Chapter 9
Other Continuous Distributions and Moments for Distributions

Chapter Outline

Key Terms

Uniform distribution	Exponential distribution
Student's *t* distribution	Moments
Degree of freedom	Coefficient of variation
Chi-square distribution	Coefficient of skewness
F distribution	Coefficient of kurtosis
F variable	Noncentral chi-square distribution

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9.1 Introduction

Two very useful continuous distributions, the normal and lognormal distributions, were discussed in Chap. 7. Because many random variables have distributions that are not normal, in this chapter, we explore five other important continuous distributions and their applications. These five distributions are the uniform distribution, Student's *t* distribution, the chi-square distribution, the *F* distribution, and the exponential distribution. All are directly or indirectly used in analyzing business and economic data. The relationship between moments and distributions is also discussed in this chapter. Finally, we explore business applications of statistical distributions in terms of the first four moments for stock rates of return.

9.2 The Uniform Distribution

The simplest continuous probability distribution is called the *uniform distribution*. This probability distribution provides a model for continuous random variables that are evenly (or randomly) distributed over a certain interval. To picture this distribution, assume that the random variable *X* can take on any value in the range from, for example, 5 to 15, as indicated in Fig. 9.1. In a uniform distribution, the probability that the variable will assume a value within a given interval is proportional to the length of the interval. For example, the probability that *X* will assume a value in the range from 6 to 8 is the same as the probability that it will assume a value in the range from 9 to 11, because these two intervals are equal in length.

The uniform distribution has the following probability density function:

$$f(X) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq X \leq b \\ 0 & \text{elsewhere} \end{cases} \tag{9.1}$$

If the foregoing condition holds, then *X* is uniformly distributed, and the shape under the density function forms a rectangle, as shown in Fig. 9.1. The rectangle's area is equal to 1, which means that *X* is sure to take on some value between $a = 5$ and $b = 15$. Mathematically, we can express this as $P(5 \leq X \leq 15) = 1$.

Figure 9.1 shows a density function for a set of values between *a* and *b*. Each density is a horizontal line segment with constant height $1/(b - a)$ over the interval from *a* to *b*. Outside the interval, $f(X) = 0$. This means that for a uniformly distributed random variable *X*, values below *a* and values above *b* are impossible. Substituting $b = 15$ and $a = 5$ into Eq. 9.1, we obtain $1/(b - a) = 1/(15 - 5) = .1$, as indicated in Fig. 9.1.

From Chaps. 5 and 7, we know that the probability that *X* will fall below a point is provided by the area under the density curve and to the left of that point. In other words, the cumulative probability distribution function, $P(X \leq x) = (x - a)/(b - a)$, is represented by this area. The cumulative function for values of *X*

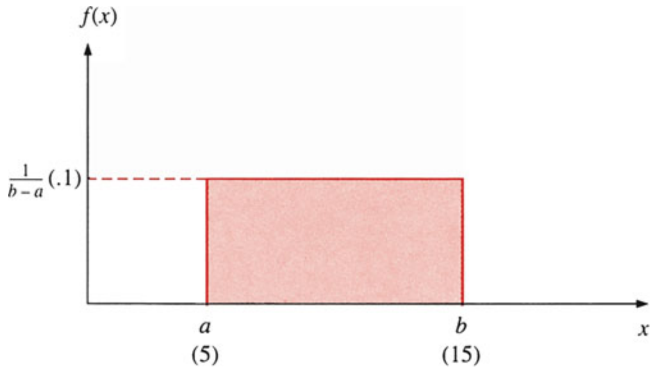

Fig. 9.1 The uniform probability distribution

between a and b is the area of the rectangle, which, again, is found by multiplying the height, $1/(b-a)$, times the base, $x-a$. To the left of a, the cumulative probabilities must be zero, whereas the probability that X lies "below points beyond b" must be 1.

The cumulative probabilities for a uniform distribution are

$$P(X \leq x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } x > b \end{cases} \tag{9.2}$$

Figure 9.2 shows the cumulative distribution function in terms of data indicated in Fig. 9.1. It presents the cumulative probabilities for $X = 5, X = 10, X = 15$, and $X = 20$ at points A, B, C, and D, respectively. Cumulative probabilities for these three points can be calculated as follows:

At point A: $P(X \leq 5) = \dfrac{5-5}{15-5} = 0$

At point B: $P(X \leq 10) = \dfrac{10-5}{15-5} = \dfrac{1}{2}$

At point C: $P(X \leq 15) = \dfrac{15-5}{15-5} = 1$

At point D: $P(X \leq 20) = P(X \leq 15) + P(15 \leq X \leq 20) = 1 + 0 = 1$

The mean and standard deviation of a uniform distribution (see Appendix 1) can be shown as

$$\mu = E(X) = \frac{a+b}{2}$$

$$\sigma_X = \frac{b-a}{\sqrt{12}} \tag{9.3}$$

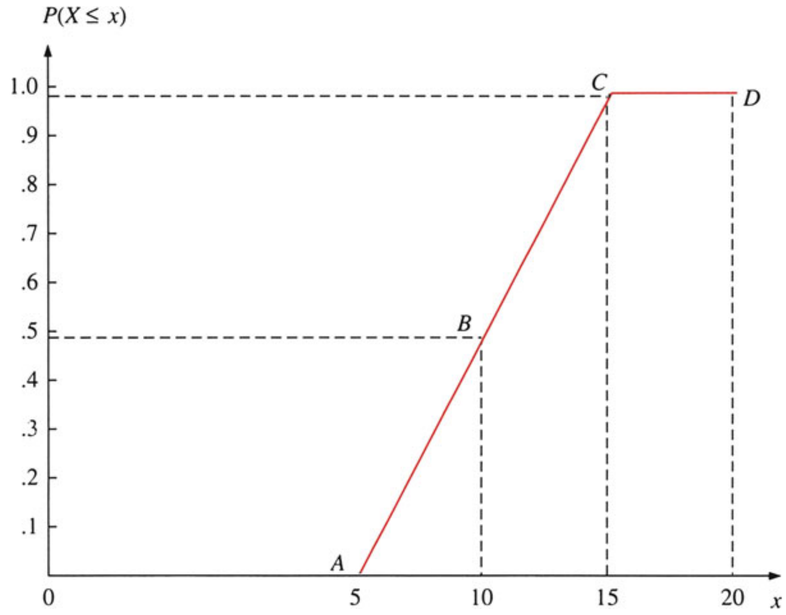

Fig. 9.2 Cumulative distribution function for the data of Fig. 9.1

Example 9.1 An Application of the Uniform Distribution in Quality Control. A quality control inspector for Gonsalves Company, which manufactures aluminum water pipes, believes that the product has varying lengths. Suppose the pipes turned out by one of the production lines of Gonsalves Company can be modeled by a uniform probability distribution over the interval 29.50–30.05 ft. The mean and standard deviation of X, the length of the aluminum water pipe, can be calculated as follows. Substituting $b = 30.05$ ft and $a = 29.50$ ft in Eq. 9.3, we obtain

$$\mu = \frac{30.05 + 29.50}{2} = 29.775 \text{ ft}$$

and

$$\sigma_X = \frac{30.05 - 29.50}{\sqrt{12}} = .1588 \text{ ft}$$

This information can be used to create a control chart to determine whether the quality of the water pipes is acceptable. The control chart and its use in statistical quality control will be discussed in Chap. 10.

Computer simulation is an application of statistics that frequently relies on the uniform distribution. In fact, the uniform distribution is the underlying mechanism for this often-complex procedure. Thus, although not so many "real-world" populations resemble this distribution as resemble the normal, the uniform distribution is important in applied statistics. For example, managers may use the uniform distribution in a simulation model to help them decide whether the company should undertake production of a new product.[1] Basic concepts of investment decision making can be found in Sect. 21.8.

9.3 Student's *t* Distribution

Student's *t* distribution was first derived by W. S. Gosset in 1908. Because Gosset wrote under the pseudonym "A Student," this distribution became known as Student's *t* distribution.

If the sampled population is normally distributed with mean μ and variance σ_X^2, the sample size n is equal to or larger than 30, and σ_X^2 is known, then from the last chapter, we know that the Z score for sample mean $\bar{X}$ defined as

$$Z = \frac{\bar{X} - \mu}{\sigma_X / \sqrt{n}} \tag{8.7}$$

which we met as Eq. 8.7, has a normal distribution with mean 0 and variance 1. Under most circumstances, however, the population variance is not known. In order for us to conduct various types of statistical analysis, we need to know what happens to Eq. 8.7 when we replace the population standard deviation σ_X by the sample standard deviation s_X. We then have the following equation for the t statistic:

$$t = \frac{\bar{X} - \mu}{s_X / \sqrt{n}} \tag{9.4}$$

Thus, the Z of Eq. 8.7 has only one source of variation: each sample has a different $\bar{X}$. Equation 9.4, however, has two sources of variation: both the sample mean $\bar{X}$ and the sample standard deviation s_X change from sample to sample. Thus, the term on the right-hand side of Eq. 9.4 follows a sampling distribution different from the normal distribution, which is the distribution followed by the term on the right-hand side of Eq. 8.7. Equation 9.4 is used only when the population from which the n sample items are drawn is normally distributed and the sample size (n) is smaller than 30.

The t distribution forms a family of distributions that are dependent on a parameter known as the *degrees of freedom*. For the t variable in Eq. 9.4, the degrees of freedom (v) are $(n - 1)$, where n is the sample size. In general, the degrees of freedom for a t statistic are the degrees of freedom associated with the sum of squares used to obtain an estimate of the variance. The variance estimate

[1] See Lee C.F.: Financial Analysis and Planning: Theory and Application, pp. 358–363. Reading, Addison-Wesley (1985)

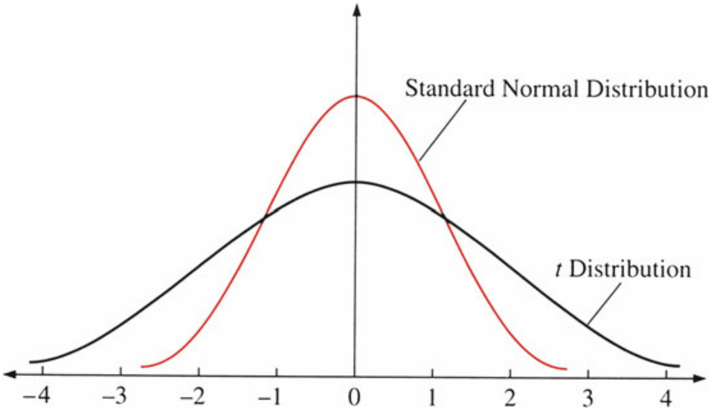

Fig. 9.3 The *t* distribution and the standard normal distribution

depends not only on the size of sample but also on how many parameters must be estimated with the sample. The more data we have, the more confidence we can have in our results; the more parameters we have to estimate, the less confidence we have. Statisticians keep track of these two factors by calculating the degrees of freedom as follows:

$$\begin{matrix}\text{Degrees of} \\ \text{freedom}\end{matrix} = \begin{matrix}\text{number of} \\ \text{observations}\end{matrix} - \begin{matrix}\text{number of parameters that must} \\ \text{be estimated beforehand}\end{matrix}$$

Here we calculate s_X by using n observations and estimating one parameter (the mean). Thus, there are $(n - 1)$ degrees of freedom.

The *t* distribution is a symmetric distribution with mean 0. Its graph is similar to that of the standard normal distribution, as Fig. 9.3 shows. However, the tail areas are greater for the *t* distribution, and the standard normal distribution is higher in the middle. The larger the number of degrees of freedom, the more closely the *t* distribution resembles the standard normal distribution. As the number of degrees of freedom increases without limit, the *t* distribution approaches the standard normal distribution. In fact, the standard normal distribution *is* a *t* distribution with an infinite number of degrees of freedom.

To determine whether the normal distribution or the Student's *t* distribution is more suitable for describing stocks' rates of return, Blattberg and Gonedes (1975, *Journal of Business*, pp. 244–280) used both daily and weekly stock rates of return for Dow Jones 30 companies to estimate the degrees of freedom for these two kinds of rates of return. They found, for example, that the degrees of freedom for Allied Chemical are 5.04 when daily data is used and 89.98 when weekly data is used. This indicates that the student's *t* distribution is more suitable for daily data for Allied Chemical, whereas the normal distribution better describes weekly data for Allied Chemical.

In addition, they found that the average degree of freedom for daily rates of return for these 30 companies is 4.79. The average degree of freedom in terms of

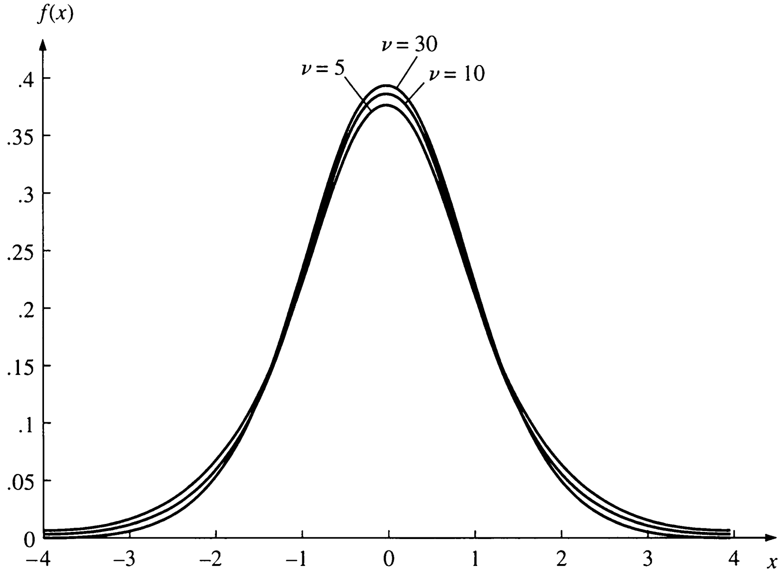

Fig. 9.4 *t* distributions of three different degrees of freedom

weekly rate of return for these 30 companies is 11.22. They concluded that Student's *t* distribution is more suitable for describing daily stock rate of return distribution, and normal distribution is more suitable for weekly rate of return distribution. Hence, *t* distribution is an important distribution for describing daily stock rate of return.

The *t* table, as presented in Table A4 at the end of the book, gives the value, t_α, such that the probability of the *t* value larger than t_α is equal to α. The percentage cutoff point t_α is defined as that point at which

$$P(t > t_\alpha) = \alpha \tag{9.5}$$

Because the distribution is symmetric around 0, only positive *t* values (upper-tail areas) are tabulated. The lower α cutoff point is $-t_\alpha$, because

$$P(t < -t_\alpha) = P(t > t_\alpha) = \alpha \tag{9.6}$$

In general, we denote a cutoff point for *t* by $t_{\alpha,v}$ where α is the probability level and v is the degrees of freedom. The number of degrees of freedom determines the shape of the *t* distribution. Figure 9.4 shows *t* distributions of varying degrees of freedom.

Example 9.2 Using the t *Distribution to Analyze Audit Sampling Information.* Let's borrow information presented in Sect. 8.7 to see how the *t* distribution can be used to do audit sampling analysis.

The sample mean and the sample variance for 30 trade accounts receivable balances are

$$\bar{X} = \$202.10 \quad \text{and} \quad s_X^2 = \$719.164$$

From Table A4, we know that the t statistics with $30 - 1 = 29°$ of freedom and $\alpha = .05$ is 1.6991. Substituting related information into Eq. 9.4, we obtain

$$1.699 = \frac{\$202.10 - \mu}{\sqrt{719.164/30}} = \frac{\$202.10 - \mu}{4.896}$$

This implies that there is a 5 % chance that the average population account receivable value will be smaller than $\$202.10 - \$(11.699)(4.896) = \$193.78$. By symmetry, there is also a 5 % chance that the average population account receivable value will be larger than $\$202.10 + \$(1.699)(4.896) = \$210.42$.

Other applications of the t distribution appear in Chaps. 10 and 11, and we will encounter more when we discuss regression analysis in Chaps. 13, 14, 15, and 16.

9.4 The Chi-Square Distribution and the Distribution of Sample Variance

In this section, we first show how a chi-square distribution can be derived from a standard normal distribution and then derive the distribution of a sample variance.

9.4.1 The Chi-Square Distribution

The *chi-square distribution* (χ^2) is a continuous distribution ordinarily derived as the sampling distribution of a sum of squares of independent standard normal variables. For instance, let $X_1, X_2, \ldots, X_n$ denote a random sample of size n from a normal distribution with mean μ and variance σ_X^2. Because these variables are not standardized, we can standardize them as

$$Z_i = \frac{X_i - \mu}{\sigma_X}$$

where Z_i is normally distributed with mean 0 and variance 1.

Now, if we define a new variable Y such that

$$Y = Z_1^2 + Z_2^2 + \cdots + Z_n^2 = \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma_X} \right)^2 \tag{9.7}$$

it can be shown that this new variable is distributed as χ^2 with n degrees of freedom.[2]

Equation 9.6 can be rewritten as[3]

$$\sum_{i=1}^{n} \frac{(X_i - \mu)^2}{\sigma_X^2} = \frac{n(\bar{X} - \mu)^2}{\sigma_X^2} + \frac{(n-1)s_X^2}{\sigma_X^2} \tag{9.8}$$

where

$$s_X^2 = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n-1}; \quad \sum_{i=1}^{n} \frac{(X_i - \mu)^2}{\sigma_X^2}$$

has an x^2 distribution with n degrees of freedom, as discussed in Eq. 9.6. In addition, from the last chapter, we know that $\bar{X}$ is normally distributed with mean μ and variance σ_X^2/n, so $\sqrt{n}(\bar{X} - \mu)/\sigma_X$ is normally distributed with mean 0 and variance 1. It can be shown that $n(\bar{X} - \mu)^2/\sigma_X^2$ has an x^2 distribution with 1° of freedom. From this information, it can be proved that

$$\frac{(n-1)s_X^2}{\sigma_X^2}$$

defined in Eq. 9.8, has a χ^2 distribution with $(n - 1)$ degrees of freedom.[4]

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[2] First, it can be proved that $(X_i - \mu)^2/\sigma_X^2$ is a χ^2 distribution with I degree of freedom. Then, by using the additive property of x^2 distribution, we can prove that $\sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma_X}\right)^2$ is also a χ^2 distribution with n degrees of freedom.

[3] Since

$$\begin{aligned}\sum_{i=1}^{n} (X_i - \mu)^2 &= \sum_{i=1}^{n} (X_i - \bar{X} + \bar{X} - \mu)^2 \\ &= \sum_{i=1}^{n} (X_i - \bar{X})^2 + 2(\bar{X} - \mu)\sum_{i=1}^{n} (X_i - \mu) + \sum_{i=1}^{n} (\bar{X} - \mu)^2 \\ &= \sum_{i=1}^{n} (X_i - \bar{X})^2 + n(\bar{X} - \mu)^2\end{aligned}$$

because

$$2(\bar{X} - \mu)\sum_{i=1}^{n} (X_i - \bar{X}) = 0 \tag{9.A}$$

by dividing Eq. 9.A by σ_X^2, we obtain Eq. 9.8.

[4] In addition to the condition described here, it is also necessary to assume that $\bar{X}$ is independent of s_X^2.

$$\frac{(n-1)s_X^2}{\sigma_X^2}$$

can be redefined as expressed in Eq. 9.9:

$$\frac{(n-1)s_X^2}{\sigma_X^2} = \sum_{i=1}^{n} \left(\frac{X_i - \bar{X}}{\sigma_X} \right)^2 \tag{9.9}$$

where s_X^2 and σ_X^2 are sample variance and population variance, respectively. The left-hand side of Eq. 9.9 implies that the ratio of sample variance to population variance, multiplied by $(n - 1)$, has a χ^2 distribution with $(n - 1)$ degrees of freedom. The χ^2 distribution defined in Eq. 9.9 can be used to describe the distribution of s_X^2, which will be discussed later in this section.

The χ^2 distribution is a skewed distribution, and only nonnegative values of the variable χ^2 are possible. It depends on a single parameter, the degrees of freedom $v = n - 1$. The χ^2 distributions for degrees of freedom 5, 10, and 30 are graphed in Fig. 9.5. The figure shows that the skewness decreases as the degrees of freedom increase. In fact, as the degrees of freedom increase to infinity, the χ^2 distribution approaches a normal distribution.[5]

Critical values of the χ^2 distributions are given in Table A5 in Appendix A.[6] They are defined by

$$P\left(\chi^2 \geq \chi^2_{\alpha,v}\right) = \alpha \tag{9.10}$$

where is that value for the χ^2 distribution with v degrees of freedom such that the area to the right (the probability of a larger value) is equal to α. For example, the upper 5 % point for χ^2 with 10 degree of freedom, $\chi^2_{.05,10}$, is 18.307 (see Fig. 9.6 and Table A5). In other words, $P(\chi^2 > 18.307) = .05$. In addition, $P(\chi^2 < 18.307) = 1 - .05 = .95$.

The mean and variance of this distribution are equal to the number of degrees of freedom and twice the number of degrees of freedom. That is,

[5] Johnson, W. L., Katz S.: In Continuous Univariate Distribution I, pp. 170–181. Houghton Mifflin, Boston, 1970, show that a normalized x^2 distribution approaches a standard normal distribution when the number of degrees of freedom approaches infinity. The normalized statistic is defined as $\left(\chi_v^2 - v\right)/\sqrt{2v}$.

[6] We can approximate $x^2{}_\alpha$ by the formula

$$\chi_\alpha^2 = v\left(1 - \frac{2}{9v} + z_\alpha\sqrt{\frac{2}{9v}}\right)^3$$

where v = degrees of freedom and z_α = standard normal value (from Table A.3).

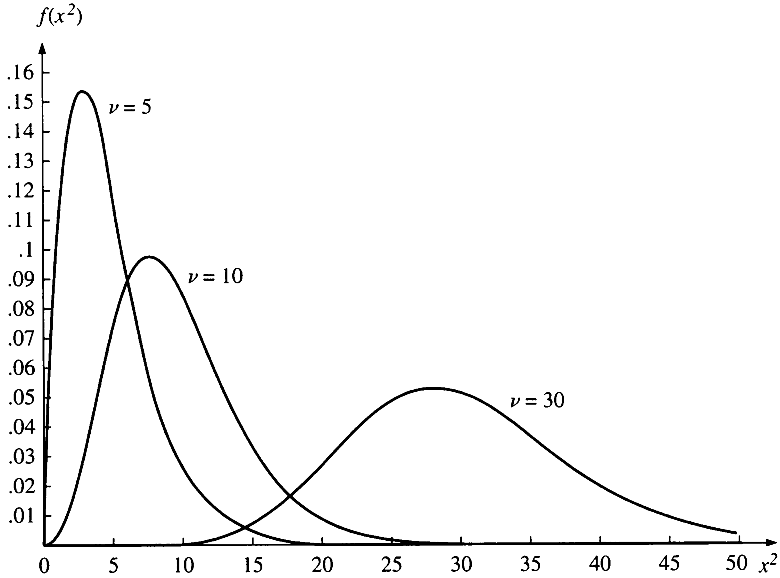


Fig. 9.5 The χ^2 distributions with three different degrees of freedom

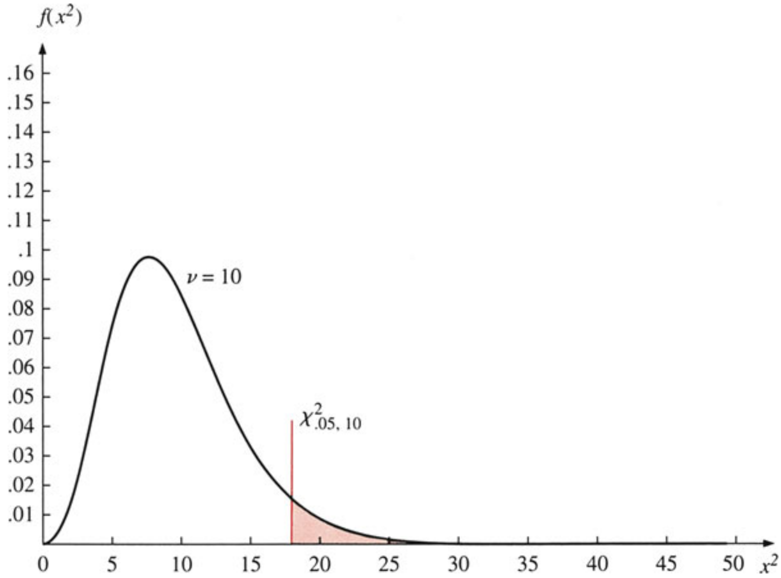


Fig. 9.6 The χ^2 distribution with 10° of freedom

$$E\left(\chi_v^2\right) = v, \quad \text{and} \quad \mathrm{Var}\left(\chi_v^2\right) = 2v \tag{9.11}$$

where v is the degree of freedom of a χ^2 distribution.

9.4.2 *The Distribution of Sample Variance*

The properties of the χ^2 distribution can be used to find the mean and variance of the sampling distribution of the sample variance (s_X^2).

9.4.2.1 The Mean of s_X^2

From the definition of the mean for a χ^2 distribution, we obtain

$$E\left[\frac{(n-1)s_X^2}{\sigma_X^2}\right] = n - 1$$

Because $E(a \cdot X) = a \cdot E(X)$, we have

$$\frac{(n-1)}{\sigma_X^2} E\left(s_X^2\right) = n - 1$$

Thus,[7]

$$E\left(s_X^2\right) = \sigma_X^2 \tag{9.12}$$

Equation 9.12 implies that the mean of the sample variance is equal to the population variance.

9.4.2.2 The Variance of s_X^2

On the basis of the definition of the variance for a χ^2 distribution, we have

$$\mathrm{Var}\left[\frac{(n-1)S_X^2}{\sigma_X^2}\right] = 2(n-1)$$

Because $\mathrm{Var}(aX) = a^2 \cdot \mathrm{Var}(X)$, we have

$$\frac{(n-1)^2}{\sigma_X^4} \mathrm{Var}\left(s_X^2\right) = 2(n-1)$$

so

$$\mathrm{Var}\left(s_X^2\right) = \frac{2\sigma_X^4}{n-1} \tag{9.13}$$

[7] This result suggests why $\sum_{i=1}^{n} (X_i - \bar{X})^2/n - 1$ instead of $\sum_{i=1}^{n} (X_i - \bar{X})^2/n$ is an unbiased estimator for the population variance, σ_X^2. Unbiased estimators will be discussed in Chap. 10.

This is the variance of the sample variance. In sum, if X is normally distributed, then the mean and variance of s_X^2 are σ_X^2 and $2\sigma_X^4/(n-1)$, respectively. We will explore applications of the χ^2 distribution and the distribution of sample variance in Chaps. 10 and 11 when we discuss confidence intervals and hypothesis testing for population variances.

Drawing on the concepts of the χ^2 distribution and the normal distribution, we can interpret the t distribution by rewriting Eq. 9.4′ as

$$t = \frac{(\bar{X}-\mu)/(\sigma_X/\sqrt{n})}{s_X/\sigma_X} \tag{9.4'}$$

In Eq. 9.4′, $(\bar{X}-\mu)/(\sigma_X/\sqrt{n})$ is normally distributed with mean 0 and variance 1; it is a standard normal distribution. S_x/σ_x is a square root of a χ^2-distributed variable with $(n-1)$ degrees of freedom divided by $v = n-1$. Hence, a t distribution with v degrees of freedom is the ratio between a standard normal variable and a transformed χ^2 variable:

$$t_v = \frac{Z}{\sqrt{\chi_v^2/v}} \tag{9.14}$$

9.5 The *F* Distribution

Some problems revolve around the value of a single population variance, but often it is a comparison of the variances of two populations that is of interest. This will be discussed in Chaps. 13, 14, and 15. In addition, we may want to know whether the means of three or more populations are equal. This will be discussed in Chap. 12. The *F distribution* is used to make inferences about these kinds of issues.

Assume two populations, each having a normal distribution. We draw two independent random samples with sample sizes n_X and n_Y and population variances σ_X^2 and σ_Y^2. From each sample, we can compute sample variances S_X^2 and S_Y^2. Then, the random variable of Eq. 9.15 follows a distribution known as the F distribution:

$$F = \frac{S_X^2/\sigma_X^2}{S_Y^2/\sigma_Y^2} \tag{9.15}$$

Equation 9.15 can be rewritten as

$$F = \frac{\chi_{v_1}^2(X)/(n_X-1)}{\chi_{v_2}^2(Y)/(n_Y-1)} \tag{9.14'}$$

where $\chi_{v_1}^2(X) = (n_X-1)S_X^2/\sigma_X^2$ and $\chi_{v_2}^2(Y) = (n_Y-1)S_Y^2/\sigma_Y^2$; $v_1 = n_x - 1$; $v_2 = n_y - 1$.

In other words, a random variable formed by the ratio of two independent chi-square variables, each divided by its degrees of freedom, is called an *F variable*.

The F distribution has an asymmetric probability density function defined only for nonnegative values. It should be observed that the F distribution is completely determined by two parameters, v_1 and v_2, which are degrees of freedom. These density functions with different sets of degrees of freedom are illustrated in Fig. 9.7.

The cutoff points $F_{v_1,v_2,\alpha}$, for α equal to .05, .025, .01, and .005, are provided in Table A6 at the end of this book. For example, in the case of 10 numerator degrees of freedom and six denominator degrees of freedom,

$$F_{10,6,.05} = 4.06 \quad F_{10,6,.025} = 5.46$$
$$F_{10,6,.01} = 7.87 \quad F_{10,6,.005} = 10.25$$

MINITAB output for $F_{10,\,6}$ is presented in Fig. 9.8. Hence,

$$P(F_{10.6} > 4.06) = .05 \qquad P(F_{10.6} > 5.46) = .025$$
$$P(F_{10.6} > 7.87) = .01 \qquad P(F_{10.6} > 10.25) = .005$$

These probabilities also can be calculated by using MINITAB as shown here.

```
MTB > SET C1
DATA> 4.06 5.46 7.87 10.25
DATA> END
MTB > CDF C1;
SUB > F 10 6.
```

Cumulative Distribution Function

```
F distribution with 10 DF in numerator and 6 DF in denominator
x P( X <= x)
4.0600 0.9500
5.4600 0.9750
7.8700 0.9900
10.2500 0.9950
```

By subtracting 1 from .95, we obtain .05; by subtracting 1 from .975, we obtain .025; by subtracting 1 from .99, we obtain .01; finally, by subtracting 1 from .9950, we obtain .005. In practice, we usually place the larger sample variance in the numerator. The four significance levels listed here are the cutoff points that are often used to test the hypothesis of equality of population variances, which will be discussed in Chaps. 11 and 12. When the population variances are equal, Eq. 9.15 becomes

$$F = \frac{S_X^2}{S_Y^2} \tag{9.16}$$

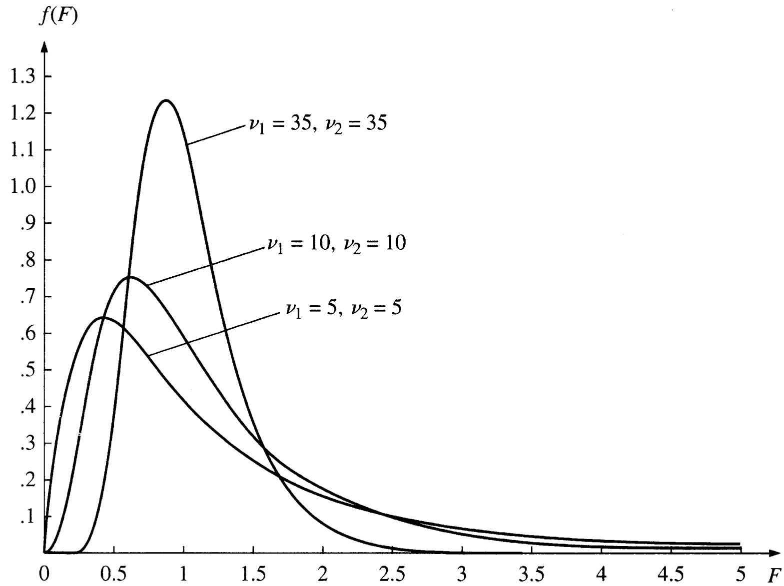


Fig. 9.7 *F* distributions with three different sets of degrees of freedom

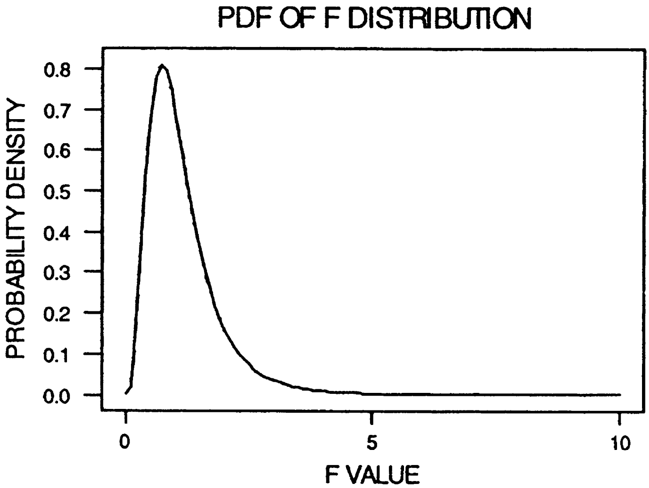


Fig. 9.8 MINITAB output for $F_{10,\ 6}$

The right-hand side of Eq. 9.16 is the ratio of two sample variances.

Applications of the *F* distribution will be discussed in Chaps. 11 and 12 and in the chapters related to regression analysis.

9.6 The Exponential Distribution (Optional)

The *exponential distribution* is related to the Poisson distribution, which, as we noted in Chap. 6, is often applied to occurrences of an event over time. The Poisson distribution is the distribution of the number of occurrences of an event in a given time interval of length t. The single parameter of the Poisson distribution is λ, the intensity of the process. Think of the number as the average occurrence of the event being counted. For example, say, the average arrival rate of customers at the Brownell Bank is 5 per 100 s. Suppose that instead of the number of occurrences in a given time period, we are interested in the amount of time until the first customer arrives at the bank. This is a problem to be solved by the exponential distribution instead of the Poisson distribution. As another example, if the number of traffic accidents in an interval of time follows the Poisson distribution, the length of time from one accident to another follows the exponential distribution. The exponential distribution can also be applied to (1) the length of time that must pass before the first incoming telephone call and (2) the length of time someone must wait for a cab in a given location, such as Penn Station in New York City.

Denoting the mean rate at which events occur over time by λ and denoting the time until the first event occurs by t, we can use the Poisson probability density function to derive the exponential probability density function (PDF).[8] It is

$$\begin{aligned} f(t) &= \lambda e^{-\lambda t}, \quad t \geq 0 \\ &= 0, \quad\quad t<0 \end{aligned} \tag{9.17}$$

where $\lambda > 0$ is the only parameter.

From Eq. 9.38 we know that the cumulative probability function is given by

$$\begin{aligned} F(t) = P(T \leq t) &= 1 - e^{-\lambda t}, \quad t \geq 0 \\ &= 0, \quad\quad t<0 \end{aligned} \tag{9.18}$$

where T is a random variable representing time and t is a specific value.

Figure 9.9 represents four exponential functions for which λ equals 3, 2, 1, and $\frac{1}{2}$. From Appendix 2, we know that

$$E(T) = \frac{1}{\lambda} \tag{9.19}$$

$$\mathrm{Var}(T) = \frac{1}{\lambda^2} \tag{9.20}$$

[8] See Appendix 2 for the derivation.

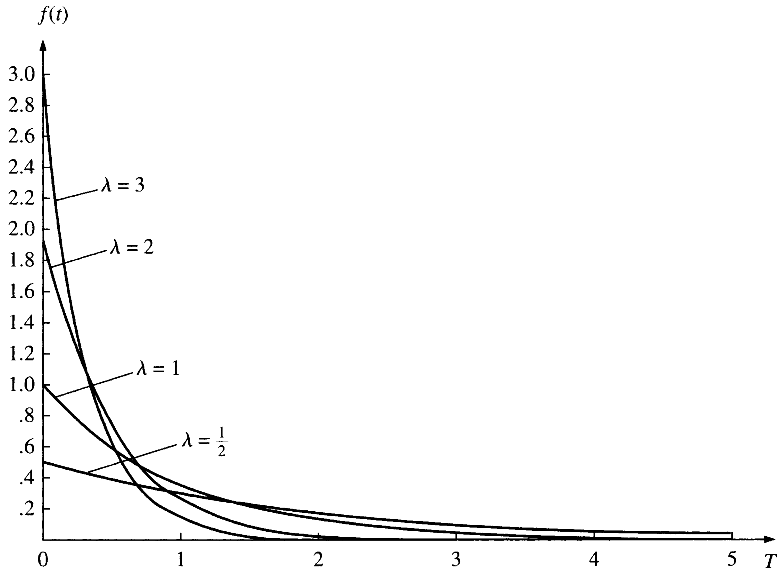


Fig. 9.9 Four exponential density functions specified by four alternative values of λ

Example 9.3 "No More Than 8 Items in This Line, Please!". Under fairly plausible assumptions about the behavior of clerks at supermarket check-out counters, it is possible to show that the time T (in minutes) a customer spends at a check-out counter is a random variable with the exponential distribution described by Eq. 9.17.

Suppose a supermarket check-out counter has a mean number of customers per minute $= \frac{1}{3}$; that is, $\lambda = \frac{1}{3}$. Our task is to find the probability that the length of time between a pair of customer arrivals is less than 6 min.

Substituting $\lambda = \frac{1}{3}$ and $t = 6$ into Eq. 9.18, we obtain $F(T \leq 6) = 1 - e^{-6/3}$. And referring to Table A7 of Appendix A (or to a hand calculator), we find $P(T < 6) = 1 - .1353 = .8647$. Thus, the probability that the service time available between two customer arrivals at the check-out counter will be less than 6 min is approximately .86. Alternatively, the probability .8647 can be obtained by MINITAB as shown here:

```
MTB > CDF 6;
SUBC> EXPONENTIAL 3.
```

Cumulative Distribution Function

```
Exponential with mean = 3.00000
x            P(X <= x)
6.0000         0.8647
```

9.7 Moments and Distributions (Optional)

The properties of a distribution can be described in many ways, but the most popular approach is by means of a set of measurements called moments. *Moments* describe the central tendency, degree of dispersion, asymmetry, peakedness, and many other aspects of a distribution. This section discusses only the first four moments of a distribution; they are the most important statistical characteristics.

The first k moments can be defined either as

$$\mu'_k = E\left(X^k\right) \tag{9.21}$$

or

$$\mu_k = E\left[(X-\mu)^k\right] \tag{9.22}$$

Equation 9.21 defines the k moments about the origin, and Eq. 9.22 defines the moments about the population mean μ. (The relationship between μ'_k and μ'_k is discussed in Appendix 3.) The *population mean* is the first moment about the origin. We obtain the first moment of a distribution about the origin by letting $k = 1$ in Eq. 9.21. It is defined as follows:

$$\mu'_1 = E(X) = \mu$$

This is the population mean of X. Following Eq. 4.1, we can define μ for a discrete variable as

$$\mu = \sum_{i=1}^{N} X_i/N \tag{4.2}$$

where N is the total number of observations in the population. The sample mean $\bar{X}$ associated with μ can be defined as

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} \tag{4.1}$$

where n is the sample size.

9.7.1 The Second Moment and the Coefficient of Variation

The second moment about the mean, the *variance*, is a measure of the dispersion of the random variable around the mean. The larger the variance, the more dispersed the distribution. Letting $k = 2$ in Eq. 9.22, we obtain

$$\mu_2 = \sigma_X^2 = E[X - E(X)]^2$$

This is the population variance of X. Following Eq. 4.5, we can define the population variance for a discrete variable as

$$\sigma_X^2 = \sum_{i=1}^{N} (X_i - \mu)^2/N \tag{4.5}$$

The sample variance (s_X^2) associated with X can be defined as

$$s_X^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2/(n-1) \tag{4.7}$$

Following Eq. 4.12, we can define the sample *coefficient of variation* (CV) as

$$\text{CV} = \frac{s_X}{\bar{X}} \tag{4.12}$$

9.7.2 The Third Moment and the Coefficient of Skewness

The third moment about the mean – *skewness*, which characterizes the asymmetry of the distribution – is given by

$$\mu_3 = E[X - E(X)]^3$$

Following Eq. 4.15, we can define the population skewness for a discrete variable as

$$\mu_3 = \sum_{i=1}^{N} (X_i - \mu)^3/N \tag{4.15}$$

Following Eq. 4.16, we can define the *coefficient of skewness* (CS), which is a relative measure of asymmetry, as

$$\text{CS} = \frac{\mu_3}{\sigma^3} \tag{4.16}$$

Following Eq. 4.16a, we can define the sample coefficient of skewness (SCS) as

$$\text{SCS} = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^3/(n-1)}{s_X^3} \tag{4.16a}$$

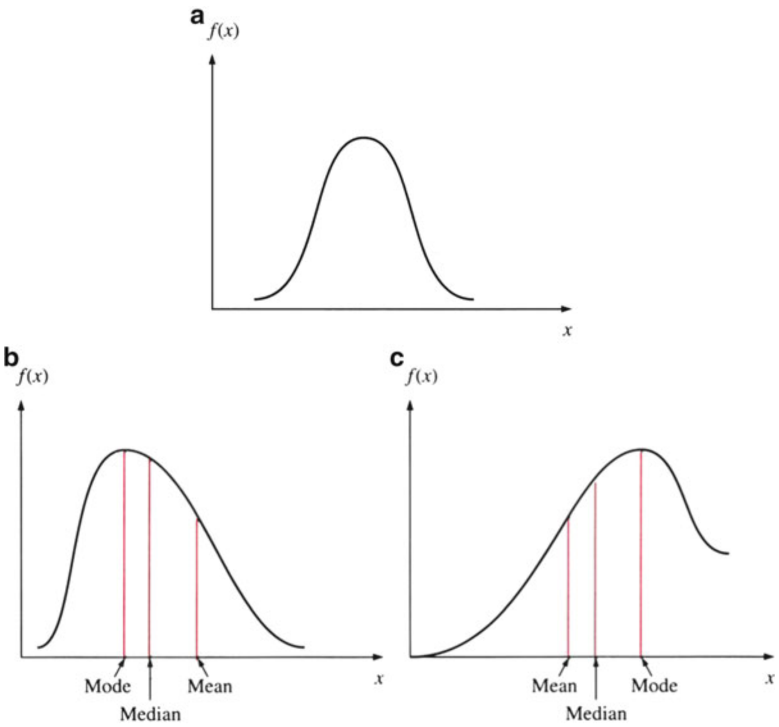


Fig. 9.10 (**a**) Zero skewness, (**b**) Positive skewness, and (**c**) negative skewness

where

$$s_X^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2 / (n - 1)$$

Figures 9.10a, b, and c present graphs of distributions with differing degrees of symmetry. Figure 9.10a shows a symmetrical distribution – that is, a distribution with zero skewness. Note that the symmetrical distribution's measures of central tendency (the mean, median, and mode) all coincide. We can also see that the half of the distribution above the mode is a mirror image of the half of the distribution below the mode.

Figure 9.10b presents a distribution that is said to be positively skewed because the distribution tapers off more slowly to the right of the mode than to the left. It is clear that the mean, median, and mode do not coincide. Here, the mode is smaller than the median and the mean.

Figure 9.10c presents a distribution that is said to be negatively skewed because the distribution tapers off more slowly to the left of the mode than to the right. Once again, the mean, median, and mode do not coincide. Here the median and mean lie to the left of the mode.

9.7.3 Kurtosis and the Coefficient of Kurtosis

The fourth moment about the mean – *kurtosis*, which characterizes the degree of peakedness – is defined by

$$\mu_4 = E[X - E(X)]^4$$

For discrete variables, the population kurtosis can be defined as

$$\mu_4 = \sum_{i=1}^{N} (X_i - \mu)^4 / N \tag{9.23}$$

and can be estimated in terms of sample data as follows:

$$\text{Sample kurtosis} = \sum_{i=1}^{n} (X_i - \bar{X})^4 / n$$

The relative peakedness of a distribution is expressed by the ratio of the fourth moment to the square of the second moment. It is called *coefficient of kurtosis* (CK):

$$\text{CK} = \mu_4 / \mu_2^2 \tag{9.24}$$

This ratio measures the degree of peakedness relative to the level of dispersion. Using sample information, we can estimate the coefficient of kurtosis by

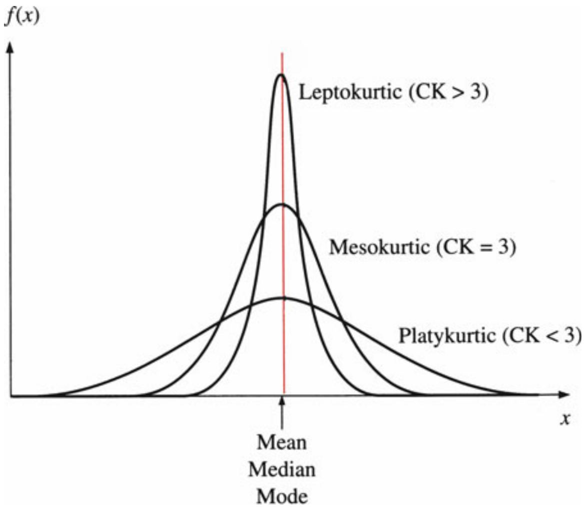
$$\text{SCK} = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^4 / (n - 1)}{s_X^4} \tag{9.25}$$

Of two distributions having the same dispersion, the one with the larger kurtosis ratio has more observations concentrated near the mean and also at the tails of the distribution (at the expense of the intermediate area).

9.7.4 Skewness and Kurtosis for Normal and Lognormal Distributions

The bell-shaped normal curve is characterized by the *mesokurtic* shape: a value of 3 for the coefficient of kurtosis as defined in Eq. 9.25. Distributions with values of the kurtosis ratio greater than 3 are *leptokurtic*. These distributions are more peaked than the standard mesokurtic (normal curve) shape. Distributions with values of the

Fig. 9.11 Three types of kurtosis



coefficient of kurtosis less than 3 are *platykurtic* – flatter in shape than the standard normal distribution. Each of these types of coefficients of kurtosis is illustrated in Fig. 9.11. Sometimes, the sample *coefficient of kurtosis* (SCK) can be redefined as

$$\mathrm{SCK}' = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^4}{\left[\Sigma (X_i - \bar{X})^2\right]^2} - 3 \tag{9.26}$$

The value for the redefined CK for a normal distribution is 0 instead of 3.

If X is lognormally distributed, then from Sect. 7.6, the mean and variance of X can be defined as

$$\mu'_1 = \mu_X = e^{\mu + 1/2\sigma^2} \tag{7.6}$$

$$\mu_2 = \sigma_X^2 = e^{2\mu + \sigma^2}\left(e^{\sigma^2} - 1\right) \tag{7.7}$$

where $\mu = E(\log X)$ and $\sigma^2 = \mathrm{Var}(\log X)$.

From Eqs. 7.6 and 7.7, the coefficient of variation (η) for X can be defined as

$$\eta = \left(e^{\sigma^2} - 1\right)^{1/2} \tag{9.27}$$

The third and fourth moments about the mean for lognormal distributions are

$$\mu_3(\text{skewness of } X) = (\mu_X)^3\left(\eta^6 + 3\eta^4\right) \tag{9.28}$$

$$\mu_3(\text{kurtosis of } X) = (\mu_X)^4\left(\eta^{12} + 6\eta^{10} + 15\eta^8 + 16\eta^6 + 3\eta^4\right) \tag{9.29}$$

where $\eta^2 = e^{\sigma^2} - 1$. (See Appendix 4 for the derivation of Eqs. 9.28 and 9.29.)

Substituting μ_1, μ_3, and μ_4 into Eqs. 4.16 and 9.24, we obtain the following equations for the coefficient of skewness (CS) and the coefficient of kurtosis (CK):

$$\text{CS} = \eta^3 + 3\eta \tag{9.30}$$

$$\text{CK} = \eta^8 + 6\eta^6 + 15\eta^4 + 16\eta^6 \tag{9.31}$$

where $\eta^2 = e^{\sigma^2} - 1$.

From Eqs. 9.28, 9.29, 9.30, and 9.31, we know that the coefficient of variation is the key variable in determining the magnitude of both skewness and kurtosis for a lognormal distribution.

In the next section, we will see how Eqs. 4.1, 4.7, 4.12, 4.16a, and 9.26 are applied with data on stock rates of return.

9.8 Analyzing the First Four Moments of Rates of Return of the 30 DJI Firms

In Table 9.1, we have listed the first four moments of the monthly returns of the 30 companies included in the Dow Jones Industrial (DJI) Average. These moments describe the central tendency, variability, asymmetry, and peakedness of the monthly return distributions between January 1990 and December 2009, inclusive. The mean column gives us a measure of central tendency. The average mean of these 30 companies is .0013. The highest monthly return mean was from McDonald's, followed by Disney, Verizon Inc., and United Technologies Corp. The lowest performances were for Bank of America and Alcoa, which had returns of −.037 and −.02, respectively.

The measure of variability is given by the standard deviation. The average standard deviation was .0813. The two companies that showed the highest variability were Bank of America and Alcoa. The lowest variability was achieved by Johnson & Johnson, followed by McDonald's.

In fact, we usually observe that higher rates of return are associated with higher levels of risk. Note that these companies that generated high rates of return tend to have high variability. The principle is simple: the higher the return you seek, the more risk you have to take. There is a trade-off between risk and return, which will be discussed in Chap. 21 in some detail.

The skewness can be used to evaluate the stock's upside potential and downside risk. Positive skewness indicates the upside potential for a stock, because such a stock has a greater probability of very large payoffs. On the other hand, negative

Table 9.1 Statistical estimates for the Dow Jones 30 industrial firms (January 1990–December 2009)

		Monthly statistical estimates					
	Company name	*n*	*Mean*	*Standard deviation*	*Skewness*	*Kurtosis*	*Coefficient of variation*
1	3M Co.	120	0.0019	0.0665	−0.4896	0.549	35.762
2	Alcoa Inc.	120	−0.0211	0.1765	−3.0010	13.522	−8.373
3	American Express	120	−0.0094	0.1230	0.1616	4.211	−13.032
4	AT&T	120	0.0055	0.0616	−1.0458	0.834	11.191
5	Bank of America	120	−0.0370	0.2225	−2.5486	10.608	−6.021
6	Boeing	120	−0.0036	0.0965	−1.0373	1.596	−26.544
7	Caterpillar Inc	120	0.0029	0.1279	−1.9207	6.756	43.496
8	Chevron	120	0.0087	0.0624	−0.7818	0.619	7.168
9	Cisco	120	−0.0010	0.0892	−0.6297	0.612	−86.495
10	Coca-Cola	120	0.0093	0.0510	−0.9161	3.869	5.483
11	E.I. du Pont de Nemours	120	0.0024	0.0872	−0.7714	1.938	36.005
12	Exxon	120	0.0048	0.0520	−0.4104	−0.067	10.810
13	General Electric	120	−0.0135	0.1126	−1.0687	2.134	−8.322
14	Hewlett-Packard	120	0.0044	0.0752	−1.0900	1.269	17.139
15	Home Depot	120	−0.0028	0.0768	−0.3816	0.091	−27.759
16	Intel	120	−0.0041	0.0855	−0.9769	0.878	−20.822
17	IBM	120	0.0093	0.0603	−1.6744	5.742	6.494
18	Johnson & Johnson	120	0.0019	0.0443	−0.9569	1.848	22.763
19	JPMorgan and Chase	120	−0.0016	0.0991	−0.6226	0.947	−61.561
20	Kraft Foods	120	0.0030	0.0623	−1.2024	2.413	20.474
21	McDonald’s	120	0.0173	0.0461	−0.2755	−0.159	2.667
22	Merck	120	0.0086	0.0778	−0.2400	0.473	9.049
23	Microsoft	120	0.0058	0.0802	0.0533	0.482	13.939
24	Pfizer	120	0.0012	0.0638	−0.1491	0.028	51.410
25	Procter and Gamble	120	0.0051	0.0489	−0.3265	0.172	9.660
26	Traveler’s Companies Inc.	120	0.0072	0.0537	−0.1024	1.817	7.451
27	United Technologies Group	120	0.0094	0.0616	−0.4312	−0.035	6.564
28	Verizon	120	0.0095	0.0571	−0.0976	−0.544	5.986
29	Walmart	120	0.0051	0.0473	−0.3584	1.539	9.333
30	Walt Disney	120	0.0109	0.0700	−0.3271	0.961	6.431
	Mean		0.0013	0.0813	−0.7873	2.170	2.678

skewness is associated with downside risk; it indicates that the stock has a greater probability of very small payoffs.[9] There are 28 companies in Table 9.1 exhibit the downside risk associated with negative skewness. The others, American Express and Microsoft, exhibit the upside potential associated with positive skewness.

[9] This is so because a positively skewed distribution has more observations above the mode and a negatively skewed distribution more observations below.

The kurtosis column shows that 26 companies here have a leptokurtic distribution (kurtosis ratio > 0)[10]; these companies have more monthly returns concentrated near the mean. Only four companies have a distribution close to platykurtic (kurtosis ratio < 0): Exxon with $SCK' = -.067$, McDonald's with $SCK' = -.159$, United Technologies Corp with $SCK' = -.035$, and Verizon with $SCK' = -.544$.

The last column, showing the coefficient of variation, enables us to compare monthly returns for the different companies. Remember that the coefficient of variation is a unitless figure that expresses the standard deviation as a percentage of the mean. High coefficients of variation show volatile monthly returns. The companies that show high volatility are Pfizer, 3M, Caterpillar, and E.I du Pont de Nemours. The companies with the lowest volatility are Cisco, JPMorgan Chase, Home Depot, and Boeing.

9.9 Summary

In this chapter, we discussed five continuous distributions. Four of these – Student's t distribution and the exponential, F, and χ^2 distributions – are closely related to the normal distribution discussed in Chap. 7. These five distributions, along with the normal and lognormal distributions, are the primary distributions we will use throughout the rest of the text for conducting statistical analyses such as determination of confidence intervals, hypothesis testing, and goodness-of-fit tests.

In Chaps. 11, 12, 13, 14, and 15, we will begin to apply these distributions in alternative statistical analyses.

Questions and Problems

1. Briefly discuss the cumulative distribution function of the uniform distribution presented in Fig. 9.2.
2. Briefly discuss the relationship between the Poisson distribution and the exponential distribution.
3. X is normally distributed, and the sample variance $s^2 = 20$ is calculated from 20 observations. Calculate $E(s^2)$ and $\text{Var}(s^2)$.
4. W is a normally distributed random variable with mean 0 and variance 1, and V is a χ^2-distributed random variable with degrees of freedom $(n - 1)$. How can both t and F distributions be defined in terms of the variables W and V?
5. Briefly discuss how F statistics can be used to test the difference between two sample variances.
6. Briefly discuss how mean, variance, skewness, kurtosis, and the coefficient of variation can be used to analyze stock rates of return.

[10] We use Eq. 9.26 to calculate the coefficient of kurtosis.

7. Suppose a random variable X can take on only values in the range from 2 to 10 and that the probability that the variable will assume any value within any interval in this range is the same as the probability that X will assume another value in another interval of similar width in the range. What is the distribution of X? Draw the probability density function for X.
8. Use the information given in question 7 to find $P(3 \leq X \leq 7)$.
9. Use the information given in question 7 to find $P(X \leq 8)$.
10. Use the information given in question 7 to find $P(X < 2 \text{ or } X > 10)$.
11. Draw the cumulative distribution function for the distribution given in question 7.
12. Suppose a random variable X is best described by a uniform distribution with $a = 8$ and $b = 20$.

    (a) Find $f(x)$.
    (b) Find $F(x)$.
    (c) Find the mean and variance of X.

13. Suppose a random variable Y is best described by a uniform distribution with $a = 3$ and $b = 32$.

    (a) Find $f(y)$.
    (b) Find $F(y)$.
    (c) Find the mean and variance of Y.

14. A very observant art thief (who should probably be teaching statistics instead) notices that the frequency of security guards passing by a museum is uniformly distributed between 15 and 60 min. Therefore, if X denotes the time (in minutes) before the guard passes by, the probability density function of X is

$$f_x(x) = \begin{cases} 1/(60-15) & \text{for } 15<x<60 \\ 0 & \text{for all other values of } x \end{cases}$$

    (a) Draw the probability density function.
    (b) Find and draw the cumulative distribution function.

15. Use the information given in question 14.

    (a) Find the probability that the guard passes by within 35 min of the thief's arrival.
    (b) Find the probability that the guard does not pass by within 30 min.
    (c) Find the probability that the guard passes by between 30 and 45 min after the thief's arrival.

16. An art dealer at an auction believes that the bid on a certain painting will be a uniformly distributed random variable between \$500 and \$2,000.

    (a) What is the probability density function for this random variable?
    (b) Find the probability that the painting will sell for less than \$675.
    (c) Find the probability that the painting will sell for more than \$1,000.

17. Suppose X has an exponential distribution with $\lambda = 5$. Find the following probabilities:

   (a) $P(X > 4)$
   (b) $P(X > .7)$
   (c) $P(X > .50)$

18. Suppose X has an exponential distribution with $\lambda = 4$. Find the following probabilities:

   (a) $P(X \leq .3)$
   (b) $P(X \leq .5)$
   (c) $P(X \leq 1.6)$

19. Suppose X has an exponential distribution with $\lambda = \frac{1}{3}$. Find the following probabilities:

   (a) $P(3 \leq X \leq 5)$
   (b) $P(5 \leq X \leq 10)$
   (c) $P(2 \leq X \leq 1)$

20. Suppose the random variable X is best approximated by an exponential distribution with $\lambda = 8$. Find the mean and the variance of X.
21. Suppose the random variable Y is best approximated by an exponential distribution with $\lambda = 3$. Find the mean and the variance of Y.
22. Briefly compare the normal distribution discussed in Chap. 7 with the t distribution discussed in this chapter.
23. Find t_α for the following:

   (a) $\alpha = .05$ and $v = 10$
   (b) $\alpha = .025$ and $v = 4$
   (c) $\alpha = .10$ and $v = 7$

24. Find the value t_0 such that

   (a) $P(t \geq t_0) = .025$, where $v = 6$
   (b) $P(t \geq t_0) = .05$, where $v = 12$
   (c) $P(t \leq t_0) = .10$, where $v = 9$

25. Find the value t_0 such that

   (a) $P(t \leq t_0) = .10$, where $v = 25$
   (b) $P(t \geq t_0) = .025$, where $v = 14$
   (c) $P(t \leq t_0) = .01$, where $v = 17$

26. Find the following probabilities for the t distributions.

   (a) $P(t > 3.078)$ if $v = 1$
   (b) $P(t < 1.943)$ if $v = 6$
   (c) $P(t > 2.492)$ if $v = 24$

27. Find the following probabilities for the t distributions.

    (a) $P(t > 1.734)$ if $v = 18$
    (b) $P(t > 1.943)$ if $v = 6$
    (c) $P(t < 1.645)$ if $v = \infty$

28. Find the following $\chi^2_{\alpha,v}$ values.

    (a) $\alpha = .05$ and $v = 25$
    (b) $\alpha = .025$ and $v = 5$
    (c) $\alpha = .10$ and $v = 50$
    (d) $\alpha = .01$ and $v = 60$

29. Find the following $\chi^2_{\alpha,v}$ values.

    (a) $\alpha = .025$ and $v = 30$
    (b) $\alpha = .0l$ and $v = 70$
    (c) $\alpha = .10$ and $v = 10$
    (d) $\alpha = .01$ and $v = 20$

30. Find the following probabilities.

    (a) $P(\chi^2 > 10.8564)$ when $v = 24$
    (b) $P(\chi^2 < 10.8564)$ when $v = 24$
    (c) $P(\chi^2 < 48.7576)$ when $v = 70$
    (d) $P(\chi^2 > 59.1963)$ when $v = 90$

31. Find the following probabilities.

    (a) $P(\chi^2 \leq 3.84146)$ when $v = 1$
    (b) $P(\chi^2 \geq 15.9871)$ when $v = 10$
    (c) $P(\chi^2 < 140.169)$ when $v = 100$
    (d) $P(\chi^2 > 1.61031)$ when $v = 5$

32. Find the following $F_{v_1,v_2,\alpha}$ values.

    (a) $v_1 = 8$, $v_2 = 10$, and $\alpha = .01$
    (b) $v_1 = 3$, $v_2 = 11$, and $\alpha = .005$
    (c) $v_1 = 12$, $v_2 = 9$, and $\alpha = .05$
    (d) $v_1 = 24$, $v_2 = 19$, and $\alpha = .025$

33. Find the following $F_{v_1,v_2,\alpha}$ values.

    (a) $v_1 = 10$, $v_2 = 10$, and $\alpha = .05$
    (b) $v_1 = 15$, $v_2 = 3$, and $\alpha = .01$
    (c) $v_1 = 12$, $v_2 = 15$, and $\alpha = .025$
    (d) $v_1 = 20$, $v_2 = 10$, and $\alpha = .005$

34. Find the probabilities, given v_1 and v_2 as shown.

    (a) $v_1 = 1$ and $v_2 = 3$; $P(F > 17.44)$
    (b) $v_1 = 3$ and $v_2 = 1$; $P(F > 864.2)$

(c) $v_1 = 3$ and $v_2 = 1$; $P(F < 215.7)$
(d) $v_1 = 30$ and $v_2 = 12$; $P(F < 4.33)$

35. Using the MINITAB program, find the probabilities, given v_1 and v_2 as shown.

(a) $v_1 = 120$ and $v_2 = 120$; $P(F > 1.35)$
(b) $v_1 = 00$ and $v_2 = \infty$; $P(F > 1.00)$
(c) $v_1 = 6$ and $v_2 = 17$; $P(F < 3.28)$
(d) $v_1 = 3$ and $v_2 = 23$; $P(F > 4.76)$

36. Find the probability that an exponentially distributed random variable X with mean $1/\lambda = 8$ will take on the values:

(a) Between 2 and 7
(b) Less than 9
(c) Greater than 6
(d) Between 1 and 15

37. Suppose the lifetime of a television picture tube is distributed exponentially with a standard deviation of 1,400 h. Find the probability that the tube will last:

(a) More than 3,000 h
(b) Less than 1,000 h
(c) Between 1,000 and 2,000 h

38. Suppose the time you wait at a bank is exponentially distributed with mean $1/\lambda = 12$ min. What is the probability that you will wait between 10 and 20 min?
39. Suppose the length of time people wait at a fast-food restaurant is distributed exponentially with a mean of 1/7 min. Use MINITAB to answer the following questions.

(a) What percentage of people will be served within 4 min?
(b) What percentage of people will be served between 3 and 8 min after they arrive?
(c) What percentage of people will wait more than 9 min?

40. Suppose the length of time a student waits to register for courses is distributed exponentially with a mean of 1/15 min.

(a) What percentage of students will register within 10 min?
(b) What percentage of students will register after waiting between 10 and 20 min?
(c) What percentage of students will wait more than 20 min to register?

41. Suppose a random variable is distributed as an x^2 distribution with n degrees of freedom. Consider the probability $P(x^2 \leq 9)$. Explain the relationship between the probability and the degrees of freedom.
42. Suppose a random variable is distributed as Student's t distribution with $(n - 1)$ degrees of freedom. Consider the probability $P(t \geq .7)$. Explain the relationship between the probability and the degrees of freedom.

43. The incomes of families in a town are assumed to be uniformly distributed between $15,000 and $85,000. What is the probability that a randomly selected family will have an income above $40,000?
44. At an antiques auction, the winning bids were found to be uniformly distributed between $500 and $2,500. What is the probability that a winning bid was less than $1,000? What is the probability that a winning bid was between $750 and $1,500?
45. The manager of a department store notices that the amount of time a customer must wait before being helped is distributed uniformly between 1 and 4 min. Find the mean and variance of the time a customer must wait to be helped.
46. A quality control expert for the Healthy Time Cereal Company notices that in a 16-oz package of cereal, the amount in the box is uniformly distributed between 15.3 and 17.1 oz. Find the mean and standard deviation for the weight of this cereal in a package of cereal.
47. The shelf life of hearing aid batteries is found to be approximated by an exponential distribution with a mean of 1/12 day. What fraction of the batteries would be expected to have a shelf life greater than 9 days?
48. A computer programmer has decided to use the exponential distribution to evaluate the reliability of a computer program. After 10 programming errors were found, the time (measured in days) to find the next error was determined to be exponentially distributed with a $\lambda = .25$.

    (a) Graph this distribution.
    (b) Find the mean time required to find the 11th error.

49. Use the information given in question 48 to find the probability that it will take more than 5 days to find the 11th error. Find the probability that it will take between 3 and 10 days to find the 11th error.
50. An advertising executive believes that the length of time a television viewer can recall a commercial is distributed exponentially with a mean of .25 days. Find how long it will take for 75 % of the viewing audience to forget the commercial.
51. Use the information given in question 50 to find the proportion of viewers who will be able to recall the commercial after 7 days.
52. An investment advisor believes that the rate of return for Horizon Company's stock is uniformly distributed between 3 % and 12 %. Find the probability that the return will be greater than 5 %. Find the probability that the return will be between 6 % and 8 %.
53. The mean life of a computer's hard disk is found to be exponentially distributed with a mean of 12,000 h. Find the proportion of hard disks that will have a life greater than 20,000 h.
54. Suppose the life of a car battery is assumed to be uniformly distributed between 3.9 and 7.3 years. Find the mean and variance of the life of a car battery.
55. Use the information given in question 54 to find the probability that the life of the car battery will be greater than 5 years. Find the probability that the life of the battery will be between 4 and 6 years.

56. The chief financial officer at Venture Corporation believes that an investment in a new project will have a cash flow in year one that is uniformly distributed between \$1 million and \$10 million. What is the probability that the cash flow in year one will be greater than \$1.7 million?
57. A hospital collects data on the number of emergency room patients in during a certain period. It is estimated that in an hour, the average number of emergency room patients to arrive is 1.2. If the time between two consecutive arrivals of patients follows an exponential distribution, what is the probability that a patient will show up in the next hour?
58. The campus bus at Haverford College is scheduled to arrive at the business school at 8:00 a.m. Usually, the bus arrives at the bus stop during the interval 7:56–8:03. Assume that the arrival time follows a uniform distribution.

    (a) What is the probability that the bus arrives at the business school before 8:00?
    (b) What is the average arrival time?
    (c) What is the standard deviation of arrival time?

59. A gas station's owner found that about two cars come into the station every minute. If the arrival time follows an exponential distribution, what is the probability that the next car will arrive in 1.5 min?
60. A college professor gives a standardized test to her students every semester. She finds that the students' grades follow a uniform distribution with 100 points as the maximum and 65 points as the minimum.

    (a) Find the mean score.
    (b) Compute the standard deviation of the score.
    (c) If the passing grade is 70, what percentage of students will fail the course?

61. Suppose the weight of a football team is uniformly distributed with a minimum weight of 175 lb and a maximum weight of 285 lb.

    (a) Find the mean weight of the team.
    (b) Compute the standard deviation of the weight.
    (c) Find the percentage of players with a weight of less than 195 lb.

62. Briefly explain how the mean, standard deviation, coefficient of variation, and skewness can be used to analyze the returns of IBM and Boeing in Table 9.1.
63. A bank manager finds that about six customers enter the bank every 5 min. If the customer arrival time follows an exponential distribution, what is the probability that the next customer will arrive in 2 min?
64. Suppose the life of a steel-belted radial tire is uniformly distributed between 30,000 and 45,000 miles.

    (a) Find the mean tire life.
    (b) Find the standard deviation of tire life.
    (c) What percentage of these tires will have a life of more than 40,000 miles?

65. Briefly discuss the relationship among t, χ^2, and F distributions.

66. Given $v_1 = 5$ and $\alpha = .05$, find v_2 for the following F values.

    (a) 5.05
    (b) 3.33
    (c) 2.53

67. In their study, Vardeman and Ray (*Technometrics*, May 1985, pp. 145–150) found that the number of accidents per hour at an industrial plant is exponentially distributed with a mean $\lambda = .5$. Use the formula $f(t) = \lambda e^{-\lambda t}$ to determine each of the following.

    (a) $f(1)$
    (b) $f(4)$
    (c) $E(t)$

68. Suppose there is a sample of 30 items drawn from a normal population. Find the probability that the sample variance exceeds 36.6869.
69. Suppose there are two independent normal populations with population variances $\sigma^2{}_1 = 4.5$ and $\sigma^2{}_2 = 2.5$, respectively. Two random samples of sizes $s^2{}_1$ and $s^2{}_2$, respectively, are drawn from the two normal populations with sample variances $s^2{}_1$ and $s^2{}_2$, respectively.

    (a) What is the probability that the ratio $s^2{}_1/s^2{}_2$ is greater than 4.230?
    (b) What is the probability that the ratio $s^2{}_1/s^2{}_2$ is greater than 6.066?
    (c) What is the probability that the ratio $s^2{}_1/s^2{}_2$ is less than 0.5263?

70. A random sample of size 7 is drawn from a population with population variance $\sigma^2 = 2.5$.

    (a) Determine the probability that the variance of the sample is greater than 7.008.
    (b) Determine the probability that the population mean is less than 0.3634.

71. The following random sample is taken from a normal population.

94	72	43	69	28	63	93	54	77	58

    (a) If the population man is $\mu = 60$, what is t statistics for the sample?
    (b) If the population man is $\mu = 55$, what is t statistics for the sample?
    (c) What is the degree of freedom of the t statistics in (a)?

Project II: Project for Probability and Important Distributions

1. Use rates of return data presented in Table 2.4 to do the following:

    (a) Use either MINITAB or Microsoft Excel to calculate:

        1. Mean
        2. Standard deviation

(continued)

Project II: (continued)

3. Coefficient of variation
4. Skewness
5. Kurtosis

(b) Analyze the statistical results of (a).
(c) Use both the standard deviation for JNJ and Merck calculated in (a) and the following information to calculate the call option and put option values for JNJ and Merck:

$$S = \$50 \quad X = 45 \quad r = 6\% \quad T = .6$$

2. Use MINITAB and the statistical estimates for JNJ and Merck obtained in (a) to calculate the mean and the variance of a portfolio with the following weights:

1. $w_1 = .4$ and $w_2 = .6$
2. $w_1 = .2$ and $w_2 = .8$
3. $w_1 = .3$ and $w_2 = .7$
4. $w_1 = .1$ and $w_2 = .9$

3. Download monthly adjusted close price data of JNJ from Yahoo Finance during the period from January 2005 to current month:

(a) Calculate monthly rates of return of JNJ.
(b) Redo 1a–c.

Appendix 1: Derivation of the Mean and Variance for a Uniform Distribution

On the basis of the definitions of $E(X)$ and $E(X^2)$ for a continuous variable given in Appendix 1 of Chap. 7, we can derive the mean and the variance of a uniform distribution as follows. First, substituting Eq. 9.1 into Eq. 7.22, we get

$$\begin{aligned} E(X) &= \int_a^b xf(x)dx = \int_a^b \frac{x}{b-a}dx \\ &= \frac{1}{b-a} \cdot \frac{x^2}{2}\bigg|_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2} \end{aligned} \tag{9.32}$$

Then, substituting Eq. 9.1 into Eq. 7.25 yields

$$E(X^2) = \int_a^b x^2 f(x)dx = \frac{1}{b-a}\int_a^b x^2 dx = \frac{1}{b-a}\cdot\frac{x^3}{3}\bigg|_a^b$$
$$= \frac{b^3-a^3}{3(b-a)} = \frac{(b-a)(b^2+ab+a^2)}{3(b-a)} = \frac{b^2+ab+a^2}{3} \tag{9.33}$$

Finally, substituting Eqs. 9.32 and 9.33 into the definition of variance given in Eq. 7.24, we obtain

$$\sigma_X^2 = E(X^2) - [E(X)]^2 = \frac{b^2+ab+a^2}{3} - \left(\frac{a+b}{2}\right)^2$$
$$= \frac{4b^2+4ab+4a^2-3b^2-6ab-3a^2}{12} = \frac{(b-a)^2}{12} \tag{9.34}$$

This implies that $\sigma_X = (b-a)/\sqrt{12}$.

The following example shows how the formulas for both the mean and the variance of a continuous variable, as discussed in Appendix 1 of Chap. 7, can be applied for a uniform distribution.

Example 9.4 Calculating the Mean and Variance of a Uniform Distribution. Let us look at an example of a continuous random variable in terms of the uniform distribution. Consider the density function of Eq. 9.35 as depicted in Fig. 9.12:

$$f(x) = \begin{cases} 1.55 - .06x & \text{if } 20 \le x \le 25 \\ 0 & \text{otherwise} \end{cases} \tag{9.35}$$

For every value of x between 20 and 25, we get $f(x) > 0$, and for every x value outside of this range, we have $f(x) = 0$. Therefore, for every x, we have $f(x) \ge 0$. Furthermore, the area under the curve equals 1:

$$\int_{20}^{25}(1.55 - .06x)dx = 1.55x\bigg|_{20}^{25} - \frac{.06x^2}{2}\bigg|_{20}^{25} = 1$$

This confirms that $f(x)$ is a density function. Now, let us calculate the expected value and variance of X:

$$E(X) = \int_{20}^{25} xf(x)dx = \int_{20}^{25} x(1.55 - .06x)dx$$
$$= \frac{1.55x^2}{2}\bigg|_{20}^{25} - \frac{.06x^3}{3}\bigg|_{20}^{25}$$
$$= 174.375 - 152.5 = 21.875$$

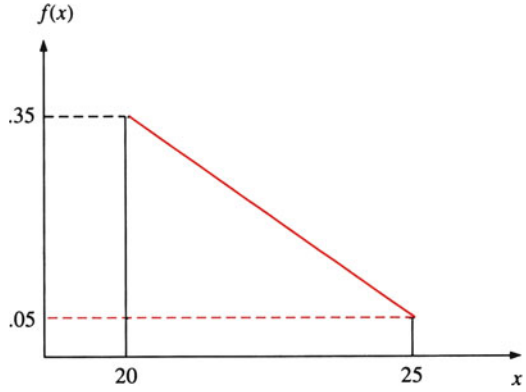


Fig. 9.12 The density function $f(x)$

Next, let us calculate $E(X^2)$:

$$\begin{aligned} E(X^2) &= \int_{20}^{25} x^2(1.55 - .06x)dx \\ &= \frac{1.55x^3}{3}\bigg|_{20}^{25} - \frac{.06x^4}{4}\bigg|_{20}^{25} \\ &= 3939.583 - 3459.375 = 480.208 \end{aligned}$$

From this result, we obtain

$$V(X) = E(X^2) - (EX)^2 = 480.208 - (21.875)^2 = 1.692$$

To find the probability, such as $P(22 \le X \le 24.5)$, we calculate

$$\begin{aligned} P(22 \le X \le 24.5) &= \int_{22}^{24.5} f(x)dx = \int_{22}^{24.5} (1.55 - .06x) \\ &= 1.55x\bigg|_{22}^{24.5} - \frac{.06x^2}{2}\bigg|_{22}^{24.5} \\ &= 3.875 - 3.4875 = .3875 \end{aligned}$$

Appendix 2: Derivation of the Exponential Density Function

The cumulative distribution function (CDF) for the first event to occur in time interval t can be written as

$$\begin{aligned} P(T \le t) &= P(\text{wait until next arrival} \le t) \\ &= P(\text{at least one arrival in time } t) \\ &= 1 - P(\text{non-arrival in time } t) \end{aligned} \tag{9.36}$$

where T is the random variable of which t is a specific value. P(non-arrival in time t) can be obtained by letting $x = 0$ in the Poisson function as defined in Eq. 6.16. We obtain P(non-arrival in time interval [0, t]) as

$$f(0) = P(T \geq t) = \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = e^{-\lambda t} \text{ for } t \geq 0$$
$$= 0 \quad \text{for } t < 0 \tag{9.37}$$

where λ denotes the mean rate at which events occur over time. Substituting Eq. 9.37 into Eq. 9.36, we obtain the CDF as

$$F(t) = P(T \leq t) = 1 - e^{-\lambda t} \tag{9.38}$$

If we differentiate $F(t)$ with respect to t, we obtain the PDF as[11]

$$f(t) = \lambda e^{-\lambda t}, \quad t \geq 0$$
$$= 0, \quad t < 0 \tag{9.39}$$

The probability that the waiting time lies between a and b is

$$P(a)) = \int_a^b \lambda e^{-\lambda t} \tag{9.40}$$

From the definition of $E(t)$ in Appendix 1 of Chap. 7, we obtain

$$E(T) = \int_{-\infty}^{\infty} tf(t)dt = \lambda \int_0^{\infty} te^{-\lambda t}dt$$

The integral can be evaluated by parts. Let $U = t$ and $dv = e^{-\lambda t}\, dt$, so $dU = dt$ and $v = -e^{-\lambda t}/\lambda$. Then

$$E(T) = \lambda\left[\left(-te^{-\lambda t}/\lambda\right)_0^{\infty} + \frac{1}{\lambda}\int_0^{\infty} e^{-\lambda t}dt\right]$$
$$= \lambda\left[(-0+0) + \frac{1}{\lambda^2}(-0+1)\right] = \frac{1}{\lambda}$$

[11] This is because

$$\frac{dF(t)}{dt} = \frac{d\left(1 - e^{-\lambda t}\right)}{dt} = 0 - \left[\frac{d(-\lambda t)}{dt}\right] e^{-\lambda t} = \lambda e^{-\lambda t}$$

Similarly, we can prove that

$$\mathrm{Var}(T) = \frac{1}{\lambda^2} \tag{9.41}$$

This appendix shows how a mean value formula of a continuous variable, which was discussed in Appendix 1 of Chap. 7, can be applied to an exponential distribution.

Example 9.5 The Average Time Required to Find the Next Computer Program Error. In finding and correcting errors in a computer program (debugging) and determining the program's reliability, Schick and others have noted the importance of the distribution of the time until the next program error is found. The cumulative exponential probability function of Eq. 9.37 is most useful in analyzing this problem.

By using the computer debugging data supplied by the US Navy, Schick (1974, *Decision Sciences*, Vol. 5, pp. 529–544) estimated the value of λ. After 26 of 31 program errors were found, Schick estimated λ to be .042. Accordingly, $1/\lambda = 23.8$ days. This means that the average time it would take to find 1 of the remaining errors (the 27th error) would be about 24 days. From this information, we can estimate, for example, that the probability of taking 50 or more days to find the next error is

$$P(T \geq 50) = e^{-(.042)(50)} = e^{-2.1} = .1125.$$

The second equality is obtained by using Table A7 in Appendix A.

Example 9.6 The Probability of Truck Arrivals. Rutgers Trucking Company had 15,600 trucks to unload at the receiving warehouse during the last calendar year. The warehouse was open from 8 a.m. to 8 p.m. each weekday. There was no noticeable pattern of truck arrivals each day. It is known that approximately five trucks arrived to unload cargo each hour. What is the probability that on September 20, 1991, the first truck arrived between 8:15 and 8:30 a.m.?

To use exponential distribution to solve this problem, we first use a time interval of 15 min (8:15–8:30) for which $\lambda = (5/60)(15) = 1.25$.

Substituting $\lambda = 1.25$, $a = 1$, and $b = 2$ into Eq. 9.40,[12] we obtain the probability that the first truck arrived between 8:15 and 8:30 a.m.:

$$\begin{aligned} P(1<T<2) &= \int_1^2 e^{-1.25t}(1.25\mathrm{d}t) = -e^{-1.25t}\Big|_1^2 \\ &= -e^{-2.5} + e^{-1.25} = .2 \end{aligned}$$

[12] We regard 15 min as 1 time unit that can be expressed as a time interval between $a = 1$ and $b = 2$.

Appendix 3: The Relationship Between the Moment About the Origin and the Moment About the Mean

Let $k = 1$ in Eq. 9.22. Then

$$\mu_1 = E(X - \mu'_1) = E(X) - \mu'_1 = 0$$

This implies that the first moment about the population mean is zero.
Alternatively, if we let $k = 2$ in Eq. 9.22 and let $\mu_1 = \mu_1$, we obtain

$$\begin{aligned} \mu_2 &= E(X - \mu'_1)^2 = E\left(X^2 - 2X\mu'_1 + \mu'_1\right) \\ &= E\left(X^2\right) - 2\mu'_1 E(X) + \mu'^2_1 = \mu'_2 - \mu'^2_1 \end{aligned} \tag{9.42}$$

where μ'_2 and μ'_1 are second and first moments, respectively. Equation 9.42 is identical to Eq. 7.24 in Appendix 1 of Chap. 7. It is a shortcut formula to calculate variance.

Now, if we *let* $k = 3$ in Eq. 9.23 and substitute $\mu_1 = \mu'_1$, we obtain

$$\begin{aligned} \mu_3 =& E\left(X - \mu'_1\right)^3 = E\left(X^3 - 3X^2\mu'_1 + 3X\mu'^2_1 - \mu'^3_1\right) \\ =& E(X^3) - 3\mu'_1 E(X^2) + 3\mu'^2_1 E(X) - \mu'^3_1 \\ =& \mu'_3 - 3\mu'_1\mu'_2 + 2\mu'^3_1 \end{aligned} \tag{9.43}$$

where μ'_1 and μ'_2 are defined in Eq. 9.42 and μ'_3 is the third moment about the origin.

Finally, letting $k = 4$ in Eq. 9.22 and substituting $\mu_1 = \mu_1$, we obtain

$$\begin{aligned} \mu_4 &= E\left(X - \mu'_1\right)^4 \\ &= E\left(X^4 - 4X^3\mu'_1 + 6X^2\mu'^2_1 - 4E(X)\mu'^3_1 + \mu'^4_1\right) \\ &= E\left(X^4\right) - 4E\left(X^3\right)\mu'_1 + 6E\left(X^2\right)\mu'^2_1 - 4E(X)\mu'^3_1 + \mu'^4_1 \\ &= \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2\mu'^2_1 - 3\mu'^4_1 \end{aligned} \tag{9.44}$$

where μ'_1, μ'_2, and μ'_3 have been defined in Eq. 9.43 and μ'_4 is the fourth moment about the origin.

In Appendix 4, Eqs. 9.42, 9.43, and 9.44 will be used to derive variance, skewness, and kurtosis of the lognormal distribution.

Appendix 4: Derivations of Mean, Variance, Skewness, and Kurtosis for the Lognormal Distribution

Following Aitchison and Brown (1963), we express the moments about the origin for the lognormal distribution as

$$\mu'_k = e^{k\mu+1/2k^2\sigma^2}, \quad k = 1,\ 2, \cdots \tag{9.45}$$

In accordance with definitions given in Appendix 3, the mean, variance skewness, and kurtosis of a lognormal distribution can be derived as follows.

Mean

Substituting $k = 1$ into Eq. 9.45 yields

$$\mu'_1 = e^{\mu+1/2\sigma^2}$$

This is Eq. 7.6.

Variance

Substituting $\mu'_2 = e^{2\mu+2\sigma^2}$ and $\mu'_1 = e^{\mu+1/2\sigma^2}$ into Eq. 9.42 in Appendix 3, we obtain

$$e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2} = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right)$$

This is Eq. 7.7.

Skewness

Substituting μ'_1, μ'_2, and $\mu'_3 = e^{3\mu+9/2\sigma^2}$ into Eq. 9.43 gives

$$\begin{aligned}\mu_3 =& (\mu'_1)^3\left[e^{3\sigma^2} - 3e^{\sigma^2} + 2\right]\\ =& (\mu'_1)^3\left[\left(e^{3\sigma^2} - 3e^{2\sigma^2} + 3e^{\sigma^2} - 1\right) + 3\left(e^{2\sigma^2} - 2e^{\sigma^2} + 1\right)\right]\\ =& (\mu'_1)^3\left(\eta^6 + 3\eta^4\right)\end{aligned}$$

where $\eta^2 = e^{\sigma 2} - 1$. This is Eq. 9.28.

Kurtosis

Substituting μ'_1, μ'_2, μ'_3, and $\mu'_4 = e^{3\mu+8\sigma^2}$ into Eq. 9.44, we get

$$\mu_4 = (\mu'_1)^4\left[e^{6\sigma^2} - 4e^{3\sigma^2} + 6e^{\sigma^2} - 3\right]$$

By considerable mathematical rearrangement of terms, it can be shown that

$$\mu_4 = (\mu'_1)^4\left[\eta^{12} + 6\eta^{10} + 15\eta^8 + 16\eta^6 + 3\eta^4\right]$$

where $\eta^2 = e^{\sigma^2} - 1$. This is Eq. 9.29.

Appendix 5: Noncentral χ^2 and the Option Pricing Model

From Eq. 9.6, we know that $Y = \sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma_x}\right)^2$ is distributed as χ^2 with n degree of freedom. This is a central χ^2 distribution. It can be shown that $Y' = \sum_{i=1}^{n} X_i^2$ is distributed as noncentral χ^2 with n degree of freedom and a noncentral parameter[13]

$$\lambda = \frac{1}{2}\sum \mu_i^2.$$

If $\mu = 0$, the distribution of Y' reduces to the central χ^2 distribution.

The option pricing model defined in Appendix 2 of Chap. 7 assumed that the variance of stock rate of return (σ^2) is constant. If the variance of stock rate of return is a function of stock price per share, $\sigma^2 S^{\beta-2}$, then the option pricing model defined in Eq. 7.35 can be generalized as[14]

$$C = S\left[1 - \chi^2\left(2n; 2 + \frac{2}{2-\beta}, 2m\right)\right] - Xe^{-r(T-t)}\left[\chi^2\left(2m; \frac{2}{\beta-2}, 2n\right)\right] \quad (\beta<2) \qquad (9.46)$$

[13] See Robert V.H. Allen T.C.: Introduction to Mathematical Statistics 4th Edition, pp. 288–290. Macmillan, New York, (1978)

[14] The derivation of this formula can be found in Mark S.: Computing the constant elasticity of variance option pricing formula. J. Finance. **44**, 211–220 (1989)

$$C = S\left[1 - \chi^2\left(2m; \frac{2}{2-\beta}, 2n\right)\right] - Xe^{-r(T-t)}\left[\chi^2\left(2n; 2 + \frac{2}{\beta-2}, 2m\right)\right] \quad ((\beta<2) \quad (9.47)$$

where T = time of expiration of option, t = current time, and r = risk-free rate. $\chi^2\,(W, V, \lambda)$ is the cumulative noncentral chi-square distribution function with W, V, and, λ being the upper limit of the integral, degree of freedom, and noncentrality, respectively. In addition m, n, and K can be defined as

$$m = KS^{2-\beta}e^{(2-\beta)\mu(T-t)}$$

$$n = KS^{2-\beta}$$

$$K = \frac{2\mu}{\sigma^2(2-\beta)(e^{(2-\beta)\mu(T-t)} - 1)} \quad (9.48)$$

Now, we discuss three possible special cases associated with Eqs. 9.46 and 9.47.

(a) If $\beta = 2$, both m and n approach infinity. Then it can be shown that both Eqs. 9.46 and 9.47 reduce to the well-known Black–Scholes formula as defined in Appendices 2 and 3 of Chap. 7.
(b) If $\beta = 1$, it can be shown that Eqs. 9.46 and 9.47 reduce to

$$C = \left(S - Xe^{-r(T-t)}\right)N(y_1) + \left(S + Xe^{-r(T-t)}\right)N(y_2) + v[n(y_1) - n(y_2)] \quad (9.49)$$

where

$$v = \sigma\sqrt{\frac{1 - e^{-2r(T-t)}}{2r}}$$

$$y_1 = \frac{S - Xe^{-r(T-t)}}{v}$$

$$y_2 = \frac{-S - Xe^{-r(T-t)}}{v}$$

$N(y_1)$ and $N(y_2)$ = cumulative standardized normal distribution function in terms of y_1 and y_2, respectively.
$n(y_1)$ and $n(y_2)$ = standardized normal density function in terms of y_1 and y_2, respectively.

(c) If $\beta = 0$, it can be shown that Eqs. 9.46 and 9.47 reduce to

$$C = SN[q(4)] - Xe^{-r(T-t)}N[q(0)] \quad (9.50)$$

where

$$q(w) = \frac{1 + h(h-1)\left(\frac{w+2y}{(w+y)^2}\right) - h(h-1)(2-h(1-3h)\left(\frac{(w+2y)^2}{2(w+y)^4}\right) - \left(\frac{z}{(w+y)}\right)^h}{\left\{2h^2\left(\frac{w+2y}{(w+y)^2}\right)(1-(1-h)(1-3h)\left(\frac{w+2y}{(w+y)^2}\right)\right\}^{\frac{1}{2}}}$$

$$h(w) = 1 - \frac{2}{3}(w+y)(w+3y)(w+2y)^{-2}$$

$$y = \frac{4rS}{\sigma^2(1 - e^{-r(T-t)})} \quad \text{and} \quad z = \frac{4rX}{\sigma^2(e^{-r(T-t)} - 1)}$$

The elasticity of variance ($\sigma^2 S^{\beta-2}$) with respect to stock price per share S is

$$\eta_s = \left[\frac{\partial\left(\sigma^2 S^{\beta-2}\right)}{\partial S}\right]\left[\frac{S}{\sigma^2 S^{\beta-2}}\right] = \left[\frac{(\beta-2)\sigma^2 S^{\beta-2})}{S}\right]\left[\frac{S}{\sigma^2 S^{\beta-2}}\right] = \beta - 2 \qquad (9.51)$$

This implies that the option pricing model defined in Eqs. 9.46 and 9.47 is a constant elasticity of variance (CEV) type of OPM.

The CEV type of option pricing model can be reduced to the following special models[15]:

(a) $\beta = 2$, Eqs. 9.46 and 9.47 reduce to the Black–Scholes model.
(b) $\beta = 1$, Eqs. 9.46 and 9.47 reduce to the absolute model as defined in Eq. 9.49.
(c) $\beta = 0$, Eqs. 9.46 and 9.47 reduce to the square root model as defined in Eq. 9.50.

From Appendix 2 of Chap. 6, Appendices 2 and 3 of Chap. 7 and this appendix, we can conclude that the binomial, normal, lognormal, and noncentral χ^2 distributions are basic statistical distributions needed for understanding alternative option pricing models.

[15] See Beckers S.: The constant elasticity of variance model and its implications for option pricing. J. Finance. **35**, 661–673 (1980)