to time T and

$$\mathbb{E}^x \left[\int_0^T \int_{\mathbb{R}_0} |f_i(t, X(t), \pi_0(t), \pi_1(t, z), \theta_0(t), \theta_1(t, z), z, \omega)| \mu(\mathrm{d}z)\mathrm{d}t + |g_i(X(T), \omega)| \right]$$
$$< \infty, \quad i = 1, 2.$$

The partial information non-zero-sum stochastic differential game problem we consider is the following:

Problem 1.1. Find $(\pi^*, \theta^*) \in \mathscr{A}_\Pi \times \mathscr{A}_\Theta$ (if it exists) such that

 (i) $J_1(\pi, \theta^*) \leq J_1(\pi^*, \theta^*)$ for all $\pi \in \mathscr{A}_\Pi$,
 (ii) $J_2(\pi^*, \theta) \leq J_2(\pi^*, \theta^*)$ for all $\theta \in \mathscr{A}_\Theta$.

Such a control (π^*, θ^*) is called a *Nash equilibrium* (if it exists). The intuitive idea is that there are two players, players I and II. While player I controls π, player II controls θ. Given that each player knows the equilibrium strategy chosen by the other player, none of the players has anything to gain by changing only his or her own strategy only (i.e., by changing unilaterally). Note that since we allow b, σ, γ, f and g to be stochastic processes and also because our controls are required to be $\mathscr{E}_t$-adapted, this problem is not of Markovian type and hence cannot be solved by dynamic programming. Our paper is related to the recent paper [1, 10], where a maximum principle for stochastic differential games with partial information and a mean-field maximum principle are dealt with, respectively. However, the approach in [1] needs the solution of the backward stochastic differential equation (BSDE) for the adjoint processes. This is often a difficult point, particularly in the partial information case. In the current paper, we use Malliavin calculus techniques to obtain a maximum principle for this general non-Markovian stochastic differential game with partial information, without the use of BSDEs.

2 The General Maximum Principle for the Stochastic Differential Games

In this section we base on Malliavin calculus to solve Problem 1.1. We assume the following:

(A1) For all $s, r, t \in (0, T)$, $t \leq r$, and all bounded $\mathcal{E}_t$-measurable random variables $\alpha = \alpha(\omega)$, $\xi = \xi(\omega)$ the controls $\beta_\alpha(s) := (0, \beta_\alpha^i(s))$ and $\eta_\xi(s) := (0, \eta_\xi^i)$, $i = 1, 2$, with

$$\beta_\alpha^i(s) = \alpha^i(\omega)\chi_{[t,r]}(s) \text{ and } \eta_\xi^i(s) = \xi^i(\omega)\chi_{[t,r]}(s); \quad s \in [0, T],$$

belong to $\mathscr{A}_\Pi$ and $\mathscr{A}_\Theta$, respectively. Also, we will denote the transposed of the vectors β and η by β^*, η^*, respectively.

(A2) For all $\pi, \beta \in \mathscr{A}_\Pi$; $\theta, \eta \in \mathscr{A}_\Theta$ with β and η are bounded, there exists $\delta > 0$ such that the controls $\pi(t) + y\beta(t)$ and $\theta(t) + \upsilon\eta(t)$, $t \in [0, T]$, belong to $\mathscr{A}_\Pi$ and $\mathscr{A}_\Theta$, respectively, for all $\upsilon \in (-\delta, \delta)$, and such that the families

$$\left\{ \frac{\partial f_1}{\partial x}(t, X^{(\pi+y\beta,\theta)}(t), \pi + y\beta, \theta, z) \frac{d}{dy} X^{(\pi+y\beta,\theta)}(t) \right.$$
$$\left. + \nabla_\pi f_1(t, X^{(\pi+y\beta,\theta)}(t), \pi + y\beta, \theta, z)\beta^*(t) \right\}_{y \in (-\delta, \delta)},$$

$$\left\{ \frac{\partial f_2}{\partial x}(t, X^{(\pi,\theta+\upsilon\eta)}(t), \pi, \theta + \upsilon\eta, z) \frac{d}{dy} X^{(\pi,\theta+\upsilon\eta)}(t) \right.$$
$$\left. + \nabla_\theta f_2(t, X^{(\pi,\theta+\upsilon\eta)}(t), \pi(t), \theta + \upsilon\eta, z)\eta^*(t) \right\}_{\upsilon \in (-\delta, \delta)}$$

are $\lambda \times \nu \times P$-uniformly integrable and the families

$$\left\{ g_1'(X^{\pi+y\beta}(T)) \frac{d}{dy} X^{(\pi+y\beta,\theta)}(T) \right\}_{y \in (-\delta, \delta)},$$

$$\left\{ g_2'(X^{\pi,\theta+\upsilon\eta}(T)) \frac{d}{dy} X^{(\pi,\theta+\upsilon\eta)}(T) \right\}_{\upsilon \in (-\delta, \delta)}$$

are P-uniformly integrable.

In the following, $D_t F$ denotes the *Malliavin derivative with respect to $B(\cdot)$ (at t)* of a given (Malliavin differentiable) random variable $F = F(\omega)$; $\omega \in \Omega$. Similarly, $D_{t,z}F$ denotes the *Malliavin derivative with respect to $\widetilde{N}(\cdot, \cdot)$ (at t, z)* of F. We let $\mathbb{D}_{1,2}$ denote the set of all random variables which are Malliavin differentiable with respect to both $B(\cdot)$ and $N(\cdot, \cdot)$. We will use the following *duality formula* for Malliavin derivatives:

$$\mathbb{E}\left[F\int_0^T \varphi(t)\mathrm{d}B(t)\right] = \mathbb{E}\left[\int_0^T \varphi(t)D_t F\mathrm{d}t\right], \tag{22.7}$$

$$\mathbb{E}\left[F\int_0^T \int_{\mathbb{R}_0} \psi(t,z)\tilde{N}(\mathrm{d}t,\mathrm{d}z)\right] = \mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} \psi(t,z)D_{t,z} F\nu(\mathrm{d}z)\mathrm{d}t\right] \tag{22.8}$$

valid for all Malliavin differentiable F- and all $\mathcal{F}_t$-predictable processes φ and ψ such that the integrals on the right converge absolutely. We also need the following basic properties of Malliavin derivatives:

If $F \in \mathbb{D}_{1,2}$ is $\mathcal{F}_s$-measurable, then

$$D_t F = D_{t,z} F = 0, \text{ for all } t > s. \tag{22.9}$$

(Fundamental theorem)

$$D_t\left(\int_0^T \varphi(s)\delta B(s)\right) = \int_0^T D_t\varphi(s)\delta B(s) + \varphi(t) \qquad \text{for a.a. } (t,\omega), \tag{22.10}$$

where $\int_0^T u(s)\delta B(s)$ denotes Skorohod integral of u with respect to $B(\cdot)$. (See [11], p. 35–38 for a definition of Skorohod integrals and for more details.)

$$D_{t,z}\left(\int_0^T \int_{\mathbb{R}} \psi(s,y)\tilde{N}(\mathrm{d}s,\mathrm{d}y)\right) = \int_0^T \int_{\mathbb{R}} D_{t,z}\psi(s,y)\tilde{N}(\mathrm{d}s,\mathrm{d}y) + \psi(t,z), \tag{22.11}$$

provided that all terms involved are well defined. We refer to [3], [5], [6], [7], [10] and [11] for more information about the Malliavin calculus for Lévy processes and its applications.

(A3) For all $(\pi,\theta) \in \mathcal{A}_\Pi \times \mathcal{A}_\Theta$, we assume the following processes, $i = 1,2$:

$$K_i(t) = g_i'(X(T)) + \int_t^T \int_{\mathbb{R}_0} \frac{\partial f_i}{\partial x}(s, X(s), \pi, \theta, z_1)\mu(\mathrm{d}z_1)\mathrm{d}s, \tag{22.12}$$

$$H_i^0(s, x, \pi, \theta) = K_i(s)b(s, x, \pi_0, \theta_0) + D_s K_i(s)\sigma(s, x, \pi_0, \theta_0)$$
$$+ \int_{\mathbb{R}_0} D_{s,z} K_i(s)\gamma(s, x, \pi, \theta, z)\nu(\mathrm{d}z), \tag{22.13}$$

$$G(t,s) := \exp\left(\int_t^s \left\{\frac{\partial b}{\partial x}(r, X(r), \pi_0(r), \theta_0(r))\right.\right.$$
$$\left.\left.- \frac{1}{2}\left(\frac{\partial \sigma}{\partial x}\right)^2(r, X(r), \pi_0(r), \theta_0(r))\right\}\mathrm{d}r\right)$$

$$+ \int_t^s \frac{\partial \sigma}{\partial x}(r, X(r), \pi_0(r), \theta_0(r)) \mathrm{d}B(r)$$

$$+ \int_t^s \int_{\mathbb{R}_0} \ln\left(1 + \frac{\partial \gamma}{\partial x}(r, X(r^-), \pi(r^-, z), \theta(r^-, z), z)\right) \tilde{N}(\mathrm{d}r, \mathrm{d}z)$$

$$+ \int_t^s \int_{\mathbb{R}_0} \left\{ \ln\left(1 + \frac{\partial \gamma}{\partial x}(r, X(r), \pi, \theta, z)\right)\right.$$

$$\left. - \frac{\partial \gamma}{\partial x}(r, X(r), \pi, \theta, z)\right\} \nu(\mathrm{d}z)\mathrm{d}r\bigg), \tag{22.14}$$

$$F_i(t, s) := \frac{\partial H_i^0}{\partial x}(s)G(t, s), \tag{22.15}$$

$$p_i(t) = K_i(t) + \int_t^T \frac{\partial H_i^0}{\partial x}(s, X(s), \pi_0(s), \pi_1(s, z), \theta_0(s), \theta_1(s, z))G(t, s)\mathrm{d}s, \tag{22.16}$$

$$q_i(t) = D_t p_i(t), \tag{22.17}$$

$$r_i(t, z) = D_{t,z} p_i(t), \tag{22.18}$$

all exist for $0 \le t \le s, z \in \mathbb{R}_0$.

We now define the *Hamiltonians* for this general problem as follows:

Definition 2.1 (The General Stochastic Hamiltonian). The general stochastic Hamiltonians for the stochastic differential game in Problem 1.1 are the functions

$$H_i(t, x, \pi, \theta, \omega) : [0, T] \times \mathbb{R} \times U \times K \times \Omega \to \mathbb{R}, \quad i = 1, 2,$$

defined by

$$H_i(t, x, \pi, \theta, \omega)$$

$$= \int_{\mathbb{R}_0} f_i(t, x, \pi, \theta, z, \omega)\mu(\mathrm{d}z) + p_i(t)b(t, x, \pi_0, \theta_0, \omega) + q_i(t)\sigma(t, x, \pi_0, \theta_0, \omega)$$

$$+ \int_{\mathbb{R}_0} r_i(t, z)\gamma(t, x, \pi, \theta, z, \omega)\nu(\mathrm{d}z), \quad i = 1, 2, \tag{22.19}$$

where $\pi = (\pi_0, \pi_1)$ and $\theta = (\theta_0, \theta_1)$.

Remark 2.1. In the classical case, the Hamiltonian $H_i^* : [0, T] \times \mathbb{R} \times U \times K \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \to \mathbb{R}$ is defined by

$$H_i^*(t, x, \pi, \theta, p, q, r) = \int_{\mathbb{R}_0} f_i(t, x, \pi, \theta)\mu(\mathrm{d}z) + p_i\, b(t, x, \pi_0, \theta_0) + q_i\, \sigma(t, x, \pi_0, \theta_0)$$

$$+ \int_{\mathbb{R}_0} r_i(t, z)\gamma(t, x, \pi, \theta, z)\nu(\mathrm{d}z), \tag{22.20}$$

where $\mathcal{R}$ is the set of functions $r_i : \mathbb{R} \times \mathbb{R}_0 \to \mathbb{R}$; $i = 1, 2$; see [12]. Thus the relation between H_i^* and H_i is that

$$H_i(t, x, \pi, \theta, \omega) = H_i^*(t, x, \pi, \theta, p(t), q(t), r(t, \cdot)), \quad i = 1, 2, \tag{22.21}$$

where $p(\cdot), q(\cdot)$ and $r(\cdot, \cdot)$ are given by Eqs. (22.16)–(22.18).

Theorem 2.1 (Maximum principle for non-zero-sum games).

(i) *Let $(\hat{\pi}, \hat{\theta}) \in \mathscr{A}_\Pi \times \mathscr{A}_\Theta$ be a Nash equilibrium with corresponding state process $\hat{X}(t) = X^{(\hat{\pi}, \hat{\theta})}(t)$, i.e.,*

$$J_1(\pi, \hat{\theta}) \leq J_1(\hat{\pi}, \hat{\theta}), \qquad \textit{for all } \pi \in \mathscr{A}_\Pi,$$

$$J_2(\hat{\pi}, \theta) \leq J_2(\hat{\pi}, \hat{\theta}), \qquad \textit{for all } \theta \in \mathscr{A}_\Theta.$$

Assume that the random variables $\frac{\partial f_i}{\partial x}$ and $F_i(t, s)$, $i = 1, 2$, belong to $\mathbb{D}_{1,2}$. Then

$$\mathbb{E}^x[\nabla_\pi \hat{H}_1(t, X^{(\pi, \hat{\theta})}(t), \pi, \hat{\theta}, \omega)|_{\pi=\hat{\pi}} \,|\mathscr{E}_t] = 0, \tag{22.22}$$

$$\mathbb{E}^x[\nabla_\theta \hat{H}_2(t, X^{(\hat{\pi}, \theta)}(t), \hat{\pi}, \theta, \omega)|_{\theta=\hat{\theta}} \,|\mathscr{E}_t] = 0, \tag{22.23}$$

for a.a. t, ω.

(ii) *Conversely, suppose that there exists $(\hat{\pi}, \hat{\theta}) \in \mathscr{A}_\Pi \times \mathscr{A}_\Theta$ such that Eqs. (22.22) and (22.23) hold. Then*

$$\frac{\partial}{\partial y} J_1(\hat{\pi} + y\beta, \hat{\theta})\Big|_{y=0} = 0 \quad \textit{for all } \beta,$$

$$\frac{\partial}{\partial \upsilon} J_2(\hat{\pi}, \hat{\theta} + \upsilon\eta)\Big|_{\upsilon=0} = 0 \quad \textit{for all } \eta.$$

In particular, if

$$\pi \to J_1(\pi, \hat{\theta}) \qquad \textit{and} \qquad \theta \to J_2(\hat{\pi}, \theta), \tag{22.24}$$

are concave, then $(\hat{\pi}, \hat{\theta})$ is a Nash equilibrium.

Proof. (i) Suppose $(\hat{\pi}, \hat{\theta}) \in \mathscr{A}_\Pi \times \mathscr{A}_\Theta$ is a Nash equilibrium. Since (i) and (ii) hold for all π and θ, $(\hat{\pi}, \hat{\theta})$ is a directional critical point for $J_i(\pi, \theta)$, $i = 1, 2$,

in the sense that for all bounded $\beta \in \mathscr{A}_\Pi$ and $\eta \in \mathscr{A}_\Theta$, there exists $\delta > 0$ such that $\hat{\pi} + y\beta \in \mathscr{A}_\Pi$, $\hat{\theta} + \upsilon\eta \in \mathscr{A}_\Theta$ for all $y, \upsilon \in (-\delta, \delta)$. Then we have

$$
\begin{aligned}
&= \frac{\partial}{\partial y} J_1\left(\hat{\pi} + y\beta, \hat{\theta}\right)\Big|_{y=0} \\
&= \mathbb{E}^x\left[\int_0^T \int_{\mathbb{R}_0}\left\{\frac{\partial f_1}{\partial x}(t, \hat{X}(t), \hat{\pi}_0(t), \hat{\pi}_1(t,z), \hat{\theta}_0(t), \hat{\theta}_1(t,z), z)\frac{\mathrm{d}}{\mathrm{d}y}X^{(\hat{\pi}+y\beta, \hat{\theta})}(t)\Big|_{y=0}\right.\right. \\
&\quad \left. +\nabla_\pi f_1(t, X^{(\pi, \hat{\theta})}(t), \pi_0(t), \pi_1(t,z), \hat{\theta}_0(t), \hat{\theta}_1(t,z), z)\Big|_{\pi=\hat{\pi}}\beta^*(t)\right\}\mu(\mathrm{d}z)\mathrm{d}t \\
&\quad \left. +g_1'(\hat{X}(T))\frac{\mathrm{d}}{\mathrm{d}y}X^{(\hat{\pi}+y\beta, \hat{\theta})}(T)\Big|_{y=0}\right] \\
&= \mathbb{E}^x\left[\int_0^T \int_{\mathbb{R}_0}\left\{\frac{\partial f_1}{\partial x}(t, \hat{X}(t), \hat{\pi}_0(t), \hat{\pi}_1(t,z), \hat{\theta}_0(t), \hat{\theta}_1(t,z), z)Y(t)\right.\right. \\
&\quad \left. +\nabla_\pi f_1(t, X^{(\pi, \hat{\theta})}(t), \pi_0(t), \pi_1(t,z), \hat{\theta}_0(t), \hat{\theta}_1(t,z), z)\Big|_{\pi=\hat{\pi}}\beta^*(t)\right\} \\
&\quad \left. \times\mu(\mathrm{d}z)\mathrm{d}t + g_1'(\hat{X}(T))Y(T)\right],
\end{aligned}
\tag{22.25}
$$

where

$$
\begin{aligned}
Y(t) =Y^{(\beta)}(t) &= \frac{\mathrm{d}}{\mathrm{d}y}X^{(\hat{\pi}+y\beta, \hat{\theta})}(t)|_{y=0} \\
&= \int_0^t\left\{\frac{\partial b}{\partial x}(s, \hat{X}(s), \hat{\pi}_0(s), \hat{\theta}_0(s))Y(s)\right. \\
&\quad \left. +\nabla_\pi b(s, X^{(\pi, \hat{\theta})}(s), \pi_0(s), \hat{\theta}_0(s))\Big|_{\pi=\hat{\pi}}\beta^*(s)\right\}\mathrm{d}s \\
&\quad +\int_0^t\left\{\frac{\partial \sigma}{\partial x}(s, \hat{X}(s), \hat{\pi}_0(s), \hat{\theta}_0(s))Y(s)\right. \\
&\quad \left. +\nabla_\pi \sigma(s, X^{(\pi, \hat{\theta})}(s), \pi_0(s), \hat{\theta}_0(s))\Big|_{\pi=\hat{\pi}}\beta^*(s)\right\}\mathrm{d}B(s) \\
&\quad +\int_0^t \int_{\mathbb{R}_0}\left\{\frac{\partial \gamma}{\partial x}(s, \hat{X}(s^-), \hat{\pi}(s^-), \hat{\theta}(s^-), z)Y(s)\right. \\
&\quad \left. +\nabla_\pi \gamma(s, X^{(\pi, \hat{\theta})}(s^-), \pi(s^-), \hat{\theta}(s^-), z)\Big|_{\pi=\hat{\pi}}\beta^*(s)\right\}\tilde{N}(\mathrm{d}s, \mathrm{d}z). \tag{22.26}
\end{aligned}
$$

If we use the shorthand notation

$$
\frac{\partial f_1}{\partial x}(t, \hat{X}(t), \hat{\pi}, \hat{\theta}, z) = \frac{\partial f_1}{\partial x}(t,z), \quad \nabla_\pi f_1(t, X^{(\pi, \hat{\theta})}(t), \pi, \hat{\theta}, z)|_{\pi=\hat{\pi}} = \nabla_\pi f_1(t,z),
$$

and similarly for $\frac{\partial b}{\partial x}$, $\nabla_\pi b$, $\frac{\partial \sigma}{\partial x}$, $\nabla_\pi \sigma$, $\frac{\partial \gamma}{\partial x}$ and $\nabla_\pi \gamma$, we can write

$$
\begin{cases}
\mathrm{d}Y(t) = (\frac{\partial b}{\partial x}(t)Y(t)+\nabla_\pi b(t)\beta^*(t))\mathrm{d}t+(\frac{\partial \sigma}{\partial x}(t)Y(t) + \nabla_\pi \sigma(t)\beta^*(t))\mathrm{d}B(t) \\
\qquad + \int_{\mathbb{R}_0}(\frac{\partial \gamma}{\partial x}(t)Y(t) + \nabla_\pi \gamma(t,z)\beta^*(t))\tilde{N}(\mathrm{d}t,\mathrm{d}z); \\
Y(0) \quad = 0.
\end{cases}
$$

(22.27)

By the duality formulas (22.7) and (22.8) and the Fubini theorem, we get

$$
\mathbb{E}^x\left[\int_0^T \int_{\mathbb{R}_0} \frac{\partial f_1}{\partial x}(t,z)Y(t)\mu(\mathrm{d}z)\mathrm{d}t\right]
$$

$$
= \mathbb{E}^x\left[\int_0^T \int_{\mathbb{R}_0} \left\{\int_0^t \left(\frac{\partial f_1}{\partial x}(t,z)\left[\frac{\partial b}{\partial x}(s)Y(s) + \nabla_\pi b(s)\beta^*(s)\right]\right.\right.\right.
$$

$$
+ D_s \frac{\partial f_1}{\partial x}(t,z)\left[\frac{\partial \sigma}{\partial x}(s)Y(s) + \nabla_\pi \sigma(s)\beta^*(s)\right]
$$

$$
+ \int_{\mathbb{R}_0} D_{s,z_1}\frac{\partial f_1}{\partial x}(t,z)\left[\frac{\partial \gamma}{\partial x}(s,z_1)Y(s)\right.
$$

$$
\left.\left.\left.+ \nabla_\pi \gamma(s,z_1)\beta^*(s)\right]\nu(\mathrm{d}z_1)\right)\mathrm{d}s\right\}\mu(\mathrm{d}z)\mathrm{d}t\right]
$$

$$
= \mathbb{E}^x\left[\int_0^T \left\{\left(\int_s^T \int_{\mathbb{R}_0} \frac{\partial f_1}{\partial x}(t,z)\mu(\mathrm{d}z)\mathrm{d}t\right)\left[\frac{\partial b}{\partial x}Y(s) + \nabla_\pi b(s)\beta^*(s)\right]\right.\right.
$$

$$
+ \left(\int_s^T \int_{\mathbb{R}_0} D_s \frac{\partial f_1}{\partial x}(t,z)\mu(\mathrm{d}z)\mathrm{d}t\right)\left[\frac{\partial \sigma}{\partial x}Y(s) + \nabla_\pi \sigma\beta^*(s)\right]
$$

$$
+ \int_{\mathbb{R}_0}\left(\int_s^T \int_{\mathbb{R}_0} D_{s,z_1}\frac{\partial f_1}{\partial x}(t,z)\mu(\mathrm{d}z)\mathrm{d}t\right)
$$

$$
\left.\times\left[\frac{\partial \gamma}{\partial x}Y(s) + \nabla_\pi \gamma\beta^*(s)\right]\nu(\mathrm{d}z_1)\right\}\mathrm{d}s\right].
$$

(22.28)

Changing notation $s \to t$ and $z_1 \to z$ this becomes

$$
\mathbb{E}^x\left[\int_0^T \int_{\mathbb{R}_0} \frac{\partial f_1}{\partial x}(t,z)Y(t)\mu(\mathrm{d}z)\mathrm{d}t\right]
$$

$$
= \mathbb{E}^x\left[\int_0^T \left\{\left(\int_t^T \int_{\mathbb{R}_0} \frac{\partial f_1}{\partial x}(s,z_1)\mu(\mathrm{d}z_1)\mathrm{d}s\right)\left[\frac{\partial b}{\partial x}(t)Y(t) + \nabla_\pi b(t)\beta^*(t)\right]\right.\right.
$$

$$
+ \left(\int_t^T \int_{\mathbb{R}_0} D_t \frac{\partial f_1}{\partial x}(s,z_1)\mu(\mathrm{d}z_1)\mathrm{d}s\right)\left[\frac{\partial \sigma}{\partial x}(t)Y(t) + \nabla_\pi \sigma(t)\beta^*(t)\right]
$$

$$+ \int_{\mathbb{R}_0} \Big(\int_t^T \int_{\mathbb{R}_0} D_{t,z} \frac{\partial f_1}{\partial x}(s, z_1) \mu(\mathrm{d}z_1) \mathrm{d}s \Big)$$

$$\times \Big[\frac{\partial \gamma}{\partial x}(t, z) Y(t) + \nabla_\pi \gamma(t, z) \beta^*(t) \Big] \nu(\mathrm{d}z) \Big\} \mathrm{d}t \Big]. \tag{22.29}$$

On the other hand, by the duality formulas (22.7) and (22.8), we get

$$\mathbb{E}^x \Big[g_1'(\hat{X}(T)) Y(T) \Big] = \mathbb{E}^x \Big[g_1'(\hat{X}(T)) \Big(\int_0^T \{ \frac{\partial b}{\partial x}(t) Y(t) + \nabla_\pi b(t) \beta^*(t) \} \, \mathrm{d}t$$

$$+ \int_0^T \{ \frac{\partial \sigma}{\partial x}(t) Y(t) + \nabla_\pi \sigma(t) \beta^*(t) \} \, \mathrm{d}B(t)$$

$$+ \int_0^T \int_{\mathbb{R}_0} \Big\{ \frac{\partial \gamma}{\partial x}(t, z) Y(t) + \nabla_\pi \gamma(t, z) \beta(t) \Big\} \tilde{N}(\mathrm{d}t, \mathrm{d}z) \Big) \Big]$$

$$= \mathbb{E}^x \Big[\int_0^T \Big\{ g_1'(\hat{X}(T)) \frac{\partial b}{\partial x}(t) Y(t) + g_1'(\hat{X}(T)) \nabla_\pi b(t) \beta^*(t)$$

$$+ D_t(g_1'(\hat{X}(T))) \frac{\partial \sigma}{\partial x}(t) Y(t) + D_t(g_1'(\hat{X}(T))) \nabla_\pi \sigma(t) \beta^*(t)$$

$$+ \int_{\mathbb{R}_0} [D_{t,z}(g_1'(\hat{X}(T))) \frac{\partial \gamma}{\partial x}(t, z) Y(t)$$

$$+ D_{t,z}(g_1'(\hat{X}(T))) \nabla_\pi \gamma(t, z) \beta^*(t)] \nu(\mathrm{d}z) \Big\} \mathrm{d}t \Big].$$

We recall that

$$\hat{K}_1(t) := g_1'(\hat{X}(T)) + \int_t^T \int_{\mathbb{R}_0} \frac{\partial f_1}{\partial x}(s, z_1) \mu(\mathrm{d}z_1) \, \mathrm{d}s,$$

and combining Eqs. (22.27)–(22.29), we get

$$\mathbb{E}^x \Big[\int_0^T \Big\{ \hat{K}_1(t) \Big(\frac{\partial b}{\partial x}(t) Y(t) + \nabla_\pi b(t) \beta^*(t) \Big)$$

$$+ D_t \hat{K}_1(t) \Big(\frac{\partial \sigma}{\partial x}(t) Y(t) + \nabla_\pi \sigma(t) \beta^*(t) \Big)$$

$$+ \int_{\mathbb{R}_0} D_{t,z} \hat{K}_1(t) \Big(\frac{\partial \gamma}{\partial x}(t, z) Y(t) + \nabla_\pi \gamma(t, z) \beta^*(t) \Big) \nu(\mathrm{d}z)$$

$$+ \int_{\mathbb{R}_0} \nabla_\pi f_1(t, z) \beta^*(t) \mu(\mathrm{d}z) \Big\} \mathrm{d}t \Big] = 0. \tag{22.30}$$

Now apply this to $\beta = \beta_\alpha \in \mathscr{A}_\Pi$ of the form $\beta_\alpha(s) = \alpha \chi_{[t,t+h]}(s)$, for some $t, h \in (0, T), t + h \leq T$, where $\alpha = \alpha(\omega)$ is bounded and $\mathscr{E}_t$-measurable. Then $Y^{(\beta_\alpha)}(s) = 0$ for $0 \leq s \leq t$. Hence Eq. (22.30) becomes

$$A_1 + A_2 = 0, \tag{22.31}$$

where

$$A_1 = \mathbb{E}^x\left[\int_t^T \left\{\hat{K}_1(s)\frac{\partial b}{\partial x}(s) + D_s\hat{K}_1(s)\frac{\partial \sigma}{\partial x}(s)\right.\right.$$
$$\left.\left. + \int_{\mathbb{R}_0} D_{s,z}\hat{K}_1(s)\frac{\partial \gamma}{\partial x}(s)v(\mathrm{d}z)\right\}Y^{(\beta_\alpha)}(s)\mathrm{d}s\right],$$

$$A_2 = \mathbb{E}^x\left[\left\{\int_t^{t+h}\left(\hat{K}_1(s)\nabla_\pi b(s) + D_s\hat{K}_1(s)\nabla_\pi \sigma(s)\right.\right.\right.$$
$$\left. + \int_{\mathbb{R}_0} D_{s,z}\hat{K}_1(s)\nabla_\pi \gamma(s,z)v(\mathrm{d}z)\right.$$
$$\left.\left.\left. + \int_{\mathbb{R}_0} \nabla_\pi f_1(s,z)\mu(\mathrm{d}z)\right)\mathrm{d}s\right\}\alpha\right].$$

Note that, by Eq. (22.26), with $Y(s) = Y^{(\beta_\alpha)}(s)$ and $s \geq t + h$,

$$\mathrm{d}Y(s) = Y(s^-)\left\{\frac{\partial b}{\partial x}(s)\mathrm{d}s + \frac{\partial \sigma}{\partial x}(s)\mathrm{d}B(s) + \int_{\mathbb{R}_0} \frac{\partial \gamma}{\partial x}(s^-,z)\tilde{N}(\mathrm{d}s,\mathrm{d}z)\right\},$$

for $s \geq t + h$. Hence, by the Itô formula,

$$Y(s) = Y(t+h)G(t+h,s); \qquad s \geq t+h, \qquad (22.32)$$

where, in general, for $s \geq t$,

$$G(t,s) = \exp\left(\int_t^s\left\{\frac{\partial b}{\partial x}(r) - \tfrac{1}{2}\left(\frac{\partial \sigma}{\partial x}\right)^2(r)\right\}\mathrm{d}r + \int_t^s \frac{\partial \sigma}{\partial x}(r)\mathrm{d}B(r)\right.$$
$$+ \int_t^s\int_{\mathbb{R}_0} \ln\left(1 + \frac{\partial \gamma}{\partial x}(r^-,z)\right)\tilde{N}(\mathrm{d}r,\mathrm{d}z)$$
$$\left. + \int_t^s\int_{\mathbb{R}_0}\left\{\ln\left(1 + \frac{\partial \gamma}{\partial x}(r,z)\right) - \frac{\partial \gamma}{\partial x}(r,z)\right\}v(\mathrm{d}z)\mathrm{d}r\right). \qquad (22.33)$$

Note that $G(t,s)$ does not depend on h. Put

$$H_1^0(s,x,\pi,\theta) = K_1(s)b(s,x,\pi_0,\theta_0) + D_s K_1(s)\sigma(s,x,\pi_0,\theta_0)$$
$$+ \int_{\mathbb{R}_0} D_{s,z}K_1(s)\gamma(s,x,\pi,\theta,z)v(\mathrm{d}z), \qquad (22.34)$$

and $\hat{H}_1^0(s) = H_1^0(s,\hat{X}(s),\hat{\pi},\hat{\theta})$. Then

$$A_1 = \mathbb{E}^x\left[\int_t^T \frac{\partial \hat{H}_1^0}{\partial x}(s)Y(s)\mathrm{d}s\right].$$

Differentiating with respect to h at $h = 0$ we get

$$\frac{\mathrm{d}}{\mathrm{d}h}A_1\big|_{h=0} = \frac{\mathrm{d}}{\mathrm{d}h}\mathbb{E}^x\left[\int_t^{t+h}\frac{\partial\hat{H}_1^0}{\partial x}(s)Y(s)\mathrm{d}s\right]_{h=0}$$

$$+\frac{\mathrm{d}}{\mathrm{d}h}\mathbb{E}^x\left[\int_{t+h}^T\frac{\partial\hat{H}_1^0}{\partial x}(s)Y(s)\mathrm{d}s\right]_{h=0}. \qquad (22.35)$$

Since $Y(t) = 0$ we see that

$$\frac{\mathrm{d}}{\mathrm{d}h}\mathbb{E}^x\left[\int_t^{t+h}\frac{\partial\hat{H}_1^0}{\partial x}(s)Y(s)\mathrm{d}s\right]_{h=0} = 0. \qquad (22.36)$$

Therefore, by Eq. (22.31),

$$\frac{\mathrm{d}}{\mathrm{d}h}A_1\big|_{h=0} = \frac{\mathrm{d}}{\mathrm{d}h}\mathbb{E}^x\left[\int_{t+h}^T\frac{\partial\hat{H}_1^0}{\partial x}(s)Y(t+h)G(t+h,s)\mathrm{d}s\right]_{h=0}$$

$$= \int_t^T\frac{\mathrm{d}}{\mathrm{d}h}\mathbb{E}^x\left[\frac{\partial\hat{H}_1^0}{\partial x}(s)Y(t+h)G(t+h,s)\right]_{h=0}\mathrm{d}s$$

$$= \int_t^T\frac{\mathrm{d}}{\mathrm{d}h}\mathbb{E}^x\left[\frac{\partial\hat{H}_1^0}{\partial x}(s)G(t,s)Y(t+h)\right]_{h=0}\mathrm{d}s. \qquad (22.37)$$

On the other hand, Eq. (22.26) gives

$$Y(t+h) = \alpha\int_t^{t+h}\left\{\nabla_\pi b(r)\mathrm{d}r + \nabla_\pi\sigma\mathrm{d}B(r) + \int_{\mathbb{R}_0}\nabla_\pi\gamma(r^-,z)\tilde{N}(\mathrm{d}r,\mathrm{d}z)\right\}$$

$$+ \int_t^{t+h}Y(r^-)\left\{\frac{\partial b}{\partial x}(r)\mathrm{d}r + \frac{\partial\sigma}{\partial x}(r)\mathrm{d}B(r) + \int_{\mathbb{R}_0}\frac{\partial\gamma}{\partial x}(r^-,z)\tilde{N}(\mathrm{d}r,\mathrm{d}z)\right\}. \qquad (22.38)$$

Combining this with Eqs. (22.36) and (22.37), we have

$$\frac{\mathrm{d}}{\mathrm{d}h}A_1\big|_{h=0} = \Lambda_1 + \Lambda_2, \qquad (22.39)$$

where

$$\Lambda_1 = \int_t^T\frac{\mathrm{d}}{\mathrm{d}h}\mathbb{E}^x\left[\frac{\partial\hat{H}_1^0}{\partial x}(s)G(t,s)\alpha\int_t^{t+h}\left\{\nabla_\pi b(r)\mathrm{d}r + \nabla_\pi\sigma(r)\mathrm{d}B(r)\right.\right.$$

$$\left.\left. + \int_{\mathbb{R}_0}\nabla_\pi\gamma(r^-,z)\tilde{N}(\mathrm{d}r,\mathrm{d}z)\right\}\right]_{h=0}\mathrm{d}s \qquad (22.40)$$

and

$$\Lambda_2 = \int_t^T \frac{\mathrm{d}}{\mathrm{d}h} \mathbb{E}^x \left[\frac{\partial \hat{H}_1^0}{\partial x}(s) G(t,s) \int_t^{t+h} Y(r^-) \left\{ \frac{\partial b}{\partial x}(r) \mathrm{d}r + \frac{\partial \sigma}{\partial x}(r) \mathrm{d}B(r) \right. \right.$$

$$\left. \left. + \int_{\mathbb{R}_0} \frac{\partial \gamma}{\partial x}(r^-, z) \tilde{N}(\mathrm{d}r, \mathrm{d}z) \right\} \right]_{h=0} \mathrm{d}s. \tag{22.41}$$

By the duality formulae (22.7) and (22.8), we have

$$\Lambda_1 = \int_t^T \frac{\mathrm{d}}{\mathrm{d}h} \mathbb{E}^x \left[\alpha \int_t^{t+h} \left\{ \nabla_\pi b(r) F_1(t,s) + \nabla_\pi \sigma(r) D_r F_1(t,s) \right. \right.$$

$$\left. \left. + \int_{\mathbb{R}_0} \nabla_\pi \gamma(r,z) D_{r,z} F_1(t,s) \nu(\mathrm{d}z) \right\} \mathrm{d}r \right]_{h=0} \mathrm{d}s$$

$$= \int_t^T \mathbb{E}^x \left[\alpha \left\{ \nabla_\pi b(t) F_1(t,s) + \nabla_\pi \sigma(t) D_t F_1(t,s) \right. \right.$$

$$\left. \left. + \int_{\mathbb{R}_0} \nabla_\pi \gamma(t,z) D_{t,z} F_1(t,s) \nu(\mathrm{d}z) \right\} \right] \mathrm{d}s. \tag{22.42}$$

Since $Y(t) = 0$ we see that

$$\Lambda_2 = 0. \tag{22.43}$$

We conclude that

$$\frac{\mathrm{d}}{\mathrm{d}h} A_1 \Big|_{h=0} = \Lambda_1$$

$$= \int_t^T \mathbb{E}^x \left[\alpha \left\{ F_1(t,s) \nabla_\pi b(t) + D_t F_1(t,s) \nabla_\pi \sigma(t) \right. \right.$$

$$\left. \left. + \int_{\mathbb{R}_0} D_{t,z} F_1(t,s) \nabla_\pi \gamma(t,z) \nu(\mathrm{d}z) \right\} \right] \mathrm{d}s. \tag{22.44}$$

Moreover, we see directly that

$$\frac{\mathrm{d}}{\mathrm{d}h} A_2 \Big|_{h=0} = \mathbb{E}^x \left[\alpha \left\{ \hat{K}_1(t) \nabla_\pi b(t) + D_t \hat{K}_1(t) \nabla_\pi \sigma(t) \right. \right.$$

$$\left. \left. + \int_{\mathbb{R}_0} \{ D_{t,z} \hat{K}_1(t) \nabla_\pi \gamma(t,z) + \nabla_\pi f_1(t,z) \} \nu(\mathrm{d}z) \right\} \right]. \tag{22.45}$$

Therefore, differentiating Eq. (22.30) with respect to h at $h = 0$ gives the equation

$$\mathbb{E}^x \left[\alpha \left\{ \left(\hat{K}_1(t) + \int_t^T F_1(t,s) \mathrm{d}s \right) \nabla_\pi b(t) + D_t \left(\hat{K}_1(t) + \int_t^T F_1(t,s) \mathrm{d}s \right) \nabla_\pi \sigma(t) \right. \right.$$

$$\left. \left. + \int_{\mathbb{R}_0} D_{t,z} \left(\hat{K}_1(t) + \int_t^T F_1(t,s) \mathrm{d}s \right) \nabla_\pi \gamma(t,z) + \nabla_\pi f_1(t,z) \nu(\mathrm{d}z) \right\} \right] = 0. \tag{22.46}$$

We can reformulate this as follows: If we define, as in Eq. (22.16),

$$\hat{p}_1(t) = \hat{K}_1(t) + \int_t^T F_1(t,s)\mathrm{d}s = \hat{K}_1(t) + \int_t^T \frac{\partial \hat{H}_1^0}{\partial x}(s)G(t,s)\mathrm{d}s, \quad (22.47)$$

then Eq. (22.44) can be written:

$$\mathbb{E}^x\Big[\nabla_\pi\Big\{\int_{\mathbb{R}_0} f_1(t,\hat{X}(t),\pi,\hat{\theta},z)\mu(\mathrm{d}z) + \hat{p}_1(t)b(t,\hat{X}(t),\pi_0,\hat{\theta}_0)$$

$$+ D_t\hat{p}_1(t)\sigma(t,\hat{X}(t),\pi_0,\hat{\theta}_0)$$

$$+ \int_{\mathbb{R}_0} D_{t,z}\hat{p}_1(t)\gamma(t,\hat{X}(t),\pi,\hat{\theta},z)\nu(\mathrm{d}z)\Big\}_{\pi=(\pi_0(t),\pi_1(t,z))}\alpha\Big] = 0.$$

Since this holds for all bounded $\mathscr{E}_t$-measurable random variable α, we conclude that

$$\mathbb{E}^x\Big[\nabla_\pi\hat{H}_1(t,X^{(\pi,\hat{\theta})}(t),\pi,\hat{\theta})\,|_{\pi=\hat{\pi}(t)}|\,\mathscr{E}_t\Big] = 0.$$

Similarly, we have

$$0 = \frac{\partial}{\partial \upsilon}J_2(\hat{\pi},\hat{\theta} + \upsilon\eta)\Big|_{\upsilon=0}$$

$$= \mathbb{E}^x\Big[\int_0^T \int_{\mathbb{R}_0}\Big\{\frac{\partial f_2}{\partial x}(t,X^{(\hat{\pi},\theta)}(t),\hat{\pi}(t,z),\hat{\theta}(t,z),z)D(t)$$

$$+ \nabla_\theta f_2(t,X^{(\hat{\pi},\theta)}(t),\hat{\pi}(t,z),\theta(t,z),z)\Big|_{\theta=\hat{\theta}}\eta(t)\Big\}\mu(\mathrm{d}z)\mathrm{d}t + g_2'(\hat{X}(T))D(T)\Big],$$

$$(22.48)$$

where

$$D(t) = D^{(\eta)}(t) = \frac{\mathrm{d}}{\mathrm{d}\upsilon}X^{(\hat{\pi},\hat{\theta}+\upsilon\eta)}(t)\Big|_{\upsilon=0}$$

$$= \int_0^t\Big\{\frac{\partial b}{\partial x}(s,\hat{X}(s),\hat{\pi}_0(s),\hat{\theta}_0(s))D(s)$$

$$+ \nabla_\theta b(s,X^{(\hat{\pi}_0,\theta_0)}(s),\hat{\pi}_0(s),\theta_0(s))\Big|_{\theta=\hat{\theta}}\eta^*(s)\Big\}\mathrm{d}s$$

$$+ \int_0^t\Big\{\frac{\partial\sigma}{\partial x}(s,\hat{X}(s),\hat{\pi}_0(s),\hat{\theta}_0(s))Y(s)$$

$$+ \nabla_\theta\sigma(s,X^{(\hat{\pi},\theta)}(s),\hat{\pi}_0(s),\theta_0(s))\Big|_{\theta=\hat{\theta}}\eta^*(s)\Big\}\mathrm{d}B(s)$$

$$+ \int_0^t\int_{\mathbb{R}_0}\Big\{\frac{\partial\gamma}{\partial x}(s,\hat{X}(s^-),\hat{\pi}(s^-,z),\hat{\theta}(s^-),z)D(s)$$

$$+ \nabla_\theta\gamma(s,X^{(\hat{\pi},\theta)}(s^-),\hat{\pi}(s^-,z),\theta(s^-,z),z)\Big|_{\theta=\hat{\theta}}\eta^*(s)\Big\}\tilde{N}(\mathrm{d}s,\mathrm{d}z).$$

$$(22.49)$$

Define

$$D(s) = D(t + h)G(t + h, s); \quad s \geq t + h,$$

where $G(t, s)$ is defined as in Eq. (22.32). By using similar arguments as above, we get

$$\mathbb{E}^x \left[\nabla_\theta \hat{H}_2(t, X^{(\hat{\pi}, \theta)}(t), \hat{\pi}, \theta) \mid_{\theta = \hat{\theta}(t)} \mid \mathcal{E}_t \right] = 0.$$

This completes the proof of (i).

(ii) Conversely, suppose that there exists $(\hat{\pi}, \hat{\theta}) \in \mathscr{A}_\Pi \times \mathscr{A}_\Theta$ such that Eqs. (22.22) and (22.23) hold. Then by reversing the above arguments, we obtain that Eq. (22.31) holds for all $\beta_\alpha(s, \omega) = \alpha(\omega)\chi_{(t,t+h]}(s) \in \mathscr{A}_\Pi$, where

$$A_1 = \mathbb{E}^x \left[\int_t^T \left\{ \hat{K}_1(s) \frac{\partial b}{\partial x}(s) + D_s \hat{K}_1(s) \frac{\partial \sigma}{\partial x}(s) \right. \right.$$
$$\left. \left. + \int_{\mathbb{R}_0} D_{s,z} \hat{K}_1(s) \frac{\partial \gamma}{\partial x}(s)\nu(dz) \right\} Y^{(\beta_\alpha)}(s) ds \right],$$

$$A_2 = \mathbb{E}^x \left[\left\{ \int_t^{t+h} \left(\hat{K}_1(s)\nabla_\pi b(s) + D_s \hat{K}_1(s)\nabla_\pi \sigma(s) \right. \right. \right.$$
$$\left. \left. \left. + \int_{\mathbb{R}_0} D_{s,z} \hat{K}_1(s)\nabla_\pi \gamma(s, z)\nu(dz) + \int_{\mathbb{R}_0} \nabla_\pi f_1(s, z)\mu(dz) \right) ds \right\} \alpha \right],$$

for some $t, h \in [0, T]$ with $t + h \leq T$ and some bounded $\mathcal{E}_t$-measurable α. Similarly,

$$A_3 + A_4 = 0 \tag{22.50}$$

for all $\eta_\xi(s, \omega) = \xi(\omega)\chi_{(t,t+h]}(s) \in \mathscr{A}_\Theta$, where

$$A_3 = \mathbb{E}^x \left[\int_t^T \left\{ \hat{K}_2(s) \frac{\partial b}{\partial x}(s) + D_s \hat{K}_2(s) \frac{\partial \sigma}{\partial x}(s) \right. \right.$$
$$\left. \left. + \int_{\mathbb{R}_0} D_{s,z} \hat{K}_2(s) \frac{\partial \gamma}{\partial x}(s)\nu(dz) \right\} Y^{(\eta_\xi)}(s) ds \right],$$

$$A_4 = \mathbb{E}^x \left[\left\{ \int_t^{t+h} \left(\hat{K}_2(s)\nabla_\theta b(s) + D_s \hat{K}_2(s)\nabla_\theta \sigma(s) \right. \right. \right.$$
$$\left. \left. \left. + \int_{\mathbb{R}_0} D_{s,z} \hat{K}_2(s)\nabla_\theta \gamma(s, z)\nu(dz) + \int_{\mathbb{R}_0} \nabla_\theta f_2(s, z)\mu(dz) \right) ds \right\} \alpha \right],$$

for some $t, h \in [0, T]$ with $t + h \leq T$ and some bounded $\mathcal{E}_t$-measurable ξ. Hence, these equalities hold for all linear combinations of β_α and η_ξ. Since all bounded $\beta \in \mathscr{A}_\Pi$ and $\eta \in \mathscr{A}_\Theta$ can be approximated pointwise boundary in

(t, ω) by such linear combinations, it follows that Eqs. (22.31) and (22.50) hold for all bounded $(\beta, \eta) \in \mathscr{A}_\Pi \times \mathscr{A}_\Theta$. Hence, by reversing the remaining part of the proof above, we conclude that

$$\frac{\partial}{\partial y} J_1(\hat{\pi} + y\beta, \hat{\theta})\Big|_{y=0} = 0,$$

$$\frac{\partial}{\partial \upsilon} J_2(\hat{\pi}, \hat{\theta} + \upsilon\eta)\Big|_{\upsilon=0} = 0,$$

for all β and η.

$\square$

3   Zero-Sum Games

Suppose that the given performance functional of player I is the negative of the player II, i.e.,

$$J_1(u_0, u_1) = \mathbb{E}^x\Big[\int_0^T \int_{\mathbb{R}_0} f(t, X(t), u_0(t), u_1(t, z), z, \omega)\mu(\mathrm{d}z)\mathrm{d}t + g(X(T), \omega)\Big]$$
$$= -J_2(u_0, u_1), \tag{22.51}$$

where $\mathbb{E}^x = \mathbb{E}^x_P$ denotes the expectation with respect to P given that $X(0) = x$. Suppose that the controls $u_0(t)$ and $u_1(t, z)$ have the form as in Eqs. (22.5) and (22.6). Let $\mathscr{A}_\Pi$ and $\mathscr{A}_\Theta$ denote the given family of controls $\pi = (\pi_0, \pi_1)$ and $\theta = (\theta_0, \theta_1)$ such that they are contained in the set of càdlàg $\mathscr{E}_t$-adapted controls, Eq. (22.1) has a unique strong solution up to time T and

$$\mathbb{E}^x\Big[\int_0^T \int_{\mathbb{R}_0} |f(t, X(t), \pi_0(t), \pi_1(t, z), \theta_0(t), \theta_1(t, z), z, \omega)|\mu(\mathrm{d}z)\mathrm{d}t$$

$$+ |g(X(T), \omega)|\Big] < \infty. \tag{22.52}$$

Then the partial information zero-sum stochastic differential game problem is the following:

Problem 3.1. Find $\Phi_\mathscr{E} \in \mathbb{R}$, $\pi^* \in \mathscr{A}_\Pi$ and $\theta^* \in \mathscr{A}_\Theta$ (if it exists) such that

$$\Phi_\mathscr{E} = \inf_{\theta \in \mathscr{A}_\Theta} (\sup_{\pi \in \mathscr{A}_\Pi} J(\pi, \theta)) = J(\pi^*, \theta^*) = \sup_{\pi \in \mathscr{A}_\Pi} (\inf_{\theta \in \mathscr{A}_\Theta} J(\pi, \theta)). \tag{22.53}$$

Such a control (π^*, θ^*) is called an *optimal control* (if it exists). The intuitive idea is that while player I controls π, player II controls θ. The actions of the players are antagonistic, which means that between players I and II, there is a payoff $J(\pi, \theta)$

which is a reward for player I and a cost for Player II. Note that since we allow b, σ, γ, f and g to be stochastic processes and also because our controls are $\mathcal{E}_t$-adapted, this problem is not of Markovian type and hence cannot be solved by dynamic programming.

Theorem 3.1 (Maximum principle for zero-sum games).

(i) *Suppose* $(\hat{\pi}, \hat{\theta}) \in \mathscr{A}_\Pi \times \mathscr{A}_\Theta$ *is a directional critical point for* $J(\pi, \theta)$ *in the sense that for all bounded* $\beta \in \mathscr{A}_\Pi$ *and* $\eta \in \mathscr{A}_\Theta$, *there exists* $\delta > 0$ *such that* $\hat{\pi} + y\beta \in \mathscr{A}_\Pi$, $\hat{\theta} + \upsilon\eta \in \mathscr{A}_\Theta$ *for all* $y, \upsilon \in (-\delta, \delta)$ *and*

$$c(y, \upsilon) := J(\hat{\pi} + y\beta, \hat{\theta} + \upsilon\eta), \quad y, \upsilon \in (-\delta, \delta)$$

has a critical point at 0, *i.e.,*

$$\frac{\partial c}{\partial y}(0, 0) = \frac{\partial c}{\partial \upsilon}(0, 0) = 0. \tag{22.54}$$

Then

$$\mathbb{E}^x[\nabla_\pi \hat{H}(t, X^{(\pi, \hat{\theta})}(t), \pi, \hat{\theta}, \omega)|\mathcal{E}_t]_{\pi = \hat{\pi}} = 0, \tag{22.55}$$

$$\mathbb{E}^x[\nabla_\theta \hat{H}(t, X^{(\hat{\pi}, \theta)}(t), \hat{\pi}, \theta, \omega)|\mathcal{E}_t]_{\theta = \hat{\theta}} = 0 \quad \text{for a.a. } t, \omega, \tag{22.56}$$

where

$$\hat{X}(t) = X^{(\hat{\pi}, \hat{\theta})}(t),$$

$$\hat{H}(t, \hat{X}(t), \pi, \theta) = \int_{\mathbb{R}_0} f(t, \hat{X}(t), \pi, \theta, z)\mu(dz) + \hat{p}(t)b(t, \hat{X}(t), \pi_0, \theta_0)$$

$$+ \hat{q}(t)\sigma(t, \hat{X}(t), \pi_0, \theta_0) + \int_{\mathbb{R}_0} \hat{r}(t, z)\gamma(t, \hat{X}(t^-), \pi, \theta, z)\nu(dz), \tag{22.57}$$

with

$$\hat{p}(t) = \hat{K}(t) + \int_t^T \frac{\partial \hat{H}^0}{\partial x}(s, \hat{X}(s), \hat{\pi}(s), \hat{\theta}(s)) \, \hat{G}(t, s) ds, \tag{22.58}$$

$$\hat{K}(t) = K^{(\hat{\pi}, \hat{\theta})}(t) = g'(\hat{X}(T)) + \int_t^T \int_{\mathbb{R}_0} \frac{\partial f}{\partial x}(s, \hat{X}(s), \hat{\pi}(s, z), \hat{\theta}(s, z), z)\mu(dz)ds, \tag{22.59}$$

$$\hat{H}^0(s, \hat{X}, \hat{\pi}, \hat{\theta}) = \hat{K}(s)b(s, \hat{X}, \hat{\pi}_0, \hat{\theta}_0) + D_s\hat{K}(s)\sigma(s, \hat{X}, \hat{\pi}_0, \hat{\theta}_0)$$

$$+ \int_{\mathbb{R}_0} D_{s,z}\hat{K}(s)\gamma(s, \hat{X}, \hat{\pi}, \hat{\theta}, z)\nu(dz), \tag{22.60}$$

$$\hat{G}(t,s) := \exp\Big(\int_t^s \Big\{\frac{\partial b}{\partial x}(r, \hat{X}(r), \hat{\pi}_0(r), \hat{\theta}_0(r))$$

$$- \frac{1}{2}\Big(\frac{\partial \sigma}{\partial x}\Big)^2 (r, \hat{X}(r), \hat{\pi}_0(r), \hat{\theta}_0(r))\Big\}dr$$

$$+ \int_t^s \frac{\partial \sigma}{\partial x}(r, \hat{X}(r), \hat{\pi}_0(r), \hat{\theta}_0(r))dB(r)$$

$$+ \int_t^s \int_{\mathbb{R}_0} \ln\Big(1 + \frac{\partial \gamma}{\partial x}(r, \hat{X}(r^-), \hat{\pi}(r^-, z), \hat{\theta}(r^-, z), z)\Big)\tilde{N}(dr, dz)$$

$$+ \int_t^s \int_{\mathbb{R}_0} \Big\{\ln\Big(1 + \frac{\partial \gamma}{\partial x}(r, \hat{X}(r), \hat{\pi}, \hat{\theta}, z)\Big)$$

$$- \frac{\partial \gamma}{\partial x}(r, \hat{X}(r), \hat{\pi}, \hat{\theta}, z)\Big\}\nu(dz)dr\Big);$$

$$\hat{q}(t) := D_t \hat{p}(t),$$

and

$$\hat{r}(t, z) := D_{t,z}\hat{p}(t). \tag{22.61}$$

(ii) *Conversely, suppose that there exists $(\hat{\pi}, \hat{\theta}) \in \mathscr{A}_\Pi \times \mathscr{A}_\Theta$ such that Eqs. (22.55) and (22.56) hold. Furthermore, suppose that g is an affine function, H is concave in π and convex in θ. Then $(\hat{\pi}, \hat{\theta})$ satisfies Eq. (22.54).*

4 Application: Worst-Case Scenario Optimal Portfolio Under Partial Information

We illustrate the results in the previous section by looking at an application to robust portfolio choice in finance:

Consider a financial market with the following two investment possibilities:

1. A *risk free asset*, where the unit price $S_0(t)$ at time t is

$$dS_0(t) = r(t)S_0(t)dt; \quad S_0(0) = 1; \quad 0 \le t \le T,$$

where $T > 0$ is a given constant.

2. A *risky asset*, where the unit price $S_1(t)$ at time t is given by

$$\begin{cases} dS_1(t) = S_1(t^-)[\theta(t)dt + \sigma_0(t)dB(t) + \int_{\mathbb{R}_0} \gamma_0(t, z)\tilde{N}(dt, dz)], \\ S_1(0) > 0, \end{cases} \tag{22.62}$$

where r, θ, σ_0 and γ_0 are predictable processes such that

$$\int_0^T \{| \ \theta(s) \ | + \sigma_0^2(s) + \int_{\mathbb{R}_0} \gamma_0^2(s,z) \nu(dz)\} ds < \infty \quad \text{a.s.}$$

We assume that θ is adapted to a given subfiltration $\mathscr{E}_t$ and that

$$\gamma_0(t,z,\omega) \geq -1 + \delta \qquad \text{for all } t, z, \omega \in [0,T] \times \mathbb{R}_0 \times \Omega,$$

for some constant $\delta > 0$.

Let $\pi(t) = \pi(t, \omega)$ be a *portfolio*, representing the amount invested in the risky asset at time t. We require that π be càdlàg and $\mathscr{E}_t$-adapted and self–financing and hence that the corresponding wealth $X(t) = X^{(\pi,\theta)}(t)$ at time t is given by

$$\begin{cases} dX(t) = [X(t) - \pi(t)]r(t)dt + \pi(t^-)[\theta(t)dt + \sigma_0(t)dB(t) \\ \qquad + \int_{\mathbb{R}_0} \gamma_0(t,z)\tilde{N}(dt,dz)] \\ X(0) = \ x > 0. \end{cases} \tag{22.63}$$

Let us assume that the mean relative growth rate $\theta(t)$ of the risky asset is not known to the trader, but subject to uncertainty. We may regard θ as a *market scenario* or a *stochastic control* of the market, which is playing against the trader. Let $\mathscr{A}_\Pi^\varepsilon$ and $\mathscr{A}_\Theta^\varepsilon$ denote the set of admissible controls π, θ, respectively. *The worst-case partial information scenario optimal problem* for the trader is to find $\pi^* \in \mathscr{A}_\Pi^\varepsilon$ and $\theta^* \in \mathscr{A}_\Theta^\varepsilon$ and $\Phi \in \mathbb{R}$ such that

$$\Phi = \inf_{\theta \in \mathscr{A}_\Theta^\varepsilon} (\sup_{\pi \in \mathscr{A}_\Pi^\varepsilon} \mathbb{E}[U(X^{(\pi,\theta)}(T))])$$

$$= \mathbb{E}[U(X^{(\pi^*,\theta^*)(T)})], \tag{22.64}$$

where $U : [0, \infty) \to \mathbb{R}$ is a given utility function, assumed to be concave, strictly increasing and $\mathcal{C}^1$ on $(0, \infty)$. We want to study this problem by using Theorem 3.1. In this case we have

$$b(t,x,\pi,\theta) = \pi(\theta - r(t)) + xr(t), \quad K(t) = U'(X^{(\pi,\theta)}(T)), \tag{22.65}$$

and

$$\begin{aligned} H_0(t,x,\pi,\theta) = & U'(X^{(\pi,\theta)}(T))[\pi(\theta - r(t)) + xr(t)] \\ & + D_t(U'(X^{(\pi,\theta)}(T)))\pi\sigma_0(t) \\ & + \int_{\mathbb{R}_0} D_{t,z}(U'(X^{(\pi,\theta)}(T)))\pi\gamma_0(t,z)\nu(dz), \end{aligned} \tag{22.66}$$

and

$$p(t) = U'(X^{(\pi,\theta)}(T))\left[1 + \int_t^T r(s)G(t,s)ds\right],$$

where

$$G(t,s) = \exp\left(\int_t^s r(v)dv\right).$$

Hence,

$$\int_t^T r(s)G(t,s)ds = \Big|_t^T \exp\left(\int_t^s r(v)dv\right) = \exp\left(\int_t^T r(v)dv\right) - 1$$

and

$$p(t) = U'(X^{(\pi,\theta)}(T)) \exp\left(\int_t^T r(s)ds\right). \tag{22.67}$$

With this value for $p(t)$ we have

$$H(t, X^{(\pi,\theta)}(t), \pi, \theta) = p(t)[\pi(\theta - r(t)) + r(t)X(t)]$$

$$+ D_t p(t)\pi\sigma_0(t) + \int_{\mathbb{R}_0} D_{t,z} p(t)\pi\gamma_0(t,z)v(dz). \tag{22.68}$$

Hence Eq. (22.55) becomes

$$\mathbb{E}\left[\frac{\partial H}{\partial \pi}(t, \hat{X}(t), \pi, \hat{\theta})\Big|\mathcal{E}_t\right]_{\pi=\hat{\pi}(t)} = \mathbb{E}\left[p(t)(\hat{\theta} - r(t)) + D_t p(t)\sigma_0(t)\right.$$

$$\left. + \int_{\mathbb{R}_0} D_{t,z} p(t)\gamma_0(t,z)v(dz)\Big|\mathcal{E}_t\right] = 0 \quad (22.69)$$

and Eq. (22.56) becomes

$$\mathbb{E}\left[\frac{\partial H}{\partial \theta}(t, \hat{X}(t), \hat{\pi}, \theta)\Big|\mathcal{E}_t\right]_{\theta=\hat{\theta}(t)} = \mathbb{E}[p(t)\hat{\pi}(t) \mid \mathcal{E}_t] = \mathbb{E}[p(t) \mid \mathcal{E}_t]\hat{\pi}(t) = 0.$$

$$\tag{22.70}$$

Since $p(t) > 0$ we conclude that

$$\hat{\pi}(t) = 0. \tag{22.71}$$

This implies that

$$\hat{X}(t) = x \exp\left(\int_0^t r(s)ds\right) \tag{22.72}$$

and

$$\hat{p}(t) = U'\left(x \exp\left(\int_0^T r(s)ds\right)\right) \exp\left(\int_t^T r(s)ds\right); \quad t \in [0, T]. \tag{22.73}$$

Substituting this into Eq. (22.69), we get

$$\hat{\theta}(t) = \frac{\mathbb{E}\left[\hat{p}(t)r(t) - D_t\hat{p}(t)\sigma_0(t) - \int_{\mathbb{R}_0} D_{t,z}\hat{p}(t)\gamma_0(t,z)\nu(\mathrm{d}z)\Big|\mathscr{E}_t\right]}{\mathbb{E}[\hat{p}(t)|\mathscr{E}_t]}. \quad (22.74)$$

We have proved the following theorem:

Theorem 4.1 (Worst-case scenario optimal portfolio under partial information). *Suppose there exists a solution* $(\pi^*, \theta^*) \in (\mathscr{A}_\Pi^\varepsilon, \mathscr{A}_\Theta^\varepsilon)$ *of the stochastic differential game Eq. (22.64). Then*

$$\pi^* = \hat{\pi} = 0, \quad (22.75)$$

and

$$\theta^* = \hat{\theta} \text{ is given by } (22.74). \quad (22.76)$$

In particular, if $r(s)$ *is deterministic, then*

$$\pi^* = 0 \quad and \quad \theta^*(t) = r(t). \quad (22.77)$$

Remark 4.1. (i) If $r(s)$ is deterministic, then Eq. (22.77) states that the worst-case scenario is when $\hat{\theta}(t) = r(t)$, for all $t \in [0, T]$, i.e., when the normalized risky asset price

$$\mathrm{e}^{-\int_0^t r(s)\mathrm{d}s} S_1(t)$$

is a martingale. In such a situation the trader might as well put all her money in the risk free asset, i.e., choose $\pi(t) = \hat{\pi}(t) = 0$. This trading strategy remains optimal if $r(s)$ is not deterministic, but now the worst-case scenario $\hat{\theta}(t)$ is given by the more complicated expression (22.74).

(ii) This is a new approach to, and a partial extension of, Theorem 2.2 in [13] and Theorem 4.1 in the subsequent paper [1]. Both of these papers consider the case with deterministic $r(t)$ only. On the other hand, in these papers the scenario is represented by a probability measure and not by the drift.

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